

1. PROOF OF THEOREM 2.2

There are 11 diagrams in Shulman's definition of symmetric monoidal double category, which we check now for decorated cospan double categories.

Diagram 1 of 11. Set up but not solved.

M_1 is given by $a \rightarrow m_1 \leftarrow b$ with $x_1 \in F(m_1)$.
 M_2 is given by $b \rightarrow m_2 \leftarrow c$ with $x_2 \in F(m_2)$.
 M_3 is given by $c \rightarrow m_3 \leftarrow d$ with $x_3 \in F(m_3)$.
 N_1 is given by $a' \rightarrow n_1 \leftarrow b'$ with $y_1 \in F(n_1)$.
 N_2 is given by $b' \rightarrow n_2 \leftarrow c'$ with $y_2 \in F(n_2)$.
 N_3 is given by $c' \rightarrow n_3 \leftarrow d'$ with $y_3 \in F(n_3)$.

$$\begin{array}{ccc}
 ((M_1 \otimes N_1) \odot (M_2 \otimes N_2)) \odot (M_3 \otimes N_3) & \xrightarrow{\chi \odot 1} & ((M_1 \odot M_2) \otimes (N_1 \odot N_2)) \odot (M_3 \otimes N_3) \\
 \downarrow \alpha & & \downarrow \chi \\
 (M_1 \otimes N_1) \odot ((M_2 \otimes N_2) \odot (M_3 \otimes N_3)) & & ((M_1 \odot M_2) \odot M_3) \otimes ((N_1 \odot N_2) \odot N_3) \\
 \downarrow 1 \odot \chi & & \downarrow \alpha \otimes \alpha \\
 (M_1 \otimes N_1) \odot ((M_2 \odot M_3) \otimes (N_2 \odot N_3)) & \xrightarrow{\chi} & (M_1 \odot (M_2 \odot M_3)) \otimes (N_1 \odot (N_2 \odot N_3))
 \end{array}$$

Decorations:

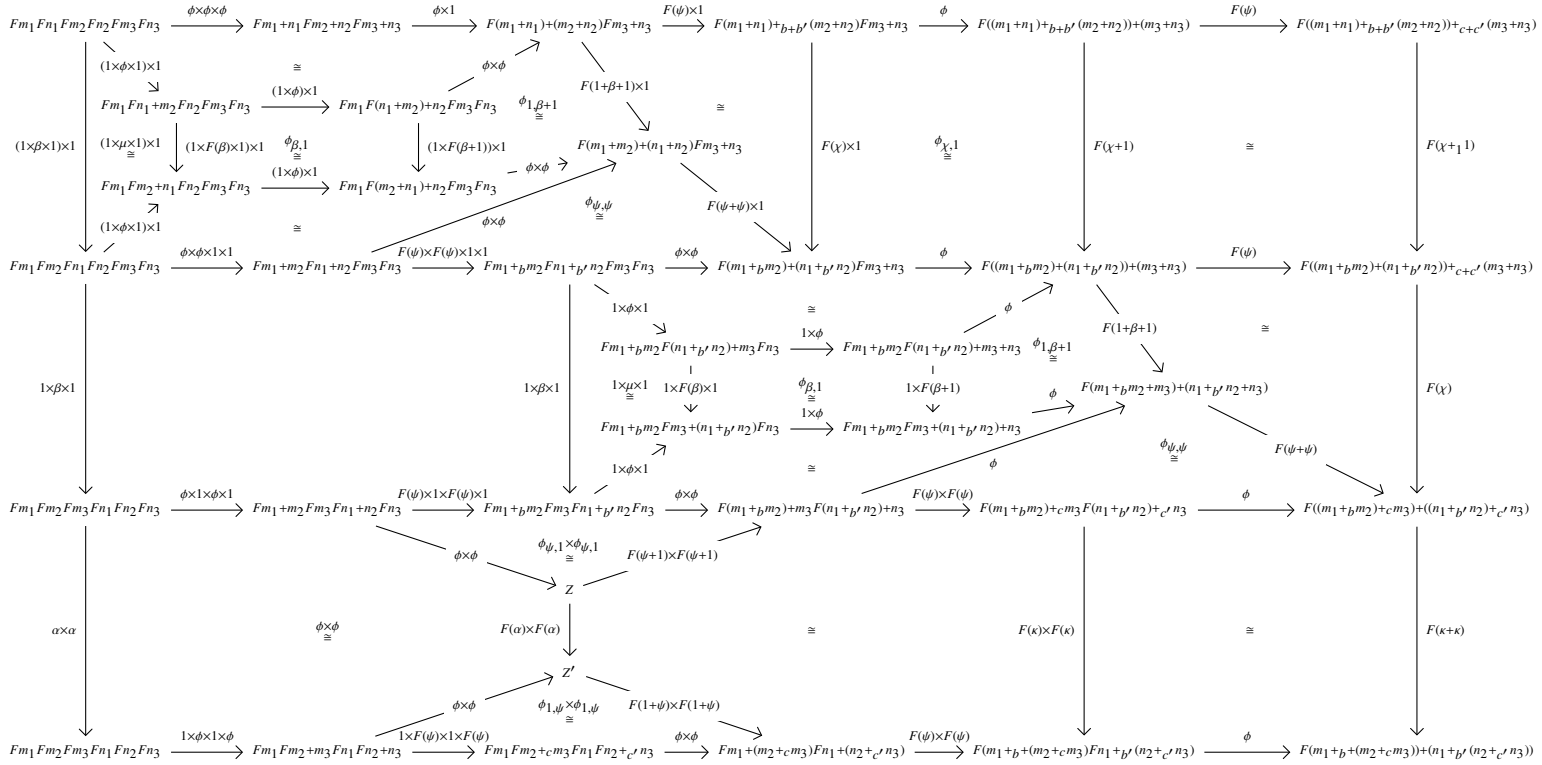
(1) Right and then down:

$$\begin{aligned}
 & F(\psi) \phi_{(m_1+n_1)+b+b', (m_2+n_2), m_3+n_3} (F(\psi) \phi_{m_1+n_1, m_2+n_2} (\phi_{m_1, n_1} (x_1, y_1), \phi_{m_2, n_2} (x_2, y_2)), F(\psi) \phi_{m_3, n_3} (x_3, y_3)) \in F(((m_1+n_1)+b+b', (m_2+n_2))+_{c+c'} (m_3+n_3)) \\
 & F(\psi) \phi_{(m_1+b, m_2)+(n_1+b', n_2), m_3+n_3} (\phi_{m_1+b, m_2, n_1+b', n_2} ((F(\psi) \phi_{m_1, m_2} (x_1, x_2), F(\psi) \phi_{n_1, n_2} (y_1, y_2))), \phi_{m_3, n_3} (x_3, y_3)) \in F(((m_1+b, m_2)+(n_1+b', n_2))+_{c+c'} (m_3+n_3)) \\
 & \phi_{(m_1+b, m_2)+c, m_3, (n_1+b', n_2)+c', n_3} ((F(\psi) \phi_{m_1+b, m_2, m_3} (F(\psi) \phi_{m_1, m_2} (x_1, x_2), x_3), (F(\psi) \phi_{n_1+b', n_2, n_3} (F(\psi) \phi_{n_1, n_2} (y_1, y_2), y_3))) \in F(((m_1+b, m_2)+c, m_3)+(n_1+b', n_2)+c', n_3)) \\
 & \phi_{m_1+b, (m_2+c, m_3), n_1+b', (n_2+c', n_3)} (F(\psi) \phi_{m_1, m_2+c, m_3} ((x_1, F(\psi) \phi_{m_2, m_3} (x_2, x_3)), (y_1, F(\psi) \phi_{n_2, n_3} (y_2, y_3))) \in F((m_1+b, (m_2+c, m_3))+(n_1+b', (n_2+c', n_3)))
 \end{aligned}$$

