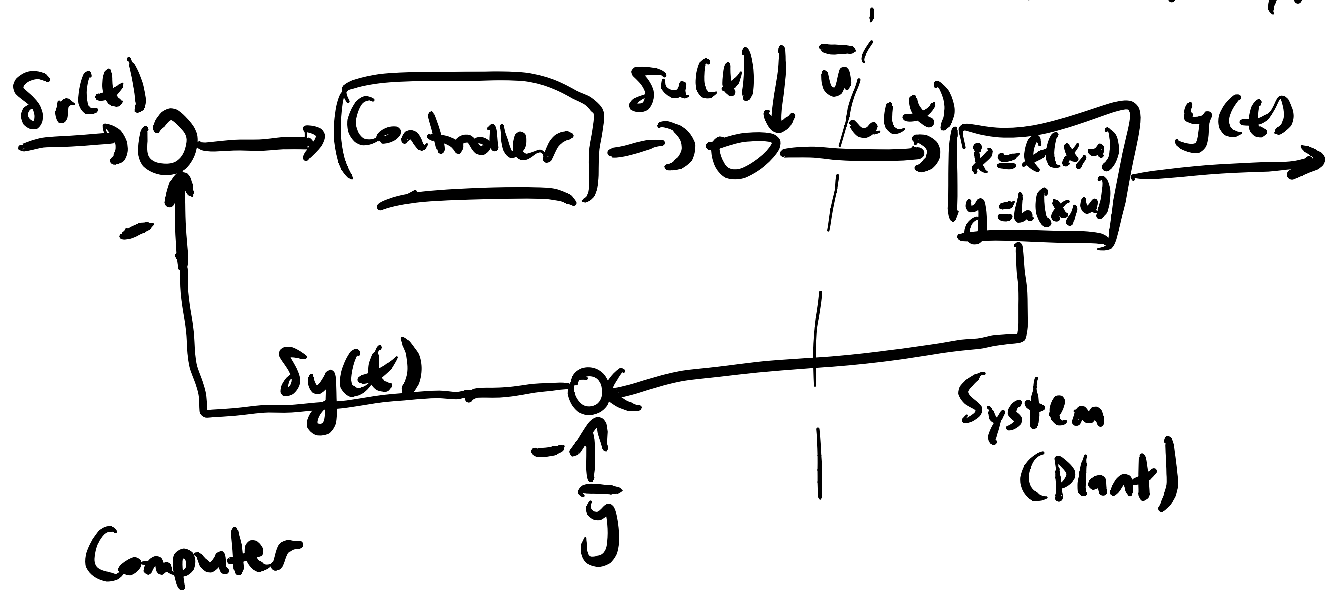
Linearization: $\delta\dot{x} = A\,\delta x + B\,\delta u$

$\delta y = C\,\delta x + D\,\delta u$

A, B, C, D Jacobians of f, h evaluated at eq. cfg.



2-stage system

1) get to eq. cfg.

2) controller takes over

2.8 Transfer Functions

The transfer function (TF) of this system is the ratio $\frac{Y(s)}{U(s)}$ where all Laplace transforms are taken with zero initial conditions.

e.g. 2.8.3 Recall the mass-spring-damper

$$M\ddot{q} = u - Kq - c(\dot{q})$$

If the damper is non-linear, then the system does not have a TF. If $c(\dot{q}) = b\dot{q}$, b is a constant, then

$$s^2 M Q(s) = U(s) - KQ(s) - bsQ(s)$$

$$\Rightarrow \frac{Q(s)}{U(s)} = \frac{1}{s^2 M + K + bs} \qquad \blacktriangle$$

<u>Other examples (Table 2.2):</u>

<u>Description</u>	<u>Block Diagram</u>	<u>TF</u>
Gain	$u \to \boxed{K} \to y$	K
Integrator	$u \to \boxed{\int_0^t} \to y$	$\frac{1}{s}$
Double Integrator	$u \to \boxed{\int_0^t} \to \boxed{\int_0^t} \to y$	$\frac{1}{s^2}$
Differentiator	$u \to \boxed{\frac{d}{dt}} \to y$	s → Improper
Time delay	$u \to \boxed{T \text{ sec delay}} \to y$	e^{-sT} → irrational

<u>Definition 2.8.1</u>: A TF $G(s)$ is <u>rational</u> (real-rational) if

$$G(s) = \frac{b_m s^m + \dots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}, \quad a_i, b_i \in \mathbb{R}$$

A rational $G(s)$ is <u>proper</u> if $n \geq m$.

