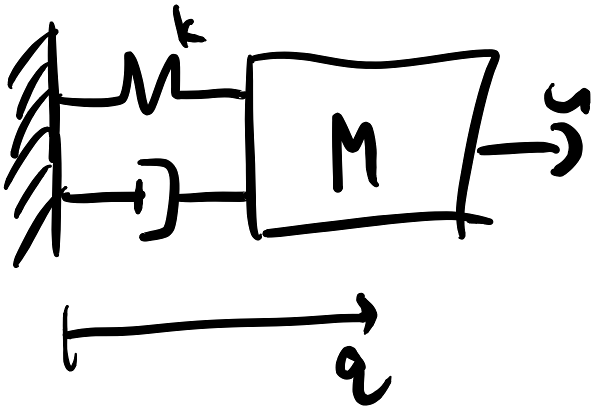



$q \in \mathbb{R}$, position of mass

$M$, mass in kg

$u$, applied force

$q=0$ at spring equilibrium pos.

<u>e.g. 2.1.1 (mass - spring - damper)</u>

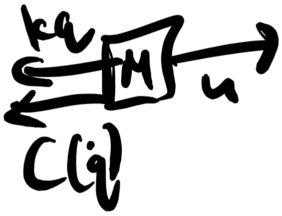
<u>Newton's 2nd Law:</u>

$$M\ddot{q} = \sum F$$

spring: $kx$ (linear)

damper: possibly nonlinear, models friction

$$C(\dot{q})$$

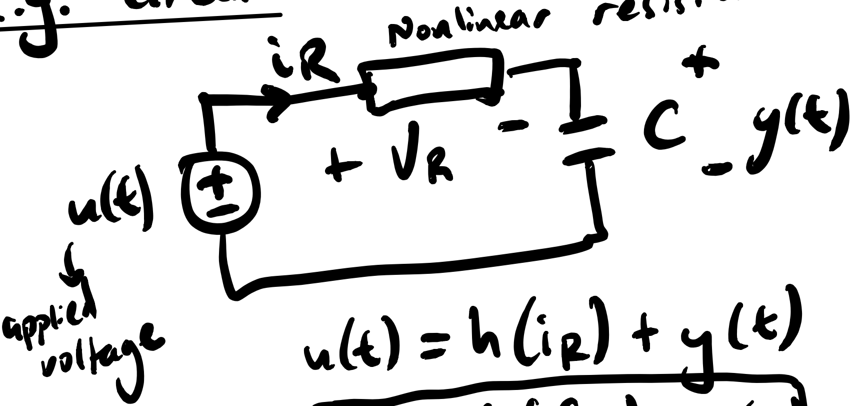



$$M\ddot{q} = u - kq - C(\dot{q})$$

$$M\ddot{q} + kq + C(\dot{q}) = u \rightarrow 2^{nd} \text{ order ODE}$$

If the damper is linear (i.e. $C(\dot{q}) = b\dot{q}$, $b$ const.), the ODE (and system) becomes linear. $\triangle$

<u>e.g. circuit</u>




$V_R = h(i_R)$

$\hookrightarrow h : \mathbb{R} \rightarrow \mathbb{R}$

KVL

$$u(t) = h(i_R) + y(t)$$

$$\boxed{u(t) = h(C\dot{y}) + y(t)}$$

If resistor was linear, 1st order ODE would be linear

$$(h(i) = Ri)$$

Compare w/ e.g. 2.3.4

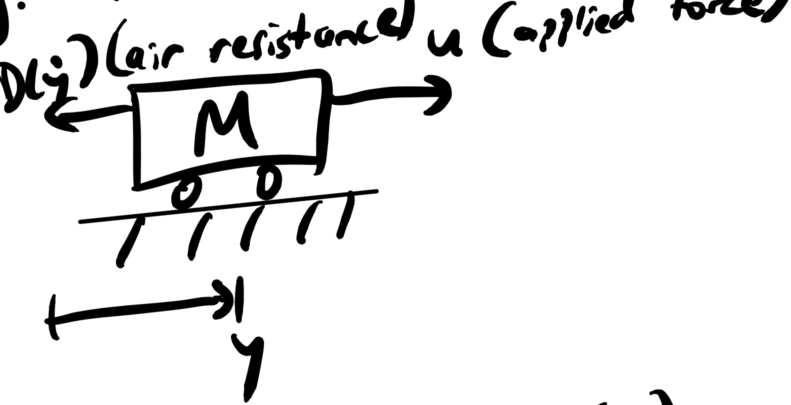
Comments:

- See 2.3 in notes for examples
- expectation is that we can model very simple mechanical and electrical systems. (1Q on MT)

## 2.4 State Space Models

State-space models are a way of expressing mathematical models in a standard form.

e.g. 2.4.1.




Newton: $M\ddot{y} = u - D(\dot{y})$

Put this model into standard form by defining two so-called <u>state variables</u>:

$$\text{State vars} \begin{cases} x_1(t) := y(t) & \text{(position)} \\ x_2(t) := \dot{y}(t) & \text{(velocity)} \end{cases}$$

We can write this system as:

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{M}u - \frac{D(x_2)}{M} \\ y &= x_1 \end{aligned}\right\} \begin{aligned} &\text{2 ODEs = state equation} \\ &\text{algebraic part = output equation} \end{aligned}$$

These equations have the form $\dot{x} = f(x, u)$
$y = h(x)$

non-linear state space model

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ $f(x,u) = \begin{bmatrix} x_2 \\ \frac{u}{M} - \frac{D(x_2)}{M} \end{bmatrix}$, $h(x) = x_1$

The function $f$ is linear if $D$ is, $h$ is linear.

As a special case, suppose force due to air resistance were linear.

i.e. $D(x_2) = d x_2$, $d$ constant.

Then $f$ becomes linear and can be written

$$f(x,u) = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\frac{d}{M} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}}_{B} u$$

and we get the linear special case

$$\boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}}, \quad C = [1 \quad 0]$$

LTI state space model

Generalizing this example, we can say that an important class of systems have models of the form:

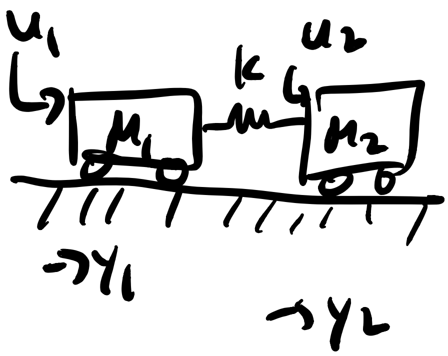
$$\dot{x} = f(x,u) \qquad f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$$
$$y = h(x,u) \qquad h: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$$

The model is nonlinear, there are $m$ control inputs $u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \in \mathbb{R}^m$. There are $p$ outputs $[y_1, \dots, y_p] \in \mathbb{R}^p$

and there are $n$ state variables, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

e.g. previous example $n=2 \quad m=1 \quad p=1$

e.g.

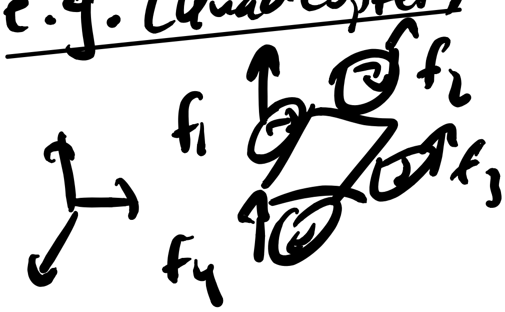



$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2 \Rightarrow m=2$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2 \Rightarrow p=2$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \Rightarrow n=4$$
$$:= \begin{bmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \end{bmatrix}$$

e.g. (Quadcopter)




Claim:

4 inputs (thrusts) $\quad m=4$

3 outputs $\quad p=3$

12 states $\quad n=12$