

$M_1$  is given by  $a \rightarrow m_1 \leftarrow b$  with  $x_1 \in F(m_1)$ .  
 $M_2$  is given by  $b \rightarrow m_2 \leftarrow c$  with  $x_2 \in F(m_2)$ .  
 $M_3$  is given by  $c \rightarrow m_3 \leftarrow d$  with  $x_3 \in F(m_3)$ .  
 $N_1$  is given by  $a' \rightarrow n_1 \leftarrow b'$  with  $y_1 \in F(n_1)$ .  
 $N_2$  is given by  $b' \rightarrow n_2 \leftarrow c'$  with  $y_2 \in F(n_2)$ .  
 $N_3$  is given by  $c' \rightarrow n_3 \leftarrow d'$  with  $y_3 \in F(n_3)$ .

Decorations:

$$\begin{aligned}
& F(\psi)(\phi_{m_1(n_1+1)} +_{b+b'} \phi_{m_2(n_2)m_3+n_3}) (F(\psi)\phi_{m_1+n_1, m_2+n_2}(\phi_{m_1, n_1}(x_1, y_1), \phi_{m_2, n_2}(x_2, y_2)), F(\psi)\phi_{m_3, n_3}(x_3, y_3)) \in F(((m_1+n_1) +_{b+b'} (m_2+n_2)) +_{c+c'} (m_3+n_3)) \\
& F(\psi)(\phi_{(m_1+b)m_2} +_{(n_1+b', n_2)m_3+n_3} (\phi_{m_1+b, m_2, n_1+b', n_2} (F(\psi)\phi_{m_1, m_2}(x_1, x_2), F(\psi)\phi_{n_1, n_2}(y_1, y_2))), \phi_{m_3, n_3}(x_3, y_3)) \in F(((m_1+b)m_2) +_{(n_1+b', n_2)} (m_3+n_3)) \\
& \phi_{(m_1+b)m_2} +_{c(m_3, (n_1+b', n_2)+_{c', n_3}} ((F(\psi)\phi_{m_1+b, m_2, n_3} (F(\psi)\phi_{m_1, m_2}(x_1, x_2), x_3), (F(\psi)\phi_{n_1+b', n_2, n_3} (F(\psi)\phi_{n_1, n_2}(y_1, y_2), y_3))) \in F(((m_1+b)m_2) +_{c(m_3)} +_{((n_1+b', n_2)+_{c', n_3}))} \\
& \phi_{m_1+b, (m_2+c)m_3, n_1+b', (n_2+c', n_3)} (F(\psi)\phi_{m_1, m_2+c, m_3}((x_1, F(\psi)\phi_{m_2, m_3}(x_2, x_3)), (y_1, F(\psi)\phi_{n_2, n_3}(y_2, y_3)))) \in F((m_1+b(m_2+c)m_3) +_{(n_1+b', (n_2+c', n_3))}
\end{aligned}$$
$$\begin{aligned}
& F(\psi)\phi_{m_1+n_1+b+b'}(m_2+n_2)m_3+n_3)(F(\psi)\phi_{m_1+n_1+m_2+m_3}(\phi_{m_1,j_1}(x_1,y_1),\phi_{m_2,j_2}(x_2,y_2)),F(\psi)\phi_{m_3,j_3}(x_3,y_3)))\in F(((m_1+n_1)+b+b'((m_2+n_2))+c+c'((m_3+n_3))) \\
& F(\psi)\phi_{m_1+n_1,(m_2+n_2))+c+c'((m_3+n_3))}((\phi_{m_1,j_1}(x_1,y_1),F(\psi)\phi_{m_2+n_2,m_3+n_3}(\phi_{m_2,j_2}(x_2,y_2),\phi_{m_3,j_3}(x_3,y_3)))))\in F((m_1+n_1)+b+b'((m_2+n_2))+c+c'((m_3+n_3))) \\
& F(\psi)\phi_{m_1+n_1,(m_2+cm_3)+(n_2+c',n_3)}(\phi_{m_1,j_1}(x_1,y_1),\phi_{m_2+c,m_3,j_2+c',n_3}((F(\psi)\phi_{m_2,m_3}(x_2,x_3),F(\psi)\phi_{n_2,n_3}(y_2,y_3))))\in F(((m_1+n_1)+b+b'((m_2+c,m_3)+(n_2+c',n_3))) \\
& \phi_{m_1+b(m_2+c,m_3),j_1+b',(n_2+c',n_3)}(F(\psi)\phi_{m_1+m_2+c,m_3}((x_1,F(\psi)\phi_{m_2,m_3}(x_2,x_3)),(y_1,F(\psi)\phi_{n_2,n_3}(y_2,y_3))))\in F((m_1+b(m_2+c,m_3))+(n_1+b',(n_2+c',n_3)))
\end{aligned}$$

