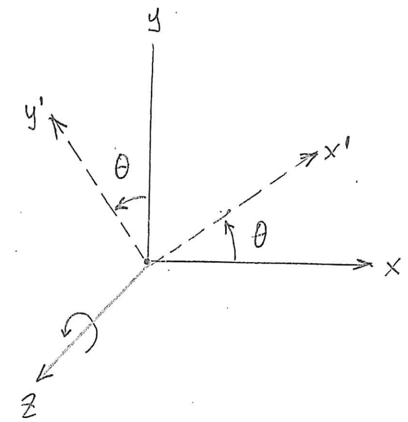
(1)



(a) $$\lambda = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) $$\lambda^T = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(-\theta) & \sin(-\theta) & 0 \\ -\sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since $\cos(-\theta) = \cos\theta$. $\sin(-\theta) = -\sin\theta$ λ^T is a rotation by $(-\theta)$ or a clockwise rotation about the z-axis

(c) $$\lambda\lambda^T = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta & 0 \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{I}$$

$\vec{A}' = \lambda\vec{A}$ or

(d) $$\begin{pmatrix} A_x' \\ A_y' \\ A_z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} A_x\cos\theta + A_y\sin\theta \\ -A_x\sin\theta + A_y\cos\theta \\ A_z \end{pmatrix}$$

$$A_x' = A_x\cos\theta + A_y\sin\theta$$
$$A_y' = -A_x\sin\theta + A_y\cos\theta$$
$$A_z' = A_z$$

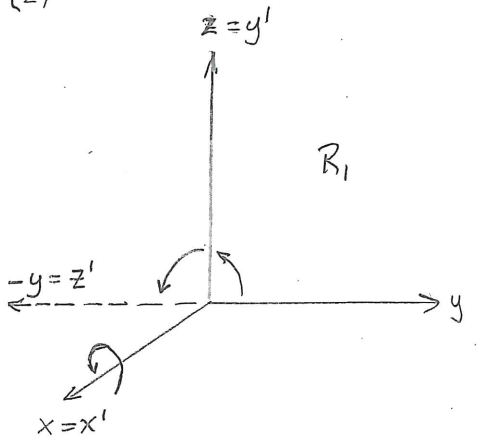
$$|A'|^2 = (A_x')^2 + (A_y')^2 + (A_z')^2 = (A_x\cos\theta + A_y\sin\theta)^2 + (-A_x\sin\theta + A_y\cos\theta)^2 + A_z^2$$

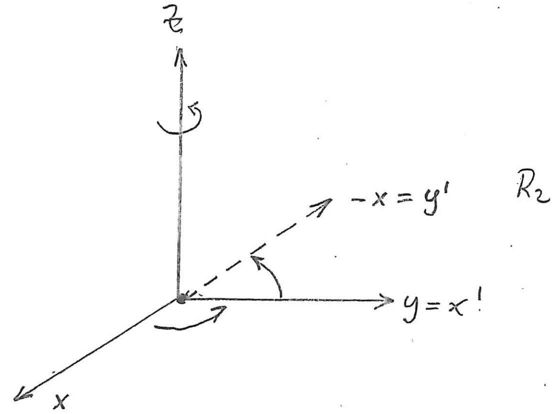
$$= (A_x^2\cos^2\theta + \cancel{2A_xA_y\sin\theta\cos\theta} + A_y^2\sin^2\theta) + A_x^2\sin^2\theta - \cancel{2A_xA_y\sin\theta\cos\theta} + A_y^2\cos^2\theta + A_z^2$$

$$= (A_x^2 + A_y^2)(\cos^2\theta + \sin^2\theta) + A_z^2 = A_x^2 + A_y^2 + A_z^2 = |A|^2$$

$\therefore$ $|A'| = |A|$ the magnitude remains invariant

(2)





$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a)

$$R'' = R_2 R_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} x'' = z \\ y'' = -x \\ z'' = -y \end{array}$$

