Stochastic Financial Models

1 Utility and Mean-Variance analysis

– contingent claims —— r.v. X

– utility function —— non-decreasing

Fact. *Y is preferred to X iff* $\mathbb{E}(U(X)) \leq \mathbb{E}(U(Y))$

– indifferent

– risk neutral

– risk averse

– concave

– strictly concave

Proposition 1.1. *risk averse iff U concave*

– CARA with parameter γ —— $\gamma \in (0, \infty)$, $U(X) = CARA_\gamma(x) = -\exp(-\gamma x)$

– CRRA with parameter R —— $R \in (0,1) \bigcup (1, \infty)$, $U(X) = CRRA_R(x) = \begin{cases} \frac{x^{1-R}}{1-R} & \text{if } x > 0 \\ -\infty & \text{otherwise} \end{cases}$

– CRRA with parameter 1 —— $U(X) = CRRA_1(x) = \begin{cases} \log x & \text{if } x > 0 \\ -\infty & \text{otherwise} \end{cases}$

– constant absolute risk aversion —— CARA

– constant relative risk aversion —— CRRA

Fact (Arrow-Pratt coefficient of absolute risk aversion)**.** *$\omega + X$ preferred to ω iff* $\frac{2\mathbb{E}(X)}{\mathbb{E}(X^2)} \geq -\frac{U''(\omega)}{U'(\omega)}$

Fact (Arrow-Pratt coefficient of relative risk aversion)**.** *$\omega(1+X)$ preferred to ω iff* $\frac{2\mathbb{E}(X)}{\mathbb{E}(X^2)} \geq -\frac{\omega U''(\omega)}{U'(\omega)}$

– available claim $\mathcal{A}$

– reservation bid price $\pi_b(Y)$ —— $\sup \pi$ st $\mathbb{E}(U(X + Y - \pi)) > \mathbb{E}(U(X^*))$

– reservation ask price $\pi_a(Y)$ —— $\inf \pi$ st $\mathbb{E}(U(X - Y + \pi)) > \mathbb{E}(U(X^*))$

Proposition 1.2 (Ask above, bid below). *$\mathcal{A}$ convex, then $\pi_b(Y) \leq \pi_a(Y)$*

Setting 1. *$\mathcal{A}$ affine space, U differentiable, strictly concave*

- marginal price $\pi_m(Y)$ —— $\pi_m(Y) = \frac{\mathbb{E}(U'(X^*)Y)}{\mathbb{E}(U'(X^*))}$
- single-period asset price model
- numeraire
- riskless bond
- interest rate —— $r > -1$
- state-price density ρ —— $S_0^i = \mathbb{E}(S_1^i \rho)$
- wealth ω_0
- portfolio θ

Example. *no bond*

$$\begin{aligned} &\textit{given} && \mathbb{E}(\theta \cdot S_1) = \theta \cdot \mu, var(\theta \cdot S_1) = \theta^T V \theta \\ &\textit{minimize} && var(\theta \cdot S_1) \\ &\textit{subject to} && \theta \cdot S_0 = \omega_0, \mathbb{E}(\theta \cdot S_1) = \omega_1 \end{aligned}$$

- mean-variance-efficient frontier —— $\{\theta^*(\omega_1)\}$
- minimum variance portfolio $\theta_{\min^*}$ —— minimise var over ω_1

Example. *with bond*

$$\begin{aligned} &\textit{minimise} && \theta^T V \theta \\ &\textit{subject to} && \theta^0 + \theta S_0 = \omega_0, \theta^0(1+r) + \theta\mu = \omega_1 \end{aligned}$$

Then , $\theta^* = \lambda \theta_m^*$

- market portfolio θ_m^* —— $A(\mu - (1+r)S_0)$, $A = V^{-1}$

Setting 2. *S_1 Gaussian, U CARA*

Example. *no bond*

$$\begin{aligned} &\textit{maximise} && \mathbb{E}(U(\theta S_1)) \\ &\textit{subject to} && \theta S_0 = \omega_0 \end{aligned}$$

Example. *with bond*

$$\begin{aligned} &\textit{maximise} && \mathbb{E}(U(\bar{\theta}\bar{S}_1)) \\ &\textit{subject to} && \bar{\theta}\bar{S}_0 = \omega_0 \end{aligned}$$

then, $\theta^* = \gamma^{-1}\theta_m^*$

- beta/sensitivity β^i —— $\beta^i = \frac{cov(S_1^i, \theta_m^* S_1)}{var(\theta_m^* S_1)}$
- μ^m —— $\theta_m^* \mu$
- S_0^m —— $\theta_m^* S_0$

Proposition 1.3. $\mu^i - (1+r)S_0^i = \beta^i(\mu^m - (1+r)S_0^m)$

- capitalization-weights of the relevant market index

Setting 3.

