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Some properties of Fib as a fibred 2-category

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Abstract

We consider some basic properties of the 2-category **Fib** of fibrations over arbitrary bases, exploiting the fact that it is fibred over **Cat**. We show a factorisation property for adjunctions in **Fib**, which has direct consequences for fibrations, e.g. a characterisation of limits and colimits for them. We also consider oplax colimits in **Fib**, with the construction of Kleisli objects as a particular example. All our constructions are based on an elementary characterisation of **Fib** as a 2-fibration. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The purpose of this paper is to examine some aspects of the 2-category **Fib** of fibrations over arbitrary bases as a 2-fibration over **Cat**, via the 2-functor which maps a fibration to its base category. The main point is that, as well as having the usual cartesian-vertical factorisation property for its morphisms, like any ordinary fibration, it also has an analogous factorisation for its 2-cells, which is essential to analyse its 2-dimensional structure.

An intrinsic property of 2-fibrations is a factorisation property for the adjunctions in the fibred 2-category (Theorem 4.3), which is our main contribution. We show some consequences of this fundamental property for ordinary fibrations themselves, qua objects of **Fib**, namely the existence of (co)limits and cartesian closure for the total category of a fibration as related to its fibrewise structure, see Section 4.1.

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In Section 4.2, as an instance of the construction of limits in the fibre 2-categories of a 2-fibration, we describe comma objects in **Fib**($\mathbb{B}$) (fibrations over the given base $\mathbb{B}$). This description is applied in Section 4.3 to give an abstract 2-categorical proof of Bénabou's characterisation of fibrations over fibrations. Our final application of the cartesian-vertical factorisation of 2-cells in **Fib** is a construction of oplax colimits for it, based on those of its base, **Cat**, and its fibres **Fib**(_), the 2-categories of fibrations over a given base. An instance of this is the construction of Kleisli objects for comonads in **Fib** in Section 5.

The above results show the value of regarding **Fib** as a 2-fibration in order to give a non-elementary study of its properties. Incidentally, the motivation to study such properties of **Fib** was to give a category-theoretic account of certain phenomena arising in logic and computer science, taking the point of view that a fibration is the proper abstract counterpart of a (constructive) predicate logic, over the simple theory corresponding to its base category. See [4], which also contains fuller details of the main constructions occurring in this paper. Such categorical-logic applications have been elaborated in [5–7].

The reader may consult [12, 16] for the relevant 2-categorical concepts involved in the present paper. Relevant background material for fibred categories can be found in [2, 8, 13]. A warning should be made about [3], which refers to a 2-fibration as a fibration in the 2-category 2-**Cat**. This concept is weaker than the one we consider here.

2. Fibred 2-categories

We introduce the relevant notions of cartesian 1-cells and 2-cells appropriate to characterise (in elementary terms) 2-fibrations.

Definition 2.1 (*Cartesian* 1-*cell*). Let $P:\mathscr{E}\to\mathscr{B}$ be a 2-functor.

(i) A 1-cell $f:X\to Y$ in $\mathscr{E}$ is 1-*cartesian* if it is cartesian in the usual sense for the underlying functor $P_0:\mathscr{E}_0\to\mathscr{B}_0$, i.e. for any 1-cell $h:Z\to Y$ with $Ph=Pf\circ u$ for some given 1-cell $u:PZ\to PX$, there is a unique 1-cell $\hat{h}:Z\to X$ with $P\hat{h}=u$ and $f\circ\hat{h}=h$.

(ii) A 1-cell $f:X\to Y$ in $\mathscr{E}$ is 2-*cartesian* if it is 1-cartesian and for any 2-cell $\alpha:g\Rightarrow h:Z\to Y$ such that

$$\begin{tikzcd} PZ \arrow[r, bend left, "Pg"] \arrow[r, bend right, "Ph"'] \arrow[r, phantom, "\Downarrow P\alpha"] & PY \end{tikzcd} \quad = \quad \begin{tikzcd} PZ \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\sigma"] & PX \arrow[r, "Pf"] & PY \end{tikzcd}$$

there is a unique 2-cell $\phi:\hat{g}\Rightarrow\hat{h}:Z\to X$ such that $f\phi=\alpha$ and $P\phi=\sigma$. Here, the 1-cells $\hat{g}$ and $\hat{h}$ are uniquely determined because f is 1-cartesian.

Note that the universal property with respect to 2-cells in (ii) above does not imply that for 1-cells in (i). However, it implies that any 1-cell with codomain Y whose projection factors through Pf, has a factorisation through f, unique up to isomorphism.

Definition 2.2 (*Cartesian 2-cell*). Let $P:\mathscr{E}\to\mathscr{B}$ be a 2-functor.

(i) A 2-cell $\sigma: f\Rightarrow g: X\to Y$ in $\mathscr{E}$ is *1-cartesian* if, for any 2-cell $\alpha: h\Rightarrow k: Z\to Y$ for which

$$\begin{tikzcd} PZ \arrow[rr, bend left, "Ph"] \arrow[dr, "u"'] & \Downarrow\gamma & PY \\ & PX \arrow[ur, bend left, "Pf"] \arrow[ur, bend right, "Pg"'] \arrow[ur, phantom, "\Downarrow P\sigma"] & \end{tikzcd} \quad = \quad \begin{tikzcd} PZ \arrow[rr, bend left, "Ph"] \arrow[dr, "u"'] & \Downarrow P\alpha & PY \\ & PX \arrow[ur, "Pg"'] & \end{tikzcd}$$

for some given 2-cell $\gamma: Ph\Rightarrow Pf\circ u: PZ\to PY$, there is a unique 2-cell $\phi: h\Rightarrow f\circ v$, with $v: Z\to X$ and $Pv=u$, such that $P\phi=\gamma$ and $\alpha=\sigma v\circ\phi$, as shown in the diagram below.

$$\begin{tikzcd} Z \arrow[rr, bend left, "h"] \arrow[dr, "v"'] & \Downarrow\phi & Y \\ & X \arrow[ur, bend left, "f"] \arrow[ur, bend right, "g"'] \arrow[ur, phantom, "\Downarrow\sigma"] & \end{tikzcd} \quad = \quad \begin{tikzcd} Z \arrow[rr, bend left, "h"] \arrow[dr, "v"'] & \Downarrow\alpha & Y \\ & X \arrow[ur, "g"'] & \end{tikzcd}$$

(ii) A 2-cell $\sigma: f\Rightarrow g: X\to Y$ is *2-cartesian* if it is 1-cartesian and its codomain $g: X\to Y$ is a 2-cartesian 1-cell.

In the Appendix we give an adjoint characterisation of 1-cartesian 2-cells, which clarifies the fact that 2-cartesian 2-cells strengthen the universal property of 1-cartesian 2-cells in a 2-dimensional way.

Definition 2.3 (*2-fibration*). A 2-functor $P:\mathscr{E}\to\mathscr{B}$ is a *2-fibration* if

(i) For any object X in $\mathscr{E}$ and any 1-cell $u: I\to PX$ in $\mathscr{B}$, there is a 2-cartesian 1-cell $\overline{u}: u^*(X)\to X$ with $P\overline{u}=u$.

(ii) For any object X in $\mathscr{E}$ and any 2-cell $\delta: u\Rightarrow v: I\to PX$ in $\mathscr{B}$, there is a 2-cartesian 2-cell $\sigma: f\Rightarrow g: Y\to X$ with $P\sigma=\delta$.

As usual, we may assume a given choice of the relevant cartesian 1-cells and 2-cells for a 2-fibration. In an ordinary fibration, every morphism factorises as a vertical one (i.e. which projects to the identity) followed by a cartesian one. The same is true about 2-cells in a 2-fibration, as shown in the following proposition.

Proposition 2.4. *Let $P:\mathscr{E}\to\mathscr{B}$ be a 2-fibration. Every 2-cell $\alpha:h\Rightarrow k:X\to Y$ in $\mathscr{E}$ can be expressed as a pasting composite*

$$
\begin{tikzcd}
& X'' \arrow[dr, "\overline{Ph}"] & \\
X \arrow[ur, "\hat{h}"] \arrow[dr, "\hat{k}"'] \arrow[r, phantom, "\Downarrow\hat{\alpha}"] & {} \arrow[r, phantom, "\Downarrow\sigma"] & Y \\
& X' \arrow[uu, "\langle\sigma\rangle"] \arrow[ur, "\overline{Pk}"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y
\end{tikzcd}
$$

where $\sigma:\overline{Ph}\circ\langle\sigma\rangle\Rightarrow\overline{Pk}$ is 2-cartesian (over $P\sigma=P\alpha$) and $P\hat{\alpha}=\mathrm{id}_{\mathrm{id}_{PX}}$.

Proof. Let $\sigma:\widetilde{Ph}\Rightarrow\overline{Pk}:X'\to Y$ be a 2-cartesian 2-cell over $P\alpha$, and $\overline{Ph}:X''\to Y$ be a 2-cartesian 1-cell over Ph. Since σ is 2-cartesian, there is a unique 2-cell $\phi:h\Rightarrow\widetilde{Ph}\circ\hat{k}:X\to Y$ such that $\sigma\hat{k}\circ\phi=\alpha$ with $P\phi=\mathrm{id}_{\mathrm{id}_{Ph}}$. Since $\overline{Ph}$ is 2-cartesian, there is a unique $\hat{\alpha}:\hat{h}\Rightarrow l:X\to X''$ such that $\overline{Ph}\hat{\alpha}=\phi$ and $P\hat{\alpha}=\mathrm{id}_{\mathrm{id}_{PX}}$. Finally, $l=\langle\sigma\rangle\circ\hat{k}$ because $\overline{Ph}$ is 2-cartesian and $\langle\sigma\rangle$ is the unique factorisation $\widetilde{Ph}=\overline{Ph}\langle\sigma\rangle$. $\square$

The above vertical-cartesian factorisation puts in evidence the role of the 2-dimensional aspect of the universal property of a 2-cartesian 1-cell. Such factorisation will be used repeatedly from Section 3 onwards.

As we would expect, the notion of cartesian 2-cell enjoys several closure properties, summarised in the following proposition (cf. [9, Lemma 2.1] where analogous properties are proved for the case of **Fib** relative to a 2-category with pullbacks).

Proposition 2.5. *Consider a 2-functor $P:\mathscr{E}\to\mathscr{B}$.*

(i) *Let $\sigma:f\Rightarrow g:X\to Y$ be a 2-cell in $\mathscr{E}$ such that g is 1-cartesian (respectively, 2-cartesian) and σ is an isomorphism. Then σ is a 1-cartesian (respectively 2-cartesian) 2-cell.*

(ii) *Given 1-cartesian (respectively 2-cartesian) 2-cells $\sigma:g\circ j\Rightarrow d:X\to Y$ and $\gamma:f\Rightarrow g:Z\to Y$ with $g:Z\to Y$ 1-cartesian, the pasting composite*

$$
\begin{tikzcd}
& Z \arrow[dr, bend left, "f"] \arrow[dr, "g"'] \arrow[dr, phantom, bend left=20, "\Downarrow\gamma"] & \\
X \arrow[ur, "j"] \arrow[rr, bend right, "d"'] \arrow[rr, phantom, "\Downarrow\sigma"] & & Y
\end{tikzcd}
$$

is 1-cartesian (respectively 2-cartesian).

