(1)

$$\vec{F} = f_0 (y\hat{i} + x\hat{j}) \qquad d\vec{s} = dx\,\hat{i} + dy\,\hat{j} = dx\,\hat{i} + \frac{dy}{dx}dx\,\hat{j}$$

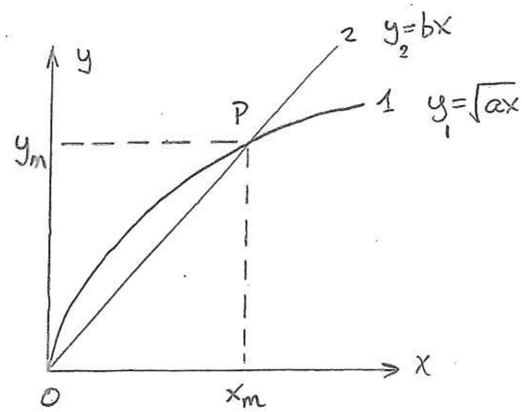
Trajectory of particle 1 $\quad y_1 = \sqrt{ax} \qquad \frac{dy_1}{dx} = \frac{1}{2}\sqrt{\frac{a}{x}}$

Trajectory of particle 2 $\quad y_2 = bx \qquad \frac{dy_2}{dx} = b$

the work performed is

$$W = \int_0^P \vec{F}\cdot d\vec{s} = \int_0^P f_0(y\hat{i} + x\hat{j})\cdot\left(dx\hat{i} + \frac{dy}{dx}dx\hat{j}\right) = \int_{0_x}^{P_x} f_0\left(y\,dx + x\frac{dy}{dx}dx\right)$$

$$= f_0\int_{0_x}^{P_x}\left(y + x\frac{dy}{dx}\right)dx$$



(a) trajectories meet when

$y_1 = \sqrt{ax} = y_2 = bx$ or squaring both sides

$ax = b^2x^2 \Rightarrow x(b^2x - a) = 0$

$x = 0 \quad x = a/b^2$

$y = 0 \quad y = \sqrt{a(a/b^2)} = b\left(\frac{a}{b^2}\right) = \frac{a}{b} \qquad (x_m, y_m) = \left(\frac{a}{b^2}, \frac{a}{b}\right)$

(b) $$W_1 = f_0\int_0^{a/b^2}\left(\sqrt{ax} + \frac{x}{2}\sqrt{\frac{a}{x}}\right)dx = f_0\int_0^{a/b^2}\frac{3}{2}\sqrt{ax}\,dx = f_0\,\frac{3}{2}\sqrt{a}\,\frac{x^{3/2}}{3/2}dx\Big|_0^{a/b^2}$$

$$= f_0\sqrt{a}\,x^{3/2}\Big|_0^{a/b^2} = f_0\sqrt{a}\,\frac{a^{3/2}}{b^3} = f_0\frac{a^2}{b^3}$$

(c) $$W_2 = \int_0^{a/b^2}(bx + bx)dx = f_0\int_0^{a/b^2}2bx\,dx = f_0\,bx^2\Big|_0^{a/b^2} = f_0\,b\frac{a^2}{b^4} = f_0\frac{a^2}{b^3}$$

(d) $W_1 = W_2$ since $\vec{F}$ depends only on position it is a conservative force.

(2) $\vec{A} = 3\hat{e}_1 - 2\hat{e}_2 + \hat{e}_3 \qquad \vec{B} = -2\hat{e}_1 + \hat{e}_2 + 3\hat{e}_3$

$$\vec{A}+\vec{B} = (3-2)\hat{e}_1 + (-2+1)\hat{e}_2 + (1+3)\hat{e}_3 = \hat{e}_1 - \hat{e}_2 + 4\hat{e}_3$$

(a) $\vec{A}-\vec{B} = (3-(-2))\hat{e}_1 + (-2-1)\hat{e}_2 + (1-3)\hat{e}_3 = \underline{5\hat{e}_1 - 3\hat{e}_2 - 2\hat{e}_3} = \vec{C}$

$$|\vec{A}-\vec{B}| = \left(C_x^2 + C_y^2 + C_z^2\right)^{1/2} = (25+9+4)^{1/2} = (38)^{1/2} = \underline{\underline{\sqrt{38}}}$$

(b) component of $\vec{B}$ along $\vec{A}$ $\quad B_A = \vec{B}\cdot\hat{n}_A = \vec{B}\cdot\left(\frac{\vec{A}}{|A|}\right) = \frac{\vec{B}\cdot\vec{A}}{|A|}$ $\quad \hat{n}_A$ = unit vector along $\vec{A}$

$$|A| = \sqrt{A_x^2 + A_y^2 + A_z^2} = (9+4+1)^{1/2} = \sqrt{14}$$

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z = -3\cdot 2 + (-2\cdot 1) + 3\cdot 1 = -5 \qquad B_A = \frac{\vec{A}\cdot\vec{B}}{|A|} = \frac{-5}{\sqrt{14}}$$

