$$\begin{tikzcd} E(m)\otimes E(n) \arrow[r, "b"] \arrow[d, "\mu"'] \arrow[dr, phantom, "?"] & E(n)\otimes E(m) \arrow[d, "\mu"] \\ E(m\otimes n) \arrow[r, "E(b)"'] & E(n\otimes m) \end{tikzcd}$$

$$E : {}_L\mathbb{C}sp(sF) \longrightarrow F\mathbb{C}sp$$

$$(a,\iota_a)\to(m,x)\leftarrow(b,\iota_b) \longmapsto (a\to m\leftarrow b,\ x\in F(m))$$

$\boxed{\mu\circ b}$

$$a+a' \xrightarrow{\iota+\iota'} m+n \xleftarrow{o+o'} b+b', \quad 1 \xrightarrow{xy} FmFn \xrightarrow{\phi_{m,n}} F(m+n)$$

$\downarrow b$ (statement of 2.2)

$$a'+a \xrightarrow[\iota'+\iota]{} n+m \xleftarrow[o'+o]{} b'+b, \quad 1 \xrightarrow{yx} FnFm \xrightarrow[\phi_{n,m}]{} F(n+m)$$

$\downarrow \mu$ (p.14)

$$a'+a \xrightarrow[\iota'+\iota]{} n+m \xleftarrow[o'+o]{} b'+b, \quad 1 \xrightarrow{yx} FnFm \xrightarrow[F(\iota'_n)F(\iota'_m)]{} F(n+m)F(n+m) \xrightarrow[\phi_{n+m,n+m}]{} F(n+m+n+m) \xrightarrow{F\nabla} F(n+m)$$

This map between decorated cospans consists of

$$\begin{tikzcd} a+a' \arrow[r] \arrow[d] & m+n \arrow[d, "b"] & b+b' \arrow[l] \arrow[d] \\ a'+a \arrow[r] \arrow[d, equal] & n+m \arrow[d, equal] & b'+b \arrow[l] \arrow[d, equal] \\ a'+a \arrow[r] & n+m & b'+b \arrow[l] \end{tikzcd}$$

on cospans, and of

$$\begin{tikzcd} 1 \arrow[r, "xy"] \arrow[dr, "yx"'] & FmFn \arrow[r, "\phi_{m,n}"] \arrow[d, "b"] \arrow[dr, phantom, "u_{m,n}\cong"] & F(m+n) \arrow[d, "F(b)"] \\ & FnFm \arrow[r, "\phi_{n,m}"'] \arrow[d, "F(\iota'_n)F(\iota'_m)"] \arrow[dr, phantom, "\phi_{\iota'_n,\iota'_m}\cong"] & F(n+m) \arrow[d, "F(\iota'_n+\iota'_m)"] \arrow[dr, equal, "\cong"] \\ & F(n+m)F(n+m) \arrow[r, "\phi_{n+m,n+m}"'] & F(n+m+n+m) \arrow[r, "F\nabla"'] & F(n+m) \end{tikzcd}$$

$\bigstar$

on decorations.

$\boxed{E(b)\circ\mu}$

$$a+a' \longrightarrow m+m' \longleftarrow b+b', \quad 1 \xrightarrow{xy} FmFn \xrightarrow{\phi_{m,n}} F(m+n)$$

$\downarrow \mu$ (p.14)

$$a+a' \longrightarrow m+m' \longleftarrow b+b', \quad 1 \xrightarrow[xy]{} FmFn \xrightarrow[F(\iota_m)F(\iota_n)]{} F(m+n)F(m+n) \xrightarrow[\phi_{m+n,m+n}]{} F(m+n+m+n) \xrightarrow{F\nabla} F(m+n)$$

$\downarrow E(b)$

$$a'+a \longrightarrow m'+m \longleftarrow b'+b, \quad 1 \xrightarrow[yx]{} FnFm \xrightarrow[F(\iota'_n)F(\iota'_m)]{} F(n+m)F(n+m) \xrightarrow[\phi_{n+m,n+m}]{} F(n+m+n+m) \xrightarrow{F\nabla} F(n+m)$$

which is the same map of cospans as above, and on decorations

$$
\begin{tikzcd}
1 \arrow[r, "xy"] \arrow[dd, bend right, "yx"'] & F_m F_n \arrow[r, "\phi_{m,n}"] \arrow[d, "F(\iota_m)F(\iota_n)"'] & F(m+n) \arrow[d, "\phi_{\iota_m,\iota_n}\cong F(\iota_m+\iota_n)"] \arrow[rr, equal, "\cong"] & & F(m+n) \arrow[dd, "F(b)"] \\
& F(m+n)F(m+n) \arrow[r, "\phi_{m+n,m+n}"'] \arrow[d, phantom, "\circledast"] & F(m+n+m+n) \arrow[rru, "F(\nabla)"'] & & \odot \\
& F_n F_m \arrow[r, "F(\iota'_n)F(\iota'_m)"'] & F(n+m)F(n+m) \arrow[r, "\phi_{n+m,n+m}"'] & F(n+m+n+m) \arrow[r, "F\nabla"'] & F(n+m)
\end{tikzcd}
$$

