Look up $\hat{a}$ + $\hat{a}^{+}$ in terms of $\hat{x}$ + $\hat{p}$ in my slides.

Final Exam (Take-Home), Physics 3315, Fall 2025. Name: ______________________________

(120 pts. maximum total.) Due December 4, 2025, in class.

$\rightarrow \hat{x} = \sqrt{\cdots}\,[\hat{a}^{+} + \hat{a}]$; $\hat{p} = i\sqrt{\cdots}\,[\hat{a}^{+} - \hat{a}]$. $\langle\psi(0)| = \frac{1}{\sqrt{2}}[\langle 0| + \langle 1|]$

1. Suppose a quantum harmonic oscillator with the Hamiltonian: $\hat{H} = \hat{p}^2/2m + \frac{1}{2}m\omega_0^2\hat{x}^2 = \hbar\omega_0\left[\hat{a}^\dagger\hat{a} + \frac{1}{2}\right]$ is initially, at time $t = 0$, in the state: $|\psi(0)\rangle = \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle]$, where $|1\rangle = \hat{a}^\dagger|0\rangle$. Equivalently, $\psi(t=0) = \frac{1}{\sqrt{2}}[\psi_0 + \psi_1]$, where $\psi_1 = \hat{a}^\dagger\psi_0$, and ψ_0 is the ground state.

$\langle 1|0\rangle = 0$

$\langle 1|1\rangle = 1$

$\langle 0|0\rangle = 1$.

$\hat{a}^{+}|0\rangle = |1\rangle$

$\hat{a}|1\rangle = |0\rangle$.

- **a.** (**15 pts.**) Starting with the expressions for $\hat{a}^\dagger$ and $\hat{a}$ in terms of $\hat{x}$ and $\hat{p}$, derive expressions for $\hat{x}$ and $\hat{p}$ in terms of $\hat{a}^\dagger$ and $\hat{a}$. Use these results to compute the position and momentum expectation values $\langle x\rangle$ and $\langle p\rangle$, respectively, at time $t = 0$. Use the orthonormality of the energy eigenstates and properties of the creation and annihilation operators to determine the expectation values – don't compute brute force integrals. $\langle x\rangle = \langle\psi|\hat{x}|\psi\rangle$.
- **b.** (**15 pts.**) Compute the same expectation values for times $t > 0$ and compare your results with the oscillatory behavior you'd expect for the classical harmonic oscillator. Don't compute brute force integrals, as in part (a).

$\langle x\rangle \sim \sqrt{\ }\cos\omega_0' t$

$\langle p\rangle \sim \hbar\sqrt{\ }\sin\omega_0' t$. $\llcorner d$

$\omega_0' = \omega - \omega_0$

2. A particle of mass m is confined inside a cubic box with edge of length a.

- **a.** (**8 pts.**) Show (e.g., with a table) that there are N different wavefunctions (energy eigenstates) that have energy $E = 14\pi^2\hbar^2/(2ma^2)$ and determine the integer N (hint: $N < 10$).
- **b.** (**10 pts.**) Write down the properly normalized wavefunctions. Take $V = 0$ for $0 < x, y, z < a$.
- **c.** (**2 pts.**) How many quantum numbers are needed to fully label an energy eigenstate for this and other 3D problems? n_x, n_y, n_z.

3. (**10 pts.**) Using the solutions in Thornton (Tables 7.1 & 7.2) or another suitable source, write down all the wavefunctions for the 3p level of hydrogen. Identify the wavefunctions by their quantum numbers. Read textbook. $R_{nl}(r)\,Y_{lm}(\theta, \phi)$.

4. (**10 pts.**) In units of Bohr radius, using the appropriate wavefunctions in Thornton or another suitable source, find the expectation value of the radial position of the electron in the hydrogen atom in the 2p state(s). You can find a relevant integral in an appendix of Thornton.

Read Thornton.

5. (**10 pts.**) Taking the z-direction as your preferred basis, and using the appropriate Pauli (spin) matrix, show that the states, $|\pm\rangle_y = \frac{1}{\sqrt{2}}\binom{1}{\pm i}$, satisfy the eigenvalue equations: $\hat{S}_y|\pm\rangle_y = \pm\frac{\hbar}{2}|\pm\rangle_y$.

$\hat{S}_y = \frac{\hbar}{2}\hat{\sigma}_y$ ← Look up.

Use the rules for matrix multiplication.

6. **(10 pts.)** Note that the total spin operator is a vector operator: $\hat{\mathbf{S}} = \mathbf{i}\hat{S}_x + \mathbf{j}\hat{S}_y + \mathbf{k}\hat{S}_z$, where $\hat{S}_x = \frac{\hbar}{2}\hat{\sigma}_x$, etc. (using Pauli matrices) and **i**, **j**, and **k** are unit vectors along the *x*-, *y*-, and *z*-directions, respectively. Also note that a unit vector **n**, at an angle θ with respect to the *z*-axis and (for its projection onto the *x*-*y* plane) an angle ϕ with respect to the *x*-axis, can be written as: $\mathbf{n} = \mathbf{i}\sin\theta\cos\phi + \mathbf{j}\sin\theta\sin\phi + \mathbf{k}\cos\theta$. Setting $\hat{S}_\mathbf{n} = \hat{\mathbf{S}}\cdot\mathbf{n}$, show that:

$$\hat{S}_\mathbf{n} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}.$$

$\hat{S}_n \cdot \vec{n} = \frac{\hbar}{2}\left[\sin\theta\cos\phi\,\hat{\sigma}_x + \text{etc.}\right]$

7. **(10 pts.)** Using the same $\hat{\mathbf{S}}$ as in the above problem, show that

$$\hat{\mathbf{S}}^2 = \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = s(s+1)\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $s = ½$ is the spin quantum number.

eg. either $\hat{S}_n \cdot \hat{S}_n = \hat{S}_n^2$.

or

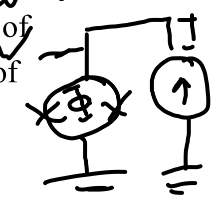
$\hat{S}_n^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z$.

8. **(10 pts.)** Two-dimensional free Fermion gas. Consider at a gas of non-interacting Fermions in a plane of width *L*. Let *m* represent the mass of each particle. Assuming two Fermions occupy each filled state to accommodate both spins, derive an expression for the relation between the Fermi energy E_F and the density of particles *n* (number of particles per unit area: $n = N/L^2$, where *N* is the total number of particles). See my slides on 3D free fermi gas.

9. **(5 pts.)** A laser emits 5.50 x 10^{18} photons per second, using a transition from an excited state with energy 1.25 eV to a lower state with energy 0.10 eV. What is the laser's power output in watts, and its wavelength in μm?

feynman, vol III Ch. 21.

10. **(5 pts.)** A SQUID magnetometer is designed and fabricated to have a magnetic flux sensitivity of $10^{-5}\,\Phi_0$, where Φ_0 is the flux quantum. If the SQUID magnetometer's pickup loop has an area of 0.1 cm², what is its magnetic field sensitivity in fT?



Flux-locked ~~for~~ loop. (John Clarke)

Do a Google search of SQUID magnetometer.

$\Phi_0 = \frac{h}{2e}$.

$\boxed{\text{MEG}}$ Fetal MCG