AN EQUIVALENCE OF COMPOSITIONAL FRAMEWORKS

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Abstract. The first two authors have developed a compositional framework well-suited for studying networks that are built out of finite sets equipped with extra stuff. This framework, which goes by the name of 'structured cospans', utilizes double categories where the objects are seen as inputs and outputs, morphisms are 'open networks', and 2-morphisms are maps between open networks. In this setup a functor $L\colon \mathbf{C} \to \mathbf{D}$, which is typically a left adjoint, is used to replace the objects and vertical 1-morphisms of a given double category $\mathbb{D}$ with the objects and morphisms, respectively, of the category $\mathbf{C}$. Horizontal 1-cells are then cospans in $\mathbf{D}$ of a particular form with 2-morphisms given by maps of these cospans. Fong has also developed a similar framework which also uses cospans and goes by the name of 'decorated cospans'. In this setup, a lax monoidal functor $F\colon \mathbf{C} \to \mathbf{E}$ is used to 'decorate' the apices of cospans in $\mathbf{C}$ with elements of $\mathbf{E}$ giving the cospans extra structure. Using a slight variation of Fong's framework, we prove that these two frameworks are equivalent in the situation where a left adjoint can be obtained from a lax monoidal pseudofunctor using a well known construction of Grothendieck.

1. Introduction

HELLO ALL!

Networks are playing an increasingly prominent role in our understanding of the world and as a result, methods of characterizing and studying networks have become evermore necessary. Applied category theory provides such an avenue and much work has been done towards this effort [1, 3, 4, 5, 6, 9, 12, 13, 14, 17]. One of the more recent frameworks developed suitable for studying networks is Fong's 'decorated cospans' [12]. A **cospan** in any category is a diagram of the form:

$$\begin{tikzcd} & c & \\ a \arrow[ur, "i"] & & b \arrow[ul, "o"'] \end{tikzcd}$$

We call c the **apex** of the cospan and a and b the **feet** of the cospan. The morphisms i and o are called the **legs**. Cospans are ideal for realizing networks and more generally 'open' systems where here open means each system or network comes with prescribed inputs and outputs which are represented by the feet of the cospan a and b, respectively. Two open systems viewed as cospans such that the outputs of the first and the inputs of the second coincide can then naturally be composed via pushout. For example, we can compose the above cospan with the following:

$$\begin{tikzcd} & f & \\ b \arrow[ur, "i'"] & & e \arrow[ul, "o'"'] \end{tikzcd}$$

to obtain a new cospan whose left foot is the left foot of the first, whose right foot is the right foot of the second, and whose apex and legs are given by the pushout.

$$\begin{tikzcd} & & c +_b f & & \\ & & c + f \arrow[u, "\psi"] & & \\ & c \arrow[uur, "\psi j_c"] \arrow[ur, "j_c"'] & & f \arrow[uul, "\psi j_f"'] \arrow[ul, "j_f"] & \\ a \arrow[ur, "i"] & & b \arrow[ul, "o"'] \arrow[ur, "i'"] & & e \arrow[ul, "o'"'] \end{tikzcd}$$

In Fong's theory of decorated cospans, given a category $\mathbf{C}$ with finite colmits, a symmetric lax monoidal functor $F\colon (\mathbf{C}, +, 0) \to (\mathbf{D}, \otimes, I)$ is used to 'decorate' the apex of a cospan in $\mathbf{C}$ with an element of the image of its apex under the functor F, in the case of the first cospan above, a morphism $d\colon I \to F(c)$. Here $F(c)$ is the collection of all F-decorations on the object c and the morphism d is selecting a particular one. Thus a decorated cospan is then a pair:

$$\begin{tikzcd} & c & \\ a \arrow[ur, "i"] & & b \arrow[ul, "o"'] \end{tikzcd} \qquad I \xrightarrow{d} F(c)$$

To obtain a decoration on the composition of two composable F-decorated cospans:

$$\begin{tikzcd} & c & \\ a \arrow[ur, "i"] & & b \arrow[ul, "o"'] \end{tikzcd} \qquad \begin{tikzcd} & f & \\ b \arrow[ur, "i'"] & & e \arrow[ul, "o'"'] \end{tikzcd}$$

$$I \xrightarrow{d_1} F(c) \qquad\qquad I \xrightarrow{d_2} F(f)$$

we use the natural map from a coproduct to a pushout as well as the natural map that comes as part of the structure of a lax monoidal functor:

$$1 \xrightarrow{\lambda^{-1}} 1 \times 1 \xrightarrow{d_1 \times d_2} F(c) \times F(f) \xrightarrow{\phi_{c,f}} F(c+f) \xrightarrow{F(j_{c,f})} F(c +_b f).$$

From this symmetric lax monoidal functor $F\colon \mathbf{C} \to \mathbf{D}$, Fong then constructs a symmetric monoidal category $F\mathrm{Cospan}(\mathbf{C})$ which has:

(i) objects as those of $\mathbf{C}$ and
(ii) morphisms as isomorphism classes of F-decorated cospans, where an F-decorated cospan is given as above, and two F-decorated cospans are in the same isomorphism class if there exists an isomorphism between the apices such that the following diagrams commute:

$$\begin{tikzcd} & c \arrow[dd, "f"] & \\ a \arrow[ur, "i"] \arrow[dr, "i'"'] & & b \arrow[ul, "o"'] \arrow[dl, "o'"] \\ & c' & \end{tikzcd} \qquad \begin{tikzcd} & F(c) \arrow[dd, "F(f)"] \\ I \arrow[ur, "d"] \arrow[dr, "d'"'] & \\ & F(c') \end{tikzcd}$$

There are some subtleties to this approach. To illustrate, consider the example where $F\colon \mathbf{FinSet} \to \mathbf{Set}$ is the symmetric lax monoidal functor that assigns to a finite set N the (large) set of all possible graph structures on the finite set N, where a graph structure on N is given by a diagram in $\mathbf{Set}$ of the form:

$$\begin{tikzcd} E \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & N. \end{tikzcd}$$

Let $N = \{v_1, v_2\}$ be a two element set. Then one element of the set $F(N)$, in other words, one possible decoration, which is a graph structure, on the set N is given by a single edge e whose source and target are v_1 and v_2, respectively.

$$v_1 \xrightarrow{e} v_2$$

Denote this element of $F(N)$ as d. We could label the edge e something else, for instance, e', and this single edge would another element of $F(N)$.

$$v_1 \xrightarrow{e'} v_2$$

Denote this second element of $F(N)$ as d'. These two graphs constitute distinct isomorphism classes in the symmetric monoidal category $F\mathrm{Cospan}(\mathbf{FinSet})$, as $f = \mathrm{id}_N$ and thus $F(f) = \mathrm{id}_{F(N)}$, but clearly $d \neq d'$. The issue here is that the functor F is decorating each finite set N not only with extra structure, but with extra *stuff*. Moreover, the finite set N is being decorated with *different* stuff in each instance and this difference in stuff is not entirely detected by the function $F(f)$. This phenomenon doesn't occur when the decorations only involve extra structure, and this has been utilized in other works [5, 6, 17].

One approach to remedying this nuisance is to instead view F as a functor $F\colon \mathbf{C} \to \mathbf{Cat}$ and then consider $F(c)$ not as a *set* of decorations on the element c but as a *category* of decorations on the element c. Then one can construct a similar symmetric monoidal category similar to Fong's, which by similarity and an abuse of notation we also call $F\mathrm{Cospan}(\mathbf{C})$. This symmetric monoidal category will have:

(i) objects as those of $\mathbf{C}$ and
(ii) morphisms as isomorphism classes of cospans of $\mathbf{C}$ together with an element of image of the apex under the functor F.

$$\begin{tikzcd} & c & \\ a \arrow[ur, "i"] & & b \arrow[ul, "o"'] \end{tikzcd} \qquad d \in F(c)$$

Except now, given another morphism:

$$\begin{tikzcd} a \arrow[dr, "i'"'] & & b \arrow[dl, "o'"] \\ & c' & \end{tikzcd} \qquad d' \in F(c')$$

these two morphisms are in the same isomorphism class if the following diagrams commute

$$\begin{tikzcd} & c \arrow[dd, "f"'] & \\ a \arrow[ur, "i"] \arrow[dr, "i'"'] & & b \arrow[ul, "o"'] \arrow[dl, "o'"] \\ & c' & \end{tikzcd} \qquad \begin{tikzcd} & F(c) \arrow[dd, "F(f)"] \\ \mathbf{1} \arrow[ur, "d"] \arrow[dr, "d'"'] & \\ & F(c') \end{tikzcd}$$

and there exists an isomorphism $\iota\colon F(f)(d) \to d'$ in the category $F(c')$.

Note that the existence of an isomorphism $\iota\colon F(f)(d) \to d'$ is equivalent to the triangle above on the right commuting up to a 2-isomorphism.

$$\begin{tikzcd}
 & F(c) \arrow[dd, "F(f)"] \\
\mathbf{1} \arrow[ur, "d"] \arrow[dr, "d'"'] \arrow[r, phantom, "\iota \Swarrow"] & \\
 & F(c')
\end{tikzcd}$$

In other words, the isomorphism ι is a *natural* isomorphism. We will typically just write this as $\iota\colon F(f)(d) \to d'$ to conserve space.

In this new category, the added structure of an isomorphism in the category of decorations shapes the isomorphism classes to look as one would expect. Fong's symmetric monoidal category has been extended to a bicategory by the first author [9] and this was one of the original illuminations of the above obstacle. In this bicategory, objects and morphisms are given as in Fong's category modulo taking cospans up to isomorphism class and 2-morphisms are given by a pair of commuting diagrams:

$$\begin{tikzcd}
 & c \arrow[dd, "f"] & \\
a \arrow[ur, "i"] \arrow[dr, "i'"'] & & b \arrow[ul, "o"'] \arrow[dl, "o'"] \\
 & c' &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & F(c) \arrow[d, "F(f)"] \\
\mathbf{1} \arrow[ur, "d"] \arrow[r, "d'"'] & F(c')
\end{tikzcd}$$

In this case, there would be no 2-morphism from:

$$v_1 \xrightarrow{e} v_2$$

to:

$$v_1 \xrightarrow{e'} v_2.$$

Morally speaking, these two simple single-edged graphs should belong to the same isomorphism class, as there is a natural isomorphism of graphs between them.

In the present work, we construct a symmetric monoidal 'double category' $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$. Double categories were first introduced by Ehresmann [10, 11] and symmetric monoidal double categories by Shulman [15]. They have long been used in topology and other branches of pure mathematics [7, 8]. More recently they have been used to study open dynamical systems [14] and open Markov processes [1]. While a mere category has only objects and morphisms, a double category has a few more types of entities:

$$\begin{tikzcd}
A \arrow[r, "M"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow a"] & B \arrow[d, "g"] \\
C \arrow[r, "N"'] & D
\end{tikzcd}$$

We call A, B, C and D 'objects', f and g 'vertical 1-morphisms', M and N 'horizontal 1-cells', and a a '2-morphism'. We can compose vertical 1-morphisms to get new vertical 1-morphisms and compose horizontal 1-cells to get new horizontal 1-cells. We can compose the 2-morphisms in two ways: horizontally by setting squares side by side, and vertically by setting one on top of the other. In a 'strict' double category all these forms of composition are associative. In a 'pseudo' double category, horizontal 1-cells compose in a weakly associative manner: that is, the associative law holds only up to an invertible 2-morphism, called the 'associator', which obeys a coherence law. The double category $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$ that we construct in this paper has:

(i) objects as those of $\mathbf{C}$,
(ii) vertical 1-morphisms as morphisms of $\mathbf{C}$,
(iii) horizontal 1-cells as F-decorated cospans of $\mathbf{C}$, which are pairs:

$$\begin{tikzcd} & c & \\ a \arrow[ur, "i"] & & b \arrow[ul, "o"'] \end{tikzcd} \qquad d \in F(c)$$

and

(iv) 2-morphisms as pairs of commuting diagrams:

$$\begin{tikzcd} a \arrow[r, "i"] \arrow[d, "f"'] & c \arrow[d, "h"] & b \arrow[l, "i"'] \arrow[d, "g"] \\ a' \arrow[r, "i'"] & c' & b' \arrow[l, "o'"'] \end{tikzcd} \qquad \begin{tikzcd} & F(c) \arrow[dd, "F(h)"'] \\ 1 \arrow[ur, "d"] \arrow[dr, "d'"'] & \\ & F(c') \end{tikzcd}$$

together with a morphism $\iota\colon F(h)(d) \to d'$ in $F(c')$.

We can then obtain a symmetric monoidal bicategory $H(\mathbb{F}\mathrm{Cospan}(\mathbf{C}))$ as the 'horizontal bicategory' of the symmetric monoidal double category $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$ using a result of Shulman [15]. This bicategory will have:

(i) objects as those of $\mathbf{C}$,
(ii) morphisms as pairs:

$$\begin{tikzcd} & c & \\ a \arrow[ur, "i"] & & b \arrow[ul, "o"'] \end{tikzcd} \qquad d \in F(c)$$

and

(iii) 2-morphisms as pairs of commuting diagrams:

$$\begin{tikzcd} & c \arrow[dd, "f"] & \\ a \arrow[ur, "i"] \arrow[dr, "i'"'] & & b \arrow[ul, "o"'] \arrow[dl, "o'"] \\ & c' & \end{tikzcd} \qquad \begin{tikzcd} & F(c) \arrow[dd, "F(f)"] \\ \mathbf{1} \arrow[ur, "d"] \arrow[dr, "d'"'] & \\ & F(c') \end{tikzcd}$$

together with a morphism $\iota\colon F(f)(d) \to d'$ of $F(c')$.

Then we can obtain the above improved symmetric monoidal category as a decategorification of the symmetric monoidal bicategory $H(\mathbb{F}\mathrm{Cospan}(\mathbf{C}))$.

The outline for the paper is as follows: In the second section we present the definition of double category and symmetric monoidal double category, both of which can be found in earlier works [15]. In the third section, we construct the symmetric monoidal double category $F\mathrm{Cospan}(\mathbf{C})$ mentioned above as well as a result of Shulman that allows us to obtain a symmetric monoidal bicategory whose decategorification is an improved version of Fong's symmetric monoidal category. In the fourth section we investigate when a symmetric lax monoidal pseudofunctor gives rise to a left adjoint via the Grothendieck construction. In the fifth section we review another compositional framework which uses cospans equipped with extra structure due to Baez and the first author, namely 'structured cospans' and prove that the double categorical versions of decorated cospans and structured cospans are equivalent, and in the final section, we present a few examples that can be realized with either framework and exhibit the equivalences.

2. A review of double categories and symmetric monoidal double categories

This section is meant to serve as a review of double categories and to present the definitions of double category, symmetric monoidal double category and a few more ideas that will be used later in the paper. Nothing in this section is new and can be found in the work of Shulman [15].

Before formally defining 'pseudo double category', it is helpful to have the following picture in mind. A pseudo double category has 2-morphisms shaped like:

$$\begin{tikzcd} A \arrow[r, "M"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow a"] & B \arrow[d, "g"] \\ C \arrow[r, "N"'] & D \end{tikzcd}$$

We call A, B, C and D **objects** or **0-cells**, f and g **vertical 1-morphisms**, M and N **horizontal 1-cells** and a a **2-morphism**. Note that a vertical 1-morphism is a morphism between 0-cells and a 2-morphism is a morphism between horizontal 1-cells. We will denote both kinds of morphisms and horizontal 1-cells as a single arrow, namely '$\to$'. We follow the notation of Shulman [15] with the following definitions.

Definition 2.1. A **pseudo double category** $\mathbb{D}$, or **double category** for short, consists of a category of objects $\mathbf{D_0}$ and a category of arrows $\mathbf{D_1}$ with the following functors

$$U\colon \mathbf{D_0} \to \mathbf{D_1}$$
$$S, T\colon \mathbf{D_1} \rightrightarrows \mathbf{D_0}$$
$$\odot\colon \mathbf{D_1} \times_{\mathbf{D_0}} \mathbf{D_1} \to \mathbf{D_1} \text{ (where the pullback is taken over } \mathbf{D_1} \xrightarrow{T} \mathbf{D_0} \xleftarrow{S} \mathbf{D_1})$$

such that

$$S(U_A) = A = T(U_A)$$
$$S(M \odot N) = SN$$
$$T(M \odot N) = TM$$

equipped with natural isomorphisms

$$\alpha\colon (M \odot N) \odot P \xrightarrow{\sim} M \odot (N \odot P)$$
$$\lambda\colon U_B \odot M \xrightarrow{\sim} M$$
$$\rho\colon M \odot U_A \xrightarrow{\sim} M$$

such that $S(\alpha), S(\lambda), S(\rho), T(\alpha), T(\lambda)$ and $T(\rho)$ are all identities and that the coherence axioms of a monoidal category are satisfied. Following the notation of Shulman, objects of $\mathbf{D_0}$ are called **0-cells** and morphisms of $\mathbf{D_0}$ are called **vertical 1-morphisms**. Objects of $\mathbf{D_1}$ are called **horizontal 1-cells** and morphisms of $\mathbf{D_1}$ are called **2-morphisms**. The morphisms of $\mathbf{D_0}$, which are vertical 1-morphisms, will be denoted $f\colon A \to C$ and we denote a 1-cell M with $S(M) = A, T(M) = B$ by $M\colon A \to B$. Then a 2-morphism $a\colon M \to N$ of $\mathbf{D_1}$ with $S(a) = f, T(a) = g$ would look like:

$$\begin{tikzcd}
A \arrow[r, "M"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow a"] & B \arrow[d, "g"] \\
C \arrow[r, "N"'] & D
\end{tikzcd}$$

The key difference between a 'strict' double category and a pseudo double category is that in a pseudo double category, horizontal composition is associative and unital only up to natural isomorphism. Equivalently, as a double category can be viewed as a category internal to **Cat**, we can view a pseudo double category as a category 'weakly' internal to **Cat**. We will sometimes omit the word pseudo and simply say double category.

Definition 2.2. A 2-morphism where f and g are identities is called a **globular 2-morphism**.

Definition 2.3. Let $\mathbb{D}$ be a pseudo double category. Then the **horizontal bicategory** of $\mathbb{D}$, which we denote as $H(\mathbb{D})$, is the bicategory consisting of objects of $\mathbb{D}$, morphisms that are horizontal 1-cells of $\mathbb{D}$ and 2-morphisms that are globular 2-morphisms of $\mathbb{D}$.

Definition 2.4. A **monoidal double category** is a double category equipped the following structure.

(i) $\mathbf{D_0}$ and $\mathbf{D_1}$ are both monoidal categories.
(ii) If I is the monoidal unit of $\mathbf{D_0}$, then U_I is the monoidal unit of $\mathbf{D_1}$.
(iii) The functors S and T are strict monoidal, i.e. $S(M \otimes N) = SM \otimes SN$ and $T(M \otimes N) = TM \otimes TN$ and S and T also preserve the associativity and unit constraints.
(iv) We have globular isomorphisms

$$\chi\colon (M_1 \otimes N_1) \odot (M_2 \otimes N_2) \xrightarrow{\sim} (M_1 \odot M_2) \otimes (N_1 \odot N_2)$$

and

$$\mu\colon U_{A \otimes B} \xrightarrow{\sim} (U_A \otimes U_B)$$

such that the following diagrams commute:

(v) The following diagrams commute expressing the constraint data for the double functor $\otimes$.

$$\begin{tikzcd}
((M_1 \otimes N_1) \odot (M_2 \otimes N_2)) \odot (M_3 \otimes N_3) \arrow[r, "\chi \odot 1"] \arrow[d, "\alpha"'] & ((M_1 \odot M_2) \otimes (N_1 \odot N_2)) \odot (M_3 \otimes N_3) \arrow[d, "\chi"] \\
(M_1 \otimes N_1) \odot ((M_2 \otimes N_2) \odot (M_3 \otimes N_3)) \arrow[d, "1 \odot \chi"'] & ((M_1 \odot M_2) \odot M_3) \otimes ((N_1 \odot N_2) \odot N_3) \arrow[d, "\alpha \otimes \alpha"] \\
(M_1 \otimes N_1) \odot ((M_2 \odot M_3) \otimes (N_2 \odot N_3)) \arrow[r, "\chi"] & (M_1 \odot (M_2 \odot M_3)) \otimes (N_1 \odot (N_2 \odot N_3))
\end{tikzcd}$$

$$\begin{tikzcd}
(M \otimes N) \odot U_{C \otimes D} \arrow[r, "1 \odot \mu"] \arrow[d, "\rho"'] & (M \otimes N) \odot (U_C \otimes U_D) \arrow[d, "\chi"] \\
M \otimes N & (M \odot U_C) \otimes (N \odot U_D) \arrow[l, "\rho \otimes \rho"']
\end{tikzcd}
\qquad
\begin{tikzcd}
U_{A \otimes B} \odot (M \otimes N) \arrow[r, "\chi \odot 1"] \arrow[d, "\lambda"'] & (U_A \otimes U_B) \odot (M \otimes N) \arrow[d, "\chi"] \\
M \otimes N & (U_A \odot M) \otimes (U_B \odot N) \arrow[l, "\lambda \otimes \lambda"']
\end{tikzcd}$$

(vi) The following diagrams commute expressing the associativity isomorphism for $\otimes$ is a transformation of double categories.

$$\begin{tikzcd}
((M_1 \otimes N_1) \otimes P_1) \odot ((M_2 \otimes N_2) \otimes P_2) \arrow[r, "\alpha \odot \alpha"] \arrow[d, "\chi"'] & (M_1 \otimes (N_1 \otimes P_1)) \odot (M_2 \otimes (N_2 \otimes P_2)) \arrow[d, "\chi"] \\
((M_1 \otimes N_1) \odot (M_2 \otimes N_2)) \otimes (P_1 \odot P_2) \arrow[d, "\chi \otimes 1"'] & (M_1 \odot M_2) \otimes ((N_1 \otimes P_1) \odot (N_2 \otimes P_2)) \arrow[d, "1 \otimes \chi"] \\
((M_1 \odot M_2) \otimes (N_1 \odot N_2)) \otimes (P_1 \odot P_2) \arrow[r, "\alpha"] & (M_1 \odot M_2) \otimes ((N_1 \odot N_2) \otimes (P_1 \odot P_2))
\end{tikzcd}$$

$$\begin{tikzcd}
U_{(A \otimes B) \otimes C} \arrow[r, "U_\alpha"] \arrow[d, "\mu"'] & U_{A \otimes (B \otimes C)} \arrow[d, "\mu"] \\
U_{A \otimes B} \otimes U_C \arrow[d, "\mu \otimes 1"'] & U_A \otimes U_{B \otimes C} \arrow[d, "1 \otimes \mu"] \\
(U_A \otimes U_B) \otimes U_C \arrow[r, "\alpha"] & U_A \otimes (U_B \otimes U_C)
\end{tikzcd}$$

(vii) The following diagrams commute expressing that the unit isomorphisms for $\otimes$ are transformations of double categories.

$$\begin{tikzcd}
(M \otimes U_I) \odot (N \otimes U_I) \arrow[r, "\chi"] \arrow[d, "r \odot r"'] & (M \odot N) \otimes (U_I \odot U_I) \arrow[d, "1 \otimes \rho"] \\
M \odot N & (M \odot N) \otimes U_I \arrow[l, "r"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& U_A \otimes U_I \arrow[d, "r"] \\
U_{A \otimes I} \arrow[ur, "\mu"] \arrow[r, "U_r"'] & U_A
\end{tikzcd}$$

$$\begin{tikzcd}
(U_I \otimes M) \odot (U_I \otimes N) \arrow[r, "\chi"] \arrow[d, "\ell \odot \ell"'] & (U_I \odot U_I) \otimes (M \odot N) \arrow[d, "\lambda \otimes 1"] \\
M \odot N & U_I \otimes (M \odot N) \arrow[l, "\ell"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & U_I \otimes U_A \arrow[d, "\ell"] \\
U_{I \otimes A} \arrow[ur, "\mu"] \arrow[r, "U_\ell"'] & U_A
\end{tikzcd}$$

A **braided monoidal double category** is a monoidal double category such that:

(viii) $\mathbb{D}_0$ and $\mathbb{D}_1$ are braided monoidal categories.
(ix) The functors S and T are strict braided monoidal functors.
(x) The following diagrams commute expressing that the braiding is a transformation of double categories.

$$\begin{tikzcd}
(M_1 \odot M_2) \otimes (N_1 \odot N_2) \arrow[r, "\beta"] \arrow[d, "\chi"'] & (N_1 \odot N_2) \otimes (M_1 \odot M_2) \arrow[d, "\chi"] \\
(M_1 \otimes N_1) \odot (M_2 \otimes N_2) \arrow[r, "\beta \odot \beta"] & (N_1 \otimes M_1) \odot (N_2 \otimes M_2)
\end{tikzcd}
\qquad
\begin{tikzcd}
U_A \otimes U_B \arrow[d, "\beta"'] & U_{A \otimes B} \arrow[l, "\mu"'] \arrow[d, "U_\beta"] \\
U_B \otimes U_A & U_{B \otimes A} \arrow[l, "\mu"]
\end{tikzcd}$$

Finally, a **symmetric monoidal double category** is a braided monoidal double category $\mathbb{D}$ such that:

(xi) $\mathbb{D}_0$ and $\mathbb{D}_1$ are symmetric monoidal.

Definition 2.5. Let $\mathbb{D}$ be a double category and $f \colon A \to B$ a vertical 1-morphism. A **companion** of f is a horizontal 1-cell $\hat{f} \colon A \to B$ together with 2-morphisms

$$\begin{tikzcd}
A \arrow[r, "\widehat{f}"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d, "1"] \\
B \arrow[r, "U_B"'] & B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[r, "U_A"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, "f"] \\
A \arrow[r, "\widehat{f}"'] & B
\end{tikzcd}$$

such that the following equations hold.

$$\begin{tikzcd}
A \arrow[r, "U_A"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, "f"] \\
A \arrow[r, "\widehat{f}"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d, "1"] \\
B \arrow[r, "U_B"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "U_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow U_f"] & A \arrow[d, "f"] \\
B \arrow[r, "U_B"'] & B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[r, "U_A"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow"] & A \arrow[r, "\widehat{f}"] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d, "1"] \\
A \arrow[r, "\widehat{f}"'] & B \arrow[r, "U_B"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "\widehat{f}"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow \mathrm{id}_{\widehat{f}}"] & B \arrow[d, "1"] \\
A \arrow[r, "\widehat{f}"'] & B
\end{tikzcd}$$

A **conjoint** of f, denoted $\check{f} \colon B \to A$, is a companion of f in the double category $\mathbb{D}^{h\cdot\mathrm{op}}$ obtained by reversing the horizontal 1-cells, but not the vertical 1-morphisms, of $\mathbb{D}$.

In a pseudo double category, the second equation above requires an insertion of unit isomorphisms to make sense due to horizontal composition only holding up to isomorphism.

Definition 2.6. We say that a double category is **fibrant** if every vertical 1-morphism has both a companion and a conjoint and **isofibrant** if every vertical 1-isomorphism has both a companion and a conjoint.