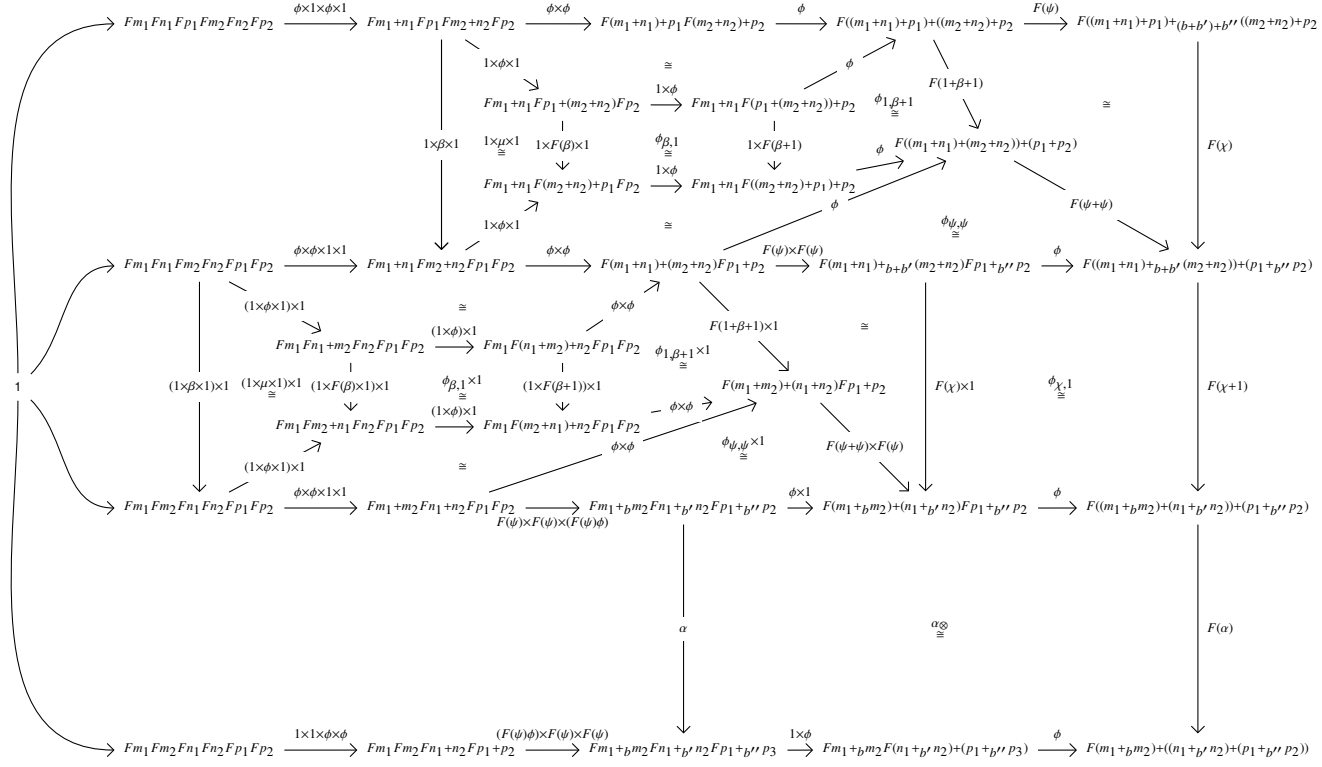
The diagram illustrates a complex commutative structure involving functors  $F$  and natural transformations  $\alpha$ ,  $\phi$ ,  $\psi$ , and  $\chi$ . The nodes are arranged in a grid-like fashion, with arrows indicating the direction of the transformations. The diagram is organized into several rows and columns, with arrows indicating the direction of the transformations. Key nodes include expressions like  $Fm_1 Fm_2 Fm_3 Fn_3$ ,  $F(m_1+n_1)+b+m_2 Fm_3+n_3$ , and  $F((m_1+b)m_2)+(n_1+b'n_2)+(m_3+n_3)$ . The diagram illustrates the coherence of these transformations, showing that different paths between the same nodes yield the same result.

