MATH2070/2970 Optimisation Notes

Contents

1 Introduction to Optimisation

In general, a mathematical optimisation problem looks like

$$\begin{aligned} &\min_{\mathbf{x}\in\mathbb{R}} f(\mathbf{x}), \\ &\text{s.t. } \mathbf{x} \in \Omega, \end{aligned}$$

where $\mathbf{x} \in \mathbb{R}^n$ are the decision variables, the objective function is $f : \mathbb{R}^n \to \mathbb{R}$ and the constraints are represented by the set $\Omega \subseteq \mathbb{R}^n$.

This represents the constraints geometrically (the *feasible region* of space).

Usually we will write constraints as equations/inequalities, which implies a feasible region (all x satisfying the (in)equations).

Simple Example. Find the dimensions of a cylinder with surface area S which maximises the volume V.

Solution. The decision variables are the cylinder dimensions are radius $r \in \mathbb{R}$ and height $h \in \mathbb{R}$. The objective is to maximise $V = \pi r^2 h$. The constraints are $2\pi r^2 + 2\pi rh = S$ for some *parameter* $S \geq 0$ (given input value, not being optimised), plus the *hidden constraints* $r \geq 0$ and $h \geq 0$ (*not given explicitly, implied by context*). We write the problem mathematically as

$$\begin{aligned} &\max_{r,h\in\mathbb{R}} V(r,h) = \pi r^2 h, \\ &\text{s.t. } 2\pi r^2 + 2\pi rh = S, \\ &\qquad r, h \geq 0. \end{aligned}$$

We write the decision variables under the max (or min), it helps to distinguish them from parameters. The region is given by

$$\Omega = \{(r,h) \in \mathbb{R}^2 : 2\pi r^2 + 2\pi rh = S, r \geq 0, h \geq 0\}.$$

We usually write min rather than max (change max to min by taking negative of objective function).

2 Linear Programming

First category of optimisation problem is *linear programming* (LP). There are many different qays to write an LP. We will look at LPs written in *standard form*

$$\begin{aligned}
&\max_{\mathbf{x}\in\mathbb{R}^n} Z = \mathbf{c}^T\mathbf{x} = \sum_{i=1}^{n} c_i x_i \\
&\text{s.t. } a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1 \\
&\qquad \vdots \\
&\qquad a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m, \\
&\qquad x_1, x_2, \ldots, x_n \geq 0.
\end{aligned}$$

In standard form, we assum that all RHS values $b_i \geq 0$. Note max not min, direction of inequalities. We sometimes write the constraints as $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ (vector inequalities meant component-wise). $\mathbf{c}$ is the *cost vector* (c_i are cost coefficients), A is the *constraint matrix*, b is the *resource vector*. The feasible region $\Omega \subset \mathbb{R}^n$ is

$$\Omega = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b} \quad \text{and} \quad \mathbf{x} \geq \mathbf{0}\}.$$

We will call any $\mathbf{x} \in \Omega$ a *feasible point*. If $\Omega \neq \emptyset$, we say the LP is *feasible* or the *constraints are consistent*.

An *optimum* (i.e. a solution) is any feasible point $\mathbf{x}^* \in \Omega$ such that $\mathbf{c}^T\mathbf{x}^* \geq \mathbf{c}^T\mathbf{x}$ for all $\mathbf{x} \in \Omega$. The corresponding $Z^* = \mathbf{c}^T\mathbf{x}^*$ is the optimal value. Since Z is linear, *any optimum must be on the boundary of the feasible set* Ω provided $\mathbf{c} \neq \mathbf{0}$. If $\mathbf{c} = \mathbf{0}$ then we have not really got an optimisation problem. If we are at a point $\mathbf{x} \in \Omega$ not on the boundary, we can take a small step $\mathbf{x} + \epsilon\mathbf{s}$ in any direction $\mathbf{s} \in \mathbb{R}^n$ and stay in Ω, provided $\epsilon > 0$ is sufficiently small. Trying $\mathbf{s} = \mathbf{c}$ gives

$$Z_{\text{new}} = \mathbf{c}^T(\mathbf{x} + \epsilon\mathbf{c}) = \mathbf{c}^T\mathbf{x} + \epsilon\mathbf{c}^T\mathbf{c} = (\mathbf{c}^T\mathbf{x}) + \epsilon\sum_{i=1}^{n} c_i^2 > \mathbf{c}^T\mathbf{x} = Z_{\text{old}}$$

where the strict inequality requires $\mathbf{c} \neq \mathbf{0}$. So, the point $\mathbf{x} + \epsilon\mathbf{c}$ is in Ω and has a larger objective value than $\mathbf{x}$, i.e., $\mathbf{x}$ cannot be a solution.

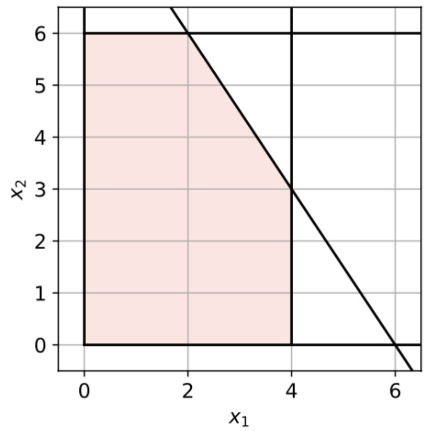
As a result, we will always look for solutions by moving around the boundary of the feasible region.

2.1 Graphical Solution

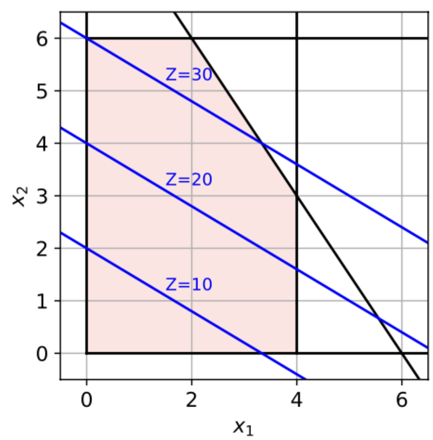
In the simple case of two decision variables ($n = 2$), we can plot our feasible region graphically.

Example 2.1.1. Consider the problem

$$\begin{aligned}
&\max_{\mathbf{x}\in\mathbb{R}^2} Z = 3x_1 + 5x_2, \\
&\text{s.t. } x_1 \leq 4, \\
&\qquad 3x_1 + 2x_2 \leq 18, \\
&\qquad 2x_2 \leq 12, \\
&\qquad x_1, x_2 \geq 0.
\end{aligned}$$



As shown, the solution(s) is on the boundary of this region. To find the maximum of the objective, here $Z = 3x_1 + 5x_2$, we can look at the *contour lines* of Z (lines along which Z is constant).



To be more careful, we need to check the values of Z on the boundary of the feasible region. Actually, we just need to check the corners (there is always at least one optimum at a corner). In the above example,

$$
\begin{aligned}
Z(0,0) &= 0 \\
Z(0,6) &= 30 \\
Z(2,6) &= 36 \\
Z(4,3) &= 27 \\
Z(4,0) &= 12
\end{aligned}
$$

Maximum is $Z^* = 36$ at $\mathbf{x}^* = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$.

This graphical method gives us a few insights.

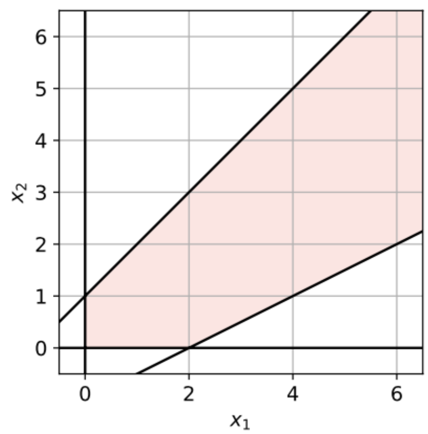
Theorem 2.1.1. The feasible region of a linear program is *convex*. In other words, let Ω be a feasible region of any linear program, for all $\mathbf{x}, \mathbf{y} \in \Omega$,

$$t\mathbf{x} + (1-t)\mathbf{y} \in \Omega$$

for all $t \in [0, 1]$.

The feasible region can be empty. For example, $x_1 \leq 5$ and $x_7 \geq 7$ (not in standard form, but can be converted). In this case, we say the LP is infeasible.

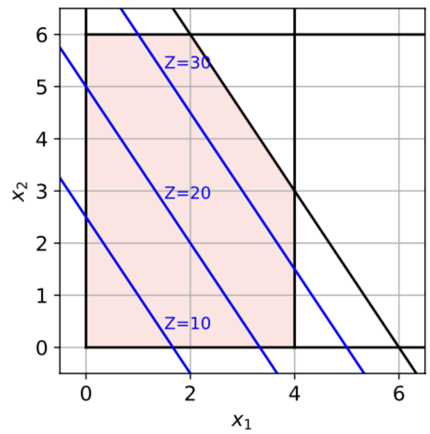
The feasible region could be unbounded.



Note: this region is still convex. For unbounded problems, there may still be a (finite) optimum at a corner point. But we may be able to find $Z \to +\infty$ inside the feasible region. In this case we say the LP is unbounded.

There may be more than one solution to the LP. For example, change the objective in Example 2.1.1 to $Z = 6x_1 + 4x_2$, so contour lines are

$$x_2 = -\frac{3}{2}x_1 + \frac{Z}{4}.$$



For example, $Z = 10$ gives contour line

$$x_2 = -\frac{3}{2}x_1 + \frac{5}{2}.$$

Contour lines are parallel to a boundary segment. In this case, the solutions lie on the boundary $3x_1 + 2x_2 = 18$ (constraint holds with equality), between $(2, 6)$ and $(4, 3)$.

Algebraically, pick a parameter t for the boundary line, for example $(x_1, x_2) = (t, 9 - \frac{3}{2}t)$. Find the values of t that satisfy all other constraints:

$$\begin{aligned} x_1 \le 4 &\implies t \le 4 \\ 2x_2 \le 12 &\implies t \ge 2 \\ x_1 \ge 0 &\implies t \ge 0 \\ x_2 \ge 0 &\implies t \le 6 \end{aligned}$$

The solution line is all values of t satisfying all constraints, i.e. $(x_1^*, x_2^*) = (t, 9 - \frac{3}{2}t)$ for all $2 \le t \le 4$. All these points give $Z^* = 36$.

2.2 Corner Points

To find a general procedure for solving LPs, we know

- any optimum is always on the boundary of the feasible region;
- the feasible region is a *polytope* (high-dimensional polygon/polyhedron)
- There is always at least one optimum at a corner (if the problem has finite solution).
  - If there are multiple optimum corner points, the line segment/face/etc. between them represents infinitely many solutions.

So, our algorithm for solving LPs should look at feasible corner points. If all corner points adjacent to the current one have worse (i.e. smaller) Z value, then you are at an optimum. (In general, this means we do not have to check all corners: only look at adjacent corners which increase Z.) The resulting algorithm is the *simplex algorithm*.

The simplex method:

1. Pick a starting feasible corner point.
2. If there is an adjacent feasible corner point which increases Z, go there.
3. If there is no adjacent feasible corner point which increases Z, stop (solution found).

The boundary of the feasible region corresponds to areas where the inequality constraints are *active* (hold with equality).

This is easy to check for constraints $x_i \geq 0$: just see if $x_i = 0$ or $x_i > 0$.

For the other constraints, $A\mathbf{x} \leq \mathbf{b}$, it will be useful to have some new variables which represent the *distance to equality*. Rewrite the first constraint as

$$a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1 \longrightarrow a_{11}x_1 + \cdots + a_{1n}x_n + x_{n+1} = b_1, \quad \text{and} \quad x_{n+1} \geq 0.$$

This is, we define $x_{n+1} := b_1 - (a_{11}x_1 + \cdots + a_{1n}x_n) \geq 0$. Do this for each of the m constraints $A\mathbf{x} \leq \mathbf{b}$ to get m new *slack variables* $x_{n+1}, \ldots, x_{n+m} \geq 0$. We go from $A\mathbf{x} \leq \mathbf{b}$ with $\mathbf{x} \geq \mathbf{0}$ to (after adding slacks) $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ (for a larger vector $\mathbf{x}$ and wider matrix A).

Now, every boundary edge of the feasible region is given by $x_i = 0$ for some i (possibly a slack variable).

Recall, a *hyperplane* in $\mathbb{R}^n$ is the set of all solutions to an equation $a_1x_1 + \cdots + a_nx_n = b$ (at least one a_i nonzero). Hyperplanes divide $\mathbb{R}^n$ into two half-spaces:

- A hyperplane in $\mathbb{R}$ is a point.
- A hyperplane in $\mathbb{R}^2$ is a line.
- A hyperplane in $\mathbb{R}^3$ is a plane, etc.

If we look at intersections of hyperplanes, in the typical case (e.g. no repeated equations, inconsistent equations,...)

- In $\mathbb{R}^2$, two lines intersect at a point.
- In $\mathbb{R}^3$, two planes intersect at a line, but three hyperplanes intersect at a point.

Here, we have $n+m$ hyperplanes $x_i = 0$ in $\mathbb{R}^n$ (including slack variables). This corresponds to the feasible set being the intersection of $n+m$ half-spaces (a *convex polytope*).

A corner of the feasible set is the intersection of n hyperplanes. Intersection of n out of $n+m$ hyperplanes: $\binom{n+m}{n} = \frac{(n+m)!}{n!m!}$ total corners.

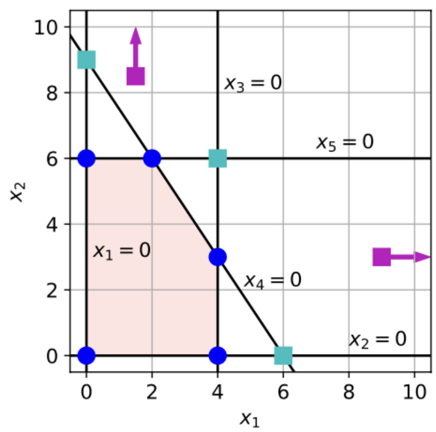
But not all corner points are feasible. For Example 2.1.1, $n = 2$ and $m = 3$ gives $\binom{5}{2} = 10$ total corners but feasible region only has 5 corners.

