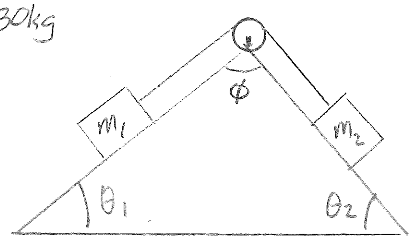
(1)

$m_1 = 15\,kg \quad m_2 = 30\,kg$

$\mu_1 = 0.4 \quad \mu_2 = 0.2$

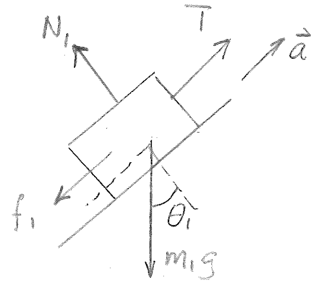


since $(5m)^2 = (3m)^2 + (4m)^2 \quad \phi = 90°$

$\cos\theta_1 = 4/5 \quad \sin\theta_1 = 3/5$

$\cos\theta_2 = 3/5 \quad \sin\theta_2 = 4/5$

Free body diagram m_1



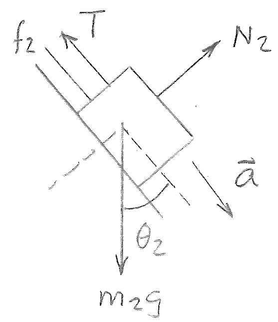
Forces on m_1

$$N_1 - m_1 g\cos\theta_1 = 0 \qquad (1)$$

$$T - f_1 - m_1 g \sin\theta_1 = m_1 a \qquad (2)$$

$$f_1 = \mu_1 N_1 = \mu_1 m_1 g\cos\theta_1 \qquad (3)$$

Free body diagram m_2



$$N_2 - m_2 g\cos\theta_2 = 0 \qquad (4)$$

$$m_2 g \sin\theta_2 - f_2 - T = m_2 a \qquad (5)$$

$$f_2 = \mu_2 m_2 g\cos\theta_2 \qquad (6)$$

add (2) and (5)

$$m_2 g\sin\theta_2 - m_1 g\sin\theta_1 - f_1 - f_2 = (m_1 + m_2)a \qquad \text{use (3) and (6)}$$

$$m_2 g\sin\theta_2 - m_1 g\sin\theta_1 - \mu_1 m_1 g\cos\theta_1 - \mu_2 m_2 g\cos\theta_2 = (m_1 + m_2)a$$

$$g\,[m_2\sin\theta_2 - m_1\sin\theta_1 - \mu_1 m_1\cos\theta_1 - \mu_2 m_2\cos\theta_2) = (m_1 + m_2)a$$

$$a = g\,\frac{[m_2\sin\theta_2 - m_1\sin\theta_1 - \mu_1 m_1\cos\theta_1 - \mu_2 m_2\cos\theta_2]}{m_1 + m_2}$$

$$= g\,[30(4/5) - 15(3/5) - 0.4(15)(4/5) - (0.2)(30)3/5]/(45) = \underline{\underline{1.44\ \tfrac{m}{s^2}}}$$

(b) $\Delta x = 1.0\,m = \tfrac{1}{2}at^2 \qquad t = \sqrt{\frac{2\Delta x}{a}} = \sqrt{\frac{2(1.0\,m)}{1.44\ m/s^2}} = \underline{\underline{1.18\,s}}$