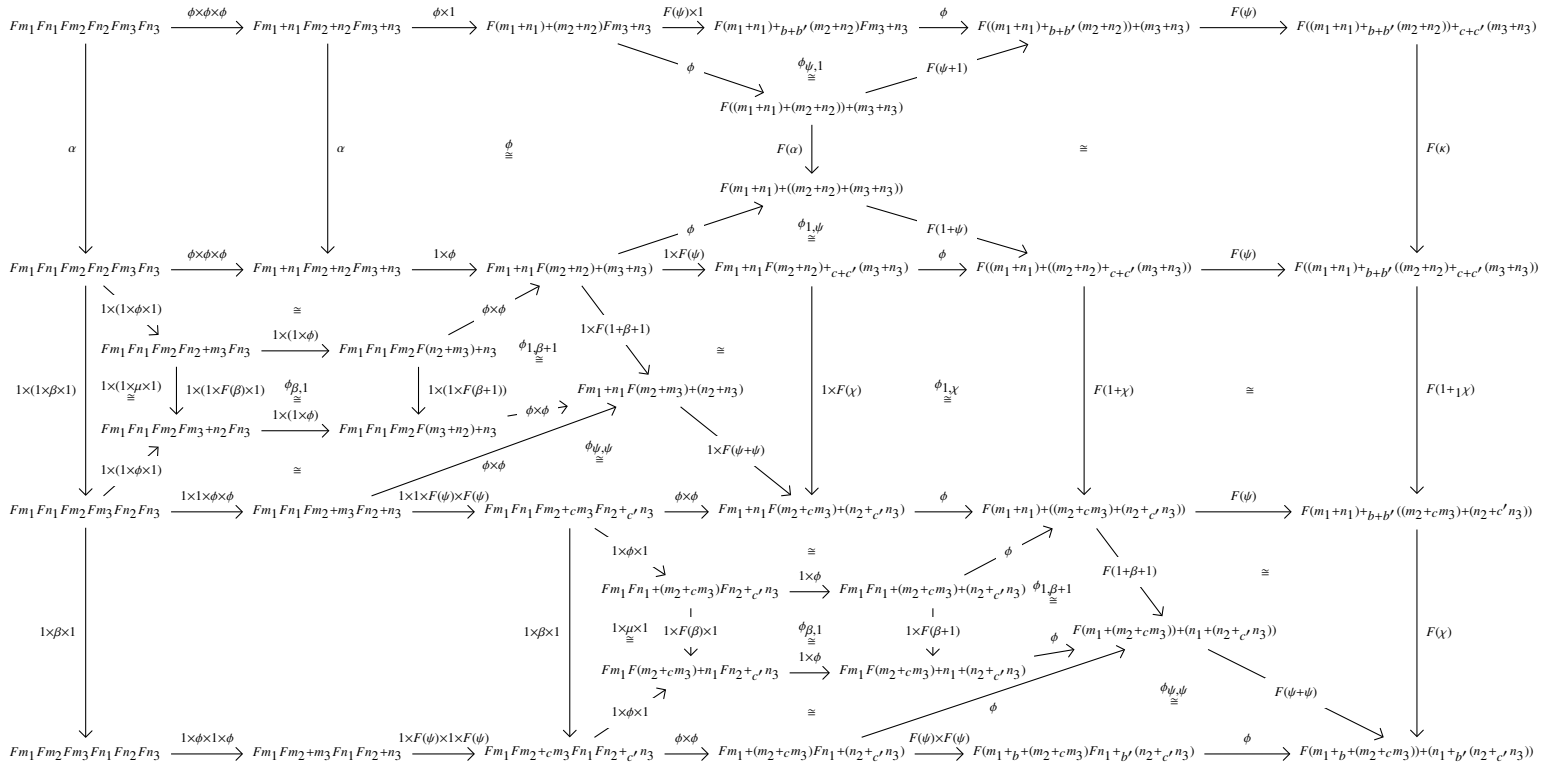
(2) Down and then right:

$$\begin{aligned}
 & F(\psi) \phi_{(m_1+n_1)+b+b', (m_2+n_2), m_3+n_3} (F(\psi) \phi_{m_1+n_1, m_2+n_2} (\phi_{m_1, n_1} (x_1, y_1), \phi_{m_2, n_2} (x_2, y_2)), F(\psi) \phi_{m_3, n_3} (x_3, y_3)) \in F(((m_1+n_1)+b+b', (m_2+n_2))+_{c+c'} (m_3+n_3)) \\
 & F(\psi) \phi_{m_1+n_1, (m_2+n_2)+c+c', (m_3+n_3)} ((\phi_{m_1, n_1} (x_1, y_1), F(\psi) \phi_{m_2+n_2, m_3+n_3} (\phi_{m_2, n_2} (x_2, y_2), \phi_{m_3, n_3} (x_3, y_3)))) \in F((m_1+n_1)+b+b', ((m_2+n_2)+c+c', (m_3+n_3))) \\
 & F(\psi) \phi_{m_1+n_1, (m_2+c, m_3)+(n_2+c', n_3)} (\phi_{m_1, n_1} (x_1, y_1), \phi_{m_2+c, m_3, n_2+c', n_3} ((F(\psi) \phi_{m_2, m_3} (x_2, x_3), F(\psi) \phi_{n_2, n_3} (y_2, y_3))) \in F((m_1+n_1)+b+b', ((m_2+c, m_3)+(n_2+c', n_3))) \\
 & \phi_{m_1+b, (m_2+c, m_3), n_1+b', (n_2+c', n_3)} (F(\psi) \phi_{m_1, m_2+c, m_3} ((x_1, F(\psi) \phi_{m_2, m_3} (x_2, x_3)), (y_1, F(\psi) \phi_{n_2, n_3} (y_2, y_3))) \in F((m_1+b, (m_2+c, m_3))+(n_1+b', (n_2+c', n_3)))
 \end{aligned}$$

Right and then down (omitting morphisms emanating out of 1 on the left due to space restrictions):



Down and then right (omitting morphisms emanating out of 1 on the left due to space restrictions):



Down and then right:

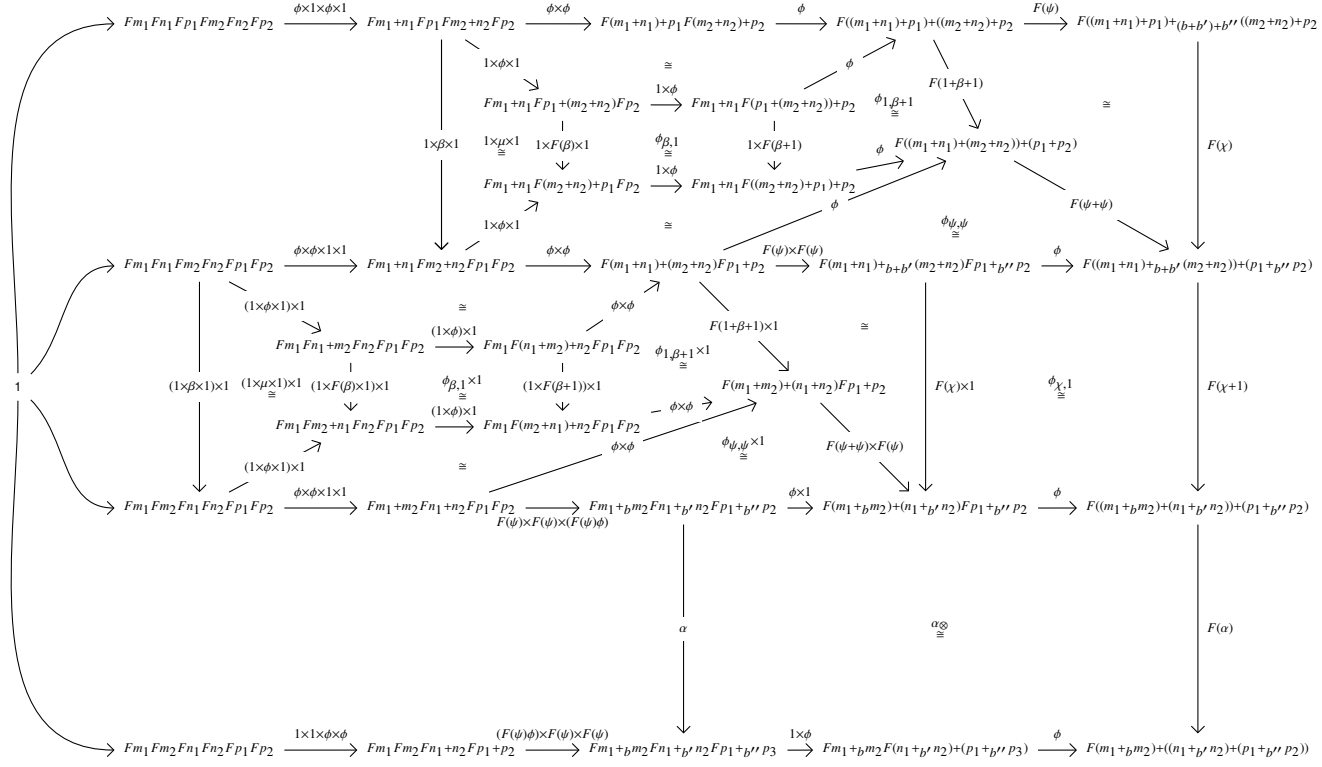


Diagram 5 of 11. Set up but not solved.

$$\begin{array}{ccccc}
 & a, b, c \in (\mathbf{A}, +, 0) & & & \\
 \perp_{(a+b)+c} \in F((a+b)+c) & U_{(a+b)+c} & \xrightarrow{\mu_\beta} & U_{a+(b+c)} & \perp_{a+(b+c)} \in F(a+(b+c)) \\
 & \downarrow \mu & & \downarrow \mu & \\
 \phi_{a+b,c}(\perp_{a+b}, \perp_c) \in F((a+b)+c) & U_{a+b} \otimes U_c & & U_a \otimes U_{b+c} & \phi_{a,b+c}(\perp_a, \perp_{b+c}) \in F(a+(b+c)) \\
 & \downarrow \mu & & \downarrow \mu & \\
 \phi_{a+b,c}(\phi_{a,b}(\perp_a, \perp_b), \perp_c) \in F((a+b)+c) & (U_a \otimes U_b) \otimes U_c & \xrightarrow{\beta'} & U_a \otimes (U_b \otimes U_c) & \phi_{a,b+c}(\perp_a, \phi_{b,c}(\perp_b, \perp_c)) \in F(a+(b+c))
 \end{array}$$

All the left cospans are $(a + b) + c \rightarrow (a + b) + c \leftarrow (a + b) + c$ and all the right cospans are $a + (b + c) \rightarrow a + (b + c) \leftarrow a + (b + c)$.

Right and then down:

$$\begin{array}{ccccc}
 & & F(!_{(a+b)+c}) & & \\
 & \nearrow \phi_0 & \xrightarrow{\quad} & \searrow F(\alpha) & \\
 & F(0) & & F((a+b)+c) & \\
 & \nearrow \phi_0 & \xrightarrow{\quad} & \searrow F(\alpha) & \\
 & F(0) & & F(a+(b+c)) & \\
 & \nearrow \phi_0 \times \phi_0 & \xrightarrow{\quad} & \searrow F(\text{id}) & \\
 & F(0) \times F(0) & \xrightarrow{F(!_a) \times F(!_{b+c})} & F(a) \times F(b+c) & \xrightarrow{\phi_{a,b+c}} & F(a+(b+c)) \\
 & \nearrow \phi_0 \times (\phi_0 \times \phi_0) & \xrightarrow{\quad} & \searrow F(\text{id}) & \\
 & F(0) \times (F(0) \times F(0)) & \xrightarrow{F(!_a) \times (F(!_b) \times F(!_c))} & F(a) \times (F(b) \times F(c)) & \xrightarrow{1 \times \phi_{b+c}} & F(a) \times F(b+c) & \xrightarrow{\phi_{a,b+c}} & F(a+(b+c))
 \end{array}$$

Down and then right:

$$\begin{array}{ccccc}
 & & F(!_{(a+b)+c}) & & \\
 & \nearrow \phi_0 & \xrightarrow{\quad} & \searrow F(\text{id}) & \\
 & F(0) & & F((a+b)+c) & \\
 & \nearrow \phi_0 & \xrightarrow{\quad} & \searrow F(\text{id}) & \\
 & F(0) & & F(a+(b+c)) & \\
 & \nearrow \phi_0^2 \times \phi_0 & \xrightarrow{\quad} & \searrow F(\alpha) & \\
 & (F(0) \times F(0)) \times F(0) & \xrightarrow{(F(!_a) \times F(!_b)) \times F(!_c)} & (F(a) \times F(b)) \times F(c) & \xrightarrow{\phi_{a,b} \times 1} & F(a+b) \times F(c) & \xrightarrow{\phi_{a+b,c}} & F(a+(b+c)) \\
 & \nearrow \phi_0 \times (\phi_0 \times \phi_0) & \xrightarrow{\quad} & \searrow F(\alpha) & \\
 & F(0) \times (F(0) \times F(0)) & \xrightarrow{F(!_a) \times (F(!_b) \times F(!_c))} & F(a) \times (F(b) \times F(c)) & \xrightarrow{1 \times \phi_{b+c}} & F(a) \times F(b+c) & \xrightarrow{\phi_{a,b+c}} & F(a+(b+c))
 \end{array}$$

Diagrams 6 and 8 of 11. Set up but not solved.

$M \in \mathbf{FCsp}$ is given by $a \xrightarrow{i} m \xleftarrow{o} b$ with $x \in F(m)$.

$N \in \mathbf{FCsp}$ is given by $b \xrightarrow{i'} n \xleftarrow{o'} c$ with $y \in F(n)$.

