Applied Probability

1 Continuous time Markov Chains

1.1 Poisson process

– Poisson process with intensity λ

(i) $N(0) = 0$, $N(s) \leq N(t)$ for $s < t$

(ii) $\mathbb{P}(N(t+h) = n+m | N(t) = n) = \begin{cases} \lambda h + o(h) & \text{if } m = 1 \\ o(h) & \text{if } m > 1 \\ 1 - \lambda h + o(h) & \text{if } m = 0 \end{cases}$

(iii) $N(t) - N(s)$ independent of $(N(k))_{k \leq s}$

Theorem 1.1. $N(t) \sim Poi(\lambda t)$

Proof. derive differential equation, then generating function □

– $p_j(t) = \mathbb{P}(N(t) = j)$

– Generating function $G(s,t) = \sum p_j(t) s^j$

– Arrival time T_n

– interarrival time U_n

Theorem 1.2.

(i) *$U_i \sim Exp(\lambda)$*

(ii) *U_i independent*

Proof. use $N(t)$ Poisson □

Fact. $N(t) \geq j \iff T_j \leq t$

– order statistics

Theorem 1.3. *$T_1, \ldots, T_n$ conditional on $\{N(t) = n\}$ same as joint distribution of order statistics of n i.i.d. $Uniform[0,t]$*

Proof. U to T, then calculate density □

1.2 Birth process

– birth process with birth rates $\lambda_0, \lambda_1, \dots$

(i) $N(0) = 0$, $N(s) \leq N(t)$ for $s < t$

(ii) $\mathbb{P}(N(t+h) = n+m | N(t) = n) = \begin{cases} \lambda_n h + o(h) & \text{if } m = 1 \\ o(h) & \text{if } m > 1 \\ 1 - \lambda_n h + o(h) & \text{if } m = 0 \end{cases}$

(iii) $N(t) - N(s)$ independent of $(N(k))_{k \leq s}$

Example.

(i) ***Poisson process****:* $\lambda_n = \lambda$

(ii) ***Simple birth****:* $\lambda_n = n\lambda$

(iii) ***Simple birth with immigration****:* $\lambda_n = n\lambda + \nu$

– $T_\infty = \lim T_n = \sum^\infty U_i$

– non-explosive / honest —— $\mathbb{P}(T_\infty = \infty) = 1$

Theorem 1.4. *birth process N, $\lambda_n > 0$, then non-explosive* $\iff \sum_n \frac{1}{\lambda_n} = \infty$

Lemma 1.5. *$U_n \sim Exp(\lambda_n)$, independent, then* $\mathbb{P}(T_\infty < \infty) = \begin{cases} 0 & \text{if } \sum_n \frac{1}{\lambda_n} = \infty \\ 1 & \text{if } \sum_n \frac{1}{\lambda_n} < \infty \end{cases}$

– forward system of equations: $p'_{ij}(t) = \lambda_{j-1} p_{i,j-1}(t) - \lambda_j p_{ij}(t)$

– backward system of equations: $p'_{ij}(t) = \lambda_i p_{i+1,j}(t) - \lambda_i p_{ij}(t)$

Theorem 1.6.

(i) *forward system has unique solution* $\{p_{ij}(t)\}$

(ii) $\{p_{ij}(t)\}$ *satisfy backward system*

Theorem 1.7. $\{p_{ij}(t)\}$ *unique solution of forward equations,* $\{\pi_{ij}(t)\}$ *any solution of backward equations, then* $p_{ij}(t) \leq \pi_{ij}(t)$

Fact. $\sum_j p_{ij}(t) = 1 \iff \mathbb{P}(T_\infty > t) = 1$

– weak Markov property

– stopping time

- strong Markov property
- right continuity
- stationary independent increments

  (i) $N(t) - N(s)$ only depends on $t - s$

  (ii) $\{N(t_i) - N(s_i)\}$ independent where $s_1 \leq t_1 \leq \cdots \leq s_n \leq t_n$

1.3 Continuous Markov Chain

Setting 1. $(X(t))$ *takes values in countable* S

- Markov property ——
  $\mathbb{P}(X(t_n) = j | X(t_{n-1}) = i_{n-1}, \ldots, X(t_1) = i_1) = \mathbb{P}(X(t_n) = j | X(t_{n-1}) = i_{n-1})$
- continuous-time Markov chain —— right-continuous, Markov property
- transition probability $p_{ij}(s,t) = \mathbb{P}(X(t) = j | X(s) = i)$
- homogeneous —— $p_{ij}(s,t) = p_{ij}(0, t-s)$
- transition semigroup $(P_t)_{ij} = p_{ij}(t)$
- stochastic semigroup

  (i) $P_0 = I$

  (ii) P_t stochastic —— non-negative entries, row sum 1

  (iii) (Chapman-Kolmogorov) $P_{s+t} = P_s P_t$

Setting 2. $(X(t))$ *homogeneous Markov chain*

Theorem 1.8. P_t *stochastic semigroup*

- $\mathbb{P}_i$ —— probability measure conditional on $X(0) = i$
- $\mathbb{E}_i$
- t-historical —— events given by $\{X(s) : s < t\}$
- t-future —— events given by $\{X(s) : s > t\}$
- stopping time T —— $\{T \leq t\}$ given by $\{X(s) : s \leq t\}$

