(1) for $F(t) = F_0 \sin(\omega t)$ assume a solution $x(t) = A\sin(\omega t - \phi)$

then $\dot{x}(t) = A\omega\cos(\omega t-\phi)$ $\quad \ddot{x}(t) = -A\omega^2\sin(\omega t-\phi)$

then $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\sin(\omega t)$ becomes

$$-A\omega^2\sin(\omega t-\phi) + 2A\gamma\omega\cos(\omega t-\phi) + A\omega_0^2\sin(\omega t-\phi) = (F_0/m)\sin(\omega t)$$

or $(\omega_0^2-\omega^2)A\sin(\omega t-\phi) + 2A\gamma\omega\cos(\omega t-\phi) = (F_0/m)\sin(\omega t)$

now $\sin(\omega t-\phi) = \sin(\omega t)\cos\phi - \cos(\omega t)\sin\phi$

$\cos(\omega t-\phi) = \cos(\omega t)\cos\phi + \sin(\omega t)\sin\phi$ therefore

$$A(\omega_0^2-\omega^2)\left[\sin(\omega t)\cos\phi - \cos(\omega t)\sin\phi\right] + 2A\gamma\omega\left[\cos(\omega t)\cos\phi + \sin(\omega t)\sin\phi\right] = F_0/m\,\sin(\omega t)$$

collect coefficients of $\sin(\omega t)$ and $\cos(\omega t)$

$$A\left[(\omega_0^2-\omega^2)\cos\phi + 2\gamma\omega\sin\phi\right]\sin(\omega t) = (F_0/m)\sin(\omega t) \qquad (1)$$

$$A\left[-(\omega_0^2-\omega^2)\sin\phi + 2\gamma\omega\cos\phi\right]\cos(\omega t) = 0 \qquad (2)$$

equation (2) leads to $+(\omega_0^2-\omega^2)\sin\phi = 2\gamma\omega\cos\phi$ $\quad \boxed{\tan\phi = \frac{2\omega\gamma}{(\omega_0^2-\omega^2)}}$

equation (1) squared & equation (2) squared becomes

$$A^2\left[(\omega_0^2-\omega^2)^2\left[\cos^2\phi + \sin^2\phi\right] + 4\gamma^2\omega^2\left[\cos^2\phi + \sin^2\phi\right]\right] = F_0^2/m^2 \quad \text{or}$$

$$\boxed{A\left[(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2\right]^{1/2} = F_0/m}$$

these are exactly the same results when $F(t) = F_0\cos(\omega t)$

(1) alternative

consider a driving force

$$\vec{F} = F_0 e^{i\omega t} = F_0 \cos(\omega t) + i F_0 \sin(\omega t)$$

find a particular solution of the form $x_p(t) = A e^{i(\omega t - \phi)}$

$$= A\left[\cos(\omega t-\phi) + i \sin(\omega t - \phi)\right]$$

that satisfies $\ddot{x}_p + 2\gamma \dot{x}_p + \omega_0^2 x_p = \frac{F_0}{m} e^{i\omega t}$ (1)

then $Re[x_p]$ gives $Re[F]$ and $Im[x_p]$ gives $Im[F]$

$\dot{x}_p = i\omega A e^{i(\omega t-\phi)}$ $\quad \ddot{x}_p = -\omega^2 A e^{i(\omega t - \phi)}$ then eq (1) becomes

$$-A\omega^2 e^{i(\omega t-\phi)} + 2\gamma i A\omega e^{i(\omega t-\phi)} + \omega_0^2 A e^{i(\omega t-\phi)} = \frac{F_0}{m} e^{i\omega t} \quad \text{or}$$

$$A\left[(\omega_0^2-\omega^2) + 2\gamma\omega i\right] e^{-i\phi} = \frac{F_0}{m}$$

$$A\left[(\omega_0^2-\omega^2) + 2\gamma\omega i\right] = \frac{F_0}{m} e^{i\phi} = \frac{F_0}{m}\left[\cos\phi + i\sin\phi\right]$$ separate into Real and Imaginary equations

