11/1/2021

$\boxed{\text{Theorem 3.2}}$ $\mathbb{F} : {}_L\mathbb{C}\mathrm{sp}(\int F) \longrightarrow F\mathbb{C}\mathrm{sp}$

symmetric monoidal double iso

① $\mathbb{F}(m \otimes m' \otimes m'') \leftleftarrows \leftleftarrows \mathbb{F}(m) \otimes \mathbb{F}(m') \otimes \mathbb{F}(m'')$ ✓

② $\mathbb{F}(m \odot m' \odot m'') \leftleftarrows \leftleftarrows \mathbb{F}(m) \odot \mathbb{F}(m') \odot \mathbb{F}(m'')$ . . .

③
$$\begin{tikzcd} \mathbb{F}(M \otimes N) \arrow[d, "\mathbb{F}(\beta)"'] & \mathbb{F}(M) \otimes \mathbb{F}(N) \arrow[l, "\mathbb{F}_\otimes"'] \arrow[d, "\beta"] \arrow[dl, phantom, "?"] \\ \mathbb{F}(N \otimes M) & \mathbb{F}(N) \otimes \mathbb{F}(M) \arrow[l, "\mathbb{F}_\otimes"] \end{tikzcd}$$

USED TO BE AD-HOC — NOT ANYMORE!! ✓ (*) = WHAT I WROTE!!!!

④
$$\begin{tikzcd} (\mathbb{F}(M_2) \otimes \mathbb{F}(N_2)) \odot (\mathbb{F}(M_1) \otimes \mathbb{F}(N_1)) \arrow[r, "\mathbb{F}_\otimes \odot \mathbb{F}_\otimes"] \arrow[d, "\xi"'] \arrow[dr, phantom, "?"] & \mathbb{F}(M_2 \otimes N_2) \odot \mathbb{F}(M_1 \otimes N_1) \arrow[d, "\mathbb{F}_\odot"] \\ (\mathbb{F}(M_2) \odot \mathbb{F}(M_1)) \otimes (\mathbb{F}(N_2) \odot \mathbb{F}(N_1)) \arrow[d, "\mathbb{F}_\odot \otimes \mathbb{F}_\odot"'] & \mathbb{F}((M_2 \otimes N_2) \odot (M_1 \otimes N_1)) \arrow[d, "\mathbb{F}(\xi)"] \\ \mathbb{F}(M_2 \odot M_1) \otimes \mathbb{F}(N_2 \odot N_1) \arrow[r, "\mathbb{F}_\otimes"] & \mathbb{F}((M_2 \odot M_1) \otimes (N_2 \odot N_1)) \end{tikzcd}$$

① Monoidal Grothendieck

$\boxed{2}$ Monoidal opfibrations $U : (X, \otimes) \to (A, +)$

[Sh, 2008] $\Updownarrow$ 2-equivalence

$\boxed{3}$ Pseudofunctors $F : A \longrightarrow \underline{\mathrm{MonCat}}$

$\boxed{2} \Rightarrow \boxed{3} \Rightarrow \boxed{2}$

$U \longmapsto F_U \longmapsto U_{F_U} \cong U$: monoidal isomorphism

$X \to A$ $\qquad \int F \longrightarrow A$

$(\overbrace{\int F}^{X}, \otimes) \longrightarrow (A, +) \longmapsto F : A \to \mathrm{MonCat} \longmapsto (\int F, \bar{\otimes}) \longrightarrow (A, +)$

monoidal opfib $\qquad a \longmapsto Fa \qquad (m, x) \bar{\otimes} (n, y) =$

$(m, x) \otimes (n, y) = \underline{(m+n, x \otimes y)}$ $\quad \otimes_a : Fa \times Fa \xrightarrow{\otimes} F(a+a) \xrightarrow{F\nabla} Fa \quad = (m+n, F(\iota_m) x \otimes_{m+n} F(\iota_n) y)$

monoidal opfib

$$\begin{tikzcd} (\int F, \otimes) \arrow[d, "U"] \\ (A,+) \end{tikzcd} \qquad \begin{tikzcd} \int F \times \int F \arrow[r, "\otimes"] \arrow[d, "U\times U"] \arrow[dr, phantom, "\cong"] & \int F \arrow[d, "U"] \\ A \times A \arrow[r, "+"'] & A \end{tikzcd}$$

$\boxed{\otimes \text{ is cocart} = \text{preserves cocartesian liftings}}$

$= (m+n, F\nabla(F(\iota_m)x \otimes F(\iota_n)y))$

K

$(\int F, \otimes) \xrightarrow{\sim} (\int F, \bar{\otimes})$ monoidal isomorphism, due to the 2-equivalence.

(strong!) with oplaxator

id on objects $K(x \otimes y) \to Kx \,\bar{\otimes}\, Ky$

$$\begin{tikzcd} 1 \arrow[r, "x \times y"] & Fm \times Fn \arrow[rr, "\otimes"] \arrow[d, "F\iota_m \times F\iota_n"] \arrow[drr, phantom, "\cong"] & & F(m+n) \arrow[d, "F(\iota_m+\iota_n)"] \arrow[dr, phantom, "\cong"] \arrow[drr, equal] & & \\ & F(m+n)\times F(m+n) \arrow[rr, "\otimes"'] & & F(m+n+m+n) \arrow[rr, "F\nabla"'] & & F(m+n) \end{tikzcd}$$