A rational $G(s)$ is <u>strictly proper</u> if $n > m$.

<u>Definition 2.8.2</u>: A complex number $p \in \mathbb{C}$ is a <u>pole</u> of G if $\lim_{s \to p} G(s) = \infty$.

$z \in \mathbb{C}$ is a <u>zero</u> of G if $\lim_{s \to z} G(s) = 0$

If G is rational and the numerator and denominator are coprime, then:

- poles of G = roots of denominator
- zeros of G = roots of numerator

<u>2.8.1. Obtaining a TF from a state model</u>

Start with $\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$
$y = Cx + Du$

Take Laplace transforms w/ zero initial conditions.

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

$$X(s) = \begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix}$$

$$(sI - A)X(s) = BU(s)$$
$$X(s) = (sI - A)^{-1} BU(s)$$

$$\Rightarrow Y(s) = C(sI-A)^{-1} BU(s) + DU(s)$$
$$= [C(sI-A)^{-1}B + D]\,U(s)$$

↳ TF → $\mathbb{C}^{p \times m}$

Remark 2.8.3.: The TF obtained from a state-space model is always rational and proper.

$$\text{state-space model} \xrightarrow{\text{ss2tf}} \text{rational, proper transfer function}$$

$$\text{TF model} \xrightarrow{?} \text{state-space model}$$

(only if TF is rational and proper, never unique)

$$u(t) \longrightarrow \boxed{\begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx + Du \end{array}} \longrightarrow y(t)$$

$$G(s) = C(sI - A)^{-1}B + D$$

e.g. 2.8.6 Recall linearized model of pendulum at upright position. $(\bar{x}, \bar{u}) = \left(\begin{bmatrix} \pi \\ 0 \end{bmatrix}, 0\right)$

$$\delta \dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{1.5g}{\ell} & 0 \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ 3/M\ell^2 \end{bmatrix} \delta u$$

$$\delta_y = \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x$$

So $G(s) = C(sI - A)^{-1} B$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ -\frac{1.5g}{\ell} & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 3/M\ell^2 \end{bmatrix}$$

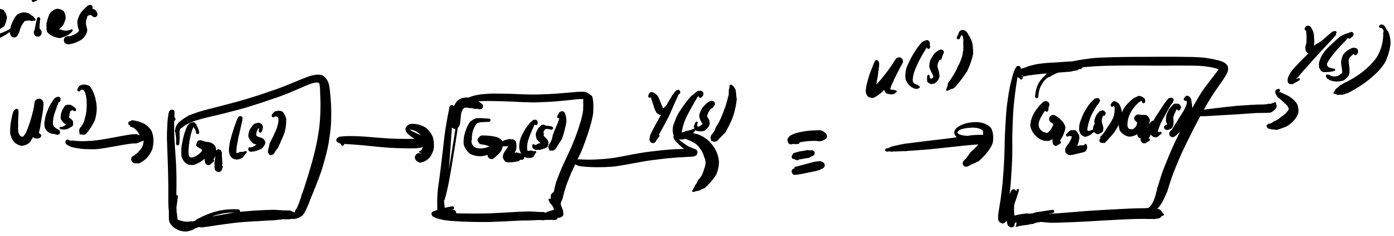
$$(sI-A)^{-1} = \frac{\overbrace{\text{adj}(sI-A)}^{n\times n}}{\underbrace{\det(sI-A)}_{\text{polynomial}}}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\begin{bmatrix} s & 1 \\ 1.5g/\ell & s \end{bmatrix}}{s^2 - \frac{1.5g}{\ell}} \begin{bmatrix} 0 \\ 3/M\ell^2 \end{bmatrix}$$

$$\Rightarrow G(s) = \frac{\frac{3}{M\ell^2}}{s^2 - \frac{1.5g}{\ell}}$$

2.9. Block Diagram Manipulations

(i) Series



(ii) Parallel