3. A symmetric monoidal double category of decorated cospans

In this section we build the symmetric monoidal double category mentioned in the introduction. A double category $\mathbb{D} = (\mathbb{D}_0, \mathbb{D}_1)$ consists of a category of objects $\mathbb{D}_0$ and a category of arrows $\mathbb{D}_1$. We obtain $\mathbb{D}$ as a comma object in the 2-category **Dbl** of monoidal double categories, monoidal double functors and monoidal double transformations by obtaining each of $\mathbb{D}_0$ and $\mathbb{D}_1$ as comma categories in **Cat**.

If **C** is a category with finite colimits, there is a well-known symmetric monoidal pseudo double category $\mathbb{C}\mathrm{ospan}(\mathbf{C})$ which has:

(i) objects as those of **C**,
(ii) vertical 1-morphisms as morphisms of **C**,
(iii) horizontal 1-cells as cospans in **C**, and
(iv) 2-morphisms as maps of cospans in **C**.

$$\begin{tikzcd} c_1 \arrow[r] \arrow[d] & d \arrow[d] & c_2 \arrow[l] \arrow[d] \\ c_1' \arrow[r] & d' & c_2' \arrow[l] \end{tikzcd}$$

The symmetric monoidal category $(\mathbf{Cat}, \times, \mathbf{1})$ consisting of categories and functors can naturally be viewed as a one object bicategory using a standard level shift. This one object bicategory, which we denote as $\mathrm{BD}(\mathbf{Cat})$, will have:

(i) one object $\{\star\}$,
(ii) objects of **Cat** as morphisms, and
(iii) morphisms of **Cat** as 2-morphisms.

We can then view this one object bicategory as a one object pseudo double category $\mathbb{B}\mathrm{D}(\mathbf{Cat})$, which will have:

(i) one object $\{\star\}$,
(ii) only an identity for vertical 1-morphisms,
(iii) morphisms of $\mathrm{BD}(\mathbf{Cat})$, which are objects of **Cat**, as horizontal 1-cells, and
(iv) 2-morphisms of $\mathrm{BD}(\mathbf{Cat})$, which are morphisms of **Cat**, as 2-morphisms.

This one object pseudo double category $\mathbb{B}\mathrm{D}(\mathbf{Cat})$ will serve as the apex of a cospan:

$$\begin{tikzcd} & \mathbb{B}\mathrm{D}(\mathbf{Cat}) & \\ \mathbf{1} \arrow[ur, "\mathbb{G}"] & & \mathbb{C}\mathrm{ospan}(\mathbf{C}) \arrow[ul, "\mathbb{F}"'] \end{tikzcd}$$

which we will use to construct a comma object in **Dbl**, the 2-category of double categories, double functors and double transformations. In order for the double comma object to exist in **Dbl**, namely a pseudo comma double category, it is necessary that $\mathbb{F}$ and $\mathbb{G}$ be lax and oplax double functors, respectively. This was made known to the first author by Shulman. This result is true in general, but we will only prove it for this particular case that we are interested in.

The oplax double functor $\mathbb{G} \colon 1 \to \mathbb{B}\mathrm{D}(\mathbf{Cat})$ is trivially defined modulo $\mathbb{G}$ mapping the identity horizontal 1-cell to the monoidal unit of $\mathbb{B}\mathrm{D}(\mathbf{Cat})$ which is given by the terminal category **1**. The category **C** has finite colimits and can be viewed as a 2-category with only trivial 2-morphisms. Then from a symmetric lax monoidal pseudofunctor $F \colon \mathbf{C} \to \mathbf{Cat}$, we can obtain symmetric lax monoidal lax double functor $\mathbb{F}\colon \mathbb{C}\mathrm{ospan}(\mathbf{C}) \to \mathbb{B}\mathrm{D}(\mathbf{Cat})$ which is defined as the pseudofunctor F on horizontal 1-cells and 2-morphisms; note that $\mathbb{B}\mathrm{D}(\mathbf{Cat})$ has a single object and an identity vertical 1-morphism. Thus $\mathbb{F}$ will map a cospan in $\mathbf{C}$:

$$a \to c \leftarrow b$$

to $F(c) \in \mathbf{Cat}$ and

Theorem 3.1. *Let $F\colon \mathbf{C} \to \mathbf{Cat}$ be a symmetric lax monoidal pseudofunctor. Then there exists a symmetric (lax) monoidal lax double functor $\mathbb{F}\colon \mathbb{C}\mathrm{ospan}(\mathbf{C}) \to \mathbb{B}\mathrm{D}(\mathbf{Cat})$ which is defined:*

- *(i) trivially on objects as $\mathbb{B}\mathrm{D}(\mathbf{Cat})$ only has a single object,*
- *(ii) trivially on vertical 1-morphisms as $\mathbb{B}\mathrm{D}(\mathbf{Cat})$ only has a single identity vertical 1-morphism,*
- *(iii) as the pseudofunctor $F\colon \mathbf{C} \to \mathbf{Cat}$ on horizontal 1-cells, meaning that*

$$(c_1 \xrightarrow{i} c \xleftarrow{o} c_2) \mapsto F(c)$$

and

- *(iv) as the pseudofunctor $F\colon \mathbf{C} \to \mathbf{Cat}$ on 2-morphisms, meaning that*

$$\begin{tikzcd} c_1 \arrow[r] \arrow[d] & c \arrow[d, "f"'] & c_2 \arrow[l] \arrow[d] \\ c_1' \arrow[r] & c' & c_2' \arrow[l] \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} F(c) \arrow[d, "F(f)"] \\ F(c') \end{tikzcd}$$

Proof. The functor $\mathbb{F}\colon \mathbb{C}\mathrm{ospan}(\mathbf{C}) \to \mathbb{B}\mathrm{D}(\mathbf{Cat})$ is defined trivially on objects and vertical 1-morphisms as the object category of $\mathbb{B}\mathrm{D}(\mathbf{Cat})$ is trivial. Given two composable horizontal 1-cells M and N in $\mathbb{C}\mathrm{ospan}(\mathbf{C})$ given respectively by:

$$\begin{tikzcd} & c & \\ c_1 \arrow[ur, "i"] & & c_2 \arrow[ul, "o"'] \end{tikzcd} \qquad \begin{tikzcd} & c' & \\ c_2 \arrow[ur, "i'"] & & c_3 \arrow[ul, "o'"'] \end{tikzcd}$$

we have that $\mathbb{F}(M) = F(c)$ and $\mathbb{F}(N) = F(c')$. The composite $M \odot N$ is given by the pushout:

$$\begin{tikzcd} & c +_{c_2} c' & \\ c_1 \arrow[ur, "j\psi_c i"] & & c_3 \arrow[ul, "j\psi_{c'} o"'] \end{tikzcd}$$

and so $\mathbb{F}(M \odot N) = F(c +_{c_2} c')$. The comparison constraint globular 2-morphism $\mathbb{F}_{M,N}\colon \mathbb{F}(M) \odot \mathbb{F}(N) \to \mathbb{F}(M \odot N)$ is then given by:

$$\begin{tikzcd} \star \arrow[r, "F(c) \times F(c')"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\Downarrow F(j)\phi_{c,c'}"] & \star \arrow[d, "\mathrm{id}"] \\ \star \arrow[r, "F(c +_{c_2} c')"'] & \star \end{tikzcd}$$

where $\phi_{c,c'} \colon F(c) \times F(c') \to F(c+c')$ is one of the natural transformations of the symmetric lax monoidal pseudo functor F and $j \colon c+c' \to c+_{c_2}c'$ is the natural map from the coproduct to the pushout.

Given an object c in $\mathbb{C}\mathrm{ospan}(\mathbf{C})$, $\mathbb{F}(c)$ is the single object $\star$ of $\mathbb{B}\mathrm{D}(\mathbf{Cat})$, and so $1_{\mathbb{F}(c)}$ is the identity horizontal 1-cell on $\star$. Similarly, 1_c is the identity cospan on c and then $\mathbb{F}(1_c) = F(c)$. The identity comparison is then the globular 2-morphism $\mathbb{F}_c \colon 1_{\mathbb{F}(c)} \to \mathbb{F}(1_c)$ given by:

$$\begin{tikzcd}
\star \arrow[r] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\Downarrow"] & \star \arrow[d, "\mathrm{id}"] \\
\star \arrow[r, "F(c)"] & \star
\end{tikzcd}$$

Finish this... □

Proposition 3.2. *Let* **1** *denote the trivial double category with a single object and only identities. Then there exists a symmetric lax monoidal oplax double functor* $\mathbb{G} \colon \mathbf{1} \to \mathbb{B}\mathrm{D}(\mathbf{Cat})$ *which is defined:*

(i) trivially on objects, as both **1** *and* $\mathbb{B}\mathrm{D}(\mathbf{Cat})$ *have a single object,*
(ii) trivially on vertical 1-morphisms, as both **1** *and* $\mathbb{B}\mathrm{D}(\mathbf{Cat})$ *only have a single identity vertical 1-morphism,*
(iii) the identity horizontal 1-cell of **1** *maps to the monoidal unit of* $(\mathbb{B}\mathrm{D}(\mathbf{Cat}), \times, \mathbf{1})$*, which is the terminal category* **1** *with a single object and identity morphism, and*
(iv) the identity 2-morphism maps to the identity functor $! \colon \mathbf{1} \to \mathbf{1}$ *on the terminal category.*

We recall the definition of *comma category* which will be crucial for the next step.

Definition 3.3. Let $f \colon \mathbf{C} \to \mathbf{E}$ and $g \colon \mathbf{D} \to \mathbf{E}$ be functors with a common codomain. Then the **comma category** determined by f and g is the category (f/g) whose:

(i) objects are triples (c, d, α) where $c \in \mathbf{C}, d \in \mathbf{D}$ and $\alpha \colon f(c) \to g(d)$ is a morphism in $\mathbf{E}$, and whose
(ii) morphisms from (c_1, d_1, α_1) to (c_2, d_2, α_2) are pairs (β, γ) where $\beta \colon c_1 \to c_2$ and $\gamma \colon d_1 \to d_2$ are morphisms in $\mathbf{C}$ and $\mathbf{D}$, respectively, such that the following diagram commutes

$$\begin{tikzcd}
f(c_1) \arrow[r, "f(\beta)"] \arrow[d, "\alpha_1"'] & f(c_2) \arrow[d, "\alpha_2"] \\
g(d_1) \arrow[r, "g(\gamma)"] & g(d_2)
\end{tikzcd}$$

Definition 3.4. Let $f \colon \mathbf{C} \to \mathbf{E}$ and $g \colon \mathbf{D} \to \mathbf{E}$ be pseudofunctors between bicategories with a common codomain. Then the **(weak) comma category** determined by f and g is the category (f/g) whose:

(i) objects are triples (c, d, α) where $c \in \mathbf{C}, d \in \mathbf{D}$ and $\alpha \colon f(c) \to g(d)$ is a morphism in $\mathbf{E}$, and whose
(ii) morphisms from (c_1, d_1, α_1) to (c_2, d_2, α_2) are triples (β, γ, ι) where $\beta \colon c_1 \to c_2$ and $\gamma \colon d_1 \to d_2$ are morphisms in $\mathbf{C}$ and $\mathbf{D}$, respectively and ι is a 2-morphism in $\mathbf{E}$, such that the following diagram commutes up to the 2-morphism ι:

$$\begin{tikzcd}
f(c_1) \arrow[r, "f(\beta)"] \arrow[d, "\alpha_1"'] \arrow[dr, phantom, "\iota \Swarrow"] & f(c_2) \arrow[d, "\alpha_2"] \\
g(d_1) \arrow[r, "g(\gamma)"] & g(d_2)
\end{tikzcd}$$

Using this definition we want to construct a 'pseudo comma double category' from the lax double functor $\mathbb{F}\colon \mathbb{C}\mathrm{ospan}(\mathbf{C}) \to \mathbb{B}\mathrm{D}(\mathbf{Cat})$ and the oplax double functor $\mathbb{G}\colon \mathbf{1} \to \mathbb{B}\mathrm{D}(\mathbf{Cat})$. First, we define how this pseudo comma double category $(\mathbb{G}/\mathbb{F}) \coloneqq \mathbb{F}\mathrm{Cospan}(\mathbf{C})$ is obtained. **Should this be a proposition or a definition? Ask Dr. Baez and Christina for their opinions.**

Definition 3.5. Let $\mathbb{G}\colon \mathbb{C} \to \mathbb{E}$ be an oplax double functor and $\mathbb{F}\colon \mathbb{D} \to \mathbb{E}$ be a lax double functor where $\mathbb{C} = (\mathbf{C}_0, \mathbf{C}_1), \mathbb{D} = (\mathbf{D}_0, \mathbf{D}_1)$ and $\mathbb{E} = (\mathbf{E}_0, \mathbf{E}_1)$ are pseudo double categories with $\mathbf{C}_0, \mathbf{C}_1$ the category of objects and category of arrows of the double category $\mathbb{C}$, respectively, and likewise for $\mathbf{D}_0, \mathbf{D}_1, \mathbf{E}_0$ and $\mathbf{E}_1$. Then the **pseudo comma double category** determined by the double functors $\mathbb{G}$ and $\mathbb{F}$ is the pseudo double category $(\mathbb{G}/\mathbb{F})$ consisting of a category of objects $(\mathbb{G}/\mathbb{F})_0$ and category of arrows $(\mathbb{G}/\mathbb{F})_1$ where the category of objects $(\mathbb{G}/\mathbb{F})_0$ is the comma category obtained from the functors $\mathbb{G}_0\colon \mathbf{C}_0 \to \mathbf{E}_0$ and $\mathbb{F}_0\colon \mathbf{D}_0 \to \mathbf{E}_0$ and $(\mathbb{G}/\mathbb{F})_1$ is the comma category obtained from the functors $\mathbb{G}_1\colon \mathbf{C}_1 \to \mathbf{E}_1$ and $\mathbb{F}_1\colon \mathbf{D}_1 \to \mathbf{E}_1$.

Theorem 3.6. *Let $\mathbb{G}\colon \mathbf{1} \to \mathbb{B}\mathrm{D}(\mathbf{Cat})$ be the oplax double functor previously defined and let $\mathbb{F}\colon \mathbb{F}\mathrm{Cospan}(\mathbf{C}) \to \mathbb{B}(\mathbf{Cat})$ be the lax double functor previously defined.*

$$\begin{tikzcd}
& \mathbb{B}\mathrm{D}(\mathbf{Cat}) & \\
\mathbf{1} \arrow[ur, "\mathbb{G}"] & & \mathbb{C}\mathrm{ospan}(\mathbf{C}) \arrow[ul, "\mathbb{F}"'] \\
& (\mathbb{G}/\mathbb{F}) \arrow[ul, "p"] \arrow[ur, "q"'] &
\end{tikzcd}$$

Then there exists a category $(\mathbb{G}/\mathbb{F})_0 \coloneqq (\mathbb{G}_0/\mathbb{F}_0)$ obtained as a comma category from the functors $\mathbb{F}_0\colon \mathbb{C}\mathrm{ospan}(\mathbf{C})_0 \to \mathbb{B}\mathrm{D}(\mathbf{Cat})_0$ and $\mathbb{G}_0\colon \mathbf{1} \to \mathbb{B}\mathrm{D}(\mathbf{Cat})_0$ which has:

- *(i) objects as those of* $\mathbf{C}$ *and*
- *(ii) morphisms as those of* $\mathbf{C}$*,*

and there exists a category $(\mathbb{G}/\mathbb{F})_1 \coloneqq (\mathbb{G}_1/\mathbb{F}_1)$ obtained as a comma category from the functors $\mathbb{F}_1\colon \mathbb{C}\mathrm{ospan}(\mathbf{C})_1 \to \mathbb{B}\mathrm{D}(\mathbf{Cat})_1$ and $\mathbb{G}_1\colon \mathbf{1} \to \mathbb{B}\mathrm{D}(\mathbf{Cat})_1$ which has:

- *(i) objects as F-decorated cospans of* $\mathbf{C}$*, which are pairs:*

$$\begin{tikzcd}
& c & \\
a \arrow[ur, "i"] & & b \arrow[ul, "o"']
\end{tikzcd} \qquad d \in F(c)$$

and

(ii) morphisms as maps of cospans of **C***:*

$$\begin{tikzcd} a \arrow[r] \arrow[d, "f"'] & c \arrow[d, "h"'] & b \arrow[l] \arrow[d, "g"'] & d \in F(c) \\ a' \arrow[r] & c' & b' \arrow[l] & d' \in F(c') \end{tikzcd}$$

together with a morphism $\iota\colon F(h)(d) \to d'$ *in* $F(c')$.

Proof. The category $(\mathbb{G}/\mathbb{F})_0$ will be the comma category obtained from the following cospan:

$$\begin{tikzcd} & \mathbb{B}\mathrm{D}(\mathbf{Cat})_0 & \\ \mathbf{1} \arrow[ur, "\mathbb{G}_0"] & & \mathbb{C}\mathrm{ospan}(\mathbf{C})_0 = \mathbf{C} \arrow[ul, "\mathbb{F}_0"'] \end{tikzcd}$$

Thus $(\mathbb{G}/\mathbb{F})_0$ will be the comma category which has objects given by triples $(\star, c, \alpha\colon \mathbb{G}(\star) \to \mathbb{F}(c))$ where $\star$ is the single object of $\mathbf{1}$, c is a object of $\mathbf{C}$ and $\alpha\colon \mathbb{G}(\star) \to \mathbb{F}(c)$ is a morphism in $\mathbb{B}\mathrm{D}(\mathbf{Cat})_0$. The pseudo double category $\mathbb{B}\mathrm{D}(\mathbf{Cat})$ only has a single object $\star$ and identity vertical 1-morphism, so the triple $(\star, c, \alpha\colon \mathbb{G}(\star) \to \mathbb{F}(c))$ amounts to an object of $\mathbf{C}$ together with trivial structure. Given another object $(\star, c', \alpha'\colon \mathbb{G}(\star) \to \mathbb{F}(c'))$, a morphism from the first triple to the second triple is a pair (β, γ) where $\beta = \mathrm{id}_\star$ is a morphism in $\mathbf{1}$ and $\gamma\colon c \to c'$ is a morphism in $\mathbf{C}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{G}(\star) \arrow[r, "\mathbb{G}(\mathrm{id}_\star)"] \arrow[d, "\alpha"'] & \mathbb{G}(\star) \arrow[d, "\alpha'"] \\ \mathbb{F}(c) \arrow[r, "\mathbb{F}(\gamma)"] & \mathbb{F}(c') \end{tikzcd}$$

The double category $\mathbb{B}\mathrm{D}(\mathbf{Cat})$ only has a single object and identity vertical 1-morphism, so this diagram is trivial, as all four objects are the same and all four morphisms are identities. Thus, morphisms are given by morphisms of $\mathbf{C}$ with added trivial structure, and so the category $(\mathbb{G}/\mathbb{F})_0$ has objects and morphisms of $\mathbf{C}$, and we have an isomorphism of categories $(\mathbb{G}/\mathbb{F})_0 \cong \mathbf{C}$.

Next we build the category $(\mathbb{G}/\mathbb{F})_1$ which will be the comma category obtained from the following cospan:

$$\begin{tikzcd} & \mathbb{B}\mathrm{D}(\mathbf{Cat})_1 & \\ \mathbf{1} \arrow[ur, "\mathbb{G}_1"] & & \mathbb{C}\mathrm{ospan}(\mathbf{C})_1 \arrow[ul, "\mathbb{F}_1"'] \end{tikzcd}$$

Objects are given by triples

$$(1_\star, a \to c \leftarrow b, d\colon \mathbb{G}(1_\star) \to \mathbb{F}(a \to c \leftarrow b))$$

where $1_\star$ is the identity horizontal 1-cell on $\star$ and $a \to c \leftarrow b$ is a cospan in $\mathbf{C}$. The double functor $\mathbb{G}$ maps the identity horizontal 1-cell to the monoidal unit of $\mathbf{Cat}$ under $\times$, namely the terminal category $\mathbf{1}$, and the double functor $\mathbb{F}$ acts as the functor F on horizontal 1-cells, meaning that $\mathbb{F}(a \to c \leftarrow b) = F(c)$. Thus the above triple is a cospan in $\mathbf{C}$ together with an element $d\colon \mathbf{1} \to F(c)$, and this is precisely an F-decorated cospan, namely a pair

$$((a \to c \leftarrow b), d \in F(c)).$$

Given two objects $((a \to c \leftarrow b), d \in F(c))$ and $(a' \to c' \leftarrow b'), d' \in F(c'))$, a morphism from the first to the second is a triple (δ, h, ι) that makes the following diagram commute:

$$\begin{tikzcd}
\mathbb{G}(1_\star) \arrow[r, "d"] \arrow[d, "\mathbb{G}(\delta)"'] \arrow[dr, phantom, "\iota \swarrow"] & \mathbb{F}(a \to c \leftarrow b) \arrow[d, "\mathbb{F}(h)"] \\
\mathbb{G}(1_\star) \arrow[r, "d'"] & \mathbb{F}(a' \to c' \leftarrow b')
\end{tikzcd}$$

Simplifying things a bit, $\mathbb{G}(\delta)$ must be the identity 2-morphism of $\mathbf{1}$ and h is a morphism of $\mathbb{C}\mathrm{ospan}(\mathbf{C})$, namely a triple of morphisms (h_1, h, h_2) in $\mathbf{C}$ that makes the following diagram commute:

$$\begin{tikzcd}
a \arrow[r] \arrow[d, "h_1"'] & c \arrow[d, "h"'] & b \arrow[l] \arrow[d, "h_2"'] \\
a' \arrow[r] & c' & b' \arrow[l]
\end{tikzcd}$$

Thus the above diagram containing the 2-morphism ι becomes:

$$\begin{tikzcd}
 & F(c) \arrow[d, "F(h)"] \\
\mathbf{1} \arrow[ur, "d"] \arrow[dr, "d'"'] \arrow[r, phantom, "\iota \swarrow"] & {} \\
 & F(c')
\end{tikzcd}$$

which is just a morphism $\iota\colon F(h)(d) \to d'$ in $F(c')$. Thus a morphism of $(\mathbb{G}/\mathbb{F})_1$ is a map of cospans in $\mathbf{C}$ together with a natural transformation that that makes the above triangle commute.

Theorem 3.7. *Let* $\mathbf{C}$ *be a category with finite colimits and* $F\colon \mathbf{C} \to \mathbf{Cat}$ *a lax monoidal pseudofunctor. Then there exists a double category* $\mathbb{F}\mathrm{Cospan}(\mathbf{C}) := (\mathbb{G}/\mathbb{F})$ *which has:*

(i) objects as those of $\mathbf{C}$*,*
(ii) vertical 1-morphisms as morphisms of $\mathbf{C}$*,*
(iii) horizontal 1-cells as pairs:

$$\begin{tikzcd}
 & c & \\
a \arrow[ur, "i"] & & b \arrow[ul, "o"']
\end{tikzcd} \qquad d \in F(c)$$

and
(iv) 2-morphisms as maps of cospans in $\mathbf{C}$

$$\begin{tikzcd}
a \arrow[r] \arrow[d, "f"'] & c \arrow[d, "h"'] & b \arrow[l] \arrow[d, "g"'] & d \in F(c) \\
a' \arrow[r] & c' & b' \arrow[l] & d' \in F(c')
\end{tikzcd}$$

together with a morphism $\iota\colon F(h)(d) \to d'$ in $F(c')$.

The unit structure functor $U\colon (\mathbb{G},\mathbb{F})_0 \to (\mathbb{G},\mathbb{F})_1$ is defined on objects as:

$$c \mapsto \qquad \begin{tikzcd} & c & \\ c \arrow[ur, "1"] & & c \arrow[ul, "1"'] \end{tikzcd} \qquad I \in F(c)$$

where $I \in F(c)$ is the trivial decoration on c given by the composition of the unique map $F(!)\colon F(0) \to F(c)$ and the morphism $\phi\colon \mathbf{1} \to F(0)$ which comes from the symmetric lax monoidal pseudofunctor $F\colon \mathbf{C} \to \mathbf{Cat}$. For morphisms, the structure functor U is defined as:

$$\begin{tikzcd} c \arrow[d, "f"'] \\ c' \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} c \arrow[r] \arrow[d, "f"'] & c \arrow[d, "f"'] & c \arrow[l] \arrow[d, "f"'] \\ c' \arrow[r] & c' & c' \arrow[l] \end{tikzcd} \qquad \begin{matrix} I \in F(c) \\ \\ I' \in F(c') \end{matrix}$$

together with the morphism $\iota_{!_f} = F(f)F(!)\phi\colon \mathbf{1} \to F(c')$. We also have source and target structure functors $s,t\colon (\mathbb{G}/\mathbb{F})_1 \to (\mathbb{G}/\mathbb{F})_0$ where the source of a horizontal 1-cell

$$\begin{tikzcd} & c & \\ a \arrow[ur, "i"] & & b \arrow[ul, "o"'] \end{tikzcd} \qquad d \in F(c)$$

is the object a in $\mathbf{C}$ and the source of a 2-morphism

$$\begin{tikzcd} a \arrow[r] \arrow[d, "f"'] & c \arrow[d, "h"'] & b \arrow[l] \arrow[d, "g"'] \\ a' \arrow[r] & c' & b' \arrow[l] \end{tikzcd} \qquad \begin{matrix} d \in F(c) \\ \\ d' \in F(c') \end{matrix}$$

$$\iota\colon F(h)(d) \to d'$$

is the source of the underlying map of cospans in $\mathbf{C}$, namely the morphism f in $\mathbf{C}$; the target structure functor is defined similarly. The functors satisfy the equations

$$SU(c) = 1(c) = TU(c)$$

for all objects c of $\mathbf{C}$.

Given two composable horizontal 1-cells:

$$\begin{tikzcd} & c & \\ a \arrow[ur, "i"] & & b \arrow[ul, "o"'] \end{tikzcd} \qquad \begin{tikzcd} & c' & \\ b \arrow[ur, "i'"] & & e \arrow[ul, "o'"'] \end{tikzcd}$$

$$d \in F(c) \qquad\qquad d' \in F(c')$$

the composite is given by:

$$\begin{tikzcd}
 & & c +_b c' & & \\
 & & c + c' \arrow[u, "\psi"] & & \\
 & c \arrow[ur, "j"] & & c' \arrow[ul, "j'"'] & \\
a \arrow[ur, "i"] \arrow[uuurr, "\psi j i"] & & b \arrow[ul, "o"'] \arrow[ur, "i'"] & & e \arrow[ul, "o'"'] \arrow[uuull, "\psi j' o'"']
\end{tikzcd}$$

with the corresponding decoration of the apex $\hat{d} \in F(c +_b c')$ given by:

$$1 \xrightarrow{\lambda^{-1}} 1 \times 1 \xrightarrow{d \times d'} F(c) \times F(c') \xrightarrow{\phi_{c,c'}} F(c + c') \xrightarrow{F(\psi)} F(c +_b c')$$

where $\psi \colon c + c' \to c +_b c'$ is the natural map from the coproduct to the pushout and $\phi_{c,c'} \colon F(c) \times F(c') \to F(c + c')$ is the natural transformation of the symmetric lax monoidal pseudofunctor $F \colon \mathbf{C} \to \mathbf{Cat}$. Denoting the first and second of these horizontal 1-cells as M and N, respectively, the source and target structure functors satisfy the equations $S(N \odot M) = S(M)$ and $T(N \odot M) = T(N)$.

