(1) $m\frac{dv}{dt} = -\frac{k}{x^2}$ use chain rule $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$

$mv\frac{dv}{dx} = -\frac{k}{x^2}$ separate variables

$mv\,dv = -\frac{k}{x^2}dx$ integrate to an arbitrary v and x from x_0 to x $v_0 = 0$ to v

$$m\int_0^v v'dv' = -\int_{x_0}^{x}\frac{k}{x'^2}dx' \Rightarrow \frac{mv^2}{2} = \frac{k}{x}\Big|_{x_0}^{x} = k\left(\frac{1}{x} - \frac{1}{x_0}\right)$$ therefore

$$v^2 = \frac{2k}{m}\left(\frac{1}{x} - \frac{1}{x_0}\right) \quad \text{or} \quad v = \sqrt{\frac{2k}{m}\left(\frac{1}{x} - \frac{1}{x_0}\right)} = \frac{dx}{dt}$$ seperate variables

$$\frac{dx}{\sqrt{\frac{1}{x} - \frac{1}{x_0}}} = \sqrt{\frac{2k}{m}}\,dt$$ multiply left hand side by $\frac{x}{x}$

$$\int_{x_0}^{0}\frac{x\,dx}{\sqrt{x - \frac{x^2}{x_0}}} = \int_0^t\sqrt{\frac{2k}{m}}\,dt$$ look up integral on LHS in tables etc

$$\frac{x_0^{3/2}}{2}\sin^{-1}\left(\frac{2x}{x_0} - 1\right) - x_0\left(x - \frac{x^2}{x_0}\right)^{1/2}\Big|_{x_0}^{0} = \sqrt{\frac{2k}{m}}\,t$$

$$\frac{x_0^{3/2}}{2}\left[\sin^{-1}(+1) - \sin^{-1}(-1)\right] - x_0(0 - 0) = \sqrt{\frac{2k}{m}}\,t$$ solve for t

$$\frac{x_0^{3/2}}{2}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]\sqrt{\frac{m}{2k}} = t$$

$$= \pi\sqrt{\frac{x_0^3 m}{8k}} = t$$

(2)

$F = cvx$ so Newton's 2nd Law becomes

$$m\frac{dv}{dt} = m\left(\frac{dx}{dt}\right)\left(\frac{dv}{dx}\right) = cvx$$

$$m\cancel{v}\frac{dv}{dx} = c\cancel{v}x \quad \Rightarrow \text{ separate variables}$$

$$m\,dv = cx\,dx \quad \text{integrate with } v = v_0 \text{ at } x = x_0 = 0 \text{ at } t=0$$

$$m\int_{v_0}^{v} dv' = c\int_0^x x'\,dx'$$

$$m(v - v_0) = c\left(\tfrac{1}{2}x^2\right) \quad \text{solve for } v$$

$$\boxed{v = \frac{c}{2m}\left(x^2 + \frac{2mv_0}{c}\right)} \quad \text{set } v = \frac{dx}{dt} \text{ and separate variables}$$

$$\frac{dx}{\left(x^2 + \frac{2mv_0}{c}\right)} = \frac{c}{2m}\,dt \quad \text{integrate}$$

$$\int_0^x \frac{dx'}{\left(x'^2 + \frac{2mv_0}{c}\right)} = \int_0^t \frac{c}{2m}\,dt' \quad \text{but } \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$\sqrt{\frac{c}{2mv_0}}\tan^{-1}\left(\sqrt{\frac{c}{2mv_0}}\,x\right) = \frac{c}{2m}t \quad \text{solve for } x$$

$$\sqrt{\frac{c}{2mv_0}}\,x = \tan\left(\sqrt{\frac{v_0 c}{2m}}\,t\right)$$

$$x = \sqrt{\frac{2mv_0}{c}}\tan\left(\sqrt{\frac{v_0 c}{2m}}\,t\right)$$

(3)

