

7/1/2021

- 1 Lex monoidal pseudo functor $(F, \phi, \phi_0): (A, +) \rightarrow (Cat, \times)$
- 2 Monoidal opfibration $U: (X, \otimes) \rightarrow (A, +)$
- 3 Pseudo functor $F: A \rightarrow MonCat$

ALSO IN SHULMAN

Let's focus on 2 \Leftrightarrow 3 for starters. The equivalence is framed as follows

$$U: (X, \otimes) \rightarrow (A, +) \xrightarrow{\quad} F: A \rightarrow MonCat$$

\uparrow global \downarrow fiber

$a \mapsto Fa := X_a$ with mon str
 $\otimes_a := X_a \times X_a \xrightarrow{\otimes} X_{a+a} \xrightarrow{\nabla!} X_a$
 \uparrow fiberwise

$$F \text{ with mon str} \xleftarrow{\quad} F: A \rightarrow MonCat$$

$$(a, x) \otimes (a', x') = (a+a', F(1_a) \times \otimes_{a+a'} F(1_{a'}) x')$$

$$\text{Then } (X, \otimes) \rightarrow (A, +) \xrightarrow{\quad} A \rightarrow MonCat \xrightarrow{\quad} (\int F, \otimes) \rightarrow (A, +)$$

produces an isomorphic monoidal opfibration, where

$$\begin{array}{ccc}
 \overset{\text{fiber}}{F_m \times F_{m'}} & \xrightarrow{\otimes} & F(m+m') \\
 \downarrow F(1_m) \times F(1_{m'}) & \cong & \downarrow F(1_{m+m'}) \\
 F(m+m') \times F(m+m') & \xrightarrow{\otimes} & F(m+m' + m+m') \xrightarrow{F(\nabla)} F(m+m')
 \end{array}$$

COULD WRITE THIS AS
 $A_m \times A_{m'} \xrightarrow{\otimes} A_{m+m'}$
 etc.



where the LHS isomorphism is due to \otimes being a comonoidal functor, namely it preserves comonoidal liftings. $[f! \times \otimes g! y \cong (f+g)!(x \otimes y)]$

SO SUCH A PASTED ISO ESTABLISHES THE 2-EQUIVALENCE $2 \Leftrightarrow 3$

two isomorphic monoidal categories!!

$$(X, \otimes_X) \text{ and } (\int F, \otimes_{\int F})$$

$$(a+a'', F(j)(F(l) \times \otimes F(l') \times \otimes F(j)) x'')$$

E.g. the associator $\alpha: (x \otimes y) \otimes z \xrightarrow{\sim} x \otimes (y \otimes z)$ and the induced connect via the isomorphism \star

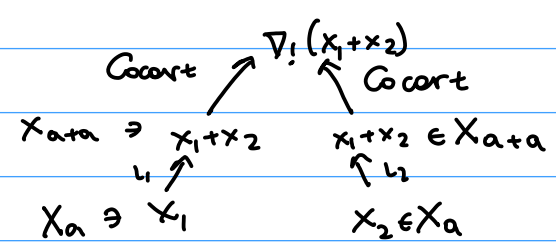
$$(a+a'') F(j) \times \otimes F(l) \times \otimes F(l') \times \otimes F(j) x''$$

Now this 2-equivalence restricts to one where \oplus_x is a coproduct, and \oplus_a give coproducts in fibers. How? As follows.

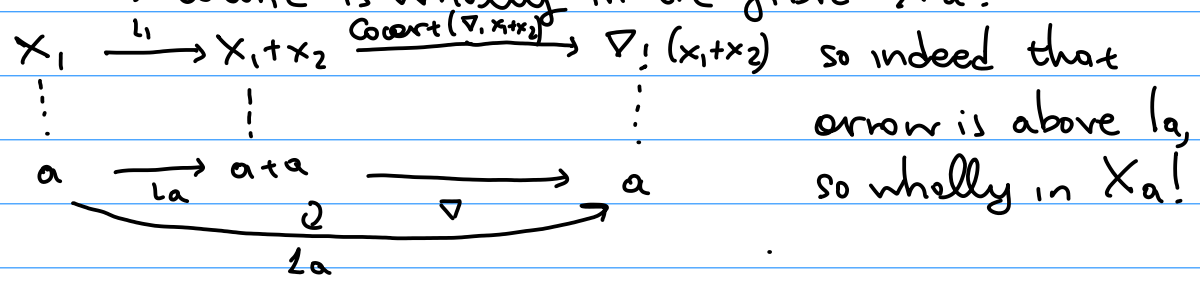
$$U: (X, +) \rightarrow (A, +) \longmapsto F: A \rightarrow \text{CoCart} \quad \hookleftarrow \text{global}$$

