18/1/2021

Thm 3.2

Axiom

$$\begin{tikzcd}
 & {} \arrow[dr] & \\
\Phi(M)\circ\Phi(N)\circ\Phi(P) \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\cong"] & & \Phi(M\circ N\circ P) \\
 & {} \arrow[ur] &
\end{tikzcd}$$

Can do (...) by brute force (axioms of lax mon pseudofun) since all maps are now known, using axiom $\Phi(M)\otimes\Phi(N)\otimes\Phi(P) \rightrightarrows \Phi(M\otimes N\otimes P)$, but once again prefer a higher-level solution.

OPP. CONVENTION FROM LAST NOTES

① Recall that the tensor axiom ended up being a result of an isomorphism of monoidal categories $(SF, \bar{\otimes}) \cong (SF, \otimes)$ where $x \otimes y = \phi_{m,n}(x,y)$ and $x \bar{\otimes} y = (F\nabla)\phi_{m+n,m+n}(F(l_m)x, F(l_n)y)$

and the isomorphism was an id-on-objects functor w. (strong) laxator

$$\begin{tikzcd}
f_m\times f_n \arrow[r, "\phi_{m,n}"] \arrow[d, "f_{l_m}\times f_{l_n}"'] \arrow[dr, phantom, "\overset{\phi_{l_m,l_n}}{=}"] & f(m+n) \arrow[d, "F(l_m+l_n)"] \arrow[dr, equal, bend left] \arrow[dr, phantom, "\cong"] & & =: \theta_{x,y} \\
F(m+n)\times F(m+n) \arrow[r, "\phi_{m+n,m+n}"'] & F(m+n+m+n) \arrow[r, "F\nabla"'] & F(m+n) &
\end{tikzcd}$$

≀ BY NATURALITY OF θ

So we already have that

$$\begin{tikzcd}
\phi(\phi(x,y),2) \arrow[r, "\phi(\theta,1)"] \arrow[dr, bend right=10, "\theta"'] \arrow[ddd, "w"'] & \phi((F\nabla)\phi(F(l_m)x,F(l_n)y),2) \arrow[d, phantom, "\wr"] \arrow[rr, "\theta"] & & (F\nabla)\phi(F(k_{m+n})(F\nabla)\phi(F(l_m)x,F(l_n)y),F(k_p)2) \arrow[ddd, "\alpha"] \\
 & (F\nabla)\phi(F(k_{m+n})\phi(x,y),F(k_p)2) \arrow[urr, "(F\nabla)\phi(F(k_{m+n})\theta{,}1)"'] & & \\
 & {} \arrow[rrd, "(F\nabla)\phi(1{,}F(k_{n+p})\theta)"] \arrow[d, phantom, "\wr"] & & \\
\phi(x,\phi(y,2)) \arrow[ur, "\theta"] \arrow[r, "\phi(1,\theta)"'] & \phi(x,F\nabla)\phi(F(l_n)y,F(l_p)2)) \arrow[rr, "\theta"'] & & (F\nabla)\phi(F(k_m)x,F(k_{n+p})(F\nabla)\phi(F(l_n)y,F(l_p)2))
\end{tikzcd}$$

$\circledast_1$

for maps as shown here

$$\begin{tikzcd}
 & & m+n+p & & \\
 & m+n \arrow[ur, "k_{m+n}"] & & n+p \arrow[ul, "k_{n+p}"'] & \\
m \arrow[ur, "l_m"] \arrow[uurr, bend left, "k_m"] & & n \arrow[ul, "l_n"'] \arrow[u, "k_n"'] \arrow[ur, "l_n"'] & & p \arrow[ul, "l_p"] \arrow[uull, bend right, "k_p"']
\end{tikzcd}$$

where the top right expression essentially is $F(k_{m+n})(F(l_m)x \otimes_{m+n} F(l_n)y) \otimes_{m+n+p} F(k_p)2$

($F(k_{m+n})$ is strong monoidal and pseudo) $\cong (F(k_m)x \otimes_{m+n+p} F(k_n)y) \otimes_{m+n+p} F(k_p)2$

and the bottom right expression essentially is $F(k_m)x\otimes_{m+n+p}(F(k_{p+n})(F(l_n)y \otimes_{n+p} F(l_p)2)$

$\cong F(k_m)\otimes_{m+n+p}(F(k_n)y \otimes_{m+n+p} F(k_p)2)$

so α is the $\overset{\text{essentially}}{\vee}$ associator $(F(k_m)x\otimes F(k_n)y)\otimes F(k_p)2 \cong F(k_m)x\otimes(F(k_n)y\otimes F(k_p)2)$

INSIDE THE FIBER $F(m+n+p)$ [can write explicitly, don't need to].

