Proof of axiom of Appendix B

14/12/2020

Need to show that the two large diagrams at current page 30 contain the same 2-isomorphisms.
First of all, we need to include diagram (37) inside both of them, by expanding diagram (15). call it θ
left is equal, right is up to coherent iso
Also removed leftmost + rightmost parts of huge diagrams.

$$
\begin{tikzcd}
Fm_1Fn_1Fm_2Fn_2 \arrow[r, "\Phi\Phi"] \arrow[d, "1b1"'] \arrow[dr, phantom, "\overset{\theta}{\cong}\ (37)"] & F(m_1+n_1)F(m_2+n_2) \arrow[rr, "\Phi"] & & F(m_1+n_1+m_2+n_2) \arrow[r, "F\psi"] \arrow[d, "F(1+b+1)"] \arrow[dr, phantom, "\cong \text{ F applied to canonical map}"] & F((m_1+n_1)+_{b+b'}(m_2+n_2)) \arrow[d, "F\chi"] \\
Fm_1Fm_2Fn_1Fn_2 \arrow[r, "\Phi\Phi"'] \arrow[dd, "b"'] \arrow[ddr, phantom, "?"] & F(m_1+m_2)F(n_1+n_2) \arrow[rr, "\Phi"] \arrow[dr, "F\psi F\psi"'] \arrow[dd, "b"] \arrow[ddr, phantom, "?"] & & F(m_1+m_2+n_1+n_2) \arrow[r, "F(\psi+\psi)"] \arrow[d, phantom, "\overset{\Phi_{\psi,\psi}}{\cong}\ (30)"] & F((m_1+_b m_2)+(n_1+_{b'} n_2)) \arrow[dd, "Fb"] \\
& & F(m_1+_b m_2)F(n_1+_b n_2) \arrow[urr, "\phi"] \arrow[d, "b"] \arrow[rr, phantom, "\overset{u}{\cong}\ (31)"] & & \\
Fn_1Fn_2Fm_1Fm_2 \arrow[r, "\Phi\Phi"'] & F(n_1+n_2)F(m_1+m_2) \arrow[r, "F\psi F\psi"'] & F(n_1+_b n_2)F(m_1+_b m_2) \arrow[rr, "\phi"'] & & F((n_1+_b n_2)+(m_1+_b m_2))
\end{tikzcd}
$$

$$\| \ ?$$

$$
\begin{tikzcd}
Fm_1Fn_1Fm_2Fn_2 \arrow[r, "\Phi\Phi"] \arrow[d, "bb"'] \arrow[dr, phantom, "\overset{u\times u}{\cong}\ (31)"] & F(m_1+n_1)F(m_2+n_2) \arrow[r, "\Phi"] \arrow[d, "FbFb"] \arrow[dr, phantom, "\overset{\Phi_{b,b}}{\cong}\ (30)"] & F(m_1+n_1+m_2+n_2) \arrow[r, "F\psi"] \arrow[d, "F(b+b)"] \arrow[dr, phantom, "\cong \text{ canonical}"] & F((m_1+n_1)+_{b+b'}(m_2+n_2)) \arrow[d, "F(b+_b b)"] \\
Fn_1Fm_1Fn_2Fm_2 \arrow[r, "\Phi\Phi"'] \arrow[dd, "1b1"'] \arrow[ddr, phantom, "\overset{\theta}{\cong}\ (37)"] & F(n_1+m_1)F(n_2+m_2) \arrow[r, "\Phi"'] & F(n_1+m_1+n_2+m_2) \arrow[r, "F\psi"'] \arrow[d, "F(1+b+1)"] \arrow[dr, phantom, "\cong \text{ canonical}"] & F((n_1+m_1)+_{b'+b}(n_2+m_2)) \arrow[dd, "F\hat{\chi}"] \\
& & F(n_1+n_2+m_1+m_2) \arrow[dr, "F(\psi+\psi)"] \arrow[d, phantom, "\overset{\Phi_{\psi,\psi}}{\cong}"] & \\
Fn_1Fn_2Fm_1Fm_2 \arrow[r, "\Phi\Phi"'] & F(n_1+n_2)F(m_1+m_2) \arrow[ur, "\Phi"] \arrow[r, "F\psi F\psi"'] & F(n_1+_b n_2)F(m_1+_b m_2) \arrow[r, "\phi"'] & F((n_1+_b n_2)+(m_1+_b m_2))
\end{tikzcd}
$$

We see that the yellow parts have a lot of canonical maps from colimit properties: we basically want to "get rid" of the ψ's if possible, since they will not be deduced by sylleptic lax monoidal pseudofunctor axioms!!