Where the $\circledast$ piece is the "canonical isomorphism" between the two sums, namely $F(\iota_m)x + F(\iota_n)y$ in $F(m+n)$
$F(\iota'_n)y + F(\iota'_m)x$ in $F(n+m)$

via $(m+n, F(\iota_m)x + F(\iota_n)y) \overset{\text{can}}{\cong} (n+m, F(\iota'_n)y + F(\iota'_m)x)$ in $\int F$

Where the stress notation corresponds to

$$
\begin{tikzcd}
m+n \arrow[r, "b"] & n+m \\
m \arrow[u, "\iota_m"] \arrow[ur, bend right=60, "\iota'_m"'] & n \arrow[ul, "\iota_n"] \arrow[u, "\iota'_n"']
\end{tikzcd}
$$

namely $\left.\begin{aligned} b\circ\iota_m &= \iota'_m \\ b\circ\iota_n &= \iota'_n \end{aligned}\right\}$ (1)

$\boxed{Q1}$ I can fill $\circledast$ with some naturally selected 2-cells, in particular doing so allows me to prove the desired axiom. How do I know that IT IS THE CANONICAL ISO? As said before, I don't see how ~~to write~~ actually the inclusions, to see if it commutes with them. Just because it is a reasonable choice? I'm missing something here.

$\boxed{Q2}$ I was hoping to get this again from universal properties. The target decoration is part of a sum in the Grothendieck category $\int F$, namely $(n+m, F(\iota'_n)y + F(\iota'_m)x)$. Can we convince ourselves that both $y\circ b$, $F(b)\circ y$ are THE CANONICAL isos that commute with some corresponding inclusions? Could I see such an argument in detail, if there is one?

(NOTE) In the proof $E(m)\otimes E(m')\otimes E(m'') \rightrightarrows E(m\otimes m'\otimes m'')$ we KNEW that the intermediate maps were canonical, or con⊗l. Here we know so for γ's, but not sure about b's. I HAVE A FEELING IT SHOULD WORK???

$$\text{(✻)}\quad
\begin{tikzcd}[column sep=small]
F_mF_n \arrow[r, "F(1_m)F(1_n)"] \arrow[dr, "F(1_m')F(1_n')"'] \arrow[dd] & F(m+n)F(m+n) \arrow[r, "\phi_{m+n,m+n}"] \arrow[d, "F(b)F(b)"] \arrow[dr, phantom, "\cong \phi_{b,b}"] & F(m+n+m+n) \arrow[r, "F\nabla"] \arrow[d, "F(b+b)"] \arrow[dr, phantom, "\cong"] & F(m+n) \arrow[d, "F(b)"'] \arrow[dd, bend left, "F(b)"] \\
\arrow[r, phantom, "\cong ①[F \text{ is a pseudofun}]"] & F(n+m)F(n+m) \arrow[r, "\phi_{n+m,n+m}"'] \arrow[d, "b"] \arrow[dr, phantom, "u_{n+m,n+m}\ \cong"] & F(n+m+n+m) \arrow[r, "F\nabla"'] \arrow[d, "F(b)"] \arrow[dr, phantom, "\cong"] & F(n+m) \arrow[d, equal] \\
F_nF_m \arrow[r, "F(1_n')F(1_m')"'] & F(n+m)F(n+m) \arrow[r, "\phi_{n+m,n+m}"'] & F(n+m+n+m) \arrow[r, "F\nabla"'] & F(n+m)
\end{tikzcd}$$

$1 \xrightarrow{\gamma_y} b$, $2$, $yx$, $2$

where two isos on the right are by pseudofun of $F$, naturality of $\nabla$, univ. props.

We want to show that with the above filler, ✩ $=$ ●. We start from ✩ and we use two properties of a braided lax monoidal pseudofunctor.

$\boxed{\text{u is a modification}}$

$$\begin{tikzcd}[column sep=small]
F_mF_n \arrow[r, "F(1_m')F(1_n')"] \arrow[d, "\phi_{m,n}"'] \arrow[dr, "b"] & F(n+m)F(n+m) \arrow[d, "b"] \\
F(m+n) \arrow[r, phantom, "u_{m,n}"] \arrow[d, "F(b)"'] & F_nF_m \arrow[r, "F(1_n')F(1_m')"] \arrow[dl, "\phi_{n,m}"] & F(n+m)F(n+m) \arrow[d, "\phi_{n+m,n+m}"] \\
F(n+m) \arrow[rr, "F(1_n'+1_m')"'] & \phi_{1_n',1_m'} & F(n+m+n+m)
\end{tikzcd}
\quad 2\ \ b \text{ IS NATURAL IN CAT}$$

$$=\quad
\begin{tikzcd}[column sep=small]
F_mF_n \arrow[r, "F(1_m')F(1_n')"] \arrow[d, "\phi_{m,n}"'] & F(n+m)F(n+m) \arrow[d, "\phi_{n+m,n+m}"'] \arrow[dr, "b"] & \\
F(m+n) \arrow[r, "F(1_m'+1_n')"] \arrow[d, "F(b)"'] & F(n+m+n+m) \arrow[dr, "F(b)"] & F(n+m) \arrow[d, "\phi_{n+m,n+m}"] \\
F(n+m) \arrow[rr, "F(1_n'+1_m')"'] & & F(n+m+n+m)
\end{tikzcd}$$

