Master of Technology

Computational Intelligence II

Fuzzy Inference

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Reference for day 4 and 5

Reference book

- Michael Negnevitsky, **Artificial Intelligence: A Guide to Intelligent Systems**, Pearson Education, 2011. (tutorial level)
- Timothy J. Ross, **Fuzzy Logic with Engineering Applications**, Wiley, 2010. (comprehensive)

Course

- ECE/CS/ME 539, Introduction to Artificial Neural Network and Fuzzy Systems (Year 2018), available at website http://homepages.cae.wisc.edu/~ece539/
- EP33FLO, Fuzzy Logic (Year 2018), available at website http://cmp.felk.cvut.cz/~navara/fl/

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Toy example: A fuzzy rule

Antecedent
(IF statement/premise)

IF	Speed is fast
THEN	Stopping_distance is long

Consequent
(THEN statement/premise)

Linguistic variable

Linguistic value

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Other examples of fuzzy rule

Multiple antecedents (inputs)

IF	Service is excellent
	OR
	Food is delicious
THEN	Tip is generous

Multiple consequents (outputs)

IF	Temperature is hot
THEN	Hot_water is reduced
	AND
	Cold_water is increased

Note: We focus on **single** consequent (output) in this course.

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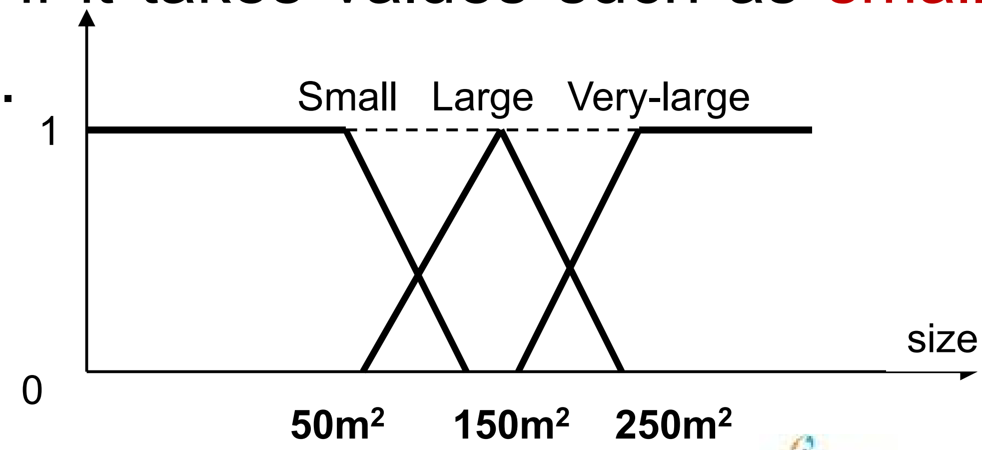
Linguistic variable and value

- Linguistic variable
  - » A variable that takes values which are not numbers but words or sentences in natural or artificial language.
- Example:
  - » Speed is a linguistic variable if it takes values such as slow, fast, very fast, and so on
  - » Size is a linguistic variable if it takes values such as small, large, very large, and so on.



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Linguistic variable and value

- Hedge modifies the shape of fuzzy sets.
  - » All-purpose modifiers, such as very, quite or extremely.
  - » Truth-values, such as quite true or mostly false.
  - » Probabilities, such as likely or not very likely.
  - » Quantifiers, such as most, several or few.
  - » Possibilities, such as almost impossible or quite possible.

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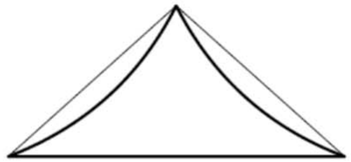
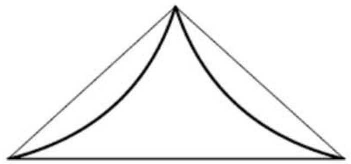
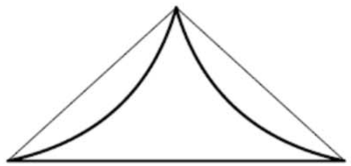
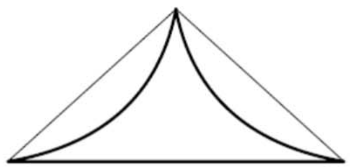
Linguistic variable and value

Examples of hedges in fuzzy logic. Note: there are many other different formulations.

Hedge	**Mathematical expression**	**Graphical representation**
A little	$[\mu_A(x)]^{1.3}$	
Slightly	$[\mu_A(x)]^{1.7}$	
Very	$[\mu_A(x)]^{2}$	
Extremely	$[\mu_A(x)]^{3}$	

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Fuzzy IF-THEN rule

- A general form of a fuzzy if-then rule (also known as *fuzzy rule*, *fuzzy implication*, or *fuzzy conditional statement*) (multi-input-single-output):

  R^i: IF x is A_i, …, y is B_i, THEN z = C_i

  where,

  » x, …, y, and z are linguistic variables

  » A_i, …, B_i, and C_i are the linguistic values of x, …, y, and z, and defined by fuzzy subsets in the universe of discourses X, …, Y, and Z, respectively.

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Fuzzy IF-THEN rule

- Example:

  IF the speed is high, THEN apply the brake a little

"the speed is high"	—	antecedent
"apply the brake a little"	—	consequent
speed and apply-brake	—	linguistic variables
high and a little	—	linguistic values

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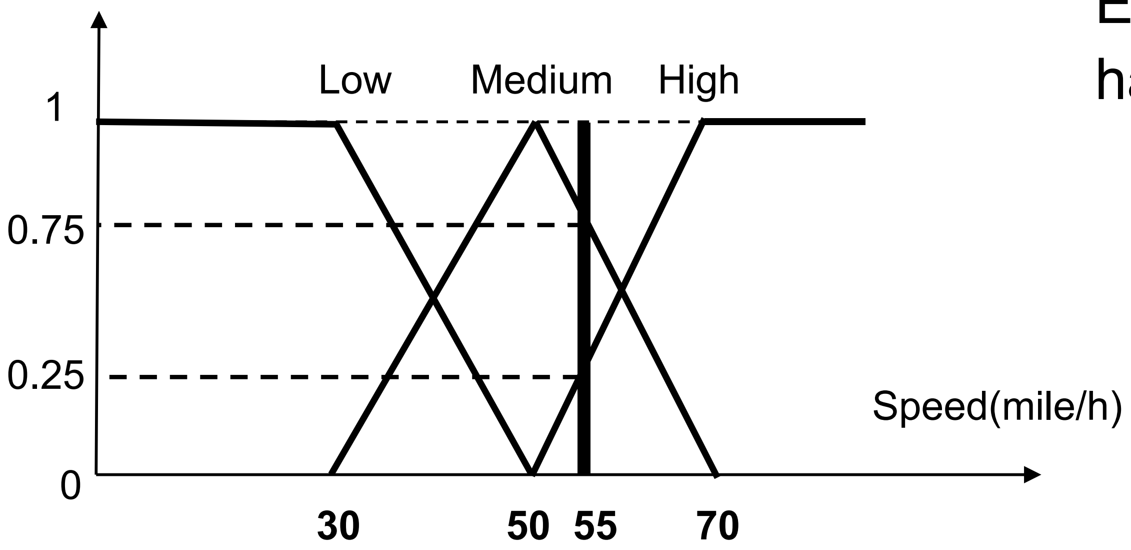
Fuzzy IF-THEN rule

- The speed 55mile/h can be represented by a fuzzy singleton and then matched with fuzzy subsets low, medium, and high.



Example: The fuzzy singleton (55mile/h) has matching degrees

- » 0 with fuzzy subset *low*
  - ✓ $\mu_{low}(55) = 0$
- » 0.75 with fuzzy subset *medium*
  - ✓ $\mu_{medium}(55) = 0.75$
- » 0.25 with fuzzy subset *high*
  - ✓ $\mu_{high}(55) = 0.25$

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Classical IF-THEN rule and Fuzzy IF-THEN rule

A classical IF-THEN rule uses binary logic

Rule: 1	Rule: 2
IF speed is > 100	IF speed is < 40
THEN stopping_distance is long	THEN stopping_distance is short

The variable *speed* can have any numerical value between 0 and 220 km/h, but the linguistic variable *stopping_distance* can take either value *long* or *short*.
In other words, classical rules are expressed in the black-and-white language of Boolean logic.

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Classical IF-THEN rule and Fuzzy IF-THEN rule

Rule: 1	Rule: 2
IF speed is fast	IF speed is slow
THEN stopping_distance is long	THEN stopping_distance is short

In fuzzy IF-THEN rule, the linguistic variable *speed* also has the range (the universe of discourse) between 0 and 220 km/h, but this range includes fuzzy sets, such as *slow*, *medium* and *fast*. The universe of discourse of the linguistic variable *stopping_distance* can be between 0 and 300 m and may include such fuzzy sets as *short*, *medium* and *long*.

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Fuzzy logic: Proposition

- User defines a set of propositional symbols, like P and Q.
  - » P means "It is hot"
  - » Q means "It is humid"
  - » R means "It is raining"
- Example
  - » $(P \wedge Q) \rightarrow R$
    "If it is hot and humid, then it is raining"
  - » $Q \rightarrow P$
    "If it is humid, then it is hot"
  - » Q
    "It is humid."

And	$\wedge$
Or	$\vee$
Not	$\neg$
Imply	$\rightarrow$
Equivalent	$\leftrightarrow$

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Fuzzy logic

- A logical proposition P
- Truth of proposition T(P)
- In binary (two-valued) logic $T(P) \in \{T, F\}$
  - » *T* and *F* are true and false, respectively
  - » They are usually represented by 1 and 0 (positive logic) or 0 and 1 (negative logic)
  - » P1 = “Mr. A is a man” T(P1) = *true*
  - » P2 = “Mrs. B is a teacher” T(P2) = *false*

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Fuzzy logic: Logical operation

- Suppose T(A) and T(B) are numerical truth values of propositions A and B, respectively T(A), T(B) $\in$ [0, 1].

Negation T(NOT A) = 1 – T(A)

Conjunction T(A AND B) = T(A) $\wedge$ T(B) = min{T(A), T(B)}

Disjunction T(A OR B) = T(A) $\vee$ T(B) = max{T(A), T(B)}

Implication T(A$\rightarrow$B) = T(NOT A) $\vee$ T(B) = max{1–T(A), T(B)}

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Fuzzy logic: Logical operation

- Example:

  T(“speed is high”) = 0.8, T(“gasoline level is low”) = 0.7

  T(“speed is high” AND “gasoline level is low”) = min(0.8, 0.7) = 0.7

  T(“weather is hot”) = 0.8,T(“luggage is heavy ”) = 0.5

  T(“weather is hot” OR “luggage is heavy”) = max(0.8, 0.5) = 0.8

  T(“John is tall”) = 0.7

  T(“John is NOT tall”) = 1 – 0.7 = 0.3

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Fuzzy reasoning

- Fuzzy reasoning also known as *approximate reasoning*, is an inference procedure that drives conclusion from a set of fuzzy if-then rules and known facts. Linguistic values of a linguistic variable are fuzzy sets in the universe of discourse.
- Challenge: R: IF *x* is *small* THEN *y* is *large*
  - » Input: *x* is relatively-small
  - » Output: *y* = ?
- The real input does not match the antecedent/premise of the rule precisely, so the output of the reasoning will be obtained on an "approximate" basis.

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Fuzzy reasoning

- Modus ponens (in classical two-valued logic)

$$\begin{array}{ll} \text{rule:} & P \rightarrow Q \\ \text{fact:} & \underline{P \quad\quad\quad} \\ & Q \end{array}$$

where P and Q are binary propositions, $T(P), T(Q) \in \{0, 1\}$.

- Generalized modus ponens

$$\begin{array}{l} P \rightarrow Q \\ \underline{P' \quad\quad\quad} \\ Q' \end{array}$$

where P, P', Q and Q' are fuzzy propositions.
P and P', Q and Q' are *approximately matching*

* With the given rule $P \rightarrow Q$ and given fact P', there are **different** patterns of inference to get an approximate conclusion Q'

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Fuzzy inference method

- Compositional inference (<u>focus in this course</u>)
  - » An *implication operator* to build the relation given by the rule
  - » A *T-norm operator* to compose the input with the relation
  - » A *maximum operator* to combine the support provided by each of the rules
- Compatibility-modification inference
  - » Determines the degree to which the antecedent of a rule is satisfied (a similarity measure matching function is needed)
  - » This measure of satisfaction is then used to modify the rule's consequent.

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Compositional inference

Question: Given	***Rule*** (premise 1)	**X is A → Y is B**
	Fact (premise 2)	**X is A**
Calculate	What is **Y?**	

- Solution:

$$\frac{\begin{array}{c} X \text{ is } A \\ (X, Y) \text{ is } R \end{array}}{Y \text{ is } A \circ R}$$

where $A \circ R$ indicates the max-min composition of a fuzzy set A and a fuzzy relation R.

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Compositional inference

Question: Given	***Rule*** (premise 1)	**X is A → Y is B**
	Fact (premise 2)	**X is A'**
Calculate	What is **Y?**	

- Solution:

$$\frac{\begin{array}{c} X \text{ is } A' \\ (X, Y) \text{ is } R \end{array}}{Y \text{ is } A' \circ R}$$

where $A' \circ R$ indicates the max-min composition of a fuzzy set A' and a fuzzy relation R.

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Example

- Given two fuzzy sets

  A = small = {1/1, 0.4/2, 0/3}

  B = large = {0/1, 0.4/2, 1/3}

which are defined on the universes of discourse U=V={1,2,3}
and

  fuzzy rule: IF A THEN B

and

  A' = more or less small = {1/1, 0.4/2, 0.2/3}

Question: What is B' ?

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Example

The fuzzy implication "IF *A* THEN *B*" is interpreted as $A \times B + \neg A \times V$, where "×" means "min", "+" means "max", and "¬" means "1-".

$$\mu_R(x,y) = \max\left[\min(\mu_A(x), \mu_B(y)), \min(1-\mu_A(x), 1)\right]$$

Build the fuzzy implication via relationship $R_{A\rightarrow B}$

$$R_{A\rightarrow B} = \begin{bmatrix} 0 & 0.4 & 1 \\ 0.6 & 0.6 & 0.6 \\ 1 & 1 & 1 \end{bmatrix}$$

Recall

$A = [1, 0.4, 0]$

$B = [0, 0.4, 1]$

$\neg A = [0, 0.6, 1]$

$V=[1, 1, 1]$

$$0.4 = \max\left[\min(1,0.4), \min(1-1,1)\right] = \max[0.4,0]$$

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Example

Apply the compositional inference method to calculate

$$B' = A' \circ R_{A \to B} = [1 \quad 0.4 \quad 0.2] \circ \begin{bmatrix} 0 & 0.4 & 1 \\ 0.6 & 0.6 & 0.6 \\ 1 & 1 & 1 \end{bmatrix} = [0.4 \quad 0.4 \quad 1]$$

$$0.4 = \max\left[\min(1,0)\,,\min(0.4,0.6),\min(0.2,1)\right] = \max[0,0.4,0.2]$$

where "$\circ$" is the compositional operator using *max-min* rule.

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Exercise

- Example: Evaluate a new invention to determine its commercial potential. Our metrics are the "uniqueness" of the invention, denoted by a universe of novelty scales, X = {1, 2, 3, 4}, and the "market size" of the invention, denoted on a universe of scaled market sizes, Y = {1, 2, 3, 4, 5, 6}. A new invention has just received scores of "medium uniqueness," denoted by fuzzy set A and "medium market size," denoted fuzzy set B. The implication is defined as, IF A THEN B. **Calculate** what market size would be associated with a uniqueness score of "almost high uniqueness" A'.

$$A = \text{medium uniqueness}=\{0.6/2, 1/3, 0.2/4\}$$

$$B = \text{medium market size}=\{0.4/2, 1/3, 0.8/4, 0.3/5\}$$

$$A' = \text{almost high uniqueness}=\{0.5/1, 1/2, 0.3/3, 0/4\}$$

Your answer is ________.