For this example, we have original decision variables $x_1, x_2 \geq 0$ and new slack variables

$$\begin{aligned} x_1 \leq 4 &\implies x_3 := 4 - x_1 \geq 0 \\ 3x_1 + 2x_2 \leq 18 &\implies x_4 := 18 - 3x_1 - 2x_2 \geq 0 \\ 2x_2 \leq 12 &\implies x_5 := 12 - 2x_2 \geq 0 \end{aligned}$$

The boundaries of the feasible region are where one (or more) constraints hold with equality; i.e. $x_i = 0$ for some $i = 1, 2, 3, 4, 5$.

For example, $x_3 = 0$ is the boundary line $x_1 = 4$.



Of the 10 possible pairs of active constraints $x_i = 0$:

- 5 corner points are feasible: e.g. $x_1 = x_5 = 0$ gives $(0, 6)$. Check $(x_2, x_3, x_4) = (6, 4, 6) \geq 0$.
- 3 corner points are infeasible: e.g. $x_3 = x_5 = 0$ gives $(4, 6)$. Here, $x_4 = 18 - 3x_1 - 2x_2 = -6 < 0$.
- 2 corner points do not exist: e.g. $x_1 = x_3 = 0$ requires $x_1 = 0$ and $x_1 = 4$.

After adding slacks, the resource constraints $A\mathbf{x} \leq \mathbf{b}$ are now equalities $A\mathbf{x} = \mathbf{b}$, essentially by definition of the slack variables.

From Gauss-Jordan elimination, any point satisfying $A\mathbf{x} = \mathbf{b}$ (m constraints, $n + m$ unknowns) means we can write m of the variables (basic variables) as an affine combination of the remaining n (non-basic variables).

Any corner point has n values $x_i = 0$: pick these to be the non-basic variables and this corresponds to a basic solution to $A\mathbf{x} = \mathbf{b}$. That is:

- Basic solutions to $A\mathbf{x} = \mathbf{b} \iff$ corner points.

- Feasible corner point $\iff$ basic solution with $\mathbf{x} \geq \mathbf{0}$.

E.g. $(4, 3)$ has $x_3 = x_4 = 0$ (non-basic variables) with $(x_1, x_2, x_5) = (4, 3, 6)$ (basic variables).

Definition 2.2.1 (Adjacent Feasible Corner Point). Two feasible corner points are *adjacent* if they have all the same basic/non-basic variables except one.

For Example 2.1.1, $(4, 3)$ has $x_3 = x_4 = 0$ (non-basic variables) and $(x_1, x_2, x_5) = (4, 3, 6)$ basic variables.

- It is adjacent to $(4, 6)$ with $(x_1, x_2, x_4) = (4, 6, -6)$ (basic variables), which is not feasible.
- It is not adjacent to $(0, 0)$ with $(x_3, x_4, x_5) = (4, 18, 12)$ (basic variables).

2.3 Simplex Method

Assume the problem is in standard form

$$\max_{\mathbf{x} \in \mathbb{R}^n} Z = \mathbf{c}^T \mathbf{x} \quad \text{s.t.} \quad A\mathbf{x} \leq \mathbf{b}, \ \ \mathbf{x} \geq \mathbf{0}$$

with $\mathbf{b} \geq \mathbf{0}$.

1. Pick any feasible corner point as a starting point (our standard form allows $\mathbf{x} = \mathbf{0}$).
2. Move to an adjacent feasible corner point: pick one variable to enter the basis and one variable to leave (choice must ensure Z increases and our new point is a feasible corner point).
3. Continue until no adjacent feasible corner point increases Z (solution found).

Changing the basis will require updating Gauss-Jordan elimination. The values in the matrix will help to decide which variables should enter/leave the basis.

We now look at these steps in detail.

2.3.1 Initialisation

We have decision variables $x_1, \ldots, x_n$. Write objective $Z = \mathbf{c}^T \mathbf{x}$ as

$$Z - c_1 x_1 - \cdots - c_n x_n = 0.$$

Our constraints with slack variables $x_{n+1}, \ldots, x_{n+m}$ are

$$\begin{aligned} a_{11} x_1 + \cdots + a_{1n} x_n + x_{n+1} &= b_1 \\ &\vdots \\ a_{m1} x_1 + \cdots + a_{mn} x_n + x_{n+m} &= b_m \end{aligned}$$

To find a feasible corner point, set all decision variables $x_1 = \cdots = x_n = 0$ (and so slacks are $x_{n+1} = b_1, \ldots, x_{n+m} = b_m$). This is feasible (all $x_i \geq 0$) since we assume $\mathbf{b} \leq \mathbf{0}$. So we start with the tableau:

Z	x_1	x_2	$\cdots$	x_n	x_{n+1}	x_{n+2}	$\cdots$	x_{n+m}	$\mathbf{b}$
1	$-c_1$	$-c_2$	$\cdots$	$-c_n$	0	0	$\cdots$	0	0
0	a_{11}	a_{12}	$\cdots$	a_{1n}	1	0	$\cdots$	0	b_1
0	a_{21}	a_{22}	$\cdots$	a_{2n}	0	1	$\cdots$	0	b_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
0	a_{m1}	a_{m2}	$\cdots$	a_{mn}	0	0	$\cdots$	1	b_m

At the start, we have basic variables $x_{n+1}, \ldots, x_{n+m}$ (slacks) and non-basic variables $x_1 = \cdots = x_n = 0$ (decision variables).

Note: the columns corresponding to the basic variables are all zeros (except a single 1). We have already done elimination on these columns.

2.3.2 Iteration

We now need a way to pick which variables enter/leave the basis. There are many ways to choose this. We will look at one simple choice.

Largest coefficient rule: the non-basic variable with largest coefficient c_i enters the basis. That is, the best chance of increasing Z if x_i increases ($\frac{\partial Z}{\partial x_i}$ is largest).

Note: Tableau shows negative c_i values. Pick the column with largest *negative* entry in Z row of the tableau.

We keep the increasing new basic variable until we reach the next corner. That is, *remove the basic variable that becomes zero first* (so $\mathbf{x} \geq \mathbf{0}$ always satisfied).

This idea can be formally stated as follows.

Smallest ratio rule: If x_i is entering the basis, choose basic variable x_j such that $a_{ji} > 0$ and

$$\frac{b_j}{a_{ji}}$$

is smallest.

That is, look down column i (excluding Z row), for row j check $a_{ji} > 0$. For these rows only, calculate b_j/a_{ji} and pick row with the smallest value.

Once we have picked the row, the basic variable leaving is the one whose column has a 1 in that row.

2.3.3 Stopping

At each iteration, the Z row changes (writing Z in terms of current non-basic variables). Call resulting $\bar{c}_i$ the modified cost coefficients. *We stop when all* $\bar{c} \leq 0$ (i.e. all Z row entries are ≥ 0).

If we cannot change any basic variable $x_i = 0$ to $x_i > 0$ without decreasing Z, we are at a solution.

Example 2.3.1. We continue with the problem

$$\begin{aligned} \max_{\mathbf{x}\in\mathbb{R}^2} \; & Z = 3x_1 + 5x_2, \\ \text{s.t.} \; & x_1 \leq 4, \\ & 3x_1 + 2x_2 \leq 18, \\ & 2x_2 \leq 12, \\ & x_1, x_2 \geq 0. \end{aligned}$$

1. Initialisation

The slacks are $x_3 = 4 - x_1$, $x_4 = 18 - 3x_1 - 2x_2$, and $x_5 = 12 - 2x_2$. So our initial tableau is

Z	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
1	-3	-5	0	0	0	0
0	1	0	1	0	0	4
0	3	2	0	1	0	18
0	0	2	0	0	1	12

The basic variables are $(x_3, x_4, x_5) = (4, 18, 12)$ and non-basic variables are $x_1 = x_2 = 0$.

2. Iteration

x_2 has the largest coefficient, hence it enters the basis.

For rows with $a_{ji} > 0$, pick the one with smallest ratio b_j/a_{ji}.

Z	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$	Ratio
1	-3	-5	0	0	0	0	
0	1	0	1	0	0	4	$-$
0	3	2	0	1	0	18	$18/2 = 9$
0	0	2	0	0	1	12	$12/2 = 6$

Corresponding basis variable (column of identity) is x_5, so x_5 leaves the basis.

After performing tow operations

$$\begin{aligned} R_4 &\leftarrow \frac{1}{2} R_4 \\ R_1 &\leftarrow R_1 + 5R_4 \\ R_3 &\leftarrow R_3 - 2R_4 \end{aligned}$$

the tableau becomes

Z	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$	Ratio
1	-3	0	0	0	$5/2$	30	
0	1	0	1	0	0	4	$4/1 = 4$
0	3	0	0	1	-1	6	$6/3 - 2$
0	0	1	0	0	$1/2$	6	$-$

Largest negative entry in Z row is -3: x_1 enters the basis, we then calculate the ratios, which indicates that the third row has smallest ratio, so x_4 leaves the basis.

After row operations, the tableau becomes

Z	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
1	0	0	0	1	$3/2$	36
0	0	0	1	$-1/3$	$1/3$	2
0	1	0	0	$1/3$	$-1/3$	2
0	0	1	0	0	$1/2$	6

In our new basis, $Z = 36 - x_4 - \frac{3}{2}x_5$, so increasing x_4 or x_5 (from 0) decreases Z. We stop as all $\bar{c}_i \leq 0$.

3. Stopping

Basic variables are x_1, x_2 and x_3, non-basic variables are x_4 and x_5. So the optimal objective is $Z^* = 36$ when $x_4^* = x_5^* = 0$.

Constraint rows tell us the basic variables

$$\begin{aligned} x_3^* - \frac{1}{3}x_4^* + \frac{1}{3}x_5^* &= 2, \\ x_1^* + \frac{1}{3}x_4^* - \frac{1}{3}x_5^* &= 2, \\ x_2^* + \frac{1}{2}x_5^* &= 6. \end{aligned}$$

So $x_1^* = 2$, $x_2^* = 6$, $x_3^* = 2$ (look at entries in $\mathbf{b}$ column).

Overall, our solution is $(x_1^*, x_2^*) = (2, 6)$ with objective $Z^* = 36$.

Simplex Method for LPs in Standard Form:

1. Add slack variables $x_{n+1}, \ldots, x_{n+m}$ to the m inequalities $A\mathbf{x} \leq \mathbf{b}$.
2. Initialise $x_1 = \cdots = x_n = 0$ (decision variables = non-basic variables) and $(x_{n+1}, \ldots, x_{n+m}) = (b_1, \ldots, b_m)$ (slack variables = basic variables).

3. At each iteration:
   - Non-basic variable x_i with most negative Z-row entry enters the basis.
   - Basic variable x_i with $a_{ji} > 0$ and minimal ratio b_j/a_{ji} leaves the basis.
   - Eliminate x_i column using x_j row as a pivot.
4. Continue until all Z-row coefficients are ≥ 0.

2.4 Issues in Simplex Algorithm

There are some special situations that might cause problems:

1. Ties (same modified cost coefficients or ratios)
2. No variable leaves the basis
3. Multiple optimal solutions

2.4.1 Ties

Smallest index rule: for any tie (enter or leave), always pick the variable with the smallest index.

2.4.2 No variable leaves

Consider the problem

$$\begin{aligned} \max_{\mathbf{x} \in \mathbb{R}^2} \ & Z = 3x_1 + 5x_2 \\ \text{s.t.} \ & -x_1 + x_2 \leq 4, \\ & x_1 - x_2 \leq 2, \\ & x_1, x_2 \geq 0. \end{aligned}$$

After adding slack variables,

$$\begin{aligned} Z - 3x_1 - 5x_2 &= 0 \\ -x_1 + x_2 + x_3 &= 4 \\ x_1 - x_2 + x_4 &= 2 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Following the usual method, we arrive at

Z	x_1	x_2	x_3	x_4	$\mathbf{b}$	Ratio
1	-8	0	5	0	20	
0	-1	1	1	0	4	–
0	0	0	1	1	6	–

x_1 enters the basis, but no variable can leave.

If no variable can leave the basis, the problem is unbounded ($Z^* \to \infty$).

2.4.3 Multiple Solutions

If a non-basic variable has zero modified cost coefficient (at the solution), there are multiple optimal solutions. Parametrise them by setting $x_i^* = t_i$ for each of these variables. For example,

Z	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
1	0	0	0	2	0	36
0	1	0	1	0	0	4
0	0	1	$-3/2$	$1/2$	0	3
0	0	0	3	-1	1	6

The non-basic variables are x_3 and x_4 with $Z^* = 36 - 2x_4^*$. We get optimal $Z^* = 36$ when $x_4^* = 0$, but any choice of x_3^* gives this value (provided ≥ 0 constraints satisfied). Let all zero cost non-basic variables be equal to a parameter: here $x_3^* = t$ for some $t \in \mathbb{R}$. Then constraints give

$$x_1^* + x_3^* = 4, \quad x_2^* - \frac{3}{2}x_3^* + \frac{1}{2}x_4^* = 3, \quad 3x_3^* - x_4^* + x_5^* = 6.$$

With $x_3^* = t$ and $x_4^* = 0$, we get

$$x_1^* = 4 - t, \quad x_2^* = 3 + \frac{3}{2}t, \quad x_5^* = 6 - 3t.$$

Valid range of t is wahtever enssures all $x_i^* \geq 0$. (e.g. $x_1^* \geq 0$ requires $t \geq 4$)

Here, valid range is $0 \leq t \leq 2$, so the set of optimal soluions is

$$(x_1^*, x_2^*) = \left(4 - t, 3 + \frac{3}{2}t\right), \quad t \in [0, 2].$$

2.5 Non-Standard LPs

Recall that simplex method will solve any problem in standard form

$$\max_{\mathbf{x}\in\mathbb{R}^n} Z = \mathbf{c}^T\mathbf{x} \quad \text{s.t.} \quad A\mathbf{x} \leq \mathbf{b}, \;\; \mathbf{x} \geq \mathbf{0} \quad (\mathbf{b} \geq \mathbf{0}).$$

What happens if our problem is not in this form? We try to change it to standard form.

