18/1/2021

Thm 3.2
Axiom $\Phi(M)\circ\Phi(N)\circ\Phi(P) \overset{\cong}{\rightrightarrows} \Phi(M\circ N\circ P)$

Can do $(\ldots)$ by brute force (axioms of lax mon pseudofun) since all maps are now known, using axiom $\Phi(M)\otimes\Phi(N)\otimes\Phi(P) \rightrightarrows \Phi(M\otimes N\otimes P)$, but once again prefer a higher-level solution.

OPP. CONVENTION FROM LAST NOTES

① Recall that the tensor axiom ended up being a result of an isomorphism of monoidal categories $(\int F, \bar{\otimes}) \cong (\int F, \otimes)$ where $x\otimes y = \phi_{m,n}(x,y)$ and $x\bar{\otimes} y = (F\nabla)\phi_{m+n,m+n}(F(\iota_m)x, F(\iota_n)y)$

and the isomorphism was an id-on-objects functor w. (strong) laxator

$$\begin{tikzcd}
Fm\times Fn \arrow[r, "\phi_{m,n}"] \arrow[d, "F\iota_m\times F\iota_n"'] \arrow[dr, phantom, "\overset{\Phi_{\iota_m,\iota_n}}{=}"] & F(m+n) \arrow[d, "F(\iota_m+\iota_n)"] \arrow[drr, equal] \arrow[dr, phantom, "\cong"] & & \\
F(m+n)\times F(m+n) \arrow[r, "\phi_{m+n,m+n}"'] & F(m+n+m+n) \arrow[rr, "F\nabla"'] & & F(m+n)
\end{tikzcd} =: \theta_{x,y}$$

So we already have that

BY NATURALITY OF θ

$$\begin{tikzcd}[column sep=small]
\phi(\phi(x,y),z) \arrow[r, "{\phi(\theta,1)}"] \arrow[d, "w"'] \arrow[dr, "\theta"] & \phi((F\nabla)\phi(F(\iota_m)x,F(\iota_n)y),z) \arrow[r, "\theta"] \arrow[d, phantom, "\cong"] & (F\nabla)\phi(F(k_{m+n})(F\nabla)\phi(F(\iota_m)x,F(\iota_n)y),F(k_p)z) \arrow[dd, "\alpha"] \\
 & (F\nabla)\phi(F(k_{m+n})\phi(x,y),F(k_p)z) \arrow[ur, "{(F\nabla)\phi(F(k_{m+n})\theta,1)}"'] & \\
\phi(x,\phi(y,z)) \arrow[r, "{\phi(1,\theta)}"'] \arrow[rr, bend left, "\theta"] & \phi(x,(F\nabla)\phi(F(\ell_n)y,F(\ell_p)z)) \arrow[r, "\theta"'] & (F\nabla)\phi(F(k_m)x,F(k_{n+p})(F\nabla)\phi(F(\ell_n)y,F(\ell_p)z))
\end{tikzcd}$$

$(F\nabla)\phi(1,F(k_{n+p})\theta)$ $\quad$ (*)

for maps as shown here

$$\begin{tikzcd}[column sep=small]
 & & m+n+p & & \\
 & m+n \arrow[ur, "k_{m+n}"] & & n+p \arrow[ul, "k_{n+p}"'] & \\
m \arrow[ur, "\iota_m"] \arrow[uurr, bend left, "k_m"] & & n \arrow[ul, "\iota_n"'] \arrow[ur, "\ell_n"] \arrow[uu, "k_n"] & & p \arrow[ul, "\ell_p"'] \arrow[uull, bend right, "k_p"']
\end{tikzcd}$$

where the top right expression essentially is $F(k_{m+n})(F(\iota_m)x\otimes_{m+n}F(\iota_n)y)\otimes_{m+n+p}F(k_p)z$

$(F(k_{m+n})$ is strong monoidal and pseudo$)$ $\cong (F(k_m)x\otimes_{m+n+p}F(k_n)y)\otimes_{m+n+p}F(k_p)z$

and the bottom right expression essentially is $F(k_m)x\otimes_{m+n+p}(F(k_{n+p})(F(\ell_n)y\otimes_{n+p}F(\ell_p)z)$

$\cong F(k_m)x\otimes_{m+n+p}(F(k_n)y\otimes_{m+n+p}F(k_p)z)$

so α is the essentially associator $(F(k_m)x\otimes F(k_n)y)\otimes F(k_p)z \cong F(k_m)x\otimes(F(k_n)y\otimes F(k_p)z)$

INSIDE THE FIBER $F(m+n+p)$ [can write explicitly, don't need to].