$U_0 \in \mathbf{FCsp}$ is given by $0 \xrightarrow{!} 0 \xleftarrow{!} 0$ with $\perp_0 \in F(0)$.

$$\begin{array}{ccc}
 \begin{array}{l} 0+a \rightarrow (0+m)+_{0+b}(0+n) \leftarrow 0+c \\ (\perp_0 \oplus y) \odot (\perp_0 \oplus x) \in F((0+m)+_{0+b}(0+n)) \end{array} & \begin{array}{c} (U_0 \otimes N) \odot (U_0 \otimes M) \xrightarrow{\chi} (U_0 \odot U_0) \otimes (N \odot M) \\ \lambda_N \odot \lambda_M \downarrow \quad \quad \quad \downarrow \lambda_{U_0} \otimes \text{id}_{N \odot M} \\ N \odot M \xleftarrow{\lambda_{N \odot M}} U_0 \otimes (N \odot M) \end{array} & \begin{array}{l} 0+a \rightarrow (0+0)+(m+b)n \leftarrow 0+c \\ (\perp_0 \odot \perp_0) \oplus (y \odot x) \in F((0+0)+(m+b)n) \end{array} \\
 \begin{array}{l} a \rightarrow m+b n \leftarrow c \\ y \odot x \in F(m+b n) \end{array} & & \begin{array}{l} 0+a \rightarrow 0+(m+b n) \leftarrow 0+c \\ \perp_0 \oplus (y \odot x) \in F(0+(m+b n)) \end{array}
 \end{array}$$

Down:

$$\begin{array}{ccccccc}
 & (F(0) \times F(m)) \times (F(0) \times F(n)) & \xrightarrow{(F(!_0) \times 1) \times (F(!_0) \times 1)} & (F(0) \times F(m)) \times (F(0) \times F(n)) & \xrightarrow{\phi_{0,m} \times \phi_{0,n}} & F(0+m) \times F(0+n) & \xrightarrow{\phi_{0+m,0+n}} F((0+m)+(0+n)) \xrightarrow{F(\psi)} F((0+m)+_{0+b}(0+n)) \\
 & \searrow^{\phi_{0,m} \times \phi_{0,n}} & & \searrow^{\phi_{!_0,1} \times \phi_{!_0,1}} & & \searrow^{\phi_{!_0,1} \times \phi_{!_0,1}} & \\
 & & & F(0+m) \times F(0+n) & \cong & F(!_0+1) \times F(!_0+1) & \\
 & & & \searrow^{\phi_{!_0,1} \times \phi_{!_0,1}} & & \searrow^{\phi_{!_0,1} \times \phi_{!_0,1}} & \\
 & & & F(\ell) \times F(\ell) & \xrightarrow{\phi_{\ell,\ell}} & F(\ell+\ell) & \cong & F(\ell+\ell) \\
 & & & \searrow^{\phi_{\ell,\ell}} & & \searrow^{\phi_{\ell,\ell}} & \\
 & & & F(m) \times F(n) & \xrightarrow{\phi_{m,n}} & F(m+n) & \xrightarrow{F(\psi)} & F(m+b n)
 \end{array}$$

Right, down and then left:

$$\begin{array}{ccccccc}
 & (F(0) \times F(m)) \times (F(0) \times F(n)) & \xrightarrow{(F(!_0) \times 1) \times (F(!_0) \times 1)} & (F(0) \times F(m)) \times (F(0) \times F(n)) & \xrightarrow{\phi_{0,m} \times \phi_{0,n}} & F(0+m) \times F(0+n) & \xrightarrow{\phi_{0+m,0+n}} F((0+m)+(0+n)) \xrightarrow{F(\psi)} F((0+m)+_{0+b}(0+n)) \\
 & \searrow^{\phi_{0,m} \times \phi_{0,n}} & & \searrow^{\phi_{!_0,1} \times \phi_{!_0,1}} & & \searrow^{\phi_{!_0,1} \times \phi_{!_0,1}} & \\
 & & & F(0+m) \times F(0+n) & \cong & F(!_0+1) \times F(!_0+1) & \\
 & & & \searrow^{\phi_{!_0,1} \times \phi_{!_0,1}} & & \searrow^{\phi_{!_0,1} \times \phi_{!_0,1}} & \\
 & & & F(\ell) \times F(\ell) & \xrightarrow{\phi_{\ell,\ell}} & F(\ell+\ell) & \cong & F(\ell+\ell) \\
 & & & \searrow^{\phi_{\ell,\ell}} & & \searrow^{\phi_{\ell,\ell}} & \\
 & & & F(m) \times F(n) & \xrightarrow{\phi_{m,n}} & F(m+n) & \xrightarrow{F(\psi)} & F(m+b n)
 \end{array}$$

(Scratch work) As a check, let us compute the decoration maps $\lambda_N : U_0 \otimes N \rightarrow N$ and $\lambda_M : U_0 \otimes M \rightarrow M$ and compose them via \odot . The maps λ_N and λ_M are given explicitly in the paper.

$$\lambda_M: U_0 \otimes M \rightarrow M$$

$$\begin{array}{ccccc}
 & F(0) \times F(m) & \xrightarrow{F(!_0) \times 1} & F(0) \times F(m) & \xrightarrow{\phi_{0,m}} & F(0+m) \\
 & \searrow \phi_{0,m} & & \nearrow \phi_{0,m} & & \nearrow F(\ell) \\
 & & \cong & & & \\
 & & F(0+m) & \xrightarrow{\phi_{!_0,1} \cong} & F(!_0+1) & \xrightarrow{\cong} & F(0+m) \\
 & & & & \searrow F(\ell) & & \searrow F(\ell) \\
 & & & & & & F(m) \\
 & & & & & & \downarrow F(\ell) \\
 & & & & & & F(m)
 \end{array}$$

$(\phi_0 \times x)$ $\xrightarrow{1}$ x $\xrightarrow{1}$

$$\lambda_N: U_0 \otimes N \rightarrow N$$

$$\begin{array}{ccccc}
 & F(0) \times F(n) & \xrightarrow{F(!_0) \times 1} & F(0) \times F(n) & \xrightarrow{\phi_{0,n}} & F(0+n) \\
 & \searrow \phi_{0,n} & & \nearrow \phi_{0,n} & & \nearrow F(\ell) \\
 & & \cong & & & \\
 & & F(0+n) & \xrightarrow{\phi_{!_0,1} \cong} & F(!_0+1) & \xrightarrow{\cong} & F(0+n) \\
 & & & & \searrow F(\ell) & & \searrow F(\ell) \\
 & & & & & & F(n) \\
 & & & & & & \downarrow F(\ell) \\
 & & & & & & F(n)
 \end{array}$$

$(\phi_0 \times y)$ $\xrightarrow{1}$ y $\xrightarrow{1}$

To construct $\lambda_N \odot \lambda_M: (U_0 \otimes N) \odot (U_0 \otimes M) \rightarrow N \odot M$, we first tensor the above two diagrams:

$$\begin{array}{ccccc}
 & (F(0) \times F(m)) \times (F(0) \times F(n)) & \xrightarrow{(F(!_0) \times 1) \times (F(!_0) \times 1)} & (F(0) \times F(m)) \times (F(0) \times F(n)) & \xrightarrow{\phi_{0,m} \times \phi_{0,n}} & F(0+m) \times F(0+n) \\
 & \searrow \phi_{0,m} \times \phi_{0,n} & & \nearrow \phi_{0,m} \times \phi_{0,n} & & \nearrow F(\ell) \times F(\ell) \\
 & & \cong & & & \\
 & & F(0+m) \times F(0+n) & \xrightarrow{\phi_{!_0,1} \times \phi_{!_0,1} \cong} & F(!_0+1) \times F(!_0+1) & \xrightarrow{\cong} & F(0+m) \times F(0+n) \\
 & & & & \searrow F(\ell) \times F(\ell) & & \searrow F(\ell) \times F(\ell) \\
 & & & & & & F(m) \times F(n) \\
 & & & & & & \downarrow F(\ell) \times F(\ell) \\
 & & & & & & F(m) \times F(n)
 \end{array}$$

$(\phi_0 \times x) \times (\phi_0 \times y)$ $\xrightarrow{1}$ $x \times y$ $\xrightarrow{1}$