Types of non-standard elements (while still being an LP):

- min rather than max.
- Resource elements $b_i < 0$.
- Resource constraints $A\mathbf{x} \leq \mathbf{b}$ constraints are $=$ or $\geq$ instead.
- Different sign constraints: $x_i \leq 0$ or $x_i \in \mathbb{R}$ (unconstrained).

All these can be handled, but some require an extended version of the simplex algorithm (two-phase).

Some of these are easy to deal with.

Min vs Max. $\min_{\mathbf{x}} Z = \mathbf{c}^T\mathbf{x}$ is equivalent to $\max_{\mathbf{x}} \tilde{Z} = (-\mathbf{c}^T)\mathbf{x}$, so just change the sign of $\mathbf{c}$. Then $Z^* = -\tilde{Z}^*$ with same optimal point $\mathbf{x}^*$.

Negative decision variables. If $x_i \leq 0$, define a new variable $\tilde{x}_i = -x_i$, so $\tilde{x}_i \geq 0$. Replace $c_i x_i$ with $(-c_i)\tilde{x}_i$ in objective, same in $A\mathbf{x} \leq \mathbf{b}$ constraints.

No sign constraints. If $x_i \in \mathbb{R}$ allowed, define two new variables $\hat{x}_i \geq 0$ and $\tilde{x} \geq 0$ such that $x_i = \hat{x}_i - \tilde{x}_t$. Replace $c_i x_i$ with $c_i\hat{x}_i + (-c_i)\tilde{x}_i$ in objective (similar for constraints).

Idea: $\hat{x}_i = \max(x_i, 0)$ is positive part of x_i, $\tilde{x}_i = \max(-x_i, 0)$ is negative part.

Negative resource elements. Suppose $b_i < 0$ for some constraint i. Then

$$\sum_{k=1}^{n} a_{ik}x_k \leq b_i \iff -\sum_{k=1}^{n} a_{ik}x_k \geq -b_i.$$

So $b_i < 0$ with $\leq$ constraint is equivalent to $b_i > 0$ with $\geq$ constraint.

Greater-than-or-equal-to constraint. For every $\geq$ and $=$ constraint, add a *surplus variable* (for $\geq$) and an extra *artificial variable*.

For example,

$$\begin{aligned}
\max_{\mathbf{x}\in\mathbb{R}^n} \; & Z = 3x_1 + 5x_2 \\
\text{s.t.} \; & x_1 \le 4 \\
& 3x_1 + 2x_2 \ge 18 \\
& 2x_2 \le 12 \\
& x_1, x_2 \ge 0.
\end{aligned}$$

After adding slack variables x_3, x_5, surplus variable x_4, and artificial variable $\overline{x}_6$, the problem becomes

$$\begin{aligned}
Z - 3x_1 - 5x_2 &= 0 \\
x_1 + x_3 &= 4 \\
3x_1 + 2x_2 - x_4 + \overline{x}_6 &= 18 \\
2x_2 + x_5 &= 12 \\
x_1, x_2, x_3, x_4, x_5, \overline{x}_6 &\ge 0.
\end{aligned}$$

Here, take decision and surplus variables as non-basic ($= 0$), slack and artificial variables as basic (≥ 0) in the initial tableau.

Unlike slack and surplus variables, adding artificial variables changes the LP we are solving.

With slack and surplus variables inside $\mathbf{x}$, we have

$$\max_{\mathbf{x}} Z = \mathbf{c}^T\mathbf{x} \quad \text{s.t.} \quad A\mathbf{x} = \mathbf{b}, \;\; \mathbf{x} \ge \mathbf{0}.$$

We cannot easily find a feasible corner point of this problem.

Adding artificial variables (call them $\overline{\mathbf{x}}$), we have

$$\max_{\mathbf{x},\overline{\mathbf{x}}} Z = \mathbf{c}^T\mathbf{x} \quad \text{s.t.} \quad A\mathbf{x} + \overline{\mathbf{x}} = \mathbf{b}, \quad \mathbf{x}, \overline{\mathbf{x}} \ge \mathbf{0}.$$

They are only the same problem if $\overline{\mathbf{x}} = \mathbf{0}$.

If we have a feasible corner point $(\mathbf{x}, \overline{\mathbf{x}})$ of the first problem with $\overline{\mathbf{x}} = \mathbf{0}$, then $\mathbf{x}$ is also a feasible corner point of the first problem. But, we can find a feasible corner point of the second problem very easy.

Lemma 2.5.1. Consider two LPs

$$\max_{\mathbf{x}\in\mathbb{R}^n} Z = \mathbf{c}^T\mathbf{x} \quad \text{s.t.} \quad A\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \ge \mathbf{0}. \tag{*}$$

and

$$\max_{\mathbf{x}\in\mathbb{R}^n, \overline{\mathbf{x}}\in\mathbb{R}^m} W = -\sum_{i=1}^{m} \overline{\mathbf{x}}_i \quad \text{s.t.} \quad A\mathbf{x} + \overline{\mathbf{x}} = \mathbf{b}, \quad \mathbf{x}, \overline{\mathbf{x}} \ge \mathbf{0} \tag{**}$$

Then (*) is feasible (i.e. constraint set nonempty) if and only if $W^* = 0$ for (**).

Proof. Since $\overline{x}_i \ge 0$, we know $W^* \le 0$ and $W^* = 0$ if and only if $\overline{\mathbf{x}}^* = \mathbf{0}$ (with some $\mathbf{x}^*$). $\overline{\mathbf{x}}^* = \mathbf{0}$ is feasible if and only if $A\mathbf{x}^* = \mathbf{b}$ and $\mathbf{0}^* \ge \mathbf{0}$ (i.e. $\mathbf{x}^*$ for (**) is feasible for (*)).

2.5.1 Two-Phase Simplex Method

Lemma 2.5.1 motivates the following idea (two-phase simplex method):

- (Phase 1) Solve problem (**) using simplex method to get $(\mathbf{x}^*, \overline{\mathbf{x}}^*)$ and W^* (initial point easy to find).
- If $W^* \neq 0$ then original problem (*) is infeasible—stop.
- If $W^* = 0$ (i.e. $\overline{\mathbf{x}}^* = \mathbf{0}$), then $\mathbf{x}^*$ is a feasible corner point for (*).
- (Phase 2) Solve (1) using simplex method with $\mathbf{x}^*$ as the initial point.

This approach works for all LPs, including those not in standard form (after adding slack, surplus and artificial variables).

Example 2.5.2. Consider the problem

$$\begin{aligned} \max_{\mathbf{x} \in \mathbb{R}^2} \ & Z = 3x_1 + 5x_2 \\ \text{s.t.} \ & x_1 \leq 4 \\ & 3x_1 + 2x_2 \geq 18 \\ & 2x_2 \leq 12 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Start by adding slack variables to $\leq$ constraints, surplus & artificial variables to $\geq$ constraint.

$$\begin{aligned} \max_{\mathbf{x} \in \mathbb{R}^5, \overline{x}_6 \in \mathbb{R}} \ & Z = 3x_1 + 5x_2 \\ \text{s.t.} \ & x_1 + x_3 = 4 \\ & 3x_1 + 2x_2 - x_4 + \overline{x}_6 = 18 \\ & 2x_2 + x_5 = 12, \\ & x_1, x_2, x_3, x_4, x_5, \overline{x}_6 \geq 0. \end{aligned}$$

Since we have artificial variables, our usual starting point is infeasible. We have to solve the phase 1 problem instead.

$$\begin{aligned} \max_{\mathbf{x} \in \mathbb{R}^5, \overline{x}_6 \in \mathbb{R}} \ & W = -\overline{x}_6 \\ \text{s.t.} \ & x_1 + x_3 = 4 \\ & 3x_1 + 2x_2 - x_4 + \overline{x}_6 = 18 \\ & 2x_2 + x_5 = 12 \\ & x_1, x_2, x_3, x_4, x_5, \overline{x}_6 \geq 0 \end{aligned}$$

Here, we have a good initial point, where non-basic variables (decision & surplus) are $x_1 = x_2 = x_4 = 0$ and basic variables (slack & artificial) $(x_3, x_5, \overline{x}_6) = (4, 12, 18)$. We always need W to write in terms of non-basic variables,

$$W = -\overline{x}_6 = 3x_1 + 2x_2 - x_4 - 18.$$

To speed up phase 2, it helps to include the original objective Z in the tableau as well (not used except for elimination).

W	Z	x_1	x_2	x_3	x_4	x_5	$\overline{x}_6$	$\mathbf{b}$	Ratio
1	0	-3	-2	0	1	0	0	-18	
0	1	-3	-5	0	0	0	0	0	
0	0	1	0	1	0	0	0	4	4
0	0	3	2	0	-1	0	1	18	6
0	0	0	2	0	0	1	0	12	–

Here, x_1 enters, x_3 leaves the basis. Eliminate x_1 column using pivot from x_3 row (including putting zero in Z row, useful for later).

After iterations, we arrive at

W	Z	x_1	x_2	x_3	x_4	x_5	$\overline{x}_6$	$\mathbf{b}$
1	0	0	0	0	0	0	1	0
0	1	0	0	$-9/2$	$-5/2$	0	$5/2$	27
0	0	1	0	1	0	0	0	4
0	0	0	1	$-3/2$	$-1/2$	0	$1/2$	3
0	0	0	0	3	1	1	-1	6

W row ≥ 0, phase 1 is done. So the solution to phase 1 is

- Non-basic variables $x_3, x_4, \overline{x}_6$ with $\overline{x}_6 = 0$ (x_3 and x_4 are parameters since zero cost coefficient).
- Basic variables x_1, x_2, x_5.
- Optimal objective value $W^* = 0$, phase 1 succeeded (original LP is feasible).

Phase 1 found a feasible corner point for the original problem

$$(x_1, x_2, x_3, x_4, x_5) = (4, 3, 0, 0, 6)$$

with $Z = 27$.

We can just continue with the same tableau (dropping W row and $\overline{x}_6$ column as not needed).

Z	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$	Ratio
1	0	0	$-9/2$	$-5/2$	0	27	
0	1	0	1	0	0	4	4
0	0	1	$-3/2$	$-1/2$	0	3	–
0	0	0	3	1	1	6	2

we keep perform simplex algorithm and iterate. Finally, phase 2 arrives at

Z	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
1	0	0	3	0	5/2	42
0	1	0	1	0	0	4
0	0	1	0	0	1/2	6
0	0	0	3	1	1	6

The solution to the original problem is $(x_1^*, x_2^*) = (4, 6)$ with $(x_3^*, x_4^*, x_5^*) = (0, 6, 0)$ and $Z^* = 42$.

2.5.2 Big-M Method

There is an alternative to the two-phase simplex method (when we have artificial variables), called the *big-M method*.

Instead of $\max_{\mathbf{x}} Z = \mathbf{c}^T\mathbf{x}$, change the objective to

$$\max_{\mathbf{x},\overline{\mathbf{x}}} \overline{Z} = Z + MW = \mathbf{c}^T\mathbf{x} - M\sum_{i=1}^{m} \overline{x}_i,$$

where M is some large positive number.

This combines our two objectives from phases 1 and 2, trying to maximise them simultaneously. Picking M large means we really care about making $\overline{x}_i = 0$.

To use big-M method, add slack, surplus and artificial variables as before. For Example 2.5.1, the problem starts as

$\overline{Z}$	x_1	x_2	x_3	x_4	x_5	$\overline{x}_6$	$\mathbf{b}$
1	-3	-5	0	0	0	M	0
0	1	0	1	0	0	0	4
0	3	2	0	-1	0	1	18
0	0	2	0	0	1	0	12

But, we always start by writing the objective $\overline{Z}$ in terms of non-basic variables x_1, x_2, x_4. In this case, $\overline{Z} = 3x_1 + 5x_2 - M\overline{x}_6$ is written as

$$\overline{Z} = 3x_1 + 5x_2 - M(18 - 3x_1 - 2x_2 + x_4) = (3 + 3M)x_1 + (5 + 2M)x_2 - Mx_4 - 18M.$$

The first row of the tableau is then actually

$\overline{Z}$	x_1	x_2	x_3	x_4	x_5	$\overline{x}_6$	$\mathbf{b}$
1	$-(3+3M)$	$-(5+2M)$	0	M	0	0	$-18M$

Advantages of the big-M method: Only one run of simplex method needed.

Disadvantages of the big-M method: Do not know M until the end (pick M when finished to make all $\overline{x}_i = 0$). Each step of simplex is more complicated (including unknown variable).

2.6 Duality

LPs come in pairs, that is, for every LP (called the *primal* problem), there is a different LP (called the *dual*). Then primal and dual problems are closely related.

Example 2.6.1 (Motivating example). A gardener wants to prepare a fertiliser by mixing 2 ingredients: GROW and THRIVE. Each kg of GROW contains 4g of nitrogen and 4g of phosphate, while each kg of THRIVE contains 10g of nitrogen and 2g of phosphate. The final product must contain at least 100g of nitrogen and at least 60g of phosphate. If GROW costs \$1.00/kg and THRIVE costs \$1.50/kg, how many kg of each ingredients should be mixed if the manufacturer wants to minimise the cost?