Given three composable horizontal 1-cells M_1, M_2 and M_3:

$$\begin{tikzcd}
 & c & \\
a \arrow[ur, "i"] & & b \arrow[ul, "o"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & c' & \\
b \arrow[ur, "i'"] & & e \arrow[ul, "o'"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & c'' & \\
e \arrow[ur, "i''"] & & f \arrow[ul, "o''"']
\end{tikzcd}$$

$$d_1 \in F(c) \qquad\qquad d_2 \in F(c') \qquad\qquad d_3 \in F(c'')$$

we get a natural isomorphism $a \colon (M_1 \odot M_2) \odot M_3 \to M_1 \odot (M_2 \odot M_3)$ which is a globular 2-morphism given by a map of cospans $(\mathrm{id}_a, \sigma, \mathrm{id}_f)$:

$$\begin{tikzcd}
a \arrow[r] \arrow[d, "\mathrm{id}_a"'] & (c +_b c') +_e c'' \arrow[d, "\sigma"'] & f \arrow[l] \arrow[d, "\mathrm{id}_f"'] & d \in F((c +_b c') +_e c'') \\
a \arrow[r] & c +_b (c' +_e c'') & f \arrow[l] & d' \in F(c +_b (c' +_e c''))
\end{tikzcd}$$

with elements of the image of the cospan's apices under the pseudofunctor F given by:

$$d := 1 \xrightarrow{\zeta_1} F(c +_b c') \times F(c'') \xrightarrow{F(\phi_{c +_b c', c''})} F((c +_b c') + c'') \xrightarrow{F(j_{c +_b c', c''})} F((c +_b c') +_e c'')$$

$$\zeta_1 = d_3 \lambda^{-1} F(j_{c,c'}) \phi_{c,c'} (d_1 \times d_2) \lambda^{-1}$$

and

$$d' := 1 \xrightarrow{\zeta_2} F(c) \times F(c' +_e c'') \xrightarrow{F(\phi_{c, c' +_e c''})} F(c + (c' +_e c'')) \xrightarrow{F(j_{c, c' +_e c''})} F(c +_b (c' +_e c''))$$

$$\zeta_2 = d_1 \lambda^{-1} F(j_{c',c''}) \phi_{c',c''} (d_2 \times d_3) \lambda^{-1}$$

together with an isomorphism $\iota_a\colon F(\sigma)(d) \to d'$. We also have left and right unitors where given a horizontal 1-cell M:

$$\begin{tikzcd} & c & \\ a \arrow[ur, "i"] & & b \arrow[ul, "o"'] \end{tikzcd} \qquad d \in F(c)$$

if we, say, compose with the identity horizontal 1-cell of b on the right:

$$\begin{tikzcd} & c & \\ a \arrow[ur, "i"] & & b \arrow[ul, "o"'] \end{tikzcd} \qquad \begin{tikzcd} & b & \\ b \arrow[ur, "1"] & & b \arrow[ul, "1"'] \end{tikzcd}$$

$$d \in F(c) \qquad\qquad !_b \in F(b)$$

where $!_b = F(!)\phi\colon \mathbf{1} \to F(b)$ is the trivial decoration on b. Composing these then gives:

$$\begin{tikzcd} & c +_b b & \\ a \arrow[ur, "j\psi_c i"] & & b \arrow[ul, "j\psi_b"'] \end{tikzcd} \qquad \hat{d} \in F(c +_b b)$$

where $\psi_b\colon b \to c{+}b$ is the natural map into the coproduct and likewise for ψ_c and $j\colon c{+}b \to c{+}_b b$ is the natural map from the coproduct to the pushout. The decoration $\hat{d}\colon \mathbf{1} \to F(c{+}_b b)$ is given by:

$$1 \xrightarrow{\lambda^{-1}} 1 \times 1 \xrightarrow{d \times !_b} F(c) \times F(b) \xrightarrow{\phi_{c,b}} F(c+b) \xrightarrow{F(j_{c,b})} F(c +_b b).$$

We then have that the right unitor of the original horizontal 1-cell M is given by the globular 2-morphism $(\mathrm{id}_a, \gamma, \mathrm{id}_b)$ from the above composite to M:

$$\begin{tikzcd} a \arrow[r, "j\psi_c i"] \arrow[d, "\mathrm{id}_a"'] & c +_b b \arrow[d, "\gamma"] & b \arrow[l, "j\psi_b"'] \arrow[d, "\mathrm{id}_b"] \\ a \arrow[r, "i"] & c & b \arrow[l, "o"'] \end{tikzcd} \qquad \begin{array}{c} \hat{d} \in F((c +_b b) \\ \\ \\ d \in F(c) \end{array}$$

where $\gamma\colon c +_b b \xrightarrow{\sim} c$ together with an isomorphism $\iota_\rho\colon F(\gamma)(\hat{d}) \to d$. The left unitor is similar. The source and target functor applied to the left and right unitors and associators yield identities. The left and right unitors together with the associator satisfy the standard pentagon and triangle identities of a monoidal category or bicategory. Finally, for the interchange law, given four 2-morphisms:

$$\begin{tikzcd} a \arrow[r, "i_1"] \arrow[d, "f"'] & c \arrow[d, "h_1"'] & b \arrow[l, "o_1"'] \arrow[d, "g"'] & d_1 \in F(c) & b \arrow[r, "i_2"] \arrow[d, "g"'] & e \arrow[d, "h_2"'] & x \arrow[l, "o_2"'] \arrow[d, "k"'] & d_2 \in F(e) \\ a' \arrow[r, "i_1'"] & c' & b' \arrow[l, "o_1'"'] & d_1' \in F(c') & b' \arrow[r, "i_2'"] & e' & x' \arrow[l, "o_2'"'] & d_2' \in F(e') \end{tikzcd}$$

$$\iota_1 \colon F(h_1)(d_1) \to d_1' \qquad \iota_2 \colon F(h_2)(d_2) \to d_2'$$

$$\begin{tikzcd} a' \arrow[r, "i_1'"] \arrow[d, "f'"'] & c' \arrow[d, "h_1'"'] & b' \arrow[l, "o_1'"'] \arrow[d, "g'"'] & d_1' \in F(c') & b' \arrow[r, "i_2'"] \arrow[d, "g'"'] & e' \arrow[d, "h_2'"'] & x' \arrow[l, "o_2'"'] \arrow[d, "k'"'] & d_2' \in F(e') \\ a'' \arrow[r, "i_1''"] & c'' & b'' \arrow[l, "o_1''"'] & d_1'' \in F(c'') & b'' \arrow[r, "i_2''"] & e'' & x'' \arrow[l, "o_2''"'] & d_2'' \in F(e'') \end{tikzcd}$$

$$\iota_1' \colon F(h_1')(d_1') \to d_1'' \qquad \iota_2' \colon F(h_2')(d_2') \to d_2''$$

if we first compose horizontally we obtain:

$$\begin{tikzcd} a \arrow[r, "j\psi_a i_1"] \arrow[d, "f"'] & c +_b e \arrow[d, "h_1 +_g h_2"'] & x \arrow[l, "j\psi_x o_2"'] \arrow[d, "k"'] & d_{1,2} \in F(c +_b e) \\ a' \arrow[r, "j\psi_{a'} i_1'"] & c' +_{b'} e' & x' \arrow[l, "j\psi_{x'} o_2'"'] & d_{1,2}' \in F(c' +_{b'} e') \end{tikzcd}$$

$$\iota_{1,2} \colon F(h_1 +_g h_2)(d_{1,2}) \to d_{1,2}'$$

$$\begin{tikzcd} a' \arrow[r, "j\psi_{a'} i_1'"] \arrow[d, "f'"'] & c' +_{b'} e' \arrow[d, "h_1' +_{g'} h_2'"'] & x' \arrow[l, "j\psi_{x'} o_2'"'] \arrow[d, "k'"'] & d_{1,2}' \in F(c' +_{b'} e') \\ a'' \arrow[r, "j\psi_{a''} i_1''"] & c'' +_{b''} e'' & x'' \arrow[l, "j\psi_{x''} o_2''"'] & d_{1,2}'' \in F(c'' +_{b''} e'') \end{tikzcd}$$

$$\iota_{1,2}' \colon F(h_1' +_{g'} h_2')(d_{1,2}') \to d_{1,2}''.$$

To obtain the morphism of decorations for a horizontal composite, we have as initial data:

$$\begin{tikzcd} & F(c) \arrow[d, "F(h_1)"] \\ \mathbf{1} \arrow[ur, "d_1"] \arrow[dr, "d_1'"'] \arrow[r, phantom, "\iota_1 \Swarrow"] & {} \\ & F(c') \end{tikzcd} \qquad \begin{tikzcd} & F(e) \arrow[d, "F(h_2)"] \\ \mathbf{1} \arrow[ur, "d_2"] \arrow[dr, "d_2'"'] \arrow[r, phantom, "\iota_2 \Swarrow"] & {} \\ & F(e') \end{tikzcd}$$

and then multiply these viewed as two 2-morphisms in **Cat** which results in:

$$\begin{tikzcd} & & F(c) \times F(e) \arrow[r, "\phi_{c,e}"] \arrow[dd, "F(h_1) \times F(h_2)"] & F(c+e) \arrow[r, "F(j_{c,e})"] \arrow[dd, "F(h_1+h_2)"] & F(c +_b e) \arrow[dd, "F(h_1 +_g h_2)"] \\ \mathbf{1} \arrow[r, "\lambda^{-1}"] & \mathbf{1} \times \mathbf{1} \arrow[ur, "d_1 \times d_2"] \arrow[dr, "d_1' \times d_2'"'] \arrow[r, phantom, "\iota_1 \times \iota_2 \Swarrow"] & {} & & \\ & & F(c') \times F(e') \arrow[r, "\phi_{c',e'}"] & F(c'+e') \arrow[r, "F(j_{c',e'})"] & F(c' +_{b'} e') \end{tikzcd}$$

where the middle square commutes since F is a lax monoidal pseudofunctor and the right square commutes as the underlying diagram commutes. The objects $d_{1,2}$ and $d_{1,2}'$ are given by top and bottom composite of arrows and the morphism of decorations $\iota_{1,2}$ is given by composing $\iota_1 \times \iota_2$ with the two commuting squares. Returning the interchange law, composing the two horizontal compositions above vertically then results in:

$$\begin{tikzcd}
a \arrow[r, "j\psi_a i_1"] \arrow[d, "f'f"'] & c +_b e \arrow[d, "(h'_1 +_{g'} h'_2)(h_1 +_g h_2)"'] & x \arrow[l, "j\psi_x o_2"'] \arrow[d, "k'k"'] & d_{1,2} \in F(c +_b e) \\
a'' \arrow[r, "j\psi_{a''} i''_1"] & c'' +_{b''} e'' & x'' \arrow[l, "j\psi_{x''} o''_2"'] & d''_{1,2} \in F(c'' +_{b''} e'')
\end{tikzcd}$$
$$(\iota'_{1,2}\iota_{1,2})\colon F((h'_1 +_{g'} h'_2)(h_1 +_g h_2))(d_{1,2}) \to d''_{1,2}.$$

F being pseudo might do something here... The vertical composite of two morphisms of decorations is straightforward. On the other hand, if we first compose vertically we obtain:

$$\begin{tikzcd}
a \arrow[r, "i_1"] \arrow[d, "f'f"'] & c \arrow[d, "h'_1 h_1"'] & b \arrow[l, "o_1"'] \arrow[d, "g'g"'] & d_1 \in F(c) \\
a'' \arrow[r, "i''_1"] & c'' & b'' \arrow[l, "o''_1"'] & d''_1 \in F(c'')
\end{tikzcd}
\qquad
\begin{tikzcd}
b \arrow[r, "i_2"] \arrow[d, "g'g"'] & e \arrow[d, "h'_2 h_2"'] & x \arrow[l, "o_2"'] \arrow[d, "k'k"'] & d_2 \in F(e) \\
b'' \arrow[r, "i''_2"] & e'' & x'' \arrow[l, "o''_2"'] & d''_2 \in F(e'')
\end{tikzcd}$$
$$\iota'_1\iota_1\colon F(h'_1 h_1)(d_1) \to d''_1 \qquad\qquad \iota'_2\iota_2\colon F(h'_2 h_2)(d_2) \to d''_2$$

and then composing horizontally results in:

$$\begin{tikzcd}
a \arrow[r, "j\psi_a i_1"] \arrow[d, "f'f"'] & c +_b e \arrow[d, "(h'_1 h_1) +_{g'g} (h'_2 h_2)"'] & x \arrow[l, "j\psi_x o_2"'] \arrow[d, "k'k"'] & d_{1,2} \in F(c +_b e) \\
a'' \arrow[r, "j\psi_{a''} i''_1"] & c'' +_{b''} e'' & x'' \arrow[l, "j\psi_{x''} o''_2"'] & d''_{1,2} \in F(c'' +_{b''} e'')
\end{tikzcd}$$
$$(\iota'_1\iota_1, \iota'_2\iota_2)\colon F((h'_1 h_1) +_{g'g} (h'_2 h_2))(d_{1,2}) \to d''_{1,2}.$$

As is usual concerning the interchange law of double categories of this nature, only the 'interior' of the two compositions appears different, but the two morphisms $(h'_1 +_{g'} h'_2)(h_1 +_g h_2)\colon c +_b e \to c'' +_{b''} e''$ and $(h'_1 h_1) +_{g'g} (h'_2 h_2)\colon c +_b e \to c'' +_{b''} e''$ are the same universal map realized in two different ways. The two morphisms of decorations are two compositions of 2-morphisms in **Cat** for which the interchange law already holds, and as a result, the morphisms

$$(\iota'_{1,2}\iota_{1,2})\colon F((h'_1 +_{g'} h'_2)(h_1 +_g h_2))(d_{1,2}) \to d''_{1,2}$$

and

$$(\iota'_1\iota_1, \iota'_2\iota_2)\colon F((h'_1 h_1) +_{g'g} (h'_2 h_2))(d_{1,2}) \to d''_{1,2}$$

are also the same. □

Theorem 3.8. *Let* $\mathbf{C}$ *be a category with finite colimits and* $F\colon \mathbf{C} \to \mathbf{Cat}$ *a symmetric lax monoidal pseudofunctor. Then the double category* $F\mathrm{Cospan}(\mathbf{C})$ *is symmetric monoidal.*

Proof. First we note that the category of objects $F\mathrm{Cospan}(\mathbf{C})_0 = \mathbf{C}$ is symmetric monoidal under binary coproducts and the left and right unitors, associators and braidings are given as natural maps. The category of arrows $F\mathrm{Cospan}(\mathbf{C})_1$ has:

(i) objects as pairs:

$$\begin{tikzcd} & c & \\ a \arrow[ur, "i"] & & b \arrow[ul, "o"'] \end{tikzcd} \qquad d \in F(c)$$

and

(ii) morphisms as maps of cospans in $\mathbf{C}$

$$\begin{tikzcd} a \arrow[r, "i"] \arrow[d, "f"'] & c \arrow[d, "h"] & b \arrow[l, "o"'] \arrow[d, "g"] \\ a' \arrow[r, "i'"] & c' & b' \arrow[l, "o'"'] \end{tikzcd} \qquad \begin{matrix} d \in F(c) \\ \\ d' \in F(c') \end{matrix}$$

together with elements of the images of the apices under the pseudofunctor F and a morphism $\iota\colon F(h)(d) \to d'$.

Given two objects M_1 and M_2 of $\mathbb{F}\mathrm{Cospan}(\mathbf{C})_1$:

$$\begin{tikzcd} & c_1 & \\ a_1 \arrow[ur, "i_1"] & & b_1 \arrow[ul, "o_1"'] \end{tikzcd} \qquad \begin{tikzcd} & c_2 & \\ a_2 \arrow[ur, "i_2"] & & b_2 \arrow[ul, "o_2"'] \end{tikzcd}$$

$$d_1 \in F(c_1) \qquad\qquad d_2 \in F(c_2)$$

their tensor product $M_1 \otimes M_2$ is given by taking the coproducts of the cospans of $\mathbf{C}$

$$\begin{tikzcd} & c_1 + c_2 & \\ a_1 + a_2 \arrow[ur, "i_1 + i_2"] & & b_1 + b_2 \arrow[ul, "o_1 + o_2"'] \end{tikzcd} \qquad d_{M_1 \otimes M_2} \in F(c_1 + c_2)$$

and where the element of the image of the apex under F is obtained using the natural transformation of the symmetric lax monoidal pseudofunctor F :

$$1 \xrightarrow{\lambda^{-1}} 1 \times 1 \xrightarrow{d_1 \times d_2} F(c_1) \times F(c_2) \xrightarrow{\phi_{c_1,c_2}} F(c_1 + c_2).$$

The monoidal unit is given by:

$$\begin{tikzcd} & 0 & \\ 0 \arrow[ur, "!"] & & 0 \arrow[ul, "!"'] \end{tikzcd} \qquad ! \in F(0)$$

where 0 is the monoidal unit of $\mathbf{C}$ and $!\colon \mathbf{1} \to F(0)$ is the morphism which is part of the structure of the symmetric lax monoidal pseudofunctor $F\colon \mathbf{C} \to \mathbf{Cat}$. Tensoring an object with the monoidal unit, say, on the left:

$$\begin{tikzcd} & 0 & \\ 0 \arrow[ur, "!"] & & 0 \arrow[ul, "!"'] \end{tikzcd} \qquad \begin{tikzcd} & c & \\ a \arrow[ur, "i"] & & b \arrow[ul, "o"'] \end{tikzcd}$$

$$! \in F(0) \qquad\qquad d \in F(c)$$

results in:

$$\begin{tikzcd} & 0+c & \\ 0+a \arrow[ur, "!+i"] & & 0+b \arrow[ul, "!+o"'] \end{tikzcd} \qquad d' \in F(0+c)$$

where $d' \in F(0+c)$ is given by

$$1 \xrightarrow{\lambda^{-1}} 1 \times 1 \xrightarrow{\phi \times d} F(0) \times F(c) \xrightarrow{\phi_{0,d}} F(0+c).$$

The left unitor is then a morphism in $\mathbb{F}\mathrm{Cospan}(\mathbf{C})_1$ given by:

$$\begin{tikzcd} 0+a \arrow[r, "!+i"] \arrow[d, "\ell"'] & 0+c \arrow[d, "\ell"'] & 0+b \arrow[l, "!+o"'] \arrow[d, "\ell"'] \\ a \arrow[r, "i"] & c & b \arrow[l, "o"'] \end{tikzcd} \qquad \begin{matrix} d' \in F(0+c) \\ \\ d \in F(c) \end{matrix}$$

where ℓ is the left unitor of $(\mathbf{C}, +, 0)$, together with the morphism $\iota_\lambda \colon F(\ell)(d') \to d$. The right unitor is similar.

Given three objects M_1, M_2 and M_3 in $\mathbb{F}\mathrm{Cospan}(\mathbf{C})_1$:

$$\begin{tikzcd} & c_1 & \\ a_1 \arrow[ur, "i_1"] & & b_1 \arrow[ul, "o_1"'] \end{tikzcd} \qquad \begin{tikzcd} & c_2 & \\ a_2 \arrow[ur, "i_2"] & & b_2 \arrow[ul, "o_2"'] \end{tikzcd} \qquad \begin{tikzcd} & c_3 & \\ a_3 \arrow[ur, "i_3"] & & b_3 \arrow[ul, "o_3"'] \end{tikzcd}$$

$$d_1 \in F(c_1) \qquad\qquad d_2 \in F(c_2) \qquad\qquad d_3 \in F(c_3)$$

tensoring the first two and then the third results in $(M_1 \otimes M_2) \otimes M_3$:

$$\begin{tikzcd} & (c_1+c_2)+c_3 & \\ (a_1+a_2)+a_3 \arrow[ur, "(i_1+i_2)+i_3"] & & (b_1+b_2)+b_3 \arrow[ul, "(o_1+o_2)+o_3"'] \end{tikzcd} \qquad d_{(M_1 \otimes M_2) \otimes M_3} \in F((c_1+c_2)+c_3)$$

where $d_{(M_1 \otimes M_2) \otimes M_3} \colon \mathbf{1} \to F((c_1+c_2)+c_3)$ is given by:

$$1 \xrightarrow{((d_1 \times d_2) \times d_3)} (F(c_1) \times F(c_2)) \times F(c_3) \xrightarrow{\phi_{c_1,c_2} \times 1} F(c_1+c_2) \times F(c_3) \xrightarrow{\phi_{c_1+c_2,c_3}} F((c_1+c_2)+c_3)$$

whereas tensoring the last two and then the first results in $M_1 \otimes (M_2 \otimes M_3)$:

$$\begin{tikzcd} & c_1 + (c_2 + c_3) & \\ a_1 + (a_2 + a_3) \arrow[ur, "i_1 + (i_2 + i_3)"] & & b_1 + (b_2 + b_3) \arrow[ul, "o_1 + (o_2 + o_3)"'] \end{tikzcd} \qquad d_{M_1\otimes(M_2\otimes M_3)} \in F(c_1 + (c_2 + c_3))$$

where $d_{M_1\otimes(M_2\otimes M_3)} \colon \mathbf{1} \to F(c_1 + (c_2 + c_3))$ is given by:

$$1 \xrightarrow{d_1\times(d_2\times d_3)} F(c_1) \times (F(c_2) \times F(c_3)) \xrightarrow{1\times\phi_{c_2,c_3}} F(c_1) \times F(c_2 + c_3) \xrightarrow{\phi_{c_1,c_2+c_3}} F(c_1 + (c_2 + c_3)).$$

If we let a denote the associator of $(\mathbf{C}, +, 0)$, the associator of $\mathbb{F}\mathrm{Cospan}(\mathbf{C})_1$ is then a map of cospans in $\mathbf{C}$ from $(M_1 \otimes M_2) \otimes M_3)$ to $M_1 \otimes (M_2 \otimes M_3)$ given by:

$$\begin{tikzcd} (a_1 + a_2) + a_3 \arrow[r, "(i_1 + i_2) + i_3"] \arrow[d, "a"'] & (c_1 + c_2) + c_3 \arrow[d, "a"'] & (b_1 + b_2) + b_3 \arrow[l, "(o_1 + o_2) + o_3"'] \arrow[d, "a"'] & d_{(M_1\otimes M_2)\otimes M_3} \in F((c_1 + c_2) + c_3) \\ a_1 + (a_2 + a_3) \arrow[r, "i_1 + (i_2 + i_3)"] & c_1 + (c_2 + c_3) & b_1 + (b_2 + b_3) \arrow[l, "o_1 + (o_2 + o_3)"'] & d_{M_1\otimes(M_2\otimes M_3)} \in F(c_1 + (c_2 + c_3)) \end{tikzcd}$$

together with the morphism $\iota_a \colon F(a)(d_{(M_1\otimes M_2)\otimes M_3}) \to d_{M_1\otimes(M_2\otimes M_3)}$. If we denote the above associator simply as a and the left and right unitors as λ and ρ, respectively, then given four objects in $F\mathrm{Cospan}(\mathbf{C})_1$, say M_1, M_2, M_3 and M_4:

$$\begin{tikzcd} & c_1 & \\ a_1 \arrow[ur, "i_1"] & & b_1 \arrow[ul, "o_1"'] \end{tikzcd} \qquad \begin{tikzcd} & c_2 & \\ a_2 \arrow[ur, "i_2"] & & b_2 \arrow[ul, "o_2"'] \end{tikzcd}$$

$$d_1 \in F(c_1) \qquad\qquad d_2 \in F(c_2)$$

$$\begin{tikzcd} & c_3 & \\ a_3 \arrow[ur, "i_3"] & & b_3 \arrow[ul, "o_3"'] \end{tikzcd} \qquad \begin{tikzcd} & c_4 & \\ a_4 \arrow[ur, "i_4"] & & b_4 \arrow[ul, "o_4"'] \end{tikzcd}$$

$$d_3 \in F(c_3) \qquad\qquad d_4 \in F(c_4)$$

then the following pentagon of underlying cospans commutes:

$$\begin{tikzcd}
 & & (M_1 \otimes M_2) \otimes (M_3 \otimes M_4) \arrow[drr, "a"] & & \\
((M_1 \otimes M_2) \otimes M_3) \otimes M_4 \arrow[urr, "a"] \arrow[dr, "a \otimes 1"'] & & & & M_1 \otimes (M_2 \otimes (M_3 \otimes M_4)) \\
 & (M_1 \otimes (M_2 \otimes M_3)) \otimes M_4 \arrow[rr, "a"] & & M_1 \otimes ((M_2 \otimes M_3) \otimes M_4) \arrow[ur, "1 \otimes a"'] &
\end{tikzcd}$$

as well as the following pentagon of corresponding decorations in the category $F(c_1 + (c_2 + (c_3 + c_4)))$:

$$\begin{tikzcd}
 & F(a)(d_{(M_1 \otimes M_2) \otimes (M_3 \otimes M_4)}) \arrow[dr, "\iota_a"] & \\
F(aa)(d_{((M_1 \otimes M_2) \otimes M_3) \otimes M_4}) \arrow[ur, "F(a)(\iota_a)"] \arrow[d, "F((1 \otimes a)a)(\iota_{a \otimes 1})"'] & & d_{M_1 \otimes (M_2 \otimes (M_3 \otimes M_4))} \\
F((1 \otimes a)a)(d_{(M_1 \otimes (M_2 \otimes M_3)) \otimes M_4}) \arrow[rr, "F(1 \otimes a)(\iota_a)"] & & F(1 \otimes a)(d_{M_1 \otimes ((M_2 \otimes M_3) \otimes M_4)}) \arrow[u, "\iota_{1 \otimes a}"']
\end{tikzcd}$$

since

$$F(aa)(d_{((M_1 \otimes M_2) \otimes M_3) \otimes M_4}) = F((1 \otimes a)a(a \otimes 1))(d_{((M_1 \otimes M_2) \otimes M_3) \otimes M_4})$$

as the above underlying diagram of maps of cospans commutes.

Similarly, if we denote the left and right unitors as λ and ρ, respectively, then the following triangle of underlying maps of cospans commutes:

$$\begin{tikzcd}
 & M_1 \otimes M_2 & \\
(M_1 \otimes U_0) \otimes M_2 \arrow[ur, "\rho \otimes 1"] \arrow[rr, "a"] & & M_1 \otimes (U_0 \otimes M_2) \arrow[ul, "1 \otimes \lambda"']
\end{tikzcd}$$

as well as the following triangle of corresponding decorations in the category $F(c_1 + c_2)$:

$$\begin{tikzcd}
 & d_{M_1 \otimes M_2} & \\
F(\rho \otimes 1)(d_{(M_1 \otimes U_0) \otimes M_2}) \arrow[ur, "\iota_{\rho \otimes 1}"] \arrow[rr, "F(\iota_a)"] & & F(1 \otimes \lambda)(d_{M_1 \otimes (U_0 \otimes M_2)}) \arrow[ul, "\iota_{1 \otimes \lambda}"']
\end{tikzcd}$$

since

$$F(\rho \otimes 1)(d_{(M_1 \otimes U_0) \otimes M_2}) = F(a \otimes (1 \otimes \lambda))(d_{(M_1 \otimes U_0) \otimes M_2})$$

as the above underlying diagram of maps of cospans commutes.

