UNIVERSITY OF CALGARY

Mathematical Methods in Physics

MATH 433

Assignment 2

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1 Discrete Differentiator and Integrator

Explain why the filter F, defined by $y_t = x_t - x_{t-1}$ is a discrete differentiator.

Suppose we wish to design a discrete-time system to differentiate a continuous-time signal. Given an input x_t, the output y_t of the filter should be the derivative of x_t. Let $x[n]$ and $y[n]$ denote samples of the signals x_t and y_t. These are the input and output of the digital, discrete-time differentiator (from here on out, we will refer to digital, discrete-time systems simply as discrete systems)[1].
If T is the sampling interval, then

$$x[n] = x(nT) \qquad \text{and} \qquad y[n] = y(nT) \tag{1}$$

Since y is the derivative of x, we have $y_t = \frac{d}{dt}x_t$. Setting $t = nT$ and using (1):

$$y(nT) = \frac{d}{dt}x_t\bigg|_{t=nT} = \lim_{T\to 0}\frac{x(nT) - x((n-1)T)}{T}$$

$$y[n] = \lim_{T\to 0}\frac{x[n] - x[n-1]}{T}$$

The sampling interval T cannot be zero, however for T sufficiently small

$$y[n] \approx \frac{x[n]}{T} - \frac{x[n-1]}{T}$$

Setting the sampling interval $T = 1$

$$y[n] = x[n] - x[n-1]$$

In terms of the analog continuous-time input and output signals of the differentiator, we have

$$y_t = x_t - x_{t-1}$$

Explain why the filter G, defined by $y_t = x_t + y_{t-1}$ is a discrete integrator.

In calculus, the integral is introduced by approximating its value by summing the areas of rectangles. We can approximate $y_t = \int_{-\infty}^{t} x(t')dt'$, where x_t is the continuous-time input signal of our integrator, using the right endpoint rule (also known as the backwards staircase approximation[1]). This approximation sums the areas of rectangular strips that lie to the left of $x(nT)$. From calculus, we know that as $T \to 0$ the following expression is in fact the definition of the integral.

$$y(nT) = \lim_{T\to 0}\sum_{k=-\infty}^{n} T\, x(kT)$$

For sufficiently small T and using (1) we have

$$y[n] = T \sum_{k=-\infty}^{n} x[k]$$

This equation is equivalent to the recursive equation

$$y[n] - y[n-1] = T\, x[n]$$

Again, we cannot have a sampling interval of zero, however assuming the approximation holds for our desired range of precision, we can set $T = 1$ and rearrange for:

$$y[n] = x[n] + y[n-1]$$

In terms of the continuous-time input and output signals of the integrator, we have

$$y_t = x_t + y_{t-1}$$

Consider the two filters H_1, which takes the output of F and uses it as the input of G, and H_2, which takes the output of G and uses it as the input of F. Determine the output of H_1 and H_2 if we give the unit impulse δ_t as input.

Consider the Unit Impulse Sequence $\delta[n]$.

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

To stay consistent with the conventions seen in textbooks[1]: When the input to a discrete-time filter is $x[n] = \delta[n]$, the filter output is the *unit impulse response*, $y[n] = h[n]$.
The following is an example calculation for the unit impulse as input to the filter F:

$$y[0] = x[0] - x[-1]$$

$$h[0] = \delta[0] - \delta[-1] = 1 - 0 = 1$$

The following table gives the output of H_1 (output of G_1) when the unit impulse is the input (input of filter F).

Filter H_1 - Output of F as Input of G					
n	**-2**	**-1**	**0**	**1**	**2**
Unit Impulse	0	0	1	0	0
Output of F	0	0	1	-1	0
Output of G	0	0	1	0	0

We see that the output is the same signal that was used as input - the unit impulse.
Similarly, we can determine the output of H_2 given that the input is the unit impulse. Recall that H_2 uses the output of G and uses it as the input of F.

