Probability and Measure

1 Boolean Algebras and Finitely Additive Measures

- Boolean algebra $\mathcal{B}$
  - $\varnothing \in \mathcal{B}$
  - stable under finite union
  - stable under complementation

Example.

(i) *trivial Boolean algebra*

(ii) *discrete Boolean algebra*

(iii) *family of constructable sets*

- constructable sets —— finite union of locally closed sets from topological space
- locally closed sets —— $O \cap C$ where O open, C closed
- finitely additive measure, m
  - $m(\varnothing) = 0$
  - $m(E \sqcup F) = m(E) + m(F)$
- sub-additive —— $m(E \cup F) \leq m(E) + m(F)$
- monotone —— $E \subset F \Rightarrow m(E) \leq m(F)$

Fact. *finitely additive measure is sub-additive and monotone*

- counting measure

2 Jordan Measure on $\mathbb{R}^d$

- box $B = I_1 \times \cdots \times I_d$
- elementary subset —— finite union of boxes
- volume of box, $|B|$
- $\mathcal{E}(B)$ —— family of elementary subsets of box B

Proposition 2.1. *Fixed B, then*

(i) $\mathcal{E}(B)$ *Boolean algebra*

(ii) *every* $E \in \mathcal{E}(B)$ *finite union of disjoint boxes*

(iii) *volume well defined*

– $m(E) = \sum |B_i|$ for $E = \bigsqcup B_i$

Fact. *m finitely additively measure on* $(B, \mathcal{E}(B))$

– Jordan measurable —— For all $\epsilon > 0$, $\exists$ elementary $E \subset A \subset F$ st $m(F \backslash E) < \epsilon$

Fact. *Jordan measurable subsets bounded*

– $m(A)$ for Jordan measurable A ——

$$m(A) = \inf\{m(F) : A \subset F, F \text{ elementary}\} = \sup\{m(F) : A \supset E, E \text{ elementary}\}$$

– $\mathcal{J}(B)$ —— family of Jordan measurable subsets of box B

Proposition 2.2. *Fixed* B*, then*

(i) $\mathcal{J}(B)$ *Boolean algebra*

(ii) *m finitely additive measure on* $(B, \mathcal{J}(B))$

Fact. $E \subset$ *finite interval* $[a, b] \subset \mathbb{R}$*, then* E *Jordan measurable iff* $\mathbb{1}_E(x)$ *Riemann integrable*

3 Lebesgue measurable sets

– Lebesgue outer-measure —— $E \subset \mathbb{R}^d$,

$$m^*(E) = \inf\{\sum |B_n| : E \subset \bigcup B_n \text{ boxes}\}$$

Fact. m^* *translation invariant*

– Lebesgue measurable —— For $\epsilon > 0$, $\exists C = \bigcup B_n$, $E \subset C$ st

$$m^*(C \backslash E) < \epsilon$$

– Lebesgue measurable —— $\exists$ B Borel st $m^*(B \bigtriangleup E) < \epsilon$

– $\mathcal{L}$ —— family of Lebesgue measurable sets

Fact. $\mathcal{L}$ *translation invariant, scales naturally*

Fact. *Jordan measurable* $\Rightarrow$ *Lebesgue measurable*

Proposition 3.1.

(i) m^* *extends* m

(ii) $\mathcal{L}$ *Boolean algebra, stable under countable unions*

(iii) *m^* countably additive on $(\mathbb{R}^d, \mathcal{L})$*

Lemma 3.2. *m^**

(i) *monotone —— $A \subset B \Rightarrow m^*(A) \leq m^*(B)$*

(ii) *countably sub-additive —— $m^*\left(\bigcup A_n\right) \leq \sum m^*(A_n)$*

Fact. *Jordan measure countably additive on Jordan measurable set*

– continuity property —— E_n non-increasing, empty intersection $\Rightarrow \lim m(E_n) = 0$

Lemma 3.3. *Jordan measure has continuity property on elementary sets*

Lemma 3.4. *Elementary sets E_n decreasing, $A = \bigcap E_n$, then*

(i) *A Lebesgue measurable*

(ii) *$m(E_n) \to m^*(A)$*

Fact. *countable intersection of elementary sets Lebesgue measurable*

Corollary 3.5. *open and closed subsets Lebesgue measurable*

– null set —— $m^*(E) = 0$

Lemma 3.6. *null set Lebesgue measurable*

Proposition 3.7. *E Lebesgue measurable, then $\exists$ closed C, open O st*

(i) *$C \subset E \subset O$*

(ii) *$m^*(O \backslash C) < \epsilon$*

Fact. *E can be written as $\left(\bigcup C_n\right) \sqcup N$ or $\left(\bigcap O_n\right) \backslash N$*

Example. *Vitali's counter example —— E set of representatives $E = \{x + \mathbb{Q}\} \subset [0, 1]$*

(i) *m^* not additive on all subsets of $\mathbb{R}^d$*

(ii) *E not Lebesgue measurable*

4 Abstract Measure Theory

– σ-algebra —— Boolean algebra, stable under countable unions

– measurable space, $(X, \mathcal{A})$

– measure μ ——

(i) $\mu(\varnothing) = 0$

(ii) countably additive

– measure space, $(X, \mathcal{A}, \mu)$

Example.

