Project Compression with Nonlinear Cost Functions

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Abstract: This paper presents three mixed-integer linear programming models to assist project managers in making decisions to compress project completion time under realistic activity time-cost relationship assumptions. The models assume nonlinear activity time-cost functions that are rational functions that can be convex or concave. A user of the models needs to estimate an activity time between normal and crash times where the rate of increase in the cost of performing an activity changes significantly. An efficient piecewise linearization method is presented through which nonlinear cost functions can be approximated in a mixed-integer linear programming model. Each of the models focuses on a different objective a project manager may pursue, such as minimizing project completion subject to a crash budget constraint, or minimizing total project cost, or minimizing total cost under late completion penalties or minimizing total cost with early completion bonuses of a project contract. The paper also uses a simple example which has activities that have all the different types of cost functions discussed in the paper to demonstrate how each model can be used for a project manager's different objectives.

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Introduction

Project planning and scheduling is an important undertaking at nearly all levels of an organization. Cost overruns and delays are the results of inadequate execution of this function. To assist project managers in their task, a number of analytical techniques have been proposed. However, most of these suggested procedures are based on the frequently unattainable assumption that the pertinent cost functions of the project are linear. Furthermore, it has been acknowledged in the literature that scheduling projects with nonlinear cost functions, a situation frequently found in practice, leads to highly complex problems. This high degree of complexity may explain why only a small number of procedures have been suggested for these cases. The motivation behind this paper is to provide a practical approach for managers in making project compression decisions in an optimal manner when some of the activity cost functions are continuous, bounded, and nonlinear (convex or concave). The practicality of the approach stems from the fact that only a basic understanding of mixed-integer linear programming is required in addition to developing a computer program (e.g., using Visual Basic for Applications) to implement the piecewise linear approximation procedure of cost functions explained below. The approach can be used in managing any construction project as well as nonconstruction projects.

This paper proposes an approach to optimal project compression that uses rational functions to represent activity time-cost relationships that are continuous and bounded. The type of function used is very flexible; depending on its parameters, it can represent a convex or concave activity cost function. A mixed-integer programming model will be developed that will enable the project manager to finish the project on time and without cost overruns, in the presence of highly complex but realistic cost relationships. Other models with a different focus (e.g., minimizing project cost with potential late completion penalties or early completion bonuses) that can be developed from this basic model will also be presented.

Background and Literature Review

The successful completion of large-scale projects demands thoughtful planning, scheduling and coordinating a large number of interconnected tasks. A number of techniques, such as critical path method (CPM) and program evaluation and review technique (PERT), have been in use since the 1950s to assist in these endeavors. Early on, both practitioners and researchers in project scheduling realized the importance of CPM as a useful tool for project costing and comparing alternate schedules based on cost. Fundamentally, CPM enables the project manager to answer questions such as (1) how long is the completion time of the project; (2) what is its cost; (3) which are the critical activities that need to be completed on time to avoid project delays; and (4) what would be the additional cost be if its completion time were to be expedited (shortened). To answer these questions, a number of mathematical models have been developed. The degree of complexity of these models depends upon the assumptions the project manager makes about the nature of the cost of expediting the tasks in the project.

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A vast amount of literature has been generated based on the assumption of linear time-cost relationship. However, in many instances, the activity cost functions are not linear. For example, it may be that an activity of the project involves the shipment of an item from a source to a destination. This can be accomplished by a number of ground modes (truck, rail, etc.) where the assumption of a linear expediting cost function is reasonable. But if air transportation modes (24-hour delivery, overnight delivery, etc.) are also available, shipment of the item could be accomplished much faster than by ground transportation, though at a considerably higher cost. Clearly, in this situation, the expedited cost function for all transportation options is not linear.

One of the earliest approaches for tackling nonlinear compression costs was by Meyer and Shaffer (1965) who used a mixed integer programming model for solving the time-cost trade-off problem in CPM type networks. Their models incorporated time-cost curves "that are nonincreasing and bounded, piecewise linear, and continuous but nonconvex; bounded but defined only at discrete points; bounded but discontinuous." Berman (1964) indicated that "continuous functions permit a more general solution of the resource allocation problem and offer the greater sensitivity necessary to study the effects on the decision arising from uncertainty in the time of completion of an activity." Berman (1964) presented a conceptual model for a PERT network that has continuous concave upward time-cost functions.

Falk and Horowitz (1972) pointed out that both convex and concave activity cost curves are possible in practice. They developed an algorithm that sets up and solves a sequence of maximal flow sub problems which yields a global solution to the original problem. Salem and Elmaghraby (1984) approximated nonlinear activity cost functions by continuous piecewise linear and convex functions that are used in a linear program. The optimal solution of the linear program is guaranteed to give an error of approximation less than or equal to a priori selected tolerance. Vrat and Kriengkrairut (1986) developed a goal programming model for project compression where some activity cost functions are strictly convex. They approximated each convex function by only two linear segments. Foldes and Soumis (1993) presented a reformulation of a time-cost tradeoff problem as a network flow optimization problem with nonlinear cost functions. Kuyumcu and Garcia-Diaz (1994) developed a solution methodology for linearizing convex or concave activity time-cost functions, which is based on Bender's decomposition approach.

Deckro et al. (1995) developed a quadratic programming model as well as a goal programming formulation of the problem. They assumed that as activity time decreases from normal time the cost will increase at an increasing rate. Another approach based on the assumption of quadratic cost functions was by Wei and Wang (2003). They proposed three methods for project compression by linearizing quadratic activity time-cost functions. Elmaghraby and Pulat (1979) developed an algorithm for project compression in which some events have due dates. Their algorithm assumes linear penalty for tardiness of key events and a linear cost of compression. Yang (2007) developed an evolutionary computation technique, particle swarm optimization, to solve project crashing problems with any type of activity cost function. Mitchell and Klastorin (2007) consider the problem of compressing a complex project when task durations are random variables (e.g., negative exponential), in order to minimize the expected total, indirect, and incentive costs.

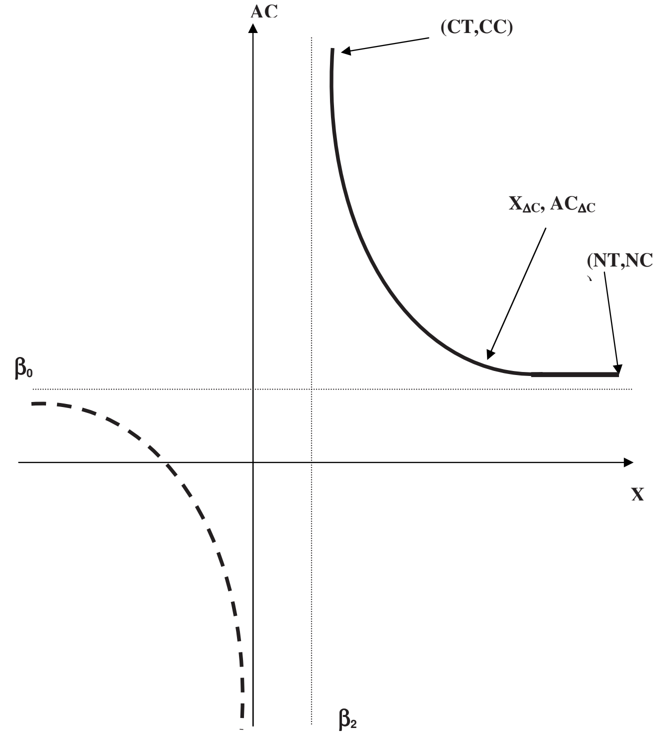

Fig. 1. Activity cost curve that concaves up

Nonlinear Activity Time-Cost Functions

In this paper a cost function of the following form for each activity j is assumed:

$$\mathrm{AC}_j = \frac{\beta_{1j}}{X_j - \beta_{2j}} + \beta_{0j}.$$

As it will be shown, depending on the signs of β_{1j} and β_{2j} and their magnitude with respect to X_j, this rational cost function may have a graph that concaves up ($\mathrm{AC}_j'' > 0$) or concaves down ($\mathrm{AC}_j'' < 0$), or it may be linear ($\mathrm{AC}_j'' = 0$).

Derivation of the Parameters of the Time-Cost Function

In this section, the parameters of the nonlinear activity cost function are derived. Below are the definitions of variables and parameters that will appear in the derivation: β_{0j}, β_{1j}, β_{2j}= parameters of the function; NT_j=normal time for activity j; CT_j=crash (compressed) time for activity j; NC_j=Normal cost for activity j; CC_j=crash (compressed) cost for activity j; X_j=completion time for activity j; AC_j=direct cost for activity j; AC_j=known nonincreasing convex or concave continuous function in the domain of X_j ($\mathrm{CT}_j \le X_j \le \mathrm{NT}_j$), whose second derivative may be a function of completion time (Figs. 1 and 2); a circumstance that provides an accurate description of many time-cost tradeoff relationships in project scheduling. AC_j can also be linear; X_{0j}=a point in the domain of X_j where AC_j accelerates noticeably, as determined by the analyst, ($\mathrm{CT}_j < X_{0j} < \mathrm{NT}_j$); and AC_{0j}=Direct cost for X_{0j}, $\mathrm{NC}_j < \mathrm{AC}_{0j} < \mathrm{CC}_j$.

Given three points, to be determined by the analyst, $(\mathrm{NT}_j, \mathrm{NC}_j)$, $(\mathrm{CT}_j, \mathrm{CC}_j)$, and $(X_{0j}, \mathrm{AC}_{0j})$, the parameters of the

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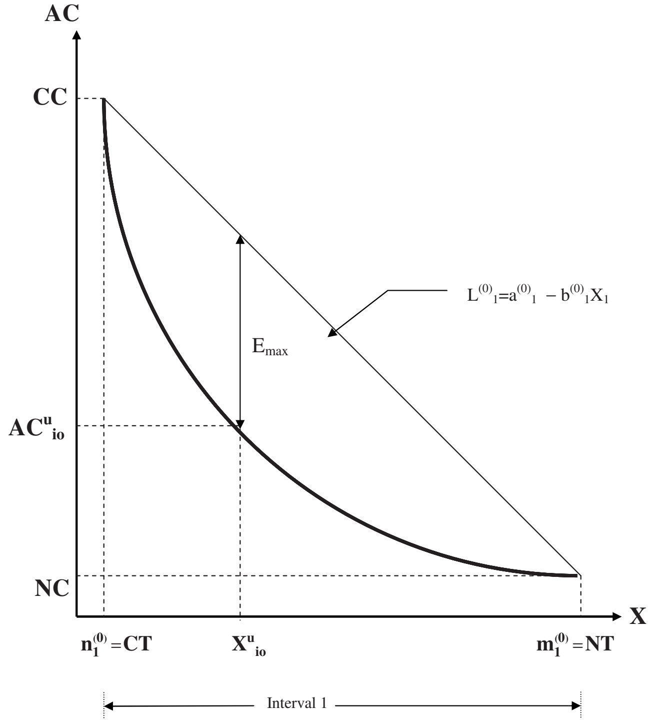

Fig. 2. First interval consisting of the entire cost curve

model can be estimated so that the cost curve will pass through the given points. Thus, omitting activity index j

$$\mathrm{CC} = \frac{\beta_1}{\mathrm{CT} - \beta_2} + \beta_0 \tag{1}$$

$$\mathrm{AC}_0 = \frac{\beta_1}{X_0 - \beta_2} + \beta_0 \tag{2}$$

$$\mathrm{NC} = \frac{\beta_1}{\mathrm{NT} - \beta_2} + \beta_0 \tag{3}$$

Eqs. (1)–(3) can be written as Eqs. (4)–(6), respectively

$$\mathrm{CC(CT)} - (\mathrm{CC})\beta_2 = \beta_1 + \beta_0\mathrm{CT} - \beta_0\beta_2 \tag{4}$$

$$(\mathrm{AC}_0)X_0 - (\mathrm{AC}_0)\beta_2 = \beta_1 + \beta_0 X_0 - \beta_0\beta_2 \tag{5}$$

$$\mathrm{NC(NT)} - (\mathrm{NC})\beta_2 = \beta_1 + \beta_0\mathrm{NT} - \beta_0\beta_2. \tag{6}$$

Subtracting Eq. (5) from Eq. (4), an expression for β_2 can be obtained

$$\mathrm{CC(CT)} - (\mathrm{CC})\beta_2 - (\mathrm{AC}_0)X_0 + (\mathrm{AC}_0)\beta_2 = \beta_0\mathrm{CT} - \beta_0 X_0$$

or

$$(\mathrm{AC}_0 - \mathrm{CC})\beta_2 = \beta_0(\mathrm{CT} - X_0) - \mathrm{CC(CT)} + (\mathrm{AC}_0)X_0$$

and

$$\beta_2 = \frac{\beta_0(\mathrm{CT} - X_0) - \mathrm{CC(CT)} + (\mathrm{AC}_0)X_0}{(\mathrm{AC}_0 - \mathrm{CC})} \tag{7}$$

Similarly, subtracting Eq. (5) from Eq. (6) another expression for β_2 is obtained:

$$\mathrm{NC(NT)} - (\mathrm{AC}_0)X_0 + (\mathrm{AC}_0)\beta_2 - (\mathrm{NC})\beta_2 = \beta_0\mathrm{NT} - \beta_0 X_0$$

or

$$(\mathrm{NT} - X_0)\beta_0 = \mathrm{NC(NT)} - (\mathrm{AC}_0)X_0 + (\mathrm{AC}_0 - \mathrm{NC})\beta_2$$

and

$$\beta_2 = \frac{\beta_0(\mathrm{NT} - X_0) - \mathrm{NC(NT)} + (\mathrm{AC}_0)X_0}{(\mathrm{AC}_0 - \mathrm{NC})} \tag{8}$$

Setting the right-hand-sides of Eqs. (7) and (8) equal to each other and cross multiplying

$$\beta_0(\mathrm{CT} - X_0)(\mathrm{AC}_0 - \mathrm{NC}) - \mathrm{CC(CT)}(\mathrm{AC}_0 - \mathrm{NC}) + (\mathrm{AC}_0)X_0(\mathrm{AC}_0 - \mathrm{NC}) = \beta_0(\mathrm{NT} - X_0)(\mathrm{AC}_0 - \mathrm{CC}) - \mathrm{NC(NT)}(\mathrm{AC}_0 - \mathrm{CC}) + (\mathrm{AC}_0)X_0(\mathrm{AC}_0 - \mathrm{CC})$$

or

$$\beta_0[(\mathrm{CT} - X_0)(\mathrm{AC}_0 - \mathrm{NC}) - (\mathrm{NT} - X_0)(\mathrm{AC}_0 - \mathrm{CC})] = [\mathrm{CC(CT)} - (\mathrm{AC}_0)X_0](\mathrm{AC}_0 - \mathrm{NC}) - [\mathrm{NC(NT)} - (\mathrm{AC}_0)X_0](\mathrm{AC}_0 - \mathrm{CC})$$

and

$$\beta_0 = \frac{[\mathrm{CC(CT)} - (\mathrm{AC}_0)X_0](\mathrm{AC}_0 - \mathrm{NC}) - [\mathrm{NC(NT)} - (\mathrm{AC}_0)X_0](\mathrm{AC}_0 - \mathrm{CC})}{[(\mathrm{CT} - X_0)(\mathrm{AC}_0 - \mathrm{NC}) - (\mathrm{NT} - X_0)(\mathrm{AC}_0 - \mathrm{CC})]} \tag{9}$$

Finally, an expression for β_1 can be obtained from Eq. (4)

$$\beta_1 = (\mathrm{CT} - \beta_2)(\mathrm{CC} - \beta_0). \tag{10}$$

Since $X_j > 0$ and $\mathrm{AC}_j \geq 0$ the model is considered in the first quadrant of the Cartesian plane. For simplicity, the subscript j will be dropped and the shape of the graph of the activity cost function will be investigated by checking its first and second derivatives

$$\mathrm{AC}' = -\beta_1(X - \beta_2)^{-2} \quad \text{and} \quad \mathrm{AC}'' = 2\beta_1(X - \beta_2)^{-3}$$

AC concaves up at a point X if

$$\beta_1 > 0 \ \text{and} \ X > \beta_2 \tag{10a}$$

$$\beta_1 < 0 \ \text{and} \ X < \beta_2 \tag{10b}$$

and concaves down at a point X if

$$\beta_1 < 0 \ \text{and} \ X > \beta_2 \tag{10c}$$

$$\beta_1 > 0 \ \text{and} \ X < \beta_2 \tag{10d}$$

Consider the case where AC concaves up (Fig. 1). As can be seen from the graph, β_o is the horizontal asymptote and β_2 is the vertical asymptote and $\beta_o < \mathrm{CC}$, $\beta_2 < \mathrm{CT}$. Therefore $\beta_1 > 0$ as can be confirmed from Eq. (10). Additionally, $X > \beta_2$ for $\mathrm{CT} \leq X$

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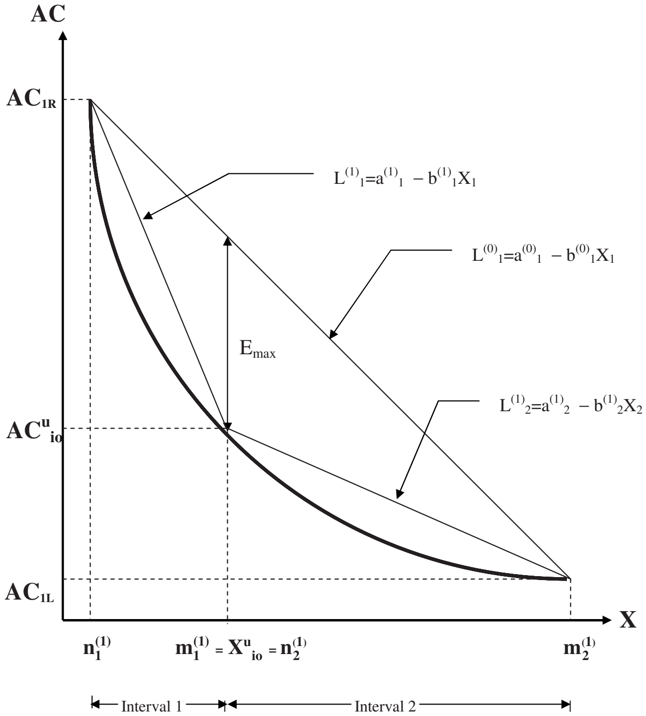

Fig. 3. Splitting the first interval

$\leq$NT. Thus, condition Eq. (10*a*) for AC to concave up is satisfied whereas condition Eq. (10*b*) is not feasible.

The graph of an AC curve that concaves down would be obtained when the entire graph of Fig. 1 is transposed into the first quadrant of the Cartesian plane, where the left side would be represented as the solid line. It can be deduced from such a graph, $\beta_o > \text{CC}$ and $\beta_2 > \text{CT}$, where β_o and β_2=horizontal and vertical asymptotes, respectively. Therefore, $\beta_1 > 0$, as can be confirmed from Eq. (10). Also, $X < \beta_2$ for $\text{CT} \leq X \leq \text{NT}$. Thus, condition Eq. (10*d*) for AC to concave down is satisfied whereas condition Eq. (10*c*) is not feasible.

Linear Approximation of the Time-Cost Function

Next, linear approximation of the activity cost curve is illustrated which will be used in the mathematical models presented below. First, consider the case where the cost curve for activity j concaves up (Fig. 3). Again, for simplicity the subscript j will be dropped. In this case, the activity cost function is strictly convex and can be approximated with a series of linear functions. For example, consider the equation of a line segment $L_i = a_i - b_i X_i$, where a_i and b_i=y-intercept and slope, respectively, of the line approximating a segment of the activity cost curve. This line passes through the end points of interval i (for Fig. 2, $i=1$), and will always be above the curve; therefore it will always overestimate the correct cost in that interval (the opposite is true for cost curves that concave down). The first interval is the entire cost curve whose end points have the coordinates $[(\text{NT}_j, \text{NC}_j), (\text{CT}_j, \text{CC}_j)]$.

For a cost curve that concaves up, the error of estimation, E_i, will be given by

$$E_i = L_i - \text{AC}_i = a_i - b_i X_i - \frac{\beta_1}{X_i - \beta_2} - \beta_0 \tag{11}$$

The value of X_i at which maximum error (X^u_{io}) occurs can be determined as follows:

$$\frac{dE_i}{dX_i} = -b_i + \frac{\beta_1}{(X_i - \beta_2)^2} = 0$$

$$-b_i(X_i - \beta_2)^2 + \beta_1 = 0$$

$$(X_i - \beta_2)^2 = \frac{\beta_1}{b_i}$$

$$X^u_{io} = \sqrt{\frac{\beta_1}{b_i}} + \beta_2$$

It can be shown that this indeed is the activity duration at which the maximum error occurs: $d^2E_i/dX_i^2 = -2\beta_1/(X_i - \beta_2)^3$, substituting X^u_{io} into this equation and simplifying $d^2E_i/dX_i^2 = -2\beta_1/[\sqrt{\beta_1/b_i} + \beta_2 - \beta_2]^3 = -2\beta_1/(\beta_1)^{3/2}/(b_i)^{3/2} = -2\sqrt{(b_i)^3}/\sqrt{\beta_1} < 0$, since both $b_i > 0$ and $\beta_1 > 0$.

For a cost curve that concaves up, the activity cost (AC^u_{io}) that corresponds to X^u_{io} can now be calculated

$$\text{AC}^u_{io} = \frac{\beta_1}{X^u_{io} - \beta_2} + \beta_0 = \frac{\beta_1}{\left(\sqrt{\frac{\beta_1}{b_i}} + \beta_2\right) - \beta_2} + \beta_0 = \sqrt{\beta_1 b_i} + \beta_0$$

and, the maximum error ($E^u_{i\,\max}$) for interval i will be

$$E^u_{i\,\max} = L_i - \text{AC}^u_{io} = a_i - b_i X^u_{io} - \text{AC}^u_{io} = a_i - b_i\left(\sqrt{\frac{\beta_1}{b_i}} + \beta_2\right) - (\sqrt{\beta_1 b_i} + \beta_0) = a_i - \sqrt{\beta_1 b_i} - \beta_2 b_i - \sqrt{\beta_1 b_i} - \beta_0 = a_i - 2\sqrt{\beta_1 b_i} - \beta_2 b_i - \beta_0$$

In a similar fashion, it can be shown that for activities whose cost curves concave down, the value of X_i at which the maximum error (X^d_{io}) occurs is $X^d_{io} = \beta_2 - \sqrt{\beta_1/b_i}$, while its corresponding activity cost (AC^d_{io}) and its maximum error ($E^d_{i\,\max}$), for interval i, can be shown to be

$$\text{AC}^d_{io} = \beta_0 - \sqrt{\beta_1 b_i}, \quad \text{and} \quad E^d_{i\,\max} = |a_i + 2\sqrt{\beta_1 b_i} - \beta_2 b_i - \beta_0|$$

The maximum error can be reduced to any finite number by increasing the number of line segments. Once the maximum error a decision maker is willing to tolerate (TE) is determined, the next task is to split each interval into as many subintervals of activity times (X) as necessary so that the magnitude of the error in estimating AC is not more than TE. An efficient way is to split the intervals at the point where the error E_i is at maximum, thereby reducing overestimation by the greatest amount.

Without loss of generality, consider an activity whose cost curve concaves up. Let the first interval be labeled as interval 1, which is approximated by the line segment $L_1^{(p)} = a_1 - b_1 X_1$, where the superscript p represents the iteration number and is set equal to 0 at the beginning of the process (Fig. 2). Also, some values will be identified by the iteration at which they are calculated. For example, $b_1^{(2)}$ represents the slope of the line that approximates the curve in subinterval 1 at the second iteration. As shown in Fig. 2, abscissa of the end points of each interval will be represented with letters n and m. Therefore, the coordinates of the end points of any interval at iteration p are

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$$\left(n_1^{(p)}, \frac{\beta_1}{n_1^{(p)}-\beta_2}+\beta_0\right) \quad \text{and} \quad \left(m_1^{(p)}, \frac{\beta_1}{m_1^{(p)}-\beta_2}+\beta_0\right) \qquad (12)$$

and the parameters of the line passing through these points $L_i = a_i - b_i X_i$, are

$$b_i^{(p)} = \left|\frac{AC_{iL}^{(p)} - AC_{iR}^{(p)}}{n_i^{(p)} - m_i^{(p)}}\right| \quad and \quad a_i^{(p)} = \left|\frac{AC_{iL}^{(p)} - AC_{iR}^{(p)}}{n_i^{(p)} - m_i^{(p)}}\right| X_i + L_i^{(p)} \qquad (13)$$

where subscripts L and R=left and right end of the relevant interval; and X_i=feasible activity time in this interval, i.e., $n_i \leq X_i \leq m_i$.

Next, the first interval is split into two for an improved approximation of the activity cost function; two subintervals are created using X_{io}^u as the split point (Fig. 3). The parameters of the two new equations $L_1^{(1)}$ and $L_2^{(1)}$ can be determined according to Eqs. (12) and (13). Also, note that the left end of the leftmost subinterval and the right end of the rightmost subinterval will always remain the same regardless of the number of subintervals created. These are the end points that defined the first interval at iteration 0. Furthermore, it should be noted that $AC_{1R}=AC_{io}^u=AC_{2L}$. Next, $a_1^{(1)}$ and $b_1^{(1)}$ for line segment $L_1^{(1)}$ should be computed. The parameter $b_1^{(1)}$ can be computed according to Eq. (13). To determine $a_1^{(1)}$ values of X and $L_1^{(1)}$ should be substituted in Eq. (13); any of the end points of the relevant interval can be used. The right end point for the first interval will be used. This is the left end point of the second interval, whose coordinates are $(X_{io}^u,\ AC_{io}^u)$. Thus

$$a_i^{(1)} = \left|\frac{AC_{1L}^{(1)} - AC_{io}^u}{n_i^{(p)} - m_i^{(p)}}\right| \left(\sqrt{\frac{\beta_1}{b_i}} + \beta_2\right) + AC_{io}^u \qquad (14)$$

Similarly, the slope and y-intercept for the line segment approximating the second subinterval can be calculated, for example, using the coordinates of its left end, whose coordinates are $(X_{io}^u,\ AC_{io}^u)$. This process of splitting activity cost intervals into subintervals can be repeated until the maximum error in every subinterval is below TE.

Model I

$$\text{Min } Z = ES_{FIN} + PT \qquad (15)$$

subject to:

$$L_{ij} = a_{ij}Y_{ij} - b_{ij}X_{ij} \qquad (16)$$

$$\sum_{i=1}^{e_j} Y_{ij} = 1 \qquad (17)$$

$$X_j = \sum_{i=1}^{e_j} X_{ij} \qquad j = 1,2,\ldots,p; \qquad (18)$$

$$X_{ij} \geq n_{ij}Y_{ij} \qquad i = 1,2,\ldots,e_j; \qquad (19)$$

$$X_{ij} \leq m_{ij}Y_{ij} \qquad k > j \qquad (20)$$

$$X_j \leq NT_j \qquad (21)$$

$$X_j \geq CT_j \qquad (22)$$

$$ES_k - ES_j - X_j \geq 0 \qquad (23)$$

$$ES_k - ES_j - X_j + MZ_{kj} \leq M \qquad (24)$$

$$\sum_{j \in I(k)} Z_{kj} = 1 \qquad (25)$$

k = all activities that have immediate predecessors and all $j \in I(k)$. Where $I(k)$ = the set of activities j that are immediate predecessors to activity k.

$$LS_k - LS_j + X_k \leq 0 \qquad (26)$$

$$LS_k - LS_j + X_k - MW_{kj} \geq -M \qquad (27)$$

$$\sum_{j \in J(k)} W_{kj} = 1 \qquad (28)$$

k = all activities that have immediate successors and all $j \in J(k), j < k$. Where $J(k)$ = the set of activities j that are immediate successors to activity k

$$ES_{FIN} = LS_{FIN} \qquad (29)$$

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$$\sum_{\text{All } j} \sum_{i=1}^{e_j} L_{ij} - \sum_{\text{All } j} \text{NC}_j - (\text{PT})(\text{CB}) \leq 0 \tag{30}$$

$$\text{PT} \leq 1.0 \tag{31}$$

$$L_{ij},\ X_j,\ X_{ij},\ ES_j,\ LS_j \geq 0 \ \text{ for all } i \text{ and } j; \quad \text{PT} \geq 0 \tag{32}$$

$$Y_{ij} = 0,\ 1, \quad \text{integers for all } i \text{ and } j \tag{33}$$

$$\text{Z}_{kj},\ \text{W}_{kj} = 0,\ 1, \quad \text{integers for all } k \text{ and } j \tag{34}$$

L_{ij}=linear approximation of the direct cost of activity j if its completion time falls in interval i; X_{ij}=completion time for activity j in interval i; a_{ij}, b_{ij}=y-intercept and slope, respectively, of the approximating line for activity j, passing through the end points of the interval i; p=number of activities in the network; e_j=number of intervals into which AC_j has been partitioned; n_{ij}, m_{ij}=lower and upper end points of interval i; ES_j=earliest start time for activity j; LS_j=latest start time for activity j; ES_{FIN}=earliest start time for the last activity (Finish) of the project; $X_{\text{FIN}}=0$; LS_{FIN}=latest start time for the last activity (Finish) of the project; PT=percent of crash budget to be used; CB=crash (compression) budget; M=a very large positive number; Y_{ij}=auxiliary variables for activity j: $Y_{ij}=1$, if $X_j \in [n_{ij},\ m_{ij}]$; $Y_{ij}=0$, otherwise.

The objective Eq. 15 in this model is to minimize the completion time and percentage of crash budget used. Eq. 16 represents a linear approximation of the cost curve of activity j in interval i. Eq. 17 ensures that only one of the intervals is selected to represent the cost of activity j. Similarly, Eq.18 guarantees that only one completion time is selected for each activity. Eqs. 19 and 20 together ascertain that each potential completion time in interval i is within the bounds of the selected interval (i.e., if $Y_{ij}=1$). Eqs. 21 and 22 secure that completion time to be selected for activity j is within the bounds defined by the normal and crash times.

Eqs. 23–28 are included in the model to ensure correct computation of earliest and latest start times of all activities and determine the optimal compression schedule [see Moussourakis and Haksever (2007)]. Eq. 23 requires that the earliest start time (ES) of an activity is at least as great as its immediate predecessor's ES plus its activity time. Eq. 24 with the binary variable Z_{kj} makes sure that only one of the immediate predecessors determines the ES of activity k. This is accomplished in conjunction with Eqs. 23 and 25 for an activity that has immediate predecessors; the correct Z_{kj} is selected because any other Z_{kj}, $j \in I(k)$, set equal to one will create infeasibility in Eq. 23. Eq. 25 makes certain that only one Z_{ji} is assigned a value of one. Eqs. 26–28 perform a similar function in determining latest start time (LS) of activities with the aid of binary variable W_{kj}. While Eq. 26 requires that LS of an activity k is no greater than the LS of a successor activity minus time of activity k, Eqs. 27 and 28 secure that only one of the successor activities determines the LS of k. Once again, the correct W_{kj} is selected, because any other W_{kj}, $j \in I(k)$, set equal to one will create infeasibility in Eq. 26 for k.

Since the sum of all L_{ij} is equal to the sum of normal and crash costs of all activities, Eq. 30 is introduced to make sure that the crash budget is not exceeded. Finally, Eq. 31 is to limit total crash cost to at most 100% of the available crash budget.

The aforementioned model can readily include additional nonprecedence constraints on its variables. For instance, if an activity (e.g., activity k) is not permitted to start before a specific date such a requirement can be satisfied by the additional constraint $\text{ES}_k \geq T$.

Computer Implementation

Two types of software are required for the implementation of Model I: one, a linear programming code that can also handle integer variables, and two, a program that approximates the number of orders function by piecewise linearization. Express Solver Engine of Frontline Systems Inc. is used for the first task. The second software can be developed relatively easily by anyone with basic programming skills; for this purpose a computer code, SplitV4, in Visual Basic for Applications (VBA) for Excel has been developed. This program splits the number of activity cost functions into subintervals that are approximated by linear equations.

Example: Next, an example project will be used to demonstrate the implementation and results from Model I as well as two other models presented below. The project has seven activities; data for the activities are given at the top of Table 1. Each activity in this project has a different shape cost function; only one of the cost functions is linear (Fig. 4).

First, a crash budget of \$0 was assumed; the model determined 154 weeks as the project completion time when all activities are performed at their normal times. Then a crash budget of \$400,000 was assumed and the model found it optimal to use the entire budget and compressed some activities for a project completion in 88.26 weeks. Table 1 shows part of the output from the solution of Model I for the example. An alternative use of this model is to increase the right-hand-side of Eq. 31) to a sufficiently large number and let the model determine the size of the crash budget that will be needed to minimize the project completion time. For example, when the problem was solved with a crash budget of \$400,000 the optimal solution indicated that the crash budget will need to be increased by a multiple of 2.44 (=PT) to complete the project in 59 weeks.

Model II

Model I can be considered as the basic model from which other models are developed, each with a different focus. For example, if the project manager's main concern is to minimize the total

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Table 1. Model I Implementation and Solution of Example Problem

A	B	C	D	E	F	G	FIN	PT	89.26 Wks	Objective function=ES_{FIN}+PT
18	30	36	25	29	32	39		1.00		Normal time (NT_j) in Wks
4	11	13	20	26	28	32				Accelerated time ($X_{\Delta Cj}$) in Wks
2	8	6	10	22	9	18				Crash time (CT_j) in Wks
20	65	105	100	20	50	155				Normal cost (NC_j) in \$1,000
75	100	130	160	190	295	245				Accelerated cost ($AC_{\Delta Cj}$) in \$1,000
175	250	275	280	200	320	270				Crash cost (CC_j) in \$1,000
0	1	1	1	2	1	4				No. of immediate predecessors
	A	A	A	B, C	E	B, C, D, F				Immediate predecessor ID
6.7829	57.9235	95.5854	0	208.3077	325.9276	288.5484				=BETA0 CRASH BUDGET= \$400,000
229.5054	161.6381	298.0780	0	60.8379	139.3268	452.3413				=BETA1
0.6357	7.1585	4.3386	0	29.3231	32.5049	42.3871				=BETA2
4.71	11.54	11.54	25.00	22.00	32.00	18.00	0.00			X (activity time)
0.00	4.71	4.71	4.71	16.26	38.26	70.26	88.26			ES (earliest start)
0.00	4.71	4.71	45.26	16.26	38.26	70.26	88.26			LS (latest start)
13.29	18.46	24.46	0.00	7.00	0.00	21.00	0.00			Number of weeks crashed

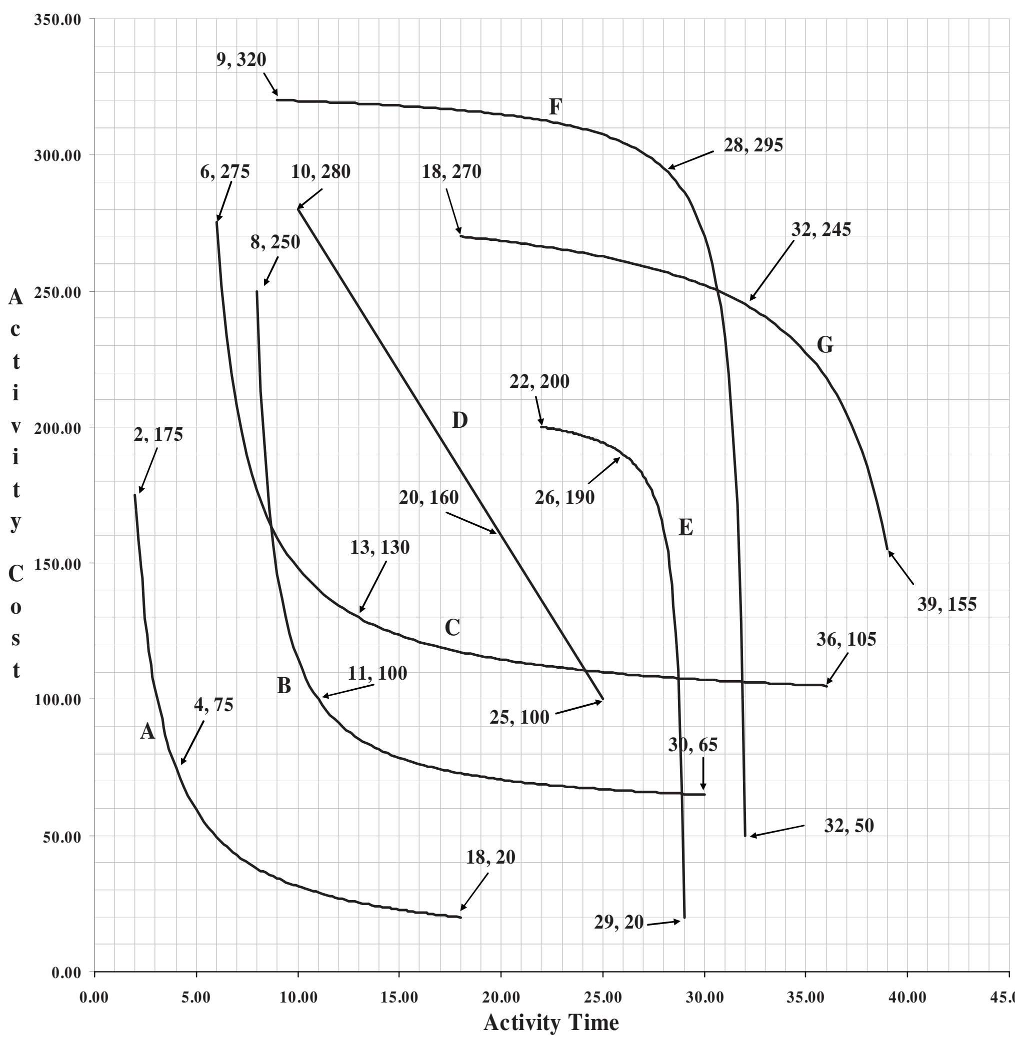

Fig. 4. Cost curves of example project

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Table 2. Model II Implementation and Solution of Example Problem

A	B	C	D	E	F	G	FIN	\$3,115,715.00	=objective Function=$\Sigma\Sigma L_{ij}$+IC(ES_{FIN})		
3.21	10.06	10.06	25.00	22.00	9.00	18.00	0.00		X (activity time) in weeks	D=	65 weeks
0.00	3.21	3.21	3.21	13.27	35.27	44.27	62.27		Earliest start in weeks	Indir. Cost=	\$30,000
0.00	3.21	3.21	19.27	13.27	35.27	44.27	62.27		Latest start in weeks		
14.79	19.94	25.94	0.00	7.00	23.00	21.00	0.00		No. of weeks crashed		

project cost (total direct and indirect costs), an alternative model can be formed with the following objective function

$$\text{Min } Z=\sum_{\text{All } j}\sum_{i=1}^{e_j} L_{ij}+(\text{IC})\text{ES}_{\text{FIN}}$$

and the following constraint replacing Eqs. 30 and 31

$$\text{ES}_{\text{FIN}} \le D$$

where IC=indirect cost per time period; and D is the target (desired) project completion deadline. For example, when $D=65$ and IC=\$30,000 for the example problem, solution of Model II resulted in an optimal project completion time of 62.27 weeks and total cost of \$3,115,715 (Table 2).

Model III

A possible third model can be developed when the focus is on minimizing total project cost in the presence of contractual penalties (rewards) for completing the project late (early). Specifically, a penalty will be paid by the project management when $\text{ES}_{\text{FIN}}-D>0$ and a reward, or bonus, will be earned when $D-\text{ES}_{\text{FIN}}>0$. Then, an objective function can be formulated to represent the total cost, total indirect cost, late completion penalty, and early completion reward by introducing two new variables, P_E and R_E, as follows:

$$\text{Min TC}=\sum_{j=1}^{k}\sum_{i=1}^{e_j} L_{ij}+(\text{IC})\text{ES}_{\text{FIN}}+C_P P_E-C_R R_E$$

s.t.

Eqs. 16–29

$$D-\text{ES}_{\text{FIN}} \le V_1 M \quad (35)$$

$$D-\text{ES}_{\text{FIN}} \ge -V_2 M \quad (36)$$

$$P_E \ge (\text{ES}_{\text{FIN}}-D) \quad (37)$$

$$R_E \le (D-\text{ES}_{\text{FIN}})+V_2 M \quad (38)$$

$$R_E \le V_1 M \quad (39)$$

$$V_1+V_2=1 \quad (40)$$

P_E, $R_E \ge 0$, V_1, $V_2=0,1$, integers, and nonnegativity restrictions Eq. 32, binary requirements Eqs. 33 and 34.

Where P_E=number of time units of late project completion; R_E=number of time units of early project completion; C_P=penalty per time units of late project completion; and C_R=reward per time units of early project completion.

Eqs. (35), (36), and (40) determine whether the project is to be completed late with a penalty ($V_1=0$, $V_2=1$) or early with a bonus ($V_1=1$, $V_2=0$). Eq. (37) together with the nonnegativity constraint PE$\ge$0 determines the number of time units the project will be completed late. Eqs. (38) and (39), and the nonnegativity constraint RE$\ge$0 determine the number of time units the project will be completed early.

When a target date D of 65 weeks, an indirect cost of \$30,000 per week, a penalty of \$100,000 per week, and a bonus of \$25,000 per week were assumed, Model III indicated an optimal solution of 60.91 weeks for project completion at a total cost of \$3,027,420 (Table 3). In this case the project management would receive \$102,250 as a bonus for completing the project 4.09 weeks earlier than the target date.

Conclusion

This paper has presented a mathematical model that can accommodate rational functions to represent activity time-cost relationships that are continuous and bounded. The type of function used is very flexible; depending on its parameters, it can represent a convex or concave cost function. For this type of function, in addition to the normal (maximum) and crash (minimum) activity times, a user needs to determine a third activity time at which the cost of crashing starts to change dramatically. The model is based on piecewise linear approximation of nonlinear activity cost functions. A mixed-integer programming model has been developed that enables the project manager to finish the project on time and without cost overruns, in the presence of highly complex but realistic cost relationships. The first version of the model focuses on completing the project as early as possible and under a crash budget constraint. Also presented were two different versions of the basic model; one, when the project manager focuses on minimizing total project cost, and two, when the objective is to minimize the total cost when the project is subject to late completion penalties or early completion bonuses. It is believed these models

Table 3. Model III Implementation and Solution of Example Problem

A	B	C	D	E	F	G	FIN	\$3,027,420.00	Objective Func.=$\Sigma\Sigma L_{IJ}$+(IC)(ES_{FIN})+$C_P P_E$-$C_R R_E$	
2.83	9.08	9.08	25.00	22.00	9.00	18.00	0.00	X (activity time) in weeks	D=	65 Wks
0.00	2.83	2.83	2.83	11.91	33.91	42.91	60.91	Earliest start in weeks	Indir.cost=	\$30,000
0.00	2.83	2.83	17.91	11.91	33.91	42.91	60.91	Latest start in weeks	Penalty=	\$100,000
15.17	20.92	26.92	0.00	7.00	23.00	21.00	0.00	No. of weeks crashed	Bonus=	\$25,000

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may help the project manager make rational decisions about project compression under realistic assumptions.

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