MATH2921: Vector Calculus and Differential Equations (Adv)

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Contents

1 Curve, Surface and Derivatives

1.1 Dot and Cross Products

Definition 1.1.1 (Dot product). If $\vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$, then the dot product of $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \cdot \vec{b} = \sum_{k=1}^{n} a_k b_k.$$

The dot product obeys many of th laws that hold for ordinary products of a real numbers. These are stated in the following theorem.

Properties of the Dot Product. If $\vec{a}, \vec{b}, \vec{c} \in \mathbb{F}^n$ and $\lambda \in \mathbb{F}$, then

1. $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
4. $(\lambda \vec{a}) \cdot \vec{b} = \lambda(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (\lambda \vec{b})$
5. $\vec{0} \cdot \vec{a} = 0$

These can be easily proven. The dot product between $\vec{a}$ and $\vec{b}$ also can be given a geometric interpretation in terms of the angle θ between $\vec{a}$ and $\vec{b}$, where $0 \leq \theta \leq \pi$. The formula in the following theorem is used commonly.

Theorem 1.1.2. If θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, then

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta.$$

This can be proven using the cosine rule.

Now, given two nonzero vectors $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, it is very useful to be able to find a nonzero vector $\vec{c}$ that is perpendicular/orthogonal to both $\vec{a}$ and $\vec{b}$. $\vec{c}$ can be obtained through the cross product between $\vec{a}$ and $\vec{b}$.

Definition 1.1.3 (Cross product). If $\vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$, then the cross product of $\vec{a}$

and $\overrightarrow{b}$ is the vector

$$\overrightarrow{a} \times \overrightarrow{b} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$$

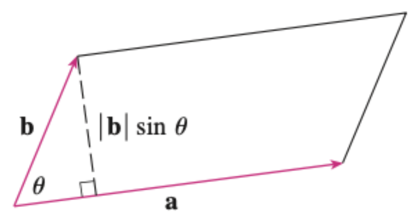
Theorem 1.1.4. If θ is the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$ (so $0 \leq \theta \leq \pi$), then

$$\|\overrightarrow{a} \times \overrightarrow{b}\| = \|\overrightarrow{a}\|\|\overrightarrow{b}\| \sin\theta.$$

This yields the following.

Corollary 1.1.5. Two nonzero vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ are parallel if and only if $\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{0}$.

The geometric interpretation of Theorem 1.1.4 can be seen by looking at the figure below.



If $\overrightarrow{a}$ and $\overrightarrow{b}$ are represented by directed line segments with the same initial point, then they determine a parallelogram with base $\|\overrightarrow{a}\|$, altitude $\|\overrightarrow{b}\| \sin\theta$, and area

$$A = \|\overrightarrow{a}\|(\|\overrightarrow{b}\| \sin\theta) = \|\overrightarrow{a} \times \overrightarrow{b}\|.$$

1.2 Vectors and Curves

A *vector-valued function*, or *vector function*, is simply a function that maps a set of real numbers to a set of vectors.

The limit of a vector function $\overrightarrow{r}$ is defined by taking the limits of its component functions as follows.

If $\overrightarrow{r}(t) = \begin{pmatrix} f(t) \\ g(t) \\ h(t) \end{pmatrix}$, then

$$\lim_{t \to a} \overrightarrow{r}(t) = \begin{pmatrix} \lim_{t \to a} f(t) \\ \lim_{t \to a} g(t) \\ \lim_{t \to a} h(t) \end{pmatrix}$$

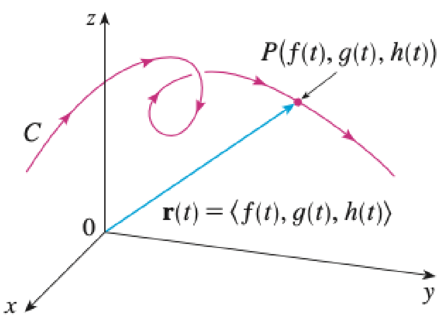
$\overrightarrow{r}$ is *continuous at* a if

$$\lim_{t \to a} \overrightarrow{r}(t) = \overrightarrow{r}(a).$$

Suppose that f, g and h are continuous real-valued functions on an interval I. Then the set C of all points (x, y, z) in space, where

$$x = f(t), \quad y = g(t), \quad z = h(t) \tag{1}$$

and t varies throughout I, is called a *space curve*. The equations in (1) are called *parametric equations of* C and t is called a *parameter*.



We can think of C as being traced out by a moving particle whose position at time t is $(f(t), g(t), h(t))$. If we now consider the vector function

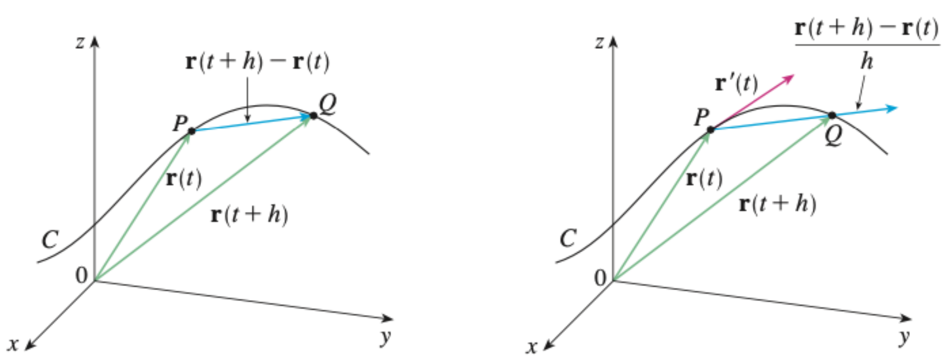
$$\overrightarrow{r}(t) = \begin{pmatrix} f(t) \\ g(t) \\ h(t) \end{pmatrix},$$

then $\overrightarrow{r}(t)$ is the position vector of the point $P(f(t), g(t), h(t))$ on C.

The *derivative* $\overrightarrow{r}'$ of a vector function $\overrightarrow{r}$ is defined in much the same way as for real-valued functions:

$$\frac{d\overrightarrow{r}}{dt} = \overrightarrow{r}'(t) = \lim_{h \to 0} \frac{\overrightarrow{r}(t+h) - \overrightarrow{r}(t)}{h}$$

if this limit exists. The geometric significance is shown in the figures below.



(a) The secant vector $\overrightarrow{PQ}$ (b) The tangent vector $\mathbf{r}'(t)$

If the points P and Q have position vectors $\overrightarrow{r}(t)$ and $\overrightarrow{r}(t+h)$, then $\overrightarrow{PQ}$ represents the vector $\overrightarrow{r}(t+h) - \overrightarrow{r}(t)$, which can therefore be regarded as a secant vector. If $h > 0$, the scalar multiple $(1/h)(\overrightarrow{r}(t+h) - \overrightarrow{r}(t))$ has the same direction as $\overrightarrow{r}(t+h) - \overrightarrow{r}(t)$. As $h \to 0$, it appears that this vector approaches a vector that lies on the tangent line. For this reason, the vector $\overrightarrow{r}'(t)$ is called the *tangent vector* to the curve defined by $\overrightarrow{r}$ at the position P, provided that $\overrightarrow{r}'(t)$ exists and $\overrightarrow{r}'(t) \neq \overrightarrow{0}$. The *tangent line* to C at P is defined to be the line through P parallel to the tangent vector $\overrightarrow{r}'(t)$. We will also have occasion to consider the *unit tangent vector*,

$$\overrightarrow{T}(t) = \frac{\overrightarrow{r}'(t)}{\|\overrightarrow{r}'(t)\|}.$$

The following theorem gives us a convenient method for computing the derivative of a vector function.

Theorem 1.2.1. If $\overrightarrow{r}(t) = \begin{pmatrix} f(t) \\ g(t) \\ h(t) \end{pmatrix}$, where f, g, h are differentiable, then

$$\overrightarrow{r}'(t) = \begin{pmatrix} f'(t) \\ g'(t) \\ h'(t) \end{pmatrix}.$$

Proof.

$$\begin{aligned}
\overrightarrow{r}'(t) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}[\overrightarrow{r}(t+\Delta t) - \overrightarrow{r}(t)] \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\begin{pmatrix} f(t+\Delta t) \\ g(t+\Delta t) \\ h(t+\Delta t) \end{pmatrix} - \begin{pmatrix} f(t) \\ g(t) \\ h(t) \end{pmatrix} \right] \\
&= \lim_{\Delta t \to 0} \begin{pmatrix} \frac{f(t+\Delta t)-f(t)}{\Delta t} \\ \frac{g(t+\Delta t)-g(t)}{\Delta t} \\ \frac{h(t+\Delta t)-h(t)}{\Delta t} \end{pmatrix} \\
&= \begin{pmatrix} \lim_{\Delta t \to 0} \frac{f(t+\Delta t)-f(t)}{\Delta t} \\ \lim_{\Delta t \to 0} \frac{g(t+\Delta t)-g(t)}{\Delta t} \\ \lim_{\Delta t \to 0} \frac{h(t+\Delta t)-h(t)}{\Delta t} \end{pmatrix} \\
&= \begin{pmatrix} f'(t) \\ g'(t) \\ h'(t). \end{pmatrix}
\end{aligned}$$

The next theorem shows that the differentiation formulas for real-valued functions have their counterparts for vector-valued functions.

Theorem 1.2.2 (Differentiation Rule). Suppose $\overrightarrow{u}$ and $\overrightarrow{v}$ are differentiable vector-valued functions, c is a scalar, and f is a real-valued function. Then:

1. $\frac{d}{dt}[\overrightarrow{u}(t) + \overrightarrow{v}(t)] = \overrightarrow{u}'(t) + \overrightarrow{v}'(t)$

2. $\frac{d}{dt}[c\overrightarrow{u}(t)] = c\overrightarrow{u}'(t)$

3. $\frac{d}{dt}[f(t)\overrightarrow{u}(t)] = f'(t)\overrightarrow{u}(t) + f(t)\overrightarrow{u}'(t)$

4. $\frac{d}{dt}[\overrightarrow{u}(t) \cdot \overrightarrow{v}(t)] = \overrightarrow{u}'(t) \cdot \overrightarrow{v}(t) + \overrightarrow{u}(t) \cdot \overrightarrow{v}'(t)$

5. $\frac{d}{dt}[\overrightarrow{u}(t) \times \overrightarrow{v}(t)] = \overrightarrow{u}'(t) \times \overrightarrow{v}(t) + \overrightarrow{u}(t) \times \overrightarrow{v}'(t)$

6. $\frac{d}{dt}[\overrightarrow{u}(f(t))] = f'(t)\overrightarrow{u}'(f(t))$ (Chain Rule)

The *definite integral* of a continuous vector function $\overrightarrow{r}(t)$can be defined in much the same way as for real-valued functions except that the integral is a vector. But then we can express the integral of $\overrightarrow{r}$ in terms of the integrals of its component functions f, g and h as follows.

$$\begin{aligned}
\int_a^b \overrightarrow{r}(t)\,dt &= \lim_{n\to\infty} \sum_{k=1}^{n} \overrightarrow{r}(t_k^*)\Delta t \\
&= \lim_{n\to\infty} \begin{pmatrix} \sum_{k=1}^{n} f(t_k^*)\Delta t \\ \sum_{k=1}^{n} g(t_k^*)\Delta t \\ \sum_{k=1}^{n} h(t_k^*)\Delta t \end{pmatrix}
\end{aligned}$$

and so

$$\int_a^b \overrightarrow{r}(t)\,dt = \begin{pmatrix} \int_a^b f(t)\,dt \\ \int_a^b g(t)\,dt \\ \int_a^b h(t)\,dt \end{pmatrix}.$$

This means that we can evaluate an integral of a vector function by integrating each component function. We can extend the Fundamental Theorem of Calculus to continuous vector functions as follows:

$$\int_a^b \overrightarrow{r}(t)\,dt = \overrightarrow{R}(t)\Big|_a^b = \overrightarrow{R}(b) - \overrightarrow{R}(a)$$

where $\overrightarrow{R}$ is an anti-derivative of $\overrightarrow{r}$.

1.3 Partial Derivatives and the Chain Rule

Definition 1.3.1 (Partial Derivative). If f is a function of n variables, its *partial derivative* with respect to x_i is defined by

$$\frac{\partial f}{\partial x_i} = f_{x_i}(x_1, \ldots, x_n) = \lim_{h\to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}.$$

The partial derivatives f_{x_i} and f_{x_j} are also functions of n variables, so we can consider their partial derivatives $(f_{x_i})_{x_i}$, $(f_{x_i})_{x_j}$, $(f_{x_j})_{x_i}$ and $(f_{x_j})_{x_j}$, which are called the *second partial derivatives* of f.

We use the following notations:

$$(f_{x_i})_{x_i} = f_{x_ix_i} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial^2 f}{\partial x_i^2}$$
$$(f_{x_i})_{x_j} = f_{x_ix_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial^2 f}{\partial x_j \partial x_i}$$
$$(f_{x_j})_{x_i} = f_{x_jx_i} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial^2 f}{\partial x_i \partial x_j}$$
$$(f_{x_j})_{x_j} = f_{x_jx_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial^2 f}{\partial x_j^2}$$

Theorem 1.3.2 (Clairant's Theorem). Suppose that f is defined on a disk D that contains the point $\overrightarrow{a}$. If the functions $f_{x_ix_j}$ and $f_{x_jx_i}$ are both continuous on D, then

$$f_{x_ix_j}(\overrightarrow{a}) = f_{x_jx_i}(\overrightarrow{a}).$$

Theorem (The Chain Rule). Suppose that u is a differentiable function of the n variables $x_1, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, \ldots, t_m$. Then u is a function of $t_1, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \sum_{k=1}^{n} \frac{\partial u}{\partial x_k}\frac{\partial x_k}{\partial t_i}$$

for each $i = 1, 2, \ldots, m$.

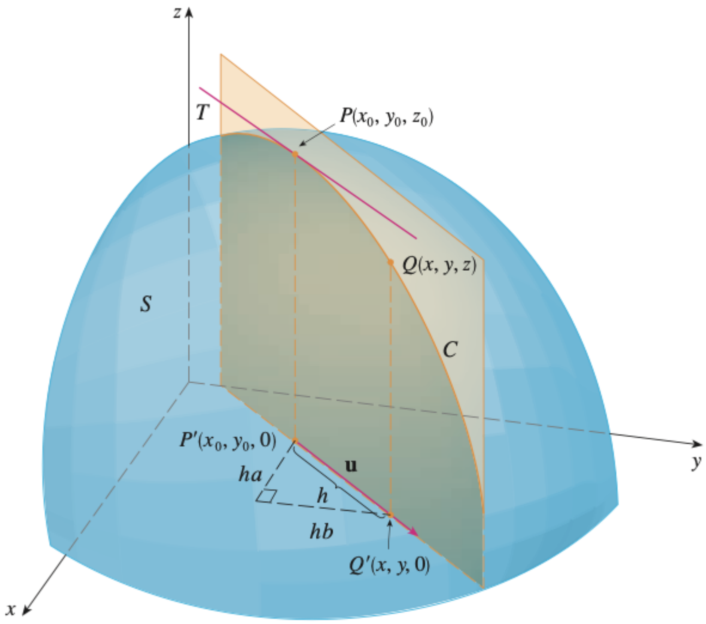
1.4 Directional Derivatives and the Gradient Vector

Recall that if $z = f(x, y)$ then

$$f_x(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$
$$f_y(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

and represent the rates of change of z in the x- and y-directions, that is, in the directions of the standard bases $\overrightarrow{i}$ and $\overrightarrow{j}$.

Suppose that we now want to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\overrightarrow{u} = \begin{pmatrix} a \\ b \end{pmatrix}$. To do so, we consider the surface S with the equation $z = f(x, y)$ (the graph of f) and we let $z_0 = f(x_0, y_0)$. Then the point $P(x_0, y_0, z_0)$ lies on S. The vertical plane that passes through P in the direction of $\overrightarrow{u}$ intersects S in a curve C. The slope of the tangent line T to C at the point P is the rate of change of z in the direction of $\overrightarrow{u}$.



If $Q(x, y, z)$ is another point on C and P', Q' are the projections of P, Q onto the xy-plane, then the vector $\overrightarrow{P'Q'}$ is parallel to $\overrightarrow{u}$ and so

$$\overrightarrow{P'Q'} = h\overrightarrow{u} = \begin{pmatrix} ha \\ hb \end{pmatrix}$$

for some scalar h. Therefore $x - x_0 = ha$, $y - y_0 = hb$, so $x = x_0 + ha$, $y = y_0 + hb$, and

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}.$$

If we take the limit as $h \to 0$, we obtain the rate of change of z (with respect to distance) in the direction of $\overrightarrow{u}$, which is called the directional derivative of f in the direction of $\overrightarrow{u}$.

Definition 1.4.1 (Directional derivative). The *directional derivative* of f at (x_0, y_0) in the direction of an unit vector $\overrightarrow{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ is

$$D_{\overrightarrow{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

When computing the directional derivative of a function defined by a formula, we generally use the following theorem.

Theorem 1.4.2. If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\overrightarrow{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ and

$$D_{\overrightarrow{u}} f(x, y) = f_x(x, y)a + f_y(x, y).$$

Proof. If we define a function g of the single variable h by

$$g(h) = f(x_0 + ha, y_0 + hb)$$

then, by the definition of a derivative, we have

$$\begin{aligned} g'(0) &= \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \\ &= D_{\overrightarrow{u}} f(x_0, y_0). \end{aligned}$$

On the other hand, we can write $g(h) = f(x, y)$, where $x = x_0 + ha$ and $y = y_0 + hb$, so the Chain Rule gives

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = f_x(x, y)a + f_y(x, y)b.$$

If we now put $h = 0$, then $x = x_0$, $y = y_0$ and

$$g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b.$$

Thus, we see that

$$D_{\overrightarrow{u}} f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b.$$

Definition 1.4.3 (The Gradient Vector). If f is a function of two variables x and y, then the *gradient* of f is the vector function $\overrightarrow{\nabla} f$ defined by

$$\overrightarrow{\nabla} f(x, y) = \begin{pmatrix} f_x(x, y) \\ f_y(x, y) \end{pmatrix}$$

With this, we can rewrite the equation for the directional derivative of a differentiable function as

$$D_{\overrightarrow{u}} f(x, y) = \overrightarrow{\nabla} f(x, y) \cdot \overrightarrow{u}$$

The same ideas apply for functions of more variables.

1.5 Tangent Planes to Level Surfaces

Suppose S is a surface with the equation $F(x, y, z) = k$, that is, it is a level surface of a function F of three variables, and let $P(x_0, y_0, z_0)$ be a point on S. Let C be any curve that lies on the surface S and passes through P. Recall that the curve C is described by a continuous vector-valued function $\overrightarrow{r}(t) = \begin{pmatrix} x(t) & y(t) & z(t) \end{pmatrix}^T$. Let t_0 be the parameter value corresponding to P, that is, $\overrightarrow{r}(t_0) = \begin{pmatrix} x_0 & y_0 & z_0 \end{pmatrix}^T$. Since C lies on S, any point $(x(t), y(t), z(t))$ must satisfy the equation of S, that is

$$F(x(t), y(t), z(t)) = k. \tag{2}$$

If x, y and z are differentiable functions of t and F is also differentiable, then we can use the Chain Rule to differentiate both sides of (2) as

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0. \tag{3}$$

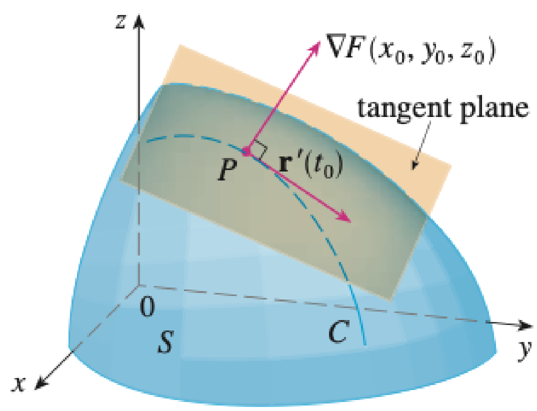
Note that (3) can also be written as

$$\vec{\nabla} F \cdot \vec{r}\,'(t) = 0.$$

In particular, when $t = t_0$ we have $\vec{r}(t_0) = \begin{pmatrix} x_0 & y_0 & z_0 \end{pmatrix}^T$, so

$$\vec{\nabla} F(x_0, y_0, z_0) \cdot \vec{r}\,'(t_0) = 0.$$

It means that the gradient vector at P, $\vec{\nabla} F(x_0, y_0, z_0)$ is perpendicular to the tangent vector $\vec{r}\,'(t_0)$ to any curve C on S that passes through P.



If $\vec{\nabla} F(x_0, y_0, z_0) \neq \vec{0}$, it is therefore natural to define the *tangent plane to the level surface* $F(x, y, z) = k$ *at* $P(x_0, y_0, z_0)$ as the plane that passes through P and has normal vector $\vec{\nabla} F(x_0, y_0, z_0)$. Using the standard equation of a plane, we can write the equation of this tangent plane as

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 \tag{4}$$

The *normal line* to S at P is the line passing through P and perpendicular to the tangent plane. The direction of the normal line is therefore given by $\vec{\nabla} F(x_0, y_0, z_0)$ and so, its symmetric equations are

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}. \tag{5}$$

In the special case in which the equation of a surface S is of the form $z = f(x, y)$, we can rewrite the equation as

$$F(x, y, z) = f(x, y) - z = 0$$

and regard S as a level surface (with $k = 0$) of F. Then

$$\begin{aligned} F_x(x_0, y_0, z_0) &= f_x(x_0, y_0) \\ F_y(x_0, y_0, z_0) &= f_y(x_0, y_0) \\ F_z(x_0, y_0, z_0) &= -1 \end{aligned}$$

so equation (4) becomes

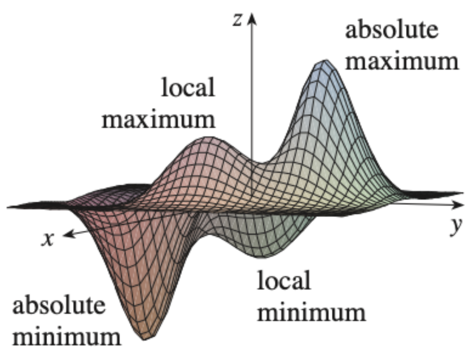
$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

2 Critical Points and Optimisation

2.1 Critical Points and Optimisation

Definition 2.1.1 (Local Extrema). A function of two variables has a *local maximum* (or *local minimum*) at (a, b) if $f(x, y) \leq f(a, b)$ (or $f(x, y) \geq f(a, b)$) for all points (x, y) in some disk with centre (a, b).

If the inequalities in Definition 2.1.1 hold for all points (x, y) in the domain of f, then f has an *absolute/global maximum* (or *absolute/global minimum*) at (a, b).



Theorem 2.1.2. If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = f_y(a, b) = 0$.

We need to be able to determine whether or not a function has an extreme value at a critical point.

Theorem 2.1.3 (Second Derivative Test). Suppose the second partial derivatives of f are continuous on a disk with centre (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Consider the Hessian matrix of f at (a, b)

$$D^2 f(a, b) = \begin{pmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{pmatrix}.$$

Let λ_1 and λ_2 be eigenvalues of $D^2 f(a, b)$ such that $\lambda_1 \leq \lambda_2$.

(a) If $\lambda_1 \leq \lambda_2 < 0$, then $f(a, b)$ is a *local maximum.*

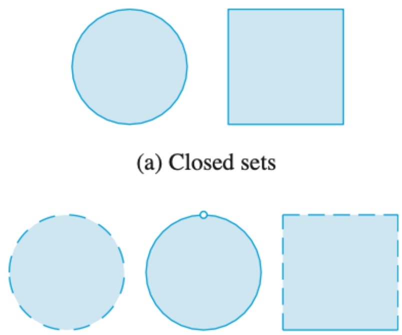
(b) If $0 < \lambda_1 \leq \lambda_2$, then $f(a, b)$ is a *local minimum.*

(c) If $\lambda_1 < 0 < \lambda_2$, then (a, b) is a *saddle point.*

(d) If $\lambda_1 \cdot \lambda_2 = 0$, then the test is inconclusive.

Global Extrema Values

For a function f on one variable, the Extreme Value Theorem says that if f is continuous on a closed interval $[a, b]$, then f has a global minimum value and a global maximum value. There is a similar situation for functions of two variables. Just as a closed interval contains its endpoints, a *closed set* in $\mathbb{R}^2$ is one that contains all its boundary points.


(a) Closed sets

(b) Sets that are not closed

A *bounded set* in $\mathbb{R}^2$ is one that is contained within some disk. In other words, it is finite in extent. Then, in terms of closed and bounded sets, we can state the following counterpart of the Extreme Value Theorem in two dimensions.

Theorem 2.1.4 (Extreme Value Theorem for Functions of Two Variables). If f is continuous on a closed, bounded set $D \subset \mathbb{R}^2$, then f attains a global maximum value $f(x_1, y_1)$ and a global minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

To find the extreme values guaranteed by Theorem 2.1.4, use the following method.

To find the global maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the global maximum value; the smallest of these values is the absolute minimum value.

2.2 Method of Lagrange Multipliers

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ (assuming that these extreme values exist and $\overrightarrow{\nabla} g \neq \overrightarrow{0}$ on the surface $g(x, y, z) = k$):

(a) Find all values of x, y, z, and λ such that

$$\overrightarrow{\nabla} f(x, y, z) = \lambda \overrightarrow{\nabla} g(x, y, z)$$

and

$$g(x, y, z) = k.$$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimu value of f.

If we write the vector equation $\overrightarrow{\nabla} f = \lambda \overrightarrow{\nabla} g$ in terms of components, then the equations in step (a) becomes

$$f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad f_z = \lambda g_z, \quad g(x, y, z) = k.$$

This is a system of four equations in the four unknowns x, y, z, and λ, but it is not necessary to find explicit values for λ.

