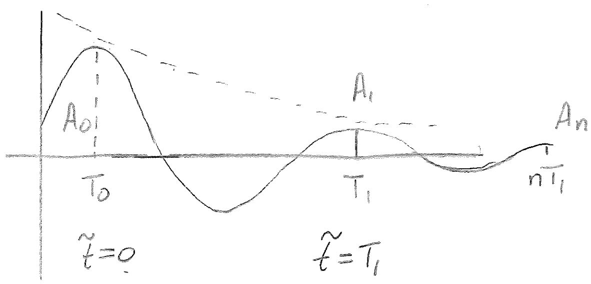
(1)



$A_n = \frac{1}{e} A_0$ where $x(t) = A_0 e^{-\gamma \tilde{t}} \sin(\omega_1 \tilde{t} + \phi)$ where $\tilde{t} = t - T_0$

at $\tilde{t} = 0$ $\phi = \pi/2$ so that $x(\tilde{t}=0) = A_0$

at $\tilde{t} = nT_1$ $x(nT_1) = x_n = A_0 e^{-\gamma(nT_1)} = A_n$

therefore $A_n = A_0 e^{-\gamma n T_1}$ $\Big\}$ or $\Big\{$ $A_n/A_0 = e^{-\gamma n T_1}$

but $A_n = 1/e\, A_0 = e^{-1} A_0$ $\Big\}$ $A_n/A_0 = e^{-1}$

therefore $\gamma n T_1 = 1$ $\boxed{\gamma = \frac{1}{nT_1}}$

the angular frequency of the damped oscillator is $\omega_1 = \sqrt{\omega_0^2 - \gamma^2}$ $\omega_0^2 = \omega_1^2 + \gamma^2$

$T_1 = \frac{2\pi}{\omega_1}$ $T_0 = \frac{2\pi}{\omega_0}$ but $\gamma = \frac{1}{nT_1} = \frac{\omega_1}{2\pi n}$ so $\omega_0^2 = \omega_1^2\left(1 + \frac{1}{4\pi^2 n^2}\right)$

$$\frac{T_1}{T_0} = \frac{2\pi/\omega_1}{2\pi/\omega_0} = \frac{\omega_0}{\omega_1} = \sqrt{\frac{1}{\omega_1^2}\omega_1^2\left(1 + \frac{1}{4\pi^2 n^2}\right)} = \sqrt{1 + \frac{1}{4\pi^2 n^2}}$$

— use the binomial expansion $\underline{(1+x)^n \approx 1 + nx} + \frac{1}{2} n(n-1)x^2 + \cdots$

where $n = 1/2$, $x = \frac{1}{4\pi^2 n^2}$ thus keeping the first two terms

$$\frac{T_1}{T_0} = \left(1 + \frac{1}{4\pi^2 n^2}\right)^{1/2} \cong 1 + \frac{1}{2}\left(\frac{1}{4\pi^2 n^2}\right) = 1 + \frac{1}{8\pi^2 n^2}$$

alternative

begin with $\gamma = \frac{1}{nT_1}$ $\quad T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{\sqrt{\omega_0^2 - \gamma^2}}$ therefore

$$T_1 = \frac{2\pi}{\sqrt{\omega_0^2 - \frac{1}{n^2 T_1^2}}}$$ or square both sides

$$T_1^2 = \frac{4\pi^2}{\omega_0^2 - \frac{1}{n^2 T_1^2}}$$

$$T_1^2 \left(\omega_0^2 - \frac{1}{n^2 T_1^2} \right) = 4\pi^2$$

$$T_1^2 \omega_0^2 - \frac{1}{n^2} = 4\pi^2$$

$$\omega_0^2 T_1^2 = 4\pi^2 + \frac{1}{n^2} = 4\pi^2 \left(1 + \frac{1}{n^2 4\pi^2} \right)$$ therefore

$$T_1 = \frac{2\pi}{\omega_0} \left(1 + \frac{1}{4\pi^2 n^2} \right)^{1/2}$$ but $T_0 = \frac{2\pi}{\omega_0}$ so

$$\frac{T_1}{T_0} = \left(1 + \frac{1}{4\pi^2 n^2} \right)^{1/2}$$

(2) $\cosh x = \dfrac{e^x+e^{-x}}{2}$ and $\sinh x = \dfrac{e^x-e^{-x}}{2}$ add and subtract to obtain