Filter H_2 - Output of G as Input of F					
n	**-2**	**-1**	**0**	**1**	**2**
Unit Impulse	0	0	1	0	0
Output of G	0	0	1	1	1
Output of F	0	0	1	0	0

Explain why this should be viewed as a discrete version of the fundamental theorem of calculus.
Recall that the fundamental theorem of calculus states:

$$\text{If} \quad F(t) = \int_a^t f(t')dt' \quad \text{then} \quad F'(t) = f(t)$$
$$\text{and} \quad \frac{d}{dt}\int_a^t f(t')dt' = f(t)$$

If we consider the continuous-time input to be $f(t)$ then filter H_2 will first integrate it, and then differentiate it. The resulting output is the same of the original input. This is exactly what the fundamental theorem of calculus states.
It is possible to rewrite the theorem, such that we first differentiate, then integrate. Again, the output is the same as the input. Both filters can thus be viewed as a discrete version of the fundamental theorem of calculus.

2 Frequency Response of LTI System

Suppose that a given filter has the frequency response

$$\boldsymbol{\mathcal{H}(e^{i\omega}) = \frac{1 - \frac{1}{2}e^{-i\omega} + e^{-i3\omega}}{1 + \frac{1}{2}e^{-i\omega} + \frac{3}{4}e^{-i3\omega}}}$$

Give a defining equation of the filter.
The general form the difference equation for a linear time-invariant (LTI) system is[2]

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} x[n-k] \qquad a_0 = 1$$

If the input is a complex sinusoid, $x[n] = e^{i\omega n}$, then the output has the form $y[n] = \mathcal{H}(e^{i\omega})e^{i\omega n}$. Combining these equations

$$\mathcal{H}(e^{i\omega}) = \frac{\sum_{k=0}^{N} b_k e^{-j\omega k}}{\sum_{k=0}^{M} a_k e^{-i\omega k}}$$

In this form we can see that the numerator and denominator of $\mathcal{H}$ provide the coefficients and corresponding indices for the input and output of the difference equation. Thus,

$$1 - \frac{1}{2}e^{-i\omega} + e^{-i3\omega} \qquad \Rightarrow \qquad x[n] - \frac{1}{2}x[n-1] + x[n-3]$$

$$1 + \frac{1}{2}e^{-i\omega} + \frac{3}{4}e^{-i3\omega} \qquad \Rightarrow \qquad y[n] + \frac{1}{2}y[n-1] + \frac{3}{4}y[n-3]$$

Putting this together, to form the difference equation

$$y[n] + \frac{1}{2}y[n-1] + \frac{3}{4}y[n-3] = x[n] - \frac{1}{2}x[n-1] + x[n-3]$$

In terms of the continuous-time input and output signals of the filter

$$y_t + \frac{1}{2}y_{t-1} + \frac{3}{4}y_{t-3} = x_t - \frac{1}{2}x_{t-1} + x_{t-3}$$

Plot the zeros and poles of the transfer function in the complex plane
Expressing the transfer function as a function of z, and multiplying both the numerator and denominator by $\boldsymbol{z^3}$ we have

$$\mathcal{H}(z) = \frac{1 - \frac{1}{2}z^{-1} + z^{-3}}{1 + \frac{1}{2}z^{-1} + \frac{3}{4}z^{-3}} = \frac{z^3 - \frac{1}{2}z^2 + 1}{z^3 + \frac{1}{2}z^2 + \frac{3}{4}}$$

Both the numerator and denominator are in irreducible form. The poles are the values of z that make the denominator zero, and the zeros are the values of z that make the numerator go to zero. Matlab was used to calculate the poles and the zeros of this transfer function.

Zeros and Poles of $\mathcal{H}(z)$			
Zeros	$0.6790 + 0.8392i$	$0.6790 - 0.8392i$	$-0.8581 + 0.0000i$
Poles	$-1.1094 + 0.0000i$	$0.3047 + 0.7637i$	$0.3047 - 0.7637i$

Matlab was used to plot these values in the complex plane using the following code[3].