3 Multiple Integrals

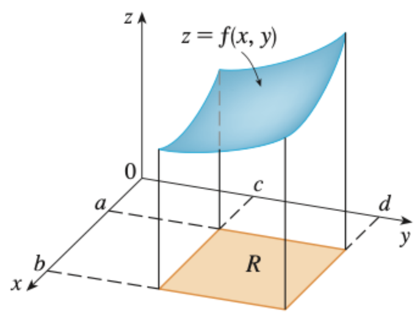
3.1 Double Integrals

Consider a function f of two variables defined on a closed rectangle

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ c \leq y \leq d\}$$

and we first suppose that $f(x, y) \geq 0$. The graph of f is a surface with equation $z = f(x, y)$. let S be the solid that lies above R and under the graph of f, that is,

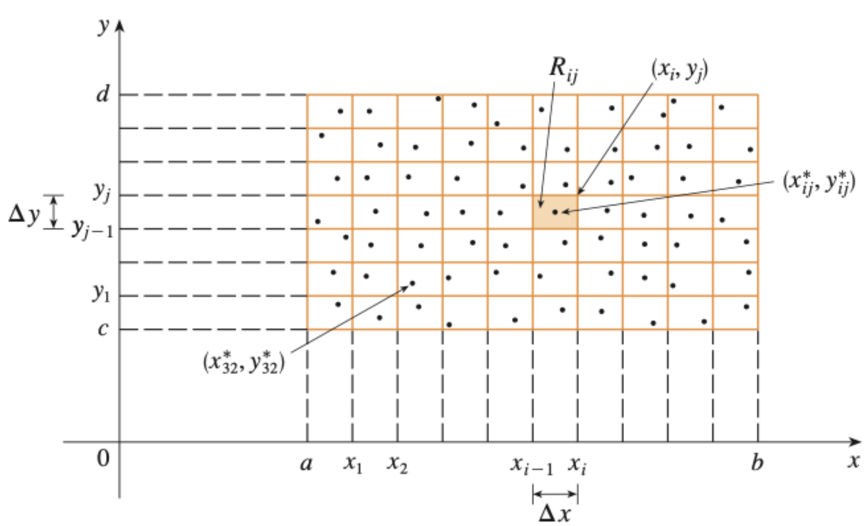
$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z \leq f(x, y),\ (x, y) \in R\}.$$



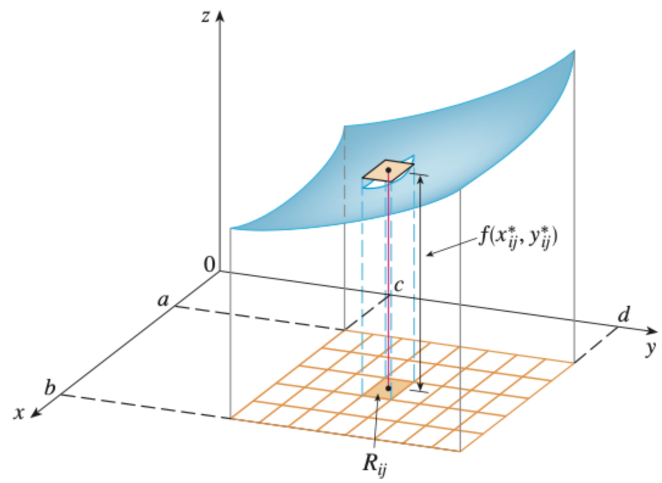
Our goal is to find the volume of S. The first step is to divide the rectangle R into subrectangles. We do this by dividing the interval $[a, b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b-a)/m$ and dividing $[c, d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = (d - c)/n$. By drawing lines parallel to the coordinate axes through the endpoints of these subintervals, as in following figure, we form the subrectangels

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \leq x \leq x_i,\ y_{j-1} \leq y \leq y_j\}$$

each with area $\Delta A = \Delta x \Delta y$.



If we choose a *sample point* (x_{ij}^*, y_{ij}^*) in each R_{ij}, then we can approximate the part of S that lies above each R_{ij} by a thin rectangular box (or "column") with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$.

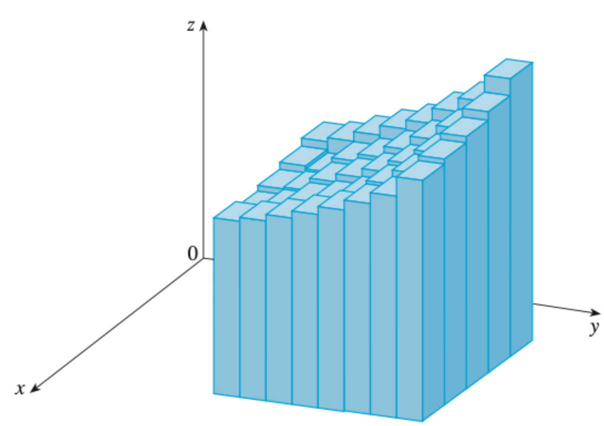


The volume of this box is the height of the box times the area of the base rectangle:

$$f(x_{ij}^*, y_{ij}^*)\Delta A$$

If we follow this procedure for all the rectangles and add the volumes of the corresponding boxes, we get an approximation to the total volume of S:

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A \tag{6}$$



This double sum means that for each subrectangle we evaluate f at the chosen point and multiply by the area of the subrectangle, and then we add the results.

Our intuition tells us that the approximation in (6) becomes better as m and n become larger and so we would expect that

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A.$$

Definition 3.1.1 (Double Integral). The *double integral* of f over the rectangle R is

$$\iint_R f(x,y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

if this limit exists.

3.2 Iterated Integrals

Theorem 3.2.1 (Fubini's Theorem). If f is continuous on the rectangle

$$R = \{(x,y) \,|\, a \le x \le b,\ c \le y \le d\},$$

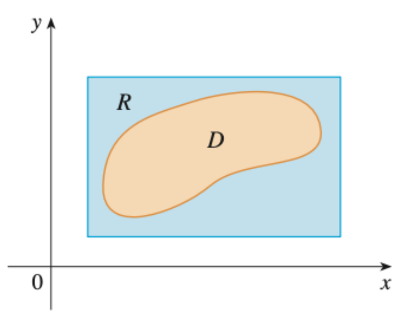
then

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Double Integrals over General Regions

Suppose that D is a bounded region, which means that D can be enclosed in a rectangular region R as shown below.



Then we define a new function F with domain R by

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D \end{cases} \tag{7}$$

If F is integrable over R, then we define the double integral of f over D by

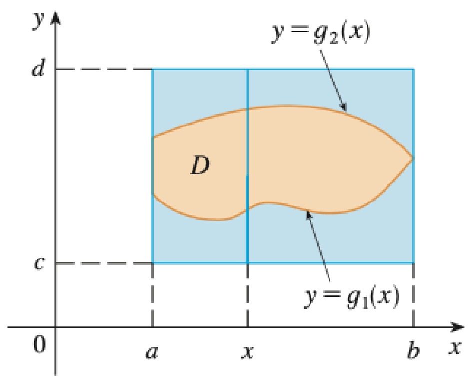
$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA. \tag{8}$$

Now we consider two types of regions.

A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

where g_1 and g_2 are continuous on $[a,b]$. In order to evaluate $\iint_D f(x,y)\,dA$ when D is a region of type I, we choose a rectangle $R = [a,b] \times [c,d]$ that contains D, as in the figure below, and we let F be the function given by equation (7); that is, F agrees with f on D and F is 0 outside D.



Then, by Fubini's Theorem,

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA = \int_a^b \int_c^d F(x,y)\,dy\,dx.$$

Observe that $F(x,y) = 0$ if $y < g_1(x)$ or $y > g_2(x)$ because (x,y) then lies outside D. Therefore

$$\int_c^d F(x,y)\,dy = \int_{g_1(x)}^{g_2(x)} F(x,y)\,dy = \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$$

because $F(x,y) = f(x,y)$ when $g_1(x) \le y \le g_2(x)$. Thus, we have the following formula that enables us to evaluate the double integral as an iterated integral. If f is continuous on a type I region D such that

$$D = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

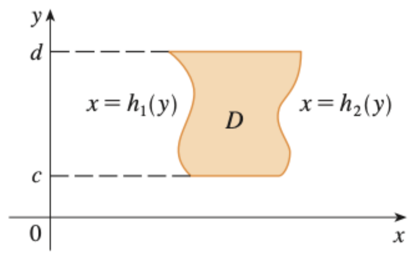
then

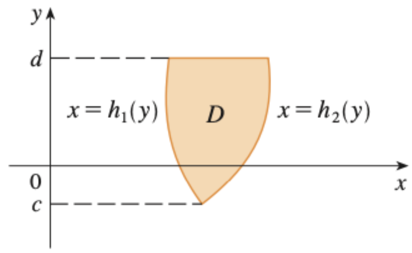
$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx. \tag{9}$$

We also consider plane region of **type II**, which can be expressed as

$$D = \{(x,y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$$

where h_1 and h_2 are continuous. Examples of D are illustrated below.





Using the same methods that were used in establishing (9), we can show that

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Properties of Double Integrals.

1. If $f(x, y) \geq g(x, y)$ for all $(x, y) \in D$, then

$$\iint_D f(x, y)\, dA \geq \iint_D g(x, y)\, dA.$$

2. Consider a region D that is neither type I nor type II but can be expressed as a union of regions of type I and type II. If $D = D_1 \cup D_2$, where D_1 and D_2 do not overlap except perhaps on their boundaries, then

$$\iint_D f(x, y)\, dA = \iint_{D_1} f(x, y)\, dA + \iint_{D_2} f(x, y)\, dA.$$

3. The area of a region D can be obtained through

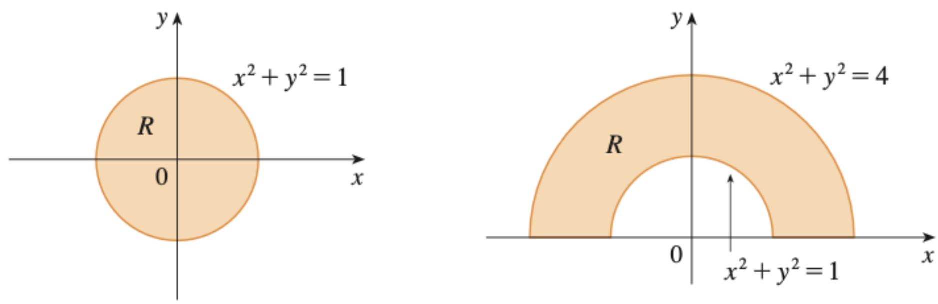
$$A(D) = \iint_D 1\, dA.$$

4. If $m \leq f(x, y) \leq M$ for all $(x, y) \in D$, then

$$mA(D) \leq \iint_D f(x, y)\, dA \leq MA(D).$$

3.3 Double Integrals in Polar Coordinates

Suppose that we want to evaluate a double integral $\iint_R f(x,y)\,dA$ where R is one of the regions in the figures below.



(a) $R = \{(r, \theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$ (b) $R = \{(r, \theta) \mid 1 \le r \le 2, 0 \le \theta \le \pi\}$

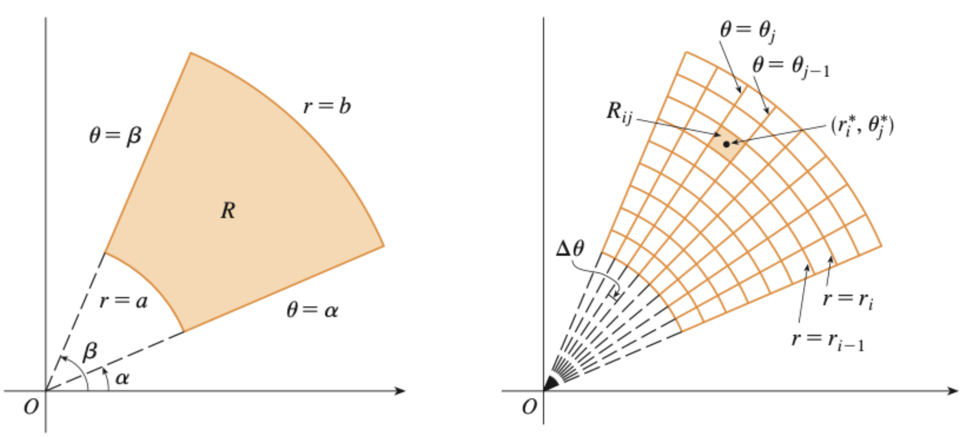
In either case, the description of R in terms of rectangular coordinates is rather complicated, but R is easily described using polar coordinates. Recall that the polar coordinates (r,θ) of a point are related to the rectangular coordinates (x,y) by the equations

$$r^2 = x^2 + y^2, \quad x = r\cos\theta, \quad y = r\sin\theta.$$

The regions in above figure are special cases of a *polar rectangle*

$$R = \{(r,\theta) \,|\, a \le r \le b, \alpha \le \theta \le \beta\}$$

which is shown below.



In order to compute the double integral $\iint_R f(x,y)\,dA$, where R is a polar rectangle, we divide the interval $[a,b]$ into m subintervals $[r_{i-1}, r_i]$ of equal width $\Delta r = (b-a)/m$ and we divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{j-1}, \theta_j]$ of equal width $\Delta\theta = (\beta - \alpha)/n$. Then the circles $r = r_i$ and the rays $\theta = \theta_j$ divide the polar rectangle R into small polar rectangles R_{ij}.

The "centre" of the polar subrectangle

$$R_{ij} = \{(r,\theta) \,|\, r_{i-1} \le r \le r_i, \theta_{j-1} \le \theta \le \theta_j\}$$

has polar coordinates

$$r_i^* = \frac{r_{i-1}+r_i}{2}, \;\; \theta_j^* = \frac{\theta_{j-1}+\theta_j}{2}.$$

We compute the area of R_{ij} using the fact that the area of a sector of a circle with radius r and central angle θ is $\frac{1}{2}r^2\theta$. Subtracting the areas of two such sectors, each of which has central angle $\Delta\theta = \theta_j - \theta_{j-1}$, we find that the area of R_{ij} is

$$\begin{aligned}\Delta A_i &= \frac{1}{2}r_i^2\Delta\theta - \frac{1}{2}r_{i-1}^2\Delta\theta = \frac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta \\ &= \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta = r_i^*\Delta r\Delta\theta\end{aligned}$$

The rectangular coordinates of the centre of R_{ij} are $(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)$, so a typical Riemann sum is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\Delta A_i = \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)r_i^*\Delta r\Delta\theta. \tag{10}$$

If we write $g(r,\theta) = rf(r\cos\theta, r\sin\theta)$, then the Riemann sum in (10) can be written as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, \theta_j^*)\Delta r\Delta\theta$$

which is a Riemann sum for the double integral

$$\int_\alpha^\beta \int_a^b g(r,\theta)\,dr\,d\theta.$$

Therefore, we have

$$\begin{aligned}\iint_R f(x,y)\,dA &= \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\Delta A_i \\ &= \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} g(r_i^*, \theta_j^*)\Delta r\Delta\theta = \int_\alpha^\beta\int_a^b g(r,\theta)\,dr\,d\theta \\ &= \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)r\,dr\,d\theta.\end{aligned}$$

Theorem 3.3.1. If f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)r\,dr\,d\theta.$$

We can extend to the following theorem.

Theorem 3.3.2. If f is continuous on a polar region of the form

$$D = \{(r, \theta) \,|\, \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$

then

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta.$$

3.4 Triple Integrals

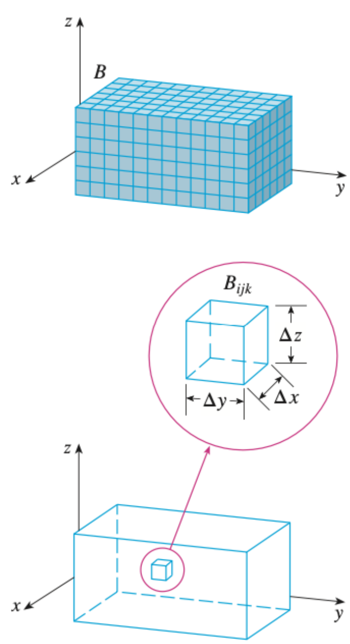
Just as we defined single integrals for functions of one variable and double integrals for functions of two variables, so we can define triple integrals for functions of three variables. We begin with the simplest case where f is defined on a rectangular box;

$$B = \{(x, y, z) \,|\, a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\}$$

The first step is to divide B into sub-boxes. We do this by dividing the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal width Δx, dividing $[c, d]$ into m subintervals of width Δy, and dividing $[r, s]$ into n subintervals of width Δz. The planes through the endpoints of these subintervals parallel to the coordinate planes divide the box B into lmn sub-boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

which are shown in the figure below. Each sub-box has volume $\Delta V = \Delta x \Delta y \Delta z$.



Then we form the *triple Riemann sum*

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V \tag{11}$$

where the sample point $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ is in B_{ijk}. By analogy with the definition of a double integral, we define the triple integral as the limit of the triple Riemann sums in (11).

Definition 3.4.1 (Triple Integral). The *triple integral* of f over the box B is

$$\iiint_B f(x, y, z)\, dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V$$

if this limit exists.

Again, the triple integral always exists if f is continuous. We can choose the sample point to be any point in the sub-box, but if we choose it to be the point (x_i, y_j, z_k) we get a simpler-looking expression for the triple integral:

$$\iiint_B f(x, y, z)\, dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_i, y_j, z_k)\Delta V$$

Just as for double integrals, the practical method for evaluating triple integrals is to express them as iterated integrals as follows.

Theorem 3.4.2 (Fubini's Theorem for Triple Integrals). If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz.$$

Now we define the *triple integral over a general bounded region* E in three-dimensional space (a solid) by much the same procedure that we used for double integrals. We enclose E in a box B of the type given by

$$B = \{(x, y, z) \,|\, a \le x \le b, c \le y \le d, r \le z \le s\}.$$

Then we define F so that it agrees with f on E but is 0 for points in B that are outside E. By definition,

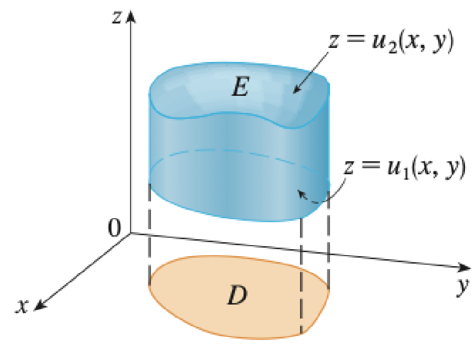
$$\iiint_E f(x, y, z)\, dV = \iiint_B F(x, y, z)\, dV.$$

This integral exists if f is continuous and the boundary of E is reasonably smooth. The triple integral has essentially the same properties as the double integral.

We restrict our attention to continuous functions f and to certain simple types of regions. A solid region E is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

$$E = \{(x, y, z) \,|\, (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\} \tag{12}$$

where D is the projection of E onto the xy-plane as shown in the figure below.



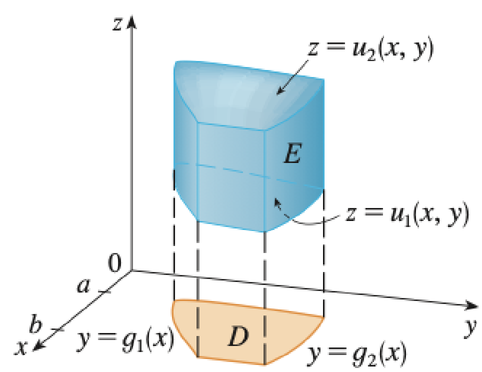
Notice that the upper boundary of the solid E is the surface with equation $z = u_2(x, y)$, while the lower boundary is the surface $z = u_1(x, y)$.

By the same sort of argument for the double integral, it can be shown that if E is a type 1 region given by equation (12), then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right) dA. \tag{13}$$

The meaning of the inner integral on the right side of equation (13) is that x and y are held fixed, and therefore $u_1(x, y)$ and $u_2(x, y)$ are regarded as constants, while $f(x, y, z)$ is integrated with respect to z.

In particular, if the projection D of E onto the xy-plane is a type I plane region (shown in the figure below),



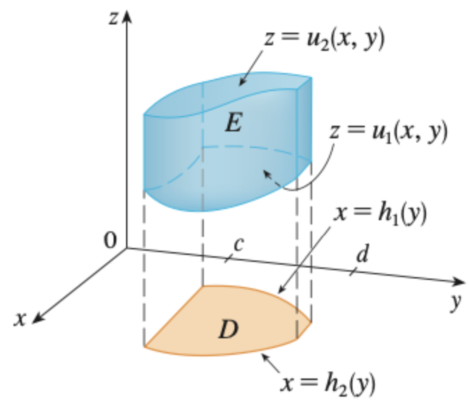
then

$$E = \{(x, y, z) \mid a \le x \le b, g_1(x) \le y \le g_2(x), u_1(x, y) \le z \le u_2(x, y)\}$$

and equation (13) becomes

$$\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dy\, dx. \tag{14}$$

If, on the other hand, D is a type II plane region (as shown below),



then

$$E = \{(x, y, z) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y), u_1(x, y) \leq z \leq u_2(x, y)\} \tag{15}$$

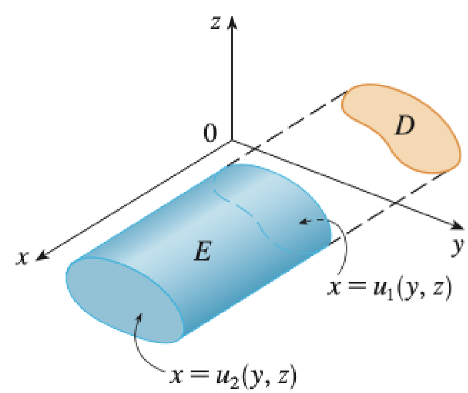
and equation (13) becomes

$$\iiint_E f(x, y, z)\, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dx\, dy \tag{16}$$

A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$$

where, this time, D is the projection of E onto the yz-plane (see the figure below).



The back surface is $x = u_1(y, z)$, the front surface is $x = u_2(y, z)$, and we have

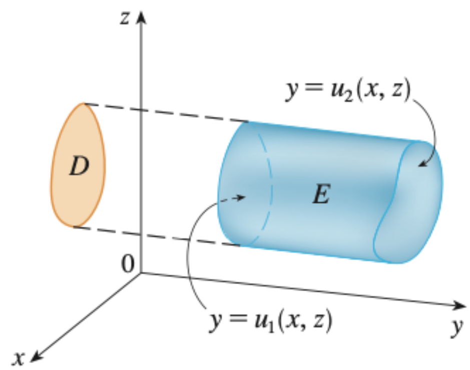
$$\iiint_E f(x, y, z)\, dV = \iint_D \Big(\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx \Big)\, dA. \tag{17}$$

Finally, a **type 3** region is of the form

$$E = \{(x, y, z) \mid (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$$

where D is the projection of E onto the xz-plane, $y = u_1(x,z)$ is the left surface, and $y = u_2(x,z)$ is the right surface (see the figure below). For this type of region we have

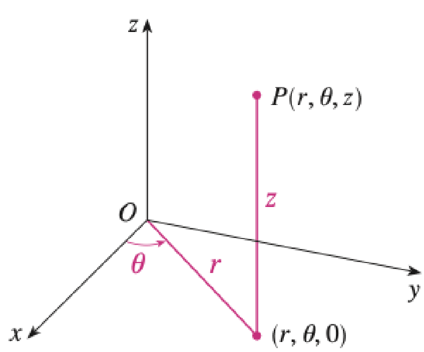
$$\iiint_E f(x,y,z)\,dV = \iint_D \Big(\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy \Big)\,dA. \tag{18}$$



In each of equations (17) and (18) there may be two possible expressions for the integral depending on whether D is a type I or type II plane region (and corresponding to equations (14) and (15)).

3.5 Triple Integrals in Cylindrical Coordinates

In the cylindrical coordinate system, a point P in three-dimensional space is represented by the ordered triple (r, θ, z), where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P.



To convert from cylindrical to rectangular coordinates, we use the equations

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$$

whereas to convert from rectangular to cylindrical coordinates, we use

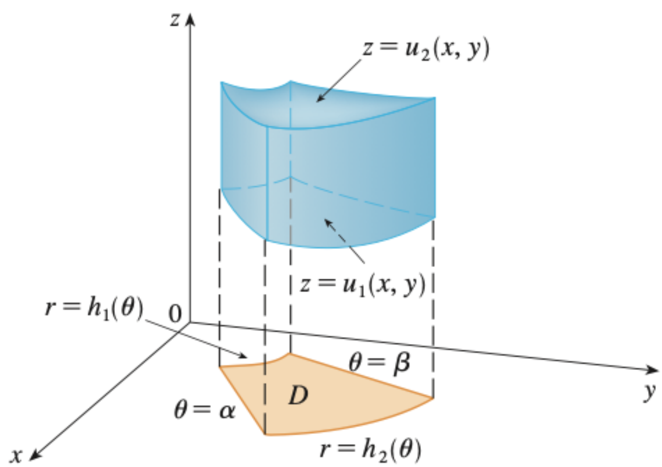
$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}, \quad z = z.$$

Suppose that E is a type 1 region whose projection D onto the xy-plane is conveniently described in polar coordinates. In particular, suppose that f is continuous and

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

where D is given in polar coordinates by

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}.$$



We know from (13) that

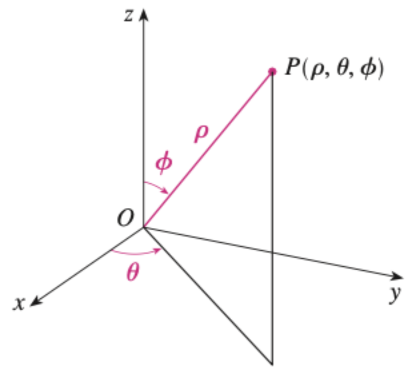
$$\iiint_E f(x, y, z)\, dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right) dA. \tag{19}$$

But we also know how to evaluate double integrals in polar coordinates. In fact, combining equation (19) with Theorem 3.3.2, we obtain

$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta) r\, dz\, dr\, d\theta. \tag{20}$$

3.6 Triple Integrals in spherical Coordinates

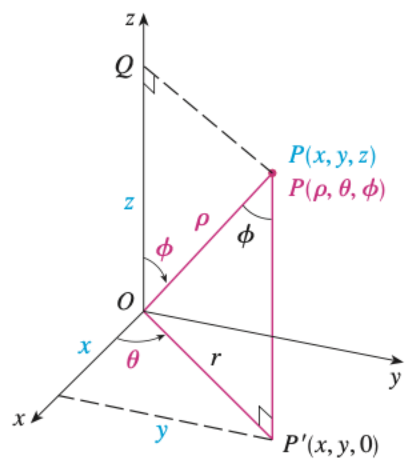
The spherical coordinates (ρ, θ, ϕ) of a point P in space are shown below, where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the line segment OP.



Note that

$$\rho \geq 0, \quad 0 \leq \phi \leq \pi.$$

The relationship between rectangular and spherical coordinates can be seen from the figure below.



From triangles OPQ and OPP' we have

$$z = \rho \cos \phi, \quad r = \rho \sin \phi.$$

But $x = r \cos \theta$ and $y = r \sin \theta$, so to convert from spherical to rectangular coordinates, we use the equations

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi.$$

Also, the distance formula shows that

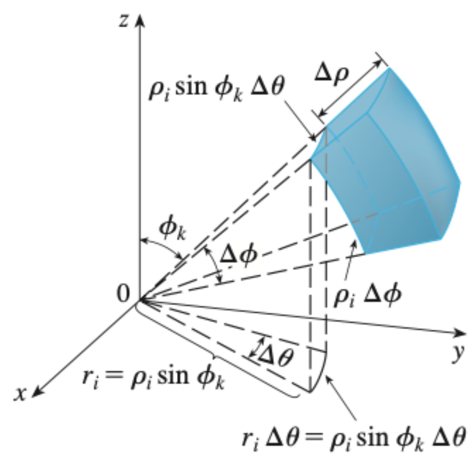
$$\rho^2 = x^2 + y^2 + z^2.$$

We use this equation in converting from rectangular to spherical coordinates.