② From the monoidal Grothendieck construction, we also have that starting from the monoidal opfibration $(\int F, \phi)$, the fibers are monoidal via $\xrightarrow{\phi}\xrightarrow{F\nabla}$ (20) already used above, but also that each reindexing functor becomes strong monoidal via (21).

In our setting, consider the map $\psi : m+n \longrightarrow m+_b n$ in A. Then the functor $F\psi : F(m+n) \longrightarrow F(m+_b n)$ becomes strong mon via

$$\begin{tikzcd}
F(m+n)\times F(m+n) \arrow[r, "\phi"] \arrow[d, "F\psi\times F\psi"'] \arrow[rr, dashed, bend left, "\otimes_{m+n}"] \arrow[dr, phantom, "\cong \phi_{\psi,\psi}"] & F(m+n+m+n) \arrow[r, "F\nabla"] \arrow[d, "F(\psi+\psi)"] \arrow[dr, phantom, "\cong n"] & F(m+n) \arrow[d, "F\psi"] & =: J_{x,y} \\
F(m+_b n)\times F(m+_b n) \arrow[r, "\phi"'] \arrow[rr, dashed, bend right, "\otimes_{m+_b n}"'] & F((m+_b n)+(m+_b n)) \arrow[r, "F\nabla"'] & F(m+_b n) &
\end{tikzcd}$$

[WHICH IS THE BOTTOM PART OF THE FACTORIZATION UNDER 33]

(we are now in the oplax direction)

Since $F\psi$ is strong monoidal, it already satisfies the following

$$\begin{tikzcd}
F\psi((r\otimes_{m+n+p}s)\otimes_{m+n+p}t) \arrow[r, "J"] \arrow[d, "F\psi(\alpha)"'] \arrow[dr, phantom, "\circledast_2"] & F\psi(r\otimes_{m+n+p}s)\otimes_{m+_b n+_c p} F\psi(t) \arrow[r, "J\otimes 1"] & (F\psi(r)\otimes_{m+_b n+_c p} F\psi(s))\otimes_{m+_b n+_c p} F\psi(t) \arrow[d, "\alpha"] \\
F\psi(r\otimes_{m+n+p}(s\otimes_{m+n+p}t)) \arrow[r, "J"] & F\psi(r)\otimes_{m+_b n+_c p} F\psi(s\otimes_{m+n+p}t) \arrow[r, "1\otimes J"] & F\psi(r)\otimes(F\psi(s)\otimes_{m+_b n+_c p} F\psi(t))
\end{tikzcd}$$

for $\psi : m+n+p \longrightarrow m+_b n+_c p$ and <u>any</u> three objects r, s, t in $F(m+n+p)$.

■ Using diagrams ① + ②, we will prove the desired result!

Notice that ☆ that we are trying to verify is, for $\boxed{\xi := J\circ F\psi(\theta)}$ ⟶ whiskering the factorization under (33),

$$\begin{tikzcd}
(F\psi)\phi((F\psi)\phi(x,y),z) \arrow[r, "(F\psi)\phi(\xi{,}1)"] \arrow[dr, "\xi"'] \arrow[dddd, "\alpha"', "(8)"] & (F\psi)\phi((F\nabla)\phi(F(v_m)x,F(v_n)y),z) \arrow[r, "\xi"] \arrow[d, phantom, "\wr"] & (F\nabla)\phi(F(j_{m+_b n})(F\nabla)\phi(F(v_m)x,F(v_n)y),F(j_p)z) \arrow[dddd, "\alpha"] \\
& (F\nabla)\phi(F(j_{m+_b n})(F\psi)\phi(x,y),F(j_p)z) \arrow[ur, "(F\nabla)\phi(F(j_{m+_b n})\xi{,}1)"'] & \\
& \bigstar & \\
& (F\nabla)\phi(F(j_m)x,F(j_{n+_c p})(F\psi)\phi(y,z)) \arrow[d, phantom, "\wr"] \arrow[dr, "(F\nabla)\phi(1{,}F(j_{n+_c p})\xi)"] & \\
(F\psi)\phi(x,(F\psi)\phi(y,z)) \arrow[ur, "\xi"] \arrow[r, "(F\psi)\phi(1{,}\xi)"'] & (F\psi)\phi(x,(F\nabla)\phi(F(v_n)y,F(v_p)z)) \arrow[r, "\xi"'] & (F\nabla)\phi(F(j_m)x,F(j_{n+_c p})(F\nabla)\phi(F(v_n)y,F(v_p)z))
\end{tikzcd}$$

