Fact: $\sim$ is an equivalence relation.

Def^n: Let X be the set of decimal expansions. Let $\mathbb{R}$ be the set of equivalence classes of X under $\sim$.

e.g. $0.72\bar{9} \in X$

$[0.72\bar{9}] = \{0.72\bar{9}, 0.73\}$

Convention: Avoid repeated 9s

We write 0.73 instead of $[0.72\bar{9}]$ or $[0.73]$.

To visualize $\mathbb{R}$ as a line we have two problems.

Ⓐ Orderability

Ⓑ Completeness (no gaps)

Problem A:

Take $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}_{\geq 0}$. Define $\frac{a}{b} < \frac{c}{d}$

$\Leftrightarrow ad < cb$

Let $q_1, q_2 \in \mathbb{Q}_{<0}$. Define $q_1 < q_2 \Leftrightarrow -q_1 > -q_2$

Finally, if $q_1 > 0$ and $q_2 < 0$, $q_1 > q_2$.

Definition: Let $x = a_0 . a_1 a_2 \ldots \in \mathbb{R}$ and

$y = b_0 . b_1 \cdots \in \mathbb{R}$. We say $x < y$ if $\exists N \geq 0$ s.t. $a_i = b_i \; \forall i < N$ and $a_N < b_N$.

e.g. $x = 27.3268\overline{4}$, $y = 27.32691$, $x < y$

Prop. (Archimedean Property): If $x, y \in \mathbb{R}_+$. Then $\exists n \in \mathbb{N}$ s.t. $nx > y$

Proof: Consider $\frac{y}{x} \in \mathbb{R}$. Since there is no largest natural number, $\exists n \in \mathbb{N}$ s.t. $n > \frac{y}{x}$
$\Rightarrow nx > y$.

HW: Use this prop. to show that $\forall \varepsilon > 0 \; \exists k \in \mathbb{N}$ s.t. $10^{-k} < \varepsilon$. (Hint: work with 10^k)

Problem B: (Completeness)

Prop: Let $x \in \mathbb{R}$. Let $\varepsilon > 0$. $\exists q \in \mathbb{Q}$ s.t. $|x - q| < \varepsilon$.

Proof: Let $\varepsilon > 0$ be given. Say $x = a_0 . a_1 a_2 \cdots$.
Take $k \in \mathbb{N}$ s.t. $10^{-k} < \varepsilon$. Let $q = a_0 . a_1 a_2 \cdots a_k \in \mathbb{Q}$.

$$\begin{aligned} |x - q| &= |0.000 \cdots 0 a_{k+1} \cdots| \\ &= 0.000 \cdots 0 a_{k+1} \cdots \\ &\leq 0.000 \cdots 1 \\ &= 10^{-k} \\ &< \varepsilon. \quad \blacksquare \end{aligned}$$

2.3. Least Upper Bound Principle

Definition

A set $X \subseteq \mathbb{R}$ is bounded above if $\exists M \in \mathbb{R}$ s.t. $x \leq M$ for all $x \in X$. We call M an upper bound for X.

Moreover, a set $Y \subseteq \mathbb{R}$ is bounded below if $\exists M \in \mathbb{R}$ s.t. $M \leq y$ for all $y \in Y$.

↳ lower bound.

A set is said to be bounded if it is both bounded above and below.

Definition

Suppose $X \neq \emptyset$ is bounded above.

An upper bound L is the least upper bound for X if whenever M is an upper bound for X,

$M \geq L$

Notation: $L = \sup X$

(Supremum)

Similar mirror defn for greatest lower bound,

$\inf(X)$.

(Infemum)