# STRUCTURED VERSUS DECORATED COSPANS

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ABSTRACT. One goal of applied category theory is to understand open systems. We compare two ways of describing open systems as cospans equipped with extra data. First, given a functor $L: A \to X$, a 'structured cospan' is a diagram in X of the form $L(a) \to x \leftarrow L(b)$. If A and X have finite colimits and $L$ preserves them, it is known that there is a symmetric monoidal double category whose objects are those of A and whose horizontal 1-cells are structured cospans. Second, given a pseudofunctor $F: A \to \mathbf{Cat}$, a 'decorated cospan' is a diagram in A of the form $a \to m \leftarrow b$ together with an object of $F(m)$. Generalizing the work of Fong, we show that if A has finite colimits and $F: (A, +) \to (\mathbf{Cat}, \times)$ is symmetric lax monoidal, there is a symmetric monoidal double category whose objects are those of A and whose horizontal 1-cells are decorated cospans. We prove that under certain conditions, these two constructions become isomorphic when we take $X = \int F$ to be the Grothendieck category of $F$. We illustrate these ideas with applications to electrical circuits, Petri nets, dynamical systems and epidemiological modeling.

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## 1. INTRODUCTION

An 'open system' is any sort of system that can interact with the outside world. Experience has shown that open systems are nicely modeled using cospans [14, 18, 40]. A cospan in some category A is a diagram of this form:

$$\begin{array}{ccc} & m & \\ {}^{i}\nearrow & & \nwarrow^{o} \\ a & & b \end{array}$$
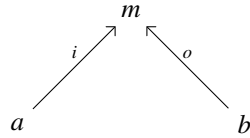
We call $m$ the **apex**, $a$ and $b$ the **feet**, and $i$ and $o$ the **legs** of the cospan. The apex describes the system itself. The feet describe 'interfaces' through which the system can interact with the outside world. The legs describe how the interfaces are included in the system. If the category A has finite colimits, we can compose cospans using pushouts: this describes the operation of attaching two open systems together in series by identifying one interface of the first with one of the second. We can also 'tensor' cospans using coproducts: this describes setting open systems side by side, in parallel. Via these operations we obtain a symmetric monoidal double category with cospans in A as its horizontal 1-cells [13, 38].

However, we often want the system itself to have more structure than its interfaces. This led Fong to develop a theory of 'decorated' cospans [17]. Given a category A with finite colimits, a symmetric lax

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monoidal functor $F\colon (\mathsf{A}, +) \to (\mathsf{Set}, \times)$ can be used to equip the apex $m$ of a cospan in $\mathsf{A}$ with some extra data: an element $x \in F(m)$, which we call a **decoration**. Thus a **decorated cospan** is a pair:

$$a \xrightarrow{\ i\ } m \xleftarrow{\ o\ } b, \qquad\qquad x \in F(m).$$

Fong proved that there is a symmetric monoidal category with objects of $\mathsf{A}$ as its objects and equivalence classes of decorated cospans as its morphisms. Such categories were used to describe a variety of open systems: electrical circuits, Markov processes, chemical reaction networks and dynamical systems [4, 5, 7].

Unfortunately, many applications of decorated cospans were flawed. The problem is that while Fong's decorated cospans are good for decorating the apex $m$ with an element of a set $F(m)$, they are unable to decorate it with an object of a category. An example would be equipping a finite set $m$ with edges making its elements into the nodes of a graph. We would like the following 'open graph' to be a decorated cospan where the apex is the finite set $m = \{n_1, n_2, n_3, n_4\}$:



We might hope to do this using a symmetric lax monoidal functor $F\colon (\mathsf{FinSet}, +) \to (\mathsf{Set}, \times)$ assigning to each finite set $m$ the set of all graphs with $m$ as their set of nodes. But this hope is doomed, for reasons painstakingly explained in [1, Section 5]. There is really a *category* of graphs with $m$ as their set of nodes— and surprisingly, trying to treat this category as a mere set does not work, despite all the tricks one might try.

Here we present a solution to this problem. Instead of basing the theory of decorated cospans on a symmetric lax monoidal functor $F\colon (\mathsf{A}, +) \to (\mathsf{Set}, \times)$, we use a symmetric lax monoidal pseudofunctor $F\colon (\mathsf{A}, +) \to (\mathbf{Cat}, \times)$. In **??** and Theorem 2.2, we use this data to construct a symmetric monoidal double category $F\mathbb{C}\mathbf{sp}$ in which:

- an object is an object of $\mathsf{A}$,
- a vertical 1-morphism is a morphism of $\mathsf{A}$,
- a horizontal 1-cell from $a$ to $b$ is a decorated cospan:

$$a \xrightarrow{\ i\ } m \xleftarrow{\ o\ } b, \qquad x \in F(m),$$

- a 2-morphism is a **map of decorated cospans**: that is, a commutative diagram

$$
\begin{array}{ccccc}
a & \xrightarrow{\ i\ } & m & \xleftarrow{\ o\ } & b \qquad x \in F(m)\\
{\scriptstyle f}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g}\\
a' & \xrightarrow{\ i'\ } & m' & \xleftarrow{\ o'\ } & b' \qquad x' \in F(m')
\end{array}
$$

together with a morphism $\tau\colon F(h)(x) \to x'$ in $F(m')$.

In fact another solution to the problem is already known: the theory of structured cospans [1, 14]. Given a functor $L\colon \mathsf{A} \to \mathsf{X}$, a **structured cospan** is a cospan in $\mathsf{X}$ whose feet come from a pair of objects in $\mathsf{A}$:

$$
\begin{array}{ccc}
& x & \\
& \nearrow \quad \nwarrow & \\
L(a) & & L(b).
\end{array}
$$

This is another way of letting the apex have more structure than the feet. When $\mathsf{A}$ and $\mathsf{X}$ have finite colimits and $L$ preserves them, there is a symmetric monoidal double category $_L\mathbb{C}\mathbf{sp}(\mathsf{X})$ where:

- an object is an object of $\mathsf{A}$,
- a vertical 1-morphism is a morphism of $\mathsf{A}$,

- a horizontal 1-cell from $a$ to $b$ is a diagram in $X$ of this form:

$$L(a) \xrightarrow{\ i\ } x \xleftarrow{\ o\ } L(b)$$

- a 2-morphism is a commutative diagram in $X$ of this form:

$$
\begin{array}{ccc}
L(a) & \xrightarrow{\ i\ } x \xleftarrow{\ o\ } & L(b) \\
{\scriptstyle L(f)}\Big\downarrow & \Big\downarrow{\scriptstyle \alpha} & \Big\downarrow{\scriptstyle L(g)} \\
L(a') & \xrightarrow[\ i'\ ] x' \xleftarrow[\ o'\ ] & L(b')
\end{array}
$$

Many of the flawed applications of decorated cospans have been fixed using structured cospans [1, Section 6], but not every decorated cospan double category is equivalent to a structured cospan double category. Here we give sufficient conditions for a decorated cospan double category to be equivalent—and in fact, isomorphic—to a structured cospan double category.

Suppose $A$ has finite colimits and $F\colon (A, +) \to (\mathbf{Cat}, \times)$ is a symmetric lax monoidal pseudofunctor. Then each category $F(a)$ for $a \in A$ becomes symmetric monoidal, and $F$ becomes a pseudofunctor $F\colon A \to \mathbf{SymMonCat}$. Using the Grothendieck construction, $F$ also gives an opfibration $U\colon X \to A$ where $X = \int F$. Let $\mathbf{Rex}$ be the 2-category of categories with finite colimits, functors preserving finite colimits, and natural transformations. We show that if $F\colon A \to \mathbf{SymMonCat}$ factors through $\mathbf{Rex}$ as a pseudofunctor, the opfibration $U\colon X \to A$ is also a right adjoint. From the accompanying left adjoint $L\colon A \to X$, we construct a symmetric monoidal double category $_L\mathbb{C}\mathbf{sp}(X)$ of structured cospans. In Theorem 3.2 we prove that this structured cospan double category $_L\mathbb{C}\mathbf{sp}(X)$ is isomorphic to the decorated cospan double category $F\mathbb{C}\mathbf{sp}$. In fact, they are isomorphic as symmetric monoidal double categories.

This result shows that under certain conditions, structured and decorated cospans provide equivalent ways of describing open systems. We illustrate this with applications to electrical circuits, Petri nets, dynamical systems, and epidemiological modeling. In particular, we describe a map from open Petri nets with rates to open dynamical systems that takes advantage of *both* structured and decorated cospans.

**Outline.** In Section 2, we construct the symmetric monoidal double category $F\mathbb{C}\mathbf{sp}$ and we define maps between decorated cospan double categories. In Section 3, we briefly review the structured cospans framework and prove that the double categorical versions of decorated cospans and structured cospans are isomorphic under suitable conditions. In Section 4, we establish the isomorphism between structured and decorated cospans at the level of bicategories and categories (via decategorification). Finally, in Section 5, we describe applications, and show that open dynamical system can be described using decorated cospans but not structured cospans.

**Conventions.** In this paper, we use a sans-serif font like $C$ for categories, boldface like $\mathbf{B}$ for bicategories or 2-categories, and blackboard bold like $\mathbb{D}$ for double categories. For double categories with names having more than one letter, like $\mathbb{C}\mathbf{sp}(X)$, only the first letter is in blackboard bold. In this paper, 'double category' means 'pseudo double category', as in Definition A.3. A double category $\mathbb{D}$ has a category of objects and a category of arrows, and we call these $\mathbb{D}_0$ and $\mathbb{D}_1$ despite the fact that they are categories. Vertical composition in our double categories is strictly associative, while horizontal composition need not be. We use $(C, \otimes)$ to stand for a monoidal or perhaps symmetric monoidal category with $\otimes$ as its tensor product.

## 2. Decorated cospans

In this section we build symmetric monoidal double categories of decorated cospans, and then study the functoriality of this construction. Theorem 2.1 explains how to construct a double category of decorated cospans from a lax monoidal pseudofunctor $F\colon (A, +) \to (\mathbf{Cat}, \times)$ whenever $A$ is a cocartesian. Theorem 2.2 gives conditions under which this double category is symmetric monoidal. The definition of lax monoidal pseudofunctor is recalled in Appendix A.1, and the definition of symmetric monoidal double category is recalled in Definition A.3.

In what follows, we define a **cocartesian monoidal category** $(A, +)$ to be a symmetric monoidal category where the monoidal structure is given by chosen binary coproducts, with a chosen initial object serving as the unit for the monoidal structure. Any category with finite coproducts can be made into a cocartesian monoidal category by arbitrarily choosing finite coproduct and a unit object.

**Theorem 2.1.** *Let $(A, +)$ be cocartesian monoidal and $F: (A, +) \to (\mathbf{Cat}, \times)$ a lax monoidal pseudofunctor. Then there exists a double category $F\mathbb{C}\mathbf{sp}$ in which*

- *an object is an object of $A$,*
- *a vertical 1-morphism is a morphism of $A$,*
- *a horizontal 1-cell is an $F$-**decorated cospan**, that is, a diagram in $A$ of the form*

$$a \xrightarrow{\ i\ } m \xleftarrow{\ o\ } b,$$

  *together with a **decoration** $x \in F(m)$,*
- *a 2-morphism is a **map of $F$-decorated cospans**, that is, a commutative diagram in $A$ of the form*

$$
\begin{array}{ccccc}
a & \xrightarrow{\ i\ } & m & \xleftarrow{\ o\ } & b \qquad x \in F(m)\\
{\scriptstyle f}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g}\\
a' & \xrightarrow[\ i'\ ]{} & m' & \xleftarrow[\ o'\ ]{} & b' \qquad x' \in F(m')
\end{array}
\tag{1}
$$

  *together with a **decoration morphism** $\tau: F(h)(x) \to x'$ in $F(m')$.*

**Theorem 2.2.** *Let $(A, +)$ be a cocartesian monoidal category and let $(F, \phi, \phi_0): (A, +) \to (\mathbf{Cat}, \times)$ be a symmetric lax monoidal pseudofunctor. Then the double category $F\mathbb{C}\mathbf{sp}$ is symmetric monoidal, where the tensor product*

- *of two objects $a$ and $b$ is their coproduct $a + b$ in $A$,*
- *of two vertical 1-morphisms $f: a \to b$ and $f': a' \to b'$ is $f + f': a + a' \to b + b'$ in $A$,*
- *of two horizontal 1-cells $(a \to m \leftarrow b, x \in F(m))$ and $(c \to n \leftarrow d, y \in F(n))$ is*

$$
\begin{array}{c}
m + n \\
{\scriptstyle i_1+i_2}\nearrow \quad \nwarrow {\scriptstyle o_1+o_2} \\
a + c \qquad\qquad b + d,
\end{array}
\qquad x \otimes y := \phi_{m,n}(x, y) \in F(m + n)
\tag{2}
$$

- *of two 2-morphisms $\alpha$ and $\beta$ is:*

$$
\begin{array}{ccc}
\begin{array}{ccccc}
a & \xrightarrow{i_1} & m & \xleftarrow{o_1} & b \\
{\scriptstyle f}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\
a' & \xrightarrow[i'_1]{} & m' & \xleftarrow[o'_1]{} & b'
\end{array}
\ \otimes\
\begin{array}{ccccc}
c & \xrightarrow{i_2} & n & \xleftarrow{o_2} & d \\
{\scriptstyle f'}\downarrow & & {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle g'} \\
c' & \xrightarrow[i'_2]{} & n' & \xleftarrow[o'_2]{} & d'
\end{array}
& = &
\begin{array}{ccccc}
a + c & \xrightarrow{i_1 + i_2} & m + n & \xleftarrow{o_1 + o_2} & b + d \\
{\scriptstyle f+f'}\downarrow & & {\scriptstyle h+h'}\downarrow & & \downarrow{\scriptstyle g+g'} \\
a' + c' & \xrightarrow[i'_1 + i'_2]{} & m' + n' & \xleftarrow[o'_1 + o'_2]{} & b' + d'
\end{array}
\end{array}
$$

$$
\tau_\alpha: F(h)(x) \to x' \text{ in } F(m') \qquad \tau_\beta: F(h')(y) \to y' \text{ in } F(n') \qquad \tau_{\alpha \otimes \beta}: F(h+h')(\phi_{m,n}(x,y)) \to \phi_{m',n'}(x',y') \text{ in } F(m'+n')
$$

  *with decoration morphism $\tau_{\alpha \otimes \beta}$ given by the following diagram:*

$$
\begin{array}{ccc}
& & F(m) \times F(n) \xrightarrow{\ \phi_{m,n}\ } F(m + n) \\
& {\scriptstyle x\times y}\nearrow & \quad\Big\downarrow \qquad\qquad \Big\downarrow{\scriptstyle F(h+h')} \\
1 & \Downarrow\tau_\alpha\times\tau_\beta \quad {\scriptstyle F(h)\times F(h')}\ \ {\overset{\phi_{h,h'}}{\underset{(14)}{\cong}}} & \\
& {\scriptstyle x'\times y'}\searrow & \quad\Big\downarrow \\
& & F(m') \times F(n') \xrightarrow[\ \phi_{m',n'}\ ]{} F(m' + n')
\end{array}
$$

We prove both these theorems using the work of Shulman [42], which we recall here. First we need a few concepts. Opfibrations are recalled in Appendix A.2, and fibrations are defined dually, with a functor $\Phi: C \to A$ being a fibration if and only if $\Phi^{\mathrm{op}}: C^{\mathrm{op}} \to A^{\mathrm{op}}$ is an opfibration. A functor $\Phi: C \to A$ is a **bifibration** if it is both a fibration and an opfibration. If $C$ and $A$ are monoidal categories, a **monoidal** bifibration $\Phi: C \to A$ is a bifibration that is also a strict monoidal functor. Similarly, if $C$ and $A$ are symmetric monoidal, a **symmetric monoidal** bifibration is a bifibration $\Phi: C \to A$ that is a symmetric strict monoidal functor.

If $\Phi\colon \mathsf{C} \to \mathsf{A}$ an opfibration, for any morphism $f\colon a \to b$ in $\mathsf{A}$ there is a covariant reindexing functor

$$f_!\colon \mathsf{C}_a \to \mathsf{C}_b$$

between the fibers of $\Phi$, as recalled in Appendix A.2. Similarly, if $\Phi$ is a fibration there is a contravariant reindexing functor

$$f^*\colon \mathsf{C}_b \to \mathsf{C}_a.$$

Thus, in a bifibration we have both covariant and contravariant reindexing functors, and in fact $f^*$ is left adjoint to $f_*$ [42, Proposition 3.9]. Using this one can easily show that for any commutative square in $\mathsf{A}$

$$
\begin{array}{ccc}
a & \xrightarrow{\ h\ } & b \\
{\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\
c & \xrightarrow{\ f\ } & d
\end{array}
$$

the following square commutes up to a specified natural transformation, built

$$
\begin{array}{ccc}
\mathsf{C}_a & \xleftarrow{\ h^*\ } & \mathsf{C}_b \\
{\scriptstyle k_!}\downarrow & \quad\Large\searrow\,\theta & \downarrow{\scriptstyle g_!} \\
\mathsf{C}_c & \xleftarrow{\ f^*\ } & \mathsf{C}_d
\end{array}
$$

If $\theta$ is a natural isomorphism whenever the original square in $\mathsf{A}$ is a pushout, we say $\Phi$ is **Beck–Chevalley**. (Shulman says 'strongly co-Beck–Chevalley'.)

