e.g. $\sup (0,1) = 1$
$\max (0,1)$ dne

Remark: We don't insist $\sup X \in X$ ($\inf X \in X$)

Remark: If X is not bounded above, we write $\sup X = +\infty$.
" below $\Rightarrow \inf X = -\infty$

Convention:

$\sup \emptyset = -\infty$
$\inf \emptyset = +\infty$

e.g. $A = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\}$

$f(x) = \frac{x}{x+1}$ $\quad f'(x) = \frac{(x+1)(1) - x(1)}{(x+1)^2}$

$= \frac{1}{(x+1)^2} \rightarrow f$ increasing for $x \geq 0$

$f(1) \leq f(2) \leq \ldots \quad \lim_{x \to \infty} f(x) = 1$

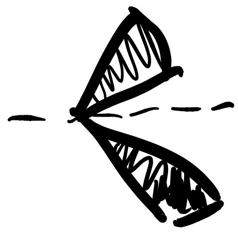
$\Rightarrow \sup A = 1$
$\Rightarrow \inf A = f(1) = \frac{1}{2}$ (min in this case)

e.g. $B = \{-(x-1)^2 + 5 : x \in \mathbb{R}\}$

$\sup B = 5$

$\inf B = -\infty$

<u>e.g.</u> $C = \{ y \in \mathbb{R}_{\geq 0} : \frac{1}{2} < \cos y < \frac{\sqrt{3}}{2} \}$



$\sup C = +\infty$ (keep going around)

$\inf C = \frac{\pi}{6}$

<u>e.g.</u> $A, B \subseteq \mathbb{R}_{>0}$ non-empty bd above

$AB = \{ ab : a \in A, b \in B \}$

Prove that $\sup AB = (\sup A)(\sup B)$

<u>Proof</u>: Take $ab \in AB$.

$ab \leq (\sup A)(\sup B)$

$\Rightarrow \sup A \sup B$ is an upper bound for AB

$\therefore \ \sup(AB) \leq \sup A \sup B$

Let $a \in A$ and $b \in B$ arbitrary.

Note $a = ab\frac{1}{b} \leq \sup(AB)\frac{1}{b}$

$\Rightarrow \sup(AB)\frac{1}{b}$ is an upper bound for a.

$\therefore \sup A \leq \sup(AB)\frac{1}{b}$

$\rightarrow \ b \leq \frac{\sup(AB)}{\sup(A)} \Rightarrow \sup(B) \leq \frac{\sup(AB)}{\sup(A)}$

$\Rightarrow \sup A \sup B \leq \sup AB.$ $\blacksquare$

<u>Theorem (Least Upper Bound Principle):</u>

Every non-empty subset $X \subseteq \mathbb{R}$ which is bounded above has a supremum.

Every non-empty $X \subseteq \mathbb{R}$ bounded below has an infemum.

(Proof omitted from lecture and exams)

e.g. $(0, \sqrt{2}) \cap \mathbb{Q} =: A$

$a \in A \Rightarrow a \leq 100$

$\sup A \notin \mathbb{Q}$ (Not in superset so sup DNE)

<u>2.4. Limits</u>

<u>Notation:</u> $(a_n)_{n=1}^{\infty} = (a_1, a_2, \ldots)$

(a_n)

<u>Definition:</u> A number $L \in \mathbb{R}$ is the <u>limit</u> of a sequence (a_n) if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow |a_n - L| < \varepsilon$.

We say $\{a_n\}$ <u>converges</u> to L and we write

$$\lim_{n \to \infty} \{a_n\} = L \quad \text{or} \quad a_n \to L.$$