(c) $T = m_1 a + f_1 + m_1 g \sin\theta_1 = m_1 [a + \mu_1 g \cos\theta_1 + g \sin\theta_1)$

$$= 15\,\text{kg}\left[1.44\,\frac{m}{s^2} + (0.4)\,9.8\,\frac{m}{s^2}(0.8) + 9.8\,\frac{m}{s^2}(0.6)\right]$$

$$= 156.8\ N$$

as a check - alternative equation

$T = m_2 g \sin\theta_2 - f_2 - m_2 a$

$$= m_2\left[g \sin\theta_2 - \mu_2 g \cos\theta_2 - a\right]$$

$$= 30\,\text{kg}\left[\left(9.8\,\frac{m}{s^2}\right)(0.8) - (0.2)\left(9.8\,\frac{m}{s^2}\right)(0.6) - 1.44\,\frac{m}{s^2}\right]$$

$= 156.8\ N$ - verified

(2) In all cases $\frac{dv}{dt} = \frac{F(t)}{m}$ or $\int_{v_0}^{v} dv' = v - v_0 = \frac{1}{m}\int_{t_0}^{t} F(t')\,dt$

(a) $$v - v_0 = \frac{1}{m}\int_{t_0}^{t}\left(f_0 + \frac{F_0}{t}\right)dt = \frac{1}{m}\left[f_0 t' + F_0 \ln(t')\right]_{t_0}^{t}$$

$$= \frac{f_0}{m}(t - t_0) + \frac{F_0}{m}\ln(t/t_0) \quad \text{then}$$

$v = \frac{dx}{dt} = \frac{f_0}{m}(t-t_0) + \frac{F_0}{m}\ln(t/t_0) + v_0$ seperate variables and integrate

$$\int_{x_0}^{x} dx = \frac{1}{m}\int_{t_0}^{t}\left[f_0(t-t_0) + \frac{F_0}{m}\ln(t/t_0)\right]dt + v_0(t-t_0) \qquad \int \ln x\,dx = x\ln x - x$$

$$x - x_0 = \frac{1}{m}\left[f_0 \frac{1}{2}(t-t_0)^2 + \frac{F_0}{m}t_0\left(\frac{t}{t_0}\ln\left(\frac{t}{t_0}\right) - \frac{t}{t_0}\right)\right]_{t_0}^{t} + v_0(t-t_0)$$

$$= \frac{1}{m}\left[f_0 \frac{1}{2}(t-t_0)^2 + \frac{F_0}{m}\left(t\ln\left(\frac{t}{t_0}\right) - t - t_0\cancel{\ln}(1) + t_0\right]\right. + v_0(t-t_0)$$

$$x = x_0 + \frac{1}{m}\left[\frac{f_0}{2}(t-t_0)^2 + \frac{F_0}{m}\left(t\ln\left(\frac{t}{t_0}\right) - t + t_0\right)\right] + v_0(t-t_0)$$

(b) $$v - v_0 = \frac{1}{m}\int_{t_0}^{t} F_0 e^{-kt}\,dt = -\frac{F_0}{mk}e^{-kt}\Big|_{t_0}^{t} = \frac{F_0}{mk}\left(e^{-kt_0} - e^{-kt}\right)$$

$v = \frac{dx}{dt} = v_0 + \frac{F_0}{mk}\left(e^{-kt_0} - e^{-kt}\right)$ seperate variables and integrate

$$\int_{x_0}^{x} dx = \int_{t_0}^{t} v_0\,dt + \int_{t_0}^{t}\frac{F_0}{mk}\left(e^{-kt_0} - e^{-kt}\right)dt$$

$$x - x_0 = v_0(t-t_0) + \frac{F_0}{mk}e^{-kt_0}(t-t_0) + \frac{F_0}{mk^2}e^{-kt}\Big|_{t_0}^{t}$$

$$x = x_0 + v_0(t-t_0) + \frac{F_0}{mk}e^{-kt_0}(t-t_0) + \frac{F_0}{mk^2}\left(e^{-kt} - e^{-kt_0}\right)$$

$$= x_0 + v_0(t-t_0) + \frac{F_0}{mk}e^{-kt_0}\left[t - t_0 - \frac{1}{k}\right] + \frac{F_0}{mk^2}e^{-kt}$$

