1. Proof of Theorem 2.2

There are 11 diagrams in Shulman's definition of symmetric monoidal double category, which we check now for decorated cospan double categories.

Diagram 1 of 11. Set up but not solved.
M_1 is given by $a \to m_1 \leftarrow b$ with $x_1 \in F(m_1)$.
M_2 is given by $b \to m_2 \leftarrow c$ with $x_2 \in F(m_2)$.
M_3 is given by $c \to m_3 \leftarrow d$ with $x_3 \in F(m_3)$.
N_1 is given by $a' \to n_1 \leftarrow b'$ with $y_1 \in F(n_1)$.
N_2 is given by $b' \to n_2 \leftarrow c'$ with $y_2 \in F(n_2)$.
N_3 is given by $c' \to n_3 \leftarrow d'$ with $y_3 \in F(n_3)$.

$$\begin{array}{ccc}
((M_1 \otimes N_1) \odot (M_2 \otimes N_2)) \odot (M_3 \otimes N_3) & \xrightarrow{\chi \odot 1} & ((M_1 \odot M_2) \otimes (N_1 \odot N_2)) \odot (M_3 \otimes N_3) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \chi} \\
(M_1 \otimes N_1) \odot ((M_2 \otimes N_2) \odot (M_3 \otimes N_3)) & & ((M_1 \odot M_2) \odot M_3) \otimes ((N_1 \odot N_2) \odot N_3) \\
\downarrow{\scriptstyle 1 \odot \chi} & & \downarrow{\scriptstyle \alpha \otimes \alpha} \\
(M_1 \otimes N_1) \odot ((M_2 \odot M_3) \otimes (N_2 \odot N_3)) & \xrightarrow{\chi} & (M_1 \odot (M_2 \odot M_3)) \otimes (N_1 \odot (N_2 \odot N_3))
\end{array}$$

Decorations:

(1) Right and then down:

$$F(\psi)\phi_{(m_1+n_1)+_{b+b'}(m_2+n_2),m_3+n_3}(F(\psi)\phi_{m_1+n_1,m_2+n_2}(\phi_{m_1,n_1}(x_1,y_1),\phi_{m_2,n_2}(x_2,y_2)),F(\psi)\phi_{m_3,n_3}(x_3,y_3)) \in F(((m_1+n_1)+_{b+b'}(m_2+n_2))+_{c+c'}(m_3+n_3))$$

$$F(\psi)\phi_{(m_1+_b m_2)+(n_1+_{b'} n_2),m_3+n_3}(\phi_{m_1+_b m_2,n_1+_{b'} n_2}((F(\psi)\phi_{m_1,m_2}(x_1,x_2),F(\psi)\phi_{n_1,n_2}(y_1,y_2))),\phi_{m_3,n_3}(x_3,y_3)) \in F(((m_1+_b m_2)+(n_1+_{b'} n_2))+_{c+c'}(m_3+n_3))$$

$$\phi_{(m_1+_b m_2)+_c m_3,(n_1+_{b'} n_2)+_{c'} n_3}((F(\psi\phi_{m_1+_b m_2,m_3}(F(\psi)\phi_{m_1,m_2}(x_1,x_2),x_3),(F(\psi)\phi_{n_1+_{b'} n_2,n_3}(F(\psi)\phi_{n_1,n_2}(y_1,y_2),y_3))) \in F(((m_1+_b m_2)+_c m_3)+((n_1+_{b'} n_2)+_{c'} n_3))$$

$$\phi_{m_1+_b(m_2+_c m_3),n_1+_{b'}(n_2+_{c'} n_3)}(F(\psi)\phi_{m_1,m_2+_c m_3}((x_1,F(\psi)\phi_{m_2,m_3}(x_2,x_3)),(y_1,F(\psi)\phi_{n_2,n_3}(y_2,y_3))) \in F((m_1+_b(m_2+_c m_3))+(n_1+_{b'}(n_2+_{c'} n_3)))$$

(2) Down and then right:

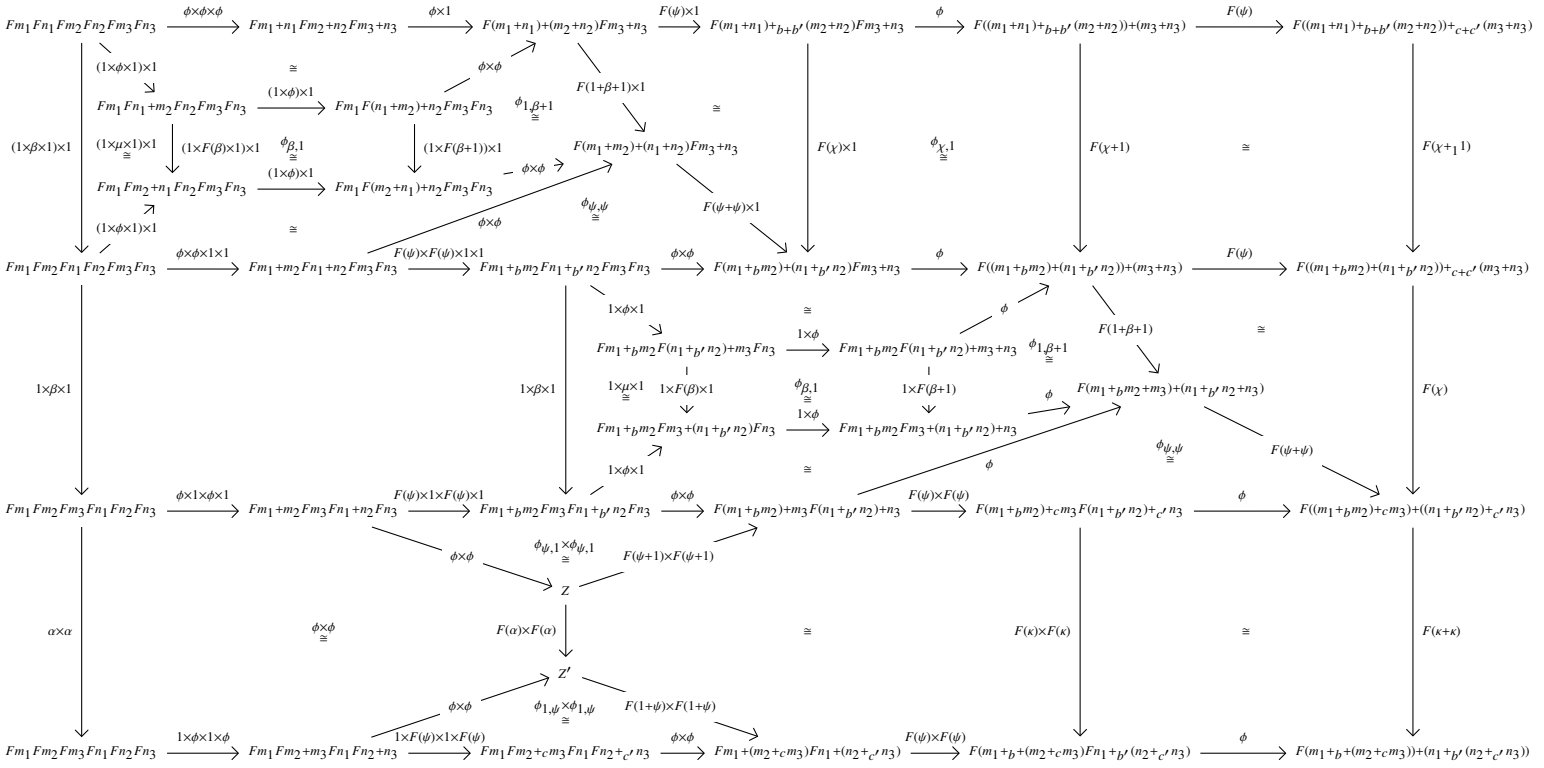
$$F(\psi)\phi_{(m_1+n_1)+_{b+b'}(m_2+n_2),m_3+n_3}(F(\psi)\phi_{m_1+n_1,m_2+n_2}(\phi_{m_1,n_1}(x_1,y_1),\phi_{m_2,n_2}(x_2,y_2)),F(\psi)\phi_{m_3,n_3}(x_3,y_3)) \in F(((m_1+n_1)+_{b+b'}(m_2+n_2))+_{c+c'}(m_3+n_3))$$

$$F(\psi)\phi_{m_1+n_1,(m_2+n_2)+_{c+c'}(m_3+n_3)}((\phi_{m_1,n_1}(x_1,y_1),F(\psi)\phi_{m_2+n_2,m_3+n_3}(\phi_{m_2,n_2}(x_2,y_2),\phi_{m_3,n_3}(x_3,y_3)))) \in F((m_1+n_1)+_{b+b'}((m_2+n_2)+_{c+c'}(m_3+n_3)))$$

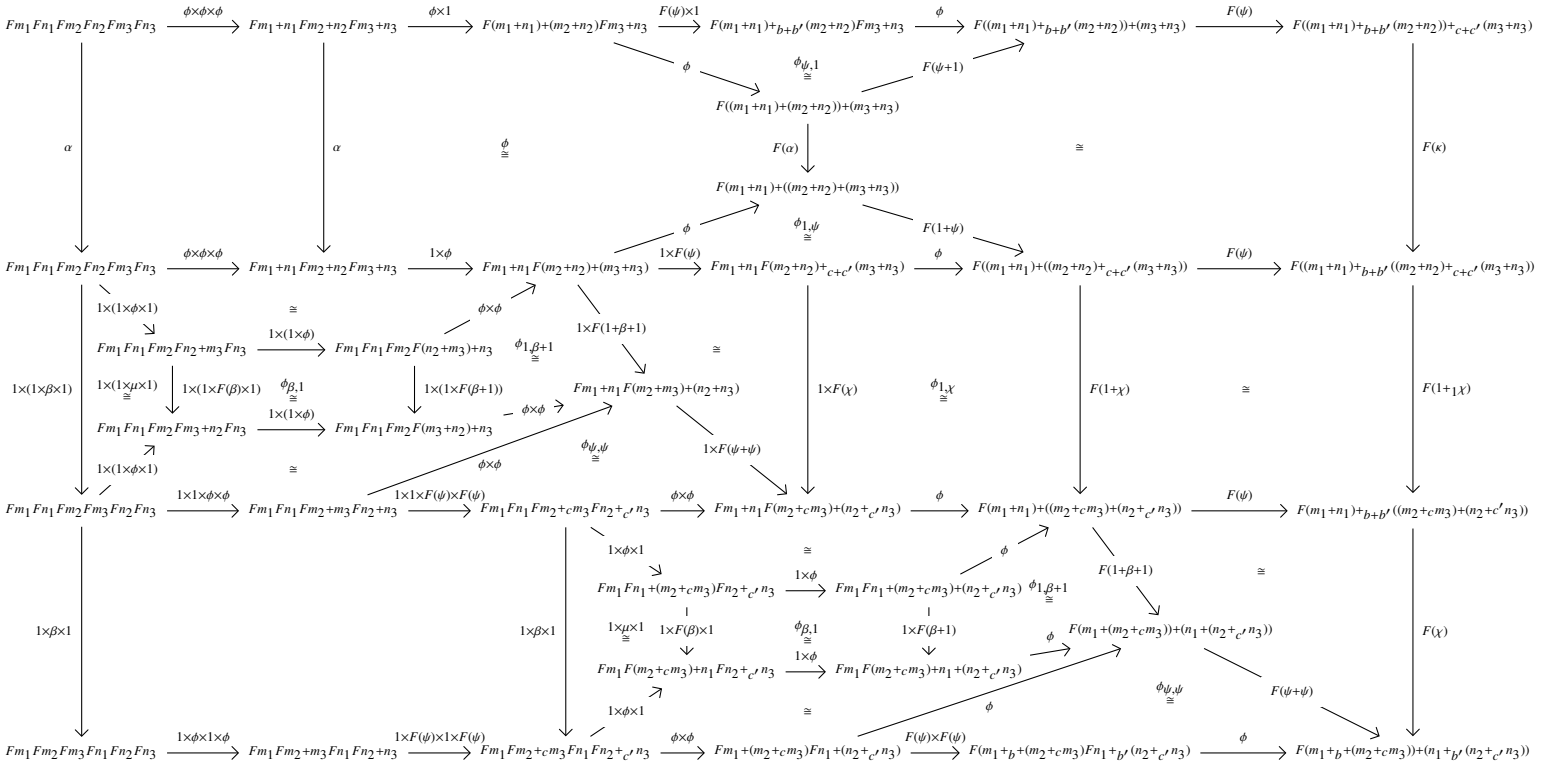
$$F(\psi)\phi_{m_1+n_1,(m_2+cm_3)+(n_2+_{c'} n_3)}(\phi_{m_1,n_1}(x_1,y_1),\phi_{m_2+_c m_3,n_2+_{c'} n_3}((F\psi)\phi_{m_2,m_3}(x_2,x_3),F(\psi)\phi_{n_2,n_3}(y_2,y_3))) \in F((m_1+n_1)+_{b+b'}((m_2+_c m_3)+(n_2+_{c'} n_3)))$$

$$\phi_{m_1+_b(m_2+_c m_3),n_1+_{b'}(n_2+_{c'} n_3)}(F(\psi)\phi_{m_1,m_2+_c m_3}((x_1,F(\psi)\phi_{m_2,m_3}(x_2,x_3)),(y_1,F(\psi)\phi_{n_2,n_3}(y_2,y_3))) \in F((m_1+_b(m_2+_c m_3))+(n_1+_{b'}(n_2+_{c'} n_3)))$$

Right and then down (omitting morphisms emanating out of 1 on the left due to space restrictions):



Down and then right (omitting morphisms emanating out of 1 on the left due to space restrictions):



Diagrams 2 and 3 of 11. Set up but not solved.

$M \in F\mathbb{C}\mathbf{sp}$ is given by $a \xrightarrow{i} m \xleftarrow{o} a'$ with $x \in F(m)$.

$N \in F\mathbb{C}\mathbf{sp}$ is given by $b \xrightarrow{i'} n \xleftarrow{o'} b'$ with $y \in F(n)$.

$U_a \in F\mathbb{C}\mathbf{sp}$ is given by $a \xrightarrow{\mathrm{id}} a \xleftarrow{\mathrm{id}} a$ with $\perp_a \in F(a)$.

$a{+}b \to (a{+}b){+}_{a+b}(m{+}n) \leftarrow a'{+}b'$
$F(\psi)\phi_{a+b,m+n}(\phi_{m,n}(x,y),\perp_{a+b}) \in F((a{+}b){+}_{a+b}(m{+}n))$

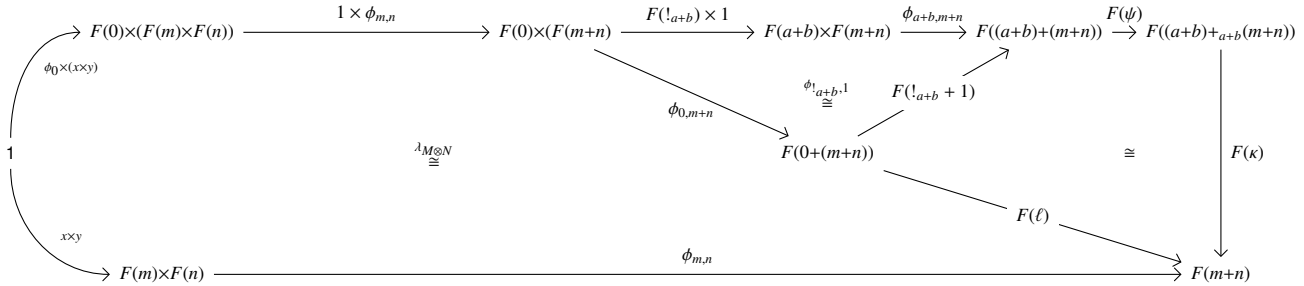
$$\begin{array}{ccc} (M\otimes N)\odot(U_{a+b}) & \xrightarrow{\mathrm{id}\odot\mu} & (M\otimes N)\odot(U_a\otimes U_b) \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle \chi} \\ M\otimes N & \xleftarrow{r\otimes r} & (M\odot U_a)\otimes(N\odot U_b) \end{array}$$

$a{+}b \to (a{+}b){+}_{a+b}(m{+}n) \leftarrow a'{+}b'$
$F(\psi)\phi_{a+b,m+n}(\phi_{m,n}(x,y),\phi_{a,b}(\perp_a,\perp_b)) \in F((a{+}b){+}_{a+b}(m{+}n))$

$a{+}b \to m{+}n \leftarrow a'{+}b'$
$\phi_{m,n}(x,y) \in F(m{+}n)$

$a{+}b \to (a{+}_a m){+}(b{+}_b n) \leftarrow a'{+}b'$
$\phi_{a+_a m,b+_b n}(F(\psi)\phi_{a,m}(\perp_a,x),F(\psi)\phi_{b,n}(\perp_b,y)) \in F((a{+}_a m){+}(b{+}_b n))$

Down:



Right, down and then left:

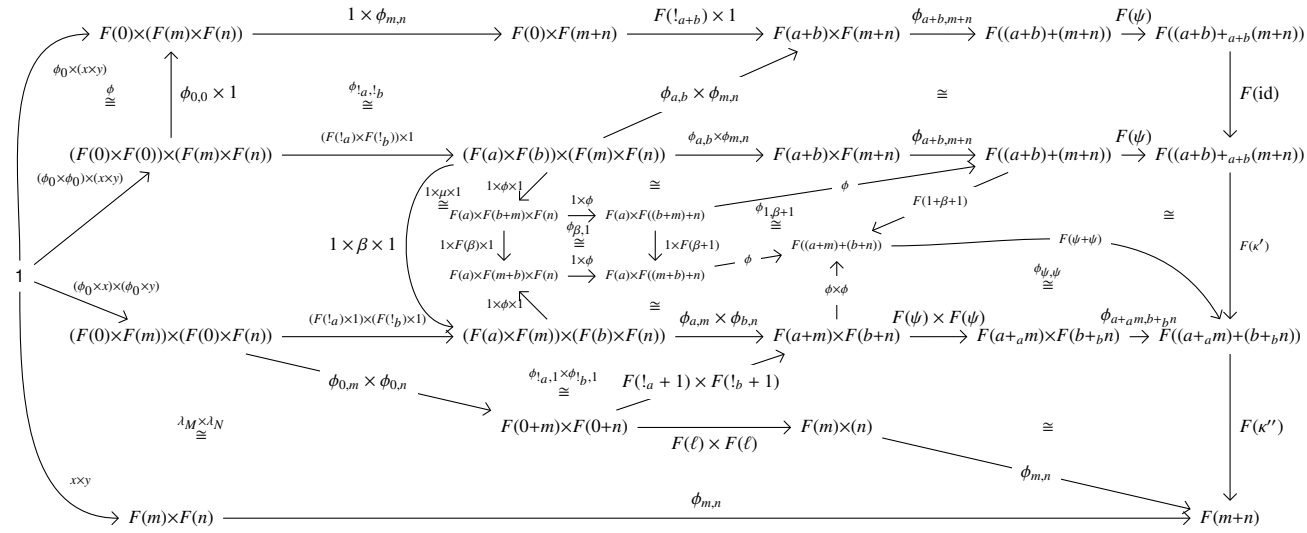
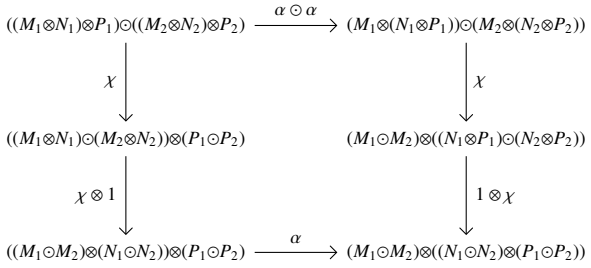


Diagram 4 of 11. Set up but not solved.
M_1 is given by $a \to m_1 \leftarrow b$ with $x_1 \in F(m_1)$.
M_2 is given by $b \to m_2 \leftarrow c$ with $x_2 \in F(m_2)$.
N_1 is given by $a' \to n_1 \leftarrow b'$ with $y_1 \in F(n_1)$.
N_2 is given by $b' \to n_2 \leftarrow c'$ with $y_2 \in F(n_2)$.
P_1 is given by $a'' \to p_1 \leftarrow b''$ with $z_1 \in F(p_1)$.
P_2 is given by $b'' \to p_2 \leftarrow c''$ with $z_2 \in F(p_2)$.