If x_1, x_2 are the kg of GROW and THRIVE, the corresponding LP is

$$\min_{\mathbf{x}\in\mathbb{R}^2} Z = x_1 + 1.5x_2 \quad \text{s.t.} \quad 4x_1 + 10x_2 \geq 100, \quad 4x_1 + 2x_2 \geq 60, \quad x_1, x_2 \geq 0.$$

We previously found the optimal solution $(x_1^*, x_2^*) = (12.5, 5)$ with minimum cost $Z^* = \$20$.

Now let us imagine we are a seller of nitrogen and phosphate (to the manufacturer of GROW and THRIVE). What is our business strategy?

We set the price (per g) of both chemicals to maximise revenue (from guaranteed sale of 100g N and 60g P to the manufacturer). But the manufacturer will only buy from us if our prices are not too high, so they can sell GROW and THRIVE to the gardener without making a loss.

Let y_1, y_2 be our price per g of nitrogen and phosphate. We solve

$$\begin{aligned}
&\max_{\mathbf{y}\in\mathbb{R}^2} V = 100y_1 + 60y_2 \\
&\text{s.t. } 4y_1 + 4y_2 \leq 1 \quad \text{(cost to make GROW)} \\
&\qquad 10y_1 + 2y_2 \leq 1.5 \quad \text{(cost to make THRIVE)} \\
&\qquad y_1, y_2 \geq 0
\end{aligned}$$

This is the *dual* of the gardeneer's (cost minimisation) problem.

The two problems are

$$\begin{aligned}
&\min_{\mathbf{x}\in\mathbb{R}^2} Z = x_1 + 1.5x_2 \\
&\text{s.t. } 4x_1 + 10x_2 \geq 100 \\
&\qquad 4x_1 + 2x_2 \geq 60 \\
&\qquad x_1, x_2 \geq 0
\end{aligned}$$

and

$$\begin{aligned}
&\max_{\mathbf{y}\in\mathbb{R}^2} V = 100y_1 + 60y_2 \\
&\text{s.t. } 4y_1 + 4y_2 \leq 1 \\
&\qquad 10y_1 + 2y_2 \leq 1.5 \\
&\qquad y_1, y_2 \geq 0
\end{aligned}$$

If one is the primal, the other is the dual (i.e. the dual of the dual is the primal). Think about the units of each quantity (dimensional analysis):

- $Z = \mathbf{c}^T\mathbf{x}$ is \$ and $\mathbf{x}$ is units of product, so $\mathbf{c}$ is \$ per unit of product
- $V = \mathbf{b}^T\mathbf{y}$ is \$ and $\mathbf{b}$ is units of resource, so $\mathbf{y}$ is \$ per unit of resource.

Here, the dual variables $\mathbf{y}$ measure the monetary value of the resources in the primal.

2.6.1 Deriation of Dual Problem

We can derive the general form of the dual problem in a different way: trying to estimate the solution of the primal problem.

Consider Example 2.1.1,

$$\begin{aligned} &\max_{\mathbf{x}\in\mathbb{R}^2} Z = 3x_1 + 5x_2, \\ &\text{s.t. } x_1 \le 4, \\ &\qquad 3x_1 + 2x_2 \le 18, \\ &\qquad 2x_2 \le 12, \\ &\qquad x_1, x_2 \ge 0. \end{aligned}$$

Suppose we do not know the solution and cannot be bothered to find it. To estimate how large Z^* will be, the first constraints give

$$Z^* = 3x_1^* + 5x_2^* \le 3(4) + 5(6) = 42.$$

This is not the only option: the second and third constraints give

$$Z^* = 3x_1^* + 5x_2^* = (3x_1^* + 2x_2^*) + 3x_2^* \le (18) + 3(6) = 36.$$

We get a better estimate this way. What are all the possible estimates we could get using this idea, and which one gives the best estimate?

Try to upper bound with any linear combination of constraints (weights y_1, y_2, y_3):

$$Z^* = 3x_1^* + 5x_2^* \le y_1(x_1^*) + y_2(3x_1^* + 2x_2^*) + y_3(2x_2^*) \le 4y_1 + 18y_2 + 12y_3.$$

Since all $x_i^* \ge 0$, the first inequality only holds if

$$y_1 + 3y_2 \ge 3 \quad \text{and} \quad 2y_2 + 2y_3 \ge 5.$$

The second inequality holds if all $y_i \ge 0$.

Then, pick y_i to get the best (i.e. smallest) estimate for $Z^* \le 4y_1 + 18y_2 + 12y_3$.

$$\min_{\mathbf{y}\in\mathbb{R}^2} 4y_1 + 18y_2 + 12y_3 \quad \text{s.t.} \quad y_1 + 3y_2 \ge 3, \quad 2y_2 + 2y_3 \ge 5, \quad y_1, y_2, y_3 \ge 0 \quad \text{(dual)}$$

In general, suppose we have an LP in standard form

$$\max_{\mathbf{x}\in\mathbb{R}^n} Z = \mathbf{c}^T\mathbf{x} \quad \text{s.t.} \quad A\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}.$$

There are m constraints $A\mathbf{x} \leq \mathbf{b}$. Take linear combination of these constraints with weights $\mathbf{y} \in \mathbb{R}^m$. Upper bound for Z is $Z \leq V$ where

$$Z = \mathbf{c}^T\mathbf{x} \leq \sum_{i=1}^{m} y_i(a_{i1}x_1 + \cdots + a_{in}x_n) \leq \mathbf{b}^T\mathbf{y} =: V.$$

Given $\mathbf{x} \geq \mathbf{0}$, the first inequality holds if (compare coefficients of x_j)

$$\sum_{i=1}^{m} y_i a_{ij} \geq c_j, \quad \forall j = 1, \ldots, n \iff A^T\mathbf{y} \geq \mathbf{c}.$$

Given $A\mathbf{x} \leq \mathbf{b}$, the second inequality holds if $\mathbf{y} \geq \mathbf{0}$. Choose weight $\mathbf{y}$ to get smallest upper bound V gives the dual

$$\min_{\mathbf{y}\in\mathbb{R}^m} V = \mathbf{b}^T\mathbf{y} \quad \text{s.t.} \quad A^T\mathbf{y} \geq \mathbf{c}, \quad \mathbf{y} \geq \mathbf{0}.$$

So the standard form for our primal and dual LPs are

$$\max_{\mathbf{x}\in\mathbb{R}^n} Z = \mathbf{c}^T\mathbf{x} \quad \text{s.t.} \quad A\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}$$

and

$$\min_{\mathbf{y}\in\mathbb{R}^m} V = \mathbf{b}^T\mathbf{y} \quad \text{s.t.} \quad A^T\mathbf{y} \geq \mathbf{c}, \quad \mathbf{y} \geq \mathbf{0}.$$

Note: the dual of the dual is the primal.

2.6.2 Weak Duality

The above argument gives us a useful result, called *weak duality*. It says that any feasible linear combination $\mathbf{y}$ makes V an upper bound for any Z (we proved for Z^*, the largest possible Z).

Theorem 2.6.2 (Weak Duality). If $\mathbf{x}$ and $\mathbf{y}$ are any feasible points for the primal and dual problems respectively, then

$$Z = \mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y} = V.$$

For any feasible $\mathbf{x}$ and $\mathbf{y}$, we know $Z \leq V$. So, this must be true for the solutions $\mathbf{x}^*$ and $\mathbf{y}^*$; i.e. the optimal values satisfy $Z^* \leq V^*$.

How large is this *duality gap*? Does any upper bound V^* give exactly the true maximum Z^*?

Theorem 2.6.3. If $\mathbf{x}^*$ and $\mathbf{y}^*$ are feasible for the primal and dual problems respectively such that $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$, then they are both solutions to their respective problems and $Z^* = V^*$.

Proof. Using weak duality and $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$, for any primal feasible $\mathbf{x}$ we have

$$\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}^* = \mathbf{c}^T\mathbf{x}^*,$$

so $Z = \mathbf{c}^T\mathbf{x}$ is maximised at $\mathbf{x}^*$ (i.e. $\mathbf{x}^*$ is optimal and $Z^* = \mathbf{c}^T\mathbf{x}^*$). Similarly, for any dual feasible $\mathbf{y}$ we have $\mathbf{b}^T\mathbf{y} \geq \mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$ so $\mathbf{y}^*$ is optimal.

2.6.3 Strong Duality

Does any upper bound V^* give exactly the true maximum Z^*?

The above theorem answers the opposite question (if $Z = V$ then we are at a solution). If we are at a solution, then do we have $Z = V$?

The answer is yes. This is called *strong duality.* We actually have a little bit more.

Theorem 2.6.4 (Strong Duality). For any primal/dual pair of LPs:

(i) If either problem has a solution, then so does the other, and $Z^* = V^*$.

(ii) If one problem is unbounded, then the other is infeasible.

Proof. (i) Primal problem is in standard form, so we first add slack variables

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij}x_j.$$

Suppose we have run the (one-phase) simplex method and found a solution $\mathbf{x}^*$. The top row of the tableau has non-negative coefficients and the last column has Z^*:

$$Z + \sum_{k=1}^{n+m} (-\overline{c}_k)x_k = Z^* \quad \text{with} \quad \overline{c}_k \leq 0$$

for any feasible $\mathbf{x}$ (remember Z row has negative cost coefficients, $\overline{c}_k = 0$ for basic variables). Define $y_i^* = -\overline{c}_{n+i}$ for $i = 1, \ldots, m$.

Claim: $\mathbf{y}^*$ is feasible for the dual problem and $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$. If this is true, then we are done: $\mathbf{y}^*$ is the dual solution and $V^* = Z^*$ (from previous theorem). From above, for any feasible $\mathbf{x}$

$$\mathbf{c}^T\mathbf{x} = Z = Z^* + \sum_{k=1}^{n+m} \overline{c}_k x_k = Z^* + \sum_{k=1}^{n} \overline{c}_k x_k + \sum_{i=1}^{m} \overline{c}_{n+i}x_{n+i}.$$

But $y_i^* = -\overline{c}_{n+i}$ and we have slacks $x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij}x_j$, so

$$\mathbf{c}^T\mathbf{x} = Z^* + \sum_{k=1}^{n} \overline{c}_k x_k - \sum_{i=1}^{m} y_i^* \left(b_i - \sum_{j=1}^{n} a_{ij}x_j \right).$$

Rearrange to collect coefficients of $x_1, \dots, x_n$ and (all other terms):

$$\sum_{j=1}^{n} c_j x_j = \left(Z^* - \sum_{i=1}^{m} y_i^* b_i \right) + \sum_{j=1}^{n} \left(\bar{c}_j + \sum_{i=1}^{m} a_{ij} y_i^* \right) x_j.$$

This holds for all primal feasible $\mathbf{x}$, so the constant term must be zero and the x_j coefficients must be equal. That is,

$$Z^* - \sum_{i=1}^{m} y_i^* b_i = 0$$
$$\bar{c}_j + \sum_{i=1}^{m} a_{ij} y_i^* = c_j$$

for all $j = 1, \dots, n$. The first equation if $\mathbf{b}^T \mathbf{y}^* = Z^*$. The second equation (in vector form) is $\bar{\mathbf{c}} + A^T \mathbf{y}^* = \mathbf{c}$. Since $\bar{\mathbf{c}} \leq \mathbf{0}$ (negative cost coefficients at solution), we have $A^T \mathbf{y}^* \geq \mathbf{c}$.

Lastly, $y_i^* = -\bar{c}_{n+i}$ and $\bar{\mathbf{c}} \leq \mathbf{0}$ gives $\mathbf{y}^* \geq \mathbf{0}$, so $\mathbf{y}^*$ is feasible for the dual problem.

So, $\mathbf{y}^*$ is dual feasible and $Z^* = \mathbf{b}^T \mathbf{y}^*$, so the claim is true (and above theorem says $\mathbf{y}^*$ is solution of dual, so $Z^* = \mathbf{b}^T \mathbf{y}^* = V^*$).

(ii) If the primal is unbounded, there exist feasible points $\mathbf{x}_1, \mathbf{x}_2, \dots$ with $\mathbf{c}^T \mathbf{x}_k \to \infty$. To find a contradiction, suppose the dual is feasible (let $\mathbf{y}$ be any feasible point).

Since $\mathbf{c}^T \mathbf{x}_k \to \infty$, there is some k with $\mathbf{c}^T \mathbf{x}_k > \mathbf{b}^T \mathbf{y}$.

But $\mathbf{x}_k$ and $\mathbf{y}$ are feasible, so this violates weak duality (contradiction). Hence the dual must be infeasible.

- If either the primal or dual are feasible, then so is the other, and $Z^* = V^*$.
- If the primal is unbounded then the dual is infeasible.
- If the dual is unbounded then the primal is infeasible.
- Or, the primal and dual are both infeasible.

The simplex method can identify if a problem is unbounded (no variable leaves the basis). Even better, since $y_i^* = -\bar{c}_{n+i}$, the value of $\mathbf{y}^*$ is given in the final simplex tableau: y_i^* is the Z-row tableau entry for slack x_{n+i}. Simplex method solves the primal and dual simultaneously.

Example 2.6.5. Consider the problem

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^2} \ & Z = 3x_1 + 2x_2 \\ \text{s.t.} \ & 8x_1 + 3x_2 \geq 24 \\ & 5x_1 + 6x_2 \geq 30 \\ & 2x_1 + 9x_2 \geq 18 \\ & x_1, x_2 \geq 0. \end{aligned}$$

To solve this, we need to add surplus and artificial variables to handle the $\geq$ constraints. Solving requires the two-phase simplex method. But the dual problem is in standard form (do not need phase 1):

$$\begin{aligned} \max_{\mathbf{y} \in \mathbb{R}^3} \; & V = 24y_1 + 30y_2 + 18y_3 \\ \text{s.t. } & 8y_1 + 5y_2 + 2y_3 \leq 3 \\ & 3y_1 + 6y_2 + 9y_3 \leq 2 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

Adding slack variables we get

$$\begin{aligned} V - 24y_1 - 30y_2 - 18y_3 &= 0 \\ 8y_1 + 5y_2 + 2y_3 + y_4 &= 3 \\ 3y_1 + 6y_2 + 9y_3 + y_5 &= 2 \\ y_1, y_2, y_3, y_4, y_5 &\geq 0. \end{aligned}$$

As usual, our initial feasible corner point starts with the non-basic variables being the decision variables ($y_1 = y_2 = y_3 = 0$) and basic variables are the slacks $(y_4, y_5) = (3, 2)$.