In the spherical coordinate system the counterpart of a rectangular box is a spherical wedge

$$E = \{(\rho, \theta, \phi) \mid a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}$$

where $a \ge 0$ and $\beta - \alpha \le 2\pi$, and $d - c \le \pi$. Although we defined triple integrals by dividing solids into small boxes, it can be shown that dividing a solid into small spherical wedges always gives the same result. So we divide E into smaller spherical wedges E_{ijk} by means of equally spaced spheres $\rho = \rho_i$, half-planes $\theta = \theta_j$, and half-cones $\phi = \phi_k$.



This figure shows that E_{ijk} is approximately a rectangular box with dimensions $\Delta\rho$, $\rho_i\Delta\phi$ (arc of a circle with radius ρ_i, angle $\Delta\phi$), and $\rho_i \sin\phi_k\Delta\theta$ (arc of a circle with radius $\rho_i \sin\phi_k$, angle $\Delta\theta$). So an approximation to the volume of E_{ijk} is given by

$$\Delta V_{ijk} \approx (\Delta\rho)(\rho_i\Delta\phi)(\rho_i \sin\phi_k\Delta\theta) = \rho_i^2 \sin\phi_k \Delta\rho\Delta\theta\Delta\phi.$$

In fact, it can be shown, with the aid of the Mean Value Theorem, that the volume of E_{ijk} is given exactly by

$$\Delta V_{ijk} = \tilde{\rho}_i^2 \sin\tilde{\phi}_k \Delta\rho\Delta\theta\Delta\phi$$

where $(\tilde{\rho}_i, \tilde{\theta}_j, \tilde{\phi}_k)$ is some point in E_{ijk}. Let $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ be the rectangular coordinates of this point. Then

$$\begin{aligned}
\iiint_E f(x, y, z)\, dV &= \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V_{ijk} \\
&= \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\tilde{\rho}_i \sin\tilde{\phi}_k \cos\tilde{\theta}_j, \tilde{\rho}_i \sin\tilde{\phi}_k \sin\tilde{\theta}_j, \tilde{\rho}_i \cos\tilde{\phi}_k)\tilde{\rho}_i^2 \sin\tilde{\phi}_k \Delta\rho\Delta\theta\Delta\phi
\end{aligned}$$

But this sum is a Riemann sum for the function

$$F(\rho, \theta, \phi) = f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi.$$

Consequently, we have arrived at the following formula for triple integration in spherical coordinates.

$$\iiint_E f(x, y, z)\, dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \tag{21}$$

where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \,|\, a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}.$$

This formula can be extended to include more general spherical regions such as

$$E = \{(\rho, \theta, \phi) \,|\, \alpha \leq \theta \leq \beta, c \leq \phi \leq d, g_1(\theta, \phi) \leq \rho \leq g_2(\theta, \phi)\}.$$

In this case the formula is the same as in (21) except that the limits of integration for ρ are $g_1(\theta, \phi)$ and $g_2(\theta, \phi)$.

3.7 Change of Variables in Multiple Integrals

A change of variables can also be useful in double integrals. An example of this is conversion to polar coordinates. The new variables r and θ are related to the old variables x and y by the equations $x = r\cos\theta$ and $y = r\sin\theta$ and the change of variable formula can be written as

$$\iint_R f(x, y)\, dA = \iint_S f(r\cos\theta, r\sin\theta) r \, dr \, d\theta$$

where S is the region in the $r\theta$-plane that corresponds to the region R in the xy-plane.

More generally, we consider a change of variables that is given by a transformation T from the uv-plane to the xy-plane:

$$T(u, v) = (x, y)$$

where x and y are related to u and v by the equations

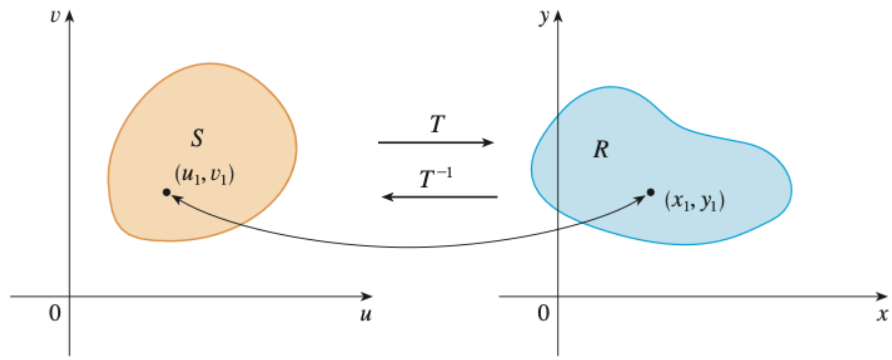
$$x = g(u, v), \quad y = h(u, v) \tag{22}$$

or, as we sometimes write,

$$x = x(u, v), \quad y = y(u, v).$$

We usually assume that T is a C^1 transformation, which means that g and h have continuous first-order partial derivatives.

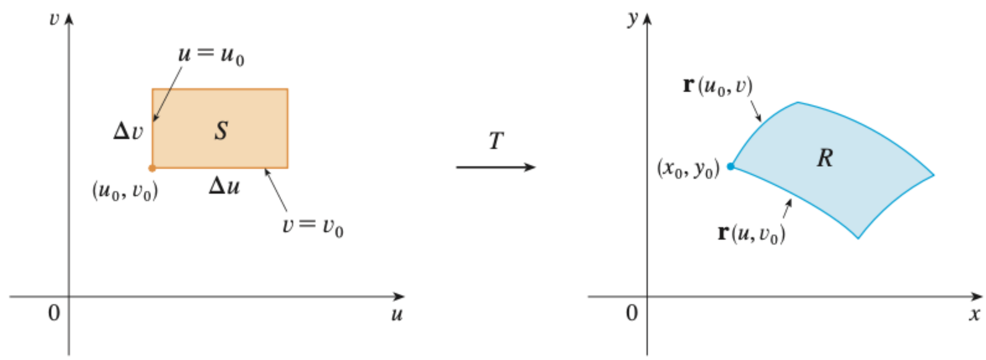
A transformation T is really just a function whose domain and range are both subsets of $\mathbb{R}^2$. If $T(u_1, v_1) = (x_1, y_1)$, then the point (x_1, y_1) is called the *image* of the point (u_1, v_1). If no two points have the same image, T is called *one-to-one*. The next figure shows the effect of a transformation T on a region S in the uv-plane. T transforms S into a region R in the xy-plane called the *image of* S, consisting of the images of all points in S.



If T is a one-to-one transformation, then it has an *inverse transformation* T^{-1} from the xy-plane to the uv-plane and it may be possible to solve equation (22) for u and v in terms of x and y:

$$u = G(x, y), \quad v = H(x, y).$$

Now let's see how a change of variables affects a double integral. We start with a small rectangle S in the uv-plane whose lower left corner is the point (u_0, v_0) and whose dimensions are Δu and Δv.



The image of S is a region R in the xy-plane, one of whose boundary points is $(x_0, y_0) = T(u_0, v_0)$. The vector

$$\overrightarrow{r}(u, v) = \begin{pmatrix} g(u, v) \\ h(u, v) \end{pmatrix}$$

is the position vector of the image of the point (u, v). The equation of the lower side of S is $v = v_0$, whose image curve is given by the vector function $\overrightarrow{r}(u, v_0)$. The tangent vector at (x_0, y_0) to this image curve is

$$\overrightarrow{r}_u = \begin{pmatrix} g_u(u_0, v_0) \\ h_u(u_0, v_0) \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \end{pmatrix}.$$

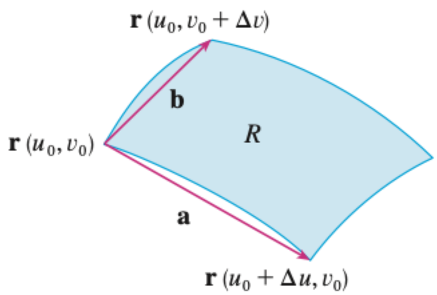
Similarly, the tangent vector at (x_0, y_0) to the image curve of the left side of S (namely, $u = u_0$) is

$$\overrightarrow{r}_v = \begin{pmatrix} g_v(u_0, v_0) \\ h_v(u_0, v_0) \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \end{pmatrix}.$$

We can approximate the image region $R = T(S)$ be a parallelogram determined by the secant vectors

$$\overrightarrow{a} = \overrightarrow{r}(u_0 + \Delta u, v_0) - \overrightarrow{r}(u_0, v_0), \quad \overrightarrow{b} = \overrightarrow{r}(u_0, v_0 + \Delta v) - \overrightarrow{r}(u_0, v_0)$$

shown in the below figure.



But

$$\vec{r}_u = \lim_{\Delta u \to 0} \frac{\vec{r}(u_0 + \Delta u, v_0) - \vec{r}(u_0, v_0)}{\Delta u}$$

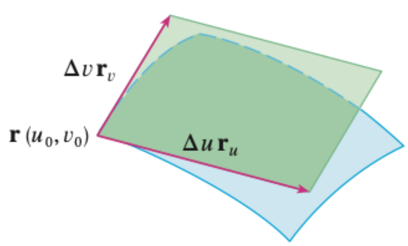
and so

$$\vec{r}(u_0 + \Delta u, v_0) - \vec{r}(u_0, v_0) \approx \Delta u\, \vec{r}_u.$$

Similarly,

$$\vec{r}(u_0, v_0 + \Delta v) - \vec{r}(u_0, v_0) \approx \Delta v\, \vec{r}_v.$$

This means that we can approximate R by a parallelogram determined by the vectors $\Delta u\, \vec{r}_u$ and $\Delta v\, \vec{r}_v$.



Therefore we can approximate the area of R by the area of this parallelogram, which is

$$\|(\Delta u\, \vec{r}_u) \times (\Delta v\, \vec{r}_v)\| = \|\vec{r}_u \times \vec{r}_v\| \Delta u \Delta v. \tag{23}$$

Computing the cross product, we obtain

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \vec{k} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \vec{k}.$$

The determinant that arises in this calculation is called the *Jacobian* of the transformation and is given a special notation.

Definition 3.7.1 (Jacobian). The *Jacobian* of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

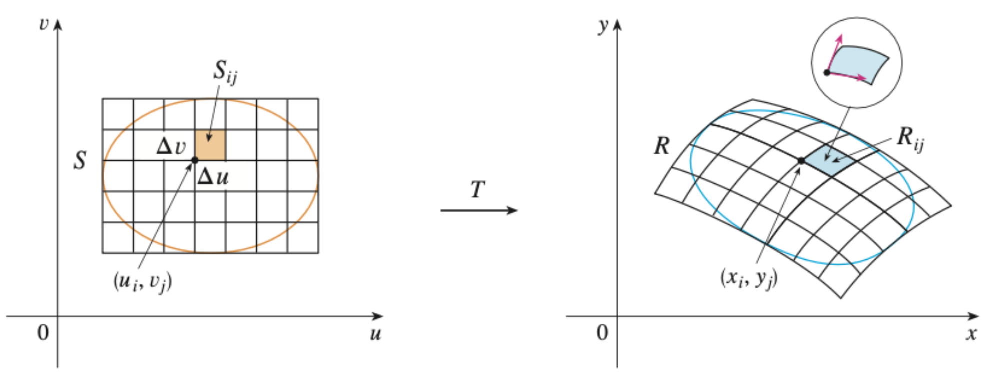
$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

With this notation we can use equation (23) to give an approximation to the area ΔA of R:

$$\Delta A \approx \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \Delta u \Delta v \tag{24}$$

where the Jacobian is evaluated at (u_0, v_0).

Next we divide a region S in the uv-plane into rectangles S_{ij} and call their images in the xy-plane R_{ij}.



Applying the approximation (24) to each R_{ij}, we approximate the double integral of f over R as follows:

$$\begin{aligned} \iint_R f(x, y)\, dA &\approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j) \Delta A \\ &\approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(g(u_i, v_j), h(u_i, v_j)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \Delta u \Delta v \end{aligned}$$

where the Jacobian is evaluated at (u_i, v_j). Notice that this double sum is a Riemann sum for the integral

$$\iint_S f(g(u, v), h(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv.$$

The foregoing argument suggests that the following theorem is true.

Theorem 3.7.2 (Change of Variables in a Double Integral). Suppose that T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

There is a similar change of variables formula for triple integrals. Let T be a transformation that maps a region S in uvw-space onto a region R in xyz-space by means of the equations

$$x = g(u,v,w), \quad y = h(u,v,w), \quad z = k(u,v,w).$$

The Jacobian of T is the following 3×3 determinant:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}. \tag{25}$$

Under hypotheses similar to those in Theorem 3.7.2, we have the following formula for triple integrals:

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(x(u,v,w),y(u,v,w),z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

3.8 Applications of Multiple Integrals

The coordinates (x_c, y_c) of the centre of mass of a lamina occupying the region D and having density function $\rho(x,y)$ are

$$x_c = \frac{1}{m}\iint_D x\rho(x,y)\,dA, \quad y_c = \frac{1}{m}\iint_D y\rho(x,y)\,dA$$

where the mass m is given by

$$m = \iint_D \rho(x,y)\,dA.$$

4 Curved Objects

4.1 Arc Length

Recall that the length of a plane curve with parametric equations $x = f(t)$, $y = g(t)$, $a \le t \le b$, as the limit of lengths of inscribed polygons and, for the case where f' and g' are continuous, we arrived at the formula

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{26}$$

The length of a space curve is defined in exactly the same way. Suppose that the curve has the vector equation

$$\overrightarrow{r}(t) = \begin{pmatrix} f(t) \\ g(t) \\ h(t) \end{pmatrix}, \quad a \le t \le b,$$

where f', g', and h' are continuous. If the curve is traversed exactly once as t increases from a to b, then it can be shown that its length is

$$\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \end{aligned} \tag{27}$$

Notice that both of the arc length formulas (26) and (27) can be put into the more compact form

$$L = \int_a^b \|\overrightarrow{r}'(t)\|\, dt$$

because, for plane curves $\overrightarrow{r}(t) = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$,

$$\|\overrightarrow{r}'(t)\| = \left\| \begin{pmatrix} f'(t) \\ g'(t) \end{pmatrix} \right\| = \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

and for space curves $\overrightarrow{r}(t) = \begin{pmatrix} f(t) \\ g(t) \\ h(t) \end{pmatrix}$,

$$\|\overrightarrow{r}'(t)\| = \left\| \begin{pmatrix} f'(t) \\ g'(t) \\ h'(t) \end{pmatrix} \right\| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}.$$

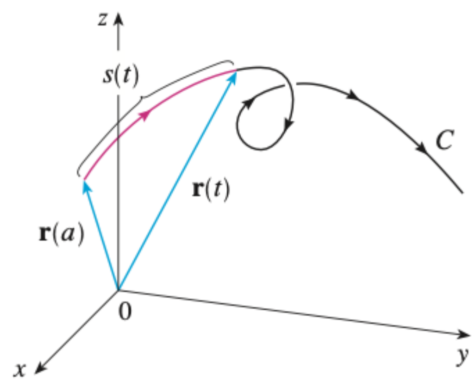
Now suppose that C is a curve given by a vector function

$$\overrightarrow{r}(t) = \begin{pmatrix} f(t) \\ g(t) \\ h(t) \end{pmatrix}, \quad a \le t \le b$$

where $\vec{r}'$ is continuous and C is traversed exactly once as t increases from a to b. We define its *arc length function* s by

$$s(t) = \int_a^t \|\vec{r}'(u)\| \, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} \, du. \tag{28}$$

Thus $s(t)$ is the length of the part of C between $\vec{r}(a)$ and $\vec{r}(t)$.



If we differentiate both sides of equation (28) using the Fundamental Theorem of Calculus, we obtain

$$\frac{ds}{dt} = \|\vec{r}'(t)\|. \tag{29}$$

It is often useful to parametrise a curve with respect to arc length because arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system. If a curve $\vec{r}(t)$ is already given in terms of a parameter t and $s(t)$ is the arc length function given by equation (28), then we may be able to solve for t as a function of s, $t = t(s)$. Then the curve can be reparametrised in terms of s by substituting for t, $\vec{r} = \vec{r}(t(s))$. Thus, if $s = 3$ for instance, $\vec{r}(t(3))$ is the position vector of the point 3 units of length along the curve from its starting point.

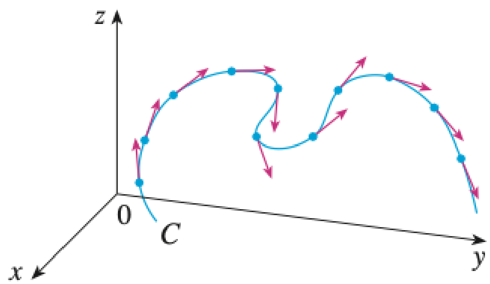
4.2 Curvature

A parametrisation $\vec{r}(t)$ is called *smooth* on an interval I if $\vec{r}'$ is continuous and $\vec{r}'(t) \neq \vec{0}$ on I. A curve is called *smooth* if it has a smooth parametrisation. A smooth curve has no sharp corners or cusps; when the tangent vector turns, it does so continuously.

If C is a smooth curve defined by the vector function $\vec{r}$, recall that the unit tangent vector $\vec{T}(t)$ is given by

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

and indicates the direction of the curve.



From the above figure we can see that $\overrightarrow{T}(t)$ changes direction very slowly when C is fairly straight, but it changes direction more quickly when C bends or twists more sharply.

The curvature of C at a given point is a measure of how quickly the curve changes direction at that point. Specifically, we define it to be the magnitude of the rate of change of the unit tangent vector with respect to arc length. (We use arc length so that the curvature will be independent of the parametrisation.)

Definition 4.2.1 (Curvature). The *curvature* of a curve is

$$k = \left\| \frac{d\overrightarrow{T}}{ds} \right\|$$

where $\overrightarrow{T}$ is the unit tangent vector.

The curvature is easier to compute if it is expressed in terms of the parameter t instead of s, so we use the Chain Rule to write

$$\frac{d\overrightarrow{T}}{dt} = \frac{d\overrightarrow{T}}{ds}\frac{ds}{dt} \quad \text{and} \quad k = \left\| \frac{d\overrightarrow{T}}{ds} \right\| = \left\| \frac{d\overrightarrow{T}/dt}{ds/dt} \right\|$$

But $ds/dt = \|\overrightarrow{r}'(t)\|$ from equation (29), so

$$k(t) = \frac{\|\overrightarrow{T}'(t)\|}{\|\overrightarrow{r}'(t)\|}. \tag{30}$$

Theorem 4.2.2. The curvature of the curve given by the vector function $\overrightarrow{r}$ is

$$k(t) = \frac{\|\overrightarrow{r}''(t) \times \overrightarrow{r}''(t)\|}{\|\overrightarrow{r}'(t)\|^3}.$$

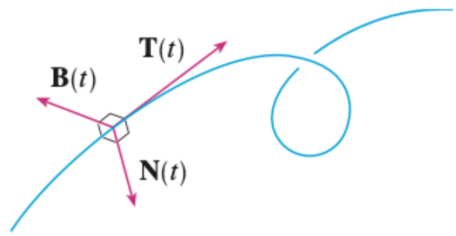
4.3 The Normal and Binormal Vectors

At a given point on a smooth space curve $\overrightarrow{r}(t)$, there are many vectors that are orthogonal to the unit tangent vector $\overrightarrow{T}(t)$. We single out one by observing that, because $\|\overrightarrow{T}(t)\| = 1$ for all t, we have $\overrightarrow{T}(t) \cdot \overrightarrow{T}'(t) = 0$, so $\overrightarrow{T}'(t)$ is orthogonal to $\overrightarrow{T}(t)$. Note that $\overrightarrow{T}'(t)$ is itself not a unit vector. But at any point where $k \neq 0$ we can define the *principal unit normal vector* $\overrightarrow{N}(t)$ (or simply *unit normal*) as

$$\overrightarrow{N}(t) = \frac{\overrightarrow{T}'(t)}{\|\overrightarrow{T}'(t)\|}.$$

The vector $\overrightarrow{B}(t) = \overrightarrow{T}(t) \times \overrightarrow{N}(t)$ is called the *binormal vector*. It is perpendicular to both $\overrightarrow{T}$ and $\overrightarrow{N}$ and is also a unit vector.

We can think of the normal vector as indicating the direction in which the curve is turning at each point.



Summary. We summarise here the formulas for unit tangent, unit normal and binormal vectors, and curvature.

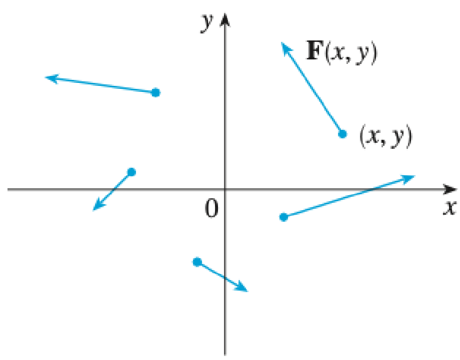
- Unit Tangent:
$$\overrightarrow{T}(t) = \frac{\overrightarrow{r}'(t)}{\|\overrightarrow{r}'(t)\|}$$
- Unit Normal:
$$\overrightarrow{N}(t) = \frac{\overrightarrow{T}'(t)}{\|\overrightarrow{T}'(t)\|}$$
- Binormal Vector:
$$\overrightarrow{B}(t) = \overrightarrow{T}(t) \times \overrightarrow{N}(t)$$
- Curvature:
$$k = \left\| \frac{d\overrightarrow{T}}{ds} \right\| = \frac{\|\overrightarrow{T}'(t)\|}{\|\overrightarrow{r}'(t)\|} = \frac{\|\overrightarrow{r}'(t) \times \overrightarrow{r}''(t)\|}{\|\overrightarrow{r}'(t)\|^3}$$

5 Line Integrals, Green's Theorem and Flux

5.1 Vector Fields

Definition 5.1.1 (Vector Field on $\mathbb{R}^2$). Let D be a set in $\mathbb{R}^2$ (a plane region). A *vector field on* $\mathbb{R}^2$ is a function $\overrightarrow{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\overrightarrow{F}(x, y)$.

The best way to picture a vector field is to draw the arrow representing the vector $\overrightarrow{F}(x, y)$ starting at the point (x, y), however it is impossible to do this for all points (x, y), but we can gain a reasonable impression of $\overrightarrow{F}$ by doing it for a few representative points in D as shown below.

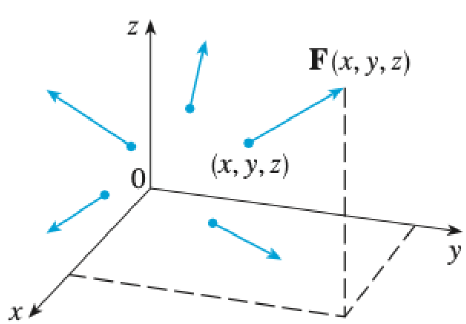


Since $\overrightarrow{F}(x, y)$ is a two-dimensional vector, we can write it in terms of its *component functions* P and Q as follows:

$$\overrightarrow{F}(x, y) = \begin{pmatrix} P(x, y) \\ Q(x, y) \end{pmatrix}.$$

Notice that P and Q are scalar functions of two variables and are sometimes called *scalar fields* to distinguish them from vector fields.

Definition 5.1.2 (Vector Field on $\mathbb{R}^3$). Let E be a subset of $\mathbb{R}^3$. A *vector field on* $\mathbb{R}^3$ is a function $\overrightarrow{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\overrightarrow{F}(x, y, z)$.



We can express a vector field $\overrightarrow{F}$ in $\mathbb{R}^3$ in terms of its component functions P, Q, and R as

$$\overrightarrow{F}(x,y,z) = \begin{pmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{pmatrix}.$$

We can define continuity of vector fields and show that $\overrightarrow{F}$ is continuous if and only if its component functions P, Q, and R are continuous.

We sometimes identify a point (x,y,z) with its position vector $\overrightarrow{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and write $\overrightarrow{F}(\overrightarrow{x})$ instead of $\overrightarrow{F}(x,y,z)$. Then $\overrightarrow{F}$ becomes a function that assigns a vector $\overrightarrow{F}(\overrightarrow{x})$ to a vector $\overrightarrow{x}$.

Gradient Fields

If f is a scalar function of two variables, recall that its gradient $\overrightarrow{\nabla} f$ (or $\operatorname{grad} f$) is defined by

$$\overrightarrow{\nabla} f(x,y) = \begin{pmatrix} f_x(x,y) \\ f_y(x,y) \end{pmatrix}.$$

Therefore $\overrightarrow{\nabla} f$ is really a vector field on $\mathbb{R}^2$ and is called a *gradient vector field*. Likewise, if f is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$ given by

$$\overrightarrow{\nabla} f(x,y,z) = \begin{pmatrix} f_x(x,y,z) \\ f_y(x,y,z) \\ f_z(x,y,z) \end{pmatrix}$$

A vector field $\overrightarrow{F}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\overrightarrow{F} = \overrightarrow{\nabla} f$. In this situation f is called a **potential function** for $\overrightarrow{F}$.

5.2 Line Integrals of Scalar Functions

We start with a plane curve C given by the parametric equations

$$x = x(t), \quad y = y(t) \tag{31}$$

for $t \in [a,b]$, or equivalently, by the vector equation $\overrightarrow{r}(t) = \begin{pmatrix} x(t) & y(t) \end{pmatrix}^T$, and we assume that C is a smooth curve ($\overrightarrow{r}'$ is continuous and $\overrightarrow{r}'(t) \neq \overrightarrow{0}$). If we divide the parameter interval $[a,b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width and we let $x_i = x(t_i)$ and $y_i = y(t_i)$, then the corresponding points $P_i(x_i, y_i)$ divide C into n subarcs with lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$. We choose any point $P_i^*(x_i^*, y_i^*)$ in the ith subarc. (This corresponds to a point $t_i^* \in [t_{i-1}, t_i]$). Now if f is any function of two variables whose domain includes the curve C, we evaluate f at the point (x_i^*, y_i^*), multiply by the length Δs_i of the subarc, and form the sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta s_i$$

which is similar to a Riemann sum. Then we take the limit of these sums and make the following definition by analogy with a single integral.

Definition 5.2.1. If f is defined on a smooth curve C given by equation (31), then the *line integral of f along C* is

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

if this limit exists.