Re: $A(\omega_0^2-\omega^2) = \frac{F_0}{m}\cos\phi$ (2) $\qquad$ Im: $2A\gamma\omega = \frac{F_0}{m}\sin\phi$ (3)

(a) divide eq (3) by eq (2)

$$\frac{\sin\phi}{\cos\phi} = \tan\phi = \frac{2\gamma\omega}{\omega_0^2-\omega^2} \qquad \phi = \tan^{-1}\left[\frac{2\gamma\omega}{\omega_0^2-\omega^2}\right]$$

(b) add eq (1) square to eq (2) squared

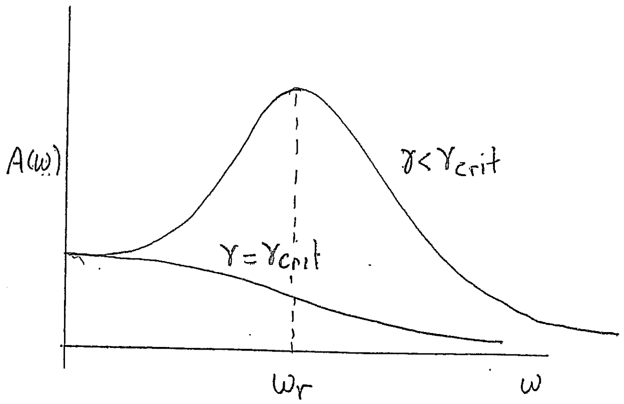
$$A^2\left[(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2\right] = \frac{F_0^2}{m^2}(\cos^2\phi + \sin^2\phi) = F_0^2/m^2 \quad \text{or}$$

$$A = \frac{F_0/m}{\left[(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2\right]^{1/2}}$$

therefore $A\cos(\omega t - \phi)$ is a solution for $F = F_0\cos(\omega t)$

$iA\sin(\omega t - \phi)$ is a solution for $iF = iF_0\sin(\omega t)$

(2)



$x(t) = A\cos(\omega t - \phi)$

$v(t) = -A\omega\sin(\omega t - \phi)$

(a) $A(\omega) = \dfrac{F_0/m}{[(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2]^{1/2}}$ is Amplitude for $x(t)$

$$\frac{dA}{d\omega} = \frac{F_0/m\left[-\frac{1}{2}\left(2(-2)(\omega_0^2-\omega^2)\omega + 8\gamma^2\omega\right)\right]}{[(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2]^{3/2}} = 0 \quad \text{solve for } \omega$$

then $-4(\omega_0^2 - \omega_r^2)\omega_r + 8\gamma^2\omega_r = 0$

$2\gamma^2\omega_r = \omega_r(\omega_0^2 - \omega_r^2)$ or $\omega_r^2 = \omega_0^2 - 2\gamma^2$ and $\omega_r = 0$

$$\boxed{\omega_r = \sqrt{\omega_0^2 - 2\gamma^2}\ ,\ 0}$$

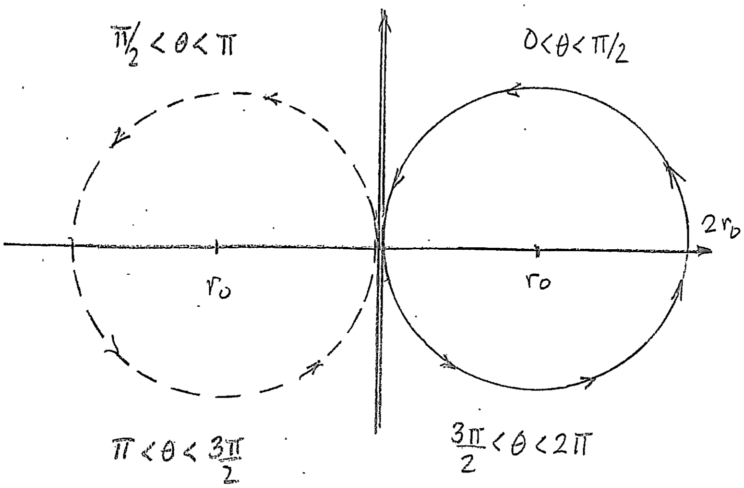
(b) $\omega A(\omega) = \dfrac{\omega F_0/m}{[(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2]^{1/2}}$ is the amplitude for $v(t)$