V	y_1	y_2	y_3	y_4	y_5	RHS
1	-24	-30	-18	0	0	0
0	8	5	2	1	0	3
0	3	6	9	0	1	2

Skipping the working, simplex takes two steps to find a solution

V	y_1	y_2	y_3	y_4	y_5	RHS
1	-9	0	27	0	$5/2$	10
0	$11/2$	0	$-11/2$	1	$-5/6$	$4/3$
0	$1/2$	1	$3/2$	0	$1/6$	$1/3$

$\longrightarrow$

V	y_1	y_2	y_3	y_4	y_5	RHS
1	0	0	18	$18/11$	$40/11$	$134/11$
0	1	0	-1	$2/11$	$-5/33$	$8/33$
0	0	1	2	$-1/11$	$8/33$	$7/33$

Reading off as usual, the dual solution if

$$(y_1^*, y_2^*, y_3^*) = \left(\frac{8}{33}, \frac{7}{33}, 0\right) \quad \text{with} \quad V^* = \frac{134}{11}.$$

What is the primal solution? Look at the Z-row entries for the slack variables.

$$(x_1^*, x_2^*) = \left(\frac{18}{11}, \frac{40}{11}\right).$$

We could calculate Z^* using $Z = 3x_1 + 2x_2$, but there is no need. We automatically know $Z^* = V^* = 134/11$ from string duality.

For non-standard LPs, the following table illustrates the types of constraints we get in primal vs dual.

Primal	Dual
$\max_{\mathbf{x}} \mathbf{c}^T\mathbf{x}$	$\min_{\mathbf{y}} \mathbf{b}^T\mathbf{y}$
$A\mathbf{x} \leq \mathbf{b}$	$\mathbf{y} \geq \mathbf{0}$
$A\mathbf{x} \geq \mathbf{b}$	$\mathbf{y} \leq \mathbf{0}$
$A\mathbf{x} = \mathbf{b}$	$\mathbf{y}$ any sign
$\mathbf{x} \geq \mathbf{0}$	$A^T\mathbf{y} \geq \mathbf{c}$
$\mathbf{x} \leq \mathbf{0}$	$A^T\mathbf{y} \leq \mathbf{c}$
$\mathbf{x}$ any sign	$A^T\mathbf{y} = \mathbf{c}$

(mix of LHS constraints $\longleftrightarrow$ mix of RHS constraints)

3 Nonlinear Optimisation

Now we focus on nonlinear optimisation. This covers optimisation problems where the objective and/or some constraints are nonlinear. We will first look at the unconstrained case,

$$\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x})$$

for some nonlinear function $f : \mathbb{R}^n \to \mathbb{R}$, and then at the constrained case

$$\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{x} \in \Omega$$

for some feasible region Ω. Just like for LPs, we will formulate our constraints as equations or inequalities

$$\begin{aligned} &\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}) \\ &\text{s.t. } g_i(\mathbf{x}) = 0, \quad i = 1, \ldots, k \\ &\qquad h_j(\mathbf{x}) \leq 0, \quad j = k+1, \ldots, m \end{aligned}$$

There are many areas where nonlinear optimisation occurs. In this course we will see it in

- **Fitting probability distributions.** Suppose we have data $x_1, \ldots, x_n$ coming from some probability distribution with parameters θ. We can estimate a good θ using

  $$\min_{\theta} -\sum_{i=1}^{n} \log f(x_i, \theta)$$

  where $f(x, \theta)$ is the probability density function.

- **Determining optimal investment strategies** (mean-variance portfolio theory). We will model the problem of choosing how much money x_i to invest in stock $i = 1, \ldots, n$ as a *quadratic program* (QP),

  $$\min_{\mathbf{x}\in\mathbb{R}^n} \mathbf{x}^T Q\mathbf{x} + \mathbf{c}^T\mathbf{x} \quad \text{s.t.} \quad A\mathbf{x} = \mathbf{b}.$$

- **Natural gas supply networks.** When natural gas flows through a network of pipes (connecting different cities, for example), each pipe i has a pressure p_i and gas flow rate q_i. We also have extra decision variables $z_i \in \{0, 1\}$ indicating if a compressor (pump) is active in a given pipe. The physics of the network gives constraint equations

  $$A\mathbf{q} - \mathbf{d} = \mathbf{0}, \quad A^T\mathbf{p}^2 + K\mathbf{q}^{2.8359} = \mathbf{0}, \quad B^T\mathbf{q} + \mathbf{z}^T\mathbf{c}(\mathbf{p}, \mathbf{q}) = \mathbf{0},$$

  for some matrices A, K and B, vector $\mathbf{d}$ and nonlinear functions $\mathbf{c}(\mathbf{p}, \mathbf{q})$. There are usually bound constraints on $p_i \in [p_{\min}, p_{\max}]$ and $q_i \in [q_{\min}, q_{\max}]$ too. Our objective here might be to minimise operating cost, minimise total gas pressure (safety), or many other possibilities.

3.1 Unconstrained Optimisation

Let us start by looking at the simple case of optimising a function in 1D (without constraints):

$$\min_{x\in\mathbb{R}} f(x)$$

Theorem 3.1.1 (Taylor's Theorem). Suppose $f : \mathbb{R} \to \mathbb{R}$ is $C^{p+1}(\mathbb{R})$ ($p+1$ times continuously differentiable). For every a and x in the domain of f, there exists ξ between a and x such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots + \frac{1}{p!}f^{(p)}(a)(x-a)^p + \frac{1}{(p+1)!}f^{(p+1)}(\xi)(x-a)^{p+1}.$$

Intuition for Taylor's Theorem is that if f is smooth, and x is close enough to a, then $f(x)$ looks like a (low degree) polynomial.

This gives us a way to fully understand extrema (maxima/minima) of f. For example, if $f'(a) = 0$ and $f''(a) > 0$ then

$$f(x) = f(a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(\xi)(x-a)^3.$$

If x is close to a, then $(x-a)^3 << (x-a)^2$, so

$$f(x) \approx f(a) + \frac{1}{2}f''(a)(x-a)^2 > f(a)$$

so a is a minimiser of f (at least compared to nearby points; "local minimiser").

Terminology: the optimal decision variable $\mathbf{x}^*$ is called a *maximiser/minimiser*, the value $f(\mathbf{x}^*)$ is called the *maximum/minimum*.

In general, Taylor's Theorem allows us to say that for some integer m:

- If $f'(a) = f''(a) = \cdots = f^{(2m-1)}(a) = 0$ and $f^{(2m)}(a) < 0$, then a is a local maximiser.
- If $f'(a) = f''(a) = \cdots = f^{(2m-1)}(a) = 0$ and $f^{(2m)}(a) > 0$, then a is a local minimiser.
- If $f'(a) = f''(a) = \cdots = f^{(2m)}(a) = 0$ and $f^{(2m+1)}(a) \neq 0$, then a is an inflection point.

These only tell us about the local behaviour of f. It does not tell us if a is a maximiser/minimiser compared to all possible values $f(x)$ (*global maximiser/minimiser*), only values nearby.

Similar ideas work in higher dimensions, but it gets very complicated very quickly (as m increases).

To look at optimisation in $\mathbb{R}^n$, we need a version of Taylor's Theorem in higher dimensions. Suppose we care about $\mathbf{a} \in \mathbb{R}^n$ and another point $\mathbf{x} \in \mathbb{R}^n$. Let's join the two points via a line (and assume the whole line is in the domain of f, i.e. domain is convex), the line is

$$\mathbf{a} + t(\mathbf{x} - \mathbf{a}), \quad t \in [0,1].$$

Look at the value of f along this line

$$g(t) := f(\mathbf{a} + t(\mathbf{x} - \mathbf{a})) = f(\mathbf{a} + t\mathbf{h})$$

where $\mathbf{h} = \mathbf{x} - \mathbf{a}$ and apply Taylor's Theorem to g. We first compute the derivatives of g.

$$\begin{aligned} g'(t) &= \frac{d}{dt} f(\mathbf{a} + t\mathbf{h}) = \frac{d}{dt} f(a_1 + th_1, \ldots, a_n + th_n) \\ &= h_1 \frac{\partial f(\mathbf{a} + t\mathbf{h})}{\partial x_1} + \cdots + h_n \frac{\partial f(\mathbf{a} + t\mathbf{h})}{\partial x_n} \\ &= \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial f(\mathbf{a}+t\mathbf{h})}{\partial x_1} \\ \vdots \\ \frac{\partial f(\mathbf{a}+t\mathbf{h})}{\partial x_n} \end{pmatrix} \end{aligned}$$

and so $g'(t) = \mathbf{h}^T \nabla f(\mathbf{a} + t\mathbf{h})$. For second derivatives, we have

$$\begin{aligned} g''(t) &= h_1 \frac{d}{dt} \frac{\partial f(\mathbf{a} + t\mathbf{h})}{\partial x_1} + \cdots + h_n \frac{d}{dt} \frac{\partial f(\mathbf{a} + t\mathbf{h})}{\partial x_n} \\ &= \sum_{i=1}^{n} h_i \left[\sum_{j=1}^{n} h_j \frac{\partial^2 f(\mathbf{a} + t\mathbf{h})}{\partial x_i \partial x_j} \right] \end{aligned}$$

Again, we can write this succinctly. Define the *Hessian* matrix

$$\nabla^2 f(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 f(\mathbf{x})}{\partial x_1^2} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_n^2} \end{pmatrix} \in \mathbb{R}^{n \times n}$$

then we get $g''(t) = \mathbf{h}^T \nabla^2 f(\mathbf{a} + t\mathbf{h})\mathbf{h}$.

For higher derivatives, we have the same basic pattern, for example

$$g'''(t) = \sum_{i,j,k=1}^{n} \frac{\partial^3 f(\mathbf{a} + t\mathbf{h})}{\partial x_i \partial x_j \partial x_k} h_i h_j h_k =: \nabla^3 f(\mathbf{a} + t\mathbf{h}) \begin{pmatrix} \mathbf{h} & \mathbf{h} & \mathbf{h} \end{pmatrix}$$

and so on.

Applying Taylor's Theorem in 1D (with $t = 1$ so $\mathbf{a} + t\mathbf{h} = \mathbf{x}$) we get *multivariate Taylor's Theorem*,

$$\begin{aligned} f(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a})^T (\mathbf{x} - \mathbf{a}) + \frac{1}{2!} (\mathbf{x} - \mathbf{a})^T \nabla^2 f(\mathbf{a}) (\mathbf{x} - \mathbf{a}) + \cdots \\ + \frac{1}{p!} \nabla^p f(\mathbf{a}) \begin{pmatrix} \mathbf{x} - \mathbf{a} & \mathbf{x} - \mathbf{a} & \mathbf{x} - \mathbf{a} \end{pmatrix} + R_p(\mathbf{x}) \end{aligned}$$

where $R_p(\mathbf{x})$ is the remainder.

For our purposes in optimisation, it is most convenient to only look up to the quadratic terms, that is, if $\mathbf{x} \approx \mathbf{a}$ then

$$f(\mathbf{x}) \approx f(\mathbf{a}) + \nabla f(\mathbf{a})^T (\mathbf{x} - \mathbf{a}) + \frac{1}{2!} (\mathbf{x} - \mathbf{a})^T \nabla^2 f(\mathbf{a}) (\mathbf{x} - \mathbf{a}).$$

Now we go back to the optimisation problem

$$\min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}).$$

Definition 3.1.2 (Global minimiser). A point $\mathbf{x}^* \in \mathbb{R}^n$ is a *global minimiser* of f if $\mathbf{x} \in \mathbb{R}^n$, $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$.

By thinking back to the 1D case, we could only use derivatives to check if a point is larger/smaller than nearby points.

Definition 3.1.3 (Local minimiser). A point $\mathbf{x}^* \in \mathbb{R}^n$ is a *local minimiser* of f if $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x}$ sufficiently close to $\mathbf{x}^*$.

For maximiser, both definitions are same but with $\geq$.

3.1.1 First-Order Optimality

To relate local minimisers/maximisers to derivatives, we need to think about how f behaves when we move in a given direction away from a point.

Given a point $\mathbf{x}$, a direction $\mathbf{p}$ is a *descent direction* if $f(\mathbf{x} + \alpha\mathbf{p}) < f(\mathbf{x})$ for all $\alpha > 0$ sufficiently small. By definition, if $\mathbf{x}^*$ is a local minimiser (maximiser), no vector is a descent (ascent) direction. From Taylor's Theorem, if α is sufficiently small then $\mathbf{x} + \alpha\mathbf{p}$ is close to $\mathbf{x}$, so

$$f(\mathbf{x} + \alpha\mathbf{p}) \approx f(\mathbf{x}) + \alpha\nabla f(\mathbf{x})^T\mathbf{p}.$$

So, if $\nabla f(\mathbf{x})^T\mathbf{p} < 0$ then $\mathbf{p}$ is a descent direction (ascent if > 0). (If $\nabla f(\mathbf{x})^T\mathbf{p} = 0$, not sure)

Geometrically, $\mathbf{p}$ is ascent if it forms an acute angle with $\nabla f(\mathbf{x})$, and descent if it forms an acute angle with $-\nabla f(\mathbf{x})$.

This gives us the first key way to characterise local extrema.

Theorem 3.1.4. If f is continuously differentiable and $\mathbf{x}^*$ is a local maximiser or minimiser, then $\nabla f(\mathbf{x}^*) = \mathbf{0}$.

