MAT186 Calculus I
Assignment 3

Instructions:

Please read the **MAT186 Assignment Policies & FAQ** document for details on submission policies, collaboration rules and academic integrity, and general instructions.

1. **Submissions are only accepted by** Gradescope. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
2. **Submit solutions using only this template pdf**. Your submission should be a single pdf with your full written solutions for each question. If your solution is not written using this template pdf (scanned print or digital) then your submission will not be assessed. Organize your work neatly in the space provided. Do not submit rough work.
3. **Show your work and justify your steps** on every question but do not include extraneous information. Put your final answer in the box provided, if necessary. We recommend you write draft solutions on separate pages and afterwards write your polished solutions here on this template.
4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for this assignment.

Academic Integrity Statement:

Full Name: Kenny Guo

Student number: 1010 207 312

Full Name: __________

Student number: __________

I confirm that:

- I have read and followed the policies described in the document **MAT186 Assignment Policies & FAQ**.
- In particular, I have read and understand the rules for collaboration, and permitted resources on assignments as described in subsection II of the the aforementioned document. I have not violated these rules while completing and writing this assignment.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while completing and writing this assignment.

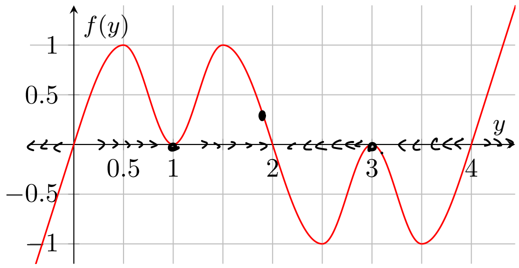
By signing this document, I agree that the statements above are true.

Signatures: 1) Kenny Guo

2) __________

1. Consider the autonomous ordinary differential equation, $y'(t) = f(y(t))$, with the graph of $f(y)$ vs. y shown below.

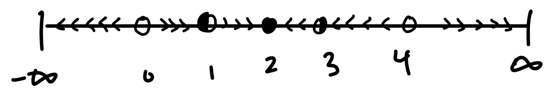
$y'(t) = 0$
$f(y(t)) = 0$
$f(y) = 0$



(a) Identify all equilibrium points for this differential equation and classify them as stable, semi-stable or unstable.

Equilibrium points when $f(y) = 0$

$f(y) = 0$ at $y = 0, 1, 2, 3, 4$



$-\infty < f(y) < 0$: $(-)$
$0 \le f(y) < 1$: $(+)$
$1 < f(y) < 2$: $(+)$
$2 < f(y) < 3$: $(-)$
$3 < f(y) < 4$: $(-)$
$4 < f(y) < \infty$: $(+)$

Stable : $y = 2$

unstable: $y = 0, 4$
semi-stable: $y = 1, 3$

(b) If $y(0) = 1.8$, does the solution $y(t)$ contain an inflection point at some time $t > 0$? If so, at approximately what value of y does it occur?

$y'(t) = f(y(t))$

$y(0) = 1.8$

In order for $y(t)$ to have an inflection point, $y''(t_i) = 0$, for some t_i

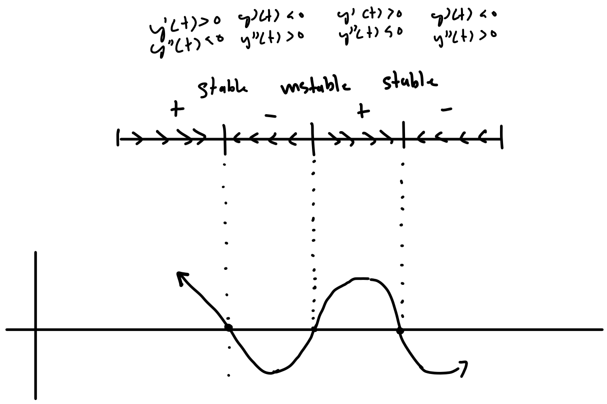
or $y'(t)$ reaches a local max/min

Given this initial value problem, $y'(1.8) \approx 0.25$. This means that $y'(t) > 0$, $t > 0$

we can see from the graph that $y''(t) < 0$, for $t > 0$

This means that $y(t)$ is concave down for all $t > 0$. y will get close to, but not equal to 2. Thus, there is no inflection point for the solution $y(t)$

1 (c) Consider the differential equation $y'(t) = g(y(t))$, where $g(y)$ is a continuous function. Is it possible for this differential equation to have *exactly* 2 equilibrium points both of which are stable?



For the function $y'(t)$, stable equilibrium is defined to be when solutions in neighbouring intervals approach the equilibrium. In other words, the function must first intersect the x axis. More specifically, two conditions must also be satisfied.

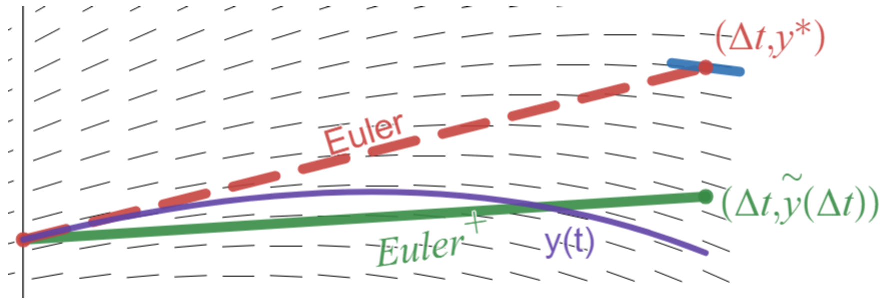
① For t values that are close to the left of this x-intercept, the function must have $y'(t) > 0$, and $y''(t) < 0$ (concave down).

② For t values that are close to the left of this x-intercept, the function must have $y'(t) < 0$, and $y''(t) > 0$ (concave up)

After one equilibrium point, the function will have $y'(t) < 0$. In order for the second stable equilibrium point to exist, the function must cross the x axis so that $y'(t) > 0$, as defined by the condition above.

Thus, this exceeds a total of 2 stable equilibrium points and we conclude a differential equation cannot have exactly 2 equilibrium points.

2. Consider the initial value problem $\begin{cases} y'(t) = y \\ y(0) = 1. \end{cases}$ $\frac{dy}{dt} = y$



To compute $\tilde{y}(\Delta t)$, Euler's method computes the slope at $(t, y) = (0, y(0))$ $(0,1)$ and follows a line with that slope until $t = \Delta t$. Between $t = 0$ and $t = \Delta t$, the true solution's slope may have changed significantly, but Euler's method doesn't update its slope until the next step. Our goal in this problem is to improve this method, so, we propose a new scheme, Euler$^+$.

1. Starting at $y(0)$, take one Euler step to find an approximation of $y(\Delta t)$. Call this approximation y^* (shown in red).
2. Compute the slope at $(\Delta t, y^*)$ (shown in blue).
3. Our approximation of $y(\Delta t)$ is given by $\tilde{y}(\Delta t) = y(0) + \Delta t S$, where S is the average of the slopes at $(0, y(0))$ and $(\Delta t, y^*)$ (shown in green).

(a) Write down an equation for an approximation for $y(1)$ using Euler$^+$. There should be no Δt and no $y(0)$ in your final answer. *Hint*: Take n steps of length Δt where $n\Delta t = 1$.

① at $t=0$, $y=1$

$y(0) = 1$ $(0,1)$

$t=0$, $y=1$

$y'(0) = 1$

$$\ell(\Delta t) = y'(0)(\Delta t) + y(0) = (1)(\Delta t) + 1 = \Delta t + 1$$

$$y^* = \Delta t + 1$$

$$\tilde{y}(\Delta t) = y(0) + \Delta t S = 1 + (\Delta t)\left(\frac{1 + \Delta t + 1}{2}\right) = 1 + \Delta t\left(\tfrac{1}{2}t + 1\right)$$

$\Delta t\left(1 + \frac{\Delta t}{2}\right) + 1$

$\Delta t + \frac{1}{2}\Delta t^2 + 1$

$$\tilde{y}(\Delta t) = \tfrac{1}{2}\Delta t^2 + \Delta t + 1$$

$$y^* = \tfrac{1}{2}\Delta t^2 + \Delta t + 1 = \frac{(\Delta t + 1)^2 + 1}{2}$$

② $$\ell(2\Delta t) = \left(\frac{(\Delta t+1)^2+1}{2}\right)(\Delta t) + \frac{(\Delta t+1)^2+1}{2} = \left(\frac{(\Delta t+1)^2+1}{2}\right)(\Delta t+1)$$

$$\tilde{y}(2\Delta t) = y(\Delta t) + \Delta t S = \frac{(\Delta t+1)^2+1}{2} + \Delta t\left(\frac{\left(\frac{(\Delta t+1)^2+1}{2}\right)(\Delta t+1)+1}{2}\right)$$

$$= \frac{(\Delta t+1)^2+1}{2} + \frac{\left(\frac{(\Delta t+1)+1}{2}\right)(\Delta t+1)(\Delta t)+\Delta t}{2}$$

$$= \frac{(\Delta t+1)^2+1}{2}\left(1 + \frac{\Delta t^2 + 2\Delta t}{2}\right)$$

$$= \left(\frac{(\Delta t+1)^2+1}{2}\right)\left(\tfrac{1}{2}\Delta t^2 + \Delta t + 1\right)$$

$$y^* = \left(\frac{(\Delta t+1)^2+1}{2}\right)^2$$

③ $$\ell(3\Delta t) = \left(\frac{(\Delta t+1)^2+1}{2}\right)^2 \Delta t + \left(\frac{(\Delta t+1)^2+1}{2}\right)^2 = \left(\frac{(\Delta t+1)^2+1}{2}\right)^2 (t+1)$$

$$\tilde{y}(3\Delta t) = \left(\frac{(\Delta t+1)^2+1}{2}\right)^2 + \Delta t\left(\frac{\left(\frac{(\Delta t+1)^2+1}{2}\right)^2(t+1)+1}{2}\right)$$

$$= \left(\frac{(\Delta t+1)^2+1}{2}\right)^2\left(\frac{\Delta t^2 + 2\Delta t}{2} + 1\right)$$

$$= \left(\frac{(\Delta t+1)^2+1}{2}\right)^2\left(\tfrac{1}{2}\Delta t^2 + \Delta t + 1\right)$$

$$= \left(\frac{(\Delta t+1)^2+1}{2}\right)^3$$

∴ we conclude,

$$\tilde{y}(n\Delta t) = \left(\frac{(\Delta t+1)^2+1}{2}\right)^n$$

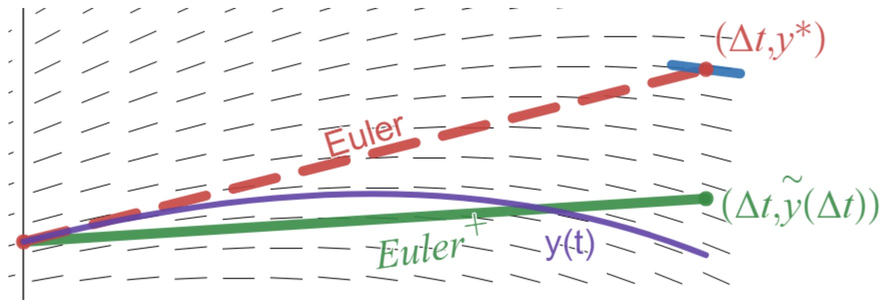
for $n\Delta t = 1$

$\Delta t = \frac{1}{n}$

$$\tilde{y}(1) = \left(\tfrac{1}{2}\Delta t^2 + \Delta t + 1\right)^n$$

$$\boxed{= \left(\frac{1}{2n^2} + \frac{1}{n} + 1\right)^n}$$

2. Consider the initial value problem $\begin{cases} y'(t) = y \\ y(0) = 1. \end{cases}$



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(b) The exact value of $y(1) = e$. By evaluating $\tilde{y}(1)$ at $n = 1, 2, 4, 8, \ldots$ is the error for Euler$^+$ proportional to the step size Δt (as we observed with Eu;er's method in PCE D4).

n	step size ($\Delta t = \frac{1}{n}$)	$y(1) = \left(\frac{1}{2n^2} + \frac{1}{n} + 1\right)^n$	error ($e - y(1)$)
1	1	2.5	0.218
2	$\frac{1}{2}$	2.64	0.0777
4	$\frac{1}{4}$	2.69	0.0234
8	$\frac{1}{8}$	2.711	0.00644

$\therefore$ Based on testing, the error for Euler+ is proportional to the step size, because as step size decreases, the error decreases

3. A bacterial culture is growing in a huge Petri dish in a laboratory. The area of the culture on the surface of the dish is measured at irregular intervals. Unfortunately, the data storing the exact times of each observation gets corrupted, but some data providing the time between pairs of observations (Observations (a) and (b)) was preserved. The preserved data is as follows:

Observation (a): cm^2	Time between Obs(a) and Obs(b):	Observation (b): cm^2
Obs. 1(a): 5	1 hour 0.65	Obs. 1(b): 5.65
Obs. 2(a): 10	2 hours 1.1	Obs. 2(b): 11.1
Obs. 3(a): 25	3 hours 0.75	Obs. 3(b): 25.75
Obs. 4(a): 30	2 hours 0.3	Obs. 4(b): 30.3

(a) Your job is to find a suitable differential equation which describes the change in area of the bacterial culture. Letting $y(t)$ denote the area of the culture at time t, you narrow it down to three possible differential equations:

- $y'(t) = ay + b.$
- $y'(t) = ay^2 + by + c$
- $y'(t) = ay^3 + by^2 + cy + d,$

where a, b, c and d are constants. Determine which of these differential equations best describes the data shown above and find the corresponding constants which match the differential equation to the data.

use $y'(t) = ay^3 + by^2 + cy + d$ to have the most options

let t be in hours

$\alpha(t) = y'(t)(\Delta t) + y(t)$

$5.65 = (a(5)^3 + b(5)^2 + c(5) + d)(1) + 5$

$5.65 = 125a + 25b + 5c + d + 5$

(1) $0.65 = 125a + 25b + 5c + d$

$11.1 = (a(10)^3 + b(10)^2 + c(10) + d)(2) + 10$

$11.1 = 2000a + 200b + 20c + 2d + 10$

(2) $1.1 = 2000a + 200b + 20c + 2d$

$25.75 = (a(25)^3 + b(25)^2 + c(25) + d)(3) + 25$

(3) $0.75 = 46875a + 1250b + 75c + 3d$

$30.3 = (a(30)^3 + b(30)^2 + c(30) + d)(2) + 30$

(4) $0.3 = 54000a + 1800b + 60c + 2d$

$$\left.\begin{aligned} 0.65 &= 125a + 25b + 5c + d \\ 1.1 &= 2000a + 200b + 20c + 2d \\ 0.75 &= 46875a + 1250b + 75c + 3d \\ 0.3 &= 54000a + 1800b + 60c + 2d \end{aligned}\right\} \text{ using 4x4 matrix calculator } \rightarrow \begin{aligned} a &\approx 0 \\ b &\approx 0 \\ c &= -0.02 \\ d &= 0.75 \end{aligned}$$

$\therefore$ $y'(t) = ay + b$ is the best model with $a = -0.02$ and $b = 0.75$

for $y'(t) = -0.02y + 0.75$

3. A bacterial culture is growing in a huge Petri dish in a laboratory. The area of the culture on the surface of the dish is measured at irregular intervals. Unfortunately, the data storing the exact times of each observation gets corrupted, but some data providing the time between pairs of observations (Observations (a) and (b)) was preserved. The preserved data is as follows:

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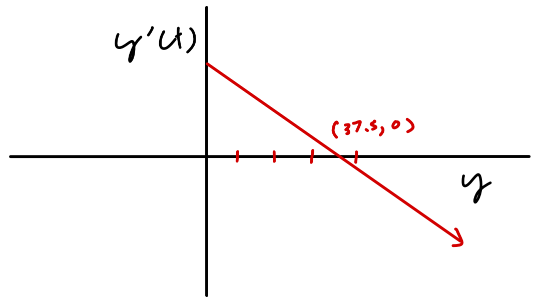
(b) Based on part (a), do you expect the area of the culture to continue growing towards an infinite area, or, do you expect it to approach a fixed size? If you expect the area to continue growing towards infinity, explain why, or, if you expect it to converge to a fixed size, determine what the final size is.

Based on given data, the graph of the derivative of t is a line that the graph on the left, assuming dom $y = t \in [0, \infty)$



$y'(t) = -0.02y + 0.75$

$0 = -0.02y + 0.75$

$y = 37.5\ cm^2$

We do not know the exact time that the data for area of the culture corresponds to. However, from the graph, we know:

for $y < 37.5$, $y'(t) > 0$ and $y''(t) \leq 0$ meaning y approaches 37.5 if $y < 37.5$

and,

for $y > 37.5$, $y'(t) \leq 0$ and $y''(t) > 0$ meaning y approaches 37.5 if $y > 37.5$

This means that $y = 37.5$ is a stable equilibrium solution. We know this is the only equilibrium solution and our data shows the area is not at the equilibrium solution since $y'(t) \neq 0$. Thus, all values of $t > 0$ for $y(t)$ will converge close to the value of $37.5\ cm^2$

$\therefore$ The final size will be approaching $37.5\ cm^2$