Lecture 11 Summary

- $\dot{x} = Ax,\ x(0) = x_0 \Rightarrow x(t) = e^{tA}x_0,\ e^{tA} = \mathcal{L}^{-1}\{(sI-A)^{-1}\}$
- $\dot{x} = Ax$ is asymptotically stable if $x(t) \to 0$ for every $x(0)$

$\Leftrightarrow e^{tA} \to 0 \Leftrightarrow$ all eigenvalues of A have neg. real part

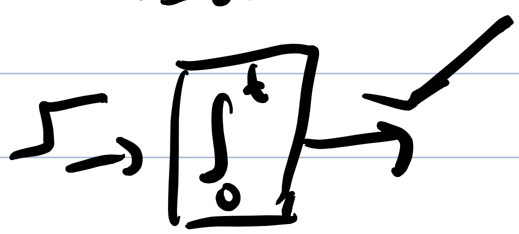
- $Y(s) = G(s)U(s)$ is BIBO stable if $\|u\|_\infty$ finite $\Rightarrow \|y\|_\infty$ finite

$\Leftrightarrow g(t) = \mathcal{L}^{-1}(G(s))$ is absolutely integrable

$\Leftrightarrow$ all poles of G have negative real part

BIBO stable?

e.g. $\frac{1}{s+1}$ ✓



$\frac{1}{s}$ ✗

$\frac{1}{(s+3)^2}$ ✓

$\frac{s-1}{(s+2)(s+3)}$ ✓ zero has no effect

$\frac{1}{s-1}$ ✗

- if $G(s)$ is only proper, and not strictly proper, then use long division to write

$$G(s) = G_1(s) + \text{constant}, \quad G_1 \text{ strictly proper}$$

- Impulse response is $g(t) = g_1(t) + \text{constant} \cdot \delta(t)$
- Thm remains true with (ii) changed to g_1 absolutely integrable

e.g. $G(s) = \dfrac{(s+1)(s+2)}{(s+3)(s+4)} = \dfrac{-4s-10}{(s+3)(s+4)} + 1$

<u>Theorem 3.5.5</u>

If G is improper, then G is unstable.

<u>Proof:</u> Write $G(s) = G_1(s) + G_2(s)$ where G_1 is strictly proper and $G_2(s)$ is a polynomial in s.

So $Y(s) = G_1(s)U(s) + G_2(s)U(s)$

- if G_1 has poles with $Re(s) \geq 0$, then esult follows from the last thm 3.5.4.
- if G_1 has all poles with $Re(s) < 0$, take

$u(t) = \sin(t^2)$

- any derivative of u has polynomial terms in t, which are unbounded, e.g.

$$\frac{du}{dt} = 2t\cos(t^2)$$

But $y_2(t) = \mathcal{L}^{-1}\{G_2(s)U(s)\}$ is a linear combination of u and its derivatives, so it is unbounded. $\blacksquare$

3.5.1. Stability of state-space models and BIBO-stability

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$\xrightarrow{ss2tf}$ $Y(s) = [C(sI-A)^{-1}B + D]U(s)$

$$= \left\{ C\frac{\text{adj}(sI-A)}{\det(sI-A)}B + D \right\} U(s)$$

$n \times n$

n^{th} order poly

eigenvalues of A = roots of $\det(sI-A)$

$\supseteq$ poles of TF

Asymptotic stability $\Rightarrow$ BIBO stability

e.g. 3.5.5 (mass-spring)

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

$\sigma(A) = \{\pm j2\}$ $\Rightarrow$ Not A.S.

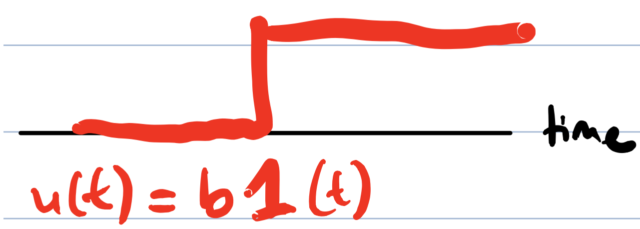
$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 4 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

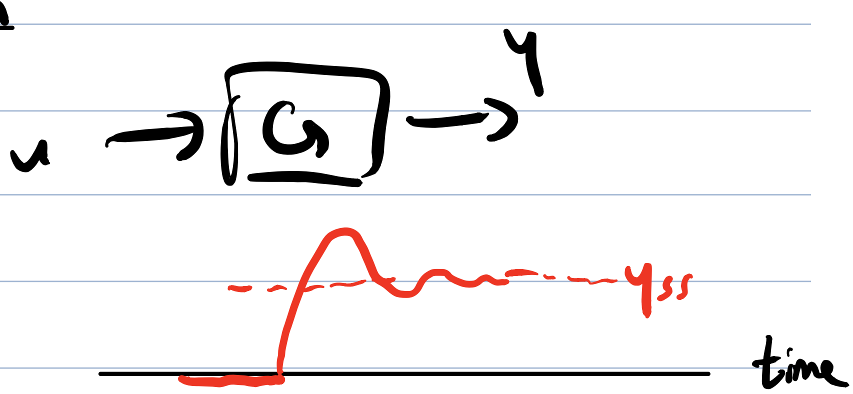
$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{\begin{bmatrix} s & 1 \\ -4 & s \end{bmatrix}}{s^2+4} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2+4} \Rightarrow \text{system not BIBO stable}$$

- in this example, the polynomials $C\,\mathrm{adj}(sI-A)B = 1$ and $\det(sI-A) = s^2+4$ are coprime.
  $\Rightarrow$ eigs of A = poles of TF $\blacktriangle$

3.6. Steady State Gain





Steady-state gain of $G = \dfrac{y_{ss}}{b}$.

Theorem 3.6.1. (Final Value Theorem)
Let $F(s) = \mathcal{L}\{f(t)\}$ for a signal $f(t)$. Assume F is rational.
(a) If $F(s)$ has all its poles with negative real part except for a single pole at $s=0$, then:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \qquad (*)$$

(b) If F has repeated poles at $s=0$, then $f(t)$ does not converge.

(c) If F has poles with $Re(s) \geq 0$, other then at $s=0$, then $f(t)$ does not converge.

e.g. $f(t)$	$\lim_{t\to\infty} f(t)$	$F(s)$	$\lim_{s\to 0} sF(s)$	$(*)$ applies
e^{-t}	0	$1/s+1$	0	✓
$1(t)$	1	$1/s$	1	✓
t	∞	$1/s^2$	∞	✗
te^{-t}	0	$1/(s+1)^2$	0	✓
e^t	∞	$1/s-1$	0	✗
$\cos(\omega t)$	DNE	$s/s^2+\omega^2$	0	✗

To apply $(*)$, we have to check $sF(s)$ has bad poles