Down and then right (omitting morphisms emanating out of 1 on the left due to space restrictions):





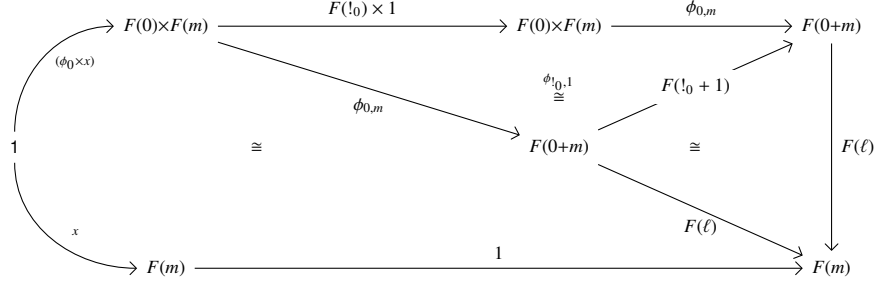
Down and then right:



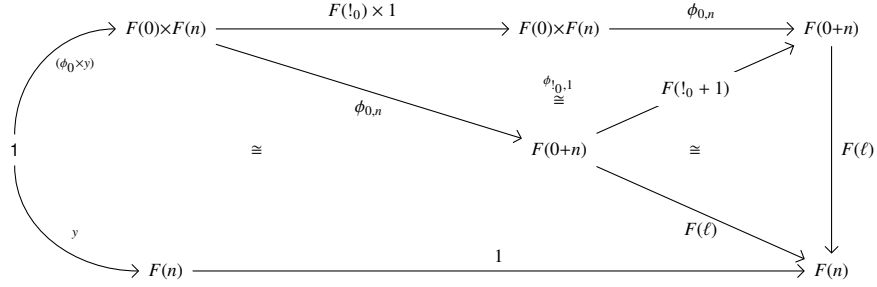




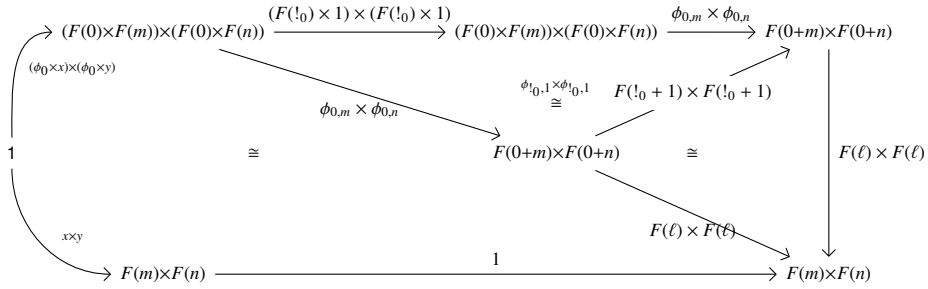
$$\lambda_M: U_0 \otimes M \rightarrow M$$



