$y = f(x)$, $f(x) = \begin{bmatrix} x_1 x_2 - 1 \\ x_3^2 - 2x_1 x_3 \end{bmatrix}$, $\bar{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$\bar{y} = f(\bar{x}) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

<u>Multivariable Taylors:</u>

$$f(x) = f(\bar{x}) + \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} (x - \bar{x}) + \text{higher order terms}$$

where $\left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}}$ = Jacobian of f evaluated at $\bar{x}$

$$= \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix}\right|_{x=\bar{x}}$$

$$= \left.\begin{bmatrix} x_2 & x_1 & 0 \\ -2x_3 & 0 & 2x_3 - 2x_1 \end{bmatrix}\right|_{x=\bar{x}}$$

$$= \begin{bmatrix} -1 & 1 & 0 \\ -4 & 0 & 2 \end{bmatrix}$$

$$\underbrace{y-\bar{y}}_{\delta y} \approx \underbrace{\frac{\partial f}{\partial x}\Big|_{x=\bar{x}}}_{A} \underbrace{(x-\bar{x})}_{\delta x}$$

$\delta y = A\,\delta x$ } linear approximation near $\bar{x}$. △

<u>By direct extension:</u>

$$f(x,u) \approx f(\bar{x},\bar{u}) + \frac{\partial f}{\partial x}\Big|_{x=\bar{x}}(x-\bar{x}) + \frac{\partial f}{\partial u}\Big|_{u=\bar{u}}(u-\bar{u})$$

near $(\bar{x},\bar{u})$

Let's apply this to
$$\begin{aligned}\dot{x} &= f(x,u) \\ y &= h(x,u)\end{aligned} \qquad (*)$$

<u>Definition:</u> A constant pair $(\bar{x},\bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$ is an <u>equilibrium configuration</u> of $(*)$ if

$f(\bar{x},\bar{u}) = \vec{0}$.

<u>e.g. 2.5.4</u>: Pendulum

$$\bar{x} = \begin{bmatrix} x_2 \\ \frac{3}{M\ell^2}u - \frac{1.5g}{\ell}\sin(x_1) \end{bmatrix}$$

$y = x_1$

Find equilibrium configurations corresponding to the pendulum being upright

$X_1 = \pi$ at the upright position. Solve

$$\begin{bmatrix} \bar{x}_2 \\ \frac{3}{M\ell^2}\bar{u} - \frac{1.5g}{\ell}\sin(\pi) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} \bar{x}_2 = 0 \\ \bar{u} = 0 \end{matrix}$$

$(\bar{x}_1 = \pi)$

The equilibrium config. is $(\bar{x}, \bar{u}) = \left(\begin{bmatrix} \pi \\ 0 \end{bmatrix}, 0\right)$ Δ

Assume the nonlinear state space model has an eq. config. $(\bar{x}, \bar{u})$.

$$f(x,u) = \underbrace{f(\bar{x}, \bar{u})}_{0} + \underbrace{\left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}}}_{A}(x-\bar{x}) + \underbrace{\left.\frac{\partial f}{\partial u}\right|_{u=\bar{u}}}_{B}(u-\bar{u}) + \text{higher order terms}$$

Consider system solutions "near" $(\bar{x}, \bar{u})$.

$$\delta x(t) := x(t) - \bar{x}$$
$$\delta u(t) := u(t) - \bar{u}$$

$$\dot{\delta x} = \dot{x} - 0$$
$$= f(x,u) - 0$$
$$\dot{\delta x} \doteq A\delta x + B\delta u \rightarrow \text{linearized model}$$

The output equation $y = h(x,u)$ is linearized in the same way.

$\delta y(t) = y - h(\bar{x}, \bar{u})$

$$\delta y \doteq \underbrace{\left.\frac{\partial h}{\partial x}\right|_{x=\bar{x}}}_{C} \delta x + \underbrace{\left.\frac{\partial h}{\partial u}\right|_{u=\bar{u}}}_{D} (u-\bar{u})$$

<u>Summary</u>: Linearizing a non-linear state space model

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

1. Select a desired equilibrium configuration $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$

   $\bar{x} \in \mathbb{R}^n$, $\bar{u} \in \mathbb{R}^m$, s.t. $f(\bar{x}, \bar{u}) = 0$, $\bar{y} = h(\bar{x}, \bar{u}) =$ desired output

2. Compute Jacobians A, B, C, D of f and h at $(\bar{x}, \bar{u})$.

3. Linearized system is
$$\delta\dot{x} = A\,\delta x + B\,\delta u$$
$$\delta y = C\,\delta x + D\,\delta u$$

e.g. 2.5.5 (Pendulum)

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{3}{M\ell^2}u - \frac{1.5g}{\ell}\sin(x_1)$$
$$y = x_1$$

The eq. config. corresponding to the upright position:

$\bar{x} = (\pi, 0)$ $\quad \bar{u} = 0$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} = \left.\begin{bmatrix} 0 & 1 \\ -\frac{1.5g}{\ell^2}\cos(x_1) & 0 \end{bmatrix}\right|_{x=\bar{x}} = \begin{bmatrix} 0 & 1 \\ 1.5g/\ell^2 & 0 \end{bmatrix}$$

$$B = \left.\frac{\partial f}{\partial u}\right|_{u=\bar{u}} = \left.\begin{bmatrix} 0 \\ 3/M\ell^2 \end{bmatrix}\right|_{u=0} = \begin{bmatrix} 0 \\ 3/M\ell^2 \end{bmatrix}$$

$$C = \frac{\partial h}{\partial x}\bigg|_{x=\bar{x}} = (1, 0) \qquad D = \frac{\partial h}{\partial u}\bigg|_{u=\bar{u}} = 0$$

Linearized model:

$$\delta\dot{x} = \begin{bmatrix} 0 & 1 \\ 1.5g/\ell & 0 \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ 3/me^2 \end{bmatrix} \delta u$$

$$\delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \delta x + [0]\,\delta u$$

$$\delta x = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} - \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

$$\delta u = u(t) - 0$$

2.7: Laplace transforms (Not covered in class)

2.8: Transfer Functions