Ken Sible — July 25, 2018

$$f(\mathbf{x}) = f(\mathbf{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\bigg|_{\mathbf{x}=\mathbf{a}} (x_i - a_i) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i x_j}\bigg|_{\mathbf{x}=\mathbf{a}} (x_i - a_i)(x_j - a_j) + \dots$$

$$\begin{aligned}
\Psi(u_{i+1}, v_j) &= \Psi(u_i, v_j) + \Psi_u(u_i, v_j)\Delta u + \frac{1}{2}\Psi_{uu}(u_i, v_j)\Delta u^2 + \dots \\
\Psi(u_i, v_{j+1}) &= \Psi(u_i, v_j) + \Psi_v(u_i, v_j)\Delta v + \frac{1}{2}\Psi_{vv}(u_i, v_j)\Delta v^2 + \dots \\
\Psi(u_{i+1}, v_{j+1}) &= \Psi(u_i, v_j) + \Psi_u(u_i, v_j)\Delta u + \Psi_v(u_i, v_j)\Delta v \\
&\quad + \frac{1}{2}\left[\Psi_{uu}(u_i, v_j)\Delta u^2 + 2\Psi_{uv}(u_i, v_j)\Delta u \Delta v + \Psi_{vv}(u_i, v_j)\Delta v^2\right] + \dots
\end{aligned}$$

$$\Psi(u_{i+1}, v_{j+1}) - \Psi(u_{i+1}, v_j) - \Psi(u_i, v_{j+1}) = \Psi_{uv}(u_i, v_j)\Delta u \Delta v - \Psi(u_i, v_j) + \dots$$

$$\Psi_{uv}(u_i, v_j) = \frac{\Psi(u_{i+1}, v_{j+1}) - \Psi(u_{i+1}, v_j) - \Psi(u_i, v_{j+1}) + \Psi(u_i, v_j)}{\Delta u \Delta v} + O(\Delta u, \Delta v)$$

$$-4\Psi_{uv} = V_l(r)\Psi$$

$$\Psi(u_{i+1}, v_{j+1}) - \Psi(u_{i+1}, v_j) - \Psi(u_i, v_{j+1}) + \Psi(u_i, v_j) = -\frac{\Delta u \Delta v}{4} V_l(r)\Psi(u_i, v_j)$$

$$\Psi(u_{i+1}, v_{j+1}) = \Psi(u_{i+1}, v_j) + \Psi(u_i, v_{j+1}) - \Psi(u_i, v_j) - \frac{\Delta u \Delta v}{4} V_l(r)\Psi(u_i, v_j)$$

$$\Psi(u_i, v_j) \to \frac{1}{2}\left[\Psi(u_{i+1}, v_j) + \Psi(u_i, v_{j+1})\right] + O(\Delta u, \Delta v)$$

$$\Psi(u_{i+1}, v_{j+1}) = \Psi(u_{i+1}, v_j) + \Psi(u_i, v_{j+1}) - \Psi(u_i, v_j) - \frac{\Delta u \Delta v}{8} V_l(r_c)\left[\Psi(u_{i+1}, v_j) + \Psi(u_i, v_{j+1})\right]$$

$$\boxed{\Psi_N = \Psi_W + \Psi_E - \Psi_S - \frac{\Delta u \Delta v}{8} V_l(r_c)\left[\Psi_W + \Psi_E\right] + O(\Delta u^2 \Delta v, \Delta u \Delta v^2)}$$