IS5152 Data-driven Decision Making Technologies
SEMESTER II 2018-2019
Assignment 2 – Suggested solution

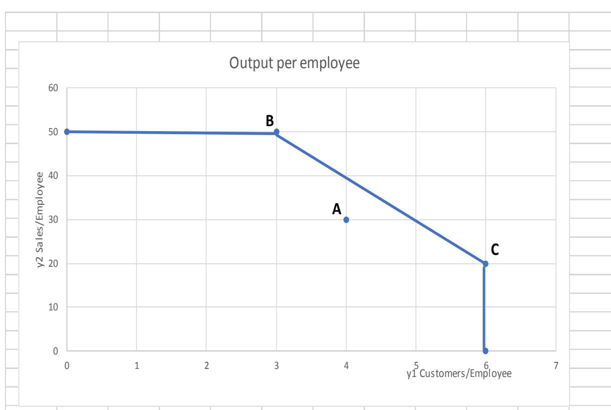
1. (10 points) Consider the data below from 3 well-known convenience shops which include the following information:

   - Input x_1: the number of employees (normalized to 1),
   - Output y_1: the number of customers per employee,
   - Output y_2: the amount of daily sales per employee.

   The number of customers and the amount of daily sales per employee are given in the table below:

Shop	Employees (x_1)	Customers (y_1)	Sales (y_2)
A	1	4	30
B	1	3	50
C	1	6	20

   (a) (2 points) By plotting the efficient frontier for the 3 shops, identify the shop(s) that is/are not efficient.



Shop A is not efficient.

(b) (2 points) Pick an inefficient shop you identify in part (a). For this shop, state a linear program (**LP**) that represents an input oriented CCR model.

$$\max\ 4t_1 + 30t_2$$

subject to

$$\begin{aligned}
(4t_1 + 30t_2)/w_1 &\leq 1\\
(3t_1 + 50t_2)/w_1 &\leq 1\\
(6t_1 + 20t_2)/w_1 &\leq 1\\
w_1 &= 1\\
t_1 &\geq 0\\
t_2 &\geq 0
\end{aligned}$$

or equivalently,

$$\max\ \ 4t_1 + 30t_2$$

subject to

$$\begin{aligned}
4t_1 + 30t_2 &\leq 1\\
3t_1 + 50t_2 &\leq 1\\
6t_1 + 20t_2 &\leq 1\\
t_1 &\geq 0\\
t_2 &\geq 0
\end{aligned}$$

(c) (3 points) Show that the LP you find in part (b) has an optimal objective function that is strictly less than 1.

KT conditions: feasibility +

$$\begin{aligned}
-4 + 4\lambda_1 + 3\lambda_2 + 6\lambda_3 - v_1 &= 0\\
-30 + 30\lambda_1 + 50\lambda_2 + 20\lambda_3 - v_2 &= 0\\
(4t_1 + 30t_2 - 1)\lambda_1 &= 0\\
(3t_1 + 50t_2 - 1)\lambda_2 &= 0\\
(6t_1 + 20t_2 - 1)\lambda_3 &= 0\\
t_1 v_1 &= 0\\
t_2 v_2 &= 0\\
\lambda_1, \lambda_2, \lambda_3, v_1, v_2 &\geq 0
\end{aligned}$$

Suppose $\lambda_1 = 0$ (because Shop A is not efficient), $\lambda_2, \lambda_3 > 0$, then

$$\begin{aligned} 3t_1 + 50t_2 - 1 &= 0 \\ 6t_1 + 20t_2 - 1 &= 0 \\ t_1 &= 1/8 \\ t_2 &= 1/80 \end{aligned}$$

We have

$$\begin{aligned} v_1 = v_2 &= 0 \\ -4 + 3\lambda_2 + 6\lambda_3 &= 0 \\ -30 + 50\lambda_2 + 20\lambda_3 &= 0 \\ \lambda_2 &= 10/24 \\ \lambda_3 &= 11/24 \end{aligned}$$

All optimality conditions are satisfied, objective function value $= 4t_1 + 30t_2 = \lambda_2 + \lambda_3 = 7/8 < 1$.

