Graph theory

1 Basic Definition

Topics covered in Exam.

2016-Paper4-16G *Handshaking*

2016-Paper1-16G *Euler's formula, planar graph inequality with girth*

2017-Paper3-15H *bipartite*

- Graph
- Vertices
- Edges
- Order of a graph
- Size of a graph
- Isomorphic

Example. P_n, K_n, C_n, E_n

- Subgraph
- Induced subgraph
- Connected
- Component
- Forest
- Tree

Theorem 1.1. *Following equiv:*

(i) *Tree*

(ii) *Minimal connected*

(iii) *Maximal acyclic*

Theorem 1.2. *Connented iff has spanning tree*

Lemma 1.3 (Handshaking lemma)**.** $\sum d(v) = 2e(G)$

– Leaf

– Minimum degree $\delta(G)$

– Maximum degree $\Delta(G)$

Theorem 1.4. *Order ≥ 2, then at least 2 leaves*

Corollary 1.5. *Tree order n-1*

Corollary 1.6. *Following equiv:*

(i) *Tree*

(ii) *Connected of order n size n-1*

(iii) *Acyclic of order n size n-1*

Theorem 1.7 (Caley). n^{n-2} *labelled trees of order n*

– r-partite

– Bipartite

Theorem 1.8. *Bipartite iff no odd cycles*

– Eulerian tour

– Eulerian

Theorem 1.9. *Eulerian iff $|G| > 1$, connected, d(v) even*

– Planar

– Plane graph

– Face

Lemma 1.10. *d(v) even, then partitioned into cycles*

Lemma 1.11. *e boundary of two faces iff contained in cycle*

Theorem 1.12 (Euler). *Connected plane graph, order n, size m, faces f*

$$n - m + f = 2$$

– Bridge

Fact. *f_i number of faces of length i*

$$\sum i f_i = 2m$$

– Girth

Theorem 1.13. *G connected, bridgeless, planar graph*

$$e(G) \leq \frac{g}{g-2}(n-2)$$

G planar graph

$$e(G) \leq 3(n-2)$$

– omplete bipartite graph $K_{p,q}$

Theorem 1.14 (Kuratowski). *Planar iff not contain subdivision of $K_{3,3}$ or K_5*

– dual graph

Fact. *Simple if 3-connected*

2 Matching and Connectivity

Topics covered in Exam.

2016-Paper4-16G Menger

2017-Paper2-15H Hall

– Matching

– Independent

– 1-factor (1 regular spanning subgraph)

– n-regular

Theorem 2.1 (Hall). *G bipartite with vertex classes X, Y*

Matching from X to Y iff $|\Gamma(A)| \geq |A|$ for all subsets $A \subset X$

Theorem 2.2 (defect form).

$|X| - d$ independent edges iff $|\Gamma(A)| \geq |A| - d$

Theorem 2.3 (polygamous version). *1 to 2 matching iff $|\Gamma(A)| \geq 2|A|$*

– Distinct representatives

Corollary 2.4. *Set of distinct representative iff for all subset $S \in [n], |\bigcup_{i \in S} Y_i| \geq |S|$*

– K-connected (be careful with K_n)

– Vertex connectivity $\kappa(G)$

– Local connectivity $\kappa(a, b; G)$

– Edge connectivity $\lambda(G), \lambda(a, b; G)$

Fact.

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

– Vertex disjoint (set of paths)

Theorem 2.5 (Menger)**.** *$ab \notin E(G)$, then*

$$\exists \kappa(a, b; G) \text{ vertex disjoint } a - b \text{ paths}$$

– graph contraction

– contration

Corollary 2.6. *$\kappa(G) \geq k, X, Y$ disjoint subsets $|X|, |Y| \geq k$*
Then exists set of k completely vertex disjoint X - Y paths

Theorem 2.7 (edge form of Menger)**.** *$a, b \in V(G)$, then*

$$\exists \text{ set of } \lambda(a, b; G) \text{ edge-disjoint } a - b \text{ paths}$$

– Line graph

Fact. *Menger impies Hall*

3 Extremal Graph Theory

Topics covered in Exam.

