Master of Technology

Computational Intelligence II

Fuzzy Sets

Dr. Zhu Fangming
isszfm@nus.edu.sg

GU Zhan (Sam)
issgz@nus.edu.sg

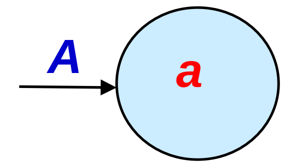
Institute of Systems Science
National University of Singapore

Objectives

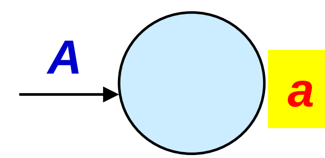
- To provide introduction to the basic concepts of fuzzy sets.
- To provide detailed description of fuzzy number, extension principle and fuzzy relation.

Fuzzy Sets vs. Crisp Sets

- Crisp sets
  - » for an individual *a*, $a \in A$ means
    - ♦ *a* is a member of the set *A*



  - » while $a \notin A$ means
    - ♦ *a* is *not* a member of the set *A*



  - » There are *only* two possible relationships between the individual *a* and the set *A*:
    - ♦ $a \in A$ (membership = 1 → 100% belonging) or
    - ♦ $a \notin A$ (membership = 0 → 0% belonging)

Fuzzy Sets vs. Crisp Sets (cont.)

- *Fuzzy sets*
  - » There is no full/zero membership in general
    - ♦ Is 45-yrs *old* or *young*?
    - ♦ Is 70/100 marks a *good* or *poor* exam result?
- Object dependent vs. subject dependent
  - » Crisp: e.g. ice, water, steam defined on temperature
  - » Fuzzy: e.g. cold, warm
- Sharp vs. Unsharp boundary
  - » Crisp: e.g. teenage (between 13 and 19 years old)
  - » Fuzzy: e.g. young, old

Defining Fuzzy Subset

- A fuzzy set is completely characterized by its *membership function* (MF).
  - » Let X be the *universe of discourse* (*universal set*), the *fuzzy subset* $A \subseteq X$ is defined as the set of ordered pairs

$$A = \{(x, \mu_A(x))\}, \qquad x \in X$$

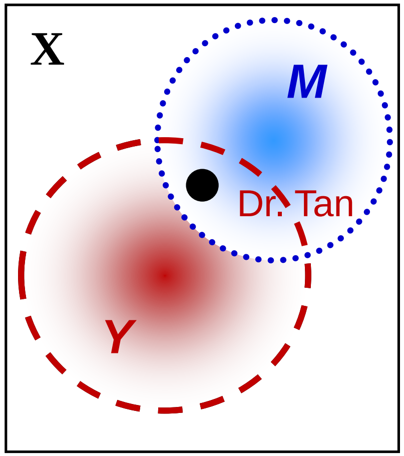
where $0 \leq \mu_A(x) \leq 1$ or $\mu_A(x) \in [0, 1]$ is the membership function (MF) that assigns to each element $x \in X$ a number in the closed unit interval [0, 1]

$$\mu_A: \quad X \rightarrow [0, 1]$$

☝ *In defining a membership function, the universal set X is always assumed to be a classical (crisp) set.*

Fuzzy Subset

- When there is no confusion caused, a fuzzy subset is also called *fuzzy set*.
- Example:
  - » universe of discourse (universal set):
    - ♦ X = ISS-lecturers
  - » fuzzy subsets:
    - ♦ *M* = middle-aged lecturers
    - ♦ *Y* = young lecturers
  - » Dr. Tan is 35 years old:
    - ♦ $\mu_M(\text{Tan}) = 0.45$ $\mu_Y(\text{Tan}) = 0.40$



Crisp Subset as Special Fuzzy Subset

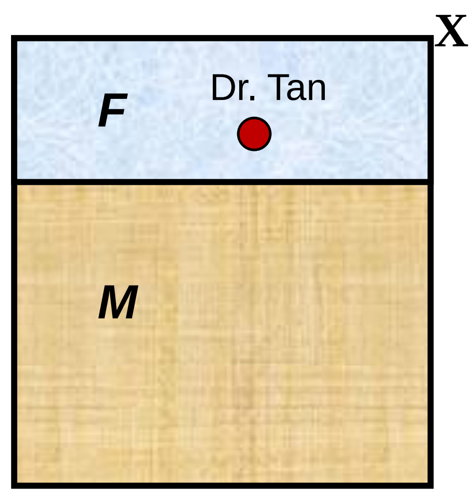
- A crisp subset can be considered as a special case of fuzzy subset when the membership degree is either 1 or 0:

$$\mu_A(x) \in \{0, 1\}$$

- Example:
  - » Crisp subsets:

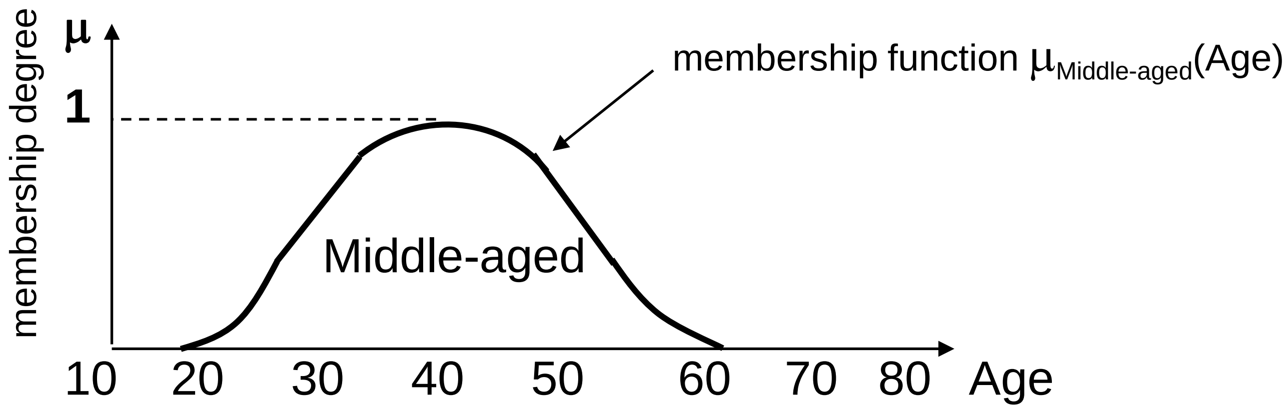
    M = Male lecturers

    F = Female lecturers



Notations of Fuzzy Set

- A fuzzy set *A* can be represented in several ways
  - » *Graphical representation*



Notations of Fuzzy Set (cont.)

» *List representation* (for discrete universe)

$A = \{<x_1, \mu_A(x_1)>, <x_2, \mu_A(x_2)>, \ \dots <x_n, \mu_A(x_n)>\}$

or $A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \dots + \mu_A(x_n)/x_n$

small-integer = {<0, 1>, <1,0.94>, <2,0.8>, …, <100,0>}

small-integer = 1/0 + 0.94/1 + 0.8/2 + 0.64/3 + …+ 0/100

(*) The symbol "/" stands for the correspondence between an element in the universal set and its membership grade in the fuzzy set. The symbol "+" merely connects the elements.

Notations of Fuzzy Set (cont.)

» *Function representation* (for continuous universe)

$$A = \int_X \mu_A(x) \,/\, x$$

(*) The symbol "$\int$" indicates the union of the elements in A.

» The generalized notation commonly used in the literature has the form:

$$A = \Sigma A(x)/x, \qquad x \in X$$

An important feature:

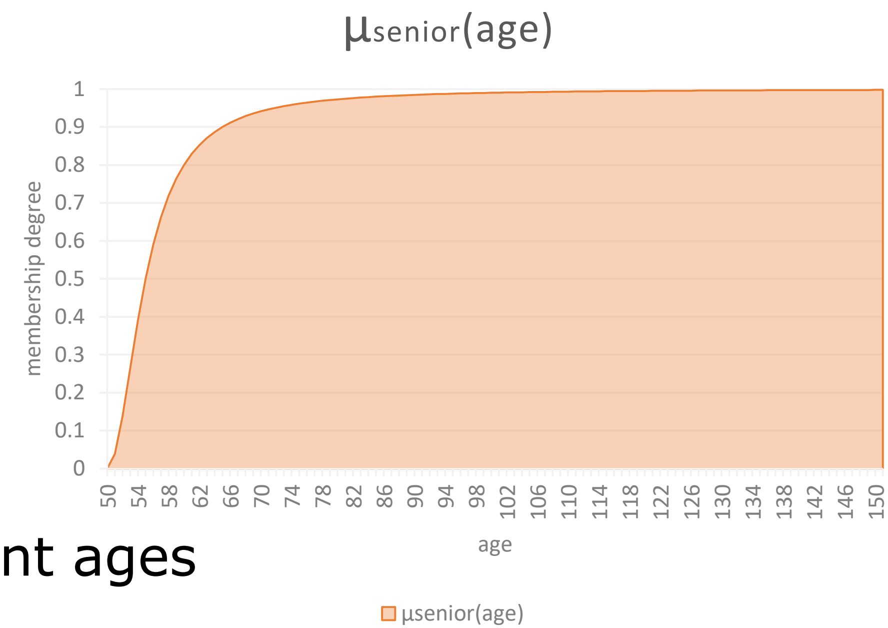
probability: $\Sigma_X p(x_i) = 1$

fuzzy set: $\Sigma_X \mu(x_i) \neq 1$

Notations of Fuzzy Set (cont.)

- Example:
  - » Zadeh's *Senior* membership function for 51 ≤ age ≤ 150

$$\mu_{senior}(age) = \frac{1}{1+\left(\frac{5}{age-50}\right)^2}$$



  - » Membership degrees of different ages

$\mu_{senior}(55) = 0.5$

$\mu_{senior}(60) = 0.8$

$\mu_{senior}(70) \approx 0.94$

Formulation and Parameterization of MF

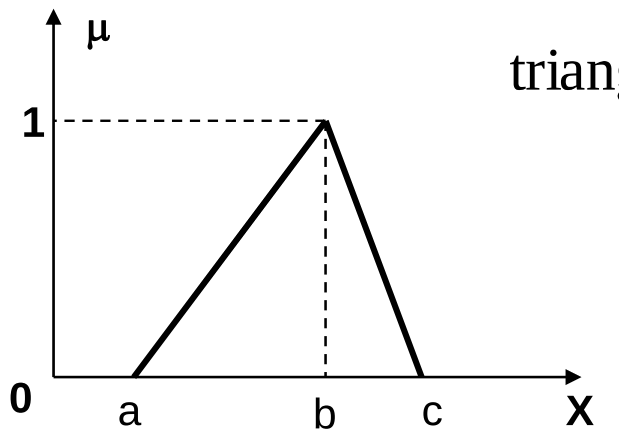
- In most cases of fuzzy sets, it would be impractical to list all the pairs defining a membership function.

  » A more convenient and concise way to define an MF is to express it as a mathematical formula (such as the *senior* MF by Zadeh), if possible.

  » Parameterized functions commonly used to define MFs
    - ♦ Triangular
    - ♦ Trapezoidal
    - ♦ Gaussian
    - ♦ Bell
    - ♦ Sigmoidal

Parameterized MFs

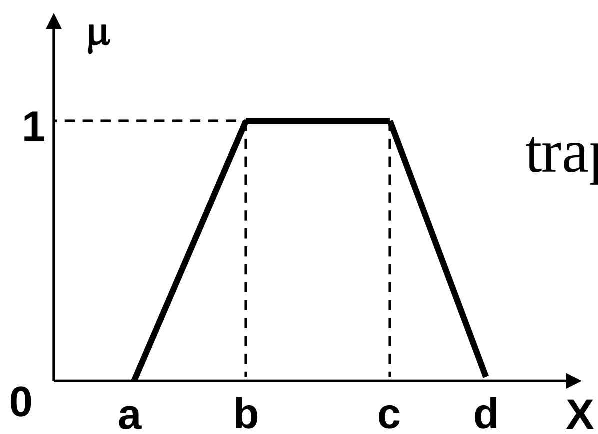
- Triangular MFs
  - » specified by three parameters {*a*, *b*, *c*} (*a* < *b* < *c*)



$$\text{triangular}(x; a, b, c) = \begin{cases} 0 & x \le a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ \dfrac{c-x}{c-b} & b \le x \le c \\ 0 & c \le x \end{cases}$$

Parameterized MFs (cont.)

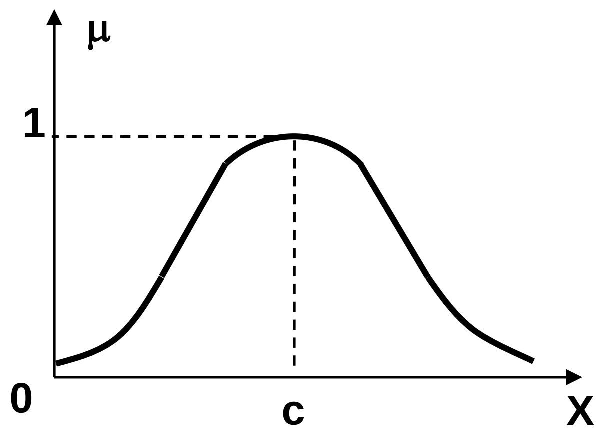
- Trapezoidal MFs
  - » specified by four parameters $\{a, b, c, d\}$ ($a < b \leq c < d$)



$$\text{trapezoid}(x; a, b, c, d) = \begin{cases} 0 & x \leq a \\ \dfrac{x-a}{b-a} & a \leq x \leq b \\ 1 & b \leq x \leq c \\ \dfrac{d-x}{d-c} & c \leq x \leq d \\ 0 & d \leq x \end{cases}$$

Parameterized MFs (cont.)

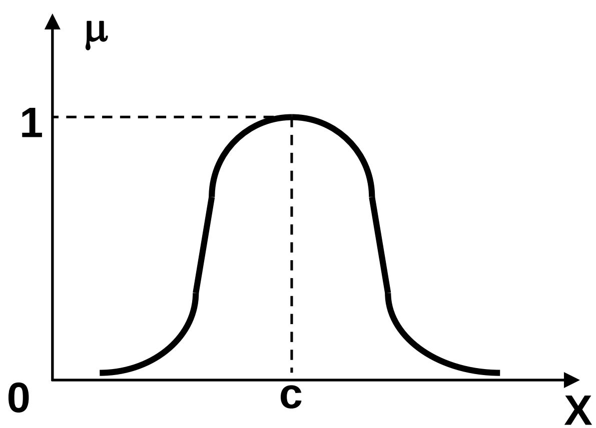
- Gaussian MFs
  - » specified by two parameters $\{c, \sigma\}$



$$\text{gaussian}(x; \mathrm{c}, \sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$

Parameterized MFs (cont.)

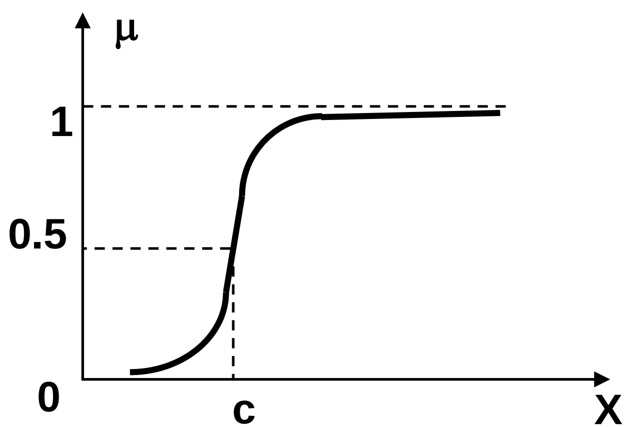
- Bell MFs
  - » specified by three parameters {*a*, *b*, *c*}
  - » *b* is usually positive. (If *b* is negative, the shape of this MF becomes an upside-down bell)



$$\text{bell}(x;a,b,c) = \frac{1}{1+\left|\frac{x-c}{a}\right|^{2b}}$$

Parameterized MFs (cont.)