Decorations:

(1) Right and then down:

$$F(\psi)\phi_{(m_1+n_1)+p_1,(m_2+n_2),p_2}(\phi_{m_1+n_1,p_1}(\phi_{m_1,n_1}(x_1,y_1),z_1),\phi_{m_2+n_2,p_2}(\phi_{m_2,n_2}(x_2,y_2),z_2))\in F(((m_1+n_1)+p_1)+_{(b+b')+b''}((m_2+n_2)+p_2))$$

$$F(\psi)\phi_{m_1+(n_1+p_1),m_2+(n_2+p_2)}(\phi_{m_1,n_1+p_1}(x_1,\phi_{n_1,p_1}(y_1,z_1)),\phi_{m_2,n_2+p_2}(x_2,\phi_{n_2,p_2}(y_2,z_2)))\in F((m_1+(n_1+p_1))+_{b+(b'+b'')}(m_2+(n_2+p_2)))$$

$$\phi_{m_1+_bm_2,(n_1+p_1)+_{b'+b''}(n_2+p_2)}(F(\psi)\phi_{m_1,m_2}(x_1,x_2),F(\psi)(\phi_{n_1+p_1,n_2+p_2}(\phi_{n_1,p_1}(y_1,z_1),\phi_{n_2,p_2}(y_2,z_2)))\in F((m_1+_bm_2)+((n_1+p_1)+_{b'+b''}(n_2+p_2)))$$

$$\phi_{m_1+_bm_2,(n_1+_{b'}n_2)+(p_1+_{b''}p_2)}(F(\psi)\phi_{m_1,m_2}(x_1,x_2),\phi_{n_1+_{b'}n_2,p_1+_{b''}p_2}(F(\psi)\phi_{n_1,n_2}(y_1,y_2),F(\psi)\phi_{p_1,p_2}(z_1,z_2)))\in F((m_1+_bm_2)+((n_1+_{b'}n_2)+(p_1+_{b''}p_2)))$$

(2) Down and then right:

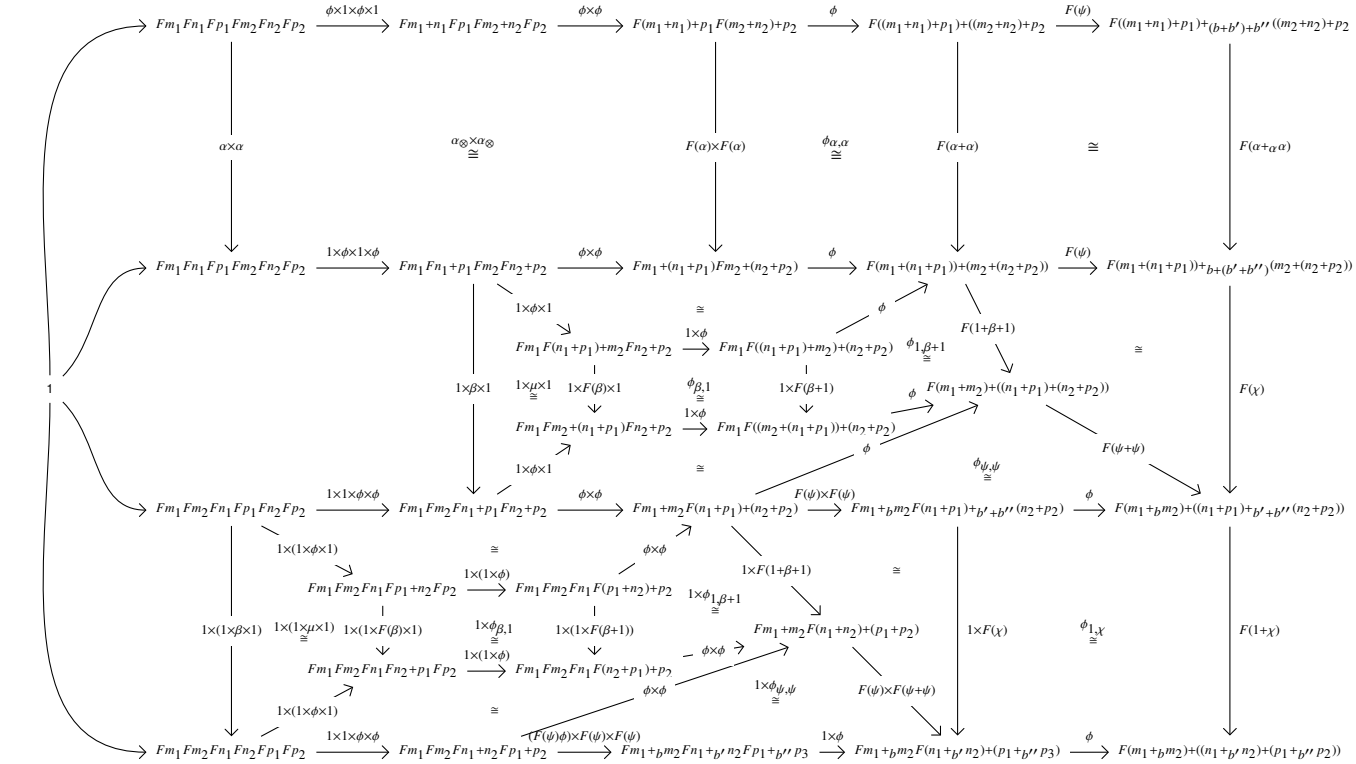
$$F(\psi)\phi_{(m_1+n_1)+p_1,(m_2+n_2),p_2}(\phi_{m_1+n_1,p_1}(\phi_{m_1,n_1}(x_1,y_1),z_1),\phi_{m_2+n_2,p_2}(\phi_{m_2,n_2}(x_2,y_2),z_2))\in F(((m_1+n_1)+p_1)+_{(b+b')+b''}((m_2+n_2)+p_2))$$

$$\phi_{(m_1+n_1)+_{b+b'}(m_2+n_2),p_1+_{b''}p_2}(F(\psi)\phi_{m_1+n_1,m_2+n_2}(\phi_{m_1,n_1}(x_1,y_1),\phi_{m_2,n_2}(x_2,y_2)),F(\psi)\phi_{p_1,p_2}(z_1,z_2))\in F(((m_1+n_1)+_{b+b'}(m_2+n_2))+(p_1+_{b''}p_2))$$

$$\phi_{(m_1+_bm_2)+(n_1+_{b'}n_2),p_1+_{b''}p_2}(\phi_{m_1+_bm_2,n_1+_{b'}n_2}(F(\psi)\phi_{m_1,m_2}(x_1,x_2),F(\psi)\phi_{n_1,n_2}(y_1,y_2)),F(\psi)\phi_{p_1,p_2}(z_1,z_2))\in F(((m_1+_bm_2)+(n_1+_{b'}n_2))+(p_1+_{b''}p_2))$$

$$\phi_{m_1+_bm_2,(n_1+_{b'}n_2)+(p_1+_{b''}p_2)}(F(\psi)\phi_{m_1,m_2}(x_1,x_2),\phi_{n_1+_{b'}n_2,p_1+_{b''}p_2}(F(\psi)\phi_{n_1,n_2}(y_1,y_2),F(\psi)\phi_{p_1,p_2}(z_1,z_2)))\in F((m_1+_bm_2)+((n_1+_{b'}n_2)+(p_1+_{b''}p_2)))$$

Right and then down:



Down and then right:

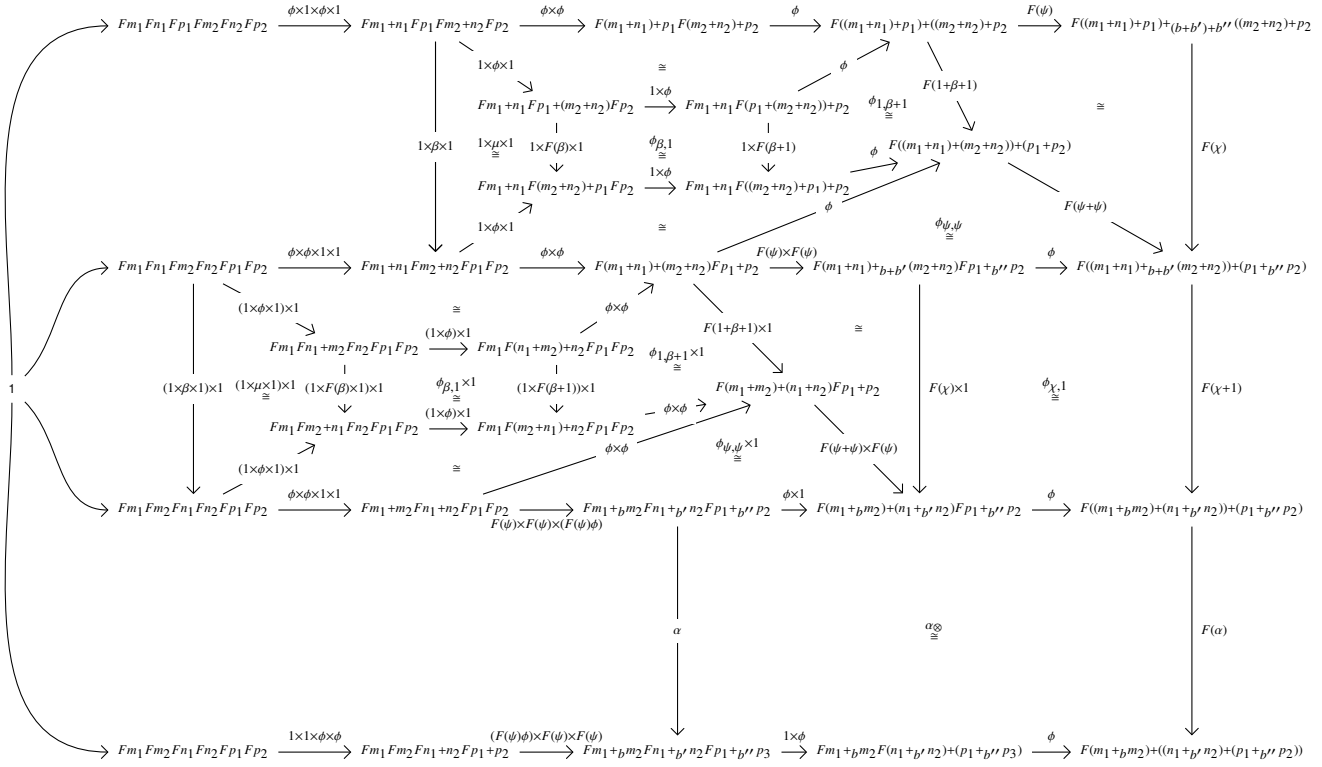
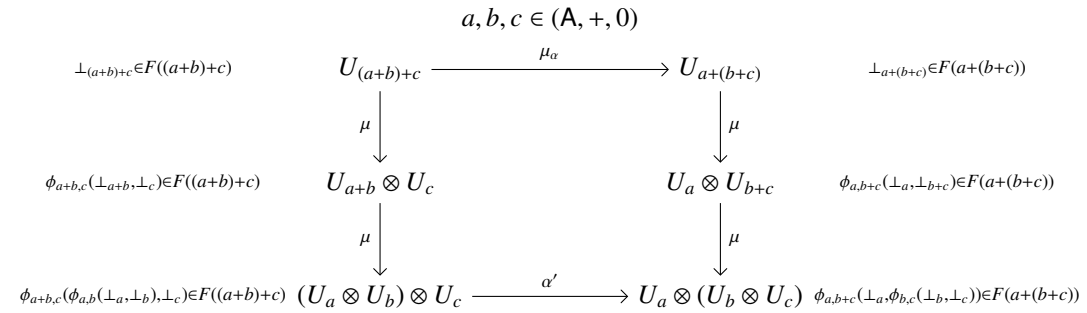
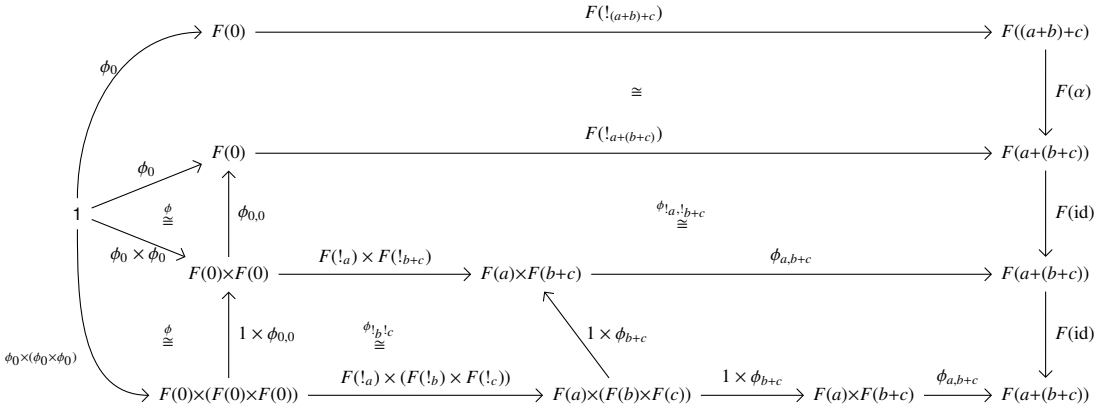


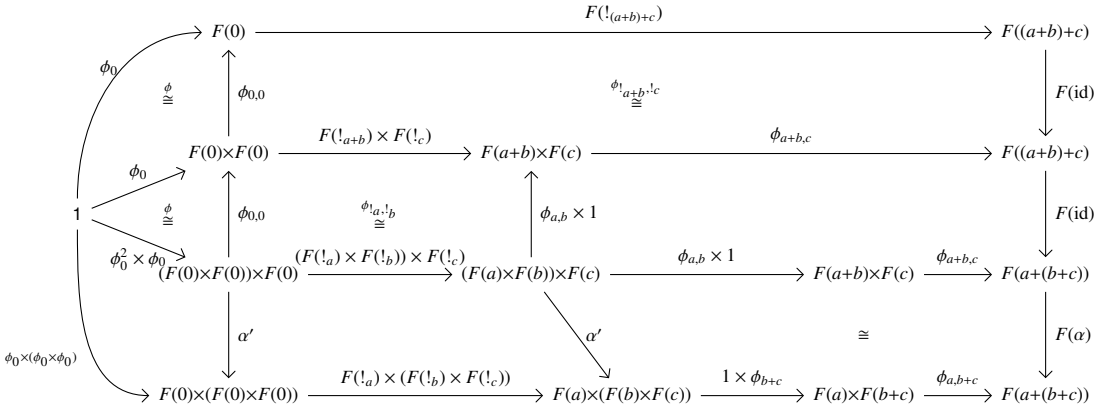
Diagram 5 of 11. Set up but not solved.



All the left cospans are $(a + b) + c \to (a + b) + c \leftarrow (a + b) + c$ and all the right cospans are $a + (b + c) \to a + (b + c) \leftarrow a + (b + c)$.
Right and then down:



Down and then right:

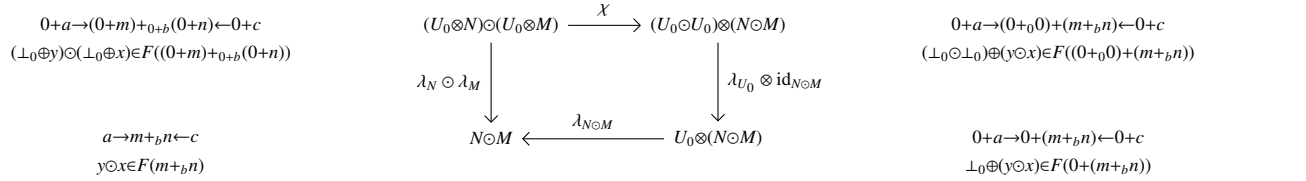


Diagrams 6 and 8 of 11. Set up but not solved.

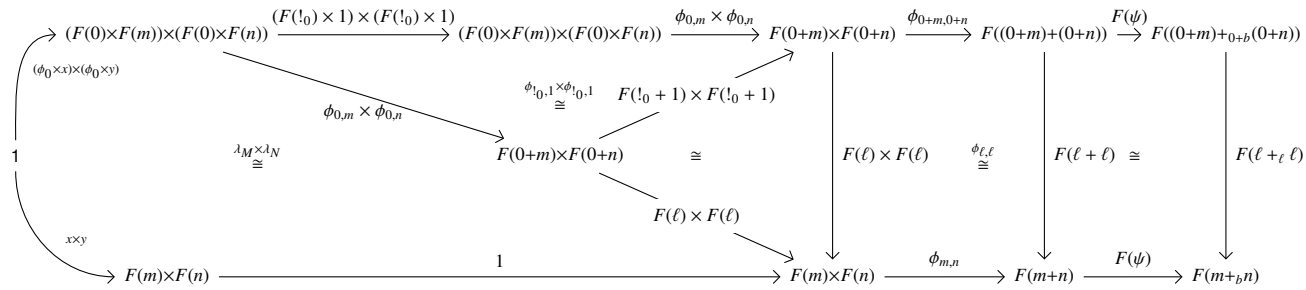
$M \in F\mathbb{C}\mathbf{sp}$ is given by $a \xrightarrow{i} m \xleftarrow{o} b$ with $x \in F(m)$.

$N \in F\mathbb{C}\mathbf{sp}$ is given by $b \xrightarrow{i'} n \xleftarrow{o'} c$ with $y \in F(n)$.

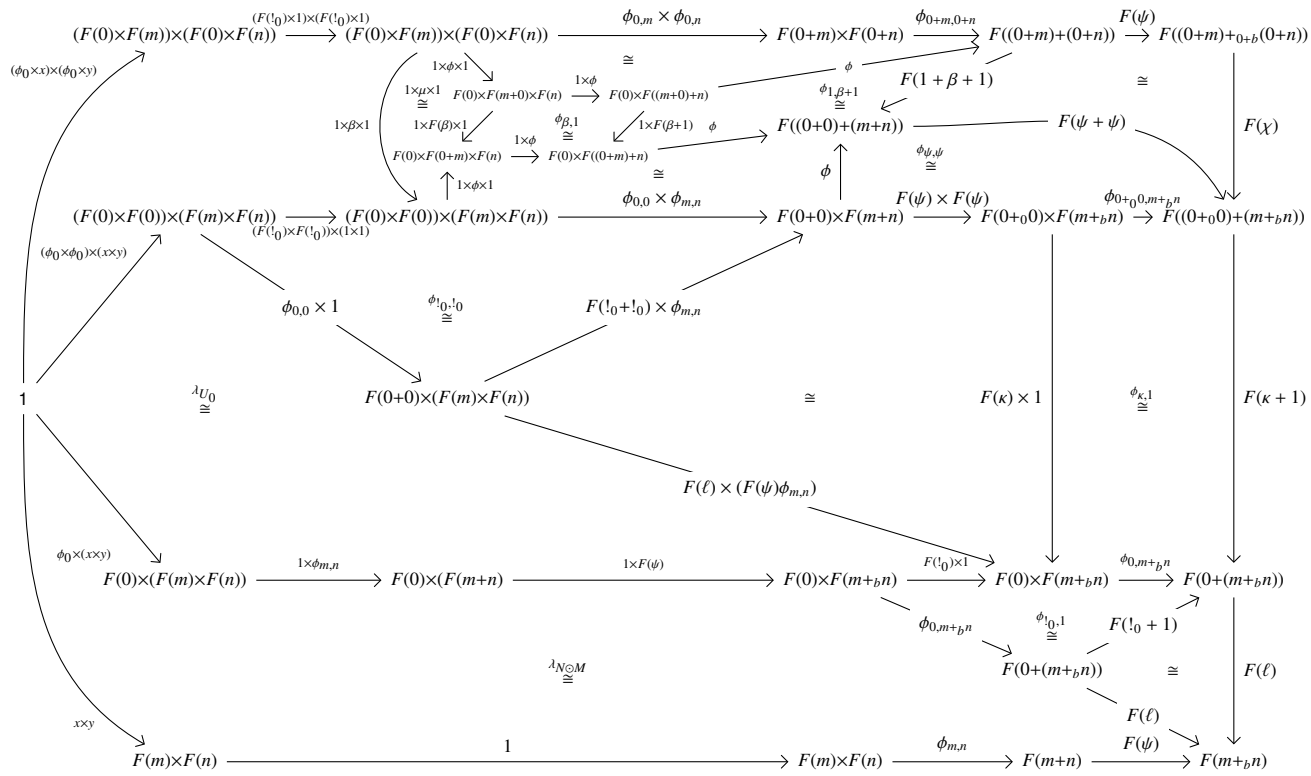
$U_0 \in F\mathbb{C}\mathbf{sp}$ is given by $0 \xrightarrow{!} 0 \xleftarrow{!} 0$ with $\perp_0 \in F(0)$.



