(1)

(a) $\omega_0 = \sqrt{K/m} = \sqrt{\dfrac{10^{-3}\ N/m}{10^{-1}\ kg}} = \sqrt{\dfrac{10^{-2}}{s^2}} = 0.1\ s^{-1}$

$f_0 = \dfrac{\omega_0}{2\pi} = 0.016\ s^{-1} \qquad T_0 = 1/f_0 = 62.8\ s$

(b) $A = 3.0\ cm = 0.03\ m$

$E_{TOT} = \frac{1}{2} k A^2 = (0.5)\left(10^{-3}\ \frac{N}{m}\right)(0.03m)^2 = 4.5 \times 10^{-7}\ J$

(c) using $T + V = E_{TOT}$ maximum velocity occurs when $V = 0$

$\frac{1}{2} m v_{max}^2 = \frac{1}{2} k A^2$

$v_{max} = \sqrt{\frac{k}{m}}\ A = \omega_0 A = (0.1\ s^{-1})(0.03m) = 0.003\ m/s$

$= 3\ mm/s$

(d) $v_0 = 0.1\ m/s$ when $V = 0$ then

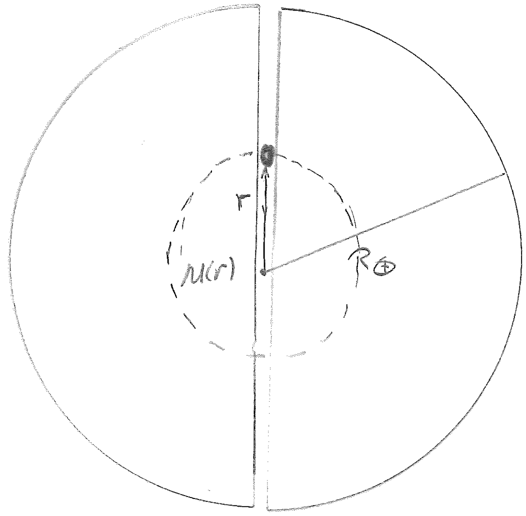
using $T + V = E_{TOT} = \frac{1}{2} k x_{max}^2$ therefore

$\frac{1}{2} m v_0^2 = \frac{1}{2} k x_{max}^2$

$x_{max} = \sqrt{\frac{m}{k}}\ v_0 = \left(\frac{1}{0.1\ s^{-1}}\right)\left(0.1\ \frac{m}{s}\right) = 1.0\ m$

(e) $V_{max} = E_{TOT} = \frac{1}{2} k v_{max}^2 = (0.5)\left(10^{-3}\ \frac{N}{m}\right)(1.0\ m)^2 = 5 \times 10^{-4}\ J$

(2)



drill a hole through the Earth
drop an object of mass m from rest at the surface

Newton's 2nd law

$$m\frac{d^2r}{dt^2} = F_G(r) = -\frac{GM(r)m}{r^2}$$

but $M(r) = \rho V(r) = \rho\left(\frac{4}{3}\pi r^3\right)$

compute $\rho = \dfrac{M_\oplus}{\frac{4}{3}\pi R_\oplus^3}$

then $M(r) = \dfrac{M_\oplus}{\cancel{\frac{4}{3}\pi} R_\oplus^3}\cancel{\frac{4}{3}\pi} r^3 = \dfrac{M_\oplus r^3}{R_\oplus^3}$

the equation of motion is

$$m\frac{d^2r}{dt^2} = -\frac{GM_\oplus r^3}{R_\oplus^3}\left(\frac{m}{r^2}\right) = -m\frac{GM_\oplus r}{R_\oplus^3}$$

but $g = 9.8\ m/s = g(R_\oplus) = \dfrac{GM_\oplus}{R_\oplus^2}$ is the gravitational acceleration at the Earth's surface

then $m\dfrac{d^2r}{dt^2} = -m\left(\dfrac{GM_\oplus}{R_\oplus^2}\right)\dfrac{r}{R_\oplus} = -mg\dfrac{r}{R_\oplus}$

$\dfrac{d^2r}{dt^2} = -\left(\dfrac{g}{R_\oplus}\right)r$ is the equation for a SHO where $\omega_0^2 = \dfrac{g}{R_\oplus}$

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{R_\oplus}{g}} = 2\pi\left[\frac{6380\ km}{9.80\ m/s^2}\right]^{1/2} = \underline{5068s} = \underline{84.5\ min}$$

(3)

use $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$ $\qquad \sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$

(a) $\cos^2\theta + \sin^2\theta = \frac{1}{4}\left[\cancel{e^{2i\theta}} + 2 + \cancel{e^{-2i\theta}}\right] - \frac{1}{4}\left[\cancel{e^{2i\theta}} - 2 + \cancel{e^{-2i\theta}}\right]$

