- This exam is open-book and open-note. You may not use any resources other than your book and class notes. You may not use your problem sets or solutions. You may not use the Internet, but you should keep your e-mail open in case I need to send you an announcement via e-mail. If you have a question, I am in a Zoom meeting:

  Section C (Tuesday): `washington.zoom.us/j/95663676863`.

  Please keep yourself muted and private message me questions.

- Since the exam is open-note, you may use any theorems that we have stated **and proved** in the notes. If you are using a theorem, make sure to cite it with a theorem number, name, or description. For example, you could cite the Cancellation Theorem with

  - "by the Cancellation Theorem," or
  - "by Theorem 39 in the notes," or
  - "by Theorem X.Y in the textbook," or even
  - "in a theorem, we proved that if $a, b, c \in \mathbb{R}$ and $a + c = b + c$, then $a = b$."

  The important thing is to acknowledge that you are using a result that we have already proved, in a way that the reader knows that you know what you are talking about.

- When writing up your solutions, you must follow the following guidelines:

  1. You must write your exam on physical sheets of paper. You may not use your computer/tablet.
  2. In the top left corner of the first page, write your name. Underneath your name, write your student ID. Underneath this, you should write your solution to problem 1.
  3. You must start each new problem on a **new page**. (If a problem has multiple parts, you do not need to start a new page for each part. Only new problems must be started on new pages.)
  4. Each problem and part must be clearly numbered. You **do not** have to copy down the statement of the problem.
  5. You must scan your exam as a **black and white pdf.** (I suggest the Genius Scan app on your phone).

- Submit your exam under the "Final Exam" assignment on Gradescope. You will have **one hour and fifty minutes** for this exam, with an additional fifteen minutes to scan and upload your exam. This means your exam needs to be on Gradescope 4:35pm. **No late submissions will be accepted.** You have been warned.

- If you are having issues during the submission process, join the Zoom meeting and let me know immediately.

Each problem is worth 10 points, for a total of 80 points.

1. For each $r \in \mathbb{Q}$, let A_r be the set containing all real numbers *other than* r.
   (a) Write A_r in set builder notation.

   Determine the following, and prove that your answer is correct.

   (b) $\displaystyle\bigcup_{r\in\mathbb{Q}} A_r$.

   (c) $\displaystyle\bigcap_{r\in\mathbb{Q}} A_r$.

2. In this problem, you may use any theorems stated in the notes, even if we did not include a proof in the notes.

   Prove that if A and B are countable sets then $A \times B$ is countable.

3. Suppose that $S \subseteq \mathbb{Z}$ with $|S| = 10$. Show that there exist two distinct integers $a, b \in S$ such that $a - b$ is a multiple of 9.

   [Hint: Consider the integers in S modulo 9.]

4. For each positive integer n, define
$$S_n = \sum_{i=1}^{n} \frac{i}{(i+1)!}.$$
   (a) Give a table for the first four values S_1, S_2, S_3, S_4.
   (b) Give an explicit formula for S_n.
   [By "explicit formula", I mean something you could substitute into to compute any term.]
   (c) Prove that your formula in part (b) is correct.

5. In this problem, all sets are subsets of some universal set U. Determine if the following statements are true or false. If the statement is true, give a complete proof. If the statement is false, give an explicit counterexample described **both** in words as well as in a clearly labeled Venn diagram.
   (a) For all sets A, B, and C, if $B \cap C \subseteq A$ then $(A \setminus B) \cap (A \setminus C) = \emptyset$.
   (b) For all sets A, B, and C, $(A \setminus B) \setminus C = A \setminus (B \cup C)$.

6. Let Q denote the set of all quadratic polynomials in one variable x with coefficients in $\mathbb{R}$
$$Q = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\}.$$
   Define a function $g : Q \to \mathbb{R}^2$ which maps a polynomial $f(x) \in Q$ to the ordered pair $(f(0), f(1))$.

   For example, if $f(x) = 3x^2 + 2x - 3$, then $g(f) = (3, 2)$.
   (a) Is g injective? Prove it.
   (b) Is g surjective? Prove it.

7. Let $A = \{a, b, c\}$.

   Suppose R and S are two **equivalence** relations on A. Recall that we can view R and S as subsets of $A \times A$. Then $R \cup S \subseteq A \times A$ so $R \cup S$ is also a relation on A. For each of the following questions, if the statement is true, give a proof. If the statement is false, give a counterexample. [Any counterexamples may be given by giving subsets of $A \times A$ for R, S, and $R \cup S$.]

   (a) Is $R \cup S$ reflexive?

   (b) Is $R \cup S$ symmetric?

   (c) Is $R \cup S$ transitive?

8. Suppose that $f : A \to B$ is an injective function.

   (a) Prove that if X, Y are sets satisfying $X \subseteq Y \subseteq A$ then $f(Y \setminus X) = f(Y) \setminus f(X)$.

   (b) State the converse to the statement in part (a).

   (c) Determine whether the converse is true. If you claim it is true, then prove it. If you claim it is false, then give a counterexample showing a case where it fails.