<u>Recall:</u> D_4 - dihedral group,
- non-abelian
- 8 elements $(|D_4| = 8)$

- $D_n = \{\text{symmetries of regular } n\text{-gon}\}$ has order $2n$

<u>Order of an element in a group:</u>

Let $(G, \cdot)$ be a group and let $a \in G$ arbitrary.
Then, for $k \in \mathbb{Z}$, a^k is defined as follows:

$$a^k = \begin{cases} \underbrace{a \cdot a \cdot \ldots \cdot a}_{k \text{ times}} & k > 0 \\ e & k = 0 \\ \underbrace{a^{-1} \cdot a^{-1} \ldots a^{-1}}_{k \text{ times}} & k < 0 \end{cases}$$

$a^k \in G$ by closure and associativity

<u>Exercise:</u> Proof laws of exponents hold in G

$a \in G$, $m, n \in \mathbb{Z}$

$$a^m \cdot a^n = a^{m+n}$$
$$(a^n)^{-1} = a^{-n} = (a^{-1})^n$$

<u>Definition (order of an element):</u> Let G be a group. Let $a \in G$. Then the order of a, denoted $\text{ord}(a)$ is the smallest positive integer k such that $a^k = e$. If such a k does not exist, we say $\text{ord}(a) = \infty$

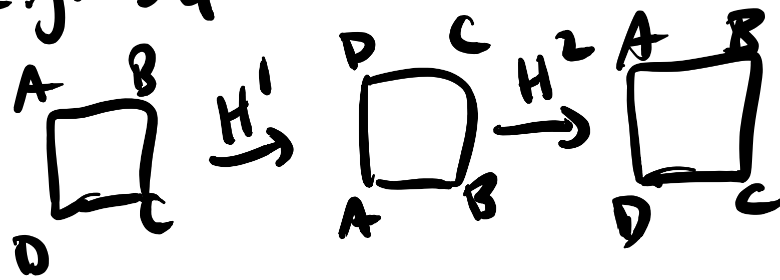
(order of a is infinite)

e.g. $U(12) = \{1, 5, 7, 11\}$ $\quad 5^1 = 5$

$5^2 = 1$

$\mathrm{ord}(5) = 2$

e.g. $D_4 \ni H$.



$= R_0 = e$

$\Rightarrow \mathrm{ord}(H) = 2$

<u>Recall $(\mathbb{Z}, +)$</u>:

No k s.t. $2^k = 0$ since $\underbrace{2 + \cdots + 2}_{k \text{ times}} > 0$

Symmetric Groups

S_n for $n \geq 3$ $\quad$ focus on S_3 today

<u>Definition (Permutation of a set)</u>

Let $A \neq \emptyset$. A permutation σ of/on A is a bijection on A, $\sigma: A \to A$.

Suppose A is a finite set; take $|A| = 4$

$A = \{1, 2, 3, 4\}$

$\alpha : A \to A$

$\alpha(1) = 2 \quad \alpha(2) = 1 \qquad \alpha(3) = 3 \qquad \alpha(4) = 4$

also
$\alpha(\circ) = \circ$ (map everything to itself)

Let A be a set with $|A| = n$, $\{1, 2, \dots, n\}$.
Then $S_n = \{\sigma \mid \sigma$ is a permutation of $A\}$.

$$= \{\sigma \mid \sigma : \{1, \dots, n\} \to \{1, \dots, n\} \cdot \sigma \text{ is a permutation}\}$$

Consider S_3.
$|S_3| = ?$

$\Rightarrow |S_3| = 3! = 6$

$1 \to 3$ choices
$2 \to 2$ choices
$3 \to 1$ choice

defined as $n \cdot (n-1) \cdots 2 \cdot 1$

In general $|S_n| = n!$

S_3, group operation: Composition of functions

$\alpha \circ \beta$ is a bijection

$(S_3, \circ)$ is closed

identity function : identity permutation

$$\epsilon = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

Let $\alpha = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ α^3 is the identity ϵ

(b) $\text{ord}(\alpha) = 3$

$\alpha \cdot \alpha = \alpha^2 \in S_3$

$= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

$S_3 = \{\epsilon, \alpha, \alpha^2, \beta, \alpha \cdot \beta, \beta \cdot \alpha\}$

$\beta = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} \in S_3$

$\alpha \cdot \beta = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} \in S_3$

$\beta \cdot \alpha = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \neq \alpha \cdot \beta \Rightarrow$ group is non-abelian

$\in S_3$

$\text{ord}(\alpha) = 3$ $\text{ord}(\beta \cdot \alpha)$

$\text{ord}(\beta) = 2$

All $S_n \to$ Non-abelian group