- Sigmoidal MFs
  - » specified by two parameters $\{a, c\}$
  - » a controls the slope at the crossover point $x = c$



$$\mathrm{sig}(x; a, c) = \frac{1}{1 + \exp[-a(x - c)]}$$

Exercise: Defining Fuzzy Sets

Defining fuzzy sets

— bank's interest rate: *high, low*

— salary: *high, low*

— driving distance: *short, long*

» Determine the universe of discourse

- ♦ continuous or discrete
- ♦ range

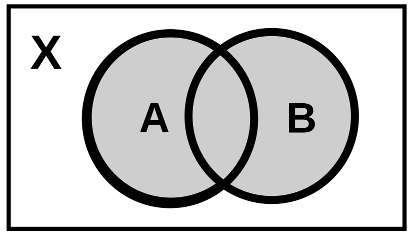
» Determine membership function

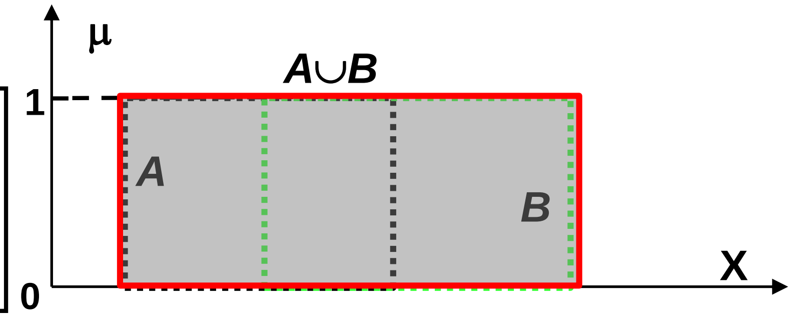
- ♦ mathematical formula (linear or non-linear)
- ♦ list representation

Basic Set Operations: on Crisp Sets

- Union of A and B

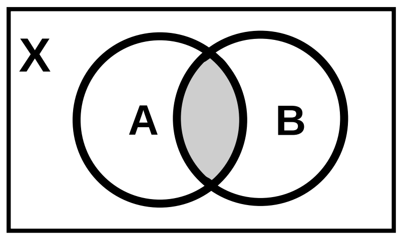
$A \cup B = \{x \mid x \in A \;\; \textit{or} \;\; x \in B\}$

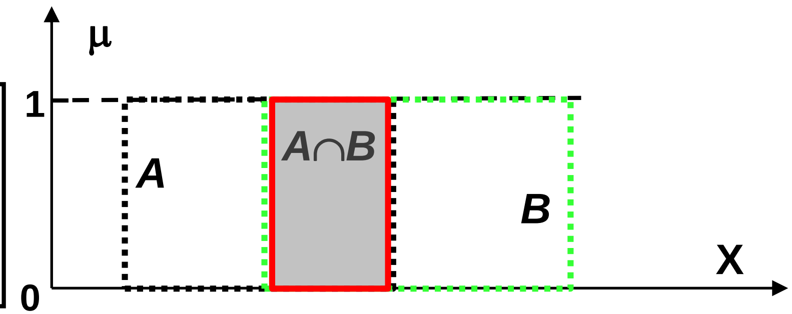




- Intersection of A and B

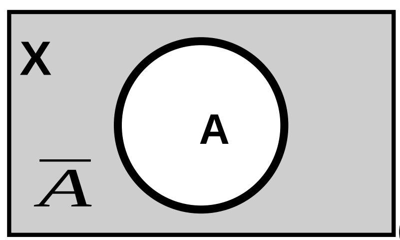
$A \cap B = \{x \mid x \in A \;\; \textit{and} \;\; x \in B\}$

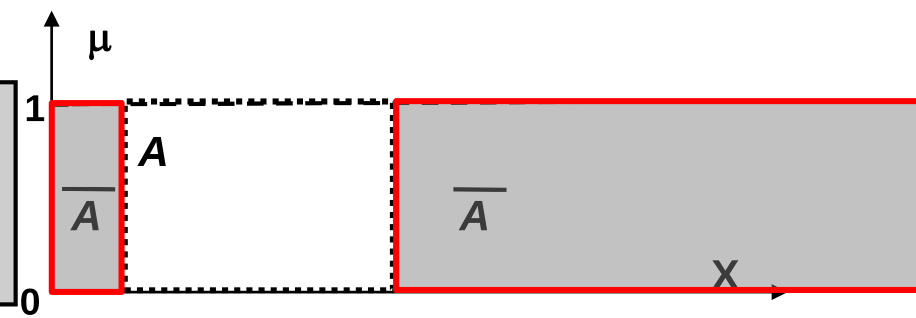




- Complement

$\overline{A} = \{x \mid x \in X, x \notin A\}$

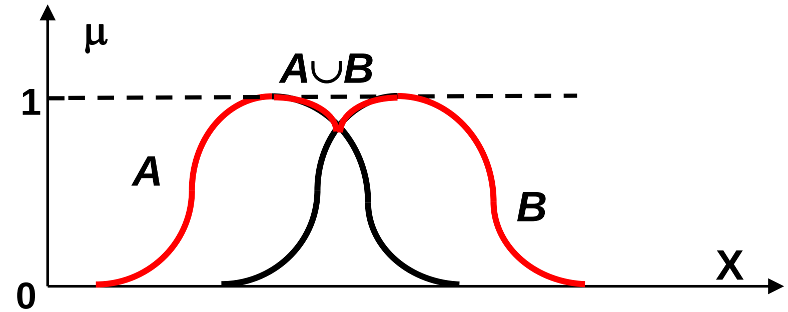




Basic Set Operations: on Fuzzy Sets

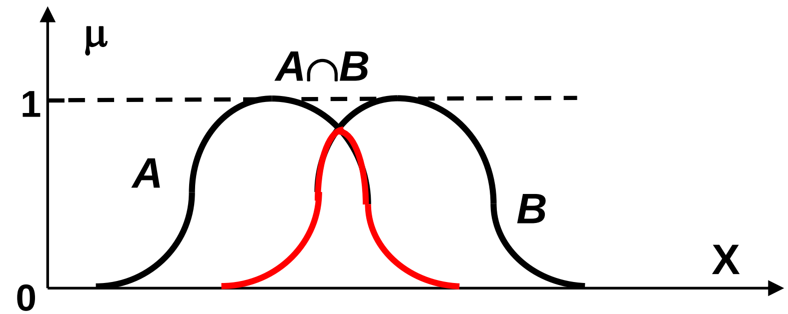
- *Fuzzy Union*

$$A \cup B = \int_X \boldsymbol{max}[\mu_A(x), \mu_B(x)] \;/\; x$$



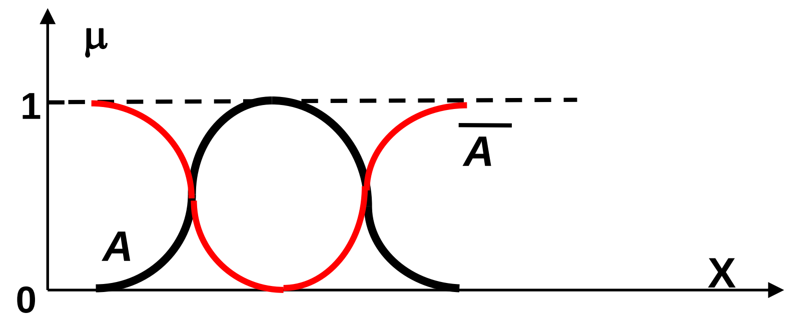
- *Fuzzy Intersection*

$$A \cap B = \int_X \boldsymbol{min}[\mu_A(x), \mu_B(x)] \;/\; x$$



- *Fuzzy Complement*

$$\overline{A} = \int_X [1 - \mu_A(x)] \;/\; x$$



Basic Set Operations: on Fuzzy Sets (cont.)

- Example:

  Universal set of ages:

  $X = \{5, 10, 20, 30, 40, 50, 60, 70, 80\}$

  » *Fuzzy sets*

  young = 1/5+1/10+.8/20+.5/30+.2/40+.1/50+0/60+0/70+0/80

  old = 0/5+0/10+.1/20+.2/30+.4/40+.6/50+.8/60+1/70+1/80

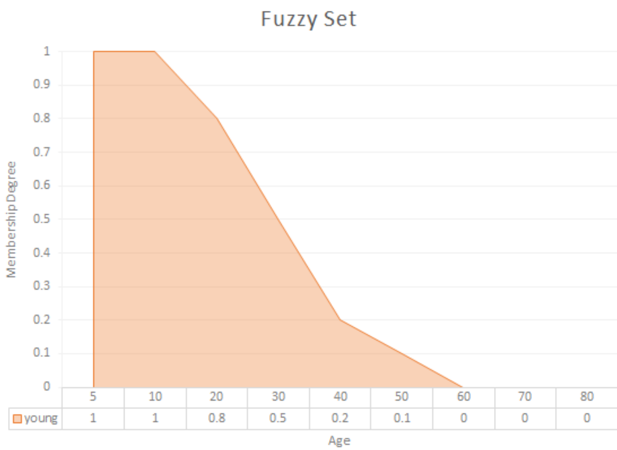
  adult = 0/5+0/10+.8/20+1/30+1/40+1/50+1/60+1/70+1/80

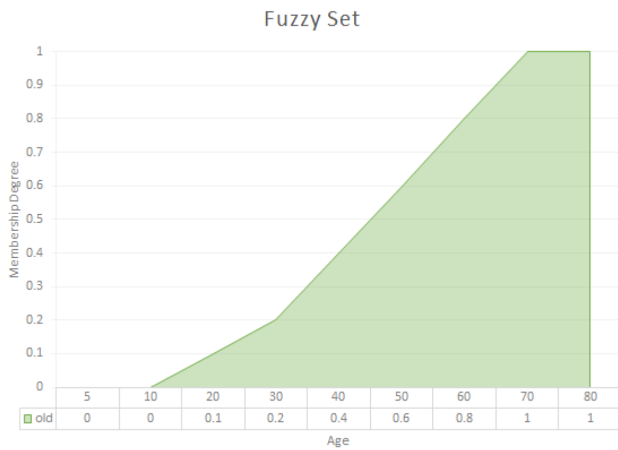
  not-young = 0/5+0/10+.2/20+.5/30+.8/40+.9/50+1/60+1/70+1/80

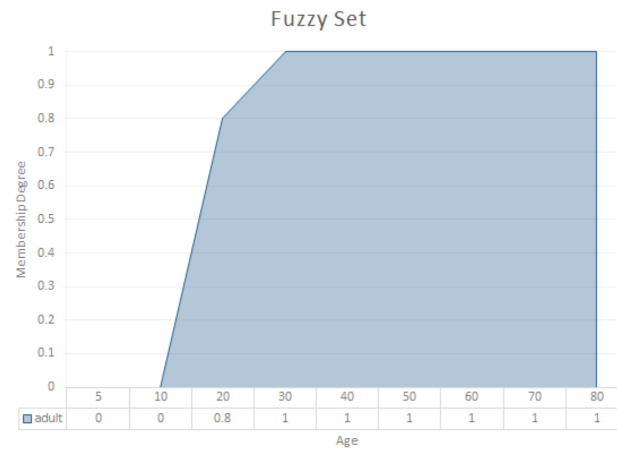
  young∩*old* = 0/5+0/10+.1/20+.2/30+.2/40+.1/50+0/60+0/70+0/80

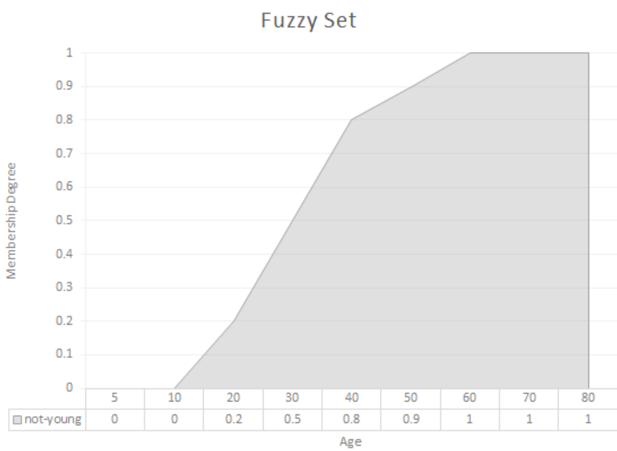
  young∪*old* = 1/5+1/10+.8/20+.5/30+.4/40+.6/50+.8/60+1/70+1/80

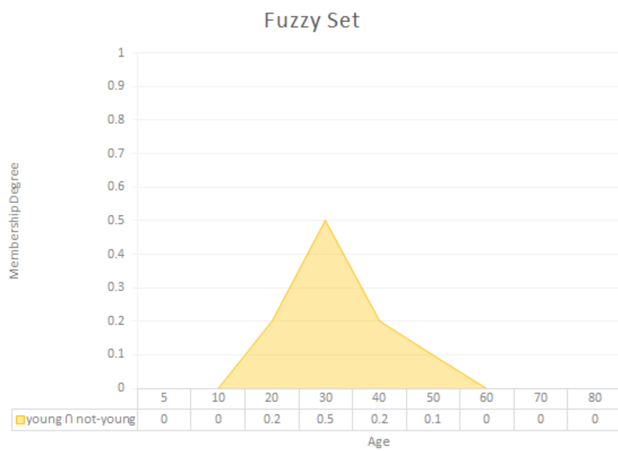
Basic Set Operations: on Fuzzy Sets (cont.)