Listing 1: M-File Code For Pole-Zero Plot

```
b = [1, -0.5, 0, 1];
a = [1, 0.5, 0, 0.75];
zplane(b,a);
```

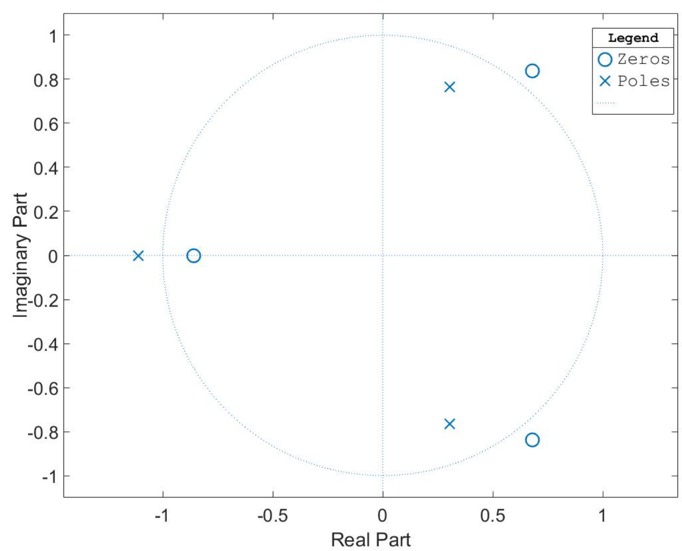

Figure 1: Pole-Zero Plot of the Transfer Function $\mathcal{H}(z)$

Explain why the filter is or is not stable
In general, to test for external (BIBO) stability for causal systems, we simply check if all the poles lie within the unit circle. If there is even one outside of the circle, then the impulse response $h[n]$ includes at least one term that is an exponentially growing term and the system is therefore unstable. We note that the above system (which is clearly causal) has a pole at $z = -1.1094$, which is outside the unit circle and therefore the system is unstable.

Plot the frequency response of the filter

Matlab was used to plot the frequency response of the filter, with the following code[3].

Listing 2: M-File Code For Frequency Response

```
b = [1, -0.5, 0, 1];
a = [1, 0.5, 0, 0.75];
[H,w] = freqz(b,a,2001);
plot(w/pi, db(H));
xlabel('Normalized Frequency (\times\pi rad/sample)');
ylabel('Magnitude (dB)');
grid on;
```

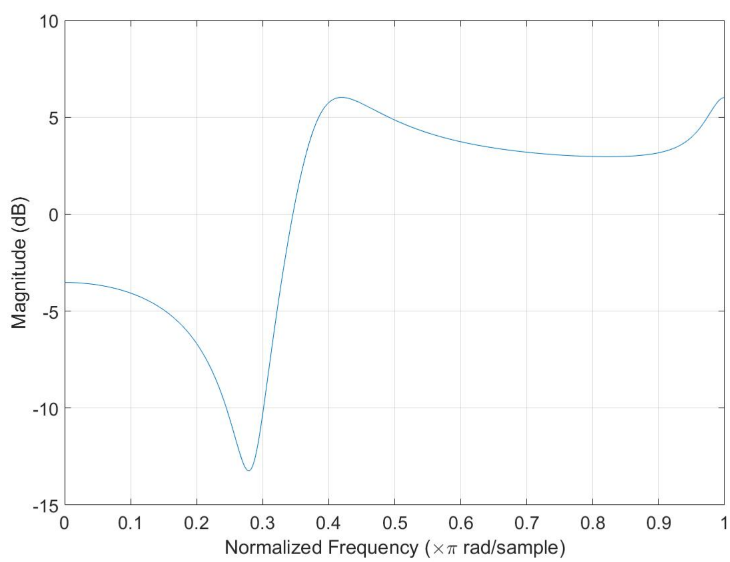

Figure 2: Frequency Response of Transfer Function$\mathcal{H}(z)$

3 Linear Homogeneous Equations with Constant Coefficients

Consider the sequence y_t, and suppose it satisfies the homogeneous difference equation

$$\sum_{k=0}^{N} a_k \, y_{t-k} = 0 \tag{2}$$

The goal here is to find the general solution of (2). Suppose the solutions of (2) have the form z^t. Substituting this into (2) yields

$$z^N + a_1 z^{N-1} + a_2 z^{N-2} + \cdots + a_N = 0$$

This is known as the characteristic equation for (2) where its roots, z, are known as the characteristic roots. Since $a_N \neq 0$, it can be shown that none of the roots are equal to zero.