For a monoidal product of objects $M_1 \otimes M_2$ in $\mathbb{F}\mathrm{Cospan}(\mathbf{C})_1$, the source and target structure functors $S, T \colon \mathbb{F}\mathrm{Cospan}(\mathbf{C})_1 \to \mathbb{F}\mathrm{Cospan}(\mathbf{C})_0$ satisfy the following equations:

$$S(M_1 \otimes M_2) = S(M_1) \otimes S(M_2)$$

$$T(M_1 \otimes M_2) = T(M_1) \otimes T(M_2).$$

For two objects M_1 and M_2 in $\mathbb{F}\mathrm{Cospan}(\mathbf{C})_1$, we have a braiding $\beta_{M_1,M_2} \colon M_1 \otimes M_2 \to M_2 \otimes M_1$ given by:

$$\begin{tikzcd}
a_1 + a_2 \arrow[r, "i_1 + i_2"] \arrow[d, "\beta_{a_1,a_2}"'] & c_1 + c_2 \arrow[d, "\beta_{c_1,c_2}"'] & b_1 + b_2 \arrow[l, "o_1 + o_2"'] \arrow[d, "\beta_{b_1,b_2}"'] & d_{M_1 \otimes M_2} \in F(c_1 + c_2) \\
a_2 + a_1 \arrow[r, "i_2 + i_1"] & c_2 + c_1 & b_2 + b_1 \arrow[l, "o_2 + o_1"'] & d_{M_2 \otimes M_1} \in F(c_2 + c_1)
\end{tikzcd}$$

$$\iota_{\beta_{M_1,M_2}} \colon F(\beta_{c_1,c_2})(d_{M_1 \otimes M_2}) \xrightarrow{\sim} d_{M_2 \otimes M_1}.$$

This braiding makes the following triangle of underlying cospans commute:

$$\begin{tikzcd}
 & M_1 \otimes M_2 & \\
M_1 \otimes M_2 \arrow[ur, "1"] \arrow[rr, "\beta_{M_1,M_2}"] & & M_2 \otimes M_1 \arrow[ul, "\beta_{M_2,M_1}"']
\end{tikzcd}$$

as well as the following diagram of corresponding decorations in the category $F(c_2 + c_1)$:

$$\begin{tikzcd}
 & F(\beta_{c_1,c_2})(d_{M_1 \otimes M_2}) & \\
F(\beta_{c_1,c_2})(d_{M_1 \otimes M_2}) \arrow[ur, "1"] \arrow[rr, "\iota_{\beta_{M_1,M_2}}"] & & d_{M_2 \otimes M_1} \arrow[ul, "F(\beta_{c_1,c_2})(\iota_{\beta_{M_2,M_1}})"']
\end{tikzcd}$$

since $F(\beta_{c_1,c_2}\beta_{c_2,c_1})(d_{M_2 \otimes M_1}) = d_{M_2 \otimes M_1}$. Thus $\mathbb{F}\mathrm{Cospan}(\mathbf{C})_1$ is also symmetric monoidal.

Now, given four horizontal 1-cells M_1, M_2, N_1 and N_2 respectively by:

$$\begin{tikzcd} & c_1 & \\ a_1 \arrow[ur, "i_1"] & & b_1 \arrow[ul, "o_1"'] \end{tikzcd} \qquad \begin{tikzcd} & e_1 & \\ b_1 \arrow[ur, "i'_1"] & & x_1 \arrow[ul, "o'_1"'] \end{tikzcd}$$

$$d_{M_1} \in F(c_1) \qquad d_{M_2} \in F(e_1)$$

$$\begin{tikzcd} & c_2 & \\ a_2 \arrow[ur, "i_2"] & & b_2 \arrow[ul, "o_2"'] \end{tikzcd} \qquad \begin{tikzcd} & e_2 & \\ b_2 \arrow[ur, "i'_2"] & & x_2 \arrow[ul, "o'_2"'] \end{tikzcd}$$

$$d_{N_1} \in F(c_2) \qquad d_{N_2} \in F(e_2)$$

we have that $(M_1 \otimes N_1) \odot (M_2 \otimes N_2)$ is given by:

$$\begin{tikzcd} & (c_1 + c_2) +_{b_1+b_2} (e_1 + e_2) & \\ a_1 + a_2 \arrow[ur, "j\psi(i_1 + i_2)"] & & x_1 + x_2 \arrow[ul, "j\psi(o'_1 + o'_2)"'] \end{tikzcd}$$

$$d_{(M_1 \otimes N_1) \odot (M_2 \otimes N_2)} \in F((c_1 + c_2) +_{b_1+b_2} (e_1 + e_2))$$

and $(M_1 \odot M_2) \otimes (N_1 \odot N_2)$ is given by:

$$\begin{tikzcd} & (c_1 +_{b_1} e_1) + (c_2 +_{b_2} e_2) & \\ a_1 + a_2 \arrow[ur, "(j\psi i_2) + (j\psi i_1)"] & & x_1 + x_2 \arrow[ul, "(j\psi o'_2) + (j\psi o'_1)"'] \end{tikzcd}$$

$$d_{(M_1 \odot M_2) \otimes (N_1 \odot N_2)} \in F((c_1 +_{b_1} e_1) + (c_2 +_{b_2} e_2))$$

where ψ and j are the natural maps into a coproduct and from a coproduct into a pushout, respectively. We then get a globular 2-morphism

$$\chi\colon (M_1 \otimes N_1) \odot (M_2 \otimes N_2) \to (M_1 \odot M_2) \otimes (N_1 \odot N_2)$$

given by:

$$\hat{d}_1 \in F((c_1 + c_2) +_{b_1+b_2} (e_1 + e_2))$$

$$\begin{tikzcd} a_1 + a_2 \arrow[r, "j\psi(i_1 + i_2)"] \arrow[d, "1"'] & (c_1 + c_2) +_{b_1+b_2} (e_1 + e_2) \arrow[d, "\hat{\chi}"'] & b_1 + b_2 \arrow[l, "j\psi(o'_1 + o'_2)"'] \arrow[d, "1"'] \\ a_1 + a_2 \arrow[r, "(j\psi i_2) + (j\psi i_1)"] & (c_1 +_{b_1} e_1) + (c_2 +_{b_2} e_2) & b_1 + b_2 \arrow[l, "(j\psi o'_2) + (j\psi o'_1)"'] \end{tikzcd}$$

$$\hat{d}_2 \in F((c_1 +_{b_1} e_1) + (c_2 +_{b_2} e_2))$$

$$\iota_{\hat{\chi}}\colon F(\hat{\chi})(d_{(M_1 \otimes N_1) \odot (M_2 \otimes N_2)}) \to d_{(M_1 \odot M_2) \otimes (N_1 \odot N_2)}.$$

For two objects $a, b \in \mathbf{C}$, U_{a+b} is given by:

$$\begin{tikzcd} & a+b & \\ a+b \arrow[ur, "1_{a+b}"] & & a+b \arrow[ul, "1_{a+b}"'] \end{tikzcd} \qquad !_{a+b} \in F(a+b)$$

where

$$!_{a+b} \colon 1 \xrightarrow{\lambda^{-1}} 1 \times 1 \xrightarrow{\phi \times \phi} F(0) \times F(0) \xrightarrow{F(!_a) \times F(!_b)} F(a) \times F(b) \xrightarrow{\phi_{a,b}} F(a+b).$$

Similarly, we have U_a and U_b given respectively by:

$$\begin{tikzcd} & a & \\ a \arrow[ur, "1_a"] & & a \arrow[ul, "1_a"'] \end{tikzcd} \qquad \begin{tikzcd} & b & \\ b \arrow[ur, "1_b"] & & b \arrow[ul, "1_b"'] \end{tikzcd}$$

$$!_a \in F(a) \qquad\qquad !_b \in F(b)$$

and then $U_a + U_b$ is given by:

$$\begin{tikzcd} & a+b & \\ a+b \arrow[ur, "1_a + 1_b"] & & a+b \arrow[ul, "1_a + 1_b"'] \end{tikzcd} \qquad !_a + !_b \in F(a+b)$$

where

$$!_a + !_b \colon 1 \xrightarrow{\lambda^{-1}} 1 \times 1 \xrightarrow{!_a \times !_b} F(a) \times F(b) \xrightarrow{\phi_{a,b}} F(a+b).$$

We then have another globular isomorphism

$$\mu_{a,b} \colon U_{a+b} \to U_a + U_b$$

given by the identity 2-morphism:

$$\begin{tikzcd} a+b \arrow[r, "1_{a+b}"] \arrow[d, "1"'] & a+b \arrow[d, "1"'] & a+b \arrow[l, "1_{a+b}"'] \arrow[d, "1"'] & !_{a+b} \in F(a+b) \\ a+b \arrow[r, "1_a + 1_b"] & a+b & a+b \arrow[l, "1_a + 1_b"'] & !_a + !_b \in F(a+b) \end{tikzcd}$$

$$\iota_{a,b} \colon !_{a+b} \xrightarrow{\sim} !_a + !_b.$$

There are a fair amount of coherence diagrams to verify, many of which are similar in flavor and make use of the two above globular ismorphisms. We check a few to give a sense of what these are like. For example, given horizontal 1-cells M_i, N_i, P_i, the following commutative diagram expresses the associativity isomorphism as a transformation of double categories.

$$
\begin{tikzcd}
((M_1 \otimes N_1) \otimes P_1) \odot ((M_2 \otimes N_2) \otimes P_2) \arrow[r, "a \odot a"] \arrow[d, "\mu"'] & (M_1 \otimes (N_1 \otimes P_1)) \odot (M_2 \otimes (N_2 \otimes P_2)) \arrow[d, "\mu"] \\
((M_1 \otimes N_1) \odot (M_2 \otimes N_2)) \otimes (P_1 \odot P_2) \arrow[d, "\mu \otimes 1"'] & (M_1 \odot M_2) \otimes ((N_1 \otimes P_1) \odot (N_2 \otimes P_2)) \arrow[d, "1 \otimes \mu"] \\
((M_1 \odot M_2) \otimes (N_1 \odot N_2)) \otimes (P_1 \odot P_2) \arrow[r, "a"] & (M_1 \odot M_2) \otimes ((N_1 \odot N_2) \otimes (P_1 \odot P_2))
\end{tikzcd}
$$

Here, a is the associator of $\mathbb{F}\mathrm{Cospan}(\mathbf{C})_1$ and μ is the first globular isomorphism above. To see that this diagram does indeed commute, we first consider this diagram with respect to only the underlying cospans of each horizontal 1-cell. For notation:

$$
M_1 = \begin{tikzcd}[column sep=small] & l & \\ k \arrow[ur] & & m \arrow[ul] \\ & d_{M_1} \in F(l) & \end{tikzcd}
\quad
N_1 = \begin{tikzcd}[column sep=small] & r & \\ q \arrow[ur] & & s \arrow[ul] \\ & d_{N_1} \in F(r) & \end{tikzcd}
\quad
P_1 = \begin{tikzcd}[column sep=small] & w & \\ v \arrow[ur] & & x \arrow[ul] \\ & d_{P_1} \in F(w) & \end{tikzcd}
$$

$$
M_2 = \begin{tikzcd}[column sep=small] & n & \\ m \arrow[ur] & & p \arrow[ul] \\ & d_{M_2} \in F(n) & \end{tikzcd}
\quad
N_2 = \begin{tikzcd}[column sep=small] & t & \\ s \arrow[ur] & & u \arrow[ul] \\ & d_{N_2} \in F(t) & \end{tikzcd}
\quad
P_2 = \begin{tikzcd}[column sep=small] & y & \\ x \arrow[ur] & & z \arrow[ul] \\ & d_{P_2} \in F(y) & \end{tikzcd}
$$

The above diagram then becomes:

$$
\begin{tikzcd}
k+m \arrow[r] & (l +_m n) + ((r +_s t) + (w +_x y)) & v+x \arrow[l] \\
k+m \arrow[r] \arrow[u] & (l +_m n) + ((r + w) +_{(s+x)} (t + y)) \arrow[u, "1 \otimes \mu"', "\iota_3"] & v+x \arrow[l] \arrow[u] \\
k+m \arrow[r] \arrow[u] & (l + (r + w)) +_{(m+(s+x))} (n + (t + y)) \arrow[u, "\mu"', "\iota_2"] & v+x \arrow[l] \arrow[u] \\
k+m \arrow[r] \arrow[u] \arrow[d] & ((l + r) + w) +_{((m+s)+x)} ((n + t) + y) \arrow[u, "a \odot a"', "\iota_1"] \arrow[d, "\mu"', "\iota_4"] & v+x \arrow[l] \arrow[u] \arrow[d] \\
k+m \arrow[r] \arrow[d] & ((l + r) +_{(m+s)} (n + t)) + (w +_x y) \arrow[d, "\mu \otimes 1"', "\iota_5"] & v+x \arrow[l] \arrow[d] \\
k+m \arrow[r] \arrow[d] & ((l +_m n) + (r +_s t)) + (w +_x y) \arrow[d, "a"', "\iota_6"] & v+x \arrow[l] \arrow[d] \\
k+m \arrow[r] & (l +_m n) + ((r +_s t) + (w +_x y)) & v+x \arrow[l]
\end{tikzcd}
$$

Here all of the vertical 1-morphisms on the left and right are identities, the middle vertical 1-morphisms are the 2-morphisms from the previous commutative diagram, and the horizontal vertical 1-morphisms pointing towards the middle are natural maps into each colimit, all of which are naturally isomorphic to each other as all the middle objects are colimits of the same diagram, namely the previous collection of cospans, taken in various ways. The above diagram of maps of cospans can then be visualized as a hexagonal prism in which all the faces commute by identifying the top and the bottom as the same. As for the decorations, each isomorphism ι_n goes from the domain under the image of the functor F applied to natural isomorphism adjacent to it to the codomain as written, meaning that, for example:

$$\iota_1 \colon F(a \odot a)(d_{((M_1 \otimes N_1) \otimes P_1) \odot ((M_2 \otimes N_2) \otimes P_2)}) \to d_{(M_1 \otimes (N_1 \otimes P_1)) \odot (M_2 \otimes (N_2 \otimes P_2))}.$$

The following diagram commutes in the category $F((l +_m n) + ((r +_s t) + (w +_x y)))$:

$$\begin{tikzcd}
F(a(\mu \otimes 1)\mu)(d_{((M_1 \otimes N_1) \otimes P_1) \odot ((M_2 \otimes N_2) \otimes P_2)}) \arrow[r, "F((1 \otimes \mu)\mu)(\iota_1)"] \arrow[d, "F(a(\mu \otimes 1))(\iota_4)"'] & F((1 \otimes \mu)\mu)(d_{(M_1 \otimes (N_1 \otimes P_1)) \odot (M_2 \otimes (N_2 \otimes P_2))}) \arrow[d, "F(1 \otimes \mu)(\iota_2)"] \\
F(a(\mu \otimes 1))(d_{((M_1 \otimes N_1) \odot (M_2 \otimes N_2)) \otimes (P_1 \odot P_2)}) \arrow[d, "F(a)(\iota_5)"'] & F(1 \otimes \mu)(d_{(M_1 \odot M_2) \otimes ((N_1 \otimes P_1) \odot (N_2 \otimes P_2))}) \arrow[d, "\iota_3"] \\
F(a)(d_{((M_1 \odot M_2) \otimes (N_1 \odot N_2)) \otimes (P_1 \odot P_2)}) \arrow[r, "\iota_6"] & d_{(M_1 \odot M_2) \otimes ((N_1 \odot N_2) \otimes (P_1 \odot P_2))}
\end{tikzcd}$$

since

$$F(a(\mu \otimes 1)\mu)(d_{((M_1 \otimes N_1) \otimes P_1) \odot ((M_2 \otimes N_2) \otimes P_2)}) = F((1 \otimes \mu)\mu(a \odot a))(d_{((M_1 \otimes N_1) \otimes P_1) \odot ((M_2 \otimes N_2) \otimes P_2)})$$

as the above underlying diagram of maps of cospans commutes.

Another requirement for a double category to be symmetric monoidal is that the braiding

$$\beta_{(-,-)} \colon \mathbb{F}\mathrm{Cospan}(\mathbf{C})_1 \times \mathbb{F}\mathrm{Cospan}(\mathbf{C})_1 \to \mathbb{F}\mathrm{Cospan}(\mathbf{C})_1 \times \mathbb{F}\mathrm{Cospan}(\mathbf{C})_1$$

be a transformation of double categories, and one of the diagrams that is required to commute is the following:

$$\begin{tikzcd}
(M_1 \odot M_2) \otimes (N_1 \odot N_2) \arrow[r, "\beta"] \arrow[d, "\mu"'] & (N_1 \odot N_2) \otimes (M_1 \odot M_2) \arrow[d, "\mu"] \\
(M_1 \otimes N_1) \odot (M_2 \otimes N_2) \arrow[r, "\beta \odot \beta"] & (N_1 \otimes M_1) \odot (N_2 \otimes M_2)
\end{tikzcd}$$

Using the same notation as the previous coherence diagram, the diagram for the underlying maps of cospans becomes:

$$\begin{tikzcd}
k+m \arrow[r] & (r+l)+_{(s+m)}(t+n) & s+u \arrow[l] \\
k+m \arrow[r] \arrow[u] & (r+_s t)+(l+_m n) \arrow[u, "\mu", "\iota_2"'] & s+u \arrow[l] \arrow[u] \\
k+m \arrow[r] \arrow[u] \arrow[d] & (l+_m n)+(r+_s t) \arrow[u, "\beta", "\iota_1"'] \arrow[d, "\mu"', "\iota_4"] & s+u \arrow[l] \arrow[u] \arrow[d] \\
k+m \arrow[r] \arrow[d] & (l+r)+_{(m+s)}(n+t) \arrow[d, "\beta\odot\beta"', "\iota_5"] & s+u \arrow[l] \arrow[d] \\
k+m \arrow[r] & (r+l)+_{(s+m)}(t+n) & s+u \arrow[l]
\end{tikzcd}$$

All the comments about the previous underlying coherence diagram of maps of cospans apply to this one. As for the decorations, the following diagram commutes in the category $F((r+l)+_{(s+m)}(t+n))$:

$$\begin{tikzcd}
F(\mu\beta)(d_{(M_1\odot M_2)\otimes(N_1\odot N_2)}) \arrow[r, "F(\mu)(\iota_1)"] \arrow[d, "F(\beta\odot\beta)(\iota_3)"'] & F(\mu)(d_{(N_1\odot N_2)\otimes(M_1\odot M_2)}) \arrow[d, "\iota_2"] \\
F(\beta\odot\beta)(d_{(M_1\otimes N_1)\odot(M_2\otimes N_2)}) \arrow[r, "\iota_4"] & d_{(N_1\otimes M_1)\odot(N_2\otimes M_2)}
\end{tikzcd}$$

since

$$F(\mu\beta)(d_{(M_1\odot M_2)\otimes(N_1\odot N_2)}) = F((\beta\odot\beta)\mu)(d_{(M_1\odot M_2)\otimes(N_1\odot N_2)})$$

as the above underlying diagram of maps of cospans commutes. The other diagrams are shown to commute similarly. □

Lemma 3.9. *The double category* $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$ *is fibrant.*

Proof. Let $f\colon c \to c'$ be a vertical 1-morphism in $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$. We can lift f to the companion horizontal 1-cell $\hat{f}$:

$$\begin{tikzcd}
& c' & & \\
c \arrow[ur, "f"] & & c' \arrow[ul, "1"'] & !_{c'} \in F(c')
\end{tikzcd}$$

and then obtain the following two 2-morphisms:

$$\begin{tikzcd}
c \arrow[r, "f"] \arrow[d, "f"'] & c' \arrow[d, "1"'] & c' \arrow[l, "1"'] \arrow[d, "1"'] & !_{c'} \in F(c') \\
c' \arrow[r, "1"] & c' & c' \arrow[l, "1"'] & !_{c'} \in F(c')
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[r, "1"] \arrow[d, "1"'] & c \arrow[d, "f"'] & c \arrow[l, "1"'] \arrow[d, "f"'] & !_{c} \in F(c) \\
c \arrow[r, "f"] & c' & c' \arrow[l, "1"'] & !_{c'} \in F(c')
\end{tikzcd}$$

$$\iota_{c'} = 1_{!_{c'}} \qquad \qquad \iota_f \colon F(f)(!_c) \to !_{c'}$$

which satisfy the equations:

$$\begin{tikzcd}
!_c \in F(c) & c \arrow[r, "1"] \arrow[d, "1"'] & c \arrow[d, "f"'] & c \arrow[l, "1"'] \arrow[d, "f"'] \\
!_{c'} \in F(c') & c \arrow[r, "f"] \arrow[d, "f"'] & c' \arrow[d, "1"'] & c' \arrow[l, "1"'] \arrow[d, "1"'] \\
!_{c'} \in F(c') & c' \arrow[r, "1"] & c' & c' \arrow[l, "1"']
\end{tikzcd}
=
\begin{tikzcd}
c \arrow[r, "1"] \arrow[d, "f"'] & c \arrow[d, "f"'] & c \arrow[l, "1"'] \arrow[d, "f"'] & !_c \in F(c) \\
c' \arrow[r, "1"] & c' & c' \arrow[l, "1"'] & !_{c'} \in F(c')
\end{tikzcd}$$

$$\iota_f \colon F(f)(!_c) \to !_{c'} \qquad \qquad \iota_f \colon F(f)(!_c) \to !_{c'}$$

$$\iota_{c'} = 1_{!_{c'}}$$

$$\begin{array}{ccc} !_c \in F(c) & !_{c'} \in F(c') & !_{c'} \in F(c') \end{array}$$

$$\begin{tikzcd}
c \arrow[r, "1"] \arrow[d, "1"'] & c \arrow[d, "f"'] & c \arrow[l, "1"'] \arrow[d, "f"'] \arrow[r, "f"] & c' \arrow[d, "1"'] & c' \arrow[l, "1"'] \arrow[d, "1"'] \\
c \arrow[r, "f"] & c' & c' \arrow[l, "1"'] \arrow[r, "1"] & c' & c' \arrow[l, "1"']
\end{tikzcd}
=
\begin{tikzcd}
c \arrow[r, "f"] \arrow[d, "1"'] & c' \arrow[d, "1"'] & c' \arrow[l, "1"'] \arrow[d, "1"'] \\
c \arrow[r, "f"] & c' & c' \arrow[l, "1"']
\end{tikzcd}$$

$$\begin{array}{ccc} !_{c'} \in F(c') & !_{c'} \in F(c') & !_{c'} \in F(c') \\ \iota_f \colon F(f)(!_c) \to !_{c'} & \iota_{c'} = 1_{!_{c'}} & \iota_{c'} = 1_{!_{c'}} \end{array}$$

The right hand sides of the above two equations are given respectively by the 2-morphisms U_f and $1_{\hat{f}}$. The conjoint of f is given by the F-decorated cospan $\check{f}$ which is just the opposite of the companion above:

$$\begin{tikzcd}
& c' & & \\
c' \arrow[ur, "1"] & & c \arrow[ul, "f"'] & !_{c'} \in F(c')
\end{tikzcd}$$

$\square$

The property of being fibrant is what allows us to lift the monoidal structure from the object category of a double category to its arrow category and obtain a symmetric monoidal bicategory. The following result, which only requires fibrancy on vertical 1-isomorphisms (isofibrancy) is due to Shulman [15]:

Theorem 3.10 (Shulman)**.** *Let $\mathbb{D}$ be an isofibrant symmetric monoidal pseudo double category. Then the horizontal bicategory $H(\mathbb{D})$ of $\mathbb{D}$ is a symmetric monoidal bicategory which has:*

(i) objects as those of $\mathbb{D}$,
(ii) morphisms as horizontal 1-cells of $\mathbb{D}$, and
(iii) 2-morphisms as globular 2-morphisms of $\mathbb{D}$.

Theorem 3.11. *There exists a symmetric monoidal bicategory $H(\mathbb{F}\mathrm{Cospan}(\mathbf{C}))$ which has:*

(i) objects as those of $\mathbf{C}$,
(ii) morphisms as pairs:

$$\begin{tikzcd} & c & \\ a \arrow[ur, "i"] & & b \arrow[ul, "o"'] \end{tikzcd} \qquad d \in F(c)$$

and
(iii) 2-morphisms as maps of cospans in $\mathbf{C}$ of the form:

$$\begin{tikzcd} & c \arrow[dd, "h"] & \\ a \arrow[ur, "i"] \arrow[dr, "i'"'] & & b \arrow[ul, "o"'] \arrow[dl, "o'"] \\ & c' & \end{tikzcd} \qquad \begin{array}{l} d \in F(c) \\ \\ d' \in F(c') \end{array}$$

together with a morphism $\iota \colon F(h)(d) \to d'$ in $F(c')$.

Proof. This follows immediately by Shulman's Theorem 3.10 above. □

This symmetric monoidal bicategory is a superior version of the symmetric monoidal bicategory constructed earlier by the second author [9]. The previous bicategory constructed by the second author suffered even more so from the issue discussed in the introduction. Given a symmetric lax monoidal functor $F \colon \mathbf{C} \to \mathbf{Set}$, a result of Fong [12] yields a symmetric monoidal category $F\mathrm{Cospan}(\mathbf{C})$ which has:

(i) objects as those of $\mathbf{C}$ and
(ii) morphisms as isomorphism classes of decorated cospans in $\mathbf{C}$.

The second author then using a result of Shulman [15] extended this to a bicategory also called $F\mathrm{Cospan}(\mathbf{C})$ which has:

(i) objects as those of $\mathbf{C}$,
(ii) morphisms as now just decorated cospans in $\mathbf{C}$, and

(iii) 2-morphisms as pairs of commuting diagrams:

$$\begin{tikzcd} & c \arrow[dd, "f"] & \\ a \arrow[ur, "i"] \arrow[dr, "i'"'] & & b \arrow[ul, "o"'] \arrow[dl, "o'"] \\ & c' & \end{tikzcd} \qquad \begin{tikzcd} & F(c) \arrow[dd, "F(f)"] \\ \mathbf{1} \arrow[ur, "d"] \arrow[dr, "d'"'] & \\ & F(c') \end{tikzcd}$$

This was discussed in the introduction and it was mentioned how in the symmetric monoidal category version of $F\mathrm{Cospan}(\mathbf{C})$, that the two single-edged graphs:

$$v_1 \xrightarrow{e} v_2$$

and

$$v_1 \xrightarrow{e'} v_2.$$

resided in distinct isomorphism classes. In the bicategorical version of $F\mathrm{Cospan}(\mathbf{C})$ where we are no longer considering decorated cospans up to isomorphism class, there is no 2-morphism between these two graphs due to the strict commutativity of the triangle to the right above. This problem does not occur in the symmetric monoidal bicategory $H(\mathbb{F}\mathrm{Cospan}(\mathbf{C}))$ and there is in fact a 2-(iso)morphism between these two graphs given by the map $\iota\colon F(f)(d) \to d'$ which maps the edge e to the edge e', where f is the underlying map of vertices.