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Master of Technology

Computational Intelligence II

Build a fuzzy system

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Objectives

- To introduce the major steps of fuzzy inference in fuzzy systems
- To understand the key tasks in modeling and building fuzzy systems through a case study
- To briefly discuss particular aspects of uncertainty handling in fuzzy expert system

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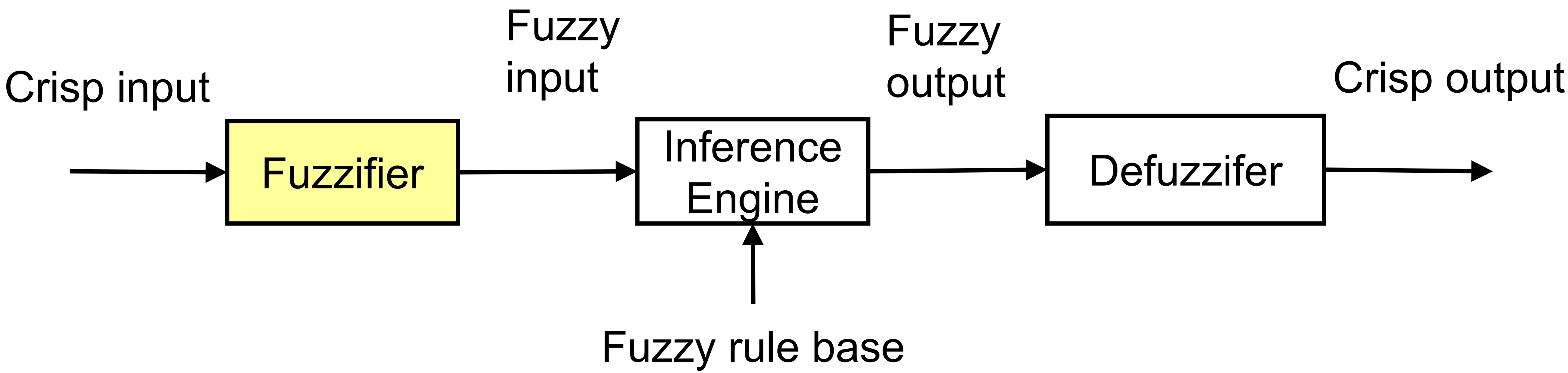
Fuzzy system



Converts the crisp input to a linguistic variable using the membership functions stored in the fuzzy knowledge base.

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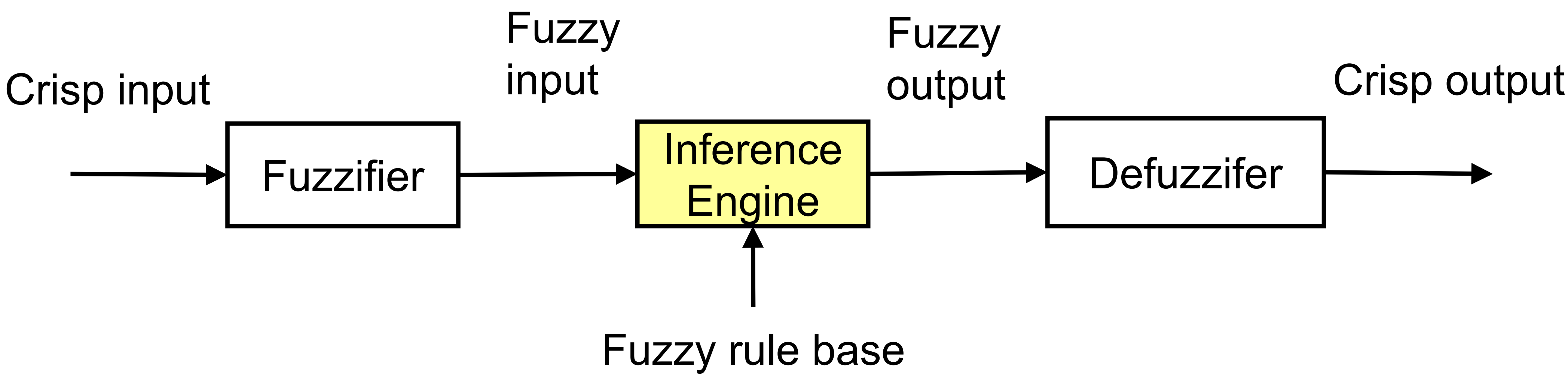
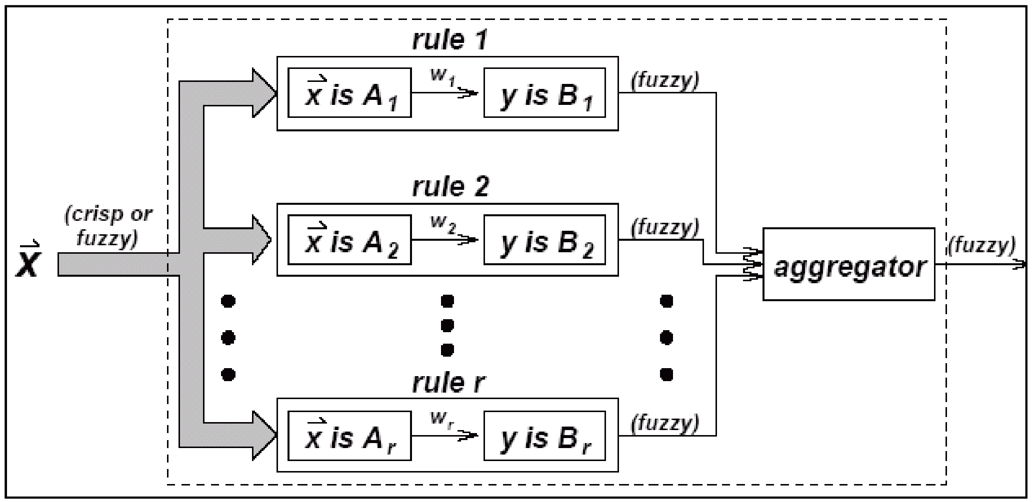
Fuzzy system



Using If-Then type fuzzy rules converts the fuzzy input to the fuzzy output.



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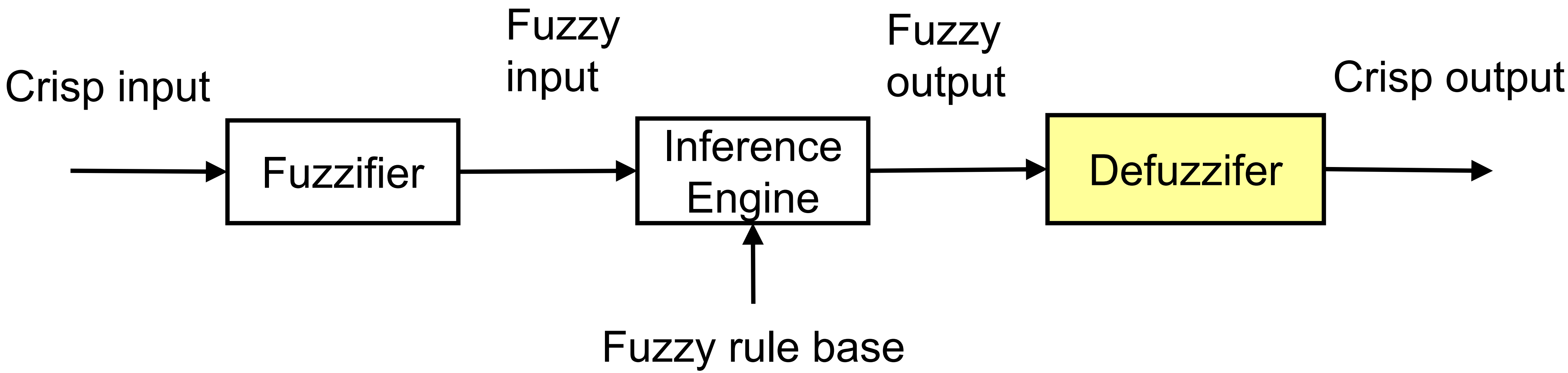
Fuzzy system



Converts the fuzzy output of the inference engine to crisp using membership functions analogous to the ones used by the fuzzifier.

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Fuzzy system: Fuzzifier

- The function of fuzzifier is to transform measurement data into valuation of a subjective value.
- It can be defined as a mapping from observed input space to labels of fuzzy sets (linguistic values) in a specified input universe of discourse.

In fuzzy control applications, the observed data are usually crisp (though they may be corrupted by noise)

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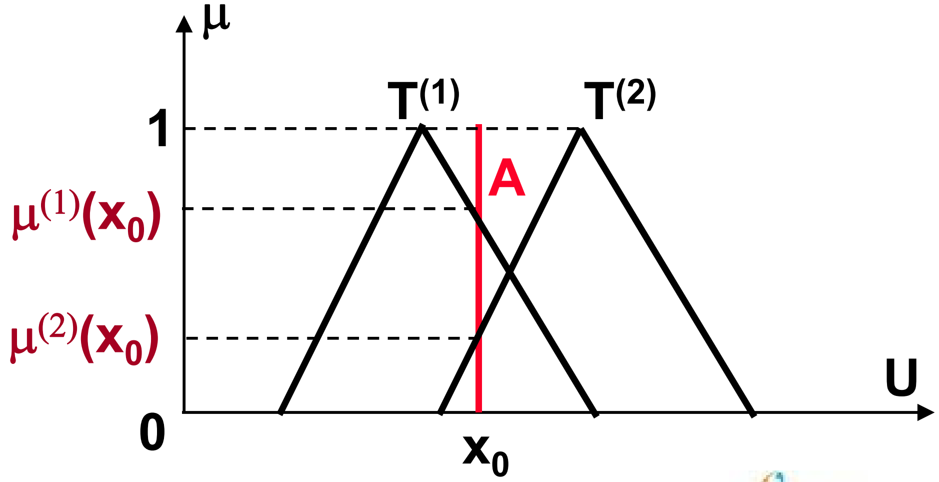
Fuzzy system: Fuzzifier

- A simple fuzzification approach is
  - » to first convert a crisp value x_0 into a fuzzy singleton A within the specified universe of discourse, i.e.:
    - ♦ the membership function of A, $\mu_A(x)$, is equal to 1 at the point x_0, and zero at other places.
  - » the fuzzy singleton is then mapped to the fuzzy sets
    - ♦ $T^{(1)}$ with degree $\mu^{(1)}(x_0)$,
    - ♦ $T^{(2)}$ with degree $\mu^{(2)}(x_0)$,



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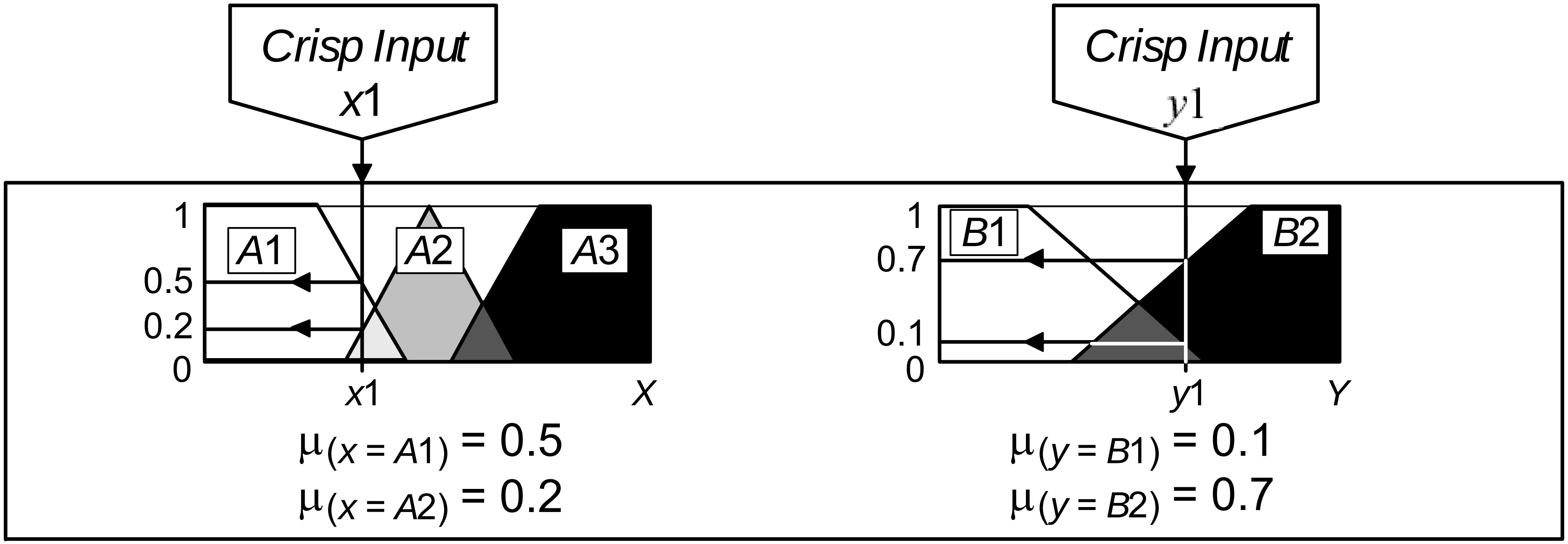
Example

An illustration: Take the crisp inputs, x1 and y1, and determine the degree to which these inputs belong to each of the appropriate fuzzy sets.



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Fuzzy system: Fuzzy rule base

- Fuzzy rule base is a collection of fuzzy IF-THEN rules (i.e.: the antecedents and/or consequents involve linguistic variables)
- It characterizes the simple input-output relation of the system.
  - » E.g.: In a fuzzy control system, the fuzzy control rules evaluate the process state at time *t* and compute and decide the control actions:
    - Input = linguistic values
    - Output = linguistic/crisp

☝ A complex fuzzy system may consist of more than one sub-inference module, each of them has a fuzzy rule base.

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Fuzzy system: Rule evaluation

- Classical if-then rules are triggered if the antecedent can be matched (=true); firing (applying) the rule then allows for inferring the consequent.
- Fuzzy if-then rules
  - » are triggered only partially: antecedent is matched with fuzzified input (if rule has multiple antecedents, apply given fuzzy operator to obtain a single number)
  - » fire only partially, consequent only holds to a certain degree

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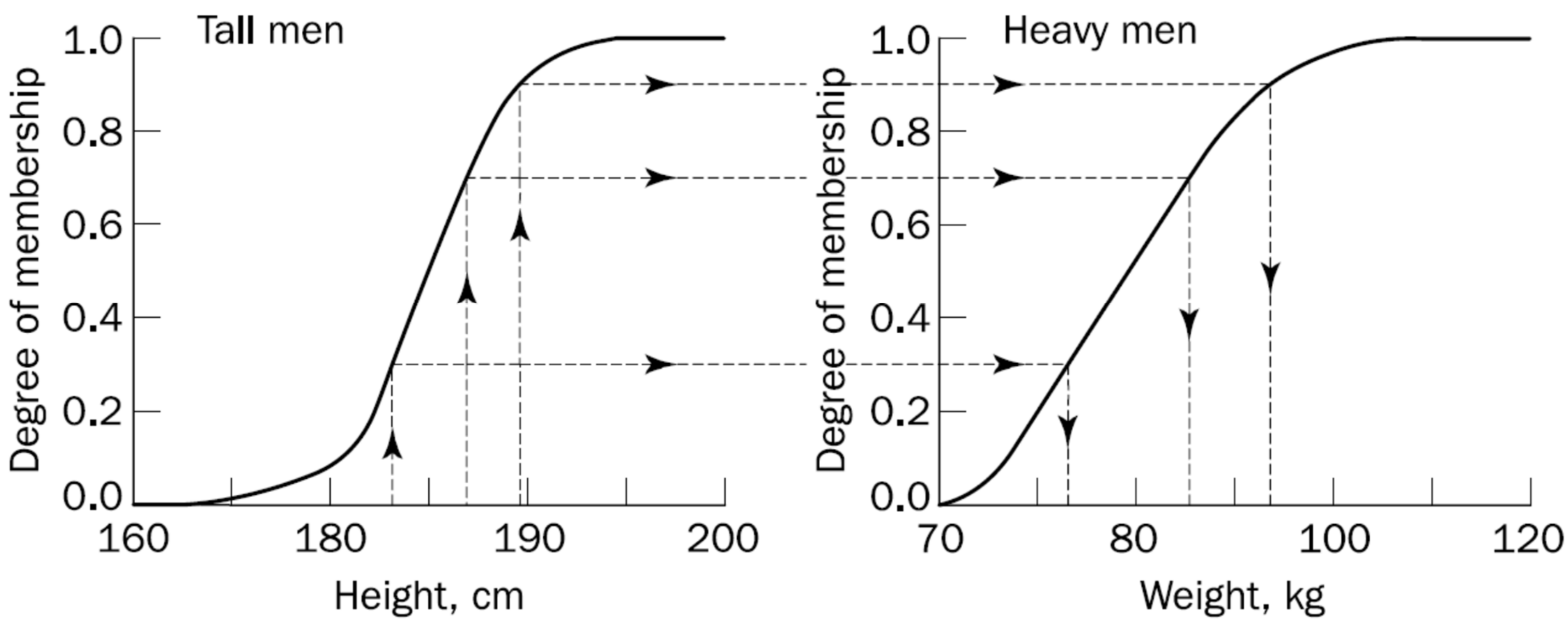
Fuzzy system: Rule evaluation

Consider the following fuzzy rule: IF height is tall THEN weight is heavy

The truth membership grade of the rule consequent can be estimated directly from a corresponding truth membership grade in the antecedent, using monotonic selection.