② From the monoidal Grothendieck construction, we also have that starting from the monoidal opfibration $(\int F, \phi)$, the fibers are monoidal via $\xrightarrow{\phi} \xrightarrow{F\nabla}$ (20) already used above, but also that each reindexing functor becomes strong monoidal via (21).

In our setting, consider the map $\psi : m+n \longrightarrow m+_b n$ in A. Then the functor $F\psi : F(m+n) \longrightarrow F(m+_b n)$ becomes strong mon via

$$\begin{tikzcd}
F(m+n)\times F(m+n) \arrow[r, "\phi"] \arrow[d, "F\psi\times F\psi"'] \arrow[rr, bend left, dashed, "\otimes_{m+n}"] \arrow[dr, phantom, "\cong \phi_{\psi,\psi}"] & F(m+n+m+n) \arrow[r, "F\nabla"] \arrow[d, "F(\psi+\psi)"] \arrow[dr, phantom, "\cong n"] & F(m+n) \arrow[d, "F\psi"] & =: J_{x,y} \\
F(m+_b n)\times F(m+_b n) \arrow[r, "\phi"'] \arrow[rr, bend right, dashed, "\otimes_{m+_b n}"'] & F((m+_b n)+(m+_b n)) \arrow[r, "F\nabla"'] & F(m+_b n) &
\end{tikzcd}$$

[WHICH IS THE BOTTOM PART OF THE FACTORIZATION UNDER 33]

(we are now in the oplax direction)

Since $F\psi$ is strong monoidal, it already satisfies the following

$$\begin{tikzcd}
F\psi((r\otimes_{m+n+p} s)\otimes_{m+n+p} t) \arrow[r, "J"] \arrow[d, "F\psi(\alpha)"'] \arrow[drr, phantom, "(*)_2"] & F\psi(r\otimes_{m+n+p} s)\otimes_{m+_b n+_c p} F\psi(t) \arrow[r, "J\otimes 1"] & (F\psi(r)\otimes_{m+_b n+_c p} F\psi(s))\otimes_{m+_b n+_c p} F\psi(t) \arrow[d, "\alpha"] \\
F\psi(r\otimes_{m+n+p}(s\otimes_{m+n+p} t)) \arrow[r, "J"] & F\psi(r)\otimes_{m+_b n+_c p} F\psi(s\otimes_{m+n+p} t) \arrow[r, "1\otimes J"] & F\psi(r)\otimes(F\psi(s)\otimes_{m+_b n+_c p} F\psi(t))
\end{tikzcd}$$

for $\psi : m+n+p \longrightarrow m+_b n+_c p$ and <u>any</u> three objects r, s, t in $F(m+n+p)$.

■ Using diagrams ① + ②, we will prove the desired result. Notice that (*), can be equivalently rewritten as

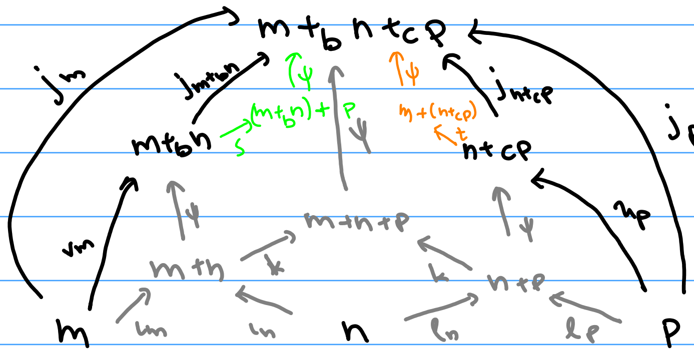
$$\begin{tikzcd}
\cdot \arrow[r, "\theta"] \arrow[d, "\alpha"'] & \cdot \arrow[r, "\Phi(\theta,1)"] & \cdot \arrow[d, "\alpha"] \\
\cdot \arrow[r, "\theta"'] & \cdot \arrow[r, "\phi(1,\theta)"'] & \cdot
\end{tikzcd}$$