Down:

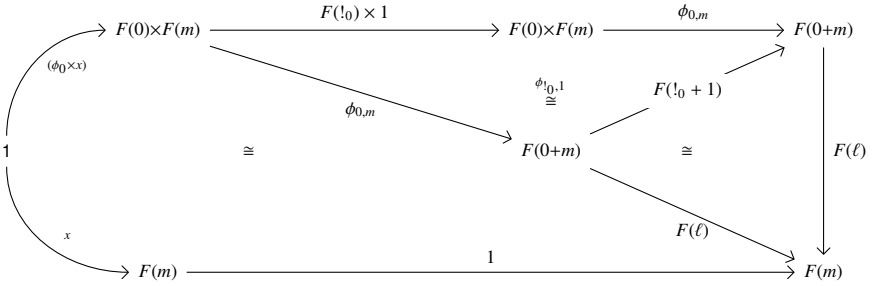


Right, down and then left:

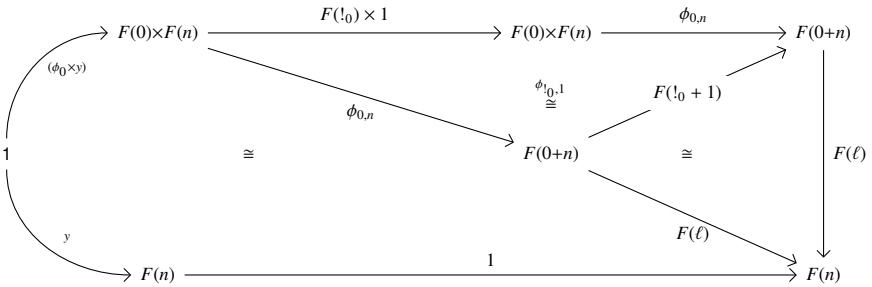


(Scratch work) As a check, let us compute the decoration maps $\lambda_N \colon U_0 \otimes N \to N$ and $\lambda_M \colon U_0 \otimes M \to M$ and compose them via $\odot$. The maps λ_N and λ_M are given explicitly in the paper.

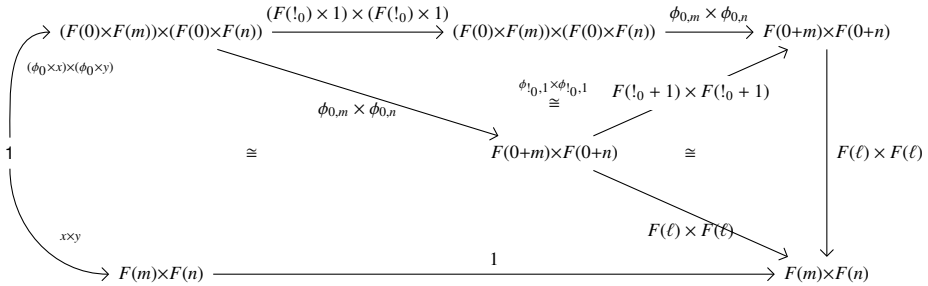
$\lambda_M \colon U_0 \otimes M \to M$



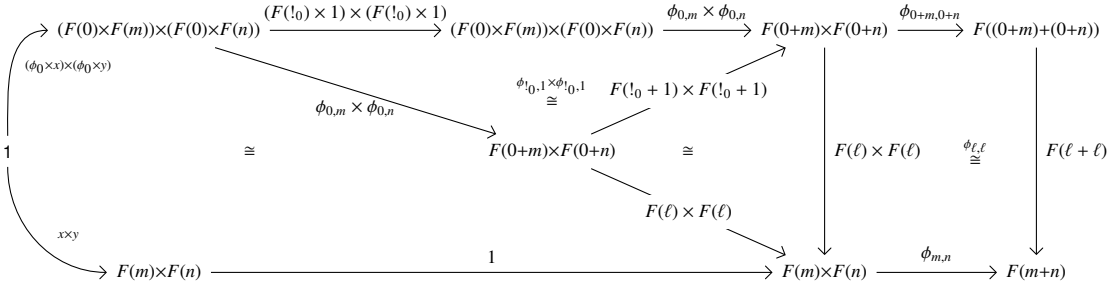
$\lambda_N \colon U_0 \otimes N \to N$



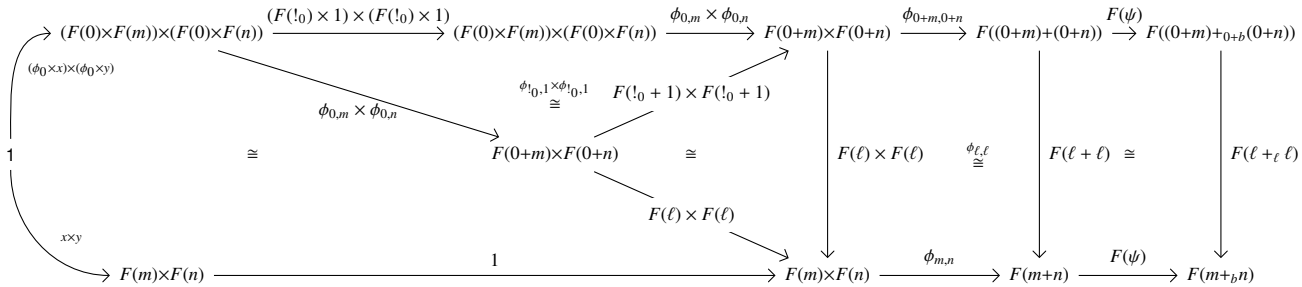
To construct $\lambda_N \odot \lambda_M \colon (U_0 \otimes N) \odot (U_0 \otimes M) \to N \odot M$, we first tensor the above two diagrams:



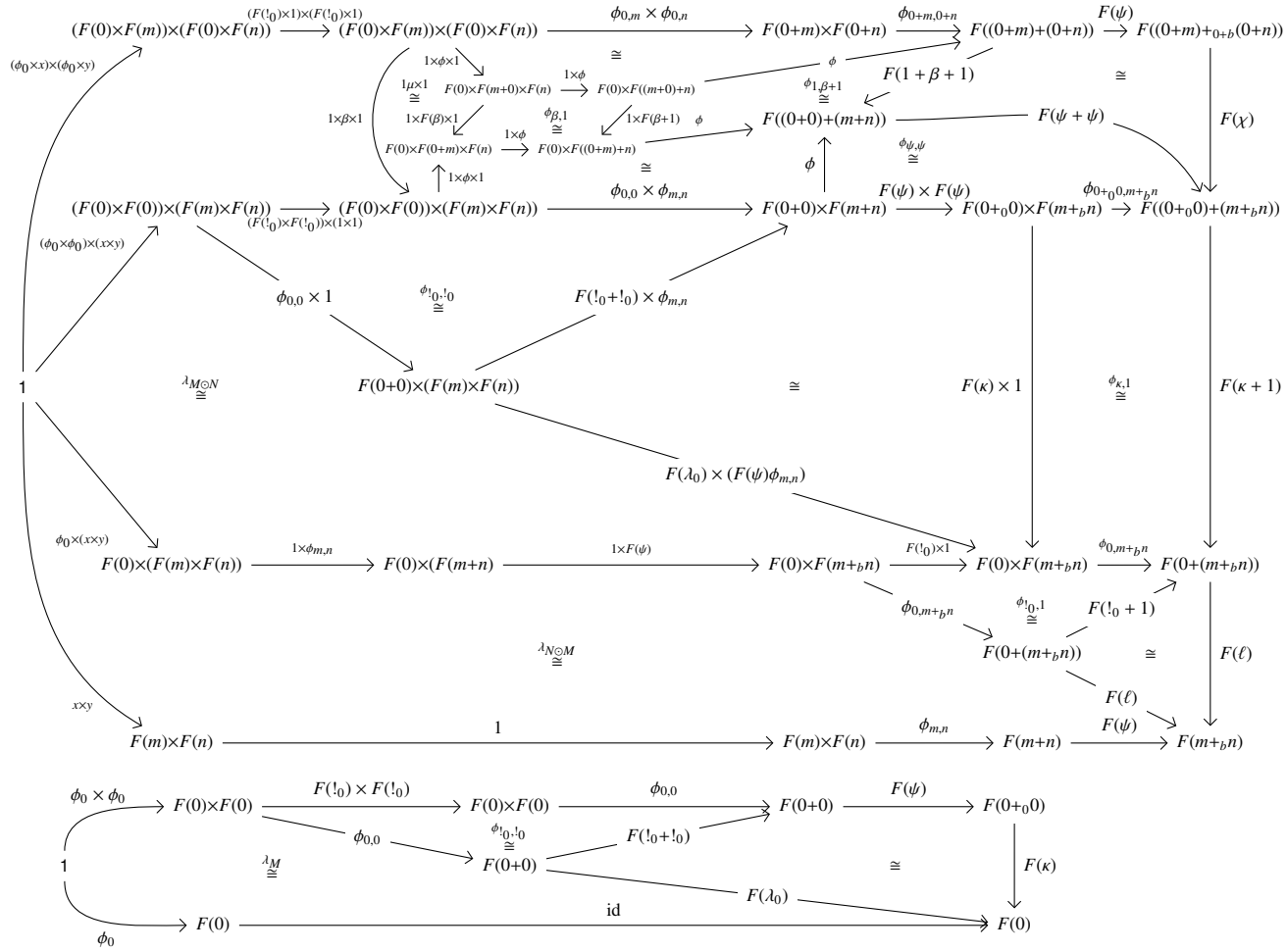
Next, we paste with a square due to pseudonaturality of ϕ:



Finally, we paste with a square due to pseudonaturality of F to obtain the map $\lambda_N \odot \lambda_M \colon (U_0 \otimes N) \odot (U_0 \otimes M) \to N \odot M$:



Next, we compute the right, down and then left route:



Diagrams 7 and 9 of 11. Should improve this, but seems okay.

$$F\colon (\mathsf{A},+,0) \to (\mathbf{Cat},\times,1)$$

$$a \in (\mathsf{A},+,0)$$

$0+a \to 0+a \leftarrow 0+a$

$\phi_{0,a}(\bot_0,\bot_a) \in F(0+a)$

$$\begin{array}{ccc} U_0 \otimes U_a & \xrightarrow{\mu} & U_{0+a} \\ & {\scriptstyle \lambda_{U_a}} \searrow & \downarrow {\scriptstyle U_{\lambda_a}} \\ & & U_a \end{array}$$

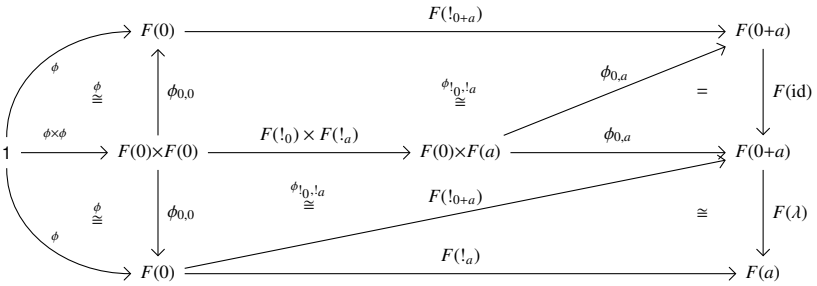
$0+a \to 0+a \leftarrow 0+a$

$\bot_{0+a} \in F(0+a)$

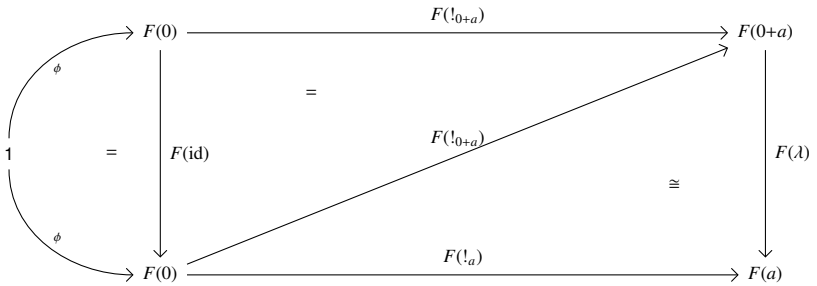
$a \to a \leftarrow a$

$\bot_a \in F(a)$

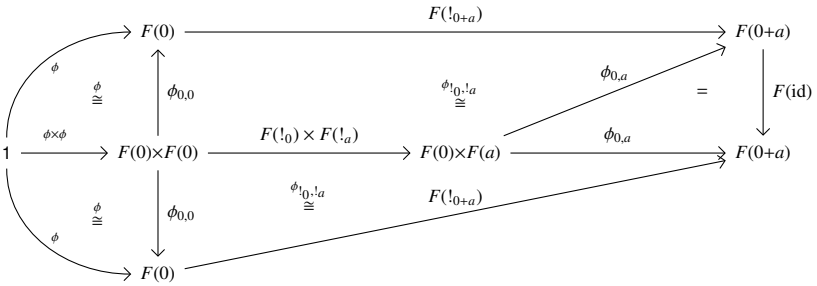
Right and then down:

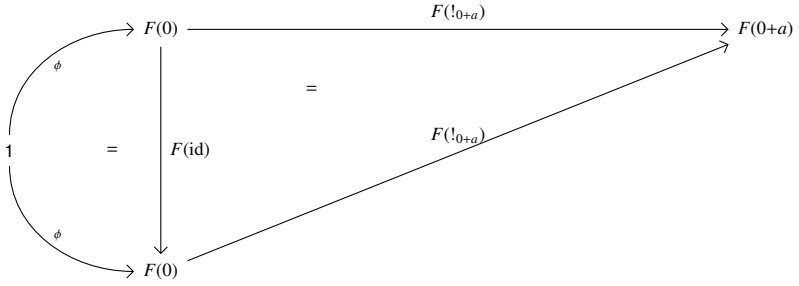


Diagonally:

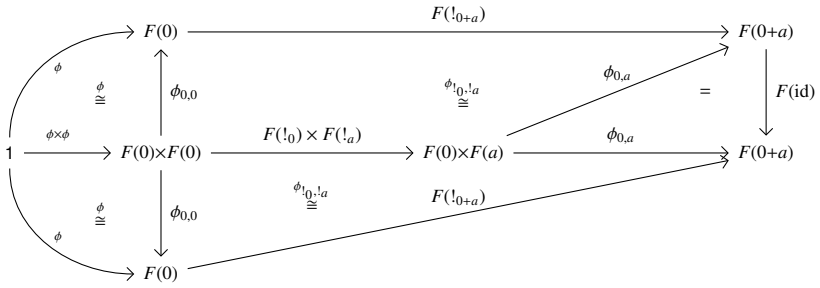


Removing the lower right ≅ which is the same in each diagram:

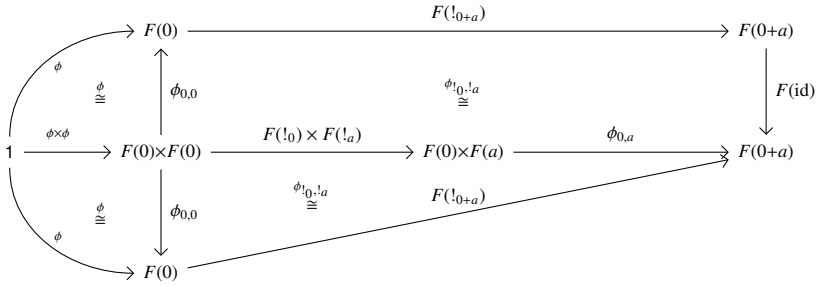




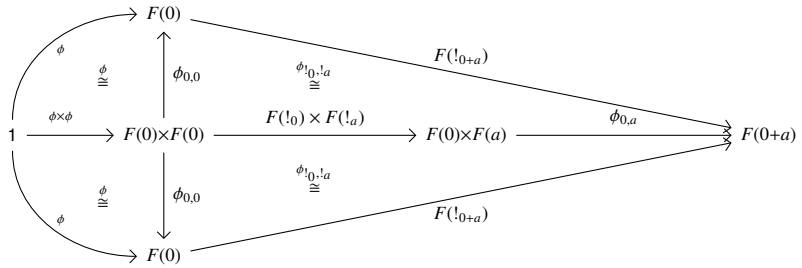
The second diagram is then an identity, so our problem reduces to showing that the following diagram is also an identity:



Removing the diagonal $\phi_{0,a}$:



This diagram is clearly the same as:



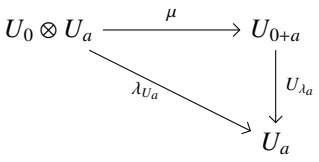
The two 2-isomorphisms in the top half of the diagram are the inverses of those in the bottom half, when read in a suitable order, and they can be shown to cancel, yielding an identity as desired.

Diagrams 7 and 9 of 11. Set up and solved.

$$F\colon (\mathsf{A},+,0) \to (\mathbf{Cat},\times,1)$$
$$a \in (\mathsf{A},+,0)$$

$0+a \to 0+a \leftarrow 0+a$

$\phi_{0,a}(\bot_0,\bot_a) \in F(0+a)$



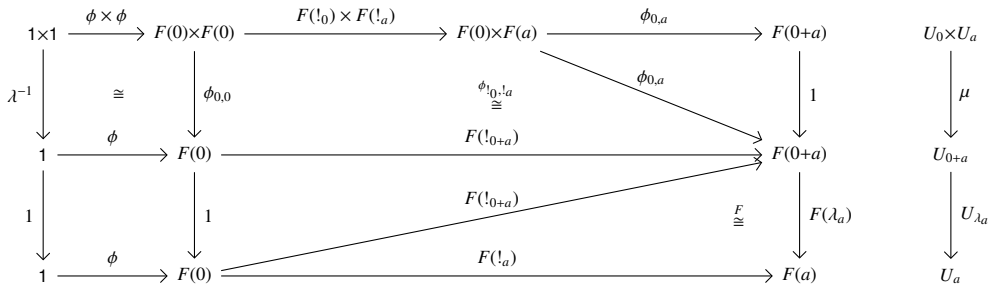
$0+a \to 0+a \leftarrow 0+a$

$\bot_{0+a} \in F(0+a)$

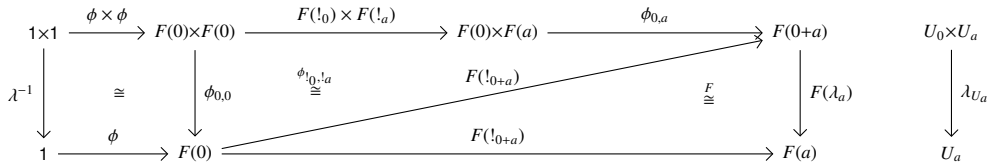
$a \to a \leftarrow a$

$\bot_a \in F(a)$

Right and then down:

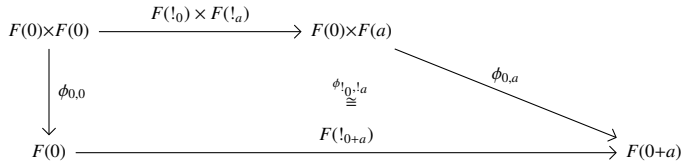


Diagonally:



If we remove the regions common to each diagram, and regions that strictly commute, these are clearly equal.

