(1)

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} = \frac{dv}{dt} + 2\gamma v = 0$$ write the last form as

$$\frac{dv}{dt} = -2\gamma v$$ separate variables $$\frac{dv}{v} = -2\gamma\, dt$$

integrate where $v = v_0$ @ $t=0$

(a) $$\int_{v_0}^{v}\left(\frac{dv'}{v'}\right) = -2\gamma\int_0^t dt \Rightarrow \ln\left(\frac{v}{v_0}\right) = -2\gamma t$$ take ln of both sides

$$\boxed{v(t) = v_0 e^{-2\gamma t}}$$ then set $v = \frac{dx}{dt}$

$$\frac{dx}{dt} = v_0 e^{-2\gamma t} \Rightarrow \int_0^x dx' = \int_0^t v_0 e^{-2\gamma t}\, dt$$

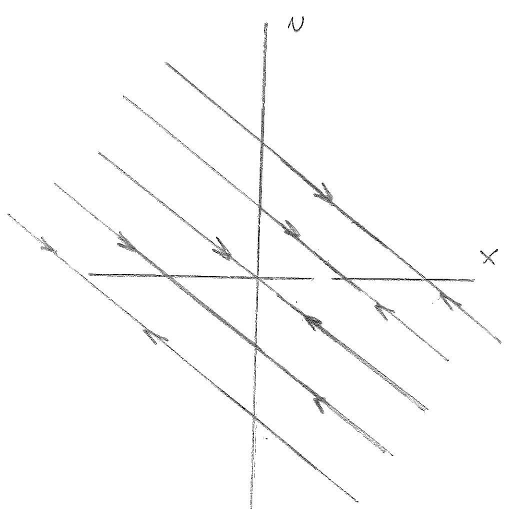
$$\boxed{x(t) = -\frac{1}{2\gamma} v_0 e^{-2\gamma t}}$$ this can be written as $x(t) = -\frac{1}{2\gamma} v(t)$

or $v(t) = -2\gamma\, x(t)$

(b) using $\frac{dx}{dt} = v$ and $\frac{dv}{dt} = -2\gamma v$ one can write

$$\frac{dv}{dx} = (dv/dt)/(dx/dt) = -2\gamma v / v = -2\gamma$$ separate variables

$$dv = -2\gamma dx \text{ or } \int_{v_0}^{v} dv = -2\gamma\int_{x_0}^{x} dx$$



$$v - v_0 = -2\gamma(x - x_0) \Rightarrow v(x) = -2\gamma x + (v_0 + 2\gamma x_0)$$

$$\boxed{v = -2\gamma x + \text{const}}$$

Phase portraits are straight lines all with slopes -2γ and different intercepts depending on x_0, v_0

(2)

if $\omega_0^2 < 0$ one can write the 2nd order differential equation as

$$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} - |\omega_0|^2 x = 0$$ assume solutions of the form $x(t) = C e^{\lambda t}$

the characteristic equation for λ is $\lambda^2 + 2\gamma\lambda - |\omega_0^2| = 0$

solutions are $\lambda_{1,2} = -\frac{2\gamma}{2} \pm \frac{1}{2}\sqrt{4\gamma^2 + 4|\omega_0^2|} = -\gamma \pm \sqrt{\gamma^2 + |\omega_0^2|}$

however $\sqrt{\gamma^2 + |\omega_0^2|} \geq |\gamma| > 0$

set $\Gamma = \sqrt{\gamma^2 + |\omega_0^2|}$ therefore

$\lambda_1 = -\gamma + \Gamma$ and $\lambda_2 = -\gamma - \Gamma$ the general solution is

$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = x_1(t) + x_2(t)$

now (a) set $\gamma > 0$ then

$\lambda_1 = -\gamma + \Gamma > 0$ and $\lambda_2 = -\gamma - \Gamma < 0$

(b) set $\gamma < 0$ then

$\lambda_1 = |\gamma| + \Gamma > 0$ and $\lambda_2 = |\gamma| - \Gamma < 0$

then $\lim_{t\to\infty} x_1 = \underline{\lim_{t\to\infty} C_1 e^{\lambda_1 t} \to \infty}$ $\lim_{t\to\infty} x_2 = \lim_{t\to\infty} C_2 e^{\lambda_2 t} \to 0$ <u>for all γ</u>

$v(t) = C_1\lambda_1 e^{\lambda_1 t} + C_2\lambda_2 e^{\lambda_2 t}$ has same properties since $\underline{e^{\lambda_1 t} \to \infty}$ and $e^{\lambda_2 t} \to 0$ for $t \to \infty$

this produces a saddle portrait since both $x(t)$ and $v(t) \to \infty$ as $t \to \infty$ and $x(t)$ & $v(t) \to \infty$ as $t \to -\infty$