$e^x = \cosh x + \sinh x$ and $e^{-x} = \cosh x - \sinh x$ therefore

$e^{-\gamma t} = \cosh(\gamma t) - \sinh(\gamma t)$ $\quad e^{\pm\Omega t} = \cosh(\Omega t) \pm \sinh(\Omega t)$

where $\Omega = \sqrt{\gamma^2-\omega_0^2}$ the solution for the overdamped oscillator is

$x(t) = e^{-\gamma t}\left(C_1 e^{\Omega t} + C_2 e^{-(\Omega t)}\right)$ and

$v(t) = e^{-\gamma t}\left[-\gamma\left(C_1 e^{\Omega t} + C_2 e^{-\Omega t}\right) + \Omega\left(C_1 e^{\Omega t} - C_2 e^{-\Omega t}\right)\right]$

$= e^{-\gamma t}\left[(\Omega-\gamma)C_1 e^{\Omega t} - (\Omega+\gamma)C_2 e^{-\Omega t}\right]$ therefore

$x(t) = \left[\cosh(\gamma t) - \sinh(\gamma t)\right]\cdot\left\{C_1\left[\cosh(\Omega t)+\sinh(\Omega t)\right] + C_2\left[\cosh(\Omega t) - \sinh(\Omega t)\right]\right\}$

$= \underline{\left[\cosh(\gamma t) - \sinh(\gamma t)\right]\cdot\left[(C_1+C_2)\cosh(\Omega t) + (C_1-C_2)\sinh(\Omega t)\right]}$

$v(t) = \left[\cosh(\gamma t) - \sinh(\gamma t)\right]\cdot\left\{(\Omega-\gamma)C_1\left[\cosh(\Omega t)+\sinh(\Omega t)\right] - (\Omega+\gamma)C_2\left[\cosh(\Omega t) - \sinh(\Omega t)\right]\right\}$

$= \underline{\left[\cosh(\gamma t) - \sinh(\gamma t)\right]\left\{\left[(\Omega-\gamma)C_1 - (\Omega+\gamma)C_2\right]\cosh(\Omega t) + \left[(\Omega-\gamma)C_1 + (\Omega+\gamma)C_2\right]\sinh(\Omega t)\right\}}$

alternative expression

$x(t) = C_1 e^{(\Omega-\gamma)t} + C_2 e^{-(\Omega+\gamma)t}$; $v(t) = (\Omega-\gamma)C_1 e^{(\Omega-\gamma)t} - (\Omega+\gamma)C_2 e^{-(\Omega+\gamma)t}$

therefore

$\underline{x(t) = C_1\left\{\cosh\left[(\Omega-\gamma)t\right] + \sinh\left[(\Omega-\gamma)t\right]\right\} + C_2\left\{\cosh\left[(\Omega+\gamma)t\right] - \sinh\left[(\Omega+\gamma)t\right]\right\}}$

$v(t) = (\Omega-\gamma)C_1\left\{\cosh\left[(\Omega-\gamma)t\right] + \sinh\left[(\Omega-\gamma)t\right]\right\}$

$-(\Omega+\gamma)C_2\left\{\cosh\left[(\Omega+\gamma)t\right] - \sinh\left[(\Omega+\gamma)t\right]\right\}$