Proof. Suppose $\nabla f(\mathbf{x}^*) \neq \mathbf{0}$. Then direction $\mathbf{p} = \nabla f(\mathbf{x}^*)$ gives

$$\nabla f(\mathbf{x}^*)^T\mathbf{p} = \nabla f(\mathbf{x}^*)^T\nabla f(\mathbf{x}^*) = \sum_{i=1}^{n}\left(\frac{\partial f(\mathbf{x}^*)}{\partial x_i}\right)^2 > 0$$

so $\mathbf{p}$ is an ascent direction. Similarly, $\mathbf{p} = -\nabla f(\mathbf{x}^*)$ is a descent direction. So, $\mathbf{x}^*$ cannot be a local maximum or minimum (contradiction).

3.1.2 Second-Order Optimality

From above, we know that local extrema are *stationary points*. A stationary point $\mathbf{x}^*$ is:

- A local minimiser if all directions from $\mathbf{x}^*$ are ascent directions.

- A local maximiser if all directions from $\mathbf{x}^*$ are descent directions.
- A *saddle point* if there are both ascent and descent directions from $\mathbf{x}^*$.

Saddle points are the equivalent of inflection points in 1D. For example, the origin is a saddle point of $f(x, y) = x^2 - y^2$.

At a stationary point, our Taylor series is

$$f(\mathbf{x}) \approx f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) + \frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T \nabla^2 f(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*).$$

Since $\nabla f(\mathbf{x}^*) = \mathbf{0}$, we need to look at the Hessian $\nabla^2 f(\mathbf{x}^*)$. Since we want to look at all $\mathbf{x}$ near $\mathbf{x}^*$, we are interested in expressions like $\mathbf{y}^T \nabla^2 f(\mathbf{x}^*)\mathbf{y}$ for different inputs $\mathbf{y}$.

For a matrix $A \in \mathbb{R}^{n \times n}$, the *quadratic form* associated with A is the quadratic function $Q(\mathbf{y}) = \mathbf{y}^T A\mathbf{y}$. The matrix A is

- *Positive definite* if $Q(\mathbf{y}) > 0$ for all $\mathbf{y} \neq \mathbf{0}$ (note $Q(\mathbf{0}) = \mathbf{0}$ always)
- *Positive semidefinite* if $Q(\mathbf{y}) \geq \mathbf{0}$ for all $\mathbf{y}$.
- *Indefinite* if $Q(\mathbf{y}) > 0$ for some $\mathbf{y}$ and $Q(\mathbf{y}) < 0$ for other $\mathbf{y}$.
- *Negative semidefinite* if $Q(\mathbf{y}) \leq 0$ for all $\mathbf{y}$.
- *Negative definite* if $Q(\mathbf{y}) < 0$ for all $\mathbf{y} \neq \mathbf{0}$.

If A is symmetric, write $A = PDP^T$, so

$$Q(\mathbf{y}) = \mathbf{y}^T PDP^T \mathbf{y} = (P^T\mathbf{y})^T D(P^T\mathbf{y}) = \mathbf{z}^T D\mathbf{z}$$

where $\mathbf{z} = P^T\mathbf{y}$. Since P^T is invertible ($P^T P = I$), $\mathbf{z}$ takes all values in $\mathbb{R}^n$ if $\mathbf{y}$ does, and $\mathbf{y} = \mathbf{0}$ if and only if $\mathbf{z} = \mathbf{0}$. Since D is diagonal with entries $\lambda_1, \ldots, \lambda_n$ (eigenvalues of A), we get

$$Q(\mathbf{y}) = \sum_{i=1}^{n} \lambda_i z_i^2.$$

Varying $\mathbf{z}$ over all vectors in $\mathbb{R}^n$ (including coordinate vectors), we get:

- A is positive definite if all $\lambda_i > 0$.
- A is positive semidefinite if all $\lambda_i \geq 0$.
- A is indefinite if some $\lambda_i > 0$ and some $\lambda_i < 0$.
- A is negative semidefinite if all $\lambda_i \leq 0$.
- A is negative definite if all $\lambda_i < 0$.

Theorem 3.1.5. Suppose f is $C^2(\mathbb{R}^n)$ and $\mathbf{x}^*$ is a stationary point (i.e. $\nabla f(\mathbf{x}^*) = \mathbf{0}$).

- If $\nabla^2 f(\mathbf{x}^*)$ is positive definite, then $\mathbf{x}^*$ is a local minimiser.
- If $\nabla^2 f(\mathbf{x}^*)$ is negative definite, then $\mathbf{x}^*$ is a local maximiser.
- If $\nabla^2 f(\mathbf{x}^*)$ is indefinite, then $\mathbf{x}^*$ is a saddle point.
- If $\nabla^2 f(\mathbf{x}^*)$ has a zero eigenvalue, then no conclusion can be drawn.

Example 3.1.6. Find and classify all stationary points of $f(x, y) = x^2 + y^2$.

Solution. First, compute gradient and Hessian:

$$\nabla f(x, y) = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \quad \text{and} \quad \nabla^2 f(x, y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Second, find stationary points

$$\nabla f(x^*, y^*) = \mathbf{0} \implies (x^*, y^*) = (0, 0).$$

Last, look at eigenvalues of Hessian $\nabla^2 f(x^*, y^*)$: eigenvalues are $\lambda = 2, 2$ (checking using $\det(A - \lambda I) = 0$, or note eigenvalues of diagonal matrix are diagonal entries). Hessian is positive definite, so point is local minimiser.

So far, we have

$$\text{Global min} \implies \text{Local min} \implies \text{Stationary point.}$$

In general, the reverse implications are not true.

For the second implication, we know that

$$\text{Stationary point} \implies \text{Positive definite Hessian} \implies \text{Local min.}$$

3.1.3 Convexity

One special class of functions are *convex* functions.

Definition 3.1.7. A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and all $t \in [0, 1]$,

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \leq t f(\mathbf{x}) + (1-t) f(\mathbf{y}).$$

For example, all linear functions, $f(x) = x^4$, $f(x) = e^x$ and $f(x) = |x|$ are convex functions.

Less obvious examples: $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ if A is symmetric & positive semidefinite, $f(\mathbf{x}) = \log(e^{x_1} + \cdots + e^{x_n})$, $f(\mathbf{x}) = \max(x_1, \ldots, x_n)$ are all convex.

An easy way to check for convexity is the following.

Theorem 3.1.8. If $f \in C^2(\mathbb{R}^n)$, then f is convex if and only if $\nabla^2 f(\mathbf{x})$ is positive semidefinite for all $\mathbf{x} \in \mathbb{R}^n$.

Theorem 3.1.9. If $f \in C^1(\mathbb{R}^n)$ and convex, then

$$\text{Global min} \Longleftrightarrow \text{Local min} \Longleftrightarrow \text{Stationary point.}$$

Proof. From our existing $\implies$ implications, we just need to show

$$\text{Global min} \Longleftarrow \text{Stationary point.}$$

Suppose $\mathbf{x}^*$ is not a global minimiser, so there exists $\mathbf{z}$ with $f(\mathbf{z}) < f(\mathbf{x}^*)$. Then we have

$$\nabla f(\mathbf{x}^*)^T(\mathbf{z} - \mathbf{x}^*) = \frac{d}{dt} f(\mathbf{x}^* + t(\mathbf{z} - \mathbf{x}^*))\Big|_{t=0} = \lim_{t\to 0} \frac{f(\mathbf{x}^* + t(\mathbf{z} - \mathbf{x}^*)) - f(\mathbf{x}^*)}{t}.$$

Using conxevity of f,

$$\nabla f(\mathbf{x}^*)^T(\mathbf{z} - \mathbf{x}^*) = \lim_{t\to 0} \frac{f((1-t)\mathbf{x}^* + t\mathbf{z}) - f(\mathbf{x}^*)}{t} \leq \lim_{t\to 0} \frac{(1-t)f(\mathbf{x}^*) + tf(\mathbf{z}) - f(\mathbf{x}^*)}{t}.$$

Hence $\nabla f(\mathbf{x}^*)^T(\mathbf{z} - \mathbf{x}^*) \leq f(\mathbf{z}) - f(\mathbf{x}^*) < 0$, so $\mathbf{p} = \mathbf{z} - \mathbf{x}^*$ is a descent direction at $\mathbf{x}^*$. Hence, $\mathbf{x}^*$ cannot be a stationary point.

3.1.4 Optimisation Algorithms

Many available, one popular choice is *linesearch* methods: given a current point $\mathbf{x}_k$, find a better point $\mathbf{x}_{k+1}$ via:

- Find a descent direction $\mathbf{p}_k$ (i.e. $\nabla f(\mathbf{x}_k)^T \mathbf{p}_k < 0$).
- Find a stepsize $\alpha_k > 0$ such that $f(\mathbf{x}_k + \alpha_k \mathbf{p}_k) < f(\mathbf{x}_k)$. Pick $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{p}_k$.

Most common types:

- $\mathbf{p}_k = -\nabla f(\mathbf{x}_k)$, direction fastest local decrease (”gradient descent”).
- $\mathbf{p}_k = -[\nabla^2 f(\mathbf{x}_k)]^{-1} \nabla f(\mathbf{x}_k)$, minimise quadratic Taylor series (”Newton’s method”). Needs Hessian to be positive definite for descent.
- $\mathbf{p}_k = -H_k^{-1} \nabla f(\mathbf{x}_k)$ for some $H_k \approx \nabla^2 f(\mathbf{x}_k)$ and positive definite (most famous are ”quasi-Newton” methods, especially ”BFGS”).

3.2 Constrained Optimisation

Now let us consider a problem with constraints for a feasible region $\Omega \subset \mathbb{R}^n$. As mentioned above, we will express our constraints algebraically, as equations/inequalities.

$$\begin{aligned} &\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}) \\ &\text{s.t. } g_i(\mathbf{x}) = 0, \quad i = 1, \dots, k \\ &\qquad h_j(\mathbf{x}) \leq 0, \quad j = k+1, \dots, m \end{aligned}$$

The feasible region is then

$$\Omega = \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) = \cdots = g_k(\mathbf{x}) = 0 \quad \text{and} \quad h_{k+1}(\mathbf{x}), \dots, h_m(\mathbf{x}) \leq 0\}.$$

Definition 3.2.1 (Global minimiser). A point $\mathbf{x}^* \in \Omega$ is a *global minimiser* if $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in \Omega$.

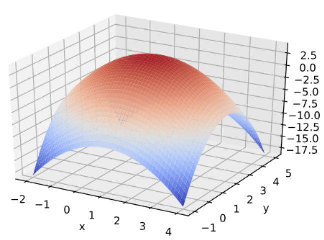
Definition 3.2.2 (Local minimiser). A point $\mathbf{x}^* \in \Omega$ is a *local minimiser* if $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in \Omega$ sufficiently close to $\mathbf{x}^*$.

These definitions are very similar to the unconstrained case (just take $\Omega = \mathbb{R}^n$). However, it is much more difficult to write necessary conditions for constrained problems.

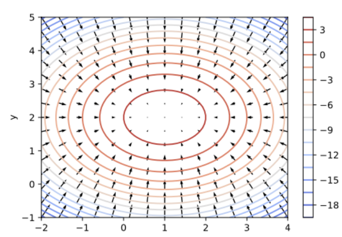
There are second-order conditions (involving $\nabla^2 f$) for constrained problems just like the unconstrained case, but we will not look at these in this course.

3.2.1 Geometric Ideas

Plot the unconstrained problem $\max_{x,y\in\mathbb{R}} f(x, y) = 4 - (x-1)^2 - 1.5(y-2)^2$.



$f(x, y)$



$f(x, y)$ and $\nabla f(x, y)$

Gradient is pointing in direction of increasing f, perpendicular to level sets.

A first-order Taylor series at $\mathbf{x}$ gives

$$f(\mathbf{y}) \approx f(\mathbf{x}) + \nabla f(\mathbf{x})^T (\mathbf{y} - \mathbf{x}).$$

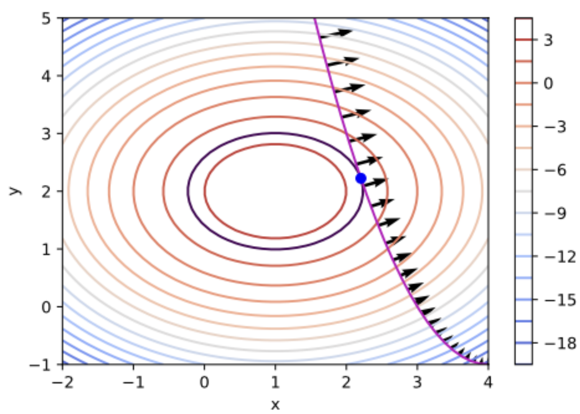
So if $\mathbf{y}$ is a point near $\mathbf{x}$ on the same level set (so $f(\mathbf{y}) = f(\mathbf{x})$), we have

$$\nabla f(\mathbf{x})^T (\mathbf{y} - \mathbf{x}) \approx 0.$$

Taking limits as $\mathbf{y} \to \mathbf{x}$, we get
$$\nabla f(\mathbf{x})^T \mathbf{v}(\mathbf{x}) = 0$$
where $\mathbf{v}(\mathbf{x})$ is the tangent to the level set at $\mathbf{x}$.

That is, ∇f always points perpendicular to (tangents of) level sets.

Now we add a single equality constraint:
$$\max_{x,y \in \mathbb{R}} f(x,y) = 4 - (x-1)^2 - 1.5(y-2)^2$$
$$\text{s.t. } g(x,y) = y - (x-4)^2 + 1 = 0$$



Observations: ∇g is perpendicular to constraint set, solution is where level set of f is parallel to constraint set.

So at the solution, ∇g is parallel to ∇f (possibly with sign difference), or $\nabla f(\mathbf{x}^*) = -\lambda \nabla g(\mathbf{x}^*)$ for some $\lambda \in \mathbb{R}$.

