1. The augmented matrix for a system of linear equations is $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & -14 & 8 \\ 2 & 3 & k^2 & k \end{array}\right]$. Determine the value(s) of k for which the system has no solutions:

   a. $k \neq 4$
   b. $k = -4$
   c. $k = 4, k = -4$
   d. $k \neq 4, k \neq -4$
   e. $k = 4$
   f. None of the above.

row reduce

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & -14 & 8 \\ 2 & 3 & k^2 & k \end{array}\right] \xrightarrow{-2R_1+R_3=R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & -14 & 8 \\ 0 & -1 & k^2-2 & k-4 \end{array}\right] \xrightarrow{R_2+R_3=R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & -14 & 8 \\ 0 & 0 & k^2-16 & k+4 \end{array}\right]$$

Take the last row

↳ $k^2-16 = k+4$

$(k-4)(k+4) = k+4$

$k=4$ will result in no solutions

$0 \neq 8$

e

2. The matrices A, B, and C are given by:

$$A = \begin{bmatrix} -2 & 4 \\ 3 & -2 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} 4 & -2 & -4 \\ 3 & 6 & -2 \end{bmatrix}, \text{and } C = \begin{bmatrix} 2 & 1 \\ 0 & -3 \\ 4 & 3 \end{bmatrix}$$

And $D = AB - 2C$. The entry $D_{2,3}$ is:

2nd row 3rd col

So we will only be concerned with row 2 and col 3 of A and B.

   a. 18
   b. -2
   c. -6
   d. -14
   e. -4
   f. $D_{2,3}$ is not defined.

$$AB = \underbrace{\begin{bmatrix} 3 & -2 \end{bmatrix}}_{A_{row2}} \cdot \underbrace{\begin{bmatrix} -4 \\ 2 \end{bmatrix}}_{B_{col3}} = -12 - 4 = -16$$

★ $C_{2,3}$ does not exist

↳ $-16 - 2C_{2,3} = -16 - 2(\text{Null}) = \text{Null}$ f

3. Let matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 1 & -3 \\ 0 & 4 & 1 \end{bmatrix}$. Find the element in the $(1,2)$ position of A^{-1}.

a. 1

b. 7

c. -12

d. 12

e. 16

f. None of the above.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ -1 & 1 & -3 & 0 & 1 & 0 \\ 0 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 + R_2 = R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-4R_2 + R_3 = R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -4 & -4 & 1 \end{array}\right]$$

$$\xrightarrow{-3R_3 + R_1 = R_1} \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 13 & 12 \\ 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \end{array}\right]$$

This is position $A^{-1}_{1,2}$, we can stop here.

d

4. The determinant of the matrix of coefficients of the system of equations

$$2x + y - 2z = 0$$
$$4y - 4z = 3$$
$$3x + 2y - 2z = -1$$

Is 12. Give the value of z in the solution set.

$$\longrightarrow A = \begin{array}{cccc} x & y & z & \text{right side} \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 0 & 4 & -4 & 3 \\ 3 & 2 & -2 & -1 \end{array}\right]$$

sub the coefficients of z with the coefficients of the right side of the equation

a. $-\frac{11}{12}$

b. $-\frac{5}{4}$

c. $-\frac{3}{2}$

d. $-\frac{5}{12}$

e. $\frac{1}{3}$

f. None of the above.

$$\begin{array}{ccc} x & y & \text{right side} \end{array}$$
$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 3 \\ 3 & 2 & -1 \end{bmatrix}$$

Now find the determinant of this new matrix.

$$\longrightarrow \det Z = 2\begin{vmatrix} 4 & 3 \\ 2 & -1 \end{vmatrix} - 1\begin{vmatrix} 0 & 3 \\ 3 & -1 \end{vmatrix} + 0$$

$$\det Z = 2(-4-6) - (-9)$$

$$\det Z = -11$$

$$z = \frac{\det Z}{\det A}$$ ← given in problem statement

$$z = \frac{-11}{12}$$

a

5. Find the values of x such that the vectors

$$v_1 = (0,1,x), \qquad v_2 = (x,0,-4), \qquad v_3 = (-2,-1,-2)$$

Are linearly independent.

a. $x = 4, x = -2$

b. x is any number except 4 and -2

c. x is any number except -4 and 2

d. x is any number except 4 and 2

e. $x = -4, x = -2$

f. All real numbers.

