1. Let F be the filter defined by
$$y_t = x_t - x_{t-1},$$
and let G be the filter defined by
$$y_t = x_t + y_{t-1}.$$
Consider the two filters H_1, which takes the output of F and uses it as the input of G, and H_2, which takes the output of G and uses it as the input of F.
   (a) Explain why F is a discrete differentiator, and G is a discrete integrator.
   (b) Determine the output of H_1 and H_2 if we give the unit impulse δ_t as input.
   (c) Explain why this should be viewed as a discrete version of the fundamental theorem of calculus

2. Suppose that a given filter has the frequency response
$$\mathscr{H}(e^{i\omega}) = \frac{1 - \frac{1}{2}e^{-i\omega} + e^{-i3\omega}}{1 + \frac{1}{2}e^{-i\omega} + \frac{3}{4}e^{-i3\omega}}.$$
   (a) Give a defining equation of the filter.
   (b) Plot the zeroes and poles of the transfer function $\mathscr{H}(z)$ in the complex plane.
   (c) Explain why the filter is or is not stable.
   (d) Using a computer, plot the frequency response of the filter. The abscissa should have units of frequency, in fractions of the sampling rate, and the ordinate should have the units of decibels.

3. Consider the sequence y_t, and suppose that it satisfies the homogeneous difference equation
$$\sum_{k=0}^{N} a_k y_{t-k} = 0$$
for each t.

(a) Show that y_t has solutions of the form

$$y_t = \sum_{m=1}^{N} A_m z_m^t$$

where the A_ms are arbitrary constants and the z_ms are the N roots of the polynomial

$$\sum_{k=0}^{N} a_k z^{-k} = 0.$$

That is to say,

$$\sum_{k=0}^{N} a_k z^{-k} = \prod_{m=1}^{N} (1 - z_m z^{-1}).$$

Note that this is the discrete analogue of the well known result from the theory of ordinary differential equations that an n^{th} order ordinary differential equation with constant coefficients has solutions of the form $e^{a_k t}$, and the numbers a_k are roots of the auxiliary polynomial.

(b) By once again examining the case of constant coefficient differential equations, deduce what happens to the solution in the case of multiple roots. Likely you will get stuck on doing the general case all at once, so make up an example that you can solve easily to see what is going on. For example, one that has two roots z_1, z_2 and try something like $z_1 = 2$ and $z_2 = 2 + \epsilon$ and letting $\epsilon \to 0$ or solving a difference equation that has auxiliary roots $z_1 = z_2 = 2$ directly.

(c) Find y_t if

$$y_t - 5y_{t-1} + 6y_{t-2} = 0$$

and $y_0 = 0$ and $y_1 = 1$.

4. The theory of gun barrels requires the Fourier series of the function

$$f(x) = \sin(\sin x).$$

(a) Find the first four nonzero Fourier coefficients. If you can not find them analytically, do the integral numerically. If you use a numerical technique, be sure to describe which one you used.

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(b) Plot the function f together with the approximation given by the first four nonzero Fourier coefficients you found.

5. (a) Plot the sum of the first 39 harmonics for the square wave

$$f(t) = \begin{cases} -1 & (2k-1)\pi < t < 2k\pi, \quad k \in \mathbf{Z}. \\ 1 & 2k\pi < t < (2k+1)\pi, \quad k \in \mathbf{Z}. \end{cases}$$

Let t range over $[0, 5\pi]$ for your plot.

(b) Plot the sum of the first 39 harmonics for the triangle wave defined by

$$w(t) = \int_0^t f(\bar{t})\, d\bar{t}.$$

Once again, let t range over $[0, 5\pi]$ for your plot.

(c) Differentiate the series for $f(t)$ term by term, and plot on $[0, 5\pi]$ the sum of the first 39 harmonics. Explain why this might have the name 'buzz'.