$$a \mapsto X_a \text{ with } X_a \times X_a \xrightarrow{+} X_{a+a} \xrightarrow{\nabla_1} X_a$$

with an induced universal cocone



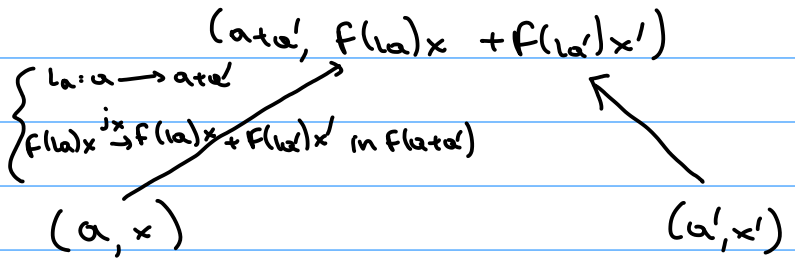
How do I know this cocone is wholly in the fibre X_a ?



$$\int F \text{ with sums} \longleftarrow F: A \rightarrow \text{CoCart}$$

$$(a, x) + (a', x') = (a+a', F(l_a)x + F(l_{a'})x')$$

with induced inclusions



Now, if again $U: (X, +) + (A, +) \longmapsto F: A \rightarrow \text{CoCart} \longmapsto (\int F, +) \rightarrow (A, +)$, the initial global sum is isomorphic to $(l_a)!x + (l_{a'})!x'$ via \star and that iso is the unique such that it commutes with the respective inclusions (the initial ones $x \xrightarrow{+} x+x$ and the final ones)

\star Moreover, the associator is well-behaved with respect to all involved inclusions $\dots \dots \dots (x+x')+x'' \xrightarrow{\alpha} x+(x'+x'')$ and the induced one after $[2] \Rightarrow [3] \Rightarrow [2]$.

Now, if we want to make [1] enter the picture, the only thing that changes in the above story is as follows:

- [1] $(F, \phi, \phi_0): (A, +) \rightarrow (G, \times)$ makes the op fib
 [2] $\int F \rightarrow A$ monoidal in a very specific way:
 $\otimes = \phi$!!!!! So instead having an arbitrary monoidal structure on the total category, we use ϕ .

So

[2] $(m, x) \otimes (m', x') = (m+m', \phi_{m, m'}^{x \otimes x'}(x, x'))$ category, we use ϕ .

\Downarrow

[3] $Fm \times Fm \xrightarrow{\phi_{m, m'}} F(m+m) \xrightarrow{F\eta} Fm$

\Downarrow

[2] $(m, x) \otimes (m', x') = (m+m', (F\eta) \phi_{m+m'} (F(lm)x + F(lm')x'))$

and the equivalence is now established via \star which becomes our very familiar

$$\begin{array}{ccc} & \xrightarrow{\phi_{m, m'}} & \\ F(lm) \downarrow & & \downarrow F(lm+m') \\ & \xrightarrow{\cong} & \\ & \xrightarrow{\phi_{m+m', m+m'}} & \xrightarrow{F\eta} \end{array}$$

"0"

Then the inclusions etc. involve ϕ 's, and also the associator in $\int F$ with $\otimes = \phi$ to begin with is "w", which then necessarily is the canonical map between respective colimits, commuting with cocores etc.....

So • This should partly respond to John's question "How do we actually prove that if a lax mon pseudofun factor through Rex, the category $\int F$ has finite colimits?"

Universal cocores etc?"

• This sketches why I don't think we have to add extra structure on lax mon pseudofun (option "3") since asking for F to factor should be equivalent enough for what we want.