Mathematical Analysis of Parametric Curve Navigation

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Github Repo: `https://github.com/kennethxiong23/bridge-o-doom`

This document encapsulates the mathematical framework for the visualization of a parametric curve, the computation of its unit tangent and normal vectors, and the determination of linear and angular velocities for a mobile robot navigating the curve. It also includes the process of graphing the robot's motion involves calculating linear speed, angular velocity, wheel velocities, and the robot's trajectory and the computation of the tangent and normal vectors to analyze the path curvature.

1 Parametric Curve and Vector Visualization

A parametric curve in the plane is defined by two functions $x(t)$ and $y(t)$, representing the coordinates of a point on the curve as a function of the parameter t. Specifically, for our curve:

$$x(t) = 0.3960\cos(2.65(1.4 + \beta t))\hat{\imath}, \tag{1}$$

$$y(t) = -0.99\sin(1.4 + \beta t)\hat{\jmath}, \tag{2}$$

where β is set to 0.2, and t ranges from 0 to 1.6. The position vector $\vec{r}(t)$ for the curve is given by $x(t) + y(t)$.

The unit tangent vector $\hat{T}$ at any point is the normalized derivative of $\vec{r}(t)$ with respect to t:

$$\hat{T} = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}. \tag{3}$$

The unit normal vector $\hat{N}$, perpendicular to $\hat{T}$, is obtained by normalizing the derivative of $\hat{T}$ with respect to t:

$$\hat{N} = \frac{d\hat{T}}{dt}\Bigg/\left\|\frac{d\hat{T}}{dt}\right\|. \tag{4}$$

2 Linear and Angular Velocities

For a mobile robot navigating the curve, its linear velocity V_n and angular velocity ω are crucial for movement and orientation. The linear velocity at any point is given by the magnitude of the derivative of $\vec{r}(t)$, representing the speed at which the robot moves along the curve. Angular velocity, on the other hand, is derived from the rate of change of the tangent vector $\hat{T}$, indicating how the robot's orientation changes along the path.

Given a series of points along the curve, the linear velocity V_n is calculated as:

$$V_n = \frac{d\vec{r}}{dt}, \tag{5}$$

where the sum is taken over the components of the velocity vector.

The angular velocity ω is determined by:

$$\omega = \frac{d\hat{T}}{dt} \times \hat{T}, \tag{6}$$

focusing on the component of ω perpendicular to the plane of motion, which dictates the robot's rate of rotation.

The calculated linear and angular velocities are then used to adjust the velocities of the robot's left and right wheels, ensuring accurate navigation along the parametric curve.

3 Mathematical Foundation of Graphing

3.1 Linear Speed and Angular Velocity

The linear speed V and angular velocity ω are computed from the differences in encoder readings for the left and right wheels over time. Given the wheelbase W_b, the linear speed V is the average of the left and right wheel speeds, and the angular velocity ω is the difference in wheel speeds divided by the wheelbase, adjusted for the time interval Δt:

$$V = \frac{\Delta L + \Delta R}{2\Delta t} \tag{7}$$

$$\omega = \frac{\Delta R - \Delta L}{W_b \Delta t} \tag{8}$$

where ΔL and ΔR are the changes in encoder readings for the left and right wheels, respectively.

3.2 Wheel Velocities

The velocities of the left and right wheels, V_L and V_R, are directly calculated from the encoder readings:

$$V_L = \frac{\Delta L}{\Delta t} \tag{9}$$

$$V_R = \frac{\Delta R}{\Delta t} \tag{10}$$

3.3 Robot Trajectory

The robot's trajectory is determined by integrating the linear speed and angular velocity over time to update the robot's position (x, y) and orientation (θ). The position update uses the average distance traveled and the change in orientation:

$$\Delta s = \frac{\Delta L + \Delta R}{2} \tag{11}$$

$$\Delta \theta = \frac{\Delta R - \Delta L}{W_b} \tag{12}$$

$$x_{new} = x_{old} + \Delta s \cos(\theta + \Delta \theta) \tag{13}$$

$$y_{new} = y_{old} + \Delta s \sin(\theta + \Delta \theta) \tag{14}$$

$$\theta_{new} = \theta_{old} + \Delta \theta \tag{15}$$

3.4 Tangent and Normal Vectors

The tangent $(\mathbf{T})$ and normal $(\mathbf{N})$ vectors along the robot's path are computed to analyze the path's curvature. The tangent vector at any point is a unit vector in the direction of the path, and the normal vector is orthogonal to the tangent vector, pointing towards the center of curvature. For discrete points along the path, these vectors are approximated from the differences in position:

$$\mathbf{T} \approx \frac{\Delta \mathbf{X}}{\|\Delta \mathbf{X}\|} \tag{16}$$

$$\mathbf{N} \approx \frac{\Delta \mathbf{T}}{\|\Delta \mathbf{T}\|} \tag{17}$$

where $\Delta \mathbf{X}$ is the change in position and $\Delta \mathbf{T}$ is the change in the tangent vector between consecutive points.

This mathematical framework underpins the visualization of the robot's movement, allowing for a detailed analysis of its behavior and path characteristics.

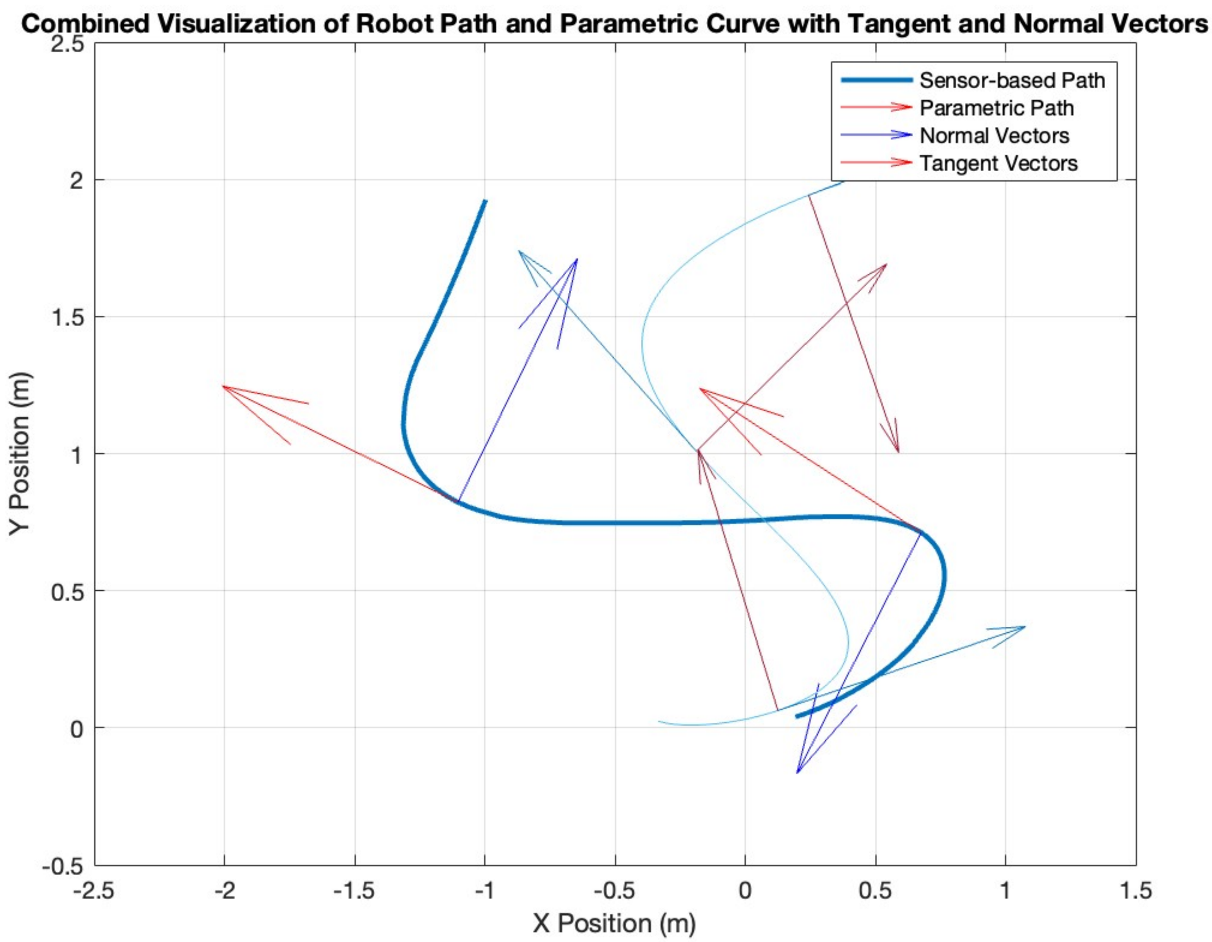

Figure 1: Detailed Combined Visualization of Robotic Path Tracking. This graph overlays the actual sensor-based path against the planned