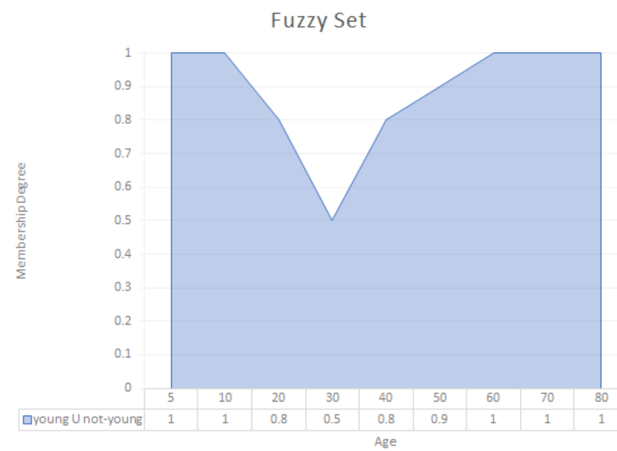






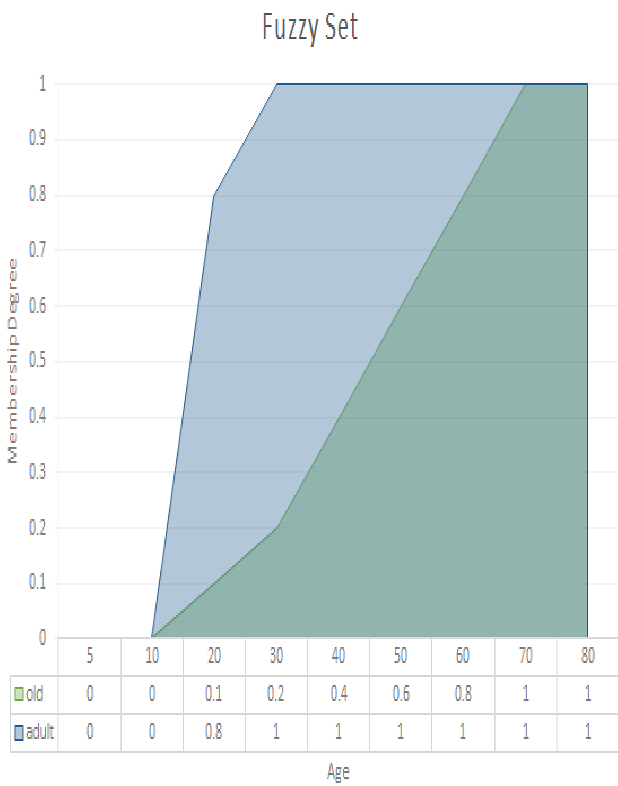




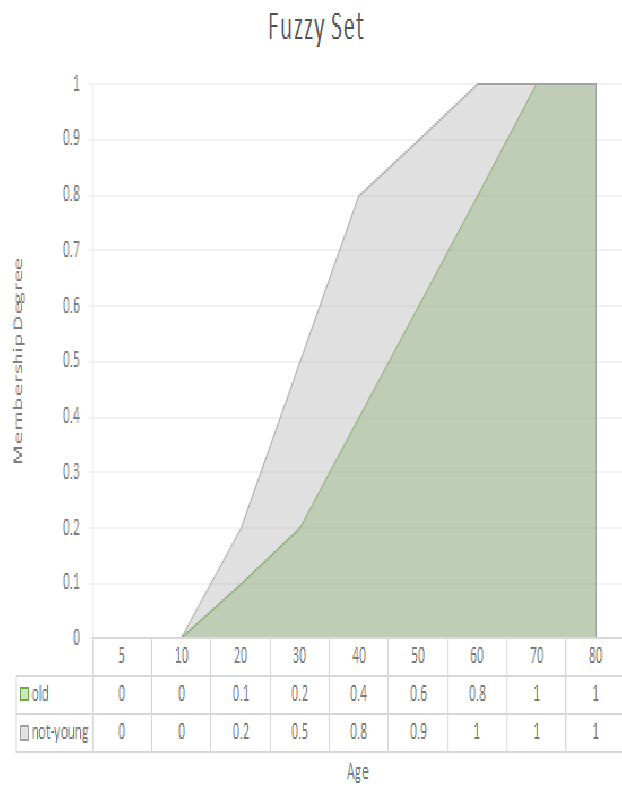


Subsets of Fuzzy Sets

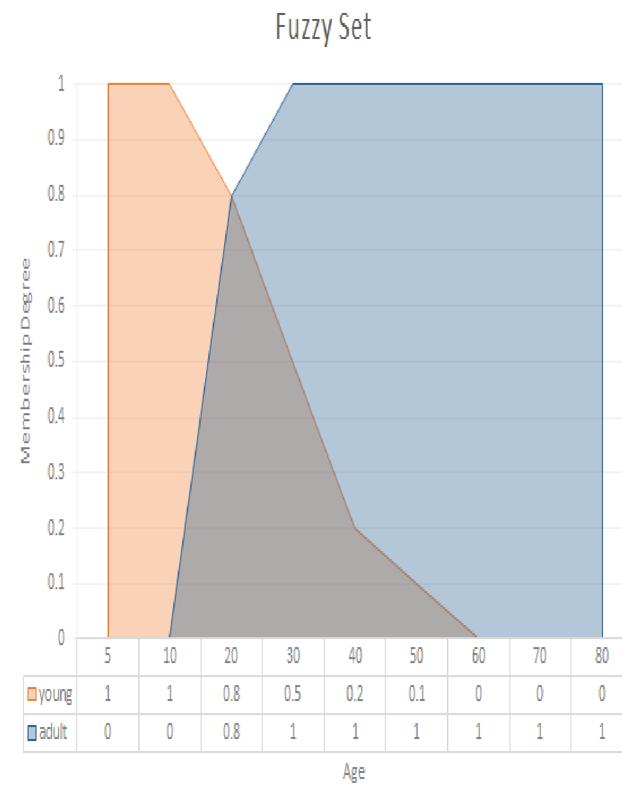
old ⊂ adult



old ⊂ not-young



young ⊄ adult



Subsets of Fuzzy Sets

- The following relationships for fuzzy sets hold:
  - » $A = B$ iff $\mu_A(x) = \mu_B(x) \quad \forall x \in X$
  - » $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x) \quad \forall x \in X$
  - » $A \subset B$ iff $A \subseteq B$ and $A \neq B$

  $\mu_A(x) \leq \mu_B(x)$ for every $x \in X$, and

  $\mu_A(x) < \mu_B(x)$ for *at least one* $x \in X$

- In the previous example

  old $\subset$ *adult* *old* $\subset$ *not-young* *young* $\not\subset$ *adult*

Properties of Set Operations

- The properties hold for both crisp sets and fuzzy sets

Commutativity	$A \cup B = B \cup A \qquad A \cap B = B \cap A$
Associativity	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$
Distributivity	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Idempotence	$A \cup A = A \qquad A \cap A = A$
Absorption by X and ∅	$A \cup X = X \qquad A \cap \varnothing = \varnothing$
Identity	$A \cup \varnothing = A \qquad A \cap X = A$
Involution	$\overline{\overline{\mathbf{A}}} = \mathbf{A}$
DeMorgan's laws	$\overline{(\mathbf{A} \cup \mathbf{B})} = \overline{\mathbf{A}} \cap \overline{\mathbf{B}} \qquad \overline{(\mathbf{A} \cap \mathbf{B})} = \overline{\mathbf{A}} \cup \overline{\mathbf{B}}$

Properties of Set Operations (cont.)

- Two special laws hold only on crisp sets

  Law of contradiction $\mathbf{A} \cap \overline{\mathbf{A}} = \varnothing$

  Law of exclude middle $\mathbf{A} \cup \overline{\mathbf{A}} = \mathbf{X}$

- In general, they **do not** hold on fuzzy sets:

$$\mathbf{A} \cap \overline{\mathbf{A}} \neq \varnothing \qquad \mathbf{A} \cup \overline{\mathbf{A}} \neq \mathbf{X}$$

- Since for any $x \in$ X and $\mu_A(x) = c$, with $0 < c < 1$

$$\mu_{\mathbf{A}\cap\overline{\mathbf{A}}}(x) = Min(c, 1-c) \neq 0$$ (0% belongs to set $\mathbf{A} \cap \overline{\mathbf{A}}$)

$$\mu_{\mathbf{A}\cup\overline{\mathbf{A}}}(x) = Max(c, 1-c) \neq 1$$ (100% belongs to set $\mathbf{A} \cup \overline{\mathbf{A}}$)

Properties of Set Operations (cont.)

Example: using the previous example on ages

young =

1/5 + 1/10 + .8/20 + .5/30 + .2/40 + .1/50 + 0/60 + 0/70 + 0/80

not-young =

0/5 + 0/10 + .2/20 + .5/30 + .8/40 + .9/50 + 1/60 + 1/70 + 1/80

young$\cap$*not-young* =

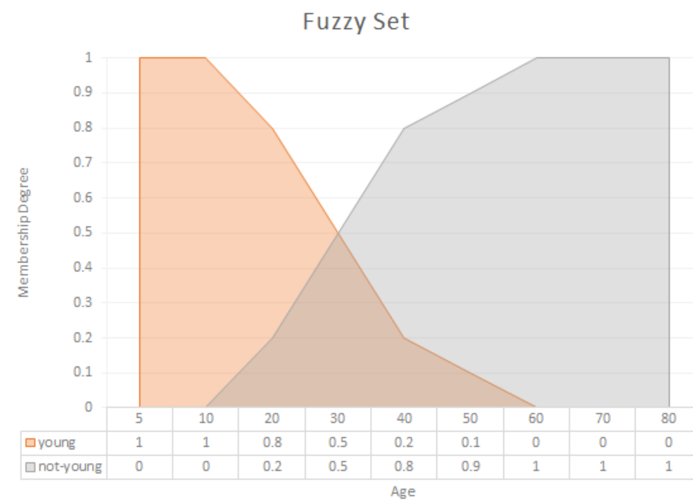
0/5 + 0/10 + .2/20 + .5/30 + .2/40 + .1/50 + 0/60 + 0/70 + 0/80

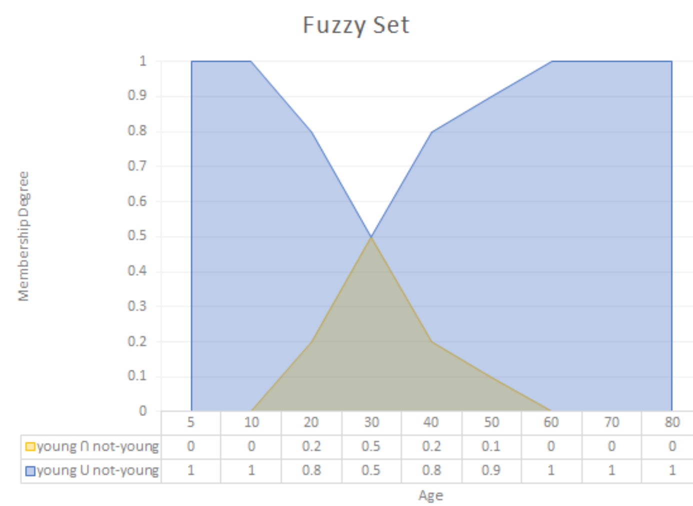
young$\cup$*not-young* =

1/5 + 1/10 + .8/20 + .5/30 + .8/40 + .9/50 + 1/60 + 1/70 + 1/80

young $\cap$ *not-young* $\neq \varnothing$ (0%)

young $\cup$ *not-young* $\neq$ X (100%)

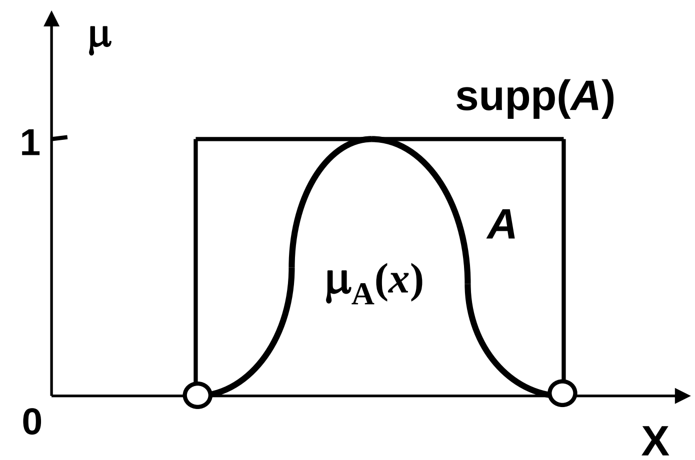




Concepts of Fuzzy Sets

Concepts of Fuzzy Sets: support

- The *support* of a fuzzy set A, supp(A), in the universal set X is the crisp set that contains all the elements $x \in$ X that have a nonzero membership grade in A. That is
  supp(A) = $\{x \mid x \in X, \mu_A(x) > 0\}$

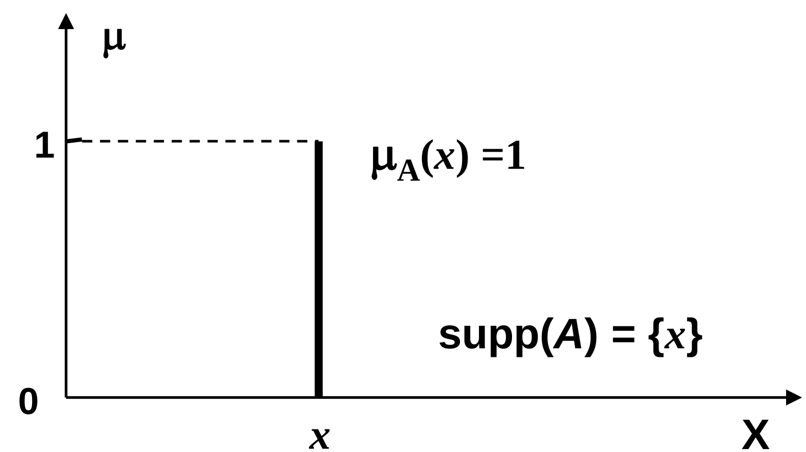


- Example:
  - » *small-integer* = 1/0 + 0.94/1 + 0.8/2 + 0.64/3 +…+ 0.02/99 + 0/100
    supp(*small-integer*) = {0, 1, 2, …, 99}

Concepts of Fuzzy Sets: singleton

Fuzzy Singleton

- a fuzzy set *A* whose support is a single point in the universal set X with $\mu_A(x) = 1$ is referred to as a *fuzzy singleton*.
  - » $A = \{(x, \mu_A(x))\} = \{(x, 1)\}$
  - » $\text{supp}(A) = \{x\}$

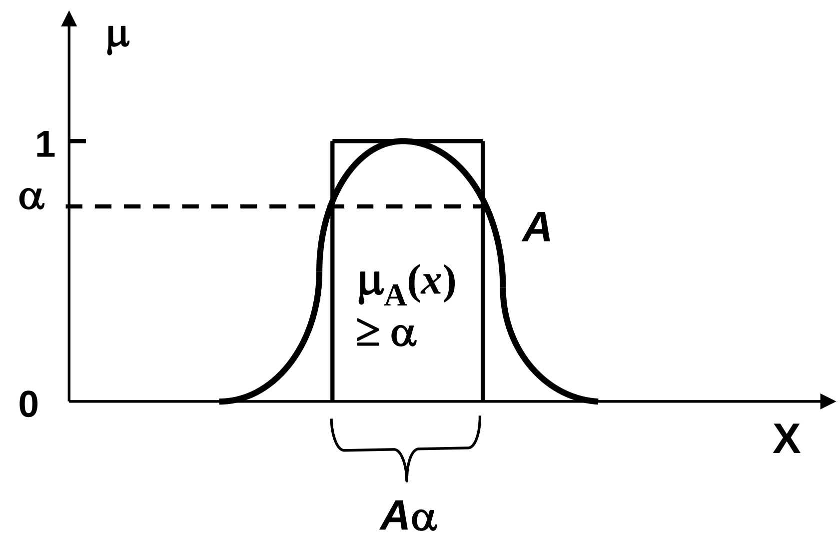


Concepts of Fuzzy Sets: α-cut

α-Level Set

- The *crisp* set of elements that belong to the fuzzy set A at least to the degree α ($0 < \alpha < 1$) is called the *α-level set* or *α-cut*

$$A_\alpha = \{x \mid x \in X, \mu_A(x) \geq \alpha\}$$



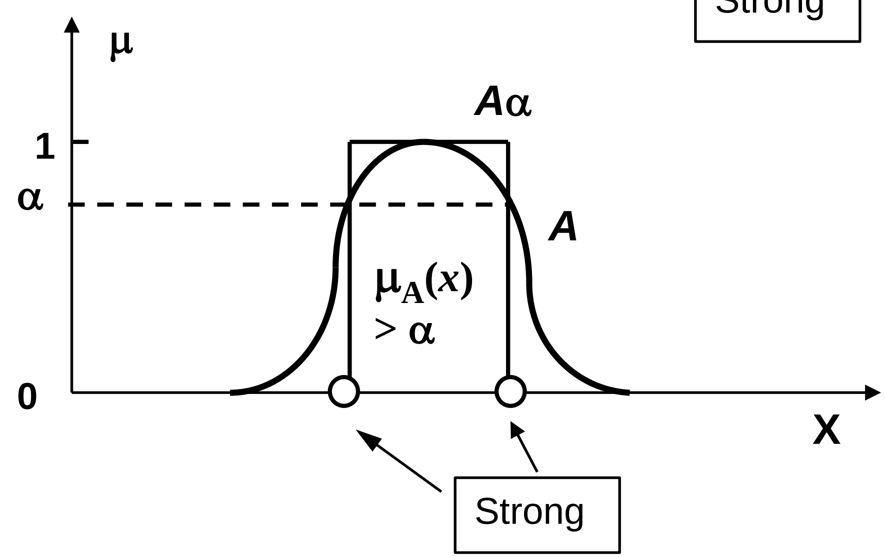
☝ *an α-cut of a fuzzy set is a crisp set*

Concepts of Fuzzy Sets: α-cut (cont.)

- A ***strong α-level set*** or ***strong α-cut*** is defined by

$$A'_\alpha = \{x \mid x \in X, \mu_A(x) > \alpha\}$$

Strong



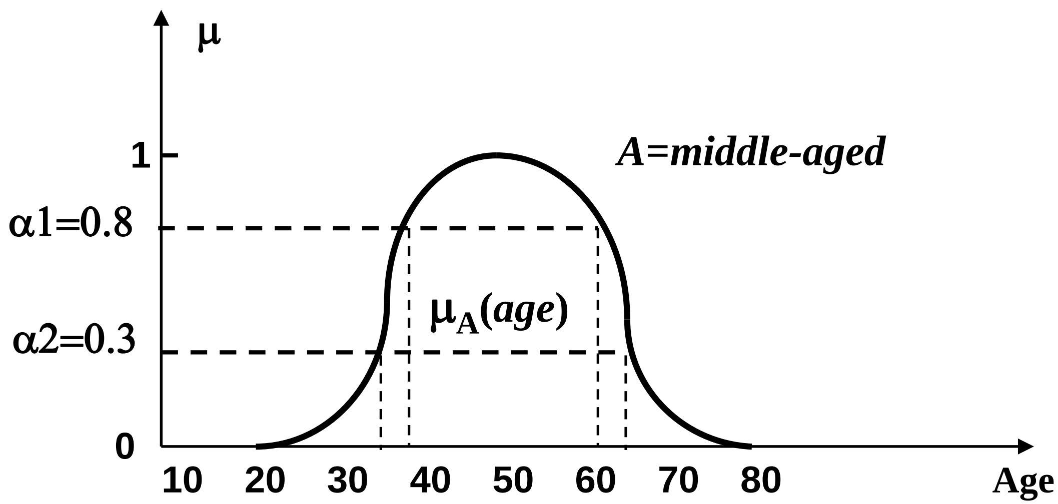
☝ The support is a special case of a strong α-level set where α = 0,

i.e. $A'_0 = \text{supp}(A)$

Concepts of Fuzzy Sets: α-cut (cont.)