**Lemma 2.3 (Shulman).** *Suppose $(\mathsf{A}, +)$ is cocartesian monoidal and $\Phi\colon (\mathsf{C}, \otimes) \to (\mathsf{A}, +)$ is a Beck–Chevalley monoidal bifibration. Then there is a double category $\mathbb{F}\mathbf{r}(\Phi)$ in which*

- *an object is an object of $\mathsf{A}$,*
- *a vertical 1-morphism is a morphism of $\mathsf{A}$,*
- *a horizontal 1-cell is a pair of objects $a, b \in \mathsf{A}$ together with an object $c \in C$ with $\Phi(c) = a + b$.*
- *a 2-morphism is a pair of morphisms $f\colon a \to a'$, $g\colon b \to b'$ in $\mathsf{A}$ together with a morphism $h\colon c \to c'$ with $\Phi(h) = f + g$.*

*If $\Phi$ is a symmetric monoidal bifibration then $\mathbb{F}\mathbf{r}(\Phi)$ becomes a symmetric monoidal double category.*

*Proof.* This is [42, Theorem 14.9]; Shulman proves the dual version in more detail in his Theorem 14.4. □

*Proof of Theorem 2.1.* We apply Lemma 2.3 to a bifibration $\Phi\colon (\mathsf{A}/U) \to \mathsf{A}$ which we now define. Let $(\mathsf{A}/U)$ be the comma category in which:

(1) an object is a triple $(a, x, a \xrightarrow{i} U(x))$, and

(2) a morphism from $(a, x, a \xrightarrow{i} U(x))$ to $(a', x', a' \xrightarrow{i'} U(x'))$ is a pair of morphisms $f\colon a \to a'$ and $\alpha\colon x \to x'$ such that the following square commutes:

$$
\begin{array}{ccc}
a & \xrightarrow{\ f\ } & a' \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\
U(x) & \xrightarrow[U(\alpha)]{} & U(x')
\end{array}
$$

The functor $\Phi$ is defined on objects by $\Phi(a, x, a \xrightarrow{i} U(x)) = a$ and on morphisms by $\Phi(f, \alpha) = f$. We have a monoidal structure on the comma category $(\mathsf{A}/U)$ where the tensor product of two objects $(a_1, x_1, a_1 \xrightarrow{i_1} U(x_1))$ and $(a_2, x_2, a_2 \xrightarrow{i_2} U(x_2))$ is given by

$$(a_1, x_1, a_1 \xrightarrow{i_1} U(x_1)) \otimes (a_2, x_2, a_2 \xrightarrow{i_2} U(x_2)) = (a_1 + a_2, x_1 + x_2, a_1 + a_2 \xrightarrow{i_1 + i_2} U(x_1 + x_2))$$

where we have left the natural isomorphism $U(x_1) + U(x_2) \cong U(x_1 + x_2)$ implicit in the above composite, for reasons of space. The monoidal unit is given by $(0_\mathsf{A}, 0_\mathsf{X}, 0_\mathsf{A} \to U(0_\mathsf{X}))$. It is easy to see that the functor $U_1$ can be made into a strict monoidal functor using this tensor product.

The functor $\Phi$ is a fibration where given a morphism $f\colon a \to b$ in $\mathsf{A}$ with $(b, y, b \xrightarrow{i} U(y))$ over $b$ in $\int F$, the cartesian lifting of $f$ is given by

$$(f, \mathrm{id})\colon (a, y, a \xrightarrow{i \circ f} U(y)) \to (b, y, b \xrightarrow{i} U(y)).$$

Denoting the cartesian lifting of a morphism $f$ by $\mathrm{cart}(f)$, the tensor product of $(\mathsf{A}/U)$ strictly preserves cartesian liftings in the sense that $\mathrm{cart}(f_1) \otimes \mathrm{cart}(f_2) = \mathrm{cart}(f_1 + f_2)$.

The functor $\Phi$ is also an opfibration where given a morphism $f\colon a \to b$ with $(a, x, a \xrightarrow{i} U(x))$ over $a$, the cocartesian lifting of $f$ is given by

$$(f, \theta)\colon (a, x, a \to U(x)) \to (b, z, b \xrightarrow{\phi} U(z))$$

where $\theta$ and $\phi$ arise from the following pushout square in $\mathsf{A}$:

$$
\begin{array}{ccc}
a & \xrightarrow{\ f\ } & b \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \phi} \\
U(x) & \dashrightarrow[\ \theta\ ] & U(x) +_a b
\end{array}
$$

and the object $z$ arises from the cocartesian lifting of $\theta$ through $U$:



Denoting the cocartesian lifting of a morphism $f$ by $\mathrm{cocart}(f)$, we have that $\mathrm{cocart}(f_1) \otimes \mathrm{cocart}(f_2) \cong \mathrm{cocart}(f_1 + f_2)$ by the universal map between two colimits of the same diagram:

$$(U(x_1) +_{a_1} b_1) + (U(x_2) +_{a_2} b_2) \cong U(x_1 + x_2) +_{a_1 + a_2} (b_1 + b_2).$$

Lastly, the monoidal bifibration $U_1$ satisfies the Beck–Chevalley condition. Given a pushout square in $\mathsf{A}$:

$$
\begin{array}{ccc}
a & \xrightarrow{\ h\ } & b \\
{\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\
c & \xrightarrow{\ f\ } & d
\end{array}
$$

we need to show that the following square commutes up to a natural isomorphism:

$$
\begin{array}{ccc}
(\mathsf{A}/U)_a & \xleftarrow{\ h^*\ } & (\mathsf{A}/U)_b \\
{\scriptstyle k_!}\downarrow & \Searrow & \downarrow{\scriptstyle g_!} \\
(\mathsf{A}/U)_c & \xleftarrow[\ f^*\ ] & (\mathsf{A}/U)_d
\end{array}
$$

Let $(b, x, b \xrightarrow{i} U(x))$ be an object of $(\mathsf{A}/U)_b$. Going left, we precompose with $h \colon a \to b$ to obtain the object $(a, x, a \xrightarrow{i \circ h} U(x))$ of $(\mathsf{A}/U)_a$. Then going down, we push forward along $k$:

$$
\begin{array}{ccc}
a & \xrightarrow{\ i \circ h\ } & U(x) \\
{\scriptstyle k}\downarrow & & \downarrow \\
c & \xrightarrow[\ \psi\ ]{} & c +_a U(x)
\end{array}
$$

to obtain the object $(c, z, c \xrightarrow{\psi} U(z))$ of $(\mathsf{A}/U)_c$, where $z \in \int F$ is given by $z = \perp_c +_{\perp_a} x$. Explicitly, $z$ is given by

$$
1 \xrightarrow{x} F(U(x)) \xrightarrow{F(\zeta)} F(0 +_0 U(x)) \xrightarrow{F(!_c +_{!_a} 1)} F(c +_a U(x))
$$

and so by construction, we have that

$$
U(z) = U(\perp_c +_{\perp_a} x) = c +_a U(x).
$$

<span style="color:red">Need to refer to a proposition where we show $U$ preserves pushouts strictly.</span> For the other route, first going down, we push the object $(b, x, b \xrightarrow{i} U(x))$ forward along $g \colon b \to d$ by taking the following pushout:

$$
\begin{array}{ccc}
b & \xrightarrow{\ i\ } & U(x) \\
{\scriptstyle g}\downarrow & & \downarrow \\
d & \xrightarrow[\ \theta\ ]{} & d +_b U(x)
\end{array}
$$

which yields the object $(d, z', d \xrightarrow{\theta} U(z'))$ of $(\mathsf{A}/U)_d$ where $z' \in \int F$ is given by $z' = \perp_d +_{\perp_b} x$. We then precompose with $f \colon c \to d$ to obtain the object $(c, z', c \xrightarrow{\theta \circ f} U(z'))$ of $(\mathsf{A}/U)_c$.

Now we seek a natural isomorphism between the two objects $(c, z, c \xrightarrow{\psi} c +_a U(x))$ and $(c, z', c \xrightarrow{\theta \circ f} d +_b U(x))$ in $(\mathsf{A}/U)_c$. It suffices to obtain a natural isomorphism between the two objects $z = \perp_c +_{\perp_a} x$ and $z' = \perp_d +_{\perp_b} x$ in $\mathsf{X}$. This pushout square:

$$
\begin{array}{ccc}
a & \xrightarrow{\ i \circ h\ } & U(x) \\
{\scriptstyle k}\downarrow & & \downarrow \\
c & \xrightarrow[\ \psi\ ]{} & c +_a U(x)
\end{array}
$$

exhibits $c +_a U(x)$ as a pushout of the span

$$
c \xleftarrow{k} a \xrightarrow{i \circ h} U(x)
$$

in $\mathsf{A}$. We may also take the first pushout square above and paste it with the original pushout square in $\mathsf{A}$:

$$
\begin{array}{ccccc}
a & \xrightarrow{\ h\ } & b & \xrightarrow{\ i\ } & U(x) \\
{\scriptstyle k}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow \\
c & \xrightarrow[\ f\ ]{} & d & \xrightarrow[\ \theta\ ]{} & d +_b U(x)
\end{array}
$$

obtaining a pushout square which exhibits $d +_b U(x)$ as a pushout of the same span in $\mathsf{A}$. The resulting natural isomorphism $\kappa$ between $c +_a U(x)$ and $d +_b U(x)$ then extends to a natural isomorphism between $z = \perp_c +_{\perp_a} x$ and $z' = \perp_d +_{\perp_b} x$ in $\mathsf{X}$:

$$
\begin{array}{ccccccc}
& & F(U(x)) & \xrightarrow{F(\zeta)} & F(0 +_0 U(x)) & \xrightarrow{F(!_c +_{!_a} 1)} & F(c +_a U(x)) \\
& {\scriptstyle x}\nearrow & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & {\scriptstyle \cong} & \downarrow{\scriptstyle F(\kappa)} \\
1 & & & & & & \\
& {\scriptstyle x}\searrow & F(U(x)) & \xrightarrow[F(\zeta)]{} & F(0 +_0 U(x)) & \xrightarrow[F(!_d +_{!_b} 1)]{} & F(d +_b U(x))
\end{array}
$$

and consequently a natural isomorphism between our two objects of $(A/U)_c$, showing that $U_1$ satisfies the Beck–Chevalley condition.                                                                                          □

*Proof of Theorem 2.2.* We apply Lemma 2.3 to $A/U \to A$: why is this a symmetric monoidal bifibration?
                                                                                                              □

Our construction gives not only decorated cospan double categories, but also maps between these: that is, symmetric monoidal double functors, as in Definition A.4. Suppose we have two categories $A, A'$ with finite colimits and two symmetric lax monoidal pseudofunctors $F: A \to \mathbf{Cat}$ and $F': A' \to \mathbf{Cat}$. Then we can obtain a map between their decorated cospan double categories, $\mathbb{H}: F\mathbb{C}\mathbf{sp} \to F'\mathbb{C}\mathbf{sp}$, from:

- a functor $H: A \to A'$ that preserves finite limits,
- a symmetric lax monoidal pseudofunctor $(E, \phi, \phi_0): \mathbf{Cat} \to \mathbf{Cat}$,
- a natural isomorphism $\theta: EF \Rightarrow F'H$:

$$
\begin{array}{ccc}
A & \xrightarrow{\ F\ } & \mathbf{Cat} \\
{\scriptstyle H}\downarrow & \quad \Swarrow\theta \quad & \downarrow{\scriptstyle E} \\
A' & \xrightarrow[\ F'\ ]{} & \mathbf{Cat}.
\end{array}
$$

The induced double functor $\mathbb{H}: F\mathbb{C}\mathbf{sp} \to F'\mathbb{C}\mathbf{sp}$ is defined as follows:

- The image of an object $a \in F\mathbb{C}\mathbf{sp}_0 = A$ is the object $H(a) \in F'\mathbb{C}\mathbf{sp}_0 = A'$.
- The image of a vertical 1-morphism $f: a \to b$ is the vertical 1-morphism $H(f): H(a) \to H(b)$.

In other words, the object component $\mathbb{H}_0$ of the double functor $\mathbb{H}$ is the functor $H$.

- The image of an $F$-decorated cospan $M$ in $F\mathbb{C}\mathbf{sp}$ is the $F'$-decorated cospan $\mathbb{H}(M)$ in $F'\mathbb{C}\mathbf{sp}$

$$
M = \left( a \xrightarrow{i} m \xleftarrow{o} b, x \in F(m) \right) \mapsto \mathbb{H}(M) = \left( H(a) \xrightarrow{H(i)} H(m) \xleftarrow{H(o)} H(b), \bar{x} \in F'(H(m)) \right) \text{ where}
$$

$$
\bar{x} := 1 \xrightarrow{\phi_0} E(1) \xrightarrow{E(x)} E(F(m)) \xrightarrow{\theta_m} F'(H(m))
$$

- The image of a map of decorated cospans $\alpha: M \to N$ in $F\mathbb{C}\mathbf{sp}$ is the map of $F'$-decorated cospans $\mathbb{H}(\alpha)$ in $F'\mathbb{C}\mathbf{sp}$

$$
\begin{array}{ccccccc}
a & \xrightarrow{\ i\ } & m & \xleftarrow{\ o\ } & b & \quad x \in F(m) & \\
{\scriptstyle f}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} & & \\
a' & \xrightarrow[\ i'\ ]{} & n & \xleftarrow[\ o'\ ]{} & b' & \quad x' \in F(n) &
\end{array}
\;\mapsto\;
\begin{array}{ccccccc}
H(a) & \xrightarrow{H(i)} & H(m) & \xleftarrow{H(o)} & H(b) & \quad \theta_m E(x)\phi_0 \in F'(H(m)) & \\
{\scriptstyle H(f)}\downarrow & & {\scriptstyle H(h)}\downarrow & & \downarrow{\scriptstyle H(g)} & & \\
H(a') & \xrightarrow[H(i')]{} & H(n) & \xleftarrow[H(o')]{} & H(b') & \quad \theta_n E(x')\phi_0 \in F'(H(n)) &
\end{array}
$$

where a decoration morphism $\tau: F(h)(x) \to x'$ is mapped to a decoration morphism as follows:

$$
\begin{array}{c}
\begin{tikzcd}
 & F(m) \\
1 \ar[ur,"x"] \ar[dr,"x'"'] & {\scriptstyle\Downarrow\tau} \; \downarrow{\scriptstyle F(h)} \\
 & F(n)
\end{tikzcd}
\end{array}
\;\mapsto\;
\begin{array}{c}
\begin{tikzcd}
 & E(F(m)) \ar[r,"\theta_m"] & F'(H(m)) \\
1 \ar[r,"\phi"] & E(1) \ar[u,"E(x)"] \ar[d,"E(x')"'] & {\scriptstyle\Downarrow E(\tau)}\; E(F(h)) \downarrow \qquad \downarrow F'(H(h)) \\
 & E(F(n)) \ar[r,"\theta_n"'] & F'(H(n))
\end{tikzcd}
\end{array}
$$

**Theorem 2.4.** *Given two categories $A$ and $A'$ with finite colimits, two symmetric lax monoidal pseudofunctors $F: A \to \mathbf{Cat}$ and $F': A' \to \mathbf{Cat}$, a finite colimit preserving functor $H: A \to A'$, a symmetric lax monoidal pseudofunctor $E: \mathbf{Cat} \to \mathbf{Cat}$ and a natural isomorphism $\theta: EF \Rightarrow F'H$ as in the following diagram:*

$$
\begin{array}{ccc}
A & \xrightarrow{\ F\ } & \mathbf{Cat} \\
{\scriptstyle H}\downarrow & \quad \Swarrow\theta \quad & \downarrow{\scriptstyle E} \\
A' & \xrightarrow[\ F'\ ]{} & \mathbf{Cat}
\end{array}
$$

*the triple $(H, E, \theta)$ induces a symmetric monoidal double functor $\mathbb{H}: F\mathbb{C}\mathbf{sp} \to F'\mathbb{C}\mathbf{sp}$ as defined above.*

*Proof.* See [14, Theorem 4.2.1].