parametric path, with the inclusion of tangent and normal vectors at key points. These vectors offer insights into the robot's orientation and directional changes, which are critical for advanced control and navigational strategies.

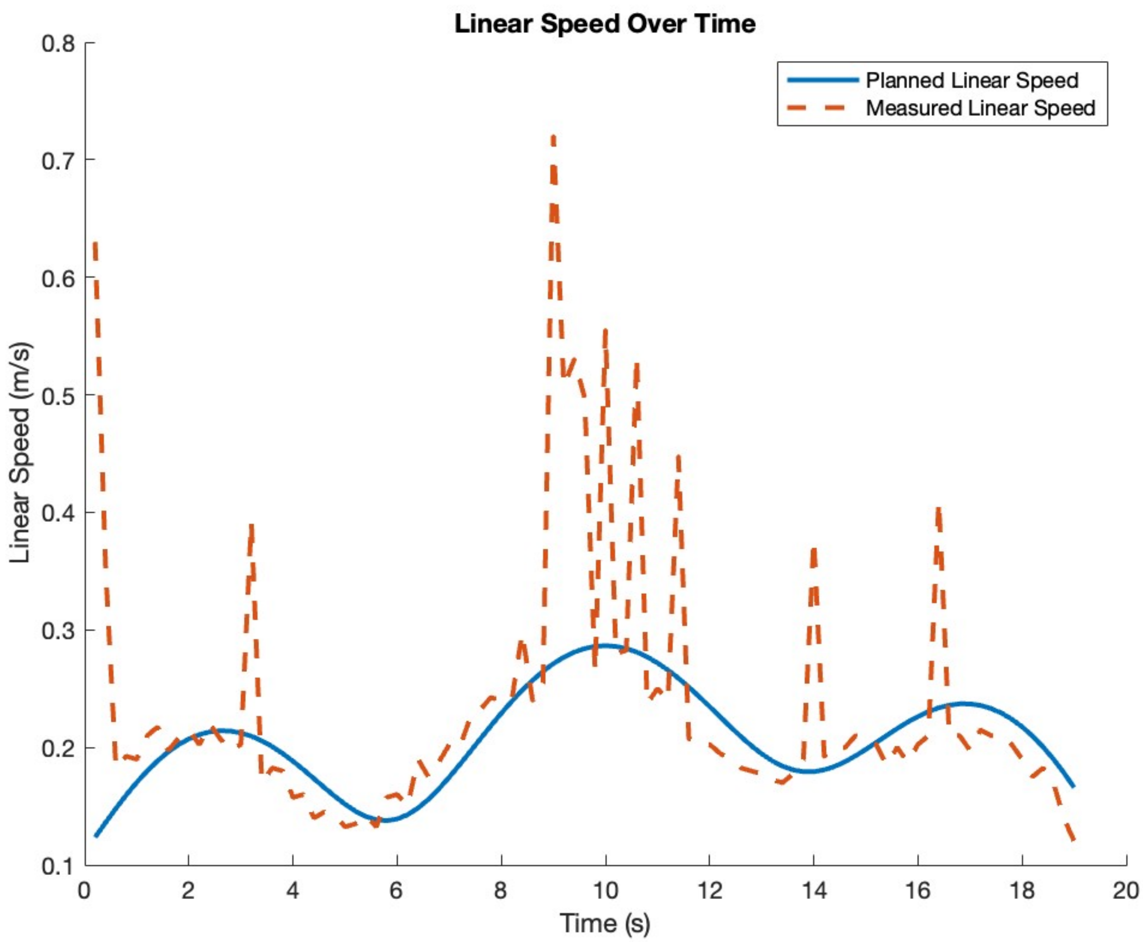

Figure 2: Linear Speed Trajectory Analysis Over Time. This graph plots the planned linear speed trajectory alongside the measured speed, offering insights into the performance and efficiency of the robot's motion. It serves as a diagnostic tool to identify speed-related discrepancies and to assess the accuracy of the robot's trajectory following capabilities.