$$0 = \frac{d}{d\omega}[A\omega] = \frac{F_0/m\left\{-\frac{1}{2}\left(-4\omega^2(\omega_0^2-\omega^2) + 8\gamma^2\omega^2\right) + (\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2\right\}}{[(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2]^{3/2}} \quad \text{solve for } \omega$$

$$2\omega_r^2\cancel{\omega_0^2} - 2\omega_r^4 - \cancel{4\gamma^2\omega_r^2} + \omega_0^4 - 2\cancel{\omega_0^2}\omega_r^2 + \omega_r^4 + \cancel{4\gamma^2\omega_r^2} = 0$$

$\omega_0^4 - \omega_r^4 = 0$ $\quad\boxed{\omega = \omega_0}$

(3)



$\theta = 0 \quad r = 2r_0$

$\theta = \pi/2 \quad r = 0$

$\theta = \frac{3\pi}{2} \quad r = 0$

$\theta = 2\pi \quad r = 2r_0$

$$r = 2r_0 \cos\theta \implies u = \frac{1}{2r_0}\left(\cos^{-1}\theta\right)$$

$$\frac{du}{d\theta} = \frac{1}{2r_0}(-1)(-\sin\theta)\cos^{-2}\theta = \frac{1}{2r_0}\,\frac{\sin\theta}{\cos^2\theta}$$

$$\frac{d^2u}{d\theta^2} = \frac{1}{2r_0}\left[\frac{\cos\theta}{\cos^2\theta} - 2\,\frac{-\sin^2\theta}{\cos^3\theta}\right] = \frac{1}{2r_0}\left[\frac{\cos^2\theta + 2\sin^2\theta}{\cos^3\theta}\right]$$

$$= \frac{1}{2r_0}\left[\frac{1+\sin^2\theta}{\cos^3\theta}\right]$$

using $\displaystyle \frac{d^2u}{d\theta^2} + u = -\frac{F(u)}{m\ell^2 u^2}$

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{2r_0}\left[\frac{1+\sin^2\theta}{\cos^3\theta}\right] + \frac{1}{2r_0}\left(\frac{1}{\cos\theta}\right) = \frac{1}{2r_0}\left[\frac{1+\sin^2\theta+\cos^2\theta}{\cos^3\theta}\right] = \frac{1}{2r_0}\,\frac{2}{\cos^3\theta}$$

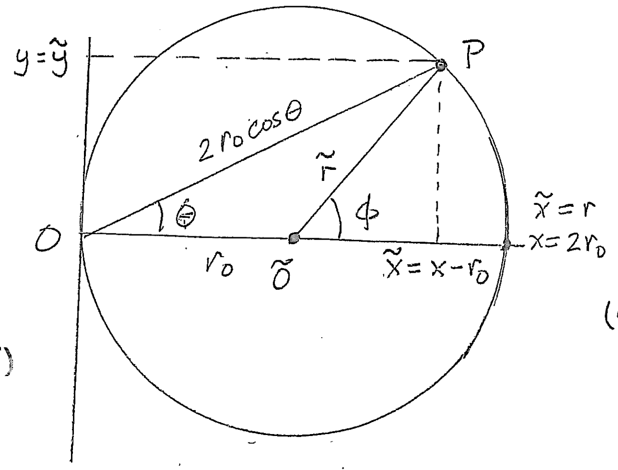
$$= \frac{1}{r_0}\,\frac{1}{\cos^3\theta} = \frac{1}{r_0}\left(8r_0^3\right)u^3 = 8r_0^2u^3$$

then $-F(u) = (8r_0^2u^3)(m\ell^2u^2) = 8r_0^2 m\ell^2 u^5$

$$\boxed{F(r) = -\frac{8m\ell^2 r_0^2}{r^5}}$$

$k = 8m\ell^2 r_0^2$

(d)