$\otimes$ is cocartesion

$$\begin{tikzcd} X \arrow[r, "K"] \arrow[d, "U"'] \arrow[dr, phantom, "\cong"] & Y \arrow[d, "P"] \\ A \arrow[r, "G"'] & B \end{tikzcd}$$

opfibs

K is cocartesion when it maps cocart lifts to cocart lifts

$x \xrightarrow{\text{cocart}} f_!x$ in X
$a \xrightarrow{f} b$ in A

$\longmapsto$

$Kx \xrightarrow{K(\text{cocart})} K(f_!x) \cong (Gf)_!(Kx)$, $Kx \xrightarrow{\text{cocart}} (Gf)_!(Kx)$ in Y
$Ga \xrightarrow{Gf} Gb$ in B

$$\begin{tikzcd} X_a \arrow[r, "K"] \arrow[d, "f_!"'] \arrow[dr, phantom, "\cong"] & Y_{Fa} \arrow[d, "(Gf)_!"] \\ X_b \arrow[r, "K"'] & Y_{Fb} \end{tikzcd}$$

This is true for any "global" monoidal structure $\otimes$ on $\int F$. For us, [using $\boxed{1} \Rightarrow \boxed{2}$ for monoidal pseudo $F : (A,+) \to (\mathrm{Cat}, \times)$], can choose this structure to be "ϕ"

$$\boxed{(m,x)\otimes(n,y) := (m+n, \phi_{m,n}(x,y))}$$

Following the above story, now $(\int F, \otimes) \xrightarrow[\sim]{K} (\int F, \bar{\otimes})$ is given by the laxator

$$\begin{tikzcd} 1 \arrow[r] & Fm \times Fn \arrow[rr, "\phi_{m,n}"] \arrow[d, "F\iota_m \times F\iota_n"'] \arrow[drr, phantom, "\phi_{\iota_m,\iota_n}"] & & F(m+n) \arrow[d, "F(\iota_m+\iota_n)"] \arrow[dr, phantom, "\cong"] \arrow[drr, equal, "{[F(1)]}"] & & \\ & F(m+n)\times F(m+n) \arrow[rr, "\phi_{m+n,m+n}"'] & & F(m+n+m+n) \arrow[rr, "F\nabla"'] & & F(m+n) \end{tikzcd}$$

$\boxed{\text{cocartesianness}}$ . . . .

Fact: K is a strong monoidal functor, therefore

$$\begin{tikzcd} K((x\otimes y)\otimes z) \arrow[r, "\theta"] \arrow[d, "K\alpha"'] & K(x\otimes y)\,\bar{\otimes}\, Kz \arrow[r, "\theta\otimes 1"] \arrow[d, phantom, "\circlearrowright"] & (Kx\,\bar{\otimes}\, Ky)\,\bar{\otimes}\, Kz \arrow[d, "\alpha"] \\ K(x\otimes(y\otimes z)) \arrow[r, "\theta"'] & Kx\,\bar{\otimes}\, K(y\otimes z) \arrow[r, "1\otimes\theta"'] & Kx\,\bar{\otimes}\,(Ky\,\bar{\otimes}\, Kz) \end{tikzcd}$$

$\otimes = \phi \quad , \quad \overline{\otimes} = F\nabla(\phi(F_L(-), F_L(-)))$

$$
\begin{tikzcd}
\phi(\phi(x,y),z) \arrow[r] \arrow[d, "w"'] & (F\nabla)\phi(F_j(\overset{x\otimes y}{\phi(x,y)}), (F_j)z) \arrow[d] \\
\phi(x,\phi(y,z)) \arrow[dr] \arrow[r, phantom, "\circlearrowleft"] & (F\nabla)\phi(F_j((F\nabla)\phi(F_L)x, F_L y)), F_j(z)) \arrow[d, "\alpha"] \\
 & (F\nabla)\phi((F_k)(x), F_k)(\phi(y,z))) \arrow[r] & (F\nabla)\phi((F_L)(x), F_k((F\nabla)\phi(F_L)y, F_L z)))
\end{tikzcd}
$$

NOT MISSING ANYMORE!

Up to some (checked) rearrangement of the top & bottom composites this gives ~~precisely~~ the desired axiom for us! Look at last Monday's notes. SO IT IS VERIFIED → either by claiming Shulman has checked → or by verifying by hand, now that α is known. $\square$.

For ② + ④, $\Phi_\odot$ is needed – so far we only talked about $\Phi_\otimes =$ "Θ". But notice: that $\Phi_\odot$ from (29) factorizes as

$\boxed{\Phi_\otimes = \Theta}$

$$
\begin{tikzcd}
Fm \times Fn \arrow[r, "\phi_{m,n}"] \arrow[d, "f_{1m}\times f_{1n}"'] \arrow[dd, bend right, "f_{\psi m}\times f_{\psi n}"'] & F(m+n) \arrow[d, "F(1_{m+n})"] \arrow[dr, equal] \arrow[l, phantom, "\cong\ \phi_{1m,1n}"] \\
F(m+n)\times F(m+n) \arrow[r, "\phi_{m+n,m+n}"] \arrow[d, "f\psi\times f\psi"'] & F(m+n+m+n) \arrow[r, "F\nabla"] \arrow[d, "F(\psi+\psi)"] \arrow[l, phantom, "\cong\ \phi_{\psi,\psi}"] & F(m+n) \arrow[d, "F\psi"] \arrow[dl, phantom, "\cong"] \\
F(m+_b n)\times F(m+_b n) \arrow[r, "\phi_{m+_b n, m+_b n}"'] & F(m+_b n + m+_b n) \arrow[r, "F\nabla"'] & F(m+_b n)
\end{tikzcd}
$$

F applied to naturality of ∇.

① should imply ② using this factorization

Then ④ looks like it could be deduced from an appropriate axiom for K, viewing $\odot$ as "$\otimes$" and interchanges as braidings. My current thoughts in a separate document.