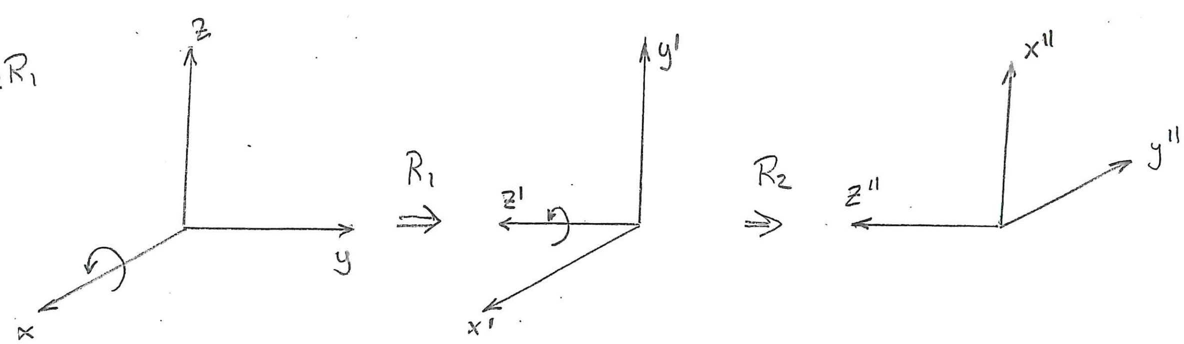
(b)

$$\bar{R}'' = R_1 R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} x'' = y \\ y'' = z \\ z'' = x \end{array}$$

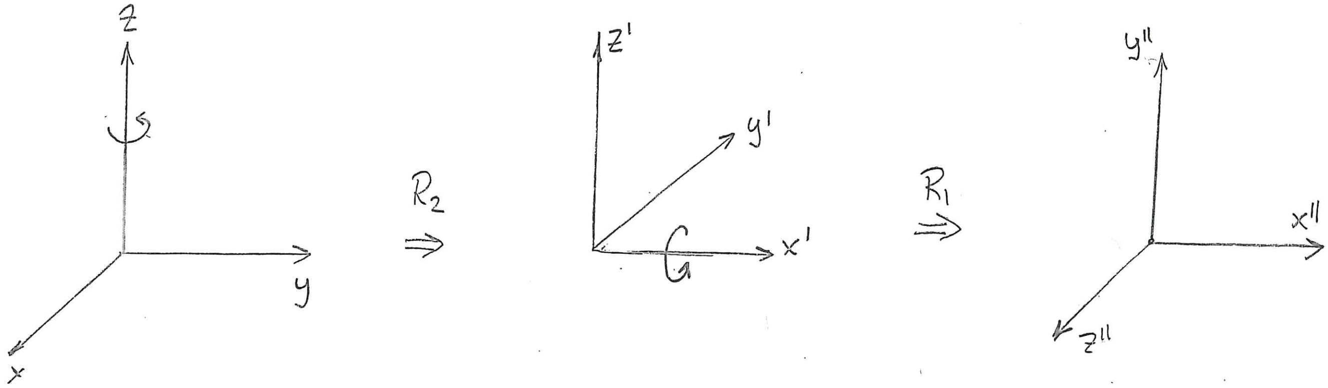
(c)

$$R_2 R_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{array}{l} x'' = -x \\ y'' = -y \\ z'' = z \end{array}$$