2016-Paper2-15G *Turan*

2017-Paper1-16H *Hamiltonian*

– Hamiltonian cycle

Theorem 3.1. *$|G| \geq 3$, every pair of non-adjacent vertices $d(x) + d(y) \geq k$*
If $k < n$, G connected, then exists path of length k
If $k = n$, exist Hamiltonian cycle

Corollary 3.2 (Dirac)**.** *$|G| = n$, $\delta \geq n/2$, then exists Hamiltonian cycle*

Theorem 3.3. *G order n, no length k path. Then $e(G) \leq \frac{k-1}{2}n$.*
Equality holds iff $k|n$, G disjoint copies of K_k

– $ex(n, F)$

– r partite Turan graph of order n, $T_r(n)$ with size $t_r(n)$

Theorem 3.4 (Turan)**.** *K_{r+1} free, order n, $e(G) \geq t_r(n)$, then $G = T_r(n)$*

Proof 1. Induction on n with base case K_r. Remove edges to G' with $t_r(n)$ edges. Delete $\delta(G')$ vertex x. $G - x$ Turan. Put back x. T_{r+1} maximal. □

Proof 2. Induction on n with base case K_r. Add edges to maximal. $K_n \subset G'$. Count edges. $G - K_n$ Turan. Put back K_n. $\square$

Theorem 3.5 (problem of Zarankiewicz)**.** *Bipartite n x n, no $K_{t,t}$*

$$z(n,t) \leq (t-1)^{1/t}(n-t+1)n^{1-1/t} + (t-1)n = O(n^{2-1/t})$$

Theorem 3.6.

$$z(n,2) \leq n/2(1+\sqrt{4n-3}), \text{ equality holds for infinitely many } n$$

Theorem 3.7.

$$ex(n, K_{2,2}) \leq n/4(1+\sqrt{4n-3})$$

Fact.

(i) $ex(n, K_{r+1}) = t_r(n)$

(ii) $ex(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$

(iii) $ex(n, C_5) = t_2(n)$

(iv) $ex(n, F) = \lfloor n^2/4 \rfloor + 2$

(v) $ex(n, P) = t_2(n) + n - 2$

- $K_r(t)$ complete r-partite graph, $K_r(t) = T_r(rt)$

Lemma 3.8. *If $n > n_1(r,t,\epsilon), |G| = n, \delta(G) \geq (1 - 1/r + \epsilon)n$ Then contains $K_{r+1}(t)$*

Theorem 3.9 (Erdos-Stone)**.** *n sufficiently large, $e(G) \geq (1 - 1/r + \epsilon)\binom{n}{2}$, then contains $K_{r+1}(t)$*

- chromatic number, $\chi(G)$

Corollary 3.10.

$$\lim_{n\to\infty} \frac{ex(n,F)}{\binom{n}{2}} = 1 - \frac{1}{\chi(F)-1}$$

4 Colouring

Topics covered in Exam.

2016-Paper3-15G *chromatic polynomial*

- Vertex colouring
- greedy algorithm

Theorem 4.1.

$$\chi(G) \leq 1 + \max_H \delta(H)$$

Corollary 4.2. $\chi(G) \leq \Delta(G) + 1$

– Block

Fact. *Tree of blocks and bridges*

Theorem 4.3 (Brooks)**.** $\chi(G) = \Delta(G) + 1$*, then G complete or odd cycle*

– clique number, $\omega(G)$

– independence number, $\alpha(G)$

Fact. $\alpha(G) = \omega(\bar{G})$

Fact. $\max\left\{\omega(G), \frac{|G|}{\alpha(G)}\right\} \leq \chi(G)$

– chromatic polynomial, $p_G(x)$ —— number of ways to colour vertices of G with colours $1, 2, \ldots, x$

Example.

(i) *complement of K_n, $p_{\bar{K}_n(x)=x^n}$*

(ii) *Tree T, $p_T(x) = x(x-1)^{n-1}$*

(iii) *complete graph, $p_{K_n}(x)x(x-1)(x-2)\cdots(x-n+1)$*

Theorem 4.4. *Any $e \in E(G)$, $p_G(x) = p_{G-e}(x) - p_{G/e}(x)$*

Fact. $p_G(x) = \prod_C p_C(x)$

Corollary 4.5.

$$p_G(x) = x^n - a_{n-1}x^{n-1} + \cdots + (-1)^n a_0$$

where $n = |G|, a_{n-1} = e(G), a_j \geq 0$ for all j, $\min\{j : a_j \neq 0\} = k$ the number of components

Fact. *G not specified by $p_G(x)$*

– k-edge-colouring

– chromatic index, $\chi'(G)$

Theorem 4.6. *Bipartite multigraph, then $\chi'(G) = \Delta(G)$*

Fact. *fail for non-bipartite graph e.g. K_3*

Theorem 4.7 (Vizing)**.** $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$

