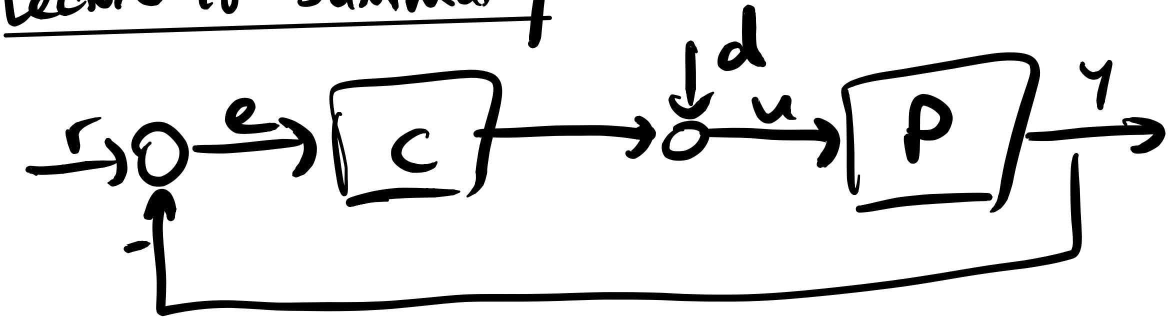
Lecture 18 Summary



I.O. stability (r, d) bounded $\Rightarrow (e, u, y)$ bounded
$(y = r - e)$

- above system has 6 TFs   IO stability $\Leftrightarrow$ all TFs are BIBO stable

- characteristic polynomial $\pi(s) = N_P N_C + D_P D_C$

$$P = \frac{N_P}{D_P}, \quad C = \frac{N_C}{D_C}$$

I.O. stability $\Leftrightarrow$ all roots of π have $Re(s) < 0$

<u>e.g.</u> pendulum revisited

$$P(s) = \frac{1}{s^2 - 1} \qquad C(s) = \frac{s+10}{s+20} \cdot 100$$

$$\pi(s) = 100(s+10) + (s^2-1)(s+20)$$
$$= s^3 + 20s^2 + 99s + 980$$

roots are -17.5 and $-1 \pm j7.3$
$\Rightarrow$ feedback loop is I.O. stable $\triangle$

Definition

The plant P and controller C have a pole-zero cancellation if there is a complex $\lambda \in \mathbb{C}$ s.t. $N_P(\lambda) = D_c(\lambda) = 0$

or $D_P(\lambda) = N_c(\lambda) = 0$

It's an unstable cancellation if $Re(\lambda) \geq 0$.

Corollary 5.2.8.

If there is an unstable pole-zero cancellation, then the feedback control system is not I.O. stable

Proof: Suppose there is an unstable pole-zero cancellation at λ, $Re(\lambda) \geq 0$.

$$\pi(\lambda) = N_c(\lambda)N_P(\lambda) + D_c(\lambda)D_P(\lambda) = 0 + 0 = 0$$

So π has a root in $\overline{\mathbb{C}^+}$ and the feedback system is not I.O. stable.

5.2.3. Internal stability and I.O. stability

It can be shown that the roots of $\pi(s)$ $\subseteq$ eigenvalues of A_{CL}.

So, internal stability $\Rightarrow$ I.O. stability

Converse is generally false but holds for many practical situations.

5.3. Routh-Hurwitz Criterion

Consider an n^{th} order poly. $\pi(s) = s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0$

Definition: $\pi(s)$ is <u>Hurwitz</u> if all its roots have $Re(s) < 0$.

- the R-H Criterion is a test that tells us if a poly. is Hurwitz without finding its roots.

5.3.1. A necessary condition for a poly to be Hurwitz

- let $\pi(s)$ have real roots $\{\lambda_1, \dots, \lambda_r\}$. and complex conjugate roots $\{\mu_1, \bar{\mu}_1, \dots, \mu_s, \bar{\mu}_s\}$.

$r + 2s = n$

- so $\pi(s) = (s-\lambda_1)\cdots(s-\lambda_r)(s-\mu_1)(s-\bar{\mu}_1)\cdots(s-\mu_s)(s-\bar{\mu}_s)$
- If π is Hurwitz, that means $-\lambda_i > 0$.
- For the complex conjugate roots:

$$(s-\mu_i)(s-\bar{\mu}_i) = s^2 + (-\mu_i - \bar{\mu}_i)s + \mu_i\bar{\mu}_i$$

$$= s^2 - 2\mathrm{Re}(\mu_i)s + |\mu_i|^2$$

- So, combining these, we see that when we expand $\pi(s)$ again, all the coefficients $a_i > 0$

<u>Proposition</u>

π is Hurwitz $\Rightarrow$ $a_i > 0$ for all $i = 1, \dots, n-1$

e.g. 5.3.1

$s^4 + 3s^3 - 2s^2 + 5s + 6$ Not Hurwitz

$s^3 + 4s + 6$ Not Hurwitz

$s^3 + 5s^2 + 9s + 1$ don't know yet $\triangle$

5.3.2 Routh's Algorithm

$\pi(s) = s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0$ ← populated from input data

s^n	1	a_{n-2}	a_{n-4}	a_{n-6}	$\dots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\dots$
s^{n-2}	$r_{2,0}$	$r_{2,1}$	$r_{2,2}$	$r_{2,3}$	$\dots$
s^{n-3}	$r_{3,0}$	$r_{3,1}$	$r_{3,2}$	$r_{3,3}$	$\dots$
$\vdots$	$\vdots$				
s^2	$r_{n-2,0}$	$r_{n-2,1}$	$\dots$		
s^1	$r_{n-1,0}$	$\dots$	$\ddots$		
s^0	$r_{n,0}$	$\dots$			

$$r_{2,0} := \frac{a_{n-1} \cdot a_{n-2} - (1)\, a_{n-3}}{a_{n-1}}$$

$$r_{2,1} := \frac{a_{n-1} \cdot a_{n-4} - (1)\, a_{n-5}}{a_{n-1}}$$

$$r_{2,2} := \frac{a_{n-1} \cdot a_{n-6} - (1)\, a_{n-7}}{a_{n-1}}, \text{ etc.}$$

4th row computed from the second and third rows using the same pattern.

$$r_{3,0} := \frac{r_{2,0} \cdot a_{n-3} - a_{n-1} \cdot r_{2,1}}{r_{2,0}}, \text{ etc.}$$

Continue along each row until you get zeros
Terminate early if we ever get a 0 in the 1st columns

Theorem 5.3.3. (Routh-Hurwitz Criterion)

(1) $\pi(s)$ is Hurwitz iff all elements in the first column have the same sign.

(2) If there are no 0s in the first

column, then the # of sign changes in the first column = # of bad roots and no roots on $j\mathbb{R}$