$S_1 = (1+R)S_0$, $S_1^m = (1+R^m)S_0^m$, $\tilde{\beta}^i = \frac{cov(R^i, R^m)}{var(R^m)}$

Fact. $\mathbb{E}(R^i) = r + \tilde{\beta}^i(\mathbb{E}(R^m) - r)$

2 Martingales

- conditional probability
- conditional expectation given event
- conditional expectation given $\mathcal{G}$, $\mathbb{E}(X|\mathcal{G})$

Theorem 2.1. *$\mathcal{G} \subset \mathcal{F}$ sub-σ-algebra, X integrable, then $\exists$ unique Y (up to a.s.) st*

(i) *Y integrable*

(ii) *Y $\mathcal{G}$-measurable*

(iii) *$\mathbb{E}(Y\mathbb{1}_A) = \mathbb{E}(X\mathbb{1}_A)$ for all $A \in \mathcal{G}$*

Fact. *also true if replace integrable by non-negative*

- $\mathbb{E}(X|Z)$ —— $\mathcal{G} = \sigma(Z)$ for r.v. Z
- $\mathbb{P}(A|\mathcal{G})$ —— $X = \mathbb{1}_A$

Fact. *$\mathcal{G} = \sigma(B_n)$ discrete, then $\mathbb{E}(X|\mathcal{G}) = \sum \mathbb{E}(X|B_n)\mathbb{1}_{B_n}$ a.s.*

Proposition 2.2. *$\mathcal{G} \subset \mathcal{F}$, X, W integrable, then*

(i) *$\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$*

(ii) *X $\mathcal{G}$-measurable, then $\mathbb{E}(X|\mathcal{G}) = X$ a.s.*

(iii) *X independent of $\mathcal{G}$, then $\mathbb{E}(X|\mathcal{G}) = E(X)$ a.s.*

(iv) *$X \geq 0$ a.s. , then $\mathbb{E}(X|\mathcal{G}) \geq 0$ a.s.*

(v) *$\mathbb{E}(\alpha X + \beta W|\mathcal{G}) = \alpha\mathbb{E}(X|\mathcal{G}) + \beta\mathbb{E}(W|\mathcal{G})$ a.s.*

Proposition 2.3 (Tower property). *$\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ sub-σ-algebra, X integrable, then* $\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = \mathbb{E}(X|\mathcal{H})$ *a.s.*

Proposition 2.4 (Taking out what is known). *$\mathcal{G} \subset \mathcal{F}$ sub-σ-algebra, X integrable, Z $\mathcal{G}$-measurable, ZX integrable, then*
$\mathbb{E}(ZX|\mathcal{G}) = Z\mathbb{E}(X|\mathcal{G})$ *a.s.*

Proposition 2.5 (Averaging over independent variables). *X_1, X_2 r.v. in $(E_1, \mathcal{E}_1), (E_2, \mathcal{E}_2)$, $\mathcal{G} \subset \mathcal{F}$, X_1 $\mathcal{G}$-measurable, X_2 independent of $\mathcal{G}$, F non-negative, let $f = \mathbb{E}(F(\cdot, X_2))$, then* $\mathbb{E}(F(X_1, X_2)|\mathcal{G}) = f(X_1)$ *a.s.*

- filtration $(\mathcal{F}_n)$
- random process
- (X_n) adapted to $(\mathcal{F}_n)$
- natural filtration $(\mathcal{F}_n^X)$ —— $\mathcal{F}_n^X = \sigma(X_k : 0 \le k \le n)$
- martingale ——
  - (Adapted) X_n F_n-measurable
  - (Intergrable) $\mathbb{E}(|X_n|) < \infty$
  - (Martingale property) $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$ a.s.
- supermartingale —— $\mathbb{E}(X_{n+1}|\mathcal{F}_n) \le X_n$
- submartingale —— $\mathbb{E}(X_{n+1}|\mathcal{F}_n) \ge X_n$

Fact. *Any martingale also martingale in natural filtration (natural filtration smalllest)*

- martingale (continuous-time) —— adapted, integrable, $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$ for all $s \le t$
- (X_t) continuous —— $t \mapsto X_t(\omega)$ continuous for all ω

Example. *X_n i.i.d. , $\mathcal{F}_0 = \{\varnothing, \Omega\}$, $(\mathcal{F}_n)$ natural filtration*

- *(additive martingale) X_1 inegrable, $E(X_1) = 0$, $S_0 = 0$, $S_n = \sum^n X_k$*
- *(multiplicative martingale) X_1 non-negative, $\mathbb{E}(X_1) = 1$, $Z_0 = 1$, $Z_n = \prod^n Z_k$*