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Fuzzy system: Fuzzy inference engine

- Fuzzy inference engine realizes the mechanism of fuzzy reasoning/approximate reasoning
  - » Given a fuzzy rule A $\rightarrow$ B and an input A'
    - ♦ the conclusion B' will be derived
- There are different models for fuzzy reasoning
  - » Zadeh's compositional rule of inference,
  - » Mamdani's model, … …

  using different
    - ♦ fuzzy implication, and
    - ♦ compositional operator (for calculating logical AND/OR)

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Fuzzy system: Fuzzy inference engine

- Two compositional operators are the most commonly and frequently used:
  - Max-min operation [Zadeh]
    (using *Min* for T-norm and *Max* for T-conorm)
  - Max-product operation [Kaufmann]
    (using *product* for T-norm and *Max* for T-conorm)
- Mamdani fuzzy model associated with max-min composition is the most frequently used model in fuzzy control.

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Fuzzy inference: Mamdani model

- Mamdani fuzzy model associated with max-min composition
  - » Using *min* for T-norm
  - » Using *max* for T-conorm,
  - » Using *min* for fuzzy implication.

- E.g.: IF *x* is A1 and *y* is B1 THEN z is C1
  or
  IF *x* is A2 and *y* is B2 THEN z is C2
  - » A1 AND B1 (A2 AND B2) — Min for T-norm
  - » fuzzy implication — Min
  - » C'1 OR C'2 — Max for T-conorm

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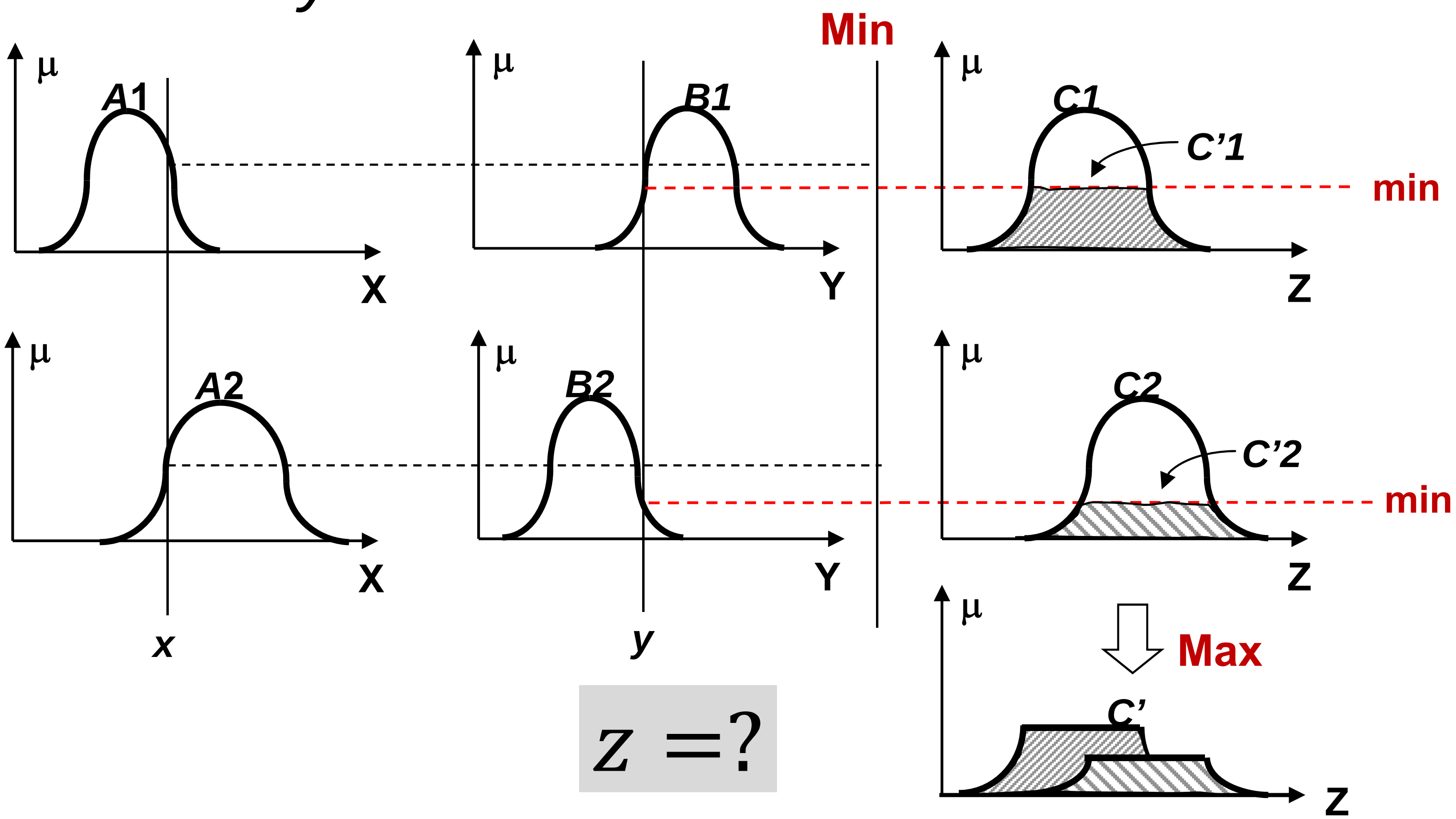
Fuzzy inference: Mamdani model

- Example:
  - » IF *x* is A1 and *y* is B1 THEN z is C1
  - » IF *x* is A2 and *y* is B2 THEN z is C2



$z = ?$

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Fuzzy system: Result aggregation

- Multiple fuzzy sets as different linguistic values are usually defined for same linguistic variable, so
  - » Given input values, multiple fuzzy rules may be fired at the same time with different strengths
- Inference result is an aggregation of outputs from the multiple fired rules

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Fuzzy system: Defuzzifier

- Defuzzifier is to perform a mapping from a space of linguistic values (decision, or actions) defined over an output universe of discourse into a space of non-fuzzy (crisp) decision action.
- Some typical methods of defuzzification:
  - » *Center of Area* (COA) method
  - » *Center of Maximum* (COM)
  - » *Mean of Maximum method* (MOM)

  (*There is no systematic procedure for choosing a defuzzification method)

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Defuzzification: COA

- Center of Area (COA) method
  - » Generates the center of gravity of the possibility distribution of a control action

for discrete universe of discourse

$$z^*_{COA} = \frac{\sum_{j=1}^{n} \mu_C(z_j) z_j}{\sum_{j=1}^{n} \mu_C(z_j)}$$

for continuous universe of discourse

$$z^*_{COA} = \frac{\int_z \mu_C(z) z dz}{\int_z \mu_C(z) dz}$$

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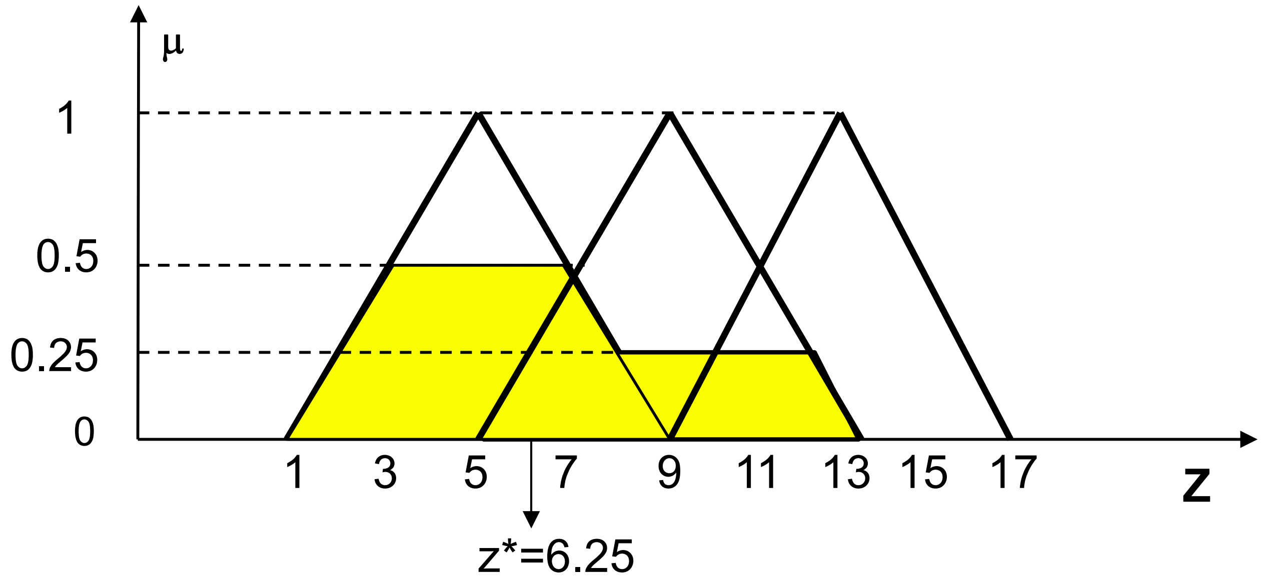
Defuzzification: COA

- Example:
  - » Given aggregated inference result



$z^*_{COA} =$

$= [1\times0 + (3+5+7)\times0.5 + (9+11)\times0.25] + 13\times0] / (0+0.5+0.5+0.5+0.25+0.25+0)$

$= 12.5 / 2 = 6.25$

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Defuzzification: COM

- Center of Maximum (COM) method
  - » Identical to COA that uses singleton membership functions.
  - » Instead of area (all z_j as in COA method), using only the typical values of each related term and balancing the weights on those representative points

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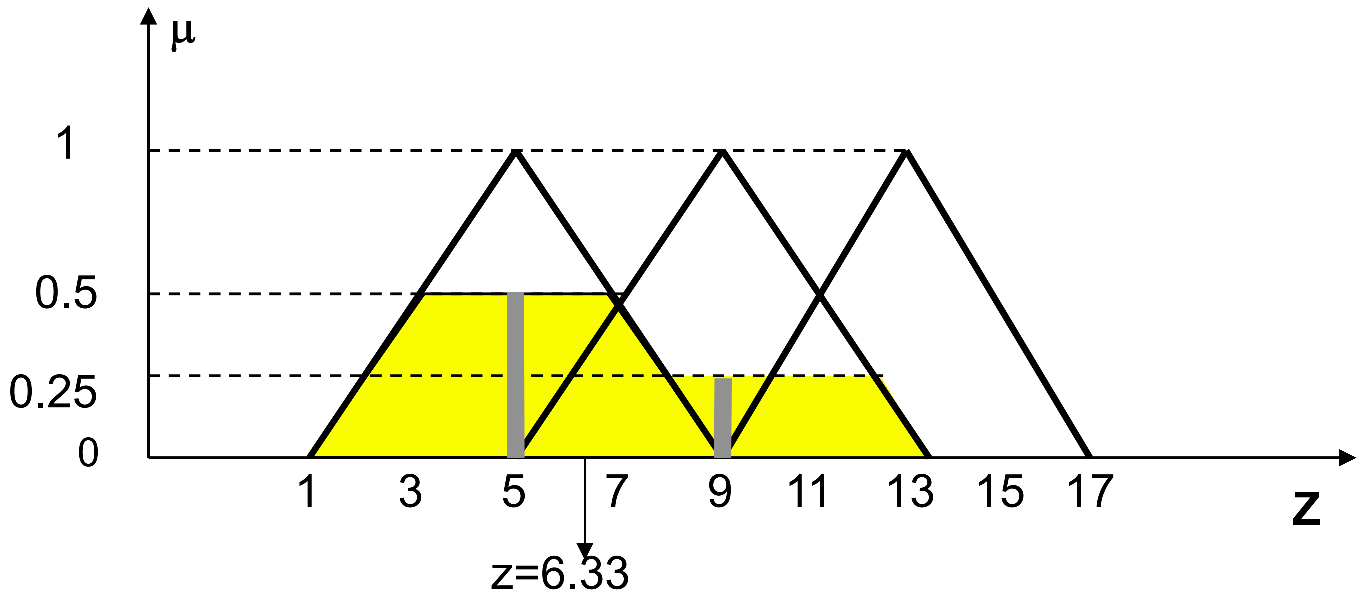
Defuzzification: COM

» Continue previous example, but using COM:

$$z^*_{COM} = (5 \times 0.5 + 9 \times 0.25) / (0.5 + 0.25) = 4.75/0.75 = 6.33$$



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Defuzzification: MOM

- Mean of Maximum (MOM) method
  - » Generates an action that represents the mean value of all actions whose membership functions reach the maximum.

$$z^*_{MOM} = \sum_{j=1}^{m} \frac{z_j}{m}$$

where z_j is the support value at which the membership function reaches the maximum value μ^* and m is the number of such support values (for continuous universe of discourse, it is the average of the maximizing z at which the MF reach a maximum μ^*)