- Example:
  - » fuzzy set A = *middle-aged*
  - » supp(A) = {20, 21, …, 79, 80}

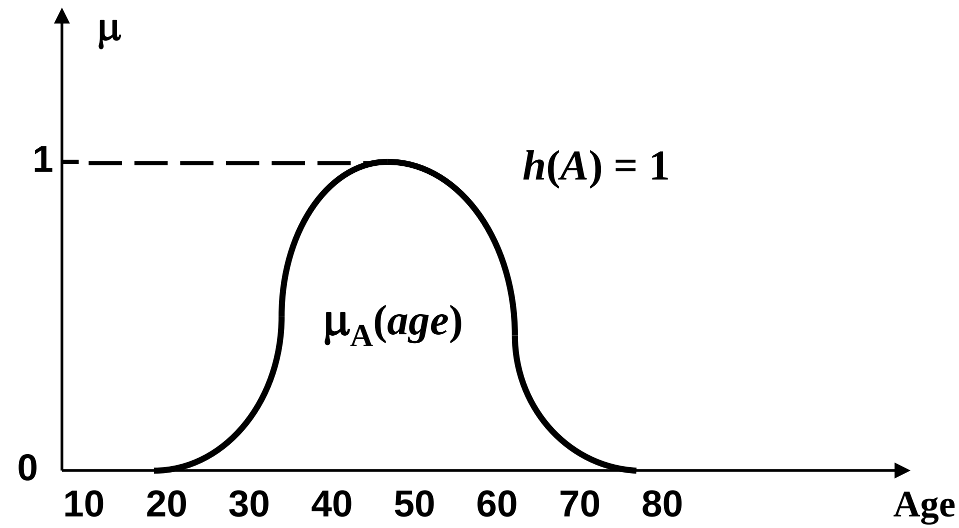
    (suppose that only integers are used)
  - » $A_{0.8}$ = {40, 41, …, 59, 60} $A_{0.3}$ = {36, 37, …, 63, 64}



Concepts of Fuzzy Sets: height

Height of fuzzy set

- The *height* of a fuzzy set, $h(A)$, is the largest membership grade attained by any element in that fuzzy set, or the largest value of α for which the α-cut is not empty



Concepts of Fuzzy Sets: normal

Normal or subnormal fuzzy set

height = 1

- A fuzzy subset A of X is called *normal*
  - » if there exists at least one element $x \in X$ such that

    $\mu_A(x) = 1.$

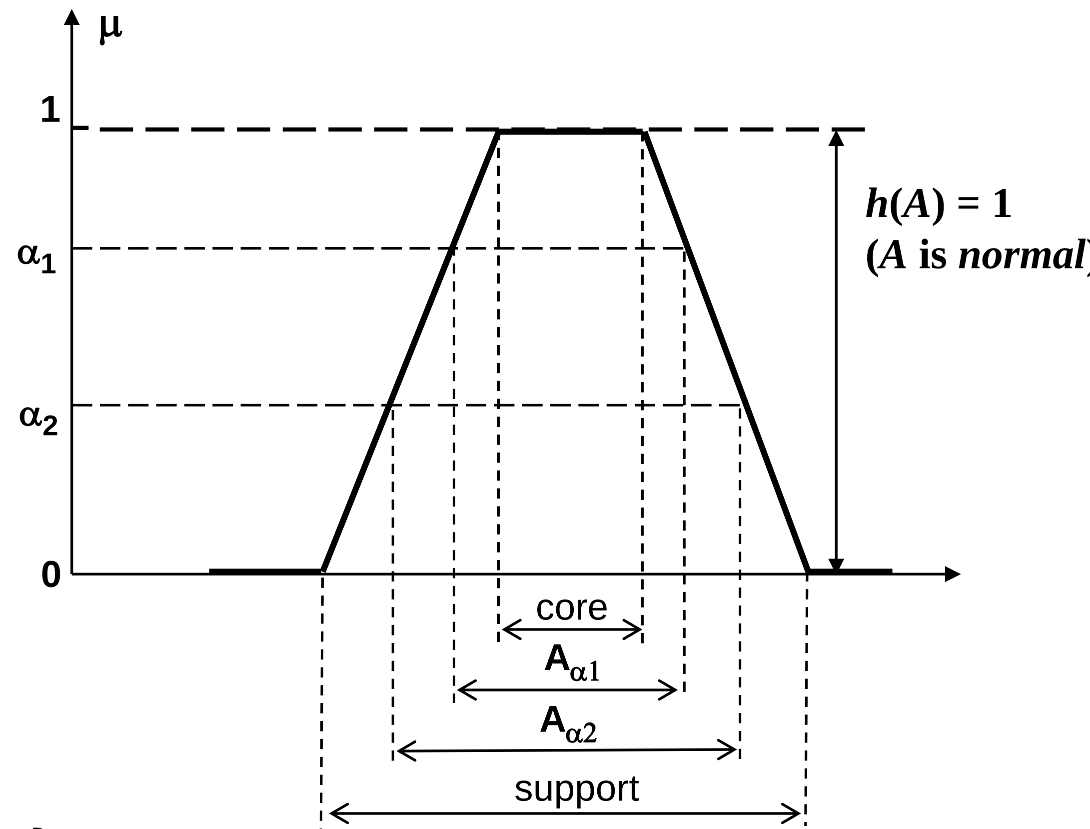
height < 1

- A fuzzy subset that is not normal is called *subnormal*.

Concepts of Fuzzy Sets: core

Core of fuzzy set

- The *core* of the fuzzy set A, core(A), is defined by the special α-cut for $\alpha = 1$:



$$\text{core}(A) =$$
$$A_1 = \{x \mid x \in X, \mu_A(x)=1\}$$

Concepts of Fuzzy Sets: convex

Convex fuzzy set

» a fuzzy set A with its corresponding universal set

$$X = [X_1, X_2]$$

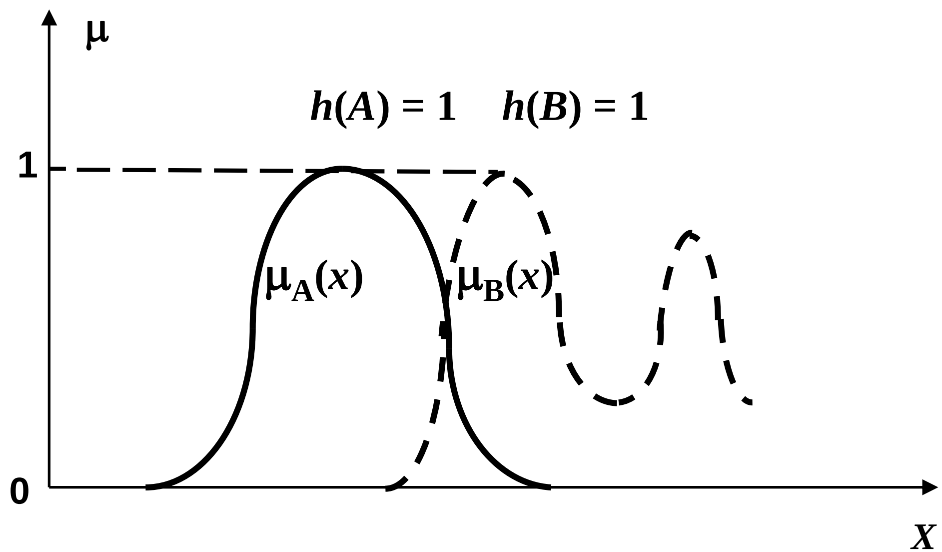
♦ for any interval $[a, b] \subseteq [X_1, X_2]$, $X_1 \leq a \leq b \leq X_2$ and any $x \in [a, b]$, A is a convex fuzzy set when the following condition is satisfied:

$$\mu_A(x) \geq \min[\mu_A(a), \mu_A(b)]$$

Concepts of Fuzzy Sets: convex (cont.)

- Example:

  A is *convex*, but *B* is *non-convex*



Concepts of Fuzzy Sets: subset

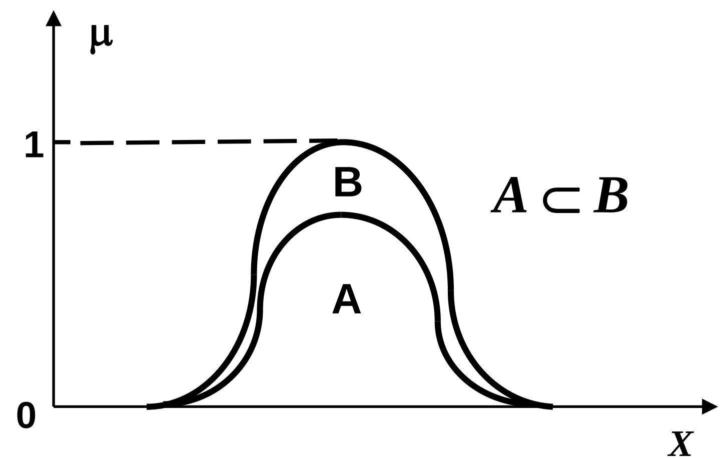
- The following relationships for fuzzy sets hold:
  - » A = B iff $\mu_A(x) = \mu_B(x) \ \forall x \in X$
  - » $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x) \ \forall x \in X$

- Fuzzy set *A* is called a *proper subset* of fuzzy set *B*, if *A* is a subset of *B* and the two sets are not equal

  $A \subset B$ iff $A \subseteq B$ and $A \neq B$

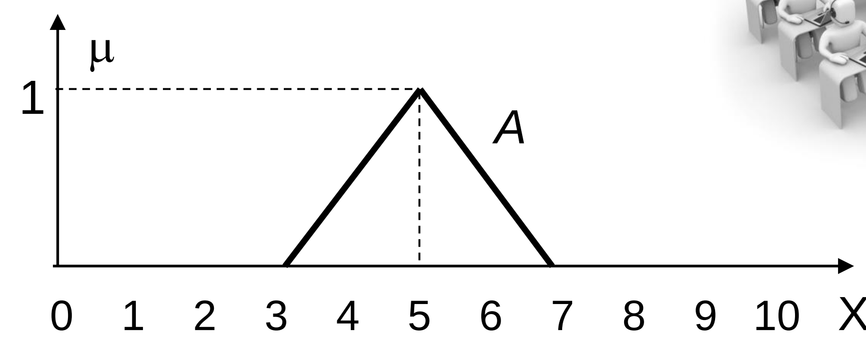
  $\mu_A(x) \leq \mu_B(x)$ for every $x \in X$, and

  $\mu_A(x) < \mu_B(x)$ for *at least one* $x \in X$



Exercise & Discussion: fuzzy set

(1) Given fuzzy set A defined on the universal set $X = [0, 10]$, find the following sets:

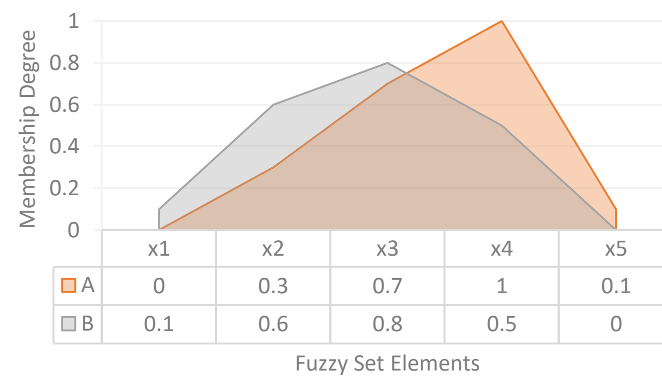


» Supp(A), Core(A)

(2) Given the universal set $X = \{x_1, x_2, x_3, x_4, x_5\}$ and two fuzzy subsets defined on X:

» $A = 0/x_1 + 0.3/x_2 + 0.7/x_3 + 1/x_4 + 0.1/x_5$,

» $B = 0.1/x_1 + 0.6/x_2 + 0.8/x_3 + 0.5/x_4 + 0/x_5$.



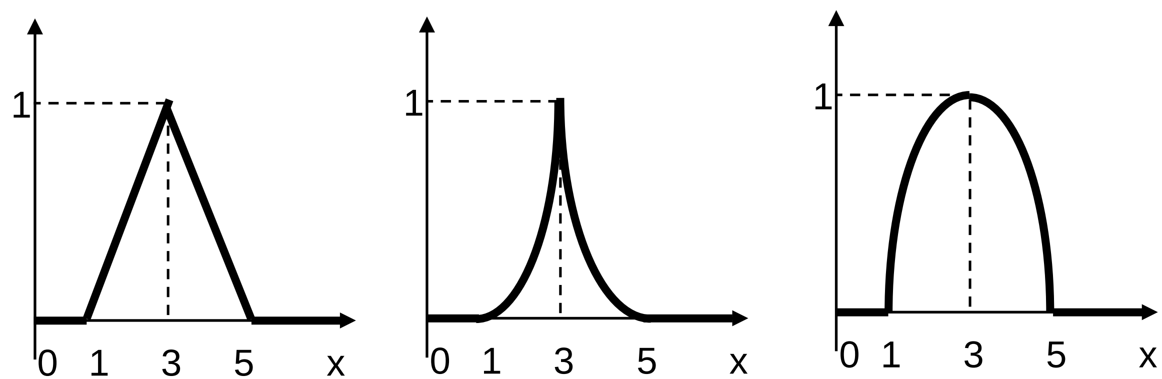
Find the following sets

» Supp(A), Core(B), $(A \cup \overline{B})_{0.4}$, $(A \cap \overline{B})'_{0.2}$

Concepts of Fuzzy Number

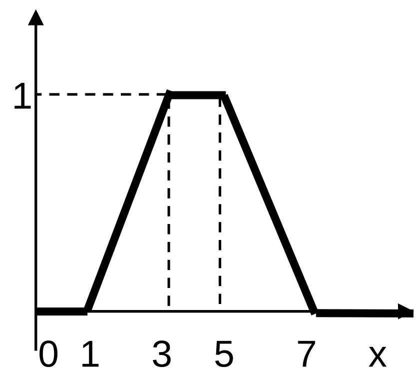
Concepts of Fuzzy Number

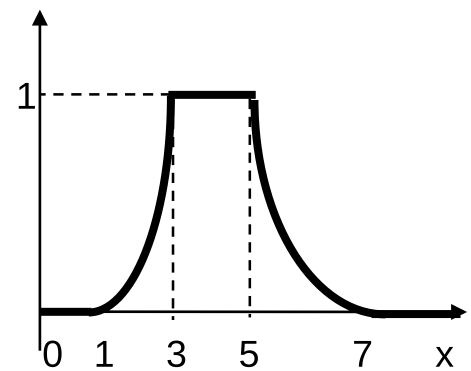
- Fuzzy numbers are fuzzy sets:
  - » the central value has membership degree 1
  - » the membership degree decreases from 1 to 0 on both sides of the central value
- Example:
  - » Possible fuzzy numbers to capture the concept "*around 3*"



Concepts of Fuzzy Number (cont.)