Example. *(X_n) Markov chain, countable state space S, transition matrix P, natural filtration $(\mathcal{F}_n)$, bounded/non-negative f on S, let*

$$Pf(x) = \sum p_{xy} f(y)$$

then if f subharmonic

$$\mathbb{E}(f(X_{n+1})|\mathcal{F}_n) = Pf(X_n) \ge f(X_n)$$

then, $(f(X_n))$ submartingale

- subharmonic —— $f(x) \leq Pf(x)$ for all x
- random time —— $T : \Omega \to \{0, 1, \dots\} \bigcup \{\infty\}$
- stopping time —— $\{T \leq n\} \in \mathcal{F}_n$

Theorem 2.6 (Optional stopping)**.** *(M_n) martingale, T bounded, stopping time, then $\mathbb{E}(M_T) = \mathbb{E}(M_0)$*

Fact. *Doob's optional sampling theorem basic form*

Theorem 2.7. *(M_n) martingale, T almost surely finite, stopping time, suppose one of the following holds:*

(i) $|M_n| \leq C$ *for all* $n \leq T$, C *constant*

(ii) $\mathbb{E}(T) \leq \infty$, $|M_n - M_{n-1}| \leq C$ *for all* $n \leq T$

Fact. *T stopping time $\Rightarrow$ $T \wedge n$ bounded stopping time*

Counter Example. *additive martingale, simple random walk, $T = \min\{n : S_n = 1\}$, then almost sure finite as recurrent, but $\mathbb{E}(T) = \infty$, $\mathbb{E}(S_T) = 1 \neq 0 = S_0$*

Counter Example. *multiplicative martingale, (X_k) i.i.d. ,$\mathbb{P}(X_1 = 0) = \mathbb{P}(X_1 = 2) = 1/2$, $T = \min\{n : Z_n = 0\}$, then $\mathbb{E}(T) = 2$, but $\mathbb{E}(Z_T) = 0 \neq 1 = Z_0$*

Theorem 2.8. *(M_n) martingale, T stopping time, then $(M_{T\wedge n})$ martingale*

- previsible —— H_n $\mathcal{F}_{n-1}$-measurable
- martingale transform of (M_n) by (H_n) —— $Y_0 = 0$, $Y_n = \sum_1^n H_k(M_k - M_{k-1})$

Theorem 2.9 (Martingale transform)**.** *(M_n) martingale, (H_n) bounded, previsible, (Y_n) martingale transform, then (Y_n) martingale*

Fact. *model stock price as (M_n), then*

(i) *optional stopping $\Rightarrow$ expected return $\mathbb{E}(M_T)$ the same no matter what stopping time*

(ii) *(H_k) amount held between time $k-1$ and k, no bounded previsible strategy gives expected gain or loss*

3 Pricing contingent claims

- asset price model $(\bar{S}_n)_{0\le n\le T}$ with numeraire
- numeraire (S_n^0) —— $S_n^0 > 0$
- discounted prices X_n^i —— $X_n^i = S_n^i/S_n^0$
- $\bar{X}_n = (1, X_n)$
- interest rate r_n —— $S_n^0 = (1+r_n)S_{n-1}^0$
- risky assets (S_n)
- portfolio $\bar{\theta}_n$
- self-financing —— $\bar{\theta}_n\bar{S}_n = \bar{\theta}_{n+1}\bar{S}_n$
- value process (V_n) —— $V_0 = \bar{\theta}_1\bar{X}_0$, $V_n = \bar{\theta}_n\bar{X}_n$
- total (discounted) value
- previsible —— if $\bar{\theta}_n$ $\mathcal{F}_{n-1}$-measurable

Setting 4.

(i) $(\mathcal{F}_n)$ *filtration generated by* $(\bar{S}_n)$, $\mathcal{F} = \mathcal{F}_T$

(ii) (S_n) *takes countable values*

(iii) (S_n^0) *deterministic process*

Proposition 3.1. (θ_n) *preivisible process, then* $\exists$ (θ_n^0) *st*

(i) (θ_n^0) *previsible*

(ii) $(\bar{\theta}_n^0)$ *self-financing with initial value* V_0

(iii) $V_T = V_0 + \sum_1^T \theta_n(X_n - X_{n-1})$

- contingent claim of maturity T —— non-negative $\mathcal{F}_T$-measurable r.v.
- European option
- call with strike price K —— $(S_T - K)^+$
- put with strike price K —— $(S_T - K)^-$
- options
- exotic options —— depending on entire path (S_n)
- barrier options ——
  - knocked out

- knocked in

– up-and-out call —— $C = \begin{cases} (S_T - K)^+ & \text{if } \max S_n < B \\ 0 & \text{otherwise} \end{cases}$

– down-and-in put —— $C = \begin{cases} (S_T - K)^- & \text{if } \min S_n \leq B \\ 0 & \text{ptherwise} \end{cases}$

– $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$, $\tilde{\mathbb{P}} \sim \mathbb{P}$ —— $\exists \rho$, st

- $\mathbb{P}(\rho > 0) = 1$
- $\tilde{\mathbb{P}}(A) = \mathbb{E}(\rho \mathbb{1}_A)$