So, using the modification axiom for u at the top diagram, and throwing away resulting parts that are the same (.....WORK....) we come down to proving
NON-TRIVIAL

$$
\text{LHS}\qquad
\begin{tikzcd}
\bullet \arrow[r, "\phi\phi"] \arrow[d, "1b1"'] & \bullet \arrow[r, "\phi"] \arrow[d, phantom, "\theta"] & \bullet \arrow[d, "F(1b1)"] \\
\bullet \arrow[r, "\phi\phi"'] \arrow[d, "b"'] \arrow[dr, phantom, "\partial"] & \bullet \arrow[r, "\phi"'] \arrow[d, "b"] \arrow[dr, phantom, "u"] & \bullet \arrow[d, "Fb"] \\
\bullet \arrow[r, "\phi\phi"'] & \bullet \arrow[r, "\phi"'] & \bullet
\end{tikzcd}
\quad \overset{F(1b1)\circ F(bb)}{\cong} \quad = \quad
\text{RHS}\qquad
\begin{tikzcd}
\bullet \arrow[r, "\phi\phi"] \arrow[d, "bb"'] \arrow[dr, phantom, "uu"] & \bullet \arrow[r, "\phi"] \arrow[d, "FbFb"] \arrow[dr, phantom, "\phi_{bb}"] & \bullet \arrow[d, "F(bb)"] \\
\bullet \arrow[r] \arrow[d, "1b1"'] & \bullet \arrow[r, "\phi"'] \arrow[d, phantom, "\theta"] & \bullet \arrow[d, "F(1b1)"] \\
\bullet \arrow[r, "\phi\phi"'] & \bullet \arrow[r, "\phi"'] & \bullet
\end{tikzcd}
$$

which was the great reduction step. I here digressed for a ~~day~~ or two, because I thought I could get this from properties of θ as a package (without breaking it down as in (37)) following some related conditions in Day-Street, but FAILED. :-(

The rest of the proof AFTER EXPANDING θ (37) uses essentially all axioms around. But before jumping into all of it, I did a sanity check which convinced me it will work! This involves checking components of the 2-cells involved, and seeing whether in principle, the LHS may give RHS using braiding axioms.

$\boxed{\text{LHS}}$ Bottom-right u is really $u_{m_1+m_2,n_1+n_2}$. Top left θ involves the component $1\times u_{n_1,m_1}\times 1$. All the rest is pseudoassociativity components, call them ω (29) as well as arrow components of ϕ (31)

$\boxed{\text{RHS}}$ Top left uu is really $u_{m_1,n_1}\times u_{m_2,n_2}$. Bottom θ involves the component $1\times u_{m_1,n_2}\times 1$. All the rest is again ω's and ϕ_{fg}'s

$\boxed{\text{KEY REMARK}}$ [Day-Street, Def.14] has the two axioms for a braided lax mon pseudomon. If you look closely, they express
1. $u_{a,b+c}$ in terms of $u_{a,b}$ and $u_{a,c}$
2. $u_{a+b,c}$ in terms of $u_{a,c}$ and $u_{b,c}$

So in principle, going from LHS to RHS using those two axioms, I would perform

$$\boxed{\text{LHS}} \quad u_{m_1+m_2,\,\overset{F\boxtimes}{n_1+n_2}},\ u_{n_1,m_1} \xrightarrow{①} u_{m_1,n_1+n_2},\ u_{m_2,n_1+n_2},\ u_{n_1,m_2}$$

$$\xrightarrow{② \times 2} u_{m_1,n_1},\ u_{m_1,n_2},\ u_{m_2,n_1},\ u_{m_2,n_2},\ u_{n_1,m_2}$$

which are 5 components of u, whereas I only have 3 in RHS. HOWEVER F is furthermore sylleptic. The relevant axiom allows me to essentially cancel u_{n_1,m_2} and u_{m_2,n_1} when appropriately composed.

∴ from geometry of diagrams, this will be possible! So indeed LHS reduces to $u_{m_1,n_1}, u_{m_1,n_2}, u_{m_2,n_2}$ LIKE RHS

So this sanity check gives very good chances of success!! The rest is (painful....) computations.