(c) $$v - v_0 = \frac{F_0}{m}\int_{t_0}^{t} \sin(\omega t')\,dt' = -\frac{F_0}{m\omega}\cos(\omega t)\Big|_{t_0}^{t}$$

$$= \frac{F_0}{m\omega}\left[\cos(\omega t_0) - \cos(\omega t)\right]$$

$$\frac{dx}{dt} = v = v_0 + \frac{F_0}{m\omega}\left[\cos(\omega t_0) - \cos(\omega t)\right]$$ separate variables and integrate

$$\int_{x_0}^{x} dx = \int_{t_0}^{t} v_0\,dt + \frac{F_0}{m\omega}\int_{t_0}^{t}\left[\cos(\omega t_0) - \cos(\omega t)\right]dt$$

$$= v_0(t-t_0) + \frac{F_0}{m\omega}\cos(\omega t_0)(t-t_0) - \frac{F_0}{m\omega^2}\sin(\omega t)\Big|_{t_0}^{t}$$

$$= \left[v_0 + \frac{F_0}{m\omega}\cos(\omega t_0)\right](t-t_0) - \frac{F_0}{m\omega^2}\left[\sin(\omega t) - \sin(\omega t_0)\right]$$

(3) $m\frac{dv}{dt} = \frac{K}{(t+\tau)^2}$ separate variables

$dv = \frac{k}{m}\frac{dt}{(t+\tau)^2}$ integrate both sides $\int_{v_0}^{v} dv = \int_0^t \frac{k}{m}\frac{dt}{(t+\tau)^2}$

or $v - v_0 = -\frac{k}{m}\frac{1}{(t+\tau)}\Big|_0^t$ or $\boxed{v = v_0 + \frac{k}{m\tau} - \frac{k}{m}\frac{1}{(t+\tau)}}$ now set $v = \frac{dx}{dt}$

$\therefore \frac{dx}{dt} = \left(v_0 + \frac{k}{m\tau}\right) - \frac{k}{m}\frac{1}{(t+\tau)}$ separate variables and integrate

$$\int_{x_0}^{x} dx = \int_0^t \left[\left(v_0 + \frac{k}{m\tau}\right) - \frac{k}{m}\frac{1}{(t+\tau)}\right]dt$$

$$x - x_0 = \left(v_0 + \frac{k}{m\tau}\right)t - \frac{k}{m}\ln(t+\tau) + \frac{k}{m}\ln(\tau)$$

$$x = x_0 + \left(v_0 + \frac{k}{m\tau}\right)t + \frac{k}{m}\ln\left(\frac{\tau}{t+\tau}\right)$$

set $\tilde{v}_0 = v_0 + k/m\tau = \frac{m v_0 \tau + k}{m\tau}$

$\tilde{x}_0 = x_0 + \frac{k}{m}\ln\tau$

(b) set $v = 0$ and solve for t

$$0 = \tilde{v}_0 - \frac{k}{m}\frac{1}{(t+\tau)} \Rightarrow (t+\tau) = \frac{k}{m}\frac{1}{\tilde{v}_0}$$

$$t = \frac{k}{m\tilde{v}_0} - \tau = \frac{k\tau - (m v_0 \tau + k)\tau}{m v_0 \tau + k} = \frac{-(m v_0 \tau^2)}{m v_0 \tau + k}$$

if $v_0 > 0$ then $t < 0$ if $v_0 < 0$ then $t > 0$

(c) $\lim_{t\to\infty} v = \lim_{t\to 0}\left[\tilde{v}_0 - \frac{k}{m}\frac{1}{(t+\tau)}\right] = \tilde{v}_0 = v_0 + \frac{k}{m\tau}$

(d) $x = \tilde{x}_0 + \tilde{v}_0 t - \frac{k}{m}\ln(t+\tau)$ as $t \to \infty$ $\tilde{v}_0 t$ increases to infinity <u>linearly</u>