Suppose $z^N,\ z^{N-1}, \ldots,\ z^1$ are distinct.

To prove that the set $\{{z_1}^t,\ {z_2}^t, \ldots,\ {z_N}^t\}$ is a fundamental set of solutions it is sufficient to show that $W(0) \neq 0$ where $W(n)$ is known as the *Vandermonde Matrix.*

$$W(0) = det \begin{bmatrix} 1 & 1 & \ldots & 1 \\ z_1 & z_2 & \ldots & z_N \\ {z_1}^2 & {z_2}^2 & \ldots & {z_N}^2 \\ \vdots & \vdots & & \vdots \\ {z_1}^{N-1} & {z_2}^{N-1} & \ldots & {z_N}^{N-1} \end{bmatrix}$$

Mathematical Induction can be used to show[4] that

$$W(0) = \prod_{1 \leq i < j \leq N} (z_j - z_i)$$

since each root is distinct, $W(0) \neq 0$ and the set $\{{z_1}^t,\ {z_2}^t, \ldots,\ {z_N}^t\}$ is a fundamental set of solutions. Recall that the general solution can be expressed as a linear combination of the fundamental set of solutions. Thus,

$$y_t = \sum_{m=1}^{N} A_m {z_m}^t \qquad \text{where } A_m s \text{ are arbitrary constants}$$

Suppose that the distinct characteristics roots are $z_1,\ z_2,\ \ldots,\ z_r$ with multiplicities $m_1,\ m_2,\ \ldots,\ m_r$ with $\sum_{i=1}^{r} m_i = N$.

Equation (2) can be expressed as

$$(E - z_1)^{m_1}(E - z_2)^{m_2} \ldots (E - z_r)^{m_r} = 0 \tag{3}$$

Where E is the *forward shift operator* which takes a term y_t and shifts it to y_{t+1}

$$E\ y_t = E\ y_{t+1}$$

Note that solutions of

$$(E - z_i)^{m_i}\ y_t = 0 \qquad i = 1,\ 2, \ldots,\ r \tag{4}$$

are also solutions of (3). We will consider a general term and determine its solution.

$$(E - z)\ y_t = y_{t+1} - z\ y_t = z^{t+1}\left[\frac{y_{t+1}}{z^{t+1}} - \frac{y_t}{z^t}\right] = z^{t+1}\Delta\left(\frac{y_t}{z^t}\right)$$

Similarly,

$$(E - z)^2\ y_t = z^{t+2}\Delta^2\left(\frac{y_t}{z^t}\right)$$

Expressing this result as a general power,

$$(E - z)^m\ y_t = z^{t+m}\ \Delta^m\left(\frac{y_t}{z^t}\right)$$

Now, it is straightforward to see that $\Delta^m y_t = 0$ has solution

$$y_t = A_1 + A_2 t + \cdots + A_m t^{m-1} \qquad \text{where } A_1,\ A_2,\ \ldots,\ A_m \text{ are constant}$$

Therefore equation (4) has a solutions in the form

$$y_t^{(i)} = (A_1 + A_2 t + \cdots + A_{m_i} t^{m_i - 1})\ z_i^{\ t}$$

Considering $i = 1,\ 2,\ \ldots,\ r$ the general solution has the form

$$\begin{aligned} y_t = \quad & z_1^{\ t}[A_1^{(1)} + A_2^{(1)} +\ \ldots\ + A_{m_1}^{\ (1)} t^{m_1 - 1}] \\ & + z_2^{\ t}[A_1^{(2)} + A_2^{(2)} +\ \ldots\ + A_{m_2}^{\ (2)} t^{m_2 - 1}] \\ & +\ \ldots \\ & + z_r^{\ t}[A_1^{(r)} + A_2^{(r)} +\ \ldots\ + A_{m_r}^{\ (r)} t^{m_r - 1}] \end{aligned}$$

Find y_t if $y_t - 5y_{t-1} + 6y_{t-2} = 0$ and $y_0 = 0$ and $y_1 = 1$.