(i) $(\mathbb{R}^d, \mathcal{L}, m)$

(ii) *$m_0(E) = m\,(A_0 \bigcap E)$ for fixed $A_0 \in \mathcal{L}$*

(iii) *$(X, 2^X, \#)$, $\#$ counting measure*

(iv) *$(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ where $\mu(I) = \sum_{i\in I} a_i$ for fixed $(a_n)_{n\geq 1}$*

Proposition 4.1. *$(X, \mathcal{A}, \mu)$ measure space*

(i) *μ monotone*

(ii) *μ countably sub-additive*

(iii) *upward monotone convergence —— E_n increasing, then $\mu(\bigcup E_n) = \lim \mu(E_n) = \sup \mu(E_n)$*

(iv) *downward monotone convergence —— $\mu(E_1) < \infty$, E_n decreasing, then $\mu(\bigcap E_n) = \lim \mu(E_n) = \inf \mu(E_n)$*

- finite —— $\mu(X) < \infty$
- σ-finite —— $X = \bigcup E_n$, $\mu(E_n) < \infty$
- probability space
- probability measure
- σ-algebra generated by $\mathcal{F}$, $\sigma(\mathcal{F})$ —— $\mathcal{F}$ family of subsets

Example.

(i) $X = \sqcup X_i$

(ii) *X countable, $\mathcal{F}$ singletons*

- Borel σ-algebra, $\mathcal{B}(X)$ —— X topological space, generated by all open subsets
- Borel sets

Fact. $\mathcal{B}(\mathbb{R}^d) \subset \mathcal{L}$

Fact. *$\mathcal{B}(\mathbb{R}^d)$ strictly smaller than $\mathcal{L}$ —— every subset of null sets is null*

Fact. *$\mathcal{B}(X)$ (σ-algebra) usually larger than family of constructable sets (Boolean algebra)*

- Boolean algebra generated by $\mathcal{F}$, $\beta(\mathcal{F})$
- explicitly described —— elements of $\beta(\mathcal{F})$ are finite unions of $F_1 \bigcap \cdots \bigcap F_n$, F_i or $\bar{F}_i$ in $\mathcal{F}$

Myth. *Borel hierarchy*

- Borel measure —— measure on $\mathcal{B}(X)$

Setting 1. *X set, $\mathcal{B}$ Boolean algebra, μ finitely additive measure*

– continuity property —— under setting 1, non-increasing (E_n), $\mu(E_1) < \infty$, empty intersection

$$\lim \mu(E_n) = 0$$

Theorem 4.2 (Caratheodory extension theorem). *Under setting 1, $\mathcal{B}$ continuity property, μ σ-finite, then μ uniquely extends to μ^* on $\sigma(B)$*

– outer-measure μ^* —— $\mu^*(E) = \inf \{\sum \mu(B_i) : E \subset \bigcup B_i, B_i \in \mathcal{B}\}$

– μ^* measurable —— $\exists \bigcup B_n := C$ st $\mu^*(C \backslash E) < \epsilon$

– completion of $\mathcal{B}$, $\mathcal{B}^*$ —— family of μ^* measurable subsets

Fact. *completion contains all null sets*

Proposition 4.3. *Under setting 1,*

(i) *$\mathcal{B}^*$ σ-algebra containing $\mathcal{B}$*

(ii) *μ^* countably additive on $\mathcal{B}^*$*

(iii) *μ^* extends μ*

Myth. *X compact metric space, μ probability measure on Borel σ-algebra $\mathcal{B}$, no atom, then $\exists$ measure preseving measurable isomorphism between $(X, \mathcal{B}^*, \mu)$ and $([0,1], \mathcal{L}, m)$*

5 Uniqueness of Measures

– π-system —— family $\mathcal{F}$

(i) contains $\varnothing$

(ii) stable under finite intersection

Proposition 5.1 (measure uniqueness). *$(X, \mathcal{A})$ measurable space, μ_1, μ_2 finite measures st*

(i) *$\mu_1 = \mu_2$ on $\mathcal{F} \bigcup \{X\}$*

(ii) *$\mathcal{F}$ π-system st $\sigma(\mathcal{F}) = \mathcal{A}$*

then $\mu_1 = \mu_2$ on $\mathcal{A}$

Fact. *For general measures, if $\exists F_n \subset \mathcal{F}$ st μ_1, μ_2 finite on F_n, $X = \bigcup F_n$, then uniqueness also holds*

Lemma 5.2 (Dynkin's lemma).

(i) *$\mathcal{F}$ π-system*

(ii) *$\mathcal{F} \subset \mathcal{C}$*

(iii) *$\mathcal{C}$ stable under complementation, disjoint countable union*

then $\sigma(\mathcal{F}) \subset \mathcal{C}$

– translation invariant —— $m(A + x) = m(A)$ for all A, x

Proposition 5.3. *Lebesgue measure unique measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ st*

(i) *translation invariant*

(ii) *$m([0,1]^d) = 1$*

6 Measurable Functions

Setting 2. $(X, \mathcal{A}), (Y, \mathcal{B})$ *measurable space*

- $f : X \to \mathbb{R}$ measurable function
- $f : X \to Y$ measurable map

Fact. *can extend to* $\{\infty\}$ *or* $\{-\infty\}$

Fact. *continuous function measurable*

Fact. $E \in \mathcal{A}$ *iff* $\mathbb{1}_E$ *measurable*

- $\mathbb{R}$-algebra

Proposition 6.1. $(f_n)_{n\geq 1}$ *measurable functions*

(i) f, g *measurable* $\Rightarrow g \circ f$ *measurable*

(ii) *Family of measurable functions form* $\mathbb{R}$*-algebra*

(iii) $\limsup f_n, \liminf f_n, \sup f_n, \inf f_n$ *measurable functions*

Proposition 6.2. $f = (f_1, f_2, \ldots, f_d)^T$*, then* f *measurable iff* f_i *measurable*

- Borel measurable (or simply Borel)