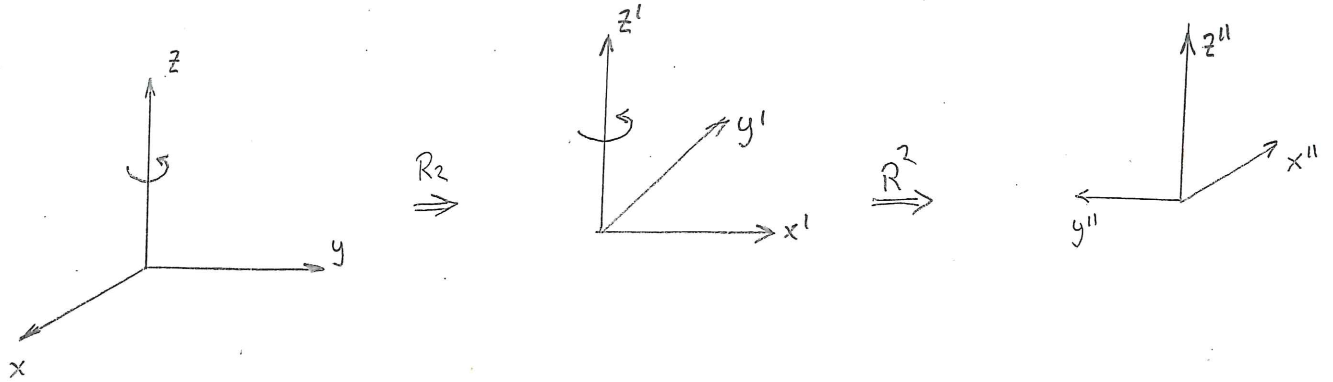
(d) $R_2 R_1$



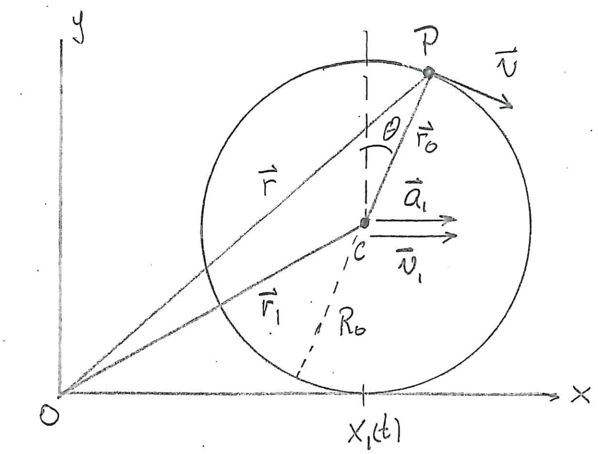
$R_1 R_2$



$R_2 R_2$



(3)



Vector addition leads to

$$\vec r = \vec r_0 + \vec r_1$$

$$\vec v = \dot{\vec r} = \dot{\vec r}_0 + \dot{\vec r}_1 = \vec v_0 + \vec v_1$$

$$\vec a = \dot{\vec v}_0 + \dot{\vec v}_1 = \ddot{\vec r}_0 + \ddot{\vec r}_1 = \vec a_0 + \vec a_1$$

rolling without slipping implies

$$x_1(t) = R_0\theta(t) \quad v_1 = \dot x_1 = R_0\dot\theta \quad \dot a_1 = \ddot x_1 = R_0\ddot\theta$$

(a) now compute $\vec r_0$, $\vec v_0$ and a_0 in terms of $\theta(t)$ $\dot\theta$ and $\ddot\theta$

$$\vec r_0 = R_0\sin\theta\,\hat i + R_0\cos\theta\,\hat j$$

$$\vec v_0 = \dot\theta R_0\cos\theta\,\hat i - \dot\theta R_0\sin\theta\,\hat j = R_0\dot\theta(\cos\,\hat i - \sin\theta\,\hat j)$$

$$\vec a_0 = = R_0\ddot\theta(\cos\theta\,\hat i - \sin\theta\,\hat j) - R_0\dot\theta^2(\sin\theta\,\hat i + \cos\theta\,\hat j)$$

$$= R_0\left[\ddot\theta\cos\theta - \dot\theta^2\sin\theta\right]\hat i - R_0\left[\ddot\theta\sin\theta + \dot\theta^2\cos\theta\right]\hat j$$

therefore

$$|a_0|^2 = R_0^2\left(\ddot\theta^2\cos^2\theta - \cancel{2\ddot\theta\dot\theta^2\cos\theta\sin\theta} + \dot\theta^4\sin^2\theta\right)$$

$$+ R_0^2\left(\ddot\theta\sin^2\theta + \cancel{2\ddot\theta\dot\theta^2\sin\theta\cos\theta} + \dot\theta^4\cos^2\theta\right) = R_0^2\ddot\theta^2 + R_0^2\dot\theta^4$$

but $a_1 = R_0\ddot\theta$ and $v_1 = R_0\dot\theta$

therefore

$$a_0 = \left(R_0^2\ddot\theta^2 + R_0^2\dot\theta^4\right)^{1/2} = \left(a_1^2 + v_1^4/R_0^2\right)^{1/2}$$