We can the decategorify this symmetric monoidal bicategory to obtain a symmetric monoidal category similar to the one obtained using Fong's result:

Corollary 3.12. *Given a symmetric lax monoidal pseudofunctor $F\colon \mathbf{C} \to \mathbf{Cat}$ where $\mathbf{C}$ is a category with finite colimits and whose monoidal structure is given by binary coproducts, there exists a symmetric monoidal category $D(H(F\mathrm{Cospan}(\mathbf{C})))$ which has:*

(i) objects as those of $\mathbf{C}$ and
(ii) morphisms as isomorphism classes of F-decorated cospans of $\mathbf{C}$, where an F-decorated cospan is given by a pair:

$$\begin{tikzcd} & c & \\ a \arrow[ur, "i"] & & b \arrow[ul, "o"'] \end{tikzcd} \qquad d \in F(c)$$

Given another F-decorated cospan:

$$\begin{tikzcd} a \arrow[dr, "i'"'] & & b \arrow[dl, "o'"] \\ & c' & \end{tikzcd} \qquad d' \in F(c')$$

these two F-decorated cospans are in the same isomorphism class if there exists an isomorphism $f\colon c \to c'$ such that following diagram commutes:

$$\begin{tikzcd} & c \arrow[dd, "f"] & \\ a \arrow[ur, "i"] \arrow[dr, "i'"'] & & b \arrow[ul, "o"'] \arrow[dl, "o'"] \\ & c' & \end{tikzcd}$$

and there exists an isomorphism $\iota\colon F(f)(d) \to d'$ in $F(c')$.

In this symmetric monoidal category, isomorphism classes are as they should morally be, and the instance of two graphs having different edge sets does not prevent them from being in the same isomorphism class due to the isomorphism ι.

4. Maps of decorated cospan double categories

Let $F\colon \mathbf{C} \to \mathbf{Cat}$ be a symmetric lax monoidal pseudofunctor. Then by Theorem 3.8 of the previous section, we get a symmetric monoidal double category $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$. This symmetric monoidal double category has:

(i) objects as those of $\mathbf{C}$,
(ii) vertical 1-morphisms as morphisms of $\mathbf{C}$,
(iii) horizontal 1-cells as pairs:

$$\begin{tikzcd} & c & \\ a \arrow[ur, "i"] & & b \arrow[ul, "o"'] \end{tikzcd} \qquad d \in F(c)$$

and

(iv) 2-morphisms as maps of cospans in $\mathbf{C}$

$$\begin{tikzcd} a \arrow[r] \arrow[d, "f"'] & c \arrow[d, "h"'] & b \arrow[l] \arrow[d, "g"'] \\ a' \arrow[r] & c' & b' \arrow[l] \end{tikzcd} \qquad \begin{matrix} d \in F(c) \\ \\ d' \in F(c') \end{matrix}$$

together with a morphism $\iota\colon F(h)(d) \to d'$ in $F(c')$.

Given another symmetric lax monoidal pseudofunctor $F'\colon \mathbf{C}' \to \mathbf{Cat}$, we can obtain another symmetric monoidal double category $\mathbb{F}'\mathrm{Cospan}(\mathbf{C}')$. Then a map from $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$ to $\mathbb{F}'\mathrm{Cospan}(\mathbf{C}')$ will be a double functor $\mathbb{H}\colon \mathbb{F}\mathrm{Cospan}(\mathbf{C}) \to \mathbb{F}'\mathrm{Cospan}(\mathbf{C}')$ whose object component is given by a finite colimit preserving functor $\mathbb{H}_0 = H\colon \mathbf{C} \to \mathbf{C}'$ and whose arrow component is given by a functor $\mathbb{H}_1$ defined on horizontal 1-cells by:

$$\begin{tikzcd}
 & c & & & & H(c) & \\
a \arrow[ur, "i"] & & b \arrow[ul, "o"'] & \mapsto & H(a) \arrow[ur, "H(i)"] & & H(b) \arrow[ul, "H(o)"'] \\
 & d \in F(c) & & & & E(d) \in F'(H(c)) &
\end{tikzcd}$$

and on 2-morphisms by:

$$\begin{tikzcd}
a \arrow[r] \arrow[d, "f"'] & c \arrow[d, "h"'] & b \arrow[l] \arrow[d, "g"'] & d \in F(c) & & H(a) \arrow[r] \arrow[d, "H(f)"'] & H(c) \arrow[d, "H(h)"'] & H(b) \arrow[l] \arrow[d, "H(g)"'] & E(d) \in F'(H(c)) \\
a' \arrow[r] & c' & b' \arrow[l] & d' \in F(c') & \mapsto & H(a') \arrow[r] & H(c') & H(b') \arrow[l] & E(d') \in F'(H(c'))
\end{tikzcd}$$

$$\iota\colon F(h)(d) \to d' \qquad\qquad E(\iota)\colon F'(H(h))(E(d)) \to E(d')$$

where $E\colon \mathbf{Cat} \to \mathbf{Cat}$ is a terminal category preserving 2-functor such that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "F"] \arrow[d, "H"'] & \mathbf{Cat} \arrow[d, "E"] \\
\mathbf{C}' \arrow[r, "F'"] & \mathbf{Cat}
\end{tikzcd}$$

Recall that we can think of the element $d \in F(c)$ as a morphism $d\colon \mathbf{1} \to F(c)$ and the morphism $\iota\colon F(h)(d) \to d'$ of $F(c')$ as a 2-morphism in $\mathbf{Cat}$:

$$\begin{tikzcd}
 & F(c) \arrow[dd, "F(h)"] \\
\mathbf{1} \arrow[ur, "d"] \arrow[dr, "d'"'] \arrow[r, phantom, "\Swarrow"] & \\
 & F(c')
\end{tikzcd}$$

Applying the terminal category preserving 2-functor $E\colon \mathbf{Cat} \to \mathbf{Cat}$ to this diagram yields:

$$\begin{tikzcd}
 & E(F(c)) \arrow[dd, "E(F(h))"] \\
E(\mathbf{1}) \cong \mathbf{1} \arrow[ur, "E(d)"] \arrow[dr, "E(d')"'] \arrow[r, phantom, "\Swarrow"] & \\
 & E(F(c'))
\end{tikzcd}$$

Then because the above square commutes, this is the same as

$$\begin{tikzcd} & F'(H(c)) \arrow[dd, "F'(H((h))"] \\ E(\mathbf{1}) \cong \mathbf{1} \arrow[ur, "E(d)"] \arrow[dr, "E(d')"'] \arrow[r, phantom, "\Swarrow"] & \\ & F'(H(c')) \end{tikzcd}$$

which is the same as a morphism $E(\iota)\colon F'(H(h))(E(d)) \to E(d')$ in $F'(H(c'))$. To check that the above recipe is functorial, given two vertically composable 2-morphisms in $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$:

$$\begin{tikzcd} a \arrow[r] \arrow[d, "f"'] & c \arrow[d, "h"] & b \arrow[l] \arrow[d, "g"] & d \in F(c) \\ a' \arrow[r] & c' & b' \arrow[l] & d' \in F(c') \end{tikzcd}$$

$$\iota\colon F(h)(d) \to d'$$

$$\begin{tikzcd} a' \arrow[r] \arrow[d, "f'"'] & c' \arrow[d, "h'"] & b' \arrow[l] \arrow[d, "g'"] & d' \in F(c') \\ a'' \arrow[r] & c'' & b'' \arrow[l] & d'' \in F(c'') \end{tikzcd}$$

$$\iota'\colon F(h')(d') \to d''$$

if we first compose these, the result is:

$$\begin{tikzcd} a \arrow[r] \arrow[d, "f'f"'] & c \arrow[d, "h'h"] & b \arrow[l] \arrow[d, "g'g"] & d \in F(c) \\ a'' \arrow[r] & c'' & b'' \arrow[l] & d'' \in F(c'') \end{tikzcd}$$

$$\iota'\iota\colon F(h'h)(d) \to d''$$

and then the image of this 2-morphism under the double functor $\mathbb{H}$ is given by:

$$\begin{tikzcd} H(a) \arrow[r] \arrow[d, "H(f'f)"'] & H(c) \arrow[d, "H(h'h)"] & H(b) \arrow[l] \arrow[d, "H(g'g)"] & E(d) \in F'(H(c)) \\ H(a'') \arrow[r] & H(c'') & H(b'') \arrow[l] & E(d'') \in F'(H(c'')) \end{tikzcd}$$

$$E(\iota'\iota)\colon F'(H(h'h))(E(d)) \to E(d'').$$

On the other hand, applying the double functor $\mathbb{H}$ first gives:

$$\begin{tikzcd} H(a) \arrow[r] \arrow[d, "H(f)"'] & H(c) \arrow[d, "H(h)"'] & H(b) \arrow[l] \arrow[d, "H(g)"'] & E(d) \in F'(H(c)) \\ H(a') \arrow[r] & H(c') & H(b') \arrow[l] & E(d') \in F'(H(c')) \end{tikzcd}$$

$$E(\iota)\colon F'(H(h))(E(d)) \to E(d')$$

$$\begin{tikzcd} H(a') \arrow[r] \arrow[d, "H(f')"'] & H(c') \arrow[d, "H(h')"'] & H(b') \arrow[l] \arrow[d, "H(g')"'] & E(d') \in F'(H(c')) \\ H(a'') \arrow[r] & H(c'') & H(b'') \arrow[l] & E(d'') \in F'(H(c'')) \end{tikzcd}$$

$$E(\iota')\colon F'(H(h'))(E(d')) \to E(d'')$$

and then composing these gives:

$$\begin{tikzcd} H(a) \arrow[r] \arrow[d, "H(f'f)"'] & H(c) \arrow[d, "H(h'h)"'] & H(b) \arrow[l] \arrow[d, "H(g'g)"'] & E(d) \in F'(H(c)) \\ H(a'') \arrow[r] & H(c'') & H(b'') \arrow[l] & E(d'') \in F'(H(c'')) \end{tikzcd}$$

$$E(\iota'\iota)\colon F'(H(h'h))(E(d)) \to E(d'').$$

This double functor $\mathbb{H}$ satisfies the equations $S\mathbb{H}_1 = HS$ and $T\mathbb{H}_1 = HT$.

Given two composable horizontal 1-cells M and N in $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$:

$$\begin{tikzcd} & c_1 & & & c_2 & \\ a_1 \arrow[ur, "i_1"] & & b \arrow[ul, "o_1"'] & b \arrow[ur, "i_2"] & & a_2 \arrow[ul, "o_2"'] \end{tikzcd}$$

$$d_1 \in F(c_1) \qquad d_2 \in F(c_2)$$

composing first gives $M \odot N$:

$$\begin{tikzcd} & c_1 +_b c_2 & \\ a_1 \arrow[ur, "\psi j_{c_1} i_1"] & & a_2 \arrow[ul, "\psi j_{c_2} o_2"'] \end{tikzcd}$$

$$d \in F(c_1 +_b c_2)$$

where

$$d\colon 1 \xrightarrow{\lambda^{-1}} 1 \times 1 \xrightarrow{d_1 \times d_2} F(c_1) \times F(c_2) \xrightarrow{\phi_{c_1,c_2}} F(c_1 + c_2) \xrightarrow{F(\psi)} F(c_1 +_b c_2).$$

The image of this horizontal 1-cell is then given by $\mathbb{H}(M \odot N)$:

$$\begin{tikzcd} & H(c_1 +_b c_2) & \\ H(a_1) \arrow[ur, "H(\psi j_{c_1} i_1)"] & & H(a_2) \arrow[ul, "H(\psi j_{c_2} o_2)"'] \end{tikzcd}$$

$$E(d) \in F'(H(c_1 +_b c_2))$$

where

$$E(d)\colon 1 \xrightarrow{E(d)} E(F(c_1 +_b c_2)) = F'(H(c_1 +_b c_2)).$$

On the other hand, the image of each horizontal 1-cell under the double functor $\mathbb{H}$ is given respectively by $\mathbb{H}(M)$ and $\mathbb{H}(N)$:

$$\begin{tikzcd} & H(c_1) & & & H(c_2) & \\ H(a_1) \arrow[ur, "H(i_1)"] & & H(b) \arrow[ul, "H(o_1)"'] & H(b) \arrow[ur, "H(i_2)"] & & H(a_2) \arrow[ul, "H(o_2)"'] \end{tikzcd}$$

$$E(d_1) \in F'(H(c_1)) \qquad E(d_2) \in F'(H(c_2))$$

Composing these then gives $\mathbb{H}(M) \odot \mathbb{H}(N)$:

$$\begin{tikzcd} & H(c_1) +_{H(b)} H(c_2) & \\ H(a_1) \arrow[ur, "\Psi j_{H(c_1)} H(i_1)"] & & H(a_2) \arrow[ul, "\Psi j_{H(c_2)} H(o_2)"'] \end{tikzcd}$$

$$d' \in F'(H(c_1) +_{H(b)} H(c_2))$$

where

$$d'\colon 1 \xrightarrow{E(d_1) \times E(d_2)} F'(H(c_1)) \times F'(H(c_2)) \xrightarrow{\Phi_{H(c_1),H(c_2)}} F'(H(c_1)+H(c_2)) \xrightarrow{F'(H(\Psi))} F'(H(c_1)+_{H(b)}H(c_2)).$$

We then have a comparison constraint:

$$\mathbb{H}_{M,N} \colon \mathbb{H}(M) \odot \mathbb{H}(N) \xrightarrow{\sim} \mathbb{H}(M \odot N)$$

given by the globular 2-isomorphism:

$$\begin{tikzcd} H(a_1) \arrow[r, "\Psi j_{H(c_1)} H(i_1)"] \arrow[d, "1"'] & H(c_1) +_{H(b)} H(c_2) \arrow[d, "\kappa^{-1}"'] & H(a_2) \arrow[l, "\Psi j_{H(c_2)} H(o_2)"'] \arrow[d, "1"'] & d' \in F'(H(c_1) +_{H(b_1)} H(c_2)) \\ H(a_1) \arrow[r, "H(\psi j_{c_1} i_1)"] & H(c_1 +_b c_2) & H(a_2) \arrow[l, "H(\psi j_{c_2} o_2)"'] & d \in F'(H(c_1 +_b c_2)) \end{tikzcd}$$

$$\iota_{\kappa^{-1}} \colon F'(\kappa^{-1})(d') \to d.$$

where κ is the isomorphism

$$\kappa\colon H(c_1 +_b c_2) \xrightarrow{\sim} H(c_1) +_{H(b)} H(c_2)$$

which comes from the functor $H\colon \mathbf{C} \to \mathbf{C}'$ preserving finite colimits. The above diagram commutes by a similar argument as to the one used in Theorem 6.7. Similarly we have a unit comparison constraint

$$\mathbb{H}_U\colon U_{\mathbb{H}(c)} \to \mathbb{H}(U_c)$$

given by the globular 2-isomorphism:

$$\begin{tikzcd}
H(c) \arrow[r, "1"] \arrow[d, "1"'] & H(c) \arrow[d, "1"] & H(c) \arrow[l, "1"'] \arrow[d, "1"] & !_{H(c)} \in F'(H(c)) \\
H(c) \arrow[r, "1"] & H(c) & H(c) \arrow[l, "1"'] & E(!_c) \in F'(H(c))
\end{tikzcd}$$

where the morphism of decorations is the identity $!_{H(c)} = E(!_c)$ as $EF = F'H$. These comparison constrains satisfy the coherence axioms of a monoidal category, namely:

$$\begin{tikzcd}
(\mathbb{H}(M) \odot \mathbb{H}(N)) \odot \mathbb{H}(P) \arrow[r, "a"] \arrow[d, "\mathbb{H}_{M,N} \odot 1"'] & \mathbb{H}(M) \odot (\mathbb{H}(N) \odot \mathbb{H}(P)) \arrow[d, "1 \odot \mathbb{H}_{N \odot P}"] \\
\mathbb{H}(M \odot N) \odot \mathbb{H}(P) \arrow[d, "\mathbb{H}_{M \odot N, P}"'] & \mathbb{H}(M) \odot \mathbb{H}(N \odot P) \arrow[d, "\mathbb{H}_{M, N \odot P}"] \\
\mathbb{H}((M \odot N) \odot P) \arrow[r, "\mathbb{H}(a')"] & \mathbb{H}(M \odot (N \odot P))
\end{tikzcd}$$

$$\begin{tikzcd}
U_{\mathbb{H}(a)} \odot \mathbb{H}(M) \arrow[r, "\mathbb{H}_U \odot 1"] \arrow[d, "\lambda"'] & \mathbb{H}(U_a) \odot \mathbb{H}(M) \arrow[d, "\mathbb{H}_{U_a, M}"] \\
\mathbb{H}(M) & \mathbb{H}(U_a \odot M) \arrow[l, "\mathbb{H}(\lambda')"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{H}(M) \odot U_{\mathbb{H}(b)} \arrow[r, "1 \odot \mathbb{H}_U"] \arrow[d, "\rho"'] & \mathbb{H}(M) \odot \mathbb{H}(U_b) \arrow[d, "\mathbb{H}_{M, U_b}"] \\
\mathbb{H}(M) & \mathbb{H}(M \odot U_b) \arrow[l, "\mathbb{H}(\rho')"']
\end{tikzcd}$$

This shows that $\mathbb{H} = (H, E)$ is a double functor. Next we show that this double functor is in fact symmetric monoidal. First, that the object component $\mathbb{H}_0 = H$ is symmetric monoidal is clear as H preserves finite colimits. As for the arrow component $\mathbb{H}_1$, given two horizontal 1-cells M_1 and M_2 in $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$:

$$\begin{tikzcd}
 & c_1 & \\
a_1 \arrow[ur, "i_1"] & & b_1 \arrow[ul, "o_1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & c_2 & \\
a_2 \arrow[ur, "i_2"] & & b_2 \arrow[ul, "o_2"']
\end{tikzcd}$$

$$d_1 \in F(c_1) \qquad\qquad d_2 \in F(c_2)$$

their tensor product $M_1 \otimes M_2$ in $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$ is given by:

$$\begin{tikzcd}
 & c_1 + c_2 & \\
a_1 + a_2 \arrow[ur, "i_1 + i_2"] & & b_1 + b_2 \arrow[ul, "o_1 + o_2"']
\end{tikzcd}$$

$$d_1 + d_2 \in F(c_1)$$

$$d_1 + d_2 \colon 1 \xrightarrow{d_1 \times d_2} F(c_1) \times F(c_2) \xrightarrow{\phi_{c_1,c_2}} F(c_1 + c_2)$$

and the image of this horizontal 1-cell under the double functor $\mathbb{H}$ is $\mathbb{H}(M_1 \otimes M_2)$ given by:

$$\begin{tikzcd} & H(c_1 + c_2) & \\ H(a_1 + a_2) \arrow[ur, "H(i_1 + i_2)"] & & H(b_1 + b_2) \arrow[ul, "H(o_1 + o_2)"'] \end{tikzcd}$$

$$E(d_1 + d_2) \in E(F(c_1 + c_2)) = F'(H(c_1 + c_2))$$

On the other hand, the image of M_1 and M_2 is given by $\mathbb{H}(M_1)$ and $\mathbb{H}(M_2)$:

$$\begin{tikzcd} & H(c_1) & \\ H(a_1) \arrow[ur, "H(i_1)"] & & H(b_1) \arrow[ul, "H(o_1)"'] \end{tikzcd} \qquad \begin{tikzcd} & H(c_2) & \\ H(a_2) \arrow[ur, "H(i_2)"] & & H(b_2) \arrow[ul, "H(o_2)"'] \end{tikzcd}$$

$$E(d_1) \in F'(H(c_1)) \qquad E(d_2) \in F'(H(c_2))$$

and their tensor product $\mathbb{H}(M_1) \otimes \mathbb{H}(M_2)$ is given by:

$$\begin{tikzcd} & H(c_1) + H(c_2) & \\ H(a_1) + H(a_2) \arrow[ur, "H(i_1) + H(i_2)"] & & H(b_1) + H(b_2) \arrow[ul, "H(o_1) + H(o_2)"'] \end{tikzcd}$$

$$E(d_1) + E(d_2) \in F'(H(c_1) + H(c_2))$$

$$E(d_1) + E(d_2) \colon 1 \xrightarrow{E(d_1) \times E(d_2)} F'(H(c_1)) \times F'(H(c_2)) \xrightarrow{\Phi_{H(c_1),H(c_2)}} F'(H(c_1) + H(c_2)).$$

We then have a natural 2-isomorphism $\mu_{M_1,M_2} \colon \mathbb{H}(M_1) \otimes \mathbb{H}(M_2) \to \mathbb{H}(M_1 \otimes M_2)$ in $\mathbb{F}'\mathrm{Cospan}(\mathbf{C}')$ given by:

$$E(d_1) + E(d_2) \in F'(H(c_1) + H(c_2))$$

$$\begin{tikzcd} H(a_1) + H(a_2) \arrow[r, "H(i_1) + H(i_2)"] \arrow[d, "\kappa"'] & H(c_1) + H(c_2) \arrow[d, "\kappa"'] & H(b_1) + H(b_2) \arrow[l, "H(o_1) + H(o_2)"'] \arrow[d, "\kappa"'] \\ H(a_1 + a_2) \arrow[r, "H(i_1 + i_2)"] & H(c_1 + c_2) & H(b_1 + b_2) \arrow[l, "H(o_1 + o_2)"'] \end{tikzcd}$$

$$E(d_1 + d_2) \in F'(H(c_1 + c_2))$$

$$\iota_\kappa \colon F'(\kappa)(E(d_1) + E(d_2)) \to E(d_1 + d_2)$$

where κ denotes the isomorphism arising from H preserving finite colimits. This natural 2-isomorphism together with the associators of $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$ and $\mathbb{F}'\mathrm{Cospan}(\mathbf{C}')$, respectively

α and α', make the following diagram commute:

$$\begin{tikzcd}
(\mathbb{H}(M_1) \otimes \mathbb{H}(M_2)) \otimes \mathbb{H}(M_3) \arrow[r, "\alpha'"] \arrow[d, "\mu_{M_1,M_2} \otimes 1"'] & \mathbb{H}(M_1) \otimes (\mathbb{H}(M_2) \otimes \mathbb{H}(M_3)) \arrow[d, "1 \otimes \mu_{M_2 \otimes M_3}"] \\
\mathbb{H}(M_1 \otimes M_2) \otimes \mathbb{H}(M_3) \arrow[d, "\mu_{M_1 \otimes M_2, M_3}"'] & \mathbb{H}(M_1) \otimes \mathbb{H}(M_2 \otimes M_3) \arrow[d, "\mathbb{H}_{M_1, M_2 \otimes M_3}"] \\
\mathbb{H}((M_1 \otimes M_2) \otimes M_3) \arrow[r, "\mathbb{H}(\alpha)"] & \mathbb{H}(M_1 \otimes (M_2 \otimes M_3))
\end{tikzcd}$$

with the corresponding diagram of decorations:

$$\begin{tikzcd}
F'(\alpha\kappa\kappa)((E(d_1) + E(d_2)) + E(d_3)) \arrow[r, "F'(\kappa\kappa)(\iota_{\alpha'})"] \arrow[d, "F'(\alpha\kappa)(\iota_\kappa + 1)"'] & F'(\kappa\kappa)(E(d_1) + (E(d_2) + E(d_3))) \arrow[d, "F'(\kappa)(1 + \iota_\kappa)"] \\
F'(\alpha\kappa)(E(d_1 + d_2) + E(d_3)) \arrow[d, "F'(\alpha)(\iota_\kappa)"'] & F'(\kappa)(E(d_1) + E(d_2 + d_3)) \arrow[d, "\iota_\kappa"] \\
F'(\alpha)(E((d_1 + d_2) + d_3)) \arrow[r, "\iota_\alpha"] & E(d_1 + (d_2 + d_3))
\end{tikzcd}$$

where

$$F'(\alpha\kappa\kappa)((E(d_1) + E(d_2)) + E(d_3)) = F'(\kappa\kappa\alpha')((E(d_1) + E(d_2)) + E(d_3)).$$

We also have that the monoidal unit of $\mathbb{F}\mathrm{Cospan}(\mathbf{C})_1$ is given by:

$$\begin{tikzcd}
& 1_{\mathbf{C}} & \\
1_{\mathbf{C}} \arrow[ur, "1"] & & 1_{\mathbf{C}} \arrow[ul, "1"']
\end{tikzcd}$$

$$!_{1_{\mathbf{C}}} \in F(1_{\mathbf{C}})$$

where $1_{\mathbf{C}}$ is the monoidal unit of the finitely cocartesian category $\mathbf{C}$. The image of this horizontal 1-cell under $\mathbb{H}$ is given by:

$$\begin{tikzcd}
& H(1_{\mathbf{C}}) & & & & 1_{\mathbf{C}'} & \\
H(1_{\mathbf{C}}) \arrow[ur, "1"] & & H(1_{\mathbf{C}}) \arrow[ul, "1"'] & = & 1_{\mathbf{C}'} \arrow[ur, "1"] & & 1_{\mathbf{C}'} \arrow[ul, "1"']
\end{tikzcd}$$

$$E(!_{1_{\mathbf{C}}}) \in F'(H(1_{\mathbf{C}})) \qquad\qquad !_{1_{\mathbf{C}'}} \in F'(1_{\mathbf{C}'})$$

since H preserves finite colimits. We then have a 2-isomorphism in $\mathbb{F}'\mathrm{Cospan}(\mathbf{C}')$ given by:

$$\mu \colon 1_{\mathbb{F}'\mathrm{Cospan}(\mathbf{C}')_1} \to \mathbb{H}(1_{\mathbb{F}\mathrm{Cospan}(\mathbf{C})_1})$$

$$\begin{tikzcd}
1_{\mathbf{C}'} \arrow[r, "1"] \arrow[d, "\kappa"'] & 1_{\mathbf{C}'} \arrow[d, "\kappa"'] & 1_{\mathbf{C}'} \arrow[l, "1"'] \arrow[d, "\kappa"'] & !_{1_{\mathbf{C}'}} \in F'(1_{\mathbf{C}'}) \\
H(1_{\mathbf{C}}) \arrow[r, "1"] & H(1_{\mathbf{C}}) & H(1_{\mathbf{C}}) \arrow[l, "1"'] & E(!_{1_{\mathbf{C}}}) \in F'(H(1_{\mathbf{C}}))
\end{tikzcd}$$

together with the morphism $\iota_\mu \colon F'(\kappa)(!_{1_{\mathbf{C}'}}) \to E(!_{1_{\mathbf{C}}})$ in $F'(H(1_{\mathbf{C}}))$. The following square then commutes:

$$\begin{tikzcd}
1_{\mathbf{C}'} \otimes \mathbb{H}(M) \arrow[r, "\mu \otimes 1"] \arrow[d, "\ell"'] & \mathbb{H}(1_{\mathbf{C}}) \otimes \mathbb{H}(M) \arrow[d, "\mu_{1_{\mathbf{C}},M}"] \\
\mathbb{H}(M) & \mathbb{H}(1_{\mathbf{C}} + M) \arrow[l, "\mathbb{H}(\ell')"']
\end{tikzcd}$$

where we have abbreviated the monoidal units of $\mathbb{F}\mathrm{Cospan}(\mathbf{C})_1$ and $\mathbb{F}'\mathrm{Cospan}(\mathbf{C}')_1$ as $1_{\mathbf{C}}$ and $1_{\mathbf{C}'}$, respectively. The diagram of corresponding decorations is given by:

$$\begin{tikzcd}
F'(\ell)(!_{1_{\mathbf{C}'}} + E(d)) \arrow[r, "F'(H(\ell')\kappa)(\iota_{\mu+1})"] \arrow[d, "\iota_\ell"'] & F'(H(\ell')\kappa)(E(!_{1_{\mathbf{C}}}) + E(d)) \arrow[d, "F'(H(\ell'))(\iota_\kappa)"] \\
E(d) & F'(H(\ell'))(E(!_{1_{\mathbf{C}}} + d)) \arrow[l, "\iota_{H(\ell')}"']
\end{tikzcd}$$

where

$$F'(\ell)(!_{1_{\mathbf{C}'}} + E(d)) = F'(H(\ell')\kappa(\mu + 1))(!_{1_{\mathbf{C}'}} + E(d)).$$

The other square involving the right unitors r and r' is similar. Note that because μ and $\mu_{(-,-)}$ are both isomorphisms, the symmetric monoidal double functor $\mathbb{H}$ is strong.