3.2.2 Lagrangian

A unified way to think about the solutions of constrained optimisation problems is to introduce a new function called a *Lagrangian*.

If we only have equality constraints:
$$\min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{g}(\mathbf{x}) = \mathbf{0}$$
where $\mathbf{g}(\mathbf{x}) = \begin{pmatrix} g_1(\mathbf{x}) & \cdots & g_m(\mathbf{x}) \end{pmatrix}^T$. The Lagrangian is defined to be
$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) := f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x}) = f(\mathbf{x}) + \boldsymbol{\lambda}^T \mathbf{g}(\mathbf{x}),$$
where $\boldsymbol{\lambda} \in \mathbb{R}^m$ is some vector of weights.

The partial derivatices of $\mathcal{L}$ are

$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{\partial f}{\partial x_j} + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \lambda_i} = g_i(\mathbf{x}).$$

So a stationary point $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ of $\mathcal{L}$ satisfies

$$\nabla_{\mathbf{x}} \mathcal{L} = \nabla f(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(\mathbf{x}^*) = \mathbf{0} \quad \text{and} \quad \nabla_{\boldsymbol{\lambda}} \mathcal{L} = \mathbf{g}(\mathbf{x}^*) = \mathbf{0}.$$

The first condition generalises what we saw earlier ($\nabla f = -\lambda \nabla g$). The second condition is feasibility, $\mathbf{x}^* \in \Omega$.

Both of these are necessary requirements for $\mathbf{x}^*$ to be a solution of the constrained problem.

3.2.3 Equality Constraints

For the equality-constrained problem

$$\min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{g}(\mathbf{x}) = \mathbf{0}$$

where $\mathbf{g}(\mathbf{x}) = \begin{pmatrix} g_1(\mathbf{x}) & \cdots & g_m(\mathbf{x}) \end{pmatrix}^T$, we have the following.

Theorem 3.2.3. Under suitable conditions (e.g. differentiability of f and all g_i), if $\mathbf{x}^*$ is a local minimiser then there exist *Lagrange multipliers* $\lambda_1^*, \ldots, \lambda_m^* \in \mathbb{R}$ such that

$$\nabla f(\mathbf{x}^*) = -\sum_{i=1}^{m} \lambda_i^* \nabla g(\mathbf{x}^*), \quad \text{and} \quad \mathbf{g}(\mathbf{x}^*) = \mathbf{0}.$$

Equivalently, $\mathbf{x}^*$ and $\boldsymbol{\lambda}^* = \begin{pmatrix} \lambda_1^* & \cdots & \lambda_m^* \end{pmatrix}^T$ form a stationary point for $\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda})$.

These conditions give $n + m$ equations in $n + m$ unknowns.

Example 3.2.4. Solve the problem

$$\min_{x,y \in \mathbb{R}} f(x, y) = \frac{1}{2}(x-1)^2 + \frac{1}{2}(y-2)^2 + 1.$$

Solution. First, write constraint as $g(x, y) = x + y - 1 = 0$. The Lagrangian is

$$\mathcal{L}(x, y, \lambda) = \frac{1}{2}(x-1)^2 + \frac{1}{2}(y-2)^2 + 1 + \lambda(x + y - 1).$$

Stationary points satisfy

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= (x-1) + \lambda = 0, \\ \frac{\partial \mathcal{L}}{\partial y} &= (y-2) + \lambda = 0, \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= x + y - 1 = 0. \end{aligned}$$

Last equation gives $y = 1 - x$, substitute into second equation to get $(1 - x - 2) + \lambda = 0$. Add this to first equation to get $2\lambda - 2 = 0$ or $\lambda = 1$, and then $x = 0$ and $y = 1$.

3.2.4 Inequality Constraints

Now we look at inequality constraints:

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n} \; & f(\mathbf{x}) \\ \text{s.t. } & \mathbf{h}(\mathbf{x}) \leq \mathbf{0} \end{aligned}$$

where $\mathbf{h}(\mathbf{x}) = \begin{pmatrix} h_1(\mathbf{x}) & \cdots & h_m(\mathbf{x}) \end{pmatrix}^T$. Just like for LPs, we can add slack variables s_i to convert $\leq$ inequalities to equalities. The main difference here is we do not want to add new constraints like $s_i \geq 0$, so instead we write $h_i(\mathbf{x}) \leq 0$ as $h_i(\mathbf{x}) + s_i^2 = 0$, not $h_i(\mathbf{x}) + s_i = 0$.

The new problem is

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n, \mathbf{s}\in\mathbb{R}^m} & f(\mathbf{x}) \\ \text{s.t. } & h_i(\mathbf{x}) + s_i^2 = 0, \quad i = 1, \ldots, m. \end{aligned}$$

This is now in equality form. The Lagrangian is

$$\mathcal{L}(\mathbf{x}, \mathbf{s}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i (h_i(\mathbf{x}) + s_i^2)$$

so the stationary conditions are given by

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_j} &= \frac{\partial f}{\partial x_j} + \sum_{i=1}^{m} \lambda_i \frac{\partial h_i}{\partial x_j}, \\ \frac{\partial \mathcal{L}}{\partial s_i} &= 2\lambda_i s_i = 0, \\ \frac{\partial \mathcal{L}}{\partial \lambda_i} &= h_i(\mathbf{x}) + s_i^2 = 0. \end{aligned}$$

Since $\mathbf{s}$ and $\boldsymbol{\lambda}$ are in $\mathbb{R}^m$, we now have $n + m + m$ equations in $n + m + m$ unknowns.

Written in vector form (as much as possible), our stationary conditions are

$$\nabla f(\mathbf{x}^*) = -\sum_{i=1}^{m} \lambda_i^* \nabla h_i(\mathbf{x}^*), \quad \lambda_i^* s_i^* = 0, \quad \text{and} \quad \mathbf{h}(\mathbf{x}^*) + (\mathbf{s}^*)^2 = \mathbf{0}$$

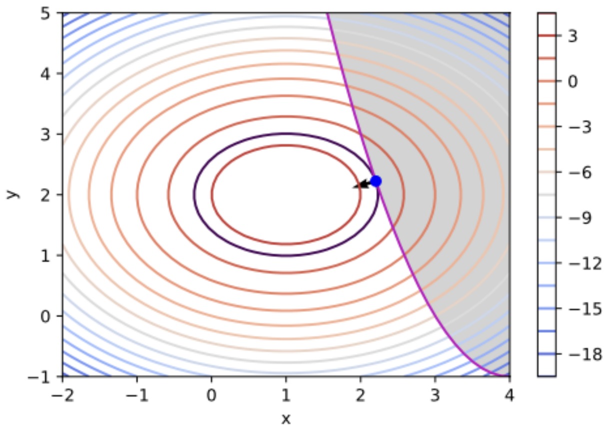
(with $(\mathbf{s}^*)^2$ meant element-wise), and the *complementary slackness* conditions $\lambda_i^* s_i^* = 0$ holding for all $i = 1, \ldots, m$.

We consider several cases.

Case 1: Unconstrained minimiser outside feasible region

Consider the problem

$$\begin{aligned}\min_{x,y\in\mathbb{R}} \; & f(x,y) = -4 + (x-1)^2 + 1.5(y-2)^2 \\ \text{s.t. } \; & h(x,y) = -y + (x-4)^2 - 1 \leq 0.\end{aligned}$$



We are not at the unconstrained solution, so $\nabla f(\mathbf{x}^*) = \mathbf{0}$, and at least one constraint must be stopping us, so $h_i(\mathbf{x}^*) = 0$ for some i (such constraints are called *active*).

There is some constraint i preventing us from decreasing f further. Consider

$$\nabla f(\mathbf{x}^*) = -\sum_{i=1}^{m} \lambda_i^* \nabla h_i(\mathbf{x}^*).$$

Plot shows $\nabla h(x^*, y^*)$ pointing in opposite direction to $\nabla f(x^*, y^*)$; to decrease f we would need to increase h (which is not allowed when $h = 0$). This gives $\lambda_i^* > 0$.

We also have $\lambda_i^* s_i^* = 0$, and so $s_i^* = 0$. The last condition is $h_i(\mathbf{x}^*) + (s_i^*)^2 = 0$ and so $h_i(\mathbf{x}^*) = 0$ (constraint i is active, exactly what we expect). Since $h_i(\mathbf{x}^*) = 0$, note that $\lambda_i^* h_i(\mathbf{x}^*) = 0$.

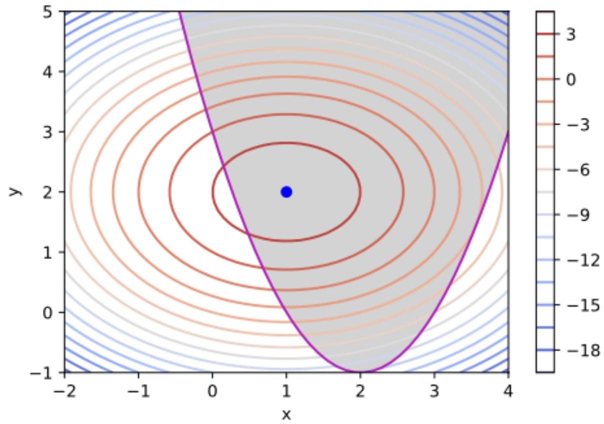
Case 2: Unconstrained minimiser inside feasible region Consider

$$\begin{aligned}\min_{x,y\in\mathbb{R}} \; & f(x,y) = -4 + (x-1)^2 + 1.5(y-2)^2 \\ \text{s.t. } \; & h(x,y) = -y + (x-2)^2 - 1 \leq 0.\end{aligned}$$

Constrained solution is equal to unconstrained solution, so $\nabla f(\mathbf{x}^*) = \mathbf{0}$. Solution is away from constraint boundary, so $h_i(\mathbf{x}^*) < 0$ for all i. If $h_i(\mathbf{x}^*) < 0$, then $h_i(\mathbf{x}^*) + (s_i^*)^2 = 0$ implies $s_i^* \neq 0$. Complementary slackness $\lambda_i^* s_i^* = 0$ implies $\lambda_i^* = 0$. Then from $\lambda_i^* = 0$ and

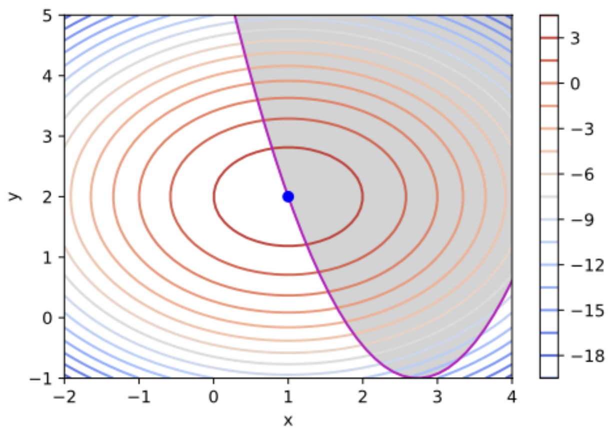
$$\nabla f(\mathbf{x}^*) = -\sum_{i=1}^{m} \lambda_i^* \nabla h_i(\mathbf{x}^*)$$

we conclude $\nabla f(\mathbf{x}^*) = \mathbf{0}$, as expected. Again note that $\lambda_i^* h_i(\mathbf{x}^*) = 0$.



Case 3: Unconstrained minimiser on boundary of feasible region

$$\min_{x,y\in\mathbb{R}} f(x,y) = -4 + (x-1)^2 + 1.5(y-2)^2$$
$$\text{s.t. } h(x,y) = -y + (x-1-\sqrt{3})^2 - 1 \le 0.$$



Unconstrained solution is still feasible, so $\nabla f(\mathbf{x}^*) = \mathbf{0}$. But it is on the boundary, so $h_i(\mathbf{x}^*) = 0$ for some i. If $h_i(\mathbf{x}^*) = 0$, we have $s_i^* = 0$, but if $h_i(\mathbf{x}^*) < 0$ then $s_i^* \neq 0$. Complementary slackness $\lambda_i^* s_i^* = 0$ implies $\lambda_i^* = 0$ if $h_i(\mathbf{x}^*) < 0$ and $\lambda_i^* \in \mathbb{R}$ otherwise. From

$$\nabla f(\mathbf{x}^*) = -\sum_{i=1}^{m} \lambda_i^* \nabla h_i(\mathbf{x}^*)$$

all the terms with $h_i(\mathbf{x}^*) < 0$ vanish (since $\lambda_i^* = 0$). From $\mathbf{x}^*$, moving inside the feasible region (so $h_i(\mathbf{x}) < 0$) must increase f. That is, decreasing h leads to increasing f, so ∇h pointing in opposite direction to ∇f. Similar to case 1, this means $\lambda_i^* \ge 0$ whenever $h_i(\mathbf{x}^*) = 0$. Again, note $\lambda_i^* h_i(\mathbf{x}^*) = 0$ in both situations.

For

$$\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{h}(\mathbf{x}) \le \mathbf{0}$$

adding slacks and finding stationary points of the Lagrangian gives

$$\nabla f(\mathbf{x}^*) = -\sum_{i=1}^{m} \lambda_i^* \nabla h_i(\mathbf{x}^*), \quad \lambda_i^* s_i^* = 0, \quad \text{and} \quad \mathbf{h}(\mathbf{x}^*) + (\mathbf{s}^*)^2 = \mathbf{0}.$$

Our analysis of 3 cases shows this is the same as

$$\begin{aligned} \nabla f(\mathbf{x}^*) &= -\sum_{i=1}^{m} \lambda_i^* \nabla h_i(\mathbf{x}^*), \\ \lambda_i^* h_i(\mathbf{x}^*) &= 0, \\ \mathbf{h}(\mathbf{x}^*) &\leq \mathbf{0}, \\ \lambda_i^* &\geq 0. \end{aligned}$$

So there are two ways of finding solutions (Lagrangian & above conditions), just like for equality constraints.

