7/1/2021

$\boxed{1}$ Lax monoidal pseudofunctor $(F, \phi, \phi_0) : (A, +) \longrightarrow (\mathrm{Cat}, \times)$

$\Updownarrow$

$\boxed{2}$ Monoidal opfibration $U : (X, \otimes) \longrightarrow (A, +)$

$\Updownarrow$

$\boxed{3}$ Pseudofunctor $F : A \longrightarrow \mathrm{MonCat}$

ALSO IN SHULMAN

Let's focus on $\boxed{2} \Leftrightarrow \boxed{3}$ for starters. The equivalence is formed as follows

$$U : (X, \otimes) \longrightarrow (A, +) \longmapsto F : A \longrightarrow \mathrm{MonCat}$$

global

fiber

$$a \longmapsto Fa := X_a \text{ with mon str}$$

$$\otimes_a := X_a \times X_a \xrightarrow{\otimes} X_{a+a} \xrightarrow{\nabla_!} X_a$$

fibermise

$$\textstyle\int F \text{ with mon str} \longleftarrow F : A \longrightarrow \mathrm{MonCat}$$

$$(a, x) \otimes (a', x') = (a + a', F(1_a)x \otimes_{a+a'} F(1_{a'})x')$$

Then $(X, \otimes) \to (A, +) \longmapsto A \to \mathrm{MonCat} \longmapsto \left(\left(\textstyle\int F, \otimes\right) \to (A, +)\right)$

produces an isomorphic monoidal opfibration, where

COULD WRITE THIS AS $A_m \times A_{m'} \xrightarrow{\otimes} A_{m+m'}$ etc.

$$\begin{tikzcd}
\underbrace{Fm \times Fm'}_{\text{fiber}} \arrow[r, "\otimes"] \arrow[d, "F(1_m) \times F(1_{m'})"'] \arrow[dr, phantom, "\cong"] & F(m+m') \arrow[d, "F(1_m + 1_{m'})"] \arrow[dr, equal] \arrow[d, phantom, shift left=8ex, "\cong"] & \\
F(m+m') \times F(m+m') \arrow[r, "\otimes"'] & F(m+m'+m+m') \arrow[r, "F(\nabla)"'] & F(m+m')
\end{tikzcd}$$

$\star$

where the LHS isomorphism is due to $\otimes$ being a cocartesian functor, namely it preserves cocartesian liftings. $[f_! x \otimes g_! y \cong (f+g)_!(x \otimes y)]$

SO SUCH A PASTED ISO ESTABLISHES THE 2-EQUIVALENCE $\boxed{2} \Leftrightarrow \boxed{3}$

two isomorphic monoidal categories!!

$(X, \otimes_X)$ and $(\textstyle\int F, \otimes_{\int F})$

$$(a + a' + a'', F(j)(F(i)x \otimes F(i')x') \otimes F(j)x'')$$
$$\cong$$
$$(a + a' + a'', F(k)x \otimes F(k)(F(i')x' \otimes F(i'')x''))$$

E.g. the associators $a : (x \otimes y) \otimes z \xrightarrow{\sim} x \otimes (y \otimes z)$ and the induced connect via the isomorphism $\star$

Now this 2-equivalence restricts to one where $\otimes_X$ is a coproduct, and $\otimes_a$ give coproducts in fibers. How? As follows.

$$U:(X,+)\longrightarrow(A,+) \quad\longmapsto\quad F: A\longrightarrow \mathrm{CoCart}$$

$$a \longmapsto X_a \text{ with } X_a\times X_a \xrightarrow[\text{global}]{+} X_{a+a} \xrightarrow{\nabla_!} X_a$$

with an induced universal cocone

$$\begin{tikzcd}[column sep=small]
 & & \nabla_!(x_1+x_2) & & \\
X_{a+a}\ni x_1+x_2 \arrow[urr, "\mathrm{Cocart}"] & & & & x_1+x_2\in X_{a+a} \arrow[ull, "\mathrm{Cocart}"'] \\
X_a \ni x_1 \arrow[u, "\iota_1"] & & & & x_2\in X_a \arrow[u, "\iota_2"']
\end{tikzcd}$$

How do I know this cocone is wholly in the fibre X_a?