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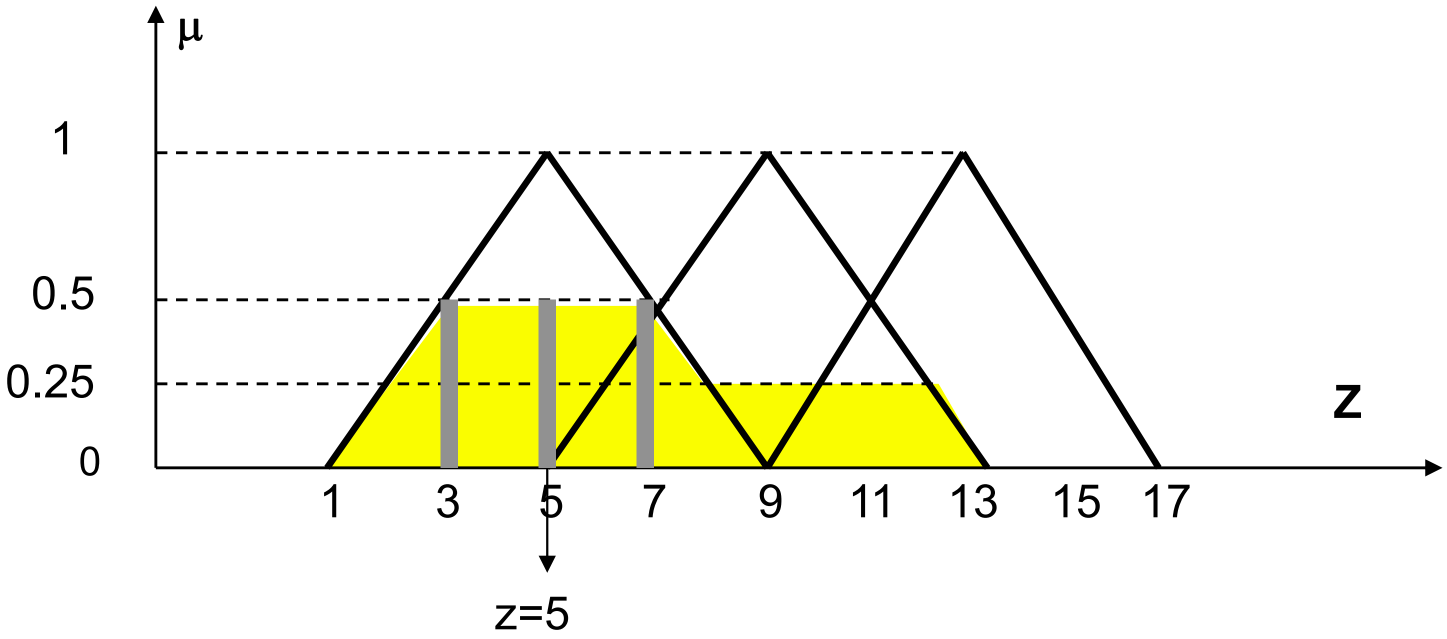
Defuzzification: MOM

» Applying MOM to the previous example:

$z^*_{MOM} = 5$



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Example 1: Fuzzy rule

- Assume two rules:
  - » Rule 1: IF x is A1 and y is B1 THEN z is C1
  - » Rule 2: IF x is A2 and y is B2 THEN z is C2

  Membership functions:

$$\mu_{A1}(x) = \begin{cases} \dfrac{x-2}{3} & 2 \le x \le 5 \\ \dfrac{8-x}{3} & 5 < x \le 8 \end{cases} \qquad \mu_{A2}(x) = \begin{cases} \dfrac{x-3}{3} & 3 \le x \le 6 \\ \dfrac{9-x}{3} & 6 < x \le 9 \end{cases}$$

$$\mu_{B1}(y) = \begin{cases} \dfrac{y-5}{3} & 5 \le y \le 8 \\ \dfrac{11-y}{3} & 8 < y \le 11 \end{cases} \qquad \mu_{B2}(y) = \begin{cases} \dfrac{y-4}{3} & 4 \le y \le 7 \\ \dfrac{10-y}{3} & 7 < y \le 10 \end{cases}$$

$$\mu_{C1}(z) = \begin{cases} \dfrac{z-1}{3} & 1 \le z \le 4 \\ \dfrac{7-z}{3} & 4 < z \le 7 \end{cases} \qquad \mu_{C2}(z) = \begin{cases} \dfrac{z-3}{3} & 3 \le z \le 6 \\ \dfrac{9-z}{3} & 6 < z \le 9 \end{cases}$$

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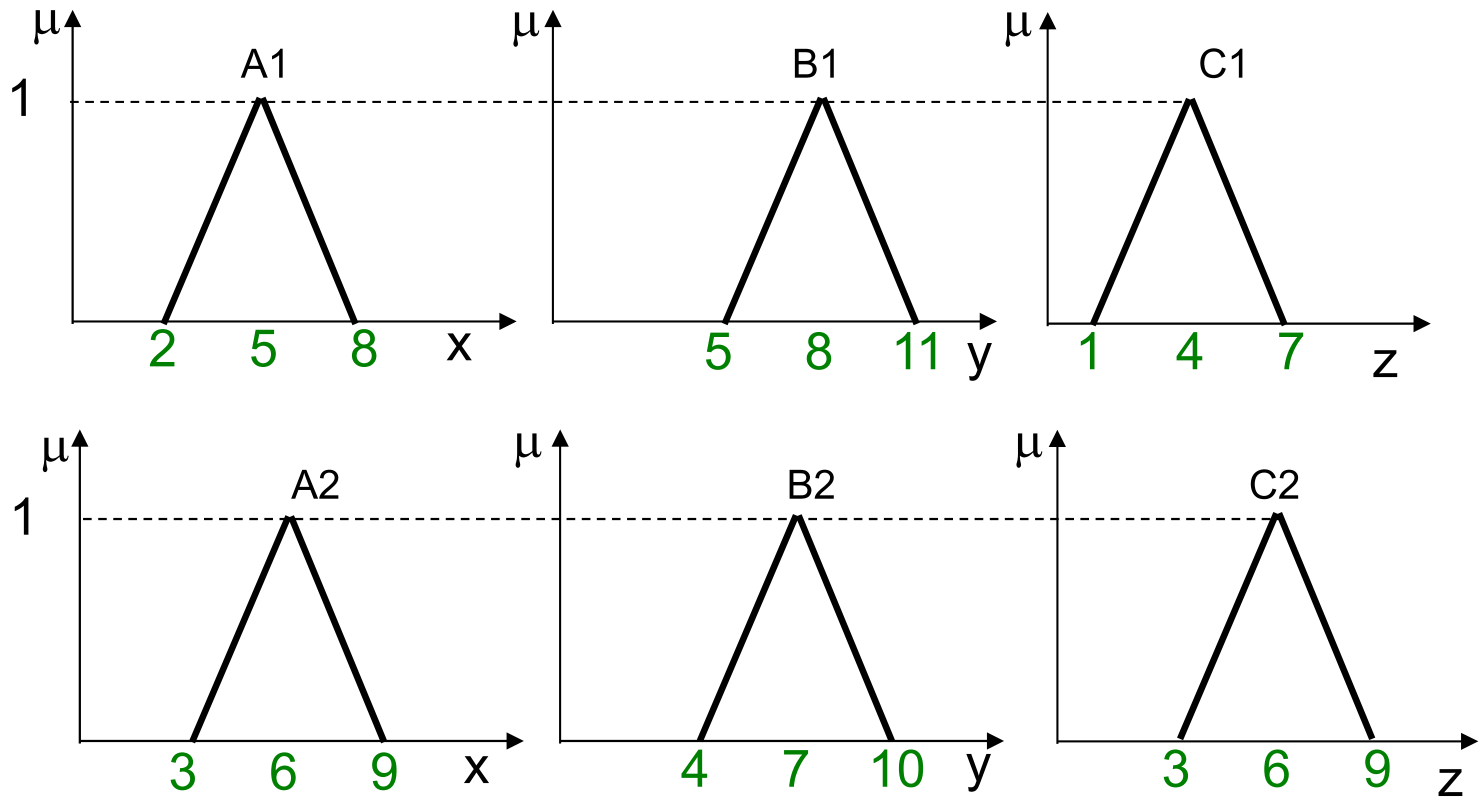
Example 1: Membership function

Example (membership functions)



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Example 1: Rule evaluation

Example

» Assume that we are reading **sensor values**

- $x_0 = 4$ and $y_0 = 8$

  $\mu_{A1}(x_0) = 2/3$ $\mu_{B1}(y_0) = 1$

  $\mu_{A2}(x_0) = 1/3$ $\mu_{B2}(y_0) = 2/3$

» Finding the strength of rules (T-norm):

- for Rule 1: $\alpha_1 = Min(\mu_{A1}(x_0), \mu_{B1}(y_0)) = 2/3$
- for Rule 2: $\alpha_2 = Min(\mu_{A2}(x_0), \mu_{B2}(y_0)) = 1/3$

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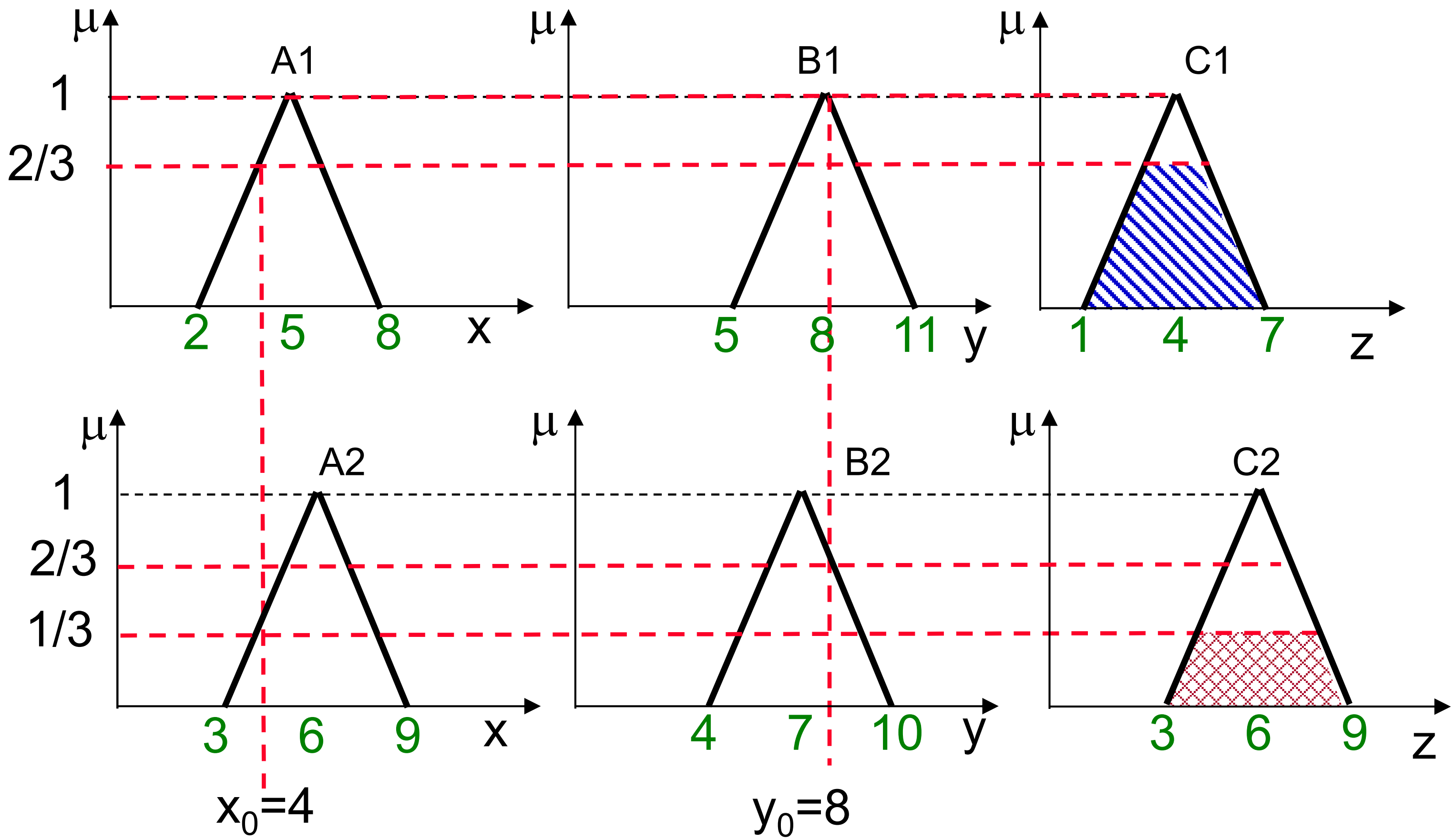
Example 1: Rule evaluation

Example

» Applying α_1 to the conclusion of Rule 1, and α_2 to the conclusion of Rule 2, respectively (implication).



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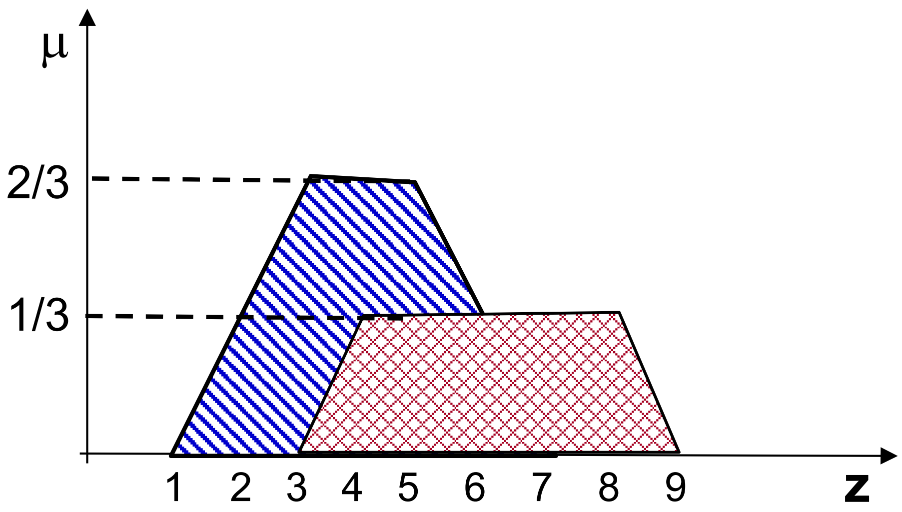
Example 1: Result aggregation

Example

» Aggregating qualified consequents from Rule 1 and Rule 2 (T-conorm)



» Defuzzification is needed to get a single crisp value output.

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Example 1: Defuzzification

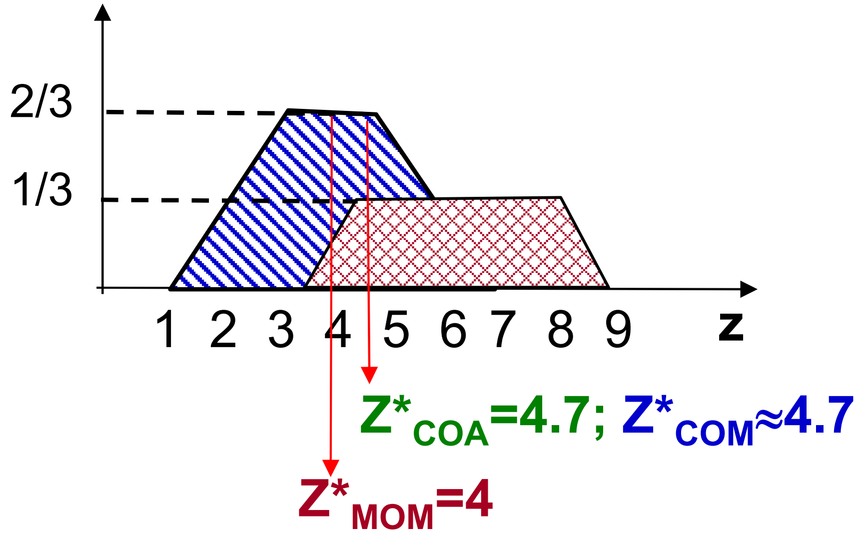
Example

» Defuzzification

$$z^*_{COA} = \frac{2\cdot\frac{1}{3}+3\cdot\frac{2}{3}+4\cdot\frac{2}{3}+5\cdot\frac{2}{3}+6\cdot\frac{1}{3}+7\cdot\frac{1}{3}+8\cdot\frac{1}{3}}{\frac{1}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 4.7$$

$$z^*_{COM} = \frac{4\cdot\frac{2}{3}+6\cdot\frac{1}{3}}{\frac{2}{3}+\frac{1}{3}} \approx 4.7$$

$$z^*_{MOM} = \frac{3+4+5}{3} = 4$$



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Example 2: Fuzzy rule

Example: a two-input -- one-output problem that includes 3 rules, 3 variables and 8 fuzzy sets.