From the symmetric lax monoidal pseudofunctor $F\colon \mathsf{A} \to \mathbf{Cat}$, we can obtain a Beck-Chevalley symmetric monoidal opfibration $\Phi\colon (\mathsf{A}/U) \to \mathsf{A}$ as in Theorem 2.1, where $U\colon \int F \to \mathsf{A}$ is the associated monoidal opfibration of the pseudofunctor $F$. We can do the same for the other symmetric lax monoidal pseudofunctor $F'\colon \mathsf{A}' \to \mathbf{Cat}$ to obtain another Beck-Chevalley symmetric monoidal opfibration $\Phi'\colon (\mathsf{A}'/U') \to \mathsf{A}'$.

$$
\begin{array}{ccc}
(\mathsf{A}/U) & & (\mathsf{A}'/U') \\
\Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow{\scriptstyle \Phi'} \\
\mathsf{A} & \xrightarrow{\;H\;} & \mathsf{A}'
\end{array}
$$

We construct a strong monoidal functor $G\colon (\mathsf{A}/U) \to (\mathsf{A}'/U')$ between the monoidal comma categories $(\mathsf{A}/U)$ and $(\mathsf{A}'/U')$ that will make the induced square commute. Such a commutative square is a morphism in the 2-category $\mathbf{FMF}$ of monoidal bifibrations satisfying the Beck-Chevalley condition which then by a 2-functor $\mathbf{Fr}\colon \mathbf{FMF} \to \mathbf{FDbl}$ due to Shulman [42, Theorem 14.11] will yield a map

$$\mathbf{Fr}(G, H)\colon \mathbf{Fr}(\Phi) \to \mathbf{Fr}(\Phi')$$

between the fibrant symmetric monoidal double categories $\mathbf{Fr}(\Phi) = F\mathbb{C}\mathbf{sp}$ and $\mathbf{Fr}(\Phi') = F'\mathbb{C}\mathbf{sp}$.

The monoidal comma category $(\mathsf{A}/U)$ has:

(1) objects as triples $(a, x, a \to U(x))$, and
(2) a morphism from $(a, x, a \to U(x))$ to $(b, y, b \to U(y))$ is a pair of morphisms $f\colon a \to b$ and $\tau\colon x \to y$ that make the following square commute:

$$
\begin{array}{ccc}
a & \xrightarrow{\;f\;} & b \\
\Big\downarrow & & \Big\downarrow \\
U(x) & \xrightarrow{\;U(\tau)\;} & U(y)
\end{array}
$$

The tensor product of two objects $(a_1, x_1, a_1 \to U(x_1))$ and $(a_2, x_2, a_2 \to U(x_2))$ is given by

$$(a_1, x_1, a_1 \to U(x_1)) \otimes (a_2, x_2, a_2 \to U(x_2)) = (a_1 + a_2, x_1 + x_2, a_1 + a_2 \to U(x_1 + x_2)).$$

The monoidal unit of $(\mathsf{A}/U)$ is given by $(0_\mathsf{A}, 0_\mathsf{X}, 0_\mathsf{A} \to U(0_\mathsf{X}))$. Those of $(\mathsf{A}'/U')$ are analogous.

On objects, the strong monoidal functor $F\colon (\mathsf{A}/U) \to (\mathsf{A}'/U')$ is defined as follows:

(1) given $(a, x, a \to U(x))$ in $(\mathsf{A}/U)$,

$$G(a, x, a \to U(x)) = (H(a), \theta_m E(x)\phi, H(a) \to U'(\theta_m E(x)\phi))$$

where if $1 \xrightarrow{x} F(m)$, then $U(x) = m$ and $\theta_m E(x)\phi$ is given by

$$1 \xrightarrow{\;\phi\;} E(1) \xrightarrow{\;E(x)\;} E(F(m)) \xrightarrow{\;\theta_m\;} F'(H(m))$$

so that $U'(\theta_m E(x)\phi) = H(m)$,

(2) given a pair of morphisms $(f, \tau)$ that make the following square commute:

$$
\begin{array}{ccc}
a & \xrightarrow{\;f\;} & b \\
\Big\downarrow & & \Big\downarrow \\
U(x) & \xrightarrow{\;U(\tau)\;} & U(y)
\end{array}
$$

we define

$$G(f, \tau) = (H(f), \theta_h E(\tau)\phi).$$

which make the following square in $(\mathsf{A}'/U')$ commute:

$$
\begin{array}{ccc}
H(a) & \xrightarrow{\;H(f)\;} & H(b) \\
\downarrow & & \downarrow \\
U'(\theta_m E(x)\phi) & \xrightarrow{\;U'(\theta_h E(\tau)\phi)\;} & U'(\theta_n E(y)\phi))
\end{array}
$$

where $\tau$ and $\theta_h E(\tau)\phi$ are given respectively by:



The laxator is given by the following natural isomorphism between $\theta_m E(x)\phi + \theta_n E(y)\phi$ and $\theta_{m+n}E(x+y)\phi$:



and we also have a natural isomorphism between $0_{\int F'}$ and $G(0_{\int F})$ given by:



One can show that this laxator and unit morphism satisfy the four required coherence laws of a symmetric strong monoidal functor. $\qquad\square$

This theorem should generalize to the case when $\theta$ is a pseudonatural equivalence, but the weaker version suffices for our application in Section 5.4.

## 3. Structured versus decorated cospans

In [1], the first two authors introduce the symmetric monoidal double category of structured cospans as a formalism to capture open networks. One of the main goals of this paper is to provide a monoidal double isomorphism between this double category and that of decorated cospans, described in detail in Section 2. We first recall the double category of structured cospans, along with a new construction of it.

**Theorem 3.1.** *Given categories* $\mathsf{A}$ *and* $\mathsf{X}$ *with finite colimits and* $L\colon \mathsf{A} \to \mathsf{X}$ *a functor preserving finite colimits, there is a symmetric monoidal double category* $_L\mathbb{C}\mathbf{sp}(\mathsf{X})$ *in which*

- *an object is an object of* $\mathsf{A}$,
- *a vertical 1-morphism is a morphism of* $\mathsf{A}$,
- *a horizontal 1-cell from* $a$ *to* $b$ *is an* $L$-**structured cospan***, that is, a diagram in* $\mathsf{X}$ *of the form*

$$
L(a) \xrightarrow{\;i\;} x \xleftarrow{\;o\;} L(b),
$$

- *a 2-morphism is a* **map of** $L$-**structured cospans***, that is, a commutative diagram in* $\mathsf{X}$ *of the form*

$$
\begin{array}{ccc}
L(a) & \xrightarrow{\;i\;} x \xleftarrow{\;o\;} & L(b) \\
{\scriptstyle L(f)}\downarrow & \;\;\alpha\;\downarrow & \downarrow{\scriptstyle L(g)} \\
L(a') & \xrightarrow[\;i'\;]{} x' \xleftarrow[\;o'\;]{} & L(b').
\end{array}
$$

*Composition of horizontal 1-cells and 2-cells is done using pushouts in* $\mathsf{X}$, *and the symmetric monoidal structure is defined using finite coproducts in* $\mathsf{A}$ *and* $\mathsf{X}$: *the tensor product of two horizontal 1-cells is*

$$
\begin{array}{ccccc}
& x & & x' & & x+x' \\
{}^{i}\nearrow \quad \nwarrow{}^{o} & & {}^{i'}\nearrow \quad \nwarrow{}^{o'} & & {}^{i+i'}\nearrow \quad \nwarrow{}^{o+o'} \\
L(a) \qquad L(b) & \otimes & L(a') \qquad L(b') & = & L(a+a') \qquad L(b+b')
\end{array}
\tag{3}
$$

*using that L preserves finite coproducts.*

*Proof.* This was proved in [1, Theorems 2.3 & 3.9], where all the structures are specified in detail. In fact, the double category structure only requires that $\mathsf{X}$ have pushouts, whereas the symmetric monoidal structure also requires that $\mathsf{X}$ and $\mathsf{A}$ have finite coproducts and that $L$ preserve these [14, Theorem 3.2.3].

   **NEW PROOF HERE.** We apply Lemma 2.3 to $L/\mathsf{X} \to \mathsf{A}$. Why is this a fibration, cofibration, Beck–Chevalley and symmetric monoidal?

   The proof is analogous to the proof of Theorem 2.1 in that we apply Lemma 2.3 to the Beck-Chevalley monoidal bifibration $U_2 \colon (L/\mathsf{X}) \to \mathsf{A}$ which we now define. Let $(L/\mathsf{X})$ be the comma category in which:

(1) an object is a triple $(a, x, L(a) \overset{I}{\to} x)$, and

(2) a morphism from $(a, x, L(a) \overset{I}{\to} x)$ to $(a', x', L(a') \overset{I'}{\to} x')$ is a pair of morphisms $f \colon a \to a'$ and $\alpha \colon x \to x'$ such that the following square commutes:

$$
\begin{array}{ccc}
L(a) & \overset{L(f)}{\longrightarrow} & L(a') \\
{\scriptstyle I}\downarrow & & \downarrow{\scriptstyle I'} \\
x & \overset{\alpha}{\longrightarrow} & x'
\end{array}
$$

The functor $U_2$ is defined on objects by $U_2(a, x, L(a) \overset{I}{\to} x) = a$ and on morphisms by $U_2(f, \alpha) = f$. The monoidal structure of $(L/\mathsf{X})$ is similar to the monoidal structure of $(\mathsf{A}/U)$ of Theorem 2.1 in that the tensor product of two objects $(a_1, x_1, L(a_1) \overset{I_1}{\to} x_1)$ and $(a_2, x_2, L(a_2) \overset{I_2}{\to} x_2)$ is given by

$$
(a_1, x_1, L(a_1) \overset{I_1}{\to} x_1) \otimes (a_2, x_2, L(a_2) \overset{I_1}{\to} x_2) = (a_1 + a_2, x_1 + x_2, L(a_1 + a_2) \overset{I_1+I_2}{\longrightarrow} x_1 + x_2)
$$

where again the natural isomorphism $L(a_1)+L(a_2) \cong L(a_1+a_2)$ is implicit. The monoidal unit is $(0_\mathsf{A}, 0_\mathsf{X}, L(0_\mathsf{A}) \to 0_\mathsf{X})$, and it is clear that $U_2$ is a strict monoidal functor using this tensor product.

   The functor $U_2$ is both a fibration and an opfibration. Given a morphism $f \colon a \to b$ in $\mathsf{A}$ with $(b, x, L(b) \overset{I}{\to} x)$ over $b$ in $\mathsf{X}$, the cartesian lifting of $f$ is given by

$$
(f, \mathrm{id}_x) \colon (a, x, L(a) \xrightarrow{I \circ L(f)} x) \to (b, x, L(b) \overset{I}{\to} x).
$$

With $(a, x, L(a) \overset{I}{\to} x)$ over $a$ in $\mathsf{X}$, the cocartesian lifting of $f$ is given by

$$
(f, \kappa) \colon (a, x, L(a) \overset{I}{\to} x) \to (b, x +_{L(a)} L(b), L(b) \overset{\theta}{\to} x +_{L(a)} L(b))
$$

where $\kappa$ and $\theta$ are as in the following diagram:

$$
\begin{array}{c}
\\ L(b) \\
{}^{L(f)}\nearrow \quad \vdots \quad \searrow{}^{L(p)} \\
L(a) \xrightarrow{\quad L(f')\quad} L(c) \\
\quad \vdots\, {}^{\theta} \\
{\scriptstyle I}\downarrow \qquad x +_{L(a)} L(b) \\
\kappa\quad\nearrow \qquad \exists!\psi \\
x \xrightarrow{\quad g'\quad} y
\end{array}
$$

Both cartesian and cocartesian liftings are preserved by the tensor product of $(L/\mathsf{X})$.

Lastly, we show that $U_2\colon (L/\mathsf{X}) \to \mathsf{A}$ satisfies the Beck-Chevalley condition. Given a pushout square in $\mathsf{A}$:

$$
\begin{array}{ccc}
a & \xrightarrow{\ h\ } & b \\
{\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\
c & \xrightarrow{\ f\ } & d
\end{array}
$$

we show that the following square commutes up to a natural isomorphism:

$$
\begin{array}{ccc}
(L/\mathsf{X})_a & \xleftarrow{\ h^*\ } & (L/\mathsf{X})_b \\
{\scriptstyle k_!}\downarrow & \cong & \downarrow{\scriptstyle g_!} \\
(L/\mathsf{X})_c & \xleftarrow{\ f^*\ } & (L/\mathsf{X})_d
\end{array}
$$

Let $(b, x, L(b) \xrightarrow{I} x)$ be an object of $(L/\mathsf{X})_b$. Going right, we precompose with $L(h)\colon L(a) \to L(b)$ to obtain the object $(a, x, L(a) \xrightarrow{I \circ L(h)} x)$ in $(L/\mathsf{X})_a$. Then, going down, we pushforward along $L(k)\colon L(a) \to L(c)$:

$$
\begin{array}{ccc}
L(a) & \xrightarrow{\ I \circ L(h)\ } & x \\
{\scriptstyle L(k)}\downarrow & & \downarrow \\
L(c) & \xrightarrow{\ \psi\ } & L(c) +_{L(a)} x
\end{array}
$$

to obtain the object $(c, L(c) +_{L(a)} x, L(c) \xrightarrow{\psi} L(c) +_{L(a)} x)$ in $(L/\mathsf{X})_c$. For the other route, first going down, we pushforward the object $(b, x, L(b) \xrightarrow{I} x)$ along $L(g)\colon L(b) \to L(d)$ by taking the following pushout:

$$
\begin{array}{ccc}
L(b) & \xrightarrow{\ I\ } & x \\
{\scriptstyle L(g)}\downarrow & & \downarrow \\
L(d) & \xrightarrow{\ \theta\ } & L(d) +_{L(b)} x
\end{array}
$$

which yields the object $(d, L(d)+_{L(b)}x, L(d) \xrightarrow{\theta} L(d)+_{L(b)}x)$ of $(L/\mathsf{X})_d$. We then precompose with $L(f)\colon L(c) \to L(d)$ to obtain the object $(c, L(d) +_{L(b)} x, L(c) \xrightarrow{\theta \circ L(f)} L(d) +_{L(b)} x)$ of $(L/\mathsf{X})_c$.

To establish a natural isomorphism between the two objects $(c, L(c) +_{L(a)} x, L(c) \xrightarrow{\psi} L(c) +_{L(a)} x)$ and $(c, L(d) +_{L(b)} x, L(c) \xrightarrow{\theta \circ L(f)} L(d) +_{L(b)} x)$ in $(L/\mathsf{X})_c$, it suffices to establish a natural isomorphism between the objects $L(c) +_{L(a)} x$ and $L(d) +_{L(b)} x$ in $\mathsf{X}$. The first pushout square above exhibits $L(c) +_{L(a)} x$ as a colimit of the span

$$
L(c) \xleftarrow{\ L(k)\ } L(a) \xrightarrow{\ I \circ L(h)\ } x
$$

in $\mathsf{X}$. On the other hand, as $L$ is a left adjoint, $L$ preserves pushout squares, and so the following square is also a pushout:

$$
\begin{array}{ccc}
L(a) & \xrightarrow{\ L(h)\ } & L(b) \\
{\scriptstyle L(k)}\downarrow & & \downarrow{\scriptstyle L(g)} \\
L(c) & \xrightarrow{\ L(f)\ } & L(d)
\end{array}
$$

which we may then paste to the left of the second above pushout square to obtain another pushout square:

$$
\begin{array}{ccccc}
L(a) & \xrightarrow{\ L(h)\ } & L(b) & \xrightarrow{\ I\ } & x \\
{\scriptstyle L(k)}\downarrow & & \downarrow{\scriptstyle L(g)} & & \downarrow \\
L(c) & \xrightarrow{\ L(f)\ } & L(d) & \xrightarrow{\ \theta\ } & L(d) +_{L(b)} x
\end{array}
$$

which exhibits $L(d) +_{L(b)} x$ also as a colimit of the same span in $\mathsf{X}$, and hence the desired natural isomorphism between our two objects of $(L/\mathsf{X})_c$. Thus $U_2$ satisfies the Beck-Chevalley condition.  □

The following theorem establishes an isomorphism between structured and decorated cospan double categories under certain conditions. Let **Rex** be the 2-category of categories with finite colimits, functors preserving finite colimits, and natural transformations. Let **SymMonCat** be the 2-category of symmetric monoidal categories, strong symmetric monoidal functors and natural transformations. Recall that for us a category $\mathsf{C} \in \textbf{Rex}$ comes with a choice of finite colimits, so it gives a specific cocartesian monoidal category $(\mathsf{C}, +)$, and this induces a 2-functor **Rex** → **SymMonCat**.

**Theorem 3.2.** *Suppose* $\mathsf{A}$ *has finite colimits and* $F \colon (\mathsf{A}, +) \to (\textbf{Cat}, \times)$ *is a symmetric lax monoidal pseudofunctor. If the corresponding pseudofunctor* $F \colon \mathsf{A} \to \textbf{SymMonCat}$ *factors through* **Rex**, *then the symmetric monoidal double categories* $F\mathbb{C}\textbf{sp}$ *of decorated cospans and* $_L\mathbb{C}\textbf{sp}(\int F)$ *of structured cospans are isomorphic, where* $L \colon \mathsf{A} \to \int F$ *is a left adjoint of the induced Grothendieck opfibration* $U \colon \int F \to \mathsf{A}$.