$$\begin{tikzcd}
x_1 \arrow[r, "\iota_1"] \arrow[d, dotted, no head] & x_1+x_2 \arrow[r, "\mathrm{Cocart}(\nabla,\, x_1+x_2)"] \arrow[d, dotted, no head] & \nabla_!(x_1+x_2) \arrow[d, dotted, no head] \\
a \arrow[r, "\iota_a"'] \arrow[rr, bend right, "1_a"'] & a+a \arrow[r, "\nabla"] \arrow[d, phantom, "\cong"] & a
\end{tikzcd}$$

so indeed that arrow is above 1_a, so wholly in X_a!

$$\textstyle\int F \text{ with sums} \quad\longleftarrow\!\!\shortmid\quad F: A\longrightarrow \mathrm{CoCart}$$

$$(a,x) \underset{\text{global}}{+} (a',x') = (a+a',\ F(\iota_a)x \underset{\text{fiberwise}}{+} F(\iota_{a'})x')$$

with induced inclusions

$$\begin{tikzcd}
 & (a+a',\ F(\iota_a)x + F(\iota_{a'})x') & \\
(a,x) \arrow[ur] & & (a',x') \arrow[ul]
\end{tikzcd}$$

$\begin{cases} \iota_a : a\longrightarrow a+a' \\ F(\iota_a)x \xrightarrow{j_x} F(\iota_a)x + F(\iota_{a'})x' \text{ in } F(a+a') \end{cases}$

Now, if again $U:(X,+)\to(A,+) \longmapsto F:A\to\mathrm{CoCart} \longmapsto (\int F,+)\to(A,+)$, the initial global sum is isomorphic to $(\iota_a)_!x+(\iota_{a'})_!x'$ via ⊛ and that iso is the unique such that it commutes with the respective inclusions (the initial ones $x \to x+x'$, $x'\to x+x'$ and the final ones)

⊛ Moreover, the associator is well-behaved with respect to all involved inclusions .... $(x+x')+x'' \xrightarrow{\alpha} x+(x'+x'')$ and the induced one after $[2] \Rightarrow [3] \Rightarrow [2]$.

in $X \cong \int F$: x, x', x''

Now, if we want to make [1] enter the picture, the <u>only</u> thing that changes in the above story is as follows: $\boxed{1}$ $(F, \phi, \phi_0) : (A, +) \longrightarrow (Cat, \times)$ makes the opfib

$\boxed{2}$ $\int F \longrightarrow A$ monoidal in a very specific way: $\otimes = \phi$!!!! So instead having an arbitrary monoidal structure on the total category, we use ϕ.

So

$$\boxed{2} \quad (m,x) \otimes (m',x') = (m+m', \overbrace{\phi_{m,m'}(x,x')}^{x \otimes x'})$$

$$\Downarrow$$

$$\boxed{3} \quad Fm \times Fm \xrightarrow{\phi_{m,m}} F(m+m) \xrightarrow{F\nabla} Fm$$

$$\Downarrow$$

$$\boxed{2} \quad (m,x) \otimes (m',x') = (m+m', (F\nabla)\, \phi_{m+m'}(F(1_m)x + F(1_{m'})x'))$$

and the equivalence is now established via ☆ which becomes our very familiar "θ"

$$\begin{tikzcd}
{} \arrow[r, "\phi_{m,m'}"] \arrow[d, "F(1_m)F(1_{m'})"'] \arrow[dr, phantom, "\cong"] & {} \arrow[d, "F(1_m+1_{m'})"] \arrow[dr, equal] \arrow[dr, phantom, "\cong"'] & \\
{} \arrow[r, "\phi_{m+m',m+m'}"'] & {} \arrow[r, "F\nabla"'] & {}
\end{tikzcd}$$

Then the inclusions etc. involve ϕ's, and also the associator in $\int F$ with $\otimes = \phi$ to begin with is "ω", which then necessarily is the canonical map between respective colimits, commuting with cocones etc......

So

- This should partly respond to John's question "How do we actually prove that if a lax mon pseudofun factors through Rex, the category $\int F$ has finite colimits? Universal cocones etc?"
- This sketches why I don't think we have to add extra structure on lax mon pseudofun (option "3") since asking for F to factor should be equivalent// enough for what we want.