Proof. (i) Given a 2-cell $\alpha:h\Rightarrow k:Z\to Y$ with a factorisation of $P\alpha=(P\sigma)u\circ\gamma$ for some 2-cell $\gamma:Ph\Rightarrow Pf\circ u:Z\to Y$, the factorisation of $Pk=Pg\circ u$ and the fact that

g is cartesian implies the existence of a unique $v : Z \to X$ such that $k = g \circ v$. Then $\phi = \sigma^{-1} v \circ \alpha$ gives the desired unique factorisation of $\alpha = \sigma v \circ \phi$.

(ii) Given a 2-cell $\alpha : h \Rightarrow k : W \to Y$ with a factorisation of $P\alpha$ as

$$\begin{tikzcd}
PW \arrow[rrd, bend left, "Ph"] \arrow[dd, "l"'] & & \\
 & PZ \arrow[u, phantom, "\Downarrow\phi"] \arrow[d, phantom, "\Downarrow P\sigma"] \arrow[r, bend left, "Pf"] \arrow[r, bend right, "Pg"'] \arrow[r, phantom, "\Downarrow P\gamma"] & PY \\
PX \arrow[ur, "Pj"] \arrow[rru, bend right, "Pd"'] & &
\end{tikzcd}$$

for some 2-cell $\phi : Ph \Rightarrow Pf \circ Pj \circ l : PZ \to PY$, the composite $(P\gamma)Pj \circ \phi$ gives a factorisation of $P\alpha$ through $P\sigma$. Since σ is 1-cartesian, there is a unique 2-cell $\theta : h \Rightarrow g \circ j \circ k' : W \to Y$ (where $k = d \circ k'$ and $Pk' = l$) such that $\sigma k' \circ \theta = \alpha$ and $P\theta = (P\gamma)(Pj)\, l \circ \phi$. In turn, this latter factorisation implies the existence of a unique 2-cell $\vartheta : h \Rightarrow f \circ r : W \to Y$ such that $P\vartheta = \phi$ and $\theta = \gamma r \circ \vartheta$. Here $r : W \to Z$ is such that $Pr = Pj \circ l$ and $gr = gjk'$, whence $r = jk'$ as g is 1-cartesian. So, ϑ is the required unique factorisation of α through the pasting $\sigma \circ (\gamma j)$. $\square$

We examine these closure properties further in order to obtain formulations which will provide alternative characterisations of the notion of 2-fibration (Theorem 2.8 below). In order to do so, we consider a class of 2-cells Σ in $\mathscr{E}$ which satisfies the conditions of Proposition 2.5, relative to either 1- or 2-cartesianness, e.g. any isomorphism with a 1-cartesian codomain is in Σ if we refer to 1-cartesianness conditions and likewise for the pasting composite in (ii). In the following proposition we use the term 'cartesian' to refer to either 1-cartesian or 2-cartesian, as appropriate for the closure conditions on Σ.

Proposition 2.6. *Let Σ be a class of 2-cells in $\mathscr{E}$ (for a 2-functor $P : \mathscr{E} \to \mathscr{B}$), satisfying the closure properties of Proposition 2.5. Then Σ satisfies*

(i) *Given $\sigma : f \Rightarrow g : X \to Y$ in Σ, with g cartesian, and a cartesian 1-cell $h : Z \to X$, then σh is in Σ.*

(ii) *Given $\gamma : f \Rightarrow g$ and $\sigma : g \Rightarrow h$ in Σ with g 1-cartesian, then the vertical composite $\sigma \circ \gamma : f \Rightarrow h$ is in Σ.*

Proof. To show (i), notice that 2.5(i) implies that $\mathrm{id}_{g \circ h}$ is in Σ, hence by 2.5(ii) the pasting composite $\mathrm{id}_{g \circ h} \circ \sigma h$ is in Σ. $\square$

An ordinary functor $p : \mathbb{E} \to \mathbb{B}$ can be shown to be a fibration in two simple equivalent ways: either by the existence of the relevant (strong) cartesian morphisms or that of *vertical* (or weak) cartesian morphisms *closed under composition*. A morphism $f : X \to Y$ in $\mathbb{E}$ is *vertical cartesian* (v-cartesian for short) if for any $g : Z \to Y$ with $pg = pf$ there is a unique (vertical) $h : Z \to X$ ($ph = \mathrm{id}$) such that $g = f \circ h$.

A similar characterisation is possible for 2-fibrations with respect to cartesian 2-cells. The corresponding notion of vertical cartesian 2-cell that we need is the following:

Definition 2.7. Let $P : \mathscr{E} \to \mathscr{B}$ be a 2-functor.

(i) A 2-cell $\sigma : f \Rightarrow g : X \to Y$ in $\mathscr{E}$ is *v-1-cartesian* if, for any other 2-cell $\alpha : h \Rightarrow k : Z \to Y$ for which $P\alpha = P\sigma$, there is a unique 2-cell $\phi : h \Rightarrow f \circ v : Z \to Y$, with $v : Z \to X$ and $Pv = \mathrm{id}_{PX}$, such that $P\phi = \mathrm{id}_{Pg}$ and $\alpha = \sigma v \circ \phi$, as shown in the diagram below.

$$\begin{tikzcd} Z \arrow[rr, bend left, "h"] \arrow[dr, "v"'] & \Downarrow\phi & Y \\ & X \arrow[ur, bend left, "f"] \arrow[ur, bend right, "g"'] \arrow[ur, phantom, "\Downarrow\sigma"] & \end{tikzcd} \quad = \quad \begin{tikzcd} Z \arrow[rr, bend left, "h"] \arrow[dr, "v"'] & \Downarrow\alpha & Y \\ & X \arrow[ur, "g"'] & \end{tikzcd}$$

(ii) A 2-cell $\sigma : f \Rightarrow g : X \to Y$ in $\mathscr{E}$ is *v-2-cartesian* if it is v-1-cartesian and its codomain $g : X \to Y$ is a 2-cartesian 1-cell.

Clearly, the above definition entails a weaker universal property than that of Definition 2.2. We now give two alternative characterisations of 2-fibrations: one in terms of v-2-cartesian 2-cells satisfying 2.5(ii) and another 'local' one, in terms of the induced functors between the hom categories being fibrations, with their cartesian morphisms (2-cells) preserved by horizontal composition.

Theorem 2.8. *Let $P : \mathscr{E} \to \mathscr{B}$. The following are equivalent:*

(i) *P is a 2-fibration*

(ii) (*v-cartesian characterisation*) *P satisfies:*

(1) *For any object X in $\mathscr{E}$ and any 1-cell $u : I \to PX$ in $\mathscr{B}$, there is a 2-cartesian 1-cell $\overline{u} : u^*(X) \to X$ with $P\overline{u} = u$.*

(2) *For any object X in $\mathscr{E}$ and any 2-cell $\delta : u \Rightarrow v : I \to PX$ in $\mathscr{B}$, there is a v-2-cartesian 2-cell $\sigma : f \Rightarrow g : Y \to X$ with $P\sigma = \delta$.*

(3) *v-2-cartesian 2-cells satisfy the closure property 2.5(ii).*

(iii) (*Local characterisation*) *P satisfies:*

(1) *For any object X in $\mathscr{E}$ and any 1-cell $u : I \to PX$ in $\mathscr{B}$, there is a 2-cartesian 1-cell $\overline{u} : u^*(X) \to X$ with $P\overline{u} = u$.*

(2) *For every pair of objects X, Y in $\mathscr{E}$, the corresponding functor $P_{X,Y} : \mathscr{E}(X,Y) \to \mathscr{B}(PX,PY)$ between the hom-categories is a fibration. Furthermore, for every 1-cell $h : Z \to X$ in $\mathscr{E}$, the precomposition functor $\mathscr{E}(h,Y) : \mathscr{E}(X,Y) \to \mathscr{E}(Z,Y)$ preserves cartesian morphisms (from $P_{X,Y}$ to $P_{Z,Y}$).*

Proof. (i) $\Rightarrow$ (ii): 2-cartesian 2-cells are v-2-cartesian and satisfy Proposition 2.5.

(ii) $\Rightarrow$ (iii): We must show every hom-functor $P_{X,Y} : \mathscr{E}(X,Y) \to \mathscr{B}(PX,PY)$ is a fibration. Consider $h : X \to Y$ and $\sigma : u \Rightarrow Ph : PX \to PY$. Let $\sigma' : \tilde{u} \Rightarrow \overline{Ph} : X' \to Y$ be a v-2-cartesian 2-cell over σ. Since $\overline{Ph}$ is a 2-cartesian 1-cell, there exists a unique

$h' : X \to X'$ such that $\overline{Ph} \circ h' = h$ and $Ph' = \mathrm{id}_P X$. We claim that

$$\begin{tikzcd} X \arrow[r, "h'"] & X' \arrow[r, bend left, "\tilde{u}"] \arrow[r, bend right, "\overline{Ph}"'] \arrow[r, phantom, "\Downarrow\sigma'"] & Y \end{tikzcd}$$

is a cartesian morphism in $\mathscr{E}(X, Y)$ over σ. For, given $\alpha : k \Rightarrow h : X \to Y$ and $\gamma : Ph \Rightarrow u : PX \to PY$, such that

$$\begin{tikzcd} PX \arrow[r, bend left=60, "Pk"] \arrow[r, "u"] \arrow[r, bend right=60, "Ph"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3, "\Downarrow\sigma"] & PY \end{tikzcd} = \begin{tikzcd} PX \arrow[r, bend left, "Pk"] \arrow[r, bend right, "Ph"'] \arrow[r, phantom, "\Downarrow P\alpha"] & PY \end{tikzcd}$$

consider $\gamma' : \widetilde{Ph} \Rightarrow \overline{u} : X'' \to Y$ v-2-cartesian over γ. Since $\overline{u}$ is 2-cartesian, there is a unique $\langle\sigma\rangle : X' \to X''$ such that $\overline{u} \circ \langle\sigma\rangle = \tilde{u}$ (and $P\langle\sigma\rangle = \mathrm{id}_{PX}$). By hypothesis, the composite

$$\begin{tikzcd} & X'' \arrow[dr, bend left, "\widetilde{Pk}"] \arrow[dr, "\overline{u}"'] \arrow[dr, phantom, "\Downarrow\gamma'"] & \\ X' \arrow[ur, "\langle\sigma\rangle"] \arrow[rr, bend right, "\overline{Ph}"'] \arrow[rr, phantom, "\Downarrow\sigma'"] & & Y \end{tikzcd}$$

is v-2-cartesian, hence there is a unique 2-cell $\phi : k \Rightarrow \widetilde{Pk} \circ \langle\sigma\rangle \circ h'$ such that $P\phi = \mathrm{id}_{Pk}$ and

$$\begin{tikzcd} X \arrow[dd, "h'"'] \arrow[rrd, bend left, "k"] \arrow[rrd, phantom, "\Downarrow\phi"] & & \\ & X'' \arrow[r, bend left, "\overline{Ph}"] \arrow[r, bend right, "\overline{u}"'] \arrow[r, phantom, "\Downarrow\gamma'"] & Y \\ X' \arrow[ur, "\langle\sigma\rangle"] \arrow[rru, bend right, "\overline{Ph}"'] \arrow[rru, phantom, "\Downarrow\sigma'"] & & \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "k"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \end{tikzcd}$$

We then see that $(\gamma'\langle\sigma\rangle h') \circ \phi$ is the required unique factorisation of α through $\sigma' h'$.