(3)

$$x(t) = Ae^{-\gamma t}\sin(\omega_1 t + \phi)$$

maxima occur at where $\frac{dx}{dt} = 0$ when $t = t_m$

$$\frac{d}{dt}x = Ae^{-\gamma t}\left[-\gamma\sin(\omega_1 t + \phi) + \omega_1\cos(\omega_1 t + \phi)\right]$$

so $\omega_1\cos(\omega_1 t + \phi) = \gamma\sin(\omega_1 t + \phi)$ solve for $t = t_m$

$$\frac{\omega_1}{\gamma} = \tan(\omega_1 t + \phi) \quad \text{or}$$

$$t_m = \frac{1}{\omega_1}\left[\tan^{-1}\left(\frac{\omega_1}{\gamma}\right) + \phi + 2n\pi\right]$$

if one maxima occurs at $t_1 = t_m$ the next maxima occurs at

$$t_2 = t_m + T_1 \quad \text{where } T_1 = \frac{2\pi}{\omega_1}$$

therefore $$\frac{x(t_2)}{x(t_1)} = \frac{x(t_m + T_1)}{x(t_m)} = \frac{Ae^{-\gamma(t_m + T)}}{Ae^{-\gamma t_m}}\frac{\sin[\omega_1(t_m + T) + \phi]}{\sin[\omega_1 t_m + \phi]}$$

but $\sin[\omega_1(t_m + T_1) + \phi] = \sin[\omega_1 t_m + \phi + 2\pi]$ since $\omega_1 T_1 = 2\pi$

$$= \sin[\omega_1 t_m + \phi]$$

therefore $$\frac{x(t_2)}{x(t_1)} = \frac{e^{-\gamma(t_m + T_1)}}{e^{-\gamma t_m}} = e^{-\gamma(t_m + T_1 - t_m)} = e^{-\gamma T_1}$$

$$= e^{-\gamma 2\pi/\omega_1}$$

since γ and T_1 are const. $e^{-\gamma T_1}$ is constant

and independent of time

(4) $E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$ is the total energy

(a) $x(t) = Ae^{-\gamma t}\sin(\omega_1 t + \phi)$ is the solution for the underdamped oscillator

$$v(t) = Ae^{-\gamma t}\left(-\gamma\sin(\omega_1 t+\phi) + \omega_1\cos(\omega_1 t+\phi)\right]$$

set $\theta(t) = \omega_1 t + \phi$ so $\frac{d\theta}{dt} = \omega_1 = \dot{\theta}$ then

$$x(t) = Ae^{-\gamma t}\sin\theta \qquad v(t) = Ae^{-\gamma t}\left(-\gamma\sin\theta + \omega_1\cos\theta\right]$$

therefore

$$E = \frac{1}{2}mA^2e^{-2\gamma t}(\gamma^2\sin^2\theta - 2\gamma\omega_1\sin\theta\cos\theta + \omega_1^2\cos^2\theta) + \frac{1}{2}kA^2e^{-2\gamma t}\sin^2\theta$$

$$= \frac{1}{2}A^2e^{-2\gamma t}\left[(\gamma^2 m + k)\sin^2\theta - 2\gamma\omega_1 m\sin\theta\cos\theta + m\omega_1^2\cos^2\theta\right]$$

$$= \frac{1}{2}mA^2e^{-2\gamma t}\left[(\gamma^2+\omega_0^2)\sin^2\theta - 2\gamma\omega_1\sin\theta\cos\theta + \omega_1^2\cos^2\theta\right] \quad \text{since } \omega_0^2 = \frac{k}{m}$$