Fact. f *measurable*

(i) $f^{-1}(L)$ *need not measurable for* $L \in \mathcal{L}$

(ii) $f(X)$ *need not measurable even for* f *continuous*

Example. ***(i)*** f *sends to trivial* σ*-algebra*

7 Integration

- simple function —— $\sum^N a_i \mathbb{1}_{A_i}$ with $a_i \geq 0$

Lemma 7.1. f *simple,* $f = \sum a_i \mathbb{1}_{A_i} = \sum b_j \mathbb{1}_{B_j}$*, then* $\sum a_i \mu(A_i) = \sum b_j \mu(B_j)$

- integral $\mu(f)$ for simple f —— $\mu(f) = \sum a_i \mu(A_i) = \int f d\mu$
- integral $\mu(f)$ for non-negative f —— $\mu(f) = \sup\{\mu(g) : g \leq f, g \text{ simple}\}$

Proposition 7.2 (positivity)**.** f, g *non-negative measurable, then*

- $f \geq g \Rightarrow \mu(f) \geq \mu(g)$
- $f \geq g$*,* $\mu(f) = \mu(g) \Rightarrow f = g$ *a.e.*
- $f = g$ almost everywhere

Lemma 7.3. $f \geq 0$*, then* $\exists$ *increasing simple functions* g_n *st* $g_n \to f$ *pointwise*

Proof. $g_n(x) = 2^{-n}\lfloor 2^n(f(x) \wedge n)\rfloor$ □

Theorem 7.4 (Monotone Convergence Theorem)**.**

(i) *(f_n) non-negative, non-decresing*

(ii) *let $f(x) = \lim f_n(x)$, the pointwise limit*

Then, $\mu(f) = \lim \mu(f_n)$

Lemma 7.5. *Fixed g simple, then $m_g(E) := \mu(\mathbb{1}_E g)$ is a measure*

Lemma 7.6 (Fatou)**.** *$f_n \geq 0$, then $\mu(\liminf f_n) \leq \liminf \mu(f_n)$*

- f^+, f^-
- μ-integrable —— $\mu(|f|) < \infty$
- intergral $\mu(f)$ for integrable f —— $\mu(f) = \mu(f^+) - \mu(f^-)$

Proposition 7.7 (Linearity of integral)**.** *f, g integrable*

(i) *$\alpha f + \beta g$ integrable*

(ii) *$\mu(\alpha f + \beta g) = \alpha\mu(f) + \beta\mu(g)$*

Fact. *Also holds for nonegative f, g, α, β*

Theorem 7.8 (Dominated Convergence Theorem)**.** *f, f_n measurable, g integrable*

(i) *$|f_n(x)| \leq g(x)$*

(ii) *$\lim f_n(x) = f(x)$*

Then,

(i) *$\lim \mu(f_n) = \mu(f)$*

(ii) *f integrable*

Fact. *condition for MCT, Fatou, DCT only need to hold μ-almost everywhere*

Corollary 7.9 (Exchange of $\int$ and $\sum$)**.**

(i) *$f_n \geq 0$, then $\mu\left(\sum^\infty f_n\right) = \sum^\infty \mu(f_n)$*

(ii) *$\sum |f_n|$ μ-integrable, then*

- *$\sum f_n$ integrable*
- *$\mu\left(\sum f_n\right) = \sum \mu(f_n)$*

Corollary 7.10 (Differentiation under $\int$ sign)**.** *U open set, $f : U \times X \to \mathbb{R}$ st*

(i) *$f(t, \cdot)$ μ-integrable*

(ii) *$f(\cdot, x)$ differentiable*

(iii) *(domination) $\exists$ integrable g st $\sup_t |\frac{\partial f}{\partial t}(t, x)| \leq g(x)$*

Then,

(i) $\frac{\partial f}{\partial t}(t,\cdot)$ *μ-integrable*

(ii) *let* $F(t) = \int_X f(t,x)d\mu(x)$*, then*

(a) *F differentiable*

(b) $F' = \int_X \frac{\partial f}{\partial t}(t,x)d\mu(x)$

Fact. *f bounded, then f Riemann integrable iff $\{x : f(x)$ not continuous$\}$ has Lebsegue measure 0*

Fact (invariance under affine map)**.** *$g \in GL_d(\mathbb{R}), f$ integrable, then* $m(f \circ g) = \frac{1}{|detg|}m(f)$

Fact. *$\phi \in C^1$, then* $\int f(\phi(x))J_\phi(x)dx = \int f(x)dx$

– Radon measure —— Borel measure, finite on every compact subset

Fact (Riesz Representation for locally compact spaces)**.**

(i) *μ Radon measure, let $\Lambda(f) = \mu(f)$, then $\Lambda \in C_c(X)'$*

(ii) *let $\Lambda \in C_c(X)'$, Λ non-negative, then $\exists$ Radon measure μ st $\Lambda(f) = \mu(f)$*

8 Product Measure

– product σ-algebra $\mathcal{A} \otimes \mathcal{B}$—— σ-algebra generated by $A \times B$ where $A \in \mathcal{A}$, $B \in \mathcal{B}$

Fact.