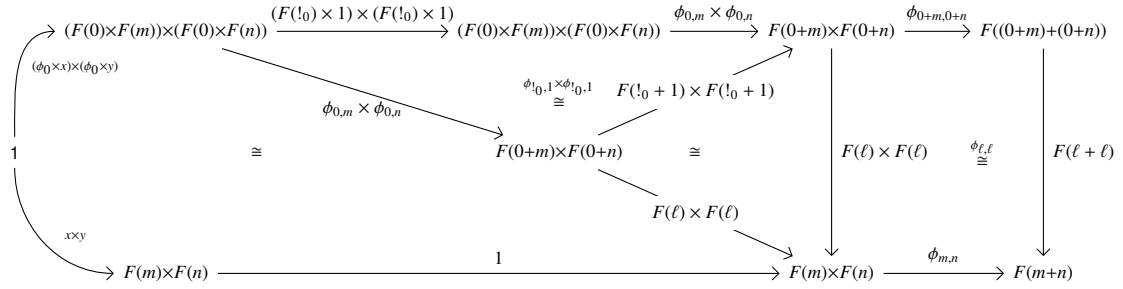
$$\lambda_N: U_0 \otimes N \rightarrow N$$



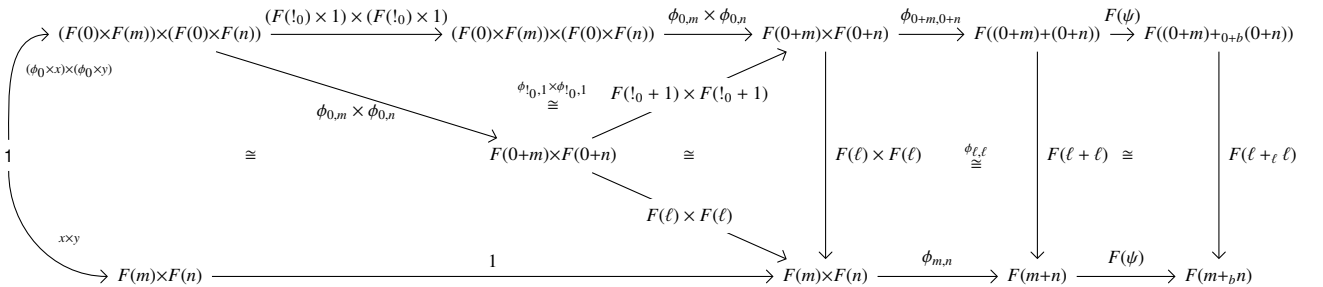
To construct  $\lambda_N \odot \lambda_M: (U_0 \otimes N) \odot (U_0 \otimes M) \rightarrow N \odot M$ , we first tensor the above two diagrams:



Next, we paste with a square due to pseudonaturality of  $\phi$ :



Finally, we paste with a square due to pseudonaturality of  $F$  to obtain the map  $\lambda_N \odot \lambda_M: (U_0 \otimes N) \odot (U_0 \otimes M) \rightarrow N \odot M$ :







**Diagrams 7 and 9 of 11. Set up and solved.**

$$\begin{array}{ccc}
 F: (\mathbf{A}, +, 0) \rightarrow (\mathbf{Cat}, \times, 1) & & \\
 a \in (\mathbf{A}, +, 0) & & \\
 0 + a \rightarrow 0 + a \leftarrow 0 + a & \begin{array}{c} U_{0+a} \xrightarrow{\mu} U_0 \otimes U_a \\ \searrow U_{\lambda_a} \quad \downarrow \lambda_{U_a} \\ U_a \end{array} & \begin{array}{c} 0 + a \rightarrow 0 + a \leftarrow 0 + a \\ \phi_{0,a}(\perp_0, \perp_a) \in F(0 + a) \\ a \rightarrow a \leftarrow a \\ \perp_a \in F(a) \end{array} \\
 \perp_{0+a} \in F(0+a) & & 
 \end{array}$$