Recall that the length of C is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

A similar type of argument can be used to show that if $f$is continuous, then the limit in Definition 5.2.1 always exists and the following formula can be used to evaluate the line integral

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt. \tag{32}$$

The value of the line integral does not depend on the parametrisation of the curve, provided that the curve is traversed exactly once as t increases from a to b.

If $s(t)$ is the length of C between $\overrightarrow{r}(a)$ and $\overrightarrow{r}(t)$, then

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

So the way to remember formula (32) is to express everything in terms of the parameter t: Use the parametric equations to express x and y in terms of t and write ds as

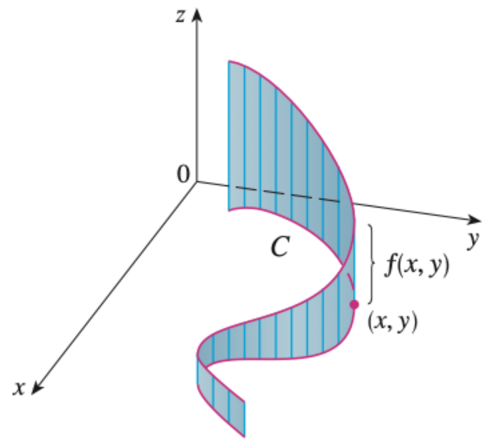
$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

In the special case where C is the line segment that joins $(a,0)$ to $(b,0)$, using x as the parameter, we can write the parametric equations of C as follows: $x = x$, $y = 0$, $a \leq x \leq b$. Formula (32) becomes

$$\int_C f(x,y)\,ds = \int_a^b f(x,0)\,dx$$

and so the line integral reduces to an ordinary single integral in this case.

Just as for an ordinary single integral, we can interpret the line integral of a *positive* function as an area. In fact, if $f(x,y) \geq 0$, $\int_C f(x,y)\,ds$ represents the area of one side of the "fence" or "curtain" in the first figure in next page, whose base is C and whose height above the point (x,y) is $f(x,y)$.



Suppose now that C is a *piecewise-smooth curve*; that is, C is a union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$, where, the initial point of C_{i+1} is the terminal point of C_i. Then we define the integral of f along C as the sum of the integrals of f along each of the smooth pieces of C:

$$\int_C f(x,y)\,ds = \sum_{i=1}^{n} \int_{C_i} f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \cdots + \int_{C_n} f(x,y)\,ds.$$

Two other line integrals are obtained by replacing Δs_i by either $\Delta x_i = x_i - x_{i-1}$ or $\Delta y_i = y_i - y_{i-1}$ in Definition 5.2.1. They are called the *line integrals of f along C with respect to x and y*:

$$\int_C f(x,y)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta x_i \tag{33}$$

$$\int_C f(x,y)\,dy = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta y_i \tag{34}$$

When we want to distinguish the original line integral $\int_C f(x,y)\,ds$ from those in equations (33) and (34), we call it the *line integral with respect to arc length.*

The following formulas say that line integrals with respect to x and y can also be evaluated by expressing everything in terms of t: $x = x(t)$, $y = y(t)$, $dx = x'(t)\,dt$, $dy = y'(t)\,dt$:

$$\int_C f(x,y)\,dx = \int_a^b f(x(t), y(t))x'(t)\,dt \tag{35}$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t), y(t))y'(t)\,dt \tag{36}$$

It frequently happens that line integrals with respect to x and y occur together. When this happens, it is customary to abbreviate by writing

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy.$$

In general, a given parametrisation $x = x(t)$, $y = y(t)$, $t \in [a,b]$, determines an *orientation* of a curve C, with the positive direction corresponding to increasing values of the parameter t. If $-C$

denotes the curve consisting of the same points as C but with the opposite orientation, then we have

$$\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx, \quad \int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy.$$

But if we integrate with respect to arc length, the value of the line integral does NOT change when we reverse the orientation of the curve,

$$\int_{-C} f(x,y)\,ds = \int_C f(x,y)\,ds.$$

This is because Δs_i is always positive, whereas Δx_i and Δy_i change sign when we reverse the orientation of C.

Line Integrals in space

We now suppose that C is a smooth space curve given by the parametric equations

$$x = x(t), \;\; y = y(t), \;\; z = z(t), \;\; t \in [a,b]$$

or by a vector equation $\overrightarrow{r}(t) = \begin{pmatrix} x(t) & y(t) & z(t) \end{pmatrix}^T$. If f is a function of three variables that is continuous on some region containing C, then we define the *line integral of f along C* (with respect to arc length) in a manner similar to that for plane curves:

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta s_i.$$

We evaluate it using a formula similar to formula (32):

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt. \tag{37}$$

Observe that the integrals in both formulas (32) and (37) can be written in the more compact vector notation

$$\int_a^b f(\overrightarrow{r}(t))\|\overrightarrow{r}'(t)\|\,dt.$$

For the special case $f(x,y,z) = 1$, we get

$$\int_C ds = \int_a^b \|\overrightarrow{r}'(t)\|\,dt = L,$$

where L is the length of the curve C.

Line integrals along C with respect to x, y, and z can also be defined. For example,

$$\begin{aligned}\int_C f(x,y,z)\,dz &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta z_i \\ &= \int_a^b f(x(t),y(t),z(t))z'(t)\,dt\end{aligned}$$

Therefore, as with line integrals in the plane, we evaluate integrals of the form

$$\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz \tag{38}$$

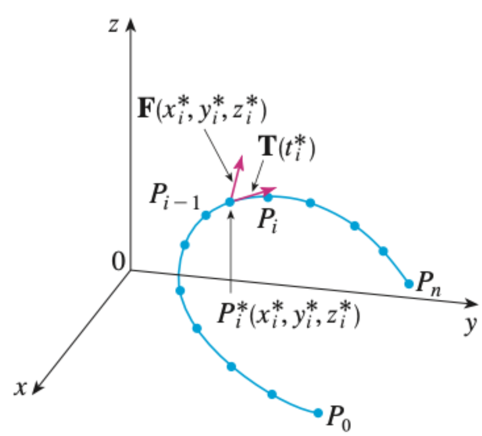
by expressing everything (x, y, z, dx, dy, dz) in terms of the parameter t.

5.3 Line Integrals of Vector Fields

The work done by a variable force $f(x)$ in moving a particle from a to b along the x-axis is $W = \int_a^b f(x)\,dx$. Then the work done by a constant force $\overrightarrow{F}$ in moving an object from a point P to another point Q in space is $W = \overrightarrow{F} \cdot \overrightarrow{D}$, where $\overrightarrow{D} = \overrightarrow{PQ}$ is the displacement vector.

Now suppose that $\overrightarrow{F} = P\overrightarrow{i} + Q\overrightarrow{j} + R\overrightarrow{k}$ is a continuous force field on $\mathbb{R}^3$, such as the gravitational field or electric force field. (A force field on $\mathbb{R}^2$ could be regarded as a special case where $R = 0$ and P and Q depend only on x and y.) We wish to compute the work done by this force in moving a particle along a smooth curve C.

We divide C into subarcs $P_{i-1}P_i$ with lengths Δs_i by dividing the parameter interval $[a, b]$ into subintervals of equal width.



Choose a point $P_i^*(x_i^*, y_i^*, z_i^*)$ on the ith subarc corresponding to the parameter value t_i^*. If Δs_i is small, then as the particle moves from P_{i-1} to P_i along the curve, it proceeds approximately in the direction of $\overrightarrow{T}(t_i^*)$, the unit tangent vector at P_i^*. Thus the work done by the force $\overrightarrow{F}$ in moving the particle from P_{i-1} to P_i is approximately

$$\overrightarrow{F}(x_i^*, y_i^*, z_i^*) \cdot [\Delta s_i \overrightarrow{T}(t_i^*)] = [\overrightarrow{F}(x_i^*, y_i^*, z_i^*) \cdot \overrightarrow{T}(t_i^*)]\Delta s_i$$

and the total work done in moving the particle along C is approximately

$$\sum_{i=1}^{n} [\overrightarrow{F}(x_i^*, y_i^*, z_i^*) \cdot \overrightarrow{T}(x_i^*, y_i^*, z_i^*)]\Delta s_i \tag{39}$$

where $\overrightarrow{T}(x,y,z)$ is the unit tangent vector at the point (x,y,z) on C. Intuitively, we see that these approximations ought to become better as n becomes larger. Therefore we define the *work* W done by the force field $\overrightarrow{F}$ as the limit of the Riemann sums in (39), namely,

$$W = \int_C \overrightarrow{F}(x,y,z) \cdot \overrightarrow{T}(x,y,z)\, ds. \tag{40}$$

Equation (40) says that *work is the line integral with respect to arc length of the tangential component of the force.*

If the curve C is given by the vector equation $\overrightarrow{r}(t) = x(t)\overrightarrow{i} + y(t)\overrightarrow{j} + z(t)\overrightarrow{k}$, then $\overrightarrow{T}(t) = \overrightarrow{r}'(t)/\|\overrightarrow{r}'(t)\|$, so using equation (37) we can rewrite (40) in the form

$$W = \int_a^b \left[\overrightarrow{F}(\overrightarrow{r}(t)) \cdot \frac{\overrightarrow{r}'(t)}{\|\overrightarrow{r}'(t)\|}\right] \|\overrightarrow{r}'(t)\|\, dt = \int_a^b \overrightarrow{F}(\overrightarrow{r}(t)) \cdot \overrightarrow{r}'(t)\, dt.$$

This integral is often abbreviated as $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$.

Definition 5.3.1. Let $\overrightarrow{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\overrightarrow{r}(t)$, $t \in [a,b]$. Then the *line integral of* $\overrightarrow{F}$ *along* C is

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \int_a^b \overrightarrow{F}(\overrightarrow{r}(t)) \cdot \overrightarrow{r}'(t)\, dt = \int_C \overrightarrow{F} \cdot \overrightarrow{T}\, ds.$$

Finally, we note the connection between line integrals of vector fields and line integrals of scalar fields. Suppose the vector field $\overrightarrow{F}$ on $\mathbb{R}^3$ is given in component form by the equation $\overrightarrow{F} = P\overrightarrow{i} + Q\overrightarrow{j} + R\overrightarrow{k}$. We use Definition 5.3.1 to compute its line integral along C:

$$\begin{aligned}
\int_C \overrightarrow{F} \cdot d\overrightarrow{r} &= \int_a^b \overrightarrow{F}(\overrightarrow{r}(t)) \cdot \overrightarrow{r}'(t)\, dt \\
&= \int_a^b (P\overrightarrow{i} + Q\overrightarrow{j} + R\overrightarrow{k}) \cdot (x'(t)\overrightarrow{i} + y'(t)\overrightarrow{j} + z'(t)\overrightarrow{k})\, dt \\
&= \int_a^b \Big(P\big(x(t),y(t),z(t)\big)x'(t) + Q\big(x(t),y(t),z(t)\big)y'(t) + R\big(x(t),y(t),z(t)\big)z'(t)\Big)\, dt
\end{aligned}$$

But this last integral is precisely the line integral in (38). Therefore we have

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \int_C P\, dx + Q\, dy + R\, dz, \quad \text{where } \overrightarrow{F} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix}.$$

5.4 The Fundamental Theorem for Line Integrals

Recall from the Fundamental Theorem of Calculus can be written as

$$\int_a^b F'(x)\,dx = F(b) - F(a) \tag{41}$$

where F' is continuous on $[a, b]$. If we think of the gradient vector $\overrightarrow{\nabla} f$ of a function f of two or three variables as a sort of derivative of f, then the following theorem can be regarded as a version of the Fundamental Theorem for line integrals.

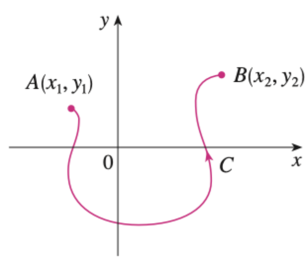
Theorem 5.4.1. Let C be a smooth curve given by the vector function $\overrightarrow{r}(t)$, $t \in [a, b]$. Let f be a differentiable function of two or three variables whose gradient vector $\overrightarrow{\nabla} f$ is continuous on C. Then

$$\int_C \overrightarrow{\nabla} f \cdot d\overrightarrow{r} = f(\overrightarrow{r}(b)) - f(\overrightarrow{r}(a)).$$

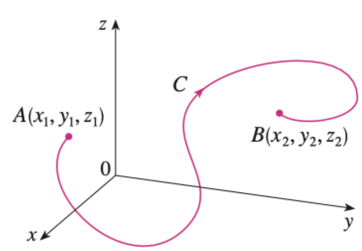
Proof.

$$\begin{aligned}\int_C \overrightarrow{\nabla} f \cdot d\overrightarrow{r} &= \int_a^b \overrightarrow{\nabla} f(\overrightarrow{r}(t)) \cdot \overrightarrow{r}'(t)\,dt = \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right) dt \\ &= \int_a^b \frac{d}{dt} f(\overrightarrow{r}(t))\,dt = f(\overrightarrow{r}(b)) - f(\overrightarrow{r}(a))\end{aligned}$$

Note that Theorem 5.4.1 says that we can evaluate the line integral of a conservative vector field (the gradient vector field of the potential function f) simply by knowing the value of f at the endpoints of C. In fact, Theorem 5.4.1 says that the line integral of $\overrightarrow{\nabla} f$ is the net change in f. If f is a function of two variables and C is a plane curve with initial point $A(x_1, y_1)$ and terminal point $B(x_2, y_2)$ as in the figure below, then the line integral in Theorem 5.4.1 becomes $f(x_2, y_2) - f(x_1, y_1)$.



(a)



(b)

If f is a function of three variables and C is a space curve joining the point $A(x_1, y_1, z_1)$ to the point $B(x_2, y_2, z_2)$, then we have

$$\int_C \overrightarrow{\nabla} f \cdot d\overrightarrow{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1).$$

Independence of Path

Suppose C_1 and C_2 are two piecewise-smooth curves (which are called *paths*) that have the same initial point A and terminal point B. We know that, in general, $\int_{C_1} \overrightarrow{F} \cdot d\overrightarrow{r} \neq \int_{C_2} \overrightarrow{F} \cdot d\overrightarrow{r}$. But one implication of Theorem 5.4.1 is that

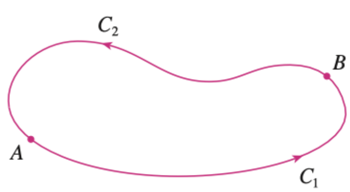
$$\int_{C_1} \overrightarrow{\nabla} f \cdot d\overrightarrow{r} = \int_{C_2} \overrightarrow{\nabla} f \cdot d\overrightarrow{r}$$

whenever $\overrightarrow{\nabla} f$ is continuous. In other words, the line integral of a *conservative* vector field depends only on the initial point and terminal point of a curve.

In general, if $\overrightarrow{F}$ is a continuous vector field with domain D, we say that the line integral $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ is **independent of path** if $\int_{C_1} \overrightarrow{F} \cdot d\overrightarrow{r} = \int_{C_2} \overrightarrow{F} \cdot d\overrightarrow{r}$ for any two paths C_1 and C_2 in D that have the same initial and terminal points. With this terminology we can say that *line integrals of conservative vector fields are independent of path.*

A curve is called **closed** if its terminal point coincides with its initial point, that is, $\overrightarrow{r}(b) = \overrightarrow{r}(a)$.

If $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ is independent of path in D and C is any closed path in D, we can choose any two points A and B on C and regard C as being composed of the path C_1 from A to B followed by the path C_2 from B to A.



Then

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \int_{C_1} \overrightarrow{F} \cdot d\overrightarrow{r} + \int_{C_2} \overrightarrow{F} \cdot d\overrightarrow{r} = \int_{C_1} \overrightarrow{F} \cdot d\overrightarrow{r} - \int_{-C_2} \overrightarrow{F} \cdot d\overrightarrow{r} = 0$$

since C_1 and $-C_2$ have the same initial and terminal points.

Conversely, if it is true that $\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = 0$ whenever C is a closed path in D, then we demonstrate independence of path as follows. Take any two paths C_1 and C_2 from A and B in D and define C to be the curve consisting of C_1 followed by $-C_2$. Then

$$0 = \int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \int_{C_1} \overrightarrow{F} \cdot d\overrightarrow{r} + \int_{-C_2} \overrightarrow{F} \cdot d\overrightarrow{r} = \int_{C_1} \overrightarrow{F} \cdot d\overrightarrow{r} - \int_{C_2} \overrightarrow{F} \cdot d\overrightarrow{r}$$

and so $\int_{C_1} \overrightarrow{F} \cdot d\overrightarrow{r} = \int_{C_2} \overrightarrow{F} \cdot d\overrightarrow{r}$. Thus we have proved the following theorem.

Theorem 5.4.2. $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ is independent of path in D if and only if $\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = 0$ for every closed path C in D.

Since we know that the line integral of any conservative vector field $\overrightarrow{F}$ is independent of path, it follows that $\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = 0$ for any closed path. The physical interpretation is that the work done by a conservative force field (such as the gravitational or electric field) as it moves an object around a closed path is 0.

The following theorem says that the *only* vector fields that are independent of path are conservative. It is stated and proved for plane curves, but there is a similar version for space curves. We assume that D is **open**, which means that for every point P in D there is a disk with centre P that lies entirely in D. (So D does not contain any of its boundary points.) In addition, we assume that D is **connected**: This means that any two points in D can be joined by a path that lies in D.

Theorem 5.4.3. Suppose $\overrightarrow{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ is independent of path in D, then $\overrightarrow{F}$ is a conservative vector field on D; that is, there exists a function f such that $\overrightarrow{\nabla} f = \overrightarrow{F}$.

Proof. Let $A(a, b)$ be a fixed point in D. We construct the desired potential function f by defining

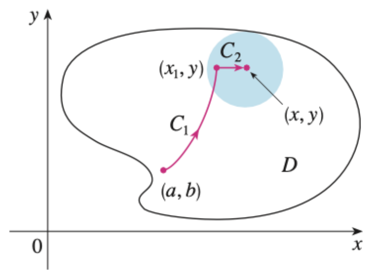
$$f(x, y) = \int_{(a,b)}^{(x,y)} \overrightarrow{F} \cdot d\overrightarrow{r}$$

for any point $(x, y) \in D$. Since $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ is independent of path, it does not matter which path C from (a, b) to (x, y) is used to evaluate $f(x, y)$. Since D is open, there exists a disk contained in D with centre (x, y). Choose any point (x_1, y) in the disk with $x_1 < x$ and let C consist of any path C_1 from (a, b) to (x_1, y) followed by the horizontal line segment C_2 from (x_1, y) to (x, y) (see first figure in the next page). Then

$$f(x, y) = \int_{C_1} \overrightarrow{F} \cdot d\overrightarrow{r} + \int_{C_2} \overrightarrow{F} \cdot d\overrightarrow{r} = \int_{(a,b)}^{(x_1,y)} \overrightarrow{F} \cdot d\overrightarrow{r} + \int_{C_2} \overrightarrow{F} \cdot d\overrightarrow{r}.$$

Notice that the first of these integrals does not depend on x, so

$$\frac{\partial}{\partial x} f(x, y) = 0 + \frac{\partial}{\partial x} \int_{C_2} \overrightarrow{F} \cdot d\overrightarrow{r}.$$



If we write $\vec{F} = \begin{pmatrix} P \\ Q \end{pmatrix}$, then

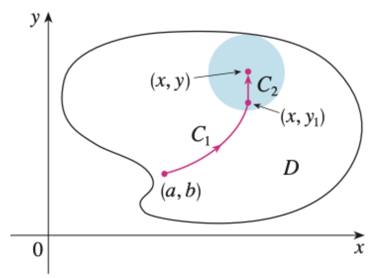
$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_2} P\,dx + Q\,dy.$$

On C_2, y is constant, so $dy = 0$. Using t as a parameter, where $t \in [x_1, x]$, we have

$$\frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial x} \int_{C_2} P\,dx + Q\,dy = \frac{\partial}{\partial x} \int_{x_1}^{x} P(t, y)\,dt = P(x, y)$$

by the Fundamental Theorem of Calculus. A similar argument, using a vertical line segment, shows that

$$\frac{\partial}{\partial y} f(x, y) = \frac{\partial}{\partial y} \int_{C_2} P\,dx + Q\,dy = \frac{\partial}{\partial y} \int_{y_1}^{y} Q(x, t)\,dt = Q(x, y).$$



Thus,

$$\vec{F} = \begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \vec{\nabla} f,$$

which says that $\vec{F}$ is conservative.

How to determine whether or not a vector field $\vec{F}$ is conservative? Suppose it is known that $\vec{F} = P\vec{i} + Q\vec{j}$ is conservative, where P and Q have continuous first-order partial derivatives. Then there is a function f such that $\vec{F} = \vec{\nabla} f$, that is,

$$P = \frac{\partial f}{\partial x} \quad \text{and} \quad Q = \frac{\partial f}{\partial y}.$$

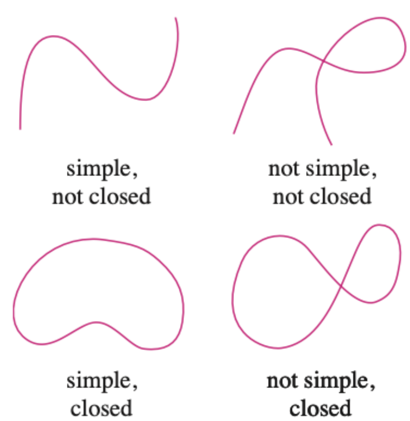
Therefore, by Clairaut's Theorem,

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}.$$

Theorem 5.4.4. If $\overrightarrow{F}(x,y) = P(x,y)\overrightarrow{i} + Q(x,y)\overrightarrow{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

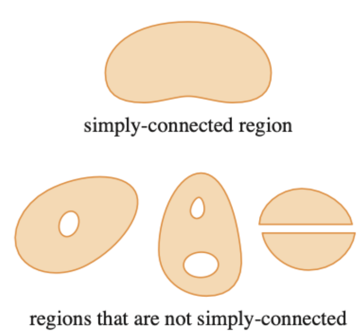
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

The converse of Theorem 5.4.4 is true only for a special type of region. To explain this, we first need the concept of a **simple curve**, which is a curve that does not intersect itself anywhere between its endpoints.



$\overrightarrow{r}(a) = \overrightarrow{r}(b)$ for a simple closed curve, but $\overrightarrow{r}(t_1) \neq \overrightarrow{r}(t_2)$ when $a < t_1 < t_2 < b$.

In Theorem 5.4.3, we needed an open connected region. For the next theorem we need a stronger condition. A **simply-connected region** in the plane is a connected region D such that every simple closed curve in D encloses only points that are in D. Notice from the figure below that, intuitively speaking, a simply-connected region contains no hole and cannot consist of two separate pieces.



In terms of simply-connected regions, we can now state a partial converse to Theorem 5.4.4 that gives a convenient method for verifying that a vector field on $\mathbb{R}^2$ is conservative.

Theorem 5.4.5. Let $\overrightarrow{F} = P\overrightarrow{i} + Q\overrightarrow{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

throughout D. Then $\overrightarrow{F}$ is conservative.

Recall that if $\overrightarrow{F} = P\overrightarrow{i} + Q\overrightarrow{j} + R\overrightarrow{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the **curl** of $\overrightarrow{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\operatorname{curl} \overrightarrow{F} = \overrightarrow{\nabla} \times \overrightarrow{F} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \begin{pmatrix} \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \end{pmatrix}.$$

Recall that the gradient of a function f of three variables is a vector field on $\mathbb{R}^3$ adn so we can compute its curl. The following theorem says that the curl of a gradient vector field is $\overrightarrow{0}$.

Theorem 5.4.6. If f is a function of three variables that has continuous second-order partial derivatives, then

$$\operatorname{curl}(\overrightarrow{\nabla} f) = \overrightarrow{0}.$$

Proof. We have

$$\operatorname{curl}(\overrightarrow{\nabla} f) = \overrightarrow{\nabla} \times (\overrightarrow{\nabla} f) = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \begin{pmatrix} \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \end{pmatrix} = \overrightarrow{0}$$

by Clairaut's Theorem.

Since a conservative vector field is one for which $\overrightarrow{F} = \overrightarrow{\nabla} f$, Theorem 5.4.6 can be rephrased as follows:

If $\overrightarrow{F}$ is conservative, then $\operatorname{curl} \overrightarrow{F} = \overrightarrow{0}$.

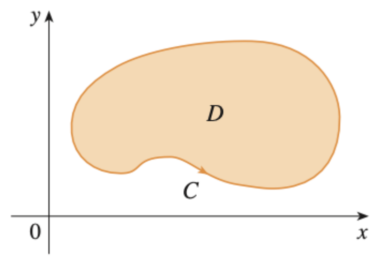
This gives us a way of verifying that a vector field is not conservative.

The converse of Theorem 5.4.6 is not true in general, but the following theorem says the converse is true if $\overrightarrow{F}$ is defined everywhere. (More generally it is true if the domain is simply-connected, that is, "has no hole.").

Theorem 5.4.7. If $\overrightarrow{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $\operatorname{curl} \overrightarrow{F} = \overrightarrow{0}$, then $\overrightarrow{F}$ is a conservative vector field.

5.5 Green's Theorem and Divergence Theorem in 2D

Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C.



(We assume that D consists of all points inside C as well as all points on C.) In stating Green's Theorem we use the convention that the **positive orientation** of a simple closed curve C refers to a single *counterclockwise* traversal of C. Thus if C is given by the vector function $\vec{r}(t)$, $t \in [a, b]$, then the region D is always on the left as the point $\vec{r}(t)$ traverses C.

Theorem 5.5.1 (Green's Theorem). Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Note: The notation

$$\oint_C P\,dx + Q\,dy$$

is sometimes used to indicate that the line integral is calculated using the positive orientation of the closed curve C. Another notation for the positively oriented boundary curve of D is ∂D, so the equation in Green's Theorem can be written as

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} P\,dx + Q\,dy.$$

Proof of Green's Theorem for the case in which D is a simple region. Notice that Green's Theorem will be proved if we can show that

$$\int_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA \tag{42}$$

and

$$\int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA. \tag{43}$$

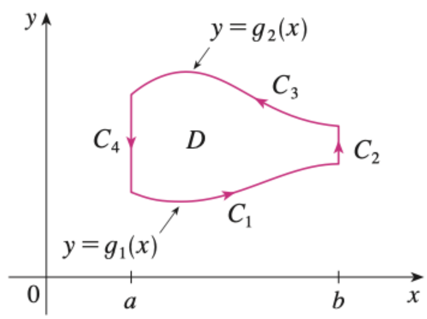
We prove equation (42) by expressing D as a type I region:

$$D = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

where g_1 and g_2 are continuous functions. This enables us to compute the double integral on the RHS of equation (42) as follows:

$$\iint_D \frac{\partial P}{\partial y}\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x, y)\, dy\, dx = \int_a^b \Big(P(x, g_2(x)) - P(x, g_1(x))\Big)\, dx \tag{44}$$

where the last step follows from the Fundamental Theorem of Calculus. Now we compute the LHS of equation (42) by breaking up C as the union of the four curves C_1, C_2, C_3, and C_4 shown below.