Next, we paste with a square due to pseudonaturality of ϕ :

$$\begin{array}{ccccc}
 & (F(0) \times F(m)) \times (F(0) \times F(n)) & \xrightarrow{(F(!_0) \times 1) \times (F(!_0) \times 1)} & (F(0) \times F(m)) \times (F(0) \times F(n)) & \xrightarrow{\phi_{0,m} \times \phi_{0,n}} & F(0+m) \times F(0+n) & \xrightarrow{\phi_{0+m,0+n}} & F((0+m)+(0+n)) \\
 & \searrow \phi_{0,m} \times \phi_{0,n} & & \nearrow \phi_{0,m} \times \phi_{0,n} & & \nearrow F(\ell) \times F(\ell) & & \nearrow F(\ell + \ell) \\
 & & \cong & & & \\
 & & F(0+m) \times F(0+n) & \xrightarrow{\phi_{!_0,1} \times \phi_{!_0,1} \cong} & F(!_0+1) \times F(!_0+1) & \xrightarrow{\cong} & F(0+m) \times F(0+n) \\
 & & & & \searrow F(\ell) \times F(\ell) & & \searrow F(\ell) \times F(\ell) \\
 & & & & & & F(m) \times F(n) \\
 & & & & & & \downarrow \phi_{m,n} \\
 & & & & & & F(m+n)
 \end{array}$$

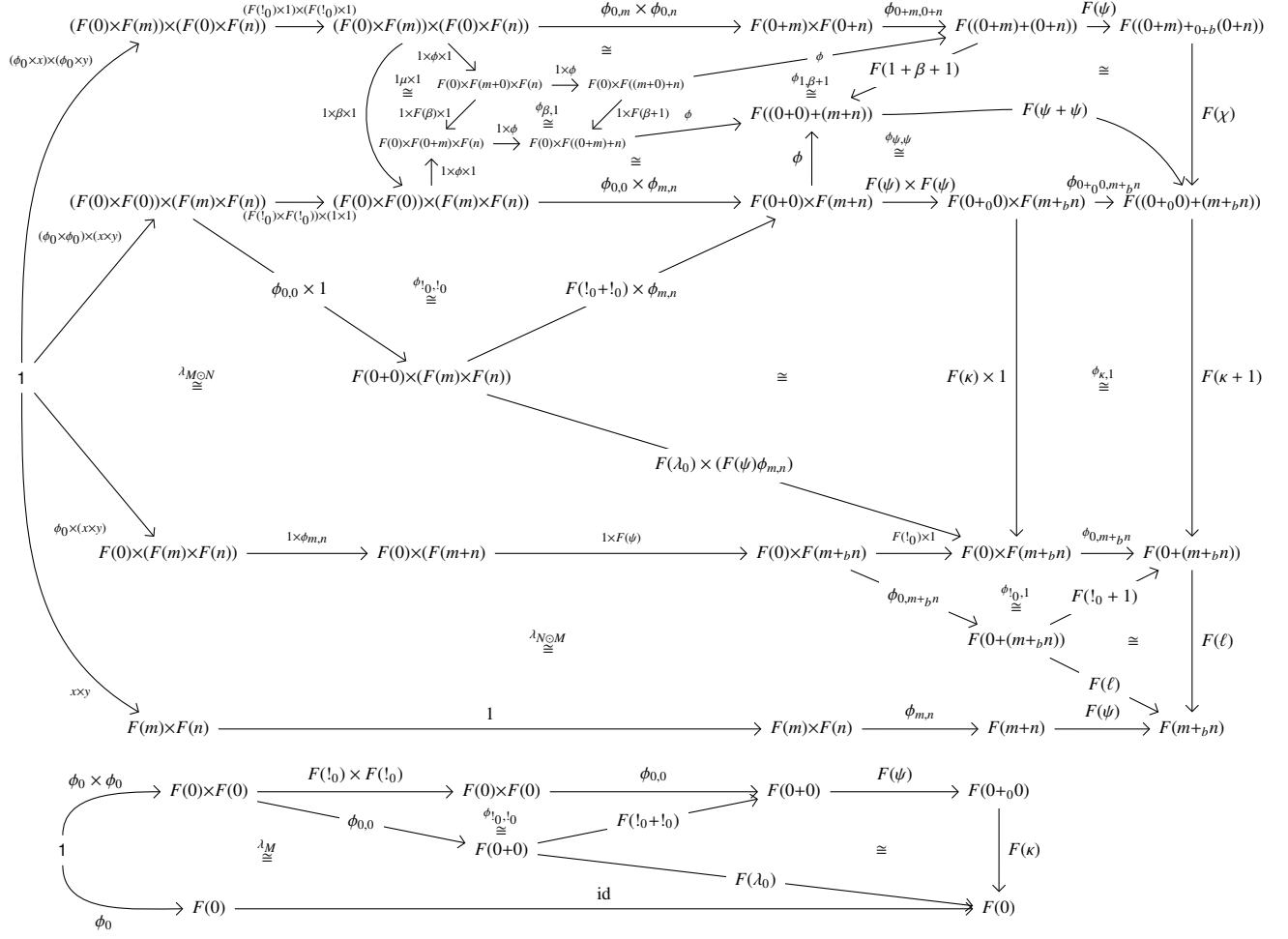
$(\phi_0 \times x) \times (\phi_0 \times y)$ $\xrightarrow{1}$ $x \times y$ $\xrightarrow{1}$

Finally, we paste with a square due to pseudonaturality of F to obtain the map $\lambda_N \odot \lambda_M: (U_0 \otimes N) \odot (U_0 \otimes M) \rightarrow N \odot M$:

$$\begin{array}{ccccc}
 & (F(0) \times F(m)) \times (F(0) \times F(n)) & \xrightarrow{(F(!_0) \times 1) \times (F(!_0) \times 1)} & (F(0) \times F(m)) \times (F(0) \times F(n)) & \xrightarrow{\phi_{0,m} \times \phi_{0,n}} & F(0+m) \times F(0+n) & \xrightarrow{\phi_{0+m,0+n}} & F((0+m)+(0+n)) & \xrightarrow{F(\psi)} & F((0+m)+_{0+b}(0+n)) \\
 & \searrow \phi_{0,m} \times \phi_{0,n} & & \nearrow \phi_{0,m} \times \phi_{0,n} & & \nearrow F(\ell) \times F(\ell) & & \nearrow F(\ell + \ell) & & \nearrow F(\ell +_{\ell} \ell) \\
 & & \cong & & & \\
 & & F(0+m) \times F(0+n) & \xrightarrow{\phi_{!_0,1} \times \phi_{!_0,1} \cong} & F(!_0+1) \times F(!_0+1) & \xrightarrow{\cong} & F(0+m) \times F(0+n) \\
 & & & & \searrow F(\ell) \times F(\ell) & & \searrow F(\ell) \times F(\ell) \\
 & & & & & & F(m) \times F(n) \\
 & & & & & & \downarrow \phi_{m,n} \\
 & & & & & & F(m+n) \\
 & & & & & & \downarrow F(\psi) \\
 & & & & & & F(m+n) \\
 & & & & & & \downarrow F(\psi) \\
 & & & & & & F(m+n) \\
 & & & & & & \downarrow F(\psi) \\
 & & & & & & F(m+n)
 \end{array}$$

$(\phi_0 \times x) \times (\phi_0 \times y)$ $\xrightarrow{1}$ $x \times y$ $\xrightarrow{1}$

Next, we compute the right, down and then left route:



Diagrams 7 and 9 of 11. Set up and solved.

