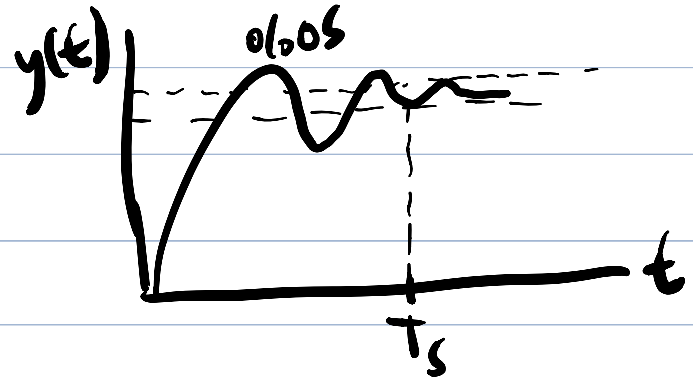
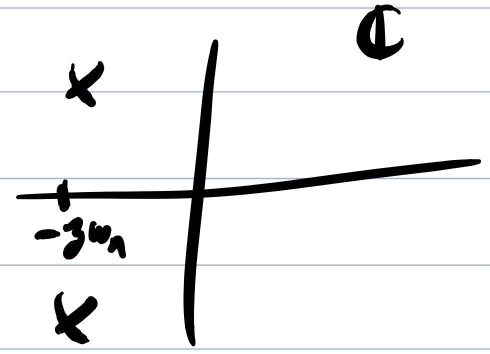
Underdamped 2nd order system

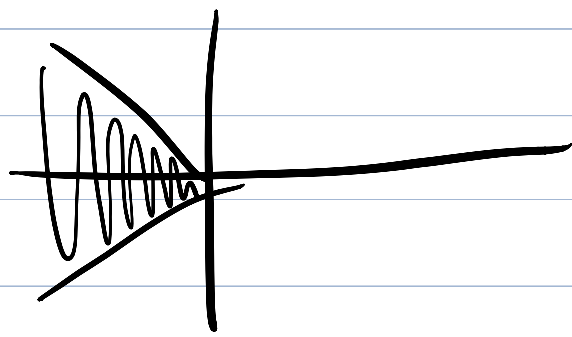
$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- metrics for quality of step response





- overshoot specs imposes constraints of argument of poles



4.3.2 Settling Time

- a crude estimate is obtained by looking at the decay rate $e^{-\zeta\omega_n t}$

$$e^{-\zeta\omega_n t} \le 0.02$$

$$\Leftrightarrow t \ge \frac{4}{\zeta\omega_n}$$

- settling time is given by $\boxed{T_s \doteq \frac{4}{\zeta\omega_n} \quad 0 < \zeta < 1}$

Higher bandwidth
ζ fixed, ω_n increasing $\iff$ Faster response

e.g. mass-spring-damper Find conditions on M, b, K_{sp} so that $T_s \le 3$ seconds

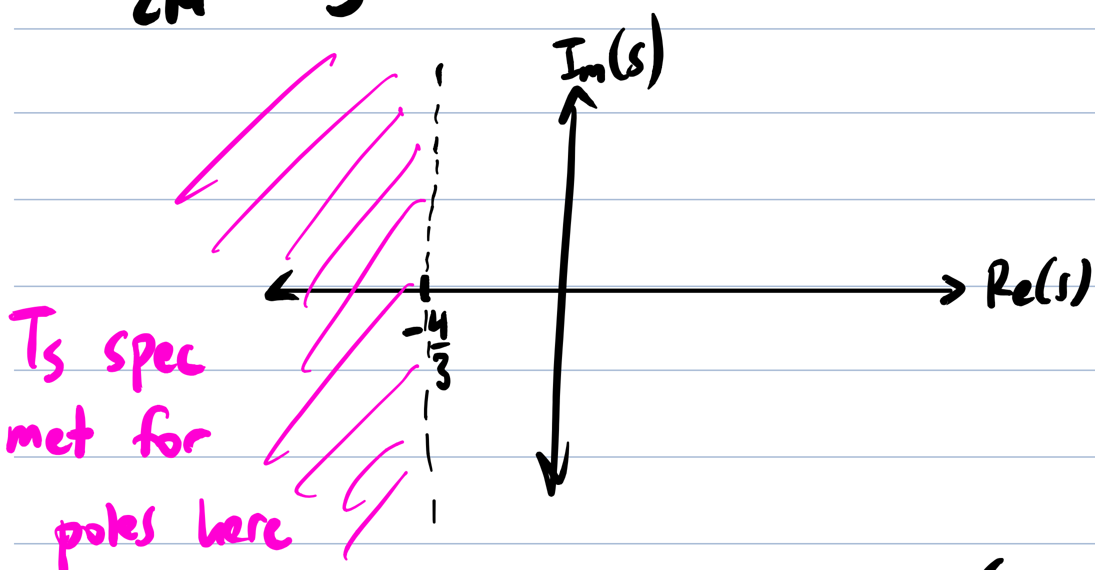
$T_s \le 3$

$\Rightarrow \frac{4}{\zeta\omega_n} \le 3$

$\Rightarrow \zeta\omega_n \ge \frac{4}{T_s^{max}} = \frac{4}{3}$ (poles further to the left)

In this example, we get:

$\frac{b}{2M} \ge \frac{4}{3}$



↳ Similar to first-order case ($4\tau = T_s$)

4.3.3. Peak Time

Smallest time T_p s.t. $\|y\|_\infty = y(T_p)$
-derived similarly to overshoot (diff. calc.)