Right and then down:



Diagonally:

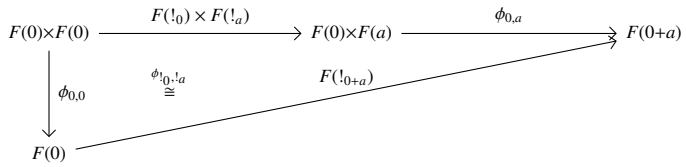


Diagram 10 of 11. Set up and solved (Christina).

$M_1 \in F\mathbb{C}\mathbf{sp}$ is given by $a \xrightarrow{i_1} m_1 \xleftarrow{o_1} b$ with $x_1 \in F(m_1)$.

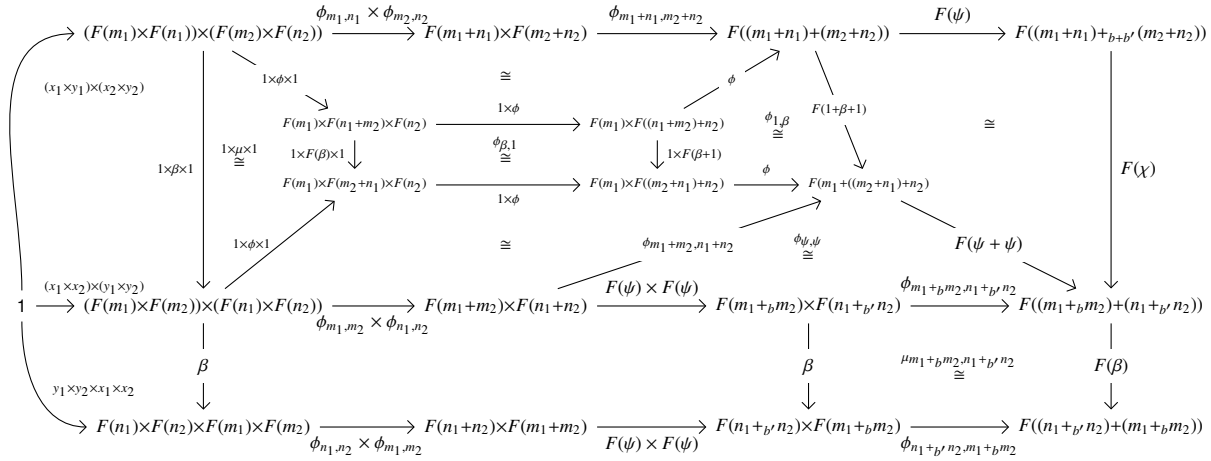
$M_2 \in F\mathbb{C}\mathbf{sp}$ is given by $b \xrightarrow{i_2} m_2 \xleftarrow{o_2} c$ with $x_2 \in F(m_2)$.

$N_1 \in F\mathbb{C}\mathbf{sp}$ is given by $a' \xrightarrow{i'_1} n_1 \xleftarrow{o'_1} b'$ with $y_1 \in F(n_1)$.

$N_2 \in F\mathbb{C}\mathbf{sp}$ is given by $b' \xrightarrow{i'_2} n_2 \xleftarrow{o'_2} c'$ with $x_2 \in F(n_2)$.

$$\begin{array}{ccc}
\underset{\phi_{m_1+n_1,m_2+n_2}(\phi_{m_1,n_1}(x_1,y_1),\phi_{m_2,n_2}(x_2,y_2))\in F((m_1+n_1)+_{b+b'}(n_1+n_2))}{a+a' \to (m_1+n_1)+_{b+b'}(m_2+n_2) \leftarrow c+c'} & (M_2\otimes N_2)\odot(M_1\otimes N_1) \xrightarrow{\beta\odot\beta} (N_2\otimes M_2)\odot(N_1\otimes M_1) & \underset{F(\psi)\phi_{n_1+m_1,n_2+m_2}(\phi_{m_1,m_1}(y_1,x_1),\phi_{n_2,m_2}(y_2,x_2))\in F((n_1+m_1)+_{b'+b}(n_1+m_2))}{a'+a \to (n_1+m_1)+_{b'+b}(n_2+m_2) \leftarrow c'+c} \\
\downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \chi} \\
\underset{\phi_{m_1+_bm_2,n_1+_{b'}n_2}(F(\psi)\phi_{m_1,m_2}(x_1,x_2),F(\psi)\phi_{n_1,n_2}(y_1,y_2))\in F((m_1+_bm_2)+(n_1+_{b'}n_2))}{a+a' \to (m_1+_bm_2)+(n_1+_{b'}n_2) \leftarrow c+c'} & (M_2\odot M_1)\otimes(N_2\odot N_1) \xrightarrow{\beta} (N_2\odot N_1)\otimes(M_2\odot M_1) & \underset{\phi_{n_1+_{b'}n_2}(F(\psi)\phi_{n_1,n_2}(y_1,y_2),F(\psi)\phi_{m_1,m_2}(x_1,x_2))\in F((n_1+_{b'}n_2)+(m_1+_bm_2))}{a'+a \to (n_1+_{b'}n_2)+(m_1+_bm_2) \leftarrow c'+c}
\end{array}$$

Down and then right:



Right and then down:

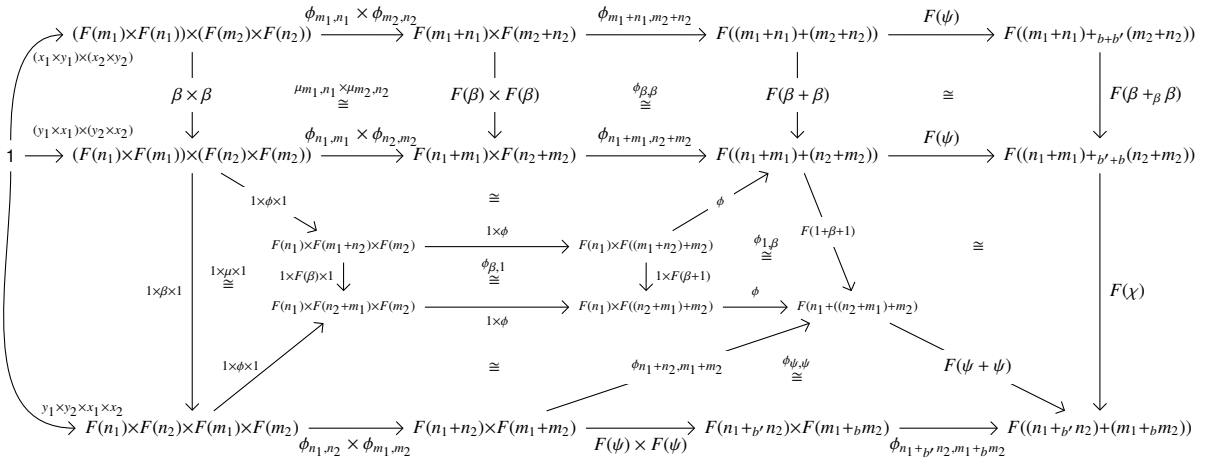


Diagram 11 of 11. Set up but not solved.

$$a, b \in (\mathsf{A}, +, 0)$$

$$\begin{array}{ccc} U_{a+b} & \xrightarrow{\mu_{a,b}} & U_a \otimes U_b \\ {\scriptstyle U_\beta}\downarrow & & \downarrow{\scriptstyle \beta'} \\ U_{b+a} & \xrightarrow{\mu_{b,a}} & U_b \otimes U_a \end{array}$$

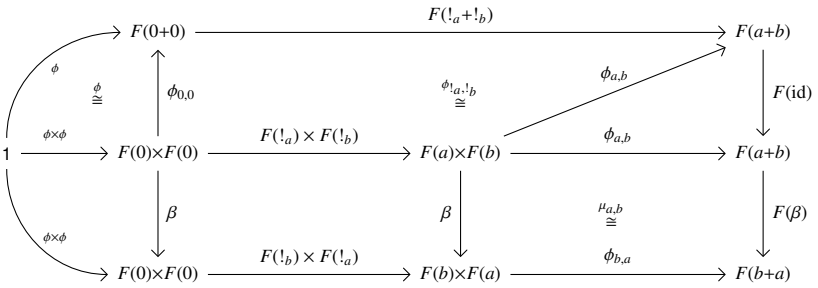
The top two underlying cospans are

$$a + b \to a + b \leftarrow a + b$$

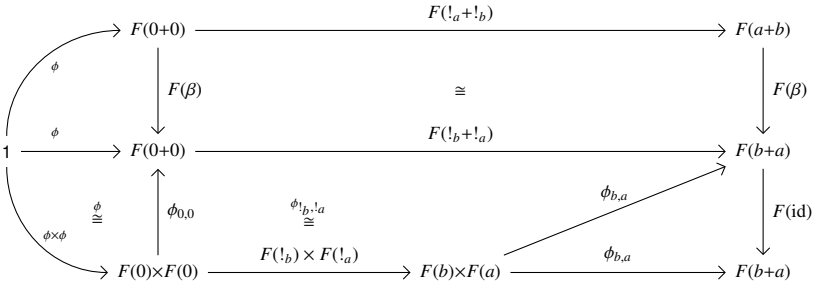
and the bottom two underyling cospans are

$$b + a \to b + a \leftarrow b + a$$

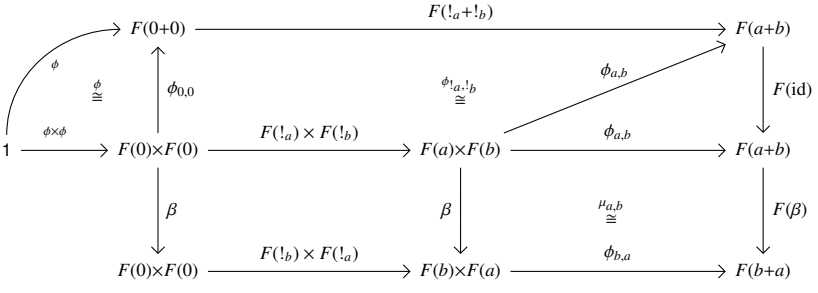
Right and then down:

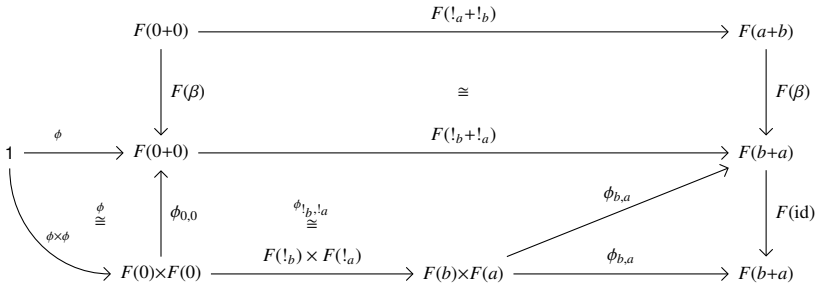


Down and then right:

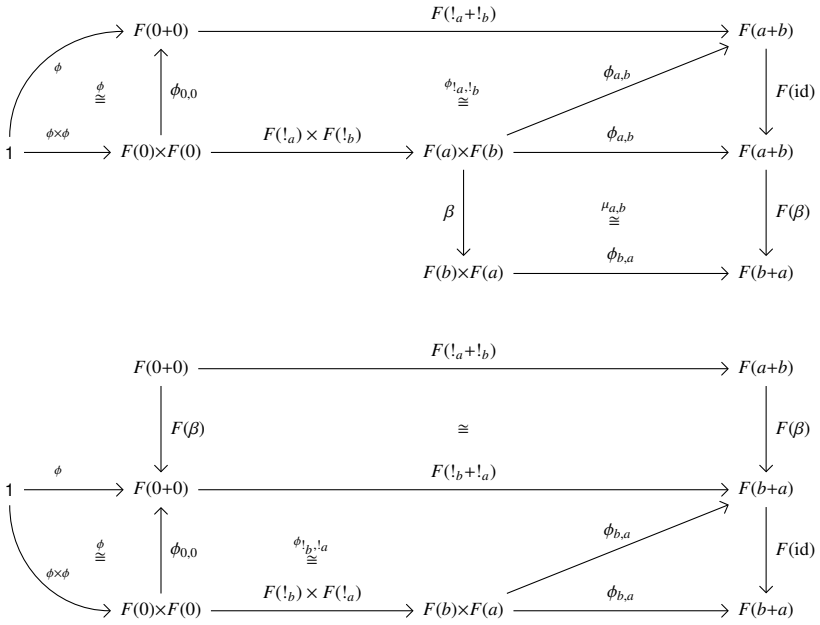


First we remove the lower left commuting quarter circle in the ‘right and then down’ diagram as well as the upper left commuting quarter circle in the ‘down and then right’ diagram:

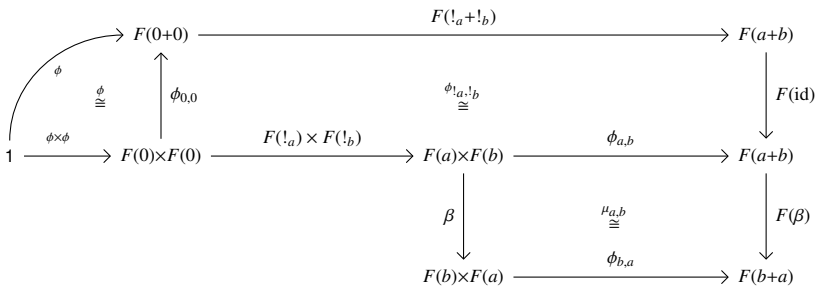


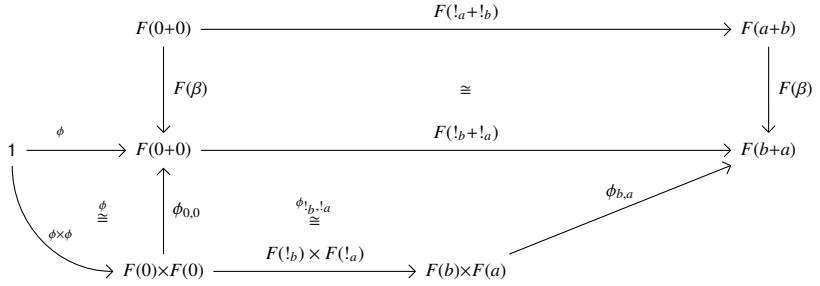


Next we remove the lower left commuting square with sides β in the first diagram:

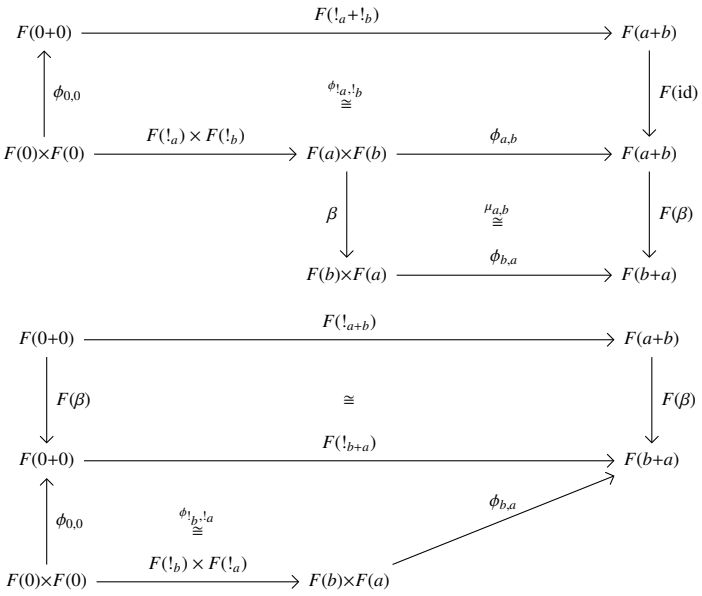


Next we assume that our pseudofunctor F is normalized, meaning that $F(\mathrm{id})$ is an identitiy, and remove the commuting triangle with sides $\phi_{a,b}$ in the first diagram and the commuting triangle with sides $\phi_{b,a}$ in the second diagram:

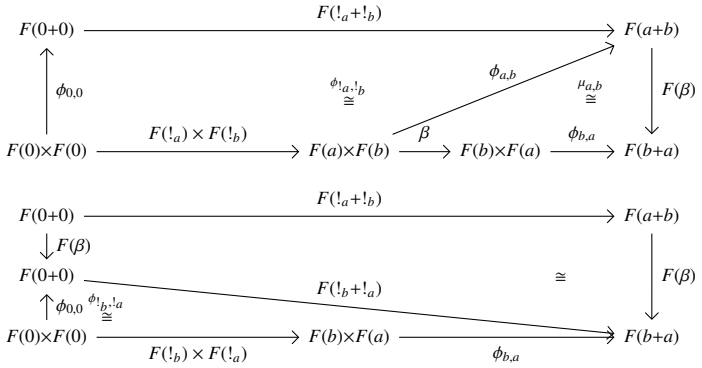




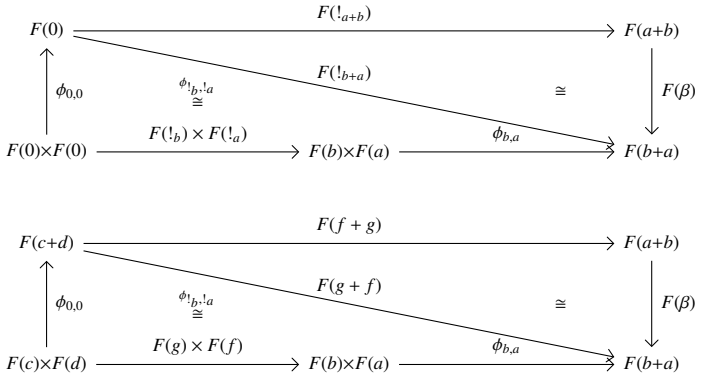
Next we remove the left quarter circle containing $\overset{\phi}{\cong}$ from each diagram:

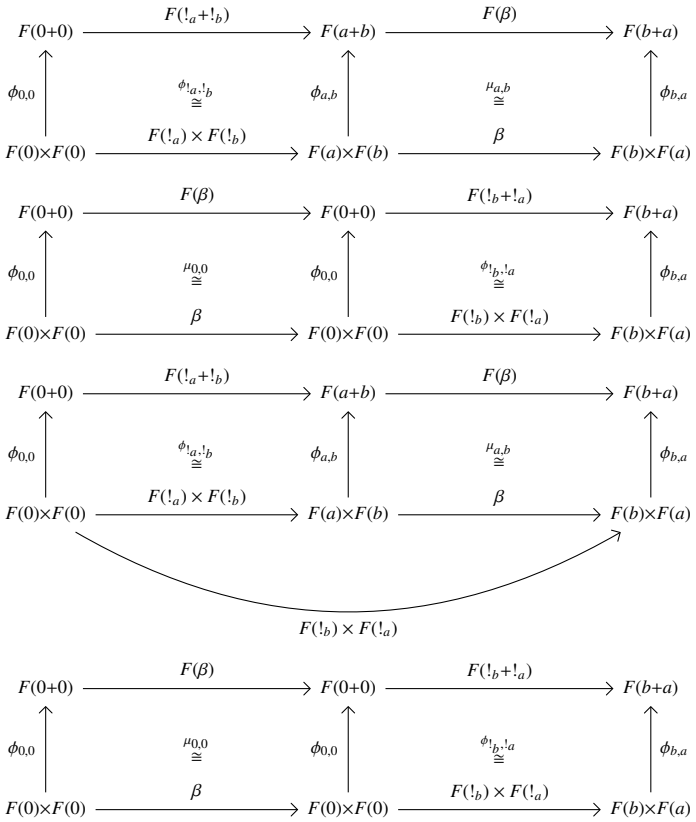


Making each diagram into a rectangle and removing the $F(\mathrm{id})$ from the first diagram:

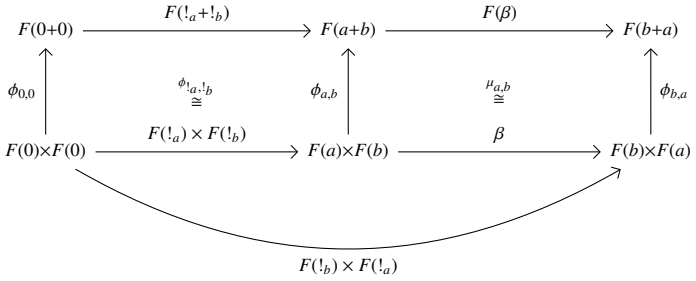


Are these two diagrams the same?