(b) $$\frac{dE}{dt} = \frac{1}{2}mA^2(-2\gamma)e^{-2\gamma t}\left[(\gamma^2+\omega_0^2)\sin^2\theta - 2\gamma\omega_1\sin\theta\cos\theta + \omega_1^2\cos^2\theta\right]$$

$$+\frac{1}{2}mA^2e^{-2\gamma t}\left[(\gamma^2+\omega_0^2)2\omega_1\sin\theta\cos\theta - 2\gamma\omega_1^2(\cos^2\theta - \sin^2\theta) - 2\omega_1^3\sin\theta\cos\theta\right]$$

$$= mA^2e^{-2\gamma t}\left[\sin^2\theta(-\gamma^3 - \gamma\omega_0^2 + \gamma\omega_1^2) + \cos^2\theta(-\gamma\omega_1^2 - \gamma\omega_1^2)\right.$$
$$\left. + \cos\theta\sin\theta\,(2\gamma^2\omega_1 + \omega_1(\gamma^2+\omega_0^2) - \omega_1^3)\right]$$

$$= -\gamma mA^2e^{-2\gamma t}\left[\sin^2\theta(\gamma^2 + \overbrace{\omega_0^2 - \omega_1^2}^{\gamma^2}) + \cos^2\theta(-2\omega_1^2) + \cos\theta\sin\theta\left(-2\gamma\omega_1 - \gamma^2\omega_1 - \omega_1(\omega_1^2+\gamma^2) + \omega_1^3\right)\right]$$

$$= -2\gamma mA^2e^{-2\gamma t}\left[\gamma^2\sin^2\theta - 2\gamma\omega_1\cos\theta\sin\theta + \omega_1^2\cos^2\theta\right] \quad \text{since } \omega_0^2 - \omega_1^2 = \gamma^2$$

(c) compute average $\frac{dE}{dt}$ over one cycle

$$\left\langle\frac{dE}{dt}\right\rangle = \frac{\int_0^{T_1}\left(\frac{dE}{dt}\right)dt}{\int_0^{T_1}dt} = \frac{1}{T_1}\int_0^{T_1}\left(\frac{dE}{dt}\right)dt$$

need to evaluate $\int_0^{T_1} \left(\frac{dE}{dt}\right) dt$ assuming $e^{-2\gamma t}$ is small and nearly constant

$$-2\gamma m A^2 \int_0^{T_1} e^{-2\gamma t}\left[\gamma^2 \sin^2(\omega_1 t+\phi) - 2\gamma\omega_1 \sin(\omega_1 t+\phi)\cos(\omega_1 t+\phi) + \omega_1^2 \cos^2(\omega_1 t+\phi)\right] dt$$

computing $\int_0^{T_1} \sin^2(\omega_1 t+\phi)dt$, $\int_0^{T_1} \cos^2(\omega_1 t+\phi)\, dt$, $\int_0^{T_1} \sin(\omega_1 t+\phi)\cos(\omega_1 t+\phi)dt$

for the integrals of $\sin^2\theta$ and $\cos^2\theta$ not

$$\int_0^{T_1}\left[\sin^2(\omega_1 t+\phi) + \cos^2(\omega_1 t+\phi)\right]dt = \int_0^{T_1} 1\, dt = T_1$$

but $\int_0^{T_1} \sin^2\theta\, dt = \int_0^{T_1} \cos^2\theta\, dt$ so

$$\int_0^{T_1} \sin^2(\omega_1 t+\phi)\, dt = \int_0^{T_1} \cos^2(\omega_1 t+\phi)\, dt = T_1/2$$

also $\int_0^{T_1} \sin(\omega_1 t+\phi)\cos(\omega_1 t+\phi)dt = \int_0^{T_1} \sin(\omega_1 t+\phi)\frac{1}{\omega_1} d(\sin(\omega_1 t+\phi))$

$$= \frac{1}{2\omega_1}\sin^2(\omega_1 t+\phi)\Big|_0^{T_1} = 0$$

therefore $\left\langle \frac{dE}{dt} \right\rangle \cong \frac{-2\gamma m A^2}{T_1} e^{-2\gamma t}\left[\gamma^2 \frac{T_1}{2} + 0 + \omega_1^2 \frac{T_1}{2}\right]$