(i) *$\{A \times B\}$ π-system*

(ii) *smallest σ-algebra st projection map measurable*

(iii) $\mathcal{B}(\mathbb{R}^{d_1}) \otimes \mathcal{B}(\mathbb{R}^{d_2}) = \mathcal{B}(\mathbb{R}^{d_1+d_2})$ *(generally not true)*

– slice, E_x —— $E_x = y : (x,y) \in E$

Lemma 8.1. *E $\mathcal{A} \otimes \mathcal{B}$-measurable, then E_x $\mathcal{B}$-measurable*

Proof. start with $A \times B$, then Dynkin's lemma □

Setting 3. *$(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ σ-finite*

Lemma 8.2. *Under setting 3, f $\mathcal{A} \otimes \mathcal{B}$-measurable, non-negative, then*

(i) *$f(x,\cdot)$ $\mathcal{B}$-measurable for every x*

(ii) $g(x) := \int f(x,y)d\nu(y)$ *$\mathcal{A}$-measurable*

– product measure $\mu \otimes \nu$

Proposition 8.3. *Under setting 3, then $\exists$ unique measure σ on $\mathcal{A} \otimes \mathcal{B}$ st $\sigma(A \times B) = \mu(A)\nu(B)$*

Theorem 8.4 (Fubini-Tonelli)**.** *Under setting 3,* *(i)* f $\mathcal{A}\otimes\mathcal{B}$ *measurable, non-negative, then* $\int_{X\times Y} f d\mu\otimes\nu = \int_X \left(\int_Y f d\nu(y)\right) d\mu(x) = \int_Y \left(\int_X f d\mu(x)\right) d\nu(y)$

(ii) f $\mu\otimes\nu$*-integrable, then*

- (a) $f(x,\cdot)$ ν*-integrable for* μ*-almost every* x
- (b) $f(\cdot,y)$ μ*-integrable for* ν*-almost every* y
- (c) *above also holds*

Fact. *justify Fubini, just need* $f(x,\cdot), f(\cdot,y)$ *integrable (????)*

9 Probability Theory

- universe Ω
- outcome ω
- events $\mathcal{F}$
- probability measure $\mathbb{P}$
- random variable X
- expectation $\mathbb{E}$
- probability space $(\Omega,\mathcal{F},\mathbb{P})$
- law of X / distribution of X —— Borel measure $\mu_X(A)=\mathbb{P}(X\in A)$ on $(\mathbb{R},\mathcal{B}(R))$
- image measure $f_*\mu$ —— $f_*\mu(A)=\mu(f^{-1}(A))$
- distribution function of X, F_X —— $F_X(t)=\mathbb{P}(X\leq t)$

Proposition 9.1. F_X

(i) *non-decreasing*

(ii) *right continuous*

(iii) F_X *determines* μ_X *uniquely*

- Lebesgue-Stieltjes measure μ_F

Proposition 9.2. *Given* F *non-decreasing, right continuous,* $\lim_{-\infty} F(t)=0$*,* $\lim_{\infty} F(t)=1$*, then* $\exists$ *unique Borel measure* μ_F *st* $F(t)=\mu_F((-\infty,t])$

Proof. One approach using Caratheodory. Another shown as follow. □

- "inverse function" g —— $g(y)=\inf(t : F(t)\geq y)$

Lemma 9.3. g

(i) *non-decreasing*

(ii) *left continuous*

(iii) *$g(y) \leq t$ iff $y \leq F(t)$*

Fact. *let m Lebesgue measure on $(0,1)$, set $\mu(A) = g_* m(A) = m(g^{-1}(A))$, then $\mu = \mu_F$*

Proposition 9.4. *μ Borel probability measure, then $\exists(\Omega, \mathcal{F}, \mathbb{R})$, r.v. X st $\mu = \mu_X$*

Fact. *Can take $\Omega, \mathcal{F}, \mu = (0,1)$, Borel σ-algebra , $\mathbb{P}$*

- density

Example.

(i) *uniform distribution*

(ii) *exponential distribution*

(iii) *gaussian distribution*

(iv) *Dirac mass*

- mean
- moment of order k
- variance

10 Independence

- events (A_i) mutually independent —— every finite $F \subset \mathbb{N}$, $\mathbb{P}(\bigcap_F A_i) = \prod_F \mathbb{P}(A_i)$

Fact. *(A_i) independent $\Rightarrow$ (B_i) independent where $B_i = A_i$ or A_i^c*

- σ-subalgebras $(\mathcal{A}_i)$ mutually independent —— $\mathcal{A}_i \subset F$, every $A_i \in \mathcal{A}_i$, (A_i) mutually independent

Fact. *$\Pi_i \subset \mathcal{A}_i$ π-system, $\sigma(\Pi_i) = \mathcal{A}_i$, then suffices just check $A_i \in \Pi_i$*

- $\sigma(X)$
- random variables (X_i) mutually independent —— $(\sigma(X_i))$ independent

Fact. *Equivalence to every finite $F \subset \mathbb{N}$*

- $\mathbb{P}(\bigcap_F X_i \leq t_i) = \prod_F \mathbb{P}(X_i \leq t_i)$
- $\mu_{(X_{i_1},\ldots,X_{i_m})} = \mu_{X_{i_1}} \otimes \cdots \otimes \mu_{X_{i_m}}$

Fact. *(X_i) independent $\Rightarrow$ $(f_i(X_i))$ independent*

Proposition 10.1. *X, Y independent, non-negative (or integrable), then $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$*

Setting 4. $\{(\Omega_i, \mathcal{F}_i, \nu_i)\}$

- cylinder set —— $A \times \prod_{i>n} \Omega_i$ where $A \subset \prod_n \Omega_i$, $A \in \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n$

- Boolean algebra of cylinder set $\mathcal{C}$
- infinite product measure

Proposition 10.2. *$\Omega = \prod \Omega_i$, $\mathcal{F} = \sigma(\mathcal{C})$, then $\exists$ unique probability measure ν on $(\Omega, \mathcal{F})$ st*

$$\nu(B) = \nu_1 \otimes \cdots \otimes \nu_n(A) \qquad \text{for every cylinder set } B$$

Fact. *more general theorem Kolmogorov extension theorem*

- $\limsup A_n$ —— $\bigcap \bigcup A_n$ (infinitely offen)

Lemma 10.3 (1st Borel Cantelli)**.** *$\sum \mathbb{P}(A_n) < \infty$, then $\mathbb{P}(\limsup A_n) = 0$*