First diagram:



Second diagram:

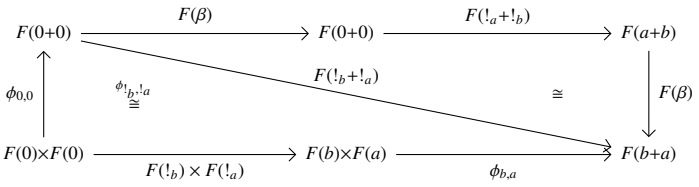
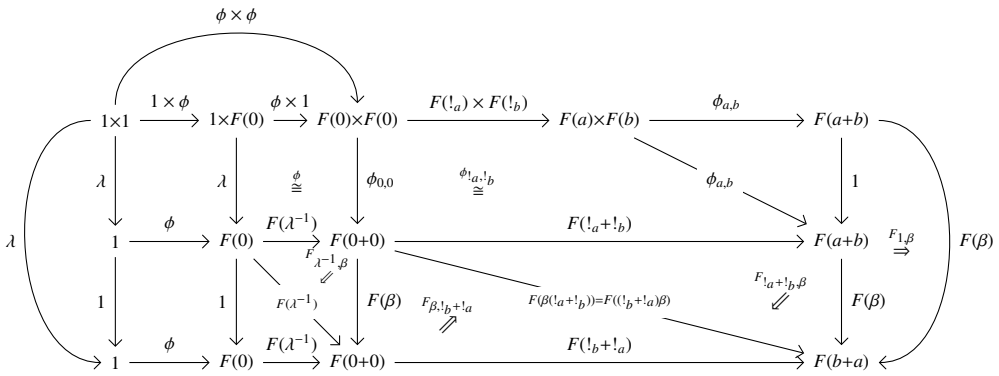


Diagram 11 of 11. Set up but not solved.

$$
\begin{array}{ccc}
a, b \in (\mathsf{A}, +, 0) & & \\
U_a \otimes U_b & \xrightarrow{\mu_{a,b}} & U_{a+b} \\
\Big\downarrow{\scriptstyle \beta'} & & \Big\downarrow{\scriptstyle U_\beta} \\
U_b \otimes U_a & \xrightarrow{\mu_{b,a}} & U_{b+a}
\end{array}
$$

Right and then down:



Down and then right:

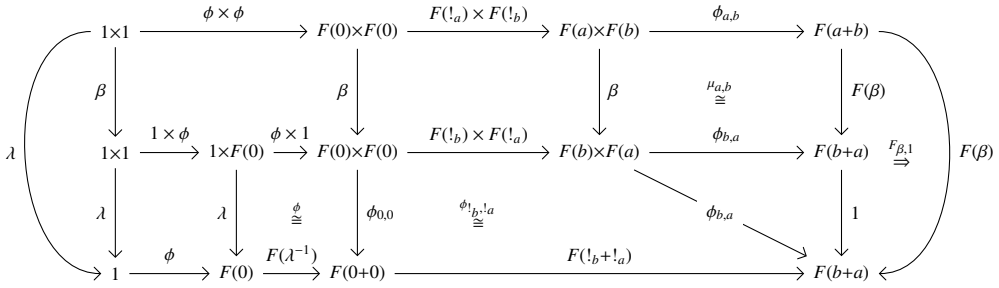
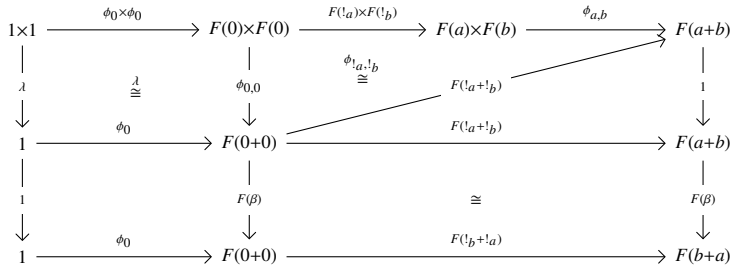


Diagram 11 of 11.

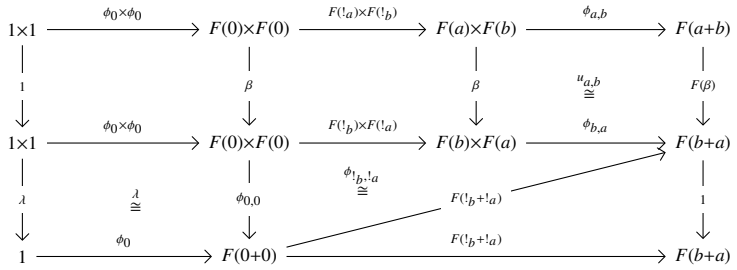
$$a, b \in (\mathsf{A}, +, 0)$$

$$\begin{array}{ccc} U_a \otimes U_b & \xrightarrow{\mu_{a,b}} & U_{a+b} \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle U_\beta} \\ U_b \otimes U_a & \xrightarrow{\mu_{b,a}} & U_{b+a} \end{array}$$

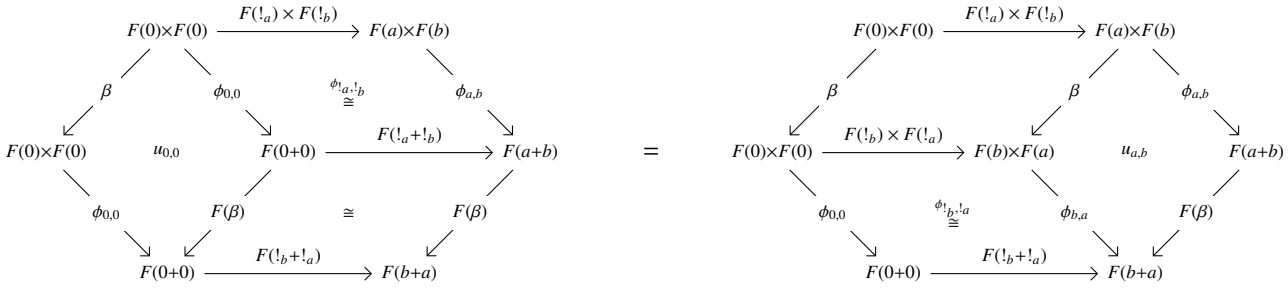
Right and then down:



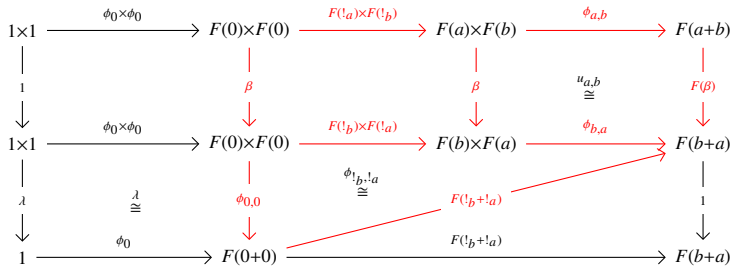
Down and then right:



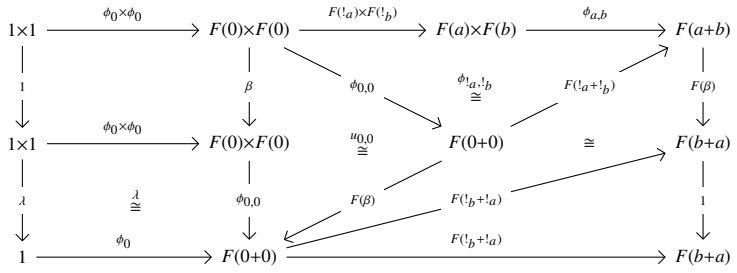
As $u\colon F(\beta)\phi \Rightarrow \phi\beta$ is an invertible modification, u satisfies the following equation:



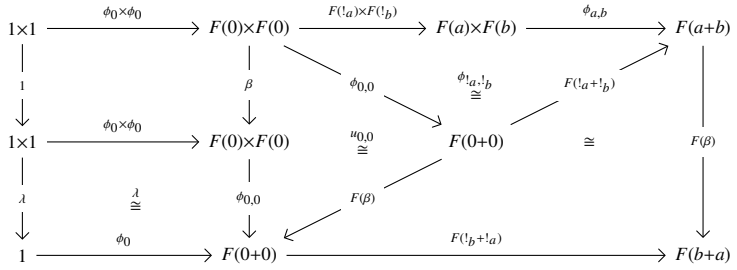
The right hand side of this equation already appears in the ”down and then right” diagram:



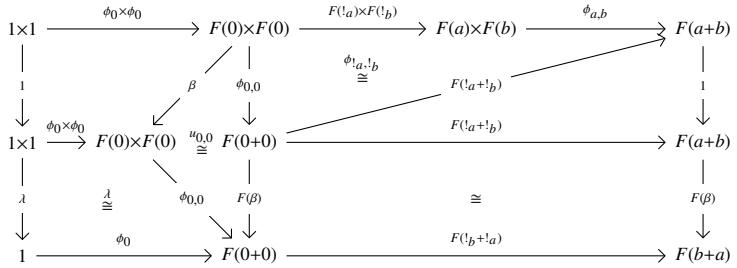
Replacing the red outlined region with the left hand side of the equation satisfied by u, the "down and then right" diagram becomes:



Next, we remove the commuting triangle in the bottom right, which results in:

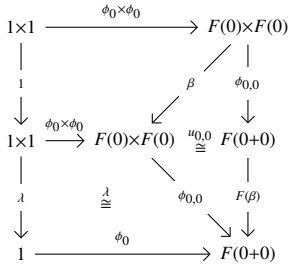


and if we slide the central $F(0+0)$ to the left and add in a commuting triangle to the top right, the diagram becomes:

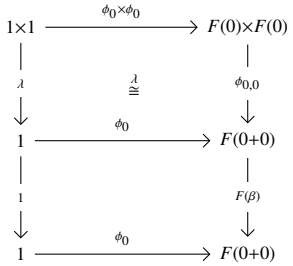


Now the right half of this diagram is identical to the right half of the "right and then down" diagram. Removing these identical right halfs, the two diagrams become:

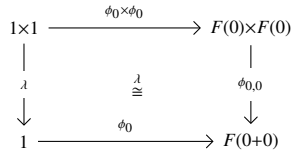
Down and then right:



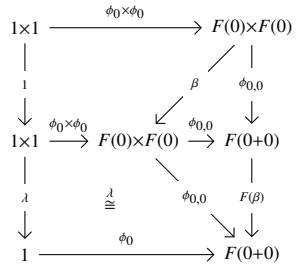
Right and then down:



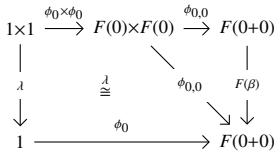
The bottom half of the "right and then down" diagram commutes and removing this bottom half leaves us with:



We now show that the "down and then right" diagram also reduces to this. First we add in a morphism $\phi_{0,0}\colon F(0)\times F(0)\to F(0+0)$ inside of the $u_{0,0}$ isomorphism:



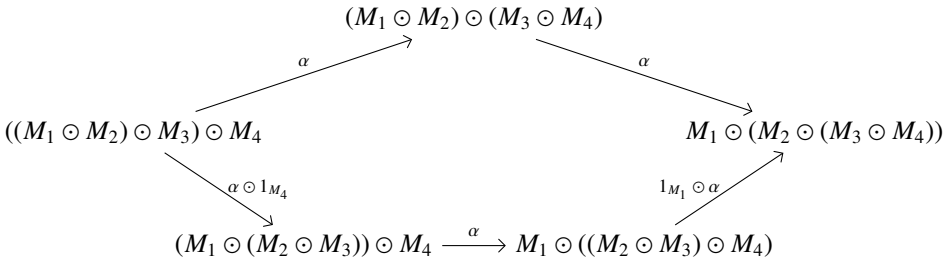
The top half of this diagram commutes as two of the three routes from 1×1 to $F(0+0)$ are identical and the third one swaps two identical objects before combining them via the laxator. Removing this top half yields:



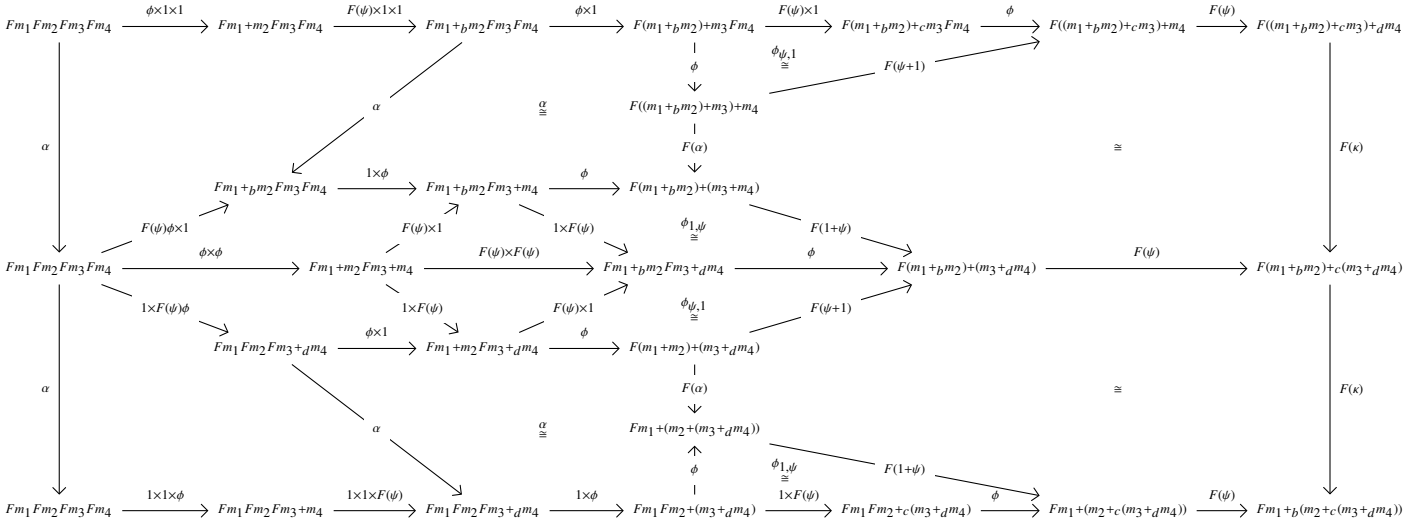
Finally, the triangle on the right when whiskered with the morphism $\phi_0\times\phi_0\colon 1\times 1\to F(0)\times F(0)$ also commutes, and removing this triangle leaves us with the isomorphism λ that the other diagram reduced to.

2. Proof of Theorem 2.1

Pentagon axiom. Set up but not solved.
M_1 is given by $a \to m_1 \leftarrow b$ with $x_1 \in F(m_1)$.
M_2 is given by $b \to m_2 \leftarrow c$ with $x_2 \in F(m_2)$.
M_3 is given by $c \to m_3 \leftarrow d$ with $x_3 \in F(m_3)$.
M_4 is given by $d \to m_4 \leftarrow e$ with $x_4 \in F(m_4)$.



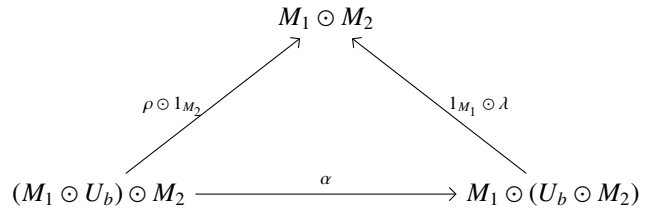
Upper route:



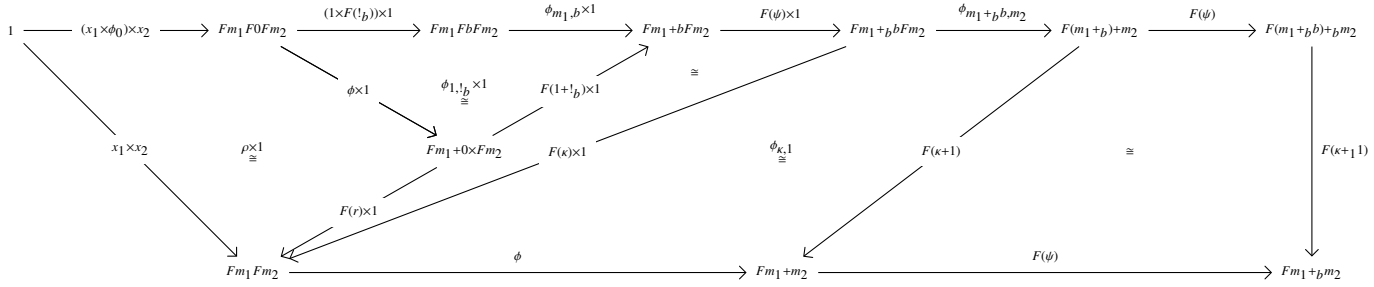
Lower route:

$Fm_1Fm_2Fm_3Fm_4$
$\phi\times1\times1$
$Fm_1{+}m_2Fm_3Fm_4$
$F(\psi)\times1\times1$
$Fm_1{+}_bm_2Fm_3Fm_4$
$\phi\times1$
$F(m_1{+}_bm_2){+}m_3Fm_4$
$F(\psi)\times1$
$F(m_1{+}_bm_2){+}_cm_3Fm_4$
ϕ
$F((m_1{+}_bm_2){+}_cm_3){+}m_4$
$F(\psi)$
$F((m_1{+}_bm_2){+}_cm_3){+}_dm_4$
$\phi\times1$
$\phi_{\psi,1}\times1$
$\cong$
$F(\psi{+}1)\times1$
$F(m_1{+}m_2){+}m_3Fm_4$
$\alpha\times1$
$\alpha\times1$
$\cong$
$F(\alpha)\times1$
$\cong$
$F(\kappa)\times1$
$\phi_{\kappa,1}$
$\cong$
$F(\kappa{+}1)$
$\cong$
$F(\kappa{+}_11)$
$Fm_1{+}(m_2{+}m_3)Fm_4$
$\phi\times1$
$\phi_{1,\psi}\times1$
$\cong$
$F(1{+}\psi)\times1$
$Fm_1Fm_2{+}m_3Fm_4$
$1\times\phi\times1$
$Fm_1Fm_2{+}_cm_3Fm_4$
$1\times F(\psi)\times1$
$Fm_1{+}(m_2{+}_cm_3)Fm_4$
$\phi\times1$
$Fm_1{+}_b(m_2{+}_cm_3)Fm_4$
$F(\psi)\times1$
$F(m_1{+}_b(m_2{+}_cm_3)){+}m_4$
ϕ
$F(\psi{+}1)$
$\phi_{\psi,1}$
$\cong$
$F(m_1{+}(m_2{+}_cm_3)){+}m_4$
$1\times\alpha\times1$
α
α
$\cong$
$F(\alpha)$
$\cong$
$F(\kappa)$
$Fm_1{+}((m_2{+}_cm_3){+}m_4)$
ϕ
$\phi_{1,\psi}$
$\cong$
$F(1{+}\psi)$
$Fm_1Fm_2Fm_3Fm_4$
$1\times\phi\times1$
$Fm_1Fm_2{+}m_3Fm_4$
$1\times F(\psi)\times1$
$Fm_1Fm_2{+}_cm_3Fm_4$
$1\times\phi$
$Fm_1F(m_2{+}_cm_3){+}m_4$
$1\times F(\psi)$
$Fm_1F(m_2{+}_cm_3){+}_dm_4$
ϕ
$Fm_1{+}((m_2{+}_cm_3){+}_dm_4)$
$F(\psi)$
$Fm_1{+}_b((m_2{+}_cm_3){+}_dm_4)$
$1\times\phi$
$1\times\phi_{\psi,1}$
$\cong$
$1\times F(\psi{+}1)$
$Fm_1F(m_2{+}m_3){+}m_4$
$1\times\alpha$
$1\times\alpha$
$\cong$
$1\times F(\alpha)$
$\cong$
$1\times F(\kappa)$
$\phi_{1,\kappa}$
$\cong$
$F(1{+}\kappa)$
$\cong$
$F(1{+}_1\kappa)$
$Fm_1Fm_2{+}(m_3{+}m_4)$
$1\times\phi$
$1\times\phi_{1,\psi}$
$\cong$
$1\times F(1{+}\psi)$
$Fm_1Fm_2Fm_3Fm_4$
$1\times1\times\phi$
$Fm_1Fm_2Fm_3{+}m_4$
$1\times1\times F(\psi)$
$Fm_1Fm_2Fm_3{+}_dm_4$
$1\times\phi$
$Fm_1Fm_2{+}(m_3{+}_dm_4)$
$1\times F(\psi)$
$Fm_1Fm_2{+}_c(m_3{+}_dm_4)$
ϕ
$Fm_1{+}(m_2{+}_c(m_3{+}_dm_4))$
$F(\psi)$
$Fm_1{+}_b(m_2{+}_c(m_3{+}_dm_4))$

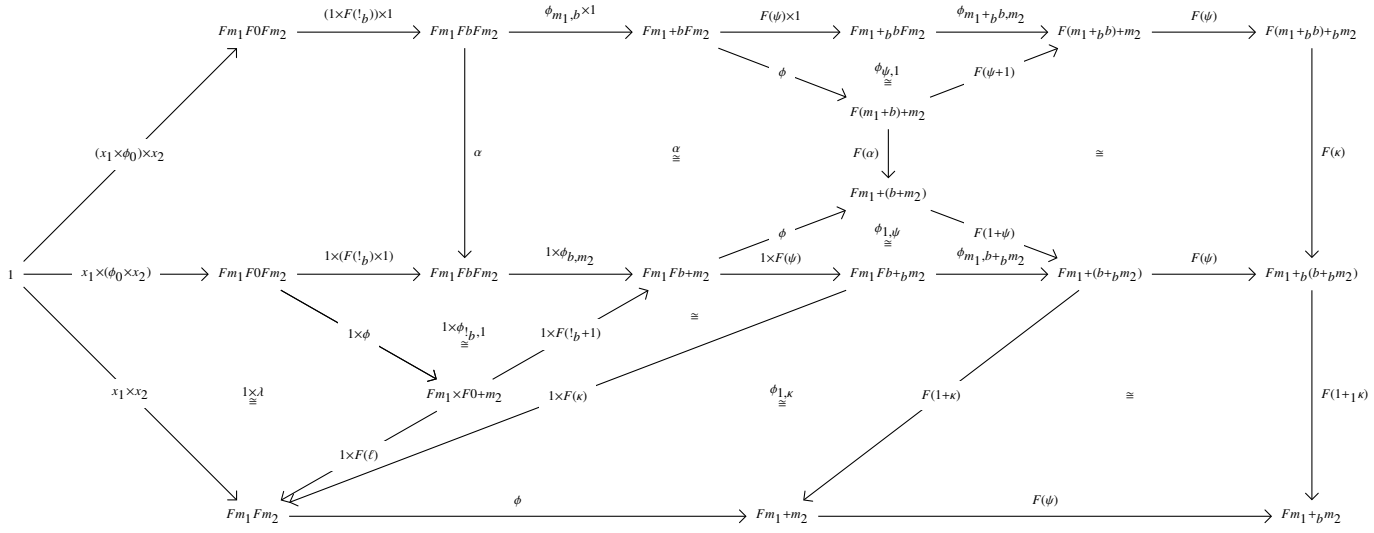
Triangle identity. Set up but not solved.
M_1 is given by $a \to m_1 \leftarrow b$ with $x_1 \in F(m_1)$.
U_b is given by $b \to b \leftarrow b$ with $\bot_b \in F(b)$.
M_2 is given by $b \to m_2 \leftarrow c$ with $x_2 \in F(m_2)$.



Up:



Across and then up:



3. Proof of Theorem 2.3

Diagram 1 of 7. Set up but not solved.

$$\begin{array}{ccc}
(\mathbb{H}(M_1)\otimes\mathbb{H}(M_2))\otimes\mathbb{H}(M_3) & \xrightarrow{\alpha'} & \mathbb{H}(M_1)\otimes(\mathbb{H}(M_2)\otimes\mathbb{H}(M_3)) \\
\downarrow{\scriptstyle \mathbb{H}_{M_1,M_2}\otimes 1} & & \downarrow{\scriptstyle 1\otimes\mathbb{H}_{M_2\otimes M_3}} \\
\mathbb{H}(M_1\otimes M_2)\otimes\mathbb{H}(M_3) & & \mathbb{H}(M_1)\otimes\mathbb{H}(M_2\otimes M_3) \\
\downarrow{\scriptstyle \mathbb{H}_{M_1\otimes M_2,M_3}} & & \downarrow{\scriptstyle \mathbb{H}_{M_1,M_2\otimes M_3}} \\
\mathbb{H}((M_1\otimes M_2)\otimes M_3) & \xrightarrow{\mathbb{H}(\alpha)} & \mathbb{H}(M_1\otimes(M_2\otimes M_3))
\end{array}$$

M_1 is given by $a_1 \xrightarrow{i_1} m_1 \xleftarrow{o_1} b_1$ with $x_1 \in F(m_1)$.
M_2 is given by $a_2 \xrightarrow{i_2} m_2 \xleftarrow{o_2} b_2$ with $x_2 \in F(m_2)$.
M_3 is given by $a_3 \xrightarrow{i_3} m_3 \xleftarrow{o_3} b_3$ with $x_3 \in F(m_3)$.
$\mathbb{H}(M_1)$ is given by $H(a_1) \xrightarrow{H(i_1)} H(m_1) \xleftarrow{H(o_1)} H(b_1)$ with $\bar{x_1} = \theta_m E(x_1)\phi_0 \in F'(H(m_1))$.
$\mathbb{H}(M_2)$ is given by $H(a_2) \xrightarrow{H(i_2)} H(m_2) \xleftarrow{H(o_2)} H(b_2)$ with $\bar{x_2} = \theta_m E(x_2)\phi_0 \in F'(H(m_2))$.
$\mathbb{H}(M_3)$ is given by $H(a_3) \xrightarrow{H(i_3)} H(m_3) \xleftarrow{H(o_3)} H(b_3)$ with $\bar{x_3} = \theta_m E(x_3)\phi_0 \in F'(H(m_3))$.
The decoration on $M_1 \otimes M_2$ is given by

$$x_1 \oplus x_2 := 1 \cong 1\times 1 \xrightarrow{x_1\times x_2} F(m_1)\times F(m_2) \xrightarrow{\phi_{m_1,m_2}} F(m_1+m_2).$$

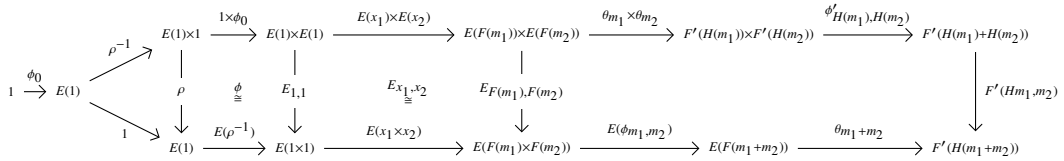
The decoration on $\mathbb{H}(M_1 \otimes M_2)$ is given by

$$x_1 \bar{\oplus} x_2 = \theta_{m_1+m_2} E(x_1 \oplus x_2)\phi_0 = 1 \cong 1\times 1 \xrightarrow{\phi_0} E(1\times 1) \xrightarrow{E(x_1\times x_2)} E(F(m_1)\times F(m_2)) \xrightarrow{E(\phi_{m_1,m_2})} E(F(m_1+m_2)) \xrightarrow{\theta_{m_1+m_2}} F'(H(m_1+m_2)).$$

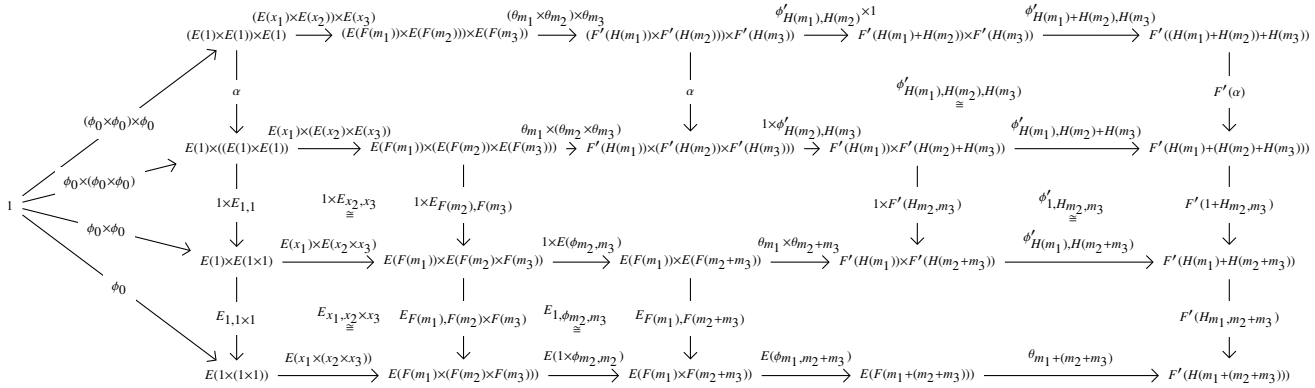
The decoration on $\mathbb{H}(M_1) \otimes \mathbb{H}(M_2)$ is given by

$$\bar{x_1} \oplus \bar{x_2} = 1 \cong 1\times 1 \xrightarrow{\phi_0\times\phi_0} E(1)\times E(1) \xrightarrow{E(x_1)\times E(x_2)} E(F(m_1))\times E(F(m_2)) \xrightarrow{\theta_{m_1}\times\theta_{m_2}} F'(H(m_1))\times F'(H(m_2)) \xrightarrow{\phi'_{H(m_1),H(m_2)}} F'(H(m_1)+H(m_2)).$$

The natural isomorphism $\mathbb{H}_\otimes \colon \mathbb{H}(M_1)\otimes\mathbb{H}(M_2) \to \mathbb{H}(M_1\otimes M_2)$ is given by:



Regarding the commuting pentagon, 1) the monoidal natural isomorphism $\theta\colon EF \to F'H$ is a strict monoidal natural isomorphism (not pseudo), and 2) the laxator of the composite $F'H$ utilizes the laxators of both F' and H.
Right and then down:



(I'm being a little reckless on the left with the ϕs)
Down and then right:

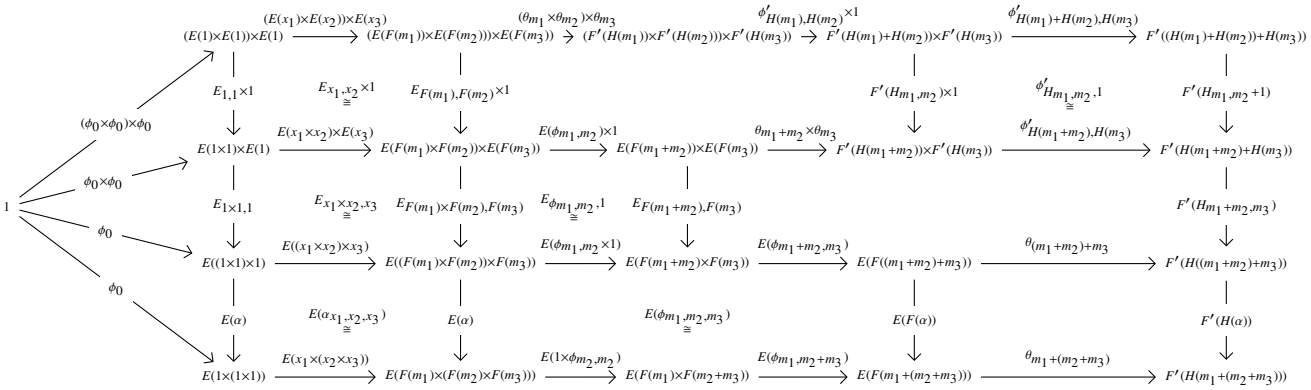
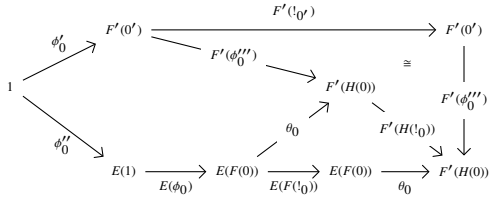
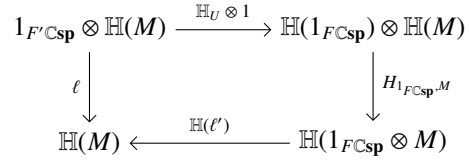
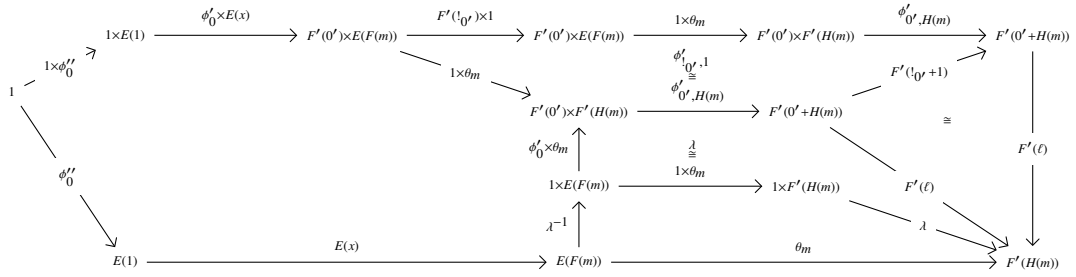


Diagram 2 of 7. Set up but not solved.
The natural isomorphism $\mathbb{H}_U : 1_{F'\mathbb{C}\mathbf{sp}} \to \mathbb{H}(1_{F\mathbb{C}\mathbf{sp}})$ is given by:





M is given by $a \xrightarrow{i} m \xleftarrow{o} b$ with $x \in F(m)$.
$\mathbb{H}(M)$ is given by $H(a) \xrightarrow{H(i)} H(m) \xleftarrow{H(o)} H(b)$ with $\bar{x} = \theta_m E(x)\phi_0 \in F'(H(m))$.
$1_{F\mathbb{C}\mathbf{sp}}$ is given by $0 \xrightarrow{1} 0 \xleftarrow{1} 0$ with $1 \xrightarrow{\phi_0} F(0) \xrightarrow{F(!_0)} F(0) \in F(0)$.
$\mathbb{H}(1_{F\mathbb{C}\mathbf{sp}})$ is given by $H(0) \xrightarrow{1} H(0) \xleftarrow{1} H(0)$ with $1 \xrightarrow{\phi_0} E(1) \xrightarrow{E(\phi_0)} E(F(0)) \xrightarrow{E(F(!_0))} E(F(0)) \xrightarrow{\theta_0} F'(H(0))$.
$1_{F'\mathbb{C}\mathbf{sp}}$ is given by $0' \xrightarrow{1} 0' \xleftarrow{1} 0'$ with $1 \xrightarrow{\phi'_0} F'(0') \xrightarrow{F'(!_{0'})} F'(0')$ where $0'$ is the monoidal unit of $(\mathsf{A}', +, 0')$.
Down:



Right, down and then left:

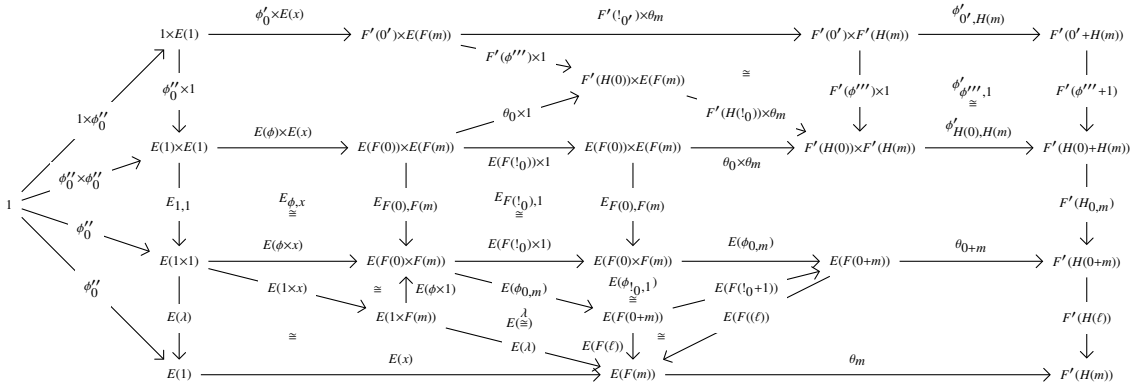


Diagram 3 of 7. Set up but not solved.

$$\begin{array}{ccc} \mathbb{H}(M)\otimes\mathbb{H}(N) & \xrightarrow{\beta} & \mathbb{H}(N)\otimes\mathbb{H}(M) \\ \downarrow{\scriptstyle \mathbb{H}_{M,N}} & & \downarrow{\scriptstyle \mathbb{H}_{N,M}} \\ \mathbb{H}(M\otimes N) & \xrightarrow{\mathbb{H}(\beta)} & \mathbb{H}(N\otimes M) \end{array}$$

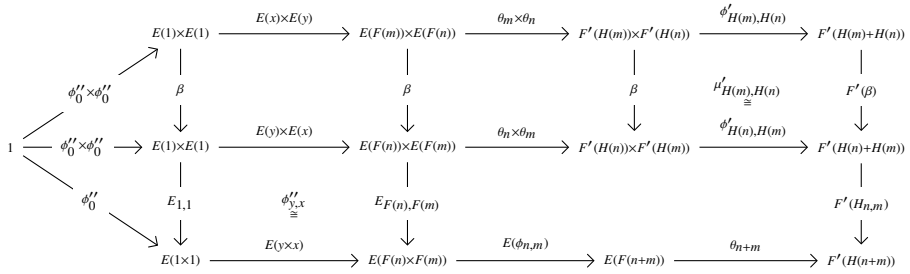
M is given by $a \xrightarrow{i} m \xleftarrow{o} b$ with $x \in F(m)$.
$\mathbb{H}(M)$ is given by $H(a) \xrightarrow{H(i)} H(m) \xleftarrow{H(o)} H(b)$ with $\bar{x} = \theta_m E(x)\phi_0 \in F'(H(m))$.
N is given by $a' \xrightarrow{i'} n \xleftarrow{o'} b'$ with $y \in F(n)$.
$\mathbb{H}(N)$ is given by $H(a') \xrightarrow{H(i')} H(n) \xleftarrow{H(o')} H(b')$ with $\bar{y} = \theta_n E(y)\phi_0 \in F'(H(n))$.
$M \otimes N$ has decoration $1 \xrightarrow{x\times y} F(m)\times F(n) \xrightarrow{\phi_{m,n}} F(m+n)$. $\mathbb{H}(M\otimes N)$ has decoration

$$1 \xrightarrow{\phi_0''} E(1) \xrightarrow{E(x\times y)} E(F(m)\times F(n)) \xrightarrow{E(\phi_{m,n})} E(F(m+n)) \xrightarrow{\theta_{m+n}} F'(H(m+n)).$$