- Example:
  - » *Fuzzy intervals* *"about 3 to 5"*(special case of fuzzy numbers)





Intuitive Arithmetic with Fuzzy Number

- Example:
  - » Addition and subtraction of triangular fuzzy numbers

A = *approximate 4* [3, 5]

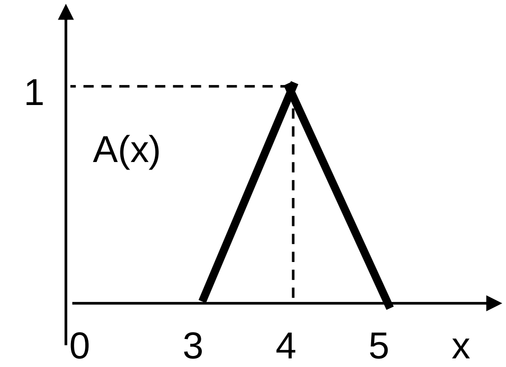
B = *approximate 2* [1, 3]

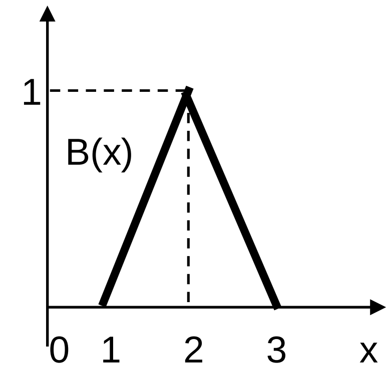
A+B = *approximate 6* [3+1, 5+3]=[4, 8]

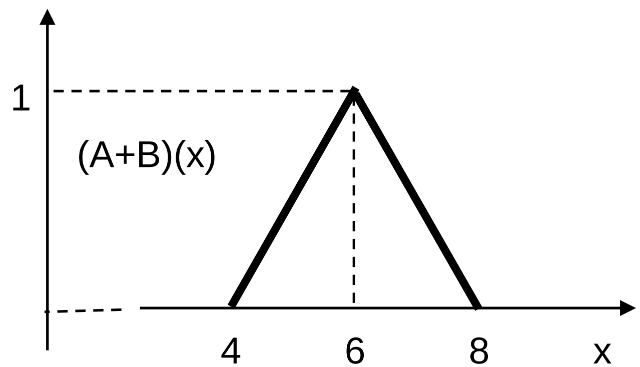
Min Min Max Max

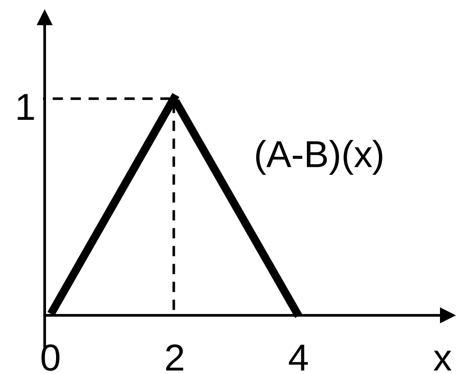
A-B = *approximate 2* [3-3, 5-1] = [0, 4]

Min Max Max Min



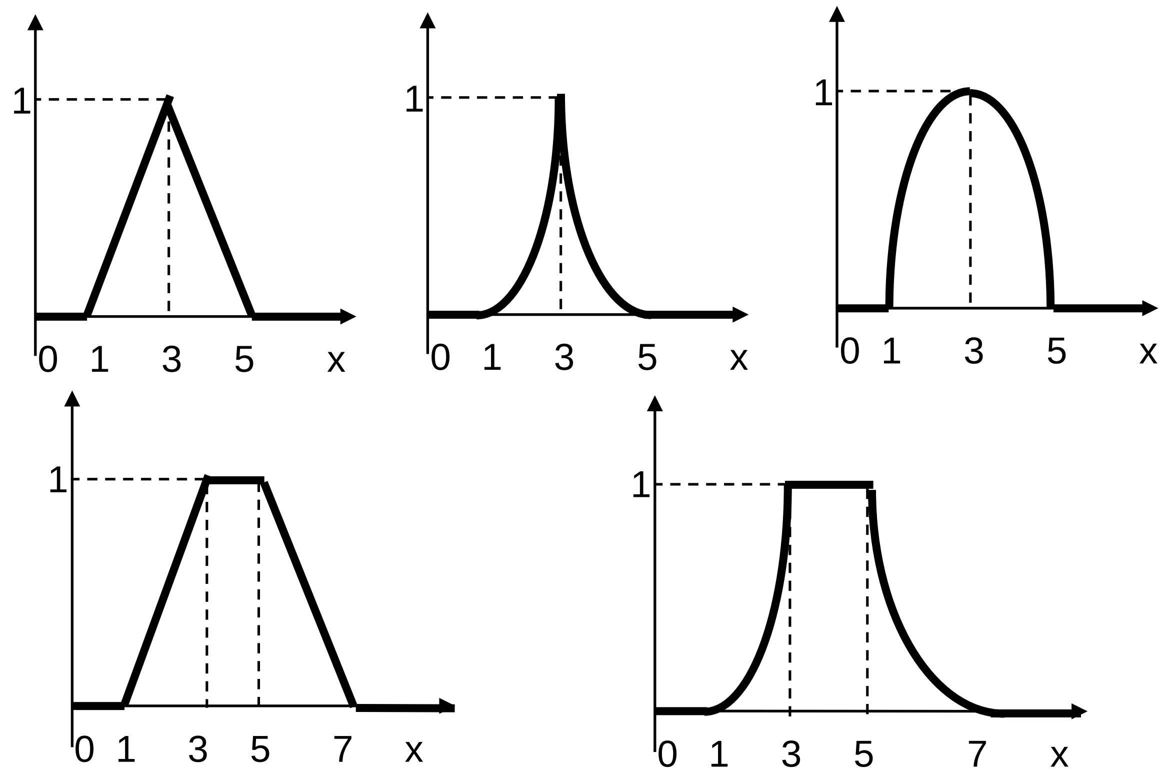






Food for Thought

- *MF Always Symmetric?*



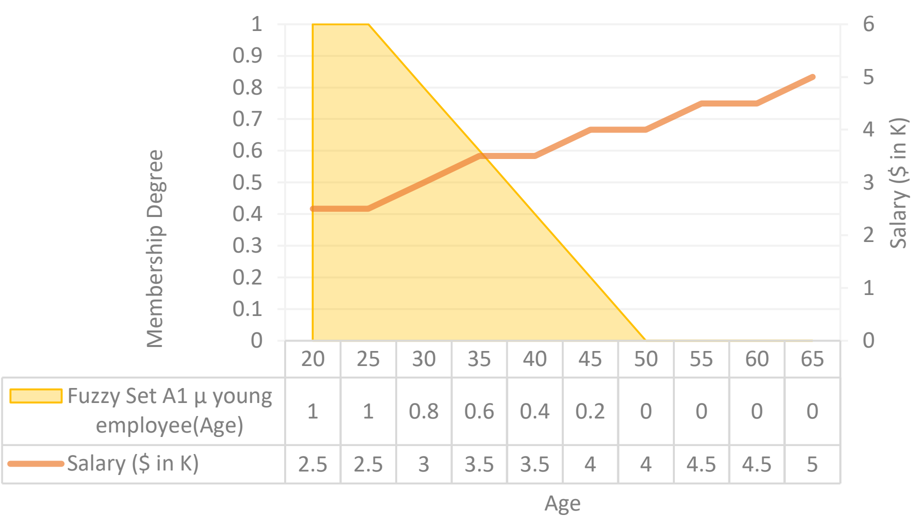
Extension Principle

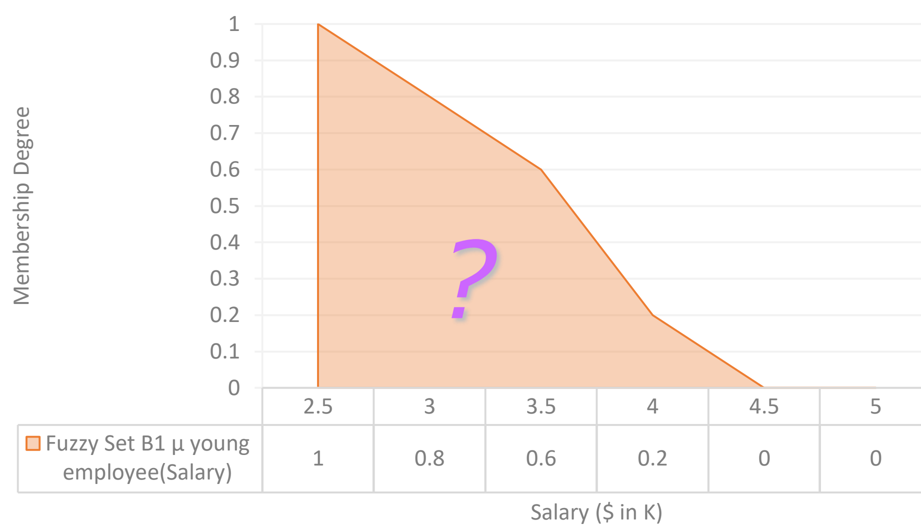
Extension Principle in Fuzzy Set Example 1

Age (in years)	20	25	30	35	40	45	50	55	60	65
Salary ($ in K)	2.5	2.5	3	3.5	3.5	4	4	4.5	4.5	5

Age (in years)	20	25	30	35	40	45	50	55	60	65
Fuzzy Set A1 $\mu_{\text{young employee}}$(Age)	1	1	0.8	0.6	0.4	0.2	0	0	0	0
young employee's age MF:	A1 = 1/20 + 1/25 + 0.8/30 + 0.6/35 + 0.4/40 + 0.2/45 + 0/50 + 0/55 + 0/60 + 0/65									
fuzzy set A1 (on Age) to represent concept young										

We know that Donald Duck is a young employee, then how much salary would he likely to earn (likely salary range with possibility)?





*Transfer **Young** employee defined on **Age** domain to **Salary** domain*

Extension Principle in Fuzzy Set Example 1 (cont.)

Salary ($ in K)	2.5	3	3.5	4	4.5	5
Fuzzy Set B1 $\mu_{\text{young employee}}$(Salary)	?	?	?	?	?	?
young employee's salary MF:	B1 = ?/2.5 + ?/3 + ?/3.5 + ?/4 + ?/4.5 + ?/5					
fuzzy set B1 (on Salary) to represent young employee's salary						

Salary = f(Age)	f(20)=2.5	f(25)=2.5	f(30)=3	f(35)=3.5	f(40)=3.5	f(45)=4	f(50)=4	f(55)=4.5	f(60)=4.5	f(65)=5

Solution:

young employee's age MF:	A1 = 1/20 + 1/25 + 0.8/30 + 0.6/35 + 0.4/40 + 0.2/45 + 0/50 + 0/55 + 0/60 + 0/65
young employee's salary MF:	B1 = 1/f(20) + 1/f(25) + 0.8/f(30) + 0.6/f(35) + 0.4/f(40) + 0.2/f(45) + 0/f(50) + 0/f(55) + 0/f(60) + 0/f(65)
	B1 = 1/2.5 + 1/2.5 + 0.8/3 + 0.6/3.5 + 0.4/3.5 + 0.2/4 + 0/4 + 0/4.5 + 0/4.5 + 0/5
	B1 = 1/2.5 ~~+ 1/2.5~~ + 0.8/3 + 0.6/3.5 ~~+ 0.4/3.5~~ + 0.2/4 ~~+ 0/4~~ + 0/4.5 ~~+ 0/4.5~~ + 0/5

Salary ($ in K)	2.5	3	3.5	4	4.5	5
Fuzzy Set B1 $\mu_{\text{young employee}}$(Salary)	1	0.8	0.6	0.2	0	0

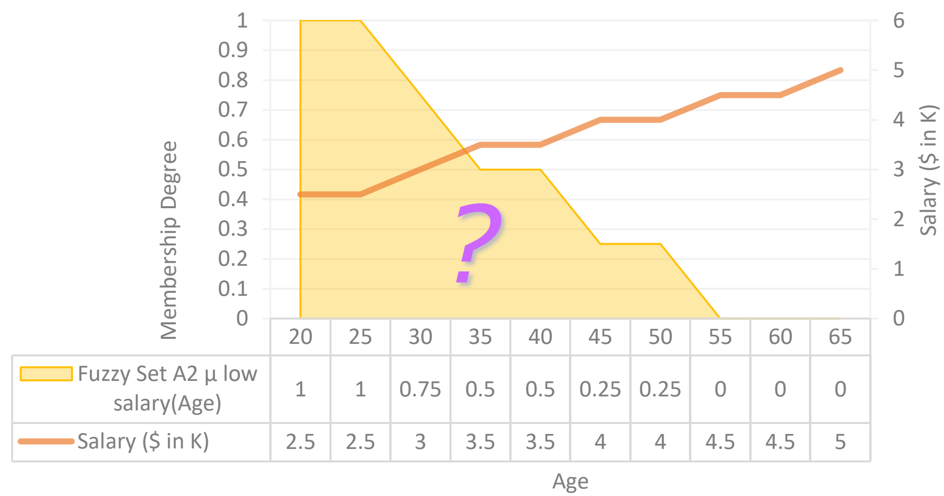
$$\mu_B(y) = \mu_{\tilde{f}(A)}(y) = \max_{x=f^{-1}(y)} [\mu_A(x)]$$

Extension Principle in Fuzzy Set Example 2

Age (in years)	20	25	30	35	40	45	50	55	60	65
Salary ($ in K)	2.5	2.5	3	3.5	3.5	4	4	4.5	4.5	5

Salary ($ in K)	2.5	3	3.5	4	4.5	5
Fuzzy Set B2 $\mu_{\text{low salary}}$(Salary)	1	0.75	0.5	0.25	0	0
low salary MF:	B2 = 1/2.5 + 0.75/3 + 0.5/3.5 + 0.25/4 + 0/4.5 + 0/5					
fuzzy set B2 (on Salary) to represent low salary						

We know that Donald Duck earns low salary, then how old is he (his likely age range with possibility)?



*Transfer **Low** salary defined on **Salary** domain to **Age** domain*

Extension Principle in Fuzzy Set Example 2 (cont.)

Age (in years)	20	25	30	35	40	45	50	55	60	65
Fuzzy Set A2 $\mu_{\text{low salary}}$(Age)	?	?	?	?	?	?	?	?	?	?

young employee's age MF: A2 = ?/20 + ?/25 + ?/30 + ?/35 + ?/40 + ?/45 + ?/50 + ?/55 + ?/60 + ?/65

fuzzy set A2 (on Age) to represent age of employees with low salary

Age = f^{-1}(Salary)	f^{-1}(2.5) = 25, 20	f^{-1}(3)=30	f^{-1}(3.5)=35, 40	f^{-1}(4)=45, 50	f^{-1}(4.5)=55, 60	f^{-1}(5)=65

Solution 1:

low salary's salary MF: B2 = 1/2.5 + 0.75/3 + 0.5/3.5 + 0.25/4 + 0/4.5 + 0/5

low salary's age MF: A2 = ?/20 + ?/25 + ?/30 + ?/35 + ?/40 + ?/45 + ?/50 + ?/55 + ?/60 + ?/65

A2 = 1/f^{-1}(2.5) + 0.75/f^{-1}(3) + 0.5/f^{-1}(3.5) + 0.25/f^{-1}(4) + 0/f^{-1}(4.5) + 0/f^{-1}(5)

A2 = (1/20 + 1/25) + 0.75/30 + (0.5/35 + 0.5/40) + (0.25/45 + 0.25/50) + (0/55 + 0/60) + 0/65

A2 = 1/20 + 1/25 + 0.75/30 + 0.5/35 + 0.5/40 + 0.25/45 + 0.25/50 + 0/55 + 0/60 + 0/65

Age (in years)	20	25	30	35	40	45	50	55	60	65
Fuzzy Set A2 $\mu_{\text{low salary}}$(Age)	1	1	0.75	0.5	0.5	0.25	0.25	0	0	0

Solution 2:

Salary = f(Age)	f(20)=2.5	f(25)=2.5	f(30)=3	f(35)=3.5	f(40)=3.5	f(45)=4	f(50)=4	f(55)=4.5	f(60)=4.5	f(65)=5

low salary's salary MF: B2 = 1/2.5 + 0.75/3 + 0.5/3.5 + 0.25/4 + 0/4.5 + 0/5

low salary's age MF: A2 = ?/20 + ?/25 + ?/30 + ?/35 + ?/40 + ?/45 + ?/50 + ?/55 + ?/60 + ?/65

A2 = B2(f(20))/20 + B2(f(25))/25 + B2(f(30))/30 + B2(f(35))/35 + B2(f(40))/40 + B2(f(45))/45 + B2(f(50))/50 + B2(f(55))/55 + B2(f(60))/60 + B2(f(65))/65

A2 = B2(2.5)/20 + B2(2.5)/25 + B2(3)/30 + B2(3.5)/35 + B2(3.5)/40 + B2(4)/45 + B2(4)/50 + B2(4.5)/55 + B2(4.5)/60 + B2(5)/65

A2 = 1/20 + 1/25 + 0.75/30 + 0.5/35 + 0.5/40 + 0.25/45 + 0.25/50 + 0/55 + 0/60 + 0/65

Extension Principle in Fuzzy Set Example (extra)

Background:

» In a company, the distribution of employees' ages and their salaries

Age (in years)	20	25	30	35	40	45	50	55	60	65
Salary ($ in K)	2.5	2.5	3.0	3.5	3.5	4.0	4.0	4.5	4.5	5.0

» f maps elements of *Age* to elements of *Salary*

» f^{-1} maps elements of *Salary* to elements of *Age*

Salary = f(Age)

Age = f^{-1}(Salary)

$f(20) = 2.5$, $f(25) = 2.5$, $f(30) = 3.0$, $f(35) = 3.5$,

$f(40) = 3.5$, … … $f(65) = 5.0$

Extension Principle in Fuzzy Set Example (extra)

Example 1

- fuzzy set *A1* (on *Age*) to represent concept *young*

  A1 =

  1/20 + 1/25 + 0.8/30 + 0.6/35 + 0.4/40 + 0.2/45 + 0/50 + 0/55 + 0/60 + 0/65

- fuzzy set *B1* (on *Salary*) to represent *young employee's salary*

$$B1 = 1/f(20) + 1/f(25) + 0.8/f(30) + 0.6/f(35) + 0.4/f(40) + 0.2/f(45) + 0/f(50) + 0/f(55) + 0/f(60) + 0/f(65)$$

$$= \underbrace{1/2.5 + 1/2.5}_{\max} + 0.8/3 + \underbrace{0.6/3.5 + 0.4/3.5}_{\max} + \underbrace{0.2/4 + 0/4}_{\max} + 0/4.5 + 0/5$$

$$= 1/2.5 + 0.8/3 + 0.6/3.5 + 0.2/4 + 0/4.5 + 0/5$$

Extension Principle in Fuzzy Set Example (extra)

Example 2

- fuzzy set *B2* (on *Salary*) to represent *low salary*

  $B2 = 1/2.5 + 0.75/3 + 0.5/3.5 + 0.25/4 + 0/4.5 + 0/5$

- fuzzy set *A2* (on *Age*) to represent *age of employees with low salary*

  $A2 = B2(f(20))/20 + B2(f(25))/25 + B2(f(30))/30 +$

  $B2(f(35))/35 + B2(f(40))/40 + B2(f(45))/45 + B2(f(50))/50 +$

  $B2(f(55))/55 + B2(f(60))/60 + B2(f(65))/65$

$= 1/20 + 1/25 + 0.75/30 + 0.5/35 + 0.5/40 + 0.25/45 + 0.25/50 + 0/55 + 0/60 + 0/65$

Extension Principle (Definition)

- *Extension Principle* (Zadeh, 1978) is one of the most important tools of fuzzy set theory.
  - » It provides a general procedure for extending crisp domains of mathematical expressions to fuzzy domains.
  - » This procedure generalizes a common point-point mapping of a function $f(\bullet)$ to a mapping between fuzzy sets.
  - » Extension principle together with other tools, such as fuzzy relation, provides a basis for fuzzy reasoning.