Lemma 10.4 (2nd Borel Cantelli)**.** *$\sum \mathbb{P}(A_n) = \infty$, (A_n) mutually independent, then $\mathbb{P}(\limsup A_n) = 1$*

Fact. *independence condition in 2nd Cantelli can be relaxed*

- *pairwise independence*
- *small correlation between events*

◇ $(\Omega, \mathcal{F}, \mathbb{P})$ probability space

- random process / stochastic process (X_n)
- n-th term of the associated filtration $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$
- tail σ-algebra $\mathcal{T} = \bigcap \sigma(X_n, X_{n+1}, \ldots)$

Theorem 10.5 (Kolmogorov 0-1 law)**.** *(X_n) mutually independent, then $\mathbb{P}(A) \in \{0, 1\}$ for all $A \in \mathcal{T}$*

- Cauchy-Schwarz —— $\mathbb{E}|XY| \leq \sqrt{\mathbb{E}(X^2)\mathbb{E}(Y^2)}$
- Markov's inequality —— $\lambda \mathbb{P}(X \geq \lambda) \leq \mathbb{E}(X)$ for $X \geq 0, \lambda \geq 0$
- Chebychev's inequality —— $\lambda^2 \mathbb{P}(|Y - \mathbb{E}(Y)| \geq \lambda) \leq Var(Y)$ for $\lambda \geq 0$

Theorem 10.6 (Strong law of large number)**.** *(X_n) i.i.d. , $\mathbb{E}|X_1| < \infty$, then $\bar{X}_n \xrightarrow[a.s.]{} \mathbb{E}X_1$*

Fact. *$\mathbb{E}(X^4) < \infty \Rightarrow \mathbb{E}((X - \mathbb{E}X)^4) < \infty$ (Jensen on X^4)*

Fact. *$\mathbb{E}(|X|^n) < \infty \Rightarrow \mathbb{E}(|X|^k) < \infty$ for $k \leq n$*

11 Convergence of Random Variables

– probability measures μ_n converge weakly —— $\forall$ bounded, continuous f, $\mu_n(f) \to \mu(f)$

⋄ sequences (X_n)

– almost surely (a.s.) —— $X_n(\omega) \to X(\omega)$ for $\mathbb{P}$ almost every ω

– in probability (in measure) —— $\mathbb{P}(\|X_n - X\| > \epsilon) \to 0$

– in law (in distribution) —— μ_{X_n} converge weakly to μ_X

Proposition 11.1. *almost surely $\Rightarrow$ in probability $\Rightarrow$ in distribution*

Fact. *$X_n \to X$ in law iff $F_{X_n}(x) \to F_X(x)$*

Fact. *To prove μ_n converge weakly to μ, suffice to check $f \in C_c^\infty$*

Counter Example.

– *weakly $\not\Rightarrow$ in prob —— i.i.d. X_n with same distribution*

– *in prob $\not\Rightarrow$ a.s. —— moving bump $\mathbb{1}_{[k/n,(k+1)/n]}$*

Proposition 11.2. *$X_n \to X$ in prob, then $\exists$ subsequence $X_{n_j} \to X$ a.s.*

– converge in L^1 —— integrable X_n, $\mathbb{E}\|X_n - X\| \to 0$

Proposition 11.3. *$L^1 \Rightarrow$ in probability*

Counter Example.

– *in prob $\not\Rightarrow$ in L^1 —— $X_n = n\mathbb{1}_{[0,1/n]}$*

– bounded —— $X_n \le C$ for constant C independent of n

Fact. *If (X_n) bounded, in prob $\Rightarrow$ in L^1*

Proof. Passing to subsequence, a.s. convergence. Then DCT ☐

– uniformly integrable (U.I.) —— integrable (X_n), $\lim_M \limsup_n \mathbb{E}(\|X_n\|\mathbb{1}_{\|X_n\|>M}) = 0$

– dominated —— $X_n \le Y$ for integrable Y, all n

Fact. *dominated $\Rightarrow$ U.I.*

– bounded in L^p —— $\sup_n \mathbb{E}\|X_n\|^p < \infty$

Fact. *bounded in L^p for $p > 1 \Rightarrow$ U.I.*

Theorem 11.4. *(X_n) integrable, then following equivalent:*

(i) *$X_n \to X$ in L^1, X integrable*

(ii) *$X_n \to X$ in prob, X_n U.I.*

Lemma 11.5. *Y integrable, (X_n) U.I. , then $(X_n + Y)$ U.I.*

12 L^p spaces

Setting 5. *$(\Omega, \mathcal{A}, \mathbb{P})$ probability space, I open interval, $X : \Omega \to I$, $\phi : I \to \mathbb{R}$*

Proposition 12.1 (Jensen's inequality)**.** *Under setting 5, X integrable, ϕ convex, then $\mathbb{E}(\phi(X)) \geq \phi(\mathbb{E}(X))$*

– convex

Lemma 12.2. *convex iff $\phi = \sup_{\mathcal{F}} l$ where $\mathcal{F}$ family of affine linear forms*

Fact. *$\phi(X)^-$ always integrable*

– L^p norm $\|f\|_p$

– L^∞ norm *essup* $|f|$

Fact. *let $g = f\mathbb{1}_{f \leq \|f\|_\infty}$, then $\sup g = essup\ |g|$*

Proposition 12.3 (Minkowski inequality)**.** *$p \in [1, \infty]$, then $\|f + g\|_p \leq \|f\|_p + \|g\|_p$*

Setting 6. $\frac{1}{p} + \frac{1}{q} = 1$

Proposition 12.4 (Holder's inequality)**.** $\int |fg| d\mu \leq \|f\|_p \|g\|_q$
Equality holds for finite p, q when $\alpha|f|^p = \beta|g|^q$ for μ-a.e.