(b) now compute the vector $\vec a_1 = \ddot{\vec r}_1 = \frac{d^2}{dt^2}(R_0\theta\,\hat i + R_0\hat j) = R_0\ddot\theta\,\hat i \Rightarrow a_1 = R_0\ddot\theta$ as before

the total acceleration vector is

$$\vec a = \vec a_0 + \vec a_1 = \left[R_0(\ddot\theta\cos\theta - \dot\theta^2\sin\theta) + R_0\ddot\theta\right]\hat i + \left[R_0(\ddot\theta\sin\theta + \dot\theta^2\cos\theta)\right]\hat j$$

or

$$\vec{a} = \left[R_0\ddot{\theta}(1+\cos\theta) - R_0\dot{\theta}^2\sin\theta\right]\hat{i} - \left[R_0\ddot{\theta}\sin\theta + R_0\dot{\theta}^2\cos\theta\right]\hat{j} \qquad \text{but } a_1 = R_0\ddot{\theta}$$

$$= \left[a_1(1+\cos\theta) - v_1^2/R_0\sin\theta\right]\hat{i} - \left[a_1\sin\theta + v_1^2/R_0\cos\theta\right]\hat{j}$$

compute the magnitude a_1

$$|a|^2 = \left[a_1(1+\cos\theta) - \frac{v_1^2}{R_0}\sin\theta\right]^2 + \left[a_1\sin\theta + v_1^2/R_0\cos\theta\right]^2$$

$$= a_1^2(1+2\cos\theta+\cos^2\theta) - 2\frac{a_1v_1^2}{R_0}(1+\cancel{\cos\theta)\sin\theta} + \frac{v_1^4}{R_0^2}\sin^2\theta$$

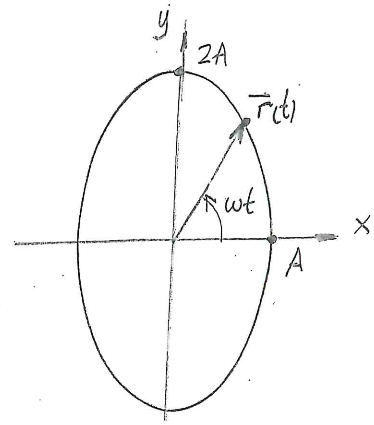
$$+ a_1^2\sin^2\theta + \cancel{\frac{2a_1v_1^2}{R_0}\sin\theta}\cos\theta + \frac{v_1^4}{R_0^2}\cos^2\theta$$

$$= a_1^2(1+2\cos\theta+1) - \frac{2a_1v_1^2}{R_0}\sin\theta + \frac{v_1^4}{R_0^2} \qquad \text{therefore}$$

$$|a| = \left\{a_1^2(2+2\cos\theta) - \frac{2a_1v_1^2}{R_0}\sin\theta + v_1^4/R_0^2\right\}^{1/2} \qquad \text{factor out } a_1$$

$$= a_1\left[2+2\cos\theta + \frac{v_1^4}{a_1^2R_0^2} - \frac{2v_1^2}{a_1R_0}\sin\theta\right]^{1/2}$$

(4)



$$\vec{r}(t) = A\cos(\omega t)\hat{\imath} + 2A\sin(\omega t)\hat{\jmath}$$

$$\vec{v}(t) = \dot{\vec{r}}(t) = -A\omega\sin(\omega t)\hat{\imath} + 2A\omega\cos(\omega t)\hat{\jmath}$$

speed $v = (\vec{v}\cdot\vec{v})^{1/2}$

$$= \left[A^2\omega^2\sin^2(\omega t) + 4A^2\omega^2\cos^2(\omega t)\right]^{1/2}$$

$$= \left\{A^2\omega^2\left[\sin^2(\omega t) + 4\cos^2(\omega t)\right]\right\}^{1/2} \Rightarrow v = A\omega\left[\sin^2(\omega t) + 4\cos^2(\omega t)\right]^{1/2}$$