– density $\rho = d\tilde{\mathbb{P}}/d\mathbb{P}$ for $\tilde{\mathbb{P}}$ wrt $\mathbb{P}$

Fact. $\tilde{\mathbb{E}}(X) = \mathbb{E}(\rho X)$

Fact. $\tilde{\mathbb{P}} \sim \mathbb{P}$ *symmetric and transitive*

Fact. $d\mathbb{P}/d\tilde{\mathbb{P}} = 1/\rho$ *a.s.*

– arbitrage for $(\bar{S}_n)_{0 \leq n \leq T}$ ——

(i) $(\bar{\theta}_n)_{1 \leq n \leq T}$ previsible, self-financing
(ii) $V_0 = 0$
(iii) $V_T \geq 0$ a.s.
(iv) $V_T > 0$ with positive probability

– $(\bar{S}_n)$ arbitrage free

Proposition 3.2. (X_n) *martingale* $\Rightarrow$ (X_n) *arbitrage free*

Fact. *proof can be simpler when* (θ_n) *bounded*

Setting 5. *single period model,* $\mathcal{F}_0 = \varnothing, \Omega$

Proposition 3.3. Y *r.v. , following equivalent:*

(i) *arbitrage free (i.e. no* θ *st* $\theta Y \geq 0$ *a.s. with* $\theta Y > 0$ *with positive prob)*

(ii) $\exists$ *equivalent probability measure* $\tilde{\mathbb{P}}$ *st* Y *integrable with* $\mathbb{E}(Y) = 0$

– equivalent martingale measure $\tilde{\mathbb{P}}$ (risk neutral measure) —— $\tilde{\mathbb{P}} \sim \mathbb{P}$, (X_n) martingale under $\tilde{\mathbb{P}}$

Theorem 3.4 ((1st) Fundamental theorem of asset pricing)**.** *following equivalent:*

(i) $(\bar{S}_n)$ *arbitrage free*

(ii) $(\bar{S}_n)$ *has equivalent martingale measure*

Setting 6. *C time-T contigent claim, $D = C/S_T^0$ discounted value*

– attainable/replicable —— $\exists$ previsible, self-financing $\bar{(\theta_n)}$ st $C = \bar{\theta}_n \bar{S}_T$

Fact. *Alternative def: $\exists V_0$ $\mathcal{F}_0$-measurable, θ_n previsible st $D = V_0 + \sum^T \theta_n (X_n - X_{n-1})$*

– fair price —— V_0

– replicating portfolio/hedging portfolio —— $\bar{\theta}_n$

– $(\bar{S}_n)$ complete —— all contingent claims attainable

Proposition 3.5. $\mathcal{F}_0 = \{\varnothing, \Omega\}$*,* $\mathcal{F}_T = \sigma(\bar{S}_1)$

(i) *C non-negative, attainable, time-T contingent claim, $\tilde{\mathbb{P}}$ equivalent martingale measure, then*
fair price $V_0 = \tilde{\mathbb{E}}(D)$, $D = C/S_T^0$

(ii) *$(\bar{S}_n)$ complete, numeraire non-random, then at most one equivalent martingale measure*

– binomial model (Cox-Ross-Rubinstein model) —— interest rate r, parameters $a < b$, R_i i.i.d. with parameter$p$

- $S_n^0 = (1+r)^n$
- $S_n = S_0 \prod (1 + R_k)$
- $\begin{cases} \mathbb{P}(R_1 = a) = 1 - p \\ \mathbb{P}(R_1 = b) = p \end{cases}$

Proposition 3.6. *binomial model has arbitrage unless $r \in (a, b)$*

Proposition 3.7. *$(\bar{S}_n)$ binamial model, $r \in (a, b)$, define*

(i) $p^* = \frac{r-a}{b-a}$

(ii) *equivalent prob measure $\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} = \left(\frac{p^*}{p}\right)^{U_T} \left(\frac{1-p^*}{1-p}\right)^{D_T}$ where $U_T = (T + S_T)/2, D_T = (T - S_T)/2$, S_T number of b*

Then under $\mathbb{P}^*$,

(i) *$R_1, \ldots, R_T$ i.i.d. ,* $\begin{cases} \mathbb{P}^*(R_1 = a) = 1 - p^* \\ \mathbb{P}^*(R_1 = b) = p^* \end{cases}$

(ii) *(X_n) martingale under $\mathbb{P}^*$*

Fact. $r = \mathbb{E}^*(R_1)$

Fact. *Binomial model arbitrage free when* $r \in (a, b)$

Setting 7. *if* $C = f(S_0, \ldots, S_T)$*, then*

$$V(C) = \frac{\mathbb{E}^*(C)}{(1+r)^T} = (1+r)^{-T} \sum f(s_0, s_1, \ldots, s_T)\mathbb{P}^*(S_1 = s_1, \ldots, S_T = s_T)$$

