- time constant τ tells us almost everything about response of 1st order system $\frac{K}{s\tau+1}$

- second order system $G(s) = \frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

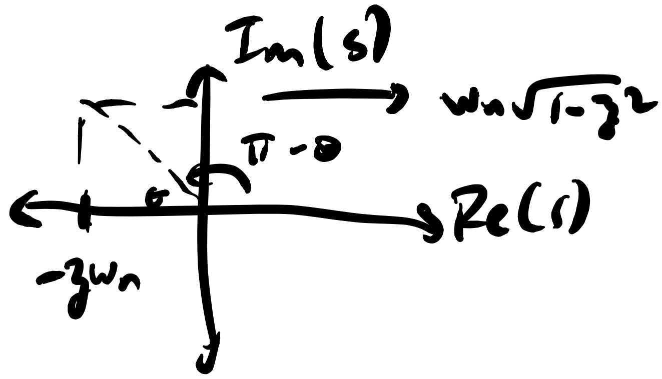
by value of damping ratio ζ

- the most interesting dynamics occur when $0 < \zeta < 1$ (complex conjugate poles)

4.2.1 Underdamped Systems $(0 < \zeta < 1)$

Poles: $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

$= \omega_n e^{\pm j(\pi-\theta)}, \quad \theta = \arccos(\zeta)$



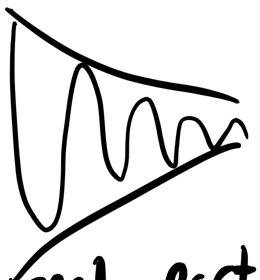
Zeros: None

Steady-State Gain: $G(0) = K$

Bandwidth: $\omega_{BW} \approx \omega_n$ (but it depends on ζ)

Impulse response: $g(t) = \mathcal{L}^{-1}\{G(s)\}$

$$= \frac{K\omega_n}{\sqrt{1-\zeta^2}} \underbrace{e^{-\zeta\omega_n t}}_{\text{decay rate}} \sin(\underbrace{\omega_n\sqrt{1-\zeta^2}\,t}_{\text{oscillatory part}})$$



decay rate

- depends on real part of poles

oscillatory part

- freq. depends on imaginary part of the poles

Observe: For fixed ζ, larger bandwidth (ω_n)

$\Leftrightarrow$ faster decay

Step Response:

$u(t) = \mathbb{1}(t) \Rightarrow U(s) = \frac{1}{s}$

$Y(s) = G(s)U(s)$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}$$

$$= K\left(1 - \frac{1}{\sqrt{1-\zeta^2}} \underbrace{e^{-\zeta\omega_n t}}_{\text{decay rate}} \sin(\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\text{freq. of oscillation}}\,t + \theta)\right)$$

$\theta = \arccos(\zeta)$

See Fig 4.9 / 4.10 in notes.

$\downarrow$ imp. (4.9) $\downarrow$ step (4.10)

Observations:

- as $\zeta \to 1$, imaginary part of the poles approach zero, response becomes less oscillatory, and there is less overshoot

- as $\zeta \to 0$, real part of the poles goes to zero, response is more oscillatory, and more overshoot

- as bandwidth increases, response gets faster
- rate of decay depends on the real part of the poles
- frequency of oscillation depends on the imaginary part of the poles

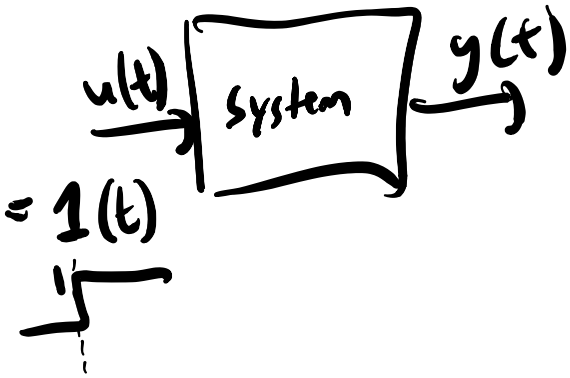
4.3. General characteristics of a step response