The characteristic equation is

$$z^2 - 5z + 6 = 0$$

Factoring this expression to determine the roots

$$0 = z^2 - 5z + 6 = (z-2)(z-3)$$

Thus, the characteristic roots are $z_1 = 2,\ z_2 = 3$. The solution is then

$$y_t = a_1 z_1^{\ t} + a_2 z_2^{\ t} = a_1 2^t + a_2 3^t$$

We can solve for the coefficients using the initial values, $y_0 = 0$ and $y_1 = 1$

$$y_0 = a_1 2^0 + a_2 3^0 = a_1 + a_2 = 0 \qquad \Longrightarrow \qquad a_1 = -a_2$$

$$y_1 = a_1 2^1 + a_2 3^1 = 2a_1 + 3a_2 = 2(-a_2) + 3a_2 = a_2 = 1$$

$$a_1 = -a_2 = -1$$

Putting this all together, the solution of the difference equation is

$$y_t = 3^t - 2^t$$

4 Fourier Series

The theory of gun barrels requires the Fourier series of the function $f(x) = sin(sinx)$

First note that $f(x)$ is an odd function, i.e. $sin(sin(-x)) = -sin(sin(x))$. We know that the Fourier Series for an odd function is

$$f(x) \approx \sum_{n=1}^{\infty} b_n sin\Big(\frac{n\pi}{T}x\Big)$$

With coefficients,

$$b_n = \frac{2}{T}\int_0^T f(x) sin\Big(\frac{n\pi x}{T}\Big)dx \qquad n = 1,\ 2,\ 3\ldots$$

Here, $T = \pi$

Note that $sin(sinx)$ converges on $-\pi \leq x \leq \pi$.

Recall that Bessel functions of the first kind have the form:

$$J_n(1) = \frac{1}{\pi}\int_0^\pi sin(sinx)sin(nx)dx \qquad n = 1,\ 3,\ 5,\ \ldots$$

Thus, we can express the general coefficients, b_n, in terms of Bessel Functions

$$\begin{aligned} b_n &= 2\ J_n(1) \qquad n = 1,\ 3,\ 5,\ \ldots \\ &= 2\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(m+n+1)}\left[\frac{1}{2}\right]^{2m+n} \qquad n = 1,\ 3,\ 5,\ \ldots \end{aligned}$$

To evaluate this sum, for the first four odd terms of n, I will used the following python code.

```python
from math import factorial
from scipy.special import gamma

def J(n, max_iter=10):
    '''
    Computes Bessel function J_n(1)
    Parameters:
        n: order of Bessel function
        max_iter: Number of iterations in summation
    '''
    m, J = 0, 0.0
    while m < max_iter:
        J += ( (-1)**m / (factorial(m) * gamma(m+n+1) ) ) * (0.5)**(2*m+n)
        m += 1
    return J

b_1, b_3, b_5, b_7 = 2*J(1), 2*J(3), 2*J(5), 2*J(7)
```

The above code returned the following values for the first four non-zero coefficients in the Fourier series of $f(x) = sin(sinx)$

$$b1 = 0.880101 \qquad b3 = 0.039127 \qquad b5 = 0.000500 \qquad b7 = 0.000003$$

Plot the Fourier Series of $f(x) = sin(sinx)$

Using the expression for b_n is the formula for the Fourier Series, with $T = \pi$

$$f(x) \approx \sum_{n=1}^{\infty} 2\, J_n(1) \sin(nx) \qquad n = 1,\ 2,\ 3\ldots$$