Furthermore, given $l : Z \to X$ in $\mathscr{E}$, we claim $\sigma' h' l : \tilde{u} h' l \Rightarrow \overline{Ph} h' l : Z \to Y$ is cartesian in $\mathscr{E}(Z, Y)$. Simply apply the above argument to $\sigma' \overline{Pl} v : \tilde{u} \overline{Pl} v \Rightarrow (\overline{Ph})(\overline{Pl}) v : Z \to Y$

for $v : Z \to Z'$ the unique factorisation of $h' \circ l$ through the 2-cartesian 1-cell $\overline{Pl} : Z' \to X$ (over Pl), using the fact that $\sigma'\overline{Pl}$ is v-2-cartesian by Proposition 2.6(i).

(iii) $\Rightarrow$ (i): The first condition is the same for both characterisations. In order to show the existence of the relevant 2-cartesian 2-cells, let X be an object of $\mathscr{E}$ and a 2-cell $\sigma : u \Rightarrow v : I \to PX$ in $\mathscr{B}$.

Let $\bar{v} : I' \to X$ be a 2-cartesian 1-cell over v. Let $\sigma' : \tilde{u} \Rightarrow \bar{v} : I' \to X$ be the cartesian lifting of σ at $\bar{v}$ for the fibration $P_{I',X} : \mathscr{E}(I', X) \to \mathscr{B}(I, PX)$. Then, σ' is the required 2-cartesian 2-cell over σ at X. For, given 2-cells $\alpha : s \Rightarrow t : Z \to X$ and $\phi : Ps \Rightarrow u \circ k : PZ \to PX$ such that

$$
\begin{tikzcd}
PZ \arrow[rr, bend left, "Ps"] \arrow[dr, "k"'] & \Downarrow\phi & PX \\
& I \arrow[ur, bend left, "u"] \arrow[ur, bend right, "v"'] \arrow[ur, phantom, "\Downarrow\sigma"] &
\end{tikzcd}
=
\begin{tikzcd}
PZ \arrow[rr, bend left, "Ps"] \arrow[dr, "k"'] & \Downarrow P\alpha & PX \\
& I \arrow[ur, "v"'] &
\end{tikzcd}
$$

with $v \circ k = t$, since $\bar{v}$ is 2-cartesian, there is a unique 1-cell $t' : Z \to I'$ such that $\bar{v} \circ t' = t$ (and $Pt' = k$). By hypothesis, $\sigma' t' : \tilde{u}t' \Rightarrow \bar{v}t' : Z \to X$ is cartesian in $\mathscr{E}(Z, X)$ (for $P_{Z,X}$). Hence, there is a unique 2-cell $\phi' : s \Rightarrow \tilde{u} \circ t' : Z \to X$ such that

$$
\begin{tikzcd}
Z \arrow[rr, bend left, "s"] \arrow[dr, "t'"'] & \Downarrow\phi' & X \\
& I' \arrow[ur, bend left, "\tilde{u}"] \arrow[ur, bend right, "\bar{v}"'] \arrow[ur, phantom, "\Downarrow\sigma'"] &
\end{tikzcd}
=
\begin{tikzcd}
Z \arrow[rr, bend left, "s"] \arrow[dr, "t'"'] & \Downarrow\alpha & X \\
& I' \arrow[ur, "\bar{v}"'] &
\end{tikzcd}
$$

and $P\phi' = \phi$, as required. $\square$

3. Fib as a fibred 2-category

In this section, we present the paradigmatic example of a 2-fibration, **Fib**, to which we devote the rest of the paper.

Cat stands for the 2-category of small categories, functors and natural transformations. For a given category $\mathbb{B}$, **Fib**($\mathbb{B}$) denotes the 2-category of fibrations over $\mathbb{B}$, fibred functors (i.e. preserving cartesian morphisms, and called cartesian functors elsewhere) and vertical natural transformations between them. We often display fibrations as $\mathbb{E} \to^{p} \mathbb{B}$. Then, a fibred functor $F : (\mathbb{E} \to^{p} \mathbb{B}) \to (\mathbb{D} \to^{q} \mathbb{B})$ is a functor $F : \mathbb{E} \to \mathbb{D}$ with $qF = p$ which preserves cartesian morphisms. A 2-cell $\alpha : F \Rightarrow F'$ between two such functors is a natural transformation satisfying $q\alpha = 1_p$, i.e. its components lie in the fibres of q.

For a fibration $\mathbb{E} \to^{p} \mathbb{B}$, given an object X in E and a morphism $u : I \to pX$ in B, we write $\bar{u}^p : u^*(X) \to X$ for a cartesian morphism over u. Whenever convenient we suppress the superscript^p.

The 2-category **Fib** has fibrations $\mathbb{E} \rightarrow^{p} \mathbb{B}$ as objects. Morphisms, called fibred 1-cells, are given by commuting squares

$$\begin{tikzcd} \mathbb{E} \arrow[r, "\tilde{K}"] \arrow[d, "p"'] & \mathbb{D} \arrow[d, "q"] \\ \mathbb{B} \arrow[r, "K"'] & \mathbb{A} \end{tikzcd}$$

where $\tilde{K}$ preserves cartesian morphisms. We write $(\tilde{K},K): p \to q$ for such a fibred 1-cell. Given fibred 1-cells $(\tilde{K},K),(\tilde{L},L): p \to q$, a *fibred 2-cell* from $(\tilde{K},K)$ to $(\tilde{L},L)$ is a pair of natural transformations $(\tilde{\sigma}:\tilde{K} \Rightarrow \tilde{L}, \sigma : K \Rightarrow L)$ with $\tilde{\sigma}$ above σ, as displayed below

$$\begin{tikzcd} \mathbb{E} \arrow[r, bend left, "\tilde{K}"{name=A}] \arrow[r, bend right, "\tilde{L}"'{name=B}] \arrow[d, "p"'] \arrow[from=A, to=B, phantom, "\Downarrow\tilde{\sigma}"] & \mathbb{D} \arrow[d, "q"] \\ \mathbb{B} \arrow[r, bend left, "K"{name=C}] \arrow[r, bend right, "L"'{name=D}] \arrow[from=C, to=D, phantom, "\Downarrow\sigma"] & \mathbb{A} \end{tikzcd}$$

and we write it as $(\tilde{\sigma},\sigma):(\tilde{K},K) \Rightarrow (\tilde{L},L)$. We have the obvious 2-functor *cod*: **Fib** $\to$ **Cat**, which maps every fibration $\mathbb{E} \rightarrow^{p} \mathbb{B}$ to its base category $\mathbb{B}$.

We recall, e.g. from [2], that fibrations are stable under pullback. This means that a functor $H : \mathbb{A} \to \mathbb{B}$ induces, assuming a choice of pullbacks, a *change-of-base* 2-functor $H^* : \mathbf{Fib}(\mathbb{B}) \to \mathbf{Fib}(\mathbb{A})$. As a consequence, we see that *cod*: **Fib** $\to$ **Cat** is an ordinary fibration, with cartesian morphisms corresponding to pullback squares. Since pullbacks in **Cat** enjoy a 2-dimensional universal property (cf. [11]), they yield 2-cartesian 1-cells for *cod*: **Fib** $\to$ **Cat**.

Before examining *cod*: **Fib** $\to$ **Cat** as a 2-fibration though, a cautionary remark seems in order. The treatment of **Fib** as a 2-fibration assumes *cloven* fibrations, i.e. with a given choice of cartesian liftings, e.g. in the proof of Proposition 3.1 below. Since we use such properties later on to infer properties about (ordinary) fibrations themselves, the reader should be aware that these latter properties are still valid without the assumption of cleavages (which, being a form of *choice*, is in general not available internally in a topos, for instance). See [1] for a deeper analysis of these issues. Nevertheless, the global point of view afforded by the 2-categorical treatment, besides its extra generality, allows for conceptually clearer statements and simpler proofs.

As is well-known, cf. [2], the concept of fibration is representable, meaning that a functor $p : \mathbb{E} \to \mathbb{B}$ is a fibration iff for every category $\mathbb{X}$, the post-composition functor $\mathbf{eCat}(\mathbb{X}, p) : \mathbf{Cat}(\mathbb{X}, \mathbb{E}) \to \mathbf{Cat}(\mathbb{X}, \mathbb{B})$ is a fibration.

Proposition 3.1. *cod*: **Fib** $\to$ **Cat** *is a 2-fibration.*

Proof. We use the local characterisation of 2-fibrations, Theorem 2.8(iii). As we mentioned above, pullbacks give the relevant 2-cartesian 1-cells. Given fibrations $\mathbb{E} \to^p \mathbb{B}$ and $\mathbb{D} \to^q \mathbb{A}$, we must show the induced functor

$$\begin{tikzcd} \mathbf{Fib}\left(\begin{array}{c} \mathbb{E} \\ \downarrow p \\ \mathbb{B} \end{array}, \begin{array}{c} \mathbb{D} \\ \downarrow q \\ \mathbb{A} \end{array} \right) \arrow[d, "cod(p,q)"] \\ \mathbf{Cat}(\mathbb{B}, \mathbb{A}) \end{tikzcd}$$

is a fibration. Let $(\tilde{G}, G): p \to q$ be a fibred 1-cell and $\sigma: F \Rightarrow G: \mathbb{B} \to \mathbb{A}$ be a natural transformation. Define $\tilde{F}: \mathbb{E} \to \mathbb{D}$ and $\sigma': \tilde{F} \Rightarrow \tilde{G}: \mathbb{E} \to \mathbb{D}$ as follows:

$$\begin{tikzcd} X \arrow[d, "f"] & & \tilde{F}X \arrow[r, dashed, "\tilde{F}f"] \arrow[d, "\sigma'_X"'] & \tilde{F}Y \arrow[d, "\sigma'_Y"] \\ Y & \mapsto \arrow[ddd, dotted, "p"] & \tilde{G}X \arrow[r, "\tilde{G}f"'] & \tilde{G}Y \\ & & & \\ pX \arrow[d, "pf"] & & FX \arrow[r, "Ff"] \arrow[d, "\sigma_X"'] & FY \arrow[d, "\sigma_Y"] \\ pY & \mapsto & GX \arrow[r, "Gf"'] & GY \end{tikzcd}$$

where σ'_X and σ'_Y are cartesian morphisms (over σ_X and σ_Y respectively) and $\tilde{F}f$ is the unique morphism making the upper square commute, induced by the universality of σ'_Y. Then $(\sigma', \sigma): (\tilde{F}, F) \Rightarrow (\tilde{G}, G)$ is cartesian in **Fib**(p, q), as is easily seen from the above pointwise construction of σ'.