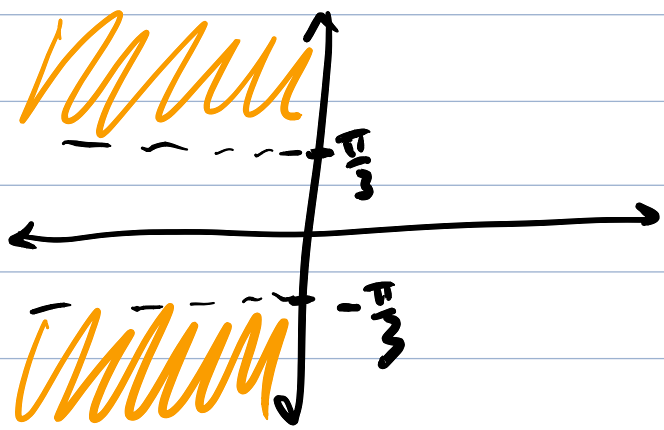
$$\boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}, \quad 0 < \zeta < 1}$$

- T_p only depends on imaginary part of the poles.

e.g. (mass-spring-damper) $\quad T_p \leq T_p^{max} = 3$ seconds

$$\Rightarrow \quad \omega_n\sqrt{1-\zeta^2} \geq \frac{\pi}{T_p^{max}}$$

$$\Rightarrow \sqrt{\frac{K_{sp}}{M} - \frac{b^2}{4M^2}} \geq \frac{\pi}{3}$$



T_p spec met for poles in this region

Summary - underdamped 2nd order systems

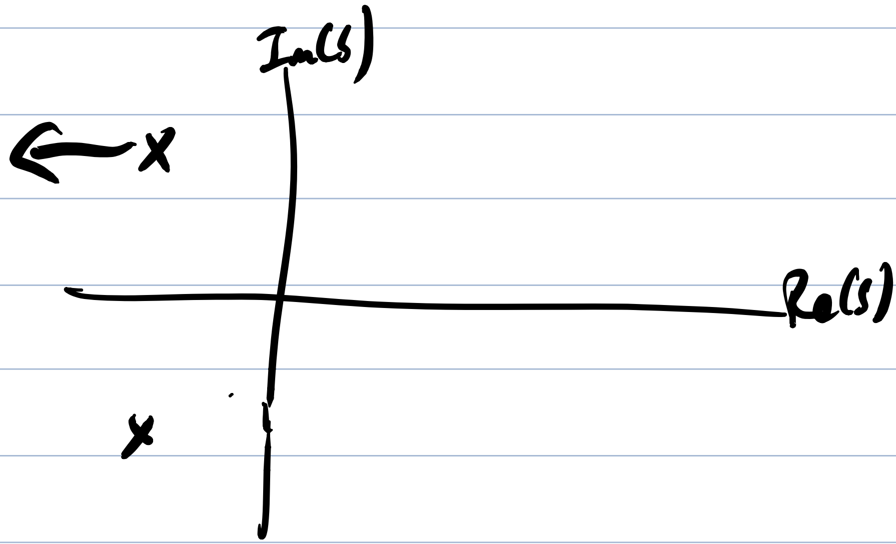
	① Re ←	② Im ↑	③ Angle ↑	④ Magnitude ↑
ω_n	+	+	/	+
ζ	+	−	+	/
%OS	−	+	−	/

T_s	–	/	–	—
T_p	/	–	+	—

poles given by $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

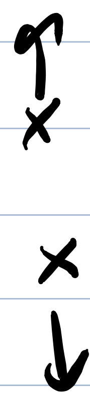
$= \omega_n \exp\left(\pm j\left(\pi-\theta\right)\right)$

where $\theta = \arccos(\zeta)$

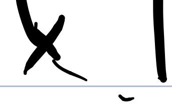
① 

Real part more negative

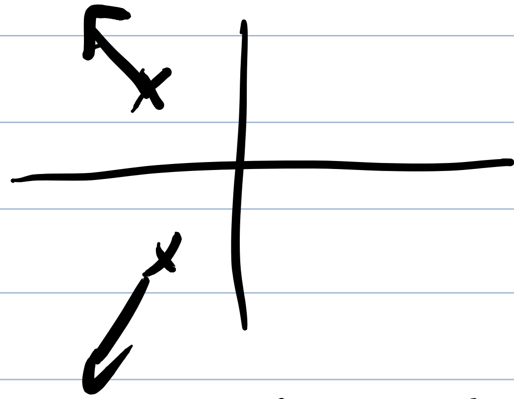
② Increase imaginary part



③ $|\measuredangle \text{ poles}| \rightarrow \pi$



④ Increase magnitude



Read 4.4 and 4.5 (*)

Chapter 3 Linear System Theory

$\dot{x} = Ax + Bu$ or $Y(s) = G(s)U(s)$

$y = Cx + Du$

3.1. Initial State Response

$\dot{x} = Ax, \quad x(0) = x_0 \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n \times m}$

Recall:

- if $n=1$ (A is a scalar), the solution to this ODE is $x(t) = e^{At}x_0$

$$e^{At} = 1 + tA + \frac{t^2}{2!}A^2 + \cdots$$

This motivates the defn of the matrix exponential

$$e^A := I + A + \frac{1}{2!}A^2 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

e.g. 3.1.1.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow e^A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Observe e^A isn't the exponential of each term in A.

e.g. 3.1.2.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\Rightarrow A^k = \begin{bmatrix} 1^k & 0 \\ 0 & 2^k \end{bmatrix}$$

$$\Rightarrow e^A = I + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 1^2 & 0 \\ 0 & 2^2 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} e^1 & 0 \\ 0 & e^2 \end{bmatrix} \blacktriangle$$

e.g. 3.1.3

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Check that $A^3 = 0$
(A is a nilpotent matrix)
$\Longleftrightarrow$ all eigenvalues are 0

$$e^A = I + A + \frac{1}{2} A^2$$

$$= \begin{bmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$ ▲