Extension Principle (Definition) Fuzzy sets

- Suppose that $f(\bullet)$ is a function from X to Y and A is a fuzzy set on X defined as

$$A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \ldots + \mu_A(x_n)/x_n$$

   » Then the extension principle states that the image of fuzzy set A under the mapping $f(\bullet)$ can be expressed as a fuzzy set B on Y

$$A \subseteq X, \qquad f : X \rightarrow Y$$

$$B = f(A), \qquad B \subseteq Y$$

Young employee on Age f-set ⊆ Age domain,
Young employee on Salary f-set = f (Young employee on Age: year),

f : Age domain → Salary domain
Young employee on Salary f-set ⊆ Salary domain

Low income on Salary f-set ⊆ Salary domain,
Low income on Age f-set = f (Low income on Salary: $),

f : Salary domain → Age domain
Low income on Age f-set ⊆ Age domain

Extension Principle (Definition) Fuzzy sets

- When f is one-to-one mapping

  $$B = \tilde{f}(A) = \mu_A(x_1)/y_1 + \mu_A(x_2)/y_2 + \ldots + \mu_A(x_n)/y_n$$

  where $y_i = f(x_i)$ is the *crisp* relation

- When f is not one-to-one mapping

  $$\mu_B(y) = \mu_{\tilde{f}(A)}(y) = \max_{x = f^{-1}(y)} [\mu_A(x)]$$

  Salary = f(Age)
  Several Age can map to one same Salary.

- Let f: $X \rightarrow Y$, where X and Y denote finite crisp sets, be a given function. Two functions may be induced from f:

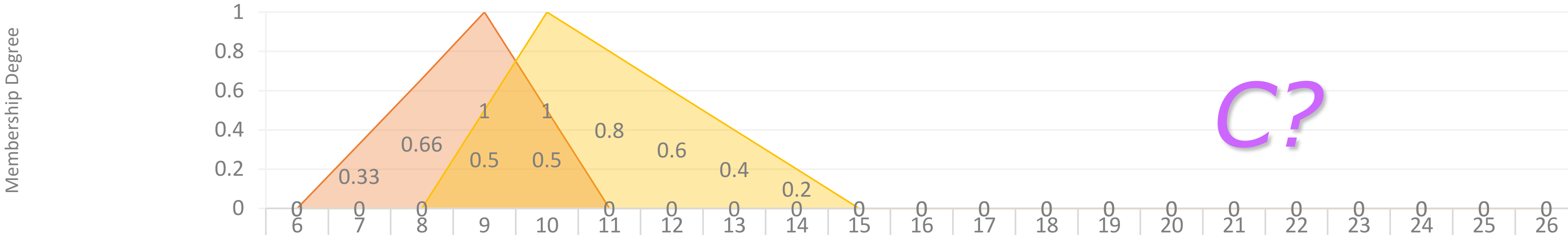
  $\tilde{f}$ — maps fuzzy sets on X to fuzzy sets on Y

  $\tilde{f}^{-1}$ — maps fuzzy sets on Y to fuzzy sets on X

Extension Principle in Fuzzy Number Example

Extension Principle in Fuzzy Number Example

Assume two fuzzy numbers represented by fuzzy sets:



	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
Set A Fuzzy Number: 9	0	0.33	0.66	1	0.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Set B Fuzzy Number: 10	0	0	0	0.5	1	0.8	0.6	0.4	0.2	0	0	0	0	0	0	0	0	0	0	0	0

Number

Set A Fuzzy Number: 9 Set B Fuzzy Number: 10

A = 0/6 + .33/7 + .66/8 + 1/9 + .5/10 + 0/11 ("*approximate 9*")

B = 0/8 + .5/9 + 1/10 + .8/11 + .6/12+ .4/13 + .2/14 + 0/15

("*approximate 10*")

C = A **+** B is another fuzzy number ("*approximate 19*")

C = ?/14 + ?/15 + ?/16 + ?/17 + ?/18 + ?/19 + ?/20

+ ?/21 + ?/22 + ?/23 + ?/24 + ?/25 + ?/26

Extension Principle in Fuzzy Number Example (cont.)

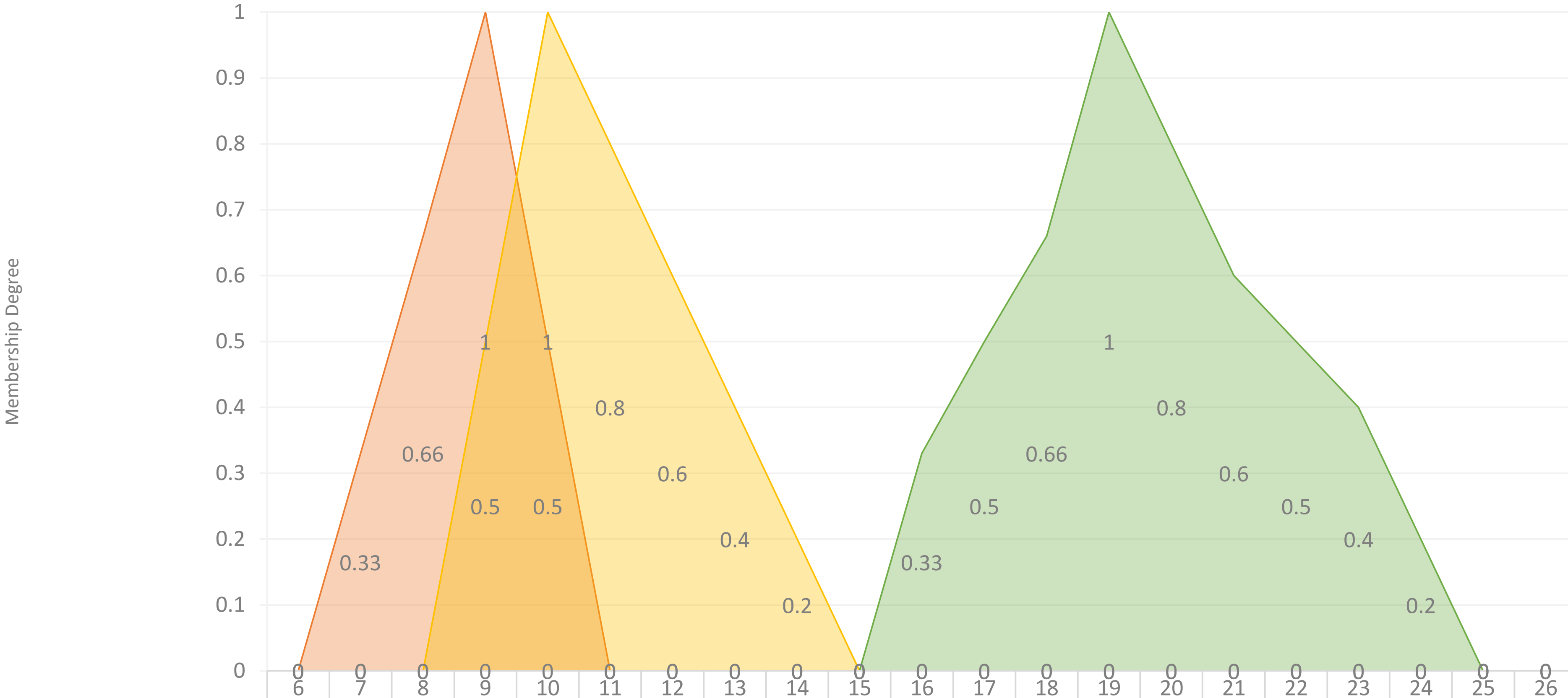
C=A+B (Plus +)	MF of Set B Fuzzy Number: 10	0	0.5	1	0.8	0.6	0.4	0.2	0
MF of Set A Fuzzy Number: 9	**Number**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
0	**6**	14	15	16	17	18	19	20	21
0.33	**7**	15	16	17	18	19	20	21	22
0.66	**8**	16	17	18	19	20	21	22	23
1	**9**	17	18	19	20	21	22	23	24
0.5	**10**	18	19	20	21	22	23	24	25
0	**11**	19	20	21	22	23	24	25	26

C(19) = Max [A(9) ∩ B(10)] = Max { Min [A(9) , B(10)] }

C=A+B (Plus +)	Max (MF1, MF2)	MF of Set B Fuzzy Number: 10							
MF of Set A Fuzzy Number: 9	MF Degreee	**0**	**0.5**	**1**	**0.8**	**0.6**	**0.4**	**0.2**	**0**
	0	0	0	0	0	0	0	0	0
	0.33	0	0.33	0.33	0.33	0.33	0.33	0.2	0
	0.66	0	0.5	0.66	0.66	0.6	0.4	0.2	0
	1	0	0.5	1	0.8	0.6	0.4	0.2	0
	0.5	0	0.5	0.5	0.5	0.5	0.4	0.2	0
	0	0	0	0	0	0	0	0	0

Number	Max of MF of Set C Fuzzy Number: 19
14	0
15	0
16	0.33
17	0.5
18	0.66
19	1
20	0.8
21	0.6
22	0.5
23	0.4
24	0.2
25	0
26	0

Extension Principle in Fuzzy Number Example (cont.)



	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
Set A Fuzzy Number: 9	0	0.33	0.66	1	0.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Set B Fuzzy Number: 10	0	0	0	0.5	1	0.8	0.6	0.4	0.2	0	0	0	0	0	0	0	0	0	0	0	0
Set C Fuzzy Number: 19	0	0	0	0	0	0	0	0	0	0	0.33	0.5	0.66	1	0.8	0.6	0.5	0.4	0.2	0	0

Number

Extension Principle in Fuzzy Number Example (cont.)

Example (cont.) Working for the sum of *A* and *B*:

x=6,y=8, 9, 10, 11, 12, 13, 14, 15, z = x+y

0/14+0/15+ 0/16+ 0/17+ 0/18+ 0/19+ 0/20+ 0/21

x = 7, y = 8, 9, 10, 11, 12, 13, 14, 15,

0/15+.33/16+.33/17+.33/18+.33/19+.33/20+.2/21+0/22

x = 8, y = 8, 9, 10, 11, 12, 13, 14, 15,

0/16 +.5/17+ .66/18+.66/19+ .6/20+ .4/21+.2/22+ 0/23

x = 9, y = 8, 9, 10, 11, 12, 13, 14, 15,

0/17+ .5/18 + 1/19 +.8/20+ .6/21+.4/22+.2/23 + 0/24

x = 10, y = 8, 9, 10, 11, 12, 13, 14, 15,

0/18 + .5/19 + .5/20 +.5/21+.5/22+.4/23+.2/24+0/25

x = 11, y = 8, 9, 10, 11, 12, 13, 14,…

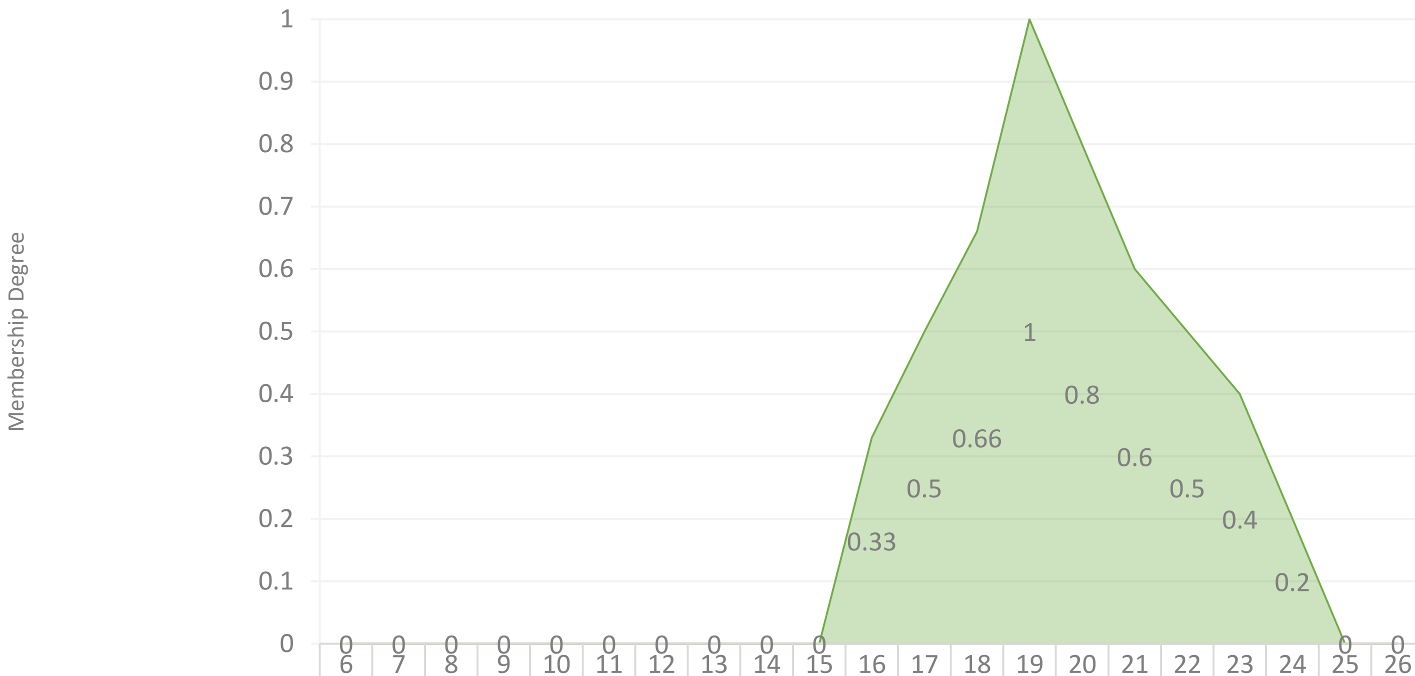
0/19 + 0/20+ 0/21+ 0/22+ 0/23+ 0/24+0/25

C =

0/14+ 0/15+.33/16+.5/17+.66/18+ 1/19+ .8/20+ .6/21+.5/22+ .4/23+.2/24+0/25

Food for Thought

- *The resulting C(19) MF is nonlinear. But is it correct? You intuition?*



	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
Set C Fuzzy Number: 19	0	0	0	0	0	0	0	0	0	0	0.33	0.5	0.66	1	0.8	0.6	0.5	0.4	0.2	0	0

Extension Principle (Definition) Fuzzy Number

Definition

x & y are fuzzy numbers defined by MF A(x) & B(y)

z = x • y (• is any binary arithmetic operation: +_x/)

$A \cap B = \int_X \mathbf{min}[\mu_A(x), \mu_B(x)] / x$

$$C(z) = \text{Max}\,[\,A(x) \cap B(y)\,]$$
$$= \text{Max}\,\{\,\text{Min}\,[\,A(x)\,,\,B(y)\,]\,\}$$

for all x, y such that x • y = z

where A and B are two fuzzy numbers, and

C = A • B is another fuzzy number, defined by MF C(z)