Right and then down:

$$\begin{array}{ccccc}
 & F(0) & \xrightarrow{F(!_{0+a})} & F(0+a) & \\
 \phi \nearrow & \uparrow \phi_{0,0} & & \searrow \phi_{0,a} & \\
 1 \xrightarrow{\phi \times \phi} & F(0) \times F(0) & \xrightarrow{F(!_0) \times F(!_a)} & F(0) \times F(a) & \xrightarrow{\phi_{0,a}} F(0+a) \\
 \phi \searrow & \downarrow \phi_{0,0} & \phi_{!_0, !_a} \cong & \downarrow \phi_{0,a} & \downarrow F(\text{id}) \\
 & F(0) & \xrightarrow{F(!_a)} & F(a) & \\
 & & & & \downarrow F(\lambda)
 \end{array}$$

Diagonally:

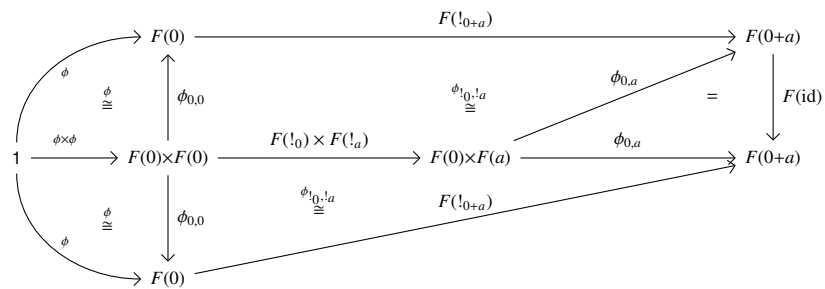
$$\begin{array}{ccccc}
 & F(0) & \xrightarrow{F(!_{0+a})} & F(0+a) & \\
 \phi \nearrow & \downarrow F(\text{id}) & & \searrow F(!_a) & \\
 1 \xrightarrow{\phi} & F(0) & \xrightarrow{F(!_a)} & F(a) & \\
 \phi \searrow & & & & \downarrow F(\lambda)
 \end{array}$$

Removing the lower right  $\cong$  which is the same in each diagram:

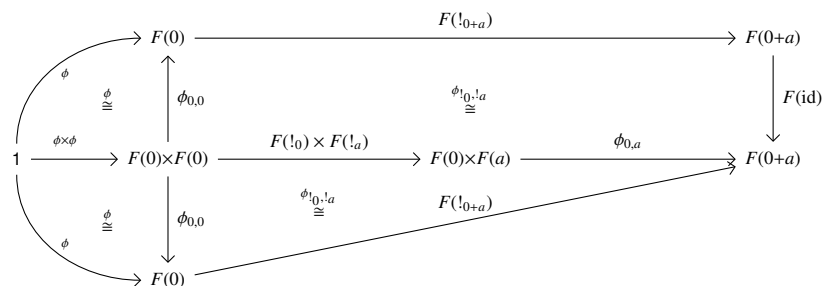
$$\begin{array}{ccccc}
 & F(0) & \xrightarrow{F(!_{0+a})} & F(0+a) & \\
 \phi \nearrow & \uparrow \phi_{0,0} & & \searrow \phi_{0,a} & \\
 1 \xrightarrow{\phi \times \phi} & F(0) \times F(0) & \xrightarrow{F(!_0) \times F(!_a)} & F(0) \times F(a) & \xrightarrow{\phi_{0,a}} F(0+a) \\
 \phi \searrow & \downarrow \phi_{0,0} & \phi_{!_0, !_a} \cong & \downarrow \phi_{0,a} & \downarrow F(\text{id}) \\
 & F(0) & \xrightarrow{F(!_a)} & F(a) & \\
 & & & & \downarrow F(\lambda)
 \end{array}$$
  

$$\begin{array}{ccccc}
 & F(0) & \xrightarrow{F(!_{0+a})} & F(0+a) & \\
 \phi \nearrow & \downarrow F(\text{id}) & & \searrow F(!_a) & \\
 1 \xrightarrow{\phi} & F(0) & \xrightarrow{F(!_a)} & F(a) & \\
 \phi \searrow & & & & \downarrow F(\lambda)
 \end{array}$$