and that ☆ that we are trying to verify is, for $\boxed{\xi := J \circ F\psi(\theta)}$ the factorization under (33), that

whiskering

$$\begin{tikzcd}[column sep=small]
(F\psi)\phi((F\psi)\phi(x,y),z) \arrow[r, "(F\psi)\phi(\xi,1)"] \arrow[dr, "\xi"'] \arrow[ddd, "\alpha"', "(8)"] & (F\psi)\phi((F\nabla)\phi(F(v_m)x, F(v_n)y), z) \arrow[r, "\xi"] \arrow[d, phantom, "\cong"] & (F\nabla)\phi(F(j_{m+_b n})(F\nabla)\phi(F(v_m)x, F(v_n)y), F(j_p)z) \arrow[ddd, "\alpha"] \\
& (F\nabla)\phi(F(j_{m+_b n})(F\psi)\phi(x,y), F(j_p)z) \arrow[ur, "(F\nabla)\phi(F(j_{m+_b n})\xi, 1)"'] \arrow[d, phantom, "\bigstar"] & \\
& (F\nabla)\phi(F(j_m)x, F(j_{n+_c p})(F\psi)\phi(y,z)) \arrow[dr, "(F\nabla)\phi(1, F(j_{n+_c p})\xi)"] \arrow[d, phantom, "\cong"] & \\
(F\psi)\phi(x,(F\psi)\phi(y,z)) \arrow[ur, "\xi"] \arrow[r, "(F\psi)\phi(1,\xi)"'] & (F\psi)\phi(x,(F\nabla)\phi(F(v_n)y, F(v_p)z)) \arrow[r, "\xi"'] & (F\nabla)\phi(F(j_m)x, F(j_{n+_c p})(F\nabla)\phi(F(v_n)y, F(v_p)z))
\end{tikzcd}$$