where ↻ commute by naturality of $\zeta = \overset{\text{natural}}{\overbrace{\zeta}} \circ \overset{\text{natural}}{\overbrace{F\psi(\theta)}}$ and the relevant maps in $(A,+)$ are denoted according to

$$\begin{tikzcd}
 & & & m+_b n+_c p & & & \\
 & m+_b n \arrow[urr, "j_{m+_b n}"] \arrow[r, "s"'] & (m+_b n)+p \arrow[ur, "\psi"] & & m+(n+_c p) \arrow[ul, "\psi"'] & n+_c p \arrow[ull, "j_{n+_c p}"'] \arrow[l, "t"'] & \\
 & m+n \arrow[u, "\psi"] \arrow[rr, "k"] & & m+n+p \arrow[uu, "\psi"] & & n+p \arrow[u, "\psi"] \arrow[ll, "k"'] & \\
m \arrow[uur, "v_m"] \arrow[ur, "i_m"'] \arrow[uuurrr, bend left, "j_m"] & & & n \arrow[ull, "i_n"] \arrow[urr, "\ell_n"'] & & & p \arrow[uul, "u_p"'] \arrow[ul, "\ell_p"] \arrow[uuulll, bend right, "j_p"']
\end{tikzcd}$$

Proof I can apply $F\psi$ to the commutative ⊛₁ ⇝

$$\begin{tikzcd}
{} \arrow[r, "F\psi(\theta)"] \arrow[d, "F\psi(\alpha)"'] & {} \arrow[r, "``F\psi(\theta)\otimes 1\text{''}"] \arrow[d, phantom, "\circledast_{11}"] & {} \arrow[d, "F\psi(\omega)"] \\
{} \arrow[r, "F\psi(\theta)"'] & {} \arrow[r, "``1\otimes F\psi(\theta)\text{''}"'] & {}
\end{tikzcd}$$

