

## 1. PROOF OF THEOREM 2.2

There are 11 diagrams in Shulman's definition of symmetric monoidal double category, which we check now for decorated cospan double categories.

### Diagram 1 of 11. Set up but not solved.

$M_1$  is given by  $a \rightarrow m_1 \leftarrow b$  with  $x_1 \in F(m_1)$ .  
 $M_2$  is given by  $b \rightarrow m_2 \leftarrow c$  with  $x_2 \in F(m_2)$ .  
 $M_3$  is given by  $c \rightarrow m_3 \leftarrow d$  with  $x_3 \in F(m_3)$ .  
 $N_1$  is given by  $a' \rightarrow n_1 \leftarrow b'$  with  $y_1 \in F(n_1)$ .  
 $N_2$  is given by  $b' \rightarrow n_2 \leftarrow c'$  with  $y_2 \in F(n_2)$ .  
 $N_3$  is given by  $c' \rightarrow n_3 \leftarrow d'$  with  $y_3 \in F(n_3)$ .

$$\begin{array}{ccc}
 ((M_1 \otimes N_1) \odot (M_2 \otimes N_2)) \odot (M_3 \otimes N_3) & \xrightarrow{\chi \odot 1} & ((M_1 \odot M_2) \otimes (N_1 \odot N_2)) \odot (M_3 \otimes N_3) \\
 \downarrow \alpha & & \downarrow \chi \\
 (M_1 \otimes N_1) \odot ((M_2 \otimes N_2) \odot (M_3 \otimes N_3)) & & ((M_1 \odot M_2) \odot M_3) \otimes ((N_1 \odot N_2) \odot N_3) \\
 \downarrow 1 \odot \chi & & \downarrow \alpha \otimes \alpha \\
 (M_1 \otimes N_1) \odot ((M_2 \odot M_3) \otimes (N_2 \odot N_3)) & \xrightarrow{\chi} & (M_1 \odot (M_2 \odot M_3)) \otimes (N_1 \odot (N_2 \odot N_3))
 \end{array}$$

Decorations:

(1) Right and then down:

$$F(\psi)\phi_{(m_1+n_1)+b+b', (m_2+n_2), m_3+n_3} (F(\psi)\phi_{m_1+n_1, m_2+n_2} (\phi_{m_1, n_1} (x_1, y_1), \phi_{m_2, n_2} (x_2, y_2)), F(\psi)\phi_{m_3, n_3} (x_3, y_3)) \in F(((m_1+n_1)+b+b', (m_2+n_2))+_{c+c'} (m_3+n_3))$$

$$F(\psi)\phi_{(m_1+b, m_2)+(n_1+b', n_2), m_3+n_3} (\phi_{m_1+b, m_2, n_1+b', n_2} ((F(\psi)\phi_{m_1, m_2} (x_1, x_2), F(\psi)\phi_{n_1, n_2} (y_1, y_2))), \phi_{m_3, n_3} (x_3, y_3)) \in F(((m_1+b, m_2)+(n_1+b', n_2))+_{c+c'} (m_3+n_3))$$

$$\phi_{(m_1+b, m_2)+c, m_3, (n_1+b', n_2)+c', n_3} ((F(\psi)\phi_{m_1+b, m_2, m_3} (F(\psi)\phi_{m_1, m_2} (x_1, x_2), x_3), (F(\psi)\phi_{n_1+b', n_2, n_3} (F(\psi)\phi_{n_1, n_2} (y_1, y_2), y_3))) \in F(((m_1+b, m_2)+c, m_3)+(n_1+b', n_2)+c', n_3))$$

$$\phi_{m_1+b, (m_2+c, m_3), n_1+b', (n_2+c', n_3)} (F(\psi)\phi_{m_1, m_2+c, m_3} ((x_1, F(\psi)\phi_{m_2, m_3} (x_2, x_3)), (y_1, F(\psi)\phi_{n_2, n_3} (y_2, y_3))) \in F((m_1+b, (m_2+c, m_3))+(n_1+b', (n_2+c', n_3)))$$

(2) Down and then right:

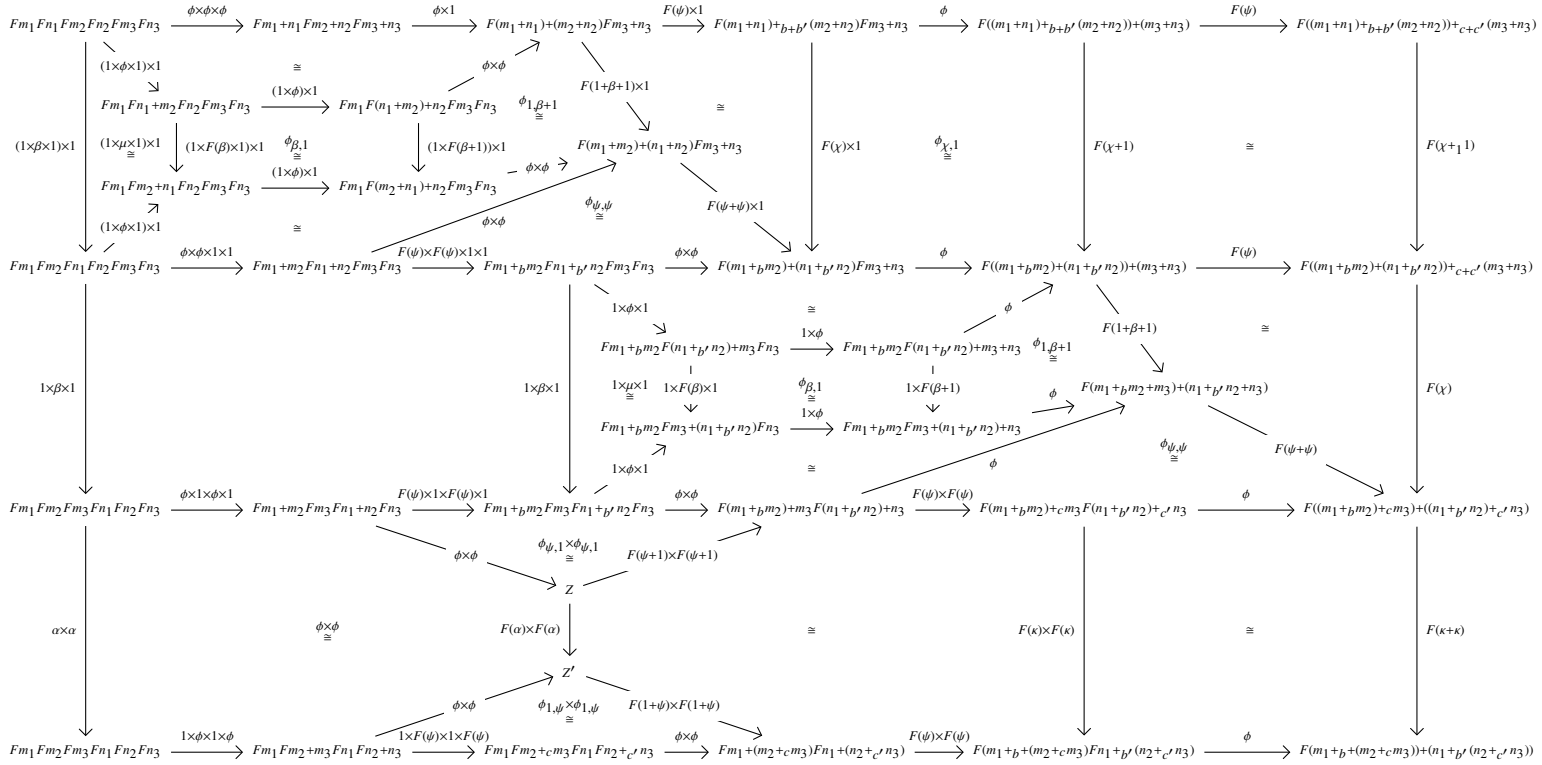
$$F(\psi)\phi_{(m_1+n_1)+b+b', (m_2+n_2), m_3+n_3} (F(\psi)\phi_{m_1+n_1, m_2+n_2} (\phi_{m_1, n_1} (x_1, y_1), \phi_{m_2, n_2} (x_2, y_2)), F(\psi)\phi_{m_3, n_3} (x_3, y_3)) \in F(((m_1+n_1)+b+b', (m_2+n_2))+_{c+c'} (m_3+n_3))$$

$$F(\psi)\phi_{m_1+n_1, (m_2+n_2)+c+c', (m_3+n_3)} ((\phi_{m_1, n_1} (x_1, y_1), F(\psi)\phi_{m_2+n_2, m_3+n_3} (\phi_{m_2, n_2} (x_2, y_2), \phi_{m_3, n_3} (x_3, y_3)))) \in F((m_1+n_1)+b+b', ((m_2+n_2)+c+c', (m_3+n_3)))$$

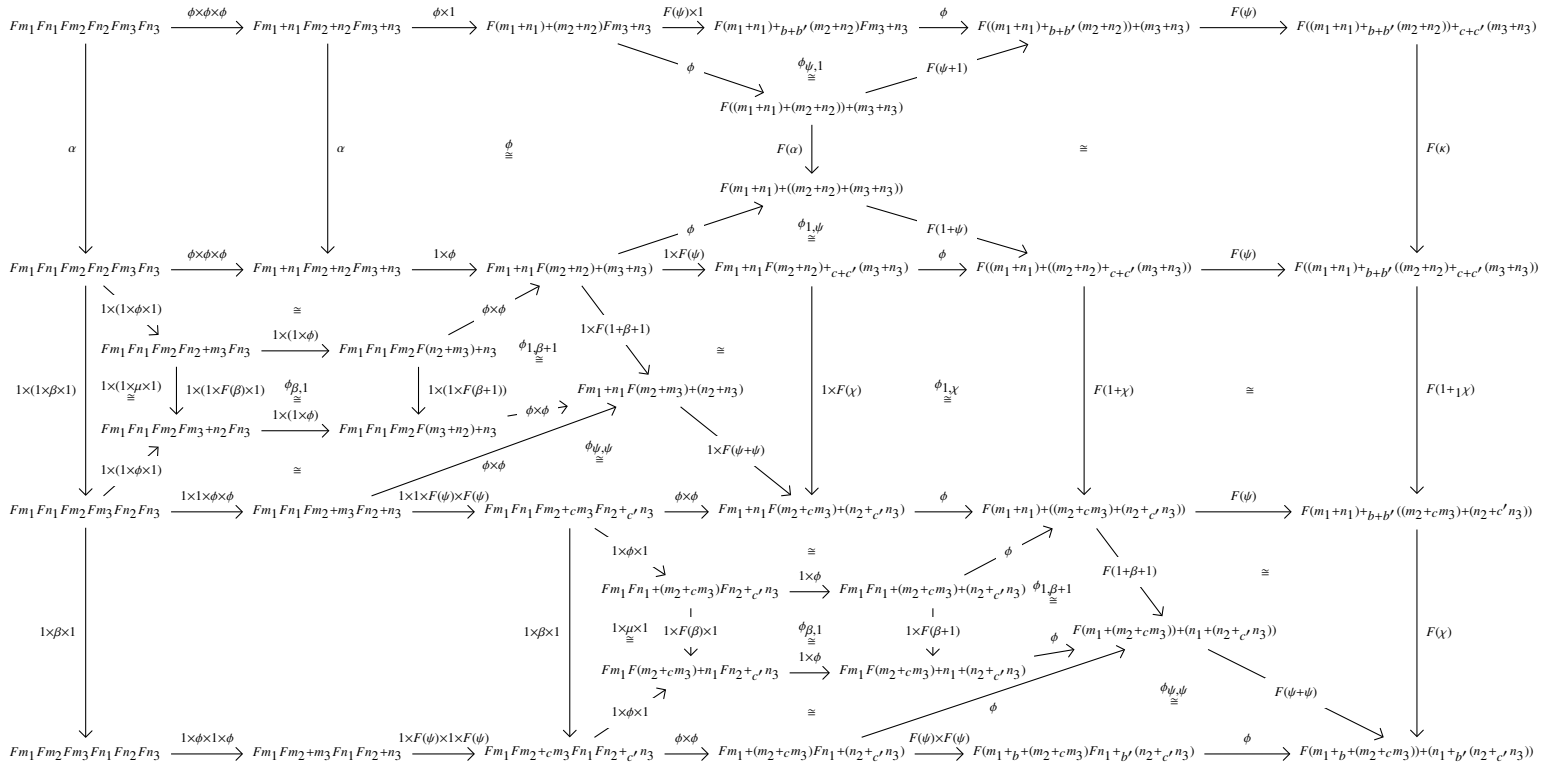
$$F(\psi)\phi_{m_1+n_1, (m_2+c, m_3)+(n_2+c', n_3)} (\phi_{m_1, n_1} (x_1, y_1), \phi_{m_2+c, m_3, n_2+c', n_3} ((F(\psi)\phi_{m_2, m_3} (x_2, x_3), F(\psi)\phi_{n_2, n_3} (y_2, y_3))) \in F((m_1+n_1)+b+b', ((m_2+c, m_3)+(n_2+c', n_3)))$$

$$\phi_{m_1+b, (m_2+c, m_3), n_1+b', (n_2+c', n_3)} (F(\psi)\phi_{m_1, m_2+c, m_3} ((x_1, F(\psi)\phi_{m_2, m_3} (x_2, x_3)), (y_1, F(\psi)\phi_{n_2, n_3} (y_2, y_3))) \in F((m_1+b, (m_2+c, m_3))+(n_1+b', (n_2+c', n_3)))$$

Right and then down (omitting morphisms emanating out of 1 on the left due to space restrictions):



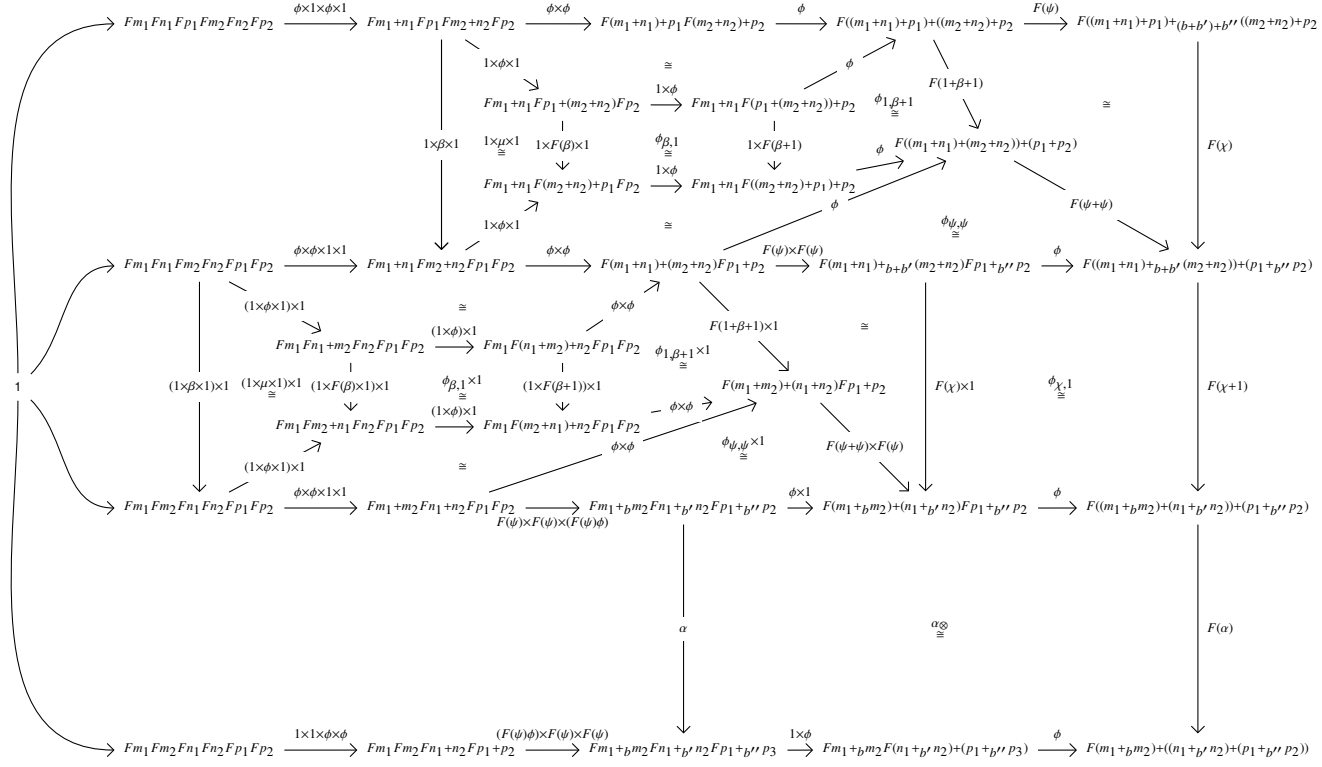
Down and then right (omitting morphisms emanating out of 1 on the left due to space restrictions):







Down and then right:



**Diagram 5 of 11. Set up but not solved.**

$$\begin{array}{ccccc}
 & a, b, c \in (\mathbf{A}, +, 0) & & & \\
 \perp_{(a+b)+c} \in F((a+b)+c) & U_{(a+b)+c} & \xrightarrow{\mu_\beta} & U_{a+(b+c)} & \perp_{a+(b+c)} \in F(a+(b+c)) \\
 & \downarrow \mu & & \downarrow \mu & \\
 \phi_{a+b,c}(\perp_{a+b}, \perp_c) \in F((a+b)+c) & U_{a+b} \otimes U_c & & U_a \otimes U_{b+c} & \phi_{a,b+c}(\perp_a, \perp_{b+c}) \in F(a+(b+c)) \\
 & \downarrow \mu & & \downarrow \mu & \\
 \phi_{a+b,c}(\phi_{a,b}(\perp_a, \perp_b), \perp_c) \in F((a+b)+c) & (U_a \otimes U_b) \otimes U_c & \xrightarrow{\beta'} & U_a \otimes (U_b \otimes U_c) & \phi_{a,b+c}(\perp_a, \phi_{b,c}(\perp_b, \perp_c)) \in F(a+(b+c))
 \end{array}$$

All the left cospans are  $(a + b) + c \rightarrow (a + b) + c \leftarrow (a + b) + c$  and all the right cospans are  $a + (b + c) \rightarrow a + (b + c) \leftarrow a + (b + c)$ .

Right and then down:

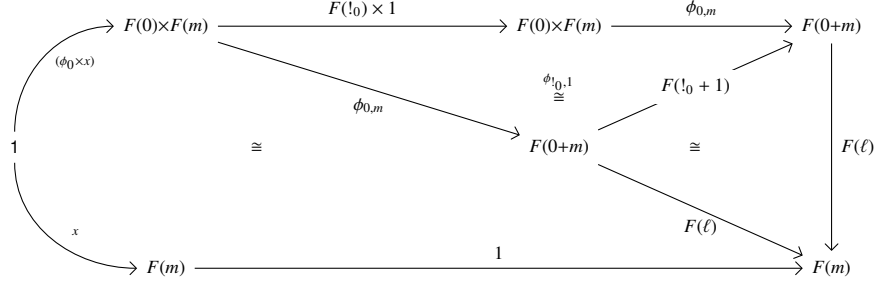
$$\begin{array}{c}
 \begin{array}{ccccc}
 & F(0) & \xrightarrow{F(!_{(a+b)+c})} & F((a+b)+c) & \\
 \phi_0 \nearrow & & & & \downarrow F(\alpha) \\
 & F(0) & \xrightarrow{F(!_{a+(b+c)})} & F(a+(b+c)) & \\
 \phi_0 \nearrow & & & & \downarrow F(\text{id}) \\
 1 & \nearrow \phi_0 & & & \\
 \phi_0 \times \phi_0 \nearrow & F(0) \times F(0) & \xrightarrow{F(!_a) \times F(!_{b+c})} & F(a) \times F(b+c) & \xrightarrow{\phi_{a,b+c}} & F(a+(b+c)) \\
 & \uparrow \phi_{0,0} & & \uparrow \phi_{!_a, !_{b+c}} & & \downarrow F(\text{id}) \\
 & & & & & F(a+(b+c)) \\
 & \uparrow \phi & & \uparrow 1 \times \phi_{b+c} & & \\
 & F(0) \times (F(0) \times F(0)) & \xrightarrow{F(!_a) \times (F(!_b) \times F(!_c))} & F(a) \times (F(b) \times F(c)) & \xrightarrow{1 \times \phi_{b+c}} & F(a) \times F(b+c) & \xrightarrow{\phi_{a,b+c}} & F(a+(b+c))
 \end{array}
 \end{array}$$

Down and then right:

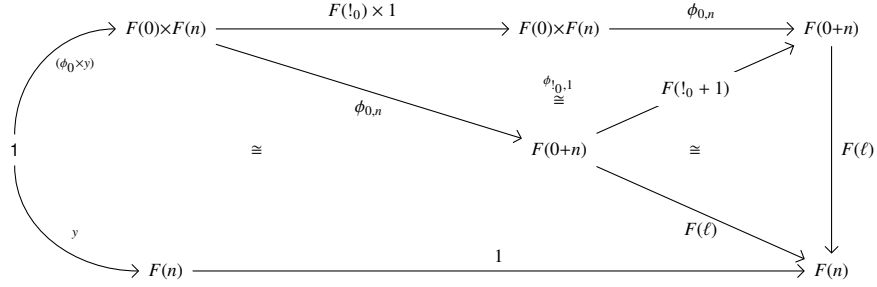
$$\begin{array}{c}
 \begin{array}{ccccc}
 & F(0) & \xrightarrow{F(!_{(a+b)+c})} & F((a+b)+c) & \\
 \phi_0 \nearrow & & & & \downarrow F(\text{id}) \\
 & F(0) & \xrightarrow{F(!_{a+(b+c)})} & F(a+(b+c)) & \\
 \phi_0 \nearrow & & & & \downarrow F(\text{id}) \\
 1 & \nearrow \phi_0 & & & \\
 \phi_0^2 \times \phi_0 \nearrow & F(0) \times F(0) & \xrightarrow{F(!_{a+b}) \times F(!_c)} & F(a+b) \times F(c) & \xrightarrow{\phi_{a+b,c}} & F(a+(b+c)) \\
 & \uparrow \phi_{0,0} & & \uparrow \phi_{!_{a+b}, !_c} & & \downarrow F(\text{id}) \\
 & & & & & F(a+(b+c)) \\
 & \uparrow \phi & & \uparrow \phi_{a,b} \times 1 & & \\
 & F(0) \times (F(0) \times F(0)) & \xrightarrow{(F(!_a) \times F(!_b)) \times F(!_c)} & (F(a) \times F(b)) \times F(c) & \xrightarrow{\phi_{a,b} \times 1} & F(a+b) \times F(c) & \xrightarrow{\phi_{a+b,c}} & F(a+(b+c)) \\
 & \downarrow \alpha' & & \downarrow \alpha' & & \downarrow F(\alpha) \\
 & F(0) \times (F(0) \times F(0)) & \xrightarrow{F(!_a) \times (F(!_b) \times F(!_c))} & F(a) \times (F(b) \times F(c)) & \xrightarrow{1 \times \phi_{b+c}} & F(a) \times F(b+c) & \xrightarrow{\phi_{a,b+c}} & F(a+(b+c))
 \end{array}
 \end{array}$$



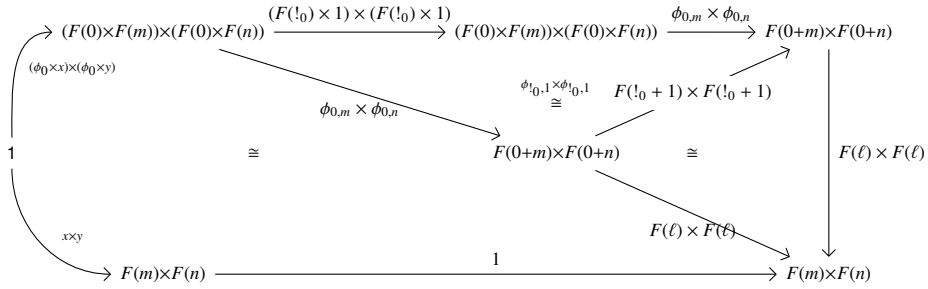
$$\lambda_M: U_0 \otimes M \rightarrow M$$



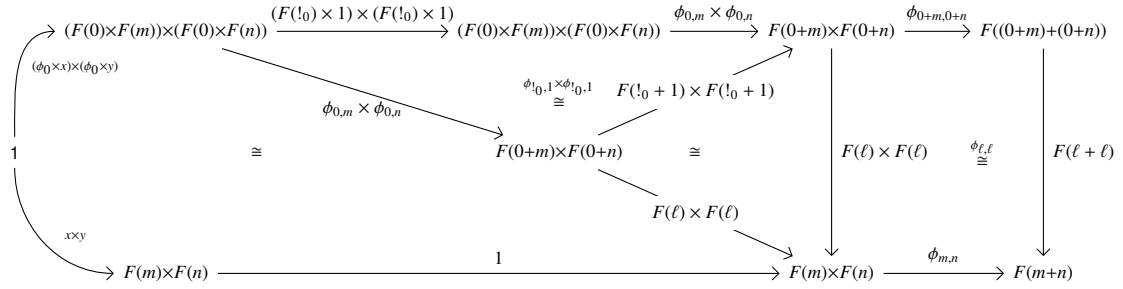
$$\lambda_N: U_0 \otimes N \rightarrow N$$



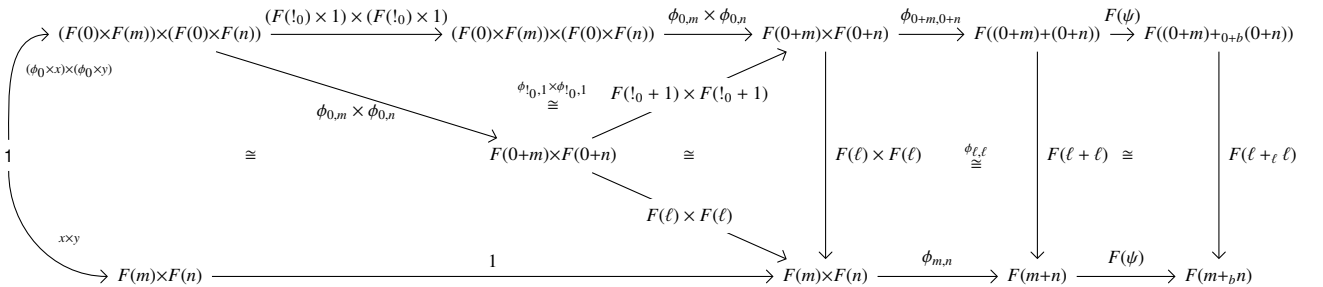
To construct  $\lambda_N \odot \lambda_M: (U_0 \otimes N) \odot (U_0 \otimes M) \rightarrow N \odot M$ , we first tensor the above two diagrams:



Next, we paste with a square due to pseudonaturality of  $\phi$ :

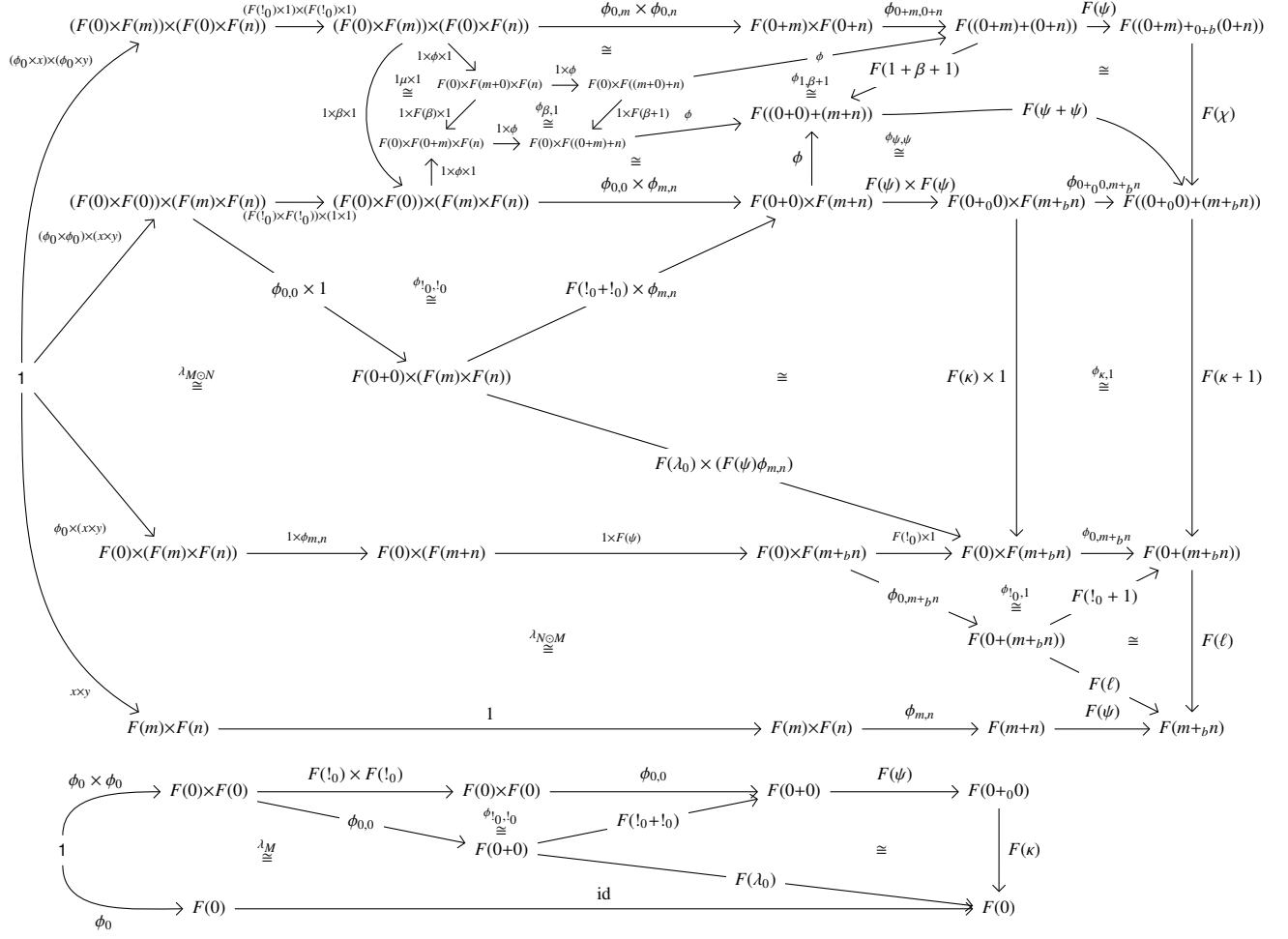


Finally, we paste with a square due to pseudonaturality of  $F$  to obtain the map  $\lambda_N \odot \lambda_M: (U_0 \otimes N) \odot (U_0 \otimes M) \rightarrow N \odot M$ :





Next, we compute the right, down and then left route:



Diagrams 7 and 9 of 11. Should improve this, but seems okay.

$$\begin{array}{ccc}
 F: (\mathbf{A}, +, 0) \rightarrow (\mathbf{Cat}, \times, 1) & & \\
 a \in (\mathbf{A}, +, 0) & & \\
 0 + a \rightarrow 0 + a \leftarrow 0 + a & U_0 \otimes U_a \xrightarrow{\mu} U_{0+a} & 0 + a \rightarrow 0 + a \leftarrow 0 + a \\
 \phi_{0,a}(\perp_0, \perp_a) \in F(0 + a) & \searrow \lambda_{U_a} \downarrow U_{\lambda_a} & \perp_{0+a} \in F(0 + a) \\
 & U_a & a \rightarrow a \leftarrow a \\
 & & \perp_a \in F(a)
 \end{array}$$

Right and then down:

$$\begin{array}{ccccc}
 & & F(!_{0+a}) & & \\
 & \nearrow \phi & & \nearrow \phi_{0,a} & \\
 1 & \xrightarrow{\phi \times \phi} & F(0) \times F(0) & \xrightarrow{F(!_0) \times F(!_a)} & F(0) \times F(a) & \xrightarrow{\phi_{0,a}} & F(0+a) & \xrightarrow{F(\text{id})} & F(0+a) \\
 & \searrow \phi & \downarrow \phi_{0,0} & \searrow \phi_{!_0, !_a} & \downarrow F(!_{0+a}) & \searrow \phi_{0,a} & \downarrow F(\lambda) & & \\
 & & F(0) & \xrightarrow{F(!_a)} & F(a) & & & & 
 \end{array}$$

Diagonally:

$$\begin{array}{ccccc}
 & & F(!_{0+a}) & & \\
 & \nearrow \phi & & \nearrow F(!_{0+a}) & \\
 1 & \xrightarrow{\phi \times \phi} & F(0) & \xrightarrow{F(!_{0+a})} & F(0+a) & \xrightarrow{F(\lambda)} & F(a) \\
 & \searrow \phi & \downarrow F(\text{id}) & \searrow F(!_a) & & & \\
 & & F(0) & \xrightarrow{F(!_a)} & F(a) & & 
 \end{array}$$

Removing the lower right  $\cong$  which is the same in each diagram:

$$\begin{array}{ccccc}
 & & F(!_{0+a}) & & \\
 & \nearrow \phi & & \nearrow \phi_{0,a} & \\
 1 & \xrightarrow{\phi \times \phi} & F(0) \times F(0) & \xrightarrow{F(!_0) \times F(!_a)} & F(0) \times F(a) & \xrightarrow{\phi_{0,a}} & F(0+a) & \xrightarrow{F(\text{id})} & F(0+a) \\
 & \searrow \phi & \downarrow \phi_{0,0} & \searrow \phi_{!_0, !_a} & \downarrow F(!_{0+a}) & \searrow \phi_{0,a} & \downarrow F(\lambda) & & \\
 & & F(0) & \xrightarrow{F(!_a)} & F(a) & & & & 
 \end{array}$$
  

$$\begin{array}{ccccc}
 & & F(!_{0+a}) & & \\
 & \nearrow \phi & & \nearrow F(!_{0+a}) & \\
 1 & \xrightarrow{\phi \times \phi} & F(0) & \xrightarrow{F(!_{0+a})} & F(0+a) & \xrightarrow{F(\lambda)} & F(a) \\
 & \searrow \phi & \downarrow F(\text{id}) & \searrow F(!_a) & & & \\
 & & F(0) & \xrightarrow{F(!_a)} & F(a) & & 
 \end{array}$$



**Diagrams 7 and 9 of 11. Set up and solved.**

$$\begin{array}{ccc}
 F: (\mathbf{A}, +, 0) \rightarrow (\mathbf{Cat}, \times, 1) & & \\
 a \in (\mathbf{A}, +, 0) & & \\
 0 + a \rightarrow 0 + a \leftarrow 0 + a & U_0 \otimes U_a \xrightarrow{\mu} U_{0+a} & 0 + a \rightarrow 0 + a \leftarrow 0 + a \\
 \phi_{0,a}(\perp_0, \perp_a) \in F(0 + a) & \searrow \lambda_{U_a} \downarrow U_{\lambda_a} & \perp_{0+a} \in F(0 + a) \\
 & & a \rightarrow a \leftarrow a \\
 & & \perp_a \in F(a)
 \end{array}$$

Right and then down:

$$\begin{array}{ccccc}
 1 \times 1 & \xrightarrow{\phi \times \phi} & F(0) \times F(0) & \xrightarrow{F(!_0) \times F(!_a)} & F(0) \times F(a) & \xrightarrow{\phi_{0,a}} & F(0+a) & U_0 \times U_a \\
 \downarrow \lambda^{-1} & \cong & \downarrow \phi_{0,0} & \phi_{!_0, !_a} \cong & \downarrow \phi_{0,a} & \searrow & \downarrow 1 & \downarrow \mu \\
 1 & \xrightarrow{\phi} & F(0) & \xrightarrow{F(!_a)} & F(0+a) & \xrightarrow{F} & F(0+a) & U_{0+a} \\
 \downarrow 1 & \phi & \downarrow 1 & F(!_a) & \searrow & \downarrow F(\lambda_a) & \downarrow & \downarrow U_{\lambda_a} \\
 1 & \xrightarrow{\phi} & F(0) & \xrightarrow{F(!_a)} & F(a) & \xrightarrow{F} & F(a) & U_a
 \end{array}$$

Diagonally:

$$\begin{array}{ccccc}
 1 \times 1 & \xrightarrow{\phi \times \phi} & F(0) \times F(0) & \xrightarrow{F(!_0) \times F(!_a)} & F(0) \times F(a) & \xrightarrow{\phi_{0,a}} & F(0+a) & U_0 \times U_a \\
 \downarrow \lambda^{-1} & \cong & \downarrow \phi_{0,0} & \phi_{!_0, !_a} \cong & \downarrow \phi_{0,a} & \searrow & \downarrow F(\lambda_a) & \downarrow \lambda_{U_a} \\
 1 & \xrightarrow{\phi} & F(0) & \xrightarrow{F(!_a)} & F(a) & \xrightarrow{F} & F(a) & U_a
 \end{array}$$

If we remove the regions common to each diagram, and regions that strictly commute, these are clearly equal.

Right and then down:

$$\begin{array}{ccc}
 F(0) \times F(0) & \xrightarrow{F(!_0) \times F(!_a)} & F(0) \times F(a) \\
 \downarrow \phi_{0,0} & \phi_{!_0, !_a} \cong & \downarrow \phi_{0,a} \\
 F(0) & \xrightarrow{F(!_a)} & F(0+a)
 \end{array}$$

Diagonally:

$$\begin{array}{ccc}
 F(0) \times F(0) & \xrightarrow{F(!_0) \times F(!_a)} & F(0) \times F(a) \\
 \downarrow \phi_{0,0} & \phi_{!_0, !_a} \cong & \downarrow \phi_{0,a} \\
 F(0) & \xrightarrow{F(!_a)} & F(0+a)
 \end{array}$$

**Diagram 10 of 11. Set up and solved (Christina).**

$M_1 \in F\mathbf{Csp}$  is given by  $a \xrightarrow{i_1} m_1 \xleftarrow{o_1} b$  with  $x_1 \in F(m_1)$ .

$M_2 \in F\mathbf{Csp}$  is given by  $b \xrightarrow{i_2} m_2 \xleftarrow{o_2} c$  with  $x_2 \in F(m_2)$ .

$N_1 \in F\mathbf{Csp}$  is given by  $a' \xrightarrow{i'_1} n_1 \xleftarrow{o'_1} b'$  with  $y_1 \in F(n_1)$ .

$N_2 \in F\mathbf{Csp}$  is given by  $b' \xrightarrow{i'_2} n_2 \xleftarrow{o'_2} c'$  with  $x_2 \in F(n_2)$ .

$$\begin{array}{ccc}
 a+a' \rightarrow (m_1+n_1)+_{b+b'}(m_2+n_2) \leftarrow c+c' & (M_2 \otimes N_2) \odot (M_1 \otimes N_1) \xrightarrow{\beta \odot \beta} (N_2 \otimes M_2) \odot (N_1 \otimes M_1) & a'+a \rightarrow (n_1+m_1)+_{b'+b}(n_2+m_2) \leftarrow c'+c \\
 F(\psi)\phi_{m_1+n_1, m_2+n_2}(\phi_{m_1, n_1}(x_1, y_1), \phi_{m_2, n_2}(x_2, y_2)) \in F((m_1+n_1)+_{b+b'}(n_1+n_2)) & \downarrow \chi & F(\psi)\phi_{n_1+m_1, n_2+m_2}(\phi_{n_1, m_1}(y_1, x_1), \phi_{n_2, m_2}(y_2, x_2)) \in F((n_1+m_1)+_{b'+b'}(n_1+m_2)) \\
 & (M_2 \odot M_1) \otimes (N_2 \odot N_1) \xrightarrow{\beta} (N_2 \odot N_1) \otimes (M_2 \odot M_1) & \\
 a+a' \rightarrow (m_1+b, m_2)+(n_1+b', n_2) \leftarrow c+c' & & a'+a \rightarrow (n_1+b', n_2)+(m_1+b, m_2) \leftarrow c'+c \\
 \phi_{m_1+b, m_2, n_1+b', n_2}(F(\psi)\phi_{m_1, m_2}(x_1, x_2), F(\psi)\phi_{n_1, n_2}(y_1, y_2)) \in F((m_1+b, m_2)+(n_1+b', n_2)) & & \phi_{n_1+b', n_2}(F(\psi)\phi_{n_1, n_2}(y_1, y_2), F(\psi)\phi_{m_1, m_2}(x_1, x_2)) \in F((n_1+b', n_2)+(m_1+b, m_2))
 \end{array}$$

Down and then right:

$$\begin{array}{c}
 \begin{array}{c}
 (x_1 \times y_1) \times (x_2 \times y_2) \\
 \downarrow 1 \times \beta \times 1 \\
 (x_1 \times x_2) \times (y_1 \times y_2) \\
 \downarrow \beta \\
 y_1 \times y_2 \times x_1 \times x_2
 \end{array}
 \end{array}$$

Diagram illustrating the "Down and then right" path in the proof. The diagram shows a complex network of nodes and arrows representing the derivation of the final result. The nodes are expressions involving  $F$ ,  $\phi$ ,  $\psi$ ,  $\chi$ ,  $\beta$ , and  $\mu$ . The arrows represent the application of various axioms and lemmas. The diagram is structured as follows:

- Top row:  $(F(m_1) \times F(n_1)) \times (F(m_2) \times F(n_2)) \xrightarrow{\phi_{m_1, n_1} \times \phi_{m_2, n_2}} F(m_1+n_1) \times F(m_2+n_2) \xrightarrow{\phi_{m_1+n_1, m_2+n_2}} F((m_1+n_1)+(m_2+n_2)) \xrightarrow{F(\psi)} F((m_1+n_1)+_{b+b'}(m_2+n_2))$
- Left side:  $(x_1 \times y_1) \times (x_2 \times y_2) \xrightarrow{1 \times \phi \times 1} F(m_1) \times F(n_1+m_2) \times F(n_2) \xrightarrow{1 \times \phi} F(m_1) \times F((n_1+m_2)+n_2) \xrightarrow{\phi_{\beta, 1}} F(m_1) \times F(m_2+n_1) \times F(n_2) \xrightarrow{1 \times \phi} F(m_1) \times F(m_2+n_1+n_2) \xrightarrow{\phi_{m_1+m_2, n_1+n_2}} F(m_1+b, m_2) \times F(n_1+b', n_2) \xrightarrow{\phi_{m_1+b, m_2, n_1+b', n_2}} F((m_1+b, m_2)+(n_1+b', n_2))$
- Right side:  $F((m_1+n_1)+(m_2+n_2)) \xrightarrow{F(1+\beta+1)} F(m_1+(m_2+n_1)+n_2) \xrightarrow{\phi_{\psi, \psi}} F(m_1+b, m_2) \times F(n_1+b', n_2) \xrightarrow{\phi_{m_1+b, m_2, n_1+b', n_2}} F((m_1+b, m_2)+(n_1+b', n_2))$
- Bottom row:  $F(n_1) \times F(n_2) \times F(m_1) \times F(m_2) \xrightarrow{\phi_{n_1, n_2} \times \phi_{m_1, m_2}} F(n_1+n_2) \times F(m_1+m_2) \xrightarrow{F(\psi) \times F(\psi)} F(n_1+b', n_2) \times F(m_1+b, m_2) \xrightarrow{\phi_{n_1+b', n_2, m_1+b, m_2}} F((n_1+b', n_2)+(m_1+b, m_2))$

Right and then down:

$$\begin{array}{c}
 \begin{array}{c}
 (x_1 \times y_1) \times (x_2 \times y_2) \\
 \downarrow \beta \times \beta \\
 (y_1 \times x_1) \times (y_2 \times x_2) \\
 \downarrow 1 \times \beta \times 1 \\
 y_1 \times y_2 \times x_1 \times x_2
 \end{array}
 \end{array}$$

Diagram illustrating the "Right and then down" path in the proof. The diagram shows a complex network of nodes and arrows representing the derivation of the final result. The nodes are expressions involving  $F$ ,  $\phi$ ,  $\psi$ ,  $\chi$ ,  $\beta$ , and  $\mu$ . The arrows represent the application of various axioms and lemmas. The diagram is structured as follows:

- Top row:  $(F(m_1) \times F(n_1)) \times (F(m_2) \times F(n_2)) \xrightarrow{\phi_{m_1, n_1} \times \phi_{m_2, n_2}} F(m_1+n_1) \times F(m_2+n_2) \xrightarrow{\phi_{m_1+n_1, m_2+n_2}} F((m_1+n_1)+(m_2+n_2)) \xrightarrow{F(\psi)} F((m_1+n_1)+_{b+b'}(m_2+n_2))$
- Left side:  $(x_1 \times y_1) \times (x_2 \times y_2) \xrightarrow{1 \times \phi \times 1} F(n_1) \times F(m_1+n_2) \times F(m_2) \xrightarrow{1 \times \phi} F(n_1) \times F((m_1+n_2)+m_2) \xrightarrow{\phi_{\beta, 1}} F(n_1) \times F(m_2+n_1) \times F(m_2) \xrightarrow{1 \times \phi} F(n_1) \times F(m_2+n_1+m_2) \xrightarrow{\phi_{n_1+n_2, m_1+m_2}} F(n_1+b', n_2) \times F(m_1+b, m_2) \xrightarrow{\phi_{n_1+b', n_2, m_1+b, m_2}} F((n_1+b', n_2)+(m_1+b, m_2))$
- Right side:  $F((m_1+n_1)+(m_2+n_2)) \xrightarrow{F(1+\beta+1)} F(m_1+(m_2+n_1)+m_2) \xrightarrow{\phi_{\psi, \psi}} F(n_1+b', n_2) \times F(m_1+b, m_2) \xrightarrow{\phi_{n_1+b', n_2, m_1+b, m_2}} F((n_1+b', n_2)+(m_1+b, m_2))$
- Bottom row:  $F(n_1) \times F(n_2) \times F(m_1) \times F(m_2) \xrightarrow{\phi_{n_1, n_2} \times \phi_{m_1, m_2}} F(n_1+n_2) \times F(m_1+m_2) \xrightarrow{F(\psi) \times F(\psi)} F(n_1+b', n_2) \times F(m_1+b, m_2) \xrightarrow{\phi_{n_1+b', n_2, m_1+b, m_2}} F((n_1+b', n_2)+(m_1+b, m_2))$

**Diagram 11 of 11. Set up but not solved.**

$$a, b \in (A, +, 0)$$

$$\begin{array}{ccc} U_{a+b} & \xrightarrow{\mu_{a,b}} & U_a \otimes U_b \\ U_\beta \downarrow & & \downarrow \beta' \\ U_{b+a} & \xrightarrow{\mu_{b,a}} & U_b \otimes U_a \end{array}$$

The top two underlying cospans are

$$a + b \rightarrow a + b \leftarrow a + b$$

and the bottom two underlying cospans are

$$b + a \rightarrow b + a \leftarrow b + a$$

Right and then down:

$$\begin{array}{ccccc} & & F(!_a + !_b) & & \\ & \nearrow \phi & & \nearrow \phi_{a,b} & \\ 1 & \xrightarrow{\phi \times \phi} & F(0) \times F(0) & \xrightarrow{F(!_a) \times F(!_b)} & F(a) \times F(b) & \xrightarrow{\phi_{a,b}} & F(a+b) & \xrightarrow{F(\text{id})} & F(a+b) \\ & \searrow \phi \times \phi & \downarrow \beta & \downarrow \beta & \downarrow \mu_{a,b} & \downarrow \phi_{b,a} & \downarrow F(\beta) & & \\ & & F(0) \times F(0) & \xrightarrow{F(!_b) \times F(!_a)} & F(b) \times F(a) & \xrightarrow{\phi_{b,a}} & F(b+a) & & \end{array}$$

Down and then right:

$$\begin{array}{ccccc} & & F(!_a + !_b) & & \\ & \nearrow \phi & & \nearrow \phi_{b,a} & \\ 1 & \xrightarrow{\phi} & F(0+0) & \xrightarrow{F(!_b + !_a)} & F(b+a) & \xrightarrow{F(\text{id})} & F(b+a) \\ & \searrow \phi \times \phi & \downarrow F(\beta) & \downarrow \phi_{b,a} & \downarrow F(\beta) & & \\ & & F(0) \times F(0) & \xrightarrow{F(!_b) \times F(!_a)} & F(b) \times F(a) & \xrightarrow{\phi_{b,a}} & F(b+a) \end{array}$$

First we remove the lower left commuting quarter circle in the ‘right and then down’ diagram as well as the upper left commuting quarter circle in the ‘down and then right’ diagram:

$$\begin{array}{ccccc} & & F(!_a + !_b) & & \\ & \nearrow \phi & & \nearrow \phi_{a,b} & \\ 1 & \xrightarrow{\phi \times \phi} & F(0) \times F(0) & \xrightarrow{F(!_a) \times F(!_b)} & F(a) \times F(b) & \xrightarrow{\phi_{a,b}} & F(a+b) & \xrightarrow{F(\text{id})} & F(a+b) \\ & \searrow \phi \times \phi & \downarrow \beta & \downarrow \beta & \downarrow \mu_{a,b} & \downarrow \phi_{b,a} & \downarrow F(\beta) & & \\ & & F(0) \times F(0) & \xrightarrow{F(!_b) \times F(!_a)} & F(b) \times F(a) & \xrightarrow{\phi_{b,a}} & F(b+a) & & \end{array}$$

$$\begin{array}{ccccc}
& F(0+0) & \xrightarrow{F(!_a + !_b)} & F(a+b) & \\
& \downarrow F(\beta) & & \downarrow F(\beta) & \\
1 & \xrightarrow{\phi} F(0+0) & \xrightarrow{F(!_b + !_a)} & F(b+a) & \\
& \uparrow \phi_{0,0} & \uparrow \phi_{!_b, !_a} & \uparrow \phi_{b,a} & \\
& F(0) \times F(0) & \xrightarrow{F(!_b) \times F(!_a)} & F(b) \times F(a) & \xrightarrow{\phi_{b,a}} F(b+a) \\
& \uparrow \phi_{0,0} & & & \\
& 1 & \xrightarrow{\phi \times \phi} & F(0) \times F(0) &
\end{array}$$

$\cong$  (between  $F(0+0) \rightarrow F(a+b)$  and  $F(0+0) \rightarrow F(b+a)$ )  
 $\cong$  (between  $F(!_b) \times F(!_a) \rightarrow F(b) \times F(a)$  and  $F(!_b) \times F(!_a) \rightarrow F(b) \times F(a)$ )  
 $\cong$  (between  $\phi_{b,a}$  and  $\phi_{b,a}$ )

Next we remove the lower left commuting square with sides  $\beta$  in the first diagram:

$$\begin{array}{ccccc}
& F(0+0) & \xrightarrow{F(!_a + !_b)} & F(a+b) & \\
& \uparrow \phi_{0,0} & & & \\
1 & \xrightarrow{\phi \times \phi} F(0) \times F(0) & \xrightarrow{F(!_a) \times F(!_b)} & F(a) \times F(b) & \xrightarrow{\phi_{a,b}} F(a+b) \\
& & \downarrow \beta & \downarrow \mu_{a,b} & \downarrow F(\beta) \\
& & F(b) \times F(a) & \xrightarrow{\phi_{b,a}} & F(b+a)
\end{array}$$

$\cong$  (between  $\phi_{a,b}$  and  $\phi_{a,b}$ )  
 $\cong$  (between  $\mu_{a,b}$  and  $\mu_{a,b}$ )

$$\begin{array}{ccccc}
& F(0+0) & \xrightarrow{F(!_a + !_b)} & F(a+b) & \\
& \downarrow F(\beta) & & \downarrow F(\beta) & \\
1 & \xrightarrow{\phi} F(0+0) & \xrightarrow{F(!_b + !_a)} & F(b+a) & \\
& \uparrow \phi_{0,0} & \uparrow \phi_{!_b, !_a} & \uparrow \phi_{b,a} & \\
& F(0) \times F(0) & \xrightarrow{F(!_b) \times F(!_a)} & F(b) \times F(a) & \xrightarrow{\phi_{b,a}} F(b+a) \\
& \uparrow \phi_{0,0} & & & \\
& 1 & \xrightarrow{\phi \times \phi} & F(0) \times F(0) &
\end{array}$$

$\cong$  (between  $F(0+0) \rightarrow F(a+b)$  and  $F(0+0) \rightarrow F(b+a)$ )  
 $\cong$  (between  $F(!_b) \times F(!_a) \rightarrow F(b) \times F(a)$  and  $F(!_b) \times F(!_a) \rightarrow F(b) \times F(a)$ )  
 $\cong$  (between  $\phi_{b,a}$  and  $\phi_{b,a}$ )

Next we assume that our pseudofunctor  $F$  is normalized, meaning that  $F(\text{id})$  is an identity, and remove the commuting triangle with sides  $\phi_{a,b}$  in the first diagram and the commuting triangle with sides  $\phi_{b,a}$  in the second diagram:

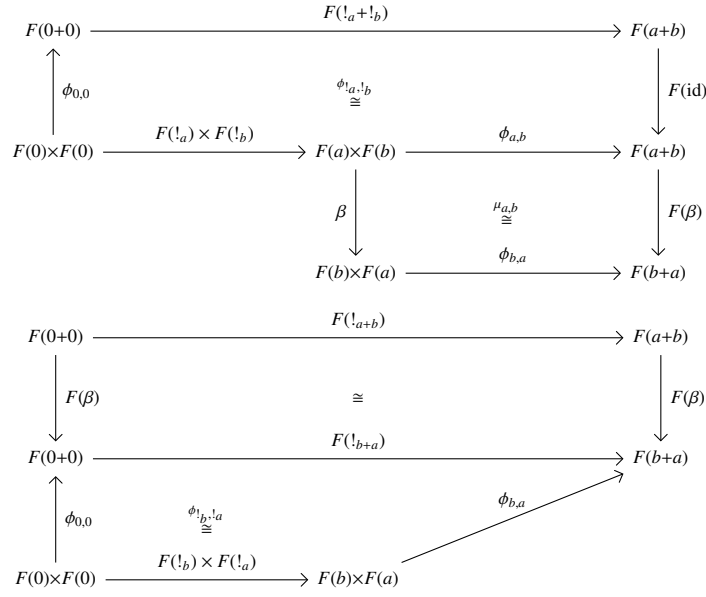
$$\begin{array}{ccccc}
& F(0+0) & \xrightarrow{F(!_a + !_b)} & F(a+b) & \\
& \uparrow \phi_{0,0} & & & \\
1 & \xrightarrow{\phi \times \phi} F(0) \times F(0) & \xrightarrow{F(!_a) \times F(!_b)} & F(a) \times F(b) & \xrightarrow{\phi_{a,b}} F(a+b) \\
& & \downarrow \beta & \downarrow \mu_{a,b} & \downarrow F(\beta) \\
& & F(b) \times F(a) & \xrightarrow{\phi_{b,a}} & F(b+a)
\end{array}$$

$\cong$  (between  $\phi_{a,b}$  and  $\phi_{a,b}$ )  
 $\cong$  (between  $\mu_{a,b}$  and  $\mu_{a,b}$ )

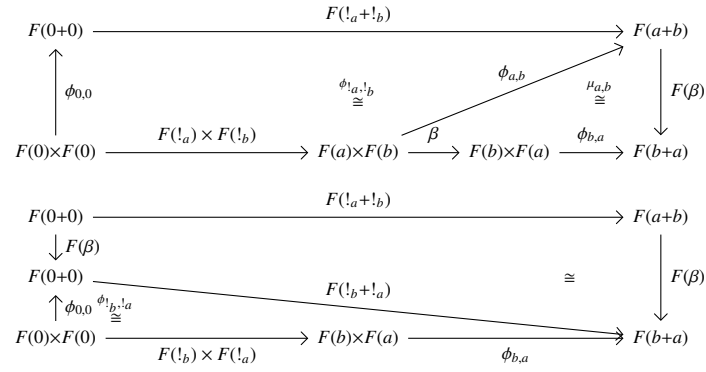
$$\begin{array}{ccccc}
& F(0+0) & \xrightarrow{F(!_a + !_b)} & F(a+b) & \\
& \downarrow F(\beta) & & \downarrow F(\beta) & \\
1 & \xrightarrow{\phi} F(0+0) & \xrightarrow{F(!_b + !_a)} & F(b+a) & \\
& \uparrow \phi_{0,0} & \uparrow \phi_{!_b, !_a} & \uparrow \phi_{b,a} & \\
& F(0) \times F(0) & \xrightarrow{F(!_b) \times F(!_a)} & F(b) \times F(a) & \xrightarrow{\phi_{b,a}} F(b+a) \\
& \uparrow \phi_{0,0} & & & \\
& 1 & \xrightarrow{\phi \times \phi} & F(0) \times F(0) &
\end{array}$$

$\cong$  (between  $F(0+0) \rightarrow F(a+b)$  and  $F(0+0) \rightarrow F(b+a)$ )  
 $\cong$  (between  $F(!_b) \times F(!_a) \rightarrow F(b) \times F(a)$  and  $F(!_b) \times F(!_a) \rightarrow F(b) \times F(a)$ )  
 $\cong$  (between  $\phi_{b,a}$  and  $\phi_{b,a}$ )

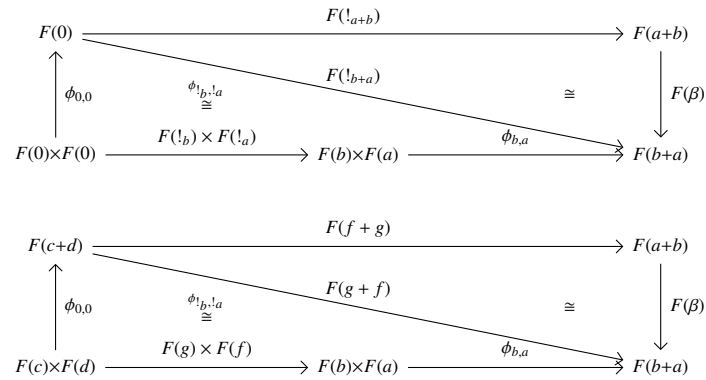
Next we remove the left quarter circle containing  $\cong$  from each diagram:



Making each diagram into a rectangle and removing the  $F(\text{id})$  from the first diagram:



Are these two diagrams the same?





$$\begin{array}{ccccc}
F(0+0) & \xrightarrow{F(!_a + !_b)} & F(a+b) & \xrightarrow{F(\beta)} & F(b+a) \\
\uparrow \phi_{0,0} & \phi_{!_a, !_b} \cong & \uparrow \phi_{a,b} & \mu_{a,b} \cong & \uparrow \phi_{b,a} \\
F(0) \times F(0) & \xrightarrow{F(!_a) \times F(!_b)} & F(a) \times F(b) & \xrightarrow{\beta} & F(b) \times F(a)
\end{array}$$
  

$$\begin{array}{ccccc}
F(0+0) & \xrightarrow{F(\beta)} & F(0+0) & \xrightarrow{F(!_b + !_a)} & F(b+a) \\
\uparrow \phi_{0,0} & \mu_{0,0} \cong & \uparrow \phi_{0,0} & \phi_{!_b, !_a} \cong & \uparrow \phi_{b,a} \\
F(0) \times F(0) & \xrightarrow{\beta} & F(0) \times F(0) & \xrightarrow{F(!_b) \times F(!_a)} & F(b) \times F(a)
\end{array}$$
  

$$\begin{array}{ccccc}
F(0+0) & \xrightarrow{F(!_a + !_b)} & F(a+b) & \xrightarrow{F(\beta)} & F(b+a) \\
\uparrow \phi_{0,0} & \phi_{!_a, !_b} \cong & \uparrow \phi_{a,b} & \mu_{a,b} \cong & \uparrow \phi_{b,a} \\
F(0) \times F(0) & \xrightarrow{F(!_a) \times F(!_b)} & F(a) \times F(b) & \xrightarrow{\beta} & F(b) \times F(a)
\end{array}$$
  

$$\begin{array}{ccccc}
F(0+0) & \xrightarrow{F(\beta)} & F(0+0) & \xrightarrow{F(!_b + !_a)} & F(b+a) \\
\uparrow \phi_{0,0} & \mu_{0,0} \cong & \uparrow \phi_{0,0} & \phi_{!_b, !_a} \cong & \uparrow \phi_{b,a} \\
F(0) \times F(0) & \xrightarrow{\beta} & F(0) \times F(0) & \xrightarrow{F(!_b) \times F(!_a)} & F(b) \times F(a)
\end{array}$$
  

$$\begin{array}{ccc}
& & \searrow \\
& & F(!_b) \times F(!_a) \\
& & \nearrow
\end{array}$$

First diagram:

$$\begin{array}{ccccc}
F(0+0) & \xrightarrow{F(!_a + !_b)} & F(a+b) & \xrightarrow{F(\beta)} & F(b+a) \\
\uparrow \phi_{0,0} & \phi_{!_a, !_b} \cong & \uparrow \phi_{a,b} & \mu_{a,b} \cong & \uparrow \phi_{b,a} \\
F(0) \times F(0) & \xrightarrow{F(!_a) \times F(!_b)} & F(a) \times F(b) & \xrightarrow{\beta} & F(b) \times F(a)
\end{array}$$
  

$$\begin{array}{ccc}
& & \searrow \\
& & F(!_b) \times F(!_a) \\
& & \nearrow
\end{array}$$

Second diagram:

$$\begin{array}{ccccc}
F(0+0) & \xrightarrow{F(\beta)} & F(0+0) & \xrightarrow{F(!_a + !_b)} & F(a+b) \\
\uparrow \phi_{0,0} & \phi_{!_b, !_a} \cong & & & \downarrow F(\beta) \\
F(0) \times F(0) & \xrightarrow{F(!_b) \times F(!_a)} & F(b) \times F(a) & \xrightarrow{\phi_{b,a}} & F(b+a)
\end{array}$$
  

$$\begin{array}{ccc}
& & \searrow \\
& & F(!_b + !_a) \\
& & \nearrow
\end{array}$$

**Diagram 11 of 11. Set up but not solved.**

$$\begin{array}{ccc}
 & a, b \in (A, +, 0) & \\
 U_a \otimes U_b & \xrightarrow{\mu_{a,b}} & U_{a+b} \\
 \beta' \downarrow & & \downarrow U_\beta \\
 U_b \otimes U_a & \xrightarrow{\mu_{b,a}} & U_{b+a}
 \end{array}$$

Right and then down:

$$\begin{array}{ccccccc}
 & & \phi \times \phi & & & & \\
 & \swarrow & \downarrow & \searrow & & & \\
 1 \times 1 & \xrightarrow{1 \times \phi} & 1 \times F(0) & \xrightarrow{\phi \times 1} & F(0) \times F(0) & \xrightarrow{F(!_a) \times F(!_b)} & F(a) \times F(b) & \xrightarrow{\phi_{a,b}} & F(a+b) \\
 \lambda \downarrow & & \lambda \downarrow & & \downarrow \phi_{0,0} & & \downarrow \phi_{a,b} & & \downarrow 1 \\
 1 & \xrightarrow{\phi} & F(0) & \xrightarrow{F(\lambda^{-1})} & F(0+0) & \xrightarrow{F(!_a + !_b)} & F(a+b) & \xrightarrow{F_{1,\beta}} & F(\beta) \\
 & & \downarrow \phi_{\lambda^{-1}, \beta} & & \downarrow F(\beta) & & \downarrow F(!_a + !_b, \beta) & & \downarrow F(\beta) \\
 1 & \xrightarrow{\phi} & F(0) & \xrightarrow{F(\lambda^{-1})} & F(0+0) & \xrightarrow{F(!_a + !_b)} & F(a+b) & \xrightarrow{F_{1,\beta}} & F(\beta) \\
 & & \downarrow \phi_{\lambda^{-1}, \beta} & & \downarrow F(\beta) & & \downarrow F(!_a + !_b, \beta) & & \downarrow F(\beta) \\
 1 & \xrightarrow{\phi} & F(0) & \xrightarrow{F(\lambda^{-1})} & F(0+0) & \xrightarrow{F(!_a + !_b)} & F(a+b) & \xrightarrow{F_{1,\beta}} & F(\beta) \\
 & & \downarrow \phi_{\lambda^{-1}, \beta} & & \downarrow F(\beta) & & \downarrow F(!_a + !_b, \beta) & & \downarrow F(\beta) \\
 1 & \xrightarrow{\phi} & F(0) & \xrightarrow{F(\lambda^{-1})} & F(0+0) & \xrightarrow{F(!_a + !_b)} & F(a+b) & \xrightarrow{F_{1,\beta}} & F(\beta)
 \end{array}$$

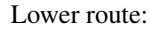
Down and then right:

$$\begin{array}{ccccccc}
 & & \phi \times \phi & & & & \\
 & \swarrow & \downarrow & \searrow & & & \\
 1 \times 1 & \xrightarrow{1 \times \phi} & 1 \times F(0) & \xrightarrow{\phi \times 1} & F(0) \times F(0) & \xrightarrow{F(!_a) \times F(!_b)} & F(a) \times F(b) & \xrightarrow{\phi_{a,b}} & F(a+b) \\
 \beta \downarrow & & \beta \downarrow & & \downarrow \phi_{0,0} & & \downarrow \phi_{a,b} & & \downarrow 1 \\
 1 & \xrightarrow{\phi} & F(0) & \xrightarrow{F(\lambda^{-1})} & F(0+0) & \xrightarrow{F(!_a + !_b)} & F(a+b) & \xrightarrow{F_{1,\beta}} & F(\beta) \\
 & & \downarrow \phi_{\lambda^{-1}, \beta} & & \downarrow F(\beta) & & \downarrow F(!_a + !_b, \beta) & & \downarrow F(\beta) \\
 1 & \xrightarrow{\phi} & F(0) & \xrightarrow{F(\lambda^{-1})} & F(0+0) & \xrightarrow{F(!_a + !_b)} & F(a+b) & \xrightarrow{F_{1,\beta}} & F(\beta) \\
 & & \downarrow \phi_{\lambda^{-1}, \beta} & & \downarrow F(\beta) & & \downarrow F(!_a + !_b, \beta) & & \downarrow F(\beta) \\
 1 & \xrightarrow{\phi} & F(0) & \xrightarrow{F(\lambda^{-1})} & F(0+0) & \xrightarrow{F(!_a + !_b)} & F(a+b) & \xrightarrow{F_{1,\beta}} & F(\beta) \\
 & & \downarrow \phi_{\lambda^{-1}, \beta} & & \downarrow F(\beta) & & \downarrow F(!_a + !_b, \beta) & & \downarrow F(\beta) \\
 1 & \xrightarrow{\phi} & F(0) & \xrightarrow{F(\lambda^{-1})} & F(0+0) & \xrightarrow{F(!_a + !_b)} & F(a+b) & \xrightarrow{F_{1,\beta}} & F(\beta)
 \end{array}$$

$M_4$  is given by  $d \rightarrow m_4 \leftarrow e$  with  $x_4 \in F(m_4)$ .

Upper route:

[illegible]



**Triangle identity. Set up but not solved.**

$M_1$  is given by  $a \rightarrow m_1 \leftarrow b$  with  $x_1 \in F(m_1)$ .

$U_b$  is given by  $b \rightarrow b \leftarrow b$  with  $\perp_b \in F(b)$ .

$M_2$  is given by  $b \rightarrow m_2 \leftarrow c$  with  $x_2 \in F(m_2)$ .

$$\begin{array}{ccc}
 & M_1 \odot M_2 & \\
 \rho \odot 1_{M_2} \nearrow & & \nwarrow 1_{M_1} \odot \lambda \\
 (M_1 \odot U_b) \odot M_2 & \xrightarrow{\alpha} & M_1 \odot (U_b \odot M_2)
 \end{array}$$

Up:

$$\begin{array}{ccccccccccccccc}
 1 & \xrightarrow{(x_1 \times \phi_0) \times x_2} & Fm_1 F0 Fm_2 & \xrightarrow{(1 \times F(\perp_b)) \times 1} & Fm_1 Fb Fm_2 & \xrightarrow{\phi_{m_1, b} \times 1} & Fm_1 + b Fm_2 & \xrightarrow{F(\psi) \times 1} & Fm_1 + b Fm_2 & \xrightarrow{\phi_{m_1 + b, b} m_2} & F(m_1 + b) + m_2 & \xrightarrow{F(\psi)} & F(m_1 + b) + b m_2 \\
 & \searrow x_1 \times x_2 & \searrow \rho \times 1 & \searrow \phi \times 1 & \searrow \phi_{1, \perp_b} \times 1 & \searrow F(1 + \perp_b) \times 1 & \searrow \cong & \searrow \phi_{\kappa, 1} & \searrow F(\kappa + 1) & \searrow \cong & \searrow F(\kappa + 1) & \searrow F(\kappa + 1) & \searrow F(\kappa + 1) \\
 & & Fm_1 Fm_2 & & Fm_1 + 0 Fm_2 & & F(\kappa) \times 1 & & Fm_1 + b Fm_2 & & F(m_1 + b) + m_2 & & F(m_1 + b) + b m_2 \\
 & & \searrow F(r) \times 1 & & \searrow \phi & & \searrow F(\psi) & & \searrow F(\psi) & & \searrow F(\psi) & & \searrow F(\psi) \\
 & & Fm_1 Fm_2 & & Fm_1 + m_2 & & Fm_1 + m_2 & & Fm_1 + b m_2 & & Fm_1 + b m_2 & & Fm_1 + b m_2
 \end{array}$$

Across and then up:

$$\begin{array}{ccccccccccccccccccc}
 & & & & Fm_1 F0 Fm_2 & \xrightarrow{(1 \times F(\perp_b)) \times 1} & Fm_1 Fb Fm_2 & \xrightarrow{\phi_{m_1, b} \times 1} & Fm_1 + b Fm_2 & \xrightarrow{F(\psi) \times 1} & Fm_1 + b Fm_2 & \xrightarrow{\phi_{m_1 + b, b} m_2} & F(m_1 + b) + m_2 & \xrightarrow{F(\psi)} & F(m_1 + b) + b m_2 \\
 & & & & \searrow \alpha & & \searrow \alpha & & \searrow \alpha & & \searrow \alpha & & \searrow \alpha & & \searrow \alpha \\
 & & & & Fm_1 F0 Fm_2 & \xrightarrow{1 \times F(\perp_b) \times 1} & Fm_1 Fb Fm_2 & \xrightarrow{1 \times \phi_{b, m_2}} & Fm_1 Fb + m_2 & \xrightarrow{1 \times F(\psi)} & Fm_1 Fb + b m_2 & \xrightarrow{\phi_{m_1, b + b} m_2} & Fm_1 + (b + b) m_2 & \xrightarrow{F(\psi)} & Fm_1 + b(b + b) m_2 \\
 & & & & \searrow 1 \times \phi & & \searrow 1 \times \phi_{\perp_b, 1} & & \searrow 1 \times F(\perp_b + 1) & & \searrow 1 \times F(\psi) & & \searrow 1 \times F(\psi) & & \searrow 1 \times F(\psi) \\
 & & & & Fm_1 Fm_2 & & Fm_1 \times F0 + m_2 & & Fm_1 \times Fb + m_2 & & Fm_1 \times Fb + b m_2 & & Fm_1 + (b + b) m_2 & & Fm_1 + b(b + b) m_2 \\
 & & & & \searrow 1 \times F(\ell) & & \searrow 1 \times F(\kappa) & & \searrow \phi_{1, \kappa} & & \searrow F(1 + \kappa) & & \searrow F(1 + \kappa) & & \searrow F(1 + \kappa) \\
 & & & & Fm_1 Fm_2 & & Fm_1 + m_2 & & Fm_1 + m_2 & & Fm_1 + b m_2 & & Fm_1 + b m_2 & & Fm_1 + b m_2
 \end{array}$$

## 3. SOME USEFUL MAPS

Given  $a \in (\mathbf{A}, +, 0)$ , the map  $U_{\lambda_a}: U_{0+a} \rightarrow U_a$  is given by:

$$\begin{array}{ccccc}
 & \phi & \rightarrow & F(0) & \xrightarrow{F(!_{0+a})} & F(0+a) \\
 & \downarrow & & \downarrow F(\text{id}) & \nearrow F(!_{0+a}) & \downarrow F(\lambda_a) \\
 1 & \downarrow & & F(0) & \xrightarrow{F(!_a)} & F(a) \\
 & \phi & \rightarrow & & & 
 \end{array}$$

$\cong$

where the  $\cong$  is given by pseudonaturality of  $F$ : we have a unique map in  $!_a: 0 \rightarrow a$  in  $\mathbf{A}$  but also a map  $\lambda_a \circ !_a: 0 \rightarrow a$  where  $\lambda_a$  is the left unitor of  $(\mathbf{A}, +, 0)$ , and so  $F(!_a) = F(\lambda_a \circ !_a) \cong F(\lambda_a)F(!_a)$ .

The left unitor  $\lambda'_{U_a}: U_0 \otimes U_a \rightarrow U_a$  is given by:

$$\begin{array}{ccccc}
 & \phi \times \phi & \rightarrow & F(0) \times F(0) & \xrightarrow{F(!_0) \times F(!_a)} & F(0) \times F(a) & \xrightarrow{\phi_{0,a}} & F(0+a) \\
 & \downarrow & & \downarrow \phi_{0,0} & \nearrow \phi_{!_0, !_a} & \nearrow F(!_a) & \nearrow \cong & \downarrow F(\lambda_a) \\
 1 & \downarrow & & F(0) & \xrightarrow{F(!_a)} & F(a) & & \\
 & \phi & \rightarrow & & & & & 
 \end{array}$$

$\cong$

where the  $\cong$  in the lower right is the same as the one in the first diagram.

For an arbitrary  $M$ , the left unitor  $\lambda'_M: U_0 \otimes M \rightarrow M$  is given by:

$$\begin{array}{ccccc}
 & \phi_0 \times x & \rightarrow & F(0) \times F(m) & \xrightarrow{F(!_0) \times 1} & F(0) \times F(m) & \xrightarrow{\phi_{0,m}} & F(0+m) \\
 & \downarrow & & \downarrow \phi_{0,m} & \nearrow \phi_{!_0, !_1} & \nearrow F(!_0 + 1) & \nearrow \cong & \downarrow F(\lambda_m) \\
 1 & \downarrow & & F(m) & \xrightarrow{\text{id}} & F(m) & & \\
 & x & \rightarrow & & & & & 
 \end{array}$$

$\cong$

For an arbitrary  $M$  given by  $a \rightarrow (m, x) \leftarrow b$ , the map  $\lambda_M: U_b \odot M \rightarrow M$  is given by:

$$\begin{array}{ccccccc}
 & x \times \phi_0 & \rightarrow & F(m) \times F(0) & \xrightarrow{1 \times F(!_b)} & F(m) \times F(b) & \xrightarrow{\phi_{m,b}} & F(m+b) & \xrightarrow{F(\psi)} & F(m+b+b) \\
 & \downarrow & & \downarrow \phi_{m,0} & \nearrow \phi_{!_1, !_b} & \nearrow F(1 + !_b) & \nearrow \cong & \downarrow F(\kappa) \\
 1 & \downarrow & & F(m) & \xrightarrow{\text{id}} & F(m) & & \\
 & x & \rightarrow & & & & & 
 \end{array}$$

$\cong$

In particular, if  $M = U_0$  above, then the map  $\lambda_{U_0}: U_0 \odot U_0 \rightarrow U_0$  is given by:

$$\begin{array}{ccccccc}
 & \phi_0 \times \phi_0 & \rightarrow & F(0) \times F(0) & \xrightarrow{F(!_0) \times F(!_0)} & F(0) \times F(0) & \xrightarrow{\phi_{0,0}} & F(0+0) & \xrightarrow{F(\psi)} & F(0+0+0) \\
 & \downarrow & & \downarrow \phi_{0,0} & \nearrow \phi_{!_0, !_0} & \nearrow F(!_0 + !_0) & \nearrow \cong & \downarrow F(\kappa) \\
 1 & \downarrow & & F(0) & \xrightarrow{\text{id}} & F(0) & & \\
 & \phi_0 & \rightarrow & & & & & 
 \end{array}$$

$\cong$