$$= -\gamma m A^2 e^{-2\gamma t}(\gamma^2+\omega_1^2) \quad \text{but } \gamma^2+\omega_1^2 = \omega_0^2$$

$$= -\gamma m A^2 \omega_0^2 e^{-2\gamma t}$$

Alternative (b) with less algebra

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right)$$

$$= mv\dot{v} + kx\dot{x}$$ since $\dot{x} = v$

$$= (m\dot{v} + kx)\, v$$ but $m\dot{v} + bv + kx = 0$ so $m\dot{v} + kx = -bv$

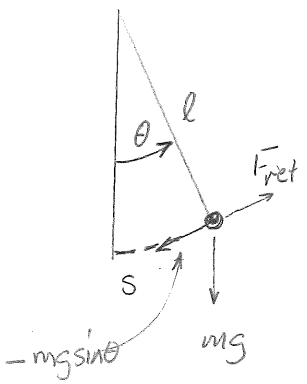
$$= -bv^2$$ therefore but $b = 2\gamma m$ and $v = Ae^{-\gamma t}[-\gamma\sin\theta + \omega_1\cos\theta]$

$$\frac{dE}{dt} = -2\gamma m\, A^2 e^{-2\gamma t}\left[\gamma^2\sin\theta - 2\gamma\omega_1\sin\theta\cos\theta + \omega_1^2\cos^2\theta\right]$$

(c) follows in same way

(5)

$F_{ret} = -2m\sqrt{\frac{g}{\ell}}\,v \qquad F_{restor} = -mg\sin\theta$ for the damped simple pendulum



Newton's 2nd law is

$$m\frac{dv}{dt} = F_{ret} + F_{rest}$$

$$= -2m\sqrt{\frac{g}{\ell}}\,v - mg\sin\theta \quad \text{but } s = \ell\theta$$

$$v = \frac{ds}{dt} = \ell\dot{\theta} \quad \frac{dv}{dt} = \ell\ddot{\theta}$$

therefore

$$\cancel{m}\ell\ddot{\theta} = -2\cancel{m}\sqrt{\frac{g}{\ell}}\,\ell\dot{\theta} - \cancel{m}g\sin\theta \quad \text{or}$$

$$\ddot{\theta} + 2\sqrt{\frac{g}{\ell}}\,\dot{\theta} + \frac{g}{\ell}\sin\theta = 0$$ now assume small angle approx. $\sin\theta \sim \theta$ then

$$\ddot{\theta} + 2\sqrt{\frac{g}{\ell}}\,\dot{\theta} + \frac{g}{\ell}\theta = 0 \quad \text{or } \gamma = \sqrt{\frac{g}{\ell}} \quad \omega_0^2 = \frac{g}{\ell} \quad \text{so } \gamma^2 = \omega_0^2$$

the system is <u>critically damped</u>

then $\theta(t) = e^{-\gamma t}(C_1 t + C_0) = e^{-\sqrt{g/\ell}\,t}(C_1 t + C_0)$ is the critically damped solution

$$\omega = \dot{\theta}(t) = e^{-\sqrt{g/\ell}\,t}\left[-\sqrt{\frac{g}{\ell}}\,C_1 t + C_1 - \sqrt{\frac{g}{\ell}}\,C_0\right]$$

initial conditions @ $t=0$ $\quad \theta(0) = \theta_0 = C_0$

$$\dot{\theta}(0) = 0 = C_1 - \sqrt{g/\ell}\,\theta_0 \Rightarrow C_1 = \sqrt{g/\ell}\,\theta_0$$

therefore

$$\theta(t) = e^{-\sqrt{g/\ell}\,t}\,\theta_0\left(\sqrt{g/\ell}\,t + 1\right) \qquad \theta(0) = \theta_0$$

$$\omega(t) = -e^{-\sqrt{g/\ell}\,t}\,\frac{g}{\ell}\,\theta_0 t \qquad \dot{\theta}(0) = 0$$