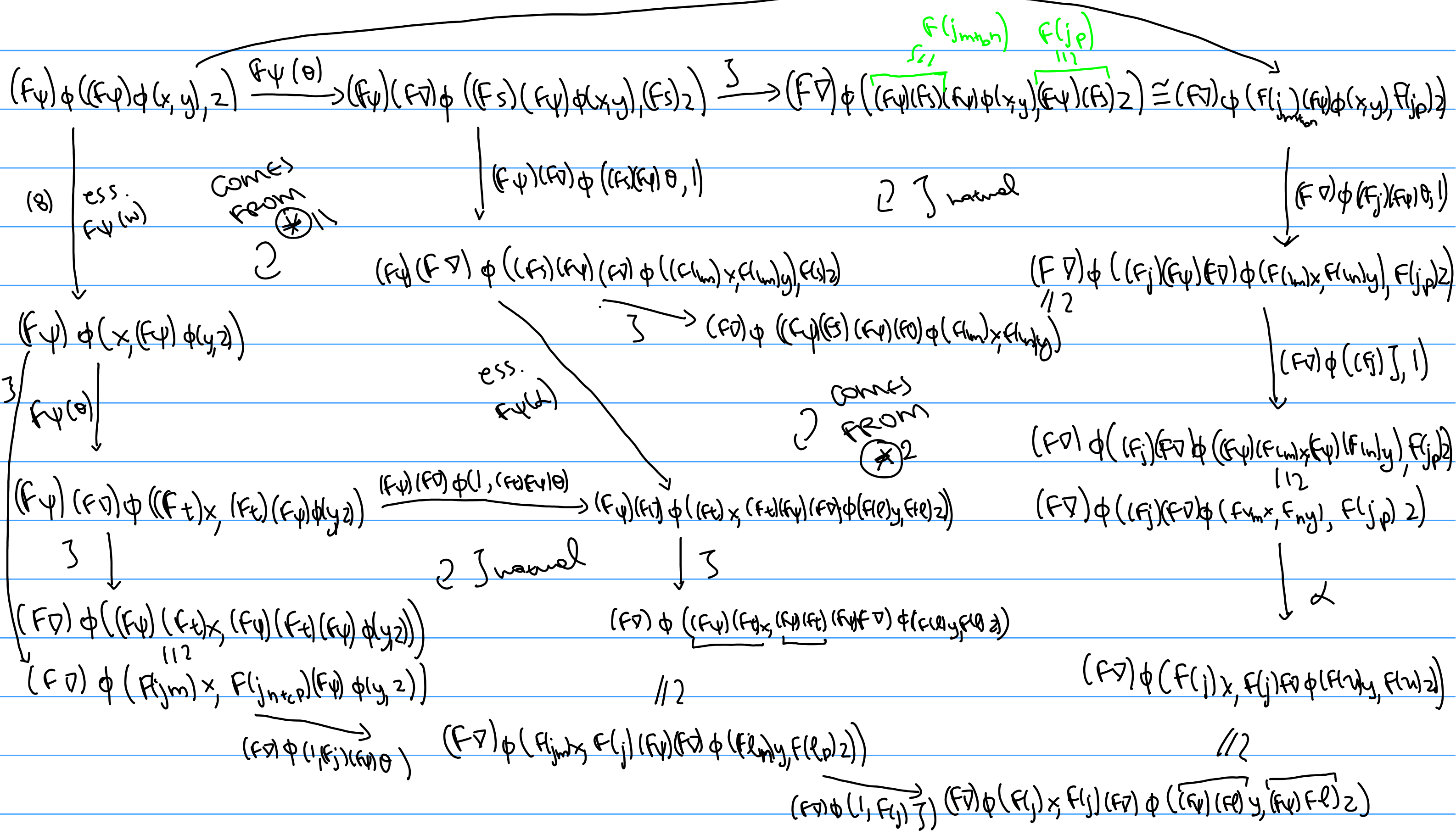
where $\wr$ commute by naturality of $\xi = \overset{\text{natural}}{\beta} \circ F\psi(\overset{\text{natural}}{\theta})$ and the relevant maps in $(A,+)$ are denoted according to



Proof I can apply $F\psi$ to the commutative $\circledast$, $\rightsquigarrow$ $F\psi(\circledast)$:

$$\begin{tikzcd} \cdot \arrow[r, "F\psi(\theta)"] \arrow[d, "F\psi(t)"'] \arrow[dr, phantom, "\circledast_{\parallel}"] & \cdot \arrow[r, "F\psi(\theta)\otimes 1"] & \cdot \arrow[d, "F\psi(t)"] \\ \cdot \arrow[r, "F\psi(\theta)"'] & \cdot \arrow[r, "1\otimes F\psi(\theta)"'] & \cdot \end{tikzcd}$$

Now expand the ξ's inside $\bigstar$



So the proof is complete. $\square$.

Final Thoughts for Axiom ④ – for any braided strong monoidal functor K, have commuting

$$\begin{tikzcd} K(x_1\otimes y_1\otimes x_2\otimes y_2) \arrow[r, "\theta"] \arrow[d, "K(1\beta 1)"'] & K(x_1\otimes y_1)\,\bar\otimes\, K(x_2\otimes y_2) \arrow[r, "\theta\bar\otimes\theta"] & K(x_1)\bar\otimes K(y_1)\bar\otimes K(x_2)\bar\otimes K(y_2) \arrow[d, "1\beta 1"] \\ K(x_1\otimes x_2\otimes y_1\otimes y_2) \arrow[r, "\theta"'] & K(x_1\otimes x_2)\,\bar\otimes\, K(y_1\otimes y_2) \arrow[r, "\theta\bar\otimes\theta"'] & K(x_1)\bar\otimes K(x_2)\bar\otimes K(y_1)\bar\otimes K(y_2) \end{tikzcd}$$

$\cdots$