Define recursive relation:

$$f_T(s_0, \ldots, s_T) = f(s_0, \ldots, s_T)$$
$$f_n(s_0, \ldots, s_n) = (1-p^*)f_{n+1}(s_0, \ldots, s_n, (1+a)s_n) + p^* f_{n+1}(s_0, \ldots, s_n, (1+b)s_n)$$

Proposition 3.8. $\mathbb{E}^*(f(S_0, \ldots, S_T)|\mathcal{F}_n) = f_n(S_0, \ldots, S_n)$, $\mathbb{E}^*(C) = f_0(S_0)$

Proposition 3.9. *Define*

$$\Delta_n(s_0, \ldots, s_{n-1}) = \frac{f_n(s_0, \ldots, s_{n-1}, (1+b)s_{n-1}) - f_n(s_0, \ldots, s_{n-1}, (1+a)s_{n-1})}{(1+r)^{T-n}(b-a)s_{n-1}}$$

then $\theta_n = \Delta_n(S_0, \ldots, S_{n-1})$ *replicating portfolio for* C

Fact. *Binomial model complete when* $r \in (a, b)$

Proposition 3.10. (W_n) *simple random walk with* $\mathbb{P}(W_1 = 1) = p$*, let* $M_T = \max W_n$*, then with* $k \leq T$*,* $2k - T \leq m \leq k$

$$\mathbb{P}(M_T = m, W_T = 2k - T) = \left(\binom{T}{k-m} - \binom{T}{k-m-1}\right) p^k (1-p)^{T-k}$$

Example. *if* $(1+a)(1+b) = 1$*, then* $S_n = S_0(1+b)^{W_n}$*, so can give fair price of* $C = F(S_T, \max S_n)$

4 Dynamic Programming

- state-space E
- action-space A

Setting 8. $F : \{0, \ldots, T-1\} \times E \times A \times [0,1] \to E$*,* (ϵ_n) *i.i.d.* *,* $\mathcal{F}_n = \sigma(\epsilon_1, \ldots, \epsilon_n)$

- adapted control $u = (u_n)_{k \leq n \leq T-1}$ —— initial time k, u_n $\mathcal{F}_n$-measurable

Setting 9. *initial state* $x \in E$*, adapted control* u*, define* $X_k = x, X_{n+1} = F(n, X_n, u_n, \epsilon_{n+1})$
Write $X_n = X_n^u(k, x)$

- expected reward —— $V^u(k, x) = \mathbb{E}\left(\left(\sum_k^{T-1} r(n, X_n^u(k,x), u_n)\right) + R(X_T^u(k,x))\right)$

- reward function r, R —— non-negative measurable function
- value function V —— $V(k,x) = \sup_u V^u(k,x)$
- u optimal control from (k,x) —— $V(k,x) = V^u(k,x)$

Proposition 4.1 (Bellman equation). *Let* $Pv(n,x,a) = \mathbb{E}\left(v(n+1, F(n,x,a,\epsilon_{n+1}))\right)$

$$\begin{aligned} v(T,x) &= R(x) \\ v(n,x) &= \sup_{a\in A}\{r(n,x,a) + Pv(n,x,a)\} \qquad n = 0,\ldots,T-1 \end{aligned}$$

Suppose $\exists a$ *st*

$$v(n,x) = r(n,x,a(n,x)) + Pv(n,x,a(n,x)) \qquad n = 0,\ldots,T-1$$

Then,

(i) $V = v$

(ii) *optimal control* $u_n^* = a(n, X_n^{u*}(k,x))$

Fact. *Possible variations:*

(i) r, R *as costs*

(ii) *mixture of costs and rewards*

(iii) *time-dependent state-space* E_n

(iv) *time-and-state-dependent action-space* $A_{n,x}$

- American call —— family of time-T contingent claim $(1+r)^{T-\tau}(S_\tau - K)^+$
- American put —— family of time-T contingent claim $(1+r)^{T-\tau}(S_\tau - K)^-$

Setting 10. (S_n) *binomial model,* $r \in (a,b)$

Fact. *complete* $\Rightarrow$ *can hedge all* C *with* $\mathbb{E}^*(C) = 0$

Example (American call). $\tau = T$ *always optimal, American and European calls equivalent*

Fact. *fair price can be founded using Bellman equation*

5 Brownian motion

- Brownian motion ——
  - $B_0 = 0$
  - $(B_{s+t} - B_s) \sim N(0,t)$, independent of $\sigma(B_r : r \le s)$
  - $t \mapsto B_t(\omega)$ continuous
- Brownian motion starting from x —— $B_0 = x$
- Gaussian process —— $\forall(t_1,\ldots,t_n), (X_{t_1},\ldots,X_{t_n})$ multivariate normal

Proposition 5.1. (B_t) *continuous process starting from 0, then following equivalent:*

