

The Hagge Circle

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In this note, I share one of my favorite results, the Hagge circle, which was first shown to me by Kapil Chandran. The proof follows from one nice observation after the other. First a useful definition.

Definition 1 (circumcevian triangle). Consider $\triangle ABC$ and a point P . Let AP, BP , and CP intersect the circumcircle of $\triangle ABC$ at A_1, B_1 , and C_1 . Then we call $\triangle A_1B_1C_1$ the **circumcevian triangle** of P w.r.t. $\triangle ABC$.

Then we can state our central result.

Theorem 2 (Hagge Circle). Let $\triangle A_1B_1C_1$ be the circumcevian triangle of a point P w.r.t. $\triangle ABC$. Let A', B' , and C' be the reflections of A_1, B_1 , and C_1 across sides BC, CA , and AB respectively. Then A', B', C' , and H are concyclic, where H is the orthocenter of $\triangle ABC$. We call this circle the **Hagge circle** of P w.r.t. $\triangle ABC$.

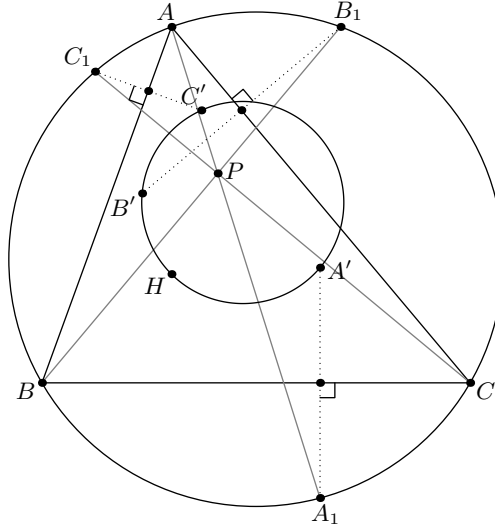


Figure 1: The Hagge Circle of P w.r.t. $\triangle ABC$

Proof. We first make the following key observation. Let Q be the isogonal conjugate of P w.r.t. $\triangle ABC$, and let $A_2B_2C_2$ be its circumcevian triangle. Then A', B' , and C' are the reflections of A_2, B_2 , and C_2 across the midpoints of sides BC, CA , and AB respectively. This allows us to define the Hagge in terms of the isogonal conjugate Q , as illustrated in Figure 2.

This turns out be the right way to frame the configuration. Let us now focus on the A -side of the diagram, which is shown in Figure 4. Let M be the midpoint of BC . Then

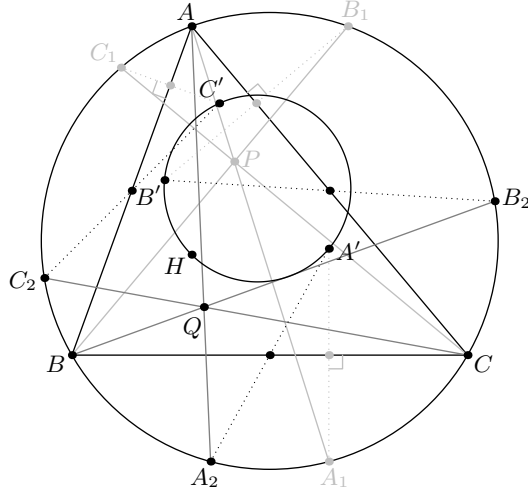


Figure 2: The Hagge Circle w.r.t. the isogonal conjugate

M is also the midpoint of $A'A_2$. By virtue of the centroid being two-thirds between a vertex and midpoint, the centroid G of $\triangle ABC$ is also the centroid of $\triangle AA'A_2$. Then the homothety $\mathcal{H}(G, -\frac{1}{2})$ sends A' to the midpoint of AA_2 and H to the circumcenter O of $\triangle ABC$.

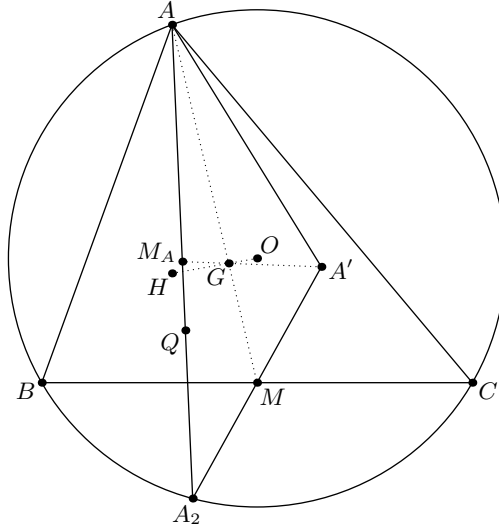


Figure 3: The A-side of the Hagge Circle

Now notice that the same transformation also sends B' and C' to the midpoints M_B and M_C of BB_2 and CC_2 respectively. Then instead of showing that A', B', C' , and H are concyclic, we can show that M_A, M_B, M_C , and O are concyclic.

