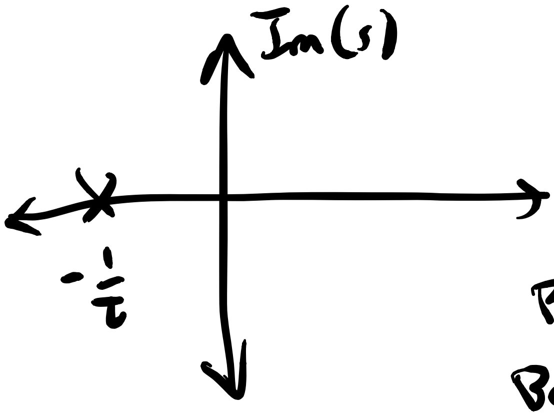
4.1

First-order

$$Y(s) = \underbrace{\frac{K}{\tau s + 1}}_{G(s)} U(s) \qquad (\tau > 0)$$



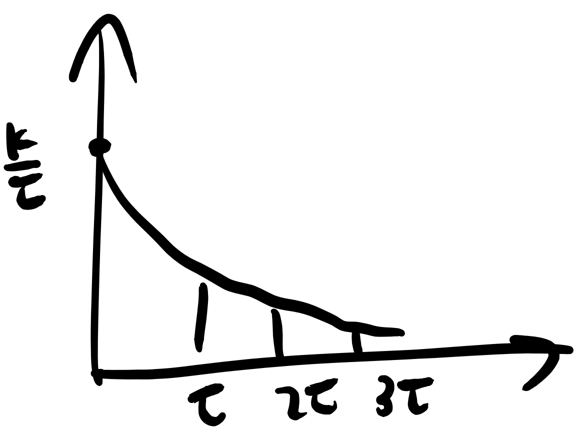
Pole location

Bandwidth $= \frac{1}{\tau}$ rad/s

Steady-state gain $G(0) = K$

Impulse response $= g(t)$

$= \mathcal{L}^{-1}\{G(s)\} = \frac{K}{\tau} e^{-t/\tau}, \quad t \geq 0$



$g(0) = K/\tau$

$g(\tau) = \left(\frac{K}{\tau}\right) e^{-1} \approx 37\% \ \frac{K}{\tau}$

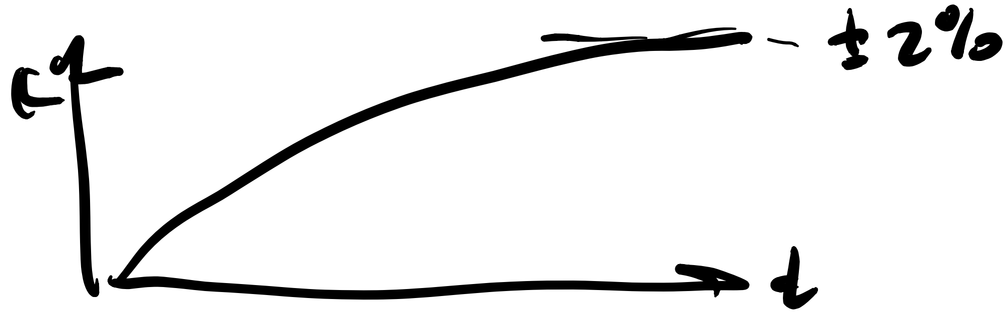
Bandwidth: $\frac{1}{\tau}$

Higher bandwidth ($1/\tau$) $\longleftrightarrow$ Faster decay

Step response

$$y(t) = \mathcal{L}^{-1}\left\{\frac{K}{\tau s+1}\cdot\frac{1}{s}\right\}$$

$$= K(1-e^{-t/\tau}),\quad t \geq 0$$



$y(0)=0 \qquad y(\tau) \doteq 0.631\,K$

$y(2\tau) \approx 0.85\,K$

$y(4\tau) \approx 0.98\,K$

Observations:

1) After $\boxed{4\tau}$ seconds, y is within 2% of its steady-state value (settling time)

2) $y(t) \leq K$ for all $t \geq 0$ (no overshoot)

3) $y(t)$ is monotonically increasing (no peaking)

Higher bandwidth $\left(\frac{1}{\tau}\right) \longleftrightarrow$ faster response

The time constant τ of the system completely determines how fast the system responds.

Bandwidth?

$$\cos(\omega t) \rightarrow \boxed{LTI} \rightarrow A\cos(\omega t + \varphi)$$

$$\frac{|G(j\omega_{BW})|}{|G(0)|} = \frac{1}{\sqrt{2}}$$

4.2 Second Order System

$$Y(s) = G(s)\,U(s), \quad G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

e.g. 4.2.1

$$M\ddot{q} = u - Kq - b\dot{q}$$

$$y = q$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{1/M}{s^2 + \frac{b}{M}s + \frac{K_{sp}}{M}}$$

$$K\frac{K_{sp}}{M} = \frac{1}{M}$$

For this system,

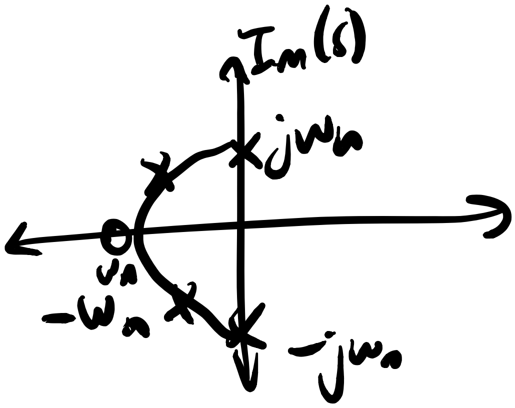
$$\omega_n = \sqrt{\frac{K_{sp}}{M}}$$

$$\zeta = \frac{b}{2\sqrt{M/K_{sp}}}$$

$$K = \frac{1}{K_{sp}}$$

<u>Pole locations</u>

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$
$$= \omega_n\left(-\zeta \pm \sqrt{\zeta^2-1}\right)$$



$\omega_n > 0$

Pole locations as ζ goes from 0 to $+\infty$

The value of ζ is used to categorize:

- G is <u>undamped</u> if $\zeta = 0$
- G is <u>underdamped</u> if $0 < \zeta < 1$
- G is <u>critically damped</u> if $\zeta = 1$.

- G is <u>overdamped</u> if $\zeta > 1$.

Step response

Terminology

ζ - damping ratio

ω_n - undamped natural frequency

K - steady state gain

4.2.1 Underdamped Systems

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

$$= \omega_n\left(-\zeta \pm \sqrt{\zeta^2-1}\right)$$

$$= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$= \omega_n e^{\pm j\sqrt{1-\zeta^2}\,(\pi-\theta)}$$

$\theta = \arccos(\zeta)$