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "\phi"] \arrow[d, "\Phi"'] & F(1_{\mathbf{C}}) \arrow[r, "F(!_c)"] & F(c) \arrow[d] \\
F'(1_{\mathbf{C}'}) \arrow[rr, "F'(!_{H(c)})"] & & F'(H(c)) = E(F(c))
\end{tikzcd}$$

show symmetric monoidal, make sure using all assumptions on the functors H and E

5. From pseudofunctors to left adjoints

In this section we investigate the necessary conditions to obtain a category $\mathbf{D}$ and a left adjoint $L \colon \mathbf{C} \to \mathbf{D}$ from a category $\mathbf{C}$ with finite colimits and a symmetric lax monoidal pseudofunctor $F \colon \mathbf{C} \to \mathbf{Cat}$. The main tool that allows us to do this will be the Grothendieck construction which we recall:

Definition 5.1. Let $\mathbf{Cat}_{\mathrm{lax},\star}$ denote the **2-category of lax-pointed categories** which has:

(i) objects as pairs $(\mathbf{C}, c)$ where $\mathbf{C}$ is a category and c is an object of $\mathbf{C}$, and
(ii) a morphism from $(\mathbf{C}, c)$ to $(\mathbf{D}, d)$ is a pair (F, f) where $F \colon \mathbf{C} \to \mathbf{D}$ is a functor and $f \colon F(c) \to d$ is a morphism in $\mathbf{D}$.

Definition 5.2. Given a pseudofunctor $F\colon \mathbf{C} \to \mathbf{Cat}$, the **Grothendieck construction of** F is given by the 'strict 2-pullback' of the following cospan:

$$
\begin{tikzcd}
 & \int F \arrow[dl, dashed, "p"'] \arrow[dr, dashed, "q"] & \\
\mathbf{C} \arrow[dr, "F"'] & & \mathbf{Cat}_{\mathrm{lax},\star} \arrow[dl, "P"] \\
 & \mathbf{Cat} &
\end{tikzcd}
$$

where $P\colon \mathbf{Cat}_{\mathrm{lax},\star} \to \mathbf{Cat}$ is the forgetful functor. This means that $\int F$ is a category which has:

(i) objects as pairs $(c, d \in F(c))$ and
(ii) a morphism from $(c, d \in F(c))$ to $(c', d' \in F(c'))$ is a pair $(f\colon c \to c', \alpha\colon F(f)(d) \to d')$.
This can also be thought of as a morphism and a 2-morphism:

$$
\begin{tikzcd}
 & F(c) \arrow[d, "F(f)"] \\
\star \arrow[ur, "d"] \arrow[dr, "d'"'] \arrow[r, phantom, "\Swarrow \alpha"] & F(c') \\
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[d, "f"'] \\
c'
\end{tikzcd}
$$

Definition 5.3. Given bicategories $\mathbf{C}$ and $\mathbf{D}$ and 2-functors (possibly lax or oplax) $(F, \phi), (G, \psi)\colon \mathbf{C} \to \mathbf{D}$, a **pseudonatural transformation** $\sigma\colon (F, \phi) \to (G, \psi)$ consists of:

(i) for each object $a \in \mathbf{C}$, a morphism $\sigma_a\colon F(a) \to G(a)$ in $\mathbf{D}$ and
(ii) for every pair of objects a and b of $\mathbf{C}$, we have natural isomorphisms:

$$
\begin{tikzcd}
\mathbf{C}(a,b) \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\sigma_{a,b} \Nearrow"] & \mathbf{D}(F(a), F(b)) \arrow[d, "(\sigma_b)_*"] \\
\mathbf{D}(G(a), G(b)) \arrow[r, "(\sigma_a)^*"'] & \mathbf{D}(F(a), G(b))
\end{tikzcd}
$$

where $(\sigma_a)^*$ and $(\sigma_b)_*$ are the functors induced by precomposition and postcompositon, respectively. Thus for each morphism $f\colon a \to b$ in $\mathbf{C}$, we have an invertible 2-morphism $\sigma_f\colon G(f)\sigma_a \xrightarrow{\sim} \sigma_b F(f)$ in $\mathbf{D}$:

$$
\begin{tikzcd}
F(a) \arrow[r, "F(f)"] \arrow[d, "\sigma_a"'] \arrow[dr, phantom, "\sigma_f \Nearrow"] & F(b) \arrow[d, "\sigma_b"] \\
G(a) \arrow[r, "G(f)"'] & G(b)
\end{tikzcd}
$$

such that for each composable pair of morphisms $f\colon a \to b$ and $g\colon b \to c$ of $\mathbf{C}$, the following diagrams commute:

$$\begin{tikzcd}
(G(g)G(f))\sigma_a \arrow[r, "a'"] \arrow[d, "\psi 1_{\sigma_a}"'] & G(g)(G(f)\sigma_a) \arrow[r, "1_{G(g)}\sigma_f"] & G(g)(\sigma_b F(f)) \arrow[r, "a'^{-1}"] & (G(g)\sigma_b)F(f) \arrow[r, "\sigma_g 1_{F(f)}"] & (\sigma_c F(g))F(f) \arrow[d, "a'"] \\
G(gf)\sigma_a \arrow[rr, "\sigma_{gf}"] & & \sigma_c F(gf) & & \sigma_c(F(g)F(f)) \arrow[ll, "1_{\sigma_c}\phi"']
\end{tikzcd}$$

$$\begin{tikzcd}
& G(1_a)\sigma_a \arrow[dr, "\sigma_{1_a}"] & \\
1_{G(a)}\sigma_a \arrow[ur, "\psi 1_{\sigma_a}"] \arrow[d, "\ell'"'] & & \sigma_a F(1_a) \\
\sigma_a \arrow[r, "\rho'^{-1}"] & \sigma_a 1_{F(a)} \arrow[ur, "1_{\sigma_a}\phi"'] &
\end{tikzcd}$$

Let $[\mathbf{C}, \mathbf{Cat}]_{\text{pseudo}}$ denote the 2-category which has:

(i) objects as pseudofunctors $(F, \phi)\colon \mathbf{C} \to \mathbf{Cat}$,
(ii) a morphism from a pseudofunctor $(F, \phi)\colon \mathbf{C} \to \mathbf{Cat}$ to another $(G, \psi)\colon \mathbf{C} \to \mathbf{Cat}$ is a pseudonatural transformation $\sigma\colon (F, \phi) \to (G, \psi)$, and
(iii) 2-morphisms are modifications.

Definition 5.4. Let $P\colon \mathbf{D} \to \mathbf{C}$ be a functor. A morphism $f\colon d_1 \to d_2$ in the category $\mathbf{D}$ is **cocartesian (with respect to the functor P)** if for any object d' in $\mathbf{D}$ and morphism $g\colon d_1 \to d'$ and every $p\colon P(d_2) \to P(d')$ such that $P(g) = pP(f)$, there exists a unique $h\colon d_2 \to d'$ such that $g = hf$ and $p = P(h)$.

$$\begin{tikzcd}
d' & & & P(d') & \\
& & \overset{P}{\mapsto} & & \\
d_2 \arrow[uu, dashed, "\exists! h"] & d_1 \arrow[l, "f"'] \arrow[uul, "g"'] & & P(d_2) \arrow[uu, "p = P(h)"] & P(d_1) \arrow[l, "P(f)"'] \arrow[uul, "P(g)"']
\end{tikzcd}$$

Definition 5.5. A functor $P\colon \mathbf{D} \to \mathbf{C}$ is a **(Grothendieck) opfibration** if for any object d in $\mathbf{D}$ and morphism $f\colon c \to P(d)$ there exists a cocartesian morphism $\phi\colon d' \to d$ such that $P(\phi) = f$.

Definition 5.6. Let $P\colon \mathbf{D} \to \mathbf{C}$ and $P'\colon \mathbf{D}' \to \mathbf{C}$ be opfibrations over a category $\mathbf{C}$. A **morphism of opfibrations** is a functor $F\colon \mathbf{D} \to \mathbf{D}'$ such that $F(p)$ is cocartesian with respect to P' if p is cocartesian with respect to P and $P = P'F$.

Let Opfib($\mathbf{C}$) denote the 2-category of opfibrations over $\mathbf{C}$ which has:

(i) objects as opfibrations $P\colon \mathbf{D} \to \mathbf{C}$,
(ii) morphisms as morphisms of opfibrations, and
(iii) a 2-morphism from one morphism of opfibrations $F\colon P \to P'$ to another $F'\colon P \to P'$ is a natural transformation $\alpha\colon F \to F'$ such that the left whiskering $P'\alpha$ is trivial.

The 2-category Opfib($\mathbf{C}$) is a 2-subcategory of $\mathbf{Cat}/\mathbf{C}$ where $\mathbf{Cat}/\mathbf{C}$ is the 2-category of functors over $\mathbf{C}$. The Grothendieck construction is a 2-functor $\int F\colon [\mathbf{C}, \mathbf{Cat}] \to \mathbf{Cat}/\mathbf{C}$ which factors through the image of the embedding Opfib($\mathbf{C}$) $\hookrightarrow \mathbf{Cat}/\mathbf{C}$:

$$\int F\colon [\mathbf{C}, \mathbf{Cat}] \to \mathrm{Opfib}(\mathbf{C}) \hookrightarrow \mathbf{Cat}/\mathbf{C}.$$

Under certain conditions, the functor resulting from the Grothendieck construction $p\colon \int F \to \mathbf{C}$ will be right adjoint to the desired left adjoint $L\colon \mathbf{C} \to \mathbf{D}$.

6. An equivalence of symmetric monoidal double categories

In this section we prove that two different frameworks which utilize symmetric monoidal double categories, namely decorated cospans and 'structured cospans', are equivalent under the conditions required from the previous section that allow for a left adjoint $L\colon \mathbf{C} \to \mathbf{D}$ to be obtained from a symmetric lax monoidal pseudofunctor $F\colon \mathbf{C} \to \mathbf{Cat}$. The double category version of decorataed cospans is given in Section 3. The other framework which also utilizes symmetric monoidal double categories, due to Baez and the first author and goes by the name of 'structured cospans', is explained below. Once again following the notation of Shulman [16], given a double category $\mathbb{C}$, we write ${}_f\mathbb{C}_g(M,N)$ for the set of 2-morphisms in $\mathbb{C}$ of the form:

$$\begin{tikzcd} A \arrow[r, "M", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow a"] & B \arrow[d, "g"] \\ C \arrow[r, "N"', "|" marking] & D \end{tikzcd}$$

We call M and N the **horizontal source and target** of the 2-morphism a, respectively, and likewise we call f and g the **vertical source and target** of the 2-morphism a, respectively. Thus ${}_f\mathbb{C}_g(M,N)$ denotes the set of 2-morphisms in $\mathbb{C}$ with horizontal source and target M and N and vertical source and target f and g.

Definition 6.1. A (possibly lax or oplax) double functor $\mathbb{F}\colon \mathbb{C} \to \mathbb{D}$ is **full** (respectively, **faithful**) if $\mathbb{F}_0\colon \mathbb{C}_0 \to \mathbb{D}_0$ is full (respectively, faithful) and each map

$$\mathbb{F}_1\colon {}_f\mathbb{C}_g(M,N) \to_{\mathbb{F}(f)} \mathbb{D}_{\mathbb{F}(g)}(\mathbb{F}(M), \mathbb{F}(N))$$

is surjective (respectively, injective).

Definition 6.2. A (possibly lax or oplax) double functor $\mathbb{F}\colon \mathbb{C} \to \mathbb{D}$ is **essentially surjective** if we can simultaneously make the following choices:

(i) For each object $d \in \mathbb{D}$, we can find an object $\hat{d} \in \mathbb{C}$ together with a vertical 1-isomorphism $\alpha_d\colon \mathbb{F}(\hat{d}) \to d$, and
(ii) For each horizontal 1-cell $N\colon d_1 \nrightarrow d_2$ of $\mathbb{D}$, we can find a horizontal 1-cell $\hat{N}\colon \hat{d}_1 \nrightarrow \hat{d}_2$ of $\mathbb{C}$ and a 2-isomorphism a_N of $\mathbb{D}$ as in the following diagram:

$$\begin{tikzcd} \mathbb{F}(\hat{d}_1) \arrow[r, "\mathbb{F}(\hat{N})", "|" marking] \arrow[d, "\alpha_{d_1}"'] \arrow[dr, phantom, "\Downarrow a_N"] & \mathbb{F}(\hat{d}_2) \arrow[d, "\alpha_{d_2}"] \\ d_1 \arrow[r, "N"', "|" marking] & d_2 \end{tikzcd}$$

Definition 6.3. A double functor $\mathbb{F}\colon \mathbb{C} \to \mathbb{D}$ is **strong** if the comparison and unit constraints are globular isomorphisms, meaning that for each composable pair of horizontal 1-cells M and N we have a natural isomorphism

$$\mathbb{F}_{M,N}\colon \mathbb{F}(M) \odot \mathbb{F}(N) \xrightarrow{\sim} \mathbb{F}(M \odot N)$$

and for each object $c \in \mathbb{C}$ a natural isomorphism

$$\mathbb{F}_c \colon \hat{U}_{\mathbb{F}(c)} \xrightarrow{\sim} \mathbb{F}(U_c).$$

Theorem 6.4 (Shulman,7.8)**.** *Given a strong double functor $\mathbb{F} \colon \mathbb{C} \to \mathbb{D}$, $\mathbb{F}$ is part of a double equivalence if and only if $\mathbb{F}$ is full, faithful and essentially surjective.*

Theorem 6.5. *Let $\mathbb{C}$ and $\mathbb{D}$ be symmetric monoidal double categories and let $\mathbb{F} \colon \mathbb{C} \to \mathbb{D}$ be a symmetric monoidal strong double functor. If $\mathbb{F}$ is part of a double equivalence, then $\mathbb{F}$ is in fact part of a symmetric monoidal double equivalence, and $\mathbb{C}$ and $\mathbb{D}$ are equivalent as symmetric monoidal double categories.*

Proof. **Prove this...is there anything to even prove?** □

Our next goal is to show that the symmetric monoidal double category $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$ of Section 3 is equivalent as a symmetric monoidal double category to the symmetric monoidal double category ${}_L\mathbb{C}\mathrm{sp}(\mathbf{D})$ obtained using structured cospans.

Theorem 6.6. *Given a category $\mathbf{D}$ with finite colimits and a category $\mathbf{C}$ with finite coproducts and a left adjoint $L \colon \mathbf{C} \to \mathbf{D}$ with $\mathbf{C}$ and $\mathbf{D}$ regarded as symmetric monoidal categories under binary coproducts, we can obtain a symmetric monoidal double category ${}_L\mathbb{C}\mathrm{sp}(\mathbf{D})$ which has:*

(i) objects given by objects of $\mathbf{C}$,
(ii) vertical 1-morphisms given by morphisms of $\mathbf{C}$,
(iii) horizontal 1-cells given by cospans of $\mathbf{D}$ of the form:

$$\begin{tikzcd} & d & \\ L(c) \arrow[ur] & & L(c') \arrow[ul] \end{tikzcd}$$

and
(iv) 2-morphisms given by maps of cospans of $\mathbf{D}$ of the form:

$$\begin{tikzcd} L(c_1) \arrow[r] \arrow[d, "L(f)"'] & d \arrow[d, "h"'] & L(c_2) \arrow[l] \arrow[d, "L(g)"'] \\ L(c_1') \arrow[r] & d' & L(c_2') \arrow[l] \end{tikzcd}$$

Composition of horizontal 1-cells and 2-morphisms is given by pushouts in $\mathbf{D}$ and tensoring of objects is under binary coproducts in $\mathbf{C}$.

Proof. See [2]. □

In the previous section, we started with a symmetric lax monoidal pseudofunctor $F \colon \mathbf{C} \to \mathbf{Cat}$ and created a symmetric monoidal double category $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$ which has:

(i) objects of $\mathbf{C}$ as objects,
(ii) morphisms of $\mathbf{C}$ as vertical 1-morphisms,

(iii) horizontal 1-cells are given by F-decorated cospans, which are pairs:

$$\begin{tikzcd} & c & \\ c_1 \arrow[ur, "i"] & & c_2 \arrow[ul, "o"'] \end{tikzcd} \qquad d \in F(c)$$

and

(iv) 2-morphisms are given by maps of cospans in $\mathbf{C}$:

$$\begin{tikzcd} c_1 \arrow[r, "i"] \arrow[d, "f"'] & c \arrow[d, "h"'] & c_2 \arrow[l, "o"'] \arrow[d, "g"'] & d \in F(c) \\ c_1' \arrow[r, "i'"] & c' & c_2' \arrow[l, "o'"'] & d' \in F(c') \end{tikzcd}$$

together with a morphism $\iota\colon F(h)(d) \to d'$ in $F(c')$.

We now show that these two symmetric monoidal double categories are equivalent.

Theorem 6.7. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be categories with finite colimits and* $F\colon \mathbf{C} \to \mathbf{Cat}$ *a symmetric lax monoidal pseudofunctor. Let* $L\colon \mathbf{C} \to \int F \cong \mathbf{D}$ *be the left adjoint of the Grothendieck construction* $R\colon \int F \to \mathbf{C}$ *of* F*. Then the symmetric monoidal double category* ${}_L\mathbb{C}\mathrm{sp}(\mathbf{D})$ *utilizing structured cospans and the symmetric monoidal double category* $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$ *utilizing decorated cospans are equivalent as symmetric monoidal double categories.*

Proof. To prove this, we define a double functor $\mathbb{E}\colon {}_L\mathbb{C}\mathrm{sp}(\mathbf{D}) \to \mathbb{F}\mathrm{Cospan}(\mathbf{C})$ as follows: the object component of the double functor $\mathbb{E}$ is given by $\mathbb{E}_0 = \mathrm{id}_{\mathbf{C}}$ as both double categories ${}_L\mathbb{C}\mathrm{sp}(\mathbf{D})$ and $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$ have objects and morphisms of $\mathbf{C}$ as objects and vertical 1-morphisms, respectively. The functor $\mathbb{E}_0$ is trivially an equivalence of categories.

Given a horizontal 1-cell of ${}_L\mathbb{C}\mathrm{sp}(\mathbf{D})$, which is a cospan in $\mathbf{D}$ of the form:

$$\begin{tikzcd} & d & \\ L(c) \arrow[ur, "i"] & & L(c') \arrow[ul, "o"'] \end{tikzcd}$$

the image of this horizontal 1-cell under the arrow component $\mathbb{E}_1$ is the pair:

$$\begin{tikzcd} & R(d) & \\ c \arrow[ur, "R(i)\eta_c"] & & c' \arrow[ul, "R(o)\eta_{c'}"'] \end{tikzcd} \qquad d \in F(R(d))$$

where $R\colon \mathbf{D} \to \mathbf{C}$ is the right adjoint to the functor $L\colon \mathbf{C} \to \mathbf{D}$ and $\eta\colon 1_{\mathbf{C}} \to RL$ is the unit of the adjunction $L \dashv R$ which is an isomorphism since L is fully faithful. **Hopefully...**

Similarly, the image of a 2-morphism in ${}_L\mathbb{C}\mathrm{sp}(\mathbf{D})$:

$$\begin{tikzcd}
L(c_1) \arrow[r, "i"] \arrow[d, "L(f)"'] & d \arrow[d, "h"'] & L(c_2) \arrow[l, "o"'] \arrow[d, "L(g)"'] \\
L(c_1') \arrow[r, "i'"] & d' & L(c_2') \arrow[l, "o'"']
\end{tikzcd}$$

is the 2-morphism in $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$ given by:

$$\begin{tikzcd}
c_1 \arrow[r, "R(i)\eta_{c_1}"] \arrow[d, "f"'] & R(d) \arrow[d, "R(h)"'] & c_2 \arrow[l, "R(o)\eta_{c_2}"'] \arrow[d, "g"'] & d \in F(R(d)) \\
c_1' \arrow[r, "R(i')\eta_{c_1'}"] & R(d') & c_2' \arrow[l, "R(o')\eta_{c_2'}"'] & d' \in F(R(d'))
\end{tikzcd}$$

together with a morphism $\iota\colon F(R(h))(d) \to d'$ in $F(R(d')) \subseteq \mathbf{D}$ which comes from the Grothendieck construction of the pseudofunctor $F\colon \mathbf{C} \to \mathbf{Cat}$. First, to see that this functor is essentially surjective, given a horizontal 1-cell in $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$:

$$\begin{tikzcd}
 & c & & d \in F(c) \\
c_1 \arrow[ur, "i"] & & c_2 \arrow[ul, "o"'] &
\end{tikzcd}$$

we can find a 2-isomorphism in $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$ whose codomain is the above horizontal 1-cell and whose domain is the image of the following horizontal 1-cell in ${}_L\mathbb{C}\mathrm{sp}(\mathbf{D})$:

$$\begin{tikzcd}
 & d & \\
L(c_1) \arrow[ur, "i'"] & & L(c_2) \arrow[ul, "o'"']
\end{tikzcd}$$

with the 2-isomorphism in $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$ given by:

$$\begin{tikzcd}
c_1 \arrow[r, "R(i')\eta_{c_1}"] \arrow[d, "1"'] & R(d) \arrow[d, "(R(e)\eta_c)^{-1}"'] & c_2 \arrow[l, "R(o')\eta_{c_2}"'] \arrow[d, "1"'] & d \in F(R(d)) \\
c_1 \arrow[r, "i"] & c & c_2 \arrow[l, "o"'] & d \in F(c)
\end{tikzcd}$$

$$\iota\colon F((R(e)\eta_c)^{-1})(d) \to d$$

where $e\colon L(c) \to d$ is given by the map from the trivial decoration on c to $d \in F(c)$. The object and arrow components $\mathbb{E}_0$ and $\mathbb{E}_1$ satisfy the equations $S\mathbb{E}_1 = \mathbb{E}_0 S$ and $T\mathbb{E}_1 = \mathbb{E}_0 T$.

To show that the double functor $\mathbb{E}$ is fully faithful, we need to show that the map

$$\mathbb{E}_1\colon {}_{f\,L}\mathbb{C}\mathrm{sp}(\mathbf{D})_g(M,N) \to_{\mathbb{E}(f)} \mathbb{F}\mathrm{Cospan}(\mathbf{C})_{\mathbb{E}(g)}(\mathbb{E}(M),\mathbb{E}(N))$$

is bijective for arbitrary vertical 1-morphisms f and g and horizontal 1-cells M and N of ${}_L\mathbb{C}\mathrm{sp}(\mathbf{D})$. Consider a 2-morphism in ${}_L\mathbb{C}\mathrm{sp}(\mathbf{D})$:

$$
\begin{tikzcd}
 & & M & & \\
 & L(c_1) \arrow[r, "i"] \arrow[d, "L(f)"'] & d \arrow[d, "\alpha"'] & L(c_2) \arrow[l, "o"'] \arrow[d, "L(g)"'] & \\
f & L(c_1') \arrow[r, "i'"] & d' & L(c_2') \arrow[l, "o'"'] & g \\
 & & N & &
\end{tikzcd}
$$

The set

$$ {}_f{}_L\mathbb{C}\mathrm{sp}(\mathbf{D})_g(M,N) $$

consists of triples

$$ (f,\alpha,g) $$

where f and g are morphisms of $\mathbf{C}$ and α is a morphism of $\mathbf{D}$. The image of the above 2-morphism under the double functor $\mathbb{E}$ is given by:

$$
\begin{tikzcd}
 & & \mathbb{E}(M) & & \\
 & & d \in F(R(d)) & & \\
 & c_1 \arrow[r, "R(i)\eta_{c_1}"] \arrow[d, "f"'] & R(d) \arrow[d, "R(\alpha)"'] & c_2 \arrow[l, "R(o)\eta_{c_2}"'] \arrow[d, "g"'] & \\
\mathbb{E}(f) & c_1' \arrow[r, "R(i')\eta_{c_1'}"] & R(d') & c_2' \arrow[l, "R(o')\eta_{c_2'}"'] & \mathbb{E}(g) \\
 & & d' \in F(R(d')) & & \\
 & & \mathbb{E}(N) & &
\end{tikzcd}
$$

together with a morphism $\iota\colon F(R(\alpha))(d) \to d'$ of $F(R(d'))$. Thus the set

$$ {}_{\mathbb{E}(f)}\mathbb{F}\mathrm{Cospan}(\mathbf{C})_{\mathbb{E}(g)}(\mathbb{E}(M),\mathbb{E}(N)) $$

consists of 4-tuples

$$ (f, R(\alpha), g, \iota). $$

The morphisms $R(\alpha)\colon R(d) \to R(d')$ and $\iota\colon F(R(\alpha))(d) \to d'$ together carry all of the information of the morphism $\alpha\colon d \to d'$ in $\mathbf{D}$ and conversely; given two objects $d = (c, d \in F(c))$ and $d' = (c', d' \in F(c'))$ of $\mathbf{D} = \int F$, a morphism from $\alpha\colon d \to d'$ is a pair

$$ (h\colon c \to c', \iota\colon F(h)(d) \to d') $$

where $h\colon c \to c'$ is given by $R(\alpha)\colon R(d) \to R(d')$. This shows that $\mathbb{E}$ is fully faithful. **At least in my favorite example...**

Next we show that the double functor $\mathbb{E}$ is strong by exhibiting natural isomorphisms

$$ \mathbb{E}_{M,N}\colon \mathbb{E}(M) \odot \mathbb{E}(N) \xrightarrow{\sim} \mathbb{E}(M \odot N) $$

for every pair of composable horizontal 1-cells M and N of ${}_L\mathbb{C}\mathrm{sp}(\mathbf{D})$ and for each object $c \in {}_L\mathbb{C}\mathrm{sp}(\mathbf{D})$ a natural isomorphism

$$ \mathbb{E}_c\colon \hat{U}_{\mathbb{E}(c)} \xrightarrow{\sim} \mathbb{E}(U_c) $$

where U and $\hat{U}$ are the unit structure functors of ${}_L\mathbb{C}\mathrm{sp}(\mathbf{D})$ and $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$, respectively. For any object c, the horizontal 1-cell $\hat{U}_{\mathbb{E}(c)}$ is given by $\hat{U}_c$ which is given by the pair:

$$\begin{tikzcd} & c & \\ c \arrow[ur, "1"] & & c \arrow[ul, "1"'] \end{tikzcd} \qquad !_c \in F(c)$$

The horizontal 1-cell U_c is given by

$$\begin{tikzcd} & L(c) & \\ L(c) \arrow[ur, "1"] & & L(c) \arrow[ul, "1"'] \end{tikzcd}$$

and so $\mathbb{E}(U_c)$ is given by the pair:

$$\begin{tikzcd} & R(L(c)) & \\ c \arrow[ur, "\eta_c"] & & c \arrow[ul, "\eta_c"'] \end{tikzcd} \qquad !_c \in F(R(L(c)))$$

Then we can obtain the natural isomorphism $\mathbb{E}_c$ as the 2-morphism

$$\begin{tikzcd} c \arrow[r, "1"] \arrow[d, "1"'] & c \arrow[d, "\eta_c"'] & c \arrow[l, "1"'] \arrow[d, "1"] \\ c \arrow[r, "\eta_c"] & R(L(c)) & c \arrow[l, "\eta_c"'] \end{tikzcd} \qquad \begin{matrix} !_c \in F(c) \\ \\ !_c \in F(R(L(c))) \end{matrix}$$

$$\iota\colon F(\eta_c)(!_c) \xrightarrow{!} L(c)$$

of $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$. Seems reasonable...

