

# Graph theory

## 1 Basic Definition

**Topics covered in Exam.**

*2016-Paper4-16G* **Handshaking**

*2016-Paper1-16G* **Euler's formula, planar graph inequality with girth**

*2017-Paper3-15H* **bipartite**

- Graph
- Vertices
- Edges
- Order of a graph
- Size of a graph
- Isomorphic

**Example.**  $P_n, K_n, C_n, E_n$

- Subgraph
- Induced subgraph
- Connected
- Component
- Forest
- Tree

**Theorem 1.1.** *Following equiv:*

- (i) *Tree*
- (ii) *Minimal connected*
- (iii) *Maximal acyclic*

**Theorem 1.2.** *Connected iff has spanning tree*

**Lemma 1.3** (Handshaking lemma).  $\sum d(v) = 2e(G)$

- Leaf
- Minimum degree  $\delta(G)$
- Maximum degree  $\Delta(G)$

**Theorem 1.4.** *Order  $\geq 2$ , then at least 2 leaves*

**Corollary 1.5.** *Tree order  $n-1$*

**Corollary 1.6.** *Following equiv:*

- (i) *Tree*
- (ii) *Connected of order  $n$  size  $n-1$*
- (iii) *Acyclic of order  $n$  size  $n-1$*

**Theorem 1.7** (Caley).  $n^{n-2}$  labelled trees of order  $n$

- r-partite
- Bipartite

**Theorem 1.8.** *Bipartite iff no odd cycles*

- Eulerian tour
- Eulerian

**Theorem 1.9.** *Eulerian iff  $|G| > 1$ , connected,  $d(v)$  even*

- Planar
- Plane graph
- Face

**Lemma 1.10.**  *$d(v)$  even, then partitioned into cycles*

**Lemma 1.11.**  *$e$  boundary of two faces iff contained in cycle*

**Theorem 1.12** (Euler). *Connected plane graph, order  $n$ , size  $m$ , faces  $f$*

$$n - m + f = 2$$

- Bridge

**Fact.**  $f_i$  number of faces of length  $i$

$$\sum i f_i = 2m$$

- Girth

**Theorem 1.13.**  *$G$  connected, bridgeless, planar graph*

$$e(G) \leq \frac{g}{g-2}(n-2)$$

*$G$  planar graph*

$$e(G) \leq 3(n-2)$$

**Fact.** *connected and bridgeless is not needed*

- complete bipartite graph  $K_{p,q}$

**Theorem 1.14** (Kuratowski). *Planar iff not contain subdivision of  $K_{3,3}$  or  $K_5$*

- dual graph

**Fact.** *Simple if 3-connected*

## 2 Matching and Connectivity

**Topics covered in Exam.**

2016-Paper4-16G **Menger**

2017-Paper2-15H **Hall**

- Matching
- Independent
- 1-factor (1 regular spanning subgraph)
- $n$ -regular

**Theorem 2.1** (Hall).  $G$  bipartite with vertex classes  $X, Y$

*Matching from  $X$  to  $Y$  iff  $|\Gamma(A)| \geq |A|$  for all subsets  $A \subset X$*

**Theorem 2.2** (defect form).

*$|X| - d$  independent edges iff  $|\Gamma(A)| \geq |A| - d$*

**Theorem 2.3** (polygamous version). *1 to 2 matching iff  $|\Gamma(A)| \geq 2|A|$*

- Distinct representatives

**Corollary 2.4.** *Set of distinct representative iff for all subset  $S \in [n], |\bigcup_{i \in S} Y_i| \geq |S|$*

- $K$ -connected (be careful with  $K_n$ ) —  $|G| > k$
- Vertex connectivity  $\kappa(G)$
- Local connectivity  $\kappa(a, b; G)$
- Edge connectivity  $\lambda(G), \lambda(a, b; G)$

**Fact.**

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

- Vertex disjoint (set of paths)