(b) $x = (2 r_0 \cos\theta) \cos\theta$

$y = (2 r_0 \cos\theta) \sin\theta$

(a) $\tilde{y} = r_0 (2 \sin\theta \cos\theta) = r_0 \sin 2\theta$

$\tilde{x} = 2 r_0 \cos^2\theta - r_0 = r_0 (2 \cos^2\theta - 1)$

but $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

$\tilde{x} = r_0 \left[2 \tfrac{1}{2} (\cos 2\theta + 1) - 1 \right] = r_0 \cos 2\theta$

$\tilde{r}^2 = \tilde{x}^2 + \tilde{y}^2 = r_0^2 \cos^2 2\theta + r_0^2 \sin^2 2\theta$

$= r_0^2$ the curve has a constant radius about $\tilde{O}$

also $\phi = 2\theta$ as expected.

(c) when the orbit passes through O $r_c = 0$ then $\underline{\underline{\theta_c = \pm \pi/2}}$ since $\underline{\cos\theta_c = 0}$

(4) Using Kepler's third law compute the semi-major axis a

$$\tau^2 = \frac{4\pi^2 a^3}{GM_\odot} \Rightarrow a = \left(\frac{GM_\odot \tau^2}{4\pi^2}\right)^{1/3} \qquad M_\odot = 1.99\times10^{30}\,kg$$

$$G = 6.67\times10^{-11}\ Nm^2/kg^2$$

$$\tau = (76\,yr)\left(\frac{365\,day}{yr}\right)\left(\frac{24\,hr}{day}\right)\left(\frac{3600\,s}{hr}\right) = 2.397\times10^9\,s$$

$$a = \left[\frac{\left(6.67\times10^{-11}\frac{N\cdot m^2}{kg^2}\right)(1.99\times10^{30}\,kg)(2.397\times10^9\,s)}{4\pi^2}\right]$$

$$= 2.68\times10^{12}\,m = 2.68\times10^{9}\,km$$

therefore

$$r_{min} = a(1-\epsilon) = (2.68\times10^9\,km)(1-0.967) = 8.8\times10^7\,km = 8.8\times10^{10}\,m$$

$$r_{max} = a(1+\epsilon) = (2.68\times10^9\,km)(1+0.967) = 5.27\times10^9\,km = 5.27\times10^{11}\,m$$

compute angular momentum L $\qquad \lambda = \frac{L^2}{m_H k}$ or $L^2 = m_H k\lambda = m_H k\, a(1-\epsilon^2)$

$$k = GM_\odot m_H \quad \text{so} \quad L = \sqrt{GM_\odot m_H^2\, a(1-\epsilon^2)} = \sqrt{GM_\odot a(1-\epsilon^2)}\ m_H$$

$$L = \left[\left(6.67\times10^{-11}\frac{N\cdot m^2}{kg^2}\right)(1.99\times10^{30}\,kg)(2.68\times10^{12}\,m)(1-0.967^2)\right]^{1/2} 3\times10^{15}\,kg$$

$$= 1.44\times10^{31}\ \frac{kg\cdot m^2}{s}$$

(5) <u>using conservation of energy</u> where $T = \frac{1}{2}mv^2$ $\quad V(r) = -k/r$

$$E = \tfrac{1}{2}mv^2 - \frac{k}{r} \quad \text{then one obtains} \quad v = \sqrt{\frac{2}{m}}\left[E + \frac{k}{r}\right]^{1/2}$$