Now expand the ζ's inside ★

$$\begin{tikzcd}[column sep=small]
(F\psi)\phi((F\psi)\phi(x,y),z) \arrow[r, "F\psi(\theta)"] \arrow[ddd, "\text{ess. } F\psi(\omega)"', "(8)"] \arrow[rrr, bend left, "\zeta"] \arrow[dr, phantom, "\text{comes from } \circledast_{11} \circlearrowleft"] & (F\psi)(F\nabla)\phi((Fs)(F\psi)\phi(x,y),(Fs)z) \arrow[r, "\zeta"] \arrow[d, "(F\psi)(F\nabla)\phi((Fs)(F\psi)\theta,1)"] \arrow[dr, phantom, "\circlearrowleft\ \zeta\ \text{natural}"] & (F\nabla)\phi(\overset{F(j_{m+_b n})}{\overbrace{(F\psi)(Fs)}}(F\psi)\phi(x,y),\overset{F(j_p)}{\overbrace{(F\psi)(Fs)}}z) & \cong (F\nabla)\phi(F(j_{m+_b n})(F\psi)\phi(x,y),F(j_p)z) \arrow[d, "(F\nabla)\phi((Fj)(F\psi)\theta,1)"] \\
 & (F\psi)(F\nabla)\phi((Fs)(F\psi)(F\nabla)\phi((F(i_m)x,F(i_n)y),F(s)z) \arrow[r, "\zeta"] \arrow[ddr, "\text{ess. } F\psi(\alpha)"] & (F\nabla)\phi((F\psi)(Fs)(F\psi)(F\nabla)\phi(F(i_m)x,F(i_n)y),(F\psi)(Fs)z) \arrow[r, "\cong"] \arrow[dd, phantom, "\circlearrowleft \text{ comes from } \circledast_2"] & (F\nabla)\phi((Fj)(F\psi)(F\nabla)\phi(F(i_m)x,F(i_n)y),F(j_p)z) \arrow[d, "(F\nabla)\phi((Fj)\zeta,1)"] \\
 & & & (F\nabla)\phi((Fj)(F\nabla)\phi((F\psi)(F(i_m))x,(F\psi)(F(i_n))y),F(j_p)z) \\
(F\psi)\phi(x,(F\psi)\phi(y,z)) \arrow[d, "F\psi(\theta)"] \arrow[ddd, bend right, "\zeta"'] & & & \wr\!\!\!\;\;\; (F\nabla)\phi((Fj)(F\nabla)\phi(F(v_m)x,F(n)y),F(j_p)z) \arrow[dd, "\alpha"] \\
(F\psi)(F\nabla)\phi((Ft)x,(Ft)(F\psi)\phi(y,z)) \arrow[d, "\zeta"] \arrow[rr, "(F\psi)(F\nabla)\phi(1,(Ft)(F\psi)\theta)"] \arrow[drr, phantom, "\circlearrowleft\ \zeta\ \text{natural}"] & & (F\psi)(Ft)\phi((Ft)x,(Ft)(F\psi)(F\nabla)\phi(F(\ell)y,F(\ell)z)) \arrow[d, "\zeta"] & \\
(F\nabla)\phi((F\psi)(Ft)x,(F\psi)(Ft)(F\psi)\phi(y,z)) \arrow[d, phantom, "\wr"] & & (F\nabla)\phi(\underbrace{(F\psi)(Ft)}x,\underbrace{(F\psi)(Ft)}(F\psi)(F\nabla)\phi(F(\ell)y,F(\ell)z)) \arrow[d, phantom, "\wr"] & (F\nabla)\phi(F(j)x,F(j)(F\nabla)\phi(F(u)y,F(u)z)) \arrow[d, phantom, "\wr"] \\
(F\nabla)\phi(F(j_m)x,F(j_{n+_c p})(F\psi)\phi(y,z)) \arrow[rr, "(F\nabla)\phi(1,(Fj)(F\psi)\theta)"'] & & (F\nabla)\phi(F(j_m)x,F(j)(F\psi)(F\nabla)\phi(F(\ell_n)y,F(\ell_p)z)) \arrow[r, "(F\nabla)\phi(1,F(j)\zeta)"'] & (F\nabla)\phi(F(j)x,F(j)(F\nabla)\phi(\overline{(F\psi)(F\ell)}y,\overline{(F\psi)F\ell}z))
\end{tikzcd}$$

So the proof is complete. □.

FINAL THOUGHTS FOR AXIOM ④ - for any $\overset{\text{braided}}{\text{strong}}$ monoidal functor K, have commuting

$$\begin{tikzcd}
K(x_1\otimes y_1\otimes x_2\otimes y_2) \arrow[r, "\theta"] \arrow[dd, "K(1\beta 1)"'] & K(x_1\otimes y_1)\,\bar{\otimes}_2\,K(x_2\otimes y_2) \arrow[r, "\theta\otimes\theta"] & K(x_1)\bar{\otimes}K(y_1)\bar{\otimes}K(x_2)\bar{\otimes}K(y_2) \arrow[dd, "1\bar{\beta}1"] \\
{} \arrow[r, "\theta"] & {} \arrow[d, "1\otimes K(\beta)"] \arrow[r, "1\otimes\theta"] & {} \arrow[d, "1\otimes K\beta\otimes 1"] \arrow[r, "1\otimes\theta\otimes 1"] & {} \\
K(x_1\otimes x_2\otimes y_1\otimes y_2) \arrow[r, "\theta", "\theta"'] & K(x_1\otimes x_2)\,\bar{\otimes}\,K(y_1\otimes y_2) \arrow[r, "1\otimes\theta"] \arrow[r, "\theta\otimes\theta"', shift right] & K(x_1)\bar{\otimes}K(x_2)\bar{\otimes}K(y_1)\bar{\otimes}K(y_2)
\end{tikzcd}$$

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