Rule: 1		Rule: 1		
IF	x is A3	IF	project_funding	is adequate
OR	y is B1	OR	project_staffing	is small
THEN	z is C1	THEN	risk	is low
Rule: 2		**Rule: 2**		
IF	x is A2	IF	project_funding	is marginal
AND	y is B2	AND	project_staffing	is large
THEN	z is C2	THEN	risk	is normal
Rule: 3		**Rule: 3**		
IF	x is A1	IF	project_funding	is inadequate
THEN	z is C3	THEN	risk	is high

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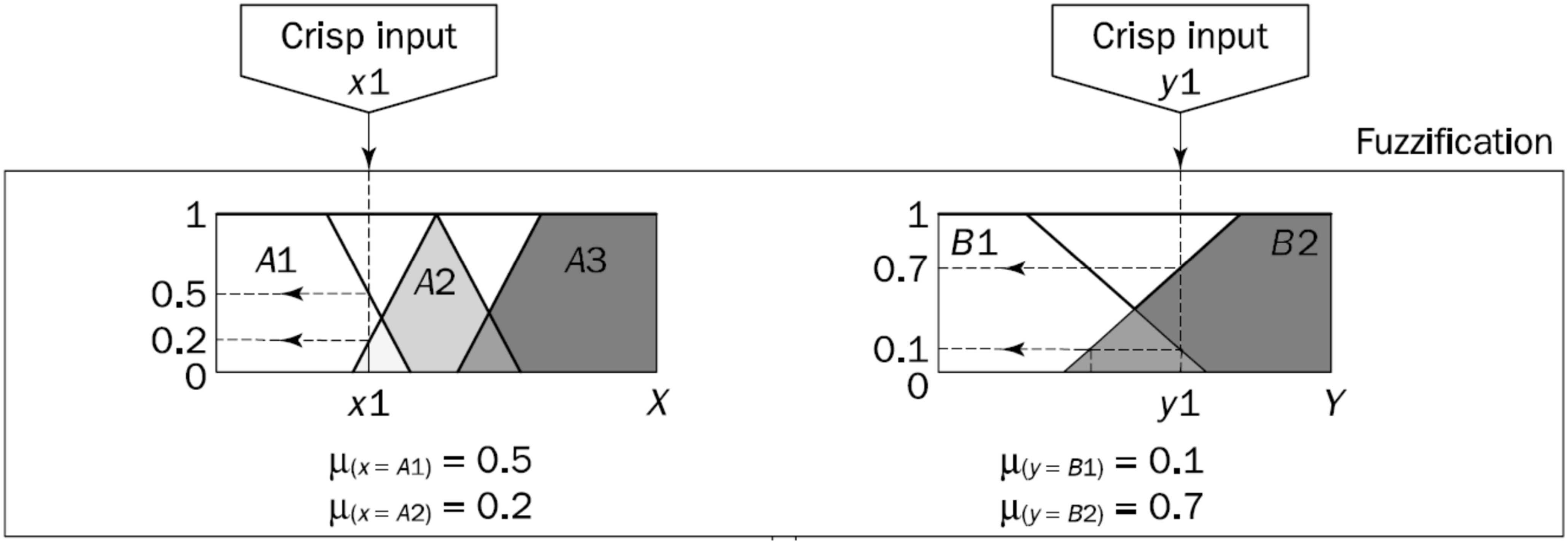
Example 2: Fuzzification

Our crisp inputs for project funding (*x*1) and project staffing (*y*1) are: *x*1 = 35% and *y*1 = 60%



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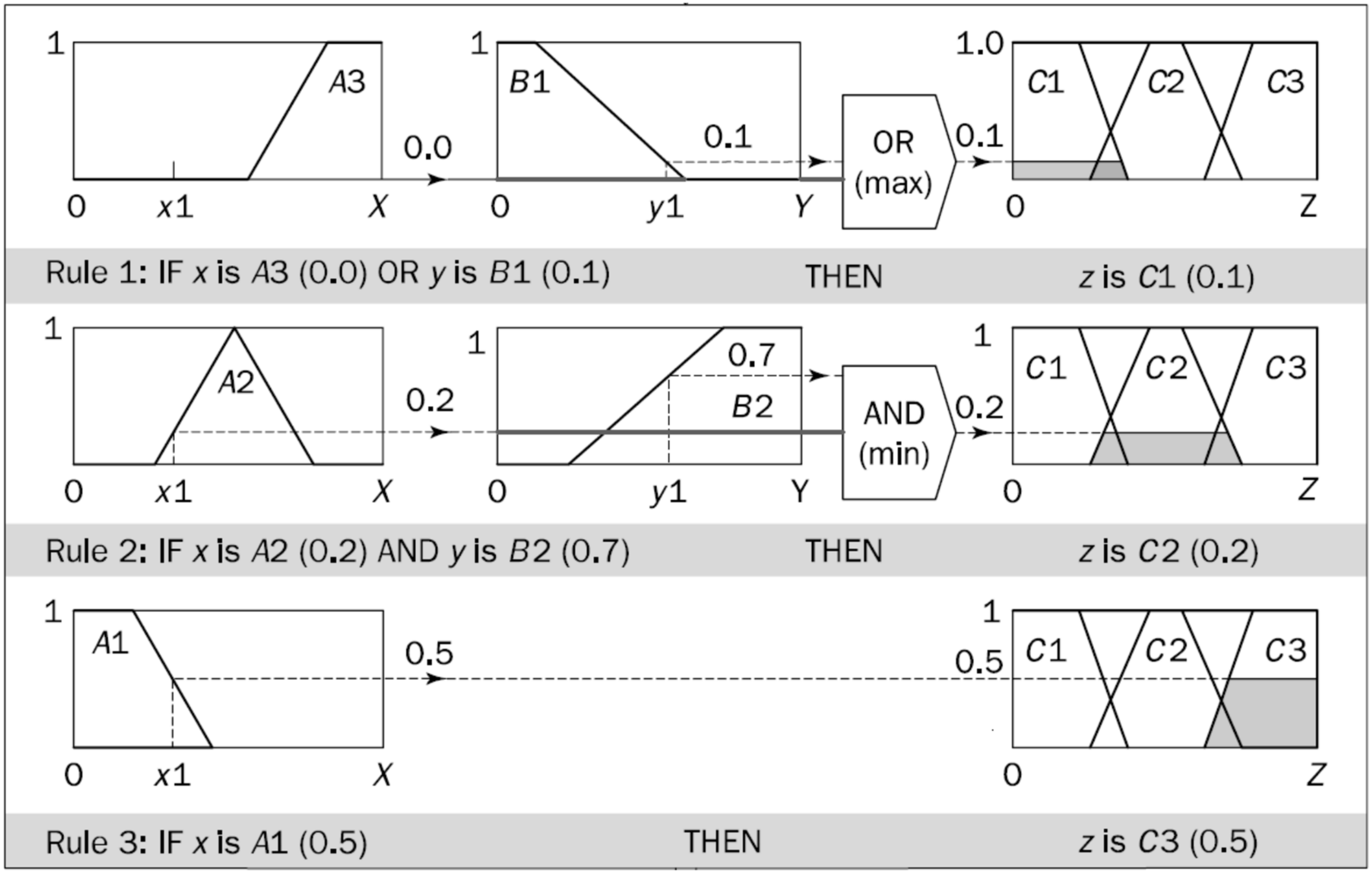
Example 2: Rule evaluation



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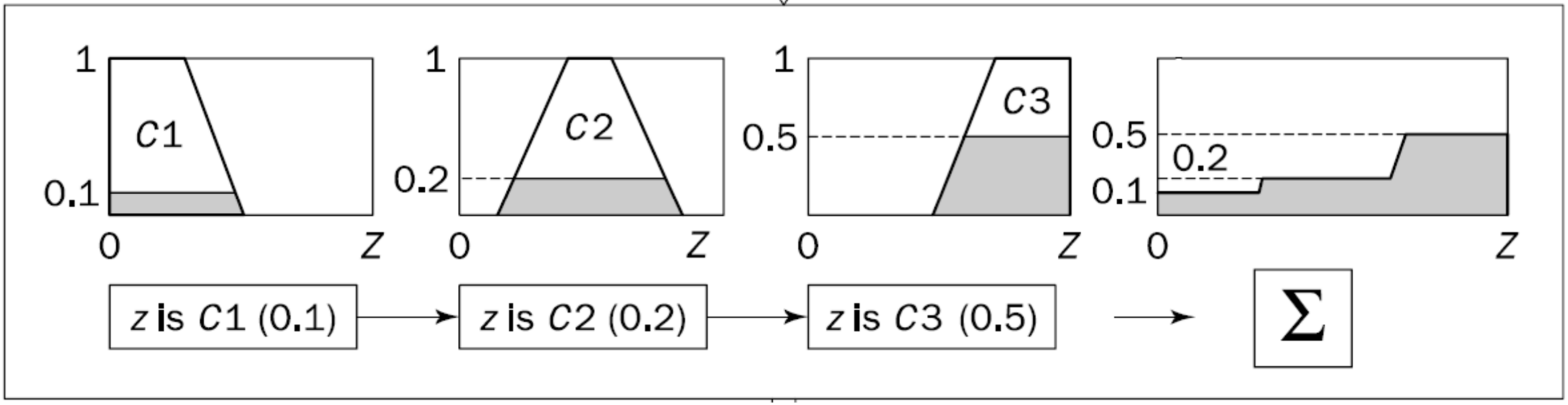
Example 2: Result aggregation

- All three rules have a consequent concerning the outcome variable *risk (z)*
- We combine three relevant clipped fuzzy sets into a single one
  - » Take for each value *z* the $\max\left(\mu_{C1}(z), \mu_{C2}(z), \mu_{C3}(z)\right)$
  - » Output is one fuzzy set for *risk*



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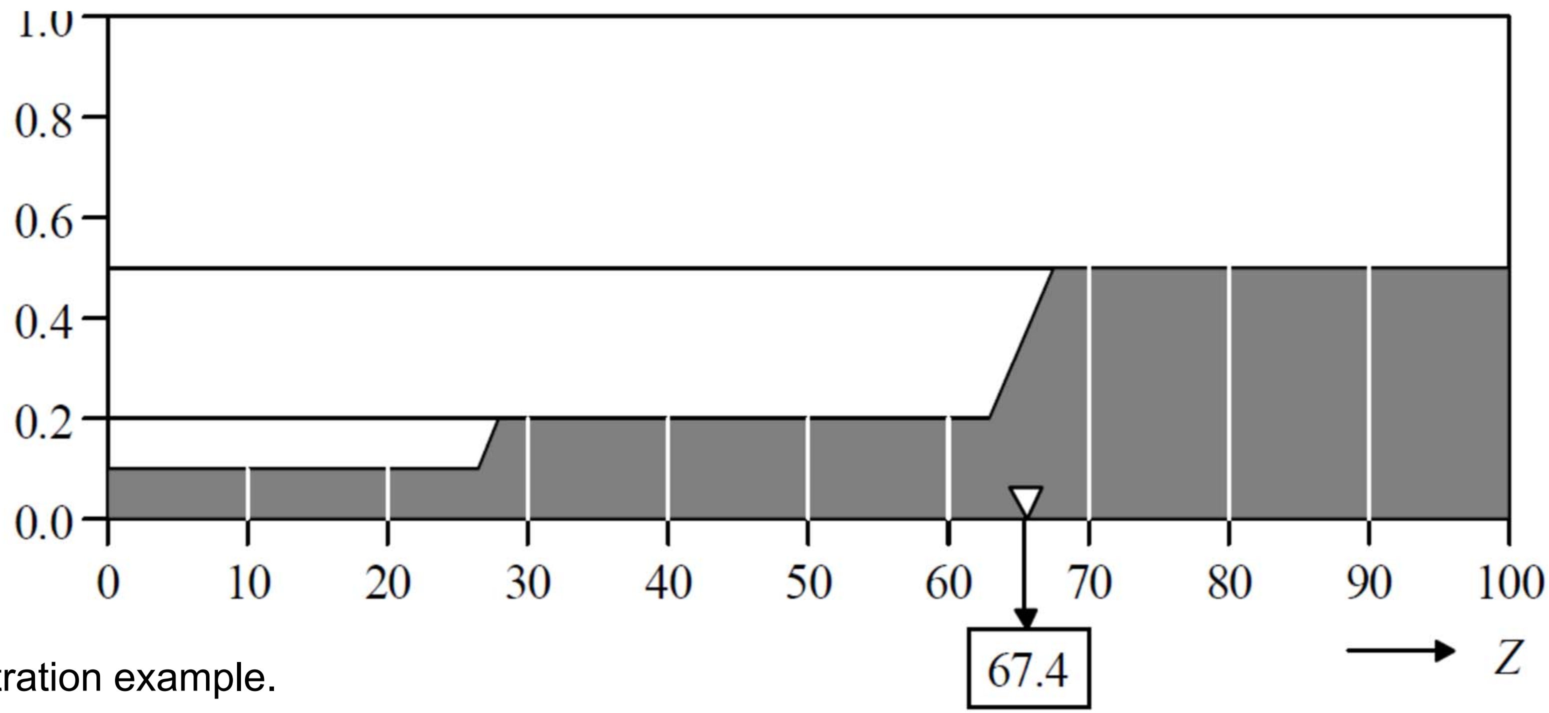
Example 2: Defuzzification

$$COG = \frac{(0+10+20)\times 0.1 + (30+40+50+60)\times 0.2 + (70+80+90+100)\times 0.5}{0.1+0.1+0.1+0.2+0.2+0.2+0.2+0.5+0.5+0.5+0.5} = 67.4$$

We approximate COG with sums and find: the project's risk is 67.4%



This is an illustration example.

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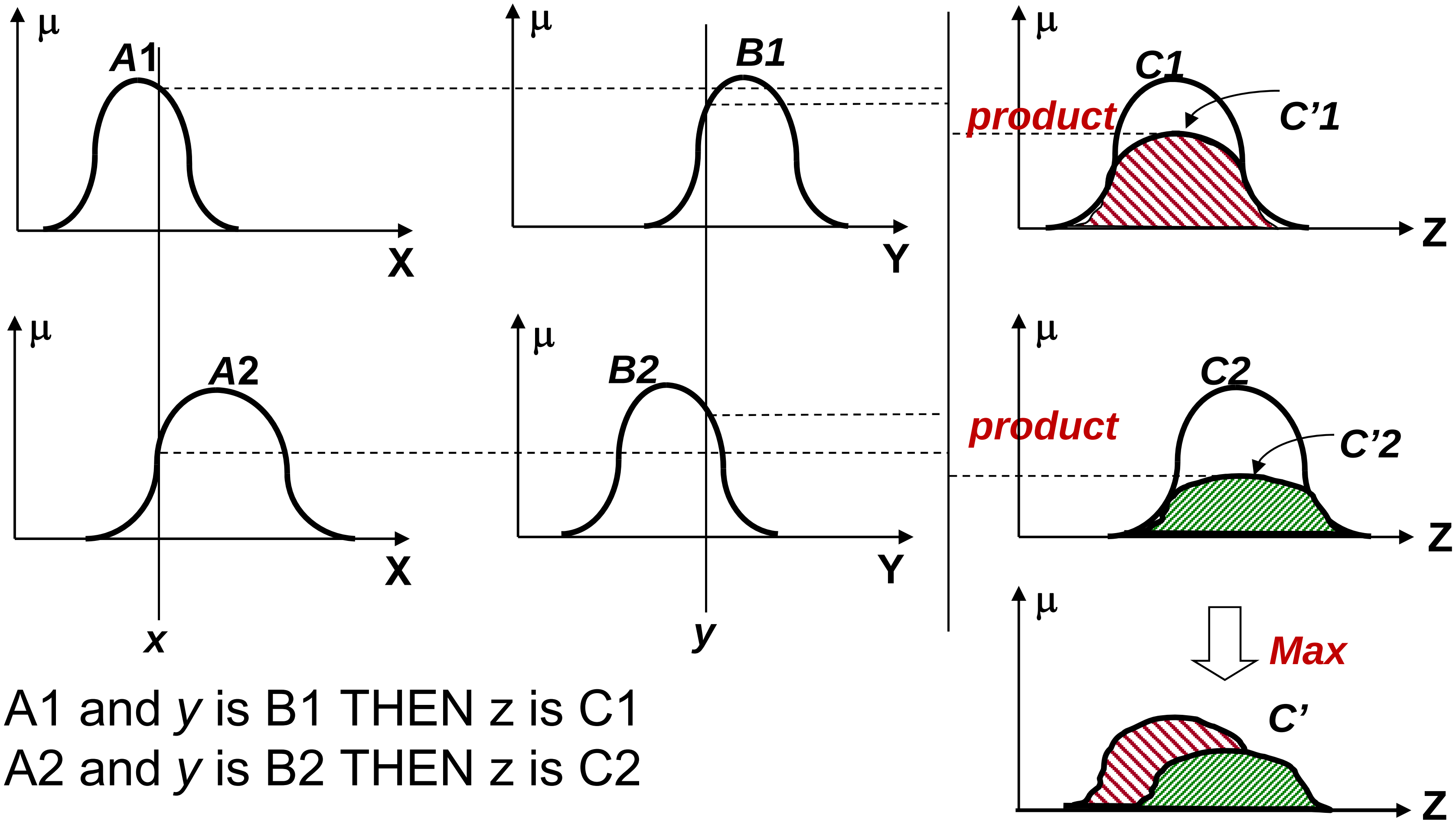
Fuzzy Inference: Mamdani model with Max-product

- Max for T-conorm and product for T-norm



IF *x* is A1 and *y* is B1 THEN z is C1
IF *x* is A2 and *y* is B2 THEN z is C2

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Fuzzy inference: Sugeno model

- Sugeno fuzzy model (also known as the *TSK model*) was proposed by Takagi, Sugeno and Kang.
  - » It is an effort to develop a systematic approach to generate fuzzy rules from a given input-output data set.
  - » A typical fuzzy rule in a Sugeno fuzzy model has the form

IF x is A and y is B THEN $z = f(x, y)$

where A and B are fuzzy sets for the antecedent,

while $z = f(x, y)$ is a crisp function for the consequent.