On C_1 we take x as the parameter and write the parametric equations as $x = x$, $y = g_1(x)$, $a \le x \le b$. Thus

$$\int_{C_1} P(x, y)\, dx = \int_a^b P(x, g_1(x))\, dx.$$

Observe that C_3 goes from right to left but $-C_3$ goes from left to right, so we can write the parametric equations of $-C_3$ as $x = x$, $y = g_2(x)$, $a \le x \le b$. Therefore

$$\int_{C_3} P(x, y)\, dx = -\int_{-C_3} P(x, y)\, dx = -\int_a^b P(x, g_2(x))\, dx.$$

On C_2 or C_4 (either of which might reduce to just a single point), x is constant, so $dx = 0$ and

$$\int_{C_2} P(x, y)\, dx = 0 = \int_{C_4} P(x, y)\, dx.$$

Hence

$$\begin{aligned}\int_C P(x, y)\, dx &= \int_{C_1} P(x, y)\, dx + \int_{C_2} P(x, y)\, dx + \int_{C_3} P(x, y)\, dx + \int_{C_4} P(x, y)\, dx \\ &= \int_a^b P(x, g_1(x))\, dx - \int_a^b P(x, g_2(x))\, dx\end{aligned}$$

Comparing this expression with the one in equation (44), we see that

$$\int_C P(x, y)\, dx = -\iint_D \frac{\partial P}{\partial y}\, dA.$$

Equation (43) can be proved in much the same way by expressing D as a type II region. Then, by adding equation (42) and (43), we obtain Green's Theorem.

Recall that if $\overrightarrow{F} = P\overrightarrow{i} + Q\overrightarrow{j} + R\overrightarrow{k}$ is a vector field on $\mathbb{R}^3$ and $\partial P/\partial x$, $\partial Q/\partial y$, and $\partial R/\partial z$ exist, then the **divergence of** $\overrightarrow{F}$ is the function of three variables defined by

$$\operatorname{div}\overrightarrow{F} = \overrightarrow{\nabla}\cdot\overrightarrow{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

If $\overrightarrow{F}$ is a vector field on $\mathbb{R}^3$, then curl $\overrightarrow{F}$ is also a vector field on $\mathbb{R}^3$. As such, we can compute its divergence. The next theorem shows that the result if 0.

Theorem 5.5.2. If $\overrightarrow{F} = P\overrightarrow{i} + Q\overrightarrow{j} + R\overrightarrow{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\operatorname{div}\operatorname{curl}\overrightarrow{F} = \overrightarrow{\nabla}\cdot(\overrightarrow{\nabla}\times\overrightarrow{F}) = 0.$$

Proof. Using the definitions of divergence and curl, we have

$$\begin{aligned}
\operatorname{div}\operatorname{curl}\overrightarrow{F} &= \overrightarrow{\nabla}\cdot(\overrightarrow{\nabla}\times\overrightarrow{F}) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \\
&= \frac{\partial^2 R}{\partial x\partial y} - \frac{\partial^2 Q}{\partial x\partial z} + \frac{\partial^2 P}{\partial y\partial z} - \frac{\partial^2 R}{\partial y\partial x} + \frac{\partial^2 Q}{\partial z\partial x} - \frac{\partial^2 P}{\partial z\partial y} \\
&= 0
\end{aligned}$$

because the terms cancel in pairs by Clairaut's Theorem.

The curl and divergence perators allow us to rewrite Green's Theorem. Suppose that the plane region D, its boundary curve C, and the functions P and Q satisfy the hypotheses of Green's Theorem. Then we consider the vector field $\overrightarrow{F} = P\overrightarrow{i} + Q\overrightarrow{j}$. Its line integral is

$$\oint_C \overrightarrow{F}\cdot d\overrightarrow{r} = \oint_C P\,dx + Q\,dy$$

and, regarding $\overrightarrow{F}$ as a vector field on $\mathbb{R}^3$ with third component 0, we have

$$\operatorname{curl}\overrightarrow{F} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\overrightarrow{k}.$$

Therefore

$$(\operatorname{curl}\overrightarrow{F})\cdot\overrightarrow{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\overrightarrow{k}\cdot\overrightarrow{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

and we can now rewrite the equation in Green's Theorem in the vector form

$$\oint_C \overrightarrow{F}\cdot d\overrightarrow{r} = \iint_D (\operatorname{curl}\overrightarrow{F})\cdot\overrightarrow{k}\,dA. \tag{45}$$

Equation (45) expresses the line integral of the tangential component of $\overrightarrow{F}$ along C as the double integral of the vertical component of curl $\overrightarrow{F}$ over the region D enclosed by C. We now derive a similar formula involving the *normal* component of $\overrightarrow{F}$.

If C is given by

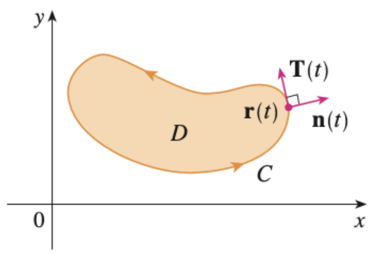
$$\overrightarrow{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad t \in [a, b]$$

then the unit tangent vector is

$$\overrightarrow{T}(t) = \frac{1}{\|\overrightarrow{r}'(t)\|} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}.$$

We can verify that the outward unit normal vector to C is given by

$$\overrightarrow{n}(t) = \frac{1}{\|\overrightarrow{r}'(t)\|} \begin{pmatrix} y'(t) \\ -x'(t) \end{pmatrix}.$$



Then, from equation (32) we have

$$\begin{aligned}
\oint_C \overrightarrow{F} \cdot \overrightarrow{n}\, ds &= \int_a^b (\overrightarrow{F} \cdot \overrightarrow{n})(t) \|\overrightarrow{r}'(t)\|\, dt \\
&= \int_a^b \left(\frac{P(x(t), y(t)) y'(t)}{\|\overrightarrow{r}'(t)\|} - \frac{Q(x(t), y(t)) x'(t)}{\|\overrightarrow{r}'(t)\|} \right) \|\overrightarrow{r}'(t)\|\, dt \\
&= \int_a^b P(x(t), y(t)) y'(t)\, dt - Q(x(t), y(t)) x'(t)\, dt \\
&= \int_C P\, dy - Q\, dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA
\end{aligned}$$

by Green's Theorem. But the integrand in this double integral is just the divergence of $\overrightarrow{F}$. So we have a second vector form of Green's Theorem,

$$\oint_C \overrightarrow{F} \cdot \overrightarrow{n}\, ds = \iint_D \operatorname{div} \overrightarrow{F}(x, y)\, dA$$

which is the Divergence Theorem in 2D. This says that the line integral of the normal component of $\overrightarrow{F}$ along C is equal to the double integral of the divergence of $\overrightarrow{F}$ over the region D enclosed by C.

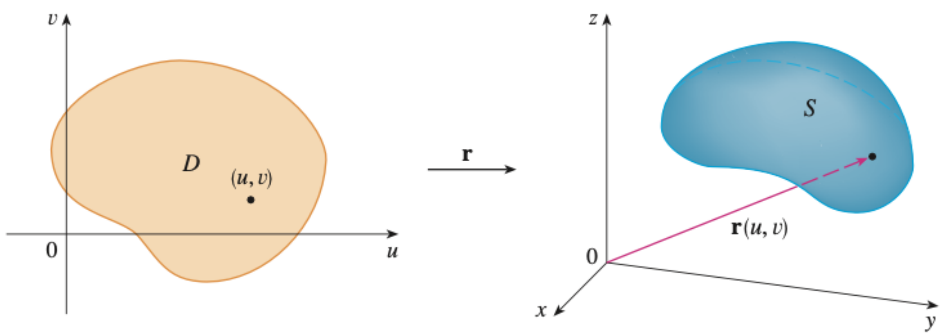
6 Surface Integrals, Stokes' and Gauss' Theorems

6.1 Parametric Surfaces and their Areas

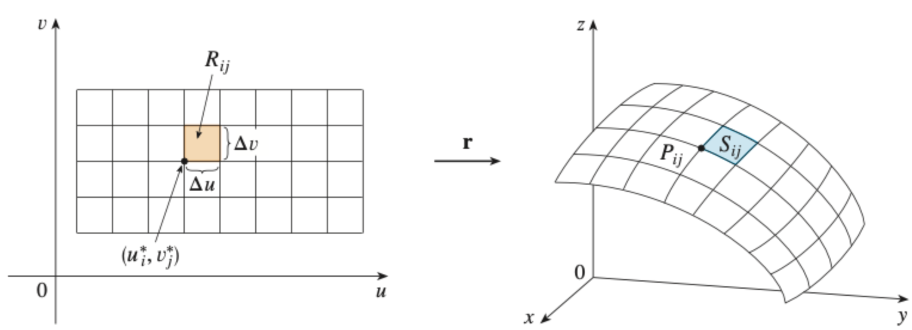
We can describe a surface by a vector function $\vec{r}(u,v)$ of two parameters u and v. We suppose that

$$\vec{r}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix} \tag{46}$$

is a vector-valued function defined on a region D in the uv-plane.



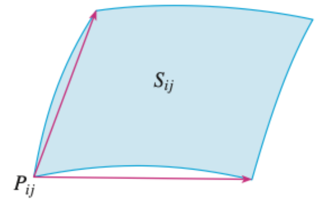
Now we define the surface area of a general parametric surface given by equation (46). For simplicity we sstart by considering a surface whose parameter domain D is a rectangle, and we divide it into subrectangles R_{ij}. Choose (u_i^*, v_j^*) to be the lower left corner of R_{ij}.



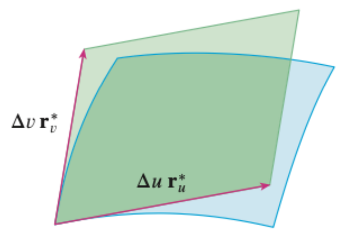
The part S_{ij} of the surface S that corresponds to R_{ij} is called a *patch* and has the point P_{ij} with position vector $\vec{r}(u_i^*, v_j^*)$ as one of its corners. Let

$$\vec{r}_u^{\,*} = \vec{r}_u(u_i^*, v_j^*) \quad \text{and} \quad \vec{r}_v^{\,*} = \vec{r}_v(u_i^*, v_j^*)$$

be the tangent vectors at P_{ij}.



(a)



(b)

Figure (a) shows how the two edges of the patch that meet at P_{ij} can be approximated by vectors. These vectors, in turn, can be approximated by the vectors $\Delta u \overrightarrow{r}_u^*$ and $\Delta v \overrightarrow{r}_v^*$ because partial derivatives can be approximated by difference quotients. So we approximate S_{ij} by the parallelogram determined by the vectors $\Delta u \overrightarrow{r}_u^*$ and $\Delta v \overrightarrow{r}_v^*$. This parallelogram is shown in (b) and lies in the tangent plane to S at P_{ij}. The area of this parallelogram is

$$\|(\Delta u \overrightarrow{r}_u^*) \times (\Delta v \overrightarrow{r}_v^*)\| = \|\overrightarrow{r}_u^* \times \overrightarrow{r}_v^*\| \Delta u \Delta v$$

and so an approximation to the area of S is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \|\overrightarrow{r}_u^* \times \overrightarrow{r}_v^*\| \Delta u \Delta v.$$

Our intuition tells us that this approximation gets better as we increase the number of subrectangles, and we recognise the double sum as a Riemann sum for the double integral $\iint_D \|\overrightarrow{r}_u \times \overrightarrow{r}_v\| \, du \, dv$. This motivates the following.

Definition 6.1.1. If a smooth parametric surface S is given by the equation

$$\overrightarrow{r}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}, \quad (u,v) \in D$$

and S is covered just once as (u,v) ranges throughout the parameter domain D, then the **surface area** of S is

$$A(S) = \iint_D \|\overrightarrow{r}_u \times \overrightarrow{r}_v\| \, dA$$

where

$$\overrightarrow{r}_u = \begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix} \quad \text{and} \quad \overrightarrow{r}_v = \begin{pmatrix} x_v \\ y_v \\ z_v \end{pmatrix}.$$

For the special case of a surface S with equation $z = f(x, y)$, where (x, y) lies in D and f has continuous partial derivatives, we take x and y as parameters. The parametric equations are

$$x = x, \quad y = y, \quad z = f(x, y)$$

so

$$\overrightarrow{r}_x = \begin{pmatrix} 1 \\ 0 \\ f_x \end{pmatrix} \quad \text{and} \quad \overrightarrow{r}_y = \begin{pmatrix} 0 \\ 1 \\ f_y \end{pmatrix}$$

and

$$\overrightarrow{r}_x \times \overrightarrow{r}_y = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \begin{pmatrix} -f_x \\ -f_y \\ 1 \end{pmatrix}. \tag{47}$$

Thus we have

$$\|\overrightarrow{r}_x \times \overrightarrow{r}_y\| = \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{1 + z_x^2 + z_y^2} \tag{48}$$

and the surface area formula in Definition 6.1.1 becomes

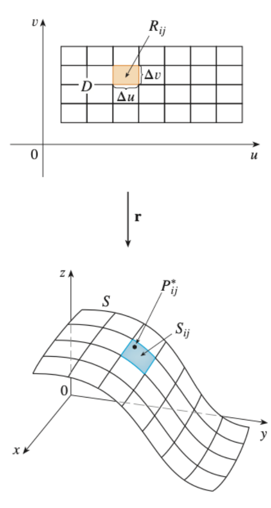
$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA. \tag{49}$$

6.2 Surface Integrals

Suppose that a surface S has a vector equation

$$\overrightarrow{r}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}, \quad (u, v) \in D.$$

We first assume that the parameter domain D is a rectangle and we divide it into subrectangles R_{ij} with dimensions Δu and Δv. Then the surface S is divided into corresponding patches S_{ij} as shown below.



We evaluate f at a point P_{ij}^* in each patch, multiply by the area ΔS_{ij} of the patch, and form the Riemann sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}.$$

Then we take the limit as the number of patches increases and define the **surface integral of f over the surface S** as

$$\iint_S f(x,y,z)\,dS = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}. \tag{50}$$

Notice the analogy with the definition of a line integral and also the analogy with the definition of a double integral.

To evaluate the surface integral in equation (50) we approximate the patch area ΔS_{ij} by the area of an approximating parallelogram in the tangent plane. In section 6.1 we made the approximation

$$\Delta S_{ij} = \|\overrightarrow{r}_u \times \overrightarrow{r}_v\|\Delta u \Delta v$$

where

$$\overrightarrow{r}_u = \begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix} \quad \text{and} \quad \overrightarrow{r}_v = \begin{pmatrix} x_v \\ y_v \\ z_v \end{pmatrix}$$

are the tangent vectors at a corner of S_{ij}. If the components are continuous and $\overrightarrow{r}_u$ and $\overrightarrow{r}_v$ are nonzero and nonparallel in the interior of D, it can be shown from (50), even when D is not a rectangle, that

$$\iint_S f(x,y,z)\,dS = \iint_D f(\overrightarrow{r}(u,v))\|\overrightarrow{r}_u \times \overrightarrow{r}_v\|\,dA. \tag{51}$$

This should be compared with the formula for a line integral:

$$\int_C f(x,y,z)\,ds = \int_a^b f(\overrightarrow{r}(t))\|\overrightarrow{r}'(t)\|\,dt.$$

Observe also that

$$\iint_S 1\,dS = \iint_D \|\overrightarrow{r}_u \times \overrightarrow{r}_v\|\,dA = A(s).$$

Formula (51) allows us to compare a surface integral by converting it into a double integral over the parameter domain D.

Any surface S with equation $z = g(x,y)$ can be regarded as a parametric surface with parametric equations

$$x = x, \quad y = y, \quad z = g(x,y)$$

and so we have

$$\overrightarrow{r}_x = \begin{pmatrix} 1 \\ 0 \\ g_x \end{pmatrix} \quad \text{and} \quad \overrightarrow{r}_y = \begin{pmatrix} 0 \\ 1 \\ g_y \end{pmatrix}.$$

Thus

$$\vec{r}_x \times \vec{r}_y = \begin{pmatrix} -g_x \\ -g_y \\ 1 \end{pmatrix} \tag{52}$$

and

$$\|\vec{r}_x \times \vec{r}_y\| = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}.$$

Therefore, in this case, formula (51) becomes

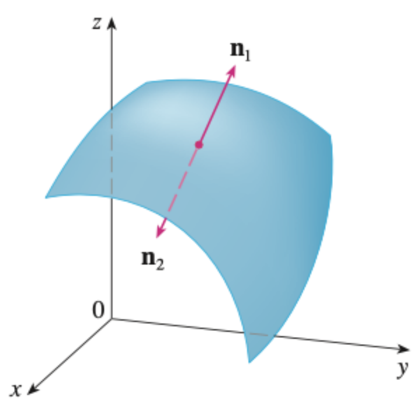
$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA. \tag{53}$$

Similar formulas apply when it is more convenient to project S onto the yz-plane or xz-plane. For instance, if S is a surface with equation $y = h(x,z)$ and D is its projection onto the xz-plane, then

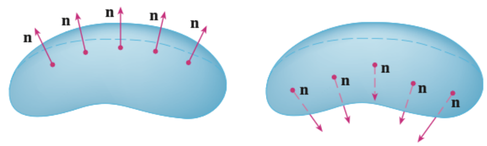
$$\iint_S f(x,y,z)\,dS = \iint_D f(x,h(x,z),z)\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\,dA.$$

6.3 Flux through the Surface

From now on we consider only orientable (two-sided) surfaces. We start with a surface S that has a tangent plane at every point $(x,y,z) \in S$ (except at any boundary point). There are two unit normal vectors $\vec{n}_1$ and $\vec{n}_2 = -\vec{n}_1$ at (x,y,z).



If it is possible to choose a unit normal vector $\vec{n}$ at every such point (x,y,z) so that $\vec{n}$ varies continuously over S, then S is called an **oriented surface** and the given choice of $\vec{n}$ provides S with an **orientation**. There are two possible orientations for any orientable surface.



For a surface $z = g(x, y)$ given as the graph of g, we use equation (52) to associate with the surface a natural orientation given by the unit normal vector

$$\vec{n} = \frac{1}{\sqrt{1 + g_x^2 + g_y^2}} \begin{pmatrix} -g_x \\ -g_y \\ 1 \end{pmatrix}. \tag{54}$$

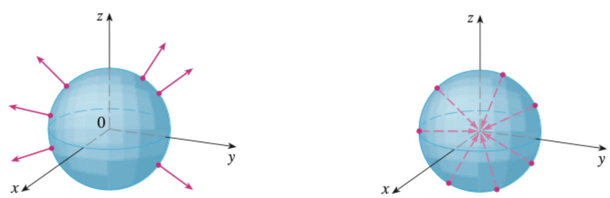
Since the $\vec{k}$-component is positive, this gives the *upward* orientation of the surface.

If S is a smooth orientable surface given in parametric form by a vector function $\vec{r}(u, v)$, then it is automatically supplied with the orientation of the unit normal vector

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} \tag{55}$$

and the opposite orientation is given by $-\vec{n}$.

For a **closed surface**, that is, a surface that is the boundary of a solid region E, the convention is that the **positive orientation** is the one for which the normal vectors point *outward* from E, and inward-pointing normals give the negative orientation.



Now suppose that S is an oriented surface with unit normal vector $\vec{n}$, and imagine a fluid with density $\rho(x, y, z)$ and velocity field $\vec{v}(x, y, z)$ flowing through S. Then the rate of flow per unit area is $\rho\vec{v}$. If we divide S into small patches S_{ij}, then S_{ij} is nearly planar and so we can approximate the mass of fluid per unit time crossing S_{ij} in the direction of the normal $\vec{n}$ by the quantity

$$(\rho\vec{v} \cdot \vec{n})A(S_{ij})$$

where ρ, $\vec{v}$, and $\vec{n}$ are evaluated at some point on S_{ij}. By summing these quantities and taking the limit we get, according to (50), the surface integral of the function $\rho\vec{v} \cdot \vec{n}$ over S,

$$\iint_S \rho\vec{v} \cdot \vec{n}\, dS = \iint_S \rho(x, y, z)\vec{v}(x, y, z) \cdot \vec{n}(x, y, z)\, dS \tag{56}$$

and this is interpreted physically as the rate of flow through S.

If we write $\overrightarrow{F} = \rho\overrightarrow{v}$, then $\overrightarrow{F}$ is also a vector field on $\mathbb{R}^3$ and the integral in equation (56) becomes

$$\iint_S \overrightarrow{F} \cdot \overrightarrow{n}\, dS.$$

A surface integral of this form occurs frequently in physics, even when $\overrightarrow{F}$ is not $\rho\overrightarrow{v}$, and is called the *surface integral* (or *flux integral*) of $\overrightarrow{F}$ over S.

Definition 6.3.1 (Surface Integral of a Vector Field). If $\overrightarrow{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\overrightarrow{n}$, then the *surface integral of* $\overrightarrow{F}$ *over* S is

$$\iint_S \overrightarrow{F} \cdot d\overrightarrow{S} = \iint_S \overrightarrow{F} \cdot \overrightarrow{n}\, dS.$$

The integral is also called the *flux* of $\overrightarrow{F}$ across S.

Definition 6.3.1 says that the surface integral of a vector field over S is equal to the surface integral of its normal component over S.

If S is given by a vector function $\overrightarrow{r}(u,v)$, then $\overrightarrow{n}$ is given by equation (55), and from Definition 6.3.1 and equation (51) we have

$$\iint_S \overrightarrow{F} \cdot d\overrightarrow{S} = \iint_S \overrightarrow{F} \cdot \frac{\overrightarrow{r}_u \times \overrightarrow{r}_v}{\|\overrightarrow{r}_u \times \overrightarrow{r}_v\|}\, dS = \iint_D \overrightarrow{F}(\overrightarrow{r}(u,v)) \cdot \frac{\overrightarrow{r}_u \times \overrightarrow{r}_v}{\|\overrightarrow{r}_u \times \overrightarrow{r}_v\|} \|\overrightarrow{r}_u \times \overrightarrow{r}_v\|\, dA$$

where D is the parameter domain. Thus we have

$$\iint_S \overrightarrow{F} \cdot d\overrightarrow{S} = \iint_D \overrightarrow{F} \cdot (\overrightarrow{r}_u \times \overrightarrow{r}_v)\, dA \tag{57}$$

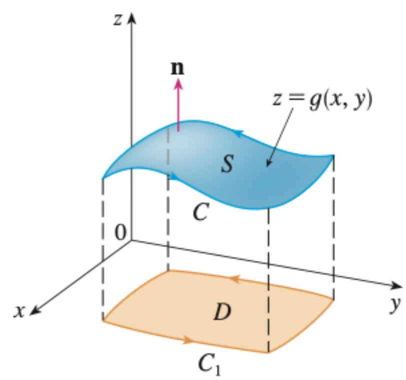
6.4 Stokes' Theorem

Stoke's Theorem can be regarded as a higher-dimensional version of Green's Theorem. Whereas Green's Theorem relates a double integral over a plane region D to a line integral around its plane boundary curve. Stoke's Theorem relates a surface integral over a surface S to a line integral around the boundary curve of S (which is a space curve).

Theorem 6.4.1 (Stoke's Theorem). Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\overrightarrow{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \iint_S \operatorname{curl} \overrightarrow{F} \cdot d\overrightarrow{S}.$$

Proof of a special case of Stokes' Theorem. We assume that the equation of S is $z = g(x, y)$, $(x, y) \in D$, where g has continuous second-order partial derivatives and D is a simple plane region whose boundary curve C_1 corresponds to C. If the orientation of S is upward, then the positive orientation of C corresponds to the positive orientation of C_1.



We are also given that $\overrightarrow{F} = P\overrightarrow{i} + Q\overrightarrow{j} + R\overrightarrow{k}$, where the partial derivatives of P, Q, and R are continuous.

Since S is a graph of a function,

$$\iint_S (\overrightarrow{\nabla} \times \overrightarrow{F}) \cdot d\overrightarrow{S} = \iint_S (\overrightarrow{\nabla} \times \overrightarrow{F}) \cdot \overrightarrow{n}\, dS = \iint_D \begin{pmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{pmatrix} \cdot \begin{pmatrix} -g_x \\ -g_y \\ 1 \end{pmatrix} dA$$

$$= \iint_D (-(R_y - Q_z)g_x - (P_z - R_x)g_y + (Q_x - P_y))\, dA \tag{58}$$

where the partial derivatives of P, Q, and R are evaluated at $(x, y, g(x, y))$. If

$$x = x(t), \quad y = y(t), \quad t \in [a, b]$$

is a parametric representation of C_1, then a parametric representation of C is

$$\overrightarrow{r}(t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \\ g(x(t), y(t)) \end{pmatrix}, \quad t \in [a, b].$$

This allows us, with the aid of the Chain Rule, to evaluate the line integral as follows:

$$\begin{aligned}
\int_C \overrightarrow{F} \cdot d\overrightarrow{r} &= \int_a^b \overrightarrow{F}(\overrightarrow{r}(t)) \cdot \overrightarrow{r}'(t)\, dt = \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt} \right) dt \\
&= \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \right) \right) dt \\
&= \int_a^b \left(\left(P + R\frac{\partial z}{\partial x} \right)\frac{dx}{dt} + \left(Q + \frac{\partial z}{\partial y} \right)\frac{dy}{dt} \right) dt \\
&= \int_{C_1} \left(P + R\frac{\partial z}{\partial x} \right) dx + \left(Q + R\frac{\partial z}{\partial y} \right) dy \\
&= \iint_D \left(\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y} \right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x} \right) \right) dA
\end{aligned}$$

where we used Green's Theorem in the last step. Then, using the Chain Rule again and remembering that P, Q, and R are functions of x, y, and z adn that z itself a function of x and y, we get

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \iint_D ((Q_x + Q_z z_x + R_x z_y + R_z z_x z_y + R z_{xy}) - (P_y + P_z z_y + R_y z_x + R_z z_y z_x + R z_{yx}))\, dA.$$

Four of the terms in this double integral cancel and the remaining six terms can be arranged to coincide with the RHS of equation (58). Therefore

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \iint_S (\overrightarrow{\nabla} \times \overrightarrow{F}) \cdot d\overrightarrow{S} = \iint_S \operatorname{curl} \overrightarrow{F} \cdot d\overrightarrow{S}.$$

6.5 Gauss' Theorem (The Divergence Theorem in 3D)

In Section 5.5, we rewrote Green's Theorem in a vector version as

$$\int_C \overrightarrow{F} \cdot \overrightarrow{n}\, ds = \iint_D \operatorname{div} \overrightarrow{F}(x,y)\, dA$$

where C is the positively oriented boundary curve of the plane region D. If we were seeking to extend this theorem to vector fields on $\mathbb{R}^3$, we might make the guess that

$$\iint_S \overrightarrow{F} \cdot \overrightarrow{n}\, dS = \iiint_E \operatorname{div} \overrightarrow{F}(x,y,z)\, dV \tag{59}$$

where S is the boundary surface of the solid region E. It turns out that equation (59) is true under appropriate hypotheses, and is called the Gauss' Theorem or the Divergence Theorem.