**Theorem 2.5** (Menger).  $ab \notin E(G)$ , then

*$\exists \kappa(a, b; G)$  vertex disjoint  $a - b$  paths*

- graph contraction
- contraction

**Corollary 2.6.**  $\kappa(G) \geq k, X, Y$  disjoint subsets  $|X|, |Y| \geq k$   
*Then exists set of  $k$  completely vertex disjoint  $X - Y$  paths*

**Theorem 2.7** (edge form of Menger).  $a, b \in V(G)$ , then

*$\exists$  set of  $\lambda(a, b; G)$  edge-disjoint  $a - b$  paths*

- Line graph

**Fact.** *Menger implies Hall*

### 3 Extremal Graph Theory

Topics covered in Exam.

2016-Paper2-15G **Turan**

2017-Paper1-16H **Hamiltonian**

- Hamiltonian cycle

**Theorem 3.1.**  $|G| \geq 3$ , every pair of non-adjacent vertices  $d(x) + d(y) \geq k$   
If  $k < n$ ,  $G$  connected, then exists path of length  $k$   
If  $k = n$ , exist Hamiltonian cycle

**Corollary 3.2** (Dirac).  $|G| = n$ ,  $\delta \geq n/2$ , then exists Hamiltonian cycle

**Theorem 3.3.**  $G$  order  $n$ , no length  $k$  path. Then  $e(G) \leq \frac{k-1}{2}n$ .  
Equality holds iff  $k|n$ ,  $G$  disjoint copies of  $K_k$

- $ex(n, F)$
- $r$  partite Turan graph of order  $n$ ,  $T_r(n)$  with size  $t_r(n)$

**Theorem 3.4** (Turan).  $K_{r+1}$  free, order  $n$ ,  $e(G) \geq t_r(n)$ , then  $G = T_r(n)$

*Proof 1.* Induction on  $n$  with base case  $K_r$ . Remove edges to  $G'$  with  $t_r(n)$  edges. Delete  $\delta(G')$  vertex  $x$ .  $G - x$  Turan. Put back  $x$ .  $T_{r+1}$  maximal.  $\square$

*Proof 2.* Induction on  $n$  with base case  $K_r$ . Add edges to maximal.  $K_n \subset G'$ . Count edges.  $G - K_n$  Turan. Put back  $K_n$ .  $\square$

**Theorem 3.5** (problem of Zarankiewicz). Bipartite  $n \times n$ , no  $K_{t,t}$

$$z(n, t) \leq (t-1)^{1/t}(n-t+1)n^{1-1/t} + (t-1)n = O(n^{2-1/t})$$

**Theorem 3.6.**

$$z(n, 2) \leq n/2(1 + \sqrt{4n-3}), \text{ equality holds for infinitely many } n$$

**Theorem 3.7.**

$$ex(n, K_{2,2}) \leq n/4(1 + \sqrt{4n-3})$$

**Fact.**

(i)  $ex(n, K_{r+1}) = t_r(n)$

(ii)  $ex(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$

(iii)  $ex(n, C_5) = t_2(n)$

(iv)  $ex(n, F) = \lfloor n^2/4 \rfloor + 2$

(v)  $ex(n, P) = t_2(n) + n - 2$

–  $K_r(t)$  complete r-partite graph,  $K_r(t) = T_r(rt)$

**Lemma 3.8.** *If  $n > n_1(r, t, \epsilon)$ ,  $|G| = n$ ,  $\delta(G) \geq (1 - 1/r + \epsilon)n$  Then contains  $K_{r+1}(t)$*

**Theorem 3.9** (Erdos-Stone).  *$n$  sufficiently large,  $e(G) \geq (1 - 1/r + \epsilon)\binom{n}{2}$ , then contains  $K_{r+1}(t)$*

– chromatic number,  $\chi(G)$

**Corollary 3.10.**

$$\lim_{n \rightarrow \infty} \frac{ex(n, F)}{\binom{n}{2}} = 1 - \frac{1}{\chi(F) - 1}$$

