Recall (2.4)

$\lim_{n\to\infty} a_n = L$ OR $a_n \to L$ if $\forall \varepsilon > 0 \cdot \exists N \in \mathbb{N} \cdot$ s.t. $(n \geq N \Rightarrow |a_n - L| < \varepsilon)$.

Remark

In this def^n, $N = N(\varepsilon)$ depends on ε.

e.g. $a_n = \frac{n}{n+1}$

Claim: $a_n \to 1$.

Proof: Let $\varepsilon > 0$ be given.

Choose $N = \lceil \frac{1}{\varepsilon} \rceil$

Suppose $n \geq N$.

$$\therefore |a_n - 1| = \left| \frac{n}{n+1} - 1 \right|$$

$$= \left| \frac{-1}{n+1} \right|$$

$$= \frac{1}{n+1}$$

$$< \frac{1}{n}$$

$$\leq \frac{1}{N} \leq \varepsilon$$

$\therefore a_n \to 1$

Remark: If $\forall \varepsilon > 0 \ \exists N \in \mathbb{R}_{\geq 0}$ s.t. $|a_n - L| < \varepsilon$

for $n \geq N$ then $a_n \to L$

$\to$ We can always take $\lceil N \rceil$

e.g. $a_n = \frac{\sqrt{n^2+1}}{n!}$

Claim: $a_n \to 0$

Proof: Let $\varepsilon > 0$ given.

Choose $N = \frac{\sqrt{2}+\varepsilon}{\varepsilon} + 1$, assume $n \geq N$.

$$\therefore |a_n - 0| = \frac{\sqrt{n^2+1}}{n!}$$

$$\leq \frac{\sqrt{2}\,n}{n!}$$

$$= \frac{\sqrt{2}}{(n-1)!}$$

$$\leq \frac{\sqrt{2}}{n-1}$$

$$\leq \frac{\sqrt{2}}{N-1}$$

$$< \frac{\sqrt{2}}{\frac{\sqrt{2}+\varepsilon}{\varepsilon} - 1} < \varepsilon$$

e.g. $\forall L \in \mathbb{R}$. $\exists$ a sequence of rationals (q_n) s.t. $q_n \to L$.

Let $L \in \mathbb{R}$. Let $\varepsilon > 0$ be given.

From before, for each $n \in \mathbb{N} \cdot \exists q_n \in \mathbb{Q}$ s.t.

$|q_n - L| < \frac{1}{n}$

As $\frac{1}{n} \to 0$, $\exists N \in \mathbb{N} \cdot \left|\frac{1}{n}\right| < \varepsilon$ when $n \geq N$.

For $n \geq N$, $|q_n - L| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$

Definition (Divergence)

If (a_n) does not converge to any $L \in \mathbb{R}$, we say it diverges.

(a_n) converges iff

$\exists L \in \mathbb{R} \cdot \forall \varepsilon > 0 \cdot \exists N \in \mathbb{N} \cdot (n \geq N \Rightarrow |a_n - L| < \varepsilon)$

(a_n) diverges

$\forall L \in \mathbb{R} \cdot \exists \varepsilon > 0 \cdot \forall N \in \mathbb{N} \cdot \exists n \geq N \cdot |a_n - L| \geq \varepsilon)$

"for any real, there is a fixed epsilon, no matter how far we go out, sequence refuses to get within ε of that real"

e.g. $a_n = (-1)^n$

FTSOC assume $a_n \to L$ for some $L \in \mathbb{R}$

For $\varepsilon = 1$, there exists $N \in \mathbb{N} \cdot |a_n - L| < 1$ for $n \geq N$.

Note that $|a_n - a_{n+1}| = 2$. Assume $n \geq N$

$\therefore |a_n - L| < 1$ and $|a_{n+1} - L| < 1$

$$\begin{aligned}
\therefore\ 2 &= |a_n - a_{n+1}| \\
&= |a_n - L + L - a_{n+1}| \\
&\leq |a_n - L| + |L - a_{n+1}| \\
&= |a_n - L| + |a_{n+1} - L| \\
&< 2 \quad \Downarrow
\end{aligned}$$

<u>Theorem (Squeeze Theorem):</u>

Suppose $a_n \leq b_n \leq c_n$ for all $n \geq M$.

If $a_n \to L$ and $c_n \to L$ then $b_n \to L$

<u>Proof:</u> Let $\varepsilon > 0$.

$\exists N_1, N_2 \in \mathbb{N}_2 \cdot n \geq N_1 \Rightarrow |a_n - L| < \varepsilon$ and

$n \geq N_2 \Rightarrow |c_n - L| < \varepsilon$.

Choose $N = \max\{N_1, N_2, M\}$. Assume $n \geq N$.

$\therefore\ L - \varepsilon < a_n \leq b_n < c_n < L + \varepsilon$

$\Rightarrow L - \varepsilon < b_n < L + \varepsilon$

$\Rightarrow |b_n - L| < \varepsilon$