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Fuzzy inference: Sugeno model

- First-order Sugeno fuzzy model
  - » when $f(x, y)$ is a first-order polynomial
- Zero-order Sugeno fuzzy model
  - » when f is a constant
- Each rule has a crisp output, the overall output is obtained via weighted average.

Example

R1: if *X* is small and *Y* is small then $z = x + y + 1$

R2: if *X* is small and *Y* is large then $z = -y + 3$

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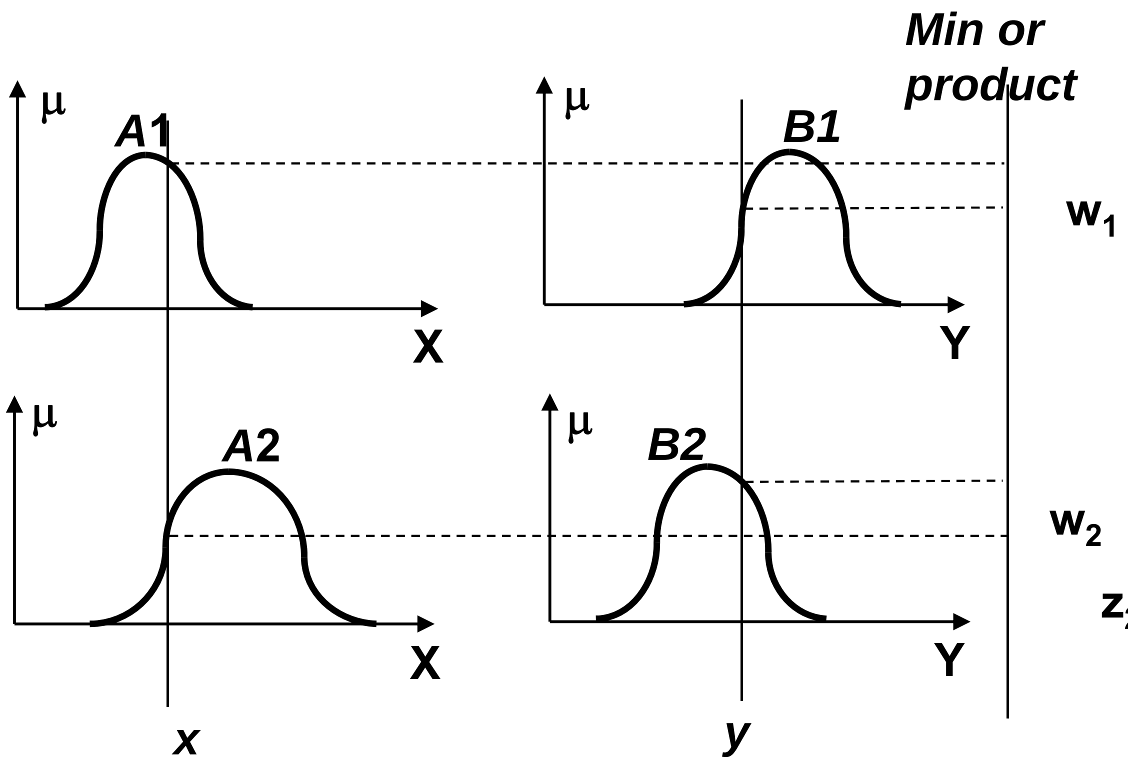
Fuzzy inference: Sugeno model

- Inference in Sugeno fuzzy model



$z_1 = p_1x + q_1y + r_1$

$z_2 = p_2x + q_2y + r_2$

Weighted average

$$z = \frac{w_1z_1 + w_2z_2}{w_1 + w_2}$$

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Mamdani model and Sugeno model

- Mamdani model is widely accepted for capturing expert knowledge. It allows us to describe the expertise in more intuitive, more human-like manner. However, Mamdani-type fuzzy inference entails a substantial computational burden.
- Sugeno model computationally effective and works well with optimisation and adaptive techniques, which makes it very attractive in control problems, particularly for dynamic nonlinear systems.

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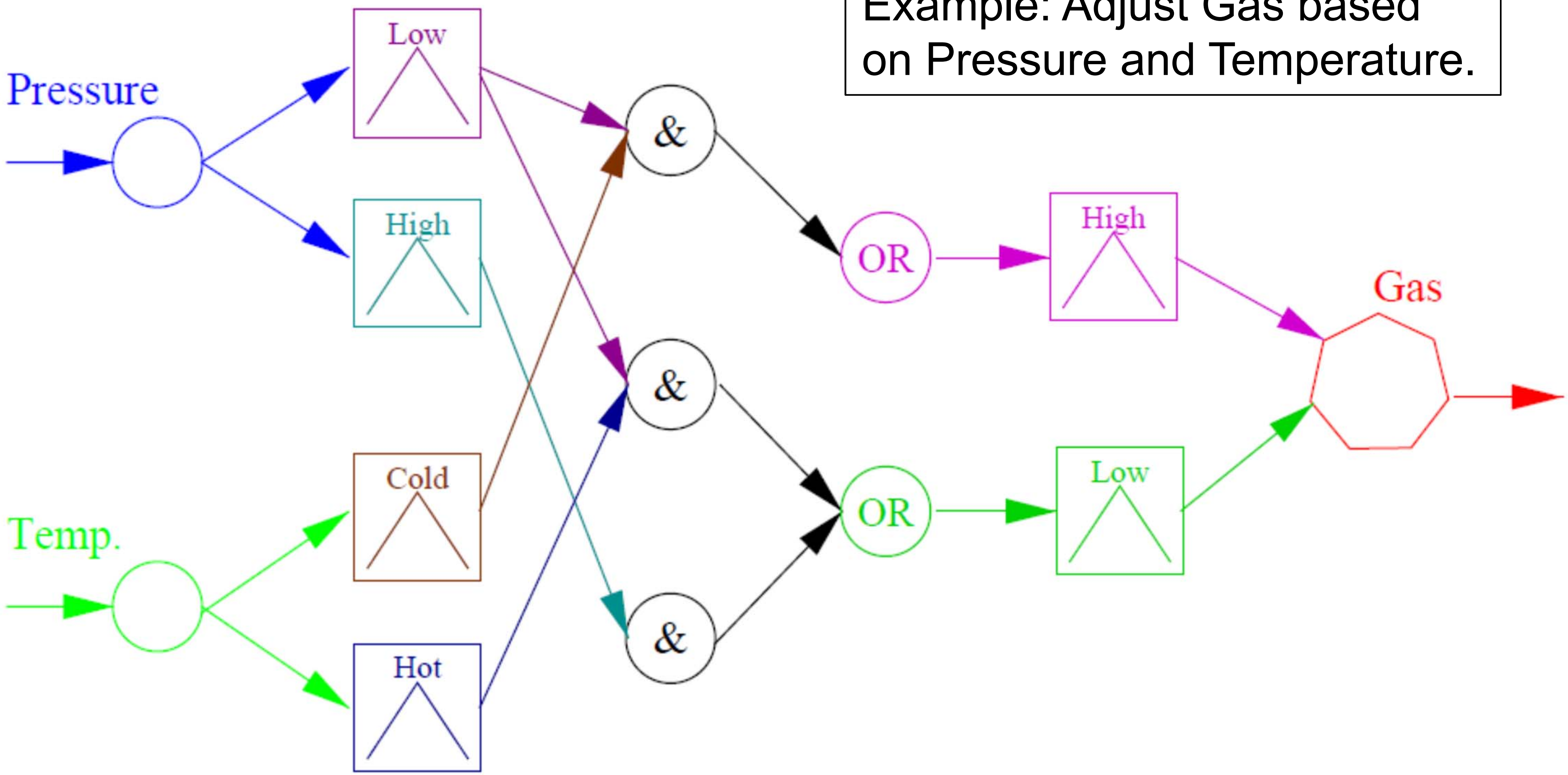
Network-like view of a fuzzy system

Example: Adjust Gas based on Pressure and Temperature.



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Master of Technology

Computational Intelligence II

Fuzzy system: Case study

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Fuzzy modelling

- In general, we design a fuzzy inference system based on the past known behaviour of a target system.
  - » The a fuzzy system is then expected to be able to reproduce the behaviour of the target system.
    - ♦ E.g.: Target system is a medical doctor
      - Fuzzy system is a fuzzy expert system for medical diagnosis
- The process called fuzzy modelling is the method for constructing a fuzzy inference system.

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Fuzzy modelling: Surface structure

- Generally, it includes four tasks
  - » Select relevant input and output variables
  - » Choose a specific type of fuzzy inference system
  - » Determine the number of linguistic terms associated with each input and output variables (fuzzy partition)
    - ♦ The meaning of linguistic terms can be determined in the second stage
  - » Design a collection of fuzzy if-then rules (fuzzy rule base) in a symbolic style:
    - ♦ i.e. IF x is A THEN y is B

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Fuzzy modelling: Deep structure

- Specifically, it may include the following tasks:
  - » Choosing an appropriate family of parameterised membership functions used in the rule base.
    - ♦ Piecewise linear: Triangular, Trapezoidal, …
    - ♦ Non-linear, differentiable : Gaussian, Bell, …
  - » Determining the parameters of membership functions used in the rule base
    - ♦ e.g.: through interviewing human experts
  - » Learning/Refining the parameters (fuzzy rules, membership functions) by other techniques:
    - ♦ e.g.: neural networks, genetic algorithms

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Fuzzy partition

- Through fuzzy partition, a fuzzy input space is divided into many overlapping fuzzy granules.
- The size (or cardinality) of a term set $|T(x_i)| = k_i$ is called the fuzzy partition of x_i. It determines
  - » the granularity of the target fuzzy inference system,
  - » also the maximum number of fuzzy rules in the system.
    - ♦ E.g.: a two-input-one-output fuzzy system.

      If $|T(x_1)| = 3$ and $|T(x_2)| = 7$, then the maximum number of fuzzy rules is $|T(x_1)| \times |T(x_2)| = 21$ (when only one rule is considered for one combination of input terms)

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Fuzzy partition

- Grid partition is the simplest method for partition
  - » It is often chosen in designing a fuzzy controller, which usually involves only several state variables as inputs to the controller
  - » Advantage:
    - ♦ Needs only a small number of membership functions for each input, such as 3, 5, 7 or 9.
  - » Disadvantage:
    - ♦ Encounters problems when we have a moderately large number of inputs. E.g. a fuzzy model with 10 inputs and only 2 membership functions on each would result 2^{10} = 1024 fuzzy if-then rules!

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Define fuzzy rules

- Example: Assume
  - » Two input variables x_1 and x_2, and one output variable y
    - ♦ each has three linguistic terms: *small*, *medium*, and *large*.
  - » Possible rules:

    IF x_1 is *small* and x_2 is *medium*

    THEN y is *large*

    IF x_1 is *medium* and x_2 is *large*

    THEN y is *small*

    … …

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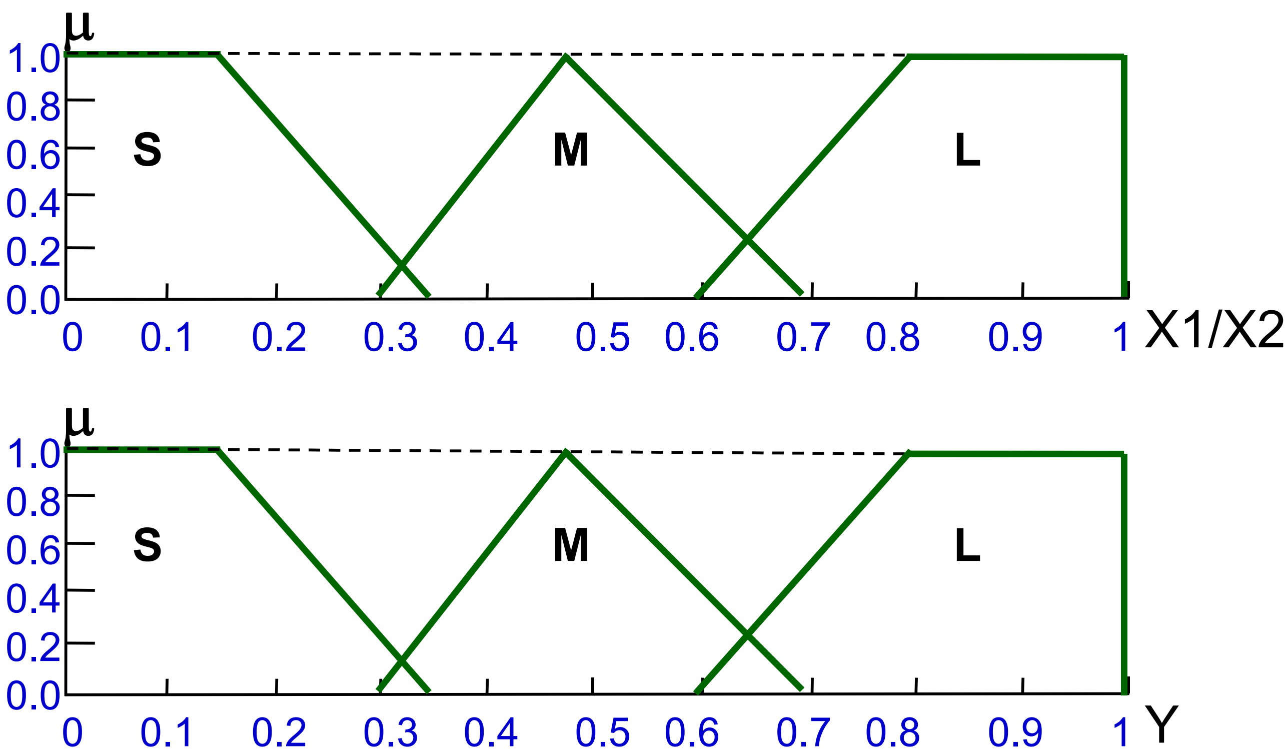
Define fuzzy rules

Example (cont.)