## 4 Colouring

**Topics covered in Exam.**

2016-Paper3-15G **chromatic polynomial**

– Vertex colouring

– greedy algorithm

**Theorem 4.1.**

$$\chi(G) \leq 1 + \max_H \delta(H)$$

**Corollary 4.2.**  $\chi(G) \leq \Delta(G) + 1$

– Block

**Fact.** *Tree of blocks and bridges*

**Theorem 4.3** (Brooks).  $\chi(G) = \Delta(G) + 1$ , then  $G$  complete or odd cycle

- clique number,  $\omega(G)$
- independence number,  $\alpha(G)$

**Fact.**  $\alpha(G) = \omega(\bar{G})$

**Fact.**  $\max \left\{ \omega(G), \frac{|G|}{\alpha(G)} \right\} \leq \chi(G)$

- chromatic polynomial,  $p_G(x)$  — number of ways to colour vertices of  $G$  with colours  $1, 2, \dots, x$

**Example.**

- (i) complement of  $K_n$ ,  $p_{\bar{K}_n}(x) = x^n$
- (ii) Tree  $T$ ,  $p_T(x) = x(x-1)^{n-1}$
- (iii) complete graph,  $p_{K_n}(x) = x(x-1)(x-2) \cdots (x-n+1)$

**Theorem 4.4.** Any  $e \in E(G)$ ,  $p_G(x) = p_{G-e}(x) - p_{G/e}(x)$

**Fact.**  $p_G(x) = \prod_C p_C(x)$

**Corollary 4.5.**

$$p_G(x) = x^n - a_{n-1}x^{n-1} + \cdots + (-1)^n a_0$$

where  $n = |G|$ ,  $a_{n-1} = e(G)$ ,  $a_j \geq 0$  for all  $j$ ,  $\min\{j : a_j \neq 0\} = k$  the number of components

**Fact.**  $G$  not specified by  $p_G(x)$

- k-edge-colouring
- chromatic index,  $\chi'(G)$

**Theorem 4.6.** Bipartite multigraph, then  $\chi'(G) = \Delta(G)$

**Fact.** fail for non-bipartite graph e.g.  $K_3$

**Theorem 4.7** (Vizing).  $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$

- list colouring,  $\chi_l(G)$

**Fact.**  $\chi_l(G) \geq \chi(G)$

**Fact.**  $\chi_l(G) \leq 1 + \max_H \delta(H)$

**Theorem 4.8** (Five Colour Theorem).  *$G$  planar, then  $\chi(G) \leq 5$*

**Theorem 4.9** (Thomasson).  *$G$  planar, then  $\chi_l(G) \leq 5$*

**Fact.** *exist graph with  $\chi_l(G) = 5$*

**Theorem 4.10** (Tait). *Four Colour Theorem holds iff  $\chi'(G) = 3$  for every cubic bridgeless planar  $G$*

#### 4.1 Graph on other surfaces

- Euler characteristic,  $E \leq 2$

**Fact.** *simply connected, then  $n - m + f = E$*

**Example.** ***Orientable surface**  $g$  handles,  $E = 2 - g$*

- $g = 1$ , torus
- $g = 2$ , double torus

***Non-orientable surfaces**, one for each  $E \leq 1$*

- $E = 1$ , projective plane
- $E = 0$ , Klein bottle

**Fact.**  $m \leq 3(n - E)$

**Theorem 4.11** (Heawood). *characteristic  $E \leq 1$ , then*

$$\chi(G) \leq H(E) = \left\lfloor \frac{7 + \sqrt{49 - 24E}}{2} \right\rfloor$$

## 5 Ramsey Theory

**Topics covered in Exam.**

2017-Paper3-15H **Ramsey numbers**

- Ramsey number,  $R(s), R(s, t)$  — smallest  $n$  st if colour edges of  $K_n$  red or blue, then exists red  $K_s$  or blue  $K_t$