Given the pointwise cartesian character of σ' it is clear that given another fibred 1-cell $(\tilde{H}, H): r \to p$ (with $S \to^r T$ a fibration), the fibred 2-cell

$$
\begin{tikzcd}
\mathbb{S} \arrow[r, "\tilde{H}"] \arrow[d, "r"'] & \mathbb{E} \arrow[r, bend left, "\tilde{F}"] \arrow[r, bend right, "\tilde{G}"'] \arrow[r, phantom, "\Downarrow\sigma'"] \arrow[d, "p"'] & \mathbb{D} \arrow[d, "q"] \\
\mathbb{T} \arrow[r, "H"'] & \mathbb{B} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\sigma"] & \mathbb{A}
\end{tikzcd}
$$

is cartesian in $\mathbf{Fib}(r, q)$. □

Notice in the above proof that cartesian morphisms in the hom categories are characterised by the fact that the top natural transformation has cartesian components. This is essentially the representability property of ordinary fibrations mentioned above. In order to provide a better understanding of the constructions to follow, we spell out the vertical-cartesian factorisation of 2-cells (given in Proposition 2.4) in this setting: given $\mathbb{D} \to^q \mathbb{A}$ and a natural transformation $\sigma: K \Rightarrow L: \mathbb{B} \to \mathbb{A}$, we have

$$
\begin{tikzcd}
& K^*(\mathbb{D}) \arrow[dd, "K^*(q)"'] \arrow[dr, "q^*(K)"] \arrow[dr, phantom, "\swarrow\sigma'_q"'] & \\
L^*(\mathbb{D}) \arrow[ur, "\langle\sigma\rangle_q"] \arrow[rr, "q^*(L)"'] \arrow[dr, "L^*(q)"'] & & \mathbb{D} \arrow[d, "q"] \\
& \mathbb{B} \arrow[r, bend left, "K"] \arrow[r, bend right, "L"'] \arrow[r, phantom, "\Downarrow\sigma"] & \mathbb{A}
\end{tikzcd}
$$

where the squares are the appropriate change-of-base pullbacks and σ'_q has cartesian components. Given a fibred 2-cell $(\tilde{\sigma}, \sigma): (\tilde{K}, K) \Rightarrow (\tilde{L}, L): p \to q$, we have a factorisation

$$
\begin{tikzcd}
\mathbb{E} \arrow[rr, "\hat{K}"] \arrow[dr, "\hat{L}"'] \arrow[ddrr, bend right, "p"'] \arrow[rr, phantom, shift right=4, "\Downarrow\hat{\sigma}\ \langle\sigma\rangle_q"] & & K^*(\mathbb{D}) \arrow[dd, "K^*(q)"'] \arrow[dr, "q^*(K)"] \arrow[dr, phantom, "\swarrow\sigma'_q"'] & \\
& L^*(\mathbb{D}) \arrow[ur] \arrow[rr, "q^*(L)"'] \arrow[dr, "L^*(q)"'] & & \mathbb{D} \arrow[d, "q"] \\
& & \mathbb{B} \arrow[r, bend left, "K"] \arrow[r, bend right, "L"'] \arrow[r, phantom, "\Downarrow\sigma"] & \mathbb{A}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathbb{E} \arrow[r, bend left, "\tilde{K}"] \arrow[r, bend right, "\tilde{L}"'] \arrow[r, phantom, "\Downarrow\tilde{\sigma}"] \arrow[dd, "p"'] & \mathbb{D} \arrow[dd, "q"] \\
& \\
\mathbb{B} \arrow[r, bend left, "K"] \arrow[r, bend right, "L"'] \arrow[r, phantom, "\Downarrow\sigma"] & \mathbb{A}
\end{tikzcd}
$$

where the components of $\hat{\sigma}$ are obtained by factoring the components $\tilde{\sigma}_X$ through the (cartesian) components of $\sigma'_{\hat{L}X}$ (X an object of $\mathbb{E}$)

In the following section we show our main intrinsic property of a 2-fibration for the case of **Fib**, namely the 'lifting' and 'factorisation' of adjoints in it.

4. Adjunctions in Fib

We now turn to study the interaction between change-of-base and fibred adjunctions, that is adjunctions in the 2-category **Fib**.

An easy 1-dimensional analogy of the characterisation of adjoints we are about to give might be helpful. Consider a fibration $\mathbb{E} \to^p \mathbb{B}$ and a morphism $f : X \to Y$ in $\mathbb{E}$. f is an isomorphism if and only if pf is an isomorphism and f is cartesian. Since cartesianness of f can be expressed as saying that $\hat{f}$ is an isomorphism, where $\hat{f}$ is the vertical factor of f through the cartesian lifting of pf, the previous statement means that isomorphisms in $\mathbb{E}$ are characterised in terms of those of $\mathbb{B}$ and those in the fibres. An analogous result holds for 1-cells admitting right adjoints within a fibred 2-category. We spell this out in **Fib**.

The following lemma establishes one important aspect of the change-of-base 2-functors with respect to adjunctions between the base categories. It is a consequence of the fact that an adjunction $\eta, \varepsilon : F \dashv G : \mathbb{B} \to \mathbb{A}$ (unit η and counit ε) induces a bi-adjunction $G^* \dashv F^* : \mathbf{Fib}(\mathbb{B}) \to \mathbf{Fib}(\mathbb{A})$ by change-of-base. The sense in which the adjunction obtained in the following lemma is cartesian will become clear in Theorem 4.3 below.

Lemma 4.1. *Given $\mathbb{E} \to^q \mathbb{B}$ and an adjunction $\eta, \varepsilon : F \dashv G : \mathbb{B} \to \mathbb{A}$, change-of-base along F yields a 'cartesian' fibred adjunction*

$$\begin{tikzcd}
F^*(\mathbb{E}) \arrow[rr, shift left, "q^*(F)"] \arrow[d, "F^*(q)"'] \arrow[rr, phantom, "\perp"] & & \mathbb{E} \arrow[ll, shift left, "\underline{G}"] \arrow[d, "q"] \\
\mathbb{A} \arrow[rr, shift left, "F"] \arrow[rr, phantom, "\perp"] & & \mathbb{B} \arrow[ll, shift left, "G"]
\end{tikzcd}$$

Proof. We will only spell out the data of the resulting adjunction, leaving the verification of the adjunction laws to the reader. Details may be found in [4]. To simplify the presentation, we assume the fibration is split, which allows us to ignore the coherent isomorphisms arising from the pseudo-functorial nature of a cleavage.

We use the following abbreviations:

$$q' = F^*(q), \quad F' = q^*(F), \quad G'' = (q')^*(G).$$

Consider

$$\begin{tikzcd}
 & G^*(F^*(q)) \arrow[dr, "F'G''"] \arrow[d, phantom, "\Downarrow \varepsilon'_q"] & \\
q \arrow[ur, "\langle \varepsilon \rangle_q"] \arrow[rr, "1_q"'] & {} & q
\end{tikzcd}$$

The right adjoint $\underline{G}$ is $G'' \circ \langle \varepsilon \rangle_q : q \to F^*(q)$, with counit $\overline{\varepsilon} = \varepsilon'_q : F'\underline{G} \Rightarrow 1_q$. The unit [1] is $\overline{\eta} = (\eta'_{q'}, \langle \varepsilon F \rangle_q)$. □

Remark 4.2. The notation $\underline{G}$ in the preceeding lemma is intended to be 'dual' to that adopted for cartesian morphisms. This is because, in this situation, the fibration $F^*(q)$ is the *direct image of p along G*. Put in other terms, the right-adjoint 1-cell $(\underline{G}, G)$ is (pseudo)cocartesian. Such cocartesian liftings obviously enjoy a 2-dimensional universal property dual to that of 2-cartesian 1-cells, which is at the base of the construction of oplax (bi)colimits for **Fib** in Section 5 below.

The following theorem gives a characterisation of adjunctions in **Fib** in terms of vertical fibred adjunctions, i.e. adjunctions in the fibre 2-categories **Fib**(_), and (ordinary) adjunctions in the base 2-category **Cat**. To express this succinctly let us introduce the following auxiliary definition. For a 2-category $\mathscr{K}$, let $\mathscr{K}_{\mathrm{ladj}}$ be the sub-2-category of $\mathscr{K}$, with the same objects and 2-cells but with only those 1-cells $f : A \to B$ which have a right adjoint $f \dashv g$. Since the composite of two such 1-cells has a right adjoint, $\mathscr{K}_{\mathrm{ladj}}$ is indeed a sub-2-category.

Theorem 4.3. *cod: $\mathbf{Fib}_{\mathrm{ladj}} \to \mathbf{Cat}_{\mathrm{ladj}}$ is a subfibration of cod: $\mathbf{Fib} \to \mathbf{Cat}$. In more detail, given $\mathbb{E} \to^p \mathbb{B}$, $\mathbb{D} \to^q \mathbb{A}$, $\eta, \varepsilon : F \dashv G : A \to B$ and a fibred 1-cell $(\tilde{F}, F) : p \to q$ as shown below*

$$\begin{tikzcd}
\mathbb{E} \arrow[r, "\tilde{F}"] \arrow[d, "p"'] & \mathbb{D} \arrow[d, "q"] \\
\mathbb{B} \arrow[r, shift left=1.5, "F"] \arrow[r, phantom, "\perp"] & \mathbb{A} \arrow[l, shift left=1.5, "G"]
\end{tikzcd}$$

[1] The full expression for the unit without the splitting assumption is $\overline{\eta} = (\eta'_{q'}, \vartheta_{GF,F}\langle \varepsilon F \rangle_q)\delta^{-1}_{F\eta,\varepsilon F}$, where $\vartheta_{GF,F} : (FGF)^* \Rightarrow (GF)^* F^*$ and $\delta_{F\eta,\varepsilon F} : \langle F\eta \rangle \langle \varepsilon F \rangle \Rightarrow \langle \varepsilon F \circ F\eta \rangle$ are the evident coherent isomorphisms. It would only obscure the presentation to include them; they all cancel appropriately to yield the triangular laws.

let $\hat{F}: p \to F^*(q)$ in **Fib**(B) *be the unique mediating functor in*

$$\begin{tikzcd}
\mathbb{E} \arrow[dr, dashed, "\hat{F}"] \arrow[drr, bend left, "\tilde{F}"] \arrow[ddr, bend right, "p"'] & & \\
& F^*(\mathbb{D}) \arrow[r, "q^*(F)"] \arrow[d, "F^*(q)"] & \mathbb{D} \arrow[d, "q"] \\
& \mathbb{B} \arrow[r, "F"'] & \mathbb{A}
\end{tikzcd}$$

Then, the following are equivalent

(i) $\exists \tilde{G}: \mathbb{D} \to \mathbb{E}. \tilde{F} \dashv \tilde{G}$ *(in* **Cat**) *such that* $(\tilde{F}, F) \dashv (\tilde{G}, G): q \to p$ *(in* **Fib**).

(ii) $\exists \hat{G}: F^*(q) \to p. \hat{F} \dashv \hat{G}$ *(in* **Fib**(B)).

Proof. (ii)$\Rightarrow$(i): This implication means that it is possible to define a 'global' fibred right adjoint $\tilde{G}$ given a vertical one $\hat{G}$ and a base one G. This is achieved by composition of adjoints.