Lemma 12.5 (Young's inequality for product)**.** *$a, b \geq 0$, then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$. Equality iff $a^p = b^q$*

– $L^p(X, \mathcal{A}, \mu)$ —— $\left\{\|f\|_p < \infty\right\}$

– $f \equiv g$ —— $f = g$ μ-a.e.

Lemma 12.6. *$\equiv$ equivalence relation, stable under addition and multiplication*

– L^p space —— $L^p(X, \mathcal{A}, \mu)/\equiv$

Proposition 12.7 (completeness of L^p spaces)**.**

(i) *L^p space with $\|\cdot\|_p$ normed vector space*

(ii) *complete (a.k.a. Banach space)*

Proposition 12.8 (Approxiamtion by simple functions)**.** *$p \in [1, \infty)$, V linear span of simple functions, then $V \bigcap L^p$ dense in L^p*

Fact. *linear span as we need $g^+ - g^-$*

Fact. *For $(\mathbb{R}^d, \mathcal{L}, m)$, $C_c^\infty(\mathbb{R}^d)$ dense in L^p*

Fact. *$\mu(X) < \infty$, then $L^{p'} \subset L^p$ for $p' \geq p$*

Fact. *X discrete, countable, then $L^{p'} \subset L^p$ for $p' \leq p$*

13 Hilbert Spaces and L^2 methods

– Hermitian inner product —— $\mathbb{C}$

– sesquilinear form —— we pick linear in first argument

– Euclidean inner product —— $\mathbb{R}$

– bilinear symmetric form

Lemma 13.1. ***(i)*** $\|\alpha x\| = |\alpha|\|x\|$

(ii) *Cauchy-Schwarz inequality* $|\langle x, y\rangle| \leq \|x\|\|y\|$

(iii) *triangle inequality* $\|x+y\| \leq \|x\| + \|y\|$

(iv) *Parallelogram identity* $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

Corollary 13.2. $(V, \|\cdot\|)$ *normed vector space*

– Hilbert space —— complete Hermitian/Euclidean vector space

– orthogonal projection —— unique $\pi_{\mathcal{C}}(x)$ st $\|x - \pi_{\mathcal{C}}(x)\| = \inf_{\mathcal{C}} \|x - c\|$

Proposition 13.3 (orthogonal projection on closed convex sets)**.** *$\mathcal{H}$ Hilbert, $\mathcal{C}$ closed convex, then $\exists$ orthogonal projection*

Corollary 13.4. *V closed vector subspace, then $\mathcal{H} = V \oplus V^\perp$*

Fact. *$V^\perp$ closed*

– bounded linear form

Fact. *bounded iff continuous*

Theorem 13.5 (Riesz representation theorem for Hilbert spaces)**.** *$\mathcal{H}$ Hilbert space, l bounded linear form, then $\exists$ unique v_0 st $l(\cdot) = \langle \cdot, v_0\rangle$*

14 Conditional Expectation

Setting 7. $(\Omega, \mathcal{F}, \mathbb{P})$ *probability space, $\mathcal{G} \subset \mathcal{F}$ σ-subalgebra, X integrable*

– conditional expectation $\mathbb{E}(X|\mathcal{G})$

Proposition 14.1. *$\exists$ (a.s.) unique conditional expectation Y st*

(i) *$\mathcal{G}$-measurable*

(ii) *integrable*

(iii) $\mathbb{E}(\mathbb{1}_A X) = \mathbb{E}(\mathbb{1}_A Y)$

Proposition 14.2.

(i) *linearity* —— $\mathbb{E}(\alpha X + \beta Y|\mathcal{G}) = \alpha\mathbb{E}(X|\mathcal{G}) + \beta\mathbb{E}(Y|\mathcal{G})$ *a.s.*

(ii) *positivity —— $X \geq 0$ a.s. , then $\mathbb{E}(X|\mathcal{G}) \geq 0$ a.s.*

(iii) *tower preperty —— $\mathcal{H} \subset \mathcal{G}$ σ-subalgebra, then $\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = \mathbb{E}(X|\mathcal{H})$ a.s.*

(iv) *independence —— X independent of $\mathcal{G}$, then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$ a.s.*

(v) *X $\mathcal{G}$-measurable, then $\mathbb{E}(X|\mathcal{G}) \leq 0$ a.s.*

(vi) *Z $\mathcal{G}$-measurable, bounded, then $\mathbb{E}(XZ|\mathcal{G}) = Z\mathbb{E}(X|\mathcal{G})$*

(vii) *MCT, Fatou, DCT holds for $\mathbb{E}(\cdot|\mathcal{G})$*

15 Fourier Transform on $\mathbb{R}^n$

– Fourier transform $\widehat{f}(u)$ —— $f \in L^1$, $\widehat{f}(u) = \int f(x)e^{i\langle u,x\rangle}dx$

Proposition 15.1.

(i) $|\widehat{f}(u)| \leq \|f\|_1$

(ii) $\widehat{f} \in C^0$

– characteristic function of $\widehat{\mu}$ —— μ finite Borel measure, $\widehat{\mu}(u) = \int e^{i\langle u,x\rangle}d\mu(x)$

Proposition 15.2.