$\ln(\tau + t)$ goes to infinity <u>logarithmically</u>

$x \to \infty$ will not cover a finite distance

(4) Write Newton's 2nd law as:

$$m\frac{dv}{dt} = -k_1 v - k_2 v^2 = -k_1 v\left(1+\frac{k_2}{k_1}v\right) \quad \text{set } \frac{k_2}{k_1} = k$$

separate variables

$$\frac{dv}{v(1+kv)} = -\frac{k_1}{m}dt$$ — use partial fractions with undetermined coefficients

$$\text{set } \frac{1}{v(1+kv)} = \frac{A}{v} + \frac{B}{(1+kv)} = \frac{A(1+kv)+Bv}{v(1+kv)} = \frac{A+(Ak+B)v}{v(1+kv)}$$

therefore $A=1$ $B+Ak=0 \Rightarrow B=-k$ then

$$\frac{dv}{v(1+kv)} = \frac{dv}{v} - \frac{k\,dv}{(1+kv)}$$ now integrate

$$\int_{v_0}^{v}\left(\frac{1}{v'} - \frac{k}{(1+kv')}\right)dv' = -\frac{k_1}{m}\int_0^t dt'$$

$$\ln\left(\frac{v}{v_0}\right) - \ln\left(\frac{1+kv}{1+kv_0}\right) = -\frac{k_1}{m}t$$ set $\frac{k_1}{m} = \alpha$ a constant

$$\ln\left[\frac{v(1+kv_0)}{v_0(1+kv)}\right] = -\alpha t$$ exponentiate both sides

$$\left(\frac{v}{1+kv}\right)\left(\frac{1+kv_0}{v_0}\right) = e^{-\alpha t} \quad \text{or} \quad \frac{1+kv}{v} = \left(\frac{1+kv_0}{v_0}\right)e^{\alpha t}$$ solve for v

$$\frac{1}{v} + k = \left(\frac{1+kv_0}{v_0}\right)e^{\alpha t} \quad \text{or} \quad \frac{1}{v} = \frac{(1+kv_0)e^{\alpha t} - kv_0}{v_0}$$ invert

$$v = \frac{v_0}{(1+kv_0)e^{\alpha t} - kv_0} = \frac{v_0 e^{-\alpha t}}{1+kv_0 - kv_0 e^{-\alpha t}}$$ therefore

$$v(t) = \frac{v_0 e^{-k_1 t/m}}{1+\frac{k_2}{k_1}v_0 - \frac{k_2}{k_1}v_0 e^{-k_1 t/m}} = \frac{k_1 v_0 e^{-k_1 t/m}}{k_1 + k_2 v_0(1-e^{k_1 t/m})}$$

when $t=0$ $\quad v = \dfrac{K_1 v_0}{K_1} = v_0$ $\quad$ as $t \to \infty$ $\quad v \to 0$

now $v = \dfrac{dx}{dt} = \dfrac{v_0 e^{-\alpha t}}{1 + k v_0 - k v_0 e^{-\alpha t}}$ $\quad$ separate variables and integrate

$$\int_0^x dx' = \int_0^t \frac{v_0 e^{-\alpha t'} dt'}{1 + k v_0 - k v_0 e^{-\alpha t'}}$$

$$x = \frac{1}{k} \int_0^t \frac{k v_0 e^{-\alpha t'} dt'}{1 + k v_0 - k v_0 e^{-\alpha t}}$$

$$= \frac{1}{k} \int_0^t \frac{\frac{k v_0}{1 + k v_0} e^{-\alpha t'} dt'}{\left[1 - \left(\frac{k v_0}{1 + k v_0}\right) e^{-\alpha t'}\right]}$$

set $\dfrac{k v_0}{1 + k v_0} = D$

$$x = \frac{1}{k} \int_0^t \frac{D e^{-\alpha t'} dt'}{1 - D e^{-\lambda t'}} = \frac{1}{k} \frac{1}{\alpha} \ln\left(1 - D e^{-\alpha t'}\right)\Big|_0^t = \frac{1}{K\alpha} \ln\left(\frac{1 - D e^{-\lambda t}}{1 - D}\right)$$