By Lemma 4.1, we get a right adjoint to $q^*(F)$, $\bar{\eta}, \bar{\varepsilon}: q^*(F) \dashv \underline{G}: q \to F^*(q)$, and therefore $\tilde{G} = \hat{G} \circ \underline{G}$ is a right adjoint to $\tilde{F}$. It only remains to verify that the unit $\tilde{\eta} = \hat{G}\bar{\eta}\hat{F} \circ \hat{\eta}$ of this adjunction, where $\hat{\eta}$ is the unit for $\hat{F} \dashv \hat{G}$, is over η:

$$p\tilde{\eta} = p\hat{G}\bar{\eta}\hat{F} \circ p\hat{\eta} = F^*(q)\bar{\eta}\hat{F} = \eta F^*(q)\hat{F} = \eta p.$$

(i)$\Rightarrow$(ii): The unit $\eta: 1_{\mathbb{B}} \Rightarrow GF$ induces $\langle \eta \rangle_p : (GF)^*(p) \to p$. Consider the following diagram:

$$\begin{tikzcd}
F^*(q) \arrow[r, "q^*(F)"] \arrow[d, "F^*(G')"'] & q \arrow[d, "G'"'] \arrow[dr, "\tilde{G}"] & \\
(GF)^*(p) \arrow[r, "(G^*(p))^*(F)"'] & G^*(p) \arrow[r, "q^*(G)"'] & p
\end{tikzcd}$$

where $G' = \langle q, \tilde{G} \rangle$ is the uniquely determined functor into the pullback. Then $\hat{G} = \langle \eta \rangle_p \circ F^*(G')$ is the desired right adjoint. The unit $\hat{\eta}: 1_p \Rightarrow \hat{G}\hat{F}$ is obtained as the vertical factor of the fibred 2-cell $(\tilde{\eta}, \eta)$. The counit $\hat{\varepsilon}: \hat{F}\hat{G} \Rightarrow 1_{F^*(p)}$ is the vertical factor of the fibred 2-cell

$$(\tilde{\varepsilon}q^*(F) \circ \tilde{F}\eta'_p F^*(G'), \varepsilon F \circ F\eta) : (\tilde{F}\hat{G}, F) \Rightarrow (q^*(F), F)$$

noting that the base 2-cell is the identity. The triangular laws are verified using the universal property of 2-cartesian 2-cells. $\square$

4.1. Fibred limits and cartesian closure

We will apply Theorem 4.3 to give a characterisation of the completeness of the total category of a fibration in terms of that of its fibres and its base category. In order to do so, we shall make use of the following simple property of the exponential 2-functor $(\_)^{\mathbb{I}}$ (for $\mathbb{I}$ a small category) in **Cat**, i.e. the 2-functor such that $\mathbb{A}^{\mathbb{I}}$ is the functor category.

Proposition 4.4. *Given a fibration $p:\mathbb{E}\to\mathbb{B}$ and a small category $\mathbb{I}$, the functor $p^{\mathbb{I}}:\mathbb{E}^{\mathbb{I}}\to\mathbb{B}^{\mathbb{I}}$ is a fibration.*

Proof. A natural transformation $\alpha:F\Rightarrow G:\mathbb{I}\to\mathbb{E}$ is $p^{\mathbb{I}}$-cartesian iff every component is p-cartesian. Thus a $p^{\mathbb{I}}$-cartesian lifting is obtained from p-cartesian liftings, pointwise. □

Remark 4.5. The above proposition actually shows that **Fib** has *cotensors*, as in **Cat**, in the sense of [11]. This means that we have the following isomorphism of categories

$$\mathbf{Fib}(q,p^{\mathbb{I}})\cong\mathbf{Cat}(\mathbb{I},\mathbf{Fib}(q,p))$$

2-natural in q.

We shall also use the following property of right adjoints in $\mathbf{Cat}/\mathbb{A}$ and $\mathbf{Cat}^{\to}$. It turns out that such right adjoints preserve cartesian morphisms.

Lemma 4.6. (i) *Given fibrations $\mathbb{E}\to^{p}\mathbb{B}$, $\mathbb{D}\to^{q}\mathbb{B}$ and a 1-cell $G:q\to p$ in $\mathbf{Cat}/\mathbb{B}$, if there is $F:p\to q$ such that $F\dashv G$ in $\mathbf{Cat}/\mathbb{B}$ then G is a $\mathbb{B}$-fibred 1-cell.*

(ii) *Given fibrations $\mathbb{E}\to^{p}\mathbb{B}$, $\mathbb{D}\to^{q}\mathbb{A}$ and a fibred 1-cell $(\tilde{G},G):q\to p$ in $\mathbf{Cat}^{\to}$, if there is $(\tilde{F},F):p\to q$ such that $(\tilde{F},F)\dashv(\tilde{G},G)$ in $\mathbf{Cat}^{\to}$ then $(\tilde{G},G)$ is a fibred 2-cell.*

Proof. The first part of the lemma is a consequence of the fact that fibrations over a given base and fibred 1-cells between them are the algebras and pseudomorphisms for a Kock-Zöberlein monad on $\mathbf{Cat}/\mathbb{B}$. The second part follows from the first and Theorem 4.3, since the construction there shows that a right adjoint in $\mathbf{Cat}^{\to}$ whose source and targets are fibrations can be factored into a pair of cocartesian and vertical right adjoints. Hence if the second factor is fibred so is the original functor. □

Definition 4.7. For any small category $\mathbb{I}$, a fibration $p:\mathbb{E}\to\mathbb{B}$ has fibred $\mathbb{I}$-limits (respectively colimits) iff the fibred functor $\widehat{\Delta}_{\mathbb{I}}:p\to\Delta_{\mathbb{I}}^{*}(p^{\mathbb{I}})$, uniquely determined in the diagram below, has a fibred right (respectively left) adjoint $\widehat{\Delta_{\mathbb{I}}} \dashv \widehat{\mathrm{Lim}_{\mathbb{I}}}$

$$\begin{tikzcd}
\mathbb{E} \arrow[dr, dashed, "\hat{\Delta}_{\mathbb{I}}"] \arrow[drr, bend left, "\tilde{\Delta}_{\mathbb{I}}"] \arrow[ddr, bend right, "p"'] & & \\
& \Delta_{\mathbb{I}}^{*}(\mathbb{E}) \arrow[r] \arrow[d, "\Delta_{\mathbb{I}}^{*}(p^{\mathbb{I}})"] & \mathbb{E}^{\mathbb{I}} \arrow[d, "p^{\mathbb{I}}"] \\
& \mathbb{B} \arrow[r, "\Delta_{\mathbb{I}}"'] & \mathbb{B}^{\mathbb{I}}
\end{tikzcd}$$

where $\Delta_{\mathbb{I}} : \mathbb{B} \longrightarrow \mathbb{B}^{\mathbb{I}}$ and $\tilde{\Delta}_{\mathbb{I}} : \mathbb{E} \longrightarrow \mathbb{E}^{\mathbb{I}}$ are the diagonal functors taking objects A to constant functors $(I \mapsto A)$.

Dually, we speak of cofibred $\mathbb{I}$-limits/colimits for a cofibration.

Remark 4.8. Similar to Remark 4.5, the fibration $\Delta_{\mathbb{I}}^{*}(p^{\mathbb{I}})$ is a cotensor in **Fib**($\mathbb{B}$), as we have

$$\mathbf{Fib}(\mathbb{B})(q, \Delta_{\mathbb{I}}^{*}(p^{\mathbb{I}})) \cong \mathbf{Cat}(\mathbb{I}, \mathbf{Fib}(\mathbb{B})(q, p)).$$

Hence, the above definition of fibred $\mathbb{I}$-limits for a fibration is analogous to the definition of $\mathbb{I}$-limits for an ordinary category, relative to its 2-categorical universe **Cat**.

Now we can characterise fibred limits as follows:

Corollary 4.9. *Let $\mathbb{I}$ be a small category and $\mathbb{E} \to^{p} \mathbb{B}$ be a fibration such that $\mathbb{B}$ has $\mathbb{I}$-limits. Then p has fibred $\mathbb{I}$-limits iff $\mathbb{E}$ has and p strictly preserves $\mathbb{I}$-limits.*

Proof. Apply Theorem 4.3 to the following data (where $p^{\mathbb{I}} : \mathbb{E}^{\mathbb{I}} \to \mathbb{B}^{\mathbb{I}}$ is a fibration by Proposition 4.4).

$$\begin{tikzcd}
\mathbb{E} \arrow[r, "\tilde{\Delta}_{\mathbb{I}}"] \arrow[d, "p"'] & \mathbb{E}^{\mathbb{I}} \arrow[d, "p^{\mathbb{I}}"] \\
\mathbb{B} \arrow[r, shift left, "\Delta_{\mathbb{I}}"] \arrow[r, phantom, "\perp"] & \mathbb{B}^{\mathbb{I}} \arrow[l, shift left, "\mathrm{Lim}_{\mathbb{I}}"]
\end{tikzcd}$$

$\mathbb{E}$ has and p strictly preserves $\mathbb{I}$-limits means precisely that the above diagram can be completed to an adjunction $(\tilde{\Delta}_{\mathbb{I}}, \Delta_{\mathbb{I}}) \dashv (\widetilde{\mathrm{Lim}}_{\mathbb{I}}, \mathrm{Lim}_{\mathbb{I}})$ in $\mathbf{Cat}^{\to}$, which by Lemma 4.6(ii) is an adjunction in **Fib**. □

In a 2-category $\mathcal{K}$ with cotensors, we can say that an object X admits $\mathbb{I}$-limits if the corresponding diagonal $\Delta : X \to X^{\mathbb{I}}$ admits a right adjoint. Then, the above corollary states that, given $\mathbb{E} \to^{p} \mathbb{B}$ where $\mathbb{B}$ has $\mathbb{I}$-limits, p has $\mathbb{I}$-limits in **Fib** if and only if it has $\mathbb{I}$-limits in **Fib**($\mathbb{B}$).

By mere duality, we have the following characterisation of colimits in a cofibred category.

Corollary 4.10. *Let $r : \mathbb{D} \to \mathbb{A}$ be a cofibration, i.e. $r^{op} : \mathbb{D}^{op} \to \mathbb{A}^{op}$ is a fibration, such that $\mathbb{A}$ has $\mathbb{I}$-colimits. Then r has cofibred $\mathbb{I}$-colimits iff $\mathbb{D}$ has and r strictly preserves $\mathbb{I}$-colimits.*

Remark 4.11. Notice that whenever a functor p has the isomorphism-lifting property [10], if it preserves given limits/colimits up to isomorphism, it is possible to 'reindex' the given limits/colimits along the isomorphism so as to get preservation 'on-the-nose'. Thus the above corollaries can be strengthened by requiring the fibration (respectively cofibration) p to preserve limits (respectively colimits) in the usual sense.

We should point out that the above characterisation of limits and colimits is well-known, e.g. from [2]. What is new here is the way they arise from our characterisation of adjunctions in **Fib**, which yields the simple proofs above.