@ $t=0$ $\quad v = A\omega[0+4]^{1/2} = 2A\omega \qquad \vec{r} = A\hat{\imath}$

$t = \frac{\pi}{2\omega} \quad v = A\omega[1+0] = A\omega \qquad \vec{r} = 2A\hat{\jmath}$

the distance from the origin is

$$|r| = (\vec{r}\cdot\vec{r})^{1/2} = A\left[\cos^2(\omega t) + 4\sin^2(\omega t)\right]^{1/2}$$

the maximum and/or minimum occurs when $\frac{dr}{dt} = 0$

$$\frac{dr}{dt} = \frac{A}{[\cos^2(\omega t) + 4\sin^2(\omega t)]^{1/2}}\left(\frac{1}{2}\right)\left[-2\sin(\omega t)\cos(\omega t) + 8\sin(\omega t)\cos(\omega t)\right]$$

$$= \frac{A[3\sin(\omega t)\cos(\omega t)]}{[\cos^2(\omega t) + 4\sin^2(\omega t)]^{1/2}} = 0$$

vanishes when $\sin(\omega t) = 0 \Rightarrow \omega t = 0$

when $\cos(\omega t) = 0 \Rightarrow \omega t = \pi/2$

min occurs when $t=0$ $\quad \vec{r}(0) = A\hat{\imath} \qquad v = 2A\omega$

max occurs when $t = \frac{\pi}{2\omega} \quad \vec{r}_0\left(\frac{\pi}{2\omega}\right) = 2A\hat{\jmath} \quad v = A\omega$

(5)

in cylindrical coordinates (ρ, θ, z) with unit vectors $(\hat{e}_\rho, \hat{e}_\theta, \hat{k})$

$$\vec{r} = \rho\hat{e}_\rho + z\hat{k} \qquad \vec{v} = \dot{\vec{r}} = \dot{\rho}\hat{e}_\rho + \rho\dot{\hat{e}} + \dot{z}\hat{k} \qquad \text{but } \dot{\hat{e}}_\rho = \dot{\theta}\hat{e}_\theta \quad \dot{\hat{e}}_\theta = -\dot{\theta}e_\rho \quad \dot{\hat{k}} = 0$$

$$= \dot{\rho}\hat{e}_\rho + \rho\dot{\theta}\hat{e}_\theta + \dot{z}\hat{k}$$

$$\vec{a} = \dot{\vec{v}} = (\ddot{\rho} - \rho\dot{\theta}^2)\hat{e}_\rho + (\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta})\hat{e}_\theta + \ddot{z}\hat{k}$$

see lecture notes and/or text book

$$\vec{J} = \dot{\vec{a}} = (\dddot{\rho} - \dot{\rho}\dot{\theta}^2 - 2\rho\dot{\theta}\ddot{\theta})\hat{e}_\rho + (\ddot{\rho} - \rho\dot{\theta}^2)\dot{\hat{e}}_\rho$$

$$+ (\dot{\rho}\ddot{\theta} + \rho\dddot{\theta} + 2\ddot{\rho}\dot{\theta} + 2\dot{\rho}\ddot{\theta})\hat{e}_\theta + (\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta})\dot{\hat{e}}_\theta + \dddot{z}\hat{k}$$

$$= (\dddot{\rho} - \dot{\rho}\dot{\theta}^2 - 2\rho\dot{\theta}\ddot{\theta})\hat{e}_\rho + \dot{\theta}(\ddot{\rho} - \rho\dot{\theta}^2)\hat{e}_\theta$$

$$+ (\dot{\rho}\ddot{\theta} + \rho\dddot{\theta} + 2\ddot{\rho}\dot{\theta} + 2\dot{\rho}\ddot{\theta})\hat{e}_\theta - \dot{\theta}(\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta})\hat{e}_\rho + \dddot{z}\hat{k}$$

$$= (\dddot{\rho} - 3\rho\dot{\theta}\ddot{\theta} - 3\dot{\rho}\dot{\theta}^2)\hat{e}_\rho + (\rho\dddot{\theta} + 3\dot{\rho}\ddot{\theta} + 3\ddot{\rho}\dot{\theta} - \rho\dot{\theta}^3)\hat{e}_\theta + \dddot{z}\hat{k}$$