$$v_{max} = \sqrt{\frac{2}{m}}\left[E + \frac{k}{r_{min}}\right]^{1/2} \qquad v_{min} = \sqrt{\frac{2}{m}}\left[E + \frac{k}{r_{max}}\right]^{1/2} \quad \text{thus}$$

$$v_{max} \cdot v_{min} = \frac{2}{m}\left[\left(E + \frac{k}{r_{min}}\right)\left(E + \frac{k}{r_{max}}\right)\right]^{1/2} \quad \text{but } E = -\frac{k}{2a}$$

$$= \frac{2}{m}\left[k^2\left(\frac{-1}{2a} + \frac{1}{r_{min}}\right)\left(-\frac{1}{2a} + \frac{1}{r_{max}}\right)\right]^{1/2}$$

$$\text{but } r_{min} = \frac{\lambda}{1+\epsilon} = \frac{a(1-\epsilon^2)}{1+\epsilon} = a(1-\epsilon)$$

$$r_{max} = \frac{\lambda}{1-\epsilon} = \frac{a(1-\epsilon^2)}{1-\epsilon} = a(1+\epsilon) \qquad \text{therefore}$$

$$v_{max} \cdot v_{min} = \frac{2k}{ma}\left[\left(\frac{1}{1-\epsilon} - \frac{1}{2}\right)\left(\frac{1}{1+\epsilon} - \frac{1}{2}\right)\right]^{1/2}$$

$$= \frac{2k}{ma}\left[\frac{1}{1-\epsilon^2} - \frac{1}{2}\left(\frac{1}{1-\epsilon} + \frac{1}{1+\epsilon}\right) + \frac{1}{4}\right]^{1/2}$$

$$= \frac{2k}{ma}\left[\frac{1}{1-\epsilon^2} - \frac{1}{2}\left(\frac{1 + \cancel{\epsilon} + 1 - \cancel{\epsilon}}{1-\epsilon^2}\right) + \frac{1}{4}\right]^{1/2}$$

$$= \frac{2k}{ma}\left[\cancel{\frac{1}{1-\epsilon^2}} - \cancel{\frac{1}{1-\epsilon^2}} + \frac{1}{4}\right]^{1/2}$$

$$= \frac{k}{ma} \qquad \text{but } \frac{k}{m} = GM \quad \text{and } GM = \frac{4\pi^2}{\tau^2}a^3 \quad \text{by Kepler's 3rd law}$$

$$= \frac{4\pi^2 a^3}{a\tau^2} = \left(\frac{2\pi a}{\tau}\right)^2$$

(5) (alternative)

use $\frac{dA}{dt} = \frac{L}{2m} \Rightarrow \frac{A_{tot}}{\tau} = \frac{\pi ab}{\tau} = \frac{L}{2m}$ but $L = mr^2\dot{\theta}$

$$\therefore \frac{\pi ab}{\tau} = \frac{\cancel{m} r_{min} v_{max}}{2\cancel{m}} = \frac{\cancel{m} r_{max} v_{max}}{2\cancel{m}}$$

since $\vec{L} = m\vec{v}\times\vec{r}$ and when $\vec{v} = \vec{v}_{max}$ and $\vec{r} = \vec{r}_{min}$, $\vec{v}_{max} \perp \vec{r}_{min}$

when $\vec{v} = \vec{v}_{min}$ and $\vec{r} = \vec{r}_{max}$ $\vec{v}_{min} \perp \vec{r}_{max}$

$\therefore |\vec{L}| = m|\vec{r}\times\vec{v}| = \underline{m r_{max} v_{min} = m r_{min} v_{max}}$ <u>conservation of angular momentum</u>

$$\therefore v_{max} = \frac{2\pi ab}{\tau r_{min}} \text{ and } v_{min} = \frac{2\pi ab}{\tau r_{max}}$$

thus $v_{max} v_{min} = \left(\frac{2\pi ab}{\tau}\right)^2 \frac{1}{r_{max}} \frac{1}{r_{min}}$ but $r_{min} = \frac{\lambda}{1+\epsilon}$ $r_{max} = \frac{\lambda}{1-\epsilon}$