(i) $|\widehat{\mu}(u)| \leq \mu(\mathbb{R}^d)$

(ii) $\widehat{\mu} \in C^0$

Example. *For Gaussian measure, let $g(x) = \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{x^2}{2}\Big)$, then $\widehat{g}(u) = \sqrt{2\pi}g(u)$*

– self-dual

– character $\chi_u(x) = e^{-i\langle u,x\rangle}$

Theorem 15.3 (Fourier Inversion Formula)**.**

(i) *μ finite Borel measure, $\widehat{\mu} \in L^1$, then*

- *$\exists$ density $\phi \in C^0$ st $d\mu = \phi(x)dx$*
- *$\phi(x) = \frac{1}{(2\pi)^d}\widehat{\widehat{\mu}}(-x)$*

(ii) *$f, \widehat{f} \in L^1$, then $f(x) = \frac{1}{(2\pi)^d}\widehat{\widehat{f}}(-x)$ a.e.*

Fact. *simple sufficient condition for $\widehat{f} \in L^1$: $f \in C^2$ and f, f', f'' integrable*

– convolution $\mu * \nu$ —— $\Phi_*(\mu \otimes \nu)$ where $\Phi(x,y) = x + y$

– convolution $f * g$ —— $f, g \in L^1$, then $f * g = \int f(x-t)g(t)dt$

Fact. *$\mu_X * \nu_Y$ equivalent to law of $X + Y$*

Fact. $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$

- G_σ —— density of $\mathcal{N}(0, \sigma^2 I_d)$
- $\tau_t(f)(x)$ —— $f(x+t)$

Proposition 15.4 (Gaussian approximation)**.** *$f \in L^p(\mathbb{R}^d)$, $p \in [1, \infty)$, then* $\lim \|f * G_\sigma - f\|_p = 0$

Lemma 15.5 (continuity of translation in L^p)**.** *$f \in L^p(\mathbb{R}^d)$, $p \in [1, \infty)$, then* $\lim \|\tau_t(f) - f\|_p = 0$

Proposition 15.6.

(i) *μ, ν Borel prob measures, then $\widehat{\mu * \nu} = \widehat{\mu}\widehat{\nu}$*

(ii) *$f, g \in L^1$, then $\widehat{f * g} = \widehat{f}\widehat{g}$*

Theorem 15.7 (Levy's criterion)**.** *X_n r.v. , then following are equivalent:*

(i) *$X_n \to X$ in law*

(ii) $\lim_n \widehat{\mu_{X_n}}(u) = \widehat{\mu_X}(u)$ *for all u*

Fact. *$\widehat{\mu_X} = \widehat{\mu_Y}$ iff $\mu_X = \mu_Y$*

Example. *$\mathcal{N}(m, \sigma^2) \to \delta_m$ weakly*

Fact. $\widehat{\mu_X}(0) = 1$

- positive definite —— $\forall u_1, \ldots, u_N \in \mathbb{R}^d$, $\forall t_1, \ldots, t_N \in \mathbb{C}$, $\sum t_i \bar{t_j} \widehat{\mu_X}(u_i - u_j)$ real and ≥ 0
- normalize —— $f(0) = 1$

Fact (Boucher's theorem)**.** *f normalized continuous positive-definite, then $\exists$ unique probability measure μ st $f = \widehat{\mu}$*

- linear isometry

Theorem 15.8 (Plancherel formula)**.** *$f, g \in L^1 \bigcap L^2$, then*

(i) $\widehat{f} \in L^2$

(ii) $\left\|\widehat{f}\right\|_2 = (2\pi)^{d/2} \|f\|_2$

(iii) $\langle \widehat{f}, \widehat{g} \rangle_{L^2} = (2\pi)^d \langle f, g \rangle_{L^2}$

(iv) $\mathcal{F} : L^1 \bigcap L^2 \to L^2$ *where* $\mathcal{F}(f) = \frac{1}{(2\pi)^{d/2}} \widehat{f}$

- *extends uniquely to linear isometry of L^2*
- $\mathcal{F} \circ \mathcal{F}(f)(x) = f(-x)$

Fact. *smoothness/decay barter*

Fact. *uncertainty principle*

Fact. *Schwarz space*

16 Gaussian random variables

– gaussian ($\mathbb{R}^d$) —— can be degenerated

– mean

– covariance matrix

– correlation coefficients

Fact. $\mathcal{N}(m, 0) := \delta_m$

Proposition 16.1. *law of Gaussian determined by mean and cov*

Proof. can assume 1-d Gaussian determined by mean and var □

Fact. *cov matrix positive semi-definite symmetric*

Proposition 16.2. *N_i i.i.d $\mathcal{N}(0,1)$, $A \in M_d(\mathbb{R})$, $b \in \mathbb{R}^d$, then*

(i) $AN + b \sim \mathcal{N}_d(b, AA^*)$

(ii) *every Gaussian $X = AN + b$ for some A, b*

Fact. *$\mathcal{N}(0, \lambda I_d)$ only Borel probability law with*

- *invariant under rotation*
- *independent coordinates*

Proposition 16.3. *$X = (X_1, \ldots, X_n)$ Gaussian vector, then following equivalent:*

(i) *X_i independent r.v.*

(ii) *X_i pairwise independent*

(iii) *Cov matrix diagonal*

Theorem 16.4 (Central Limit Theorem)**.** *(X_n) i.i.d with common law μ, finite moment of order 2, then $\sqrt{n}(\bar{X}_n - \mathbb{E}(X_1)) \to \mathcal{N}(0, Cov(X_1))$ in law*

Fact. $\widehat{\mu_Y}(tu) = \widehat{\mu_{\langle Y,u \rangle}}(t)$

17 Introduction to Ergodic Theory

Setting 8. *measurable map $T : X \to X$*

– measure preserving map —— $T_*\mu = \mu$

– measure preserving system —— measure space $(X, \mathcal{A}, \mu)$ with measure preseving map T

– T-invariant function —— measurable $f = f \circ T$

– T-invariant subset —— $T^{-1}A = A$

– invariant σ-algebra $\mathcal{I}$ —— $\{A : T^{-1}A = A\}$

Lemma 17.1. *f measurable functoin, then following equivalent:*

(i) *f T-invariant*

(ii) *f measurable wrt $\mathcal{I}$*

- egodic wrt T —— T measure preserving, $\forall A \in \mathcal{I}$, $\mu(A) = 0$ or $\mu(A^c) = 0$