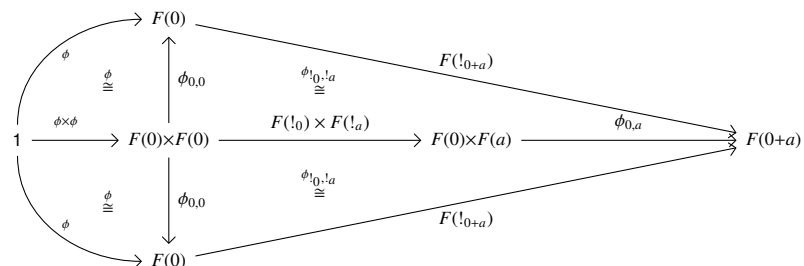
The second diagram is then an identity, so our problem reduces to showing that the following diagram is also an identity:



Removing the diagonal  $\phi_{0,a}$ :



This diagram is clearly the same as:



The two 2-isomorphisms in the top half of the diagram are the inverses of those in the bottom half, when read in a suitable order, and they can be shown to cancel, yielding an identity as desired.



**Diagram 11 of 11. Set up but not solved.**

$$a, b \in (A, +, 0)$$

$$\begin{array}{ccc} U_{a+b} & \xrightarrow{\mu_{a,b}} & U_a \otimes U_b \\ U_\beta \downarrow & & \downarrow \beta' \\ U_{b+a} & \xrightarrow{\mu_{b,a}} & U_b \otimes U_a \end{array}$$

The top two underlying cospans are

$$a + b \rightarrow a + b \leftarrow a + b$$

and the bottom two underlying cospans are

$$b + a \rightarrow b + a \leftarrow b + a$$

Right and then down:

$$\begin{array}{ccccccc} & & F(0) & \xrightarrow{F(!_{a+b})} & F(a+b) & & \\ & \nearrow \phi & \uparrow \phi_{0,0} & & \nearrow \phi_{a,b} & & \\ 1 & \xrightarrow{\phi \times \phi} & F(0) \times F(0) & \xrightarrow{F(!_a) \times F(!_b)} & F(a) \times F(b) & \xrightarrow{\phi_{a,b}} & F(a+b) \\ & \searrow \phi \times \phi & \downarrow \beta & & \downarrow \beta & & \downarrow F(\text{id}) \\ & & F(0) \times F(0) & \xrightarrow{F(!_b) \times F(!_a)} & F(b) \times F(a) & \xrightarrow{\phi_{b,a}} & F(b+a) \end{array}$$

$\phi_{!_a, !_b} \cong \mu_{a,b} \cong \phi_{b,a}$

Down and then right:

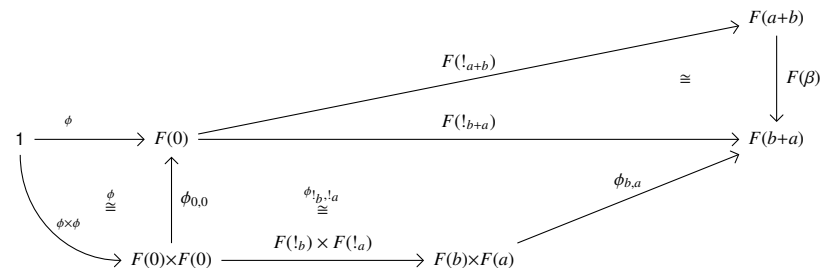
$$\begin{array}{ccccccc} & & F(0) & \xrightarrow{F(!_{a+b})} & F(a+b) & & \\ & \nearrow \phi & \downarrow \text{id} & & \downarrow F(\beta) & & \\ 1 & \xrightarrow{\phi} & F(0) & \xrightarrow{F(!_{b+a})} & F(b+a) & & \\ & \searrow \phi \times \phi & \uparrow \phi_{0,0} & & \nearrow \phi_{b,a} & & \\ & & F(0) \times F(0) & \xrightarrow{F(!_b) \times F(!_a)} & F(b) \times F(a) & \xrightarrow{\phi_{b,a}} & F(b+a) \end{array}$$