– list colouring, $\chi_l(G)$

Fact. $\chi_l(G) \geq \chi(G)$

Fact. $\chi_l(G) \leq 1 + \max_H \delta(H)$

Theorem 4.8 (Five Colour Theorem)**.** *G planar, then $\chi(G) \leq 5$*

Theorem 4.9 (Thomasson)**.** *G planar, then $\chi_l(G) \leq 5$*

Fact. *exist graph with $\chi_l(G) = 5$*

Theorem 4.10 (Tait)**.** *Four Colour Theorem holds iff $\chi'(G) = 3$ for every cubic bridgeless planar G*

4.1 Graph on other surfaces

- Euler characteristic, $E \leq 2$

Fact. *simply connected, then* $n - m + f = E$

Example. ***Orientable surface*** g *handles,* $E = 2 - g$

- *g = 1, torus*
- *g = 2, double torus*

Non-orientable surfaces*, one for each* $E \leq 1$

- *E = 1, projective plane*
- *E = 0, Klein bottle*

Fact. $m \leq 3(n - E)$

Theorem 4.11 (Heawood)**.** *characteristic* $E \leq 1$*, then*

$$\chi(G) \leq H(E) = \left\lfloor \frac{7 + \sqrt{49 - 24E}}{2} \right\rfloor$$

5 Ramsey Theory

Topics covered in Exam.

2017-Paper3-15H *Ramsey numbers*

- Ramsey number, $R(s), R(s,t)$ —— smallest n st if colour edges of K_n red or blue, then exists red K_s or blue K_t

Theorem 5.1 (Ramsey)**.** $R(s,t) \leq R(s-1,t) + R(s,t-1)$

Fact. $R(s,t) \leq \binom{s+t-2}{s-1} < 2^{s+t}$

Fact. *inequality not exact*

- $R_k(s_1, s_2, \ldots, s_k)$

Theorem 5.2. $R_k(s_1, s_2, \ldots, s_k)$ *exists*

Fact. $R_k(s_1, \ldots, s_k) \leq R_{k-1}(R(s_1, s_2), s_3, \ldots, s_k)$

Fact. $R_k(s_1, \ldots, s_k) \leq R_k(s_1 - 1, \ldots, s_k) + \cdots + R_k(s_1, \ldots, s_k - 1) - k + 2$

- r-uniform hypergraph

Theorem 5.3 (Ramsey for r-sets)**.** $R^{(r)}(s,t)$ *exists*

Theorem 5.4 (Infinite Ramsey)**.** *k colours $\mathbb{N}^{(r)}$, then exists infinite $M \subset \mathbb{N}$ s.t. $M^{(r)}$ monochromatic*

- big if $|Y| \geq \min Y$

Theorem 5.5. *exists $B(s,k,r)$ s.t. if $n \geq B(s,k,r)$, $\{s, s+1, \ldots, n\}^{(r)}$ k coloured, then exist big set $Y \subset \{s, s+1, \ldots, n\}^{(r)}$ s.t. $Y^{(r)}$ monochromatic*

6 Probabilistic Method

Theorem 6.1 (Erdos). *$s \geq 3$, then $R(s) \geq 2^{(s-1)/2}$*

Theorem 6.2. *G has independent set of size at least*

$$\sum_{v \in G} \frac{1}{d(v)+1} \geq \frac{|G|}{d+1}$$

- $\mathcal{G}(n,p)$ —— space of all $2^{\binom{n}{2}}$ labelled graph

Fact. *From*

Theorem 6.3 (Erdos). *$g \geq 3, k \geq 2$, then exists G with $\chi(G) \geq k$ ang girth at least g*

Theorem 6.4. *$n \geq t \geq 2$, then $z(n,t) > \frac{3}{4} n^{2-2/(t+1)}$*

Lemma 6.5. *$\mathbb{E}(X_n^2)/\mathbb{E}(X_n)^2 \to 1$ or $Var(X_n)/\mathbb{E}(X_n)^2 \to 0$, then any constant c,*

$$\mathbb{P}(|X_n - \mathbb{E}(X_n)| \geq c\mathbb{E}(X_n)) \to 0$$

let $c = 1$, then

$$\mathbb{P}(X_n = 0) \to 0$$

Fact. $\mathbb{P}(X = 0) \leq \frac{Var(X)}{(\mathbb{E}X)^2}$

Fact. *let $X = \sum_A I_A$, then*

- $\mathbb{E}X^2 = \sum_{A,B} \mathbb{P}(A \cap B) = \sum_{A,B} \mathbb{P}(A)\mathbb{P}(B|A)$
- $Var(X) = \sum_{A,B} \mathbb{P}(A)[\mathbb{P}(B|A) - \mathbb{P}(B)]$