We claim that M_A, M_B , and M_C all lie on the circle with diameter OQ . Indeed, since

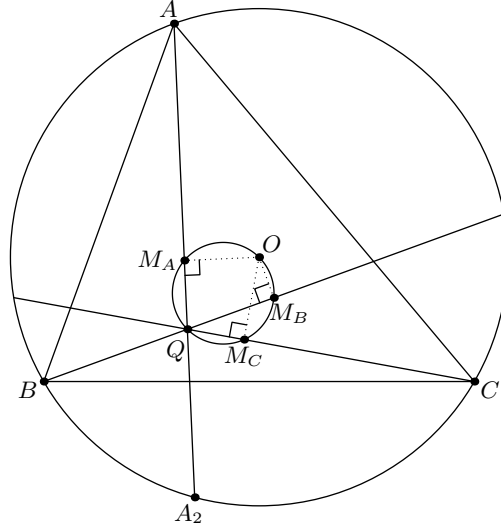


Figure 4: The Hagge Circle after $\mathcal{H}(G, -\frac{1}{2})$

M_A is the midpoint of a chord AA_2 , we know that $OM_A \perp AA_2$, so $OM_AQ = 90^\circ$. Similarly, we can show that M_B and M_C lie on the circle. \square

After framing the configuration in terms of the isogonal conjugate and applying a natural homothety centered at the centroid, we arrive at much simpler concyclicity. Moreover, this process tells us a little more about the Hagge circle.

Definition 3 (anticomplement). Let $\triangle ABC$ have centroid G . Let P' be the image of P under $\mathcal{H}(G, -2)$. Then P' is the **anticomplement** of P w.r.t. $\triangle ABC$.

Corollary 4. Let P' be the anticomplement of the isogonal conjugate of P in $\triangle ABC$. Then HP' is the diameter of the Hagge circle of P w.r.t. $\triangle ABC$.

We now consider the special case when we choose $P = I$, the incenter. The Hagge circle of I is called the **Fuhrmann circle**.

Proposition 5 (Properties of the Fuhrmann Circle). Let ω be the Fuhrmann circle of $\triangle ABC$, with center Fu . Then

- (i) HN_a is the diameter of ω where N_a is the Nagel point of $\triangle ABC$.
- (ii) $IO \parallel HN_a$.
- (iii) ω has radius IO .
- (iv) The nine-point center N is the centroid of parallelogram $OIH Fu$.
- (v) The Spieker center Sp is the centroid of parallelogram $OIFuNa$.

These properties follow more or less directly from properties of points on the Euler line and the Nagel line.

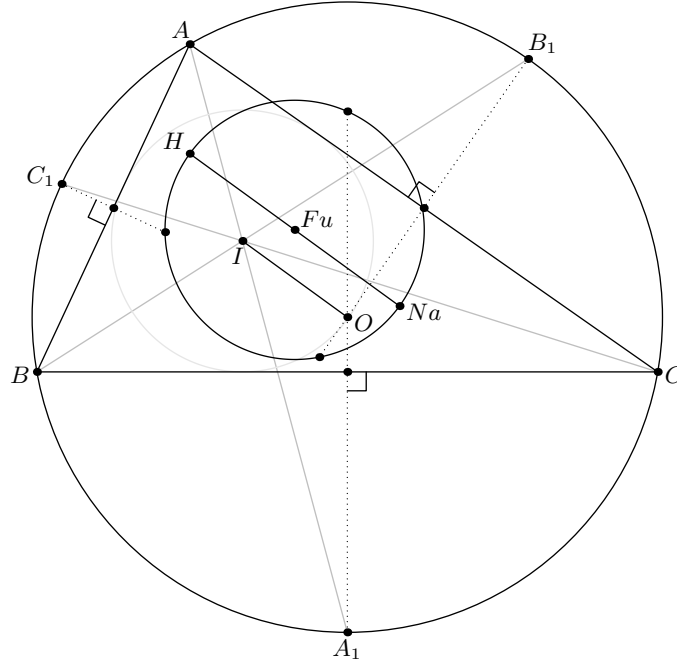


Figure 5: The Fuhrmann Circle

Proof. I is its own isogonal conjugate. Its anticomplement is the Nagel point Na . Then (i) follows from Corollary 4. We also know that $\mathcal{H}(G, -2)$ sends IO to NaH , implying (ii). This also tells us that the diameter has length $NaH = 2IO$, so the radius of ω is IO , showing (iii). (iv) follows from nine-point center being the midpoint of OH and (v) follows from the Spieker center being the midpoint of INa . \square

The Fuhrmann circle ties together the numerous results involving homothety about the centroid.

References

- [1] Weisstein, Eric W. "Fuhrmann Circle." From MathWorld—A Wolfram Web Resource. <http://mathworld.wolfram.com/FuhrmannCircle.html>