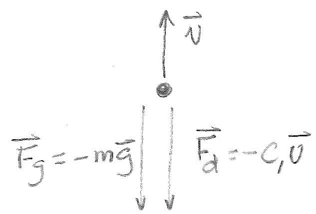
$$x(t) = \frac{k_1}{k_2} \frac{m}{k_1} \ln\left\{ \frac{1 - \frac{k v_0}{1 + k v_0} e^{-\alpha t}}{\left(1 - \frac{k v_0}{1 + k v_0}\right)} \right\}$$

$$= \frac{m}{k_2} \ln\left[1 + k v_0 - k v_0 e^{-k_1 t/m}\right] = \frac{m}{k_2} \ln\left[1 + \frac{k_2}{k_1} v_0 - \frac{k_2}{k_1} v_0 e^{-k_1 t/m}\right]$$

as $t \to 0$ $\quad x(t) = \dfrac{m}{k_2} \ln(1) = 0$

as $t \to \infty$ $\quad x(t) \to \dfrac{m}{k_2} \ln\left[1 + \dfrac{k_2 v_0}{k_1}\right\}$ $\quad$ which is finite and positive since $k_1 > 0$ $k_2 > 0$ and $v_0 > 0$

(5) The velocity is upwards and both the gravitational and damping forces downwards



then $m\dfrac{dv}{dt} = -mg - c_1 v$

$\dfrac{dv}{dt} = -\dfrac{c_1}{m}\left(v + \dfrac{mg}{c_1}\right)$ seperate variables set $\dfrac{mg}{c_1} = K$

$\dfrac{dv}{v+K} = -\dfrac{c_1}{m}\,dt$ integrate when $t=0$ $v = v_0$

$\displaystyle\int_{v_0}^{v} \frac{dv}{(v+K)} = \int_0^t -\frac{c_1}{m}\,dt \Rightarrow \ln\left(\frac{v+K}{v_0+K}\right) = -\frac{c_1}{m}t$ or after exponentiation

$$v = -K + (v_0 + K)e^{-c_1 t/m} = -\frac{mg}{c_1} + \left(v_0 + \frac{mg}{c_1}\right)e^{-c_1 t/m}$$

maximum height occurs at $v=0$ then

$\dfrac{mg}{c_1} = \left(v_0 + \dfrac{mg}{c_1}\right)e^{-c_1 t/m}$ or

$$\ln\left(\frac{mg/c_1}{v_0 + mg/c_1}\right) = -\frac{c_1 t}{m} \Rightarrow t = \frac{m}{c_1}\ln\left[\frac{mg/c_1 + v_0}{mg/c_1}\right] = \underline{\underline{\frac{m}{c_1}\ln\left[1 + \frac{c_1 v_0}{mg}\right]}}$$

now when $c_1 \to 0$ compute t using a Taylor series for

$\ln(1+x) \cong x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$ when $x = \dfrac{c_1 v_0}{mg} \ll 1$

then

$$t \simeq \frac{m}{c_1}\left[\frac{c_1 v_0}{mg} - \frac{1}{2}\left(\frac{c_1 v_0}{mg}\right)^2 + \frac{1}{3}\left(\frac{c_1 v_0}{mg}\right)^3 + \cdots\right]$$

$$\simeq \frac{v_0}{g} - \frac{1}{2}\frac{c_1 v_0^2}{mg^2} + \frac{1}{3}\left(\frac{c_1^2 v_0^3}{m^2 g^3}\right) + \cdots$$ now take $\lim\limits_{c_1 \to 0}$

$= \dfrac{v_0}{g}$ this is the time to reach maximum height with no resistive force