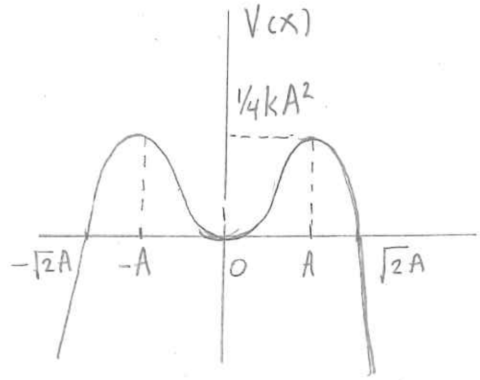
Since $F(x) = -\frac{dV}{dx}$

$$V(x) = -\int F(x)\,dx = -\int(-kx + kx^3/A^2)\,dx = \tfrac{1}{2}kx^2 - \tfrac{1}{4}k\frac{x^4}{A^2} + V_0$$

set $V_0 = 0$

$V = 0$ when $\tfrac{1}{2}kx^2 = \tfrac{1}{4}kx^4/A^2$

then $x = 0, \pm\sqrt{2}A$



(b) if $\tfrac{1}{2}mv_0^2 = T_0$ @ $x=0$ where $V(x) = 0$ then the total energy is

$$E = T_0 = \tfrac{1}{2}kA^2 \quad \text{therefore} \quad E = T + V(x) = \tfrac{1}{2}kA^2$$

in general $T = \tfrac{1}{2}kA^2 - V(x)$

$$= \tfrac{1}{2}kA^2 - \tfrac{1}{2}kx^2 + \tfrac{1}{4}k\frac{x^4}{A^2} = \underline{\tfrac{1}{2}k\left[A^2 - x^2 + \tfrac{1}{2}\frac{x^4}{A^2}\right]}$$

(c) the total energy is conserved therefore $E_{TOT} = \tfrac{1}{2}kA^2$ independent of position

(d) Determine maximum values of $V(x)$

then $\frac{dV}{dx} = kx - \frac{kx^3}{A^2} = 0 \qquad x_{equil} = 0, \pm A$

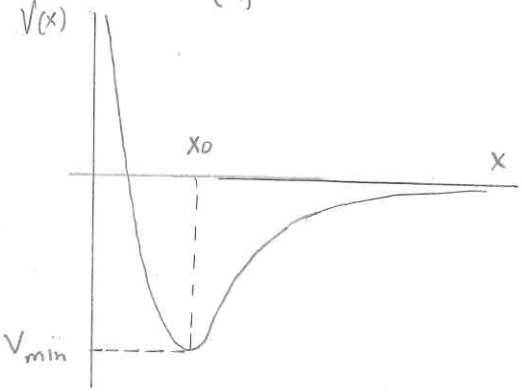
$U(0) = 0 \qquad U(\pm A) = \tfrac{1}{2}kA^2 - \tfrac{1}{4}kA^2 = \tfrac{1}{4}kA^2$

the kinetic energy T_{max} must be less than $\tfrac{1}{4}kA^2$ for turning points to exist

but $T_0 = \tfrac{1}{2}kA^2 > \tfrac{1}{4}kA^2$ there are <u>no</u> turning points

(4)

(a)



(b) $$V = V_o\left[\left(\frac{a}{x}\right)^4 - 2\left(\frac{a}{x}\right)^2\right]$$

$$\frac{dV}{dx} = V_o\left[-4\frac{a^4}{x^5} - 2(-2)\frac{a^2}{x^3}\right] = 4V_o\left[\frac{a^2}{x^3} - \frac{a^4}{x^5}\right]$$

set $\frac{dV}{dx} = 0$

$$\frac{4V_o a^2}{x^3}\left[-\frac{a^2}{x^2} + 1\right] = 0 \qquad \underline{x = \pm a} \qquad \underline{x = \infty}$$

are the equilibrium positions

at $x = a$ $\quad V = V_o\left[\left(\frac{a}{a}\right)^4 - 2\left(\frac{a}{a}\right)^2\right] = -V_o \qquad V_{min} = V_o$

(c) compute the Taylor series expansion for $V(x)$ close to $x = x_0$

need $\frac{d^2V}{dx^2} = 4V_o\left[5\frac{a^4}{x^6} - 3\frac{a^2}{x^4}\right] =$ evaluate at $x = a$

$$\left.\frac{d^2V}{dx^2}\right|_{x=a} = 4V_o\left[5\frac{1}{a^2} - 3\frac{1}{a^2}\right] = \frac{8V_o}{a^2} > 0$$

then $V(x) \sim V(x_0) + \left.\frac{dV}{dx}\right|_{x_0}(x - x_0) + \frac{1}{2}\left.\frac{d^2V}{dx^2}\right|_{x_0}(x - x_0)^2 + \cdots$ but $\left.\frac{dV}{dx}\right|_{x_0} = 0$

$$= V_o + \frac{1}{2}\left(\frac{8V_o}{a^2}\right)(x - x_0)^2 + \cdots = V_o + \frac{1}{2}k(x - x_0)^2$$

the effective spring constant $k = \frac{8V_o}{a^2}$

$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{8V_o}{ma^2}} = \frac{2}{a}\sqrt{\frac{2V_o}{m}}$$