Theorem 6.5.1 (Gauss' Theorem). Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\overrightarrow{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \overrightarrow{F} \cdot d\overrightarrow{S} = \iiint_E \operatorname{div} \overrightarrow{F}\, dV.$$

Proof. Let $\overrightarrow{F} = P\overrightarrow{i} + Q\overrightarrow{j} + R\overrightarrow{k}$. Then

$$\operatorname{div} \overrightarrow{F} = \overrightarrow{\nabla} \cdot \overrightarrow{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

so

$$\iiint_E \operatorname{div} \overrightarrow{F}\, dV = \iiint_E \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dV.$$

If $\overrightarrow{n}$ is the unit outward normal of S, then the surface integral on the LHS of the Gauss' Theorem is

$$\begin{aligned}\iint_S \overrightarrow{F} \cdot d\overrightarrow{S} = \iint_S \overrightarrow{F} \cdot \overrightarrow{n}\, dS &= \iint_S (P\overrightarrow{i} + Q\overrightarrow{j} + R\overrightarrow{k}) \cdot \overrightarrow{n}\, dS \\ &= \iint_S (P\overrightarrow{i}) \cdot \overrightarrow{n}\, dS + \iint_S (Q\overrightarrow{j}) \cdot \overrightarrow{n}\, dS + \iint_S (Q\overrightarrow{k}) \cdot \overrightarrow{n}\, dS\end{aligned}$$

Therefore, to prove the Gauss' Theorem, it suffices to prove the following three equations:

$$\iint_S (P\overrightarrow{i}) \cdot \overrightarrow{n}\, dS = \iiint_E \frac{\partial P}{\partial x}\, dV \tag{60}$$

$$\iint_S (Q\overrightarrow{j}) \cdot \overrightarrow{n}\, dS = \iiint_E \frac{\partial Q}{\partial y}\, dV \tag{61}$$

$$\iint_S (R\overrightarrow{k}) \cdot \overrightarrow{n}\, dS = \iiint_E \frac{\partial R}{\partial z}\, dV \tag{62}$$

To prove equation (62) we use the fact that E is a type 1 region:

$$E = \{(x, y, z) \,|\, (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$$

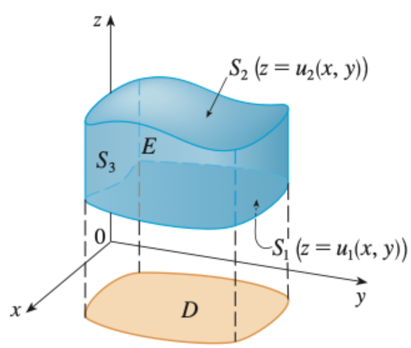
where D is the projection of E onto the xy-plane. We have

$$\iiint_E \frac{\partial R}{\partial z}\, dV = \iint_D \Big(\int_{u_1(x,y)}^{u_2(x,y)} \frac{\partial R}{\partial z}(x, y, z)\, dz \Big)\, dA$$

and therefore, by the Fundamental Theorem of Calculus,

$$\iiint_E \frac{\partial R}{\partial z}\, dV = \iint_D \big(R(x, y, u_2(x, y)) - R(x, y, u_1(x, y)) \big)\, dA. \tag{63}$$

The boundary surface S consists of three pieces: the bottom surface S_1, the top surface S_2, and possibly a vertical surface S_3, which lies above the boundary curve of D.



(It might happen that S_3 does not appear, as in the case of a sphere.) Notice that on S_3 we have $\overrightarrow{k} \cdot \overrightarrow{n} = 0$, because $\overrightarrow{k}$ is vertical and $\overrightarrow{n}$ is horizontal, and so

$$\iint_{S_3} (R\overrightarrow{k}) \cdot \overrightarrow{n}\, dS = \iint_{S_3} 0\, dS = 0.$$

Thus, regardless of whether there is a vertical surface, we can write

$$\iint_S (R\overrightarrow{k}) \cdot \overrightarrow{n}\, dS = \iint_{S_1} (R\overrightarrow{k}) \cdot \overrightarrow{n}\, dS + \iint_{S_2} (R\overrightarrow{k}) \cdot \overrightarrow{n}\, dS. \tag{64}$$

The equation of S_2 is $z = u_2(x, y)$, $(x, y) \in D$, and the outward normal $\overrightarrow{n}$ points upward, so we have

$$\iint_{S_2} (R\overrightarrow{k}) \cdot \overrightarrow{n}\, dS = \iint_D R(x, y, u_2(x, y))\, dA.$$

On S_1 we have $z = u_1(x, y)$, but here the outward normal $\overrightarrow{n}$ points downward, so we multiply by -1:

$$\iint_{S_1} (R\overrightarrow{k}) \cdot \overrightarrow{n}\, dS = -\iint_D R(x, y, u_1(x, y))\, dA.$$

Therefore equation (64) gives

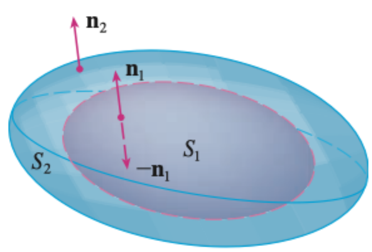
$$\iint_S (R\overrightarrow{k}) \cdot \overrightarrow{n}\, dS = \iint_D \big(R(x, y, u_2(x, y)) - R(x, y, u_1(x, y))\big)\, dA.$$

Comparison with equation (63) shows that

$$\iint_S (R\overrightarrow{k}) \cdot \overrightarrow{n}\, dS = \iiint_E \frac{\partial R}{\partial z}\, dV.$$

Equations (60) and (61) are proved in a similar manner using the expressions for E as a type 2 or type 3 region, respectively.

Although we have proved the Gauss' Theorem only for simple solid regions, it can be proved for regions that are finite unions of simple solid regions. For example, let's consider the region E that lies between the closed surfaces S_1 and S_2, where S_1 lies inside S_2. Let $\overrightarrow{n}_1$ and $\overrightarrow{n}_2$ be outward normals of S_1 and S_2 respectively. Then the boundary surface of E is $S = S_1 \cup S_2$ and its normal $\overrightarrow{n}$ is given by $\overrightarrow{n} = -\overrightarrow{n}_1$ on S_1 and $\overrightarrow{n} = \overrightarrow{n}_2$ on S_2.



Applying the Gauss' Theorem to S, we get

$$\begin{aligned}\iiint_E \operatorname{div} \overrightarrow{F} &= \iint_S \overrightarrow{F} \cdot d\overrightarrow{S} = \iint_S \overrightarrow{F} \cdot \overrightarrow{n}\, dS \\ &= \iint_{S_1} \overrightarrow{F} \cdot (-\overrightarrow{n}_1)\, dS + \iint_{S_2} \overrightarrow{F} \cdot \overrightarrow{n}_2\, dS \\ &= -\iint_{S_1} \overrightarrow{F} \cdot d\overrightarrow{S} + \iint_{S_2} \overrightarrow{F} \cdot d\overrightarrow{S}\end{aligned}$$

7 Simple ODEs and Matrix ODEs (Revision)

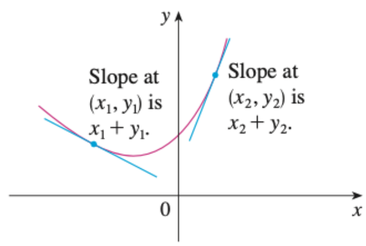
7.1 First Order ODEs

Direction Fields

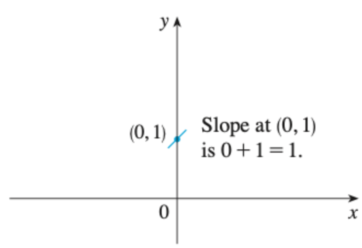
Suppose we are asked to sketch the graph of the solution of the initial-value problem

$$y' = x + y, \quad y(0) = 1.$$

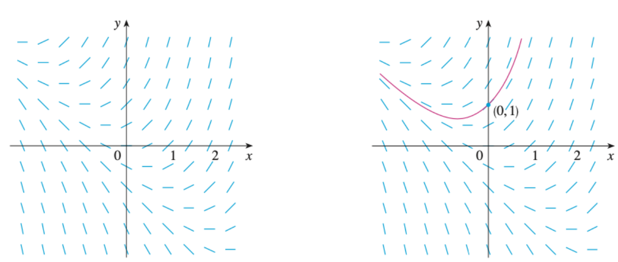
The equation tells us that the slope at any point (x, y) on the graph (called the *solution curve*) is equal to the sum of the x- and y-coordinates of the point.



In particular, because the curve passes through the point $(0, 1)$, its slope there must be $0 + 1 = 1$. So a small portion of the solution curve near the point $(0, 1)$ looks like a short line segment through $(0, 1)$ with slope 1.



As a guide to sketch the rest of the curve, we draw short line segments at a number of points (x, y) with slope $x + y$.The result if called a *directional field*.



Separable Equations

A **separable equation** is a first-order ODE in which the expression for dy/dx can be factored as a function of x times a function of y, i.e. it can be written in the form

$$\frac{dy}{dx} = g(x)f(y).$$

If $f(y) \neq 0$, we could write

$$\frac{dy}{dx} = \frac{g(x)}{h(y)} \tag{65}$$

where $h(y) = 1/f(y)$. To solve this equation we rewrite it in the differential form

$$h(y)\,dy = g(x)\,dx$$

so that we can integrate both sides of the equation

$$\int h(y)\,dy = \int g(x)\,dx$$

and obtain a solution.

First Order Linear ODE

A first-order **linear** ODE is one that can be written in the form

$$y' + p(x)y = q(x) \tag{66}$$

where p and q are continuous functions on a given interval.

Every first-order linear ODE can be solved by multiplying both sides by a suitable function $P(x)$ called an *integrating factor*. We try to find P so that the LHS of (66), when multiplied by $P(x)$, becomes the derivative of the product $P(x)y$:

$$P(x)(y' + p(x)y) = (P(x)y)' \tag{67}$$

If we can find such a function P, then equation (66) becomes

$$(P(x)y)' = P(x)q(x).$$

Integrating both sides, we would have

$$P(x)y = \int P(x)q(x)\,dx$$

so the solution would be

$$y(x) = \frac{1}{P(x)}\int P(x)q(x)\,dx. \tag{68}$$

To find such a P, we expand equation (67) and cancel terms:

$$P(x)y' + P(x)p(x)y = (P(x)y)' = P'(x)y + P(x)y',$$

so $P(x)p(x) = P'(x)$. This is a separable ODE for P, which we solve as

$$\begin{aligned}\int \frac{dP}{P} &= \int p(x)\,dx \\ \ln|P| &= \int p(x)\,dx \\ P(x) &= Ae^{\int p(x)\,dx}\end{aligned}$$

where $A = \pm e^C$. We are looking for a particular integrating factor, not the most general one, so we take $A = 1$ and use

$$P(x) = e^{\int p(x)\,dx}. \tag{69}$$

Theorem 7.1.1. To solve the first order linear ODE

$$y' + p(x)y = q(x),$$

we multiply both sides by the *integrating factor* $P(x) = e^{\int p(x)\,dx}$ and integrate both sides.

7.2 Second Order Linear ODE with Constant Coefficients

A **second-order linear ODE** has the form

$$P(x)y'' + Q(x)y' + R(x)y = G(x)$$

where P, Q, R, and G are continuous functions. In this section we assume the coefficients are all constants, i.e.,

$$ay'' + by' + cy = f(x), \tag{70}$$

for constants a, b and c, and continuous function f.

Homogeneous

We now study the case where $f(x) = 0$, for all x, in equation (70). Such equations are said to be **homogeneous**. The form of a second-order linear homogeneous ODE is

$$ay'' + by' + cy = 0. \tag{71}$$

If $f(x) \neq 0$ for some x, equation (70) is **nonhomogeneous** and is discussed later.

If we know two solutions y_1 and y_2 of a second-order linear homogeneous ODE, then the **linear combination** $y = C_1y_1 + C_2y_2$ is also a solution.

Theorem 7.2.1. If $y_1(x)$ and $y_2(x)$ are both solutions of the linear homogeneous equation (71) and C_1 and C_2 are any constants, then the function

$$y(x) = C_1y_1(x) + C_2y_2(x)$$

is also a solution of (71).

Proof. Since y_1 and y_2 are solutions of (71), we have

$$ay_1'' + by_1' + cy_1 = 0$$

and

$$ay_2'' + by_2' + cy_2 = 0.$$

Therefore, using the basic rule for differentiation, we have

$$\begin{aligned} ay'' + by' + cy &= a(C_1y_1 + C_2y_2)'' + b(C_1y_1 + C_2y_2)' + c(C_1y_1 + C_2y_2) \\ &= a(C_1y_1'' + C_2y_2'') + b(C_1y_1' + C_2y_2') + c(C_1y_1 + C_2y_2) \\ &= C_1(ay_1'' + by_1' + cy_1) + C_2(ay_2'' + by_2' + cy_2) \\ &= C_1 \cdot 0 + C_2 \cdot 0 = 0 \end{aligned}$$

Thus $y = C_1y_1 + C_2y_2$ is a solution of equation (71).

We are looking for a function y such that a constant times its second derivative y'' plus another constant times y' plus a third constant times y is equal to 0. We know that the exponential function $y = e^{rx}$ (where r is a constant) has the property that its derivative is a constant multiple of itself. If we substitute these expressions into equation (71), we see that $y = e^{rx}$ is a solution if

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

or

$$(ar^2 + br + c)e^{rx} = 0.$$

But e^{rx} is never 0. Thus $y = e^{rx}$ is a solution of (71) if r is a root of the equation

$$ar^2 + br + c = 0. \tag{72}$$

Equation (72) is called the **characteristic equation** of the ODE $ay'' + by' + cy = 0$. Notice that it is an algebraic equation that is obtained from the differential equation by replacing y'' by r^2, y' by r, and y by 1.

Sometimes the roots r_1 and r_2 of the characteristic equation can be found by factoring. In other cases they are found by using the quadratic formula:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

We distinguish three cases according to the sign of the discriminant $b^2 - 4ac$.

Case 1: $b^2 - 4ac > 0$

If the roots r_1 and r_2 of the characteristic equation $ar^2 + br + c = 0$ are real and unequal, then the general solution of (71) is

$$y = C_1e^{r_1x} + C_2e^{r_2x}.$$

Case 2: $b^2 - 4ac = 0$

If the characteristic equation $ar^2 + br + c = 0$ has only one real root r, then the general solution of $ay'' + by' + cy = 0$ is

$$y = (C_1 + C_2 x)e^{rx}.$$

Case 3: $b^2 - 4ac < 0$

If the roots of the characteristic equation $ar^2 + br + c = 0$ are the complex numbers $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, then the general solution of $ay'' + by' + cy = 0$ is

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x).$$

Nonhomogeneous

For second-order nonhomogeneous linear ODE with constant coefficients, that is, equations of the form

$$ay'' + by' + cy = f(x), \tag{73}$$

where a, b, and c are constants and f is a continuous function. The related homogeneous equation

$$ay'' + by' + cy = 0 \tag{74}$$

is called the **complementary equation**.

Theorem 7.2.2. The general solution of the nonhomogeneous ODE (73) can be written as

$$y(x) = y_p(x) + y_h(x)$$

where y_p is a particular solution of (73) and y_h is the general solution of the complementary equation (74).

Proof.

$$\begin{aligned} a(y - y_p)'' + b(y - y_p)' + c(y - y_p) &= ay'' - ay_p'' + by' - by_p' + cy - cy_p \\ &= (ay'' + by' + cy) - (ay_p'' + by_p' + cy_p) \\ &= f(x) - f(x) = 0 \end{aligned}$$

Thus, we know that $y_h = y - y_p$ and so $y(x) = y_p(x) + y_h(x)$.

7.3 Homogeneous First Order System of ODEs

A homogeneous system of ODEs that can be written in the form

$$\begin{aligned} y_1' &= a_{11}(x)y_1 + a_{12}(x)y_2 + \cdots + a_{1n}(x)y_n \\ y_2' &= a_{21}(x)y_1 + a_{22}(x)y_2 + \cdots + a_{2n}(x)y_n \\ &\vdots \\ y_n' &= a_{n1}(x)y_1 + a_{n2}(x)y_2 + \cdots + a_{nn}(x)y_n \end{aligned}$$

is called a **linear system**.

It can be written as a matrix equation

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix} = \begin{pmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

or more precisely

$$\overrightarrow{y}'(x) = A(x)\overrightarrow{y}(x).$$

For now, we assume that A only consists of constant entries. The first order linear homogeneous system

$$\overrightarrow{y}'(x) = A\overrightarrow{y}(x) \tag{75}$$

can be interpreted as a separable equation. It is helpful to compare with the equation

$$y'(x) = ay(x)$$

for a constant coefficient a, whose general solution is given by

$$y(x) = Ce^{ax}.$$

The solution of equation (75) looks similar.

Theorem 7.3.1. The solution for a homogeneous system of ODE (75) with constant coefficients is

$$\overrightarrow{y}(x) = e^{xA}\overrightarrow{y}(0).$$

Proof. Recall that the Taylor series for e^x at $x = 0$ is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Since A is of constant entries, the Taylor series of e^{xA} is

$$\sum_{k=0}^{\infty} \frac{(xA)^k}{k!} = I_n + xA + \frac{1}{2!}(xA)^2 + \frac{1}{3!}(xA)^3 + \cdots \tag{76}$$

Note that $(xA)^2 = A^2x^2 = x^2A^2$ as x is a scalar. By taking derivative of (76), we get

$$\begin{aligned}
\frac{d}{dx}e^{xA} &= \frac{d}{dx}\left(I_n + xA + \frac{1}{2!}x^2A^2 + \frac{1}{3!}x^3A^3 + \cdots\right) \\
&= A + xA^2 + \frac{1}{2!}x^2A^3 + \frac{1}{3!}x^3A^4 + \cdots \\
&= A\left(I_n + xA + \frac{1}{2!}x^2A^2 + \frac{1}{3!}x^3A^3 + \cdots\right) \\
&= Ae^{xA} \qquad (77)
\end{aligned}$$

To complete the proof, we differentiate $\vec{y}(x) = e^{xA}\vec{y}(0)$.

$$\vec{y}'(x) = [e^{xA}\vec{y}(0)]' = Ae^{xA}\vec{y}(0) = A\vec{y}(x)$$

by (77). Therefore, $\vec{y}(x) = e^{xA}\vec{y}(0)$ satisfies (75).

Calculating Matrix Exponential

Suppose an $n \times n$ matrix A of constant entries is diagonalisable, that is it can be written in the form

$$A = PDP^{-1},$$

where the diagonal entries of D are eigenvalues of A, $\lambda_1, \ldots, \lambda_n$ other entries of D are zero, and the columns of P are eigenvectors, $\vec{v}_1, \ldots, \vec{v}_n$, corresponding to the eigenvalues in D. Then

$$\begin{aligned}
e^{xA} &= Pe^{xD}P^{-1} \\
&= \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} e^{\lambda_1 x} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 x} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n x} \end{pmatrix} \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix}^{-1}.
\end{aligned}$$

8 Solving ODE I

8.1 General Principle for Second-Order Linear ODE

Definition 8.1.1 (Fundamental Set of Solutions). We say $\{y_1, y_2\}$ is a *fundamental set of solutions of the equation*

$$y'' + p(x)y' + q(x)y = 0$$

on (a, b) if every solution of the equation can be written as a linear combination of y_1 and y_2, that is, in the form $y = C_1 y_1 + C_2 y_2$.

Definition 8.1.2 (Linear Independence). We say that two functions y_1 and y_2 are *linearly independent* on (a, b) if neither is a constant multiple of the other on (a, b).

Theorem 8.1.3. Suppose p and q are continuous functions, then a set $\{y_1, y_2\}$ of solutions of

$$y'' + p(x)y' + q(x)y = 0$$

on (a, b) is a fundamental set if and only if $\{y_1, y_2\}$ is linearly independent on (a, b).

8.2 Wronskian and Abel's Formula

Consider the homogeneous second-order linear ODE

$$y'' + p(x)y' + q(x)y = 0 \tag{78}$$

on (a, b), where p and q are continuous functions on (a, b).

Definition 8.2.1 (Wronskian). Suppose $\{y_1, y_2\}$ are solutions of (78) on (a, b). We define the *Wronskian* of $\{y_1, y_2\}$ as

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2.$$

This is essential for the following theorem.

Theorem 8.2.2 (Abel's Formula). Consider the equation (78), let x_0 be any point in (a, b). Then

$$W(x) = W(x_0) e^{\int_{x_0}^{x} p(t)\,dt}, \quad x \in (a, b). \tag{79}$$

Therefore, either

- $W(x) \neq 0$ for all $x \in (a, b)$, or
- $W(x) = 0$ for all $x \in (a, b)$.

Proof. Differentiating W yields

$$W' = y_1'y_2' + y_1'y_2'' - y_1'y_2' - y_1''y_2' = y_1y_2'' - y_1''y_2. \tag{80}$$

Since y_1 and y_2 are solutions of (78),

$$y_1'' = -py_1' - qy_1 \quad \text{and} \quad y_2'' = -py_2' - qy_2.$$

Substituting these into (80) yields

$$\begin{aligned} W' &= -y_1(py_2' + qy_2) + y_2(py_1' + qy_1) \\ &= -p(y_1y_2' - y_1'y_2) - q(y_1y_2 - y_1y_2) \\ &= -p(y_1y_2' - y_1'y_2) = -pW. \end{aligned}$$

Therefore $W' + pW = 0$; that is, W is the solution of the initial value problem

$$y' + p(x)y = 0, \;\; y(x_0) = W(x_0).$$

If $W(x_0) \neq 0$, (79) implies that W has no zeros in (a,b), since an exponential is never zero. On the other hand, if $W(x_0) = 0$, (79) implies that $W(x) = 0$ for all $x \in (a,b)$.

For solutions y_1 and y_2 of (78) on (a,b), and let $W = y_1y_2' - y_1'y_2$. We can show that y_1 and y_2 are linearly independent on (a,b) if and only if $W \neq 0$ for all $x \in (a,b)$.

We first show that if $W(x_0) = 0$ for some $x_0 \in (a,b)$ then y_1 and y_2 are linearly dependent on (a,b). Let $I \subset (a,b)$ on which y_1 has no zeros. (If $y_1(x) \equiv 0$ for all $x \in (a,b)$, then y_1 and y_2 are linearly independent.) Then y_2/y_1 is defined on I, and

$$\left(\frac{y_2}{y_1}\right)' = \frac{y_1y_2' - y_1'y_2}{y_1^2} = \frac{W}{y_1^2}. \tag{81}$$

However, if $W(x_0) = 0$, by Theorem 8.2.2, $W \equiv 0$ for all $x \in (a,b)$. Therefore (81) implies that $(y_2/y_1)' \equiv 0$, so $y_2/y_1 = c$ (constant) on I. This shows that $y_2 = cy_1$ for all $x \in I$. However, we want to show that $y_2 = cy_1$ for all $x \in (a,b)$. Let $Y = y_2 - cy_1$. Then Y is a solution of (78) on (a,b) such that $Y \equiv 0$ on I, and therefore $Y' \equiv 0$ on I. Consequently, if x_0 is chosen arbitrarily in I then Y is a solution of the initial value problem,

$$y'' + p(x)y' + q(x)y = 0, \;\; y(x_0) = 0, \;\; y'(x_0) = 0,$$

which implies that $Y \equiv 0$ on (a,b). Hence, $y_2 - cy_2 \equiv 0$ on (a,b), which implies that y_1 and y_2 are not linearly independent on (a,b). Now suppose that W has no zeros on (a,b). Then y_1 cannot be identically zero on (a,b), and therefore there is a subinterval $I \subset (a,b)$ on which y_1 has no zeros. Since (81) implies that y_2/y_1 is nonconstant on I, y_2 is not a constant multiple of y_1 on (a,b). A similar argument shows that y_1 is not a constant multiple of y_2 on (a,b), since

$$\left(\frac{y_1}{y_2}\right)' = \frac{y_1'y_2 - y_1y_2'}{y_2^2} = -\frac{W}{y_2^2}$$

on any subinterval of (a,b) where y_2 has no zeros.

The next theorem summarises the relationships among the concepts discussed in this section.

Theorem 8.2.3. Let y_1 and y_2 be solutions of (78) on (a, b). Then the following statements are equivalent:

(a) The general solution of (78) on (a, b) is $y = C_1y_1 + C_2y_2$.

(b) $\{y_1, y_2\}$ is a fundamental set of solutions of (78) on (a, b).

(c) $\{y_1, y_2\}$ is linearly independent on (a, b).

(d) The Wronskian of $\{y_1, y_2\}$ is nonzero at some point in (a, b).

(e) The Wronskian of $\{y_1, y_2\}$ is nonzero at all points in (a, b).

We next introduce the notion of Fundamental Solution Matrix.

Definition 8.2.4 (Fundamental Solution Matrix for Second-Order ODE). Suppose that $\{y_1, y_2\}$ is a fundamental set of solutions of (78). We define the *fundamental solution matrix* of (78) as

$$\Phi(x) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}.$$

Notice that the determinant of the fundamental solution matrix is the Wronskian of $\{y_1, y_2\}$.

Any solution of (78) can be obtained by

$$\Phi \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} C_1y_1 + C_2y_2 \\ C_1y_1' + C_2y_2' \end{pmatrix}$$

for constants C_1 and C_2. In practice, the conditions we use to fix C_1 and C_2 are $y(x_0)$ and $y'(x_0)$, where

$$\Phi(x_0) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} y(x_0) \\ y'(x_0) \end{pmatrix},$$

thus,

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \Phi(x_0)^{-1} \begin{pmatrix} y(x_0) \\ y'(x_0) \end{pmatrix}.$$

If $\Phi(x_0) = I_2$, we call $\Phi(x_0)$ the **principal fundamental solution matrix** and such y_1 and y_2 are called **principal fundamental set of solutions**.