(3)

$$F(x) = x - x^3 \quad \text{so} \quad m\frac{d^2x}{dt^2} = x - x^3 \quad \text{can be written as}$$

(a)
$$\begin{cases} \dfrac{dx}{dt} = v \\ \dfrac{dv}{dt} = \dfrac{x - x^3}{m} = f \end{cases}$$

find equilibrium points x_e

$$\frac{dx}{dt} = 0 = v_e \qquad v_e = 0$$

$$\frac{dv}{dt} = 0 = \frac{1}{m}\, x(1 - x^2)$$

$$\boxed{x_e = 0, \pm 1}$$

equilibrium points are

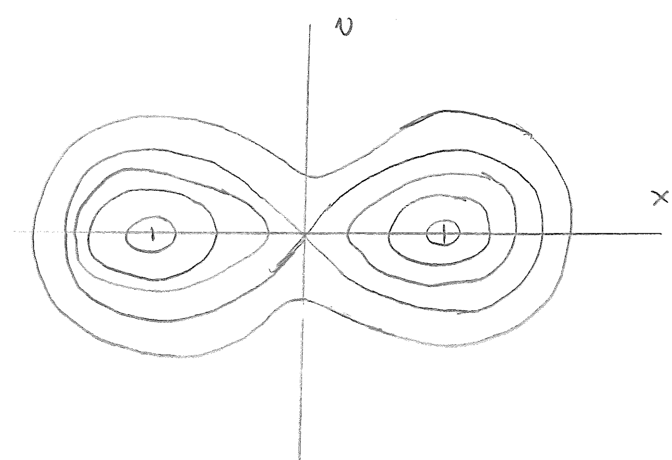
(b) $(x_e, v_e) = (0,0),\ (-1,0),\ (+1,0)$

(c) compute the perturbation equations

$$\begin{cases} \dfrac{d}{dt}(\delta x) = \delta v \\ \dfrac{d}{dt}(\delta v) = \dfrac{\partial f}{\partial x}\delta x + \cancel{\dfrac{\partial f}{\partial v}\delta v} = \dfrac{1}{m}(1 - 3x^2)\,\delta x \end{cases}$$

@ $(0,0)$ $\dfrac{d}{dt}(\delta v) = \dfrac{1}{m}\delta x$ or $\omega_0^2 = -\dfrac{1}{m} < 0$ unstable (saddle)

@ $(\pm 1, 0)$ $\dfrac{d}{dt}(\delta) = \dfrac{1}{m}(1-3)\,\delta x = -\dfrac{2}{m}\delta x$ $\quad \omega_0^2 = \dfrac{2}{m}$ stable (centre)



(4)

the equation for the undamped forced oscillator is

$$m\frac{d^2x}{dt^2} + kx = F_0 e^{-\alpha t} \quad \text{or}$$

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \frac{F_0}{m} e^{-\alpha t} \qquad \text{if } x_p(t) = Ce^{-\alpha t + \phi} = Ce^{\phi}e^{-\alpha t} = Ae^{-\alpha t}$$

the phase ϕ can be incorporated into $A = Ce^{\phi}$

then $\ddot{x}_p = \alpha^2 A e^{-\alpha t}$ so

$$\frac{d^2x_p}{dt} + \omega_0^2 x_p = (\alpha^2 + \omega_0^2) A e^{-\alpha t} = \frac{F_0}{m} e^{-\alpha t}$$

$$A = \frac{F_0/m}{\alpha^2 + \omega_0^2} \qquad \boxed{x_p = \frac{F_0}{m(\alpha^2+\omega_0^2)} e^{-\alpha t}}$$

the homogeneous solution is $x_h = A_0 \cos(\omega_0 t + \varphi)$ then the general solution is

$$x = x_h + x_p = \underline{A_0 \cos(\omega_0 t + \varphi) + \frac{F_0}{m(\alpha^2+\omega_0^2)} e^{-\alpha t}} \qquad \text{compute } v(t)$$

$$v(t) = -\omega_0 A_0 \sin(\omega_0 t + \varphi) - \frac{\alpha F_0}{m(\alpha^2+\omega_0^2)} e^{-\alpha t} \qquad \text{now set } t=0$$