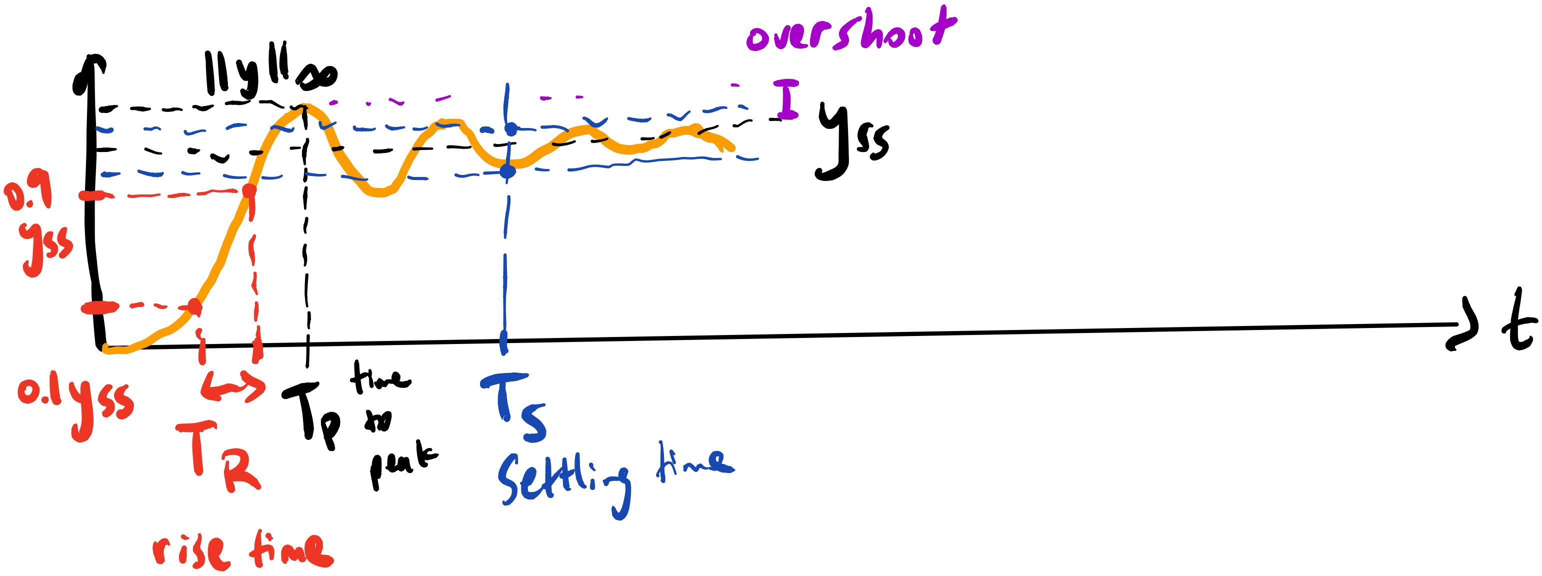
- common methods to quantify the quality of a step response.

- metrics apply to any system, not just 1st and 2nd order

- in this section, we'll use an underdamped 2nd-order system to express the metrics in terms of K, ζ, ω_n.

- the value of using a 2nd order TF is:
  - it's easy
  - we can approx. higher order systems using 2nd order models





4.3.1 Overshoot

$$\%OS := \frac{\|y\|_\infty - |y_{ss}|}{|y_{ss}|}, \quad \text{for } u(t) = \mathbb{1}(t), \; y_{ss} = G(0)$$

- for an underdamped 2^{nd} order system

$$\%OS = \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right), \quad 0 < \zeta < 1$$

↳ only depends on damping ratio, i.e. on the argument of the poles

More damping ($\zeta \to 1$) ⟷ less OS

e.g. mass-spring-damper

$$\frac{Y(s)}{U(s)} = \frac{1/M}{s^2 + \frac{b}{M}s + \frac{K_{spring}}{M}}$$

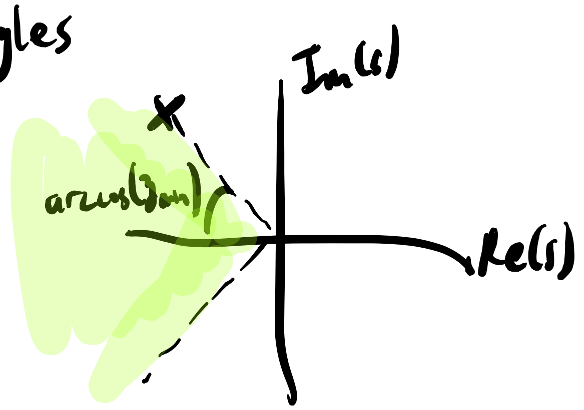
Find conditions on b, M, K_s, so that $\%OS < \%OS^{max} = 0.05$

$$\zeta = \frac{b}{2\sqrt{K_{sp}M}} \qquad \%OS \le \%OS^{max} \iff$$

$$\zeta \ge \frac{-\ln(\%OS^{max})}{\left(\pi^2 - \ln^2(\%OS^{max})\right)^{1/2}} = \zeta_{min}$$

To meet spec, we need $\frac{b}{2\sqrt{K_{sp}M}} \ge 0.6901$

- we can view this spec as a constraint on the pole angles



$\theta = \arccos(\zeta)$

So $\zeta \ge \zeta_{min}$

$\Rightarrow \theta \le \arccos(\zeta_{min})$

Poles in shaded region $\Rightarrow$ Met overshoot spec.

4.3.2 Settling Time

- the smallest time $T_s > 0$ s.t. the following holds:

$$(\forall t \ge T_s) \quad \frac{|y_{ss} - y(t)|}{|y(t)|} \le 0.02 \quad (2\% \text{ settling})$$