Theorem 6.6. *$\omega(n) \to \infty, G \in (G)(n,p)$*
If $p = \frac{\log n - \omega(n)}{n}$, G has isolated vertices a.s.
If $p = \frac{\log n + \omega(n)}{n}$, G has no isolated vertices a.s.

- *almost surely* —— $\mathbb{P}(A) \to 1$ *as* $n \to \infty$
- *threshold phenomenon*

Theorem 6.7. *let $\mu(d) = \binom{n}{d} p^{\binom{d}{2}}$, fixed p, then*
If $\mu(d) \to 0$, $G \in \mathcal{G}(n,p)$ almost certainly not contain K_d
If $\mu(d) \to \infty$, $G \in \mathcal{G}(n,p)$ almost certainly contain K_d

Corollary 6.8. *$0 < p < 1$ fixed, then clique number is $(2 + o(1)) \log_{1/p} n$ almost surely*

7 Eigenvalue Methods

Topics covered in Exam.

2017-Paper4-16H *strongly regular, Petersen graph*

– *adjacency matrix, A*

Fact. *A real symmetric*

Fact. *$(A^d)_{ij}$ number of walks length d from i to j*

– *diameter, $diam(G)$*

Fact. *$\{I, A, A^2, A^{diam(G)}\}$ linearly independent, so at least $diam(G) + 1$ distinct eigenvalues*

Fact. *$\lambda_{\min} = \min_{|x|=1} x^T Ax, \lambda_{\max} = \max_{|x|=1} x^T Ax$*

Fact. *H induced subgraph, $\lambda_{\min}(G) \leq \lambda_{\min}(H) \leq \lambda_{\max}(H) \leq \lambda_{\max}(G)$*

Fact. *bipartite G, then $A = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$. Also, characteristic polynomial is in t^2*

Theorem 7.1. *G graph*

(i) *$\delta(G) \leq \lambda_{\max} \leq \Delta(G)$*

(ii) *$|\lambda| \leq \Delta(G)$ for all λ*

If G connected, then

(i) *$\lambda_{\max} = \Delta(G)$ iff G regular, in this case multiplicity of $\lambda_{\max}$ = 1*

(ii) *$\lambda_{\min} = -\Delta(G)$ iff G regular and bipartite, in this case multiplicity of $\lambda_{\min}$ = 1*

– *orientation*

– *incidence matrix, B*

– *combinatorial Laplacian, $L = BB^T$*

Fact. *let $D = diag(d(1), \dots, d(n))$*

$$L = D - A = \begin{cases} d(i) & \text{if } i = j \\ -1 & i \to j \text{ or } j \to i \\ 0 & \text{otherwise} \end{cases}$$

Fact. *L positive semi-definite*

Fact. *let eigenvalue be $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$*
$x^T Lx = \sum_{ij \in E(G)} (x_i - x_j)^2$, hence eigenvalue $\mu_1 = 0$

Proposition 7.2. *$\mu_2 = \min\{x^T Lx/\|x\|^2 : x \neq 0, \sum x_i = 0\}$*

Theorem 7.3. *G graph, $U \subset V(G)$, then at least $\frac{\mu_2 |U||V-U|}{|G|}$ edges between U and $V - U$*

- *Moore graph, d-regular, diameter 2, order* $d^2 + 1$
- *strongly regular* (d, a, b)*:*
  - ***(i)*** *regular of degree* d
  - ***(ii)*** *every pair of adjacent vertices exactly* a *common neighbours*
  - ***(iii)*** *every pair of non-adjacent vertices exactly* b *common neighbours*

Fact. *Moore graph is* $(d, 0, 1)$

Theorem 7.4. *G strongly regular* (d, a, b)*, order* n*, then*

$$\frac{1}{2}\left(n - 1 \pm \frac{(n-1)(b-a) - 2d}{\sqrt{(a-b)^2 + 4(d-b)}}\right) \in \mathbb{N}$$

Theorem 7.5. *exist Moore graph only if* $d \in \{1, 2, 3, 7, 57\}$