How about the whole machinery for ODE homogeneous system? We want to find n linearly independent solutions for it so that we can write down all solutions. Suppose $\overrightarrow{y}_1(x), \ldots, \overrightarrow{y}_n(x)$ are solutions for

$$\overrightarrow{y}'(x) = A(x)\overrightarrow{y}(x).$$

We have the $n \times n$ solution matrix

$$\Phi(x) = \begin{pmatrix} \overrightarrow{y}_1(x) & \cdots & \overrightarrow{y}_n(x) \end{pmatrix}.$$

Since $W(x) = \det \Phi(x)$,

$$\begin{aligned}
W'(x) &= \left| \vec{y}_1 \quad \cdots \quad \vec{y}_n \right|' \\
&= \left| \vec{y}'_1 \quad \vec{y}_2 \quad \cdots \quad \vec{y}_n \right| + \left| \vec{y}_1 \quad \vec{y}'_2 \quad \cdots \quad \vec{y}_n \right| + \cdots + \left| \vec{y}_1 \quad \cdots \quad \vec{y}_{n-1} \quad \vec{y}'_n \right| \\
&= \left| A\vec{y}_1 \quad \vec{y}_2 \quad \cdots \quad \vec{y}_n \right| + \left| \vec{y}_1 \quad A\vec{y}_2 \quad \cdots \quad \vec{y}_n \right| + \cdots + \left| \vec{y}_1 \quad \cdots \quad \vec{y}_{n-1} \quad A\vec{y}_n \right| \\
&= \sum_{k=1}^{n} a_{kk} \left| \vec{y}_1 \quad \cdots \quad \vec{y}_n \right| \\
&= \operatorname{tr}(A) W.
\end{aligned}$$

Hence,

$$W'(x) = \operatorname{tr}(A(x)) W(x).$$

Suppose $W(x_0) \neq 0$ and $\vec{y}_1, \ldots, \vec{y}_n$ are linearly independent, then $\Phi(x) = \begin{pmatrix} \vec{y}_1 & \cdots & \vec{y}_n \end{pmatrix}$ is invertible anywhere according to Abel's Identity.

Now we want to obtain an expression for the derivative of Φ.

$$\begin{aligned}
\Phi'(x) &= \begin{pmatrix} \vec{y}'_1 & \cdots & \vec{y}_n \end{pmatrix} = \begin{pmatrix} A\vec{y}_1 & \cdots & A\vec{y}_n \end{pmatrix} \\
&= A \begin{pmatrix} \vec{y}_1 & \cdots & \vec{y}_n \end{pmatrix} = A\Phi(x),
\end{aligned}$$

therefore,

$$\Phi' = A\Phi. \tag{82}$$

8.3 Variation of Constants

The variation of constants is a general method for solving nonhomogeneous linear ODEs. We consider the nonhomogeneous linear system of ODE

$$\vec{y}'(x) = A(x)\vec{y}(x) + \vec{f}(x) \tag{83}$$

where $\vec{y}(x) = \begin{pmatrix} \vec{y} \\ \vec{y}' \\ \vdots \\ \vec{y}^{(n-1)} \end{pmatrix}$ and $\vec{f}(x) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f(x) \end{pmatrix}$. Suppose we have a fundamental set of solutions $\{\vec{y}_1, \ldots, \vec{y}_n\}$ for the complementary equation

$$\vec{y}'(x) = A(x)\vec{y}(x).$$

Set the fundamental solution matrix

$$\Phi(x) = \begin{pmatrix} \vec{y}_1(x) & \cdots & \vec{y}_n(x) \end{pmatrix}.$$

Firstly, we want to find out the relationship between the inverse of Φ and its derivative. We know that multiplying a matrix by its inverse results in the identity matrix (here, we assume that Φ is always invertible), i.e.

$$\Phi\Phi^{-1} = I_n.$$

Differentiate on both sides, we get

$$\Phi'\Phi^{-1} + \Phi(\Phi^{-1})' = [0]_n \tag{84}$$

where $[0]_n$ is the $n \times n$ zero matrix. By (82), we can rewrite the LHS of (84) as

$$\Phi'\Phi^{-1} + \Phi(\Phi^{-1})' = A\Phi\Phi^{-1} + \Phi(\Phi^{-1})' = A + \Phi(\Phi^{-1})',$$

which results in

$$(\Phi^{-1})' = -\Phi^{-1}A. \tag{85}$$

This helps us in solving (83). After rearrangement, (83) becomes

$$\overrightarrow{y}' - A\overrightarrow{y} = \overrightarrow{f},$$

multiplying both sides by Φ^{-1} yields

$$\Phi^{-1}\overrightarrow{y}' - \Phi^{-1}A\overrightarrow{y} = \Phi^{-1}\overrightarrow{f}.$$

By (85), we rewrite the LHS as

$$\Phi^{-1}\overrightarrow{y}' + (\Phi^{-1})'\overrightarrow{y}.$$

By the product rule, we get

$$(\Phi^{-1}\overrightarrow{y})' = \Phi^{-1}\overrightarrow{f}.$$

Integrating both sides yields

$$\Phi^{-1}\overrightarrow{y} = \int \Phi^{-1}\overrightarrow{f}\,dx.$$

Therefore, the general solution of (83) is given by

$$\overrightarrow{y}(x) = \Phi(x)\int_{x_0}^{x} \Phi(t)^{-1}\overrightarrow{f}(t)\,dt + \Phi(x)\overrightarrow{C}, \tag{86}$$

for any $\overrightarrow{C} = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}$.

This method is called the **variation of constants**.

Now, consider the case for the equation

$$\overrightarrow{y}'(x) = A\overrightarrow{y}(x) + \overrightarrow{f}(x)$$

where A only has constant entries. From section 7.3, we know that the solution of the complementary is $\overrightarrow{y}(x) = e^{xA}\overrightarrow{y}(0)$. Thus, we can set $\Phi(x) = e^{xA}$, and by (86), the solution in this case is

$$\overrightarrow{y}(x) = e^{xA}\int_{x_0}^{x} e^{-tA}\overrightarrow{f}(t)\,dt + e^{xA}\overrightarrow{C}.$$

9 Solving ODE II

9.1 Undetermined Coefficients

We first illustrate the method of undetermined coefficients for the equation

$$ay'' + by' + cy = G(x)$$

where $G(x)$ is a polynomial. It is reasonable to guess that there is a particular solution y_p that is a polynomial of the same degree as G be cause if y is a polynomial, then $ay'' + by' + cy$ is also a polynomial. We therefore substitute $y_p(x) =$ a polynomial (of the same degree as G) into the differential equation and determine the coefficients.

We summarise the method of undetermined coefficients.

Summary of the method of Undetermined Coefficients:

1. If $G(x) = e^{kx}P(x)$, where P is an nth-degree polynomial, then try $y_p(x) = e^{kx}Q(x)$, where Q is an nth-degree polynomial.
2. If $G(x) = e^{kx}P(x)\cos mx$ or $G(x) = e^{kx}P(x)\sin mx$, where P is an nth-degree polynomial, then try
$$y_p(x) = e^{kx}Q(x)\cos mx + e^{kx}R(x)\sin mx$$
where Q and R are nth-degree polynomials.

9.2 Reduction of Order

In this section we give a method for finding the general solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x), \tag{87}$$

if we know a nontrivial solution y_1 of the complementary equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0. \tag{88}$$

The **reduction of order method** reduces the task of solving (87) to solving a first-order ODE. It does not require P_0, P_1, and P_2 to be constants, or F to be of any special form.

We look for solutions of (87) in the form

$$y(x) = V(x)y_1(x) \tag{89}$$

where V is to be determined so that y satisfies (87). Substituting (89) and

$$\begin{aligned} y' &= V'y_1 + Vy_1' \\ y'' &= V''y_1 + 2V'y_1' + Vy_1'' \end{aligned}$$

into (87) yields

$$P_0(x)(V''y_1 + 2V'y_1' + Vy_1'') + P_1(x)(V'y_1 + Vy_1') + P_2(x)Vy_1 = F(x).$$

Collecting the coefficients of V, V', and V'' yields

$$(P_0y_1)V'' + (2P_0y_1' + P_1y_1)V' + (P_0y_1'' + P_1y_1' + P_2y_1)V = F. \tag{90}$$

However, the coefficient of V is zero, since y_1 is a solution of the complementary equation (88). Therefore, by letting $u = V'$, (90) reduces to a first-order linear ODE

$$Q_0(x)V' + Q_1(x)V = F \tag{91}$$

with $Q_0 = P_0y_1$ and $Q_1 = 2P_0y_1' + P_1y_1$. This can be easily solved.

9.3 Variation of Parameters

Continue to consider the equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x). \tag{92}$$

If we know a fundamental set $\{y_1, y_2\}$ of solutions of the complementary equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0. \tag{93}$$

We consider solutions of (92) and (93) on an interval (a, b), where P_0, P_1, P_2, and F are all continue and $P_0(x) \neq 0$ on (a, b). We look for a particular solution of (92) in the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{94}$$

where u_1 and u_2 are functions to be determined so that y_p satisfies (92). Since u_1 and u_2 have to satisfy only one condition (that y_p is a solution of (92)), we can impose a second condition that produces a convenient simplification, as follows.

Differentiating (94) yields

$$y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'. \tag{95}$$

As our second condition on u_1 and u_2, we require that

$$u_1'y_1 + u_2'y_2 = 0. \tag{96}$$

Then (95) becomes

$$y_p' = u_1y_1' + u_2y_2'; \tag{97}$$

that is, (96) permits us to differentiate y_p as if u_1 and u_2 are constants. Differentiating (95) yields

$$y_p'' = u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2'. \tag{98}$$

Substituting (94), (97), and (98) into (92) and collecting the coefficients of u_1 and u_2 yields

$$u_1(P_0y_1'' + P_1y_1' + P_2y_1) + u_2(P_0y_2'' + P_1y_2' + P_2y_2) + P_0(u_1'y_1' + u_2'y_2') = F.$$

The coefficients of u_1 and u_2 here are both zero because y_1 and y_2 satisfy the complementary equation. Hence, we can rewrite the last equation as

$$P_0(u_1'y_1' + u_2'y_2') = F. \tag{99}$$

Therefore, y_p in (94) satisfies (92) if

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= \frac{F}{P_0}, \end{aligned}$$

where the first equation is the same as (90). This system of linear equation can be expressed as

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ F/P_0 \end{pmatrix}.$$

Notice that the LHS is equivalent to $\Phi \begin{pmatrix} u_1' \\ u_2' \end{pmatrix}$, so

$$\begin{aligned} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} &= \Phi^{-1} \begin{pmatrix} 0 \\ F/P_0 \end{pmatrix} = \frac{1}{\det \Phi} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ F/P_0 \end{pmatrix} \\ &= \frac{1}{W} \begin{pmatrix} -Fy_2/P_0 \\ Fy_1/P_0 \end{pmatrix}. \end{aligned}$$

Therefore we obtain

$$u_1(x) = -\int \frac{y_2(x)F(x)}{W(x)P_0(x)}\,dx \tag{100}$$

$$u_2(x) = \int \frac{y_1(x)F(x)}{W(x)P_0(x)}\,dx. \tag{101}$$

The constants of integration can be taken to be zero, since any choice of u_1 and u_2 in (94) will suffice.

(100) and (101)can be derived using the fact

$$\overrightarrow{y}(x) = \Phi(x) \int_{x_0}^{x} \Phi(t)^{-1} \overrightarrow{f}(t)\,dt + \Phi(x)\overrightarrow{C}$$

that was shown before.

10 Power and Fourier Series in Solving ODE

10.1 Power Series in solving ODE

Many differential equations cannot be solved explicitly in terms of finite combinations of simple familiar functions. In such case we use the method of power series; that is, we look for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + +c_3 x^3 \cdots$$

The method is to substitute this expression into the differential equation and determine the values of the coefficients $c_0, c_1, c_2, \ldots$ This technique resembles the method of undetermined coefficients discussed before.

Example 10.1.1 Use power series to solve $y'' + y = 0$.

Solution. We assume there is a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n. \tag{102}$$

We can differentiate power series term by term, so

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}.$$

In order to compare the expressions for y and y'' more easily, we rewrite y'' as follows:

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n. \tag{103}$$

Substituting (102) and (103) into the ODE, we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

or

$$\sum_{n=0}^{\infty} [(n+2)(n+1) c_{n+2} + c_n] x^n = 0. \tag{104}$$

If two power series are equal, then the corresponding coefficients must be equal. Therefore the coefficients of x^n in equation (104) must be 0, i.e.

$$(n+2)(n+1) c_{n+2} + c_n = 0,$$

which gives a recursive relation

$$c_{n+2} = -\frac{c_n}{(n+1)(n+2)}, \quad n = 0, 1, 2, 3, \ldots \tag{105}$$

If c_0 and c_1 are known, this equation allows us to determine the remaining coefficients recursively by putting $n = 0, 1, 2, 3, \dots$ in succession.

$$\begin{aligned}
n = 0: &\quad c_2 = -\frac{c_0}{1 \times 2} \\
n = 1: &\quad c_3 = -\frac{c_1}{2 \times 3} \\
n = 2: &\quad c_4 = -\frac{c_2}{3 \times 4} = \frac{c_0}{1 \times 2 \times 3 \times 4} = \frac{c_0}{4!} \\
n = 3: &\quad c_5 = -\frac{c_3}{4 \times 5} = \frac{c_1}{2 \times 3 \times 4 \times 5} = \frac{c_1}{5!} \\
n = 4: &\quad c_6 = -\frac{c_4}{5 \times 6} = -\frac{c_0}{4!5 \times 6} = -\frac{c_0}{6!} \\
n = 5: &\quad c_7 = -\frac{c_5}{6 \times 7} = -\frac{c_1}{5!6 \times 7} = -\frac{c_1}{7!}
\end{aligned}$$

By now we see the pattern:

$$\text{For the even coefficients,} \quad c_{2n} = (-1)^n \frac{c_0}{(2n)!}.$$

$$\text{For the odd coefficients,} \quad c_{2n+1} = (-1)^n \frac{c_1}{(2n+1)!}.$$

Putting these values back into (102), we write the solution as

$$\begin{aligned}
y(x) &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots \\
&= c_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \right) + c_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots \right) \\
&= c_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}
\end{aligned}$$

Notice that there are two arbitrary constants c_0 and c_1. We recognise the series obtained is the Maclaurin series for $\cos x$ and $\sin x$. Therefore, we could write the solution as

$$y(x) = c_0 \cos x + c_1 \sin x.$$

10.2 Boundary Value Problems and Eigenvalue Problems

Consider the following problems, where $\lambda \in \mathbb{R}$ and $L > 0$:

(a) $y'' + \lambda y = 0$, $y(0) = y(L) = 0$.

(b) $y'' + \lambda y = 0$, $y'(0) = y'(L) = 0$.

(c) $y'' + \lambda y = 0$, $y'(0) = y(L) = 0$.

(d) $y'' + \lambda y = 0$, $y(0) = y'(L) = 0$.

(e) $y'' + \lambda y = 0$, $y(-L) = y(L) = 0$, $y'(-L) = y'(L)$.

In each problem, the conditions following the ODE are called *boundary conditions*. A value of λ for which the problem has a nontrivial solution is an *eigenvalue* of the problem, and the nontrivial solutions are λ-*eigenfunctions*. Note that a nonzero constant multiple of a λ-eigenfunction is again a λ-eigenfunction.

Problem (a)-(e) are called **eigenvalue problems**. Solving an eigenvalue problem means finding all its eigenvalues and associated eigenfunctions. We will take it as given here that all the eigenvalues of problems (a)-(e) are real numbers.

Theorem 10.2.1. Problems (a)-(e) have no negative eigenvalues. Moreover, $\lambda = 0$ is an eigenvalue of problems (b) and (e), with associated eigenfunction $y_0 = 1$, but $\lambda = 0$ is not an eigenvalue of problems (a), (c) or (d).

Proof. Consider Problems (a)-(d). If $y'' + \lambda y = 0$, then $y(y'' + \lambda y) = 0$, so

$$\int_0^L y(y'' + \lambda y)\, dx = 0;$$

therefore,

$$\lambda \int_0^L y^2\, dx = -\int_0^L yy''\, dx. \tag{106}$$

Integration by parts yields

$$\begin{aligned}\int_0^L yy''\, dx &= yy'\Big|_0^L - \int_0^L (y')^2\, dx \\ &= y(L)y'(L) - y(0)y'(0) - \int_0^L (y')^2\, dx. \end{aligned} \tag{107}$$

However, if y satisfies any of the boundary conditions of problems (a)-(d), then

$$y(L)y'(L) - y(0)y'(0) = 0;$$

hence (106) and (107) imply that

$$\lambda \int_0^L y^2\, dx = \int_0^L (y')^2\, dx.$$

If $y \neq 0$, then

$$\int_0^L y^2\, dx > 0.$$

Therefore $\lambda \geq 0$ and, if $\lambda = 0$, then $y' = 0$ for all $x \in (0, L)$, and y is constant on $(0, L)$. Any constant function satisfies the boundary conditions of problem (b), so $\lambda = 0$ is an eigenvalue of problem (b) and any nonzero constant function is an associated eigenfunction. However, the only constant function that satisfies the boundary conditions of problems (a), (c), or (d) is $y \equiv 0$. Therefore $\lambda = 0$ is not an eigenvalue of any of these problems.

Theorem 10.2.1. For $y'' + \lambda y = 0$, the eigenvalue problems:

(a) on $[0, L]$: $y(0) = y(L) = 0$ has eigenvalues $\lambda_n = \frac{n^2\pi^2}{L^2}$ with associated eigenfunctions $y_n = \sin(\frac{n\pi}{L}x)$ for $n = 1, 2, 3, \ldots$

(b) on $[0, L]$: $y'(0) = y'(L) = 0$ has eigenvalues $\lambda_n = \frac{n^2\pi^2}{L^2}$ with associated eigenfunctions $y_n = \cos(\frac{n\pi}{L}x)$ for $n = 0, 1, 2, \ldots$

(c) on $[0, L]$: $y'(0) = y(L) = 0$ has eigenvalues $\lambda_n = \frac{(2n-1)^2\pi^2}{4L^2}$ with associated eigenfunctions $y_n = \cos(\frac{(2n-1)\pi}{2L}x)$ for $n = 1, 2, 3, \ldots$

(d) on $[0, L]$: $y(0) = y'(L) = 0$ has eigenvalues $\lambda_n = \frac{(2n-1)^2\pi^2}{4L^2}$ with associated eigenfuctions $y_n = \sin(\frac{(2n-1)\pi}{2L}x)$ for $n = 1, 2, 3, \ldots$

(e) on $[-L, L]$: $y(-L) = y(L) = 0$, $y'(-L) = y'(L)$ has eigenvalue $\lambda_0 = 0$ with associated eigenfunction $y_0 = 1$, and eigenvalues $\lambda_n = \frac{n^2\pi^2}{L^2}$ with associated eigenfunctions $y_{1n} = \cos(\frac{n\pi}{L}x)$ and $y_{2n} = \sin(\frac{n\pi}{L}x)$ for $n = 1, 2, 3, \ldots$

Proof. (a) From Theorem 10.2.1, any eigenvalues of problem (a) must be positive. If y satisfies problem (a) with $\lambda > 0$, then

$$y = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x,$$

where C_1 and C_2 are constants. The boundary condition $y(0) = 0$ implies that $C_1 = 0$. Therefore $y = C_2 \sin\sqrt{\lambda}x$. Now the boundary condition $y(L) = 0$ implies that $C_2 \sin\sqrt{\lambda}x = 0$. To make $C_2 \sin\sqrt{\lambda}x = 0$ with $C_2 \neq 0$, we must choose $\sqrt{\lambda} = \frac{n\pi}{L}$, where n is a positive integer. Therefore $\lambda_n = \frac{n^2\pi^2}{L^2}$ is an eigenvalue and $y_n = \sin(\frac{n\pi}{L}x)$ is an associated eigenfunction.

(b)-(d) can be shown in similar way. Now we prove (e).

From Theorem 10.2.1, $\lambda_0 = 0$ is an eigenvalue of problem (e) with associated eigenfunction $y_0 = 1$, and any other eigenvalues must be positive. If y satisfies (e) with $\lambda > 0$, then

$$y = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x, \tag{108}$$

where C_1 and C_2 are constants. The boundary condition $y(-L) = y(L) = 0$ implies that

$$C_1 \cos(-\sqrt{\lambda}L) + C_2 \sin(-\sqrt{\lambda}L) = C_1 \cos\sqrt{\lambda}L + C_2 \sin\sqrt{\lambda}L. \tag{109}$$

Since

$$\cos(-\sqrt{\lambda}L) = \cos\sqrt{\lambda}L \quad \text{and} \quad \sin(-\sqrt{\lambda}L) = -\sin\sqrt{\lambda}L, \tag{110}$$

(109) implies that

$$C_2 \sin\sqrt{\lambda}L = 0. \tag{111}$$

Differentiating (108) yields

$$y' = \sqrt{\lambda}\big[-C_1 \sin\sqrt{\lambda}x + C_2 \cos\sqrt{\lambda}x\big].$$

The boundary condition $y'(-L) = y'(L)$ implies that

$$-C_1 \sin(-\sqrt{\lambda}L) + C_2 \cos(-\sqrt{\lambda}L) = -C_1 \sin\sqrt{\lambda}L + C_2 \cos\sqrt{\lambda}L$$

and (110) implies that

$$C_1 \sin\sqrt{\lambda}L = 0. \tag{112}$$

Equation (111) and (112) imply that $C_1 = C_2 = 0$ unless $\sqrt{\lambda} = \frac{n\pi}{L}$, where n is a positive integer. In this case (111) and (112) both hold for arbitrary C_1 and C_2. The eigenvalue determined in this way is $\lambda_n = \frac{n^2\pi^2}{L^2}$, and each such eigenvalue has the linearly independent associated eigenfunctions

$$\cos\frac{n\pi x}{L} \quad \text{and} \quad \sin\frac{n\pi x}{L}.$$

10.3 Periodic Functions and Fourier Series

To study the periodic functions and Fourier Series, we need to first define the orthogonality of functions.

Definition 10.3.1 (Orthogonality). We say that the functions $\phi_1, \phi_2, \ldots, \phi_n \ldots$ (finitely or infinitely many) are orthogonal on $[a, b]$ if

$$\int_a^b \phi_i(x)\phi_j(x)\,dx = 0$$

whenever $i \neq j$.

Theorem 10.3.2. Suppose functions $\phi_1, \phi_2, \phi_3, \ldots$ are orthogonal on $[a, b]$ and

$$\int_a^b \phi_n^2(x)\,dx \neq 0, \quad n = 1, 2, 3, \ldots \tag{113}$$

Let $c_1, c_2, c_3, \ldots$ be constants such that the partial sums

$$f_N(x) = \sum_{m=1}^{N} c_m\phi_m(x)$$

satisfy the inequalities

$$|f_N(x)| < M, \quad x \in [a, b], \quad N = 1, 2, 3, \ldots$$

for some constant $M < \infty$. Suppose also that the series

$$f(x) = \sum_{m=1}^{\infty} c_m\phi_m(x) \tag{114}$$

converges and is integrable on $[a, b]$. Then

$$c_n = \frac{\int_a^b f(x)\phi_n(x)\,dx}{\int_a^b \phi_n^2(x)\,dx}, \quad n = 1, 2, 3, \ldots \tag{115}$$

Proof. Multiplying (114) by ϕ_n and integrating yields

$$\int_a^b f(x)\phi_n(x)\,dx = \int_a^b \phi_n(x)\left(\sum_{m=1}^{\infty} c_m\phi_m(x)\right)dx. \tag{116}$$

It can be shown that the boundedness of the partial sums $\{f_N\}_{N=1}^{\infty}$ and the integrability of f allow us to interchange the operations of integration and summation on the RHS of (116), and rewrite (116) as

$$\int_a^b f(x)\phi_n(x)\,dx = \sum_{m=1}^{\infty} c_m \int_a^b \phi_n(x)\phi_m(x)\,dx. \tag{117}$$

Since

$$\int_a^b \phi_n(x)\phi_m(x)\,dx = 0$$

if $m \neq n$, (117) reduces to

$$\int_a^b f(x)\phi_n(x)\,dx = c_n \int_a^b \phi_n^2(x)\,dx.$$

Hence, (113) implies (115).

Theorem 10.3.2 motivates the next definition.

Definition 10.3.3 (Fourier Expansion). Suppose $\phi_1, \phi_2, \ldots, \phi_n, \ldots$ are orthogonal on $[a,b]$ and $\int_a^b \phi_n^2(x)\,dx \neq 0$, $n = 1, 2, 3, \ldots$. Let f be integrable on $[a,b]$ and define

$$c_n = \frac{\int_a^b f(x)\phi_n(x)\,dx}{\int_a^b \phi_n^2(x)\,dx}, \quad n = 1, 2, 3, \ldots. \tag{118}$$

Then the infinite series

$$\sum_{n=1}^{\infty} c_n\phi_n(x)$$

is called the *Fourier expansion of f in terms of the orthogonal set $\{\phi_n\}_{n=1}^{\infty}$*, and $c_1, c_2, \ldots, c_n, \ldots$ are called the *Fourier coefficients of f with respect to $\{\phi_n\}_{n=1}^{\infty}$*. We indicate the relationship between f and its Fourier expansion by

$$f(x) \sim \sum_{n=1}^{\infty} c_n\phi_n(x), \quad a \leq x \leq b. \tag{119}$$

We now study Fourier expansions in terms of the eigenfunctions

$$1,\ \cos\frac{\pi x}{L},\ \sin\frac{\pi x}{L},\ \cos\frac{2\pi x}{L},\ \sin\frac{2\pi x}{L}, \ldots, \cos\frac{n\pi x}{L},\ \sin\frac{n\pi x}{L}, \ldots$$

of problem (e). If f is integrable on $[-L, L]$, its Fourier expansion in terms of these functions is called the *Fourier series of f on $[-L, L]$*. Since

$$\int_{-L}^{L} 1^2\,dx = 2L,$$

$$\int_{-L}^{L} \cos^2 \frac{n\pi x}{L}\, dx = \frac{1}{2}\int_{-L}^{L} \left(1 + \cos \frac{2n\pi x}{L}\right) dx = \frac{1}{2}\left(x + \frac{L}{2n\pi} \sin \frac{2n\pi x}{L}\right)\Bigg|_{-L}^{L} = L,$$

and

$$\int_{-L}^{L} \sin^2 \frac{n\pi x}{L}\, dx = \frac{1}{2}\int_{-L}^{L} \left(1 - \cos \frac{2n\pi x}{L}\right) dx = \frac{1}{2}\left(x - \frac{L}{2n\pi} \sin \frac{2n\pi x}{L}\right)\Bigg|_{-L}^{L} = L,$$

we see from (118) that the Fourier series of f on $[-L, L]$ is

$$a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\right),$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx,$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx,$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx, \quad n = 1, 2, \ldots$$

Even and Odd Functions

Let f and g be defined on $[-L, L]$ and suppose that

$$f(-x) = f(x) \quad \text{and} \quad g(-x) = -g(x), \quad x \in [-L, L].$$

Then we say that f is an **even** function and g is an **odd** function. Note that:

- The product of two even functions or the product of two odd functions is even.
- The product of an even function and an odd function is odd.