$$\begin{array}{ccc}
 F: (\mathbf{A}, +, 0) \rightarrow (\mathbf{Cat}, \times, 1) & & \\
 a \in (\mathbf{A}, +, 0) & & \\
 0 + a \rightarrow 0 + a \leftarrow 0 + a & \begin{array}{ccc} U_{0+a} & \xrightarrow{\mu^{-1}} & U_0 \otimes U_a \\ & \searrow U_{\lambda_a} & \downarrow \lambda_{U_a} \\ & & U_a \end{array} & \begin{array}{l} 0 + a \rightarrow 0 + a \leftarrow 0 + a \\ \phi_{0,a}(\perp_0, \perp_a) \in F(0 + a) \\ a \rightarrow a \leftarrow a \\ \perp_a \in F(a) \end{array}
 \end{array}$$

Right and then down:

$$\begin{array}{ccccc}
 & & F(!_{0+a}) & & \\
 & & \longrightarrow & & \\
 \phi & & F(0) & \xrightarrow{\quad} & F(0+a) \\
 \cong & & \uparrow \phi_{0,0} & & \downarrow F(\text{id}) \\
 1 & \xrightarrow{\phi \times \phi} & F(0) \times F(0) & \xrightarrow{F(!_0) \times F(!_a)} & F(0) \times F(a) & \xrightarrow{\phi_{0,a}} & F(0+a) \\
 \cong & & \downarrow \phi_{0,0} & & \downarrow \phi_{0,a} & & \downarrow F(\lambda) \\
 \phi & & F(0) & \xrightarrow{\quad} & F(a) & & \\
 & & \downarrow \phi_{0,0} & & \downarrow F(!_a) & & \\
 & & F(0) & \xrightarrow{\quad} & F(a) & &
 \end{array}$$

Diagonally:

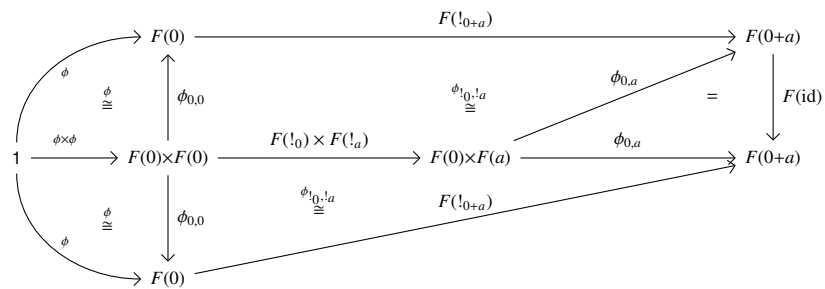
$$\begin{array}{ccccc}
 & & F(!_{0+a}) & & \\
 & & \longrightarrow & & \\
 \phi & & F(0) & \xrightarrow{\quad} & F(0+a) \\
 = & & \downarrow F(\text{id}) & & \downarrow F(\lambda) \\
 1 & \xrightarrow{\quad} & F(0) & \xrightarrow{\quad} & F(a) \\
 \phi & & \downarrow F(!_a) & & \\
 & & F(0) & \xrightarrow{\quad} & F(a) & &
 \end{array}$$

Removing the lower right \cong which is the same in each diagram:

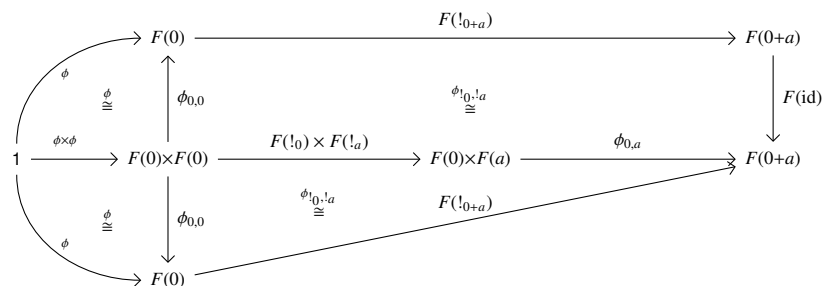
$$\begin{array}{ccccc}
 & & F(!_{0+a}) & & \\
 & & \longrightarrow & & \\
 \phi & & F(0) & \xrightarrow{\quad} & F(0+a) \\
 \cong & & \uparrow \phi_{0,0} & & \downarrow F(\text{id}) \\
 1 & \xrightarrow{\phi \times \phi} & F(0) \times F(0) & \xrightarrow{F(!_0) \times F(!_a)} & F(0) \times F(a) & \xrightarrow{\phi_{0,a}} & F(0+a) \\
 \cong & & \downarrow \phi_{0,0} & & \downarrow \phi_{0,a} & & \downarrow F(\lambda) \\
 \phi & & F(0) & \xrightarrow{\quad} & F(a) & & \\
 & & \downarrow \phi_{0,0} & & \downarrow F(!_a) & & \\
 & & F(0) & \xrightarrow{\quad} & F(a) & &
 \end{array}$$

$$\begin{array}{ccccc}
 & & F(!_{0+a}) & & \\
 & & \longrightarrow & & \\
 \phi & & F(0) & \xrightarrow{\quad} & F(0+a) \\
 = & & \downarrow F(\text{id}) & & \downarrow F(\lambda) \\
 1 & \xrightarrow{\quad} & F(0) & \xrightarrow{\quad} & F(a) \\
 \phi & & \downarrow F(!_a) & & \\
 & & F(0) & \xrightarrow{\quad} & F(a) & &
 \end{array}$$

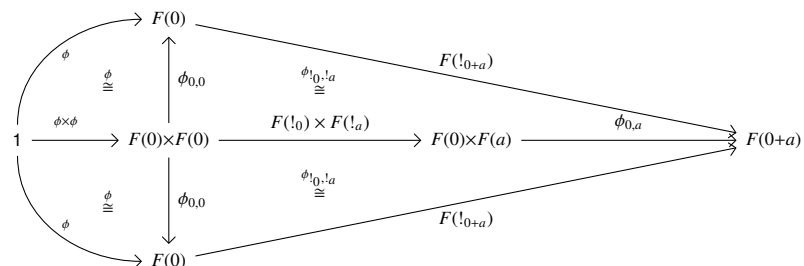
The second diagram is then an identity, so our problem reduces to showing that the following diagram is also an identity:



Removing the diagonal $\phi_{0,a}$:



This diagram is clearly the same as:



The two 2-isomorphisms in the top half of the diagram are the inverses of those in the bottom half, when read in a suitable order, and they can be shown to cancel, yielding an identity as desired.

Diagram 10 of 11. Set up and solved (Christina).

$M_1 \in F\mathbf{Csp}$ is given by $a \xrightarrow{i_1} m_1 \xleftarrow{o_1} b$ with $x_1 \in F(m_1)$.

$M_2 \in F\mathbf{Csp}$ is given by $b \xrightarrow{i_2} m_2 \xleftarrow{o_2} c$ with $x_2 \in F(m_2)$.

$N_1 \in F\mathbf{Csp}$ is given by $a' \xrightarrow{i'_1} n_1 \xleftarrow{o'_1} b'$ with $y_1 \in F(n_1)$.