(5) $V(x) = k\left[x^2 + 2\alpha \ln\left(1 - \frac{x}{a}\right)\right]$ then

(a) $F(x) = -\frac{\partial V}{\partial x} = -k\left[2x + 2\alpha \frac{1}{\left(1-\frac{x}{a}\right)}\left(-\frac{1}{a}\right)\right]$

$= -2k\left[x - \frac{\alpha}{a-x}\right] = 2k\left[\frac{\alpha}{a-x} - x\right]$

(b) equilibrium points exist where $\frac{dV}{dx} = 0$ $(\vec{F}=0)$

$\frac{\alpha}{a-x} = x$ or $\alpha = ax - x^2 \Rightarrow x^2 - ax + \alpha = 0$

solve quadratic equ for x $\underline{x = \frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 - 4\alpha}}$

with $a = 10\,cm$ $\alpha = 16\,cm^2$

$x_1 = \left(5 - \frac{1}{2}\sqrt{100-64}\right) cm = \left(5 - \frac{1}{2}6\right) = \underline{2\,cm}$

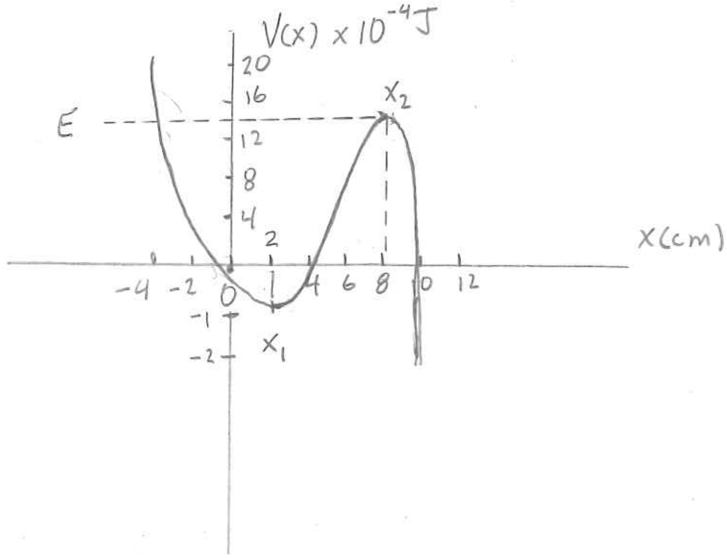
$x_2 = \left(5 + \frac{1}{2}6\right) cm = \underline{8\,cm}$

compute $\frac{d^2V}{dx^2}$ and evaluate at x_1 and x_2 to determine stability

$\frac{d^2V}{dx^2} = 2k\frac{d}{dx}\left(x - \frac{\alpha}{a-x}\right) = 2k\left[1 - \frac{\alpha}{(a-x)^2}\right] = 2\left[1 - \frac{16}{(10-x)^2}\right] J/m^2$

@ $x_1 = 2m$ $\left.\frac{d^2V}{dx^2}\right|_{x_1} = 2\left[1 - \frac{16}{64}\right] = 1.5 \frac{J}{m^2} > 0$ x_1 is stable

@ $x_2 = 8m$ $\left.\frac{d^2V}{dx^2}\right|_{x_2} = 2\left[1 - \frac{16}{4}\right] = -6 \frac{J}{m^2} < 0$ x_2 is unstable



The total energy for a system that is bounded (ie. two turning points exist) is determined by $V(x_2)$.

$$E_{tot} = V(x_2) = k\left[x_2^2 - 2\alpha \ln\left(1 - \frac{x_2}{\alpha}\right)\right] = 12.5 \times 10^{-4}\,J$$

T_{max} occurs when $x = x_1$

$$T_{max} = E_{TOT} - V(x_1) = V(x_2) - V(x_1)$$

$$V(x_1) = k\left[x_1^2 - 2\alpha \ln\left(1 - \frac{x_1}{\alpha}\right)\right] = -3.14 \times 10^{-4}\,J$$

therefore $T_{max} = V(x_2) - V(x_1)$

$$= \left[12.5 - (-3.14)\right] \times 10^{-4}\,J = 15.64 \times 10^{-4}\,J$$

$$= 1.564 \times 10^{-3}\,J$$