Fact. *egodic kind of irreducibility condition*

Lemma 17.2. *$(X, \mathcal{A}, \mu, T)$ measure preseving system, then following equivalent:*

(i) *T ergodic*

(ii) *every $\mathcal{I}$-measurable f, $f(x) \equiv a$ μ-a.e. for some a*

Setting 9 (circle rotation)**.** $X = \mathbb{R}/\mathbb{Z}$, $T(x) = x + a$

Proposition 17.3. *T ergodic iff a irrational*

Fact (Parseval's formula)**.** $f(x) = \sum \widehat{f}(n)e^{-i2\pi nx}$

Setting 10 (times 2 map on the circle)**.** $X = \mathbb{R}/\mathbb{Z}$, $T_2(x) = 2x \mod \mathbb{Z}$

Proposition 17.4. *T_2 ergodic*

18 Canonical Model for Stochastic Process

Setting 11. *(X_n) $\mathbb{R}^d$ r.v. on $(\Omega, \mathcal{F}, \mathbb{P})$ probability space*

- space of sequences $(\mathbb{R}^d)^{\mathbb{N}}$
- sample path map Φ —— $\Phi(\omega) = (X_n(\omega))$
- shift map T —— $T((x_n)_n) = (x_{n+1})_n$
- shift space
- coordinate functions x_k —— $x_k((x_n)_n) = x_k$

Setting 12. *$X = ((\mathbb{R}^d)\mathbb{N})$ endowed σ-algebra $\mathcal{A} = \sigma(x_k)$*

- cylinder set
- π_F —— F finite set of indices
- law of the stochastic process μ —— $\mu = \Phi_*\mathbb{P}$ prob measure on $(X, \mathcal{A})$
- canonical model —— $(X, \mathcal{A}, \mu, T)$

Proposition 18.1. *(X_n) stochastic process, $(X, \mathcal{A}, \mu, T)$ canonical model, then following equivalent:*

(i) *$(X, \mathcal{A}, \mu, T)$ measure preseving*

(ii) *joint law of $(X_n, X_{n+1}, \ldots, X_{n+k})$ independent of n*

- stationary

Proposition 18.2. *(X_n) i.i.d. process, then*

(i) *stationary*

(ii) *canonical model ergodic*

- Bernoulli shift $\mu = \nu^{\otimes\mathbb{N}}$ —— ν law of X_1

19 Mean Ergodic Theorem

Setting 13. $(X, \mathcal{A}, \mu, T)$ *prob measure preserving system*

- $S_n(f) = \frac{1}{n}\sum_0^{n-1} f \circ T^i$

Theorem 19.1 (Mean ergodic theorem in L^2)**.** *$f \in L^2(X, \mathcal{A}, \mu)$, then $\exists$ T-invariant $\bar{f} = \mathbb{E}(f|\mathcal{I})$ st $S_n(f) \to \bar{f}$ in L^2*

- adjoint A^* —— $\mathcal{H}$ Hilbert space, A bounded linear map, then by Riesz $\exists$ A^* st $\langle Ax, y\rangle = \langle x, A^*y\rangle$
- involutive —— $A^{**} = A$

Fact.

(i) $\|A^*\| = \|A\|$

(ii) $\|AA^*\| = \|A\|^2$

- $U(f) = f \circ T$
- co-boundaries $W = \{\phi - U\phi\}$

Fact. *$\bar{f}$ orthogonal projection onto $W^\perp = \{g = Ug\}$*

Corollary 19.2 (Mean ergodic theorem in L^p)**.** *$p \in [1, \infty)$, $f \in L^p$, then $\exists$ T-invariant $\bar{f} = \mathbb{E}(f|\mathcal{I})$ st $S_n(f) \to \bar{f}$ in L^p*

- $E_t = \{x : \sup_n S_n f(x) > t\}$

Theorem 19.3 (Maximal ergodic theorem)**.** *$f \in L^1$, then $\mu(E_t) \le \frac{1}{t}\|f\|_1$*

Lemma 19.4 (the maximal inequality)**.** *$f \in L^1$, $f_n = nS_nf$, $f_0 = 0$, $P_N = \{x : \max_{0 \le x \le N} f_n(x) > 0\}$, then $\int_{P_N} f d\mu \ge 0$*

Theorem 19.5 (Pointwise ergodic theorem)**.** *$f \in L^1$, then $S_n(f) \to \bar{f}$ μ-a.e.*

Fact. *ergodic $\Rightarrow \mathbb{E}(f|\mathcal{I}) = \mathbb{E}(f)$*

Fact. *$f = \mathbb{1}_A$, orbit $\{T^n x\}$, then time spent in almost every orbit equidistributed*

Corollary 19.6 (Strong law of large number)**.** *(X_n) i.i.d. , $\mathbb{E}(\|X_1\|) < \infty$, then $\frac{1}{n}\sum S_i \to \mathbb{E}(X_1)$ a.s.*

- $\mathcal{I}(X)$ —— family of all T-invariant probability measures
- extremal —— $\mu \in \mathcal{I}$, $\nexists \mu_1, \mu_2$ st $\mu = t\mu_1 + (1-t)\mu_2$

Proposition 19.7. *$\mu \in \mathcal{I}(X)$, then ergodic iff extremal*

20 Fun fact

- G_δ-set —— countable intersection of open sets
- locus of continuity —— set of continuous point

Fact. *locus of continuity is G_δ-set*

- Cantor function (a.k.a Devil staircase)