The 'corresponding pseudofunctor' comes from the so-called monoidal Grothendieck construction, and the conditions of this theorem relate to the existence of colimits as well as left adjoints for opfibrations. We first sketch the relevant underlying framework in detail, and then we proceed to the proof of the theorem. The basics of fibration theory needed for our purposes are recalled in Appendix A.2.

In [37], the classical Grothendieck construction is generalized to the monoidal setting: given a (symmetric) monoidal category $\mathsf{A}$, there is a bijection between **(symmetric) monoidal opindexed categories**, which are (symmetric) lax monoidal pseudofunctors $F \colon (\mathsf{A}, \otimes_A) \to (\textbf{Cat}, \times)$, and **(symmetric) monoidal opfibrations**, which are opfibrations $U \colon \mathsf{X} \to \mathsf{A}$ where $\mathsf{X}, \mathsf{A}$ are (symmetric) monoidal, $U$ is (symmetric) strict monoidal, and $\otimes_\mathsf{X}$ preserves cocartesian liftings. This bijection sends a monoidal opindexed category $F \colon (\mathsf{A}, \otimes_A) \to (\textbf{Cat}, \times)$ to the monoidal opfibration $U \colon \int F \to \mathsf{A}$. If the monoidal structure on $\mathsf{A}$ is in fact cocartesian, there is a further correspondence between these structures and ordinary pseudofunctors $F \colon \mathsf{A} \to \textbf{MonCat}$ (or to **SymMonCat** in the symmetric case).

In fact, all these correspondences lift to 2-equivalences between appropriate 2-categories.

**Lemma 3.3.** *There is a 2-equivalence between the 2-categories of monoidal opfibrations and monoidal opindexed categories, and if the base is cocartesian monoidal, there is a 2-equivalence between these and pseudofunctors into* **MonCat**. *Similarly there is a 2-equivalence between symmetric monoidal opfibrations and symmetric monoidal opindexed categories, and if the base is cocartesian monoidal, also a 2-equivalence between these and pseudofunctors into* **SymMonCat**.

*Proof.* This was shown by Moeller and the third author [37, Theorems 3.13 & 4.2]. In summary, for a cocartesian base $\mathsf{A}$ we have correspondences

$$\text{lax monoidal pseudofunctors } F \colon (\mathsf{A}, +) \to (\textbf{Cat}, \times)$$
$$\Updownarrow$$
$$\text{monoidal opfibrations } U \colon (\mathsf{X}, \otimes_\mathsf{X}) \to (\mathsf{A}, +)$$
$$\Updownarrow$$
$$\text{pseudofunctors } F \colon \mathsf{A} \to \textbf{MonCat}$$

The second equivalence was observed earlier by Shulman [42]. Moreover, symmetric lax monoidal pseudofunctors correspond to symmetric monoidal opfibrations, and those to pseudofunctors into **SymMonCat**.

In more detail, if $(\phi, \phi_0)$ is the lax monoidal structure of the pseudofunctor $F$ as recalled in Appendix A.1, the induced monoidal structure on the total Grothendieck category $\mathsf{X} = \int F$ (Definition A.1) is given by

$$\left(a, x \in F(a)\right) \otimes_\mathsf{X} \left(b, y \in F(b)\right) = \left(a + b, \phi_{a,b}(x, y) \in F(a + b)\right), \qquad I_\mathsf{X} = \left(0_A, \phi_0\right) \tag{4}$$

If $F$ is a symmetric lax monoidal pseudofunctor, the induced monoidal structure in $\int F$ is symmetric via

$$\left(\beta_{a,b}, (u_{a,b})_{x,y}\right) \colon (a + b, \phi_{a,b}(x, y)) \xrightarrow{\sim} (b + a, \phi_{b,a}(y, x))$$

where $\beta$ is the canonical symmetry for $\mathsf{A}$ and $u$ is the natural isomorphism of (15).

Moreover, each fiber $\mathsf{X}_a = F(a)$ obtains a monoidal structure via

$$\otimes_a \colon F(a) \times F(a) \xrightarrow{\phi_{a,a}} F(a + a) \xrightarrow{F(\nabla)} F(a), \quad I_a \colon 1 \xrightarrow{\phi_0} F(0) \xrightarrow{F(!_a)} F(a) \tag{5}$$

where $\nabla$ is the fold map, which is symmetric when $F$ is again via the components of $u_{a,a}$. Also, each reindexing functor $f_! = F(f)$ obtains a strong monoidal structure with

$$
\begin{array}{ccccc}
F(a) \times F(a) & \xrightarrow{\phi_{a,a}} & F(a+a) & \xrightarrow{F(\nabla)} & F(a) \\
{\scriptstyle F(f)\times F(f)}\Big\downarrow & {\scriptstyle \phi_{f,f}}\!\!\underset{(14)}{\cong} & \Big\downarrow{\scriptstyle F(f+f)} & \cong & \Big\downarrow{\scriptstyle Ff} \\
F(b) \times F(b) & \xrightarrow[\phi_{b,b}]{} & F(b+b) & \xrightarrow[F(\nabla)]{} & F(b)
\end{array}
\tag{6}
$$

$\square$

These 2-equivalences further restrict to the case when the Grothendieck category $(X, \otimes_X)$ is specifically cocartesian monoidal itself, with coproducts built up from (4). In that case, opfibrations $(X, +) \to (A, +)$ that strictly preserve coproducts and initial object bijectively correspond to pseudofunctors into the 2-category of cocartesian categories. For more details, see [37, Corollary 4.7] and the related discussion.

Finally, we can further restrict to the setting of opfibrations that also preserve pushouts, and thus all finite colimits, thanks to the following more general result.

**Lemma 3.4** (**Hermida**). *Suppose* $J$ *is a small category and* $U: X \to A$ *is an opfibration where the base* $A$ *has* $J$-*colimits. Then the following are equivalent:*

*(1) all fibers have $J$-colimits, and the reindexing functors preserve them;*

*(2) the total category $X$ has $J$-colimits, and $U$ preserves them.*

*Moreover, if $X$ has $J$-colimits and $U$ preserves them, for any choice of $J$-colimits in $A$, they can be chosen in $X$ in such a way that $U$ strictly preserves them.*

*Proof.* See [28, Corollary 4.9], and for the final statement [28, Remark 4.11]. $\square$

The first part formulates the existence of colimits *locally* in each fiber, and if we let $J$ range over all finite categories it says that the corresponding pseudofunctor $F: A \to \mathbf{Cat}$ lands in the sub-2-category $\mathbf{Rex}$. The second part formulates the existence of colimits *globally* in the total category $\int F$, and if we let $J$ range over all finite categories it says that $X$ has finite colimits and $U$ preserves all finite colimits. As an example, suppose that $A$ has pushouts and a Grothendieck opfibration $U: \int F \to A$ has fiberwise pushouts preserved by its reindexing functors. We can construct pushouts in $\int F$ as follows:

$$
\begin{array}{ccc}
 & (a +_b c, w) & \\
\nearrow & \vee & \nwarrow \\
(a, x) & & (c, z) \\
\nwarrow & & \nearrow \\
\left\{\begin{array}{ll} f: b\to a & \text{in } A \\ k: F(f)y \to x & \text{in } F(a) \end{array}\right. & & \left\{\begin{array}{ll} g: b \to c & \text{in } A \\ \ell: F(g)y \to z & \text{in } F(c) \end{array}\right. \\
& (b, y) &
\end{array}
\tag{7}
$$

where $a +_b c$ is defined using the pushout in $A$ shown at left below, and $w$ is formed as the pushout in the fiber $F(a +_b c)$ at the right below:

$$
\begin{array}{ccc}
 & a +_b c & \\
{\scriptstyle v_a}\nearrow \ \vee \ \nwarrow{\scriptstyle v_c} & & \\
a \qquad\qquad c & & \\
\nwarrow \qquad \nearrow & & \\
{\scriptstyle f} \quad b \quad {\scriptstyle g} & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & w & \\
\nearrow & \vee & \nwarrow \\
F(v_a)x & & F(v_c)z \\
\underset{F(v_a)k}{\nwarrow} & & \underset{F(v_c)\ell}{\nearrow} \\
& F(v_a)(F(f)(y)) \cong F(v_c)(F(g)(y)) &
\end{array}
$$

In a similar way we can construct the initial object and coproducts in $\int F$ from those in the fibers, namely

$$
(a, x) + (b, y) = (a + b, F(\iota_a)x + F(\iota_b)y), \qquad 0_{\int F} = (0_A, 0_{F(0_A)})
\tag{8}
$$

where $\iota_a: a \to a + b$ and $\iota_b: b \to a + b$ are the sum inclusions in $A$ and the sum on the second variable is in the fiber $F(a + b)$.

**Corollary 3.5.** *Suppose* A *has finite colimits and* $F \colon (\mathsf{A}, +) \to (\mathbf{Cat}, \times)$ *is a symmetric lax monoidal pseudo-functor for which the corresponding pseudofunctor* $F \colon \mathsf{A} \to \mathbf{SymMonCat}$ *from Lemma 3.3 factors through* **Rex**. *Then* $\int F$ *has all finite colimits and the induced opfibration* $U \colon \int F \to \mathsf{A}$ *preserves them. Moreover we can choose finite colimits in* $\int F$ *so that* $U$ *preserves them strictly.*

Regarding the assumptions of the above corollary, notice that since A is cocartesian monoidal, the lax monoidal pseudofunctor structure $(F, \phi, \phi_0)$ gives rise to a specific symmetric monoidal structure on the fibers $F(a)$ in terms of $\phi, \phi_0$, given explicitly by (5). Since we now ask that the pseudofunctor $F \colon \mathsf{A} \to \mathbf{SymMonCat}$ factors through **Rex**, this fiberwise monoidal structure is required to be cocartesian, namely (5) gives coproducts and an initial object in each $F(a)$.

On a highly related note, in what follows we are also interested in the existence of a left adjoint $L$ to an induced monoidal opfibration. The following result provides sufficient conditions for that. Following Gray [24], we say a functor has a 'left adjoint right inverse' or **lari** if it has a left adjoint where the unit of the adjunction is the identity.

**Lemma 3.6** (**Gray**). *Let* $U \colon \mathsf{X} \to \mathsf{A}$ *be an opfibration. Then* $U$ *has a lari if its fibers have initial objects that are preserved by the reindexing functors.*

*Proof.* This is [24, Proposition 4.4]. Suppose each fiber $\mathsf{X}_a$ of the opfibration $U$ has an initial object $\perp_a$ and these objects are preserved (up to isomorphism) by the reindexing functors. Define $L \colon \mathsf{A} \to \mathsf{X}$ on objects $a \in \mathsf{A}$ by $L(a) = \perp_a$. Given a morphism $f \colon a \to a'$ in A, define $L(f)$ to be the composite

$$\perp_a \xrightarrow{\mathrm{Cocart}(f, \perp_a)} f_!(\perp_a) \xrightarrow{\chi_a} \perp_{a'} \tag{9}$$

where $\mathrm{Cocart}(f, \perp_a)$ is the cocartesian lifting of $f$ to $\perp_a$ and $\chi_a$ is the unique isomorphism between two initial objects in the fiber above $a'$. The functor $L$ then becomes left adjoint to $R$ with unit $\iota_a \colon a \to U(L(a))$ being the identity, using the fact that $U(L(a)) = U(\perp_a) = a$. □

We now have all the necessary background to formally construct an isomorphism between the double category of decorated cospans and the double category of structured cospans, starting from a symmetric lax monoidal pseudofunctor $F \colon (\mathsf{A}, +) \to (\mathbf{Cat}, \times)$ whose corresponding pseudofunctor $F \colon \mathsf{A} \to \mathbf{SymMonCat}$ factors through **Rex**.

*Proof of Theorem 3.2.* Since we are assuming the pseudofunctor $F \colon \mathsf{A} \to \mathbf{SymMonCat}$ factors through **Rex**, Corollary 3.5 implies that the Grothendieck construction gives rise to a category $\int F$ with finite colimits, and we can choose these in such a way that the corresponding opfibration $U \colon \int F \to \mathsf{A}$ strictly preserves them. We do this in what follows.

By Lemma 3.6, $U$ has a left adjoint $L \colon \mathsf{A} \to \int F$ with $UL = 1_\mathsf{A}$. Diagrammatically,

$$F \colon \mathsf{A} \to \mathbf{Cat} \quad \mapsto \quad \begin{array}{c} \int F \\ U \downarrow \\ \mathsf{A} \end{array} \quad \mapsto \quad \mathsf{A} \underset{U}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \int F \tag{10}$$

describes the construction of the adjunction from the original $F$. Explicitly, the left adjoint picks the initial object in the fiber of $a$, namely $L(a) = (a, \perp_a)$ which is expressed as $F(!_a) \circ \phi_0(*)$ according to (5) for $(\phi, \phi_0)$ the monoidal structure of $F$.

As a left adjoint, this induced $L$ preserves all colimits that exist between the categories A and $\int F$, which have finite colimits, so we can construct the double category of structured cospans $_L\mathbb{C}\mathbf{sp}(\int F)$ of Theorem 3.1.

**N**EW PROOF HERE. To show that the symmetric monoidal double categories $F\mathbb{C}\mathbf{sp}$ and $_L\mathbb{C}\mathbf{sp}(\int F)$ of Theorems 2.2 and 3.1 are isomorphic, we make use of a result of Shulman [43, Theorem 14.11] which states that there exists a 2-functor **Fr**: **FMF** → **FDbl**. Here, **FMF** is the 2-category consisting of

(1) monoidal bifibrations satisfying the Beck-Chevalley condition,
(2) strong monoidal morphisms of bifibrations, and
(3) monoidal transformations of bifibrations.

and **FDbl** is the 2-category of fibrant double categories, double functors and double transformations. Given two monoidal bifibrations $\Phi \colon \mathsf{A} \to \mathsf{B}$ and $\Phi' \colon \mathsf{A}' \to \mathsf{B}'$, a **strong monoidal morphism of bifibrations** from

$\Phi$ to $\Phi'$ consists of a pair of strong monoidal functors $F_1$ and $F_2$ making the following square commute:

$$
\begin{array}{ccc}
\mathsf{A} & \xrightarrow{F_1} & \mathsf{A}' \\
{\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \Phi} \\
\mathsf{B} & \xrightarrow{F_2} & \mathsf{B}'
\end{array}
$$

and such that the identity $\Phi'F_1 = F_2\Phi$ is a monoidal transformation. The two monoidal fibrations in question are the functors $U_1$ and $U_2$ of Theorems 2.2 and 3.1, respectively, which share $\mathsf{A}$ as a common base, and thus we only need a single strong monoidal functor $F\colon (\mathsf{A}/U) \to (L/\mathsf{X})$ making the following diagram commute:

$$
\begin{array}{ccc}
(\mathsf{A}/U) & \xrightarrow[\cong]{F} & (L/\int F) \\
{\scriptstyle U_1}\searrow & & \swarrow{\scriptstyle U_2} \\
& \mathsf{A} &
\end{array}
$$

with $U_2F = U_1$ a monoidal transformation. The functor $F\colon (\mathsf{A}/U) \to (L/\int F)$ is expressed by the natural isomorphism of an adjunction, namely:

$$F(a, x, a \to U(x)) = (a, x, L(a) \to x)$$

and is the identity on pairs of morphisms making the required squares commute. It is clear that $F$ is strong monoidal via component-wise binary coproducts which are preserved by both $L$ and $U$, and that the identity $U_2F = U_1$ is a monoidal transformation. As $F$ is an isomorphism, we obtain an isomorphism

$$\mathbf{Fr}(F)\colon \mathbf{Fr}(U_1) \to \mathbf{Fr}(U_2)$$

between the two symmetric monoidal double categories $\mathbb{Fr}(U_1) = F\mathbb{C}\mathbf{sp}$ and $\mathbf{Fr}(U_2) =_L \mathbb{C}\mathbf{sp}(\int F)$ of Theorems 2.2 and 3.1, respectively.                                                                $\square$

In fact, a careful examination of the proof reveals that the assumptions of Theorem 3.2 could be weakened to state that the pseudofunctor $F\colon \mathsf{A} \to \mathbf{SymMonCat}$ factors through the 2-category $\mathbf{Cocart}$ of cocartesian monoidal categories. The reason is that when we compose two structured cospans as in **??**, the apex of the composite cospan ends up being a sum in the second variable due to the definition of the left adjoint functor $L$ which forces the feet of all cospans to involve a fiberwise initial object. However, in the most interesting examples seen so far, when $F$ factors through $\mathbf{Cocart}$ it also factors through $\mathbf{Rex}$.

## 4. Bicategorical and categorical aspects

While double categories are a natural context for studying cospans, bicategories are more familiar—and of course, *categories* are even more so! Luckily, all our results phrased in the language of double categories have analogues for bicategories and categories. We explain those here.

