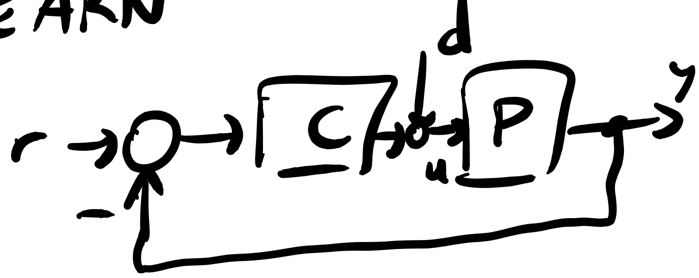
Lecture 17 Summary

- pendulum example
  - files on LEARN
- feedback stability (interconnected systems)



- internal stability
  - set exogenous signals to zero
  - find closed-loop state model $\dot{x}_{cl} = A_{cl} x_{cl}$
- feedback system is stable if $\dot{x}_{cl} = A_{cl} x_{cl}$ is A.S.

e.g. (cont'd)

$$x_{cl} = \begin{bmatrix} x_p \\ x_c \end{bmatrix} \in \mathbb{R}^3$$

$$\dot{x}_{cl} = \left[\begin{array}{c|c} \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}(100)\begin{bmatrix} 1 & 0 \end{bmatrix} & -\begin{bmatrix} 0 \\ 1 \end{bmatrix} 1000 \\ \hline -\begin{bmatrix} 1 & 0 \end{bmatrix} & -15 \end{array}\right] x_{cl}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -100 & -5 & -1000 \\ -1 & 0 & -15 \end{bmatrix} x_{cl}$$

- check eigenvalues

$$|sI - A_{cl}| = \begin{vmatrix} s & -1 & 0 \\ 100 & s+5 & 1000 \\ 1 & 0 & s+15 \end{vmatrix}$$

$$= s \begin{vmatrix} s+5 & 1000 \\ 0 & s+15 \end{vmatrix} + \begin{vmatrix} 100 & 1000 \\ 1 & s+15 \end{vmatrix}$$

$$= s(s+5)(s+15) + 100(s+15) - 1000$$

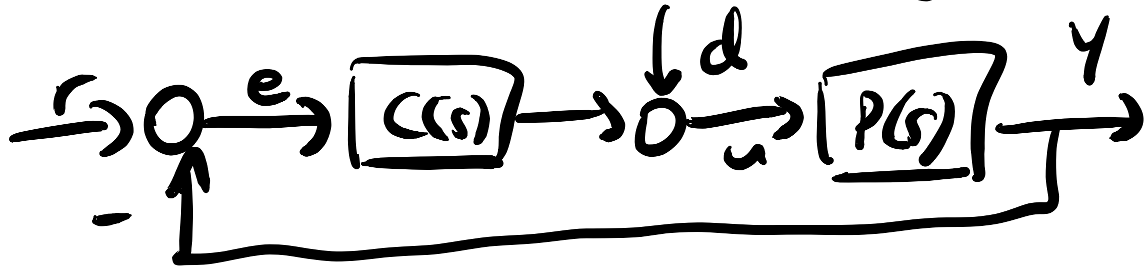
$$= (s+5)(s^2 + 15s + 100)$$

- eigenvalues of A_{cl}: $-5,\ -\frac{15}{2} \pm \frac{1}{2}\sqrt{15^2 - 400}$

and they all have negative real part

$\Rightarrow$ closed-loop system is internally stable

5.2.2. Input-output stability



System has 6 TFs from the independent signals (r, d) to (e, u, y).

$$E = R - PU$$
$$U = D + CE$$

$$\begin{bmatrix} 1 & P \\ -C & 1 \end{bmatrix} \begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} R \\ D \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} E \\ U \end{bmatrix} = \frac{1}{1+PC} \begin{bmatrix} 1 & -P \\ C & 1 \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}$$

So the 6 TFs are:

R to E: $\frac{1}{1+PC}$ D to E: $\frac{-P}{1+PC}$

R to U: $\frac{C}{1+PC}$ D to U: $\frac{1}{1+PC}$

R to Y: $\frac{PC}{1+PC}$ D to Y: $\frac{P}{1+PC}$

$$Y(s) = P(s)U(s) = \frac{PC}{1+PC} R + \frac{P}{1+PC} D$$

<u>Definition</u>: The feedback system is I.O.

stable if (e, u, y) are bounded whenever (r, d) are bounded.

<u>e.g. 5.2.5</u>

assume input distribance!

$$P(s) = \frac{1}{(s+1)(s-1)}, \quad C(s) = \frac{s-1}{s+1}$$

The 4 TFs from (r, d) to (e, u) are:

$$\begin{bmatrix} E(s) \\ U(s) \end{bmatrix} = \begin{bmatrix} \frac{(s+1)^2}{s^2+2s+2} & \frac{s+1}{(s-1)(s^2+2s+2)} \\ \frac{(s+1)(s-1)}{s^2+2s+2} & \frac{(s+1)^2}{s^2+2s+2} \end{bmatrix} \begin{pmatrix} R(s) \\ D(s) \end{pmatrix}$$

- 3 of these TFs are BIBO stable, the one from D to E is not.

$\Rightarrow$ system is not I/O stable

Observe: $\frac{Y(s)}{R(s)} = \frac{PC}{1+PC} = \frac{1}{s^2+2s+2}$ is BIBO stable.

- <u>Problem:</u> C cancels an unstable pole of P; not allowed.

- Write $P(s) = \frac{N_p}{D_p}$, $C(s) = \frac{N_c}{D_c}$.

- N_p, N_c, D_p, D_c are polynomials in s.

- $\deg(N_p) < \deg(D_p)$
- $\deg(N_c) \leq \deg(D_c)$

<u>Definition</u>: The <u>characteristic polynomial</u> of the feedback system is defined to be

$$\pi(s) = D_p D_c + N_p N_c$$

<u>e.g. 5.2.7.</u>

$$\begin{aligned}\pi(s) &= (s+1)^2(s-1) + 1(s-1) \\ &= (s-1)\left((s+1)^2 + 1\right) \\ &= (s-1)(s^2 + 2s + 2) \quad \blacktriangle\end{aligned}$$

The characteristic. poly. has a root at $s=1$, the pole we tried to cancel.

<u>Theorem 5.2.6</u>: The feedback system is I.O. stable iff its c.p. has no roots with $Re(s) \geq 0$.

<u>Proof</u>

($\Leftarrow$) Assume all roots of π are in $\mathbb{C}^-$.

Observe that

$$\begin{bmatrix} \frac{1}{1+PC} & \frac{-P}{1+PC} \\ \frac{C}{1+PC} & \frac{1}{1+PC} \end{bmatrix} = \frac{1}{\pi(s)} \begin{bmatrix} D_p D_c & -N_p D_c \\ N_c D_p & D_p D_c \end{bmatrix}$$

- So all 4 TFs are BIBO stable. (Thm 3.5.9)
- So by definition, the system is I/O stable.

($\Rightarrow$) Assume the system is I.O. stable

$\Rightarrow$ The 4 TFs above are all BIBO stable

To conclude that π has no bad roots, it suffices to show that $\pi(s)$, $D_p D_c$, $N_p D_c$, $N_c D_p$ are coprime. (Prove it! <u>Hint</u>: By assumption $N_p D_p$ and $N_c D_c$ are coprime.) $\blacksquare$