This formula was implemented in python code, and plotted with the first four odd terms of n over the range $[-3\pi : 3\pi]$.

```
import numpy as np
import matplotlib.pyplot as plt

def coef_b(n, max_iter=20):
    '''Calcultes Fourier Series'''
    return 2.0 * J(n, max_iter)

def fourier(x, iterns):
    '''Computes Fourier Series of f(x) = sin(sinx)'''
    result = 0
    for n in range(iterns):
        print(2*n+1)
        result += coef_b(2*n+1) * np.sin((2*n+1)*x)
    return result

def exact_func(x):
    '''f(x) = sin(sinx)'''
    return np.sin(np.sin(x))

x_vals = np.linspace(-3.0*np.pi, 3.0*np.pi, 100)
y_vals = fourier(x_vals, 4)
```

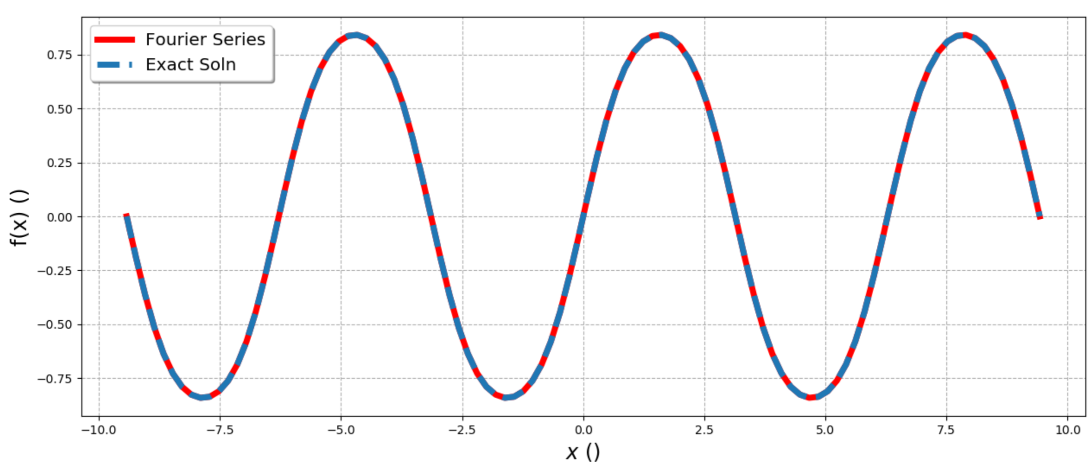

Figure 3: Fourier Series of $f(x) = sin(sin(x))$

5 Harmonics

Plot the sum of the first 39 harmonics for the square wave

$$\boldsymbol{f(t) = \begin{cases} -1 & (2k-1)\pi < t < 2k\pi \quad k \in Z \\ 1 & 2k\pi < t < (2k+1)\pi \quad k \in Z \end{cases}}$$

Recall the Fourier sine series of $f(x)$ is the Fourier series of the odd-periodic extension of $f(x)$. It is given by

$$f(x) \approx \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi}{L}x\right)$$

with coefficients

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(n\frac{\pi}{L}x\right)dx \qquad n = 1,\ 2,\ 3,\ \ldots$$

Here, $L = \pi$. Calculating the coefficients,

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi}(1)\sin(nx)dx = -\frac{2}{\pi}\frac{cos(n\pi)}{n}\Big|_0^{\pi} \\ &= -\frac{2}{\pi n}[\cos(n\pi) - cos(0)] = -\frac{2}{n\pi}[cos(n\pi) - 1] \\ &= \frac{2}{n\pi}[1 - cos(n\pi)] \\ &= \begin{cases} \frac{4}{n\pi} & \text{if n is odd} \\ 0 & \text{if n is even} \end{cases} \end{aligned}$$

Thus, the formula for the Fourier Series of the square wave is

$$f(x) \approx \sum_{n=1,3,5,\ldots} \frac{4}{n\pi}\sin(nx)$$

Matlab was used to plot the first 39 harmonics by implementing the following code.