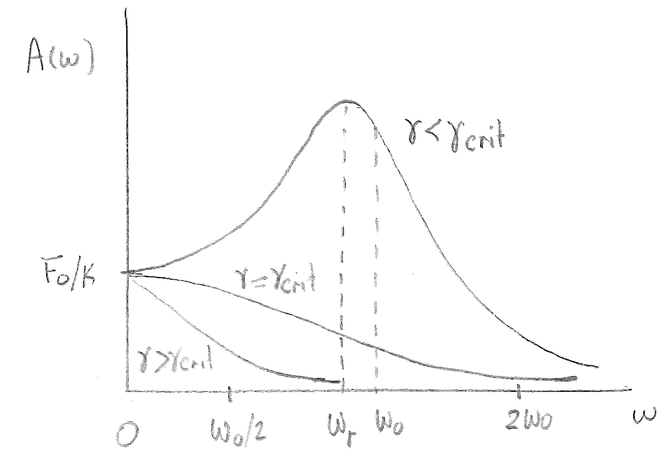
$$x(0) = A_0 \cos\varphi + \frac{F_0}{m(\alpha^2+\omega_0^2)} = x_0 \qquad A_0 \cos\varphi = x_0 - \frac{F_0}{m(\alpha^2+\omega_0^2)}$$

$$v(0) = A_0 \sin\varphi - \frac{\alpha F_0}{m(\alpha^2+\omega_0^2)} = v_0 \qquad A_0 \sin\varphi = v_0 + \frac{\alpha F_0}{m(\alpha^2+\omega_0^2)}$$

$$\frac{\sin\varphi}{\cos\varphi} = \tan\varphi = \frac{m v_0(\alpha^2+\omega_0^2) + \alpha F_0}{m x_0(\alpha^2+\omega_0^2) - F_0} \quad \text{and} \quad A_0^2 m^2(\alpha^2+\omega_0^2)^2 = (x_0 m(\alpha^2+\omega_0^2) - F_0)^2 + (v_0 m(\alpha^2+\omega_0^2) + \alpha F_0)^2$$

$$A_0 = \frac{\left[(x_0 m(\alpha^2+\omega_0^2) - F_0)^2 + (v_0 m(\alpha^2+\omega_0^2) + \alpha F_0)^2\right]^{1/2}}{m(\alpha^2+\omega_0^2)}$$

(5)



the critical γ occurs when the peak in $A(\omega)$ occurs at $\omega_r = 0$

then for all values $\omega > 0$ $\frac{dA}{d\omega} < 0$

$$A(\omega) = \frac{F_0/m}{\left[(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2\right]^{1/2}} \qquad \text{find } \frac{dA}{d\omega} = 0$$

$$\frac{dA}{d\omega} = \frac{(F_0/m)\left[-\tfrac{1}{2}\left((2)(-2)(\omega_0^2-\omega^2)\omega + 8\gamma^2\omega\right)\right]}{\left[(\omega_0^2-\omega^2) + 4\gamma^2\omega^2\right]^{3/2}} = 0 \Rightarrow -2(\omega_0^2-\omega_r^2)\omega_r + 4\gamma^2\omega_r = 0$$

or $2\gamma^2\cancel{\omega_r} = \cancel{\omega_r}(\omega_0^2-\omega_r^2)$ $\quad \omega_r^2 = \omega_0^2 - 2\gamma^2$ $\quad \boxed{\omega_r = \sqrt{\omega_0^2 - 2\gamma^2}}$

for the critical case $\omega_r = 0$ so $\boxed{\gamma_{crit} = \frac{\omega_0}{\sqrt{2}}}$

$$\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2-\omega^2}\right) = \tan^{-1}\left(\frac{\sqrt{2}\,\omega_0\omega}{\omega_0^2-\omega^2}\right) \quad \text{for the critical phase}$$

(i) $\omega = \omega_0/2$ $\quad \phi = \tan^{-1}\left(\frac{\omega_0^2/\sqrt{2}}{\omega_0^2(1-1/4)}\right) = \tan^{-1}\left(\frac{4}{3}\frac{1}{\sqrt{2}}\right) = \tan^{-1}\left(\frac{2\sqrt{2}}{3}\right) = 0.76 \text{ rad} = 43.3^\circ$

(ii) $\omega = \omega_0$ $\quad \phi = \tan^{-1}\left(\frac{\sqrt{2}\,\omega_0^2}{\omega_0^2-\omega_0^2}\right) = \tan^{-1}(\infty) = \pi/2 = 90^\circ$

(iii) $\omega = 2\omega_0$ $\quad \phi = \tan^{-1}\left(\frac{2\sqrt{2}\,\omega_0^2}{\omega_0^2(1-4)}\right) = \tan^{-1}\left(\frac{-2\sqrt{2}}{3}\right) = \pi - 0.76 \text{ rad} = 136.7^\circ$