(c) $\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{|A||B|}$ $\quad$ but $|B| = \left(B_x^2 + B_y^2 + B_z^2\right)^{1/2} = (4+1+9)^{1/2} = \sqrt{14}$

$$\cos\theta = \frac{-5}{\sqrt{14}\sqrt{14}} = -\frac{5}{14} \qquad \theta = \cos^{-1}\left(-\frac{5}{14}\right) = \underline{110.9^\circ = 1.94 \text{ radians}}$$

(d) $$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ 3 & -2 & 1 \\ -2 & 1 & 3 \end{vmatrix} = \hat{e}_1(-6-1) + \hat{e}_2(-2-9) + \hat{e}_3(3-4) = \underline{-7\hat{e}_1 - 11\hat{e}_2 - \hat{e}_3}$$

(e) $$(\vec{A}-\vec{B})\times(\vec{A}+\vec{B}) = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ 5 & -3 & -2 \\ 1 & -1 & 4 \end{vmatrix} = \hat{e}_1(-12-2) + \hat{e}_2(-2-20) + \hat{e}_3(-5+3)$$

$$= -14\hat{e}_1 - 22\hat{e}_2 - 2\hat{e}_3$$

check $(\vec{A}-\vec{B})\times(\vec{A}+\vec{B}) = \underbrace{\vec{A}\times\vec{A}}_{0} + \vec{A}\times\vec{B} - \vec{B}\times\vec{A} - \underbrace{\vec{B}\times\vec{B}}_{0} = \vec{A}\times\vec{B} - \vec{B}\times\vec{A} = 2(\vec{A}\times\vec{B})$

$$= -14\hat{e}_1 - 22\hat{e}_2 - 2\hat{e}_3 \quad ✓$$

(3)

$$\vec{B}\times\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \hat{i}\left[B_yC_z - C_yB_z\right] + \hat{j}\left[B_zC_x - C_zB_x\right] + \hat{k}\left[B_xC_y - C_xB_y\right]$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ (B_yC_z - C_yB_z) & (B_zC_x - C_zB_x) & (B_xC_y - C_xB_y) \end{vmatrix}$$

$$= \hat{i}\left[A_yB_xC_y - A_yC_xB_y - A_zB_zC_x + A_zC_zB_x\right]$$

$$+\hat{j}\left[A_zB_yC_z - A_zC_yB_z - A_xB_xC_y + A_xC_xB_y\right]$$

$$+\hat{k}\left[A_xB_zC_x - A_xC_zB_x - A_yB_yC_z + A_yC_yB_z\right]$$

$$= \hat{i}\left[B_x(A_yC_y + A_zC_z) - C_x(A_yB_y + A_zB_z)\right] + \hat{i}\left[B_xA_xC_x - C_xA_xB_x\right]$$

$$+\hat{j}\left[B_y(A_xB_x + A_zC_z) - C_y(A_xB_x + A_zB_z)\right] + \hat{j}\left[B_yA_yC_y - C_yA_yB_y\right]$$

$$+\hat{k}\left[B_z(A_xC_x + A_yC_y) - C_z(A_xB_x + A_yB_y)\right] + \hat{k}\left[B_zA_zC_z - C_zA_zB_z\right]$$

add and subtract same quantity

$$= \hat{i}\left[B_x(A_xC_x + A_yC_y + A_zC_z) - C_x(A_xB_x + A_yB_y + A_zB_z)\right]$$

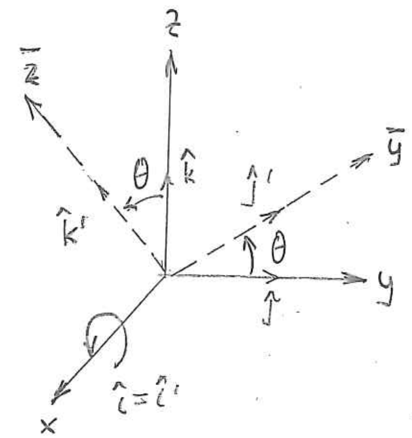
$$+\hat{j}\left[B_y(A_xC_x + A_yC_y + A_zC_z) - C_y(A_xB_x + A_yB_y + A_zB_z\right]$$

$$+\hat{k}\left(B_z(A_xC_x + A_yC_y + A_zC_z) - C_z(A_xB_x + A_yB_y + A_zB_z\right]$$

$$= (\vec{A}\cdot\vec{C})\left[B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\right] - (\vec{A}\cdot\vec{B})\left[C_x\hat{i} + C_y\hat{j} + C_z\hat{k}\right]$$

$$= (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C} = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