$= \frac{(2\pi ab)^2}{\lambda^2\tau^2}(1-\epsilon)(1+\epsilon)$ now $\lambda = (1-\epsilon^2)a$

$b = \sqrt{1-\epsilon^2}\, a$

$$\therefore v_{max} v_{min} = \left(\frac{2\pi a^2}{\tau^2}\right)^2 \frac{(1-\epsilon^2)^2}{a^2(1-\epsilon^2)^2}$$

$$= \left(\frac{2\pi a}{\tau}\right)^2$$

(5) 2nd alternative

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta \qquad v^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

using $r = \dfrac{\lambda}{\epsilon\cos\theta + 1}$ $\quad \dot{r} = \dfrac{dr}{dt} = \dfrac{dr}{d\theta}\dfrac{d\theta}{dt} = \dfrac{\epsilon\lambda\sin\theta\,\dot{\theta}}{[\epsilon\cos\theta+1]^2}$

also $\dot{\theta} = \dfrac{\ell}{r^2} = \dfrac{\ell}{\lambda^2}(\epsilon\cos\theta+1)$ then $\dot{r} = \dfrac{\epsilon\lambda\sin\theta}{(\epsilon\cos\theta+1)^2}\dfrac{\ell}{\lambda^2}(\epsilon\cos\theta+1)^2 = \epsilon\sin\theta\left(\dfrac{\ell}{\lambda}\right)$

$r\dot{\theta} = \dfrac{\ell}{r} = \dfrac{\ell}{\lambda}(\epsilon\cos\theta+1)$ therefore

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 = \frac{\ell^2}{\lambda^2}\left[\epsilon^2\sin^2\theta + \epsilon^2\cos^2\theta + 2\epsilon\cos\theta + 1\right]$$

$$= \frac{\ell^2}{\lambda^2}(\epsilon^2+1+2\epsilon\cos\theta) \qquad v = \frac{\ell}{\lambda}(\epsilon^2+1+2\epsilon\cos\theta)^{1/2}$$

find θ for v_{max}, v_{min} solve for θ where $\dfrac{dv}{d\theta} = 0$

$$\frac{dv}{d\theta} = \frac{1}{2}(\epsilon^2+1+2\epsilon\cos\theta)^{3/2}(2\epsilon)(-\sin\theta) = 0 \qquad \theta_m = 0, \pi$$

$\therefore$ $\theta = 0$ $\quad v = v_{max} = r\dot{\theta} = \dfrac{\ell}{\lambda}(\epsilon\cos(0)+1) = \dfrac{\ell}{\lambda}(\epsilon+1) = \dfrac{\ell}{\lambda}(1+\epsilon)$

$\theta = \pi$ $\quad v = v_{min} = r\dot{\theta} = \dfrac{\ell}{\lambda}(\epsilon\cos(\pi)+1) = \dfrac{\ell}{\lambda}(1-\epsilon)$

$(v_{max})(v_{min}) = \dfrac{\ell^2}{\lambda^2}(1-\epsilon^2)$ $\qquad$ but $\lambda = a(1-\epsilon^2)$ $\quad \ell = \dfrac{2A}{\tau} = \dfrac{2\pi ab}{\tau} = \dfrac{2\pi a^2\sqrt{1-\epsilon^2}}{\tau}$

$$= \frac{4\pi^2a^4(1-\epsilon^2)}{\tau^2}\frac{1}{a^2(1-\epsilon^2)^2}(1-\epsilon^2) = \frac{4\pi^2a^2}{\tau^2} = \left(\frac{2\pi a}{\tau}\right)^2$$