» Define fuzzy sets, assume the universal sets are [0, 1]



☝ For linguistic variables x_1 and x_2, different fuzzy sets may be defined for the same linguistic values: *small*, *medium*, and *large*

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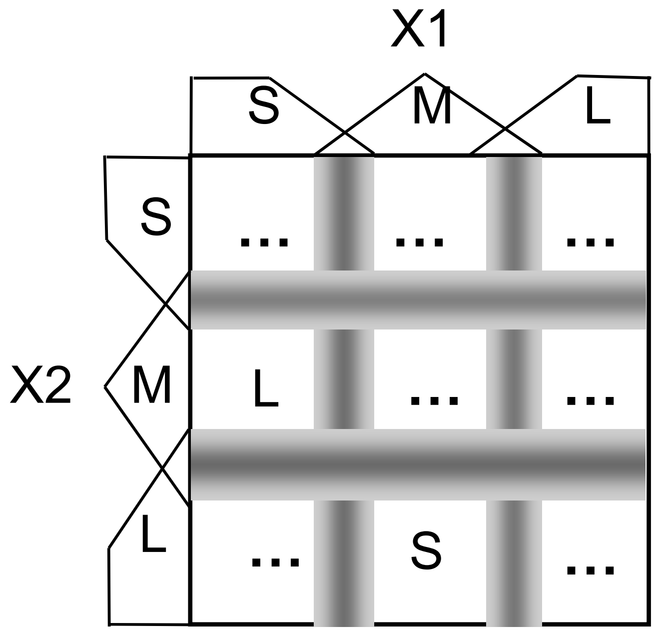
Fuzzy granules of problem space

Example (cont.)

» Construct fuzzy rules

♦ Assume the same input combination always results one same output label



☝ The membership functions of linguistic values used in a fuzzy system are often parametric functions such as triangular functions, trapezoidal functions, and bell-shaped functions.

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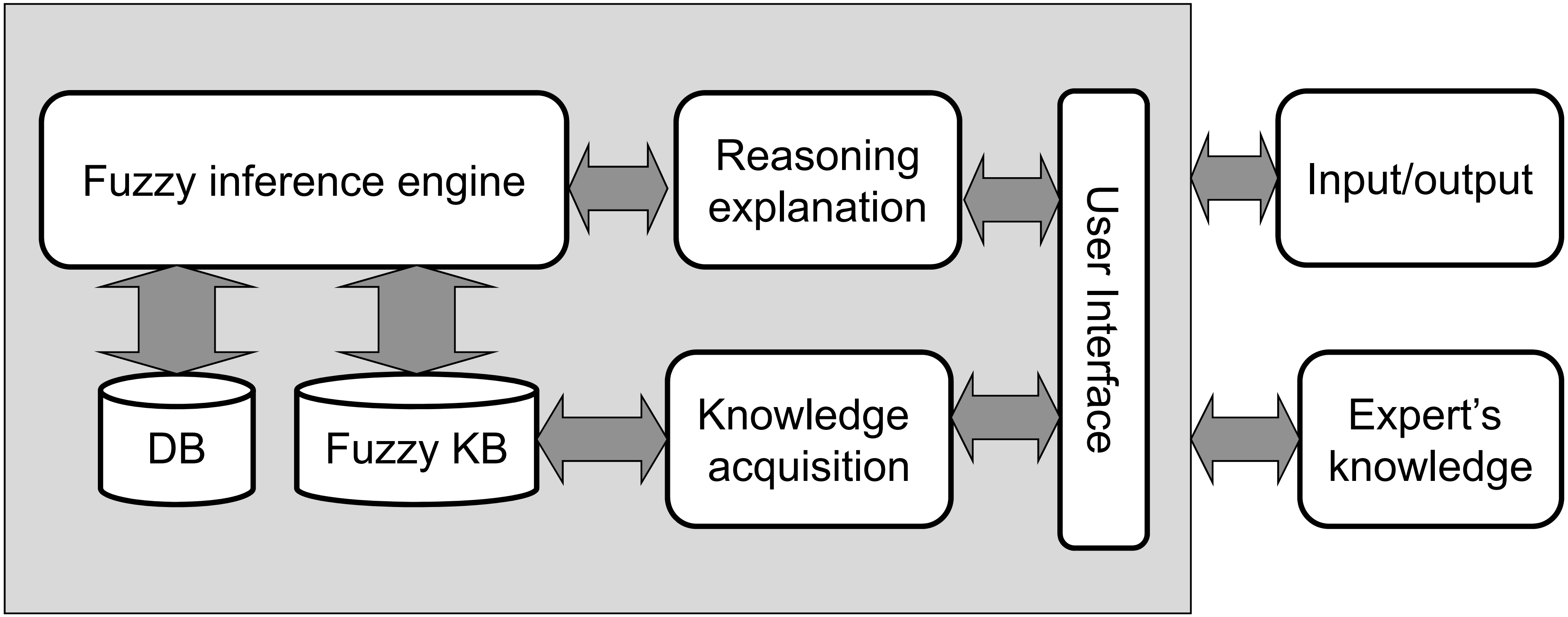
Fuzzy expert system



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Fuzzy expert system

- Fuzzy knowledge-base
  - » stores experts' knowledge as fuzzy IF-THEN rules
- Database
  - » contains input data, membership function, etc.
- Fuzzy inference engine
  - » performs approximate reasoning
- User interface
  - » man-machine interface which allows fuzzy (linguistic) input/output (including necessary processing of fuzzification, defuzzification)
- Reasoning process explanation system (optional)
  - » explains the process and results to the user
- Knowledge acquisition support system (optional)
  - » knowledge augmentation, modification, and evaluation

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Building a fuzzy expert system

1) Defining input and output variables
   - » identify what is the mapping function you need approximate
2) Deciding on fuzzy partition of the input and output spaces
   - » determine a term set at the right level of granularity
   - » define fuzzy sets for linguistic variables
3) Designing fuzzy rules
   - » describe the mapping relation
4) Designing the inference mechanism
   - » choosing a fuzzy implication
   - » choosing a compositional operator with proper T-norm and T-conorm
   - » choosing defuzzification operator
5) Implement/Evaluate/Fine tune the system

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Case study: Service center of spare parts

- Background (adopted from Michael Negnevitsky, Artificial Intelligence: A Guide to Intelligent Systems, Pearson Education, 2011.)
  - » A service centre keeps spare parts and repairs failed ones. A customer brings a failed item and receives a spare of the same type. Failed parts are repaired, placed on the shelf, and thus become spares.
  - » If the required spare is available on the shelf, the customer takes it and leaves the service centre. However, if there is no spare on the shelf, the customer has to wait until the needed item becomes available.
  - » The objective here is to advise a manager of the service centre on certain decision policies to achieve a good productivity when keeping the customers satisfied.

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Problem modeling

- Step 1:
  - » Specify the problem and define linguistic variables
    - ♦ Consider 4 linguistic variables:
      - Average waiting time (mean delay), m
      - Repair utilisation factor of the service centre, ρ
      - Number of servers, s
      - Number of spare parts required, n

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Problem modeling

- Step 1: (cont.)
  - » The number of servers, *s*, and the initial number of spares, *n*, directly affect the customer's average waiting time, and thus have major impact on the centre's performance.
    - ♦ By increasing *s* and *n*, we achieve lower values of the mean delay, but at the same time we increase the cost.
  - » To increase the productivity of the service centre, its manager will try to keep the repair utilisation factor as high as possible.
    - ♦ So the strategy is to adjust the number of spares with an acceptable productivity and average waiting time (number of servers cannot be easily changed in reality)

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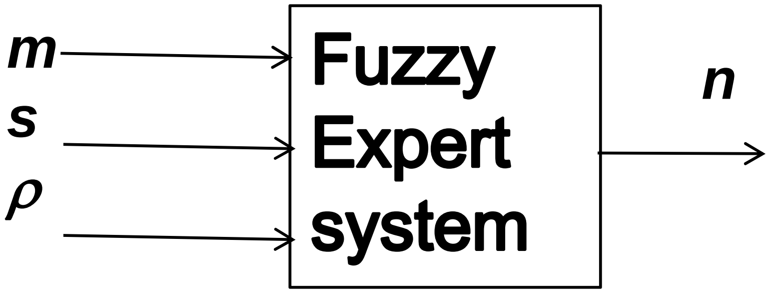
Problem modeling

- Step 1: (cont.)
  - » Decision model
    - ♦ A manager wants to determine the number of spares required to maintain the actual mean delay in customer service within an acceptable range



  - » Ranges of linguistic variables
    - ♦ All of them are normalized to [0, 1] by dividing base numerical values by the corresponding maximum magnitudes.

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Fuzzy partition

- Step 2: determine fuzzy partition and fuzzy sets
  - » For customer mean delay, *m*
    - ♦ 3 linguistic values:
      - *very-short, short, medium*
  - » For number of servers, *s*
    - ♦ 3 linguistic values:
      - small, medium, large
  - » For repair utilisation factor, ρ
    - ♦ 3 linguistic values
      - low, medium, high
  - » For number of spares, *n*
    - ♦ 7 linguistic values
      - very-small, small, rather-small, medium, rather-large, large, very-large

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Linguistic variable and value

Linguistic Variable: *Mean Delay, m*		
Linguistic Value	Notation	Numerical Range (normalised)
Very Short	VS	[0, 0.3]
Short	S	[0.1, 0.5]
Medium	M	[0.4, 0.7]

Linguistic Variable: *Number of Servers, s*		
Linguistic Value	Notation	Numerical Range (normalised)
Small	S	[0, 0.35]
Medium	M	[0.30, 0.70]
Large	L	[0.60, 1]

Linguistic Variable: *Repair Utilisation Factor*, ρ		
Linguistic Value	Notation	Numerical Range
Low	L	[0, 0.6]
Medium	M	[0.4, 0.8]
High	H	[0.6, 1]

Linguistic Variable: *Number of Spares, n*		
Linguistic Value	Notation	Numerical Range (normalised)
Very Small	VS	[0, 0.30]
Small	S	[0, 0.40]
Rather Small	RS	[0.25, 0.45]
Medium	M	[0.30, 0.70]
Rather Large	RL	[0.55, 0.75]
Large	L	[0.60, 1]
Very Large	VL	[0.70, 1]

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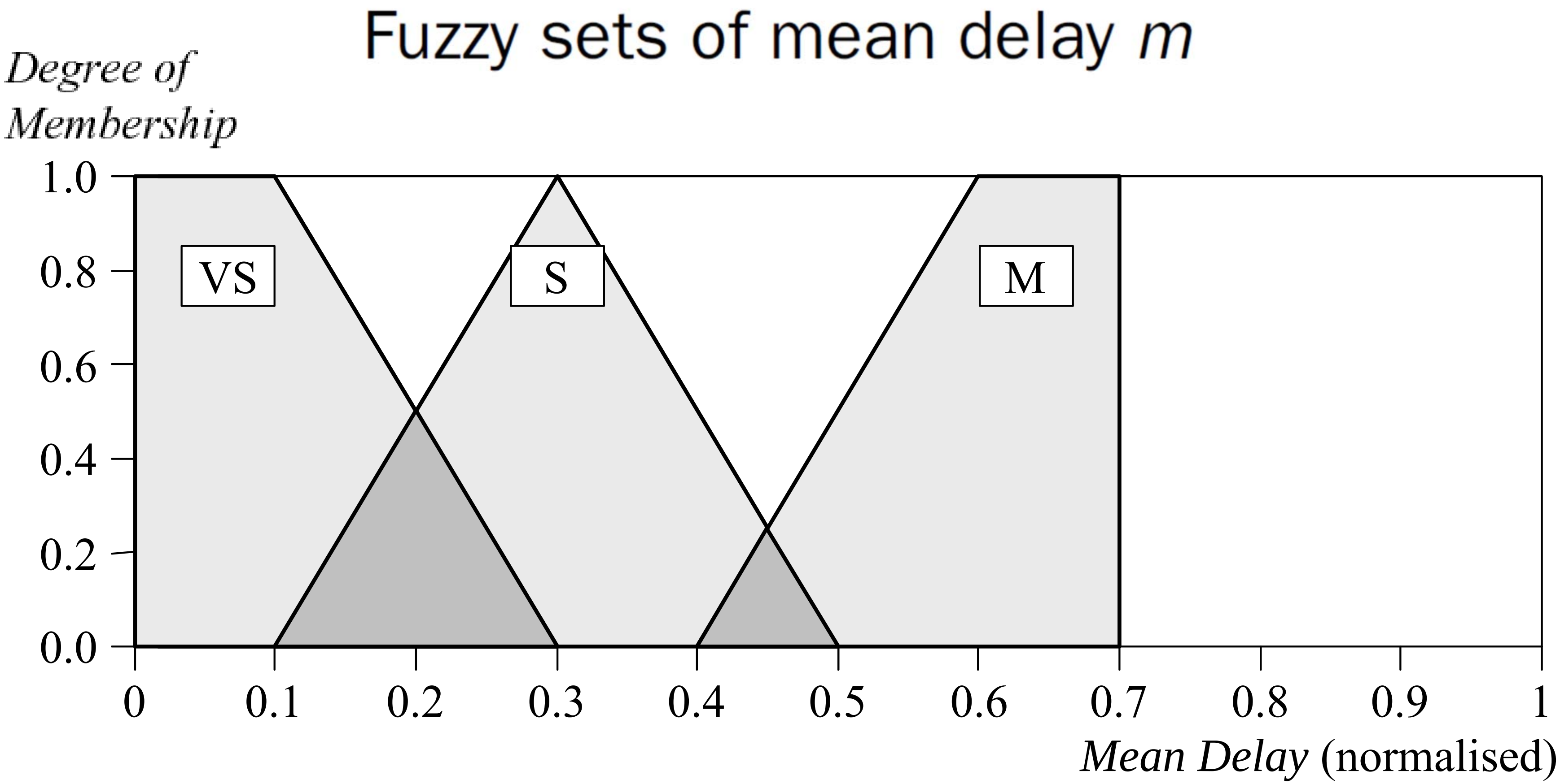
Fuzzy set