(i) (B_t) *Brownian motion*

(ii) (B_t) *zero mean Gaussian process,* $\mathbb{E}(B_s B_t) = s \wedge t$

Proposition 5.2 (Scaling property)**.** (B_t) *Brownian motion, set* $\tilde{B}_t = c^{-1} B_{c^2 t}$*, then* $(\tilde{B}_t)$ *Brownian motion*

Proposition 5.3. (B_t) *Brownian motion,* (B_t) *exit every finite interval a.s.*

– $\mathcal{F}_t = \sigma(B_s : s \in [0, t])$

– stopping time —— $\{T \leq t\} \in \mathcal{F}_t$ for all t

– $\mathcal{F}_T$ —— $A \in \mathcal{F}_\infty$ st $A \bigcap \{T \leq t\} \in \mathcal{F}_t$

Proposition 5.4 (Strong Markov property)**.** (B_t) *Brownian motion,* T *a.s. finite stopping time. Define* $\tilde{B}_t = B_{T+t} - B_T$*, then*

(i) $(\tilde{B}_t)$ *Brownian motion*

(ii) *independent of* $\mathcal{F}_T$

Proposition 5.5. (B_t) *Brownian motion, define* $T_a = \inf\{t \geq 0 : B_t = a\}$*, then*

(i) T_a *stopping time*

(ii) T_a *almost surely finite*

Theorem 5.6. $(\Omega, \mathcal{F}, \mathbb{P})$ *not discrete,* m *prob measure on* $\mathbb{R}$*, mean 0, variance 1, then* $\exists (B_t), (W_t^{(k)})$ *for all* $k \in \mathbb{N}$ *st*

(i) (B_t) *Brownian motion*

(ii) $(W_{\frac{n}{k}}^{(k)})$ *random walk with distribution* m*,* $(W_t^{(k)})$ *linear interpolation of values* $\left\{\frac{n}{k}\right\}$

(iii) $\frac{W_t^{(k)}}{\sqrt{k}} \to B_t$ *uniformly on compacts in* t *a.s.*

Fact. *combination of Wiener's Theorem and Donsker's Invariance Principle*

– Wiener measure

Proposition 5.7. *let $T \geq 0$, $c \in \mathbb{R}$, $B = (B_t)_{\{0 \leq t \leq T\}}$ brownian motion, $\tilde{B}_t = B_t + ct$, then $\forall$ measurable set $A \subset C[0,T]$, $\mathbb{P}(\tilde{B} \in A) = \mathbb{E}(\mathbb{1}_{\{B \in A\}} e^{cB_T - \frac{c^2 T}{2}})$*

Fact. *special case of Cameron-Martin theorem*

Proposition 5.8 (Reflection principle). *(B_t) Brownian motion, $a \geq 0$, set $T_a = \inf \{t \geq 0 : B_t = a\}$, define $\tilde{B}_t = \begin{cases} B_t & \text{if } t \leq T_a \\ 2a - B_t & \text{if } t > T_a \end{cases}$, then $(\tilde{B}_t)$ Brownian motion*

- maximum process —— $M_t = \sup_{\{0 \leq s \leq t\}} B_s$

Fact. *M_t same distribution as $|B_t|$*

Proposition 5.9. *T_a has density $h_a(t) = \frac{a}{\sqrt{2\pi t^3}} e^{-\frac{a^2}{2t}}$*

- $p_t(x, y)$ —— density of B_t starting at x
- $p_t^a(x, y) = p_t(x, y) - p_t(x, 2a - y)$

Proposition 5.10. *$x \leq a$, (B_t) Brownian motion with density starting from x, then $\forall$ non-negative measurable f, $\mathbb{E}_x(f(B_t)\mathbb{1}_{\{T_a > t\}}) = \int_{\infty}^{a} f(y) p_t^a(x, y) dy$*

6 Black-Scholes model

- Black-Scholes model —— $S_t^0 = e^{rt}, S_t = S_0 e^{\sigma B_t + \mu t}$
- price of riskless bond S_t^0
- interest rate r
- price of risky asset S_t
- drift μ
- volatility σ

Fact. *$(e^{\sigma B_t - \sigma^2 t/2})$ martingale*

Fact. *with $\mu^* = r - \sigma/2$, discounted asset price $(e^{rt} S_t)$ martingale*

Proposition 6.1. *(S_t^0, S_t) Black-Scholes, fix T, consider $\frac{\mathrm{d}\mathbb{P}*}{\mathrm{d}\mathbb{P}} = e^{\lambda B_T - \lambda^2 T/2}$ where $\sigma\lambda = \mu^* - \mu$, then under $\mathbb{P}^*$, discounted seet price $(e^{-rt}S_t)$ martingale*

Fact. *abuse notation in writing $\mathbb{P}^*$ instead of $\mathbb{P}$*

- time-T contingent claim C —— $\mathcal{F}_T$-measurable r.v.
- Black-Scholes price V_0 —— $V_0 = e^{-rT}\mathbb{E}^*(C)$

Fact. *fair price unique*

Example.