(d) (3 points) Show that it is possible to create a "composite" shop that is either

- using less input and producing at least the same output, or
- using the same input and producing more output

than the inefficient shop.

Create a composite of Shop B and Shop C with λ_2 and λ_3 as the weights. Its output would be

$$10/24 \begin{pmatrix} 3 \\ 50 \end{pmatrix} + 11/24 \begin{pmatrix} 6 \\ 20 \end{pmatrix} = \begin{pmatrix} 4 \\ 30 \end{pmatrix}$$

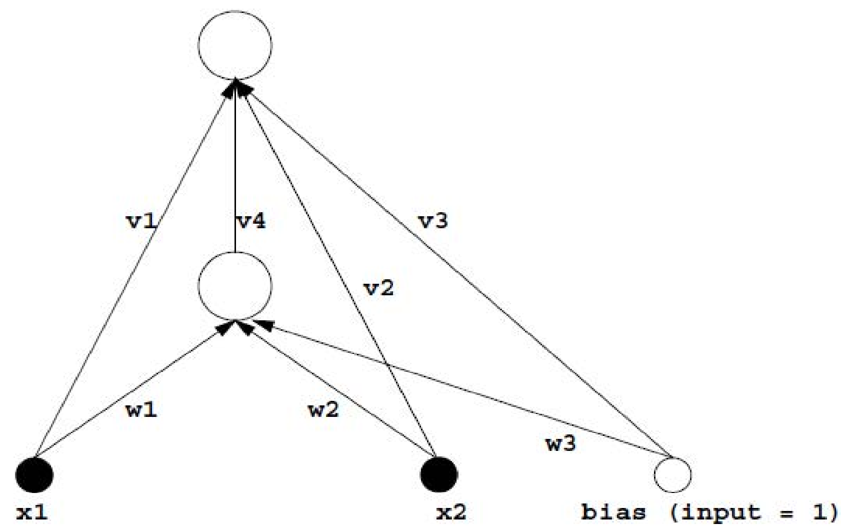
but its input would be $(10/24)(1) + (11/24)(1) = 7/8 < 1$.

2. (10 points) The XOR problem

x_1	x_2	target
-1	-1	1
-1	1	-1
1	-1	-1
1	1	1

can be solved by a neural network with one hidden unit if we allow direct connections between the input units and the output unit (see the figure below).

(a) (4 points) You are given the following weight values: $w_1 = 20, w_2 = 20, w_3 = -30, v_1 = -20, v_2 = -20, v_4 = 40$. Find the value(s) of v_3 such that all four samples in the problem are correctly classified. Assume that the hyperbolic activation function is used for both the hidden unit and the output unit.



x_1	x_2	$x_1w_1+x_2w_2+w_3$	$H=\delta(w_1x_1+w_2x_2+w_3)$	$o=\delta(v_4H+x_1v_1+x_2v_2+v_3)$
-1	-1	-70	$\delta(-70)=-1$	$\delta(-40+20+20+v_3)=\delta(v_3)$
-1	1	-30	$\delta(-30)=-1$	$\delta(-40+20-20+v_3)=\delta(v_3-40)$
1	-1	-30	$\delta(-30)=-1$	$\delta(-40-20+20+v_3)=\delta(v_3-40)$
1	1	10	$\delta(10)=1$	$\delta(40-20-20+v_3)=\delta(v_3)$

We find v_3 such that $\delta(v_3)>0$ and $\delta(v_3-40)<0$, that is, $0<v_3<40$.