$N_2 \in F\mathbf{Csp}$ is given by $b' \xrightarrow{i'_2} n_2 \xleftarrow{o'_2} c'$ with $x_2 \in F(n_2)$.

$$\begin{array}{ccc}
 a+a' \rightarrow (m_1+n_1)+_{b+b'}(m_2+n_2) \leftarrow c+c' & (M_2 \otimes N_2) \odot (M_1 \otimes N_1) \xrightarrow{\beta \odot \beta} (N_2 \otimes M_2) \odot (N_1 \otimes M_1) & a'+a \rightarrow (n_1+m_1)+_{b'+b}(n_2+m_2) \leftarrow c'+c \\
 F(\psi)\phi_{m_1+n_1, m_2+n_2}(\phi_{m_1, n_1}(x_1, y_1), \phi_{m_2, n_2}(x_2, y_2)) \in F((m_1+n_1)+_{b+b'}(n_1+n_2)) & \downarrow \chi & F(\psi)\phi_{n_1+m_1, n_2+m_2}(\phi_{n_1, m_1}(y_1, x_1), \phi_{n_2, m_2}(y_2, x_2)) \in F((n_1+m_1)+_{b'+b'}(n_1+m_2)) \\
 & (M_2 \odot M_1) \otimes (N_2 \odot N_1) \xrightarrow{\beta} (N_2 \odot N_1) \otimes (M_2 \odot M_1) & \\
 a+a' \rightarrow (m_1+b, m_2)+(n_1+b', n_2) \leftarrow c+c' & & a'+a \rightarrow (n_1+b', n_2)+(m_1+b, m_2) \leftarrow c'+c \\
 \phi_{m_1+b, m_2, n_1+b', n_2}(F(\psi)\phi_{m_1, m_2}(x_1, x_2), F(\psi)\phi_{n_1, n_2}(y_1, y_2)) \in F((m_1+b, m_2)+(n_1+b', n_2)) & & \phi_{n_1+b', n_2}(F(\psi)\phi_{n_1, n_2}(y_1, y_2), F(\psi)\phi_{m_1, m_2}(x_1, x_2)) \in F((n_1+b', n_2)+(m_1+b, m_2))
 \end{array}$$

Down and then right:

$$\begin{array}{c}
 \begin{array}{c}
 (x_1 \times y_1) \times (x_2 \times y_2) \\
 \downarrow 1 \times \beta \times 1 \\
 (x_1 \times x_2) \times (y_1 \times y_2) \\
 \downarrow \beta \\
 y_1 \times y_2 \times x_1 \times x_2
 \end{array}
 \end{array}$$

Diagram illustrating the "Down and then right" path in the proof. The diagram shows a complex network of nodes and arrows representing the derivation of the final result. The nodes are labeled with expressions involving F , ϕ , ψ , χ , and β . The arrows are labeled with $1 \times \phi \times 1$, $1 \times \mu \times 1$, $1 \times \phi \times 1$, ϕ , $\phi_{\beta, 1}$, $1 \times \phi$, ϕ_{m_1, n_1} , ϕ_{m_2, n_2} , $\phi_{m_1+n_1, m_2+n_2}$, $F(\psi)$, $F(\chi)$, $F(\beta)$, $\mu_{m_1, n_1} \times \mu_{m_2, n_2}$, $\phi_{n_1, n_2} \times \phi_{m_1, m_2}$, $F(\psi) \times F(\psi)$, $\phi_{n_1+b', n_2, m_1+b, m_2}$, and $\phi_{n_1, n_2} \times \phi_{m_1, m_2}$.

Right and then down:

$$\begin{array}{c}
 \begin{array}{c}
 (x_1 \times y_1) \times (x_2 \times y_2) \\
 \downarrow \beta \times \beta \\
 (y_1 \times x_1) \times (y_2 \times x_2) \\
 \downarrow 1 \times \beta \times 1 \\
 y_1 \times y_2 \times x_1 \times x_2
 \end{array}
 \end{array}$$

Diagram illustrating the "Right and then down" path in the proof. The diagram shows a complex network of nodes and arrows representing the derivation of the final result. The nodes are labeled with expressions involving F , ϕ , ψ , χ , and β . The arrows are labeled with $1 \times \phi \times 1$, $1 \times \mu \times 1$, $1 \times \phi \times 1$, ϕ , $\phi_{\beta, 1}$, $1 \times \phi$, ϕ_{m_1, n_1} , ϕ_{m_2, n_2} , $\phi_{m_1+n_1, m_2+n_2}$, $F(\psi)$, $F(\chi)$, $F(\beta)$, $\mu_{m_1, n_1} \times \mu_{m_2, n_2}$, $\phi_{n_1, n_2} \times \phi_{m_1, m_2}$, $F(\psi) \times F(\psi)$, $\phi_{n_1+b', n_2, m_1+b, m_2}$, and $\phi_{n_1, n_2} \times \phi_{m_1, m_2}$.

Diagram 11 of 11. Set up but not solved.

$$a, b \in (A, +, 0)$$

$$\begin{array}{ccc} U_{a+b} & \xrightarrow{\mu_{a,b}} & U_a \otimes U_b \\ U_\beta \downarrow & & \downarrow \beta' \\ U_{b+a} & \xrightarrow{\mu_{b,a}} & U_b \otimes U_a \end{array}$$

The top two underlying cospans are

$$a + b \rightarrow a + b \leftarrow a + b$$

and the bottom two underlying cospans are

$$b + a \rightarrow b + a \leftarrow b + a$$

Right and then down:

$$\begin{array}{ccccccc} & & F(0) & \xrightarrow{F(!_{a+b})} & F(a+b) & & \\ & \nearrow \phi & \uparrow \phi_{0,0} & & \nearrow \phi_{a,b} & & \\ 1 & \xrightarrow{\phi \times \phi} & F(0) \times F(0) & \xrightarrow{F(!_a) \times F(!_b)} & F(a) \times F(b) & \xrightarrow{\phi_{a,b}} & F(a+b) \\ & \searrow \phi \times \phi & \downarrow \beta & & \downarrow \beta & & \downarrow F(\text{id}) \\ & & F(0) \times F(0) & \xrightarrow{F(!_b) \times F(!_a)} & F(b) \times F(a) & \xrightarrow{\phi_{b,a}} & F(b+a) \end{array}$$

$\phi_{!_a, !_b} \cong \mu_{a,b} \cong \phi_{b,a}$

Down and then right:

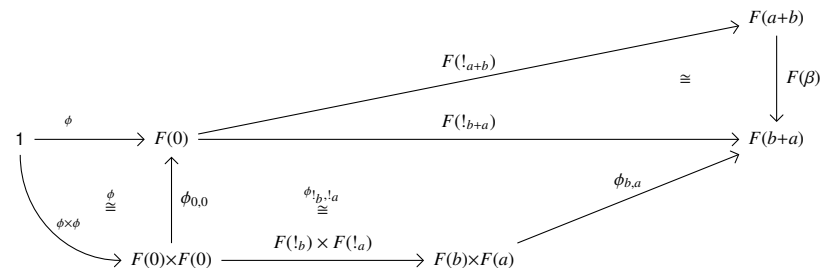
$$\begin{array}{ccccccc} & & F(0) & \xrightarrow{F(!_{a+b})} & F(a+b) & & \\ & \nearrow \phi & \downarrow \text{id} & & \downarrow F(\beta) & & \\ 1 & \xrightarrow{\phi} & F(0) & \xrightarrow{F(!_{b+a})} & F(b+a) & & \\ & \searrow \phi \times \phi & \uparrow \phi_{0,0} & & \nearrow \phi_{b,a} & & \\ & & F(0) \times F(0) & \xrightarrow{F(!_b) \times F(!_a)} & F(b) \times F(a) & \xrightarrow{\phi_{b,a}} & F(b+a) \end{array}$$