Listing 3: M-File Code For Fourier Series of Square Wave

```
N = 39; % Number of harmonics
t = linspace(0,5*pi,500); % t-vals
f = zeros(1,500); % f(t)-vals

% for i=1,3,5,7...,N
for n = 1:2:N
    b = 4.0/(pi*n); % calc coefficient
    f = f + b*sin(n*t); % add harmonic to soln
end

plot(t,f)
```

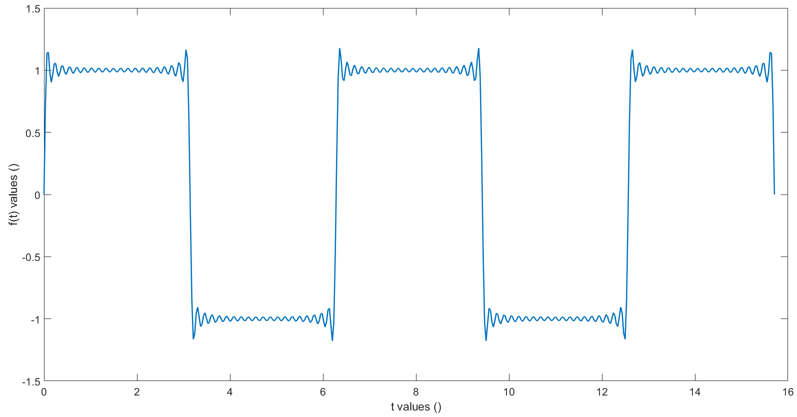

Figure 4: Fourier Series of Square Wave

Plot the sum of the first 39 harmonics for the triangle wave defined by

$$\boldsymbol{w(t) = \int_0^t f(\bar{t})d\bar{t}}$$

This problem could be solved in two ways. The first would be to simply integrate the Fourier Series of the square wave in the previous question. To get my Fourier Series practice in, I will instead integrate the function which describes the square wave, given below, and then determine the Fourier Series of this new function.

$$w(t) = \begin{cases} t & (2k-1)\pi < t < 2k\pi \\ -t & 2k\pi < t < (2k+1)\pi \end{cases}$$

Since this function is even, it can be represented as a Fourier Cosine Series.

$$w(t) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos(nt)$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} tdt = \frac{2}{\pi}\frac{t^2}{2}\Big|_0^{\pi} = \pi \qquad a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)dx = \frac{2}{n^2\pi}[cos(n\pi) - 1]$$

Thus, the Fourier Series of the triangle wave is

$$w(t) \approx \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}[cos(n\pi) - 1]cos(nt)$$

Matlab was used to plot the first 39 harmonics using the following code ($n = 2, 4, 6, \ldots$ were omitted from calculation as the result is zero).

Listing 4: M-File Code For Fourier Series of a Triangle Wave

```
N = 39; % Number of harmonics
t = linspace(0,5*pi,500); % t-vals
w = zeros(1,500); % w(t)-vals
w = w + (pi/2); % add a_0 = pi/2
```

```
% for i=1,3,5,7...,N
for n = 1:2:N
    a_n = (2/(pi*n^2))*(cos(n*pi)-1) % calc coef
    w = w + a_n*cos(n*t); % add harmonic to soln
end

plot(t,w)
```

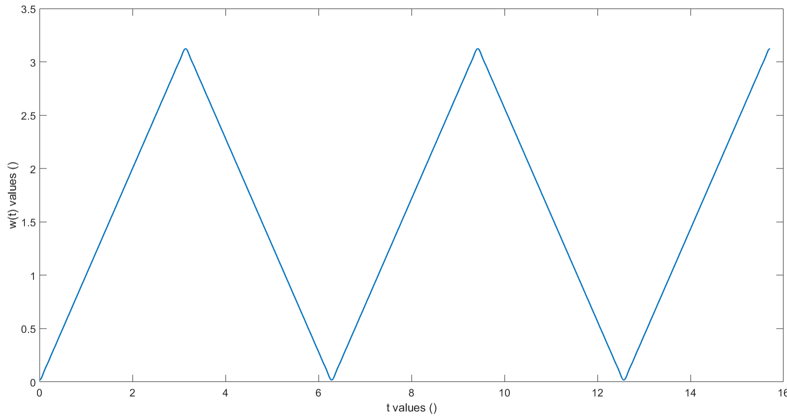

Figure 5: Fourier Series of Triangle Wave

Differentiate the series for $f(t)$ term by term, and plot on $[0, 5\pi]$ the sum of the first 39 harmonics. Explain why this might have the name buzz.
To differentiate term by term, we can simply differentiate the Fourier Series of $f(t)$ which was calculated in the first part of this problem.

