Summary Lec 10

- relationship between poles of $\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ and step response

- Read Sec 4.4 - 4.5

- Ch. 3: $\dot{x} = Ax + Bu$ or $Y(s) = G(s)U(s)$
$y = Cx + Du$

- $\frac{dx}{dt} = Ax(t)$ $\quad x(0) = x_0 \in \mathbb{R}^n$ Solve for $x(t)$

- motivated by scalar case $(n=1)$ we defined $e^A := I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots$

Replace A with tA in the above:

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \dots = \sum_{k=0}^{\infty} \frac{t^k}{k!}A^k$$ (state transition matrix)

Theorem 3.1.2.

The unique solution to $\dot{x} = Ax$ for some $x(0) = x_0 \in \mathbb{R}^n$ is $x(t) = e^{tA}x_0$

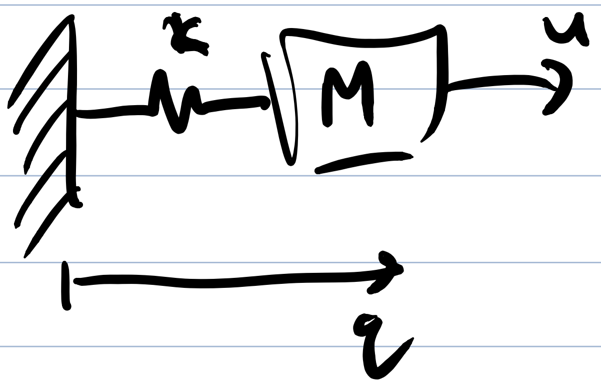
Take LT of $\dot{x} = Ax$ without assuming $x(0) = 0$.

$sX(s) - x_0 = AX(s)$

$\Rightarrow X(s) = (sI-A)^{-1} x_0$

$\Rightarrow e^{tA}$ and $(sI-A)^{-1}$ are LT pairs.

e.g. mass-spring



$$M\ddot{q} = u - kq$$

$$x := \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \quad y := q$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} u$$

$$y = [1 \; 0] x$$

Find $x(t)$ when $k=4$, $M=1$, $u=0$, and $x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$sI - A = \begin{bmatrix} s & -1 \\ 4 & s \end{bmatrix} \quad (sI-A)^{-1} = \begin{bmatrix} s & 1 \\ -4 & s \end{bmatrix} \cdot \frac{1}{s^2+4}$$

Use table to get:

$\mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\}$

$= e^{tA} = \begin{bmatrix} \cos(2t) & \frac{1}{2}\sin(2t) \\ -2\sin(2t) & \cos(2t) \end{bmatrix}$

$x(t) = \begin{bmatrix} \cos 2t & \frac{1}{2}\sin 2t \\ -2\sin 2t & \cos 2t \end{bmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$= \begin{bmatrix} 3\cos 2t + \sin 2t \\ -6\sin 2t + 2\cos 2t \end{bmatrix}$

3.4. Stability of State-Space Models

Definition: The system $\dot{x} = Ax$ is asymptotically stable if $x(t) \to 0$ as $t \to \infty$ for any initial condition

Since $x(t) = e^{tA}x_0$, system is asymptotically stable $\Leftrightarrow e^{tA} \to 0$ as $t \to \infty$

Prop. 3.4.2. $e^{tA} \to 0$ as $t \to \infty$ iff every eigenvalue of A has negative real parts.

e.g. 3.4.2 (mass-spring)

$A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$, e^{tA} as before.

So $e^{tA} \not\to 0$ as $t \to \infty$ so the system is not a.s.

<u>Check the prop.:</u>

eigs of A = roots of $\det(sI - A)$

= roots of $s^2 + 4$

$\sigma(A) = \{\pm j2\} \Rightarrow$ by Prop. system is not a.s.

Bring in friction: $\ddot{q} = u - 4q - \dot{q}$

$\Rightarrow$ (Verify!) $A = \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix}$

$$\det(sI - A) = \begin{vmatrix} s & -1 \\ 4 & s+1 \end{vmatrix} = s^2 + s + 4$$

$$\Rightarrow \sigma(A) = \left\{ \frac{-1 \pm j\sqrt{15}}{2} \right\}$$

$\blacktriangle$

$\Rightarrow$ system is a.s.

<u>3.5 BIBO Stability</u>

$Y(s) = G(s)\,U(s)$ or $y(t) = (g * u)(t)$

Definition

A real-valued signal $u(t)$ is <u>bounded</u> if there exists a constant b such that for all $t \geq 0$ $|u(t)| \leq b$

e.g. $\sin(t)$, $\sin(t^2)$, $\mathbb{1}(t)$, e^{-t} $\rightarrow$ bounded

e^{2t}, $t\cos t$, t $\rightarrow$ unbounded

If u is bounded, $\|u\|_\infty$ is the <u>least upper bound</u>.

Definition

An LTI system is <u>BIBO stable</u> if every bounded input produces a bounded output.

<u>e.g. 3.5.1.</u> $G(s) = \frac{1}{s+2}$

Impulse response $\mathcal{L}^{-1}\{G(s)\} = e^{-2t}$

$y(t) = (g * u)(t)$, $\|u\|_\infty$ assumed finite.

$$= \int_0^t e^{-2\tau} u(t-\tau)\, d\tau \leq \int_0^t e^{-2\tau} |u(t-\tau)|\, d\tau$$

$$\leq \|u\|_\infty \int^t e^{-2\tau}\, d\tau$$

$$= \frac{1}{2} \|u\|_\infty$$

$\|y\|_\infty \leq \frac{1}{2} \|u\|_\infty$ so the system is BIBO stable. ▲

Theorem 3.5.4. Assume $G(s)$ rational and strictly proper. The following are equivalent:

1. System is BIBO stable
2. Impulse response $g(t) = \mathcal{L}^{-1}\{G(s)\}$ is absolutely integrable
$$\int_0^\infty |g(t)|\,dt < \infty$$
3. Every pole of G has a negative real part