$\phi_{!_b, !_a} \cong \mu_{b,a} \cong \phi_{b,a}$

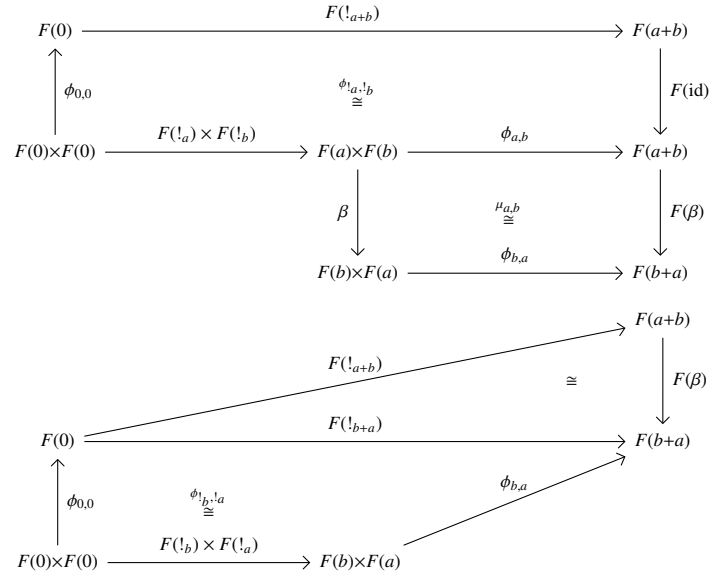
First we remove the lower left commuting quarter circle in the ‘right and then down’ diagram as well as the upper left commuting quarter circle in the ‘down and then right’ diagram:

$$\begin{array}{ccccccc} & & F(0) & \xrightarrow{F(!_{a+b})} & F(a+b) & & \\ & \nearrow \phi & \uparrow \phi_{0,0} & & \nearrow \phi_{a,b} & & \\ 1 & \xrightarrow{\phi \times \phi} & F(0) \times F(0) & \xrightarrow{F(!_a) \times F(!_b)} & F(a) \times F(b) & \xrightarrow{\phi_{a,b}} & F(a+b) \\ & \searrow \phi \times \phi & \downarrow \beta & & \downarrow \beta & & \downarrow F(\text{id}) \\ & & F(0) \times F(0) & \xrightarrow{F(!_b) \times F(!_a)} & F(b) \times F(a) & \xrightarrow{\phi_{b,a}} & F(b+a) \end{array}$$

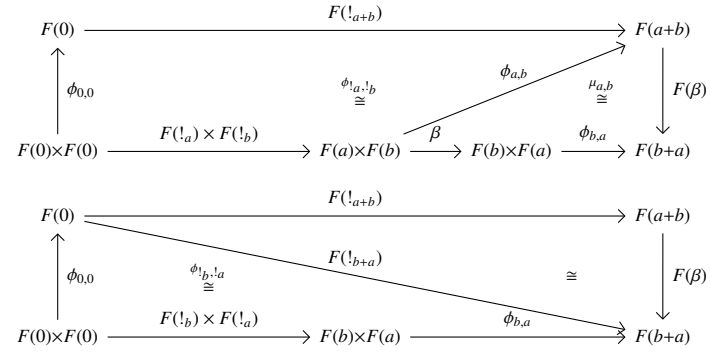
$\phi_{!_a, !_b} \cong \mu_{a,b} \cong \phi_{b,a}$



Next we remove the left quarter circle containing  $\cong$  from each diagram:



Making each diagram into a rectangle and removing the  $F(\text{id})$  from the first diagram:



Are these two diagrams the same?

### Some useful maps

Given  $a \in (\mathbf{A}, +, 0)$ , the map  $U_{\lambda_a}: U_{0+a} \rightarrow U_a$  is given by:

$$\begin{array}{ccccc}
 \phi & \xrightarrow{\quad} & F(0) & \xrightarrow{F(!_{0+a})} & F(0+a) \\
 \downarrow 1 & & \downarrow F(\text{id}) & \nearrow F(!_{0+a}) & \downarrow F(\lambda_a) \\
 \phi & \xrightarrow{\quad} & F(0) & \xrightarrow{F(!_a)} & F(a)
 \end{array}
 \quad \cong$$

where the  $\cong$  is given by pseudonaturality of  $F$ : we have a unique map in  $!_a: 0 \rightarrow a$  in  $\mathbf{A}$  but also a map  $\lambda_a \circ !_a: 0 \rightarrow a$  where  $\lambda_a$  is the left unitor of  $(\mathbf{A}, +, 0)$ , and so  $F(!_a) = F(\lambda_a \circ !_a) \cong F(\lambda_a)F(!_a)$ .

The left unitor  $\lambda'_{U_a}: U_0 \otimes U_a \rightarrow U_a$  is given by:

$$\begin{array}{ccccc}
 \phi \times \phi & \xrightarrow{\quad} & F(0) \times F(0) & \xrightarrow{F(!_0) \times F(!_a)} & F(0) \times F(a) & \xrightarrow{\phi_{0,a}} & F(0+a) \\
 \downarrow 1 & & \downarrow \phi_{0,0} & \nearrow \phi_{!_0, !_a} & \nearrow F(!_a) & \nearrow \cong & \downarrow F(\lambda_a) \\
 \phi & \xrightarrow{\quad} & F(0) & \xrightarrow{F(!_a)} & F(a)
 \end{array}$$

where the  $\cong$  in the lower right is the same as the one in the first diagram.

For an arbitrary  $M$ , the left unitor  $\lambda'_M: U_0 \otimes M \rightarrow M$  is given by:

$$\begin{array}{ccccc}
 \phi_0 \times x & \xrightarrow{\quad} & F(0) \times F(m) & \xrightarrow{F(!_0) \times 1} & F(0) \times F(m) & \xrightarrow{\phi_{0,m}} & F(0+m) \\
 \downarrow 1 & & \downarrow \phi_{0,m} & \nearrow \phi_{!_0, !_1} & \nearrow F(!_0 + 1) & \nearrow \cong & \downarrow F(\lambda_m) \\
 x & \xrightarrow{\quad} & F(m) & \xrightarrow{\text{id}} & F(m)
 \end{array}$$

For an arbitrary  $M$  given by  $a \rightarrow (m, x) \leftarrow b$ , the map  $\lambda_M: U_b \odot M \rightarrow M$  is given by:

$$\begin{array}{ccccc}
 x \times \phi_0 & \xrightarrow{\quad} & F(m) \times F(0) & \xrightarrow{1 \times F(!_b)} & F(m) \times F(b) & \xrightarrow{\phi_{m,b}} & F(m+b) & \xrightarrow{F(\psi)} & F(m+_b b) \\
 \downarrow 1 & & \downarrow \phi_{m,0} & \nearrow \phi_{!_1, !_b} & \nearrow F(1+_b) & \nearrow \cong & \downarrow F(\kappa) \\
 x & \xrightarrow{\quad} & F(m) & \xrightarrow{\text{id}} & F(m)
 \end{array}$$

In particular, if  $M = U_0$  above, then the map  $\lambda_{U_0}: U_0 \odot U_0 \rightarrow U_0$  is given by:

$$\begin{array}{ccccc}
 \phi_0 \times \phi_0 & \xrightarrow{\quad} & F(0) \times F(0) & \xrightarrow{F(!_0) \times F(!_0)} & F(0) \times F(0) & \xrightarrow{\phi_{0,0}} & F(0+0) & \xrightarrow{F(\psi)} & F(0+_0 0) \\
 \downarrow 1 & & \downarrow \phi_{0,0} & \nearrow \phi_{!_0, !_0} & \nearrow F(!_0 + !_0) & \nearrow \cong & \downarrow F(\kappa) \\
 \phi_0 & \xrightarrow{\quad} & F(0) & \xrightarrow{\text{id}} & F(0)
 \end{array}$$