Theorem 10.3.4. Suppose f is even and g is odd on $[-L, L]$. Then:

- $\int_{-L}^{L} f(x)\, dx = 2\int_0^L f(x)\, dx$
- $\int_{-L}^{L} g(x)\, dx = 0$
- $\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx = 2\int_0^L f(x) \cos \frac{n\pi x}{L}\, dx$
- $\int_{-L}^{L} g(x) \sin \frac{n\pi x}{L}\, dx = 2\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx$
- $\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx = \int_{-L}^{L} g(x) \cos \frac{n\pi x}{L}\, dx = 0$

Theorem 10.3.5. Suppose f is integrable on $[-L, L]$.

(i) If f is *even*, the Fourier series of f on $[-L, L]$ is

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L},$$

where

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx \quad \text{and} \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n \geq 1.$$

(ii) If f is *odd*, the Fourier series of f on $[-L, L]$ is

$$F(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}\,dx,$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

This can be extended to obtain the Fourier sine, cosine, mixed cosine and mixed sine series, which are the Fourier expansions in terms of eigenfunctions of eigenvalue problems (a), (b), (c) and (d) respectively.

11 Solving PDE I - Homogeneous PDE

In this section we focuses on the following forms of homogeneous PDEs:

- **Heat/Diffusion:** $u_t = a^2 u_{xx}$
- **Wave:** $u_{tt} = a^2 u_{xx}$
- **Laplace:** $u_{xx} + u_{yy} = 0$

11.1 The Heat Equation

We begin with the boundary conditions $u(0,t) = u(L,t) = 0$, and write the initial-boundary value problem as

$$\begin{aligned} u_t &= a^2 u_{xx}, \quad x \in (0,L), \quad t > 0, \\ u(0,t) &= u(L,t) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad x \in [0,L]. \end{aligned} \tag{120}$$

Our method of solving is called *separation of variables.* We begin by looking for functions of the form

$$v(x,t) = X(x)T(t)$$

that are not identically zero and satisfy

$$v_t = a^2 v_{xx}, \quad v(0,t) = v(L,t) = 0$$

for all (x,t). Since

$$v_t = XT' \quad \text{and} \quad v_{xx} = X''T,$$

$v_t = a^2 v_{xx}$ if and only if

$$XT' = a^2 X''T,$$

which we rewrite as

$$\frac{T'}{a^2 T} = \frac{X''}{X}.$$

Since LHS is independent of x while the RHS is independent of t, this equation can hold for all (x,t) only if the two sides equal the same constant, which we call a **separation constant**, and write it as $-\lambda$; thus,

$$\frac{X''}{X} = \frac{T'}{a^2 T} = -\lambda.$$

This is equivalent to

$$X'' + \lambda X = 0$$

and

$$T' = -a^2 \lambda T. \tag{121}$$

Since $v(0,t) = X(0)T(t) = 0$ and $v(L,t) = X(L)T(t) = 0$ and we do not want T to be identically zero, $X(0) = X(L) = 0$. Therefore λ must be an eigenvalue of the boundary value problem

$$X'' + \lambda X = 0, \quad X(0) = X(L) = 0, \tag{122}$$

and X must be a λ-eigenfunction. From Theorem 10.2.1, the eigenvalues of (122) are $\lambda_n = \frac{n^2\pi^2}{L^2}$ with associated eigenfunctions

$$X_n = \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

Substituting $\lambda = \frac{n^2\pi^2}{L^2}$ into (121) yields

$$T' = -\frac{n^2\pi^2a^2}{L^2}T,$$

which has the solution

$$T_n = e^{-\frac{n^2\pi^2a^2}{L^2}t}.$$

Now let

$$v_n(x,t) = X_n(x)T_n(t) = e^{-\frac{n^2\pi^2a^2}{L^2}t}\sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

Since

$$v_n(x,0) - \sin\frac{n\pi x}{L},$$

v_n satisfies (120) with $f(x) = \sin(\frac{n\pi}{L}x)$. More generally, if $\alpha_1, \ldots, \alpha_m$ are constants and

$$u_m(x,t) = \sum_{n=1}^{m}\alpha_n e^{-\frac{n^2\pi^2a^2}{L^2}t}\sin\frac{n\pi x}{L},$$

then u_m satisfies (120) with

$$f(x) = \sum_{n=1}^{m}\alpha_n\sin\frac{n\pi x}{L}.$$

This and similar works motivate the next definition.

Definition 11.1.1.

(i) The formal solution of the initial-boundary value problems:

$$\begin{aligned} u_t = a^2u_{xx}, \quad x \in (x, L), \quad t > 0,\\ u(0,t) = u(L,t) = 0, \quad t > 0,\\ u(x,0) = f(x), \quad x \in [0, L] \end{aligned}$$

is

$$u(x,t) = \sum_{n=1}^{\infty}\alpha_n e^{\frac{n^2\pi^2a^2}{L^2}t}\sin\frac{n\pi x}{L},$$

where $\sum_{n=1}^{\infty}\alpha_n\sin\frac{n\pi x}{L}$ is the Fourier sine series of f on $[0, L]$; that is,

$$\alpha_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

(ii) The formal solution of the initial-boundary value problem

$$
\begin{aligned}
u_t = a^2 u_{xx}, \quad x \in (x, L), \quad t > 0,\\
u_x(0,t) = u_x(L,t) = 0, \quad t > 0,\\
u(x,0) = f(x), \quad x \in [0, L]
\end{aligned}
$$

is

$$
u(x,t) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n e^{-\frac{n^2\pi^2 a^2}{L^2}t} \cos \frac{n\pi x}{L},
$$

where $\alpha_0 + \sum_{n=1}^{\infty} \alpha_0 \cos \frac{n\pi x}{L}$ is the Fourier cosine series of f on $[0, L]$; that is

$$
\alpha_0 = \frac{1}{L}\int_0^L f(x)\,dx \quad \text{and} \quad \alpha_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots
$$

(iii) The formal solution of the initial-boundary value problem

$$
\begin{aligned}
u_t = a^2 u_{xx}, \quad x \in (x, L), \quad t > 0,\\
u(0,t) = u_x(L,t) = 0, \quad t > 0,\\
u(x,0) = f(x), \quad x \in [0, L]
\end{aligned}
$$

is

$$
u(x,t) = \sum_{n=1}^{\infty} \alpha_n e^{-\frac{(2n-1)^2\pi^2 a^2}{4L^2}t} \sin \frac{(2n-1)\pi x}{2L},
$$

where $\sum_{n=1}^{\infty} \alpha_n \sin \frac{(2n-1)\pi x}{2L}$ is the mixed Fourier sine series of f on $[0, L]$; that is

$$
\alpha_n = \frac{2}{L}\int_0^L f(x)\sin\frac{(2n-1)\pi x}{2L}\,dx.
$$

(iv) The formal solution of the initial-boundary value problem

$$
\begin{aligned}
u_t = a^2 u_{xx}, \quad x \in (x, L), \quad t > 0,\\
u_x(0,t) = u(L,t) = 0, \quad t > 0,\\
u(x,0) = f(x), \quad x \in [0, L]
\end{aligned}
$$

is

$$
u(x,t) = \sum_{n=1}^{\infty} \alpha_n e^{-\frac{(2n-1)^2\pi^2 a^2}{4L^2}t} \cos \frac{(2n-1)\pi x}{2L},
$$

where $\sum_{n=1}^{\infty} \alpha_n \cos \frac{(2n-1)\pi x}{2L}$ is the mixed Fourier cosine series of f on $[0, L]$; that is

$$
\alpha_n = \frac{2}{L}\int_0^L f(x)\cos\frac{(2n-1)\pi x}{2L}\,dx.
$$

11.2 The Wave Equation

Consider the initial-boundary value problems of the form

$$\begin{aligned} u_{tt} = a^2 u_{xx}, \quad x \in (0, L) \quad t > 0, \\ u(0, t) = u(L, t) = 0, \quad t > 0, \\ u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad x \in [0, L] \end{aligned} \tag{123}$$

where a is a constant and f and g are given functions of x. We use separation of variables to obtain a suitable definition for the formal solution of (123). We begin by looking for functions of the form $v(x, t) = X(x)T(t)$ that are not identically zero and satisfy

$$v_{tt} = a^2 v_{xx}, \;\; v(0, t) = v(L, t) = 0$$

for all (x, t). Since

$$v_{tt} = XT'' \quad \text{and} \quad v_{xx} = X''T,$$

$v_{tt} = a^2 v_{xx}$ if and only if

$$XT'' = a^2 X''T,$$

which we rewrite as

$$\frac{T''}{a^2 T} = \frac{X''}{X}.$$

For this to hold for all (x, t), the two sides must equal the same constant; thus, set

$$\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda,$$

which is equivalent to

$$X'' + \lambda X = 0$$

and

$$T'' + a^2 \lambda T = 0. \tag{124}$$

Since $v(0, t) = X(0)T(t) = 0$ and $v(L, t) = X(L)T(t) = 0$ and we do not want T to be identically zero, $X(0) = X(L) = 0$. Therefore λ must be an eigenvalue of

$$X'' + \lambda X = 0, \quad X(0) = X(L) = 0, \tag{125}$$

and X must be a λ-eigenfunction. From Theorem 10.2.1, the eigenvalues of (125) are $\lambda_n = \frac{n^2\pi^2}{L^2}$, with associated eigenfunctions

$$X_n = \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

Substituting $\lambda = \frac{n^2\pi^2}{L^2}$ into (124) yields

$$T'' + \frac{n^2\pi^2 a^2}{L} T = 0,$$

which has the general solution

$$T_n = \alpha_n \cos \frac{n\pi a}{L} t + \frac{\beta_n L}{n\pi a} \sin \frac{n\pi a}{L} t,$$

where α_n and β_n are constants. Now let

$$v_n(x,t) = X_n(x)T_n(t) = \left(\alpha_n \cos\frac{n\pi a}{L}t + \frac{\beta_n L}{n\pi a}\sin\frac{n\pi a}{L}t\right)\sin\frac{n\pi x}{L}.$$

Then

$$\frac{\partial}{\partial t}v_n(x,t) = \left(-\frac{n\pi a}{L}\alpha_n \sin\frac{n\pi a}{L}t + \beta_n \cos\frac{n\pi a}{L}t\right)\sin\frac{n\pi x}{L},$$

so

$$v_n(x,0) = \alpha_n \sin\frac{n\pi x}{L} \quad \text{and} \quad \frac{\partial}{\partial t}v_n(x,0) = \beta_n \sin\frac{n\pi x}{L}.$$

Therefore v_n satisfies (123) with $f(x) = \alpha_n \sin\frac{n\pi x}{L}$ and $g(x) = \beta_n \cos\frac{n\pi x}{L}$. More generally, if $\alpha_1, \alpha_2, \ldots, \alpha_m$ and $\beta_1, \beta_2, \ldots, \beta_m$ are constants and

$$u_m(x,t) = \sum_{n=1}^{m}\left(\alpha_n \cos\frac{n\pi a}{L}t + \frac{\beta_n L}{n\pi a}\sin\frac{n\pi a}{L}t\right)\sin\frac{n\pi x}{L},$$

then u_m satisfies (123) with

$$f(x) = \sum_{n-1}^{m}\alpha_n \sin\frac{n\pi x}{L} \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty}\beta_n \sin\frac{n\pi x}{L}.$$

Definition 11.2.1. If f and g are piecewise smooth on $[0, L]$, then the formal solution of

$$\begin{aligned} u_{tt} = a^2 u_{xx}, \quad x \in (0,L) \quad t > 0, \\ u(0,t) = u(L,t) = 0, \quad t > 0, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \in [0,L] \end{aligned} \tag{126}$$

is

$$u(x,t) = \sum_{n=1}^{\infty}\left(\alpha_n \cos\frac{n\pi a}{L}t + \frac{\beta_n L}{n\pi a}\sin\frac{n\pi a}{L}t\right)\sin\frac{n\pi x}{L}$$

where $\sum_{n=1}^{\infty}\alpha_n \sin\frac{n\pi x}{L}$ and $\sum_{n=1}^{\infty}\beta_n \sin\frac{n\pi x}{L}$ are Fourier sine series of f and g respectively on $[0, L]$; that is

$$\alpha_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \quad \text{and} \quad \beta_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx.$$

The next theorem guarantees that the series converges not only for $x \in [0, L]$ and $t \geq 0$, but for $x \in (-\infty, \infty)$ and $t \in (-\infty, \infty)$.

Theorem 11.2.2 (D'Alembert's Solution). If f and g are piecewise smooth on $[0, L]$, then u in (126) converges for all (x, t), and can be written as

$$u(x,t) = \frac{1}{2}[S_f(x+at) + S_f(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} S_g(\tau)\,d\tau,$$

where $S_f = \sum_{n=1}^{\infty}\alpha_n \sin\frac{n\pi x}{L}$ is the Fourier sine series of f.

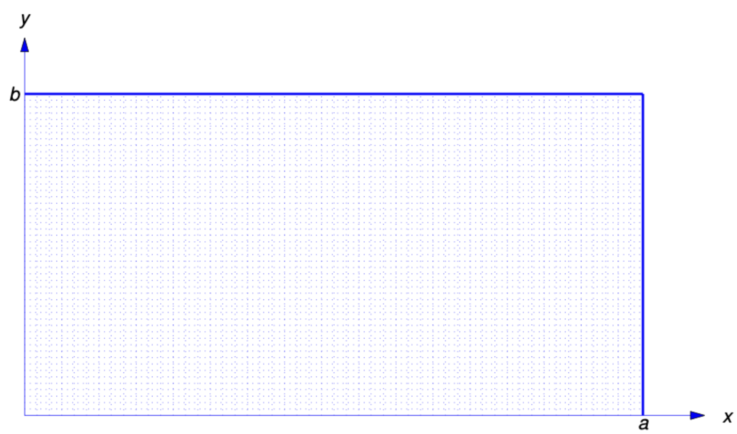
Proof for Theorem 11.1.3 is omitted (Can be done using product to sum & differences formula from high school).

11.3 Laplace Equation

Solving boundary value problems for

$$u_{xx} + u_{yy} = 0 \tag{127}$$

is beyond the scope of this course, so we consider only very simple regions. We begin by considering the rectangular region shown below.



The possible boundary conditions for this region can be written as

$$\begin{aligned}
(1-\alpha)u(x,0) + \alpha u_y(x,0) &= f_0(x), \quad x \in [0,a], \\
(1-\beta)u(x,b) + \beta u_y(x,b) &= f_1(x), \quad x \in [0,a], \\
(1-\gamma)u(0,y) + \gamma u_x(0,y) &= g_0(x), \quad y \in [0,b], \\
(1-\delta)u(a,y) + \delta u_x(a,y) &= g_1(y), \quad y \in [0,b],
\end{aligned}$$

where $\alpha, \beta, \gamma, \delta$ can each be either 0 or 1; thus, there are 16 possibilities. Let $\mathrm{BVP}(\alpha, \beta, \gamma, \delta)(f_0, f_1, g_0, g_1)$ denote the problem of finding a solution of (127) that satisfies these conditions. This is a Dirichlet problem if $\alpha = \beta = \gamma = \delta = 0$, or a Neumann problem if $\alpha = \beta = \gamma = \delta = 1$.

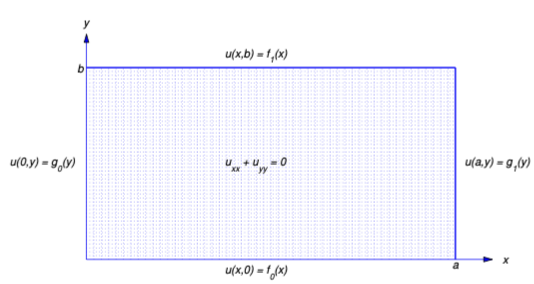


Figure 12.3.2 A Dirichlet problem

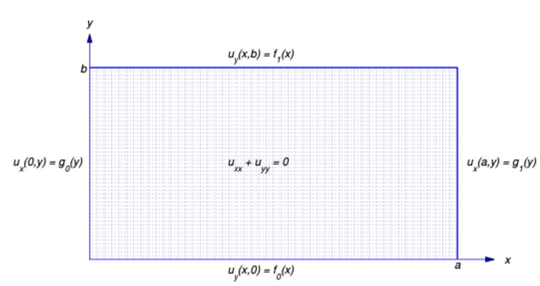


Figure 12.3.3 A Neumann problem

For given $(\alpha,\beta,\gamma,\delta)$, the sum of solutions of $\mathrm{BVP}(\alpha,\beta,\gamma,\delta)(f_0,0,0,0)$, $\mathrm{BVP}(\alpha,\beta,\gamma,\delta)(0,f_1,0,0)$, $\mathrm{BVP}(\alpha,\beta,\gamma,\delta)(0,0,g_0,0)$, and $\mathrm{BVP}(\alpha,\beta,\gamma,\delta)(0,0,0,g_1)$ is a solution of

$$\mathrm{BVP}(\alpha,\beta,\gamma,\delta)(f_0,f_1,g_0,g_1).$$

Therefore we concentrate on problems where only one of the functions f_0, f_1, g_0, g_1 is not identically zero. There are 64 problems of this form. Each has homogeneous boundary-value conditions on three sides of the rectangle, and a nonhomogeneous boundary condition on the fourth. We use separation of variables to find infinitely many functions that satisfy Laplace's equation and the three homogeneous boundary conditions in the open rectangle. We then use these solutions as building blocks to construct a formal solution of Laplace's equation that also satisfies the nonhomogeneous boundary condition. We only go through four of them.

If $v(x,y) = X(x)Y(y)$ then

$$v_{xx} + v_{yy} = X''Y + XY'' = 0$$

for all (x,y) if and only if

$$\frac{X''}{X} = -\frac{Y''}{Y} = k$$

for all (x,y), where k is a separation constant. This equation is equivalent to

$$X'' - kX = 0, \quad Y'' + kY = 0. \tag{128}$$

From here, the strategy depends upon the boundary conditions, we can use the previous sections on eigenvalue problems to solve solve each of them.

Theorem 11.3.1.

(i) The formal solution of

$$\begin{aligned} u_{xx} + u_{yy} = 0, \quad x \in (0,a), \quad y \in (0,b), \\ u(x,0) = f(x), \quad u(x,b) = 0, \quad x \in [0,a], \\ u(0,y) = u(a,y) = 0, \quad y \in [0,b]. \end{aligned}$$

is

$$u(x,y) = \sum_{n=1}^{\infty} \alpha_n \frac{\sinh \frac{n\pi(b-y)}{a}}{\sinh \frac{n\pi b}{a}} \sin \frac{n\pi x}{a}$$

where $\sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi x}{a}$ is the Fourier sine series of f on $[0,a]$; that is

$$\alpha_n = \frac{2}{a}\int_0^a f(x) \sin \frac{n\pi x}{a}\, dx, \quad n = 1,2,3,\ldots$$

(ii) The formal solution of

$$\begin{aligned} u_{xx} + u_{yy} = 0, \quad x \in (0,a), \quad y \in (0,b), \\ u(x,0) = 0, \quad u_y(x,b) = f(x), \quad x \in [0,a], \\ u_x(0,y) = u_x(a,y) = 0, \quad y \in [0,b]. \end{aligned}$$

is

$$u(x,y) = \alpha_0 y + \frac{a}{\pi}\sum_{n=1}^{\infty} \alpha_n \frac{\sinh \frac{n\pi y}{a}}{\cosh \frac{n\pi b}{a}} \cos\frac{n\pi x}{a},$$

where $\alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos\frac{n\pi x}{a}$ is the Fourier cosine series of f on $[0,a]$; that is,

$$\alpha_0 = \frac{1}{a}\int_0^a f(x)\,dx \quad \text{and} \quad \alpha_n = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}\,dx, \quad n = 1,2,3,\ldots$$

(iii) The formal solution of

$$\begin{aligned} u_{xx} + u_{yy} = 0, \quad x \in (0,a), \quad y \in (0,b), \\ u(x,0) = u_y(x,b) = 0, \quad x \in [0,a], \\ u(0,y) = g(y), \quad u_x(a,y) = 0, \quad y \in [0,b]. \end{aligned}$$

is

$$u(x,y) = \sum_{n=1}^{\infty} \alpha_n \frac{\cosh \frac{(2n-1)\pi(x-a)}{2b}}{\cosh \frac{(2n-1)\pi a}{2b}} \sin\frac{(2n-1)\pi y}{2b}$$

where $\sum_{n=1}^{\infty} \alpha_n \sin\frac{(2n-1)\pi y}{2b}$ is the mixed Fourier sine series of g on $[0,b]$; that is

$$\alpha_n = \frac{2}{b}\int_0^b g(y)\sin\frac{(2n-1)\pi y}{2b}\,dy.$$

(iv) The formal solution of

$$\begin{aligned} u_{xx} + u_{yy} = 0, \quad x \in (0,a), \quad y \in (0,b), \\ u_y(x,0) = u(x,b) = 0, \quad x \in [0,a], \\ u_x(0,y) = 0, \quad u_x(a,y) = g(y), \quad y \in [0,b]. \end{aligned}$$

is

$$u(x,y) = \frac{2b}{\pi}\sum_{n=1}^{\infty} \alpha_n \frac{\cosh \frac{(2n-1)\pi x}{2b}}{(2n-1)\sinh \frac{(2n-1)\pi a}{2b}} \cos\frac{(2n-1)\pi y}{2b},$$

where $\sum_{n=1}^{\infty} \alpha_n \cos\frac{(2n-1)\pi y}{2b}$ is the mixed Fourier cosine series of g on $[0,b]$; that is

$$\alpha_n = \frac{2}{b}\int_0^b g(y)\cos\frac{(2n-1)\pi y}{2b}\,dy.$$

12 Solving PDE II - Nonhomogeneous PDE and Transport Equations

12.1 Modification of the Unknown Function

Consider the nonhomogeneous Heat Equation:

$$\begin{aligned} u_t = u_{xx} + h(x), \quad x \in (0, L), \quad t > 0, \\ u(0,t) = a, \quad u(L,t) = b, \quad t > 0, \\ u(x,0) = f(x), \quad x \in [0, L], \end{aligned} \tag{129}$$

where a and b are constants. The method of modification for solving nonhomogeneous PDE involves using an alternative function to reduce the problem to a homogeneous PDE. Firstly, let

$$v(x,t) = u(x,t) + g(x) \tag{130}$$

for some unknown function g. To obtain suitable g, we set the following conditions

$$v_t - v_{xx} = u_t - u_{xx} - g''(x) = h(x) - g''(x) = 0, \tag{131}$$

$$v(0,t) = u(0,t) + g(0) = a + g(0) = 0, \tag{132}$$

$$v(L,t) = u(L,t) + g(L) = b + g(L) = 0, \tag{133}$$

and

$$v(x,0) = u(x,0) + g(x) = f(x) + g(x). \tag{134}$$

From (131), we get that

$$g(x) = \int \left(\int h(x)\, dx \right) dx.$$

(132) and (133) fix the constants of integrations. Thus, we can obtain such $g(x)$. Now notice that, (131), (132), (133), and (134) generated a nonhomogeneous PDE

$$\begin{aligned} v_t - v_{xx} = 0, \quad x \in (0, L), \quad t > 0, \\ v(0,t) = v(L,t) = 0, \quad t > 0, \\ v(x,0) = f(x) + g(x), \quad x \in [0, L], \end{aligned}$$

which can be solved using methods from the previous week. And we simply substituting the solution $v(x,t)$ and $g(x)$ into (130), we can obtain the solution for (129), $u(x,t)$.

12.2 Change of Variables for solving Transport Equations

The linear/uniform transport equations often appear in the form

$$\begin{aligned} u_t + C u_x = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}, \\ u(x,0) = f(x), \end{aligned} \tag{135}$$

where C is a constant. The setup of the method of change of variable is the following:

$$\zeta = x - Ct, \quad \tau = t.$$

Then we can write $u(x,t)$ as

$$u(x,t) = u(\zeta + C\tau, \tau) = v(\zeta, \tau).$$

By the chain rule, we have

$$\begin{aligned}\frac{\partial v}{\partial \tau} &= \frac{\partial}{\partial \tau} u(\zeta + C\tau, \tau) \\ &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \tau} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \tau} \\ &= u_x \cdot C + u_t = 0.\end{aligned}$$

This means that v is independent of $t = \tau$, and hence

$$v(\zeta, \tau) = v(\zeta). \tag{136}$$

Thus, for the boundary value condition $u(x,0) = f(x)$, we have $u(x,0) = v(x - C\cdot 0) = v(x) = f(x)$. So by (136), the solution for (135) is

$$u(x,t) = v(x - Ct) = f(x - Ct).$$

For the non-uniform transport PDE:

$$\begin{aligned} u_t + C(x)u_x = 0, \quad &(x,t) \in \mathbb{R} \times \mathbb{R}, \\ &u(x,0) = f(x), \end{aligned} \tag{137}$$

where $C(x)$ is a function of x. We change the setup for the change of variable method to

$$\zeta = \beta(x) - t, \quad \tau = t. \tag{138}$$

We rewrite $u(x,t)$ as

$$u(x,t) = v(\zeta, \tau).$$

From (138) we know that $x = \beta^{-1}(\zeta + \tau)$, so

$$v(\zeta, \tau) = u(\beta^{-1}(\zeta + \tau), \tau).$$

We want $v_t = 0$ to make v independent of t. That is,

$$\begin{aligned} 0 = \frac{\partial v}{\partial \tau} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \tau} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \tau} \\ &= u_t + u_x(\beta^{-1}(\zeta + \tau))' \cdot 1 \\ &= u_1 + \frac{1}{\beta'(x)} u_x \end{aligned}$$

($x = \beta^{-1}(\beta)(x)$, so $1 = (\beta^{-1})'(\beta(x)) \cdot \beta'(x) = (\beta^{-1})(\zeta + \tau)\beta'(x)$.) This indicates that we need $1/\beta'(x) = C(x)$, i.e.

$$\beta(x) = \int \frac{1}{C(x)}\, dx.$$

Thus, we can write

$$\begin{aligned} u(x,t) &= v(\zeta,\tau) = v(\zeta) \\ &= v(\beta(x) - t). \end{aligned} \tag{139}$$

For the boundary condition $u(x, 0) = f(x)$, we have $v(\beta(x)) = f(x)$, implies that

$$v(y) = f(\beta^{-1}(y))$$

by setting $y = \beta(x)$. Therefore, putting everything together, the solution for (137) is

$$u(x,t) = v(\beta(x) - t) = f(\beta^{-1}(\beta(x) - t)).$$