AN EQUIVALENCE OF COMPOSITIONAL FRAMEWORKS

JOHN C. BAEZ[*,•], KENNY COURSER[*], AND CHRISTINA VASILAKOPOULOU[*]

Abstract. The first two authors have developed a compositional framework well-suited for studying networks that are built out of finite sets equipped with extra stuff. This framework, which goes by the name of 'structured cospans', utilizes double categories where the objects are seen as inputs and outputs, horizontal 1-cells are 'open networks', and 2-morphisms are maps between open networks. In this setup a functor $L\colon \mathsf{A} \to \mathsf{X}$, which is typically a left adjoint, is used to replace the objects and vertical 1-morphisms of a given double category $\mathbb{X}$ with the objects and morphisms, respectively, of the category A. Horizontal 1-cells are then cospans in X of a particular form with 2-morphisms given by maps of these cospans. Fong has also developed a similar framework utilizing cospans to study open networks which goes by the name of 'decorated cospans'. In this setup, a lax monoidal functor $F\colon \mathsf{A} \to \mathsf{Set}$ is used to 'decorate' the apices of cospans in A with elements of Set giving the cospans extra structure. Using a slight variation of Fong's framework, we prove that these two frameworks are equivalent in the situation where a left adjoint can be obtained from a lax monoidal pseudofunctor using a well known construction of Grothendieck.

Contents

Various issues-in-progress

- Would a title like 'An equivalence of compositional frameworks for networks' or 'An equivalence of compositional network frameworks' be more informative? Otherwise, compositional framework on its own doesn't say much!
- Discuss notation for Decorated and Structured cospan double categories one last time? :)
- Notation between Theorem 3.2 and Theorem 2.1 right now does not match! Need to fix. Slightly prefer a,b,c for objects of A, then a' and b' for other objects, and x an object of the category F(c) namely the decoration. The previously used 'd' is slightly confusing around all these a's, b's and c's that are in a different category. Also I prefer a and b rather than a_1 and a_2.

[*]Department of Mathematics, University of California, Riverside CA, USA 92521

[•]Centre for Quantum Technologies, National University of Singapore, Singapore 117543

E-mail address: `baez@ucr.edu, kcour001@ucr.edu, vasilak@ucr.edu`.

1. Introduction

2. A symmetric monoidal double category of decorated cospans

In this section we build the symmetric monoidal double category $F\mathbb{C}\mathrm{sp}$ mentioned in the introduction. Then we obtain an underlying symmetric monoidal bicategory $F\mathbf{Csp}$ using a result of Shulman [24] and then finally decategorify the obtained symmetric monoidal bicategory to obtain a symmetric monoidal category $F\mathrm{Csp}$ which is a generalization of Fong's original decorated cospan category [17]. The definition of a lax monoidal pseudofunctor is recalled in Section 5.1, and the basics of double categories are recalled in Section 5.3.

Theorem 2.1. *Let A be a category with finite colimits and $F\colon \mathsf{A} \to \mathbf{Cat}$ a lax monoidal pseudofunctor. Then there exists a (pseudo) double category $F\mathbb{C}sp$ which has:*

(1) objects as those of A,
(2) vertical 1-morphisms as morphisms of A,
(3) horizontal 1-cells as F-decorated cospans in A which are pairs:

$$a_1 \xrightarrow{i} b \xleftarrow{o} a_2 \qquad d \in F(b)$$

and
(4) 2-morphisms as maps of F-decorated cospans in A

$$\begin{tikzcd} a_1 \arrow[r, "i"] \arrow[d, "f"'] & b \arrow[d, "h"'] & a_2 \arrow[l, "o"'] \arrow[d, "g"'] & d \in F(b) \\ a_1' \arrow[r, "i'"] & b' & a_2' \arrow[l, "o'"'] & d' \in F(b') \end{tikzcd} \tag{1}$$

together with a morphism $\iota\colon F(h)(d) \to d'$ in $F(b')$.

Kenny: We should try to make this theorem shorter. I agree!

Proof. The unit structure functor $U\colon F\mathbb{C}\mathrm{sp}_0 \to F\mathbb{C}\mathrm{sp}_1$ is defined on objects as:

$$a \quad \mapsto \quad a \xrightarrow{1} a \xleftarrow{1} a \qquad !_a \in F(a)$$

where $!_a \in F(a)$ is the trivial decoration on a given by the composition of the unique map $F(!)\colon F(0) \to F(a)$ and the morphism $\phi\colon 1 \to F(0)$ which comes from the structure of the lax monoidal pseudofunctor $F\colon \mathsf{A} \to \mathbf{Cat}$. For morphisms, the structure functor U is defined as:

$$\begin{tikzcd} a \arrow[d, "f"'] & & a \arrow[r, "1"] \arrow[d, "f"'] & a \arrow[d, "f"'] & a \arrow[l, "1"'] \arrow[d, "f"'] & !_a \in F(a) \\ a' & \mapsto & a' \arrow[r, "1"] & a' & a' \arrow[l, "1"'] & !_{a'} \in F(a') \end{tikzcd}$$

together with the morphism $\iota_{!_f} = F(f)F(!)\phi\colon 1 \to F(a')$. We also have source and target structure functors $S, T\colon F\mathbb{C}\mathrm{sp}_1 \to F\mathbb{C}\mathrm{sp}_0$ where the source of the horizontal 1-cell

$$a_1 \xrightarrow{i} b \xleftarrow{o} a_2 \qquad d \in F(b)$$

is the object a_1 in A and the source of the 2-morphism

$$\begin{tikzcd}
a \arrow[r, "i"] \arrow[d, "f"'] & b \arrow[d, "h"'] & a_2 \arrow[l, "o"'] \arrow[d, "g"'] & d \in F(b) \\
a' \arrow[r, "i'"] & b' & a_2' \arrow[l, "o'"'] & d' \in F(b')
\end{tikzcd}$$

$$\iota\colon F(h)(d) \to d'$$

is the source of the underlying map of cospans in A, namely the morphism f in A; the target structure functor is defined similarly. These structure functors satisfy the equations

$$SU(a) = 1(a) = TU(a)$$

for all objects a of A.

Given two composable horizontal 1-cells M and N:

$$a_1 \xrightarrow{i} b \xleftarrow{o} a_2 \qquad\qquad a_2 \xrightarrow{i'} b' \xleftarrow{o'} a_3$$

$$d \in F(b) \qquad\qquad d' \in F(b')$$

the composite $N \odot M$ is given by:

$$\begin{tikzcd}
 & & b +_{a_2} b' & & \\
 & & b + b' \arrow[u, "\psi"] & & \\
 & b \arrow[ur, "j"] & & b' \arrow[ul, "j'"'] & \\
a_1 \arrow[ur, "i"] \arrow[uuurr, bend left, "\psi j i"] & & a_2 \arrow[ul, "o"'] \arrow[ur, "i'"] & & a_3 \arrow[ul, "o'"'] \arrow[uuull, bend right, "\psi j' o'"']
\end{tikzcd}$$

with the corresponding decoration of the apex $\hat{d} \in F(b +_{a_2} b')$ given by:

$$1 \xrightarrow{\lambda^{-1}} 1 \times 1 \xrightarrow{d \times d'} F(b) \times F(b') \xrightarrow{\phi_{b,b'}} F(b + b') \xrightarrow{F(\psi)} F(b +_{a_2} b')$$

where $\psi\colon b + b' \to b +_{a_2} b'$ is the natural map from the coproduct to the pushout and $\phi_{b,b'}\colon F(b) \times F(b') \to F(b + b')$ is the natural transformation coming from the structure of the lax monoidal pseudofunctor $F\colon \mathsf{A} \to \mathbf{Cat}$. Denoting the first and second of these horizontal 1-cells as M and N, respectively, the source and target structure functors satisfy the equations $S(N \odot M) = S(M)$ and $T(N \odot M) = T(N)$.

Given three composable horizontal 1-cells M_1, M_2 and M_3:

$$a_1 \xrightarrow{i} b \xleftarrow{o} a_2 \qquad a_2 \xrightarrow{i'} b' \xleftarrow{o'} a_3 \qquad a_3 \xrightarrow{i''} b'' \xleftarrow{o''} a_4$$

$$d_{M_1} \in F(b) \qquad\qquad d_{M_2} \in F(b') \qquad\qquad d_{M_3} \in F(b'')$$

we get a natural isomorphism $a_{M_1,M_2,M_3}\colon (M_1 \odot M_2) \odot M_3 \to M_1 \odot (M_2 \odot M_3)$ which is a globular 2-morphism given by a map of cospans $(\mathrm{id}_{a_1}, \sigma, \mathrm{id}_{a_4})$:

$$\begin{tikzcd}
a_1 \arrow[r] \arrow[d, "1"'] & (b +_{a_2} b') +_{a_3} b'' \arrow[d, "a"] & a_4 \arrow[l] \arrow[d, "1"] & d_{(M_1 \odot M_2) \odot M_3} \in F((b +_{a_2} b') +_{a_3} b'') \\
a_1 \arrow[r] & b +_{a_2} (b' +_{a_3} b'') & a_4 \arrow[l] & d_{M_1 \odot (M_2 \odot M_3)} \in F(b +_{a_2} (b' +_{a_3} b''))
\end{tikzcd}$$

with the decorations on the cospan's apices given by:

$$d_{(M_1\odot M_2)\odot M_3} \coloneqq 1 \xrightarrow{\zeta_1} F(b{+_{a_2}}b') {\times} F(b'') \xrightarrow{\phi_{b+_{a_2}b',b''}} F((b{+_{a_2}}b'){+}b'') \xrightarrow{F(j_{b+_{a_2}b',b''})} F((b{+_{a_2}}b'){+_{a_3}}b'')$$

$$\zeta_1 = d_{M_3}\lambda^{-1}F(j_{b,b'})\phi_{b,b'}(d_{M_1} \times d_{M_2})\lambda^{-1}$$

and

$$d_{M_1\odot(M_2\odot M_3)} \coloneqq 1 \xrightarrow{\zeta_2} F(b) {\times} F(b'{+_{a_3}}b'') \xrightarrow{\phi_{b,b'+_{a_3}b''}} F(b{+}(b'{+_{a_3}}b'')) \xrightarrow{F(j_{b,b'+_{a_3}b''})} F(b{+_{a_2}}(b'{+_{a_3}}b''))$$

$$\zeta_2 = d_{M_1}\lambda^{-1}F(j_{b',b''})\phi_{b',b''}(d_{M_2} \times d_{M_3})\lambda^{-1}$$

together with the isomorphism $\iota_a\colon F(a)(d_{(M_1\odot M_2)\odot M_3}) \to d_{M_1\odot(M_2\odot M_3)}$. Note that the map $a\colon (b +_{a_2} b') +_{a_3} b'' \to b +_{a_2} (b' +_{a_3} b'')$ is the universal map between two colimits of the same diagram. We also have left and right unitors where given a horizontal 1-cell M:

$$a_1 \xrightarrow{i} b \xleftarrow{o} a_2 \qquad d \in F(b)$$

if we, say, compose with the identity horizontal 1-cell of a_2 on the right:

$$\begin{array}{cc}
a_1 \xrightarrow{i} b \xleftarrow{o} a_2 & \qquad a_2 \xrightarrow{1} a_2 \xleftarrow{1} a_2 \\[1em]
d \in F(b) & \qquad !_{a_2} \in F(a_2)
\end{array}$$

where $!_{a_2} = F(!)\phi\colon 1 \to F(a_2)$ is the trivial decoration on a_2, composing these then gives:

$$a_1 \xrightarrow{j\psi_b i} b +_{a_2} a_2 \xleftarrow{j\psi_{a_2}} a_2 \qquad d_{M+!_{a_2}} \in F(b +_{a_2} a_2)$$

where $\psi_b\colon b \to b + a_2$ is the natural map into the coproduct and likewise for ψ_{a_2} and $j\colon b + a_2 \to b +_{a_2} a_2$ is the natural map from the coproduct to the pushout. The decoration $d_{M+!_{a_2}}\colon 1 \to F(b +_{a_2} a_2)$ is given by:

$$1 \xrightarrow{\lambda^{-1}} 1 \times 1 \xrightarrow{d_M \times !_{a_2}} F(b) \times F(a_2) \xrightarrow{\phi_{b,a_2}} F(b + a_2) \xrightarrow{F(j_{b,a_2})} F(b +_{a_2} a_2).$$

We then have that the right unitor $R\colon M \odot 1_{a_2} \xrightarrow{\sim} M$ is given by the globular 2-morphism $(\mathrm{id}_{a_1}, r, \mathrm{id}_{a_2})$ from the above composite to M:

$$\begin{tikzcd}
a_1 \arrow[r, "j\psi_b i"] \arrow[d, "\mathrm{id}_{a_1}"'] & b +_{a_2} a_2 \arrow[d, "r"'] & a_2 \arrow[l, "j\psi_{a_2}"'] \arrow[d, "\mathrm{id}_{a_2}"'] & d_{M+!_{a_2}} \in F(b +_{a_2} a_2) \\
a_1 \arrow[r, "i"] & b & a_2 \arrow[l, "o"'] & d_M \in F(b)
\end{tikzcd}$$

where $r\colon b {+_{a_2}} a_2 \xrightarrow{\sim} b$ is a universal map together with the isomorphism $\iota_r\colon F(r)(d_{M+!_{a_2}}) \to d_M$. The left unitor is similar. The source and target functor applied to the left and right unitors and associators yield identities. The left and right unitors together with the associator satisfy the standard pentagon and triangle identities of a monoidal category or bicategory. Finally, for the interchange law, given four 2-morphisms α, β, α' and β':

$$\begin{tikzcd} a_1 \arrow[r, "i_1"] \arrow[d, "f"'] & b \arrow[d, "h_1"] & a_2 \arrow[l, "o_1"'] \arrow[d, "g"] & d_{M_1} \in F(b) \\ a_1' \arrow[r, "i_1'"] & b' & a_2' \arrow[l, "o_1'"'] & d_{M_2} \in F(b') \end{tikzcd} \qquad \begin{tikzcd} a_2 \arrow[r, "i_2"] \arrow[d, "g"'] & c \arrow[d, "h_2"] & a_3 \arrow[l, "o_2"'] \arrow[d, "k"] & d_{N_1} \in F(c) \\ a_2' \arrow[r, "i_2'"] & c' & a_3' \arrow[l, "o_2'"'] & d_{N_2} \in F(c') \end{tikzcd}$$

$$\iota_\alpha \colon F(h_1)(d_{M_1}) \to d_{M_2} \qquad \iota_\beta \colon F(h_2)(d_{N_1}) \to d_{N_2}$$

$$\begin{tikzcd} a_1' \arrow[r, "i_1'"] \arrow[d, "f'"'] & b' \arrow[d, "h_1'"] & a_2' \arrow[l, "o_1'"'] \arrow[d, "g'"] & d_{M_2} \in F(b') \\ a_1'' \arrow[r, "i_1''"] & b'' & a_2'' \arrow[l, "o_1''"'] & d_{M_3} \in F(b'') \end{tikzcd} \qquad \begin{tikzcd} a_2' \arrow[r, "i_2'"] \arrow[d, "g'"'] & c' \arrow[d, "h_2'"] & a_3' \arrow[l, "o_2'"'] \arrow[d, "k'"] & d_{N_2} \in F(b') \\ a_2'' \arrow[r, "i_2''"] & c'' & a_3'' \arrow[l, "o_2''"'] & d_{N_3} \in F(b'') \end{tikzcd}$$

$$\iota_{\alpha'} \colon F(h_1')(d_{M_2}) \to d_{M_3} \qquad \iota_{\beta'} \colon F(h_2')(d_{N_2}) \to d_{N_3}$$

if we first compose horizontally we obtain:

$$\begin{tikzcd} a_1 \arrow[r, "j\psi_{a_1} i_1"] \arrow[d, "f"'] & b +_{a_2} c \arrow[d, "h_1 +_g h_2"'] & a_3 \arrow[l, "j\psi_{a_3} o_2"'] \arrow[d, "k"'] & d_{M_1 \odot N_1} \in F(b +_{a_2} c) \\ a_1' \arrow[r, "j\psi_{a_1'} i_1'"] & b' +_{a_2'} c' & a_3' \arrow[l, "j\psi_{a_3'} o_2'"'] & d_{M_2 \odot N_2} \in F(b' +_{a_2'} c') \end{tikzcd}$$

$$\iota_{\alpha \odot \beta} \colon F(h_1 +_g h_2)(d_{M_1 \odot N_1}) \to d_{M_1' \odot N_1'}$$

$$\begin{tikzcd} a_1' \arrow[r, "j\psi_{a_1'} i_1'"] \arrow[d, "f'"'] & b' +_{a_2'} c' \arrow[d, "h_1' +_{g'} h_2'"'] & a_3' \arrow[l, "j\psi_{a_3'} o_2'"'] \arrow[d, "k'"'] & d_{M_2 \odot N_2} \in F(b' +_{a_2'} c') \\ a_1'' \arrow[r, "j\psi_{a_1''} i_1''"] & b'' +_{a_2''} c'' & a_3'' \arrow[l, "j\psi_{a_3''} o_2''"'] & d_{M_3 \odot N_3} \in F(b'' +_{a_2''} c'') \end{tikzcd}$$

$$\iota_{\alpha' \odot \beta'} \colon F(h_1' +_{g'} h_2')(d_{M_2 \odot N_2}) \to d_{M_3 \odot N_3}.$$

To obtain the morphism of decorations for a horizontal composite, we have as initial data:

$$\begin{tikzcd} & F(b) \arrow[dd, "F(h_1)"] \\ 1 \arrow[ur, "d_{M_1}"] \arrow[dr, "d_{M_2}"'] \arrow[r, phantom, "\iota_\alpha \Swarrow"] & {} \\ & F(b') \end{tikzcd} \qquad \begin{tikzcd} & F(c) \arrow[dd, "F(h_2)"] \\ 1 \arrow[ur, "d_{N_1}"] \arrow[dr, "d_{N_2}"'] \arrow[r, phantom, "\iota_\beta \Swarrow"] & {} \\ & F(c') \end{tikzcd}$$

These two 2-morphisms ι_α and ι_β are two 2-morphisms in the monoidal 2-category $(\mathbf{Cat}, \times, 1)$ and so we can tensor them which results in:

$$\begin{tikzcd} & & F(b) \times F(c) \arrow[r, "\phi_{b,c}"] \arrow[dd, "F(h_1) \times F(h_2)"] & F(b+c) \arrow[r, "F(j_{b,c})"] \arrow[dd, "F(h_1 + h_2)"] & F(b +_{a_2} c) \arrow[dd, "F(h_1 +_g h_2)"] \\ 1 \arrow[r, "\lambda^{-1}"] & 1 \times 1 \arrow[ur, "d_{M_1} \times d_{N_1}"] \arrow[dr, "d_{M_2} \times d_{N_2}"'] \arrow[r, phantom, "\iota_\alpha \times \iota_\beta \Swarrow"] & {} & & \\ & & F(b') \times F(c') \arrow[r, "\phi_{b',c'}"] & F(b' + c') \arrow[r, "F(j_{b',c'})"] & F(b' +_{a_2'} c') \end{tikzcd}$$

where the middle square commutes since F is a lax monoidal pseudofunctor and the right square commutes as the underlying diagram commutes. The decorations $d_{M_1 \odot N_1}$ and $d_{M_2 \odot N_2}$ are given respectively by top and bottom composite of arrows and the morphism of decorations $\iota_{\alpha \odot \beta}$ is given by composing $\iota_\alpha \times \iota_\beta$ with the two commuting squares, which can equivalently be viewed as a morphism in $F(b' +_{a_2'} c')$.

Returning to the interchange law, composing the two horizontal compositions above vertically then results in:

$$\begin{tikzcd} a_1 \arrow[rr, "j\psi_{a_1} i_1"] \arrow[d, "f'f"'] & & b +_{a_2} c \arrow[d, "(h_1' +_{g'} h_2')(h_1 +_g h_2)"'] & & a_3 \arrow[ll, "j\psi_{a_3} o_2"'] \arrow[d, "k'k"'] & d_{M_1 \odot N_1} \in F(b +_{a_2} c) \\ a_1'' \arrow[rr, "j\psi_{a_1''} i_1''"] & & b'' +_{a_2''} c'' & & a_3'' \arrow[ll, "j\psi_{a_3''} o_2''"'] & d_{M_3 \odot N_3} \in F(b'' +_{a_2''} c'') \end{tikzcd}$$

$$\iota_{(\alpha' \odot \beta')(\alpha \odot \beta)} \colon F((h_1' +_{g'} h_2')(h_1 +_g h_2))(d_{M_1 \odot N_1}) \to d_{M_3 \odot N_3}.$$

The vertical composite of two morphisms of decorations is straightforward. On the other hand, if we first compose vertically we obtain:

$$\begin{tikzcd} a_1 \arrow[r, "i_1"] \arrow[d, "f'f"'] & b \arrow[d, "h_1'h_1"'] & a_2 \arrow[l, "o_1"'] \arrow[d, "g'g"'] & d_{M_1} \in F(b) \\ a_1'' \arrow[r, "i_1''"] & b'' & a_2'' \arrow[l, "o_1''"'] & d_{M_3} \in F(b'') \end{tikzcd} \qquad \begin{tikzcd} a_2 \arrow[r, "i_2"] \arrow[d, "g'g"'] & c \arrow[d, "h_2'h_2"'] & a_3 \arrow[l, "o_2"'] \arrow[d, "k'k"'] & d_{N_1} \in F(c) \\ a_2'' \arrow[r, "i_2''"] & c'' & a_3'' \arrow[l, "o_2''"'] & d_{N_3} \in F(c'') \end{tikzcd}$$

$$\iota_{\alpha'\alpha} \colon F(h_1'h_1)(d_{M_1}) \to d_{M_3} \qquad \iota_{\beta'\beta} \colon F(h_2'h_2)(d_{N_1}) \to d_{N_3}$$

and then composing horizontally results in:

$$\begin{tikzcd} a_1 \arrow[rr, "j\psi_{a_1} i_1"] \arrow[d, "f'f"'] & & b +_{a_2} c \arrow[d, "(h_1'h_1) +_{g'g} (h_2'h_2)"'] & & a_3 \arrow[ll, "j\psi_{a_3} o_2"'] \arrow[d, "k'k"'] & d_{M_1 \odot N_1} \in F(b +_{a_2} c) \\ a_1'' \arrow[rr, "j\psi_{a_1''} i_1''"] & & b'' +_{a_2''} c'' & & a_3'' \arrow[ll, "j\psi_{a_3''} o_2''"'] & d_{M_3 \odot N_3} \in F(b'' +_{a_3''} c'') \end{tikzcd}$$

$$\iota_{(\alpha'\alpha) \odot (\beta'\beta)} \colon F((h_1'h_1) +_{g'g} (h_2'h_2))(d_{M_1 \odot N_1}) \to d_{M_3 \odot N_3}.$$

As is usual concerning the interchange law of double categories of this nature, only the 'interior' of the two compositions appears different, but the two morphisms $(h_1' +_{g'} h_2')(h_1 +_g h_2) \colon b +_{a_2} c \to b'' +_{a_2''} c''$ and $(h_1'h_1) +_{g'g} (h_2'h_2) \colon b +_{a_2} c \to b'' +_{a_2''} c''$ are the same universal map realized in two different ways. The two morphisms of decorations $\iota_{(\alpha' \odot \beta')(\alpha \odot \beta)}$ and $\iota_{(\alpha'\alpha) \odot (\beta'\beta)}$ are obtained as two different compositions of four 2-morphisms in **Cat**, namely horizontally then vertically and vertically then horizontally. As **Cat** is a 2-category, the interchange law for these 2-morphisms already holds, and as a result, the morphisms

$$\iota_{(\alpha' \odot \beta')(\alpha \odot \beta)} \colon F((h_1' +_{g'} h_2')(h_1 +_g h_2))(d_{M_1 \odot N_1}) \to d_{M_3 \odot N_3}$$

and

$$\iota_{(\alpha'\alpha) \odot (\beta'\beta)} \colon F((h_1'h_1) +_{g'g} (h_2'h_2))(d_{M_1 \odot N_1}) \to d_{M_3 \odot N_3}$$

are also the same. Thus the interchange law for 2-morphisms holds and $F\mathbb{C}\mathrm{sp}$ is a double category. $\square$

Theorem 2.2. *Let* A *be a category with finite colimits and* $F\colon \mathsf{A} \to \mathbf{Cat}$ *a symmetric lax monoidal pseudofunctor. Then the double category* $F\mathbb{C}\mathrm{sp}$ *is symmetric monoidal.*

Kenny: We should try to make this theorem shorter. Yes!

Proof. First we note that the category of objects $F\mathbb{C}\mathrm{sp}_0 = \mathsf{A}$ is symmetric monoidal under binary coproducts and the left and right unitors, associators and braidings are given as natural maps. The category of arrows $F\mathbb{C}\mathrm{sp}_1$ has:

(1) objects as F-decorated cospans which are pairs:

$$a_1 \xrightarrow{i} b \xleftarrow{o} a_2 \qquad d \in F(b)$$

and

(2) morphisms as maps of cospans in A

$$\begin{tikzcd} a_1 \arrow[r, "i"] \arrow[d, "f"'] & b \arrow[d, "h"] & a_2 \arrow[l, "o"'] \arrow[d, "g"] & d \in F(b) \\ a_1' \arrow[r, "i'"] & b' & a_2' \arrow[l, "o'"'] & d' \in F(b') \end{tikzcd}$$

together with a morphism $\iota\colon F(h)(d) \to d'$.

Given two objects M_1 and M_2 of $F\mathbb{C}\mathrm{sp}_1$:

$$a_1 \xrightarrow{i} b \xleftarrow{o} a_2 \qquad\qquad a_1' \xrightarrow{i'} b' \xleftarrow{o'} a_2'$$

$$d_{M_1} \in F(b) \qquad\qquad d_{M_2} \in F(b')$$

their tensor product $M_1 \otimes M_2$ is given by taking the coproducts of the cospans of A

$$a_1 + a_1' \xrightarrow{i+i'} b + b' \xleftarrow{o+o'} a_2 + a_2' \qquad d_{M_1 \otimes M_2} \in F(b+b')$$

and where the decoration on the apex is obtained using the natural transformation of the symmetric lax monoidal pseudofunctor F :

$$d_{M_1 \otimes M_2} \coloneqq 1 \xrightarrow{\lambda^{-1}} 1 \times 1 \xrightarrow{d_{M_1} \times d_{M_2}} F(b) \times F(b') \xrightarrow{\phi_{b,b'}} F(b+b').$$

The monoidal unit 0 is given by:

$$0 \xrightarrow{!} 0 \xleftarrow{!} 0 \qquad !_0 \in F(0)$$

where 0 is the monoidal unit of A and $!_0\colon 1 \to F(0)$ is the morphism which is part of the structure of the symmetric lax monoidal pseudofunctor $F\colon \mathsf{A} \to \mathbf{Cat}$. Tensoring an object with the monoidal unit, say, on the left:

$$0 \xrightarrow{!} 0 \xleftarrow{!} 0 \quad \otimes \quad a_1 \xrightarrow{i} b \xleftarrow{o} a_2$$

$$!_0 \in F(0) \qquad\qquad d_M \in F(b)$$

results in:

$$0 + a_1 \xrightarrow{!+i} 0 + b \xleftarrow{!+o} 0 + a_2 \qquad d_{0 \otimes M} \in F(0+b)$$

where $d_{0\otimes M} \in F(0+b)$ is given by

$$1 \xrightarrow{\lambda^{-1}} 1 \times 1 \xrightarrow{!_0 \times d_M} F(0) \times F(b) \xrightarrow{\phi_{0,b}} F(0+b).$$

The left unitor is then an isomorphism in $F\mathbb{C}\mathrm{sp}_1$ given by:

$$\begin{tikzcd}
0+a_1 \arrow[r, "!+i"] \arrow[d, "\ell"'] & 0+b \arrow[d, "\ell"'] & 0+a_2 \arrow[l, "!+o"'] \arrow[d, "\ell"'] & d_{0+M} \in F(0+b) \\
a_1 \arrow[r, "i"] & b & a_2 \arrow[l, "o"'] & d_M \in F(b)
\end{tikzcd}$$

where ℓ is the left unitor of $(\mathsf{A}, +, 0)$, together with the isomorphism $\iota_\lambda \colon F(\ell)(d_{0\otimes M}) \to d_M$. The right unitor is similar.

Given three objects M_1, M_2 and M_3 in $F\mathbb{C}\mathrm{sp}_1$:

$$a_1 \xrightarrow{i_1} c_1 \xleftarrow{o_1} b_1 \qquad a_2 \xrightarrow{i_2} c_2 \xleftarrow{o_2} b_2 \qquad a_3 \xrightarrow{i_3} c_3 \xleftarrow{o_3} b_3$$

$$d_{M_1} \in F(c_1) \qquad d_{M_2} \in F(c_2) \qquad d_{M_3} \in F(c_3)$$

tensoring the first two and then the third results in $(M_1 \otimes M_2) \otimes M_3$:

$$(a_1+a_2)+a_3 \xrightarrow{(i_1+i_2)+i_3} (c_1+c_2)+c_3 \xleftarrow{(o_1+o_2)+o_3} (b_1+b_2)+b_3$$

$$d_{(M_1\otimes M_2)\otimes M_3} \in F((c_1+c_2)+c_3)$$

where $d_{(M_1\otimes M_2)\otimes M_3} \colon 1 \to F((c_1+c_2)+c_3)$ is given by:

$$1 \xrightarrow{(d_{M_1}\times d_{M_2})\times d_{M_3})} (F(c_1)\times F(c_2))\times F(c_3) \xrightarrow{\phi_{c_1,c_2}\times 1} F(c_1+c_2)\times F(c_3) \xrightarrow{\phi_{c_1+c_2,c_3}} F((c_1+c_2)+c_3)$$

whereas tensoring the last two and then the first results in $M_1 \otimes (M_2 \otimes M_3)$:

$$a_1+(a_2+a_3) \xrightarrow{i_1+(i_2+i_3)} c_1+(c_2+c_3) \xleftarrow{o_1+(o_2+o_3)} b_1+(b_2+b_3)$$

$$d_{M_1\otimes(M_2\otimes M_3)} \in F(c_1+(c_2+c_3))$$

where $d_{M_1\otimes(M_2\otimes M_3)} \colon 1 \to F(c_1+(c_2+c_3))$ is given by:

$$1 \xrightarrow{d_{M_1}\times(d_{M_2}\times d_{M_3})} F(c_1)\times(F(c_2)\times F(c_3)) \xrightarrow{1\times\phi_{c_2,c_3}} F(c_1)\times F(c_2+c_3) \xrightarrow{\phi_{c_1,c_2+c_3}} F(c_1+(c_2+c_3)).$$

If we let a denote the associator of $(\mathsf{A}, +, 0)$, the associator of $F\mathbb{C}\mathrm{sp}_1$ is then a map of cospans in A from $(M_1 \otimes M_2) \otimes M_3$ to $M_1 \otimes (M_2 \otimes M_3)$ given by:

$$\begin{tikzcd}
(a_1+a_2)+a_3 \arrow[r, "(i_1+i_2)+i_3"] \arrow[d, "a"'] & (c_1+c_2)+c_3 \arrow[d, "a"'] & (b_1+b_2)+b_3 \arrow[l, "(o_1+o_2)+o_3"'] \arrow[d, "a"'] & d_{(M_1\otimes M_2)\otimes M_3} \in F((c_1+c_2)+c_3) \\
a_1+(a_2+a_3) \arrow[r, "i_1+(i_2+i_3)"] & c_1+(c_2+c_3) & b_1+(b_2+b_3) \arrow[l, "o_1+(o_2+o_3)"'] & d_{M_1\otimes(M_2\otimes M_3)} \in F(c_1+(c_2+c_3))
\end{tikzcd}$$

together with the isomorphism $\iota_a \colon F(a)(d_{(M_1\otimes M_2)\otimes M_3}) \to d_{M_1\otimes(M_2\otimes M_3)}$. If we denote the above associator simply as a and the left and right unitors as λ and ρ, respectively, then given four objects in $F\mathbb{C}\mathrm{sp}_1$, say M_1, M_2, M_3 and M_4:

$$
\begin{array}{cc}
a_1 \xrightarrow{i_1} c_1 \xleftarrow{o_1} b_1 & a_2 \xrightarrow{i_2} c_2 \xleftarrow{o_2} b_2 \\
d_{M_1} \in F(c_1) & d_{M_2} \in F(c_2) \\
 & \\
a_3 \xrightarrow{i_3} c_3 \xleftarrow{o_3} b_3 & a_4 \xrightarrow{i_4} c_4 \xleftarrow{o_4} b_4 \\
d_{M_3} \in F(c_3) & d_{M_4} \in F(c_4)
\end{array}
$$

then as $\mathbb{C}\mathrm{sp}(\mathsf{A})$ is a symmetric monoidal double category, the following pentagon of underlying cospans commutes:

$$
\begin{tikzcd}
 & (M_1 \otimes M_2) \otimes (M_3 \otimes M_4) \arrow[dr, "a"] & \\
((M_1 \otimes M_2) \otimes M_3) \otimes M_4 \arrow[ur, "a"] \arrow[d, "a \otimes 1"'] & & M_1 \otimes (M_2 \otimes (M_3 \otimes M_4)) \\
(M_1 \otimes (M_2 \otimes M_3)) \otimes M_4 \arrow[r, "a"] & M_1 \otimes ((M_2 \otimes M_3) \otimes M_4) \arrow[ur, "1 \otimes a"'] &
\end{tikzcd}
$$

as well as the following pentagon of corresponding decorations in the category $F(c_1 + (c_2 + (c_3 + c_4)))$:

$$
\begin{tikzcd}
 & F(a)(d_{(M_1 \otimes M_2) \otimes (M_3 \otimes M_4)}) \arrow[dr, "\iota_a"] & \\
F(aa)(d_{((M_1 \otimes M_2) \otimes M_3) \otimes M_4}) \arrow[ur, "F(a)(\iota_a)"] \arrow[d, "F((1 \otimes a)a)(\iota_{a \otimes 1})"'] & & d_{M_1 \otimes (M_2 \otimes (M_3 \otimes M_4))} \\
F((1 \otimes a)a)(d_{(M_1 \otimes (M_2 \otimes M_3)) \otimes M_4}) \arrow[rr, "F(1 \otimes a)(\iota_a)"] & & F(1 \otimes a)(d_{M_1 \otimes ((M_2 \otimes M_3) \otimes M_4)}) \arrow[u, "\iota_{1 \otimes a}"']
\end{tikzcd}
$$

since

$$F(aa)(d_{((M_1 \otimes M_2) \otimes M_3) \otimes M_4}) = F((1 \otimes a)a(a \otimes 1))(d_{((M_1 \otimes M_2) \otimes M_3) \otimes M_4})$$

as the corresponding pentagon in the symmetric monoidal category $(\mathsf{A}, +, 0)$ commutes.

Similarly, if we denote the left and right unitors as λ and ρ, respectively, then the following triangle of underlying maps of cospans commutes:

$$
\begin{tikzcd}
 & M_1 \otimes M_2 & \\
(M_1 \otimes 0) \otimes M_2 \arrow[ur, "\rho \otimes 1"] \arrow[rr, "a"] & & M_1 \otimes (0 \otimes M_2) \arrow[ul, "1 \otimes \lambda"']
\end{tikzcd}
$$

as well as the following triangle of corresponding decorations in the category $F(c_1 + c_2)$:

$$\begin{tikzcd} & d_{M_1 \otimes M_2} & \\ F(\rho \otimes 1)(d_{(M_1 \otimes 0) \otimes M_2}) \arrow[ur, "\iota_{\rho \otimes 1}"] \arrow[rr, "F(1 \otimes \lambda)(\iota_a)"] & & F(1 \otimes \lambda)(d_{M_1 \otimes (0 \otimes M_2)}) \arrow[ul, "\iota_{1 \otimes \lambda}"'] \end{tikzcd}$$

since

$$F(\rho \otimes 1)(d_{(M_1 \otimes 0) \otimes M_2}) = F((1 \otimes \lambda)a)(d_{(M_1 \otimes 0) \otimes M_2})$$

as the corresponding triangle in the symmetric monoidal category $(\mathsf{A}, +, 0)$ commutes.

For a tensor product of objects $M_1 \otimes M_2$ in $F\mathbb{C}\mathrm{sp}_1$, the source and target structure functors $S, T\colon F\mathbb{C}\mathrm{sp}_1 \to F\mathbb{C}\mathrm{sp}_0$ satisfy the following equations:

$$S(M_1 \otimes M_2) = S(M_1) \otimes S(M_2)$$

$$T(M_1 \otimes M_2) = T(M_1) \otimes T(M_2).$$

For two objects M_1 and M_2 in $F\mathbb{C}\mathrm{sp}_1$, we have a braiding $\beta_{M_1,M_2}\colon M_1 \otimes M_2 \to M_2 \otimes M_1$ given by:

$$\begin{tikzcd} a_1 + a_2 \arrow[r, "i_1 + i_2"] \arrow[d, "\beta_{a_1,a_2}"'] & c_1 + c_2 \arrow[d, "\beta_{c_1,c_2}"'] & b_1 + b_2 \arrow[l, "o_1 + o_2"'] \arrow[d, "\beta_{b_1,b_2}"'] & d_{M_1 \otimes M_2} \in F(c_1 + c_2) \\ a_2 + a_1 \arrow[r, "i_2 + i_1"] & c_2 + c_1 & b_2 + b_1 \arrow[l, "o_2 + o_1"'] & d_{M_2 \otimes M_1} \in F(c_2 + c_1) \end{tikzcd}$$

$$\iota_{\beta_{M_1,M_2}}\colon F(\beta_{c_1,c_2})(d_{M_1 \otimes M_2}) \xrightarrow{\sim} d_{M_2 \otimes M_1}$$

where the vertical 1-morphisms are given by braidings in $(\mathsf{A}, +, 0)$. This braiding makes the following triangle of underlying cospans commute:

$$\begin{tikzcd} & M_1 \otimes M_2 & \\ M_1 \otimes M_2 \arrow[ur, "1"] \arrow[rr, "\beta_{M_1,M_2}"] & & M_2 \otimes M_1 \arrow[ul, "\beta_{M_2,M_1}"'] \end{tikzcd}$$

as well as the following diagram of corresponding decorations in the category $F(c_1 + c_2)$:

$$\begin{tikzcd} & d_{M_1 \otimes M_2} & \\ d_{M_1 \otimes M_2} \arrow[ur, "1"] \arrow[rr, "F(\beta_{c_2,c_1})(\iota_{\beta_{M_1,M_2}})"] & & F(\beta_{c_2,c_1})(d_{M_2 \otimes M_1}) \arrow[ul, "\iota_{\beta_{M_2,M_1}}"'] \end{tikzcd}$$

since $F(\beta_{c_2,c_1}\beta_{c_1,c_2})(d_{M_1 \otimes M_2}) = d_{M_1 \otimes M_2}$. Thus $F\mathbb{C}\mathrm{sp}_1$ is also symmetric monoidal.

Now, given four horizontal 1-cells M_1, M_2, N_1 and N_2 respectively by:

$$a_1 \xrightarrow{i_1} b \xleftarrow{o_1} a_2 \qquad\qquad a_2 \xrightarrow{i_2} b' \xleftarrow{o_2} a_3$$
$$d_{M_1} \in F(b) \qquad\qquad d_{M_2} \in F(b')$$

$$a'_1 \xrightarrow{i'_1} c \xleftarrow{o'_1} a'_2 \qquad\qquad a'_2 \xrightarrow{i'_2} c' \xleftarrow{o'_2} a'_3$$
$$d_{N_1} \in F(c) \qquad\qquad d_{N_2} \in F(c')$$

we have that $(M_1 \otimes N_1) \odot (M_2 \otimes N_2)$ is given by:

$$a_1 + a'_1 \xrightarrow{j\psi(i_1 + i'_1)} (b + c) +_{a_2 + a'_2} (b' + c') \xleftarrow{j\psi(o_2 + o'_2)} a_3 + a'_3$$
$$d_{(M_1 \otimes N_1)\odot(M_2 \otimes N_2)} \in F((b + c) +_{a_2 + a'_2} (b' + c'))$$

where the decoration $d_{(M_1 \otimes N_1)\odot(M_2 \otimes N_2)} \in F((b + c) +_{a_2 + a'_2} (b' + c'))$ is given by:

$$\begin{array}{c}
1 \\
\downarrow{\scriptstyle \lambda^{-1}} \\
1 \times 1 \\
\downarrow{\scriptstyle \lambda^{-1} \times \lambda^{-1}} \\
(1 \times 1) \times (1 \times 1) \\
\downarrow{\scriptstyle (d_{M_1} \times d_{N_1}) \times (d_{M_2} \times d_{N_2})} \\
(F(b) \times F(c)) \times (F(b') \times F(c')) \\
\downarrow{\scriptstyle \phi_{b,c} \times \phi_{b',c'}} \\
F(b + c) \times F(b' + c') \\
\downarrow{\scriptstyle \phi_{b+c,b'+c'}} \\
F((b + c) + (b' + c')) \\
\downarrow{\scriptstyle F(j_{b+c,b'+c'})} \\
F((b + c) +_{a_2 + a'_2} (b' + c'))
\end{array}$$

and $(M_1 \odot M_2) \otimes (N_1 \odot N_2)$ is given by:

$$a_1 + a'_1 \xrightarrow{(j\psi i_1) + (j\psi i'_1)} (b +_{a_2} c) + (b' +_{a'_2} c') \xleftarrow{(j\psi o_2) + (j\psi o'_2)} a_3 + a'_3$$
$$d_{(M_1 \odot M_2)\otimes(N_1 \odot N_2)} \in F((b +_{a_2} c) + (b' +_{a'_2} c'))$$

where the decoration $d_{((M_1 \odot M_2) \otimes (N_1 \odot N_2))} \in F((b +_{a_2} c) + (b' +_{a_2'} c'))$ is given by:

$$\begin{array}{c} 1 \\ \Big\downarrow{\scriptstyle \lambda^{-1}} \\ 1 \times 1 \\ \Big\downarrow{\scriptstyle \lambda^{-1} \times \lambda^{-1}} \\ (1 \times 1) \times (1 \times 1) \\ \Big\downarrow{\scriptstyle (d_{M_1} \times d_{N_1}) \times (d_{M_2} \times d_{N_2})} \\ (F(b) \times F(c)) \times (F(b') \times F(c')) \\ \Big\downarrow{\scriptstyle \phi_{b,c} \times \phi_{b',c'}} \\ F(b + c) \times F(b' + c') \\ \Big\downarrow{\scriptstyle F(j_{b,c}) \times F(j_{b',c'})} \\ F(b +_{a_2} c) \times F(b' +_{a_2'} c') \\ \Big\downarrow{\scriptstyle \phi_{b +_{a_2} c, b' +_{a_2'} c'}} \\ F((b +_{a_2} c) + (b' +_{a_2'} c')) \end{array}$$

and where ψ and j are the natural maps into a coproduct and from a coproduct into a pushout, respectively. We then get a globular 2-morphism

$$\chi\colon (M_1 \otimes N_1) \odot (M_2 \otimes N_2) \to (M_1 \odot M_2) \otimes (N_1 \odot N_2)$$

given by:

$$d_{(M_1 \otimes N_1) \odot (M_2 \otimes N_2)} \in F((b + c) +_{a_2 + a_2'} (b' + c'))$$

$$\begin{tikzcd} a_1 + a_1' \arrow[r, "j\psi(i_1 + i_1')"] \arrow[d, "1"'] & (b + c) +_{a_2 + a_2'} (b' + c') \arrow[d, "\hat{\chi}"'] & a_3 + a_3' \arrow[l, "j\psi(o_2 + o_2')"'] \arrow[d, "1"'] \\ a_1 + a_1' \arrow[r, "(j\psi i_1') + (j\psi i_1)"] & (b +_{a_2} c) + (b' +_{a_2'} c') & a_3 + a_3' \arrow[l, "(j\psi o_2) + (j\psi o_2')"'] \end{tikzcd}$$

$$d_{(M_1 \odot M_2) \otimes (N_1 \odot N_2)} \in F((b +_{a_2} c) + (b' +_{a_2'} c'))$$

$$\iota_{\hat{\chi}}\colon F(\hat{\chi})(d_{(M_1 \otimes N_1) \odot (M_2 \otimes N_2)}) \to d_{(M_1 \odot M_2) \otimes (N_1 \odot N_2)}$$

where $\hat{\chi}$ is the universal map between two colimits of the same diagram. For two objects $a, b \in \mathrm{A}$, U_{a+b} is given by:

$$a + b \xrightarrow{1_{a+b}} a + b \xleftarrow{1_{a+b}} a + b$$

$$!_{a+b} \in F(a + b)$$

where

$$!_{a+b}\colon 1 \xrightarrow{\phi} F(0) \xrightarrow{F(!_{a+b})} F(a + b).$$

Similarly, we have U_a and U_b given respectively by:

$$a \xrightarrow{1_a} a \xleftarrow{1_a} a \qquad\qquad b \xrightarrow{1_b} b \xleftarrow{1_b} b$$

$$!_a \in F(a) \qquad\qquad\qquad !_b \in F(b)$$

and then $U_a + U_b$ is given by:

$$a + b \xrightarrow{1_a + 1_b} a + b \xleftarrow{1_a + 1_b} a + b$$

$$!_a + !_b \in F(a + b)$$

where

$$!_a + !_b \colon 1 \xrightarrow{\lambda^{-1}} 1 \times 1 \xrightarrow{\phi \times \phi} F(0) \times F(0) \xrightarrow{F(!_a) \times F(!_b)} F(a) \times F(b) \xrightarrow{\phi_{a,b}} F(a + b).$$

We then have another globular isomorphism

$$\mu_{a,b} \colon U_{a+b} \to U_a + U_b$$

given by the identity 2-morphism:

$$\begin{tikzcd}
a + b \arrow[r, "1_{a+b}"] \arrow[d, "1"'] & a + b \arrow[d, "1"'] & a + b \arrow[l, "1_{a+b}"'] \arrow[d, "1"'] & !_{a+b} \in F(a + b) \\
a + b \arrow[r, "1_a + 1_b"] & a + b & a + b \arrow[l, "1_a + 1_b"'] & !_a + !_b \in F(a + b)
\end{tikzcd}$$

$$\iota_{a,b} \colon !_{a+b} \xrightarrow{\sim} !_a + !_b$$

where $!_{a+b}$ and $!_a + !_b$ are both initial objects in $F(a + b)$, hence isomorphic.

There are a fair amount of coherence diagrams to verify, many of which are similar in flavor and make use of the two above globular ismorphisms. We check a few to give a sense of what these are like. For example, given horizontal 1-cells M_i, N_i, P_i, the following commutative diagram expresses the associativity isomorphism as a transformation of double categories.

$$\begin{tikzcd}
((M_1 \otimes N_1) \otimes P_1) \odot ((M_2 \otimes N_2) \otimes P_2) \arrow[r, "a \odot a"] \arrow[d, "\chi"'] & (M_1 \otimes (N_1 \otimes P_1)) \odot (M_2 \otimes (N_2 \otimes P_2)) \arrow[d, "\chi"] \\
((M_1 \otimes N_1) \odot (M_2 \otimes N_2)) \otimes (P_1 \odot P_2) \arrow[d, "\chi \otimes 1"'] & (M_1 \odot M_2) \otimes ((N_1 \otimes P_1) \odot (N_2 \otimes P_2)) \arrow[d, "1 \otimes \chi"] \\
((M_1 \odot M_2) \otimes (N_1 \odot N_2)) \otimes (P_1 \odot P_2) \arrow[r, "a"] & (M_1 \odot M_2) \otimes ((N_1 \odot N_2) \otimes (P_1 \odot P_2))
\end{tikzcd}$$

Here, a is the associator of $F\mathbb{C}\mathrm{sp}_1$ and χ is the first globular isomorphism above. To see that this diagram does indeed commute, we first consider this diagram with respect to only the underlying cospans of each horizontal 1-cell. For notation:

$$M_1 = \quad k \longrightarrow l \longleftarrow m \qquad N_1 = \quad q \longrightarrow r \longleftarrow s \qquad P_1 = \quad v \longrightarrow w \longleftarrow x$$
$$d_{M_1} \in F(l) \qquad\qquad d_{N_1} \in F(r) \qquad\qquad d_{P_1} \in F(w)$$

$$M_2 = \quad m \longrightarrow n \longleftarrow p \qquad N_2 = \quad s \longrightarrow t \longleftarrow u \qquad P_2 = \quad x \longrightarrow y \longleftarrow z$$
$$d_{M_2} \in F(n) \qquad\qquad d_{N_2} \in F(t) \qquad\qquad d_{P_2} \in F(y)$$

The above diagram then becomes:

$$
\begin{tikzcd}
k+m \arrow[r] & (l +_m n) + ((r +_s t) + (w +_x y)) & v+x \arrow[l] \\
k+m \arrow[r] \arrow[u] & (l +_m n) + ((r + w) +_{(s+x)} (t + y)) \arrow[u, "1 \otimes \chi"', "\iota_3"] & v+x \arrow[l] \arrow[u] \\
k+m \arrow[r] \arrow[u] & (l + (r + w)) +_{(m+(s+x))} (n + (t + y)) \arrow[u, "\chi"', "\iota_2"] & v+x \arrow[l] \arrow[u] \\
k+m \arrow[r] \arrow[u] \arrow[d] & ((l + r) + w) +_{((m+s)+x)} ((n + t) + y) \arrow[u, "a \odot a"', "\iota_1"] \arrow[d, "\chi"', "\iota_4"] & v+x \arrow[l] \arrow[u] \arrow[d] \\
k+m \arrow[r] \arrow[d] & ((l + r) +_{(m+s)} (n + t)) + (w +_x y) \arrow[d, "\chi \otimes 1"', "\iota_5"] & v+x \arrow[l] \arrow[d] \\
k+m \arrow[r] \arrow[d] & ((l +_m n) + (r +_s t)) + (w +_x y) \arrow[d, "a"', "\iota_6"] & v+x \arrow[l] \arrow[d] \\
k+m \arrow[r] & (l +_m n) + ((r +_s t) + (w +_x y)) & v+x \arrow[l]
\end{tikzcd}
$$

Here all of the vertical 1-morphisms on the left and right are identities, the middle vertical 1-morphisms are the 2-morphisms from the previous commutative diagram, and the horizontal vertical 1-morphisms pointing towards the middle are natural maps into each colimit, all of which are naturally isomorphic to each other as all the middle objects are colimits of the same diagram, namely the previous collection of cospans, taken in various ways. The above diagram of maps of cospans can then be visualized as a hexagonal prism in which all the faces commute by identifying the top and the bottom as the same. As for the morphisms of decorations, each isomorphism ι_n goes from the domain under the image of the functor F applied to natural isomorphism adjacent to it to the codomain as written, meaning that, for example:

$$\iota_1 \colon F(a \odot a)(d_{((M_1 \otimes N_1) \otimes P_1) \odot ((M_2 \otimes N_2) \otimes P_2)}) \to d_{(M_1 \otimes (N_1 \otimes P_1)) \odot (M_2 \otimes (N_2 \otimes P_2))}.$$

The following diagram commutes in the category $F((l +_m n) + ((r +_s t) + (w +_x y)))$:

$$
\begin{tikzcd}
F(a(\chi \otimes 1)\chi)(d_{((M_1 \otimes N_1) \otimes P_1) \odot ((M_2 \otimes N_2) \otimes P_2)}) \arrow[r, "F((1 \otimes \chi)\chi)(\iota_1)"] \arrow[d, "F(a(\chi \otimes 1))(\iota_4)"'] & F((1 \otimes \chi)\chi)(d_{(M_1 \otimes (N_1 \otimes P_1)) \odot (M_2 \otimes (N_2 \otimes P_2))}) \arrow[d, "F(1 \otimes \chi)(\iota_2)"] \\
F(a(\chi \otimes 1))(d_{((M_1 \otimes N_1) \odot (M_2 \otimes N_2)) \otimes (P_1 \odot P_2)}) \arrow[d, "F(a)(\iota_5)"'] & F(1 \otimes \chi)(d_{(M_1 \odot M_2) \otimes ((N_1 \otimes P_1) \odot (N_2 \otimes P_2))}) \arrow[d, "\iota_3"] \\
F(a)(d_{((M_1 \odot M_2) \otimes (N_1 \odot N_2)) \otimes (P_1 \odot P_2)}) \arrow[r, "\iota_6"] & d_{(M_1 \odot M_2) \otimes ((N_1 \odot N_2) \otimes (P_1 \odot P_2))}
\end{tikzcd}
$$

since

$$F(a(\chi \otimes 1)\chi)(d_{((M_1 \otimes N_1) \otimes P_1) \odot ((M_2 \otimes N_2) \otimes P_2)}) = F((1 \otimes \chi)\chi(a \odot a))(d_{((M_1 \otimes N_1) \otimes P_1) \odot ((M_2 \otimes N_2) \otimes P_2)})$$

as the above underlying diagram of maps of cospans commutes.

Another requirement for a double category to be symmetric monoidal is that the braiding

$$\beta_{(-,-)} \colon F\mathbb{C}\mathrm{sp}_1 \times F\mathbb{C}\mathrm{sp}_1 \to F\mathbb{C}\mathrm{sp}_1 \times F\mathbb{C}\mathrm{sp}_1$$

be a transformation of double categories, and one of the diagrams that is required to commute is the following:

$$\begin{tikzcd}
(M_1 \odot M_2) \otimes (N_1 \odot N_2) \arrow[r, "\beta"] \arrow[d, "\chi"'] & (N_1 \odot N_2) \otimes (M_1 \odot M_2) \arrow[d, "\chi"] \\
(M_1 \otimes N_1) \odot (M_2 \otimes N_2) \arrow[r, "\beta \odot \beta"] & (N_1 \otimes M_1) \odot (N_2 \otimes M_2)
\end{tikzcd}$$

Using the same notation as the previous coherence diagram, the diagram for the underlying maps of cospans becomes:

$$\begin{tikzcd}
k+m \arrow[r] & (r+l)+_{(s+m)}(t+n) & s+u \arrow[l] \\
k+m \arrow[r] \arrow[u] & (r+_s t)+(l+_m n) \arrow[u, "\chi"', "\iota_2"] & s+u \arrow[l] \arrow[u] \\
k+m \arrow[r] \arrow[u] \arrow[d] & (l+_m n)+(r+_s t) \arrow[u, "\beta"', "\iota_1"] \arrow[d, "\chi"', "\iota_4"] & s+u \arrow[l] \arrow[u] \arrow[d] \\
k+m \arrow[r] \arrow[d] & (l+r)+_{(m+s)}(n+t) \arrow[d, "\beta\odot\beta"', "\iota_5"] & s+u \arrow[l] \arrow[d] \\
k+m \arrow[r] & (r+l)+_{(s+m)}(t+n) & s+u \arrow[l]
\end{tikzcd}$$

All the comments about the previous underlying coherence diagram of maps of cospans apply to this one. As for the decorations, the following diagram commutes in the category $F((r+l)+_{(s+m)}(t+n))$:

$$\begin{tikzcd}
F(\chi\beta)(d_{(M_1\odot M_2)\otimes(N_1\odot N_2)}) \arrow[r, "F(\chi)(\iota_1)"] \arrow[d, "F(\beta\odot\beta)(\iota_3)"'] & F(\chi)(d_{(N_1\odot N_2)\otimes(M_1\odot M_2)}) \arrow[d, "\iota_2"] \\
F(\beta\odot\beta)(d_{(M_1\otimes N_1)\odot(M_2\otimes N_2)}) \arrow[r, "\iota_4"] & d_{(N_1\otimes M_1)\odot(N_2\otimes M_2)}
\end{tikzcd}$$

since

$$F(\chi\beta)(d_{(M_1\odot M_2)\otimes(N_1\odot N_2)}) = F((\beta\odot\beta)\chi)(d_{(M_1\odot M_2)\otimes(N_1\odot N_2)})$$

as the above underlying diagram of maps of cospans commutes. The other diagrams are shown to commute similarly. □

Lemma 2.3. *The double category $F\mathbb{C}\mathrm{sp}$ is fibrant.*

Proof. Let $f\colon c \to c'$ be a vertical 1-morphism in $F\mathbb{C}\mathrm{sp}$. We can lift f to the companion horizontal 1-cell $\hat{f}$:

$$c \xrightarrow{f} c' \xleftarrow{1} c'$$

$$!_{c'} \in F(c')$$

and then obtain the following two 2-morphisms:

$$
\begin{tikzcd}
c \arrow[r, "f"] \arrow[d, "f"'] & c' \arrow[d, "1"'] & c' \arrow[l, "1"'] \arrow[d, "1"'] & !_{c'} \in F(c') \\
c' \arrow[r, "1"] & c' & c' \arrow[l, "1"'] & !_{c'} \in F(c')
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[r, "1"] \arrow[d, "1"'] & c \arrow[d, "f"'] & c \arrow[l, "1"'] \arrow[d, "f"'] & !_c \in F(c) \\
c \arrow[r, "f"] & c' & c' \arrow[l, "1"'] & !_{c'} \in F(c')
\end{tikzcd}
$$

$$\iota_{1_{c'}} = 1_{!_{c'}} \qquad\qquad \iota_f \colon F(f)(!_c) \to !_{c'}$$

which satisfy the equations:

$$
\begin{tikzcd}
!_c \in F(c) & c \arrow[r, "1"] \arrow[d, "1"'] & c \arrow[d, "f"'] & c \arrow[l, "1"'] \arrow[d, "f"'] \\
!_{c'} \in F(c') & c \arrow[r, "f"] \arrow[d, "f"'] & c' \arrow[d, "1"'] & c' \arrow[l, "1"'] \arrow[d, "1"'] \\
!_{c'} \in F(c') & c' \arrow[r, "1"] & c' & c' \arrow[l, "1"']
\end{tikzcd}
=
\begin{tikzcd}
c \arrow[r, "1"] \arrow[d, "f"'] & c \arrow[d, "f"'] & c \arrow[l, "1"'] \arrow[d, "f"'] & !_c \in F(c) \\
c' \arrow[r, "1"] & c' & c' \arrow[l, "1"'] & !_{c'} \in F(c')
\end{tikzcd}
$$

$$\iota_f \colon F(f)(!_c) \to !_{c'} \qquad\qquad \iota_f \colon F(f)(!_c) \to !_{c'}$$

$$\iota_{c'} = 1_{!_{c'}}$$

$$!_c \in F(c) \qquad !_{c'} \in F(c') \qquad\qquad !_{c'} \in F(c')$$

$$
\begin{tikzcd}
c \arrow[r, "1"] \arrow[d, "1"'] & c \arrow[d, "f"'] & c \arrow[l, "1"'] \arrow[r, "f"] \arrow[d, "f"'] & c' \arrow[d, "1"'] & c' \arrow[l, "1"'] \arrow[d, "1"'] \\
c \arrow[r, "f"] & c' & c' \arrow[l, "1"'] \arrow[r, "1"] & c' & c' \arrow[l, "1"']
\end{tikzcd}
=
\begin{tikzcd}
c \arrow[r, "f"] \arrow[d, "1"'] & c' \arrow[d, "1"'] & c' \arrow[l, "1"'] \arrow[d, "1"'] \\
c \arrow[r, "f"] & c' & c' \arrow[l, "1"']
\end{tikzcd}
$$

$$!_{c'} \in F(c') \qquad !_{c'} \in F(c') \qquad\qquad !_{c'} \in F(c')$$

$$\iota_f \colon F(f)(!_c) \to !_{c'} \qquad \iota_{c'} = 1_{!_{c'}} \qquad\qquad \iota_{c'} = 1_{!_{c'}}$$

The right hand sides of the above two equations are given respectively by the 2-morphisms U_f and $1_{\hat{f}}$. The conjoint of f is given by the F-decorated cospan $\check{f}$ which is just the opposite of the companion above:

$$
\begin{tikzcd}
c' \arrow[r, "1"] & c' & c \arrow[l, "f"']
\end{tikzcd}
\qquad\qquad !_{c'} \in F(c')
$$

$\square$

The property of being fibrant is what allows us to lift the monoidal structure from the object category of a double category to its arrow category and obtain a symmetric monoidal bicategory. The following result, which only requires fibrancy on vertical 1-isomorphisms (isofibrancy) is due to Shulman [24]:

Theorem 2.4 (Shulman)**.** *Let $\mathbb{X}$ be an isofibrant symmetric monoidal pseudo double category. Then the horizontal bicategory $H(\mathbb{X})$ of $\mathbb{X}$ is a symmetric monoidal bicategory which has:*

(1) objects as those of $\mathbb{X}$*,*
(2) morphisms as horizontal 1-cells of $\mathbb{X}$*, and*
(3) 2-morphisms as globular 2-morphisms of $\mathbb{X}$*.*

Corollary 2.5. *Let* $(\mathsf{C}, +, 0)$ *be a category with finite colimts and* $F\colon \mathsf{C} \to \mathbf{Cat}$ *a symmetric lax monoidal pseudofunctor. Then there exists a symmetric monoidal bicategory* $F\mathbf{Csp} := H(F\mathbb{C}\mathrm{sp})$ *which has:*

(1) objects as those of A*,*
(2) morphisms as F*-decorated cospans:*

$$a \xrightarrow{i} c \xleftarrow{o} b \qquad\qquad d \in F(c)$$

and

(3) 2-morphisms as maps of cospans in A *of the form:*

$$\begin{tikzcd} & c \arrow[d, "h"] & & d \in F(c) \\ a \arrow[ur, "i"] \arrow[dr, "i'"'] & & b \arrow[ul, "o"'] \arrow[dl, "o'"] & \\ & c' & & d' \in F(c') \end{tikzcd}$$

together with a morphism $\iota\colon F(h)(d) \to d'$ *in* $F(c')$*.*

Proof. This follows immediately by Shulman's Theorem 2.4 above applied to the fibrant symmetric monoidal double category $F\mathbb{C}\mathrm{sp}$. □

This symmetric monoidal bicategory $F\mathbf{Csp}$ is a superior version of the symmetric monoidal bicategory $F\mathrm{Cospan}(\mathsf{A})$ constructed earlier by the second author [12] in that there is greater flexibility in what 2-morphisms are allowed.

We can then decategorify this symmetric monoidal bicategory to obtain a symmetric monoidal category similar to the one obtained using Fong's result, but with larger isomorphism classes:

Corollary 2.6. *Given a symmetric lax monoidal pseudofunctor* $F\colon \mathsf{A} \to \mathbf{Cat}$ *where* A *is a category with finite colimits and whose monoidal structure is given by binary coproducts, there exists a symmetric monoidal category* $F\mathrm{Csp} := D(F\mathbf{Csp})$ *which has:*

(1) objects as those of A *and*
(2) morphisms as isomorphism classes of F*-decorated cospans of* A*, where an* F*-decorated cospan is given by a pair:*

$$a \xrightarrow{i} c \xleftarrow{o} b \qquad\qquad d \in F(c)$$

Given another F*-decorated cospan:*

$$a \xrightarrow[i']{} c' \xleftarrow[o']{} b \qquad\qquad d' \in F(c')$$

these two F*-decorated cospans are in the same isomorphism class if there exists an isomorphism* $f\colon c \to c'$ *such that following diagram commutes:*

$$\begin{tikzcd} & c \arrow[d, "f"] & \\ a \arrow[ur, "i"] \arrow[dr, "i'"'] & & b \arrow[ul, "o"'] \arrow[dl, "o'"] \\ & c' & \end{tikzcd}$$

and there exists an isomorphism $\iota\colon F(f)(d) \to d'$ *in* $F(c')$.

In this symmetric monoidal category, isomorphism classes are as they should morally be, and the instance of two graphs having different edge sets does not prevent them from being in the same isomorphism class due to the isomorphism ι.

Regarding maps between symmetric monoidal double categories, consider another symmetric lax monoidal pseudofunctor $F'\colon \mathsf{A}' \to \mathbf{Cat}$a morphism from $F\mathbb{C}\mathrm{sp}$ to $F'\mathbb{C}\mathrm{sp}$ will then be a double functor $\mathbb{H}\colon F\mathbb{C}\mathrm{sp} \to F'\mathbb{C}\mathrm{sp}$ whose object component is given by a finite colimit preserving functor $\mathbb{H}_0 = H\colon \mathsf{A} \to \mathsf{A}'$ and whose arrow component is given by a functor $\mathbb{H}_1$ defined on horizontal 1-cells by:

$$\begin{array}{ccc} a \xrightarrow{i} c \xleftarrow{o} b & \mapsto & H(a) \xrightarrow{H(i)} H(c) \xleftarrow{H(o)} H(b) \\ d \in F(c) & & \theta_c E(d)\phi \in F'(H(c)) \end{array}$$

and on 2-morphisms by:

$$\begin{array}{ccccc}
\begin{tikzcd} a \arrow[r] \arrow[d, "f"'] & c \arrow[d, "h"'] & b \arrow[l] \arrow[d, "g"] \\ a' \arrow[r] & c' & b' \arrow[l] \end{tikzcd} & \begin{array}{c} d \in F(c) \\ \\ d' \in F(c') \end{array} & \mapsto & \begin{tikzcd} H(a) \arrow[r] \arrow[d, "H(f)"'] & H(c) \arrow[d, "H(h)"'] & H(b) \arrow[l] \arrow[d, "H(g)"] \\ H(a') \arrow[r] & H(c') & H(b') \arrow[l] \end{tikzcd} & \begin{array}{c} \theta_c E(d)\phi \in F'(H(c)) \\ \\ \theta_{c'} E(d')\phi \in F'(H(c')) \end{array} \\
\iota\colon F(h)(d) \to d' & & & E(\iota)\colon F'(H(h))(\theta_c E(d)\phi) \to (\theta_{c'} E(d')\phi) &
\end{array}$$

where $E\colon \mathbf{Cat} \to \mathbf{Cat}$ is a symmetric lax monoidal pseudofuctor such that the following diagram commutes up to isomorphism $\theta\colon EF \Rightarrow F'H$:

$$\begin{tikzcd} \mathsf{A} \arrow[r, "F"] \arrow[d, "H"'] \arrow[dr, phantom, "\swarrow\theta"] & \mathbf{Cat} \arrow[d, "E"] \\ \mathsf{A}' \arrow[r, "F'"'] & \mathbf{Cat} \end{tikzcd}$$

Theorem 2.7. *Given two finitely cocomplete categories* A *and* A'*, two symmetric lax monoidal pseudofunctors* $F\colon \mathsf{A} \to \mathbf{Cat}$ *and* $F'\colon \mathsf{A}' \to \mathbf{Cat}$*, a finite colimit preserving functor* $H\colon \mathsf{A} \to \mathsf{A}'$*, a symmetric lax monoidal pseudofunctor* $E\colon \mathbf{Cat} \to \mathbf{Cat}$ *and a 2-isomorphism* $\theta\colon EF \Rightarrow F'H$ *as in the following diagram, the triple* (H, E, θ) *induces a symmetric monoidal (strong) double functor* $\mathbb{H}\colon F\mathbb{C}\mathrm{sp} \to F'\mathbb{C}\mathrm{sp}$ *as defined above.*

$$\begin{tikzcd} \mathsf{A} \arrow[r, "F"] \arrow[d, "H"'] \arrow[dr, phantom, "\swarrow\theta"] & \mathbf{Cat} \arrow[d, "E"] \\ \mathsf{A}' \arrow[r, "F'"'] & \mathbf{Cat} \end{tikzcd}$$

Details of the constructions and the proof of the above theorem can be found at [13].

3. An equivalence of symmetric monoidal double categories

In [2], the first two authors introduces the symmetric monoidal double category of *structured cospans* as a formalism to capture open networks. One of the main goals of this paper is to provide a monoidal double equivalence between this double category and that of *decorated cospans*, described in detail in Section 2. Below we recall the double category structured cospans form.

Wait, why do you need to consider them as monoidal in the assumptions of the theorem?

Theorem 3.1. *Given a category* X *with finite colimits and a category* A *with finite coproducts and a finite coproduct preserving functor* $L\colon \mathsf{A} \to \mathsf{X}$ *with* A *and* X *regarded as cocartesian monoidal categories, there exists a symmetric monoidal double category* ${}_L\mathbb{C}\mathrm{sp}(\mathsf{X})$ *which has:*

- *objects given by objects of* A*,*
- *vertical 1-morphisms given by morphisms of* A*,*
- *horizontal 1-cells* $a \nrightarrow b$ *given by cospans of* X *of the form:*

$$L(a) \longrightarrow x \longleftarrow L(b) \tag{2}$$

- *2-morphisms given by maps of cospans of* X *of the form:*

$$\begin{tikzcd} L(a) \arrow[r] \arrow[d, "L(f)"'] & x \arrow[d, "\alpha"'] & L(b) \arrow[l] \arrow[d, "L(g)"] \\ L(a') \arrow[r] & x' & L(b') \arrow[l] \end{tikzcd} \tag{3}$$

Composition of horizontal 1-cells and 2-morphisms is given by pushouts in X *and tensoring of objects is under binary coproducts in* A*.*

Proof. See the first two authors' work on structured cospans [2]. □

The desired equivalence between these two different double categorical network frameworks, namely structured cospans and decorated cospans, is established by the following theorem. We denote by **fcocCat** the 2-category of finitely cocomplete categories and finitely cocontinuous functors.

Theorem 3.2. *Suppose* A *is a finitely cocomplete category and* $(F, \phi, \phi_0)\colon (\mathsf{A}, +, 0) \to (\mathbf{Cat}, \times, \mathbf{1})$ *is a symmetric lax monoidal pseudofunctor, which factors through* **fcocCat** *as an ordinary pseudofunctor. Then the symmetric monoidal double category* $F\mathbb{C}\mathrm{sp}$ *of decorated cospans (Theorem 2.1) is equivalent to the symmetric monoidal double category* ${}_{L_F}\mathbb{C}\mathrm{sp}\left(\int F\right)$ *of structured cospans (Theorem 3.1), where* $L_F\colon \mathsf{A} \to \int F$ *is the left adjoint of the induced Grothendieck opfibration* $U_F\colon \int F \to \mathsf{A}$*.*

The conditions of the above theorem relate to the existence of left adjoints for opfibrations; we first sketch this underlying framework, and then we proceed to the proof of the theorem.

The basics of fibration theory needed for our purposes are recalled in Section 5.2. In [23], the classic Grothendieck construction is generalized to the monoidal setting: lax monoidal structures on a pseudofunctor $F\colon \mathsf{A} \to \mathbf{Cat}$ bijectively correspond to monoidal structures on the total category $\int F$ such that the induced opfibration is a strict monoidal functor and $\otimes_{\int F}$ preserves liftings, see [23, Thm. 3.10]. If the monoidal category A is in fact cocartesian, there is a further correspondence of the two structures with an ordinary pseudofunctor $F\colon \mathsf{A} \to \mathbf{MonCat}$ into the 2-category of monoidal categories, strong monoidal functors and monoidal natural transformations, summarized as

$$\begin{gathered}\text{lax monoidal pseudofunctors } F\colon (\mathsf{A},+,0)\to(\mathbf{Cat},\times,1)\\ \Updownarrow\\ \text{monoidal opfibrations } U\colon (\mathsf{X},\otimes,I)\to(\mathsf{A},+,0)\\ \Updownarrow\\ \text{pseudofunctors } F\colon \mathsf{A}\to\mathbf{MonCat}\end{gathered}\tag{4}$$

The second equivalence was in fact earlier observed by Shulman in [25]. In more detail, if (ϕ,ϕ_0) is the lax monoidal structure of F, each fibre category $\mathsf{X}_a = F(a)$ obtains a monoidal structure via

$$\begin{gathered}\otimes_a\colon F(a)\times F(a)\xrightarrow{\phi_{a,a}} F(a+a)\xrightarrow{F(\nabla)} F(a)\\ I_x\colon \mathbf{1}\xrightarrow{\phi_0} F(0)\xrightarrow{F(!)} F(a)\end{gathered}\tag{5}$$

Moreover, these correspondences further restrict to the case when the Grothendieck category is specifically cocartesian monoidal itself; for more details, see [23, Cor. 4.8]. In that case, the monoidal opfibration clauses for $U\colon(\mathsf{X},+,0)\to(\mathsf{A},+,0)$ translate to the functor (strictly) preserving coproducts and the initial object, and bijectively correspond to pseudofunctors $F\colon\mathsf{A}\to\mathbf{cocartCat}$ into the 2-category of cocartesian categories, functors that preserve coproducts and initial objects and natural transformations.

Finally, using the following result we can further restrict to our current case of interest, namely opfibrations that also preserve pushouts, i.e. all finite colimits. Of course, the following statement is more general since it relates the existence of any class of colimits in the total category of an opfibration to their existence in the fibres; for a more detailed exposition and proof, see [19].

Lemma 3.3. *Suppose J is a small category and $U\colon\mathsf{X}\to\mathsf{A}$ is an opfibration. If the base A has J-colimits, the following are equivalent:*

(1) all fibres have J-colimits, and the reindexing functors preserve them;
(2) the total X has J-colimits, and U preserves them.

The first part formulates the existence of colimits *locally* in each fibre, which can equivalently be expressed by the corresponding pseudofunctor $F\colon\mathsf{A}\to\mathbf{Cat}$ landing on the sub-2-category **fcocCat** of finitely cocomplete categories and functors that preserve colimits. The second part formulates the existence of colimits *globally* in the total category $\int F$. Combining these, we can deduce the following.

Corollary 3.4. *Suppose A is a finitely cocomplete category and $F\colon(\mathsf{A},+,0)\to(\mathbf{Cat},\times,1)$ is a lax monoidal pseudofunctor. If its corresponding pseudofunctor $\mathsf{A}\to\mathbf{MonCat}$ via (4) factors through* **fcocCat***, then the Grothendieck category $\int F$ has all finite colimits and the induced opfibration $U_F\colon\int F\to\mathsf{A}$ preserves them.*

On a different but related subject, for our purposes we are interested in the existence of a left adjoint L_F to the induced monoidal opfibration $\int F\to\mathsf{A}$ by the Grothendieck construction as discussed above. The following result provides sufficient conditions for that, which go hand-in-hand with Corollary 3.4 and Lemma 3.3.

Proposition 3.5. [18, Prop. 4.4] *Let $U\colon\mathsf{X}\to\mathsf{A}$ be an opfibration. Then U is a right adjoint 'left inverse', a.k.a the unit is the identity, if (and only if) its fibers have initial objects which are preserved by the reindexing functors.*

Proof. (Sketch) The left adjoint $L\colon \mathsf{A} \to \mathsf{X}$ maps an object a to the initial object in its fibre X_a, denoted by $\perp_a$. By construction, $U(L(a)) = U(\perp_a) = a$. On morphisms $f\colon a \to a'$, $L(f)$ is defined by

$$\perp_a \xrightarrow{\mathrm{Cocart}(f,\perp_a)} f_!(\perp_a) \xrightarrow[\cong]{\chi_a} \perp_{a'} \tag{6}$$

where χ_a is the unique isomorphism between the initial objects in the fibre above a' since $f_!$ preserves them. ◻

Remark 3.6. Notice that under Lemma 3.3, if A has an initial object 0_A, the above conditions are equivalent to X having an initial object 0_X above 0_A. In this case, $\perp_a$ is precisely the cocartesian lifting of 0_X along the unique map $u_a\colon 0_\mathsf{A} \to a$ in the base category:

$$\begin{tikzcd}
0_\mathsf{X} \arrow[r, "\mathrm{Cocart}(0_\mathsf{X},u_a)"] \arrow[d, dotted] & (u_a)_!(0_X) =: \perp_a \arrow[d, dotted] & \text{in } \mathsf{X} \\
0_\mathsf{A} \arrow[r, "\exists! u_a"] & a & \text{in } \mathsf{A}
\end{tikzcd}$$

Moreover, if $U = U_F$ for a pseudofunctor $F\colon \mathsf{A} \to \mathbf{Cat}$ under Theorem 5.2, the reindexing functors $(u_a)_!$ of the opfibration are precisely $F(u_a)$, therefore $\perp_a = (a, F(u_a)(0_\mathsf{X}))$ in the Grothendieck category, for u_a the unique map from 0_A as above.

Finally, if that pseudofunctor is lax monoidal $(F, \phi, \phi_0)\colon (\mathsf{A}, +, 0) \to (\mathbf{Cat}, \times, \mathbf{1})$ to begin with, the monoidal Grothendieck construction in the cocartesian case discussed earlier (5) expresses $\perp_a$ as the image of the composite

$$\mathbf{1} \xrightarrow{\phi_0} F(0_\mathsf{A}) \xrightarrow{F(u_a)} F(a) \tag{7}$$

not sure if the below is needed after all In the opposite direction, we also have the following result. For a discussion on the unusual strict cocontinuity condition, we refer the reader to [11].

Proposition 3.7. [11, Prop. 3.3] *Suppose $U\colon \mathsf{X} \to \mathsf{A}$ is a right adjoint 'left inverse'. If X and A have chosen pushouts and initial objects and U strictly preserves them, then U is an opfibration.*

We have now laid all the necessary background to formally construct an equivalence between the double category of decorated cospans and the double category of structured cospans, constructed from a symmetric lax monoidal pseudofunctor into **Cat**.

Proof of Theorem 3.2. Recall that the double category of decorated cospans $F\mathbb{C}\mathrm{sp}$ has objects and vertical 1-morphisms those of A, horizontal 1-cells $a \nrightarrow b$ are cospans $a \longrightarrow c \longleftarrow b$ in A along with $x \in F(c)$, and 2-morphisms are cospan maps $k\colon c \to c'$ together with a morphism $(Fk)(x) \to x'$, see (1).

By Corollary 3.4, when F as an ordinary pseudofunctor factors through **fcocCat**, the (monoidal) Grothendieck construction gives rise to a finitely cocomplete $\int F$ such that the corresponding opfibration $U_F\colon (\int F, +, 0) \to (\mathsf{A}, +, 0)$ preserves all finite colimits. Since in particular it preserves the initial object, Proposition 3.5 through Lemma 3.3 applies to construct a left adjoint $L_F\colon \mathsf{A} \to \int F$ which is also a right inverse, namely $U_F L_F = \mathrm{id}_\mathsf{A}$. Explicitly, this left adjoint is given by $L(a) = (a, \perp_a)$, picking the initial object in the finitely cocomplete fibre also expressed as $F(u_a) \circ \phi_0(*)$ in (7). Diagrammatically,

$$F\colon \mathsf{X} \to \mathbf{Cat} \quad \mapsto \quad \begin{tikzcd} \int F \arrow[d, "U_F"'] \\ \mathsf{X} \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \mathsf{X} \arrow[r, bend left, "L_F"] \arrow[r, phantom, "\perp"] & \int F \arrow[l, bend left, "U_F"] \end{tikzcd}$$

roughly describes the processes between the original F and the resulting L_F.

Using this functor L_F between finitely cocomplete categories that preserves all colimits that exist (as a left adjoint), we can construct the double category of structured cospans ${}_{L_F}\mathbb{C}\mathrm{sp}\left(\int F\right)$. Its objects and vertical morphisms are again those of the category A, whereas horizontal 1-cells $a \nrightarrow b$ as in (2) are now cospans of the form $L_F(a) \longrightarrow v \longleftarrow L_F(b)$ in the Grothendieck category $\int F$. Explicitly, they consist of two pairs of morphisms

$$(a, \perp_a) \xrightarrow{\begin{cases} i\colon a \to c & \text{in } \mathsf{A} \\ !\colon F(i)(\perp_a) \to x & \text{in } F(c) \end{cases}} (c, x) \xleftarrow{\begin{cases} o\colon b \to c & \text{in } \mathsf{A} \\ !\colon F(o)(\perp_b) \to x & \text{in } F(c) \end{cases}} (b, \perp_b) \tag{8}$$

where $x \in F(c)$, according to Definition 5.1. Finally, the 2-morphisms in this double category are as described in (3), in this context fully unravelled below:

$$\begin{tikzcd}
(a,\perp_a) \arrow[rr, "\begin{cases} i\colon a\to c & \text{in } \mathsf{A} \\ !\colon F(i)(\perp_a)\to x & \text{in } F(c) \end{cases}"] \arrow[d, "\begin{cases} f\colon a\to a' & \text{in } \mathsf{A} \\ \chi_a\colon F(f)(\perp_a)\cong\perp_{a'} & \text{in } F(a') \end{cases}"'] && (c,x) \arrow[d, "\begin{cases} k\colon c\to c' & \text{in } \mathsf{A} \\ \iota\colon F(k)(x)\to x' & \text{in } F(c') \end{cases}"] && (b,\perp_b) \arrow[ll, "\begin{cases} o\colon b\to c & \text{in } \mathsf{A} \\ !\colon F(o)(\perp_b)\to x & \text{in } F(c) \end{cases}"'] \arrow[d, "\begin{cases} g\colon b\to b' & \text{in } \mathsf{A} \\ \chi_b\colon F(g)(\perp_b)\cong\perp_{b'} & \text{in } F(b') \end{cases}"] \\
(a',\perp_{a'}) \arrow[rr, "\begin{cases} i'\colon a'\to c' & \text{in } \mathsf{A} \\ !\colon F(i')(\perp_{a'})\to x' & \text{in } F(c') \end{cases}"'] && (c',x') && (b',\perp_{b'}) \arrow[ll, "\begin{cases} o'\colon b'\to c' & \text{in } \mathsf{A} \\ !\colon F(o')(\perp_{b'})\to x' & \text{in } F(c') \end{cases}"]
\end{tikzcd}$$

where the outside vertical legs come from the definition of L_F on arrows, (6). The above diagram might be a bit overwhelming, but all info is there. Choose which parts are required for a smooth proof! Also, iota and the letter i look alarmingly similar. Should we change iota to some other distinctive greek letter? The square commutativities translate to $k \circ i = i' \circ f$ and $k \circ o = o' \circ g$ in A, and also by composition in the Grothendieck category, to

$$\begin{aligned} F(k \circ i)(\perp_a) \cong Fk(Fi(\perp_a)) \xrightarrow{Fk(!)} Fk(x) \xrightarrow{\iota} x' &= \\ F(i' \circ f)(\perp_a) \cong Fi'(Ff(\perp_a)) \xrightarrow{Fi'(\chi_a)} Fi'(\perp_{a'}) \xrightarrow{!} x' & \end{aligned} \tag{9}$$

in the category $F(c')$. However, since all maps are unique in the above equality, emanating from initial objects (since reindexing functors preserve those), this gives no extra conditions for the morphisms involved; similarly for the equality including o, o'.

In order to prove that there is a double equivalence (Theorem 5.10) between $F\mathbb{C}\mathrm{sp}$ and ${}_{L_F}\mathbb{C}\mathrm{sp}\left(\int F\right)$, we define a double functor

$$\mathbb{E}\colon {}_{L_F}\mathbb{C}\mathrm{sp}\left(\textstyle\int F\right) \longrightarrow F\mathbb{C}\mathrm{sp}$$

as follows. The object component of $\mathbb{E} = (\mathbb{E}_0, \mathbb{E}_1)$ is $\mathbb{E}_0 = \mathrm{id}_{\mathsf{A}}$ since both double categories have A as their vertical category; trivially, $\mathbb{E}_0$ is an equivalence of categories. Given a horizontal 1-cell in ${}_{L_F}\mathbb{C}\mathrm{sp}\left(\int F\right)$ like (8), the arrow component $\mathbb{E}_1$ of the double functor simply maps it to the decorated cospan

$$a \xrightarrow{i} c \xleftarrow{o} b \;\text{ with decoration } x \in F(c)$$

Notice that this really is a bijective correspondence since the unique maps from the initial objects in the fibres provide no extra information. Finally, given a 2-morphism of L_F-structured cospans as described above, $\mathbb{E}_1$ maps it to the underlying cospan 2-morphism

$$\begin{tikzcd}
a \arrow[r, "i"] \arrow[d, "f"'] & c \arrow[d, "k"] & b \arrow[l, "o"'] \arrow[d, "g"] \\
a' \arrow[r, "i'"'] & c' & b' \arrow[l, "o'"]
\end{tikzcd}$$

in A along with the map $\iota\colon Fk(x) \to x'$ in $F(c')$, namely a decorated cospan 2-morphism as per (1). Once again, this is a bijective correspondence since, as discussed above, the commutativity (9) holds automatically by initiality of the domain. Christina commented out big chunks of proof. They are right below if we want to revive parts of them, if we think the reader needs deeper level of detail.

Functoriality of $\mathbb{E}_1$ can be verified, and moreover $(\mathbb{E}_0, \mathbb{E}_1)$ indeed define a strong double functor: there exist natural isomorphisms

$$\mathbb{E}(M) \odot \mathbb{E}(N) \xrightarrow{\sim} \mathbb{E}(M \odot N)$$
$$U_{\mathbb{E}(c,x)} \xrightarrow{\sim} \mathbb{E}(U_{(c,x)})$$

for any pair of composable horizontal 1-cells M and N and any object (c, x) of ${}_{L_F}\mathbb{C}\mathrm{sp}\left(\int F\right)$ in progress

For any object c, the horizontal 1-cell $\hat{U}_{\mathbb{E}(c)}$ is given by $\hat{U}_c$ which is given by the pair:

$$c \xrightarrow{1} c \xleftarrow{1} c \qquad !_c \in F(c)$$

The horizontal 1-cell U_c is given by

$$L(c) \xrightarrow{1} L(c) \xleftarrow{1} L(c)$$

and so $\mathbb{E}(U_c)$ is given by the pair:

$$c \xrightarrow{\eta_c} R(L(c)) \xleftarrow{\eta_c} c \qquad !_c \in F(R(L(c)))$$

Then we can obtain the natural isomorphism $\mathbb{E}_c\colon \hat{U}_{\mathbb{E}(c)} \xrightarrow{\sim} \mathbb{E}(U_c)$ as the 2-morphism

$$\begin{tikzcd}
c \arrow[r, "1"] \arrow[d, "1"'] & c \arrow[d, "\eta_c"'] & c \arrow[l, "1"'] \arrow[d, "1"] & !_c \in F(c) \\
c \arrow[r, "\eta_c"] & R(L(c)) & c \arrow[l, "\eta_c"'] & !_{R(L(c))} \in F(R(L(c)))
\end{tikzcd}$$

$$\iota\colon F(\eta_c)(!_c) \xrightarrow{!} !_{R(L(c))}$$

of $F\mathbb{C}\mathrm{sp}$.

Next, given composable horizontal 1-cells M and N in ${}_L\mathbb{C}\mathrm{sp}(\mathsf{X})$:

$$L(c_1) \xrightarrow{i} x \xleftarrow{o} L(c_2) \qquad L(c_2) \xrightarrow{i'} x' \xleftarrow{o'} L(c_3)$$

their images $\mathbb{E}(M)$ and $\mathbb{E}(N)$ are given by:

$$c_1 \xrightarrow{R(i)\eta_{c_1}} R(x) \xleftarrow{R(o)\eta_{c_2}} c_2 \qquad c_2 \xrightarrow{R(i')\eta_{c_2}} R(x') \xleftarrow{R(o')\eta_{c_3}} c_3$$

$$x \in F(R(x)) \qquad x' \in F(R(x'))$$

and so $\mathbb{E}(M) \odot \mathbb{E}(N)$ is given by:

$$c_1 \xrightarrow{j\psi R(i)\eta_{c_1}} R(x) +_{c_2} R(x') \xleftarrow{j\psi R(o')\eta_{c_3}} c_3$$

$$d_{\mathbb{E}(M)\odot\mathbb{E}(N)} = \hat{x} \in F(R(x) +_{c_2} R(x'))$$

$$\hat{x}\colon 1 \xrightarrow{\lambda^{-1}} 1\times 1 \xrightarrow{x\times x'} F(R(x))\times F(R(x')) \xrightarrow{\phi_{R(x),R(x')}} F(R(x)+R(x')) \xrightarrow{F(j_{R(x),R(x')})} F(R(x)+_{c_2}R(x'))$$

where ψ denotes each natural map into the coproduct and j denotes the natural map from the coproduct to the pushout. On the other hand, $M \odot N$ is given by

$$L(c_1) \xrightarrow{J\zeta i} x +_{L(c_2)} x' \xleftarrow{J\zeta o'} L(c_3)$$

where ζ is a natural map into a coproduct and J is the natural map from the coproduct to the pushout. Then $E(M \odot N)$ is given by

$$c_1 \xrightarrow{R(J\zeta i)\eta_{c_1}} R(x +_{L(c_2)} x') \xleftarrow{R(J\zeta o')\eta_{c_3}} c_3$$

$$d_{\mathbb{E}(M\odot N)} = x +_{L(c_2)} x' \in F(R(x +_{L(c_2)} x'))$$

and so $\mathbb{E}_{M,N}\colon \mathbb{E}(M) \odot \mathbb{E}(N) \xrightarrow{\sim} \mathbb{E}(M \odot N)$ is given by the 2-morphism:

$$\begin{tikzcd} c_1 \arrow[r, "j\psi R(i)\eta_{c_1}"] \arrow[d, "1"'] & R(x) +_{c_2} R(x') \arrow[d, "\sigma"] & c_3 \arrow[l, "j\psi R(o')\eta_{c_3}"'] \arrow[d, "1"] \\ c_1 \arrow[r, "R(J\zeta i)\eta_{c_1}"'] & R(x +_{L(c_2)} x') & c_3 \arrow[l, "R(J\zeta o')\eta_{c_3}"] \end{tikzcd} \qquad \begin{matrix} d_{\mathbb{E}(M)\odot\mathbb{E}(N)} \in F(R(x) +_{c_2} R(x')) \\ \\ d_{\mathbb{E}(M\odot N)} \in F(R(x +_{L(c_2)} x')) \end{matrix}$$

together with a morphism of decorations which is obtained as follows. First, as $R\colon \mathsf{X} \to \mathsf{A}$ preserves finite colimits, we have an isomorphism

$$\kappa\colon R(x) +_{R(L(c_2))} R(x') \to R(x +_{L(c_2)} x').$$

Also, since the left adjoint $L\colon \mathsf{A} \to \mathsf{X}$ is fully faithful, the unit of the adjunction $L \dashv R$ at the object c_2 gives an isomorphism $\eta_{c_2}\colon c_2 \to R(L(c_2))$ which results in an isomorphism

$$j_{\eta_{c_2}}\colon R(x) +_{c_2} R(x') \to R(x) +_{R(L(c_2))} R(x').$$

Composing these two results in an isomorphism

$$\sigma \coloneqq \kappa j_{\eta_{c_2}}\colon R(x) +_{c_2} R(x') \to R(x +_{L(c_2)} x').$$

Next, to see that the above diagram commutes, it suffices to show that for the object $c_1 \in \mathsf{A}$,

$$R(J)R(\zeta)R(i)\eta_{c_1}(c_1) = R(J\zeta i)\eta_{c_1}(c_1) \stackrel{!}{=} \sigma j\psi R(i)\eta_{c_1}(c_1) = \kappa j_{\eta_{c_2}}\psi R(i)\eta_{c_1}(c_1).$$

This follows as $R(i)\eta_{c_1}\colon c_1 \to R(x)$ and the following diagram commutes:

$$\begin{tikzcd}
R(x) \arrow[r, "\psi"] \arrow[dd, "R(\zeta)"'] & R(x)+R(x') \arrow[r, "j"] & R(x)+_{c_2}R(x') \arrow[d, "j_{\eta_{c_2}}"] \arrow[dd, bend left, "\sigma"] \\
 & & R(x)+_{R(L(c_2))}R(x') \arrow[d, "\kappa"] \\
R(x+x') \arrow[rr, "R(J)"] & & R(x+_{L(c_2)}x')
\end{tikzcd}$$

Lastly, this map of cospans comes with an isomorphism

$$\iota\colon F(\sigma)(d_{\mathbb{E}(M)\odot\mathbb{E}(N)}) \to d_{\mathbb{E}(M\odot N)}$$

in $F(R(x+_{L(c_2)}x'))$. This shows that $\mathbb{E}$ is strong, and so

Up to here

Based on the above description of its mapping, it follows that the strong double functor $\mathbb{E} = (\mathbb{E}_0, \mathbb{E}_1)$ is full, faithful and essentially surjective on objects as per Definitions 5.8 and 5.9. Therefore $\mathbb{E}\colon {}_{L_F}\mathbb{C}\mathrm{sp}\left(\int F\right) \to F\mathbb{C}\mathrm{sp}$ is a double equivalence, by Theorem 5.10. In fact, Christina and Kenny believe it is an isomorphism of double categories! Bijective on all levels of objects and morphisms...should we say?

Next, we show that if both double categories ${}_L\mathbb{C}\mathrm{sp}(\mathsf{X})$ and $F\mathbb{C}\mathrm{sp}$ are symmetric monoidal, as they are if both A and X have finite colimits, then this equivalence of double categories $\mathbb{E}\colon {}_L\mathbb{C}\mathrm{sp}(\mathsf{X}) \to F\mathbb{C}\mathrm{sp}$ will be symmetric monoidal. First, note that we have an isomorphism $\epsilon\colon 1_{F\mathbb{C}\mathrm{sp}} \to \mathbb{E}(1_{{}_L\mathbb{C}\mathrm{sp}(\mathsf{X})})$ and natural isomorphisms $\mu_{c_1,c_2}\colon \mathbb{E}(c_1)\otimes\mathbb{E}(c_2) \to \mathbb{E}(c_1\otimes c_2)$ for every pair of objects $c_1, c_2 \in {}_L\mathbb{C}\mathrm{sp}(\mathsf{X})$ both of which are given by identities since both double categories ${}_L\mathbb{C}\mathrm{sp}(\mathsf{X})$ and $F\mathbb{C}\mathrm{sp}$ have A as their category of objects and $\mathbb{E}_0 = \mathrm{id}_{\mathsf{A}}$. The diagrams utilizing these maps that are required to commute do so trivially.

For the arrow component $\mathbb{E}_1$, we have an isomorphism $\delta\colon U_{1_{F\mathbb{C}\mathrm{sp}}} \to \mathbb{E}(U_{1_{{}_L\mathbb{C}\mathrm{sp}(\mathsf{X})}})$ where the horizontal 1-cell $U_{1_{F\mathbb{C}\mathrm{sp}}}$ is given by:

$$1_{\mathsf{A}} \xrightarrow{1} 1_{\mathsf{A}} \xleftarrow{1} 1_{\mathsf{A}} \qquad !_{1_{\mathsf{A}}} \in F(1_{\mathsf{A}})$$

where $!_{1_{\mathsf{A}}} = \phi\colon 1 \to F(1_{\mathsf{A}})$ is the trivial decoration which comes from the structure of the symmetric lax monoidal pseudofunctor $F\colon \mathsf{A} \to \mathbf{Cat}$. The horizontal 1-cell $U_{1_{{}_L\mathbb{C}\mathrm{sp}(\mathsf{X})}}$ is given by:

$$L(1_{\mathsf{A}}) \xrightarrow{1} L(1_{\mathsf{A}}) \xleftarrow{1} L(1_{\mathsf{A}})$$

where here we make use of the fact that the left adjoint $L\colon (\mathsf{A}, +, 1_{\mathsf{A}}) \to (\mathsf{X}, +, 1_{\mathsf{X}})$ preserves all colimits and thus $L(1_{\mathsf{A}}) \cong 1_{\mathsf{X}}$. The horizontal 1-cell $\mathbb{E}(U_{1_{{}_L\mathbb{C}\mathrm{sp}(\mathsf{X})}})$ is then given by the pair:

$$1_{\mathsf{A}} \xrightarrow{\eta_{1_{\mathsf{A}}}} R(L(1_{\mathsf{A}})) \xleftarrow{\eta_{1_{\mathsf{A}}}} 1_{\mathsf{A}} \qquad !_{R(L(1_{\mathsf{A}}))} \in F(R(L(1_{\mathsf{A}})))$$

The isomorphism δ is then given by the 2-morphism:

$$\begin{tikzcd} 1_{\mathsf{A}} \arrow[r, "1"] \arrow[d, "1"'] & 1_{\mathsf{A}} \arrow[d, "\eta_{1_{\mathsf{A}}}"'] & 1_{\mathsf{A}} \arrow[l, "1"'] \arrow[d, "1"'] & !_{1_{\mathsf{A}}} \in F(1_{\mathsf{A}}) \\ 1_{\mathsf{A}} \arrow[r, "\eta_{1_{\mathsf{A}}}"] & R(L(1_{\mathsf{A}})) & 1_{\mathsf{A}} \arrow[l, "\eta_{1_{\mathsf{A}}}"'] & !_{R(L(1_{\mathsf{A}}))} \in F(R(L(1_{\mathsf{A}}))) \end{tikzcd}$$

$$\iota_{\eta_{1_{\mathsf{A}}}} \colon F(\eta_{1_{\mathsf{A}}})(!_{1_{\mathsf{A}}})) \to !_{R(L(1_{\mathsf{A}}))}$$

of $F\mathbb{C}\mathrm{sp}$. This is just the natural isomorphism $\mathbb{E}_{1_{\mathsf{A}}}$ from earlier.

Given two horizontal 1-cells M and N of ${}_L\mathbb{C}\mathrm{sp}(\mathsf{X})$:

$$L(c_1) \xrightarrow{i} x \xleftarrow{o} L(c_2) \qquad L(c_1') \xrightarrow{i'} x' \xleftarrow{o'} L(c_2')$$

their images $\mathbb{E}(M)$ and $\mathbb{E}(N)$ are given by:

$$c_1 \xrightarrow{R(i)\eta_{c_1}} R(x) \xleftarrow{R(o)\eta_{c_2}} c_2 \qquad c_1' \xrightarrow{R(i')\eta_{c_1'}} R(x') \xleftarrow{R(o')\eta_{c_2'}} c_2'$$

$$x \in F(R(x)) \qquad x' \in F(R(x'))$$

and so $\mathbb{E}(M) \otimes \mathbb{E}(N)$ is given by:

$$c_1 + c_1' \xrightarrow{R(i)\eta_{c_1} + R(i')\eta_{c_1'}} R(x) + R(x') \xleftarrow{R(o)\eta_{c_2} + R(o)\eta_{c_2'}} c_2 + c_2'$$

$$d_{\mathbb{E}(M)\otimes\mathbb{E}(N)} \in F(R(x) + R(x'))$$

where

$$d_{\mathbb{E}(M)\otimes\mathbb{E}(N)} \colon 1 \xrightarrow{\lambda^{-1}} 1 \times 1 \xrightarrow{x \times x'} F(R(x)) \times F(R(x')) \xrightarrow{\phi_{R(x),R(x')}} F(R(x) + R(x')).$$

On the other hand, $M \otimes N$ is given by

$$L(c_1 + c_1') \xrightarrow{(i+i')\phi^{-1}_{c_1,c_1'}} x + x' \xleftarrow{(o+o')\phi^{-1}_{c_2,c_2'}} L(c_2 + c_2')$$

and $\mathbb{E}(M \otimes N)$ is given by:

$$c_1 + c_1' \xrightarrow{R((i+i')\phi^{-1}_{c_1,c_1'})\eta_{c_1+c_1'}} R(x + x') \xleftarrow{R((o+o')\phi^{-1}_{c_2,c_2'})\eta_{c_2+c_2'}} c_2 + c_2'$$

$$d_{\mathbb{E}(M \odot N)} \in F(R(x + x')).$$

We then have a 2-isomorphism $\mu_{M,N} \colon E(M) \otimes E(N) \xrightarrow{\sim} E(M \otimes N)$ in $F\mathbb{C}\mathrm{sp}$ given by:

$$\begin{tikzcd} c_1 + c_1' \arrow[r, "R(i)\eta_{c_1} + R(i')\eta_{c_1'}"] \arrow[d, "1"'] & R(x) + R(x') \arrow[d, "\kappa"'] & c_2 + c_2' \arrow[l, "R(o)\eta_{c_2} + R(o')\eta_{c_2'}"'] \arrow[d, "1"] & d_{\mathbb{E}(M)\otimes\mathbb{E}(N)} \in F(R(x) + R(x')) \\ c_1 + c_1' \arrow[r, "R((i+i')\phi^{-1}_{c_1,c_1'})\eta_{c_1+c_1'}"] & R(x + x') & c_2 + c_2' \arrow[l, "R((o+o')\phi^{-1}_{c_2,c_2'})\eta_{c_2+c_2'}"'] & d_{\mathbb{E}(M\otimes N)} \in F(R(x + x')) \end{tikzcd}$$

$$\iota_\mu \colon F(\kappa)(d_{\mathbb{E}(M)\odot\mathbb{E}(N)}) \to d_{\mathbb{E}(M\odot N)}$$

where κ is the isomorphism which comes from $R \colon \mathsf{X} \to \mathsf{A}$ preserving finite colimits.

The isomorphisms δ and μ satisfy the left and right unitality squares, associativity hexagon and braiding square. **Kenny: I think this might be a good place to cut the monoidal stuff.**Deal! They are right below.

□

Using a result of Shulman [24], each of the isofibrant symmetric monoidal double categories $F\mathbb{C}\mathrm{sp}$ and ${}_L\mathbb{C}\mathrm{sp}(\mathsf{X})$ give rise to underlying symmetric monoidal bicategories, namely $F\mathbb{C}\mathrm{sp}$ induces a symmetric monoidal bicategory $F\mathbf{Csp} := H(F\mathbb{C}\mathrm{sp})$ which has:

(1) objects as those of A,
(2) morphisms as horizontal 1-cells of $F\mathbb{C}\mathrm{sp}$, and
(3) 2-morphisms as globular 2-morphisms of $F\mathbb{C}\mathrm{sp}$.

Likewise, ${}_L\mathbb{C}\mathrm{sp}(\mathsf{X})$ induces a symmetric monoidal bicategory $H({}_L\mathbb{C}\mathrm{sp}(\mathsf{X}))$ which has:

(1) objects as those of A,
(2) morphisms as horizontal 1-cells of ${}_L\mathbb{C}\mathrm{sp}(\mathsf{X})$, and
(3) 2-morphisms as globular 2-morphisms of ${}_L\mathbb{C}\mathrm{sp}(\mathsf{X})$.

Another result of Shulman [25] is the following:

Proposition 3.8 (Shulman, Prop. B.3)**.** *An equivalence of fibrant double categories induces a biequivalence of horizontal bicategories.*

Corollary 3.9. *The bicategories $F\mathbf{Csp}$ and $H({}_L\mathbb{C}\mathrm{sp}(\mathsf{X}))$ are biequivalent.*

Is this biequivalnce symmetric monoidal? Mike says yes, but it follows from a result that he has yet to officially publish. It's a result of a paper he's currently working on with someone.

We can also define the part of the double equivalence $\mathbb{G}\colon F\mathbb{C}\mathrm{sp} \to {}_L\mathbb{C}\mathrm{sp}(\mathsf{X})$ which goes in the other direction: again, the object component of this double functor will be $\mathbb{G}_0 = \mathrm{id}_{\mathsf{A}}$.

Given a horizontal 1-cell M of $F\mathbb{C}\mathrm{sp}$:

$$c_1 \xrightarrow{I} c \xleftarrow{O} c_2$$
$$x \in F(c)$$

the image $\mathbb{G}(M)$ is the horizontal 1-cell in ${}_L\mathbb{C}\mathrm{sp}(\mathsf{X})$ given by:

$$L(c_1) \xrightarrow{!_c L(I)} x \xleftarrow{!_c L(O)} L(c_2)$$

where $!_c\colon L(c) \to x$ is the unique morphism from the trival decoration on c given by

$$!_c := 1 \xrightarrow{\phi} F(0) \xrightarrow{F(!)} F(c)$$

to $x \in F(c)$. In other words, the trivial decoraction $!_c$ is initial in $F(c)$. Similarly, given a 2-morphism $(f, h, g, \iota)\colon M \to M'$ of $F\mathbb{C}\mathrm{sp}$:

$$x \in F(c)$$

$$\begin{tikzcd} c_1 \arrow[r, "I"] \arrow[d, "f"'] & c \arrow[d, "h"'] & c_2 \arrow[l, "O"'] \arrow[d, "g"] \\ c_1' \arrow[r, "I'"] & c' & c_2' \arrow[l, "O'"'] \end{tikzcd}$$

$$x' \in F(c')$$

$$\iota\colon F(h)(x) \to x'$$

the image $\mathbb{G}(f, h, g, \iota)$ in ${}_L\mathbb{C}\mathrm{sp}(\mathsf{X})$ is given by the 2-morphism:

$$\begin{tikzcd} L(c_1) \arrow[r, "!_c L(I)"] \arrow[d, "L(f)"'] & x \arrow[d, "\alpha"'] & L(c_2) \arrow[l, "!_c L(O)"'] \arrow[d, "L(g)"] \\ L(c_1') \arrow[r, "!_{c'} L(I')"] & x' & L(c_2') \arrow[l, "!_{c'} L(O')"'] \end{tikzcd}$$

where $\alpha\colon x \to x'$ is the morphism in the Grothendieck construction of F given by $\alpha = (h\colon c \to c', \iota\colon F(h)(x) \to x')$.

Next, we exhibit natural isomorphisms $\eta\colon \mathrm{id}_{{}_L\mathbb{C}\mathrm{sp}(\mathsf{X})} \cong \mathbb{G}\mathbb{E}$ and $\epsilon\colon \mathbb{E}\mathbb{G} \cong \mathrm{id}_{F\mathbb{C}\mathrm{sp}}$. Specifically, these are double natural isomorphisms given by double transformations [25] η and ϵ whose object and arrow components are natural isomorphisms.

First we compute the composites $\mathbb{G}\mathbb{E}$ and $\mathbb{E}\mathbb{G}$. On the object categories, both composites are id_{A} and we have natural isomorphisms $\eta\colon \mathrm{id}_{\mathsf{A}} \cong \mathbb{G}_0\mathbb{E}_0$ and $\epsilon\colon \mathbb{E}_0\mathbb{G}_0 \cong \mathrm{id}_{\mathsf{A}}$.

Given a horizontal 1-cell M in ${}_L\mathbb{C}\mathrm{sp}(\mathsf{X})$:

$$L(c_1) \xrightarrow{i} x \xleftarrow{o} L(c_2)$$

the horizontal 1-cell $\mathbb{E}(M)$ is given by:

$$c_1 \xrightarrow{R(i)\eta_{c_1}} R(x) \xleftarrow{R(o)\eta_{c_2}} c_2$$

$$x \in F(R(x))$$

and then the horizontal 1-cell $\mathbb{G}\mathbb{E}(M)$ is given by:

$$L(c_1) \xrightarrow{!_{R(x)} L(R(i)\eta_{c_1})} x \xleftarrow{!_{R(x)} L(R(o)\eta_{c_2})} L(c_2)$$

Then we can find a 2-isomorphism $\eta_M\colon M \xrightarrow{\sim} \mathbb{G}\mathbb{E}(M)$ in ${}_L\mathbb{C}\mathrm{sp}(\mathsf{X})$ given by:

$$\begin{tikzcd} L(c_1) \arrow[r, "i"] \arrow[d, "1"'] & x \arrow[d, "1"'] & L(c_2) \arrow[l, "o"'] \arrow[d, "1"] \\ L(c_1) \arrow[r, "!_{R(x)} L(R(i)\eta_{c_1})"] & x & L(c_2) \arrow[l, "!_{R(x)} L(R(o)\eta_{c_2})"'] \end{tikzcd}$$

where

$$\begin{tikzcd} L(c_1) \arrow[r, "L(\eta_{c_1})"] \arrow[rrr, bend right, "i"'] & L(R(L(c_1))) \arrow[r, "L(R(i))"] & L(R(x)) \arrow[r, "!_{R(x)}"] & x \end{tikzcd}$$

commutes.

On the other hand, given a horizontal 1-cell N in $F\mathbb{C}\mathrm{sp}$:

$$c_1 \xrightarrow{I} c \xleftarrow{O} c_2$$

$$x \in F(c)$$

the horizontal 1-cell $\mathbb{G}(N)$ is given by:

$$L(c_1) \xrightarrow{!_c L(I)} x \xleftarrow{!_c L(O)} L(c_2)$$

and then the horizontal 1-cell $\mathbb{E}\mathbb{G}(N)$ is given by:

$$c_1 \xrightarrow{R(!_c L(I))\eta_{c_1}} R(x) \xleftarrow{R(!_c L(O))\eta_{c_2}} c_2$$

$$x \in F(R(x))$$

Then we have a 2-isomorphism $\epsilon_N \colon \mathbb{E}\mathbb{G}(N) \xrightarrow{\sim} N$ in $F\mathbb{C}\mathrm{sp}$ given by:

$$x \in F(R(x))$$

$$\begin{tikzcd} c_1 \arrow[r, "R(!_c L(I))\eta_{c_1}"] \arrow[d, "1"'] & R(x) \arrow[d, "\hat{e}"'] & c_2 \arrow[l, "R(!_c L(O))\eta_{c_2}"'] \arrow[d, "1"] \\ c_1 \arrow[r, "I"] & c & c_2 \arrow[l, "O"'] \end{tikzcd}$$

$$x \in F(c)$$

$$\iota \colon F(\hat{e})(x) \to x$$

where

$$\begin{tikzcd} c_1 \arrow[r, "\eta_{c_1}"] \arrow[rrrr, bend right, "I"'] & R(L(c_1)) \arrow[r, "R(L(I))"] & R(L(c)) \arrow[r, "R(!_c)"] & R(x) \arrow[r, "\hat{e}"] & c \end{tikzcd}$$

commutes.

4. Applications

5. Appendix

Here, I basically rearranged things to make them come in a certain order, and also added a few things for monoidal bicats, maybe more is needed or not. Didn't change Kenny's notation or writings almost at all

In this chapter, we gather some well-known concepts required to make the material self-contained, as well as references to more detailed expositions.

5.1. **2-categories.** For standard 2-categorical material, we refer the reader to e.g. [21] or [14] for monoidal structures.

Recall that a pseudofunctor $F\colon \mathbf{A} \to \mathbf{B}$ between bicategories $\mathbf{A}$ and $\mathbf{B}$ is functorial up to coherent natural isomorphism, namely for composable arrows we have $F(g \circ f) \cong Fg \circ Ff$ and $F(1_a) \cong 1_{Fa}$ satisfying standard axioms. Given pseudofunctors $F, G\colon \mathbf{A} \to \mathbf{B}$, a **pseudonatural transformation** σ consists of

- for each object $a \in \mathbf{A}$, a morphism $\sigma_a\colon F(a) \to G(a)$ in $\mathbf{B}$
- for each morphism $f\colon a \to b$ in $\mathbf{A}$, an invertible natural 2-morphism $\sigma_f\colon G(f)\sigma_a \xrightarrow{\sim} \sigma_b F(f)$ in $\mathbf{B}$ compatible with composition and identities.

We denote by $[\mathsf{A}, \mathbf{Cat}]_{\mathrm{ps}}$ the 2-category of pseudofunctors, pseudonatural transformations and modifications from an ordinary category A viewed as a 2-category with trivial 2-cells, into $\mathbf{Cat}$. This is also referred to as the 2-category of *opindexed categories*, since an indexed category is a contravariant pseudofunctor into $\mathbf{Cat}$.

A **lax monoidal** pseudofunctor between monoidal bicategories $F\colon \mathbf{A} \to \mathbf{B}$ is a pseudofunctor equipped with pseudonatural transformations with components $\mu_{a,b}\colon Fa \otimes Fb \to F(a \otimes b)$ and $\mu_0\colon I \to FI$ along with coherent invertible modifications for associativity and unitality. This is commonly also called 'weak monoida' pseudofunctor. A **symmetric lax monoidal** pseudofunctor between symmetric monoidal bicategories comes with invertible $F(b) \circ \mu_{a,b} \cong \mu_{b,a} \circ b$, where b is the braiding. Hiding various details under the rug - e.g. what a symmetric monoidal bicategory is. Should be fine? Also, not sure if we need monoidal transformations?

5.2. **Fibrations.** We recall some basics regarding opfibrations and their relation to pseudofunctors into $\mathbf{Cat}$ via the Grothendieck construction. Standard material on the theory of fibrations can be found for example in [19], [8].

A functor $U\colon \mathsf{X} \to \mathsf{A}$ is a **Grothendieck opfibration** if for every $x \in \mathsf{X}$ with $U(x) = a$ and $f\colon a \to b$ in A, there exists a **cocartesian lifting** of x along f, namely a morphism β in X with domain x above f with the following universal property: for any $g\colon b \to b'$ in A and $\gamma\colon x \to y'$ in X above the composite $g \circ f$, there exists a unique $\delta\colon y \to y'$ such that $U(\delta) = g$ and $\gamma = \delta \circ \beta$ as shown below

$$\begin{tikzcd}
x \arrow[r, "\beta"'] \arrow[rr, "\gamma"] \arrow[d, dotted, mapsto] & y \arrow[r, dashed, "\exists!\delta"'] \arrow[d, dotted, mapsto] & y' \arrow[d, dotted, mapsto] & \text{in } \mathsf{X} \\
a \arrow[r, "f=U\beta"'] \arrow[rr, "g\circ f=U\gamma"] & b \arrow[r, "g"'] & b' & \text{in } \mathsf{A}
\end{tikzcd}$$

The category X is called the **total** category and A is called the **base** category of the opfibration. For any $a \in \mathsf{A}$, the **fibre category** X_a consists of all objects that map to a and vertical morphisms between them, i.e. mapping to 1_a. Assuming the axiom of choice, we may select a cocartesian arrow over each $f\colon a \to b$ in A and $x \in \mathsf{X}_a$, denoted by $\mathrm{Cocart}(f, x)\colon x \to f_!(x)$, rendering U a so-called **cloven** opfibration. This choice induces **reindexing functors** $f_!\colon \mathsf{X}_a \to \mathsf{X}_b$ between the fibre categories, which by the liftings universal property adhere to natural $(1_a)_! \cong 1_{\mathsf{X}_a}$ and $(f \circ g)_! \cong f_! \circ g_!$. If these isomorphisms are equalities, we have the notion of a **split** opfibration.

Let $\mathbf{OpFib}(\mathsf{A})$ denote the 2-subcategory of the slice 2-category $\mathbf{Cat}/\mathsf{A}$ of opfibrations over A, functors that preserve cocartesian liftings and natural transformations with vertical components. In fact, there is a 2-equivalence between opfibrations and pseudofunctors induced by the so-called *Grothendieck construction*.

Definition 5.1. For any pseudofunctor $F\colon \mathsf{A} \to \mathbf{Cat}$ where A is a category viewed as a 2-category with trivial 2-cells, the **Grothendieck category** $\int F$ has

- objects pairs $(a, x \in F(a))$ and
- a morphism from $(a, x \in F(a))$ to $(b, y \in F(b))$ is a pair $(f\colon a \to b, \delta\colon F(f)(x) \to y) \in \mathsf{A} \times F(b)$.

Might need to write the composition rule? Depending on the final proof of Big Theorem. This is an opfibred category over A via the obvious forgetful functor, with fibre categories $(\int F)_a = F(a)$ and reindexing functors $f_! = F(f)$.

The above in fact provides the one direction of the following well-known equivalence.

Theorem 5.2.

(1) Every opfibration $U\colon \mathsf{X} \to \mathsf{A}$ gives rise to a pseudofunctor $F_U\colon \mathsf{A} \to \mathbf{Cat}$.
(2) Every pseudofunctor $F\colon \mathsf{A} \to \mathbf{Cat}$ gives rise to an opfibration $U_F\colon \int F \to \mathsf{A}$.
(3) The above correspondences yield an equivalence of 2-categories

$$[\mathsf{A}, \mathbf{Cat}]_{\mathrm{ps}} \simeq \mathbf{OpFib}(\mathsf{A})$$

so that $F_{U_F} \cong F$ and $U_{F_U} \cong U$.

5.3. **Double categories.** Before formally defining 'pseudo double category', it is helpful to have the following picture in mind. A pseudo double category has 2-morphisms shaped like:

$$\begin{tikzcd} A \arrow[r, "M"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow a"] & B \arrow[d, "g"] \\ C \arrow[r, "N"'] & D \end{tikzcd}$$

We call A, B, C and D **objects** or **0-cells**, f and g **vertical 1-morphisms**, M and N **horizontal 1-cells** and a a **2-morphism**. Note that a vertical 1-morphism is a morphism between 0-cells and a 2-morphism is a morphism between horizontal 1-cells. We will denote both kinds of morphisms and horizontal 1-cells as a single arrow, namely '$\to$'. We follow the notation of Shulman [24] with the following definitions.

Definition 5.3. A **pseudo double category** $\mathbb{D}$, or **double category** for short, consists of a category of objects $\mathbf{D_0}$ and a category of arrows $\mathbf{D_1}$ with the following functors

$$U\colon \mathbf{D_0} \to \mathbf{D_1}$$
$$S, T\colon \mathbf{D_1} \rightrightarrows \mathbf{D_0}$$
$$\odot\colon \mathbf{D_1} \times_{\mathbf{D_0}} \mathbf{D_1} \to \mathbf{D_1} \text{ (where the pullback is taken over } \mathbf{D_1} \xrightarrow{T} \mathbf{D_0} \xleftarrow{S} \mathbf{D_1})$$

such that

$$S(U_A) = A = T(U_A)$$
$$S(M \odot N) = SN$$
$$T(M \odot N) = TM$$

equipped with natural isomorphisms

$$\alpha\colon (M\odot N)\odot P \xrightarrow{\sim} M\odot(N\odot P)$$
$$\lambda\colon U_B\odot M \xrightarrow{\sim} M$$
$$\rho\colon M\odot U_A \xrightarrow{\sim} M$$

such that $S(\alpha), S(\lambda), S(\rho), T(\alpha), T(\lambda)$ and $T(\rho)$ are all identities and that the coherence axioms of a monoidal category are satisfied. Following the notation of Shulman, objects of $\mathbf{D_0}$ are called **0-cells** and morphisms of $\mathbf{D_0}$ are called **vertical 1-morphisms**. Objects of $\mathbf{D_1}$ are called **horizontal 1-cells** and morphisms of $\mathbf{D_1}$ are called **2-morphisms**. The morphisms of $\mathbf{D_0}$, which are vertical 1-morphisms, will be denoted $f\colon A \to C$ and we denote a 1-cell M with $S(M) = A, T(M) = B$ by $M\colon A \to B$. Then a 2-morphism $a\colon M \to N$ of $\mathbf{D_1}$ with $S(a) = f, T(a) = g$ would look like:

$$\begin{tikzcd}
A \arrow[r, "M"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow a"] & B \arrow[d, "g"] \\
C \arrow[r, "N"'] & D
\end{tikzcd}$$

The key difference between a 'strict' double category and a pseudo double category is that in a pseudo double category, horizontal composition is associative and unital only up to natural isomorphism. Equivalently, as a double category can be viewed as a category internal to **Cat**, we can view a pseudo double category as a category 'weakly' internal to **Cat**. We will sometimes omit the word pseudo and simply say double category.

Definition 5.4. A 2-morphism where f and g are identities is called a **globular 2-morphism**.

Definition 5.5. Let $\mathbb{D}$ be a pseudo double category. Then the **horizontal bicategory** of $\mathbb{D}$, which we denote as $H(\mathbb{D})$, is the bicategory consisting of objects of $\mathbb{D}$, morphisms that are horizontal 1-cells of $\mathbb{D}$ and 2-morphisms that are globular 2-morphisms of $\mathbb{D}$.

Definition 5.6. Let $\mathbb{A}$ and $\mathbb{B}$ be pseudo double categories. A **lax double functor** is a functor $\mathbb{F}\colon \mathbb{A} \to \mathbb{B}$ that takes items of $\mathbb{A}$ to items of $\mathbb{B}$ of the corresponding type, respecting vertical composition in the strict sense and the horizontal composition up to an assigned comparison ϕ. This means that we have functors $\mathbb{F}_0\colon \mathbb{A}_0 \to \mathbb{B}_0$ and $\mathbb{F}_1\colon \mathbb{A}_1 \to \mathbb{B}_1$ such that the following equations are satisfied:

$$S\circ\mathbb{F}_1 = \mathbb{F}_0\circ S$$

$$T\circ\mathbb{F}_1 = \mathbb{F}_0\circ T$$

Sometimes for brevity, we will omit the subscripts and simply say $\mathbb{F}$. As to whether we mean $\mathbb{F}_0$ or $\mathbb{F}_1$ will be clear from context.

Also, every object a is equipped with a special globular 2-morphism $\phi_a\colon 1_{\mathbb{F}(a)} \to \mathbb{F}(1_a)$ (the identity comparison), and every horizontal composition $N_1\odot N_2$ is equipped with a special globular 2-morphism $\phi(N_1, N_2)\colon \mathbb{F}(N_1)\odot\mathbb{F}(N_2) \to \mathbb{F}(N_1\odot N_2)$ (the composition comparison), in a coherent way. This means that the following diagrams commute.

(1) For a horizontal composite, $\beta\odot\alpha$,

(10)
$$
\begin{tikzcd}
\mathbb{F}(A) \arrow[r, "\mathbb{F}(N_2)", "|" marking] \arrow[d] \arrow[dr, phantom, "\mathbb{F}(\alpha)"] & \mathbb{F}(B) \arrow[r, "\mathbb{F}(N_1)", "|" marking] \arrow[d] \arrow[dr, phantom, "\mathbb{F}(\beta)"] & \mathbb{F}(C) \arrow[d] \\
\mathbb{F}(A') \arrow[r, "\mathbb{F}(N_4)"', "|" marking] \arrow[d, "1"'] \arrow[drr, phantom, "\phi(N_3,N_4)"] & \mathbb{F}(B') \arrow[r, "\mathbb{F}(N_3)"', "|" marking] & \mathbb{F}(C') \arrow[d, "1"] \\
\mathbb{F}(A') \arrow[rr, "\mathbb{F}(N_3\odot N_4)"', "|" marking] & & \mathbb{F}(C')
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathbb{F}(A) \arrow[r, "\mathbb{F}(N_2)", "|" marking] \arrow[d, "1"'] \arrow[drr, phantom, "\phi(N_1,N_2)"] & \mathbb{F}(B) \arrow[r, "\mathbb{F}(N_1)", "|" marking] & \mathbb{F}(C) \arrow[d, "1"] \\
\mathbb{F}(A) \arrow[rr, "\mathbb{F}(N_1\odot N_2)", "|" marking] \arrow[d] \arrow[drr, phantom, "\mathbb{F}(\beta\odot\alpha)"] & & \mathbb{F}(C) \arrow[d] \\
\mathbb{F}(A') \arrow[rr, "\mathbb{F}(N_3\odot N_4)"', "|" marking] & & \mathbb{F}(C')
\end{tikzcd}
.
$$

(2) For a horizontal 1-cell $N\colon A \to B$, the following diagrams are commutative (under horizontal composition).

$$
\begin{tikzcd}
\mathbb{F}(N)\odot 1_{\mathbb{F}(A)} \arrow[r, "\rho_{\mathbb{F}(N)}"] \arrow[d, "1\odot\phi_A"'] & \mathbb{F}(N) \\
\mathbb{F}(N)\odot\mathbb{F}(1_A) \arrow[r, "\phi(N,1_A)"] & \mathbb{F}(N\odot 1_A) \arrow[u, "\mathbb{F}\rho"']
\end{tikzcd}
\qquad
\begin{tikzcd}
1_{\mathbb{F}(B)}\odot\mathbb{F}(N) \arrow[r, "\lambda_{\mathbb{F}(N)}"] \arrow[d, "\phi_B\odot 1"'] & \mathbb{F}(N) \\
\mathbb{F}(1_B)\odot\mathbb{F}(N) \arrow[r, "\phi(1_B,N)"] & \mathbb{F}(1_B\odot N) \arrow[u, "F\lambda"']
\end{tikzcd}
$$

(3) For consecutive horizontal 1-cells N_1, N_2 and N_3, the following diagram is commutative.

$$
\begin{tikzcd}
(\mathbb{F}(N_1)\odot\mathbb{F}(N_2))\odot\mathbb{F}(N_3) \arrow[r, "a'"] \arrow[d, "\phi(N_1,N_2)\odot 1"'] & \mathbb{F}(N_1)\odot(\mathbb{F}(N_2)\odot\mathbb{F}(N_3)) \arrow[d, "1\odot\phi(N_2,N_3)"] \\
\mathbb{F}(N_1\odot N_2)\odot\mathbb{F}(N_3) \arrow[d, "\phi(N_1\odot N_2,N_3)"'] & \mathbb{F}(N_1)\odot\mathbb{F}(N_2\odot N_3) \arrow[d, "\phi(N_1,N_2\odot N_3)"] \\
\mathbb{F}((N_1\odot N_2)\odot N_3) \arrow[r, "\mathbb{F}a"] & \mathbb{F}(N_1\odot(N_2\odot N_3))
\end{tikzcd}
$$

We say the double functor $\mathbb{F}$ is **strict** if the comparison constraints ϕ_a and ϕ_{N_1,N_2} are identities, **pseudo** if the comparison constraints are isomorphisms, and **oplax** if the comparison constraints go in the opposite direction.

Pretty sure the def below can be incorporated above, however got a bit lost in different notation. Kenny Fix?

Definition 5.7. A double functor $\mathbb{F}\colon \mathbb{A} \to \mathbb{X}$ is **strong** if the comparison and unit constraints are globular isomorphisms, meaning that for each composable pair of horizontal 1-cells M and N we have a natural isomorphism

$$\mathbb{F}_{M,N}\colon \mathbb{F}(M)\odot\mathbb{F}(N) \xrightarrow{\sim} \mathbb{F}(M\odot N)$$

and for each object $a \in \mathbb{A}$ a natural isomorphism

$$\mathbb{F}_a\colon \hat{U}_{\mathbb{F}(a)} \xrightarrow{\sim} \mathbb{F}(U_a).$$

Following the notation of Shulman [25], given a double category $\mathbb{A}$, we write ${}_f\mathbb{A}_g(M,N)$ for the set of 2-morphisms in $\mathbb{A}$ of the form:

$$\begin{tikzcd} A \arrow[r, "M", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow a"] & B \arrow[d, "g"] \\ C \arrow[r, "N"', "\shortmid"{marking}] & D \end{tikzcd}$$

We call M and N the **horizontal source and target** of the 2-morphism a, respectively, and likewise we call f and g the **vertical source and target** of the 2-morphism a, respectively. Thus ${}_f\mathbb{A}_g(M,N)$ denotes the set of 2-morphisms in $\mathbb{A}$ with horizontal source and target M and N and vertical source and target f and g.

Definition 5.8. A (possibly lax or oplax) double functor $\mathbb{F}\colon \mathbb{A} \to \mathbb{X}$ is **full** (respectively, **faithful**) if $\mathbb{F}_0\colon \mathbb{A}_0 \to \mathbb{X}_0$ is full (respectively, faithful) and each map

$$\mathbb{F}_1\colon {}_f\mathbb{A}_g(M,N) \to_{\mathbb{F}(f)} \mathbb{X}_{\mathbb{F}(g)}(\mathbb{F}(M),\mathbb{F}(N))$$

is surjective (respectively, injective).

Definition 5.9. A (possibly lax or oplax) double functor $\mathbb{F}\colon \mathbb{A} \to \mathbb{X}$ is **essentially surjective** if we can simultaneously make the following choices:

(1) For each object $x \in \mathbb{X}$, we can find an object $a \in \mathbb{A}$ together with a vertical 1-isomorphism $\alpha_x\colon \mathbb{F}(a) \to x$, and
(2) For each horizontal 1-cell $N\colon x_1 \nrightarrow x_2$ of $\mathbb{X}$, we can find a horizontal 1-cell $M\colon a_1 \nrightarrow a_2$ of $\mathbb{A}$ and a 2-isomorphism a_N of $\mathbb{X}$ as in the following diagram:

$$\begin{tikzcd} \mathbb{F}(a_1) \arrow[r, "\mathbb{F}(M)", "\shortmid"{marking}] \arrow[d, "\alpha_{x_1}"'] \arrow[dr, phantom, "\Downarrow a_N"] & \mathbb{F}(a_1) \arrow[d, "\alpha_{x_2}"] \\ x_1 \arrow[r, "N"', "\shortmid"{marking}] & x_2 \end{tikzcd}$$

Formally, two strong double functors between two double categories form a *double equivalence* if their two-way composite is isomorphic to the identity, better expressed, possibly in a definition environment. Similarly to ordinary equivalence of categories, cite shulman's theorem here [Shulman,7.8] allows us to use the following equivalent characterization.

Theorem 5.10. *A strong double functor $\mathbb{F}\colon \mathbb{A} \to \mathbb{X}$ is part of a double equivalence if and only if it is full, faithful and essentially surjective. on objects?*

Proposition 5.11. *Let $\mathbb{A}$ and $\mathbb{X}$ be symmetric monoidal double categories and let $\mathbb{F}\colon \mathbb{A} \to \mathbb{X}$ be a symmetric monoidal strong double functor. If $\mathbb{F}$ is part of a double equivalence, then $\mathbb{F}$ is in fact part of a symmetric monoidal double equivalence, and $\mathbb{A}$ and $\mathbb{X}$ are equivalent as symmetric monoidal double categories.*

Definition 5.12. A **monoidal double category** is a double category equipped the following structure.

(1) $\mathbf{D_0}$ and $\mathbf{D_1}$ are both monoidal categories.
(2) If I is the monoidal unit of $\mathbf{D_0}$, then U_I is the monoidal unit of $\mathbf{D_1}$.
(3) The functors S and T are strict monoidal, i.e. $S(M \otimes N) = SM \otimes SN$ and $T(M \otimes N) = TM \otimes TN$ and S and T also preserve the associativity and unit constraints.

(4) We have globular isomorphisms

$$\chi\colon (M_1 \otimes N_1) \odot (M_2 \otimes N_2) \xrightarrow{\sim} (M_1 \odot M_2) \otimes (N_1 \odot N_2)$$

and

$$\mu\colon U_{A\otimes B} \xrightarrow{\sim} (U_A \otimes U_B)$$

such that the following diagrams commute:

(5) The following diagrams commute expressing the constraint data for the double functor $\otimes$.

$$\begin{tikzcd}
((M_1 \otimes N_1) \odot (M_2 \otimes N_2)) \odot (M_3 \otimes N_3) \arrow[r, "\chi \odot 1"] \arrow[d, "\alpha"'] & ((M_1 \odot M_2) \otimes (N_1 \odot N_2)) \odot (M_3 \otimes N_3) \arrow[d, "\chi"] \\
(M_1 \otimes N_1) \odot ((M_2 \otimes N_2) \odot (M_3 \otimes N_3)) \arrow[d, "1 \odot \chi"'] & ((M_1 \odot M_2) \odot M_3) \otimes ((N_1 \odot N_2) \odot N_3) \arrow[d, "\alpha \otimes \alpha"] \\
(M_1 \otimes N_1) \odot ((M_2 \odot M_3) \otimes (N_2 \odot N_3)) \arrow[r, "\chi"] & (M_1 \odot (M_2 \odot M_3)) \otimes (N_1 \odot (N_2 \odot N_3))
\end{tikzcd}$$

$$\begin{tikzcd}
(M \otimes N) \odot U_{C\otimes D} \arrow[r, "1 \odot \mu"] \arrow[d, "\rho"'] & (M \otimes N) \odot (U_C \otimes U_D) \arrow[d, "\chi"] \\
M \otimes N & (M \odot U_C) \otimes (N \odot U_D) \arrow[l, "\rho \otimes \rho"']
\end{tikzcd}
\qquad
\begin{tikzcd}
U_{A\otimes B} \odot (M \otimes N) \arrow[r, "\chi \odot 1"] \arrow[d, "\lambda"'] & (U_A \otimes U_B) \odot (M \otimes N) \arrow[d, "\chi"] \\
M \otimes N & (U_A \odot M) \otimes (U_B \odot N) \arrow[l, "\lambda \otimes \lambda"']
\end{tikzcd}$$

(6) The following diagrams commute expressing the associativity isomorphism for $\otimes$ is a transformation of double categories.

$$\begin{tikzcd}
((M_1 \otimes N_1) \otimes P_1) \odot ((M_2 \otimes N_2) \otimes P_2) \arrow[r, "\alpha \odot \alpha"] \arrow[d, "\chi"'] & (M_1 \otimes (N_1 \otimes P_1)) \odot (M_2 \otimes (N_2 \otimes P_2)) \arrow[d, "\chi"] \\
((M_1 \otimes N_1) \odot (M_2 \otimes N_2)) \otimes (P_1 \odot P_2) \arrow[d, "\chi \otimes 1"'] & (M_1 \odot M_2) \otimes ((N_1 \otimes P_1) \odot (N_2 \otimes P_2)) \arrow[d, "1 \otimes \chi"] \\
((M_1 \odot M_2) \otimes (N_1 \odot N_2)) \otimes (P_1 \odot P_2) \arrow[r, "\alpha"] & (M_1 \odot M_2) \otimes ((N_1 \odot N_2) \otimes (P_1 \odot P_2))
\end{tikzcd}$$

$$\begin{tikzcd}
U_{(A\otimes B)\otimes C} \arrow[r, "U_\alpha"] \arrow[d, "\mu"'] & U_{A\otimes(B\otimes C)} \arrow[d, "\mu"] \\
U_{A\otimes B} \otimes U_C \arrow[d, "\mu \otimes 1"'] & U_A \otimes U_{B\otimes C} \arrow[d, "1 \otimes \mu"] \\
(U_A \otimes U_B) \otimes U_C \arrow[r, "\alpha"] & U_A \otimes (U_B \otimes U_C)
\end{tikzcd}$$

(7) The following diagrams commute expressing that the unit isomorphisms for $\otimes$ are transformations of double categories.

$$\begin{tikzcd}
(M \otimes U_I) \odot (N \otimes U_I) \arrow[r, "\chi"] \arrow[d, "r \odot r"'] & (M \odot N) \otimes (U_I \odot U_I) \arrow[d, "1 \otimes \rho"] \\
M \odot N & (M \odot N) \otimes U_I \arrow[l, "r"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & U_A \otimes U_I \arrow[d, "r"] \\
U_{A\otimes I} \arrow[ur, "\mu"] \arrow[r, "U_r"'] & U_A
\end{tikzcd}$$

$$\begin{tikzcd}
(U_I \otimes M) \odot (U_I \otimes N) \arrow[r, "\chi"] \arrow[d, "\ell \odot \ell"'] & (U_I \odot U_I) \otimes (M \odot N) \arrow[d, "\lambda \otimes 1"] \\
M \odot N & U_I \otimes (M \odot N) \arrow[l, "\ell"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & U_I \otimes U_A \arrow[d, "\ell"] \\
U_{I\otimes A} \arrow[ur, "\mu"] \arrow[r, "U_\ell"'] & U_A
\end{tikzcd}$$

A **braided monoidal double category** is a monoidal double category such that:

(8) $\mathbb{D}_0$ and $\mathbb{D}_1$ are braided monoidal categories.
(9) The functors S and T are strict braided monoidal functors.
(10) The following diagrams commute expressing that the braiding is a transformation of double categories.

$$\begin{tikzcd}
(M_1 \odot M_2) \otimes (N_1 \odot N_2) \arrow[r, "\beta"] \arrow[d, "\chi"'] & (N_1 \odot N_2) \otimes (M_1 \odot M_2) \arrow[d, "\chi"] \\
(M_1 \otimes N_1) \odot (M_2 \otimes N_2) \arrow[r, "\beta \odot \beta"] & (N_1 \otimes M_1) \odot (N_2 \otimes M_2)
\end{tikzcd}
\qquad
\begin{tikzcd}
U_A \otimes U_B \arrow[d, "\beta"'] & U_{A\otimes B} \arrow[l, "\mu"'] \arrow[d, "U_\beta"] \\
U_B \otimes U_A & U_{B\otimes A} \arrow[l, "\mu"']
\end{tikzcd}$$

Finally, a **symmetric monoidal double category** is a braided monoidal double category $\mathbb{D}$ such that:

(11) $\mathbb{D}_0$ and $\mathbb{D}_1$ are symmetric monoidal.

Definition 5.13. Let $\mathbb{D}$ be a double category and $f : A \to B$ a vertical 1-morphism. A **companion** of f is a horizontal 1-cell $\hat{f} : A \to B$ together with 2-morphisms

$$\begin{tikzcd}
A \arrow[r, "\widehat{f}"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d, "1"] \\
B \arrow[r, "U_B"'] & B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[r, "U_A"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, "f"] \\
A \arrow[r, "\widehat{f}"'] & B
\end{tikzcd}$$

such that the following equations hold.

$$\begin{tikzcd}
A \arrow[r, "U_A"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow"] & A \arrow[d, "f"] \\
A \arrow[r, "\widehat{f}"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d, "1"] \\
B \arrow[r, "U_B"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "U_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow U_f"] & A \arrow[d, "f"] \\
B \arrow[r, "U_B"'] & B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[r, "U_A"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow"] & A \arrow[r, "\widehat{f}"] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow"] & B \arrow[d, "1"] \\
A \arrow[r, "\widehat{f}"'] & B \arrow[r, "U_B"'] & B
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "\widehat{f}"] \arrow[d, "1"'] \arrow[dr, phantom, "\Downarrow \mathrm{id}_{\widehat{f}}"] & B \arrow[d, "1"] \\
A \arrow[r, "\widehat{f}"'] & B
\end{tikzcd}
\tag{11}$$

A **conjoint** of f, denoted $\check{f} : B \to A$, is a companion of f in the double category $\mathbb{D}^{h\cdot\mathrm{op}}$ obtained by reversing the horizontal 1-cells, but not the vertical 1-morphisms, of $\mathbb{D}$.

In a pseudo double category, the second equation above requires an insertion of unit isomorphisms to make sense due to horizontal composition only holding up to isomorphism.

Definition 5.14. We say that a double category is **fibrant** if every vertical 1-morphism has both a companion and a conjoint and **isofibrant** if every vertical 1-isomorphism has both a companion and a conjoint.

Definition 5.15. A **(strong) monoidal lax double functor** $\mathbb{F}\colon \mathbb{C} \to \mathbb{D}$ between monoidal double categories $\mathbb{C}$ and $\mathbb{D}$ is a lax double functor $\mathbb{F}\colon \mathbb{C} \to \mathbb{D}$ such that

- $\mathbb{F}_0$ and $\mathbb{F}_1$ are monoidal functors, meaning that there exists
  (1) an isomorphism $\epsilon\colon 1_{\mathbb{D}} \to \mathbb{F}(1_{\mathbb{C}})$
  (2) a natural isomorphism $\mu_{A,B}\colon \mathbb{F}(A) \otimes \mathbb{F}(B) \to \mathbb{F}(A \otimes B)$ for all objects A and B of $\mathbb{C}$
  (3) an isomorphism $\delta\colon U_{1_{\mathbb{D}}} \to \mathbb{F}(U_{1_{\mathbb{C}}})$
  (4) a natural isomorphism $\nu_{M,N}\colon \mathbb{F}(M) \otimes \mathbb{F}(N) \to \mathbb{F}(M \otimes N)$ for all horizontal 1-cells N and M of $\mathbb{C}$

  such that the following diagrams commute: for objects A, B and C of $\mathbb{C}$,

$$\begin{tikzcd}
(\mathbb{F}(A) \otimes \mathbb{F}(B)) \otimes \mathbb{F}(C) \arrow[r, "\alpha'"] \arrow[d, "\mu_{A,B}\otimes 1"'] & \mathbb{F}(A) \otimes (\mathbb{F}(B) \otimes \mathbb{F}(C)) \arrow[d, "1\otimes\mu_{B,C}"] \\
\mathbb{F}(A \otimes B) \otimes \mathbb{F}(C) \arrow[d, "\mu_{A\otimes B,C}"'] & \mathbb{F}(A) \otimes \mathbb{F}(B \otimes C) \arrow[d, "\mu_{A,B\otimes C}"] \\
\mathbb{F}((A \otimes B) \otimes C) \arrow[r, "\mathbb{F}\alpha"] & \mathbb{F}(A \otimes (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{F}(A) \otimes 1_{\mathbb{D}} \arrow[r, "r_{\mathbb{F}(A)}"] \arrow[d, "1\otimes\epsilon"'] & \mathbb{F}(A) \\
\mathbb{F}(A) \otimes \mathbb{F}(1_{\mathbb{C}}) \arrow[r, "\mu_{A,1_{\mathbb{C}}}"] & \mathbb{F}(A \otimes 1_{\mathbb{C}}) \arrow[u, "\mathbb{F}(r_A)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
1_{\mathbb{D}} \otimes \mathbb{F}(A) \arrow[r, "\ell_{\mathbb{F}(A)}"] \arrow[d, "\epsilon\otimes 1"'] & \mathbb{F}(A) \\
\mathbb{F}(1_{\mathbb{C}}) \otimes \mathbb{F}(A) \arrow[r, "\mu_{1_{\mathbb{C}},A}"] & \mathbb{F}(1_{\mathbb{C}} \otimes A) \arrow[u, "\mathbb{F}(\ell_A)"']
\end{tikzcd}$$

  and for horizontal 1-cells N_1, N_2 and N_3 of $\mathbb{C}$,

$$\begin{tikzcd}
(\mathbb{F}(N_1) \otimes \mathbb{F}(N_2)) \otimes \mathbb{F}(N_3) \arrow[r, "\alpha'"] \arrow[d, "\nu_{N_1,N_2}\otimes 1"'] & \mathbb{F}(N_1) \otimes (\mathbb{F}(N_2) \otimes \mathbb{F}(N_3)) \arrow[d, "1\otimes\nu_{N_2,N_3}"] \\
\mathbb{F}(N_1 \otimes N_2) \otimes \mathbb{F}(N_3) \arrow[d, "\nu_{N_1\otimes N_2,N_3}"'] & \mathbb{F}(N_1) \otimes \mathbb{F}(N_2 \otimes N_3) \arrow[d, "\nu_{N_1,N_2\otimes N_3}"] \\
\mathbb{F}((N_1 \otimes N_2) \otimes N_3) \arrow[r, "\mathbb{F}\alpha"] & \mathbb{F}(N_1 \otimes (N_2 \otimes N_3))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{F}(N_1)\otimes U_{1_{\mathbb{D}}} \arrow[r, "r_{\mathbb{F}(N_1)}"] \arrow[d, "1\otimes\delta"'] & \mathbb{F}(N_1) \\
\mathbb{F}(N_1)\otimes \mathbb{F}(U_{1_{\mathbb{C}}}) \arrow[r, "\nu_{N_1,U_{1_{\mathbb{C}}}}"] & \mathbb{F}(N_1\otimes U_{1_{\mathbb{C}}}) \arrow[u, "\mathbb{F}(r_{N_1})"']
\end{tikzcd}
\qquad
\begin{tikzcd}
U_{1_{\mathbb{D}}}\otimes \mathbb{F}(N_1) \arrow[r, "\ell_{\mathbb{F}(N_1)}"] \arrow[d, "\delta\otimes 1"'] & \mathbb{F}(N_1) \\
\mathbb{F}(U_{1_{\mathbb{C}}})\otimes \mathbb{F}(N_1) \arrow[r, "\nu_{U_{1_{\mathbb{C}}},N_1}"] & \mathbb{F}(U_{1_{\mathbb{C}}}\otimes N_1) \arrow[u, "\mathbb{F}(\ell_{N_1})"']
\end{tikzcd}$$

- $S\mathbb{F}_1 = \mathbb{F}_0 S$ and $T\mathbb{F}_1 = \mathbb{F}_0 T$ are equations between monoidal functors, and
- the composition and unit comparisons $\phi(N_1, N_2) \colon \mathbb{F}_1(N_1)\odot\mathbb{F}_1(N_2) \to \mathbb{F}_1(N_1\odot N_2)$ and $\phi_A \colon U_{\mathbb{F}_0(A)} \to \mathbb{F}_1(U_A)$ are monoidal natural transformations.

The monoidal lax double functor is **braided** if $\mathbb{F}_0$ and $\mathbb{F}_1$ are braided monoidal functors and **symmetric** if they are symmetric monoidal functors, and lax monoidal or oplax monoidal if the isomorphisms and families of natural isomorphisms of items (1)-(4) above are merely morphisms and natural transformations going in the appropriate directions.

References

[1] J. C. Baez and K. Courser, Coarse-graining open Markov processes. Available as arXiv:1710.11343.
[2] J. C. Baez and K. Courser, Structured cospans. In preparation.
[3] J. C. Baez, B. Coya and F. Rebro, Props in circuit theory. In preparation. THIS IS FINISHED!!!
[4] J. C. Baez and B. Fong, A compositional framework for passive linear networks. Available as arXiv:1504.05625.
[5] J. C. Baez, B. Fong and B. Pollard, A compositional framework for Markov processes, *Jour. Math. Phys.* **57** (2016), 033301. Available as arXiv:1508.06448.
[6] J. C. Baez and J. Master, Open Petri nets. Available as arXiv:1808.05415.
[7] J. C. Baez and B. Pollard, A compositional framework for chemical reaction networks, in preparation.
[8] F. Borceux, Handbook of categorical algebra. 2, volume 51 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1994
[9] R. Brown and C. B. Spencer, Double groupoids and crossed modules, *Cah. Top. Géom. Diff.* **17** (1976), 343–362.
[10] R. Brown, K. Hardie, H. Kamps and T. Porter, The homotopy double groupoid of a Hausdorff space, *Th. Appl. Categ.* **10** (2002), 71–93.
[11] D. Cicala and C. Vasilakopoulou, On Adjoints and Fibrations. In preparation.
[12] K. Courser, A bicategory of decorated cospans, *Theor. Appl. Cat.* **32** (2017), 995–1027. Also available as arXiv:1605.08100.
[13] K. Courser, Open Systems: a Double Categorical Perspective, PhD Thesis, University of California at Riverside (2019)
[14] B. Day and R. Street, Monoidal Bicategories and Hopf Algebroids, *Adv. Math.*, 129(1):99–157, 1997.
[15] C. Ehresmann, Catégories structurées III: Quintettes et applications covariantes, *Cah. Top. Géom. Diff.* **5** (1963), 1–22.
[16] C. Ehresmann, *Catégories et Structures*, Dunod, Paris, 1965.
[17] B. Fong, Decorated cospans, *Theor. Appl. Cat.* **30** (2015), 1096–1120. Also available as arXiv:1502.00872.
[18] J. Gray, Fibred and Cofibred categories, Proc. Conf. Categorical Algebra (La Jolla, California, 1965), 21—83, New York, 1966
[19] C. Hermida, Fibrations, logical predicates and indeterminates, PhD Thesis, University of Edinburgh, 1993.
[20] J. C. Baez and J. Master, A compositional framework for Petri nets. In preparation.
[21] G. Kelly and R. Street, Review of the elements of 2-categories, *Lecture Notes in Mathematics* vol. 40, pages 75—103, Springer Berlin/Heidelberg, 1974.
[22] E. Lerman and D. Spivak, An algebra of open continuous time dynamical systems and networks. Available as arXiv:1602.01017.
[23] J. Moeller and C. Vasilakopoulou, Monoidal Grothendieck construction. Available as arXiv:1809.00727.
[24] M. Shulman, Constructing symmetric monoidal bicategories. Available as arXiv:1004.0993.
[25] M. Shulman, Framed bicategories and monoidal fibrations. Available as arXiv:0706.1286.

[26] A. Yassine, Open Systems in Classical Mechanics. Available as arXiv:1710.11392.