As discussed for example by Shulman [43], any double category $\mathbb{D}$ has a **horizontal bicategory**, denoted $\mathbf{D}$, in which:

- objects are objects of $\mathbb{D}$,
- morphisms are horizontal 1-cells of $\mathbb{D}$,
- 2-morphisms are **globular** 2-morphisms of $\mathbb{D}$, meaning 2-morphisms whose source and target vertical 1-morphisms are identities,
- composition of morphisms is given by composition of horizontal 1-cells in $\mathbb{D}$,
- vertical and horizontal composition of 2-morphisms are given by vertical and horizontal composition of 2-morphisms in $\mathbb{D}$.

The bicategory $\mathbf{D}$ has a **decategorification**, a category $\mathsf{D}$ in which:

- objects are objects of $\mathbf{D}$,
- morphisms are isomorphism classes of morphisms of $\mathbf{D}$.

Thus, the double category $F\mathbb{C}\mathbf{sp}$ of structured cospans constructed in **??** automatically gives rise to a bicategory $F\mathsf{Csp}$, and a category $F\mathsf{Csp}$. In Theorem 2.2 we gave conditions under which the double category $F\mathbb{C}\mathbf{sp}$ becomes symmetric monoidal. We would like the bicategory $F\mathsf{Csp}$ and the category $F\mathsf{Csp}$ to become symmetric monoidal under the same conditions, and indeed this is true.

A double category is 'fibrant' if every vertical 1-morphism has a 'companion' and a 'conjoint'—concepts explained in Definition A.10. Shulman (Theorem A.12) proved that when a double category $\mathbb{D}$ is fibrant, any symmetrical monoidal structure on $\mathbb{D}$ gives one on **D**. We can apply this to decorated cospans as follows:

**Lemma 4.1.** *The double category $F\mathbb{C}\mathbf{sp}$ is fibrant.*

*Proof.* We show that any vertical 1-morphism $f\colon a \to b$ in $F\mathbb{C}\mathbf{sp}$ has a companion and a conjoint. First, we can make this horizontal 1-cell $\hat{f}$:

$$a \xrightarrow{\ f\ } b \xleftarrow{\ 1\ } b \qquad\qquad \bot_b \in F(b)$$

where $\bot_b$ is the trivial decoration as in **??**, into a companion of $f$ using the following 2-morphisms:



where the decoration morphism $\tau_f$ is the isomorphism given by the pseudofunctoriality of $F$:



These 2-morphisms satisfy the equations (19) required of a companion, involving vertical and horizontal composition of 2-morphisms in this double category:

$$
\begin{array}{c}
\perp_a \in F(a) \qquad\qquad \perp_b \in F(b) \\[4pt]
\begin{array}{ccccccccc}
a & \xrightarrow{1} & a & \xleftarrow{1} & a & \xrightarrow{f} & b & \xleftarrow{1} & b
\end{array}
\end{array}
$$

$\tau_f : F(f)(\perp_a) \to \perp_b \qquad$ diagram $\qquad \tau_b = 1_{\perp_b}$

$\perp_b \in F(b)$

$\perp_b \in F(b) \quad =$

$$\tau_{\lambda_{\hat f}} : (\perp_b \odot \perp_b) \to \perp_b$$

$$\tau_{\rho_{\hat f}} : (\perp_b \odot \perp_a) \to \perp_b$$

Note that the right hand side of the first equation is $U_f$, while the second equation involves the left and right unitors for $\odot$: these are maps from a horizontal composite of two decorated cospans to a single decorated cospan. The conjoint of $f$ is given by this horizontal 1-cell $\check f$, which is just the opposite of the companion above:

$$
b \xrightarrow{\;1\;} b \xleftarrow{\;f\;} a \qquad\qquad \perp_b \in F(b).
$$

Just as $\hat f$ obeys the equations required of a companion, $\check f$ obeys the equations required of a conjoint with similar structure 2-morphisms to those of a companion above.  $\qquad\square$

**Theorem 4.2.** *Let* $\mathsf{A}$ *be a category with finite colimits and* $F\colon (\mathsf{A}, +) \to (\mathbf{Cat}, \times)$ *a symmetric lax monoidal pseudofunctor. Then there exists a symmetric monoidal bicategory* $F\mathbf{Csp}$ *in which:*

(1) *objects are those of* $\mathsf{A}$,

(2) *morphisms are F-decorated cospans:*

$$
a \xrightarrow{\;i\;} m \xleftarrow{\;o\;} b \qquad\qquad x \in F(m),
$$

(3) *a 2-morphism is a map of cospans in* $\mathsf{A}$



$x \in F(m)$

$x' \in F(m')$

*together with a morphism* $\tau\colon F(h)(x) \to x'$ *in* $F(m')$.

*Proof.* This follows by applying Shulman's result (Theorem A.12) to the fibrant symmetric monoidal double category $F\mathbb{C}\mathbf{sp}$.  $\qquad\square$

This symmetric monoidal bicategory $F\mathbf{Csp}$ generalizes the one constructed by the second author [13]. We can decategorify $F\mathbf{Csp}$ to obtain a symmetric monoidal category generalizing the kind considered by Fong [17]:

**Corollary 4.3.** *Let* $\mathsf{A}$ *be a category with finite colimits and* $F\colon (\mathsf{A}, +) \to (\mathbf{Cat}, \times)$ *a symmetric lax monoidal pseudofunctor. Then there exists a symmetric monoidal category* $F\mathsf{Csp}$ *in which:*

(1) *objects are those of* $\mathsf{A}$

(2) *morphisms are isomorphism classes of F-decorated cospans of* $\mathsf{A}$, *where two F-decorated cospans*

$$
a \xrightarrow{\;i\;} m \xleftarrow{\;o\;} b \qquad\qquad x \in F(m)
$$

$$
a \xrightarrow{\;i'\;} m' \xleftarrow{\;o'\;} b \qquad\qquad x' \in F(m')
$$

*are isomorphic if and only if there exists an isomorphism* $f\colon m \to m'$ *in* A *such that following diagram commutes:*

$$
\begin{array}{ccc}
 & m & \\
{}^{i}\nearrow & \downarrow f & \nwarrow^{o} \\
a & & b \\
{}_{i'}\searrow & \uparrow & \swarrow_{o'} \\
 & m' &
\end{array}
$$

*and there exists an isomorphism* $\tau\colon F(f)(x) \to x'$ *in* $F(m')$.

In Theorem 3.2 we gave conditions under which the symmetric monoidal double category of *decorated* cospans $F\mathbb{C}\mathbf{sp}$ is isomorphic to the symmetric monoidal double category of *structured* cospans $_L\mathbb{C}\mathbf{sp}(\smallint F)$. We now show that under the same conditions we get an isomorphism of symmetric monoidal bicategories, and of categories.

**Theorem 4.4.** *Suppose* A *has finite colimits and* $F\colon (\mathsf{A}, +) \to (\mathbf{Cat}, \times)$ *is a symmetric lax monoidal pseudofunctor that factors through* **Rex** *as an ordinary pseudofunctor. Define the symmetric monoidal bicategory* $_L\mathbf{Csp}(\smallint F)$ *as in Theorem 3.2. Then there is an isomorphism of symmetric monoidal bicategories*

$$F\mathbf{Csp} \cong {}_L\mathbf{Csp}(\smallint F)$$

*and of symmetric monoidal categories*

$$F\mathsf{Csp} \cong {}_L\mathsf{Csp}(\smallint F).$$

*Proof.* Hansen and Shulman [27] showed that the passage from symmetric monoidal double categories to symmetric monoidal bicategories is functorial in a suitable sense. This implies that an isomorphism of symmetric monoidal double categories $\mathbb{D} \cong \mathbb{D}'$ gives an isomorphism of symmetric monoidal bicategories $\mathbf{D} \cong \mathbf{D}'$. Since the process of decategorifying a bicategory merely discards 2-morphisms and takes isomorphism classes of 1-morphisms, the isomorphism of symmetric monoidal bicategories $\mathbf{D} \cong \mathbf{D}'$ in turn induces an isomorphism of symmetric monoidal categories $\mathsf{D} \cong \mathsf{D}'$. Thus, the theorem follows from Theorem 3.2. □

## 5. APPLICATIONS

Thinking about systems and processes categorically dates back to early works by Lawvere [35], Bunge–Fiore [11], Joyal–Nielsen–Winskel [30], Katis–Sabadini–Walters [31] and others. Spivak and others have used wiring diagrams and sheaves to capture compositional features of dynamical systems, [8, 41, 45]. Another approach uses signal flow diagrams and other string diagrams [3, 9, 19] to understand systems behaviorally, following ideas of Willems [47].

Decorated cospans were introduced by Fong [17, 18] to describe open systems as cospans equipped with extra data. They were then applied to open electrical circuits [4], Markov processes [5], and chemical reaction networks [7]. Unfortunately, some of these applications were marred by technical flaws, which were later fixed using structured cospans [1]. Here we explain how to also fix them using our new decorated cospans, since they provide another solution to these problems. Below we compare the two approaches in applications to graphs, electrical circuits, Petri nets, reaction networks and dynamical systems.

In many cases, Theorem 3.2 shows that the structured and decorated cospan approaches are equivalent: Sections 5.1 to 5.3 illustrate this. However, in Section 5.4 we describe a map from 'open Petri nets with rates' to open dynamical systems. The former are conveniently described using structured cospans, while the latter can only be treated using decorated cospans. Here Theorem 3.2 serves as a bridge that lets us connect the two formalisms.

5.1. **Graphs.** One of the simplest kinds of network is a graph. For us a **graph** will be a pair of functions $s, t\colon E \to N$ where $E$ and $N$ are finite sets. We call elements of $E$ **edges** and elements of $N$ **nodes**. There is a category Graph where the objects are graphs and a morphism from the graph $s, t\colon E \to N$ to the graph $s', t'\colon E' \to N'$ is a pair of functions $f\colon E \to E', g\colon N \to N'$ such that these diagrams commute:

$$
\begin{array}{ccc}
E & \xrightarrow{s} & N \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
E' & \xrightarrow{s'} & N'
\end{array}
\qquad\qquad
\begin{array}{ccc}
E & \xrightarrow{t} & N \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
E' & \xrightarrow{t'} & N'.
\end{array}
$$

We can easily build a double category with 'open graphs' as horizontal 1-cells using the machinery of structured cospans, see [1, Section 5]. Let $L\colon \mathsf{FinSet} \to \mathsf{Graph}$ be the functor that assigns to a finite set $N$ the **discrete graph** on $N$: the graph with no edges and $N$ as its set of vertices. Both $\mathsf{FinSet}$ and $\mathsf{Graph}$ have finite colimits, and the functor $L\colon \mathsf{FinSet} \to \mathsf{Graph}$ is left adjoint to the forgetful functor $R\colon \mathsf{Graph} \to \mathsf{FinSet}$ that assigns to a graph $G$ its underlying set of vertices $R(G)$. Thus, using structured cospans and appealing to Theorem 3.1, we get a symmetric monoidal double category $_L\mathbb{C}\mathbf{sp}(\mathsf{Graph})$ in which:
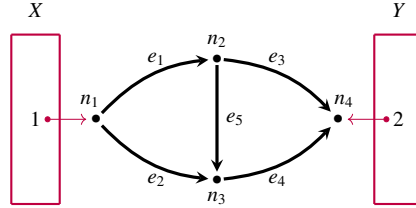
- objects are finite sets,
- a vertical 1-morphism from $X$ to $Y$ is a function $f\colon X \to Y$,
- a horizontal 1-cell from $X$ to $Y$ is an **open graph** from $X$ to $Y$, meaning a cospan in $\mathsf{Graph}$ of this form:

$$L(X) \xrightarrow{\ i\ } G \xleftarrow{\ o\ } L(Y),$$

- a 2-morphism is a commuting diagram in $\mathsf{Graph}$ of this form:

$$
\begin{array}{ccccc}
L(X_1) & \xrightarrow{\ I_1\ } & G_1 & \xleftarrow{\ O_1\ } & L(Y_1) \\
{\scriptstyle L(f)}\downarrow & & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle L(g)} \\
L(X_2) & \xrightarrow{\ I_2\ } & G_2 & \xleftarrow{\ O_2\ } & L(Y_2).
\end{array}
$$

Here is an example of an open graph:



We can also build a double category with open graphs as horizontal 1-cells using decorated cospans. For any finite set $N$, there is a category $F(N)$ where:

- an object is a **graph structure** on $N$: that is, a graph $s, t\colon E \to N$,
- a morphism from $s, t\colon E \to N$ to $s', t'\colon E' \to N$ is a morphism of graphs that is the identity on $N$: that is, a function $g\colon E \to E'$ such that these diagrams commute:



In general, decorated cospans involve a pseudofunctor to **Cat**, but in this example there is actually an honest functor $F\colon \mathsf{Set} \to \mathbf{Cat}$ that assigns to a set $N$ the above category $F(N)$. Given a function $f\colon M \to N$, we define $F(f)\colon F(M) \to F(N)$ as the functor that maps any graph structure $s, t\colon E \to M$ to the graph structure $fs, ft\colon E \to N$.

We can make $F$ into a symmetric lax monoidal pseudofunctor $F\colon (\mathsf{FinSet}, +) \to (\mathbf{Cat}, \times)$ by equipping it with suitable functors

$$\phi_{N,N'}\colon F(N) \times F(N') \to F(N + N'), \qquad \phi_0\colon 1 \to F(\emptyset).$$

The functor $\phi_0$ is uniquely determined since $F(\emptyset)$ is the terminal category. More interesting is $\phi_{N,N'}$. This functor maps a pair of graph structures $s, t\colon E \to N$ and $s', t'\colon E' \to N'$ to the graph structure $s + s', t + t'\colon E + E' \to N + N'$. In other words, it sends a pair of graph structures to their 'disjoint union'. Surprisingly, though $F$ is a functor, this choice of $\phi_{N,N'}$ does not make $F$ into a symmetric lax monoidal functor, but only a symmetric lax monoidal pseudofunctor, since it obeys the required laws only up to natural isomorphism, as in (12). See [1, Section 5] for a proof that these laws fail to hold on the nose. This fact is what necessitated a generalization of Fong's original approach to decorated cospans.

It is well known, and easy to check, that the Grothendieck category $\int F$ is isomorphic to the category $\mathsf{Graph}$. The other side of this observation is that the opfibration $U\colon \int F \to \mathsf{FinSet}$ is isomorphic to the

forgetful functor $R \colon \mathsf{Graph} \to \mathsf{FinSet}$. In fact one can check that $U \colon \int F \to \mathsf{FinSet}$ and $R \colon \mathsf{Graph} \to \mathsf{FinSet}$ are isomorphic as symmetric monoidal opfibrations, where all the categories involved are given cocartesian monoidal structures.

Starting from the symmetric lax monoidal pseudofunctor $F \colon (\mathsf{FinSet}, +) \to (\mathbf{Cat}, \times)$, Theorem 2.2 gives us a symmetric monoidal double category $F\mathbb{C}\mathbf{sp}$ in which:

- objects are sets,
- a vertical 1-morphism from $X$ to $X'$ is a function $f \colon X \to X'$,
- a horizontal 1-cell from $X$ to $Y$ is a pair

$$X \xrightarrow{\ i\ } N \xleftarrow{\ o\ } Y \qquad G \in F(N)$$

which can also be thought of as an open graph from $X$ to $Y$,
- a 2-morphism

$$
\begin{array}{ccccc}
X & \xrightarrow{\ i\ } & N & \xleftarrow{\ o\ } & Y & \quad G \in F(N) \\
{\scriptstyle f}\big\downarrow & & {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle g} & \\
X' & \xrightarrow{\ i'\ } & N' & \xleftarrow{\ o'\ } & Y' & \quad G' \in F(N')
\end{array}
$$

is a commuting diagram in $\mathsf{FinSet}$ together with a morphism $\tau \colon F(h)(G) \to G'$ in $F(N')$.

We thus have two symmetric monoidal double categories: $_L\mathbb{C}\mathbf{sp}(\mathsf{Graph})$ obtained from structured cospans and $F\mathbb{C}\mathbf{sp}$ obtained from decorated cospans. Each of these double categories has $\mathsf{FinSet}$ as its category of objects, open graphs as horizontal 1-cells, and maps of open graphs as 2-morphisms. This suggests that $_L\mathbb{C}\mathbf{sp}(\mathsf{Graph})$ and $F\mathbb{C}\mathbf{sp}$ are isomorphic as symmetric monoidal double categories—and indeed this follows from Theorem 3.2.

## 5.2. Circuits.
Structured and decorated cospans are a powerful tool for studying categories where the morphisms are electrical circuits—see [1, Section 6.1] and [2, 4]. The key idea is to use open graphs with labeled edges to describe circuits, where the labels can stand for resistors with any chosen resistance, capacitors with any chosen capacitance, or other circuit elements. The whole theory of open graphs discussed in the previous section can be recapitulated for labeled graphs. Since the abstract formalism works the same way, we can be brief. Concrete applications of this formalism are discussed in the references.

Fix a set $\mathcal{L}$ to serve as edge labels. Define an $\mathcal{L}$-**graph** to be a graph $s, t \colon E \to N$ equipped with a function $\ell \colon E \to \mathcal{L}$. There is a category $\mathsf{Graph}_\mathcal{L}$ where the objects are $\mathcal{L}$-graphs and a morphism from the $\mathcal{L}$-graph

$$\mathcal{L} \xleftarrow{\ \ell\ } E \xrightarrow[\ t\ ]{\ s\ } N$$

to the $\mathcal{L}$-graph

$$\mathcal{L} \xleftarrow{\ \ell'\ } E' \xrightarrow[\ t'\ ]{\ s'\ } N'$$

is a pair of functions $f \colon E \to E', g \colon N \to N'$ such that these diagrams commute:

$$
\begin{array}{ccc}
E & \xrightarrow{\ s\ } & N \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
E' & \xrightarrow{\ s'\ } & N'
\end{array}
\qquad
\begin{array}{ccc}
E & \xrightarrow{\ t\ } & N \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
E' & \xrightarrow{\ t'\ } & N'
\end{array}
\qquad
\begin{array}{ccc}
& & E \\
& {\scriptstyle \ell}\swarrow & \big\downarrow{\scriptstyle f} \\
\mathcal{L} & & \\
& {\scriptstyle \ell'}\nwarrow & E'.
\end{array}
$$

There is a functor $U \colon \mathsf{Graph}_\mathcal{L} \to \mathsf{FinSet}$ that takes an $\mathcal{L}$-graph to its underlying set of nodes. This has a left adjoint $L \colon \mathsf{FinSet} \to \mathsf{Graph}_\mathcal{L}$ sending any set to the $\mathcal{L}$-graph with that set of nodes and no edges. Both $\mathsf{FinSet}$ and $\mathsf{Graph}_\mathcal{L}$ have colimits, and $L$ preserves them.

This sets the stage for structured cospans: Theorem 3.1 gives us a symmetric monoidal double category $_L\mathbb{C}\mathbf{sp}(\mathsf{Graph}_\mathcal{L})$ where a horizontal 1-cell is an **open $\mathcal{L}$-graph**, also called an $\mathcal{L}$-**circuit**: that is, a cospan in $\mathsf{Graph}_\mathcal{L}$ of this form:

$$L(X) \xrightarrow{\ i\ } G \xleftarrow{\ o\ } L(Y).$$

For example, here is a $\mathcal{L}$-circuit with $\mathcal{L} = (0, \infty)$:
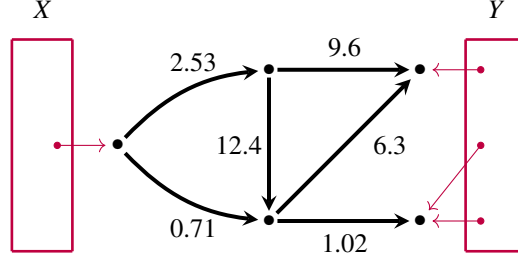


The edges here represent wires, with the positive real numbers labeling them serving to describe the resistance of resistors on the wires. The elements of the sets $X$ and $Y$ represent 'terminals': that is, points where we allow ourselves to attach a wire from another circuit.

We can now also describe $\mathcal{L}$-circuits using our new approach to decorated cospans. There is a symmetric lax monoidal pseudofunctor $F\colon (\mathsf{FinSet}, +) \to (\mathbf{Cat}, \times)$ such that for any finite set $N$, the category $F(N)$ has:

- objects being $\mathcal{L}$-**graph structures** on $N$: that is, $\mathcal{L}$-graphs where the set of nodes is $N$,
- morphisms being morphisms of $\mathcal{L}$-graphs that are the identity on the set of nodes.

This gives a symmetric monoidal double category $F\mathbb{C}\mathbf{sp}$, and using Theorem 3.2 we can show that this is isomorphic, as a symmetric monoidal double category, to $_L\mathbb{C}\mathbf{sp}(\mathsf{Graph}_{\mathcal{L}})$.

5.3. **Petri nets.** Petri nets are widely used as models of systems in engineering and computer science [20, 39]. Structured cospans have been used to define a symmetric monoidal double category of 'open Petri nets' [6], which lets us build large Petri nets out of smaller pieces. We can also use decorated cospans to create a double category of open Petri nets. Again this example is very similar to the example of open graphs.

A **Petri net** is a pair of sets $S$ and $T$ and functions $s, t\colon T \to \mathbb{N}[S]$. Here $S$ is the set of **places**, $T$ is the set of **transitions**, and $\mathbb{N}[S]$ is the underlying set of the free commutative monoid on $S$. Each transition thus has a formal sum of places as its source and target as prescribed by the functions $s$ and $t$, respectively. Here is an example:
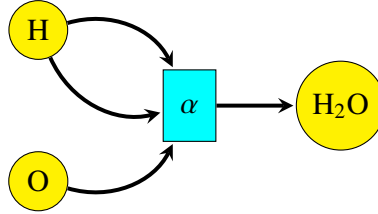


This Petri net has a single transition $\alpha$ with $2H + O$ as its source and $H_2O$ as its target.

There is a category $\mathsf{Petri}$ with Petri nets as objects, where a morphism from the Petri net $s, t\colon T \to \mathbb{N}[S]$ to the Petri net $s', t'\colon T' \to \mathbb{N}[S']$ is a pair of functions $f\colon T \to T', g\colon S \to S'$ such that the following diagrams commute:

$$
\begin{array}{ccc}
T & \xrightarrow{\ s\ } & \mathbb{N}[S] \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathbb{N}[g]} \\
T' & \xrightarrow{\ s'\ } & \mathbb{N}[S']
\end{array}
\qquad
\begin{array}{ccc}
T & \xrightarrow{\ t\ } & \mathbb{N}[S] \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathbb{N}[g]} \\
T' & \xrightarrow{\ t'\ } & \mathbb{N}[S'].
\end{array}
$$

There is a functor $R\colon \mathsf{Petri} \to \mathsf{Set}$ sending any Petri net to its set of places, and this has a left adjoint $L\colon \mathsf{Set} \to \mathsf{Petri}$ sending any set $S$ to the Petri net with $S$ as its set of places and no transitions [6, Lemma 11]. Since both $\mathsf{Set}$ and $\mathsf{Petri}$ have finite colimits and $L$ preserves them, Theorem 3.1 yields a symmetric monoidal double category $_L\mathbb{C}\mathbf{sp}(\mathsf{Petri})$ in which:

- objects are finite sets,
- vertical 1-morphisms are functions,
- horizontal 1-cells are **open Petri nets**, which are cospans in $\mathsf{Petri}$ of the form:

$$
L(X) \xrightarrow{\ i\ } P \xleftarrow{\ o\ } L(Y)
$$

- 2-morphisms are diagrams in Petri of the form:

$$
\begin{array}{ccccc}
L(X_1) & \xrightarrow{I_1} & P_1 & \xleftarrow{O_1} & L(Y_1) \\
{\scriptstyle L(f)}\downarrow & & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle L(g)} \\
L(X_2) & \xrightarrow{I_2} & P_2 & \xleftarrow{O_2} & L(Y_2).
\end{array}
$$

We can equivalently describe open Petri nets using decorated cospans. This works very much like the previous examples. There is a symmetric lax monoidal pseudofunctor $F\colon (\mathsf{FinSet}, +) \to (\mathbf{Cat}, \times)$ such that for any finite set $S$, the category $F(S)$ has:

- objects given by Petri nets whose set of places is $S$,
- morphisms given by morphisms of Petri nets that are the identity on the set of places.

This gives a symmetric monoidal double category $F\mathbb{C}\mathbf{sp}$, and using Theorem 3.2 we can show that this is isomorphic, as a symmetric monoidal double category, to $_L\mathbb{C}\mathbf{sp}(\mathsf{Petri})$.

The machinery of structured cospans has been used to provide a semantics for open Petri nets [6]: a symmetric monoidal double functor from $_L\mathbb{C}\mathbf{sp}(\mathsf{Petri})$ to a symmetric monoidal double category of 'open commutative monoidal categories'. Presumably this double functor can equivalently be obtained using the machinery of decorated cospans, with the help of Theorem 2.4. However, it should be clear by now that so far, in cases where either structured or decorated cospans can be used, structured cospans are simpler. We next turn to an example where decorated cospans are necessary.

5.4. **Petri nets with rates.** In chemistry, population biology, epidemiology and other fields, modelers use 'Petri nets with rates', where the transitions are labeled with positive real numbers called 'rate constants' [25, 33, 46]. From any Petri net with rates one can systematically construct a dynamical system. Mathematical chemists have proved deep theorems relating the topology of Petri nets with rates to the qualitative behavior of their dynamical systems [15].

Pollard and the first author showed how to construct an open dynamical system from any open Petri net with rates, thus defining a functor from a category with open Petri nets with rates as morphisms to one with open dynamical systems as morphisms [7]. They used Fong's original decorated cospans to do this. Later, structured cospans were used to promote the first of these categories to a double category [1, Section 6.16]. Here we show that the second of these categories, with open dynamical systems as morphisms, *cannot* be made into a double category using structured cospans. However, we *can* do it using decorated cospans.

First, to briefly illustrate these ideas, here is an open Petri net with rates:
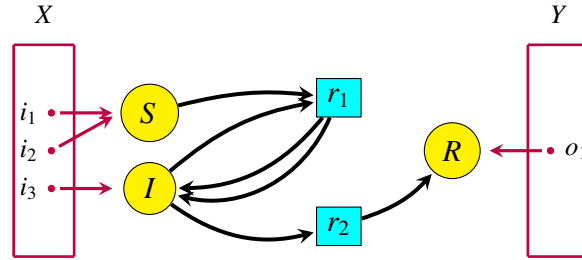


It is an open Petri net where the transitions are labeled with rate constants $r_1, r_2 > 0$. Here is the corresponding open dynamical system:

$$
\begin{aligned}
\frac{dS(t)}{dt} &= -r_1\, S(t)I(t) + I_1(t) + I_2(t) \\[2mm]
\frac{dI(t)}{dt} &= r_1\, S(t)I(t) - r_2\, I(t) + I_3(t) \\[2mm]
\frac{dR(t)}{dt} &= r_2\, I(t) - O_1(t).
\end{aligned}
\tag{11}
$$

Here $I_1(t), I_2(t), I_3(t)$ and $O_1(t)$ are arbitrary smooth functions of time, which describe inflows and outflows at the points $i_1, i_2, i_3 \in X$ and $o_1 \in Y$. If we drop these inflow and outflow terms, we obtain a dynamical system: an autonomous system of coupled nonlinear first-order ordinary differential equations. In fact these equations are a famous model of infectious disease, the 'SIR model', where $S(t)$, $I(t)$ and $R(t)$ describe the

populations of susceptible, infected and recovered individuals, respectively. The inflow and outflow terms allow individuals to enter or leave the population. This in turn lets us couple the SIR model to other models, and build larger models from smaller pieces.

Recently Halter and Patterson [26] implemented this idea in a software tool that makes it easy to build epidemiological models using structured cospans. They used this to rebuild part of the COVID-19 model that the UK has been using to make policy decisions. The advantage of using structured or decorated cospans is that one can build complex models from smaller pieces, so one can easily add or change parts, and better understand the effects of doing this.

Now we turn to the details. A **Petri net with rates** is a Petri net $s, t\colon T \to \mathbb{N}[S]$ together with a function $r\colon T \to (0, \infty)$ assigning to each transition $\tau \in T$ a positive real number called its **rate constant**. There is a category $\mathsf{Petri}_r$ whose objects are Petri nets with rates, where a morphism from

$$(0, \infty) \xleftarrow{\;\;r\;\;} T \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{N}[S]$$

to

$$(0, \infty) \xleftarrow{\;\;r'\;\;} T' \underset{t'}{\overset{s'}{\rightrightarrows}} \mathbb{N}[S']$$

is a morphism of the underlying Petri nets such that the following diagram also commutes:

$$
\begin{array}{ccc}
 & & T \\
 & \overset{r}{\swarrow} & \big\downarrow{\scriptstyle f} \\
(0, \infty) & & \\
 & \underset{r'}{\nwarrow} & \big\downarrow \\
 & & T'
\end{array}
$$

There is a functor $R\colon \mathsf{Petri}_r \to \mathsf{Set}$ sending any Petri net with rates to its set of places, and this has a left adjoint $L\colon \mathsf{Set} \to \mathsf{Petri}_r$ sending any set $S$ to the Petri net with $S$ as its set of places and no transitions [1, Lemma 6.18]. Since $\mathsf{Petri}_r$ has finite colimits [1, Lemma 6.19], it follows that there is a symmetric monoidal double category $_L\mathbb{C}\mathbf{sp}(\mathsf{Petri}_r)$ where:

- objects are finite sets,
- vertical 1-morphisms are functions,
- horizontal 1-cells are **open Petri nets with rates**, namely diagrams in $\mathsf{Petri}_r$ of the form

$$L(X) \xrightarrow{\;i\;} P \xleftarrow{\;o\;} L(Y),$$

- 2-morphisms are diagrams in $\mathsf{Petri}_r$ of the form

$$
\begin{array}{ccccc}
L(X_1) & \xrightarrow{\;I_1\;} & P_1 & \xleftarrow{\;O_1\;} & L(Y_1) \\
{\scriptstyle L(f)}\big\downarrow & & {\scriptstyle \alpha}\big\downarrow & & \big\downarrow{\scriptstyle L(g)} \\
L(X_2) & \xrightarrow{\;I_2\;} & P_2 & \xleftarrow{\;O_2\;} & L(Y_2).
\end{array}
$$

We can equivalently describe open Petri nets with rates using decorated cospans. There is a symmetric lax monoidal pseudofunctor $F\colon (\mathsf{FinSet}, +) \to (\mathbf{Cat}, \times)$ such that for any finite set $S$, the category $F(S)$ has:

- objects given by Petri nets with rates whose set of places is $S$,
- morphisms given by morphisms of Petri nets with rates that are the identity on the set of places.

This gives a symmetric monoidal double category $F\mathbb{C}\mathbf{sp}$, and using Theorem 3.2 we can show that this is isomorphic, as a symmetric monoidal double category, to $_L\mathbb{C}\mathbf{sp}(\mathsf{Petri}_r)$.

All this so far is very similar to the previous examples. More interesting is the symmetric monoidal double category of open dynamical systems. A dynamical system is a vector field from the perspective of a system of first-order ordinary differential equations. A Petri net with rates gives a special sort of dynamical system: an **algebraic** vector field on $\mathbb{R}^n$, meaning one where the components of the vector field are polynomials in the coordinates. We shall think of such a vector field as a special sort of function $v\colon \mathbb{R}^n \to \mathbb{R}^n$.

Using Fong's original approach to decorated cospans, Pollard and the first author constructed a symmetric monoidal category for which the morphisms are open dynamical systems [7, Theorem 17]. This category is constructed from a symmetric lax monoidal functor $D \colon \mathsf{FinSet} \to \mathsf{Set}$ such that:

- $D$ maps any finite set $S$ to
$$D(S) = \{v \colon \mathbb{R}^S \to \mathbb{R}^S \mid v \text{ is algebraic}\}.$$

- $D$ maps any function $f \colon S \to S'$ between finite sets to the function $D(f) \colon D(S) \to D(S')$ given as follows:
$$D(f)(v) = f_* \circ v \circ f^*$$
where the pullback $f^* \colon \mathbb{R}^{S'} \to \mathbb{R}^S$ is given by
$$f^*(c)(\sigma) = c(f(\sigma))$$
while the pushforward $f_* \colon \mathbb{R}^S \to \mathbb{R}^{S'}$ is given by
$$f_*(c)(\sigma') = \sum_{\{\sigma \in S : f(\sigma) = \sigma'\}} c(\sigma).$$

The functorality of $D$ is proved in [7, Lemma 15] while the lax symmetric monoidal stucture is given in Lemma 16 of that paper.

Since every set gives a discrete category with that set of objects, we can reinterpret $D$ as a symmetric lax monoidal pseudofunctor $D \colon (\mathsf{FinSet}, +) \to (\mathbf{Cat}, \times)$ which happens to actually be a functor. Applying Theorem 2.2 we obtain a symmetric monoidal double category $D\mathbb{C}\mathbf{sp}$ where:

- objects are finite sets,
- vertical 1-morphisms are functions,
- a horizontal 1-cell from $X$ to $Y$ is an **open dynamical system**, that is, a cospan
$$X \xrightarrow{\ i\ } S \xleftarrow{\ o\ } Y$$
in $\mathsf{FinSet}$ together with an algebraic vector field $v \in D(S)$,
- a 2-morphism from
$$X \xrightarrow{\ i\ } S \xleftarrow{\ o\ } Y, \qquad v \in D(S)$$
to
$$X' \xrightarrow{\ i'\ } S' \xleftarrow{\ o'\ } Y' \qquad v' \in D(S')$$
is a diagram

$$
\begin{array}{ccccc}
X & \xrightarrow{\ i\ } & S & \xleftarrow{\ o\ } & Y \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\
X' & \xrightarrow{\ i'\ } & S' & \xleftarrow{\ o'\ } & Y'
\end{array}
$$

in $\mathsf{FinSet}$ such that $D(h)(v) = v'$.

Next, we can define a symmetric monoidal double functor
$$\blacksquare \colon F\mathbb{C}\mathbf{sp} \to D\mathbb{C}\mathbf{sp}$$

sending any open Petri net with rates to its corresponding open dynamical system. This was already defined at the level of categories by Pollard and the first author [7, Section 7], who called it 'gray-boxing'. To boost this result to the double category level we use Theorem 2.4, taking the square in that theorem to be

$$
\begin{array}{ccc}
\mathsf{FinSet} & \xrightarrow{\ F\ } & \mathbf{Cat} \\
{\scriptstyle 1}\downarrow & \Downarrow \theta & \downarrow{\scriptstyle 1} \\
\mathsf{FinSet} & \xrightarrow{\ D\ } & \mathbf{Cat}.
\end{array}
$$

Here $\theta$ is a monoidal natural isomorphism given as follows. For any finite set $S$, $\theta_S \colon F(S) \to D(S)$ maps any Petri nets with rates
$$(0, \infty) \xleftarrow{\ r\ } T \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{N}[S]$$

to an algebraic vector field on $\mathbb{R}^S$, say $v$. This vector field is defined using a standard prescription taken from chemistry, called 'the law of mass action'. Namely, for any $c \in \mathbb{R}^s$, we set

$$v(c) = \sum_{\tau \in T} r(\tau)\,(t(\tau) - s(\tau))c^{s(\tau)}$$

where

$$c^{s(\tau)} = \prod_{i \in S} c_i^{s(\tau)_i}$$

and we think of $t(\tau), s(\tau) \in \mathbb{N}[S]$ as vectors in $\mathbb{R}^S$. This formula is explained in the paper with Pollard, where it is also shown that $\theta$ defines a monoidal natural isomorphism between functors to $(\mathsf{Set}, \times)$ [7, Theorem 18]. As such, it automatically becomes a monoidal natural isomorphism between the functors $F, D \colon (\mathsf{FinSet}, +) \to (\mathbf{Cat}, \times)$. Thus, it defines a symmetric monoidal double functor $\blacksquare \colon F\mathbb{C}\mathbf{sp} \to D\mathbb{C}\mathbf{sp}$.

In applications, this double functor lets us turn an open Petri net with rates into an open dynamical system as follows. Given a Petri net with rates $P$ and defining $v$ as above, we obtain a system of first-order ordinary differential equations for a function $c \colon \mathbb{R} \to \mathbb{R}^S$ called the **rate equation**:

$$\frac{d}{dt}c(t) = v(c(t)).$$

More generally, when $P$ is part of an open Petri net with rates

$$L(X) \xrightarrow{\ i\ } P \xleftarrow{\ o\ } L(Y),$$

we get an open dynamical system called the **open rate equation**:

$$\frac{d}{dt}c(t) = v(c(t)) + i_*(I(t)) - o_*(O(t))$$

where $I \colon \mathbb{R} \to \mathbb{R}^X$ and $O \colon \mathbb{R} \to \mathbb{R}^X$ are arbitrary smooth functions describing **inflows** and **outflows**, respectively. Applying this prescription to the open Petri net with rates shown at the start of this section one gets the differential equations (11). Other examples are worked out in [7].

We now show that the decorated cospan double category $D\mathbb{C}\mathbf{sp}$ of open dynamical systems is not isomorphic to a structured cospan double category via Theorem 3.2. Recall that in that theorem we start with the data required to build a decorated cospan category, namely a symmetric lax monoidal pseudofunctor $F \colon (\mathsf{A}, +) \to (\mathbf{Cat}, \times)$, and show that if the resulting pseudofunctor $F \colon \mathsf{A} \to \mathbf{SymMonCat}$ factors through **Rex**, then the opfibration $U \colon \mathsf{X} = \int F \to \mathsf{A}$ has a left adjoint $L \colon \mathsf{A} \to \mathsf{X}$. We then obtain an isomorphism between decorated and structured cospan double categories, $F\mathbb{C}\mathbf{sp} \cong {}_L\mathbb{C}\mathbf{sp}(\mathsf{X})$. We now show that in the case at hand, where $F = D$, the opfibration $U$ does *not* have a left adjoint. Thus, the conditions of Theorem 3.2 do not hold in this case: $F$ does not factor through **Rex**.

Taking $D$ as above, it is easy to see that in the category $\int D$

- an object is a pair $(S, v)$ where $S$ is a finite set and $v$ is an algebraic vector field $v \colon \mathbb{R}^S \to \mathbb{R}^S$,
- a morphism from $(S, v)$ to $(S', v')$ is a function $f \colon S \to S'$ such that $v' = f_* \circ v \circ f^*$.

with the usual composition of functions. The forgetful functor $U \colon \int D \to \mathsf{FinSet}$ acts as follows:

- on objects, $D(S, v) = S$,
- on morphisms, $D(f) = f$.

To show that $U$ does not have a left adjoint, we use the following well-known result:

**Lemma 5.1.** *A functor $U \colon \mathsf{A} \to \mathsf{X}$ admits a left adjoint if and only if for every $x \in \mathsf{X}$, the comma category $x \downarrow U$ has an initial object.*

Because the empty set is initial in $\mathsf{FinSet}$, the comma category $\emptyset \downarrow U$ is just $\int D$. This contains an object $(\emptyset, v_\emptyset)$, where $v_\emptyset$ is the only possible vector field on $\mathbb{R}^\emptyset$, namely, the zero vector field. The only object in $\int D$ with any morphisms to $(\emptyset, v_\emptyset)$ is $(\emptyset, v_\emptyset)$ itself, so no other object can be initial. However $(\emptyset, v_\emptyset)$ is not initial either, because it has no morphisms to an object $(S, v)$ unless $v$ is the zero vector field on $\mathbb{R}^S$. Thus by Lemma 5.1, $U$ does not have a left adjoint.

## 6. Conclusions

We have given conditions under which a decorated cospan double category is isomorphic to a structured cospan double category, in Theorem 3.2. The converse question is also interesting: is every structured cospan double category isomorphic to a decorated cospan double category? The answer is similar to the previous one: yes, under certain conditions that let us pass from an appropriate functor $L\colon \mathsf{A} \to \mathsf{X}$ to an appropriate pseudofunctor $F\colon \mathsf{A} \to \mathbf{Cat}$.

Let us now sketch the story; details will appear in a forthcoming paper [12]. Suppose the conditions hold for constructing the double category of structured cospans ${}_L\mathbb{C}\mathbf{sp}(\mathsf{X})$ as in Theorem 3.1. That is, suppose $\mathsf{A}$ and $\mathsf{X}$ have finite colimits and $L\colon \mathsf{A} \to \mathsf{X}$ preserves them. If $L$ also has a right adjoint 'left inverse' (meaning the unit is the identity) $U\colon \mathsf{X} \to \mathsf{A}$, which moreover strictly preserves the chosen pushouts, it can be shown that $U$ is an opfibration. Consequently, $U$ corresponds to a pseudofunctor $F\colon \mathsf{A} \to \mathbf{Cat}$ by the inverse Grothendieck construction, as in the first part of Theorem A.2. Furthermore, if $U$ preserves finite coproducts, $F$ acquires the structure of a lax monoidal pseudofunctor $F\colon (\mathsf{A}, +) \to (\mathbf{Cat}, \times)$ by the special case of the cocartesian monoidal Grothendieck construction discussed under Lemma 3.3. As a result, $F$ now has enough structure to induce a double category of decorated cospans $F\mathbb{C}\mathbf{sp}$ as in Theorem 2.2. Finally, it can be shown that the structured and decorated cospan double categories are isomorphic as symmetric monoidal double categories: ${}_L\mathbb{C}\mathbf{sp}(\mathsf{X}) \cong F\mathbb{C}\mathbf{sp}$.

To give a better sense of how the pseudofunctor $F\colon \mathsf{A} \to \mathbf{Cat}$ is constructed: for each object $a \in A$, $F(a)$ is defined to be the fiber of $U$ over $a$, namely the category of all objects in $x \in \mathsf{X}$ such that $U(x) = a$ and morphisms $k\colon x \to y$ such that $U(k) = 1_a$. Given a morphism $f\colon a \to b$, there is a functor $F(f)\colon F(a) \to F(b)$ that maps $x \in F(a)$ to the following pushout:

$$
\begin{array}{ccc}
La & \xrightarrow{\ Lf\ } & Lb \\
{\scriptstyle \varepsilon_x}\downarrow & \ulcorner & \downarrow \\
x & \longrightarrow & x +_{La} Lb \qquad\quad \text{in } \mathsf{X} \\
\vdots\downarrow & & \vdots\downarrow \\
a & \xrightarrow{\ f\ } & b \qquad\qquad\quad \text{in } \mathsf{A}
\end{array}
$$

where $\varepsilon_x\colon LU(x) = L(a) \to x$ is the counit of the adjunction $L \dashv U$. The fact that $U$ strictly preserves pushouts is necessary to show that the pushout is mapped, via $U$, directly down to $b$.

Even though for both Theorem 3.2 and the above result the conditions stated are only sufficient, they suggest that with work we could establish this functorial picture:
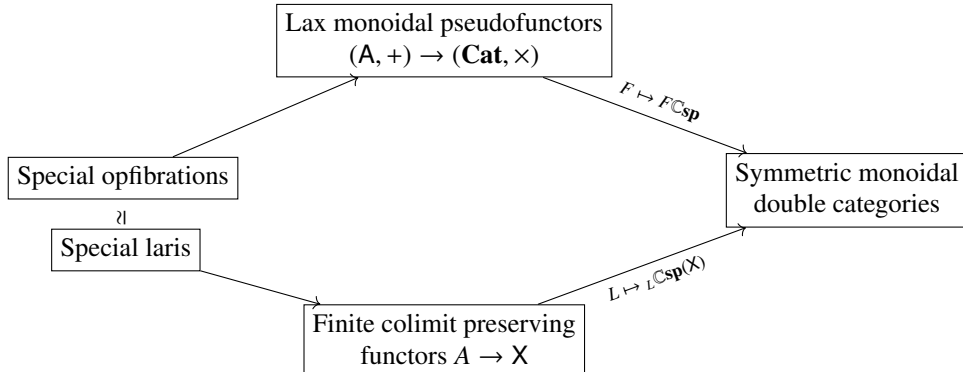


with a natural isomorphism in the middle. The connection between opfibrations and laris goes back to Gray's Lemma 3.6, but we need to specialize it to a class suitable for both the structured and decorated cospan constructions. This would imply that starting from an appropriate middle ground, these two constructions are essentially the same. We leave such considerations for future work.

## Appendix A. Definitions

In this appendix, we gather some well-known concepts required to make the material self-contained, as well as references to more detailed expositions.

A.1. **Bicategories.** For standard 2-categorical material, we refer the reader to [32]. For monoidal 2-categories see [16], and for detailed definitions concerning monoidal bicategories see [21, 36, 44]. Briefly, a **monoidal** bicategory $\mathbf{A}$ comes with a pseudofunctor $\otimes\colon \mathbf{A}\times\mathbf{A} \to \mathbf{A}$ and a unit object $I$ that are associative and unital up to coherent equivalence. A **braided** monoidal bicategory also comes with a pseudonatural equivalence $\beta_{a,b}\colon a \otimes b \to b \otimes a$ and appropriate invertible modifications obeying certain equations; it is **sylleptic** if there is an invertible modification $1_{a\otimes b} \Rightarrow \beta_{b,a} \circ \beta_{a,b}$ obeying its own equation, and **symmetric** if one further axiom holds.

A **lax monoidal** pseudofunctor (called *weak monoidal* homomorphism in some earlier references) between monoidal bicategories $F\colon \mathbf{A} \to \mathbf{B}$ is a pseudofunctor equipped with pseudonatural transformations with components $\phi_{a,b}\colon Fa \otimes Fb \to F(a \otimes b)$ and $\phi_0\colon I \to FI$ along with invertible modifications for associativity and unitality with components

$$
\begin{array}{ccc}
(Fa \otimes Fb) \otimes Fc & \xrightarrow{\;\phi_{a,b}\otimes 1\;} & F(a \otimes b) \otimes Fc \\
\downarrow & & \downarrow{\scriptstyle \phi_{a\otimes b,c}} \\
Fa \otimes (Fb \otimes Fc) & \cong & F((a \otimes b) \otimes c) \\
\downarrow{\scriptstyle 1\otimes\phi_{b,c}} & & \downarrow \\
Fa \otimes F(b \otimes c) & \xrightarrow{\;\phi_{a,b\otimes c}\;} & F(a \otimes (b \otimes c))
\end{array}
\tag{12}
$$

$$
Fa \xrightarrow{\;\sim\;} Fa \otimes I \xrightarrow{\;1\otimes\phi_0\;} Fa \otimes FI \qquad\qquad Fa \xrightarrow{\;\sim\;} I \otimes Fa \xrightarrow{\;\phi_0\otimes 1\;} FI \otimes Fa
\tag{13}
$$

with $\phi_{a,I}$ to $F(a \otimes I)$ and $\phi_{I,a}$ to $F(I \otimes a)$.

subject to coherence conditions listed in [16, Definition 2]. In particular, pseudonaturality of the monoidal structure means that it comes with isomorphisms of this form:

$$
\begin{array}{ccc}
Fa \otimes Fb & \xrightarrow{\;Ff\otimes Fg\;} & Fa' \otimes Fb' \\
\downarrow{\scriptstyle \phi_{a,b}} & \overset{\phi_{f,g}}{\underset{\cong}{}} & \downarrow{\scriptstyle \phi_{a',b'}} \\
F(a \otimes b) & \xrightarrow{\;F(f\otimes g)\;} & F(a' \otimes b')
\end{array}
\tag{14}
$$

natural in $f$ and $g$. A **braided lax monoidal** pseudofunctor between braided monoidal bicategories comes with an invertible modification with components

$$
\begin{array}{ccc}
Fa \otimes Fb & \xrightarrow{\;\phi_{a,b}\;} & F(a \otimes b) \\
\downarrow{\scriptstyle \beta_{Fa,Fb}} & \overset{u_{a,b}}{\underset{\cong}{}} & \downarrow{\scriptstyle F(\beta_{a,b})} \\
Fb \otimes Fa & \xrightarrow{\;\phi_{b,a}\;} & F(b \otimes a)
\end{array}
\tag{15}
$$

subject to two axioms found e.g. in [16, Definition 14]. A **sylleptic lax monoidal** pseudofunctor satisfies one extra condition and a **symmetric lax monoidal** pseudofunctor between symmetric monoidal bicategories is just a sylleptic one.

A.2. **Fibrations.** Basic material regarding the theory of fibrations can be found, for example, in [10, 24]. Recall that a functor $U\colon \mathbf{X} \to \mathbf{A}$ is an **opfibration** if for every $x \in \mathbf{X}$ with $U(x) = a$ and $f\colon a \to b$ in $\mathbf{A}$, there exists a **cocartesian lifting** of $f$ to $x$, namely a morphism $\beta$ in $\mathbf{X}$ with domain $x$ with $U(\beta) = f$ and the following universal property: for any $g\colon b \to b'$ in $\mathbf{A}$ and $\gamma\colon x \to y'$ in $\mathbf{X}$ above the composite $g \circ f$, there exists a unique $\delta\colon y \to y'$ such that $U(\delta) = g$ and $\gamma = \delta \circ \beta$ as shown below

$$
\begin{array}{cc}
\begin{array}{c}
x \xrightarrow{\;\beta\;} y \overset{\exists!\delta}{\dashrightarrow} y' \\[2pt]
\gamma\colon x \to y'
\end{array} & \text{in } \mathbf{X} \\[20pt]
\begin{array}{c}
a \xrightarrow{\;f=U(\beta)\;} b \xrightarrow{\;g\;} b' \\[2pt]
g\circ f = U(\gamma)
\end{array} & \text{in } \mathbf{A}
\end{array}
$$

The category $\mathsf{X}$ is called the **total** category and $\mathsf{A}$ is called the **base** category of the opfibration. For any $a \in \mathsf{A}$, the **fiber** above $a$ is the category $\mathsf{X}_a$ consisting of all objects that map to $a$ and **vertical** morphisms between them, i.e., morphisms mapping to $1_a$.