$= \frac{1}{4}\left[2-(-2)\right] = \frac{1}{4}(4) = 1$

(b) $\cos^3\theta = \frac{1}{8}\left(e^{i\theta}+e^{-i\theta}\right)^3$

$= \frac{1}{8}\left(e^{3i\theta} \underbrace{+ 3e^{i2\theta}e^{-i\theta} + 3e^{i\theta}e^{-i2\theta}} + e^{-3i\theta}\right)$

$= \frac{1}{4}\left(\dfrac{e^{3i\theta}+e^{-3i\theta}}{2} + \frac{3}{2}\left(e^{i\theta}+e^{-i\theta}\right)\right)$

$= \frac{1}{4}\left[\cos 3\theta + 3\cos\theta\right]$

(c) $= \sin^3\theta = \frac{1}{(2i)^3}\left(e^{i\theta}-e^{-i\theta}\right)^3$

$= \frac{-1}{8i}\left(e^{3i\theta} - 3e^{2i\theta}e^{-i\theta} + 3e^{-2i\theta}e^{i\theta} - e^{-i3\theta}\right)$

$= \frac{-1}{8i}\left(e^{3i\theta} - e^{-3i\theta} + 3e^{-i\theta} - 3e^{i\theta}\right)$

$= \frac{-1}{4}\left(\dfrac{e^{3i\theta}-e^{-3i\theta}}{2i} - \frac{3}{2i}\left(-e^{i\theta}+e^{-i\theta}\right)\right)$

$= \frac{1}{4}\left[-\sin 3\theta + 3\sin\theta\right]$

(3) Alternative

(a) $e^{i\theta} = \cos\theta + i\sin\theta \qquad e^{-i\theta} = \cos\theta - i\sin\theta$

$\therefore\ e^{i\theta} e^{-i\theta} = 1$ but

$$(e^{i\theta})(e^{-i\theta}) = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)$$

$$= \cos^2\theta + \cancel{i\sin\theta\cos\theta} - \cancel{i\cos\theta\sin\theta} - i^2\sin^2\theta \qquad i^2 = -1$$

$$= \cos^2\theta + \sin^2\theta \implies \cos^2\theta + \sin^2\theta = 1$$

b + c $\quad e^{i3\theta} = \cos 3\theta + i\sin 3\theta$

$$e^{i3\theta} = (e^{i\theta})^3 = (\cos\theta + i\sin\theta)^3$$

$$= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + i^3\sin^3\theta$$

$$= \cos^3\theta - 3\cos\theta\sin^2\theta + i\left(3\cos^2\theta\sin\theta - \sin^3\theta\right)$$

now equate real and imaginary parts

Real: $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta \qquad$ but $\sin^2\theta = 1 - \cos^2\theta$

$$= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta)$$

$$= 4\cos^3\theta - 3\cos\theta \implies \cos^3\theta = \tfrac{1}{4}(3\cos\theta + \cos 3\theta)$$

Imaginary: $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta \qquad$ set $\cos^2\theta = 1 - \sin^2\theta$

$$= 3\sin\theta(1 - \sin^2\theta) - \sin^3\theta$$

$$= 3\sin\theta - 4\sin^3\theta \quad \text{or}$$

$$\sin^3\theta = \tfrac{1}{4}[3\sin\theta - \sin 3\theta]$$

(4) $x(t) = Ce^{i\omega_0 t} + C^* e^{-i\omega_0 t}$ then

$$v(t) = \frac{dx}{dt} = C(i\omega_0)e^{i\omega_0 t} - C^*(i\omega_0)e^{-i\omega_0 t}$$

$$a(t) = \frac{d^2x}{dt^2} = C(i\omega_0)^2 e^{i\omega_0 t} + C^*(-i\omega_0)^2 e^{-i\omega_0 t}$$

$$= -\omega_0^2\left[Ce^{i\omega_0 t} + C^* e^{-i\omega_0 t}\right]$$

$$= -\omega_0^2\, x(t)$$

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2$$

$$= \tfrac{1}{2}m(i\omega_0)^2\left[Ce^{i\omega_0 t} - C^* e^{-i\omega_0 t}\right] + \tfrac{1}{2}k\left[Ce^{i\omega_0 t} + C^* e^{-i\omega_0 t}\right]^2$$

$$= \tfrac{1}{2}(-m\omega_0^2)\left[C^2 e^{2i\omega_0 t} - 2CC^* + (C^*)^2 e^{-2i\omega_0 t}\right]$$