(i) $C = S_T$

(ii) $C = K$

- simple replicable claim —— constant C_0, $0 = t_0 \leq \cdots \leq t_n = T$, θ_k bounded $\mathcal{F}_{t_{k-1}}$-measurable

$$e^{rT}C = C_0 + \sum_1^n \theta_k(X_{t_k} - X_{t_{k-1}})$$

(Replicating strategy) at time t_{k-1}, buy θ_k, then sell θ_k at time t_k, then buy θ_k bond

Fact. *any simple replicable claim can be replicated for cost C_0 at time 0, $V_0 = C_0$*

Fact (Brownian martingale representation theorem)**.** *every integrable $\mathcal{F}_T$-measurable contingent claim is limit in probability of simple replicable claims*

Setting 11. $S_0 = s$

Fact. $\log S_t = \log s + \sigma B_t + \mu t$

- $p(t, x, z) = \frac{1}{\sqrt{2\pi t}} e^{\frac{|x-z|^2}{2t}}$
- $p(\sigma^2 t, x(t), \cdot)$ —— density of $\log S_t$, $x(t) = \log s + \mu t$
- $\rho(t, s, \cdot)$ —— density of S_t
- $\dot{\rho}$ —— derivative in first argument
- ρ' —— derivative in second argument

Fact. $y\rho(t, s, y) = p(\sigma^2 t, x(t), z)$, $z = \log y$

Fact. $\dot{\rho} = \frac{1}{2}\sigma^2 s^2 \rho'' + rs\rho'$

Proposition 6.2. *F on $(0,\infty)$ continuous, polynomial growth, $t \in [0,T], s \in (0,\infty)$ $V(t,s)$ time-t value of time-T contingent claim $F(S_T)$, conditional on $S_t = s$ $V(t,s) = e^{-r(T-t)}\mathbb{E}^*(F(S_T)|S_t = s) = e^{-r(T-t)}\mathbb{E}(F(se^{\sigma B_{T-t}+\mu^*(T-t)}))$, then*

(i) *V continous on $(0,T) \times (0,\infty)$*

(ii) *$V \in C^{1,2}$ on $(0,T) \times (0,\infty)$*

(iii) *(Black Scholes PDE) $\mathcal{L}V = \dot{V} + \frac{1}{2}\sigma^2 s^2 V'' + rsV' - rV$ with $V(\cdot,T) = F$*

Setting 12 (Binomial approximation to BS)**.** *Consider convergence of random walk to Brownian motion, special case $(W^{(k)}_{n/k})$ simple symmetric random walk on $\{-1,1\}$*

- $1 + a_k = \exp\left(-\frac{\sigma}{\sqrt{k}} + \frac{\mu}{k}\right)$
- $1 + b_k = \exp\left(\frac{\sigma}{\sqrt{k}} + \frac{\mu}{k}\right)$
- $1 + r_k = \exp\left(\frac{r}{k}\right)$
- $S^{(k)}_t = S_0 \exp\left(\frac{\sigma W^{(k)}_t}{\sqrt{k}} + \mu t\right)$
- $S_t = S_0 \exp(\sigma B_t + \mu t)$
- $S^{(k)0}_t = S^0_t = \exp(rt)$

Fact.

(i) *(S^0_t, S_t) Black-Scholes of drift μ, volatility σ, interest rate r*

(ii) *$(S^{(k)0}_{n/k}, S^{(k)}_{n/k})$ binomial model of parameters $a_k < r_k < b_k$, $p = \frac{1}{2}$*

(iii) *$S^{(k)}_t \to S_t$ uniformly on compacts in t a.s.*

- $\mathbb{P}^{(k)*}$ —— martingale measures for binomial model
- $\mathbb{P}^*$ —— martingale measures for Black-Scholes model

Fact. *$\mathbb{E}^{(k)*}(G(S^{(k)})) \to \mathbb{E}^*(G(S))$, so can approximate fair price using Binomial model*

Setting 13. *$\mu = \mu^* = r - \frac{\sigma^2}{2}$ for convenience*

- expression $C = F(B)$ —— F on $C[0,T]$
- terminal-value option $C = f(B_T)$

Fact. *for $C = f(B_T)$, $V_0 = e^{-rT}\int f\left(\sqrt{T}y\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy$*

Fact. *May not be efficient way:*

(i) *multiple assets —— exponentially growth computational cost*

(ii) *want to compute pricing surface*

- pricing surface —— $V(t,s) = e^{-r(T-t)}\mathbb{E}^*(C|S_t = s)$

Setting 14. *terminal-value option $C = g(S_T)$, g continuous, no more than linear growth*