We conclude this section showing how to infer cartesian closure of a fibred category. In order to do so, we call a *fibred-ccc* a fibration such that every fibre is cartesian closed, and the reindexing functors preserve such structure (of course, we could give an elementary choice-free definition). Given a fibration $\mathbb{E} \to^{p} \mathbb{B}$, where the base $\mathbb{B}$ has finite products, we say that p admits *simple $\mathbb{B}$-products* if, for every I in $\mathbb{B}$, the fibred functor $\langle \pi_{\_,I} \rangle : p \to (\_ \times I)^*(p)$ – induced by the natural transformation $\pi_{\_,I} : \_ \times I \Rightarrow 1_{\mathbb{B}}$ given by first projection – has a fibred right adjoint $\Pi_I : (\_ \times I)^*(p) \to p$. This amounts to the usual formulation in terms of right adjoints to reindexing functors $\pi^*_{X,I}$, satisfying the Beck–Chevalley condition. Now we can formulate the following corollary.

Corollary 4.12. *Given $\mathbb{E} \to^{p} \mathbb{B}$ such that p is a fibred-ccc with simple $\mathbb{B}$-products, if $\mathbb{B}$ is a ccc then $\mathbb{E}$ is a ccc and p strictly preserves the cartesian closed structure.*

Proof. From Corollary 4.9 we know $\mathbb{E}$ has finite products $\tilde{\times} : \mathbb{E} \times \mathbb{E} \to \mathbb{E}$. Then, for every $X \in \mathbb{E}_A$, we must supply $X \tilde{\Rightarrow} \_ : \mathbb{E} \to \mathbb{E}$ such that the following is a fibred adjunction:

$$
\begin{tikzcd}
\mathbb{E} \arrow[r, shift left=1.5, "(\_)\tilde{\times}X"] \arrow[r, phantom, "\perp"] \arrow[d, "p"'] & \mathbb{E} \arrow[l, shift left=1.5, "X\tilde{\Rightarrow}(\_)"] \arrow[d, "p"] \\
\mathbb{B} \arrow[r, shift left=1.5, "(\_)\times A"] \arrow[r, phantom, "\perp"] & \mathbb{B} \arrow[l, shift left=1.5, "A\Rightarrow(\_)"]
\end{tikzcd}
$$

By Theorem 4.3, it is sufficient to define a fibred right adjoint $G : (\_ \times I)^*(p) \to p$ to

$$\widehat{\_\tilde{\times} X} : p \to (\_ \times I)^*(p).$$

The latter can be expressed as the following composite:

$$p \xrightarrow{\widehat{(\_)\tilde{\times} X}} ((\_) \times I)^*(p) \;=\; p \xrightarrow{\langle \pi_{-,I} \rangle} (\_ \times I)^*(p) \xrightarrow{(-)\otimes \langle \pi'_{-,I} \rangle X} ((\_) \times I)^*(p)$$

where $\otimes : p \times p \to p$ stands for the fibred product. The hypothesis guarantees the existence of fibred right adjoints for both factors, whence the right adjoint for

$$\widehat{(\_)\tilde{\times} X} : p \to ((\_) \times I)^*(p)$$

is given by $\Pi_I \circ (\langle \pi'_{\_,I} \rangle \Rightarrow \_)$, where $\Rightarrow$ stands for the fibred exponential. $\square$

Remark 4.13. It was the above corollary on cartesian closure which spurred the analysis of adjunctions in **Fib**. It provides a categorical account of logical predicates for the simply typed λ-calculus. See [4] for details, where we also provide a counterexample to the converse of the above corollary.

4.2. Comma objects in **Fib**

As a preliminary to our treatment of fibrations over fibrations in Section 4.3 below, and also as an illustration of limits in a fibred 2-category, we show that **Fib** and its fibre 2-categories **Fib**($\mathbb{B}$) admit comma objects in the sense of [17]. The construction is entirely analogous to that of ordinary limits in fibred categories in Corollary 4.9. That is, we first build comma objects in **Fib**, and then obtain them in the fibres by restriction along the diagonal. We make the details explicit in the following proposition, as we need them in Section 4.3.

Proposition 4.14. **Fib** *and* **Fib**($\mathbb{B}$) *admit comma objects.*

Proof. Comma objects in **Fib** are inherited from $\mathbf{Cat}^{\to}$, where they are computed pointwise. In more detail, given

$$\begin{tikzcd}
\mathbb{A}' \arrow[r, "\tilde{F}"] \arrow[d, "p"'] & \mathbb{C}' \arrow[d, "r"] & \mathbb{B}' \arrow[l, "\tilde{G}"'] \arrow[d, "q"] \\
\mathbb{A} \arrow[r, "F"'] & \mathbb{C} & \mathbb{B} \arrow[l, "G"]
\end{tikzcd}$$

in **Fib**, i.e. p, q and r are fibrations, and $\tilde{F}$ and $\tilde{G}$ preserve cartesian morphisms, their comma object is given by

$$
\begin{tikzcd}
\tilde{F}{\downarrow}\tilde{G} \arrow[rr, "\tilde{\pi}_1"] \arrow[dr, "\tilde{\pi}_0"'] \arrow[dd, "\langle p,q\rangle"'] & & \mathbb{B}' \arrow[dr, "\tilde{G}"] \arrow[dd, "q"' near start] \arrow[dl, phantom, "\nearrow"] & \\
& \mathbb{A}' \arrow[rr, "\tilde{F}"' near start] \arrow[dd, "p" near start] & & \mathbb{C}' \arrow[dd, "r"] \\
F{\downarrow}G \arrow[rr, "\pi_1" near start] \arrow[dr, "\pi_0"'] & & \mathbb{B} \arrow[dr, "G"] \arrow[dl, phantom, "\nearrow"] & \\
& \mathbb{A} \arrow[rr, "F"'] & & \mathbb{C}
\end{tikzcd}
$$

where the top and bottom rows are comma objects in **Cat**, i.e. the usual comma categories, and $p\downarrow q$ is the morphism between comma categories uniquely induced by the universal property of the bottom comma object from the 2-cell determined by the front, right and top faces of the cube.

The functor $p\downarrow q:\tilde{F}\downarrow\tilde{G}\rightarrow F\downarrow G$ acts thus as follows:

$$(a,\tilde{F}a \xrightarrow{h} \tilde{G}b,b)\mapsto(pa,Fpa \xrightarrow{rh} Gqb,qb)$$

with a similar action on morphisms.

The universal property of the above construction in $\mathbf{Cat}^{\rightarrow}$ follows easily from the fact that limits in a (2-)functor category are computed pointwise. It only remains to show that $p\downarrow q:\tilde{F}\downarrow\tilde{G}\rightarrow F\downarrow G$ is actually a fibration. This is shown in the following diagram:

$$
\begin{tikzcd}
b \arrow[r, "v"] & qb' & b' & & v^*(b') \arrow[r, "\bar{v}"] & b' \\
Gb \arrow[r, "Gv"] & Gqb' & \tilde{G}b' & & \tilde{G}v^*(b') \arrow[r, "\tilde{G}\bar{v}"] & \tilde{G}b' \\
Fa \arrow[r, "Fu"] \arrow[u, "h"] & Fpa' \arrow[u, "rh'"'] & \tilde{F}a' \arrow[u, "h'"'] & \mapsto & \tilde{F}u^*(a') \arrow[r, "\tilde{F}\bar{u}"] \arrow[u, "h''"] & \tilde{F}a' \arrow[u, "h'"'] \\
a \arrow[r, "u"] & pa' & a' & & u^*(a') \arrow[r, "\bar{u}"] & a'
\end{tikzcd}
$$

where $\bar{v}$ is a q-cartesian lifting of $v,\bar{u}$ is a p-cartesian lifting of u and h'' is the unique such morphism making the right square commute, with $rh''=h$. It is clear then that $\tilde{\pi}_0$ and $\tilde{\pi}_1$ preserve cartesian morphisms.

In **Fib**($\mathbb{B}$) given 1-cells $K:p\rightarrow r$ and $H:q\rightarrow r$, regard them as fibred 1-cells (in **Fib**)

$$
\begin{tikzcd}
\mathbb{A}' \arrow[r, "K"] \arrow[d, "p"'] & \mathbb{C}' \arrow[d, "r"] & \mathbb{B}' \arrow[l, "H"'] \arrow[d, "q"] \\
\mathbb{B} \arrow[r, "1"'] & \mathbb{B} & \mathbb{B} \arrow[l, "1"]
\end{tikzcd}
$$

and construct their comma object in **Fib** as above. Then, their comma object in **Fib**($\mathbb{B}$) is obtained as the left fibration in the following pullback square:

$$\begin{tikzcd} K{\downarrow}_{\mathbb{B}}H \arrow[r] \arrow[d] & K{\downarrow}H \arrow[d, "\langle p,q\rangle"] \\ \mathbb{B} \arrow[r, "\langle 1,1\rangle"'] & \mathbb{B}^{\to} \end{tikzcd}$$

where $\mathbb{B}^{\to} = 1 \downarrow 1$, and $\langle 1,1\rangle : \mathbb{B} \to \mathbb{B}^{\to}$ is the 'diagonal', uniquely induced by the identity square

$$\begin{tikzcd} \mathbb{B} \arrow[r, "1"] \arrow[d, "1"'] & \mathbb{B} \arrow[d, "1"] \\ \mathbb{B} \arrow[r, "1"'] & \mathbb{B} \end{tikzcd}$$

and thus maps every object to its identity arrow. That is, in $K \downarrow_{\mathbb{B}} H$ we only consider the vertical morphisms of the comma category $K \downarrow H$. It is clear then that it enjoys the required universal property within **Fib**($\mathbb{B}$). □

Remark 4.15. The existence of the appropriate 2-categorical limits in **Fib**($\mathbb{B}$) follows from its algebraic treatment in [17]. The construction above shows the special case of comma objects as an instance of a limit in a fibre of a 2-fibration, thereby making explicit its relation to the corresponding limit in the total 2-category, which is essential for our algebraic proof of Bénabou's characterisation of fibrations over fibrations in Section 4.3 below.

4.3. Fibrations over fibrations

The notion of fibration can be internalised in any 2-category in a representable fashion: a 1-cell $p : E \to B$ in a 2-category $\mathscr{K}$ is a fibration iff for all objects X, the functor $\mathscr{K}(X,p) : \mathscr{K}(X,E) \to \mathscr{K}(X,B)$ is a fibration in **Cat**. If the 2-category $\mathscr{K}$ admits comma objects, it is possible to give a purely algebraic internal description of fibrations in it, as in [17]. Given such a $\mathscr{K}$, for a 1-cell $p : E \to B$ consider the diagram

$$\begin{tikzcd} E \arrow[dr, "\eta_p"] \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "p"'] & & \\ & B{\downarrow}p \arrow[r] \arrow[d, "d"'] \arrow[dr, phantom, "\nearrow"] & E \arrow[d, "p"] \\ & B \arrow[r, "1"'] & B \end{tikzcd}$$

where $\eta_p : E \to B \downarrow p$ is induced by the outer identity 2-cell into the comma object $B \downarrow p$. Since all the constructs are preserved by representables, the 1-cell $d : B \downarrow p \to B$ is the free fibration on p. In fact, this construction sets up a Kock–Zöberlein monad on $\mathscr{K}/B$, whose (pseudo) algebras are fibrations over B and the algebra (pseudo) morphisms are those which preserve the fibration structure. So, p is a fibration when $\eta_p : p \to d$ has a right adjoint in $\mathscr{K}/B$. This means that the 1-cell $\eta_p : E \to B \downarrow p$ has a right adjoint $\mathscr{C}\ell : B \downarrow p \to E$ such that the counit is mapped to the identity by d.

