Week 2

A1 due, A2 posted tomorrow

2.5. Properties of Limits

Definition

We say (a_n) is <u>bounded</u> if $\{a_n : n \in \mathbb{N}\}$ is bounded.
i.e. $|a_n| \leq M \quad \forall n \in \mathbb{N}$

Proposition: If $a_n \to L$ then (a_n) is bounded.

Proof: Suppose $a_n \to L$. For $\epsilon = 1$, there exists $N \in \mathbb{N}$ s.t.
$|a_n - L| < 1$ when $n \geq N$

$\therefore \; -1 < a_n - L < 1$

$\therefore \; -1 + L < a_n < 1 + L$

Take $m = \min\{a_1, a_2, \ldots, a_{N-1}, L-1\}$,
$M = \max\{a_1, a_2, \ldots, a_{N-1}, L+1\}$

$\therefore \; m \leq a_n \leq M \quad \forall n \in \mathbb{N}$

$\therefore \; (a_n)$ is bounded.

e.g. $(a_n) = (\sin(n)) \qquad (a_n) = (-1)^n$
bounded but not convergent (converse not true)

Proposition $a_n \to L$, $b_n \to M$

1) $a_n + b_n \to L + M$ $\quad$ *

2) $\alpha a_n \to \alpha L$, $\alpha \in \mathbb{R}$

3) $a_n b_n \to LM$

4) $\frac{a_n}{b_n} \to \frac{L}{M}$ if $b_n \neq 0$ for all n and $M \neq 0$

Proof: (1) Suppose $a_n \to L$ and $b_n \to M$

Let $\varepsilon > 0$ be given.

Since $a_n \to L$, $\exists N_1 \in \mathbb{N} \cdot n \geq N_1 \Rightarrow |a_n - L| < \boxed{\varepsilon/2}$

" $b_n \to M$, $\exists N_2 \in \mathbb{N} \cdot n \geq N_2 \Rightarrow |b_n - L| < \boxed{\varepsilon/2}$

Choose $N = \max\{N_1, N_2\}$. Suppose $n \geq N$.

$|a_n + b_n - (L+M)|$

$= |(a_n - L) + (b_n - M)|$

$\leq |a_n - L| + |b_n - M|$

$< \varepsilon$. $\blacksquare$

3) Suppose $a_n \to L$ and $b_n \to M$.

Since $a_n \to L$, (a_n) is bounded $\Rightarrow \exists k \in \mathbb{R}_{\geq 0} \cdot$

$|a_n| \leq k \ \forall n \in \mathbb{N}$. Let $\varepsilon > 0$ be given.

$\exists n_1 \in \mathbb{N}$ s.t. $n \geq n_1 \Rightarrow |a_n - L| < \boxed{\varepsilon/2|M|}$

$\exists n_2 \in \mathbb{N}$ s.t. $n \geq n_2 \Rightarrow |b_n - M| < \boxed{\varepsilon/2k}$

Choose $N = \max\{n_1, n_2\}$ and suppose $n \geq N$.

$\therefore |a_n b_n - LM|$

$= |a_n b_n - a_n M + a_n M - LM|$

$\leq |a_n b_n - a_n M| + |a_n M - LM|$

$$= |a_n(b_n - M)| + |M(a_n - L)|$$
$$= |a_n||b_n - M| + |M||a_n - L|$$
$$\leq k|b_n - M| + |M||a_n - L|$$
$$< \varepsilon.$$

<u>Note:</u> If $|M| = 0$, take $|a_n - L| < 1$ instead of $\frac{\varepsilon}{2|M|}$. [redacted]

Hint to prove (4) — first prove $\frac{1}{a_n}$ limit law then use product rule

<u>2.6. Monotone Convergence Theorem</u>

<u>Definition</u>

We say (a_n) is:

① Monotone increasing if $a_n \leq a_{n+1}$ $\forall n \in \mathbb{N}$

② strictly monotone increasing if $a_n < a_{n+1}$ $\forall n \in \mathbb{N}$

③ monotone decreasing if $a_n \geq a_{n+1}$ $\forall n \in \mathbb{N}$

④ strictly m. dec. if $a_n > a_{n+1}$ $\forall n \in \mathbb{N}$

<u>Theorem (Monotone Convergence Theorem):</u>

Every monotone increasing sequence that is bounded above converges. ↳ (to Sup) Same for mon. dec. seq. bounded below. ↳ (to Inf)

<u>Definition</u>

We say (a_n) is monotone if it is monotone increasing or monotone decreasing.

Proof next time.