Next, given composable horizontal 1-cells M and N in ${}_L\mathbb{C}\mathrm{sp}(\mathbf{D})$:

$$\begin{tikzcd} & d & \\ L(c_1) \arrow[ur, "i"] & & L(c_2) \arrow[ul, "o"'] \end{tikzcd} \qquad \begin{tikzcd} & d' & \\ L(c_2) \arrow[ur, "i'"] & & L(c_3) \arrow[ul, "o'"'] \end{tikzcd}$$

their images $\mathbb{E}(M)$ and $\mathbb{E}(N)$ are given by:

$$\begin{tikzcd} & R(d) & \\ c_1 \arrow[ur, "R(i)\eta_{c_1}"] & & c_2 \arrow[ul, "R(o)\eta_{c_2}"'] \end{tikzcd} \qquad \begin{tikzcd} & R(d') & \\ c_2 \arrow[ur, "R(i')\eta_{c_2}"] & & c_3 \arrow[ul, "R(o')\eta_{c_3}"'] \end{tikzcd}$$

$$d \in F(R(d)) \qquad\qquad d' \in F(R(d'))$$

and so $\mathbb{E}(M) \odot \mathbb{E}(N)$ is given by:

$$\begin{tikzcd} & R(d) +_{c_2} R(d') & \\ c_1 \arrow[ur, "j\psi R(i)\eta_{c_1}"] & & c_3 \arrow[ul, "j\psi R(o')\eta_{c_3}"'] \end{tikzcd}$$

$$\hat{d} \in F(R(d) +_{c_2} R(d'))$$

$$\hat{d}\colon 1 \xrightarrow{\lambda^{-1}} 1\times 1 \xrightarrow{d\times d'} F(R(d))\times F(R(d')) \xrightarrow{\phi_{R(d),R(d')}} F(R(d)+R(d')) \xrightarrow{F(j_{R(d),R(d')})} F(R(d)+_{c_2}R(d'))$$

where ψ denotes each natural map into the coproduct and j denotes the natural map from the coproduct to the pushout. On the other hand, $M \odot N$ is given by

$$\begin{tikzcd} & d +_{L(c_2)} d' & \\ L(c_1) \arrow[ur, "J\zeta i"] & & L(c_3) \arrow[ul, "J\zeta o'"'] \end{tikzcd}$$

where ζ is each inclusion into the coproduct and J is the natural map from the coproduct to the pushout. Then $E(M \odot N)$ is given by

$$\begin{tikzcd} & R(d +_{L(c_2)} d') & \\ c_1 \arrow[ur, "R(J\zeta i)\eta_{c_1}"] & & c_3 \arrow[ul, "R(J\zeta o')\eta_{c_3}"'] \end{tikzcd}$$

$$d +_{L(c_2)} d' \in F(R(d +_{L(c_2)} d'))$$

and so $\mathbb{E}_{M,N}\colon \mathbb{E}(M) \odot \mathbb{E}(N) \xrightarrow{\sim} \mathbb{E}(M \odot N)$ is given by the 2-morphism:

$$\begin{tikzcd} c_1 \arrow[r, "j\psi R(i)\eta_{c_1}"] \arrow[d, "1"'] & R(d) +_{c_2} R(d') \arrow[d, "\sigma"'] & c_3 \arrow[l, "j\psi R(o')\eta_{c_3}"'] \arrow[d, "1"'] \\ c_1 \arrow[r, "R(J\zeta i)\eta_{c_1}"] & R(d +_{L(c_2)} d') & c_3 \arrow[l, "R(J\zeta o')\eta_{c_3}"'] \end{tikzcd} \qquad \begin{matrix} \hat{d} \in F(R(d) +_{c_2} R(d')) \\ \\ d +_{L(c_2)} d' \in F(R(d +_{L(c_2)} d')) \end{matrix}$$

First, if the right adjoint $R\colon \mathbf{D} \to \mathbf{C}$ is also a left adjoint, then R also preserves all colimits and we have an isomorphism

$$\kappa\colon R(d) +_{R(L(c_2))} R(d') \to R(d +_{L(c_2)} d').$$

Also, since the left adjoint $L\colon \mathbf{C} \to \mathbf{D}$ is fully faithful, the unit of the adjunction $L \dashv R$ at the object c_2 gives an isomorphism $\eta_{c_2}\colon c_2 \to R(L(c_2))$ which results in an isomorphism

$$j_{\eta_{c_2}}\colon R(d) +_{c_2} R(d') \to R(d) +_{R(L(c_2))} R(d').$$

Composing these two results in an isomorphism

$$\sigma \coloneqq \kappa j_{\eta_{c_2}}\colon R(d) +_{c_2} R(d') \to R(d +_{L(c_2)} d').$$

Next, to see that the above diagram commutes, it suffices to show that for the object $c_1 \in \mathbf{C}$,

$$R(J\zeta i)\eta_{c_1}(c_1) = R(J)R(\zeta)R(i)\eta_{c_1}(c_1) \overset{!}{=} \sigma j\psi R(i)\eta_{c_1}(c_1).$$

This follows as $R(i)\eta_{c_1}\colon c_1 \to R(d)$ and the following diagram commutes:

$$\begin{tikzcd}
R(d) \arrow[r, "\psi"] \arrow[dd, "R(\zeta)"'] & R(d)+R(d') \arrow[r, "j"] & R(d)+_{c_2} R(d') \arrow[d, "j_{\eta_{c_2}}"] \arrow[dd, bend left, "\sigma"] \\
& & R(d)+_{R(L(c_2))} R(d') \arrow[d, "\kappa"] \\
R(d+d') \arrow[rr, "R(J)"] & & R(d+_{L(c_2)} d')
\end{tikzcd}$$

Lastly, this map of cospans comes with an isomorphism $\iota\colon F(\sigma)(\hat{d}) \to (d +_{L(c_2)} d')$ in $F(R(d+_{L(c_2)} d'))$. This shows that $\mathbb{E}$ is strong, and so $\mathbb{E}\colon {}_L\mathbb{C}\mathrm{sp}(\mathbf{D}) \xrightarrow{\sim} \mathbb{F}\mathrm{Cospan}(\mathbf{C})$ is part of a double equivalence by Theorem 6.4.

Next, if both double categories ${}_L\mathbb{C}\mathrm{sp}(\mathbf{D})$ and $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$ are symmetric monoidal, as they are if both $\mathbf{C}$ and $\mathbf{D}$ have finite colimits, then this equivalence of double categories $\mathbb{E}\colon {}_L\mathbb{C}\mathrm{sp}(\mathbf{D}) \to \mathbb{F}\mathrm{Cospan}(\mathbf{C})$ will be symmetric monoidal. First note that we have an isomorphism $\epsilon\colon 1_{\mathbb{F}\mathrm{Cospan}(\mathbf{C})} \to \mathbb{E}(1_{{}_L\mathbb{C}\mathrm{sp}(\mathbf{D})})$ and natural isomorphisms $\mu_{c_1,c_2}\colon \mathbb{E}(c_1)\otimes\mathbb{E}(c_2) \to \mathbb{E}(c_1\otimes c_2)$ for every pair of objects $c_1, c_2 \in {}_L\mathbb{C}\mathrm{sp}(\mathbf{D})$ both of which are given by identities since both double categories ${}_L\mathbb{C}\mathrm{sp}(\mathbf{D})$ and $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$ have $\mathbf{C}$ as their category of objects and $\mathbb{E}_0 = \mathrm{id}_{\mathbf{C}}$. The diagrams containing these morphisms that are required to commute do so trivially.

For the arrow component $\mathbb{E}_1$, we have an isomorphism $\delta\colon U_{1_{\mathbb{F}\mathrm{Cospan}(\mathbf{C})}} \to \mathbb{E}(U_{1_{{}_L\mathbb{C}\mathrm{sp}(\mathbf{D})}})$ where the horizontal 1-cell $U_{1_{\mathbb{F}\mathrm{Cospan}(\mathbf{C})}}$ is given by:

$$\begin{tikzcd}
& 1_{\mathbf{C}} & \\
1_{\mathbf{C}} \arrow[ur, "1"] & & 1_{\mathbf{C}} \arrow[ul, "1"']
\end{tikzcd} \qquad !_{1_{\mathbf{C}}} \in F(1_{\mathbf{C}})$$

where $!_{1_{\mathbf{C}}} = \phi\colon \mathbf{1} \to F(1_{\mathbf{C}})$ is the trivial decoration which comes from the structure of the symmetric lax monoidal pseudofunctor $F\colon \mathbf{C} \to \mathbf{Cat}$. The horizontal 1-cell $U_{1_{{}_L\mathbb{C}\mathrm{sp}(\mathbf{D})}}$ is given by:

$$\begin{tikzcd}
& L(1_{\mathbf{C}}) & \\
L(1_{\mathbf{C}}) \arrow[ur, "1"] & & L(1_{\mathbf{C}}) \arrow[ul, "1"']
\end{tikzcd}$$

where here we make use of the fact that the left adjoint $L\colon (\mathbf{C}, +, 1_{\mathbf{C}}) \to (\mathbf{D}, +, 1_{\mathbf{D}})$ preserves all colimits and thus $L(1_{\mathbf{C}}) \cong 1_{\mathbf{D}}$. The horizontal 1-cell $\mathbb{E}(U_{1_{{}_L\mathbb{C}\mathrm{sp}(\mathbf{D})}})$ is given by the pair:

$$\begin{tikzcd}
& R(L(1_{\mathbf{C}})) & \\
1_{\mathbf{C}} \arrow[ur, "\eta_{1_{\mathbf{C}}}"] & & 1_{\mathbf{C}} \arrow[ul, "\eta_{1_{\mathbf{C}}}"']
\end{tikzcd} \qquad !_{1_{\mathbf{C}}} \in F(R(L(1_{\mathbf{C}}))) \cong F(1_{\mathbf{C}})$$

The isomorphism δ is then given by the 2-morphism:

$$\begin{tikzcd}
1_{\mathbf{C}} \arrow[r, "1"] \arrow[d, "1"'] & 1_{\mathbf{C}} \arrow[d, "\eta_{1_{\mathbf{C}}}"'] & 1_{\mathbf{C}} \arrow[l, "1"'] \arrow[d, "1"] & !_{1_{\mathbf{C}}} \in F(1_{\mathbf{C}}) \\
1_{\mathbf{C}} \arrow[r, "\eta_{1_{\mathbf{C}}}"] & R(L(1_{\mathbf{C}})) & 1_{\mathbf{C}} \arrow[l, "\eta_{1_{\mathbf{C}}}"'] & !_{1_{\mathbf{C}}} \in F(R(L(1_{\mathbf{C}}))) \cong F(1_{\mathbf{C}})
\end{tikzcd}$$

$$\iota_{\eta_{1_{\mathbf{C}}}} \colon F(\eta_{1_{\mathbf{C}}}(!_{1_{\mathbf{C}}})) \to !_{1_{\mathbf{C}}}$$

of $\mathbb{F}$Cospan($\mathbf{C}$).

Given two horizontal 1-cells M and N of ${}_L\mathbb{C}\mathrm{sp}(\mathbf{D})$:

$$\begin{tikzcd}
& d & & & d' & \\
L(c_1) \arrow[ur, "i"] & & L(c_2) \arrow[ul, "o"'] & L(c_1') \arrow[ur, "i'"] & & L(c_2') \arrow[ul, "o'"']
\end{tikzcd}$$

their images $\mathbb{E}(M)$ and $\mathbb{E}(N)$ are given by:

$$\begin{tikzcd}
& R(d) & & & R(d') & \\
c_1 \arrow[ur, "R(i)\eta_{c_1}"] & & c_2 \arrow[ul, "R(o)\eta_{c_2}"'] & c_1' \arrow[ur, "R(i')\eta_{c_1'}"] & & c_2' \arrow[ul, "R(o')\eta_{c_2'}"']
\end{tikzcd}$$

$$d \in F(R(d)) \qquad\qquad d' \in F(R(d'))$$

and so $\mathbb{E}(M) \otimes \mathbb{E}(N)$ is given by:

$$\begin{tikzcd}
& R(d) + R(d') & \\
c_1 + c_1' \arrow[ur, "R(i)\eta_{c_1} + R(i')\eta_{c_1'}"] & & c_2 + c_2' \arrow[ul, "R(o)\eta_{c_2} + R(o)\eta_{c_2'}"']
\end{tikzcd}$$

$$\hat{d} \in F(R(d) + R(d'))$$

where

$$\hat{d} \colon 1 \xrightarrow{\lambda^{-1}} 1 \times 1 \xrightarrow{d \times d'} F(R(d)) \times F(R(d')) \xrightarrow{\phi_{R(d),R(d')}} F(R(d) + R(d')).$$

On the other hand, $M \otimes N$ is given by

$$\begin{tikzcd}
& d + d' & \\
L(c_1 + c_1') \arrow[ur, "i + i'"] & & L(c_2 + c_2') \arrow[ul, "o + o'"']
\end{tikzcd}$$

and $\mathbb{E}(M \otimes N)$ is given by:

$$\begin{tikzcd} & R(d+d') & \\ c_1+c_1' \arrow[ur, "R(i+i')\eta_{c_1+c_1'}"] & & c_2+c_2' \arrow[ul, "R(o+o')\eta_{c_2+c_2'}"'] \end{tikzcd}$$

$$d + d' \in F(R(d + d'))$$

We then have a 2-isomorphism $\mu_{M,N} \colon E(M) \otimes E(N) \xrightarrow{\sim} E(M \otimes N)$ in $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$ given by:

$$\begin{tikzcd} c_1+c_1' \arrow[r, "R(i)\eta_{c_1}+R(i')\eta_{c_1'}"] \arrow[d, "1"'] & R(d)+R(d') \arrow[d, "\kappa"'] & c_2+c_2' \arrow[l, "R(o)\eta_{c_2}+R(o')\eta_{c_2'}"'] \arrow[d, "1"] & \hat{d} \in F(R(d)+R(d')) \\ c_1+c_1' \arrow[r, "R(i+i')\eta_{c_1+c_1'}"] & R(d+d') & c_2+c_2' \arrow[l, "R(o+o')\eta_{c_2+c_2'}"'] & d+d' \in F(R(d+d')) \end{tikzcd}$$

$$\iota_\mu \colon F(\kappa)(\hat{d}) \to d + d'$$

The isomorphisms δ and μ satisfy the left and right unitality squares, associativity hexagon and braiding square. Let M_1, M_2 and M_3 be horizontal 1-cells in ${}_L\mathbb{C}\mathrm{sp}(\mathbf{D})$ given by:

$$\begin{tikzcd} & d_1 & \\ L(c_1) \arrow[ur, "i_1"] & & L(c_1') \arrow[ul, "o_1"'] \end{tikzcd} \quad \begin{tikzcd} & d_2 & \\ L(c_2) \arrow[ur, "i_2"] & & L(c_2') \arrow[ul, "o_2"'] \end{tikzcd} \quad \begin{tikzcd} & d_3 & \\ L(c_3) \arrow[ur, "i_3"] & & L(c_3') \arrow[ul, "o_3"'] \end{tikzcd}$$

The left unitality square:

$$\begin{tikzcd} 1_{\mathbb{F}\mathrm{Cospan}(\mathbf{C})} \otimes \mathbb{E}(M_1) \arrow[r, "\delta \otimes 1"] \arrow[d, "\lambda"'] & \mathbb{E}(1_{{}_L\mathbb{C}\mathrm{sp}(\mathbf{D})}) \otimes \mathbb{E}(M_1) \arrow[d, "\mu_{1,M_1}"] \\ \mathbb{E}(M_1) & \mathbb{E}(1_{{}_L\mathbb{C}\mathrm{sp}(\mathbf{D})} \otimes M_1) \arrow[l, "\mathbb{E}(\lambda)"'] \end{tikzcd}$$

has underlying maps of cospans given by:

$$\begin{tikzcd}
c_1 \arrow[r, "R(i_1)\eta_{c_1}"] & R(d_1) & c_1' \arrow[l, "R(o_1)\eta_{c_1'}"'] & \mathbb{E}(M_1) \\
1_{\mathbf{C}} + c_1 \arrow[u, "\lambda_{\mathbf{C}}"] \arrow[d, "1"'] \arrow[r, "1 + R(i_1)\eta_{c_1}"] & 1_{\mathbf{C}} + R(d_1) \arrow[u, "\lambda_{\mathbf{C}}", "\iota_1"'] \arrow[d, "\eta_{1_{\mathbf{C}}} + 1"', "\iota_2"] & 1_{\mathbf{C}} + c_1' \arrow[u, "\lambda_{\mathbf{C}}"] \arrow[d, "1"'] \arrow[l, "1 + R(o_1)\eta_{c_1'}"'] & 1_{\mathbb{F}\mathrm{Cospan}(\mathbf{C})} \otimes \mathbb{E}(M_1) \arrow[u, "\lambda"] \arrow[d, "\delta \otimes 1"'] \\
1_{\mathbf{C}} + c_1 \arrow[d, "1"'] \arrow[r, "\eta_{1_{\mathbf{C}}} + R(i_1)\eta_{c_1}"] & R(L(1_{\mathbf{C}})) + R(d_1) \arrow[d, "\mu_{L(1_{\mathbf{C}}),d_1}"', "\iota_3"] & 1_{\mathbf{C}} + c_1' \arrow[d, "1"'] \arrow[l, "\eta_{1_{\mathbf{C}}} + R(o_1)\eta_{c_1'}"'] & \mathbb{E}(1_{L\mathbb{C}\mathrm{sp}(\mathbf{D})}) \otimes \mathbb{E}(M_1) \arrow[d, "\mu_{1,M_1}"'] \\
1_{\mathbf{C}} + c_1' \arrow[d, "\lambda_{\mathbf{C}}"'] \arrow[r, "(\mu_{L(1_{\mathbf{C}}),d_1})(\eta_{1_{\mathbf{C}}} + R(i_1)\eta_{c_1})"] & R(L(1_{\mathbf{C}}) + d_1) \arrow[d, "R(\lambda_{\mathbf{D}})"', "\iota_4"] & 1_{\mathbf{C}} + c_1' \arrow[d, "\lambda_{\mathbf{C}}"'] \arrow[l, "(\mu_{L(1_{\mathbf{C}}),d_1})(\eta_{1_{\mathbf{C}}} + R(o_1)\eta_{c_1'})"'] & \mathbb{E}(1_{L\mathbb{C}\mathrm{sp}(\mathbf{D})} \otimes M_1) \arrow[d, "\mathbb{E}(\lambda)"'] \\
c_1 \arrow[r, "R(i_1)\eta_{c_1}"] & R(d_1) & c_1' \arrow[l, "R(o_1)\eta_{c_1'}"'] & \mathbb{E}(M_1)
\end{tikzcd}$$

with the corresponding maps of decorations amounting to the following commutative diagram in $F(R(d_1))$:

$$\begin{tikzcd}
F(\lambda_{\mathbf{C}})(!_{1_{\mathbf{C}}} + d_1) \arrow[r, "F(R(\lambda_{\mathbf{D}})(\mu_{L(1_{\mathbf{C}}),d_1}))(\iota_2)"] \arrow[d, "\iota_1"'] & F(R(\lambda_{\mathbf{D}})(\mu_{L(1_{\mathbf{C}}),d_1}))(!_{R(L(1_{\mathbf{C}}))} + d_1) \arrow[d, "F(R(\lambda_{\mathbf{D}}))(\iota_3)"] \\
d_1 & F(R(\lambda_{\mathbf{D}}))(d_{!+1}) \arrow[l, "\iota_4"']
\end{tikzcd}$$

where $d_{!+1}$ is the decoration d_1 on the element $R(L(1_{\mathbf{C}}) + d_1) \in \mathbf{C}$. The above square commutes because

$$F(\lambda_{\mathbf{C}})(!_{1_{\mathbf{C}}} + d_1) = F(R(\lambda_{\mathbf{D}})(\mu_{L(1_{\mathbf{C}}),d_1})(\eta_{1_{\mathbf{C}}} + 1))(!_{1_{\mathbf{C}}} + d_1)$$

as the above diagram of maps of cospans commutes. The right unitality square is similar. The associator hexagon:

$$\begin{tikzcd}
(\mathbb{E}(M_1) \otimes \mathbb{E}(M_2)) \otimes \mathbb{E}(M_3) \arrow[r, "\mu_{M_1,M_2} \otimes 1"] \arrow[d, "a'"'] & \mathbb{E}(M_1 \otimes M_2) \otimes \mathbb{E}(M_3) \arrow[r, "\mu_{M_1 \otimes M_2, M_3}"] & \mathbb{E}((M_1 \otimes M_2) \otimes M_3) \arrow[d, "\mathbb{E}(a)"] \\
\mathbb{E}(M_1) \otimes (\mathbb{E}(M_2) \otimes \mathbb{E}(M_3)) \arrow[r, "1 \otimes \mu_{M_2,M_3}"] & \mathbb{E}(M_1) \otimes \mathbb{E}(M_2 \otimes M_3) \arrow[r, "\mu_{M_1, M_2 \otimes M_3}"] & \mathbb{E}(M_1 \otimes (M_2 \otimes M_3))
\end{tikzcd}$$

has underlying maps of cospans given by:

$$
\begin{tikzcd}
\mathbb{E}(M_1 \otimes (M_2 \otimes M_3)) \\
\mathbb{E}((M_1 \otimes M_2) \otimes M_3) \arrow[u, "\mathbb{E}(a)"] \\
\mathbb{E}(M_1 \otimes M_2) \otimes \mathbb{E}(M_3) \arrow[u, "\mu_{M_1 \otimes M_2, M_3}"] \\
(\mathbb{E}(M_1) \otimes \mathbb{E}(M_2)) \otimes \mathbb{E}(M_3) \arrow[u, "\mu_{M_1, M_2} \otimes 1"] \arrow[d, "a'"'] \\
\mathbb{E}(M_1) \otimes (\mathbb{E}(M_2) \otimes \mathbb{E}(M_3)) \arrow[d, "1 \otimes \mu_{M_2, M_3}"'] \\
\mathbb{E}(M_1) \otimes \mathbb{E}(M_2 \otimes M_3) \arrow[d, "\mu_{M_1, M_2 \otimes M_3}"'] \\
\mathbb{E}(M_1 \otimes (M_2 \otimes M_3))
\end{tikzcd}
$$

$$
\begin{tikzcd}
c_1 + (c_2 + c_3) \arrow[r, "R(i_1 + (i_2 + i_3))\eta_{c_1 + (c_2 + c_3)}"] & R(d_1 + (d_2 + d_3)) & c'_1 + (c'_2 + c'_3) \arrow[l, "R(o_1 + (o_2 + o_3))\eta_{c'_1 + (c'_2 + c'_3)}"'] \\
(c_1 + c_2) + c_3 \arrow[u, "a_{\mathbf{C}}"] \arrow[r, "R((i_1 + i_2) + i_3)\eta_{(c_1 + c_2) + c_3}"] & R((d_1 + d_2) + d_3) \arrow[u, "R(a_{\mathbf{D}})", "\iota_3"'] & (c'_1 + c'_2) + c'_3 \arrow[u, "a_{\mathbf{C}}"] \arrow[l, "R((o_1 + o_2) + o_3)\eta_{(c'_1 + c'_2) + c'_3}"'] \\
(c_1 + c_2) + c_3 \arrow[u, "1"] \arrow[r, "R(i_1 + i_2)\eta_{c_1 + c_2} + R(i_3)\eta_{c_3}"] & R(d_1 + d_2) + R(d_3) \arrow[u, "\kappa", "\iota_2"'] & (c'_1 + c'_2) + c'_3 \arrow[u, "1"] \arrow[l, "R(o_1 + o_2)\eta_{c'_1 + c'_2} + R(o_3)\eta_{c'_3}"'] \\
(c_1 + c_2) + c_3 \arrow[u, "1"] \arrow[d, "a_{\mathbf{C}}"'] \arrow[r, "(R(i_1)\eta_{c_1} + R(i_2)\eta_{c_2}) + R(i_3)\eta_{c_3}"] & (R(d_1) + R(d_2)) + R(d_3) \arrow[u, "\kappa + 1", "\iota_1"'] \arrow[d, "a_{\mathbf{C}}"', "\iota_4"] & (c'_1 + c'_2) + c'_3 \arrow[u, "1"] \arrow[d, "a_{\mathbf{C}}"'] \arrow[l, "(R(o_1)\eta_{c'_1} + R(o_2)\eta_{c'_2}) + R(o_3)\eta_{c'_3}"'] \\
c_1 + (c_2 + c_3) \arrow[d, "1"'] \arrow[r, "R(i_1)\eta_{c_1} + (R(i_2)\eta_{c_2} + R(i_3)\eta_{c_3})"] & R(d_1) + (R(d_2) + R(d_3)) \arrow[d, "1 + \kappa"', "\iota_5"] & c'_1 + (c'_2 + c'_3) \arrow[d, "1"'] \arrow[l, "R(o_1)\eta_{c'_1} + (R(o_2)\eta_{c'_2} + R(o_3)\eta_{c'_3})"'] \\
c_1 + (c_2 + c_3) \arrow[d, "1"'] \arrow[r, "R(i_1)\eta_{c_1} + R(i_2 + i_3)\eta_{c_2 + c_3}"] & R(d_1) + R(d_2 + d_3) \arrow[d, "\kappa"', "\iota_6"] & c'_1 + (c'_2 + c'_3) \arrow[d, "1"'] \arrow[l, "R(o_1)\eta_{c'_1} + R(o_2 + o_3)\eta_{c'_2 + c'_3}"'] \\
c_1 + (c_2 + c_3) \arrow[r, "R(i_1 + (i_2 + i_3))\eta_{c_1 + (c_2 + c_3)}"] & R(d_1 + (d_2 + d_3)) & c'_1 + (c'_2 + c'_3) \arrow[l, "R(o_1 + (o_2 + o_3))\eta_{c'_1 + (c'_2 + c'_3)}"']
\end{tikzcd}
$$

with the corresponding maps of decorations amounting to the following commutative diagram in $F(R(d_1 + (d_2 + d_3)))$:

$$\begin{tikzcd}
F((\kappa)(1+\kappa)(a_{\mathbf{C}}))((d_1+d_2)+d_3) \arrow[r, "F((R(a_{\mathbf{D}}))(\kappa))(\iota_1)"] \arrow[d, "F((\kappa)(1+\kappa))(\iota_4)"'] & F((R(a_{\mathbf{D}}))(\kappa))((d_1+d_2)+d_3) \arrow[d, "F(R(a_{\mathbf{D}}))(\iota_2)"] \\
F((\kappa)(1+\kappa))((d_1+d_2)+d_3) \arrow[d, "F(\kappa)(\iota_5)"'] & F(R(a_{\mathbf{D}}))((d_1+d_2)+d_3) \arrow[d, "\iota_3"] \\
F(\kappa)(d_1+(d_2+d_3)) \arrow[r, "\iota_6"] & d_1+(d_2+d_3)
\end{tikzcd}$$