(b) (6 points) Describe how the weights v_1 and w_1 will be updated using the error backpropagation method when an input (x_1, x_2) is presented to the network. Note: let the error be $E=\frac{1}{2}(d_1-o_1)^2$.

$$
\begin{aligned}
E &= \frac{1}{2}(d_1-o_1)^2 \\
&= \frac{1}{2}(d-\delta(x_1v_1+x_2v_2+v_3+v_4\delta(x_1w_1+x_2w_2+w_3)))^2 \\
\frac{dE}{dv_1} &= -(d_1-o_1)\times(1-o_1^2)\times x_1 \\
\frac{dE}{dw_1} &= -(d_1-o_1)\times(1-o_1^2)\times v_4\times(1-H^2)\times x_1 \\
\text{where } H &= \delta(x_1w_1+x_2w_2+w_3)
\end{aligned}
$$

Update weights as follows (with learning rate $\eta>0$):

$$
\begin{aligned}
v_1(k+1) &= v_1(k)-\eta\frac{dE}{v_1} \\
w_1(k+1) &= w_1(k)-\eta\frac{dE}{w_1}
\end{aligned}
$$

3. (10 points) A firm produces two types of products: A and B. Production of either product requires an average of 3 hours. The plant has a normal production capacity of 135 hours per week. According to the marketing department, in each week 30 product A and 25 product B can be sold. Profits per unit of product A and product B are \$5 and \$6, respectively. Management has set the following goals, in order of decreasing importance:

   - minimize underutilization of production capacity.
   - avoid overtime operation of the plan beyond 15 hours per week.
   - meet the demand for both products.
   - maximize profit.

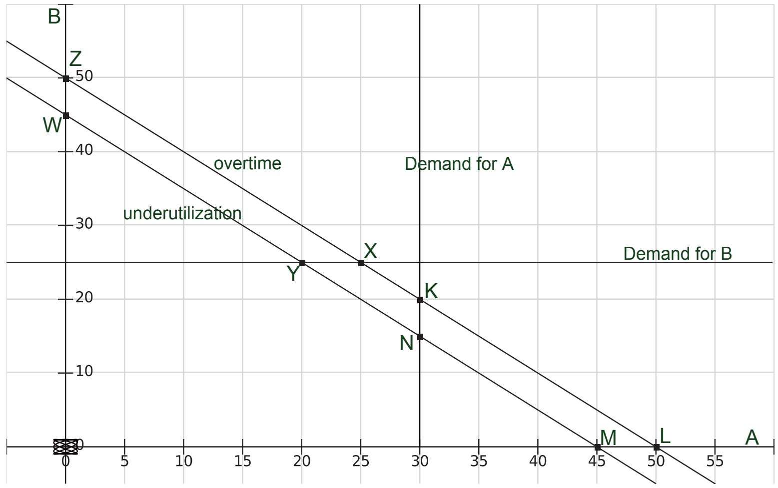
   (a) (4 points) Formulate a preemptive goal programming model for this situation. Let A = the number of product A produced, B = the number of product B produced.

   - minimize underutilization of production capacity.
     $3A + 3B + s_1^- - s_1^+ = 135$
   - avoid overtime operation of the plan beyond 15 hours per week.
     $3A + 3B + s_2^- - s_2^+ = 150$
   - meet the demand for both products.
     $A + s_3^- - s_3^+ = 30$
     $B + s_4^- - s_4^+ = 25$
   - $A, B, s_i^-, s_i^+ \geq 0, i = 1, 2, 3, 4.$

   Objective function: minimize $Z = P_1 s_1^- + P_2 s_2^+ + P_3(s_3^- + s_4^-) - (5A + 6B)$, where $P_1 >>> P_2 >>> P_3$.

   (b) (6 points) Solve the problem using the graphical method. State which goals (if any) are not achieved.