$\phi_{!_b, !_a} \cong \mu_{b,a} \cong \phi_{b,a}$

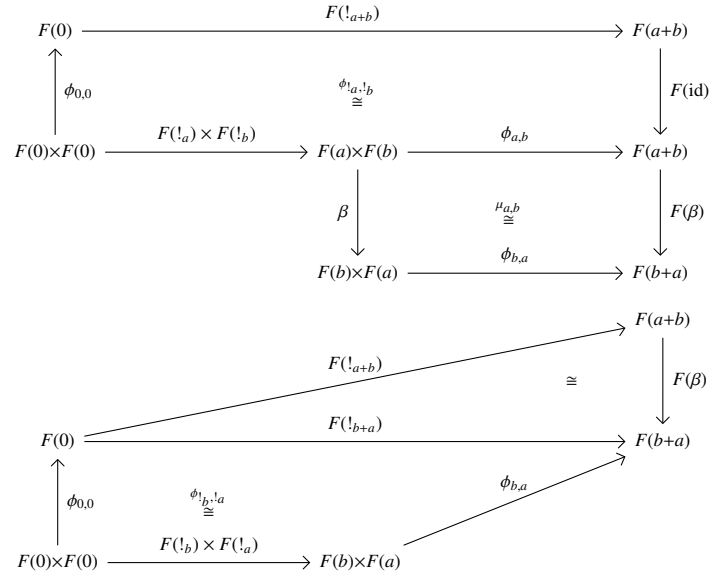
First we remove the lower left commuting quarter circle in the ‘right and then down’ diagram as well as the upper left commuting quarter circle in the ‘down and then right’ diagram:

$$\begin{array}{ccccccc} & & F(0) & \xrightarrow{F(!_{a+b})} & F(a+b) & & \\ & \nearrow \phi & \uparrow \phi_{0,0} & & \nearrow \phi_{a,b} & & \\ 1 & \xrightarrow{\phi \times \phi} & F(0) \times F(0) & \xrightarrow{F(!_a) \times F(!_b)} & F(a) \times F(b) & \xrightarrow{\phi_{a,b}} & F(a+b) \\ & \searrow \phi \times \phi & \downarrow \beta & & \downarrow \beta & & \downarrow F(\text{id}) \\ & & F(0) \times F(0) & \xrightarrow{F(!_b) \times F(!_a)} & F(b) \times F(a) & \xrightarrow{\phi_{b,a}} & F(b+a) \end{array}$$

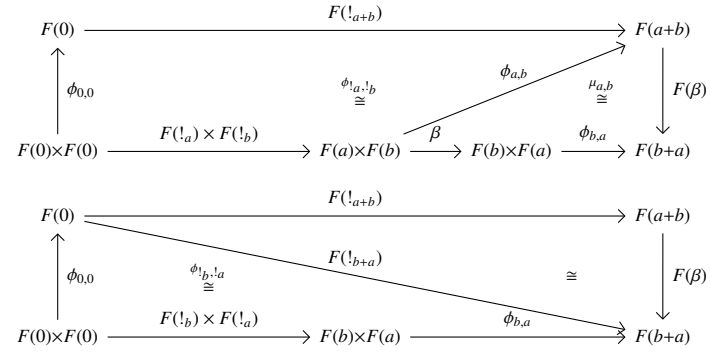
$\phi_{!_a, !_b} \cong \mu_{a,b} \cong \phi_{b,a}$



Next we remove the left quarter circle containing \cong from each diagram:



Making each diagram into a rectangle and removing the $F(\text{id})$ from the first diagram:



Are these two diagrams the same?

Some useful maps

Given $a \in (\mathbf{A}, +, 0)$, the map $U_{\lambda_a}: U_{0+a} \rightarrow U_a$ is given by:

$$\begin{array}{ccccc}
 \phi & \xrightarrow{\quad} & F(0) & \xrightarrow{F(!_{0+a})} & F(0+a) \\
 \downarrow 1 & & \downarrow F(\text{id}) & \searrow F(!_{0+a}) & \downarrow F(\lambda_a) \\
 \phi & \xrightarrow{\quad} & F(0) & \xrightarrow{F(!_a)} & F(a)
 \end{array}
 \quad \cong$$

where the \cong is given by pseudonaturality of F : we have a unique map in $!_a: 0 \rightarrow a$ in \mathbf{A} but also a map $\lambda_a \circ !_a: 0 \rightarrow a$ where λ_a is the left unitor of $(\mathbf{A}, +, 0)$, and so $F(!_a) = F(\lambda_a \circ !_a) \cong F(\lambda_a)F(!_a)$.

The left unitor $\lambda'_{U_a}: U_0 \otimes U_a \rightarrow U_a$ is given by:

$$\begin{array}{ccccc}
 \phi \times \phi & \xrightarrow{\quad} & F(0) \times F(0) & \xrightarrow{F(!_0) \times F(!_a)} & F(0) \times F(a) & \xrightarrow{\phi_{0,a}} & F(0+a) \\
 \downarrow 1 & & \downarrow \phi_{0,0} & \searrow \phi_{!_0, !_a} & \searrow F(!_a) & \searrow F(!_a) & \downarrow F(\lambda_a) \\
 \phi & \xrightarrow{\quad} & F(0) & \xrightarrow{F(!_a)} & F(a)
 \end{array}
 \quad \cong$$

where the \cong in the lower right is the same as the one in the first diagram.

For an arbitrary M , the left unitor $\lambda'_M: U_0 \otimes M \rightarrow M$ is given by:

$$\begin{array}{ccccc}
 \phi_0 \times x & \xrightarrow{\quad} & F(0) \times F(m) & \xrightarrow{F(!_0) \times 1} & F(0) \times F(m) & \xrightarrow{\phi_{0,m}} & F(0+m) \\
 \downarrow 1 & & \downarrow \phi_{0,m} & \searrow \phi_{!_0, !_1} & \searrow F(!_0 + 1) & \searrow F(\lambda_m) & \downarrow F(\lambda_m) \\
 x & \xrightarrow{\quad} & F(m) & \xrightarrow{\text{id}} & F(m)
 \end{array}
 \quad \cong$$

For an arbitrary M given by $a \rightarrow (m, x) \leftarrow b$, the map $\lambda_M: U_b \odot M \rightarrow M$ is given by:

$$\begin{array}{ccccccc}
 x \times \phi_0 & \xrightarrow{\quad} & F(m) \times F(0) & \xrightarrow{1 \times F(!_b)} & F(m) \times F(b) & \xrightarrow{\phi_{m,b}} & F(m+b) & \xrightarrow{F(\psi)} & F(m+_b b) \\
 \downarrow 1 & & \downarrow \phi_{m,0} & \searrow \phi_{!_1, !_b} & \searrow F(1+_b) & \searrow F(\lambda_m) & \searrow F(\lambda_m) & \searrow F(\kappa) & \downarrow F(\kappa) \\
 x & \xrightarrow{\quad} & F(m) & \xrightarrow{\text{id}} & F(m) & \xrightarrow{\quad} & F(m)
 \end{array}
 \quad \cong$$

In particular, if $M = U_0$ above, then the map $\lambda_{U_0}: U_0 \odot U_0 \rightarrow U_0$ is given by:

$$\begin{array}{ccccccc}
 \phi_0 \times \phi_0 & \xrightarrow{\quad} & F(0) \times F(0) & \xrightarrow{F(!_0) \times F(!_0)} & F(0) \times F(0) & \xrightarrow{\phi_{0,0}} & F(0+0) & \xrightarrow{F(\psi)} & F(0+_0 0) \\
 \downarrow 1 & & \downarrow \phi_{0,0} & \searrow \phi_{!_0, !_0} & \searrow F(!_0 + !_0) & \searrow F(\lambda_0) & \searrow F(\lambda_0) & \searrow F(\kappa) & \downarrow F(\kappa) \\
 \phi_0 & \xrightarrow{\quad} & F(0) & \xrightarrow{\text{id}} & F(0) & \xrightarrow{\quad} & F(0)
 \end{array}
 \quad \cong$$