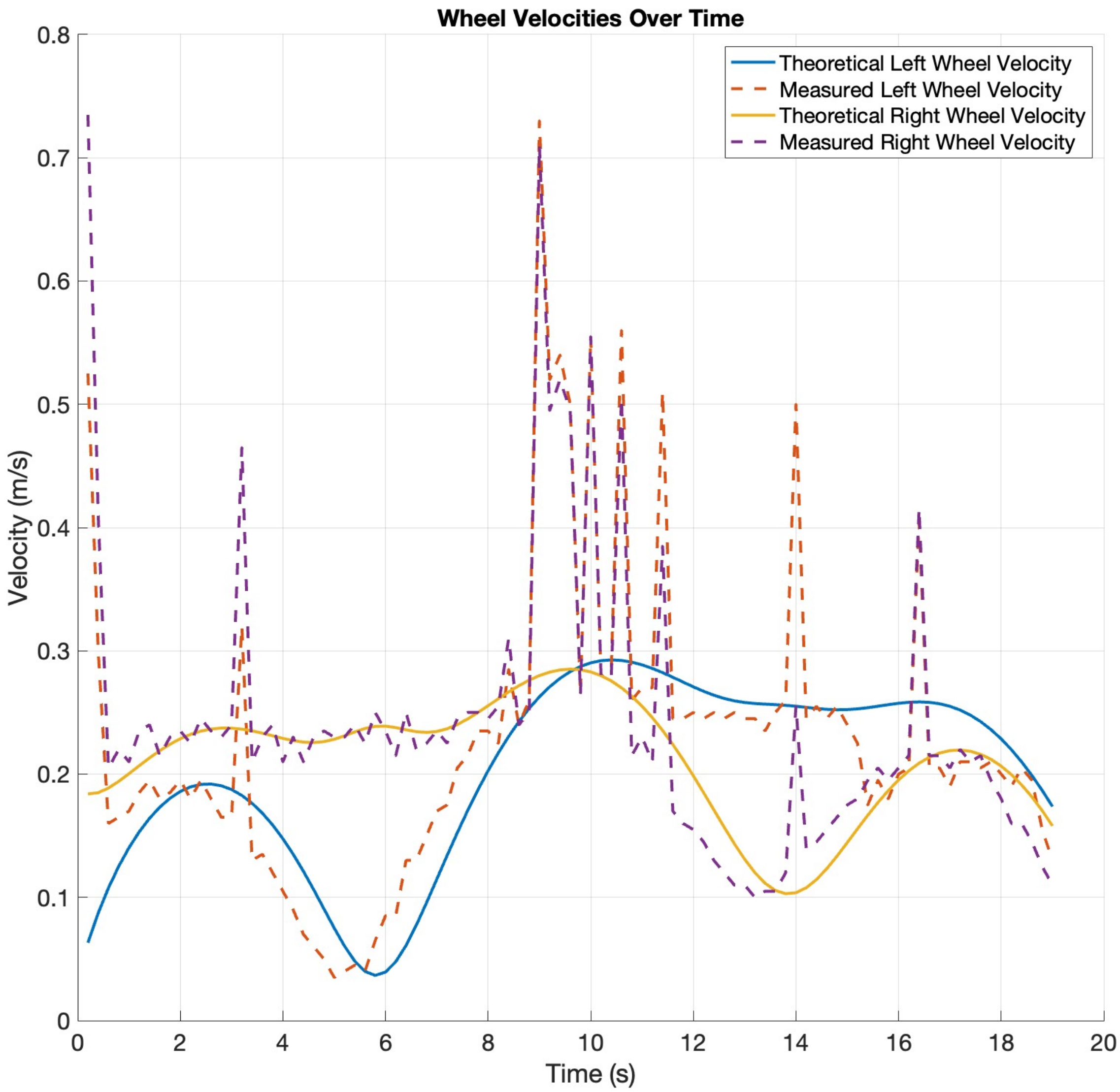

Figure 3: Individual Wheel Velocities in a Multi-Wheel Robotic System. The graph illustrates the individual speed profiles of each wheel over time, providing a detailed look into the dynamics of the robot's movement. Analyzing these profiles is crucial for ensuring synchronized wheel operation and optimizing overall system stability and traction control.