Augmented matrix

$$\hookrightarrow A = \left[\begin{array}{ccc|c} 0 & 1 & x & 0 \\ x & 0 & -4 & 0 \\ -2 & -1 & -2 & 0 \end{array}\right]$$

$$\det A = 0 - 1\begin{vmatrix} x & -4 \\ -2 & -2 \end{vmatrix} + x\begin{vmatrix} x & 0 \\ -2 & -1 \end{vmatrix}$$

$$\det A = -(-2x-(-8)) + x(-x)$$

$$\det A = 2x - 8 - x^2$$

$$\det A = -x^2 + 2x - 8 \rightarrow x^2 - 2x + 8 = 0$$

$$(x-4)(x+2) = 0$$

For the vectors to be linearly independent, $\det A \neq 0$ so $x \neq 4$, $x \neq -2$

b

6. Find the eigenvalues of $A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 3 & 4 \end{bmatrix}$.

a. $-2, 1, 3$

b. $2, -1, 3$

c. $2, 1, 3$

d. $-2, 1, -3$

e. $2, -1, -3$

f. All real numbers.

$$A - \lambda I = \left[\begin{array}{ccc|c} 2-\lambda & 2 & 0 & 0 \\ 0 & -\lambda & -1 & 0 \\ 0 & 3 & 4-\lambda & 0 \end{array}\right]$$

$$\det(A-\lambda I) = 2-\lambda\begin{vmatrix} -\lambda & -1 \\ 3 & 4-\lambda \end{vmatrix} - \cancel{0\begin{vmatrix} 2 & 0 \\ 3 & 4-\lambda \end{vmatrix}} + \cancel{0\begin{vmatrix} 2 & 0 \\ -\lambda & -1 \end{vmatrix}}$$

$$= 2-\lambda\left[-\lambda(4-\lambda) - (-3)\right]$$

$$= 2-\lambda(-4\lambda + \lambda^2 + 3)$$

$$= -8\lambda + 2\lambda^2 + 6 + 4\lambda^2 - \lambda^3 - 3\lambda$$

$$= -\lambda^3 + 6\lambda^2 - 11\lambda + 6$$

Synthetic division

guess 1	-1	6	-11	6
	$\downarrow$	-1	5	-6
	-1	5	-6	$\boxed{0}$

flipping signs

$$(\lambda-1)(\lambda^2 - 5\lambda + 6)$$

$$(\lambda-1)(\lambda-3)(\lambda-2)$$

$\lambda = 1 \quad \lambda = 3 \quad \lambda = 2$

c

7. The number $\lambda = 2 + 2i$ is an eigenvalue of $A = \begin{bmatrix} 2 & -4 \\ 1 & 2 \end{bmatrix}$. Independent eigenvectors for A are:

a. $v_1 = \binom{0}{1} + i\binom{2}{0}$, $v_2 = \binom{0}{1} - i\binom{2}{0}$

b. $v_1 = \binom{0}{2} + i\binom{1}{0}$, $v_2 = \binom{0}{2} - i\binom{1}{0}$

c. $v_1 = \binom{1}{0} + i\binom{0}{2}$, $v_2 = \binom{1}{0} - i\binom{0}{2}$

d. $v_1 = \binom{2}{0} + i\binom{0}{1}$, $v_2 = \binom{2}{0} - i\binom{0}{1}$

e. $v_1 = \binom{0}{-1} + i\binom{2}{0}$, $v_2 = \binom{0}{-1} - i\binom{2}{0}$

f. None of the above.