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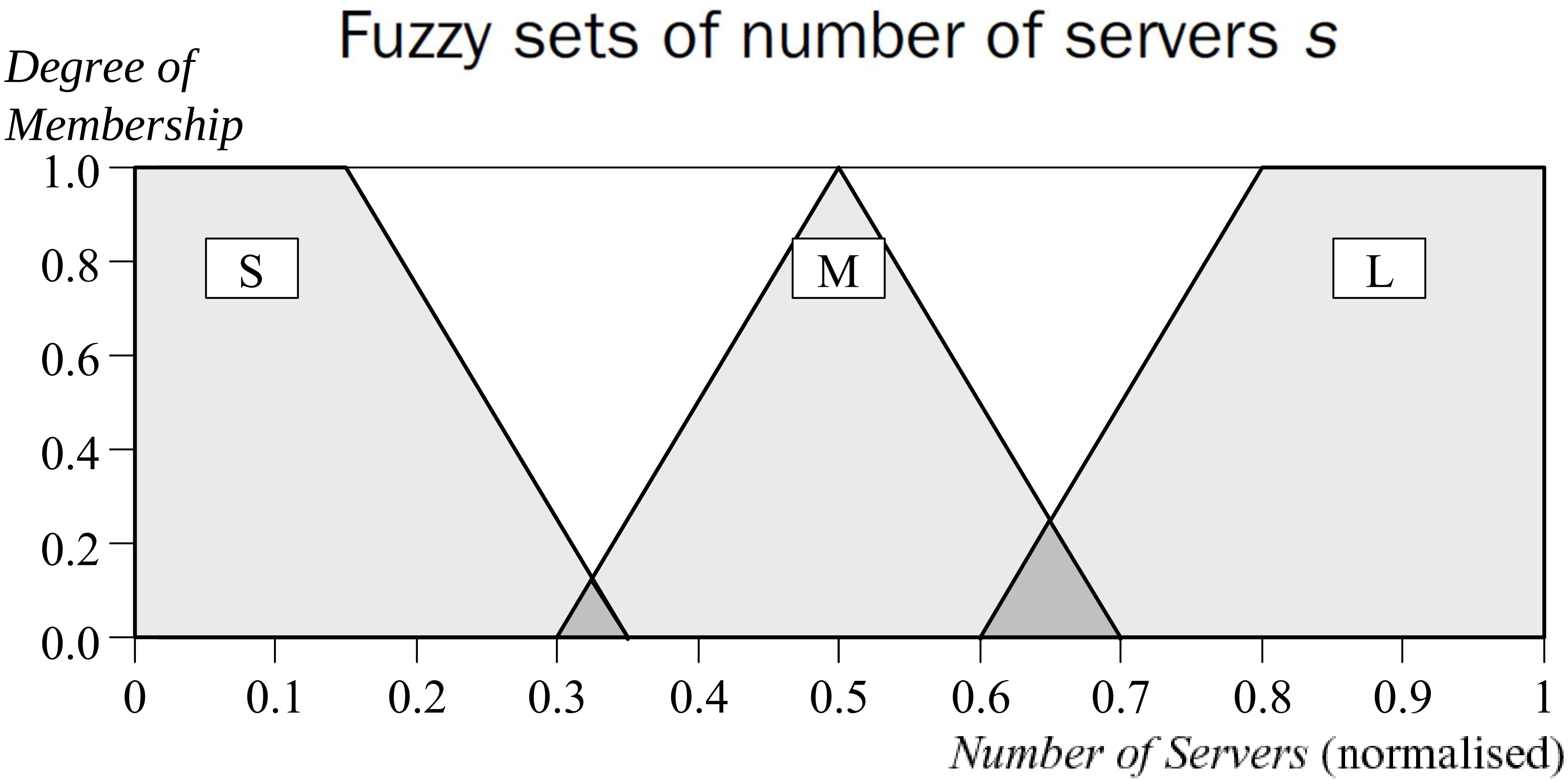
Fuzzy set



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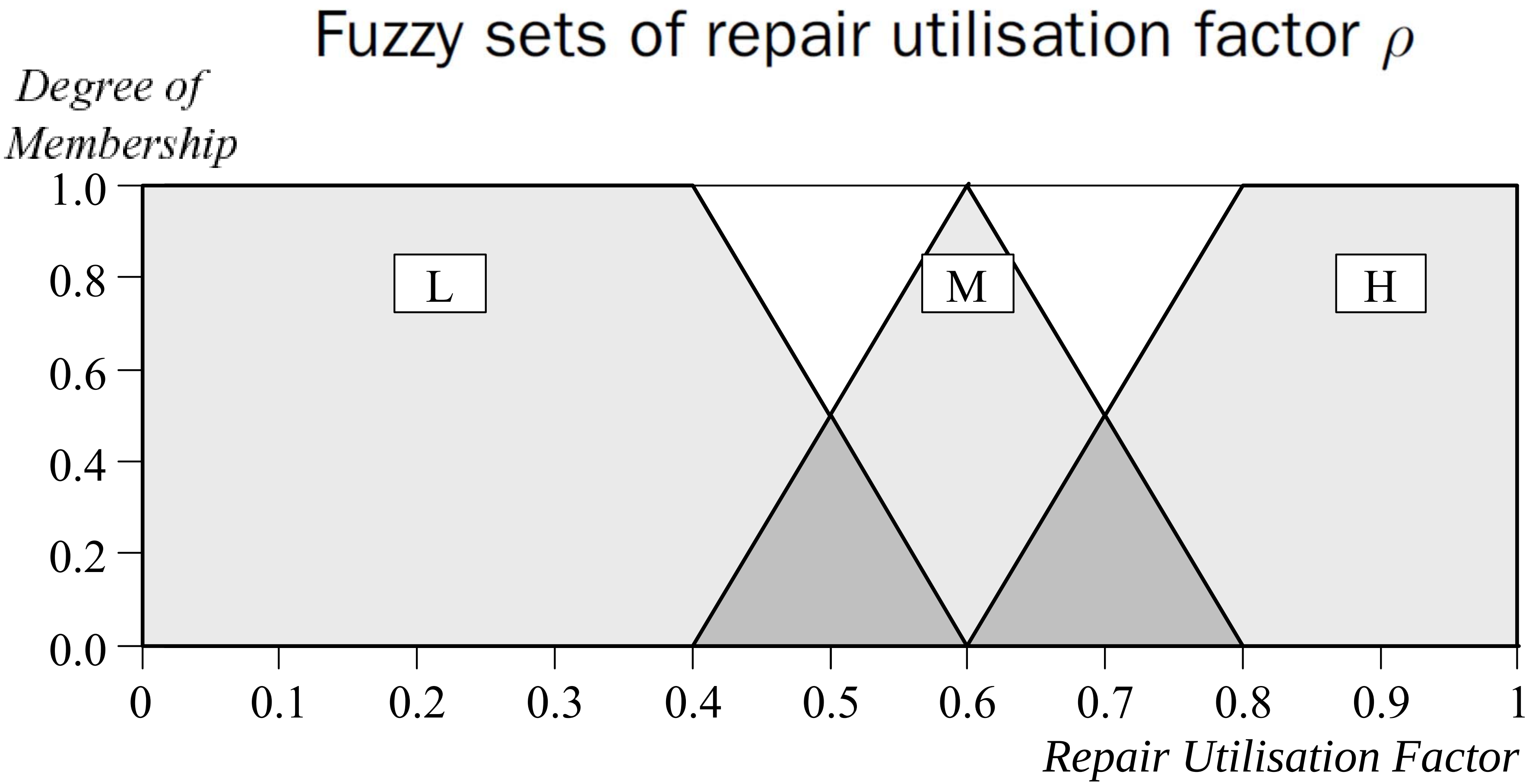
Fuzzy set



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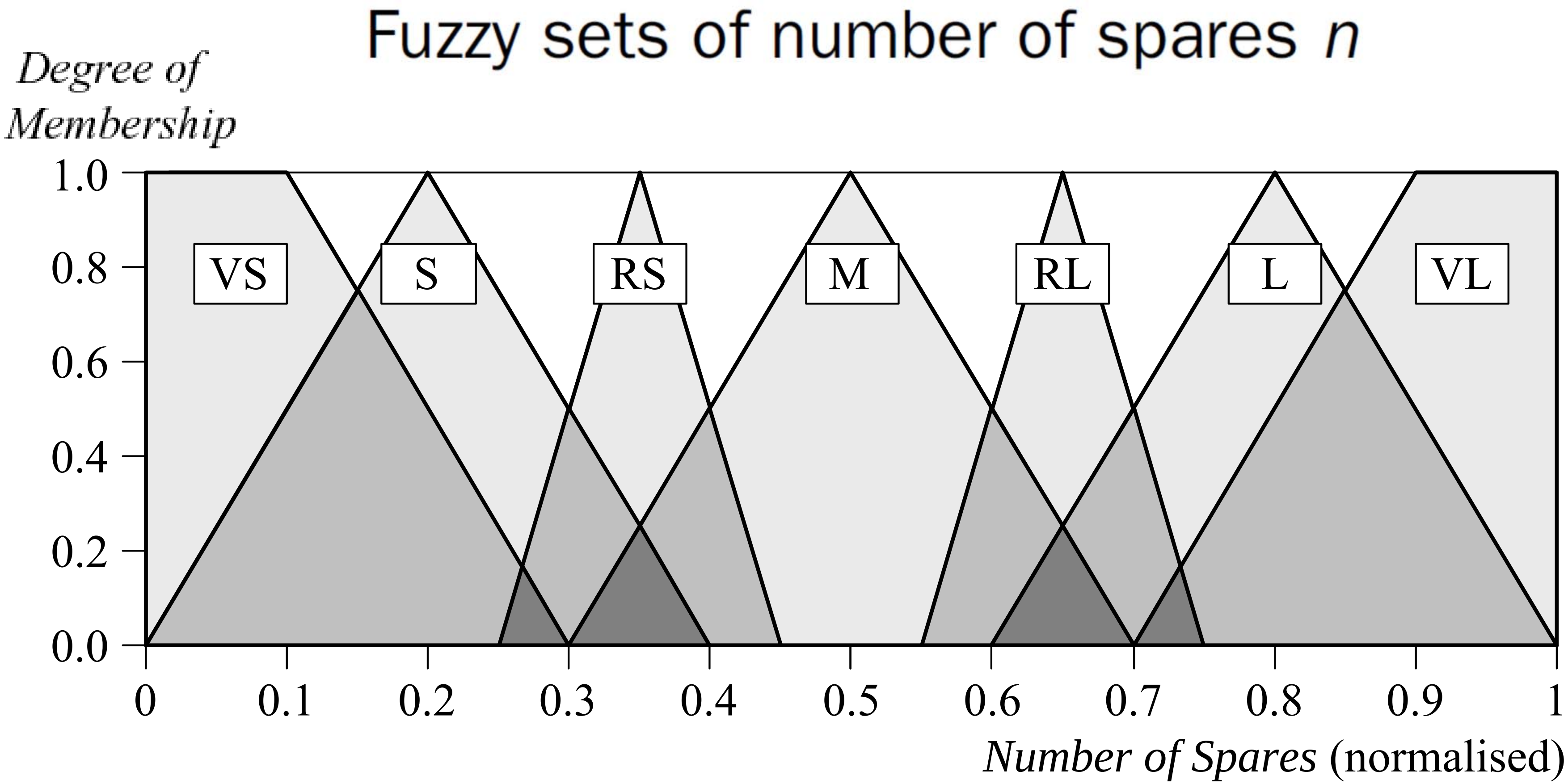
Fuzzy set



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Fuzzy rules

- Step 3: construct fuzzy rules

  Scheme-1 (total 12 rules)

  » First make use of a very basic relation between the repair utilisation factor ρ, and the number of spares n, assuming that other input variables are fixed

  - If ρ is L, then n is S
  - If ρ is M, then n is M
  - If ρ is H, then n is L

  » Then consider other 9 rules representing the relation between (m and s), and n

s \ *m*	VS	S	M
L	M	S	VS
M	RL	RS	S
S	VL	L	M

Example: If m is VS and s is S, then n is VL

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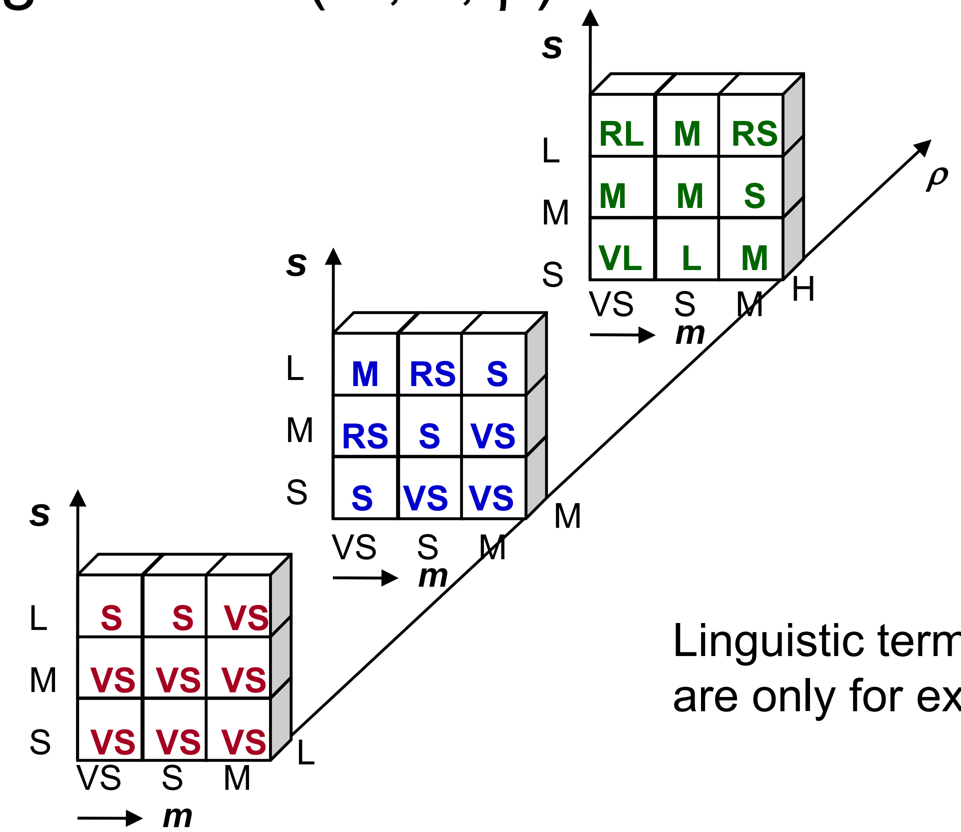
Fuzzy rules

Step 3: construct fuzzy rules (cont.)

Scheme-2 (total 27 rules)

» Consider mapping relation (m, s, ρ) → n



Linguistic terms given here are only for examples

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Fuzzy rules

Rule	*m*	*s*	ρ	*n*	Rule	*m*	*s*	ρ	*n*	Rule	*m*	*s*	ρ	*n*
1	VS	S	L	VS	10	VS	S	M	S	19	VS	S	H	VL
2	S	S	L	VS	11	S	S	M	VS	20	S	S	H	L
3	M	S	L	VS	12	M	S	M	VS	21	M	S	H	M
4	VS	M	L	VS	13	VS	M	M	RS	22	VS	M	H	M
5	S	M	L	VS	14	S	M	M	S	23	S	M	H	M
6	M	M	L	VS	15	M	M	M	VS	24	M	M	H	S
7	VS	L	L	S	16	VS	L	M	M	25	VS	L	H	RL
8	S	L	L	S	17	S	L	M	RS	26	S	L	H	M
9	M	L	L	VS	18	M	L	M	S	27	M	L	H	RS

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Inference model

Step 4: select an appropriate inference model

In general, you need to

- » Choose a fuzzy implication
- » choose a compositional operator with proper T-norm and T-conorm
- » choose a defuzzification operator

- Mamdani's fuzzy model associated with max-min composition is adopted in many fuzzy expert system for its
  - » simplicity of model
  - » high interpretability

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Implementation

- Step 5: implement the fuzzy system
  - » We may build the system using a programming language, such as Python, R, C++/C, Java, or using fuzzy logic development tool, such as fuzzy TECH.

- More fuzzy systems software, http://sci2s.ugr.es/fss

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Evaluation and fine-tuning

- Step 6: evaluate and tune the system
  - » In this stage, we want to see whether our fuzzy system meets the requirements specified at the beginning.
  - » Our system has three inputs and one output, we can have three-dimensional output surface by varying any two of the inputs and keeping other input constant
  - » Assume test data is available. We may compare the performance of Scheme-1 and Scheme-2, to select a better solution.

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Evaluation and fine-tuning

- Step 6: evaluate and tune the system
  - » If we are not satisfied with the performance, may consider further improve it with possible actions:
    - ♦ Have additional fuzzy sets (change the fuzzy partition)
      - Make the fuzzy granules finer
    - ♦ Adjust the fuzzy sets definition
      - E.g.: change the shape, the key points, the range, …
    - ♦ Modify the fuzzy rules
      - E.g.: for the same combination of input linguistic values, choose different linguistic value of output
    - ♦ Adjust weights of fuzzy rules

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Fuzzy expert system: Summary

- Fuzzy expert systems materialise uncertainty processing and fuzzy decision making by offering:
  - » Fuzzy interface which allows fuzzy input/output
  - » Fuzzy knowledge representation
    - ♦ fuzzy sets, fuzzy numbers
    - ♦ fuzzy rules with confidence or weight
  - » Fuzzy inference
    - ♦ evaluation of input data by fuzzy matching (similarity measure)
    - ♦ rule firing based on uncertainty/confidence measure
    - ♦ uncertainty transferring

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Thank you!

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