Lec 12 Summary

- BIBO stability: G proper ; G improper
  - ↳ check poles ; ↳ unstable, e.g. $\frac{1}{s}$

- $\dot{x} = Ax + Bu$, $y = Cx + Du$ asymp. stable $\Rightarrow$ $G(s) = C(sI-A)^{-1}B + D$ BIBO stable

  eigs $A \supseteq$ poles G

- F.V.T. $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$ if $sF(s)$ has no bad poles

Theorem 3.6.2

If G is BIBO stable and $u(t) = b\mathbb{1}(t)$ then $y_{ss} = b \cdot G(0)$

Proof: If $u(t) = b\mathbb{1}(t)$, then

$$y_{ss} = \int_0^\infty g(\tau)\, u(t-\tau)\, d\tau$$

$$= b \int_0^\infty g(\tau)\, d\tau$$

$$= b \int_0^\infty g(\tau)\, e^{-0\tau}\, d\tau$$

Since G is BIBO-stable, all its poles are in the open left-half plane so $s=0$ is in the ROC of $G(s)$

$\Rightarrow y_{ss} = b \cdot G(0)$

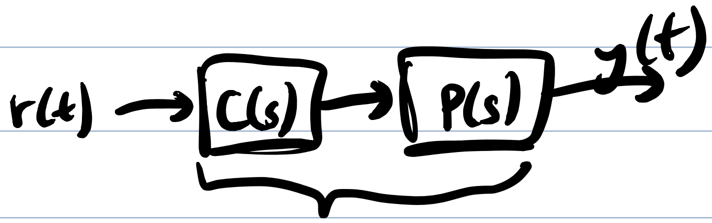
<u>Conclusion:</u> steady-state gain: $\frac{y_{ss}}{b} = G(0)$.
(LTI system)

<u>e.g.</u> $\dot{x} = -2x + u$
$y = x$
$x \in \mathbb{R}$

<u>Problem:</u> Given a constant reference $r(t) = r_0 \mathbb{1}(t)$ for y, find a controller that makes $y \to r$ as $t \to \infty$

Try an open-loop solution for now.

$$Y(s) = \underbrace{\frac{1}{s+2}}_{P(s)} U(s)$$



<u>Idea:</u> pick C s.t. system has s.s. gain of 1.

$$Y(s) = \frac{1}{s+2} C(s) \frac{r_0}{s}$$

$$y_{ss} = \lim_{t \to \infty} y(t) = \lim_{s \to 0} \frac{C(s)}{s+2} r_0 \quad \text{iff } C(s) \text{ is BIBO stable}$$

so $y_{ss} = \frac{1}{2} C(0) \cdot r_0$

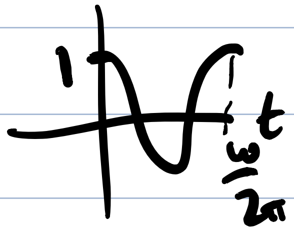
We pick $C(s) = 2 = \frac{1}{P(0)}$ (pure gain) $\triangle$

$C(s) = 2 \cdot \frac{3s+1}{4s+1}$ also works.

3.7. Frequency Response

$Y(s) = G(s) U(s)$, G BIBO stable

If $u(t) = \cos(\omega t)$, $\omega \in \mathbb{R}$, what is the steady-state output?



$\rightarrow \boxed{G(s)} \rightarrow y$

Theorem 3.7.1.

The steady-state response to $u(t) = \cos(\omega t)$ is
$y(t) = |G(j\omega)| \cos(\omega t + \angle G(j\omega))$

e.g. $\dot{x} = -10x + u \Rightarrow Y(s) = \underbrace{\frac{1}{s+10}}_{G(s)} U(s)$

$y = x$

$x \in \mathbb{R}$

Find s.s. output when $u(t) = 2\cos(3t + \frac{\pi}{6})$

- system is BIBO-stable so we can apply Thm 3.7.1

- $y(t) = A\cos(3t + \frac{\pi}{6} + \phi)$

$A = |G(j3)| \qquad \phi = \angle G(j3)$

$G(j\omega) = \frac{1}{j\omega + 10}$

$|G(j3)| \stackrel{=}{=} \left|\frac{1}{3j+10}\right| = \frac{1}{(9+100)^{1/2}} \approx 0.1$

$\angle G(j3) = \angle \frac{1}{3j+10} = 0 - \arctan(\frac{3}{10})$

$\approx -0.2915 \text{ rad} \approx -16.7^\circ$

ss output : $y(t) = 2 \cdot 0.1\cos(3t + \frac{\pi}{6} - 0.2915)$ $\triangle$

Definition 3.7.2 Assume G is BIBO-stable.

(a) the function $\mathbb{R} \to \mathbb{C}$, $\omega \mapsto G(j\omega)$ is the <u>frequency response</u>.

(b) The function $\mathbb{R} \to \mathbb{R}$, $\omega \mapsto |G(j\omega)|$ is the <u>magnitude response</u>.

(c) The function $\mathbb{R} \to (-\pi, \pi]$, $\omega \mapsto \angle G(j\omega)$ is the <u>phase</u> response

3.8. Graphical representations of the frequency response

When we graphically represent $G(j\omega)$, we will only consider $\omega \geq 0$, since for rational transfer functions, $|G(j\omega)| = |G(-j\omega)|$, and $\angle G(j\omega) = -\angle G(-j\omega)$.

Bode plot

(1) Magnitude plot

$20\log|G(j\omega)|$ vs. $\log \omega$

(2) Phase plot

$\angle G(j\omega)$ vs $\log \omega$

usually in degrees

Polar Plot

$\mathrm{Re}(G(j\omega))$ vs $\mathrm{Im}(G(j\omega))$ as functions of ω