$A-(2+2i)I = \begin{bmatrix} \cancel{2}-(\cancel{2}+2i) & -4 \\ 1 & \cancel{2}-(\cancel{2}+2i) \end{bmatrix}$

$= \begin{bmatrix} -2i & -4 \\ 1 & -2i \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -2i & 0 \\ -2i & -4 & 0 \end{array}\right]$

$\xrightarrow{2iR_1 + R_2 = R_2} \left[\begin{array}{cc|c} 1 & -2i & 0 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow$ x_2 arb, $x_1 = 2ix_2$

↳ $x_1 = 2i$ when $x_2 = 1$

Proving this becomes 0

$2i(-2i) + (-4) = 0$

$-4i^2 - 4 = 0$

$4 - 4 = 0$ ✓

$\vec{v}_1 = \binom{2i}{1}$

$\vec{v}_1 = \binom{0}{1} + i\binom{2}{0}$

$\vec{v}_2$ is the conjugate of $\vec{v}_1$

↳ $\vec{v}_2 = \binom{0}{1} - i\binom{2}{0}$

(a)

8. **This is a written question.**

Given the system of equations

$$2x_1 + 4x_2 - 5x_3 + 3x_4 = 3$$
$$x_1 + 2x_2 - 3x_3 + 2x_4 = 0$$
$$-x_1 - 2x_2 + 4x_3 - x_4 = 7$$

(a) Write the augmented matrix for the system.

(b) Reduce the augmented matrix to row-echelon form.

(c) Give the solution set of the system.

a) a) $\left[\begin{array}{cccc|c} 2 & 4 & -5 & 3 & 3 \\ 1 & 2 & -3 & 2 & 0 \\ -1 & -2 & 4 & -1 & 7 \end{array}\right]$

b) $\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & 2 & -3 & 2 & 0 \\ 2 & 4 & -5 & 3 & 3 \\ -1 & -2 & 4 & -1 & 7 \end{array}\right]$

$\xrightarrow[R_1 + R_3 = R_3]{-2R_1 + R_2 = R_2} \left[\begin{array}{cccc|c} 1 & 2 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 1 & 7 \end{array}\right] \xrightarrow{-R_2 + R_3 = R_3} \left[\begin{array}{cccc|c} 1 & 2 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 2 & 4 \end{array}\right] \xrightarrow{\frac{1}{2}R_3 = R_3}$ b) $\left[\begin{array}{cccc|c} 1 & 2 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$

c) $x_4 = 2$

$x_3 = x_4 + 3 \longrightarrow x_3 = 5$

$x_1 + 2x_2 - 3x_3 + 2x_4 = 0$

↳ $x_1 + 2x_2 - 3(5) + 2(2) = 0$

so, c) $x_1 = -2x_2 + 11$

x_2 arb (x_2 is a free variable)

$x_3 = 5$

$x_4 = 2$

9. **This is a written question.**

Given the matrix

$$A = \begin{bmatrix} 8 & -12 & 6 \\ 3 & -4 & 3 \\ 3 & -6 & 5 \end{bmatrix}$$

$\lambda_1 = 5$ is an eigenvalue of A and $\lambda_2 = 2$ is an eigenvalue of A of multiplicity 2.

(a) Find the eigenvector(s) corresponding to $\lambda_1 = 5$.

(b) Find the eigenvector(s) corresponding to $\lambda_2 = 2$.

(c) Find the general solution $x' = Ax$.

a) $A - 5I = \left[\begin{array}{ccc|c} 3 & -12 & 6 & 0 \\ 3 & -9 & 3 & 0 \\ 3 & -6 & 0 & 0 \end{array}\right] \xrightarrow[\text{row by } 1/3]{\text{mult every}} \left[\begin{array}{ccc|c} 1 & -4 & 2 & 0 \\ 1 & -3 & 1 & 0 \\ 1 & -2 & 0 & 0 \end{array}\right] \xrightarrow[-R_1+R_3=R_3]{-R_1+R_2=R_2} \left[\begin{array}{ccc|c} 1 & -4 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right]$

$\xrightarrow{-2R_2+R_3=R_3} \left[\begin{array}{ccc|c} 1 & -4 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow$ x_3 arb, $x_2 = x_3$, $x_1 = 4x_2 - 2x_3$

$\hookrightarrow \left.\begin{array}{l} x_3 = 1 \\ x_2 = 1 \\ x_1 = 2 \end{array}\right\}$ a) $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

b) $A - 2I = \left[\begin{array}{ccc|c} 6 & -12 & 6 & 0 \\ 3 & -6 & 3 & 0 \\ 3 & -6 & 3 & 0 \end{array}\right] \xrightarrow[\text{mult } R_1 \text{ by } 1/6]{\text{mult } R_2 \& R_3 \text{ by } 1/3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -2 & 1 & 0 \end{array}\right] \xrightarrow[-R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow$ x_2 arb, x_3 arb, $x_1 = 2x_2 - x_3$