$\mathbb{H}(M)\otimes\mathbb{H}(N)$ has decoration

$$1 \xrightarrow{\phi_0''\times\phi_0''} E(1)\times E(1) \xrightarrow{E(x)\times E(y)} E(F(m))\times E(F(n)) \xrightarrow{\theta_m\times\theta_n} F'(H(m))\times F'(H(n)) \xrightarrow{\phi'_{H(m),H(n)}} F'(H(m)+H(n)).$$

Right and then down:



Down and then right:

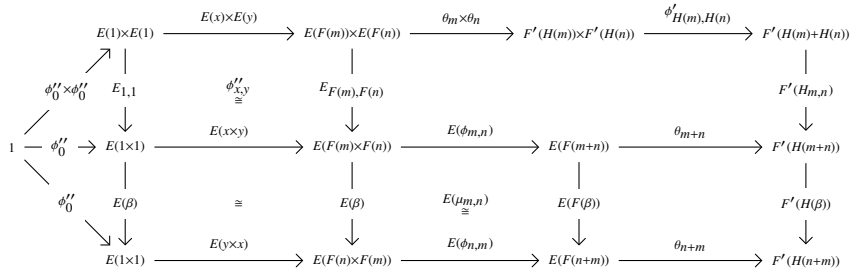
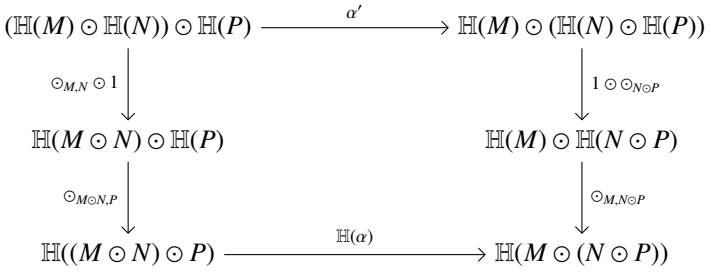
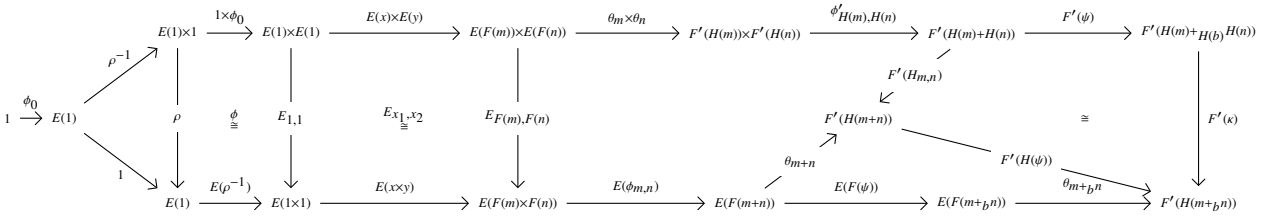


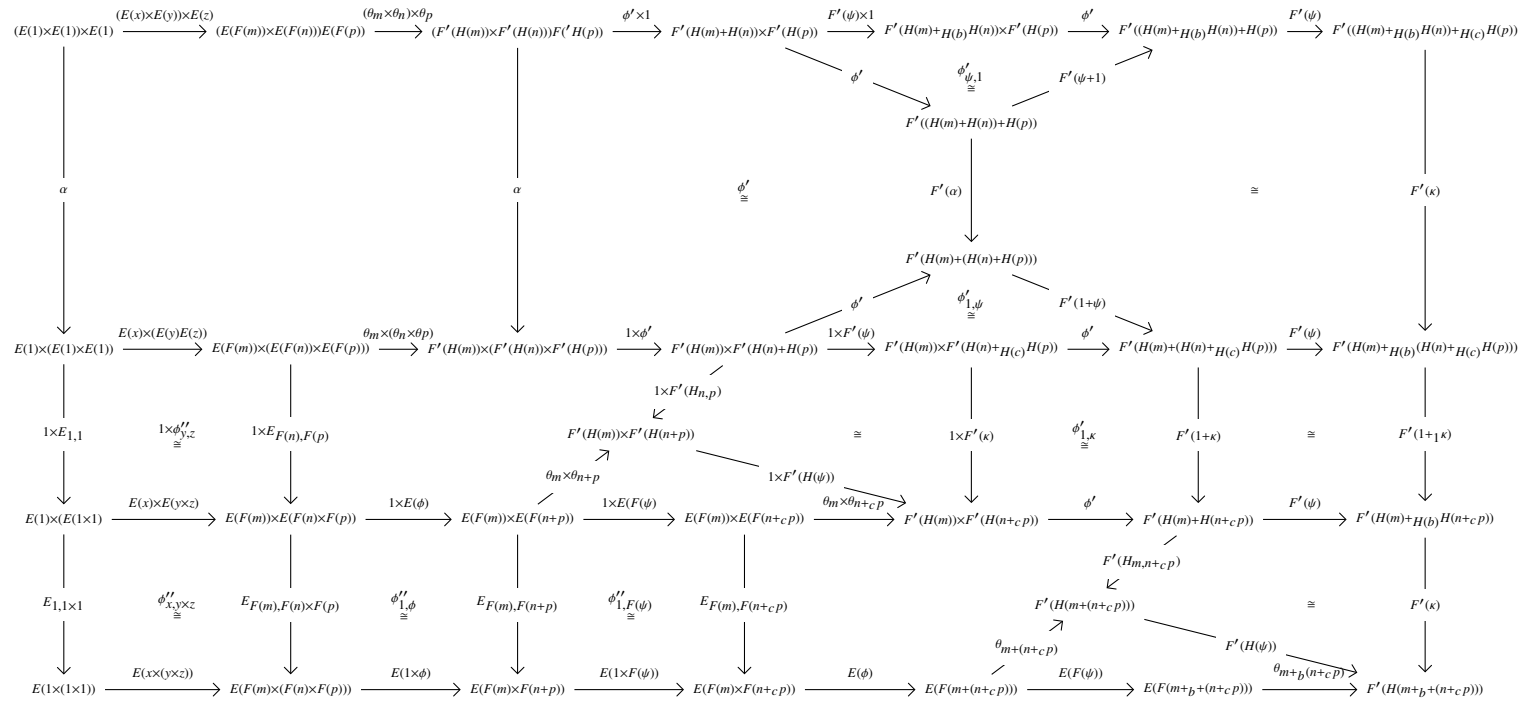
Diagram 4 of 7. Set up but not solved.



M is given by $a \xrightarrow{i} m \xleftarrow{o} b$ with $x \in F(m)$.
N is given by $b \xrightarrow{i'} n \xleftarrow{o'} c$ with $y \in F(n)$.
P is given by $c \xrightarrow{i''} p \xleftarrow{o''} d$ with $z \in F(p)$.
$\mathbb{H}(M)$ is given by $H(a) \xrightarrow{H(i)} H(m) \xleftarrow{H(o)} H(b)$ with $\bar{x} = \theta_m E(x)\phi_0 \in F'(H(m))$.
$\mathbb{H}(N)$ is given by $H(b) \xrightarrow{H(i')} H(n) \xleftarrow{H(o')} H(c)$ with $\bar{y} = \theta_n E(y)\phi_0 \in F'(H(n))$.
$\mathbb{H}(P)$ is given by $H(c) \xrightarrow{H(i'')} H(p) \xleftarrow{H(o'')} H(d)$ with $\bar{z} = \theta_p E(z)\phi_0 \in F'(H(p))$.
The map $\odot_{M,N} \colon \mathbb{H}(M)\odot\mathbb{H}(N) \to \mathbb{H}(M\odot N)$ is given by:



Right and then down:



Down and then right:

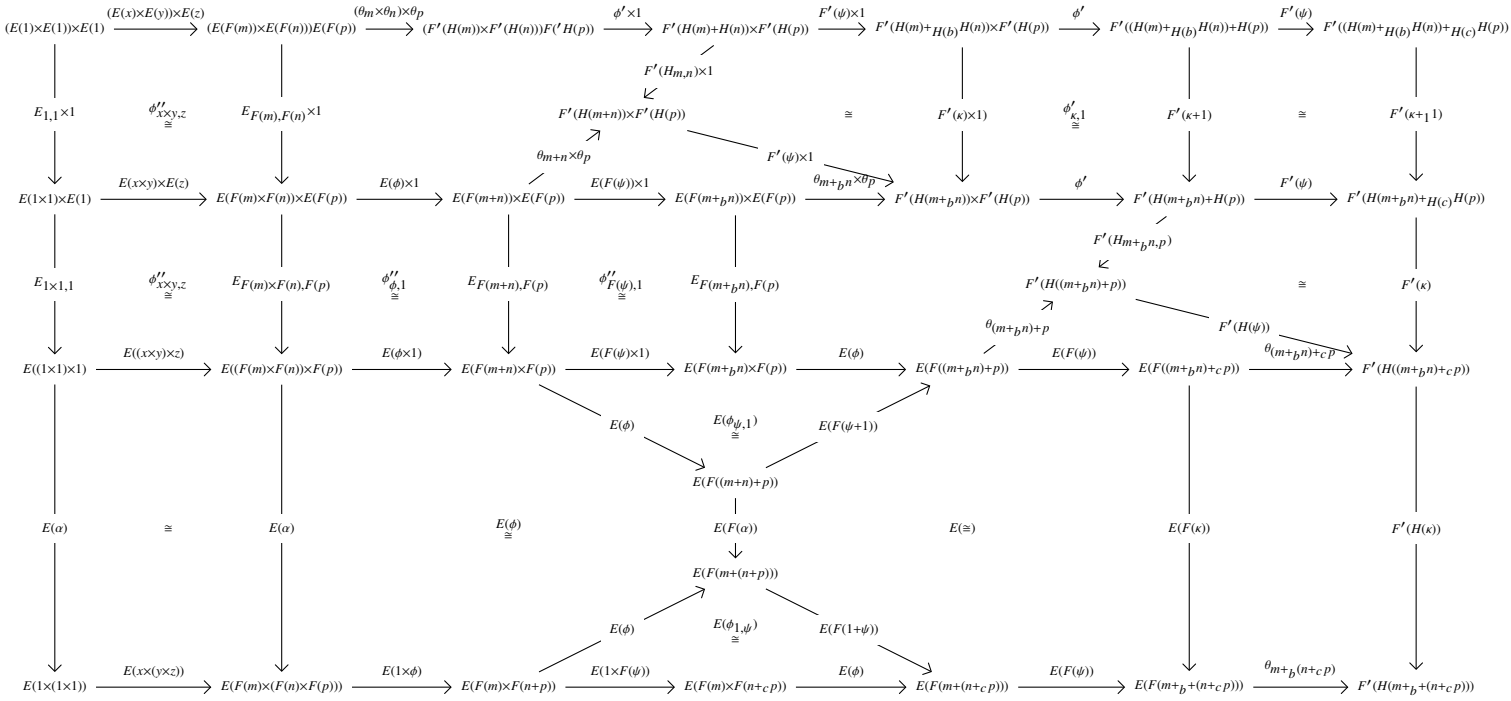
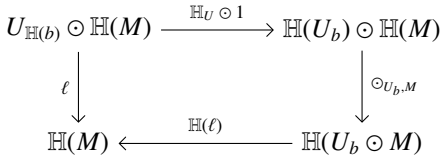
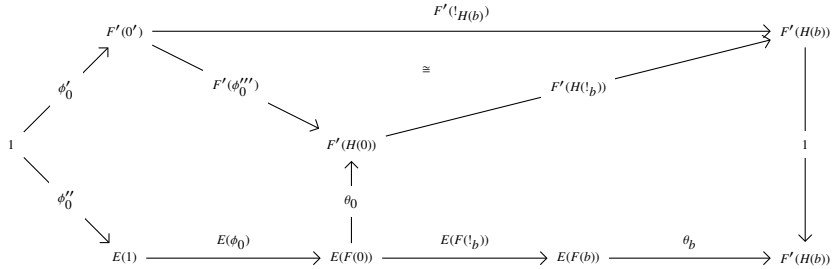


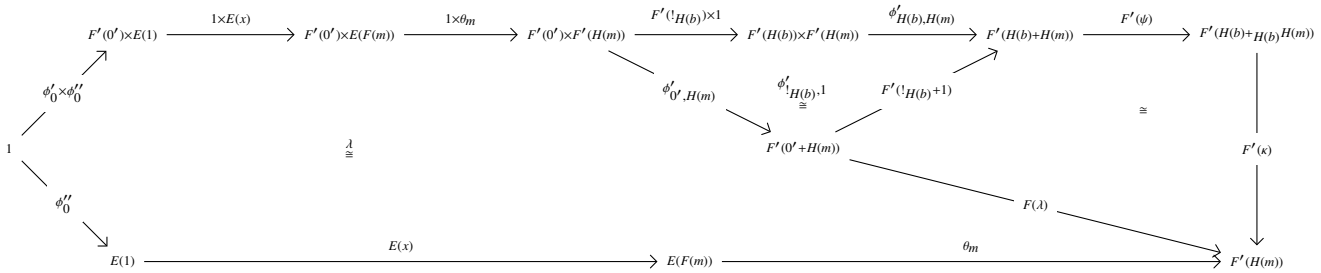
Diagram 5 of 7. Not set up and not solved.



M is given by $a \xrightarrow{i} m \xleftarrow{o} b$ with $x \in F(m)$.
U_b is given by $b \xrightarrow{1} b \xleftarrow{1} b$ with $\bot_b \in F(b)$.
$\mathbb{H}(M)$ is given by $H(a) \xrightarrow{H(i)} H(m) \xleftarrow{H(o)} H(b)$ with $\bar{x} = \theta_m E(x)\phi_0'' \in F'(H(m))$.
$U_{\mathbb{H}(b)}$ is given by $H(b) \xrightarrow{1} H(b) \xleftarrow{1} H(b)$ with $\bot_{\mathbb{H}(b)} \in F'(H(b))$.
$\mathbb{H}(U_b)$ is given by $H(b) \xrightarrow{1} H(b) \xleftarrow{1} H(b)$ with $\bar{\bot}_b = \theta_b E(\bot_b)\phi_0'' \in F'(H(b))$.
The map $\mathbb{H}_U : U_{\mathbb{H}(b)} \to \mathbb{H}(U_b)$ is given by:



The left commuting region is a naturality triangle of units for a (strict) monoidal natural isomorphism and the right commuting region is an ordinary naturality square for a (strict) monoidal natural isomorphism. Down:



Right, down and then left:

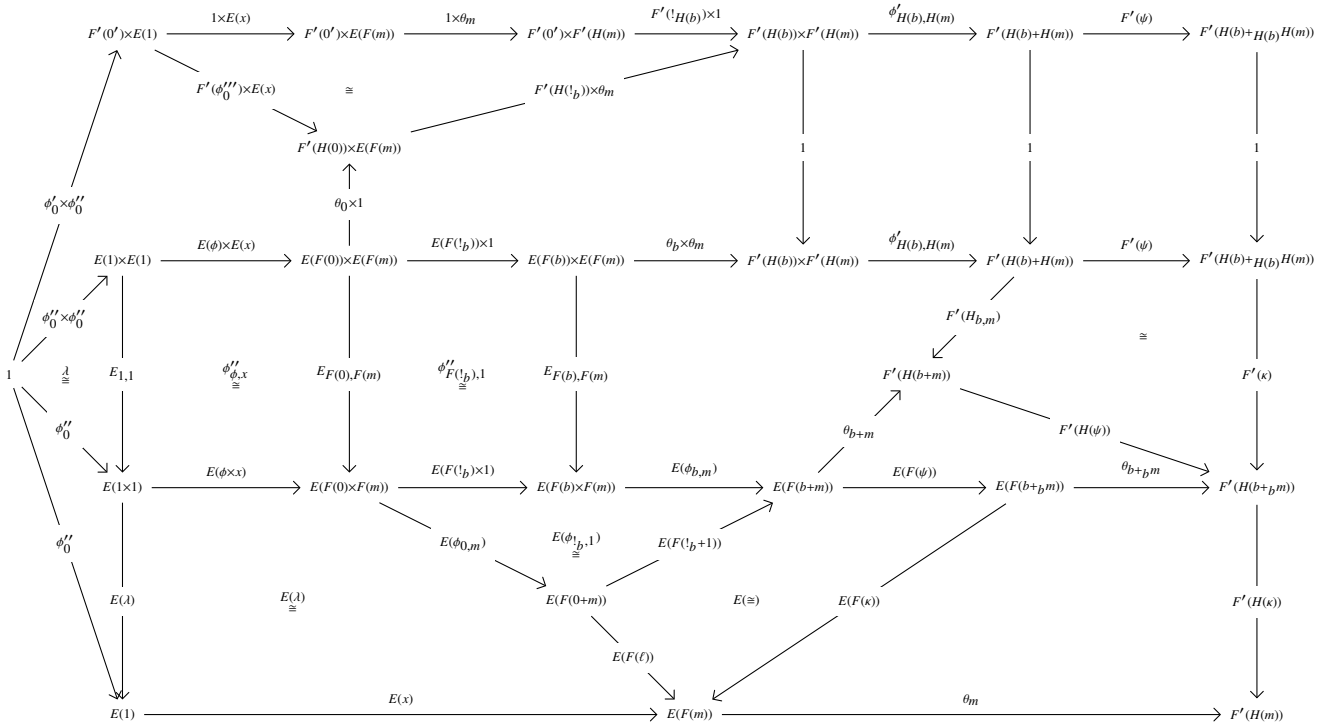
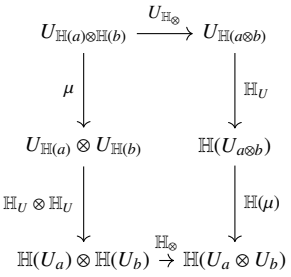
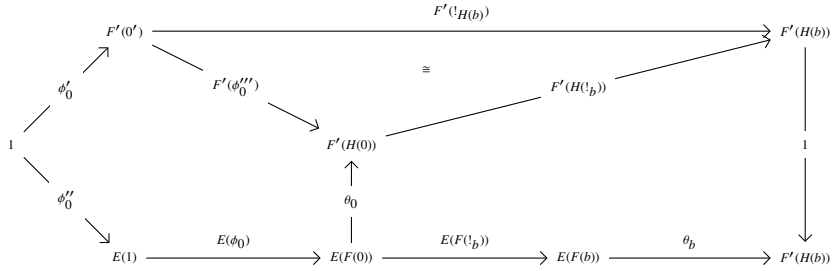


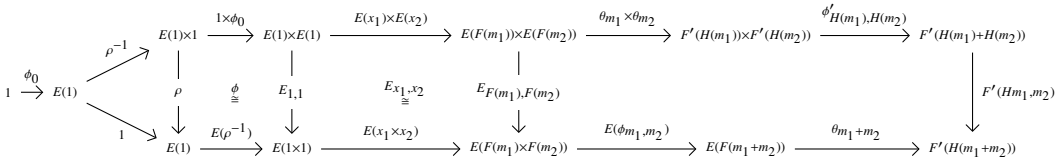
Diagram 6 of 7. Set up but not solved.