Assuming the axiom of choice, we may select a cocartesian lifting of each morphism $f \colon a \to b$ in $\mathsf{A}$ to each $x \in \mathsf{X}_a$, denoted by $\mathrm{Cocart}(f, x) \colon x \to f_!(x)$, rendering $U$ a so-called **cloven** opfibration. This choice induces **reindexing functors** $f_! \colon \mathsf{X}_a \to \mathsf{X}_b$ between the fibers, which by the lifting's universal property come equipped with natural isomorphisms $(1_a)_! \cong 1_{\mathsf{X}_a}$ and $(f \circ g)_! \cong f_! \circ g_!$. With the help of these, any cloven opfibration $U \colon \mathsf{X} \to \mathsf{A}$ gives a pseudofunctor $F \colon \mathsf{A} \to \mathbf{Cat}$, where $\mathsf{A}$ is viewed as a 2-category with trivial 2-morphisms, $F(a) = \mathsf{X}_a$ for each object $a \in \mathsf{A}$, and $F(f) = f_!$ for each morphism $f$ in $\mathsf{A}$.

In fact, there is a 2-equivalence between opfibrations and pseudofunctors induced by the so-called 'Grothendieck construction'. Let $\mathbf{OpFib}(\mathsf{A})$ denote the 2-subcategory of the slice 2-category $\mathbf{Cat}/\mathsf{A}$ of opfibrations over $\mathsf{A}$, functors that preserve cocartesian liftings, and natural transformations with vertical components.

**Definition A.1.** For any pseudofunctor $F \colon \mathsf{A} \to \mathbf{Cat}$ where $\mathsf{A}$ is a category viewed as a 2-category with trivial 2-morphisms, the **Grothendieck category** $\int F$ has

- objects pairs $(a, x \in F(a))$ and
- a morphism from $(a, x \in F(a))$ to $(b, y \in F(b))$ is a pair $(f \colon a \to b, k \colon F(f)(x) \to y) \in \mathsf{A} \times F(b)$.

The unit is $(1_a \colon a \to a, F(1_a)(x) \cong x)$ and composition of $(f, k) \colon (a, x) \to (b, y)$ and $(g, \ell) \colon (b, y) \to (c, z)$ is

$$\left( a \xrightarrow{f} b \xrightarrow{g} c, F(g \circ f)x \cong Fg(Ff(x)) \xrightarrow{Fg(k)} Fg(y) \xrightarrow{\ell} z \right) \tag{16}$$

This is an opfibered category over $\mathsf{A}$ via the obvious forgetful functor, with fibers $(\int F)_a = F(a)$ and reindexing functors $f_! = F(f)$.

The constructions sketched so far—the Grothendieck construction and the construction of a pseudofunctor into $\mathbf{Cat}$ from a cloven opfibration—are the two halves of the following equivalence.

**Theorem A.2.**

*(1) Every opfibration $\mathsf{X} \to \mathsf{A}$ gives rise to a pseudofunctor $\mathsf{A} \to \mathbf{Cat}$.*

*(2) Every pseudofunctor $\mathsf{A} \to \mathbf{Cat}$ gives rise to an opfibration $\int F \to \mathsf{A}$.*

*(3) The above correspondences yield an equivalence of 2-categories*

$$[\mathsf{A}, \mathbf{Cat}]_{\mathrm{ps}} \simeq \mathbf{OpFib}(\mathsf{A})$$

*where $[\mathsf{A}, \mathbf{Cat}]_{\mathrm{ps}}$ is the 2-category of pseudofunctors from $\mathsf{A}$ to $\mathbf{Cat}$, pseudonatural transformations, and modifications.*

*Proof.* The idea goes back to Grothendieck; a proof can be found in, for example, [29, Section 1.10]. $\square$

A.3. **Double categories.** For double categories we follow the notation of our paper on structured cospans [1], which in turn follows that of Hansen and Shulman [27, 43]. Our double categories are always 'pseudo' double categories, where composition of horizontal 1-cells is unital and associative only up to coherent isomorphism [22, 23, 42].

**Definition A.3.** A **double category** $\mathbb{D}$ consists of a **category of objects** $\mathbb{D}_0$, a **category of arrows** $\mathbb{D}_1$, functors

$$S, T \colon \mathbb{D}_1 \to \mathbb{D}_0, \quad U \colon \mathbb{D}_0 \to \mathbb{D}_1, \quad \text{and} \odot \colon \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$$

called the **source** and **target**, **unit** and **composition** functors, respectively, such that

$$S(U_A) = A = T(U_A), \quad S(M \odot N) = S(N), \quad T(M \odot N) = T(M),$$

and natural isomorphisms called the **associator**

$$\alpha_{L,M,N} \colon (L \odot M) \odot N \to L \odot (M \odot N)$$

and **left and right unitors**

$$\lambda_N \colon U_{T(N)} \odot N \to N, \qquad \rho_N \colon N \odot U_{S(N)} \to N$$

such that $S(\alpha), S(\lambda), S(\rho), T(\alpha), T(\lambda)$ and $T(\rho)$ are all identities, such that the standard coherence laws hold: the pentagon identity for the associator and the triangle identity for the left and right unitor.

Objects of $\mathbb{D}_0$ are called **objects** and morphisms of $\mathbb{D}_0$ are called **vertical 1-morphisms**. Objects of $\mathbb{D}_1$ are called **horizontal 1-cells** and morphisms of $\mathbb{D}_1$ are called **2-morphisms**. We can draw a 2-morphism $a\colon M \to N$ with $S(a) = f, T(a) = g$ as follows:

$$
\begin{array}{ccc}
A & \xrightarrow{\ M\ } & B \\
{\scriptstyle f}\downarrow & \Downarrow\alpha & \downarrow{\scriptstyle g} \\
C & \xrightarrow[\ N\ ]{} & D
\end{array}
$$

We call $M$ and $N$ the **horizontal source and target** of $a$ respectively, and call $f$ and $g$ its **vertical source and target**. A 2-morphism where $f$ and $g$ are identities is called **globular**. For example, the associator and unitors in a double category are globular 2-morphisms.

**Definition A.4.** Given double categories $\mathbb{D}$ and $\mathbb{E}$, a **double functor** $\mathbb{F}\colon \mathbb{D} \to \mathbb{E}$ consists of:

- functors $\mathbb{F}_0\colon \mathbb{D}_0 \to \mathbb{E}_0$ and $\mathbb{F}_1\colon \mathbb{D}_1 \to \mathbb{E}_1$ such that $S\mathbb{F}_1 = \mathbb{F}_0 S$ and $T\mathbb{F}_1 = \mathbb{F}_0 T$, and
- for every composable pair of horizontal 1-cells $M$ and $N$ in $\mathbb{D}$, a natural transformation $\mathbb{F}_{\odot}\colon \mathbb{F}(N) \odot \mathbb{F}(M) \to \mathbb{F}(N \odot M)$ called the **composite comparison** and for every object $a$ in $\mathbb{D}$, a natural transformation $\mathbb{F}_U\colon U_{\mathbb{F}_0(a)} \to \mathbb{F}_1(U_a)$ called the **unit comparison**. The components of each of these natural transformations are globular isomorphisms that must obey coherence laws analogous to those of a monoidal functor.

**Definition A.5.** Given double functors $\mathbb{F}, \mathbb{G}\colon \mathbb{D} \to \mathbb{E}$, a **double natural transformation** $\alpha\colon \mathbb{F} \Rightarrow \mathbb{G}$ consists of natural transformations $\alpha_0\colon \mathbb{F}_0 \Rightarrow \mathbb{G}_0$ and $\alpha_1\colon \mathbb{F}_1 \Rightarrow \mathbb{G}_1$ such that:

- $S(\alpha_M) = \alpha_{S(M)}$ and $T(\alpha_M) = \alpha_{T(M)}$ for all horizontal 1-cells $M$ of $\mathbb{D}$,
- $\alpha \circ \mathbb{F}_{\odot} = \mathbb{G}_{\odot} \circ (\alpha_M \odot \alpha_N)$ for all composable pairs $M$ and $N$ of horizontal 1-cells in $\mathbb{D}$, and
- $\alpha \circ \mathbb{F}_U = \mathbb{G}_U \circ \alpha$ for all objects $a$ of $\mathbb{D}$.

The double natural transformation $\alpha$ is a **double natural isomorphism** if both $\alpha_0$ and $\alpha_1$ are natural isomorphisms.

Let **Dbl** denote the 2-category of double categories, double functors and double transformations. One can check that **Dbl** has finite products, and in any 2-category with finite products we can define a 'pseudomonoid', which is a categorified analogue of a monoid [16]. For example, a pseudomonoid in **Cat** is a monoidal category. We can also define symmetric pseudomonoids, which in **Cat** are symmetric monoidal categories.

**Definition A.6.** A **monoidal double category** is a pseudomonoid in **Dbl**, namely it is equipped with double functors $\otimes\colon \mathbb{D} \times \mathbb{D} \to \mathbb{D}$, $I\colon 1 \to \mathbb{D}$ and invertible double transformations $\otimes \circ (1 \times \otimes) \cong \otimes \circ (\otimes \times 1)$, $\otimes \circ (1 \times I) \cong 1 \cong \otimes \circ (I \times 1)$ satisfying standard axioms.

Explicitly, a monoidal double category is a double category $\mathbb{D}$ with:

- monoidal structures on both $\mathbb{D}_0$ and $\mathbb{D}_1$ (each with tensor product denoted $\otimes$, associator $a$, left unitor $\ell$ and right unitor $r$ and unit object $I$), such that $U\colon \mathbb{D}_0 \to \mathbb{D}_1$ strictly preserves the unit objects and $S, T\colon \mathbb{D}_1 \to \mathbb{D}_0$ are strict monoidal,
- the structure of a double functor on $\otimes$: that is, invertible globular 2-morphisms

$$\chi\colon (M_2 \otimes N_2) \odot (M_1 \otimes N_1) \xrightarrow{\sim} (M_2 \odot M_1) \otimes (N_2 \odot N_1)$$

$$\mu\colon U_{A \otimes B} \xrightarrow{\sim} U_A \otimes U_B$$

obeying a list of equations that can be found after [27, Definition 2.10] and also [1, Definition A.5].

**Definition A.7.** A **symmetric monoidal double category** is a symmetric pseudomonoid in **Dbl**.

Explicitly, a symmetric monoidal double category is a monoidal double category $\mathbb{D}$ such that:

- $\mathbb{D}_0$ and $\mathbb{D}_1$ are symmetric monoidal categories, with braidings both denoted $\beta$.
- The functors $S$ and $T$ are symmetric strict monoidal functors.

- The following diagrams commute, expressing that the braiding is a transformation of double categories:

$$
\begin{array}{ccc}
(M_2 \otimes N_2) \odot (M_1 \otimes N_1) \xrightarrow{\ \beta \odot \beta\ } (N_2 \otimes M_2) \odot (N_1 \otimes M_1) & & U_A \otimes U_B \xleftarrow{\ \mu\ } U_{A \otimes B} \\
\chi \downarrow \qquad\qquad\qquad\qquad\qquad \downarrow \chi & & \beta \downarrow \qquad\qquad \downarrow U_\beta \\
(M_2 \odot M_1) \otimes (N_2 \odot N_1) \xrightarrow{\ \beta\ } (N_2 \odot N_1) \otimes (M_2 \odot M_1) & & U_B \otimes U_A \xleftarrow{\ \mu\ } U_{B \otimes A}
\end{array} \tag{17}
$$

**Definition A.8.** Given symmetric monoidal double categories $\mathbb{D}$ and $\mathbb{E}$, a **symmetric monoidal double functor** $\mathbb{F}\colon \mathbb{D} \to \mathbb{E}$ is a double functor $\mathbb{F}$ together with invertible transformations $\mathbb{F}_\otimes\colon \otimes \circ (\mathbb{F}, \mathbb{F}) \to \mathbb{F} \circ \otimes$ and $I_\mathbb{E} \to \mathbb{F} \circ I_\mathbb{D}$ that satisfy the usual coherence axioms for a symmetric monoidal functor.

Explicitly, a symmetric monoidal double functor is a double functor $\mathbb{F}\colon \mathbb{D} \to \mathbb{E}$ such that:

- $\mathbb{F}_0$ and $\mathbb{F}_1$ are symmetric monoidal functors,
- we have equalities $\mathbb{F}_0 S_\mathbb{D} = S_\mathbb{E} \mathbb{F}_1$ and $\mathbb{F}_0 T_\mathbb{D} = T_\mathbb{E} \mathbb{F}_1$ of monoidal functors, and
- the following diagrams commute, expressing that $\phi$ is a transformation of double categories:

$$
\begin{array}{ccc}
(\mathbb{F}(M_2) \otimes \mathbb{F}(N_2)) \odot (\mathbb{F}(M_1) \otimes \mathbb{F}(N_1)) \xrightarrow{\ \mathbb{F}_\otimes \odot \mathbb{F}_\otimes\ } \mathbb{F}(M_2 \otimes N_2) \odot \mathbb{F}(M_1 \otimes N_1) & & U_{\mathbb{F}(a) \otimes \mathbb{F}(b)} \xrightarrow{\ U_{\mathbb{F}_\otimes}\ } U_{\mathbb{F}(a \otimes b)} \\
\chi \downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \mathbb{F}_\odot & & \mu \downarrow \qquad\qquad \downarrow \mathbb{F}_U \\
(\mathbb{F}(M_2) \odot \mathbb{F}(M_1)) \otimes (\mathbb{F}(N_2) \odot \mathbb{F}(N_1)) \qquad\qquad \mathbb{F}((M_2 \otimes N_2) \odot (M_1 \otimes N_1)) & & U_{\mathbb{F}(a)} \otimes U_{\mathbb{F}(b)} \qquad \mathbb{F}(U_{a \otimes b}) \\
\mathbb{F}_\odot \otimes \mathbb{F}_\odot \downarrow \qquad\qquad\qquad\qquad\qquad \downarrow \mathbb{F}(\chi) & & \mathbb{F}_U \otimes \mathbb{F}_U \downarrow \qquad\qquad \downarrow \mathbb{F}(\mu) \\
\mathbb{F}(M_2 \odot M_1) \otimes \mathbb{F}(N_2 \odot N_1) \xrightarrow{\ \mathbb{F}_\otimes\ } \mathbb{F}((M_2 \odot M_1) \otimes (N_2 \odot N_1)) & & \mathbb{F}(U_a) \otimes \mathbb{F}(U_b) \xrightarrow{\ \mathbb{F}_\otimes\ } \mathbb{F}(U_a \otimes U_b)
\end{array} \tag{18}
$$

**Definition A.9.** An **isomorphism** of symmetric monoidal categories is a symmetric monoidal double functor $\mathbb{F}\colon \mathbb{D} \to \mathbb{E}$ that has an inverse.

A symmetric monoidal double functor is an isomorphism if it is bijective on objects, vertical 1-morphisms, horizontal 1-cells and 2-morphisms.

**Definition A.10.** Let $\mathbb{D}$ be a double category and $f\colon A \to B$ a vertical 1-morphism. A **companion** of $f$ is a horizontal 1-cell $\hat{f}\colon A \to B$ together with 2-morphisms

$$
\begin{array}{ccc}
A \xrightarrow{\ \hat{f}\ } B & & A \xrightarrow{\ U_A\ } A \\
f \downarrow \ \Downarrow\ \downarrow 1 & \text{and} & 1 \downarrow \ \Downarrow\ \downarrow f \\
B \xrightarrow[U_B]{} B & & A \xrightarrow[\hat{f}]{} B
\end{array}
$$

such that the following equations hold.

$$
\begin{array}{ccccc}
A \xrightarrow{\ U_A\ } A & & A \xrightarrow{\ U_A\ } A \xrightarrow{\ \hat{f}\ } B & & A \xrightarrow{\ U_A\ } A \xrightarrow{\ \hat{f}\ } B \\
1 \downarrow \Downarrow \downarrow f \quad A \xrightarrow{U_A} A & & 1 \downarrow \Downarrow f \downarrow \Downarrow \downarrow 1 & & A \xrightarrow{U_A} A \xrightarrow{\hat{f}} B \\
A \xrightarrow{\hat{f}} B \ = \ f \downarrow \Downarrow U_f \downarrow f & \text{and} & A \xrightarrow{\hat{f}} B \xrightarrow{U_B} B \ = \ 1 \downarrow \Downarrow \rho_{\hat{f}} \downarrow 1 \\
f \downarrow \Downarrow \downarrow 1 \quad B \xrightarrow[U_B]{} B & & 1 \downarrow \quad \Downarrow \lambda_{\hat{f}} \quad \downarrow 1 & & A \xrightarrow{\hat{f}} B \\
B \xrightarrow[U_B]{} B & & A \xrightarrow[\hat{f}]{} B & &
\end{array} \tag{19}
$$

A **conjoint** of $f$, denoted $\check{f}\colon B \to A$, is a companion of $f$ in the double category obtained by reversing the horizontal 1-cells, but not the vertical 1-morphisms, of $\mathbb{D}$.

**Definition A.11.** We say that a double category is **fibrant** if every vertical 1-morphism has both a companion and a conjoint.

**Theorem A.12.** [27, Theorem 1.1] *If $\mathbb{D}$ is a fibrant monoidal double category, then its horizontal bicategory $\mathbf{D}$ is a monoidal bicategory. If $\mathbb{D}$ is braided or symmetric, then so is $\mathbf{D}$.*

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