$\phi_{1_m',1_n'}$, $u_{n+m,n+m}$, $2$, $\cong$ [F applied to nat. b]

You already see that $\swarrow \searrow \longrightarrow$ $u_{m,n}$ $\phi_{1_n',1_m'}$ exists as is in ✩!

So we will replace that by the RHS

and notice we can just plug the top equality of LHS in ✩ if we want....

$$\text{✩} =
\begin{tikzcd}[column sep=small]
F_mF_n \arrow[r, "\phi_{m,n}"] \arrow[d, "F(1_m')F(1_n')"'] \arrow[dr, phantom, "\phi_{1_m',1_n'}"] & F(m+n) \arrow[r, "F(b)"] \arrow[d, "F(1_m'+1_n')"] \arrow[dr, phantom, "\cong [F \text{ pseudo}]"] & F(n+m) \arrow[d, "F(1_n'+1_m')"] \arrow[dd, bend left, phantom, "\cong"] \\
F(n+m)F(n+m) \arrow[r, "\phi_{n+m,n+m}"] \arrow[d, "b"] & F(n+m+n+m) \arrow[r, "F(B)"] & F(n+m+n+m) \arrow[d, "F\nabla"] \\
F(n+m)F(n+m) \arrow[urr, "\phi_{n+m,n+m}"'] & u_{n+m,n+m} & F(n+m)
\end{tikzcd}$$

We are very close. We wish to make some $l_m, l_n$ appear instead of those $l_m', l_n'$ we currently have. We will do so, using ① and pseudonaturality of $\phi_{f,g}$.

$\boxed{\phi \text{ is pseudonatural}}$

$$\begin{tikzcd}
F_mF_n \arrow[r, "F(l_m')F(l_n')"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\phi_{l_m',l_n'}"] & F(n+m)F(n+m) \arrow[d, "\phi"] \\
F(m+n) \arrow[r, "F(l_m'+l_n')"'] & F(n+m+n+m)
\end{tikzcd}
=
\begin{tikzcd}
F_mF_n \arrow[r, "F(l_m)F(l_n)"] \arrow[rr, bend left, "F(l_m')F(l_n')"] \arrow[d, "\phi_{m,n}"'] \arrow[dr, phantom, "\phi_{l_m,l_n}"] & F(m+n)F(m+n) \arrow[r, "F(b)F(b)"] \arrow[d, "\phi_{m+n,m+n}"] \arrow[u, phantom, "\cong \text{ pseudofun}"] \arrow[dr, phantom, "\phi_{b,b}"] & F(n+m)F(n+m) \arrow[d, "\phi_{n+m,n+m}"] \\
F(m+n) \arrow[r, "F(l_m+l_n)"'] \arrow[rr, bend right, "F(l_m'+l_n')"'] & F(m+n+m+n) \arrow[r, "F(b+b)"'] \arrow[d, phantom, "\cong"] & F(n+m+n+m)
\end{tikzcd}$$

So now we replace the red part, and we are done!

$$\begin{tikzcd}
\star = F_mF_n \arrow[rrr, "\phi"] \arrow[dr, "F(l_m)F(l_n)"] \arrow[dd, "F(l_m')(l_n')"'] & & & F(m+n) \arrow[rr, "Fb"] \arrow[d, "F(l_m+l_n)"] \arrow[dd, bend left=60, "F(l_m'+l_n')"] & & F(n+m) \arrow[ddd, equal] \\
\cong & F(m+n)F(m+n) \arrow[rr, "\phi"'] \arrow[dl, "FbFb"] \arrow[rr, phantom, "\phi_{l_m,l_n}"' near start] & & F(m+n+m+n) \arrow[d, "F(b+b)"] \arrow[r, phantom, "\cong"] & \cong & \\
F(n+m)F(n+m) \arrow[rrr, "\phi_{n+m,n+m}"'] \arrow[drr, "b"'] & \phi_{b,b} & & F(n+m+n+m) \arrow[dr, "F(b)"] & & \\
& & F(n+m)F(n+m) \arrow[rr, "\phi_{n+m,n+m}"'] \arrow[u, phantom, "u_{n+m,n+m}"] & & F(n+m+n+m) \arrow[r, "F\nabla"'] & F(n+m)
\end{tikzcd}$$

where the right hand-side iso (two isos composed basically) is the same as the iso on the very right of ⊙ [the pseudofunctor F applied to equal sums].

$\square$.