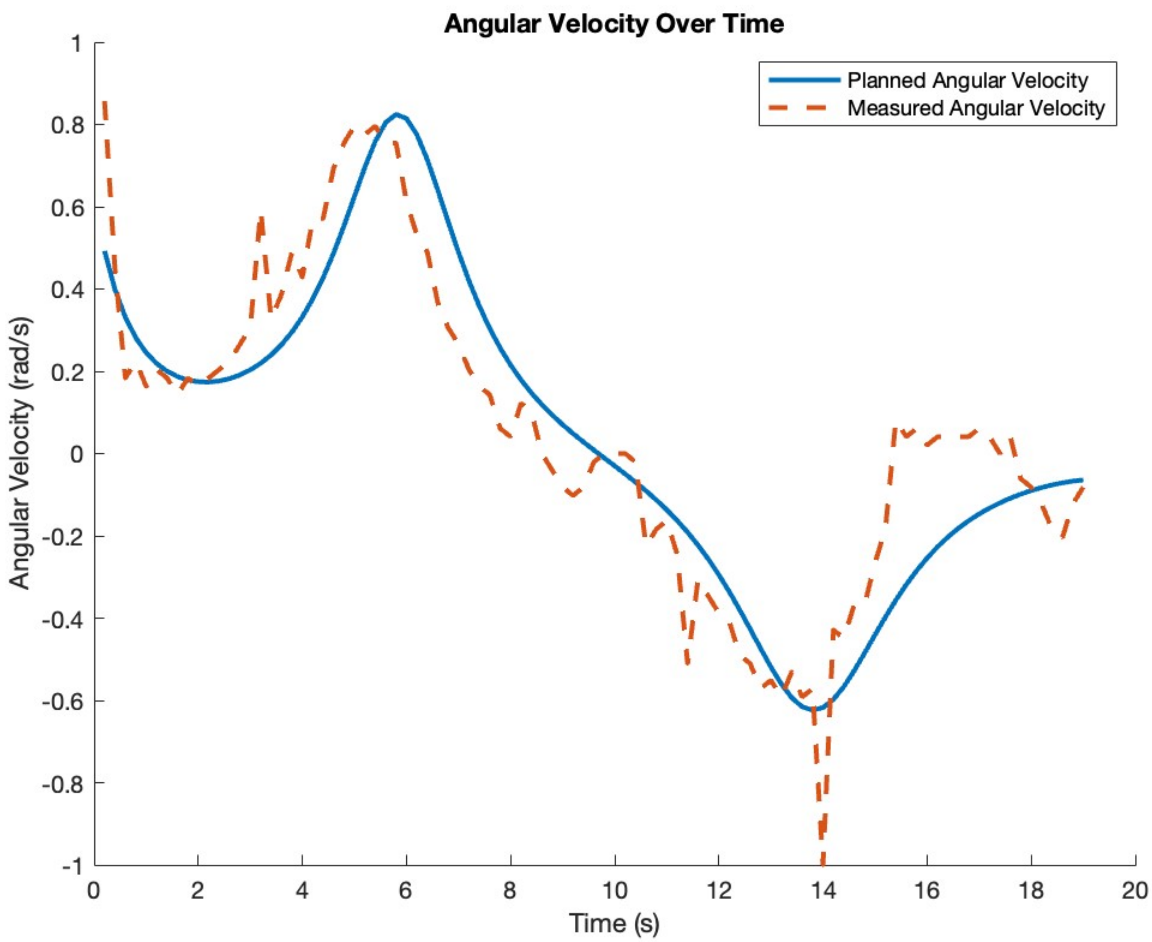

Figure 4: Comparison of Planned vs. Measured Angular Velocity Over Time. The graph provides a clear representation of the system's angular velocity as it compares theoretical values with real-time sensor data. This comparison helps in evaluating the precision of the robot's control systems and the effectiveness of its response to dynamic control inputs.