The above square commutes because

$$F((\kappa)(1+\kappa)(a_{\mathbf{C}}))((d_1+d_2)+d_3) = F((R(a_{\mathbf{D}}))(\kappa)(\kappa+1))((d_1+d_2)+d_3)$$

as the above diagram of maps of cospans commutes. Lastly, the braiding square:

$$\begin{tikzcd}
\mathbb{E}(M_1)\otimes\mathbb{E}(M_2) \arrow[r, "\beta"] \arrow[d, "\mu_{M_1,M_2}"'] & \mathbb{E}(M_2)\otimes\mathbb{E}(M_1) \arrow[d, "\mu_{M_2,M_1}"] \\
\mathbb{E}(M_1\otimes M_2) \arrow[r, "\mathbb{E}(\beta)"] & \mathbb{E}(M_2\otimes M_1)
\end{tikzcd}$$

has underlying map of cospans given by:

$$\begin{tikzcd}
c_2+c_1 \arrow[r, "R(i_2+i_1)\eta_{c_2+c_1}"] & R(d_2+d_1) & c_2'+c_1' \arrow[l, "R(o_2+o_1)\eta_{c_2'+c_1'}"'] & \mathbb{E}(M_2\otimes M_1) \\
c_2+c_1 \arrow[u, "1"] \arrow[r, "R(i_2)\eta_{c_2}+R(i_1)\eta_{c_1}"] & R(d_2)+R(d_1) \arrow[u, "\kappa", "\iota_2"'] & c_2'+c_1' \arrow[u, "1"] \arrow[l, "R(o_2)\eta_{c_2'}+R(o_1)\eta_{c_1'}"'] & \mathbb{E}(M_2)\otimes\mathbb{E}(M_1) \arrow[u, "\mu_{M_2,M_1}"] \\
c_1+c_2 \arrow[u, "\beta_{\mathbf{C}}"] \arrow[r, "R(i_1)\eta_{c_1}+R(i_2)\eta_{c_2}"] \arrow[d, "1"'] & R(d_1)+R(d_2) \arrow[u, "\beta_{\mathbf{C}}", "\iota_1"'] \arrow[d, "\kappa"', "\iota_3"] & c_1'+c_2' \arrow[u, "\beta_{\mathbf{C}}"] \arrow[l, "R(o_1)\eta_{c_1'}+R(o_2)\eta_{c_2'}"'] \arrow[d, "1"'] & \mathbb{E}(M_1)\otimes\mathbb{E}(M_2) \arrow[u, "\beta"] \arrow[d, "\mu_{M_1,M_2}"'] \\
c_1+c_2 \arrow[r, "R(i_1+i_2)\eta_{c_1+c_2}"] \arrow[d, "\beta_{\mathbf{C}}"'] & R(d_1+d_2) \arrow[d, "R(\beta_{\mathbf{D}})"', "\iota_4"] & c_1'+c_2' \arrow[l, "R(o_1+o_2)\eta_{c_1'+c_2'}"'] \arrow[d, "\beta_{\mathbf{C}}"'] & \mathbb{E}(M_1\otimes M_2) \arrow[d, "\mathbb{E}(\beta)"'] \\
c_2+c_1 \arrow[r, "R(i_2+i_1)\eta_{c_2+c_1}"] & R(d_2+d_1) & c_2'+c_1' \arrow[l, "R(o_2+o_1)\eta_{c_2'+c_1'}"'] & \mathbb{E}(M_2\otimes M_1)
\end{tikzcd}$$

with the corresponding maps of decorations amounting to the following commutative diagram in $F(R(d_2 + d_1))$:

$$\begin{tikzcd}
F((\kappa)(\beta_{\mathbf{C}}))(d_1+d_2) \arrow[r, "F(\kappa)(\iota_1)"] \arrow[d, "F(R(\beta_{\mathbf{D}}))(\iota_3)"'] & F(\kappa)(d_2+d_1) \arrow[d, "\iota_2"] \\
F(R(\beta_{\mathbf{D}}))(d_1+d_2) \arrow[r, "\iota_4"] & d_2+d_1
\end{tikzcd}$$

The above square commutes because

$$F((\kappa)(\beta_{\mathbf{C}}))(d_1 + d_2) = F((R(\beta_{\mathbf{D}}))(\kappa))(d_1 + d_2)$$

as the above diagram of maps of cospans commutes. Thus the double functor $\mathbb{E}\colon {}_L\mathbb{C}\mathrm{sp}(\mathbf{D}) \to \mathbb{F}\mathrm{Cospan}(\mathbf{C})$ is symmetric monoidal. $\square$

Using a result of Shulman [15], each of the isofibrant symmetric monoidal double categories $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$ and ${}_L\mathrm{Csp}(\mathbf{D})$ give rise to underlying symmetric monoidal bicategories, namely $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$ induces a symmetric monoidal bicategory $H(\mathbb{F}\mathrm{Cospan}(\mathbf{C}))$ which has:

(i) objects as those of $\mathbf{C}$,
(ii) morphisms as horizontal 1-cells of $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$, and
(iii) 2-morphisms as globular 2-morphisms of $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$.

Likewise, ${}_L\mathrm{Csp}(\mathbf{D})$ induces a symmetric monoidal bicategory $H({}_L\mathrm{Csp}(\mathbf{D}))$ which has:

(i) objects as those of $\mathbf{C}$,
(ii) morphisms as horizontal 1-cells of ${}_L\mathrm{Csp}(\mathbf{D})$, and
(iii) 2-morphisms as globular 2-morphisms of ${}_L\mathrm{Csp}(\mathbf{D})$.

Another result of Shulman [16] is the following:

Proposition 6.8 (Shulman, Prop. B.3)**.** *An equivalence of fibrant double categories induces a biequivalence of horizontal bicategories.*

Corollary 6.9. *The bicategories $H(\mathbb{F}\mathrm{Cospan}(\mathbf{C}))$ and $H({}_L\mathrm{Csp}(\mathbf{D}))$ are biequivalent.*

Is this biequivalnce symmetric monoidal? Probably

We can also define the part of the double equivalence $\mathbb{G}\colon \mathbb{F}\mathrm{Cospan}(\mathbf{C}) \to_L \mathrm{Csp}(\mathbf{D})$ which goes in the other direction: again, the object component of this double functor will be $\mathbb{G}_0 = \mathrm{id}_{\mathbf{C}}$.

Given a horizontal 1-cell M of $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$:

$$\begin{tikzcd} & c & \\ c_1 \arrow[ur, "I"] & & c_2 \arrow[ul, "O"'] \end{tikzcd}$$

$$d \in F(c)$$

the image $\mathbb{G}(M)$ is the horizontal 1-cell in ${}_L\mathbb{C}\mathrm{sp}(\mathbf{D})$ given by:

$$\begin{tikzcd} & d & \\ L(c_1) \arrow[ur, "L(I)"] & & L(c_2) \arrow[ul, "L(O)"'] \end{tikzcd}$$

and similarly, given a 2-morphism $(f, h, g, \iota)\colon M \to M'$ of $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$:

$$d \in F(c)$$

$$\begin{tikzcd} c_1 \arrow[r, "I"] \arrow[d, "f"'] & c \arrow[d, "h"'] & c_2 \arrow[l, "O"'] \arrow[d, "g"'] \\ c_1' \arrow[r, "I'"] & c' & c_2' \arrow[l, "O'"'] \end{tikzcd}$$

$$d' \in F(c')$$

$$\iota\colon F(h)(d) \to d'$$

the image in ${}_L\mathbb{C}\mathrm{sp}(\mathbf{D})$ is given by the 2-morphism:

$$\begin{tikzcd} L(c_1) \arrow[r, "L(I)"] \arrow[d, "L(f)"'] & d \arrow[d, "\alpha"'] & L(c_2) \arrow[l, "L(O)"'] \arrow[d, "L(g)"'] \\ L(c_1') \arrow[r, "L(I')"] & d' & L(c_2') \arrow[l, "L(O')"'] \end{tikzcd}$$

where $\alpha\colon d \to d'$ is a morphism in the Grothendieck construction of F given by $\alpha = (h\colon c \to c', \iota\colon F(h)(d) \to d')$.

Next, we exhibit natural isomorphisms $\eta\colon \mathrm{id}_{{}_L\mathbb{C}\mathrm{sp}(\mathbf{D})} \cong \mathbb{G}\mathbb{E}$ and $\epsilon\colon \mathbb{E}\mathbb{G} \cong \mathrm{id}_{\mathbb{F}\mathrm{Cospan}(\mathbf{C})}$. First we compute the composites $\mathbb{G}\mathbb{E}$ and $\mathbb{E}\mathbb{G}$. On the object categories, both composites are $\mathrm{id}_{\mathbf{C}}$ and we have natural isomorphism $\eta\colon \mathrm{id}_{\mathbf{C}} \cong \mathbb{G}_0\mathbb{E}_0$ and $\epsilon\colon \mathbb{E}_0\mathbb{G}_0 \cong \mathrm{id}_{\mathbf{C}}$.

Given a horizontal 1-cell M in ${}_L\mathbb{C}\mathrm{sp}(\mathbf{D})$:

$$\begin{tikzcd} & d & \\ L(c_1) \arrow[ur, "i"] & & L(c_2) \arrow[ul, "o"'] \end{tikzcd}$$

the horizontal 1-cell $\mathbb{E}(M)$ is given by:

$$\begin{tikzcd} & R(d) & \\ c_1 \arrow[ur, "R(i)"] & & c_2 \arrow[ul, "R(o)"'] \end{tikzcd}$$

$$d \in F(R(d))$$

and then the horizontal 1-cell $\mathbb{G}\mathbb{E}(M)$ is given by:

$$\begin{tikzcd} & d & \\ L(c_1) \arrow[ur, "L(R(i))"] & & L(c_2) \arrow[ul, "L(R(o))"'] \end{tikzcd}$$

What can we say about LR? Find a 2-iso between these in ${}_LCsp(D)$. Then we can find a 2-isomorphism $\eta\colon M \xrightarrow{\sim} \mathbb{GE}(M)$ in ${}_L\mathbb{C}\mathrm{sp}(\mathbf{C})$ given by:

$$\begin{tikzcd} L(c_1) \arrow[r, "i"] \arrow[d, "1"'] & d \arrow[d, "1"'] & L(c_2) \arrow[l, "o"'] \arrow[d, "1"'] \\ L(c_1) \arrow[r, "L(R(i))"] & d & L(c_2) \arrow[l, "L(R(o))"'] \end{tikzcd}$$

On the other hand, given a horizontal 1-cell N in $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$:

$$\begin{tikzcd} & c & \\ c_1 \arrow[ur, "I"] & & c_2 \arrow[ul, "O"'] \end{tikzcd}$$

$$d \in F(c)$$

the horizontal 1-cell $\mathbb{G}(N)$ is given by:

$$\begin{tikzcd} & d & \\ L(c_1) \arrow[ur, "L(I)"] & & L(c_2) \arrow[ul, "L(O)"'] \end{tikzcd}$$

and then the horizontal 1-cell $\mathbb{EG}(N)$ is given by:

$$\begin{tikzcd} & R(d) & \\ c_1 \arrow[ur, "R(L(I))"] & & c_2 \arrow[ul, "R(L(O))"'] \end{tikzcd}$$

$$d \in F(R(d))$$

Then we can find a 2-isomorphism $\epsilon\colon \mathbb{EG}(N) \xrightarrow{\sim} N$ in $\mathbb{F}\mathrm{Cospan}(\mathbf{C})$ given by:

$$d \in F(R(d))$$

$$\begin{tikzcd} c_1 \arrow[r, "R(L(I))"] \arrow[d, "1"'] & R(d) \arrow[d, "e"'] & c_2 \arrow[l, "R(L(O))"'] \arrow[d, "1"'] \\ c_1 \arrow[r, "I"] & c & c_2 \arrow[l, "O"'] \end{tikzcd}$$

$$d \in F(c)$$

$$\iota\colon F(e)(d) \to d$$

where $R(L(I)) = I$ and $R(L(O)) = O$ since $L\colon \mathbf{C} \to \mathbf{D} \cong \int F$ is a left adjoint with right adjoint $R\colon \int F \to \mathbf{C}$ as its inverse.

7. An example

As an example, let $L\colon \mathbf{FinSet} \to \mathbf{Graph}$ be the functor which is left adjoint to the forgetful functor $R\colon \mathbf{Graph} \to \mathbf{FinSet}$; the functor L maps a finite set N to the discrete graph with N as its set of vertices and no edges. Since **Graph** is a topos and thus has finite colimits, we can then obtain a symmetric monoidal double category ${}_L\mathbb{C}\mathrm{sp}(\mathbf{Graph})$ which has:

(i) finite sets as objects,
(ii) functions as vertical 1-morphisms,
(iii) cospans of graphs of the form

$$\begin{tikzcd} & M & \\ L(N) \arrow[ur, "i"] & & L(N') \arrow[ul, "o"'] \end{tikzcd}$$

as horizontal 1-cells, where $L(N)$ and $L(N')$ are discrete graphs on the sets N and N', respectively, M is a graph and i and o are graph morphisms, and

(iv) maps of cospans of graphs of the form

$$\begin{tikzcd} L(N_1) \arrow[r, "i"] \arrow[d, "L(f)"'] & M \arrow[d, "h"] & L(N_2) \arrow[l, "o"'] \arrow[d, "L(g)"] \\ L(N_1') \arrow[r, "i'"] & M' & L(N_2') \arrow[l, "o'"'] \end{tikzcd}$$

as 2-morphisms, where $L(f)$ and $L(g)$ are maps of discrete graphs given by the underlying functions f and g, respectively, and $h\colon M \to M'$ is a graph morphism.

On the other hand, let $F\colon \mathbf{FinSet} \to \mathbf{Cat}$ be the symmetric lax monoidal pseudofunctor that assigns to a finite set N the *category* of all graph structures whose underlying set of vertices is N. Using Theorem 3.8, we can then obtain a symmetric monoidal double category $\mathbb{F}\mathrm{Cospan}(\mathbf{FinSet})$ which has:

(i) finite sets as objects,
(ii) functions as vertical 1-morphisms,
(iii) horizontal 1-cells are given by pairs:

$$\begin{tikzcd} & N & \\ N_1 \arrow[ur, "i"] & & N_2 \arrow[ul, "o"'] \end{tikzcd} \qquad M \in F(N)$$

and

(iv) 2-morphisms as maps of cospans

$$\begin{tikzcd} N_1 \arrow[r, "i"] \arrow[d, "f"'] & N \arrow[d, "h"] & N_2 \arrow[l, "o"'] \arrow[d, "g"] & M \in F(N) \\ N_1' \arrow[r, "i'"] & N' & N_2' \arrow[l, "o'"'] & M' \in F(N') \end{tikzcd}$$

together with a graph morphism $\iota\colon F(h)(M) \to M'$ in $F(N')$.

We thus have two symmetric monoidal double categories: ${}_L\mathbb{C}\mathrm{sp}(\mathbf{Graph})$ obtained from structured cospans and $\mathbb{F}\mathrm{Cospan}(\mathbf{FinSet})$ using Theorem 3.8. These two double categories are in fact equivalent as symmetric monoidal double categories which we now show.

Define a double functor $\mathbb{E}\colon {}_L\mathbb{C}\mathrm{sp}(\mathbf{Graph}) \to \mathbb{F}\mathrm{Cospan}(\mathbf{FinSet})$ as follows: $\mathbb{E}_0 = \mathrm{id}_{\mathbf{FinSet}}$ as both double categories ${}_L\mathbb{C}\mathrm{sp}(\mathbf{Graph})$ and $\mathbb{F}\mathrm{Cospan}(\mathbf{FinSet})$ have $\mathbf{FinSet}$ as their category of objects. The identity functor on $\mathbf{FinSet}$ is trivially an equivalence. For the arrow component $\mathbb{E}_1$, given a horizontal 1-cell of ${}_L\mathbb{C}\mathrm{sp}(\mathbf{Graph})$, which is a cospan in $\mathbf{Graph}$ of the form:

$$\begin{tikzcd} & M & \\ L(N) \arrow[ur, "i"] & & L(N') \arrow[ul, "o"'] \end{tikzcd}$$

the functor $\mathbb{E}_1$ maps this horizontal 1-cell to the horizontal 1-cell of $\mathbb{F}\mathrm{Cospan}(\mathbf{FinSet})$ given by:

$$\begin{tikzcd} & R(M) & \\ N \arrow[ur, "R(i)"] & & N' \arrow[ul, "R(o)"'] \end{tikzcd} \qquad M \in F(R(M))$$

where $R\colon \mathbf{Graph} \to \mathbf{FinSet}$ is the forgetful functor, right adjoint to $L\colon \mathbf{FinSet} \to \mathbf{Graph}$. The image of a 2-morphism in ${}_L\mathbb{C}\mathrm{sp}(\mathbf{Graph})$ given by:

$$\begin{tikzcd} L(N_1) \arrow[r, "i"] \arrow[d, "L(f)"'] & M \arrow[d, "h"] & L(N_2) \arrow[l, "o"'] \arrow[d, "L(g)"] \\ L(N_1') \arrow[r, "i'"] & M' & L(N_2') \arrow[l, "o'"'] \end{tikzcd}$$

is the 2-morphism in $\mathbb{F}\mathrm{Cospan}(\mathbf{FinSet})$ given by:

$$\begin{tikzcd} N_1 \arrow[r, "R(i)"] \arrow[d, "f"'] & R(M) \arrow[d, "R(h)"] & N_2 \arrow[l, "R(o)"'] \arrow[d, "g"] \\ N_1' \arrow[r, "R(i')"] & R(M') & N_2' \arrow[l, "R(o')"'] \end{tikzcd} \qquad \begin{tikzcd} & F(R(M)) \arrow[d, "F(R(h))"] \\ 1 \arrow[ur, "M"] \arrow[r, "M'"'] & F(R(M')) \end{tikzcd}$$

together with a graph morphism $\iota_h\colon F(R(h))(M) \to M'$. Denoting a cospan by its apex, the map

$$\mathbb{E}_1\colon {}_{fL}\mathbb{C}\mathrm{sp}(\mathbf{Graph})_g(M, M') \to {}_{\mathbb{E}(f)}\mathbb{F}\mathrm{Cospan}(\mathbf{FinSet})_{\mathbb{E}(g)}(\mathbb{E}(M), \mathbb{E}(M'))$$

is bijective and thus the double functor $\mathbb{E}$ is full and faithful. To see that it is also essentially surjective, given a horizontal 1-cell in $\mathbb{F}\mathrm{Cospan}(\mathbf{FinSet})$:

$$\begin{tikzcd} & N & \\ N_1 \arrow[ur, "i"] & & N_2 \arrow[ul, "o"'] \end{tikzcd} \qquad M \in F(N)$$

there exists a horizontal 1-cell in ${}_L\mathbb{C}\mathrm{sp}(\mathbf{Graph})$:

$$\begin{tikzcd} & M & \\ L(N_1) \arrow[ur, "i'"] & & L(N_2) \arrow[ul, "o'"'] \end{tikzcd}$$

and a 2-isomorphism in $\mathbb{F}\mathrm{Cospan}(\mathbf{FinSet})$ from the image of the latter horizontal 1-cell to the former given by:

$$\begin{tikzcd} N_1 \arrow[r, "R(i')"] \arrow[d, "1"'] & R(M) \arrow[d, "1"'] & N_2 \arrow[l, "R(o')"'] \arrow[d, "1"'] & M \in F(R(M)) \\ N_1 \arrow[r, "i"] & N & N_2 \arrow[l, "o"'] & M \in F(N) \end{tikzcd}$$

$$\iota = \mathrm{id}_M \colon F(R(M)) \to M$$

which trivially commutes as $R(M) = N$ and $R(i')$ and $R(o')$ are the underlying maps of vertices of the graph morphisms i' and o' which act as the functions i and o, respectively. Given two composable horizontal 1-cells α and β of ${}_L\mathbb{C}\mathrm{sp}(\mathbf{Graph})$:

$$\begin{tikzcd} & M & & & M' & \\ L(N_1) \arrow[ur, "i_1"] & & L(N_2) \arrow[ul, "o_1"'] & L(N_2) \arrow[ur, "i_2"] & & L(N_3) \arrow[ul, "o_2"'] \end{tikzcd}$$

we have a natural isomorphism $\mathbb{E}_{\alpha,\beta} \colon \mathbb{E}(\alpha) \odot \mathbb{E}(\beta) \to \mathbb{E}(\alpha \odot \beta)$ given by:

$$\begin{tikzcd} N_1 \arrow[r, "j\psi R(i_1)"] \arrow[d, "1"'] & R(M) +_{N_2} R(M') \arrow[d] & N_3 \arrow[l, "j\psi R(o_2)"'] \arrow[d, "1"'] & M +_{L(N_2)} M' \in F(R(M) +_{N_2} R(M')) \\ N_1 \arrow[r, "U(i')"] & R(M +_{L(N_2)} M') & N_3 \arrow[l, "U(o')"'] & M +_{L(N_2)} M' \in F(R(M +_{L(N_2)} M')) \end{tikzcd}$$

The right adjoint $R \colon \mathrm{Graph} \to \mathrm{Set}$ is also a left adjoint $R \dashv K$ to the codiscrete graph functor $K \colon \mathrm{Set} \to \mathrm{Graph}$ which sends a finite set N to the complete graph with N as its underlying set of vertices. The functor $R \colon \mathrm{Graph} \to \mathrm{Set}$ thus also preserves colimits which yields an isomorphism

$$R(M +_{L(N_2)} M') \cong R(M) +_{R(L(N_2))} R(M')$$

and again we make use of the fact that the left adjoint $L \colon \mathrm{Set} \to \mathrm{Graph}$ is fully faithful and thus the unit

$$\eta_{N_2} \colon N_2 \to R(L(N_2))$$

is an isomorphism which gives

$$R(M +_{L(N_2)} M') \cong R(M) +_{R(L(N_2))} R(M') \cong R(M) +_{N_2} R(M').$$

Thus the double functor $\mathbb{E} \colon {}_L\mathbb{C}\mathrm{sp}(\mathbf{Graph}) \to \mathbb{F}\mathrm{Cospan}(\mathbf{Set})$ is strong and we have an equivalence of double categories ${}_L\mathbb{C}\mathrm{sp}(\mathrm{Graph}) \sim \mathbb{F}\mathrm{Cospan}(\mathrm{Set})$. Moreover, each of these categories is symmetric monoidal and the double functor $\mathbb{E}$ is symmetric monoidal; first, we have that the following diagrams commute.

$$\begin{tikzcd}
{}_L\mathbb{C}\mathrm{sp}(\mathrm{Graph})_1 \arrow[r, "\mathbb{E}_1"] \arrow[d, "S"'] & \mathbb{F}\mathrm{Cospan}(\mathrm{FinSet})_1 \arrow[d, "S"] \\
{}_L\mathbb{C}\mathrm{sp}(\mathrm{FinGraph})_0 = \mathrm{FinSet} \arrow[r, "\mathbb{E}_0 = \mathrm{id}_{\mathrm{FinSet}}"] & \mathbb{F}\mathrm{Cospan}(\mathrm{FinSet})_0 = \mathrm{FinSet}
\end{tikzcd}$$

$$\begin{tikzcd}
{}_L\mathbb{C}\mathrm{sp}(\mathrm{Graph})_1 \arrow[r, "\mathbb{E}_1"] \arrow[d, "T"'] & \mathbb{F}\mathrm{Cospan}(\mathrm{FinSet})_1 \arrow[d, "T"] \\
{}_L\mathbb{C}\mathrm{sp}(\mathrm{Graph})_0 = \mathrm{FinSet} \arrow[r, "\mathbb{E}_0 = \mathrm{id}_{\mathrm{FinSet}}"] & \mathbb{F}\mathrm{Cospan}(\mathrm{FinSet})_0 = \mathrm{FinSet}
\end{tikzcd}$$

Both diagrams are similar: focusing on the top one and starting with an object of ${}_L\mathbb{C}\mathrm{sp}(\mathrm{Graph})_1$ which is a cospan in Graph of the form:

$$\begin{tikzcd}
 & M & \\
L(N) \arrow[ur, "i"] & & L(N_2) \arrow[ul, "o"']
\end{tikzcd}$$

the functor $\mathbb{E}_1$ then maps this horizontal 1-cell to the horizontal 1-cell in $\mathbb{F}\mathrm{Cospan}(\mathrm{FinSet})$ which is given by the pair:

$$\begin{tikzcd}
 & R(M) & & \\
N_1 \arrow[ur, "R(i)"] & & N_2 \arrow[ul, "R(o)"'] & M \in F(R(M))
\end{tikzcd}$$

and the source of this horizontal 1-cell is the object $N_1 \in \mathrm{FinSet}$. Going the other way around the diagram also yields N_1. A morphism in ${}_L\mathbb{C}\mathrm{sp}(\mathrm{Graph})_1$ given by a map of cospans in Graph of the form:

$$\begin{tikzcd}
L(N_1) \arrow[r, "i_1"] \arrow[d, "L(f)"'] & M \arrow[d, "h"] & L(N_2) \arrow[l, "o_1"'] \arrow[d, "L(g)"] \\
L(N_1') \arrow[r, "i_2"] & M' & L(N_2') \arrow[l, "o_2"']
\end{tikzcd}$$

maps under the functor $\mathbb{E}_1$ to:

$$\begin{tikzcd}
N_1 \arrow[r, "R(i_1)"] \arrow[d, "f"'] & R(M) \arrow[d, "R(h)"] & N_2 \arrow[l, "R(o_1)"'] \arrow[d, "g"] & M \in F(R(M)) \\
N_1' \arrow[r, "R(i_2)"] & R(M') & N_2' \arrow[l, "R(o_2)"'] & M' \in F(R(M'))
\end{tikzcd}$$

together with a morphism $\iota\colon F(R(h))(M) \to M'$ in Graph which is really just the morphism $h \in \mathrm{Graph}$. The source of this morphism is then the morphism $f \in \mathrm{FinSet}$, compared with going the other way around the diagram which also results in f and thus both of these squares commute.

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