$x_2 = 1$, $x_3 = 0$, $x_1 = 2$ $\rightarrow \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

$x_2 = 0$, $x_3 = 1$, $x_1 = -1$ $\rightarrow \vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

b)

c) $\vec{x} = C_1 e^{\lambda_1 t}(\vec{v}_1) + C_2 e^{\lambda_2 t}(\vec{v}_2) + C_3 e^{\lambda_3 t}(\vec{v}_3)$

c) $\vec{x} = C_1 e^{5t}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + C_3 e^{2t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

10. **This is a written question.**

Let

$$A = \begin{bmatrix} 5 & 4 \\ -1 & 1 \end{bmatrix}$$

(a) Find the eigenvalues of A.

(b) Find the eigenvectors of A.

(c) Find the general solution of the linear differential system $x' = Ax$.

a) $A - \lambda I = \left[\begin{array}{cc|c} 5-\lambda & 4 & 0 \\ -1 & 1-\lambda & 0 \end{array}\right]$

$\det(A - \lambda I) = (5-\lambda)(1-\lambda) + 4$

$5 - 5\lambda - \lambda + \lambda^2 + 4 = 0$

$\lambda^2 - 6\lambda + 9 = 0$

$(\lambda - 3)(\lambda - 3) = 0$

a) $\lambda_1 = \lambda_2 = 3$

b) $A - 3I = \left[\begin{array}{cc|c} 2 & 4 & 0 \\ -1 & -2 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1 + R_2} \left[\begin{array}{cc|c} 2 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\xrightarrow{\frac{1}{2}R_1 = R_1} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow$ x_2 arb, $x_1 = -2x_2$

$\hookrightarrow x_2 = 1$, $x_1 = -2$

b) $\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

generalized vector $\vec{w}$

$\left[\begin{array}{cc|c} 2 & 4 & -2 \\ -1 & -2 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1 + R_2} \left[\begin{array}{cc|c} 2 & 4 & -2 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1 = R_1} \left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 0 & 0 \end{array}\right]$

x_2 arb

$x_1 = -2x_2 - 1$

$\hookrightarrow x_2 = 1$, $x_1 = -3$

b) $\vec{w} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

c) $\vec{x} = C_1 e^{\lambda_1 t}\left(\vec{v}_1\right) + C_2 e^{\lambda_2 t}\left[\left(\vec{w}\right) + t\left(\vec{v}_1\right)\right]$

c) $\vec{x} = C_1 e^{3t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + C_2 e^{3t}\left[\begin{pmatrix} -3 \\ 1 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right]$

11. **This is a BONUS question.**

If a system of n linear equations in n unknowns has infinitely many solutions, then the rank of the matrix of coefficients is n.

a. Always true.

b. Sometimes true.

c. Never true.

d. None of the above.

$$A = \left[\begin{array}{ccc|c} n_1 & n_2 & n_3 & 0 \\ n_4 & n_5 & n_6 & 0 \\ n_7 & n_8 & n_9 & 0 \end{array}\right]$$

"infinitely many solns" means that the last row of A is zero.

$$\hookrightarrow A = \left[\begin{array}{ccc|c} n_1 & n_2 & n_3 & 0 \\ n_4 & n_5 & n_6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

2 non-zero rows
rank is 2

If the last row of A is zero, then the rank of A is not n, it is n-1.

12. **This is a BONUS question.**

If the reduced row echelon form of the matrix of coefficients of a system of n linear equations in n unknowns is not I_n, then 0 is an eigenvalue of the matrix of coefficients.

a. Always true.

b. Sometimes true.

c. Never true.

d. None of the above.

After reducing, you do NOT end up with: $I_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

If row reduction does not output the identity matrix, then the rows reduced matrix will have a row of zeros.

If you have a row of zeros, your determinant is zero.

Since the determinant is a product of eigenvalues, then one of them must be zero (since the determinant is zero).