Exercise & Discussion: extension principle

- Verify the previous example of fuzzy number:

Intuitive Arithmetic with Fuzzy Number

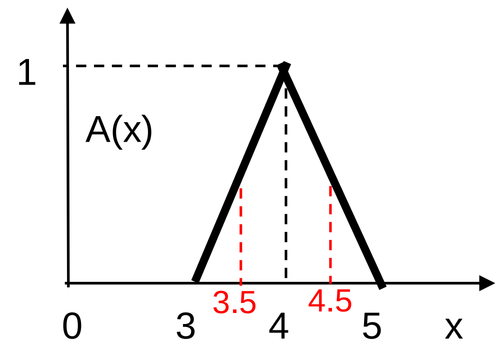
number, using extension principle

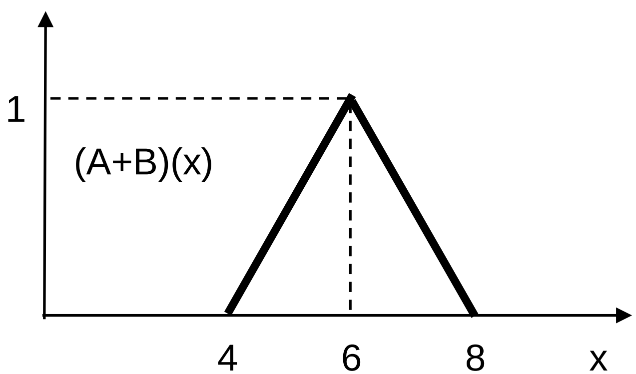
A = *approximate 4* [3, 5]

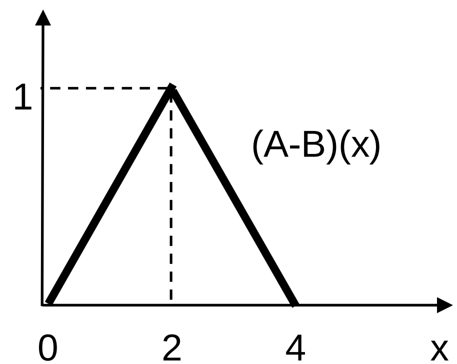
B = *approximate 2* [1, 3]

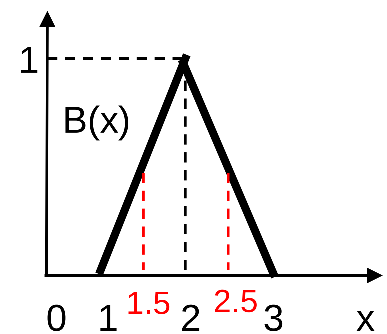
A+B = *approximate 6* [3+1, 5+3]=[4, 8]

A-B = *approximate 2* [3-3, 5-1] = [0, 4]









Fuzzy Relation

Fuzzy Relation: Example

In real life observations, Number of Admirers seems relating to the Beauty Level, at different degrees / possibility.

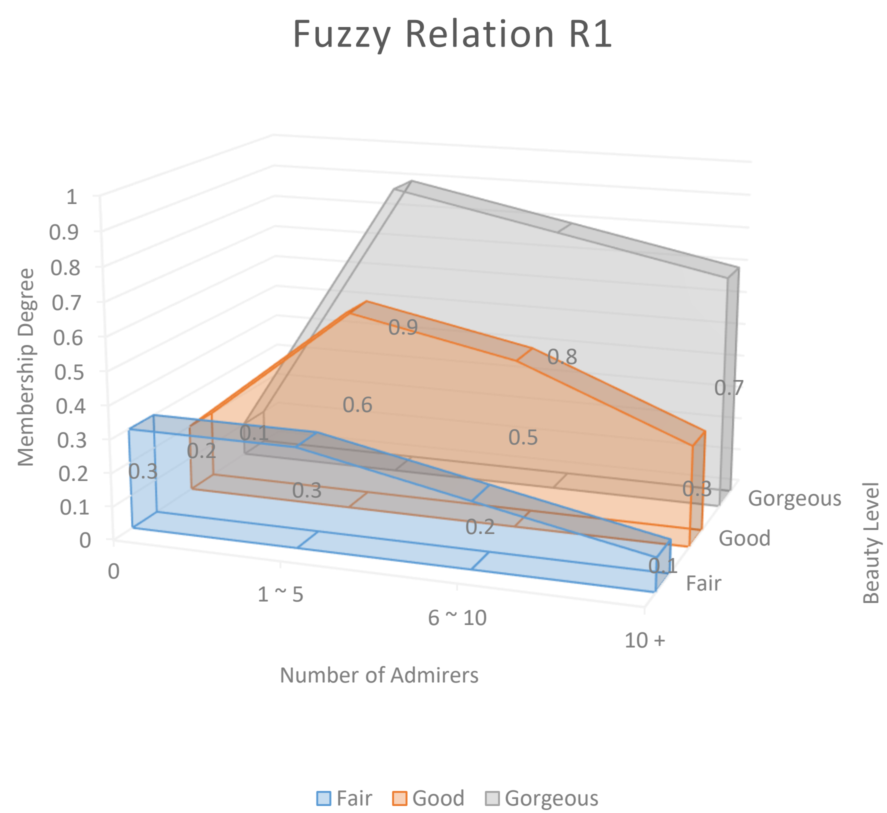
Representation of fuzzy relation

A *binary fuzzy relation* **R1** on $X \times Y$

X: Beauty Level

Y: Number of Admirers

3x4		Number of Admirers (Y)			
	Fuzzy Relation R1 (XxY)	Few 0	Some 1 ~ 5	Many 6 ~ 10	Flooding 10 +
Beauty Level (X)	Fair	0.3	0.3	0.2	0.1
	Good	0.2	0.6	0.5	0.3
	Gorgeous	0.1	0.9	0.8	0.7



Fuzzy Relation: Example: Inverse FR

fuzzy relation R1 and *inverse (transpose) fuzzy relation* $R1^{-1}$

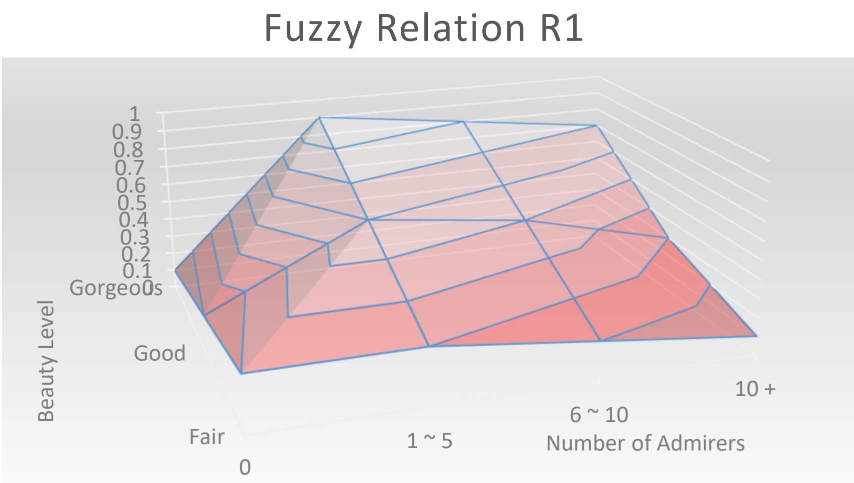
3x4		Number of Admirers (Y)			
	Fuzzy Relation R1 (XxY)	Few 0	Some 1 ~ 5	Many 6 ~ 10	Flooding 10 +
Beauty Level (X)	Fair	0.3	0.3	0.2	0.1
	Good	0.2	0.6	0.5	0.3
	Gorgeous	0.1	0.9	0.8	0.7

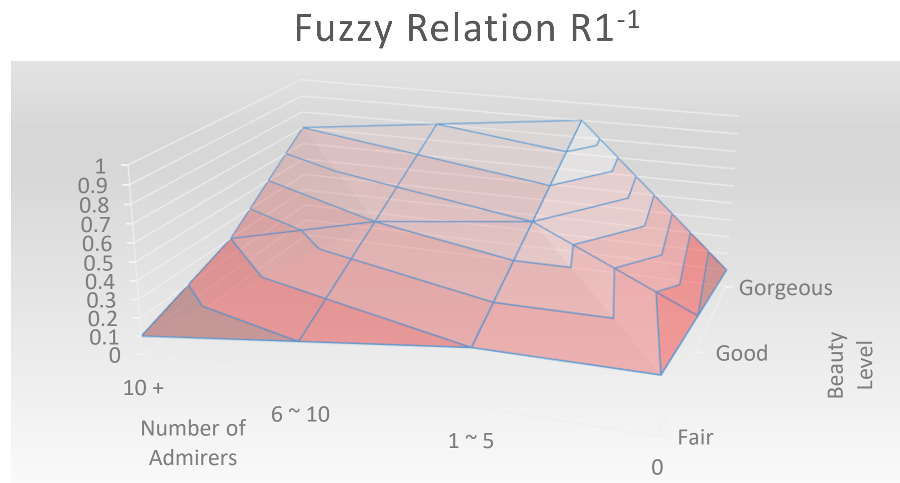
4x3		Beauty Level (X)		
	Inverse Fuzzy Relation $R1^{-1}$ (YxX)	Fair	Good	Gorgeous
Number of Admirers (Y)	0	0.3	0.2	0.1
	1 ~ 5	0.3	0.6	0.9
	6 ~ 10	0.2	0.5	0.8
	10 +	0.1	0.3	0.7

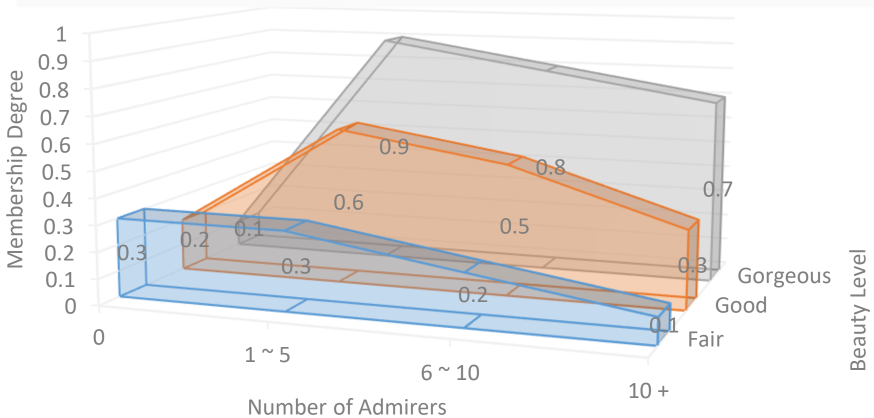
Fuzzy Relation: Example: Inverse FR

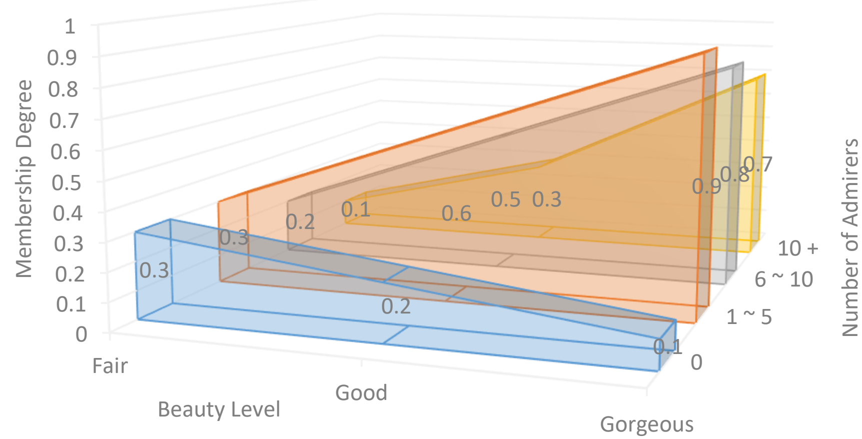
R1

R1 $^{-1}$









Fuzzy Relation: Example: Composition

Now we have two fuzzy relations R1 (XxY) & R2 (YxZ)

R1 on $X \times Y$ X: Beauty Level Y: Number of Admirers

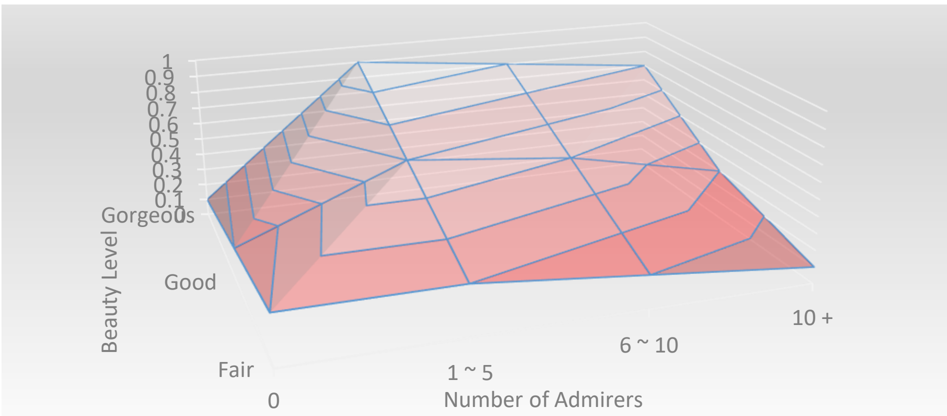
R2 on $Y \times Z$ Y: Number of Admirers Z: Weekend Activity (or Study Time: Short vs. Long)

3x4		Number of Admirers (Y)			
	Fuzzy Relation R1 (XxY)	Few 0	Some 1 ~ 5	Many 6 ~ 10	Flooding 10 +
Beauty Level (X)	Fair	0.3	0.3	0.2	0.1
	Good	0.2	0.6	0.5	0.3
	Gorgeous	0.1	0.9	0.8	0.7

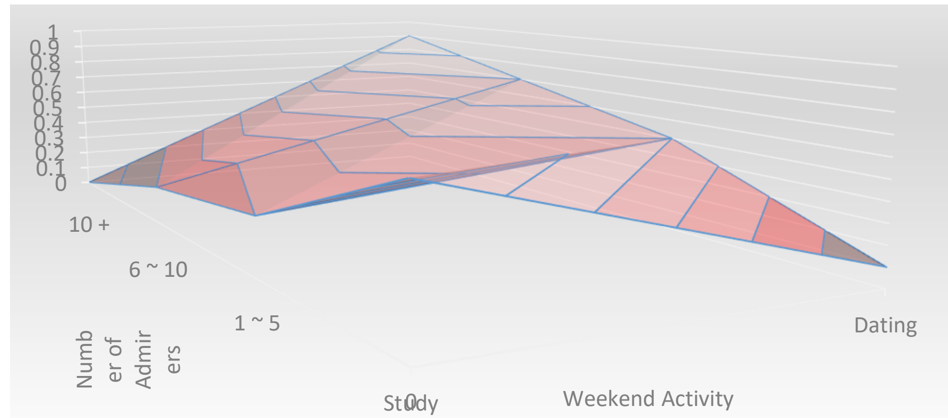
4x2		Weekend Activity (Z)	
	Fuzzy Relation R2 (YxZ)	Study	Dating
Number of Admirers (Y)	0	0.7	0.1
	1 ~ 5	0.3	0.5
	6 ~ 10	0.2	0.7
	10 +	0	0.9

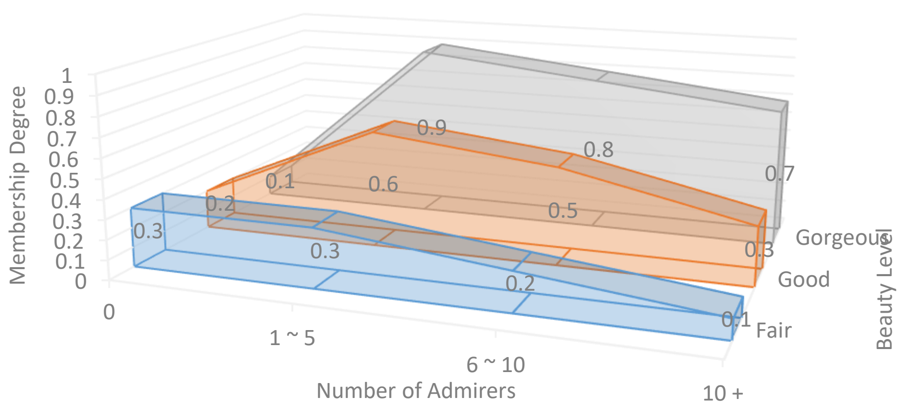
Fuzzy Relation: Example: Composition

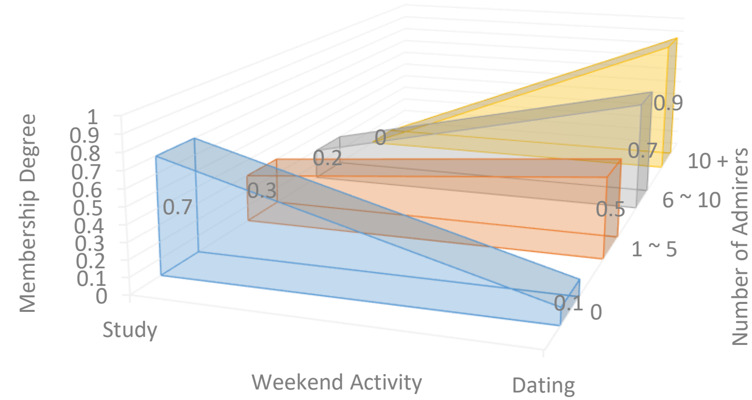
R1 (XxY)



R2 (YxZ)







Fuzzy Relation: Example: Composition

We use fuzzy relation composition to answer questions:

1. Donald Duck looks fair, is he likely to study or date in weekends?
2. Mickey Mouse usually studies in weekends, is she beautiful?