   - Points in ZWML satisfy the first 2 goals.
   - Points in ZWYX have $B \geq 25$, but $A < 30$; while those in $KLMN$ have $A \geq 30$, but $B < 25$.
   - Points in LZ all have $A + B = 50$, but only those in KX have $|30 - A| + |25 - B| = |30 - A| + |25 - (50 - A)| = |30 - A| + |A - 25| = 5$, all others have $|30 - A| + |25 - B| > 5$.
   - Profit at X(25,25) = 275 is higher than profit at K(30,20) = 270.
   - Solution: produce 25 A and 25 B, total profit = \$275.
   - The goal of meeting demand for product A is not met.



4. (10 points) A group of dentists is considering opening of a new private clinic. If the demand for dentists is high (that is, there is a favorable market for the clinic), the dentists could realize a net profit of \$1,000,000. If the market is not favorable, they would lose \$400,000. If they do not proceed at all, there will be no cost/profit. In the absence of any market data, the best the dentists can guess is that there is a 50-50 chance the clinic will be successful. The dentists may engage a market research firm to perform a study of the market, at a fee of \$50,000. The market researchers claim their past experience shows that when the markets were favorable, their study correctly predicted success 70% of the time. Thirty percent of the time the study falsely predicted a failure. On the other hand, when the market condition was unfavorable, the study was correct 80% of the time in predicting a failure. The remaining 20% of the time, it incorrectly predicted a success.

   (a) (2 points) State the Expected Value of Perfect Information.

   - EV with PI = 0.5(1,000,000) + 0.5(0) = 500,000
   - Expected value with original information $= 0.5 \times 1,000,000 + 0.5 \times (-400,000) = 300,000$
   - EV of PI = 500,000 - 300,000 = 200,000

   (b) (6 points) State the Expected Value of Sample Information.

   Define: FM = favourable market, UM = unfavorable market, FS = favorable study, US = unfavorable study

   Conditional probabilities:

$$P(FS|FM) = 0.70, \qquad P(US|UM) = 0.8$$
$$P(US|FM) = 0.30, \qquad P(FS|UM) = 0.2$$

   Original probabilities: $P(FM) = 0.5, P(UM) = 0.5$

Joint probabilities:

$$\begin{aligned}
P(FS \cap FM) = P(FS|FM) \times P(FM) &= 0.35 \\
P(FS \cap UM) = P(FS|UM) \times P(UM) &= 0.10 \\
P(US \cap FM) = P(US|FM) \times P(FM) &= 0.15 \\
P(US \cap UM) = P(US|UM) \times P(UM) &= 0.40
\end{aligned}$$

Marginal probabilities:

$$\begin{aligned}
P(US) &= 0.55 = P(US \cap FM) + P(US \cap UM) \\
P(FS) &= 0.45 = 1 - P(US)
\end{aligned}$$

Revised probabilities:

$$\begin{aligned}
P(FM|FS) = P(FM \text{ and } FS)/P(FS) &= 35/45 = \frac{7}{9} \\
P(UM|FS) = P(UM \text{ and } FS)/P(FS) &= 10/45 = \frac{2}{9} \\
P(FM|US) = P(FM \text{ and } US)/P(US) &= 15/55 = \frac{3}{11} \\
P(UM|US) = P(UM \text{ and } US)/P(US) &= 40/55 = \frac{8}{11}
\end{aligned}$$

With sample information:

If the study indicates favorable market: expected return $= \frac{7}{9} \times 1,000,000 + \frac{2}{9} \times (-400,000) = 6,200,000/9$. Open the clinic.

If the study indicates unfavorable market: expected return $= \frac{3}{11} \times 1,000,000 + \frac{8}{11} \times (-400,000) = -200,000/11$. Do not open the clinic.

Expected value with sample information $= 0.45 \times 6,200,000/9 + 0.55 \times 0 = 310,000$

Expected value with original information $= 0.5 \times 1,000,000 + 0.5 \times (-400,000) = 300,000$

Expected value of sample information = 310,000 - 300,000 = 10,000

(c) (2 points) What is the best decision that the dentists should take?

Since the expected value of sample information is less than the fee for the study, the best decision is to open the clinic without conducting a study of the market condition.