$$\frac{d\,f(t)}{dt} = \frac{d}{dt}\left[\frac{4}{n\pi}\sin(nx)\right] = \frac{4}{\pi}\cos(nx) \quad n = 1, 3, 5, \ldots$$

Matlab was used to plot the first 39 harmonics. Note that the even n are equal to zero.

Listing 5: M-File Code For $\frac{d\,f}{dt}$

```
N = 39; % Number of harmonics
t = linspace(0,5*pi,500); % t-vals
g = zeros(1,500); % g(t) = df/dt

for n = 1:2:N
    b = 4.0/pi;
    g = g + n*cos(n*t);
end

plot(t,g)
```

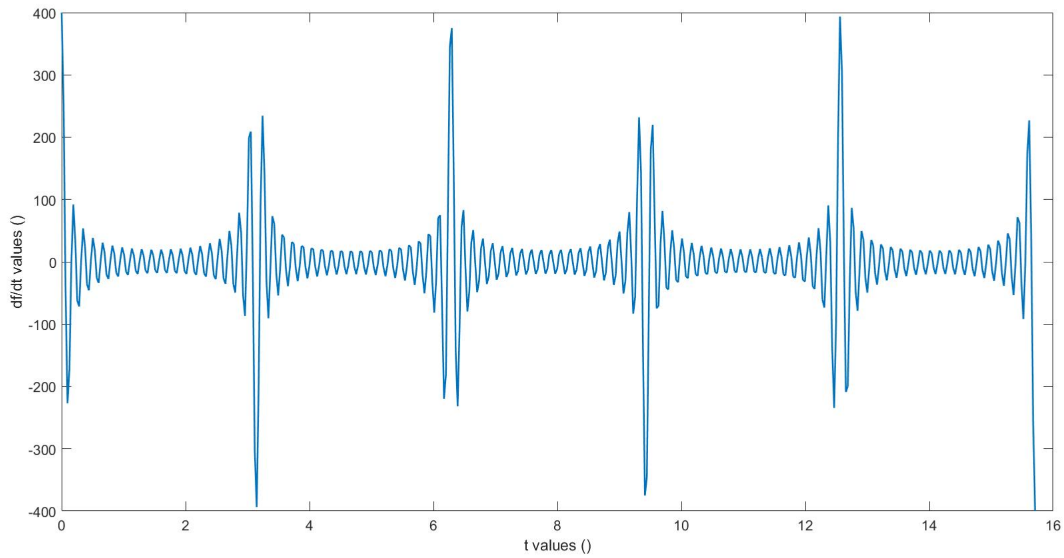

Figure 6: Derivative of Square Wave

We might expect that the derivative of a square wave is equal to zero. While this is true, differentiating the Fourier Series of a square wave, with a finite sum of terms, is not exactly zero. We see spikes when the square wave goes from positive to negative (or vise-versa). With a bit of experimentation, it was found that the amplitude of the spikes tends to zero, as the number of terms is increased. Further, the wave between the peaks flattens out, and indeed the Fourier Series goes to zero as $n \to \infty$. This might be called buzz because even though the signal is supposed to be zero there is still a signal there- just buzzing in the background.

6 References

(1) Lathi B.P. (2014) *Essentials of Digital Signal Processing*. Cambridge University Press.
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(3) Roberts M.J. (2012) *Signals and Systems. Analysis Using Transform Methods and MATLAB*. McGraw-Hill. New York, NY.
(4) Elaydi S. (2005) *An Introduction To Difference Equations*. Springer. USA.