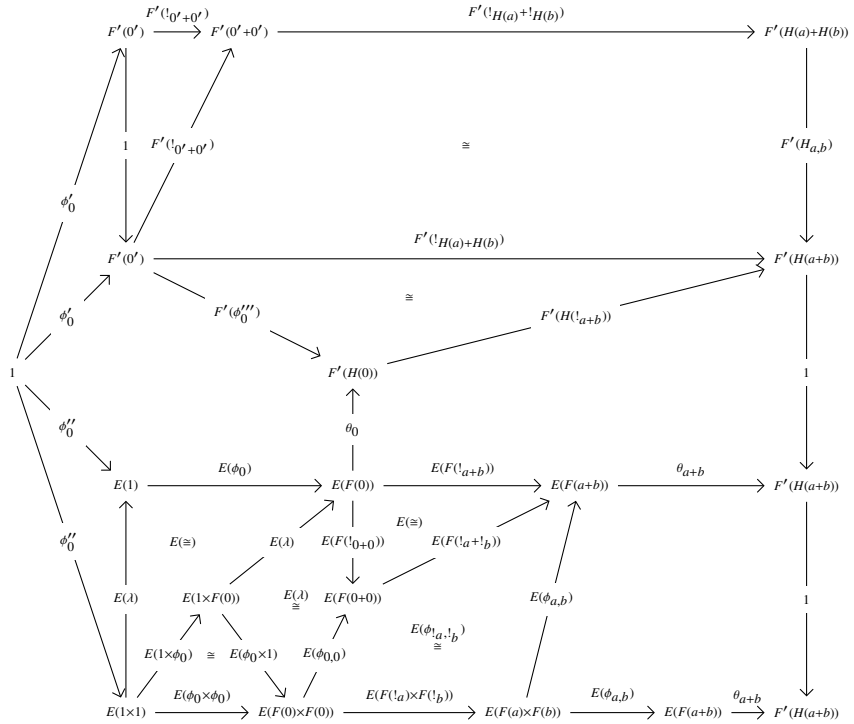
The maps $\mathbb{H}_U : U_{\mathbb{H}} \to \mathbb{H}(U)$ and $\mathbb{H}_\otimes : \mathbb{H}(U) \otimes \mathbb{H}(U) \to \mathbb{H}(U \otimes U)$ are previously defined:
The map $\mathbb{H}_U : U_{\mathbb{H}(b)} \to \mathbb{H}(U_b)$ is given by:



The map $\mathbb{H}_\otimes : \mathbb{H}(M_1) \otimes \mathbb{H}(M_2) \to \mathbb{H}(M_1 \otimes M_2)$ is given by:



Right and then down:



Down and then right:

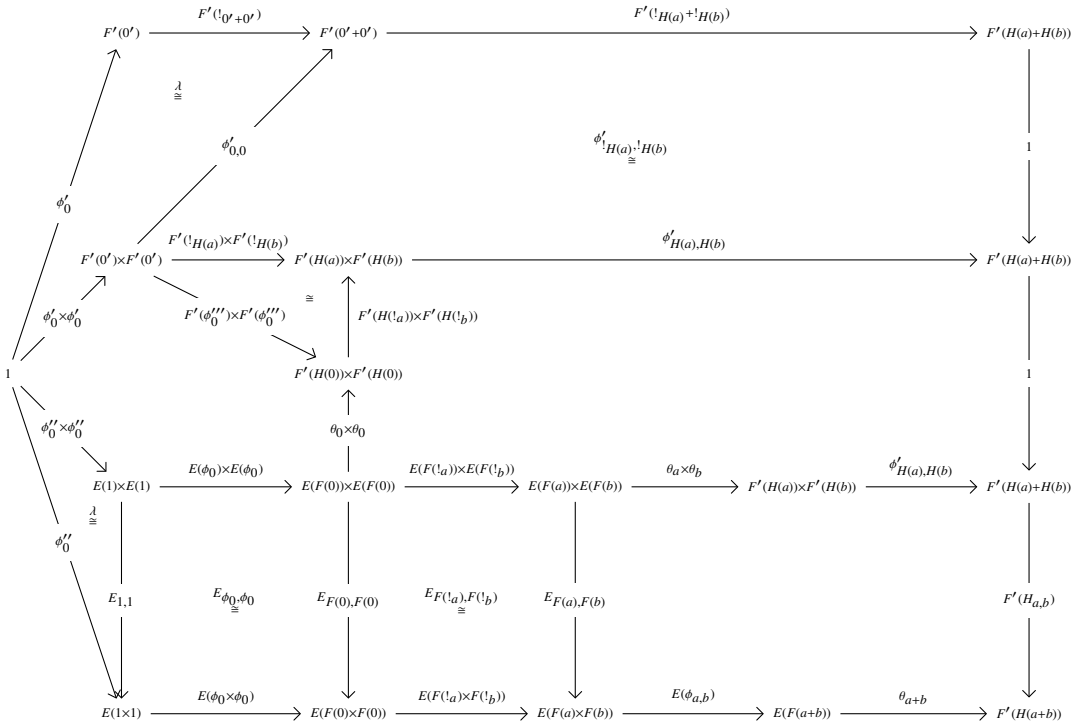
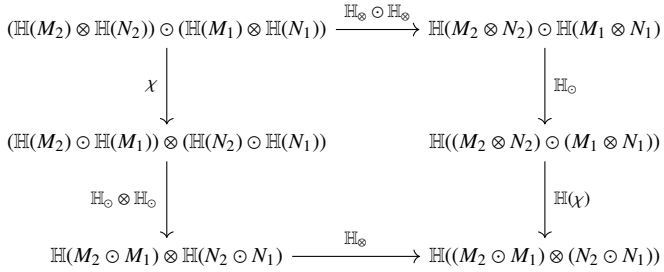
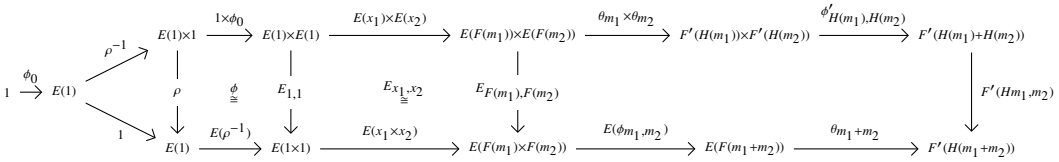


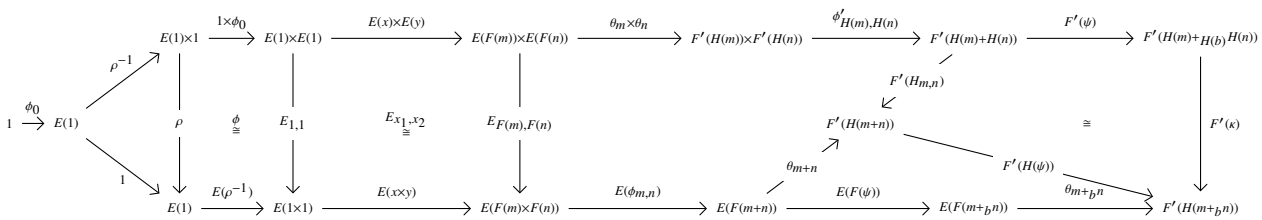
Diagram 7 of 7. Set up but not solved.



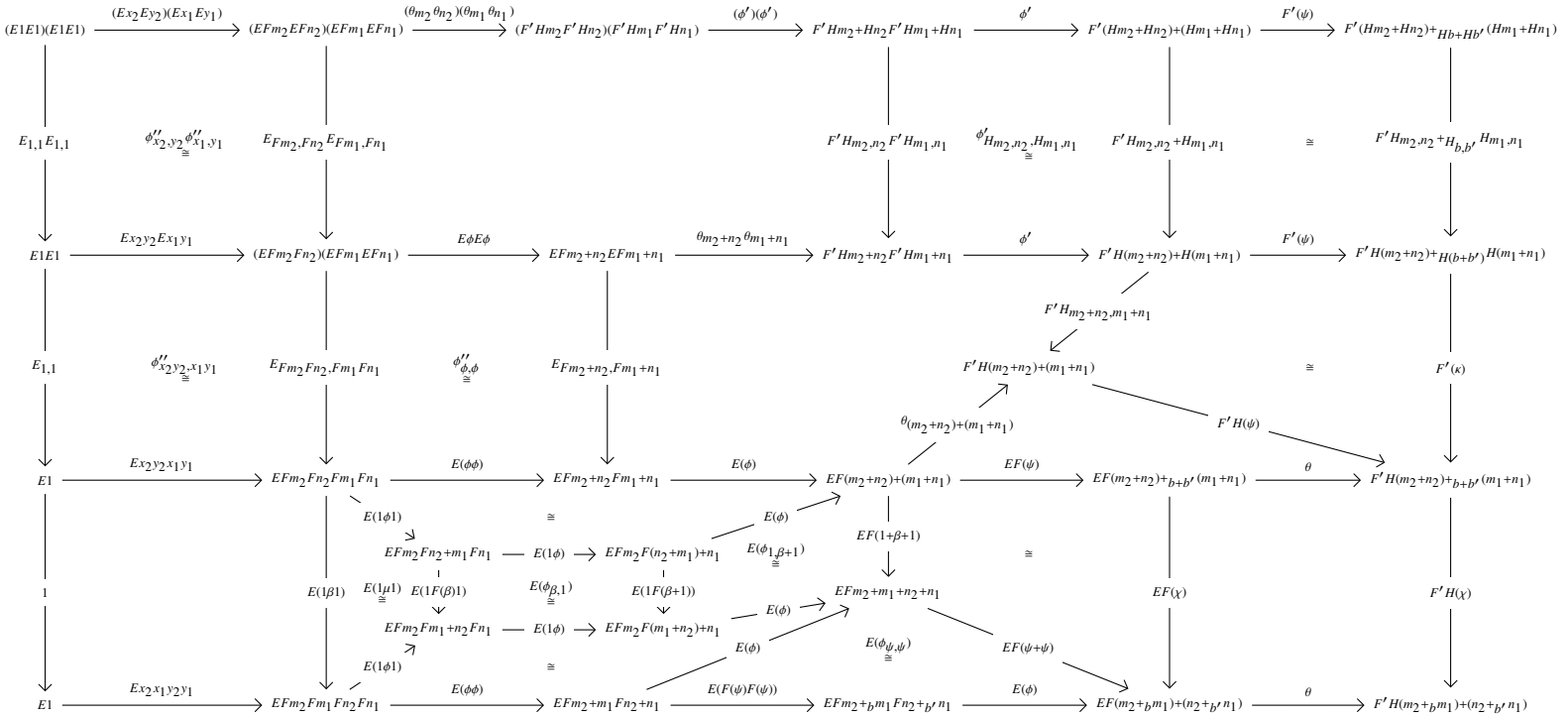
M_1 is given by $a \xrightarrow{i_1} m_1 \xleftarrow{o_1} b$ with $x_1 \in F(m_1)$.
M_2 is given by $b \xrightarrow{i_2} m_2 \xleftarrow{o_2} c$ with $x_2 \in F(m_2)$.
N_1 is given by $a' \xrightarrow{i'_1} n_1 \xleftarrow{o'_1} b'$ with $y_1 \in F(n_1)$.
N_2 is given by $b' \xrightarrow{i'_2} n_2 \xleftarrow{o'_2} c'$ with $y_2 \in F(n_2)$.
$\mathbb{H}(M_1)$ is given by $H(a) \xrightarrow{H(i_1)} H(m_1) \xleftarrow{H(o_1)} H(b_1)$ with $\bar{x_1} = \theta_{m_1}E(x_1)\phi_0 \in F'(H(m_1))$.
$\mathbb{H}(M_2)$ is given by $H(b) \xrightarrow{H(i_2)} H(m_2) \xleftarrow{H(o_2)} H(c)$ with $\bar{x_2} = \theta_{m_2}E(x_2)\phi_0 \in F'(H(m_2))$.
$\mathbb{H}(N_1)$ is given by $H(a') \xrightarrow{H(i'_1)} H(n_1) \xleftarrow{H(o'_1)} H(b')$ with $\bar{y_1} = \theta_{n_1}E(y_1)\phi_0 \in F'(H(n_1))$.
$\mathbb{H}(N_2)$ is given by $H(b') \xrightarrow{H(i'_2)} H(n_2) \xleftarrow{H(o'_2)} H(c')$ with $\bar{y_2} = \theta_{n_2}E(x_2)\phi_0 \in F'(H(n_2))$.
The maps $\mathbb{H}_\otimes$ and $\mathbb{H}_\odot$ are previously defined, and the map χ may be found in the paper.
The map $\mathbb{H}_\otimes\colon \mathbb{H}(M_1)\otimes\mathbb{H}(M_2) \to \mathbb{H}(M_1\otimes M_2)$ is given by:



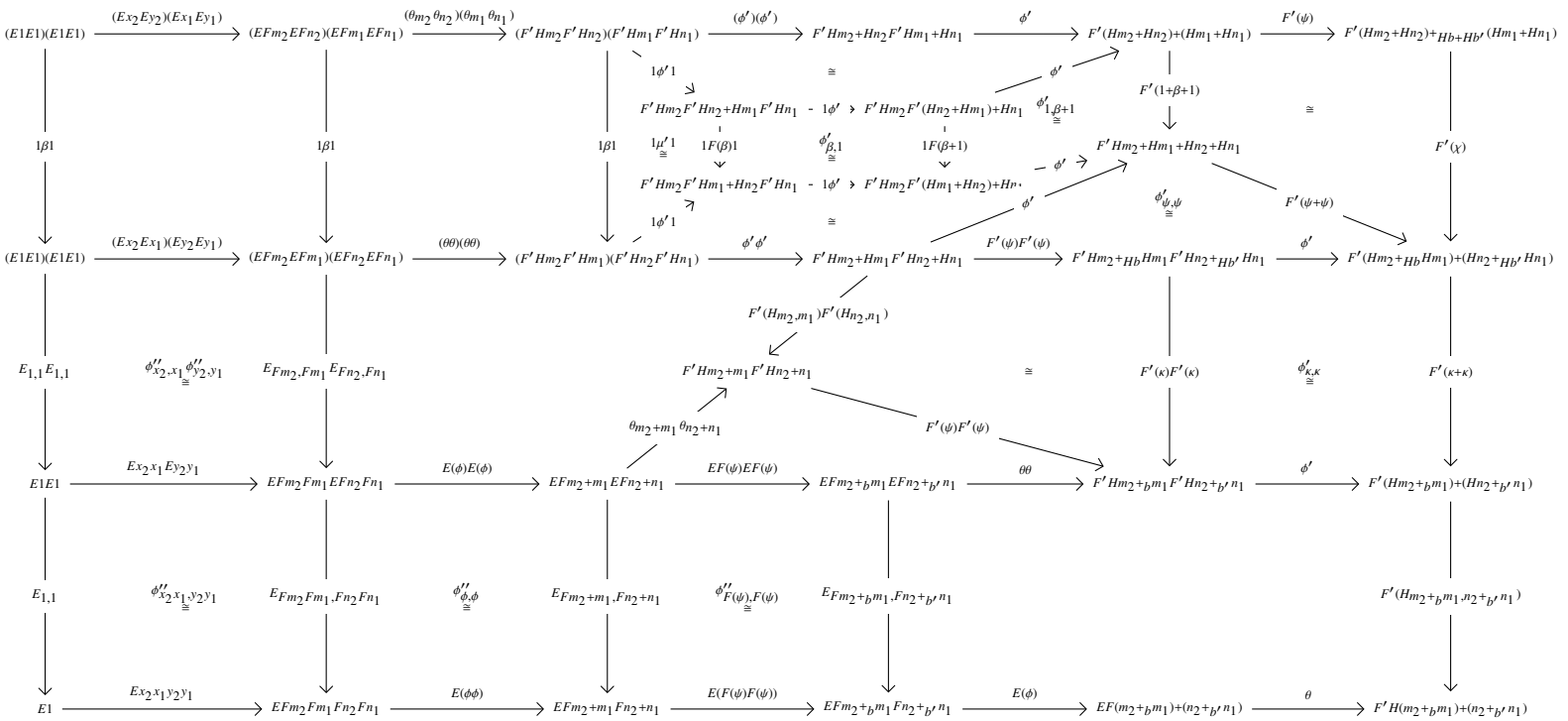
The map $\mathbb{H}_\odot\colon \mathbb{H}(M)\odot\mathbb{H}(N) \to \mathbb{H}(M\odot N)$ is given by:



Right and then down:

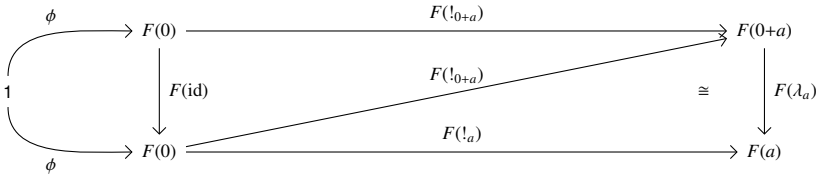


Down and then right:



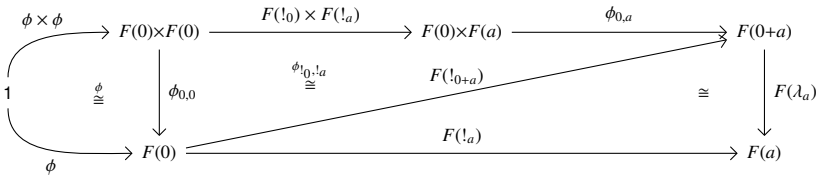
4. Some useful maps

Given $a \in (\mathsf{A}, +, 0)$, the map $U_{\lambda_a} \colon U_{0+a} \to U_a$ is given by:



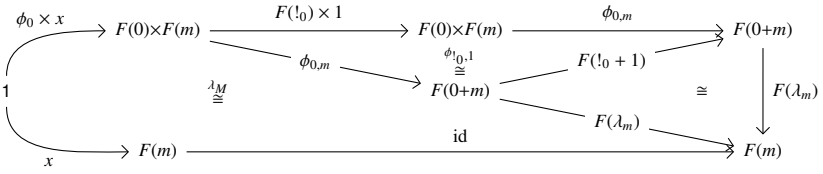
where the $\cong$ is given by pseudonaturality of F: we have a unique map in $!_a \colon 0 \to a$ in A but also a map $\lambda_a \circ !_{0+a} \colon 0 \to a$ where λ_a is the left unitor of $(\mathsf{A}, +, 0)$, and so $F(!_a) = F(\lambda_a \circ !_{0+a}) \cong F(\lambda_a)F(!_{0+a})$.

The left unitor $\lambda'_{U_a} \colon U_0 \otimes U_a \to U_a$ is given by:

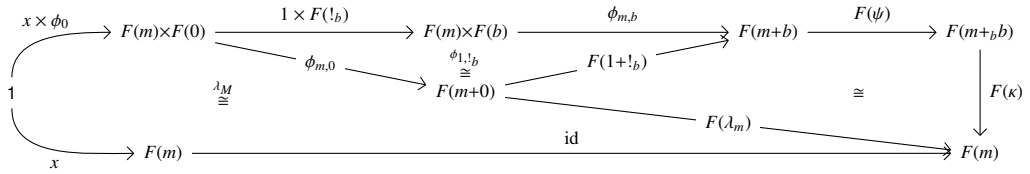
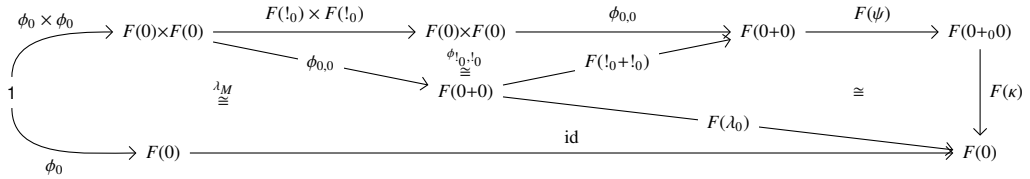


where the $\cong$ in the lower right is the same as the one in the first diagram.

For an arbitrary M, the left unitor $\lambda'_M \colon U_0 \otimes M \to M$ is given by:



For an arbitrary M given by $a \to (m, x) \leftarrow b$, the map $\lambda_M \colon U_b \odot M \to M$ is given by:



In particular, if $M = U_0$ above, then the map $\lambda_{U_0} \colon U_0 \odot U_0 \to U_0$ is given by: