4.4 Effect of Adding Poles and Zeros

4.4.1. Adding a stable pole

$$G_a(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \frac{1}{1+\tau s}, \quad \tau > 0 \text{ (augmented plant)}$$

The pole is located at $s = -\frac{1}{\tau} \in \mathbb{C}$.

As $\tau \to +\infty$, the new pole approaches $s = 0$

As $\tau \to 0$, pole approaches $-\infty$.

From G_a expr, we see that as the new pole $\to -\infty$ $(\tau \to 0)$ we recover the 2^{nd} order TF

Suppose $K = 1$ and rewrite $G_a(s)$ using PFD:

$$G_a(s) = \frac{a}{1+\tau s} + \frac{bs + c}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where $a = \dfrac{\tau^2\omega_n^2}{1 - 2\zeta\omega_n\tau + \omega_n^2\tau^2}$, $b = -\dfrac{a}{\tau}$, $c = \dfrac{\zeta\omega_n}{\tau}\left(\dfrac{1}{\tau\zeta\omega_n} - 2\right)a$

As $\tau \to \infty$ $(s \to 0)$, $b, c \to 0$ and the augmented system's response approaches a 1^{st}-order response e.g. 4.4.1 Consider the underdamped 2nd-order system

$$G(s) = \frac{8}{s^2+2s+4}$$

Here $\omega_n = 2$ and $\zeta = \frac{1}{2}$. Consider the augmented plant

$$G_a(s) = G(s)\frac{1}{1+\tau s} = \frac{8}{s^2+2s+4} \cdot \frac{1}{1+\tau s}$$

From Bode plots and step response:

- As added pole → origin (τ increases) the bandwidth decreases, the phase is more negative, and the step response is more sluggish.
- These effects become less prominent as the pole moves further into $\mathbb{C}^-$ (i.e. τ decreases)

4.4.2. Adding a left-half plane zero

$$G_a(s) = \frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \cdot (s\tau+1), \quad \tau > 0$$

zero at $s = \frac{-1}{\tau}$. As $\tau \to +\infty$, zero $\to s=0$.

$\tau \to 0 \Rightarrow$ zero $\to -\infty$. Let's assume $K=1$

and find the step response:

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{G_a(s)}{s} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} (\tau s+1) \frac{1}{s} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \cdot \frac{1}{s} \right\} + \tau \mathcal{L}^{-1}\left\{ \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \right\}$$

= step resp of $G(s)$ + $\tau \cdot$ impulse resp of $G(s)$

From this $\Rightarrow$ as zero $\to -\infty$ ($\tau \to 0$), $y(t)$ approaches standard 2nd order step response.
As zero $\to 0$ ($\tau \to +\infty$), the impulse response term becomes more noticeable in the step response.

<u>e.g. 4.4.2</u> $G(s) = \dfrac{8}{s^2+2s+4}$

$$G_a(s) = \frac{8(\tau s+1)}{s^2+2s+4}$$

<u>From Bode plots + Step Resp:</u>

- As minimum phase zero moves closer to origin, ($\tau \uparrow$), bandwidth increases, phase is more positive, and response gets faster but more oscillatory

- As zero $\to -\infty$ ($\tau \downarrow$), resp. approaches that of the prototype 2nd order system.

$\Rightarrow$ We try to keep zeros away from the origin.

4.4.3. Adding right-half-plane zero

Zeros in the CRHP are called <u>non-minimum phase</u>.

$$G_a(s) = \frac{K\omega_n^2}{\omega_n^2 + 2\zeta\omega_n s + s^2} \cdot (1 - \tau s), \quad \tau > 0$$

The added zero is at $s = \frac{1}{\tau}$. Generally, nonminimum phase zeros make the system hard to control. It slows the system down and causes the system to go in the wrong direction at first.

Suppose $K = 1$, step resp:

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s} \right\} - \tau \left\{ \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right\}$$

= Step resp of G $-$ (τ · imp. resp. of G.)

<u>e.g.</u> $G_a(s) = \frac{8(1 - \tau s)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

From Bode plots and Step Resp:

- As non min. phase $\emptyset \to$ origin ($\tau \uparrow$), bandwidth increases, phase is more negative. Step resp. goes further in the wrong direction near $t=0$

4.5 Dominant Poles and Zeros

In 4.4 we showed poles and zeros "far to the left" have little impact on the low frequency response of a system. Hence good low-order approximations can be found by appropriately neglecting less significant poles & zeros.

↳ ≥ 5 times further away from Im axis
non-dominant poles / zeros

e.g. 4.5.1. Model Reduction

$$G(s) = \frac{s+10}{(s+11)(s+12)(s^2+2s+2)} = \underbrace{\frac{s+10}{(s+11)(s+12)}}_{G_{fast}} \cdot \underbrace{\frac{1}{s^2+2s+2}}_{G_{slow}}$$

Approximate $G_{fast}(s)$ by steady state gain.

$$G_{fast}(0) = \frac{10}{132}$$

$$G(s) \approx \frac{10}{132} \frac{1}{s^2+2s+2} =: \hat{G}(s)$$

e.g. 4.5.2.

$$G(s) = 100 \frac{\left(1+\frac{1}{60}s\right)\left(1+\frac{1}{900}s\right)}{\left(1+\frac{s}{100}\right)\left(1+\frac{s}{500}\right)\left(1+\frac{s}{600}\right)\left(1+\frac{s}{1000}\right)}$$

Model the process well for $\omega \leq 600$.

$$\Rightarrow G_{fast}(s) = \frac{1+\frac{s}{900}}{1+\frac{s}{1000}}$$

$$\Rightarrow \hat{G}(s) = 100 \frac{1+\frac{1}{60}s}{\left(1+\frac{1}{100}s\right)\left(1+\frac{1}{500}s\right)\left(1+\frac{1}{600}s\right)}$$