Research Paper Outline

Topic: Adaptive Characteristic Method for Regge-Wheeler Equation and Power-Law Tails

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I. Abstract

We demonstrate that an adaptive evolution of the Regge-Wheeler equation along characteristic curves has significant advantages for obtaining the power-law tail decay that occurs after the quasi-normal ringing from binary black hole mergers. The canonical approach in the literature involves reaching the power-law tail through spectral methods, but this adaptive, characteristic approach has the benefit of rapidly jumping to the potential peak, slowing for the interaction, then taking increasingly larger steps as the evolution proceeds towards null infinity. We can obtain the power-law tail behavior more efficiently using this adaptive, characteristic method, as the propagating wave has minimal dynamics along the null geodesics outside of the interaction.

II. Spectral Method (Literature Overview)

III. Method of Lines (Arbitrary RK Order)

We implemented the method of lines for solving the Regge-Wheeler equation on an (r_*, t) grid where r_* denotes the tortoise coordinate defined as $r_* = r + 2M \ln|r/2M - 1|$ such that $r_* \to -\infty$ as $r \to 2M$. The Regge-Wheeler potential, $V_l(r_*)$, has dependence on this tortoise coordinate, so we numerically inverted the coordinate transformation using the Newton-Raphson method. We modified the fourth-order Runge-Kutta method for storing the wave function amplitudes along the initial and final characteristic slices that are later compared with those obtained using the adaptive, characteristic method. The second-order, spatial partial derivative in the Regge-Wheeler equation was calculated from the fourth-order, centered finite difference approximation during the numerical integration of the equation. We defined the initial wave function as Gaussian, and determined the center x_0 from the procedure detailed later during the error analysis section of this outline. The purpose of implementing the method of lines was for comparison with the adaptive, characteristic method, and we have control over the numerical error by setting the integration tolerance.

IV. Adaptive Characteristic Method (Step-Doubling)

We implemented the characteristic method for solving the Regge-Wheeler equation on the numerical grid with coordinates $u = t - r_*$ (retarded time) and $v = t + r_*$ (advanced time). The characteristic curves for this coordinate system are null geodesics that electromagnetic or gravitational waves propagate along. We modified this characteristic method by implementing an adaptive, step-doubling algorithm that would allow almost instant traversal to the interaction. We justified this behavior with the observation that there are minimal dynamics occurring away from the event horizon where the potential has insignificant value. In this region far from the potential peak, the Regge-Wheeler equation approximates the homogeneous wave equation, validating our assumption that an adaptive characteristic method would minimize the global error that increases with wave propagation in the method of lines. We required continuous boundary conditions for the adaptive evolution, so discrete sampling of the method of lines along the initial characteristic curve was replaced with discrete sampling of the continuous initial Gaussian. The next section of this outline argues that we could place the initial Gaussian far enough along the x-axis that the error associated with this continuous limit is smaller than the length scale where power-law tail decay occurs.

V. Error Analysis: Computational Time vs Characteristic Distance (Fixed Error)

Suppose $\Psi = \Sigma\, c_n r_*^{-n}$. We observe that $\Psi = c_0$ whenever $r_* \to \infty$. We determine this constant coefficient from polynomial curve fitting with only finite terms, obtained from numerical solutions with variable initial pulse position. For every retarded time coordinate point $u = t - r_*$, we build the wave function when $r_* \to \infty$ from these c_0 values, then take the initial pulse position as the largest initial position from the curve fitting provided that $|\Psi_{large} - c_0| < x_c$ where x_c denotes the critical tortoise coordinate where $\ln |\Psi|$ exhibits power-law tail decay. We assume that the complex wave function has an odd-parity, real component from the Regge-Wheeler potential and an even-parity, imaginary component from the Zerilli potential. For the comparison between numerical methods, the evolution proceeded using the method of lines, with fixed tolerance, and the computational time was plotted versus distance along the characteristic curve $u = t - r_* = 0$ that extends towards null infinity from the origin. We defined the effective numerical error as the l_2-norm of the difference between the method of lines solution and the characteristic solution obtained from uniform, high-resolution sampling. We then varied the tolerance of the adaptive, characteristic method such that the numerical error, previously defined, was consistent with the error computed for the method of lines. For the adaptive, characteristic method, the computational time was plotted versus distance along the characteristic curve $u = t - r_* = 0$. We showed that, with error held constant, the adaptive, characteristic method achieved the same distance along the characteristic curve as the method of lines with much shorter computational time, providing an alternative, more efficient, method for reaching power-law tail decay than the spectral approach.