(4)



(a)
$$T = \begin{pmatrix} \hat{i}'\cdot\hat{i} & \hat{i}'\cdot\hat{j} & \hat{i}'\cdot\hat{k} \\ \hat{j}'\cdot\hat{i} & \hat{j}'\cdot\hat{j} & \hat{j}'\cdot\hat{k} \\ \hat{k}'\cdot\hat{k} & \hat{k}'\cdot\hat{j} & \hat{k}'\cdot\hat{k} \end{pmatrix}$$

but $\hat{i}'\cdot\hat{i} = 1$ $\quad \hat{i}'\cdot\hat{j} = \hat{i}'\cdot\hat{k} = \hat{j}'\cdot\hat{i} = \hat{k}'\cdot\hat{i} = 0$

$\hat{j}'\cdot\hat{j} = \cos\theta \quad \hat{j}'\cdot\hat{k} = \cos(\pi/2 - \theta) = \sin\theta$

$\hat{k}'\cdot\hat{j} = \cos(\pi/2 + \theta) = -\sin\theta \quad \hat{k}'\cdot\hat{k} = \cos\theta \qquad \theta = \frac{\pi}{6}$

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\frac{\pi}{6}) & \sin(\pi/6) \\ 0 & -\sin(\pi/6) & \cos(\pi/6) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & 1/2 \\ 0 & -1/2 & \sqrt{3}/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & \sqrt{3} & 1 \\ 0 & -1 & \sqrt{3} \end{pmatrix}$$

(b) two rotations by θ gives $\theta_1 + \theta_2 = \theta_T \Rightarrow 30^\circ + 30^\circ = 60^\circ$ or $(\pi/6 + \pi/6) = \pi/3$

therefore $T\cdot T = T^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\pi/3) & \sin(\pi/3) \\ 0 & -\sin(\pi/3) & \cos(\pi/3) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & 1/2 \end{pmatrix} = 1/2 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & \sqrt{3} \\ 0 & -\sqrt{3} & 1 \end{pmatrix}$

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \underline{\underline{\bar{x} = x}} \quad \underline{\underline{\bar{y} = \tfrac{1}{2}y + \tfrac{\sqrt{3}}{2}z}} \quad \underline{\underline{\bar{z} = -\tfrac{\sqrt{3}}{2}y + \tfrac{1}{2}z}}$$

(c) rotation is counter-clockwise therefore it is <u>positive</u> by right-hand rule.

(5) method (1) use $TT^T = \mathbb{I}$ condition for orthogonality

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma & -\sigma \\ 0 & \sigma & \sigma \end{pmatrix} \qquad T^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma & \sigma \\ 0 & -\sigma & 0 \end{pmatrix} \quad \text{then}$$

$$TT^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma & -\sigma \\ 0 & \sigma & \sigma \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma & \sigma \\ 0 & -\sigma & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2\sigma^2 & 0 \\ 0 & 0 & 2\sigma^2 \end{pmatrix} = \mathbb{I}$$

$$\therefore \quad 2\sigma^2 = 1 \qquad \underline{\underline{\sigma = \pm 1/\sqrt{2}}}.$$

method (2) $T^T = T^{-1}$ for orthogonality

find T^{-1} from $TT^{-1} = T^{-1}T = \mathbb{I}$ set $T^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$

Then $$T^{-1}T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma & -\sigma \\ 0 & \sigma & \sigma \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a\sigma + b\sigma & -a\sigma + b\sigma \\ 0 & c\sigma + d\sigma & -c\sigma + d\sigma \end{pmatrix}$$

or $(a+b)\sigma = 1$ and $(b-a)\sigma = 0 \Rightarrow a = b \quad 2a\sigma = 1 \quad a = 1/2\sigma = b$

or $(c+d)\sigma = 0$ and $(d-c)\sigma = 1 \Rightarrow c = -d \quad 2d\sigma = 1 \quad d = 1/2\sigma = -c$

$$T^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2\sigma} & \frac{1}{2\sigma} \\ 0 & -1/2\sigma & 1/2\sigma \end{pmatrix}$$ but $T^T = T^{-1}$ leads to $\frac{1}{2\sigma} = \sigma \quad -\frac{1}{2\sigma} = -\sigma$

or $\sigma^2 = \frac{1}{2} \Rightarrow \underline{\underline{\sigma = \pm 1/\sqrt{2}}}$

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \Rightarrow \cos(\theta) = 1/\sqrt{2}, \quad \sin(\theta) = -1/\sqrt{2} \quad \theta = -45^\circ \qquad \cos\theta = -1/\sqrt{2}, \quad \sin\theta = +1/\sqrt{2} \quad \theta = +135^\circ$$