**Theorem 5.1** (Ramsey).  $R(s, t) \leq R(s-1, t) + R(s, t-1)$

**Fact.**  $R(s, t) \leq \binom{s+t-2}{s-1} < 2^{s+t}$

**Fact.** *inequality not exact*

–  $R_k(s_1, s_2, \dots, s_k)$

**Theorem 5.2.**  $R_k(s_1, s_2, \dots, s_k)$  exists

**Fact.**  $R_k(s_1, \dots, s_k) \leq R_{k-1}(R(s_1, s_2), s_3, \dots, s_k)$

**Fact.**  $R_k(s_1, \dots, s_k) \leq R_k(s_1 - 1, \dots, s_k) + \dots + R_k(s_1, \dots, s_k - 1) - k + 2$

–  $r$ -uniform hypergraph

**Theorem 5.3** (Ramsey for  $r$ -sets).  $R^{(r)}(s, t)$  exists

**Theorem 5.4** (Infinite Ramsey).  $k$  colours  $\mathbb{N}^{(r)}$ , then exists infinite  $M \subset \mathbb{N}$  s.t.  $M^{(r)}$  monochromatic

– big if  $|Y| \geq \min Y$

**Theorem 5.5.** exists  $B(s, k, r)$  s.t. if  $n \geq B(s, k, r)$ ,  $\{s, s+1, \dots, n\}^{(r)}$   $k$  coloured, then exist big set  $Y \subset \{s, s+1, \dots, n\}^{(r)}$  s.t.  $Y^{(r)}$  monochromatic

## 6 Probabilistic Method

**Theorem 6.1** (Erdos).  $s \geq 3$ , then  $R(s) \geq 2^{(s-1)/2}$

**Theorem 6.2.**  $G$  has independent set of size at least

$$\sum_{v \in G} \frac{1}{d(v) + 1} \geq \frac{|G|}{d + 1}$$

–  $\mathcal{G}(n, p)$  — space of all  $2^{\binom{n}{2}}$  labelled graph

**Fact.** From ??, exists graph with  $\omega(G)$  and  $\alpha(G)$  at most  $2 \log_2 n + 1$ . Hence, exist  $G$  with  $\chi(G) \geq \frac{n}{2 \log_2 n + 1}$

**Theorem 6.3** (Erdos).  $g \geq 3, k \geq 2$ , then exists  $G$  with  $\chi(G) \geq k$  and girth at least  $g$

**Theorem 6.4.**  $n \geq t \geq 2$ , then  $z(n, t) > \frac{3}{4}n^{2-2/(t+1)}$

**Lemma 6.5.**  $\mathbb{E}(X_n^2)/\mathbb{E}(X_n)^2 \rightarrow 1$  or  $\text{Var}(X_n)/\mathbb{E}(X_n)^2 \rightarrow 0$ , then any constant  $c$ ,

$$\mathbb{P}(|X_n - \mathbb{E}(X_n)| \geq c\mathbb{E}(X_n)) \rightarrow 0$$

let  $c = 1$ , then

$$\mathbb{P}(X_n = 0) \rightarrow 0$$

**Fact.**  $\mathbb{P}(X = 0) \leq \frac{\text{Var}(X)}{(\mathbb{E}X)^2}$

**Fact.** let  $X = \sum_A I_A$ , then

- $\mathbb{E}X^2 = \sum_{A,B} \mathbb{P}(A \cap B) = \sum_{A,B} \mathbb{P}(A)\mathbb{P}(B|A)$
- $\text{Var}(X) = \sum_{A,B} \mathbb{P}(A)[\mathbb{P}(B|A) - \mathbb{P}(B)]$

**Theorem 6.6.**  $\omega(n) \rightarrow \infty, G \in (G)(n, p)$

If  $p = \frac{\log n - \omega(n)}{n}$ ,  $G$  has isolated vertices a.s.

If  $p = \frac{\log n + \omega(n)}{n}$ ,  $G$  has no isolated vertices a.s.