- $f(x) = g(e^{\sigma x})$
- $u(t,x) = \mathbb{E}_x(f(B_t))$

Fact. $V(t,s) = e^{-r(T-t)}\mathbb{E}(g(se^{\sigma B_{T-t}+\mu(T-t)})) = e^{-r(T-t)}u\left(T-t, \frac{\log s+\mu(T-t)}{\sigma}\right)$

- $p_t(x,y) = \frac{1}{\sqrt{2\pi t}}e^{\frac{-(y-x)^2}{2t}}$

Fact. $u(t,x) = \int p_t(x,y)f(y)dy$

Fact. $\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial x^2} \Rightarrow \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}$

Setting 15. *assume can accurately approximate $u(t,\pm L)$ for all t*

- grid $\{(ik,jh)\} \subset [0,T]\times[-L,L]$ —— $k = \frac{T}{N}, h = \frac{L}{M}$
- grid points U_j^i —— idea $U_j^i \approx u(ik,jh)$
- FTCS (forward-in-time, central-in-space) —— $\frac{U_j^{i+1}-U_j^i}{k} = \frac{U_{j-1}^i-2U_j^i+U_{j+1}^i}{2h^2}$
  - explicit
  - first-order in time
- BTCS (backward-in-time, central-in-space) —— $\frac{U_j^{i+1}-U_j^i}{k} = \frac{U_{j-1}^{i+1}-2U_j^{i+1}+U_{j+1}^{i+1}}{2h^2}$
  - require to solve $2M\times 2M$ matrix inversion
  - better stability
  - first-order in time
- Crack-Nicolson —— $\frac{U_j^{i+1}-U_j^i}{k} = \frac{1}{2}\left(\frac{U_{j-1}^i-2U_j^i+U_{j+1}^i}{2h^2} + \frac{U_{j-1}^{i+1}-2U_j^{i+1}+U_{j+1}^{i+1}}{2h^2}\right)$
  - require to solve $2M\times 2M$ matrix inversion
  - better stability
  - second-order in time
- Monte Carlo ——
  - time-step $k = \frac{T}{N}$
  - **(i)** $(B_t^{(N)} : t = ik)$ linear interpolation of random walk with step distribution $N(0,k)$
    (ii) $(X_t : t = ik)$ simple symmetric random walk with step-size $h = \sqrt{k}$
  - generate sample $(B^{(N),i})$
  - **(Idea)** $\frac{1}{n}\sum F(B^{(N),i}) \approx \mathbb{E}(F(B^{(N)})) \to \mathbb{E}(F(B))$

- fair price for European call $EC(x, K, \sigma, r, T) = \mathbb{E}^*(e^{-rT}(S_T - K)^+)$
- $\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$
- $\Phi(a) = \int_{\infty}^{a} \phi(y)dy$
- $\bar{\Phi}(a) = 1 - \Phi(a)$
- $d^{\pm} = \frac{\log\left(\frac{x}{K}\right) + rT}{\sigma\sqrt{T}} \pm \frac{\sigma\sqrt{T}}{2}$

Proposition 6.3 (Black-Scholes formula)**.** $EC(x, K, \sigma, r, T) = x\Phi(d^+) - e^{-rT} K\Phi(d^-)$

- put-call parity —— $(S_T - K)^+ - (S_T - K)^- = S_T - K$
- fair price for forward contract —— $\mathbb{E}^*(e^{-rT}(S_T - K)) = x - e^{-rT}K$
- fair price for European put $EP(x, K, \sigma, r, T) = e^{-rT} K\bar{\Phi}(d^-) - x\bar{\Phi}(d^+)$

Setting 16. $v(x) = v(x, C, \sigma, r, T) = \mathbb{E}^*(e^{-rT}C)$

- **Sensitivities**
- Delta $\Delta = \frac{\partial v}{\partial x}$
- Gamma $\Gamma = \frac{\partial^2 v}{\partial x^2}$
- Vega $\mathcal{V} = \frac{\partial v}{\partial \sigma}$
- Rho $\rho = \frac{\partial v}{\partial r}$

Example. *C European call,* $\Delta = \Phi(d^+), \mathcal{V} = x\phi(d^+)\sqrt{T}$

Proposition 6.4.

(i) $\sigma \mapsto EC(x, K, \sigma, r, T)$ *increasing bijection*

(ii) $\lim_{\sigma \to 0} EC(x, K, \sigma, r, T) = (x - e^{-rT})^+$

(iii) $\lim_{\sigma \to \infty} EC(x, K, \sigma, r, T) = x$

- implied volatility $\sigma_{implied}(K, T)$ —— $EC(S_0, K, \sigma_{implied}(K, T), r, T) = EC_{market}(K, T)$

Example. *up-and-out call* $C = h(S_T)\mathbb{1}_{\{\sup_{0 \le t \le T} S_t < A\}}$, $h(s) = (s - K)^+$, $A \ge \max\{S_0, K\}$

7 Fun fact

Fact. (B_t) *Brownian motion, then* $tB_{\frac{1}{t}}$ *Brownian motion*