$$\frac{dx}{dt} = f(x(t), u(t)), \quad x(t) \in \mathbb{R}^n \text{ (state)}$$
$$u(t) \in \mathbb{R}^m \text{ (input)}$$

$$y(t) = h(x(t), u(t)), \quad y(t) \in \mathbb{R}^p \text{ (output)}$$

↳ standard form of state-space equations

→ linear special case

$$\frac{dx}{dt} = A\overset{\mathbb{R}^n}{x(t)} + B\overset{\mathbb{R}^m}{u(t)} \qquad A \in \mathbb{R}^{n\times n}, \; B \in \mathbb{R}^{n\times m},$$
$$C \in \mathbb{R}^{p\times n},$$

$$y(t) = Cx(t) + Du(t) \qquad D \in \mathbb{R}^{p\times m}$$

- quadrotor

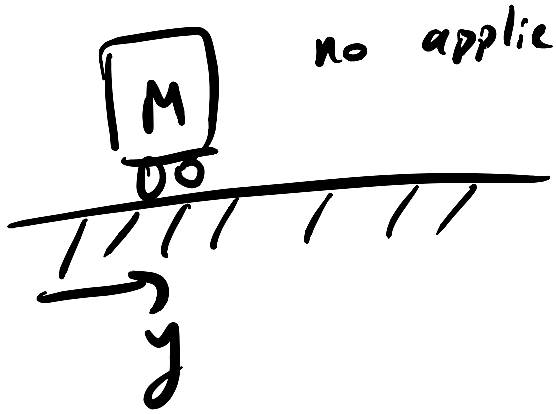
  $m = 4$ (forces on rotors)
  $p = 3$ (position of UAV)
  $n = 12$ (position, velocity, angular velocity)

<u>What is the state of a system?</u>

The state vector $x(t_0)$ encapsulates all of a system's dynamics up to time $t_0$.

Specifically: For any times $t_0 < t_1$, knowing the state at time $t_0$ and knowing $u(t)$ for $t_0 \le t \le t_1$, we can compute $x(t_1)$ and hence $y(t_1)$.

<u>e.g. 2.4.2</u>




no applied force or drag

$$M\ddot{y} = 0$$

If we try a 1-dimensional state, say $x := y \in \mathbb{R}$.
In this case, knowing $x(t_0)$ without knowing $\dot{y}(t_0)$, we cannot compute $x(t_1)$ for $t_1 > t_0$.
Same issue if we tried $x = \dot{y} \in \mathbb{R}$.

Since the governing equation is second-order, we need at least 2 initial conditions

$x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \in \mathbb{R}^2$ is the natural state.

e.g. 2.5.1.


$\downarrow g$ (gravity)

(see Ex 2.3.3)
The model of this system is:

$$\ddot{\theta} = \frac{3}{M\ell^2}\tau - 1.5\frac{g}{\ell}\sin\theta$$

Put this system into state-space form:

Take 1) $x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$ 2) $u = \tau$

3) $y = \theta$

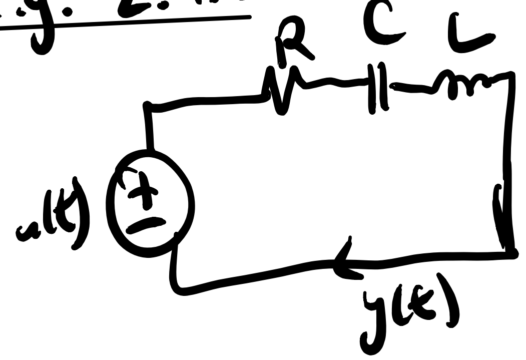
$\dot{x}_1 = x_2$

$\dot{x}_2 = \frac{3}{Ml^2} u - 1.5 \frac{g}{l} \sin x_1$

$y = x_1$ → Non linear system

Model can't be written $y = Cx + Du$ $\Delta$

<u>e.g. 2.4.5</u>




Choose $x_1 =$ Voltage across cap. $= \frac{1}{C} \int_0^t y(\tau) d\tau$

$x_2 =$ current through inductor $= y(t)$

$-u(t) + V_R(t) + V_C(t) + V_L(t) = 0$

$-u(t) + y(t) R + x_1 + L\dot{x}_2 = 0$

Capacitor dynamics: $\dot{x}_1 = \frac{1}{C} x_2$

Combine equations:

$$\left.\begin{aligned} \dot{x}_1 &= \frac{1}{C} x_2 \\ \dot{x}_2 &= -\frac{1}{L} x_1 - \frac{R}{L} x_2 + \frac{1}{L} u \\ y &= x_2 \end{aligned}\right\}$$

state-space model

→ linear system with:

$$A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad D = 0$$

## 2.5. Linearization

- In this course we will always linearize nonlinear models.
- This refers to the process of approximating a non-linear system with a linear model.

e.g. 2.5.2. Linearize $y = x^3$ at $\bar{x} = 1$.

Let $\bar{y} = (\bar{x})^3 = 1$.

Taylor series of $f(x) = x^3$ at $\bar{x} = 1$:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-\bar{x})^n \quad , \quad c_n = \frac{1}{n!} \frac{d^n f}{dx^n}\bigg|_{x=\bar{x}}$$

$$f(x) = f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + \text{higher order terms}$$

Keep only the terms for $n = 0, 1$:

$$f(x) \approx f(\bar{x}) + f'(\bar{x})(x-\bar{x})$$
$$= 1 + (3 \cdot 1^2)(x-1)$$
$$= 1 + 3x - 3 = 3x - 2$$

$$y - \bar{y} \approx \frac{df}{dx}\bigg|_{x=\bar{x}} (x - \bar{x})$$

$$\Delta y = \frac{df}{dx}\bigg|_{x=\bar{x}} \Delta x$$

$\Delta y = 3 \Delta x$ in our example

<u>e.g. 2.5.3</u> $\quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = f(x) = \begin{bmatrix} x_1 x_2 - 1 \\ x_3^2 - 2x_1 x_3 \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}$

$f: \mathbb{R}^3 \to \mathbb{R}^2$

Linearize at $\bar{x} = (1, -1, 2) \Rightarrow \bar{y} = f(\bar{x}) = (-2, 0)$