- almost surely —  $\mathbb{P}(A) \rightarrow 1$  as  $n \rightarrow \infty$
- threshold phenomenon

**Theorem 6.7.** let  $\mu(d) = \binom{n}{d} p^{\binom{d}{2}}$ , fixed  $p$ , then

If  $\mu(d) \rightarrow 0$ ,  $G \in \mathcal{G}(n, p)$  almost certainly not contain  $K_d$

If  $\mu(d) \rightarrow \infty$ ,  $G \in \mathcal{G}(n, p)$  almost certainly contain  $K_d$

**Corollary 6.8.**  $0 < p < 1$  fixed, then clique number is  $(2 + o(1)) \log_{1/p} n$  almost surely

## 7 Eigenvalue Methods

Topics covered in Exam.

2017-Paper4-16H **strongly regular, Petersen graph**

- adjacency matrix,  $A$

**Fact.**  $A$  real symmetric

**Fact.**  $(A^d)_{ij}$  number of walks length  $d$  from  $i$  to  $j$

– diameter,  $\text{diam}(G)$

**Fact.**  $\{I, A, A^2, A^{\text{diam}(G)}\}$  linearly independent, so at least  $\text{diam}(G) + 1$  distinct eigenvalues

**Fact.**  $\lambda_{\min} = \min_{|x|=1} x^T A x, \lambda_{\max} = \max_{|x|=1} x^T A x$

**Fact.**  $H$  induced subgraph,  $\lambda_{\min}(G) \leq \lambda_{\min}(H) \leq \lambda_{\max}(H) \leq \lambda_{\max}(G)$

**Fact.** bipartite  $G$ , then  $A = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$ . Also, characteristic polynomial is in  $t^2$

**Theorem 7.1.**  $G$  graph

(i)  $\delta(G) \leq \lambda_{\max} \leq \Delta(G)$

(ii)  $|\lambda| \leq \Delta(G)$  for all  $\lambda$

If  $G$  connected, then

(i)  $\lambda_{\max} = \Delta(G)$  iff  $G$  regular, in this case multiplicity of  $\lambda_{\max} = 1$

(ii)  $\lambda_{\min} = -\Delta(G)$  iff  $G$  regular and bipartite, in this case multiplicity of  $\lambda_{\min} = 1$

– orientation

– incidence matrix,  $B$

– combinatorial Laplacian,  $L = BB^T$

**Fact.** let  $D = \text{diag}(d(1), \dots, d(n))$

$$L = D - A = \begin{cases} d(i) & \text{if } i = j \\ -1 & \text{if } i \rightarrow j \text{ or } j \rightarrow i \\ 0 & \text{otherwise} \end{cases}$$

**Fact.**  $L$  positive semi-definite

**Fact.** let eigenvalue be  $\mu_1 \leq \mu_2 \leq \dots \leq \mu_n$   
 $x^T L x = \sum_{ij \in E(G)} (x_i - x_j)^2$ , hence eigenvalue  $\mu_1 = 0$

**Proposition 7.2.**  $\mu_2 = \min\{x^T L x / \|x\|^2 : x \neq 0, \sum x_i = 0\}$

**Theorem 7.3.**  $G$  graph,  $U \subset V(G)$ , then at least  $\frac{\mu_2 |U| |V-U|}{|G|}$  edges between  $U$  and  $V - U$

– Moore graph,  $d$ -regular, diameter 2, order  $d^2 + 1$

– strongly regular  $(d, a, b)$ :

- (i) regular of degree  $d$
- (ii) every pair of adjacent vertices exactly  $a$  common neighbours
- (iii) every pair of non-adjacent vertices exactly  $b$  common neighbours

**Fact.** *Moore graph is  $(d, 0, 1)$*

**Theorem 7.4.**  *$G$  strongly regular  $(d, a, b)$ , order  $n$ , then*

$$\frac{1}{2} \left( n - 1 \pm \frac{(n-1)(b-a) - 2d}{\sqrt{(a-b)^2 + 4(d-b)}} \right) \in \mathbb{N}$$

**Theorem 7.5.** *exist Moore graph only if  $d \in \{1, 2, 3, 7, 57\}$*