$$+\tfrac{1}{2}k\left[C^2 e^{2i\omega_0 t} + 2CC^* + (C^*)^2 e^{-2i\omega_0 t}\right]$$ but $\omega_0^2 = \frac{k}{m} \Rightarrow m\omega_0^2 = k$

$$= \tfrac{1}{2}k\left[\cancel{(-C^2 + C^2)}e^{2i\omega_0 t} + 4CC^* + \cancel{(-(C^*)^2 + (C^*)^2)}e^{-i2\omega_0 t}\right]$$

$$\therefore\ E = 2kCC^*$$

but $CC^* = |C|^2$ is Real and time independent

or if $C = \frac{A+iB}{2}$ $C^* = \frac{A-iB}{2}$ where A and B are real

$CC^* = \tfrac{1}{4}(A^2+B^2)$ then

$E = \tfrac{1}{2}k(A^2+B^2) = \tfrac{1}{2}m\omega_0^2(A^2+B^2)$ is time independent and Real

(5) method 1

$$x(t) = A \sin(\omega_0 t + \phi) \qquad v(t) = A \omega_0 \cos(\omega_0 t + \phi)$$

$$x_1 = A \sin(\omega_0 t_1 + \phi) \qquad v_1 = A \omega_0 \cos(\omega_0 t_1 + \phi)$$

$$x_2 = A \sin(\omega_0 t_2 + \phi) \qquad v_2 = A \omega_0 \cos(\omega_0 t_2 + \phi)$$

using $\sin^2(\omega_0 t + \phi) + \cos^2(\omega_0 t + \phi) = 1$ one obtains

$$\frac{x_1^2}{A^2} + \frac{v_1^2}{A^2 \omega_0^2} = 1 \qquad \text{and} \qquad \frac{x_2^2}{A^2} + \frac{v_2^2}{A^2 \omega_0^2} = 1$$

therefore

$$x^2 + \frac{v_1^2}{\omega_0^2} = x_2^2 + \frac{v_2^2}{\omega_0^2} \qquad \text{solve for } \omega_0$$

$$\omega_0^2 (x_1^2 - x_2^2) = v_2^2 - v_1^2 \implies \omega_0 = \left[\frac{v_2^2 - v_1^2}{x_1^2 - x_2^2} \right]^{1/2}$$

now solve for A

$$x_2^2 + \frac{v_2^2}{\omega_0^2} = A^2$$

$$x_2^2 + v_2^2 \left[\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2} \right] = A^2$$

$$\frac{x_2^2 (\cancel{v_2^2} - v_1^2) + v_2^2 (x_1^2 - \cancel{x_2^2})}{v_2^2 - v_1^2} = A^2$$

$$A = \left[\frac{x_1^2 v_2^2 - x_2^2 v_1^2}{v_2^2 - v_1^2} \right]^{1/2}$$

method 2 - use energy conservation

$$\frac{1}{2}mv_1^2 + \frac{1}{2}kx_1^2 = \frac{1}{2}kA^2 = \frac{1}{2}mv_2^2 + \frac{1}{2}kx_2^2$$ divide by $\frac{m}{2}$ set $\frac{k}{m} = \omega_0^2$

$$v_1^2 + \omega_0^2 x_1^2 = \omega_0^2 A^2$$

$$v_2^2 + \omega_0^2 x_2^2 = \omega_0^2 A^2$$

solve for ω_0^2 $\quad v_1^2 + \omega_0^2 x_1^2 = v_2^2 + \omega_0^2 x_2^2$

$$v_1^2 - v_2^2 = \omega_0^2 (x_2^2 - x_1^2) \Rightarrow \omega_0^2 = \frac{v_1^2 - v_2^2}{x_2^2 - x_1^2} = \frac{v_2^2 - v_1^2}{x_1^2 - x_2^2}$$

$$\therefore \omega_0 = \sqrt{\frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}} = \sqrt{\frac{v_2^2 - v_1^2}{x_1^2 - x_2^2}}$$

Now solve for A^2

$$A^2 = \frac{v_1^2}{\omega_0^2} + x_1^2 = \frac{v_1^2 (x_2^2 - x_1^2)}{v_1^2 - v_2^2} - \frac{x_1^2 (v_1^2 - v_2^2)}{v_1^2 - v_2^2}$$

$$= \frac{v_1^2 x_2^2 - x_1^2 v_2^2}{v_1^2 - v_2^2}$$

$$A = \left[\frac{v_1^2 x_2^2 - x_1^2 v_2^2}{v_1^2 - v_2^2} \right]^{1/2} = \left[\frac{x_1^2 v_2^2 - x_2^2 v_1^2}{v_2^2 - v_1^2} \right]^{1/2}$$