3x2		Weekend Activity (Z)	
	Fuzzy Relation R3 (XxZ)	Study	Dating
Beauty Level (X)	Fair	?	?
	Good	?	?
	Gorgeous	?	?

R3 is *Composition of two fuzzy relations* R3 = R1 ◦ R2 (XxZ)

Fuzzy Relation: Example: Composition (Max-Min)

3x4	Fuzzy Relation R1 (XxY)	Few 0	Some 1 ~ 5	Many 6 ~ 10	Flooding 10 +
		Number of Admirers (Y)			
Beauty Level (X)	Fair	0.3	0.3	0.2	0.1
	Good	0.2	0.6	0.5	0.3
	Gorgeous	0.1	0.9	0.8	0.7

∘

4x2	Fuzzy Relation R2 (YxZ)	Study	Dating
		Weekend Activity (Z)	
Number of Admirers (Y)	0	0.7	0.1
	1 ~ 5	0.3	0.5
	6 ~ 10	0.2	0.7
	10 +	0	0.9

=

3x2	Fuzzy Relation R3 (XxZ)	Study	Dating
		Weekend Activity (Z)	
Beauty Level (X)	Fair	0.3	0.3
	Good	0.3	0.5
	Gorgeous	0.3	0.7

Number of Admirers (Y) for Beauty Level (X) = Good	Min	0.7	0.3	0.2	0
		Number of Admirers (Y) for Weekend Activity (Z) = Study			
	0.2	0.2			
	0.6		0.3		
	0.5			0.2	
	0.3				0

Max - Min Composition

Let $\Re_1(x, y), (x, y \in \mathbf{X} \times \mathbf{Y})$ and $\Re_2(y, z), (y, z) \in \mathbf{Y} \times \mathbf{Z})$ be the two relations.

The max- min composition is then the fuzzy set

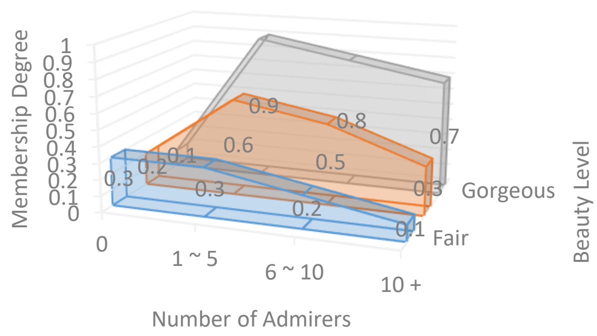
$$\Re_1 \circ \Re_2 = \{[(x,y), \max_y\{\min\{\mu\Re_1(x,y), \mu\Re_2(y.z)\}\}]\ x \in \mathbf{X}, y \in \mathbf{Y}, z \in \mathbf{Z}\}$$

T-norm

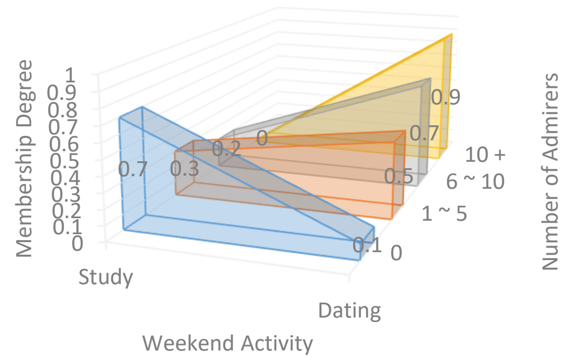
S-norm

Fuzzy Relation: Example: Composition (Max-Min)

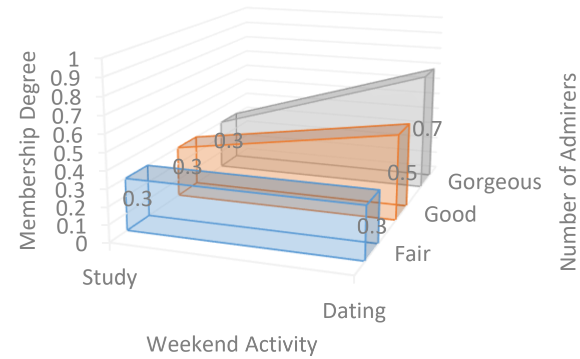
Fuzzy Relation R1: X Y

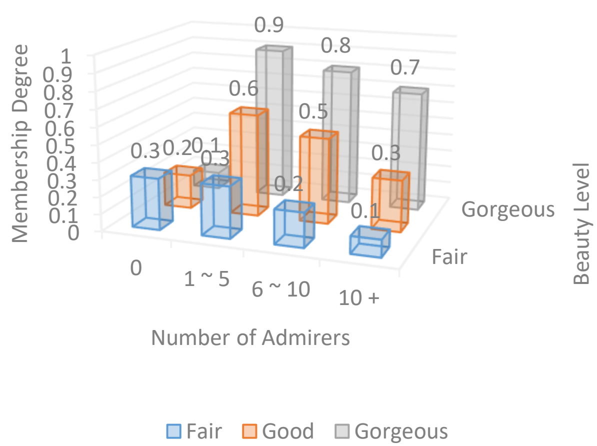


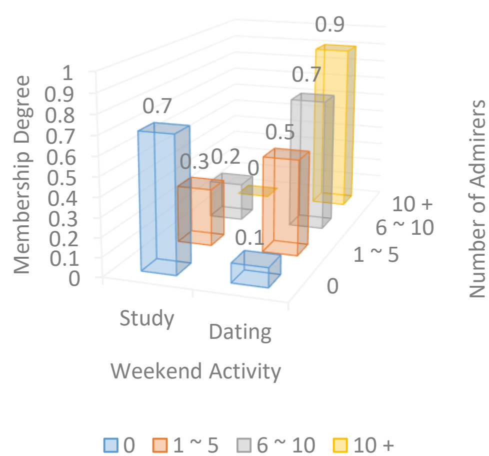
Fuzzy Relation R2: Y Z

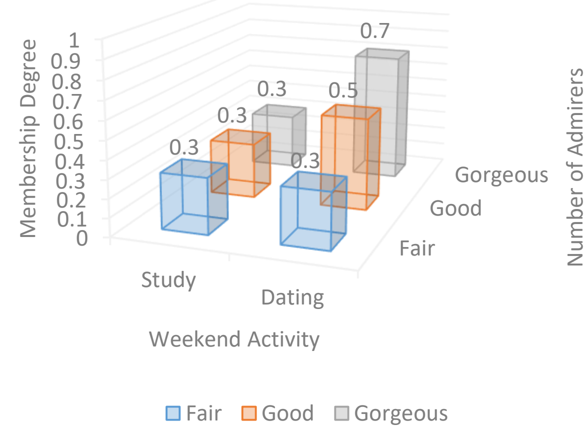


Fuzzy Relation R3: X Z









Fuzzy Relation: Example: Composition (Max-Product)

3x4		Number of Admirers (Y)			
	Fuzzy Relation R1 (XxY)	Few 0	Some 1 ~ 5	Many 6 ~ 10	Flooding 10 +
Beauty Level (X)	Fair	0.3	0.3	0.2	0.1
	Good	0.2	0.6	0.5	0.3
	Gorgeous	0.1	0.9	0.8	0.7

o

4x2		Weekend Activity (Z)	
	Fuzzy Relation R2 (YxZ)	Study	Dating
Number of Admirers (Y)	0	0.7	0.1
	1 ~ 5	0.3	0.5
	6 ~ 10	0.2	0.7
	10 +	0	0.9

=

3x2		Weekend Activity (Z)	
	Fuzzy Relation R3 (XxZ)	Study	Dating
Beauty Level (X)	Fair		
	Good		
	Gorgeous		

Max - Product Composition

Let $\Re_1$ (x, y), (x, y $\in$ **X** $\times$ **Y**) and $\Re_2$ (y, z) ,(y, z) $\in$ **Y** $\times$ **Z**) be the two relations.

The max-product composition is then the fuzzy set

$$\Re_1 \circ \Re_2 = \{[(x,y), \max_y\{\mu\Re_1(x,y) \times \mu\Re_2(y.z)\}]\ x\in \mathbf{X}, y\in \mathbf{Y}, z\in \mathbf{Z}\}$$

T-norm

S-norm

Fuzzy Relation: Self reference example

- Example:
  - » Let **R** denote a fuzzy relation on $X \times X$ that attempts to capture the relational concept *very-far-from*.
  - » X = { Beijing, Chicago, London, Moscow, New York, Paris, Sydney, Tokyo}

	B	C	L	M	N	P	S	T
B		1	0.7	0.5	1	0.7	0.6	0.1
C	1		0.5	0.9	0	0.5	1	1
L	0.7	0.5		0.3	0.5	0	1	0.7
M	0.5	0.9	0.3		0.9	0.3	0.8	0.5
N	1	0	0.5	0.9		0.5	1	1
P	0.7	0.5	0	0.3	0.5		1	0.7
S	0.6	1	1	0.8	1	1		0.6
T	0.1	1	0.7	0.5	1	0.7	0.6	

Fuzzy Relation

- *Fuzzy relations* are fuzzy subsets defined on universal sets which are *Cartesian products*.

  » Example of Cartesian product:

$$A = \{1, 2\} \qquad\qquad B = \{a, b, c\}$$

$$A \times B = \{<1,a>, <1,b>, <1,c>, <2,a>, <2,b>, <2,c>\}$$

$$B \times A = \{<a,1>, <b,1>, <c,1>, <a,2>, <b,2>, <c,2>\}$$

$$A \times A = \{<1,1>, <1,2>, <2,1>, <2,2>\}$$

Fuzzy Relation (cont.)

- Representation of fuzzy relation
  - » A *binary fuzzy relation* **R** on $X \times Y$
    - ♦ $X = \{x_1, x_2, \ldots, x_n\}$,
    - ♦ $Y = \{y_1, y_2, \ldots, y_m\}$

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ r_{21} & r_{22} & \cdots & r_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ r_{n1} & r_{n2} & \cdots & r_{nm} \end{bmatrix}$$

where $r_{ij} = R(x_i, y_j)$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$ represents the membership degree $R(x_i, y_j)$ of pair (x_i, y_j) in the fuzzy relation R, $0 \le r_{ij} \le 1$.

Operations on Binary Fuzzy Relation

- Fuzzy relations are special fuzzy sets
  - » All basic operations on fuzzy sets (complements, intersection, union, etc) are applicable to fuzzy relations as well.
- Two operations on binary fuzzy relation that are not applicable to ordinary fuzzy sets:
  - » *Inverse*
  - » *Composition*

Operations on Binary Fuzzy Relation: inverse

- *Inverse*
  - » The inverse of a fuzzy binary relation R on $X \times Y$, is a relation on $Y \times X$ defined as
    - ♦ $R^{-1}(y, x) = R(x, y)$

    for all pairs $<y, x> \in Y \times X$
  - » For any fuzzy binary relation
    - ♦ $(R^{-1})^{-1} = R$

Operations on Binary Fuzzy Relation: composition

- *Composition*
  - » The composition of two crisp binary relations
    - ♦ P on $X \times Y$ and
    - ♦ Q on $Y \times Z$
    - ☝ Cartesian products $X \times Y$ and $Y \times Z$ must share the set Y.
  - » The composition $R = P \circ Q$ consists of those pairs $<x, z>$ of the Cartesian product $X \circ Z$ that can be connected via the two given relations P and Q and *at least one* element y in Y.

Operations on Binary Fuzzy Relation: composition (cont.)

- *Max-min composition* of two binary fuzzy relations *P* and *Q*

$$R(x,z) = (P \circ Q)(x,z) = \max_{y \in Y} \min[P(x,y), Q(y,z)]$$

  - » Each connection from *x* to *z* via the two relations and a particular element $y \in Y$ is a matter of degree.
  - » This degree depends on the membership degrees *P*(*x*, *y*) and *Q*(*y*, *z*) and is determined by the smaller of those two membership degrees.
  - » There might be more than one $y \in Y$ to connect *x* to *z*. Among the chains that connect *x* to *z*, the largest degree of membership should be taken to characterise the relationship of *x* to *z*.

More Calculation: max-min composition

- Example:

  $X = \{1, 2, 3\}$, $Y = \{A, B, C, D\}$, $Z = \{a, b\}$

  Two fuzzy relations:

  R_1 = "x is relevant to y" defined on $X \times Y$

  R_2 = "y is relevant to z" defined on $Y \times Z$

$$\begin{array}{c} \text{X:} \\ 1 \\ 2 \\ 3 \end{array} \quad R_1 = \begin{array}{cccc} \text{Y: A} & \text{B} & \text{C} & \text{D} \\ \end{array} \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.7 \\ 0.4 & 0.2 & 0.8 & 0.9 \\ 0.6 & 0.8 & 0.3 & 0.2 \end{bmatrix}$$

$$R_2 = \begin{array}{cc|c} \text{Z: a} & \text{b} & \text{Y:} \end{array} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.3 \\ 0.5 & 0.6 \\ 0.7 & 0.2 \end{bmatrix} \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array}$$

$(R_1 \circ R_2)(2,a) =$

max[min(0.4,0.9), min(0.2,0.2), min(0.8,0.5), min(0.9,0.7)]

= max(0.4, 0.2, 0.5, 0.7) = 0.7

More Calculation: max-product composition

- Example:

  Max-product composition for the same example:

  $(R_1 \circ R_2)(2,a) =$

  $\max[0.4 \times 0.9,\ 0.2 \times 0.2,\ 0.8 \times 0.5,\ 0.9 \times 0.7]$

  $= \max(0.36,\ 0.04,\ 0.4,\ 0.63) = 0.63$

Appendix

Various Definitions of T-norm and T-conorm

- **T-norm**
  - » **fuzzy intersection operators are usually referred to as *T-norm* operators**
  - » **can be understood as "*fuzzy AND*"**

- **Four of the most frequently used T-norm operators**
  - » **Minimum** $T_{\min}(a,b) = \min(a,b) = a \wedge b$
  - » **Algebraic product** $T_{ap}(a,b) = ab$
  - » **Boundary product** $T_{bp}(a,b) = 0 \vee (a+b-1)$

At least 0

  - » **Drastic product** $T_{dp}(a,b) = \begin{cases} a, & if \quad b=1 \\ b, & if \quad a=1 \\ 0 & if \quad a,b<1 \end{cases}$

$$T_{dp}(a,b) \le T_{bp}(a,b) \le T_{ap}(a,b) \le T_{\min}(a,b)$$

Various Definitions of T-norm and T-conorm

- **T-conorm (S-norm)**
  - » **fuzzy union operators are usually referred to as T-conorm operators**
  - » **can be understood as "*fuzzy OR*"**

- **Four of the most frequently used T-conorm operators**
  - » **Maximum** $S_{\max}(a,b) = \max(a,b) = a \vee b$
  - » **Algebraic sum** $S_{as}(a,b) = a + b - ab$

At most 1

  - » **Boundary sum** $S_{bs}(a,b) = 1 \wedge (a+b)$
  - »
  - » **Drastic sum** $S_{ds}(a,b) = \begin{cases} a, & if \quad b = 0 \\ b, & if \quad a = 0 \\ 1 & if \quad a,b > 0 \end{cases}$

$$S_{\max}(a,b) \leq S_{as}(a,b) \leq S_{bs}(a,b) \leq S_{ds}(a,b)$$