Example 3.2.5. Solve the problem

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^3} \ & f(\mathbf{x}) = x_1(x_1 - 10) + x_2(x_2 - 50) - 2x_3 \\ \text{s.t.} \ & x_1 + x_2 \leq 10, \\ & x_3 \leq 10. \end{aligned}$$

Solution. Starting by defining $h_1(\mathbf{x}) = x_1 + x_2 - 10 \leq 0$ and $h_2(\mathbf{x}) = x_3 - 10 \leq 0$. Adding the slacks, then the Lagrangian is

$$\begin{aligned} \mathcal{L}(\mathbf{x}, \mathbf{s}, \boldsymbol{\lambda}) &= f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i (h_i(\mathbf{x}) + s_i^2) \\ &= x_1(x_1 - 10) + x_2(x_2 - 50) - 2x_3 + \lambda_1(x_1 + x_2 - 10 + s_1^2) + \lambda_2(x_3 - 10 + s_2^2) \end{aligned}$$

Next we set all partial derivatives to zero. Partial derivatives are

$$\frac{\partial \mathcal{L}}{\partial x_1} = 2x_1 - 10 + \lambda_1 = 0, \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 2x_2 - 50 + \lambda_1 = 0, \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial x_3} = -2 + \lambda_2 = 0, \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial s_i} = 2\lambda_i s_i = 0, \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = x_1 + x_2 - 10 + s_1^2 = 0, \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_2} = x_3 - 10 + s_2^2 = 0. \tag{6}$$

Firstly, (3) gives $\lambda_2 = 2$, so (4) gives $s_2 = 0$ (i.e. $x_3 = 10$ from (6)). Now, if $\lambda_1 = 0$, then (1) and (2) give $x_1 = 5$ and $x_2 = 25$. But then (5) has no solutions. So we must have $\lambda_1 \neq 0$. Since $\lambda_1 \neq 0$, (4) means $s_1 = 0$, and then (5) implies $x_1 + x_2 = 10$. This, (1) and (2) give three linear equations in x_1, x_2, and λ_1 with solution $x_1 = -5$, $x_2 = 15$, and $\lambda_1 = 20$.

3.2.5 General Case

Now let's consider the fully general case with both equality and inequality constraints

$$\begin{aligned}
\min_{\mathbf{x}\in\mathbb{R}^n} \ & f(\mathbf{x}) \\
\text{s.t.} \ & g_i(\mathbf{x}) = 0, \quad i = 1, \dots, k \\
& h_j(\mathbf{x}) \leq 0, \quad j = k+1, \dots, m.
\end{aligned}$$

As before, we can find stationary points of the Lagrangian (after adding slacks to the inequality constraints)

$$\mathcal{L}(\mathbf{x}, \mathbf{s}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \sum_{i=1}^{k} \lambda_i g_i(\mathbf{x}) + \sum_{j=k+1}^{m} \lambda_j (h_j(\mathbf{x}) + s_j^2).$$

3.2.6 KKT Conditions

Theorem 3.2.6 (KKT Conditions). Suppose $f, g_i, h_i \in C^1(\mathbb{R}^n)$, $\mathbf{x}^* \in \Omega$ is a local minimiser, and *constraint qualification holds at* $\mathbf{x}^*$. Then $\mathbf{x}^*$ is a *KKT point*, that is, there exist Lagrange multipliers $\boldsymbol{\lambda}^* \in \mathbb{R}^m$ such that

$$\begin{aligned}
\nabla f(\mathbf{x}^*) &= -\sum_{i=1}^{k} \lambda_i^* \nabla g_i(\mathbf{x}^*) - \sum_{j=k+1}^{m} \lambda_j^* \nabla h_j(\mathbf{x}^*), && \text{(stationarity)} \\
g_i(\mathbf{x}^*) &= 0, \quad \forall i = 1, \dots, k, && \text{(feasibility)} \\
h_j(\mathbf{x}^*) &\leq 0, \quad \forall j = k+1, \dots, m, && \text{(feasibility)} \\
\lambda_j^* &\geq 0, \quad \forall j = k+1, \dots, m, && \text{(dual feasibility)} \\
\lambda_j^* h_j(\mathbf{x}^*) &= 0, \quad \forall j = k+1, \dots, m. && \text{(complementary slackness)}
\end{aligned}$$

Intuition: Each Lagrange multiplier represents how much (and in what direction) each constraint is preventing us from decreasing f further.

This result only works if we have a *constraint qualification* (CQ). Roughly, this says that first-order Taylor series for g_i and h_j accurately represent the geometry of Ω. There are many CQs, such as:

- All constraints g_i and h_j are linear/affine (i.e. linear $\pm$ constant).
- Slater's CQ: If g_i are linear/affine and h_j are convex, and there exists $\mathbf{x}$ such that $h_j(\mathbf{x}) < 0$ for all j.
- LICQ:
$$\{\nabla g_i(\mathbf{x}^*) : i = 1, \dots, k\} \cup \{\nabla h_j(\mathbf{x}^*) : j \in \mathcal{A}(\mathbf{x}^*)\}$$
is a linearly independent set, where $\mathcal{A}(\mathbf{x}) = \{j = k+1, \dots, m : h_j(\mathbf{x}) = 0\}$ are the *active* inequality constraints.

In this course, we assume that every problem has a CQ, so we do not need to worry about this.

Example 3.2.7. Solve the problem

$$\begin{aligned}
\min_{x,y\in\mathbb{R}} \; & f(x,y) = \frac{1}{2}(x-1)^2 + \frac{1}{2}(y-2)^2 + 1 \\
\text{s.t.} \; & x + y = 1, \\
& x \geq \frac{1}{2}, \\
& y \leq \frac{3}{4}.
\end{aligned}$$

Start by defining $g_1(x,y) = x + y - 1 = 0$, $h_2(x,y) = \frac{1}{2} - x \leq 0$ and $h_2(x,y) = y - \frac{3}{4} \leq 0$. he KKT conditions are: find $x, y, \lambda_1, \lambda_2, \lambda_3$ such that

$$\begin{aligned}
\nabla f(x,y) &= -\lambda_1 \nabla g_1(x,y) - \lambda_2 \nabla h_2(x,y) - \lambda_3 \nabla h_3(x,y) \\
g_1(x,y) &= 0 \\
h_2(x,y), h_3(x,y) &\leq 0, \\
\lambda_2, \lambda_3 &\geq 0, \\
\lambda_2 h_2(x,y) = \lambda_3 h_3(x,y) &= 0.
\end{aligned}$$

As before, we solve the five equations in five unknowns. Computing gradients, our equations are

$$\begin{pmatrix} x-1 \\ y-2 \end{pmatrix} = -\lambda_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \lambda_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} - \lambda_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\lambda_1 + \lambda_2 \\ -\lambda_1 - \lambda_3 \end{pmatrix}, \tag{1-2}$$

$$x + y - 1 = 0, \tag{3}$$

$$\lambda_2 \left(\frac{1}{2} - x \right) = 0, \tag{4}$$

$$\lambda_3 \left(y - \frac{3}{4} \right) = 0, \tag{5}$$

plus $1/2 - x \leq 0$, $y - 3/4 \leq 0$ and $\lambda_2, \lambda_3 \geq 0$. Every is linear except for complementary slackness, so try different cases for these and see what happens.

Case 1. Suppose $\lambda_2 = \lambda_3 = 0$. Then (1), (2) and (3) are three linear equations in x, y, λ_1 with solution $x = 1$, $y = 1$, and $\lambda_1 = 1$. This violates the constraint $1/2 - x \leq 0$. No solutions in this case.

Case 2. Suppose $\lambda_2 = 0$, $\lambda_3 > 0$. Then (5) gives $y = 3/4$, and hence (3) gives $x = 1/4$. This violates the constraint $1/2 - x \leq 0$. No solutions in this case.

Case 3. Suppose $\lambda_2 > 0$, $\lambda_3 = 0$. Then (4) gives $x = 1/2$, and hence (3) gives $y = 1/2$. This and $\lambda_3 = 0$ in (2) gives $\lambda_1 = 3/2$, and finally (1) gives $\lambda_2 = 1$. Inequalities all hold, so the solution found.

Case 4. Suppose $\lambda_2, \lambda_3 > 0$. Then (4) and (5) give $x = 1/2$ and $y = 3/4$, which violates (3). No solutions in this case.

All together, there is only one KKT point, $(x^*, y^*) = (1/2, 1/2)$ with $\lambda_1^* = 3/2$, $\lambda_2^* = 1$ and $\lambda_3^* = 0$.

The KKT conditions work for any constrained problem (provided the functions are smooth and a CQ holds). For LPs

$$\max_{\mathbf{x}\in\mathbb{R}^n} Z = \mathbf{c}^T\mathbf{x} \quad \text{s.t.} \quad A\mathbf{x} \le \mathbf{b}, \quad \mathbf{x} \ge \mathbf{0},$$

constraints are all linear/affine, so a CQ holds (and all functions are linear so differentiable), hence, assumptions of KKT theorem hold. Using our notation, we have $m+n$ constraints: (where $\mathbf{a}_j^T$ is jth row of A)

$$\begin{aligned} h_j(\mathbf{x}) &= \mathbf{a}_j^T\mathbf{x} - b_j \le 0, \quad j = 1,\dots,m \\ h_{m+j}(\mathbf{x}) &= -x_j \le 0, \quad j = 1,\dots,n. \end{aligned}$$

Computing gradients (remembering to convert max to min)

$$\nabla f(\mathbf{x}) = -\mathbf{c}, \quad \nabla h_j(\mathbf{x}) = \mathbf{a}_j \;\; (j = 1,\dots,m) \quad \text{and} \quad \nabla h_{m+j}(\mathbf{x}) = -\mathbf{e}_j \;\; (j = 1,\dots,n).$$

The KKT conditions are (splitting Lagrange multipliers for $A\mathbf{x} \le \mathbf{b}$ and $\mathbf{x} \ge \mathbf{0}$ into separate variables): find $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$ and $\boldsymbol{\lambda} \in \mathbb{R}^n$ such that

$$\begin{aligned} -\mathbf{c} &= -\sum_{j=1}^m y_j\mathbf{a}_j + \sum_{j=1}^n \lambda_j\mathbf{e}_j, \\ A\mathbf{x} &\le \mathbf{b}, \quad \mathbf{x} \ge \mathbf{0}, \\ \mathbf{y}, \boldsymbol{\lambda} &\ge \mathbf{0}, \\ y_j(\mathbf{a}_j^T\mathbf{x} - b_j) &= \lambda_j(-x_j) = 0. \end{aligned}$$

The first equation is $-\mathbf{c} = -A^T\mathbf{y} + \boldsymbol{\lambda}$ or $A^T\mathbf{y} = \mathbf{c} + \boldsymbol{\lambda}$. Since $\boldsymbol{\lambda} \ge \mathbf{0}$, we could write $A^T\mathbf{y} \ge \mathbf{c}$. So we have $A^T\mathbf{y} \ge \mathbf{c}$ and $\mathbf{y} \ge \mathbf{0}$. These are the constraints for the dual problem. Actually, the connection is even closer than this. Suppose $\mathbf{x}^*, \mathbf{y}^*, \boldsymbol{\lambda}^*$ is a KKT point for the original LP, and let $\mathbf{y} \in \mathbb{R}^m$ be any other dual feasible point ($A^T\mathbf{y} \ge \mathbf{c}$ and $\mathbf{y} \ge \mathbf{0}$). Then,

$$\begin{aligned} \mathbf{b}^T\mathbf{y} &\ge (\mathbf{x}^*)^T A^T\mathbf{y} && (A\mathbf{x}^* \le \mathbf{b} \text{ and } \mathbf{y} \ge \mathbf{0}) \\ &\ge (\mathbf{x}^*)^T\mathbf{c} && (A^T\mathbf{y} \ge \mathbf{c} \text{ and } \mathbf{x}^* \ge \mathbf{0}) \\ &= (\mathbf{x}^*)^T(A^T\mathbf{y}^* + \boldsymbol{\lambda}^*) && (A^T\mathbf{y}^* - \boldsymbol{\lambda}^* = \mathbf{c}) \\ &= (\mathbf{x}^*)^T A^T\mathbf{y}^* && (\lambda_j x_j = 0) \\ &= (A\mathbf{x}^* - \mathbf{b})^T\mathbf{y}^* + \mathbf{b}^T\mathbf{y}^* && (\text{rearrange}) \\ &= \mathbf{b}^T\mathbf{y}^* && (y_j(\mathbf{a}_j^T\mathbf{x}^* - b_j) = 0) \end{aligned}$$

So, the Lagrange multipliers $\mathbf{y}^*$ for the original LP minimise $\mathbf{b}^T\mathbf{y}$ over all dual feasible $\mathbf{y}$: the Lagrange multipliers $\mathbf{y}^*$ are the solution to the dual problem. This also gives us an extra fact about duality for LPs: complementary slackness, $y_j^*(\mathbf{a}_j^T\mathbf{x}^* - b_j) = 0$ for all $j = 1,\dots,m$.

3.2.7 Convexity

Just like for unconstrained problems, we have the implications (under reasonable assumptions)

$$\text{Global min} \implies \text{Local min} \implies \text{KKT point}$$

and the implications do not usually go the other way. One key result about existence of global extrema for constrained problems is the following.

Theorem 3.2.8 (Weierstrass Extreme Value Theorem). If f is continuous and the feasible region Ω is closed and bounded, then there exists a global minimiser and maximiser of f in Ω.

So if Ω is closed and bounded, then at least one KKT point is a global minimiser and at least one is a global maximiser (check $f(\mathbf{x}^*)$ to see which ones). We will not study how to classify KKT points as local minima/maxima using Hessians in this course.

If f and Ω are convex, then

$$\text{Global min} \iff \text{Local min} \iff \text{KKT point (if CQ, smoothness).}$$