Since $\mathbf{Fib}(\mathbb{B})$ admits comma objects, we can consider fibrations in it, that is, given a fibred 1-cell f: $(\mathbb{E} \to^{p} \mathbb{B}) \to (\mathbb{D} \to^{q} \mathbb{B})$ we can ask for it to be a fibration over q. The following is a standard result of fibred category theory, due to Bénabou:

Theorem 4.16. *$f : \mathbb{E} \to \mathbb{D}$ is a fibration over q if and only if it is an ordinary fibration over $\mathbb{D}$.*

Bénabou's argument relies on a very exact application of his fibred Yoneda lemma. We give here a purely algebraic elementary proof of this result as an immediate consequence of Theorem 4.3.

According to Street's characterisation of fibrations in a 2-category above, f: $(\mathbb{E} \to^{p} \mathbb{B}) \to (\mathbb{D} \to^{q} \mathbb{B})$ is a fibration in $\mathbf{Fib}(\mathbb{B})$ if and only if $\eta_f : f \to d$ in the following diagram has a right adjoint (in $\mathbf{Fib}(\mathbb{B})/q$):

$$
\begin{tikzcd}
p \arrow[dr, "\eta_f"] \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "f"'] & & \\
& q\downarrow f \arrow[r] \arrow[d, "d"'] \arrow[dr, phantom, "\Nearrow"] & p \arrow[d, "f"] \\
& q \arrow[r, "1"'] & q
\end{tikzcd}
$$

In view of the construction of comma objects in $\mathbf{Fib}(\mathbb{B}), \eta_f$ is obtained as depicted in the following diagram:

$$
\begin{tikzcd}
\mathbb{E} \arrow[dr, dashed, "\eta_f"] \arrow[drr, bend left, "\langle f,1\rangle"] \arrow[ddr, bend right, "p"'] & & \\
& q\downarrow f \arrow[r] \arrow[d, "d"'] & \mathbb{D}\downarrow f \arrow[d, "\langle q,p\rangle"] \\
& \mathbb{B} \arrow[r, "\langle 1,1\rangle"'] & \mathbb{B}^{\to}
\end{tikzcd}
$$

where the square is a pullback and

$$
\begin{tikzcd}
\mathbb{D}\downarrow f \arrow[rr] \arrow[dr, "d_f"] \arrow[dd, "\langle q,p\rangle"'] & & \mathbb{E} \arrow[dr, "f"] \arrow[dd, "p"' near start] & \\
& \mathbb{D} \arrow[rr, "1"' near start] \arrow[dd, "q"' near start] \arrow[ur, phantom, "\Nearrow"] & & \mathbb{D} \arrow[dd, "q"] \\
\mathbb{B}^{\to} \arrow[rr, "d_1"] \arrow[dr, "d_0"'] & & \mathbb{B} \arrow[dr, "1"] & \\
& \mathbb{B} \arrow[rr, "1"'] \arrow[ur, phantom, "\Nearrow"] & & \mathbb{B}
\end{tikzcd}
$$

so that $\langle f,1\rangle:\mathbb{E}\to\mathbb{D}\downarrow f$ is induced by the universal property of the (codomain) comma object. Since $\langle 1,1\rangle:\mathbb{B}\to\mathbb{B}^{\to}$ has a right adjoint, $\langle 1,1\rangle\dashv d_0$, with identity unit and counit $\theta:\langle 1,1\rangle d_0\Rightarrow 1$, we can apply Theorem 4.3 to conclude that η_f has a right adjoint (over $\mathbb{B}$) precisely when $\langle f,1\rangle:\mathbb{E}\to\mathbb{D}\downarrow f$ has a right adjoint (over d_0). This latter means that $f:\mathbb{E}\to\mathbb{D}$ is a fibration. We must simply check that both adjoints live equivalently over $\mathbb{D}$.

To be precise, we must know the equivalence of the following:

(i) $\exists I\,.\,\eta_f\dashv I$ in **Fib**($\mathbb{B}$)/q

(ii) $\exists R\,.\,\langle f,1\rangle\dashv R$ in **Cat**/$\mathbb{D}$.

(i) $\Rightarrow$ (ii): Given the counit $\varepsilon_1:\eta_f I\Rightarrow \mathrm{id}$ such that $d'_f\varepsilon_1=\mathrm{id}$, the counit

$$\varepsilon=\varepsilon'\circ\overline{\langle 1,1\rangle}\varepsilon_1\underline{d_0}:\langle f,1\rangle R\Rightarrow \mathrm{id}$$

satisfies $d_f\varepsilon=\mathrm{id}$, where $\varepsilon':\overline{\langle 1,1\rangle}\underline{d_0}\Rightarrow\mathrm{id}$ is the counit of the adjunction obtained 'lifting' $\langle 1,1\rangle\dashv d_0$.

(ii) $\Rightarrow$(i): First, $fR=d_f$ implies that $fI=fR\overline{\langle 1,1\rangle}=d_f\overline{\langle 1,1\rangle}=d'_f$. As for the counit, let $(\tilde{\varepsilon},\varepsilon)$ be the counit of $(\tilde{R},d_0)\dashv(\langle f,1\rangle,\langle 1,1\rangle)$. Then, ε_1, the counit of $\eta_f\dashv I$ is the vertical factor of the 2-cell $\tilde{\varepsilon}\overline{\langle 1,1\rangle}$, and thus

$$d'_f\varepsilon_1=d_f\overline{\langle 1,1\rangle}\varepsilon_1=d_f\tilde{\varepsilon}=\mathrm{id}$$

as required.

The above result that a fibration over a fibration amounts to an ordinary fibration over the total category has important consequences. Let us just mention one such: the slice of a presheaf topos is again a presheaf topos $\mathscr{S}et^{\mathbb{C}^{op}}/F\equiv\mathscr{S}et^{\mathscr{E}lt(F)^{op}}$, where $\mathscr{E}lt(F)$ is the category of elements of the presheaf $F:\mathbb{C}^{op}\to\mathscr{S}et$. Bénabou's argument for this equivalence uses the above result and the following basic facts:

- We may identify a presheaf with a discrete fibration, that is, a discrete object in **Fib**($\mathbb{B}$). Recall that an object A in a 2-category $\mathscr{K}$ is *discrete* if for every object $X,\mathscr{K}(X,A)$ is a discrete category. When viewing a presheaf $F:\mathbb{C}^{op}\to\mathscr{S}et$ as a discrete fibration, its total category is precisely $\mathscr{E}lt(F)$.

- Any 1-cell with a discrete codomain is a fibration. Using Street's formalization of fibrations, this observation follows from the fact that, for any 1-cell $p: B \to A$, with A discrete, the following diagram

$$\begin{tikzcd} B \arrow[r, "1"] \arrow[d, "p"'] & B \arrow[d, "p"] \\ A \arrow[r, "1"'] & A \end{tikzcd}$$

is a comma-object square.

This presheaf slicing result is applied in [14] to show that the Yoneda embedding preserves locally cartesian closed structure.

5. Oplax colimits: Kleisli objects in Fib

Corollary 4.10 shows how to construct colimits in a category $\mathbb{D}$, for a given cofibration $r: \mathbb{D} \to \mathbb{A}$, provided the base and the fibres have colimits. Briefly put, given a diagram $D: \mathbb{I} \to \mathbb{D}$, we first construct a colimit in $\mathbb{A}$ for $rD: \mathbb{I} \to \mathbb{A}$, obtaining a colimit cocone $\eta_{rD}: rD \Rightarrow C$. Using the cocartesian liftings of the DI's (I an object in $\mathbb{I}$) along the cocone components η_{rDI}, we obtain an $\mathbb{I}$-diagram in the fibre $\mathbb{D}_C$. The colimit of this latter diagram in $\mathbb{D}_C$ gives the desired colimit in $\mathbb{D}$.

An entirely analogous construction yields oplax colimits in **Fib**, using the 2-dimensional property of the cocartesian liftings. The fibres **Fib**($\mathbb{B}$) have oplax colimits, created by the forgetful 2-functor *dom*: **Fib**($\mathbb{B}$) $\to$ **Cat**, which sends a fibration $\mathbb{E} \to^p \mathbb{B}$ to its total category $\mathbb{E}$. Clearly, the base **Cat** admits oplax colimits [11]. To be precise, the pseudo-cocartesian liftings of **Fib** imply that we would obtain only oplax bicolimits, i.e. unique up to equivalence rather than up to isomorphism. The relevant definitions of (op)lax functors and their (co)limits can be found in [16].

We illustrate this construction of oplax colimits with the simplest case, which nevertheless contains all the essential details: oplax colimits for oplax functors $G: 1 \to$ **Fib**, where 1 is the terminal (one-object) category. In this case G amounts to a comonad $(\tilde{G}, G): p \to p$ in **Fib**, and its oplax colimit to its 'Kleisli object' $p_{(\tilde{G},G)}$ [15]. To structure the proof, we state some auxiliary lemmas first, in line with the construction of colimits outlined above. We omit the laborious calculations involved in the verification that the constructions provided satisfy the relevant properties/axioms; details can be found in [4].

In any 2-category $\mathscr{K}$, given a comonad $\langle G: A \to A, \varepsilon, \delta \rangle$, we write (U, λ) for an oplax cocone for it, i.e. $U: A \to C$ and $\lambda: U \Rightarrow UG$ satisfying $U\varepsilon \circ \lambda = 1_U$ and $\lambda G \circ \lambda = U\delta \circ \lambda$. Recall, e.g. from [15], that when (U, λ) is a colimiting oplax cocone, U has a left adjoint F which gives a *resolution* for the comonad G, i.e. the adjunction $F \dashv U$ generates the comonad $\langle G: A \to A, \varepsilon, \delta \rangle$.

Lemma 5.1. *Given*

- *a comonad* $\langle (\tilde{G}, G): (\mathbb{E} \to^p \mathbb{B}) \to (\mathbb{E} \to^p \mathbb{B}), (\tilde{\varepsilon}, \varepsilon), (\tilde{\delta}, \delta) \rangle$ *for* $\mathbb{E} \to^p \mathbb{B}$,

- *a fibred oplax cocone* $\big((\tilde{L}, L) : (\mathbb{E} \to^p \mathbb{B}) \to (\mathbb{D} \to^q \mathbb{C}), (\tilde{\sigma}, \sigma)\big)$,
- *an oplax cocone* $(K : \mathbb{B} \to \mathbb{A}, v : K \Rightarrow KG)$ *for* G, *and*
- *a functor* $J : \mathbb{A} \to \mathbb{C}$ *such that* $JK = L$ *and* $Jv = \sigma$.

There is a unique oplax cocone $(L' : \mathbb{E} \to J^*(\mathbb{D}), \sigma^\dagger : L' \Rightarrow L'\tilde{G})$ *such that* $((L', K), (v, \sigma))$ *is a fibred oplax cocone for* $(\tilde{G}, G)$, $q^*(J)L' = L$ *and* $q^*(J)\sigma^\dagger = \tilde{\sigma}$.

Proof. L' and $\sigma^\dagger$ are uniquely determined by the (2-dimensional) universal property of the pullback $J^*(\mathbb{D})$. $\square$

Lemma 5.2. *Given a comonad* $(\tilde{G}, G) : (\mathbb{E} \to^p \mathbb{B}) \to (\mathbb{E} \to^p \mathbb{B})$, *let* $(U : \mathbb{B} \to \mathbb{B}_G, \lambda)$ *be the Kleisli object for the base comonad* G. *The corresponding resolution* $\eta, \varepsilon : F \dashv U : \mathbb{B} \to \mathbb{B}_G$ *for* G *induces a comonad* $\overline{G} : F^*(p) \to F^*(p)$ *on* $F^*(p)$, *the direct image of* p *along* U, *in* **Fib**$(\mathbb{B}_G)$ *(see Remark* 4.2*).*

Proof. The data for the comonad $\overline{G}$ is given by (assuming everything is split, for simplicity)

- $\overline{G} = \langle F\eta \rangle F^*(\hat{G})$. Recall that $\hat{G} : p \to G^*(p)$ is obtained by factoring $\tilde{G} : \mathbb{E} \to \mathbb{E}$ through the pullback of G and p.
- The counit is $\bar{\varepsilon} = \langle F\eta \rangle F^*(\hat{\varepsilon})$, where $\hat{\varepsilon} : \hat{G} \Rightarrow \langle \varepsilon \rangle$ is the fibred 2-cell obtained by factoring $\tilde{\varepsilon} : \tilde{G} \Rightarrow 1_E$ through $\varepsilon : G \Rightarrow 1_B$.
- The comultiplication is $\bar{\delta} = \langle F\eta \rangle F^*(\hat{\delta})$, where $\hat{\delta} : \hat{G} \Rightarrow \langle \delta \rangle \circ G^*(\hat{G}) \circ \hat{G}$ is the $\mathbb{B}$-fibred 2-cell obtained by factoring $\tilde{\delta} : \tilde{G} \Rightarrow \tilde{G}^2$ through $\delta : G \Rightarrow G^2$. $\square$

Notice that the above lemma means that we can 'factorise' the given oplax diagram, namely the comonad $(\tilde{G}, G)$ through the cocartesian lifting of the oplax colimit for the base diagram, namely the Kleisli object for the (base) comonad G. We must then construct the corresponding oplax colimit for the 'vertical' diagram so determined, and upon pasting it with the cocartesian lifting mentioned above, obtain the desired oplax colimit for $(\tilde{G}, G)$.

Lemma 5.3. *Consider a comonad* $(\tilde{G}, G) : (\mathbb{E} \to^p \mathbb{B})(\mathbb{E} \to^p \mathbb{B})$. *Let* $(U : \mathbb{B} \to \mathbb{B}_G, \lambda)$ *be the Kleisli object for the base comonad* G. *A fibred oplax cocone for* $(\tilde{G}, G)$ *over* $(U, \lambda), ((\tilde{U}, U) : (\mathbb{E} \to^p \mathbb{B}) \to (\mathbb{D} \to^q \mathbb{C}))$, *induces an oplax cocone* $(U' : F^*(p) \to q, \lambda')$ *in* **Fib**$(\mathbb{B}_G)$ *for the comonad* $\overline{G}$ *associated to the Kleisli resolution* $F \dashv U : \mathbb{B} \to \mathbb{B}_G$ (cf. *Lemma* 5.2)

Theorem 5.4. **Fib** *admits Kleisli objects for comonads.*

Proof. Given a comonad $(\tilde{G}, G) : (\mathbb{E} \to^p \mathbb{B}) \to (\mathbb{E} \to^p \mathbb{B})$, let $(U : \mathbb{B} \to \mathbb{B}_G, \lambda)$ be the Kleisli object for G and $F \dashv U$ its associated resolution. By Lemma 5.2, this resolution induces a comonad $\overline{G} : F^*(p) \to F^*(p)$ in **Fib**$(\mathbb{B}_G)$, which admits a Kleisli fibration $F^*(p)_{\overline{G}} : (F^*(\mathbb{E}))_{\overline{G}} \to \mathbb{B}_G$.

This is then the fibration corresponding to the Kleisli object for $(\tilde{G}, G)$. The corresponding oplax cocone is $((U_1U_2, U): p \to F^*(p)_{\overline{G}}, (\lambda', \lambda))$, given as follows:

- $U_1 : F^*(p) \to F^*(p)_{\overline{G}}$ is the fibred functor corresponding to the Kleisli object (U_1, λ_1) for $\overline{G}$ in $\mathbf{Fib}(\mathbb{B}_G)$. The associated resolution is $F_1 \dashv U_1$.
- $(U_2, U): p \to F^*(p)$ is the right adjoint (in **Fib**) to $(p^*(F), F): F^*(p) \to p$ (change-of-base square), given in Lemma 4.1.
- λ' is obtained from the resolution $p^*(F)F_1 \dashv U_1U_2$ for $\tilde{G}$ given by the adjunctions $F_1 \dashv U_1$ and $p^*(F) \dashv U_2$.

The universal property of the oplax cocone $((U_1U_2, U): p \to F^*(p)_{\overline{G}}, (\lambda', \lambda))$ follows from Lemmas 5.1 and 5.3: given another oplax cocone $((\tilde{L}, L) : (\mathbb{E} \to^p \mathbb{B}) \to (\mathbb{D} \to^q \mathbb{C}), (\tilde{\sigma}, \sigma))$, we get a functor $J : \mathbb{B}_G \to \mathbb{C}$ by universality of (U, λ). Applying Lemma 5.1 we get a fibred oplax cocone $((L', U): p \to J^*(q), (\sigma', \lambda))$. This cocone then yields, by Lemma 5.3, an oplax cocone for $\overline{G}: F^*(p) \to F^*(p)$, which yields the desired mediating fibred functor between $F^*(p)_{\overline{G}}$ and q (composing with $(q^*(J), J): J^*(q) \to q$). $\square$

We end up mentioning that Kleisli objects in **Fib** are applied to construct 'fibrations with indeterminates' for polymorphic λ-calculus in [4, 6].

6. Concluding remarks

The purpose of the paper was to analyse some aspects of **Fib** based on its 2-fibred structure over **Cat**. We have thus given a non-elementary analysis of those aspects, based on the existence of cartesian-vertical factorisation for fibred 2-cells. Although the characterisation of a 2-functor admitting such property was given in elementary 2-categorical terms, it is possible to give a more abstract formulation of 2-fibrations. The notion of fibration in a 2-category can be made completely internal to it, i.e. without relying on representability. This requires the presence of comma objects in the 2-category, cf. [3, 17]. The situation for 2-fibrations is much more delicate, as they are not simply 'fibrations in the 3-category 2-**Cat**'. The appropriate setting for 2-fibrations is actually 'weakly tricategorical': the formal definition of a 2-fibration in the above style takes place in (a mild variant of) Gray's 2-$\mathbf{Cat}_{\otimes}$ [3, I.4.25]. Since the latter is not a 2-category, most of the algebraic properties of adjunctions in a 2-category (or bicategory) are lost. We must therefore postpone a full-fledged formal treatment of 2-fibrations until the subtle coherence issues arising in the above setting are settled down.

Other properties of **Fib** qua 2-fibration have been studied by Bénabou, notably the appropriate version of the Beck–Chevalley condition for cocartesian liftings (or direct images), alongside a number of preservation properties of these latter, although no explicit definition of 2-fibration appears in the material available to the author.

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Appendix A. Cartesian 2-cells and right adjoints

For an ordinary functor $\mathbb{E} \to^{p} \mathbb{B}$, we can characterise cartesian morphisms in $\mathbb{E}$ with codomain X as follows (cf. [2]):

> p has cartesian liftings with codomain X if and only if the functor
> $p/X : \mathbb{E}/X \to \mathbb{B}/pX$ has a right adjoint right inverse.

The functor p/X takes a morphism f to pf. The above statement means that a cartesian morphism is a right adjoint value for p/X. We now show such a characterisation is also possible for 1-cartesian 2-cells.

Given a 2-functor $P : \mathscr{E} \to \mathscr{B}$ and an object X in $\mathscr{E}$ define the following 2-category $\mathscr{E} \Downarrow X$:

Objects. 2-cells with (0-)codomain X, $\alpha : a \Rightarrow b : Z \to X$.

1-cells: For objects $\alpha : a \Rightarrow b : Z \to X$, $\alpha' : a' \Rightarrow b' : Z' \to X$, a 1-cell consists of a 1-cell $h : Z \to Z'$ in $\mathscr{E}$, and a 2-cell $\gamma : a \Rightarrow a'h$ such that $b'h = b$ and

$$\alpha' h \circ \gamma = \alpha$$

that is, a 1-cell amounts to a lax triangle between the domain 1-cells of α, α' and strict triangle between the codomain ones, plus the obvious coherence requirement.

2-cells: For 1-cells (h, γ) and (h', γ'), a 2-cell is $\theta : h \Rightarrow h'$ such that $b'\theta = b$ and

$$(a'\theta) \circ \gamma' = \gamma : a \Rightarrow a'h'.$$

The 2-functor $P : \mathscr{E} \to \mathscr{B}$ induces $P \Downarrow X : \mathscr{E} \Downarrow X \to \mathscr{B} \Downarrow PX$, applying P to the relevant components. We write $\mathscr{K}_0$ for the underlying (1-)category of a 2-category $\mathscr{K}$. We then have

Proposition A.1. *Consider a 2-functor $P : \mathscr{E} \to \mathscr{B}$.*

(i) *P admits 1-cartesian 2-cells with codomain X if and only if the functor $(P \Downarrow X)_0 : (\mathscr{E} \Downarrow X)_0 \to (\mathscr{B} \Downarrow PX)_0$ admits a right adjoint right inverse, i.e. with identity counit.*

(ii) *Furthermore, if P admits 2-cartesian 2-cells with codomain X, the above adjunction becomes a 2-adjunction (adjunction in the 2-category $2-\mathbf{Cat}$) between $\mathscr{E} \Downarrow X$ and $\mathscr{B} \Downarrow PX$.*

Proof. For 2-cell $\alpha : a \Rightarrow b : Z \to X$ in $\mathscr{E}$ and $\sigma : f \Rightarrow g : I \to PX$ the bijection

$$\frac{(\mathscr{E} \Downarrow X)_0(\alpha, \sigma')}{(\mathscr{B} \Downarrow PX)_0(P\alpha, \sigma)}$$

means precisely that σ' is 1-cartesian over σ ($P\sigma' = \sigma$ is the requirement that the right adjoint be a right inverse).

Furthermore, this bijection extends to 2-cells if the codomain of σ' is 2-cartesian: given a 2-cell $\theta : h \Rightarrow h' : PZ \to I$ between $P\alpha$ and σ, $g\sigma = Pb$ implies there is a unique 2-cell $\theta^\dagger : h^\dagger \Rightarrow (h')^\dagger$ with $P\theta^\dagger = \theta$, where $(\_)^\dagger$ denotes the adjoint transpose. □

Although this result does not give a full characterisation of 2-cartesianness, it certainly indicates that the 2-categories $\mathscr{E} \Downarrow X$ should play a similar role in the abstract theory of 2-fibrations to the one that ordinary slices play in ordinary fibred category theory.

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