Theorem 1.9 (Extended Markov property)**.** *H t-historical, F t-future, then* $\mathbb{P}(F | X(t) = j, H) = \mathbb{P}(F | X(t) = j)$

Theorem 1.10 (Strong Markov property)**.** *T stopping time, conditional on $\{T \le T_\infty\} \bigcap \{X(T) = i\}$, then*

(i) *$(X_{T+u})_u$ continuous Markov chain start at state i*

(ii) *same transition prob*

(iii) *independent to $\{X(s) : s < T\}$*

Setting 3. $X(0) = i$

- $U_0 = \inf \{t : X(t) \neq i\}$

Fact. *right continuous* $\Rightarrow U_0 > 0$

Theorem 1.11.

(i) $U_0 \sim Exp(g_i)$

(ii) *U_0 stopping time*

Proof. Extended Markov and homogeneity to deduce memoryless □

- transition matrix $\mathbf{Y} = (y_{ij})$ —— $y_{ij} = \begin{cases} \delta_{ij} & \text{if } g_i = 0 \\ \mathbb{P}_i(X(U_0) = j) & \text{if } g_i > 0 \end{cases}$
- generator $\mathbf{G} = (g_{ij})$ —— $g_{ij} = \begin{cases} g_i y_{ij} & \text{if } j \neq i \\ -g_i & \text{if } j = i \end{cases}$

Fact. $\mathbb{P}(X(t+h) = j | X(t) = i) = g_{ij} h + o(h)$

Fact. $g_{ij} = g_i(y_{ij} - \delta_{ij})$

Theorem 1.12. *$X(0) = i$, then*

(i) *$X(U_0)$ independent of U_0*

(ii) *conditional on $X(U_0) = j$, $X^*(s) = X(U_0 + s)$ continuous-time Markov chain, same transition prob, initial state j, independent to the past*

- T_m
- holding time $U_m = T_{m+1} - T_m$
- jump chain $Y = \{Y_n\}$
- $T_\infty = \lim T_n$
- minimal process
- explode from state i —— $\mathbb{P}_i(T_\infty < \infty) > 0$

Proposition 1.13. *X minimal process, then $\boldsymbol{P}_{s+t} = \boldsymbol{P}_s$*

Proof. may go to $\{\infty\}$ □

Theorem 1.14. *$i \in S$, non-explosive from i if any of the following holds:*

(i) *S finite*

(ii) $\sup_j g_j < \infty$

(iii) *i recurrent in jump chain Y*

Proof. be dominated by Poisson process which is non-explosive □

- irreducible —— $\forall i, j, \exists t > 0, p_{ij}(t) > 0$

Theorem 1.15.

(i) *(Levy dichotomy) X irreducible, then $\forall t > 0$, $p_{ij}(t) > 0$*

(ii) *X irreducible $\iff$ Y irreducible*

Proof. look at jump chain, $g_{i_0} \cdots g_{i_n} > 0$, $p_{i_k, i_{k+1}}(t) > 0$ □

Fact. *birth process not irreducible*

- recurrent —— $\mathbb{P}(\{t : X(t) = i\} \text{ unbounded}) = 1$
- transient —— $\mathbb{P}(\{t : X(t) = i\} \text{ unbounded}) = 0$
- $R_i = \inf \{t > U_0 : X(t) = i\}$
- mean return time $m_i = \mathbb{E}(R_i)$
- positive recurrent / non-null recurrent —— $m_i < \infty$

Theorem 1.16. *continuous-time chain X, jump chain Y*

(i) *$g_i = 0$, then i recurrent for X*

(ii) *$g_i > 0$, then i recurrent for X $\iff$ recurrent for Y*

(iii) *i recurrent $\iff$ $\int p_{ii}(t)dt = \infty$*

(iv) *i transient $\iff$ $\int p_{ii}(t)dt < \infty$*

(v) *X irreducible, then every state recurrent or every state transient*

Proof. main point is no explosion. Interchange summation, then old result. ∎

- **Forward equation:** $\boldsymbol{P}_t' = \boldsymbol{P}_t \boldsymbol{G}$ with boundary condition $\boldsymbol{P}_0 = 1$
- **Backward equation:** $\boldsymbol{P}_t' = \boldsymbol{G}\boldsymbol{P}_t$ with boundary condition $\boldsymbol{P}_0 = 1$

Fact. *If states S finite, then $\boldsymbol{P}_t = e^{t\boldsymbol{G}}$*

- minimal solution —— $p_{ij}(t) \le \pi_{ij}(t)$
- sub-stochastic —— $\sum_j p_{ij}(t) < 1$

Theorem 1.17. *S countable, X minimal Markov chain with generator $\boldsymbol{G}$, then*

(i) *$\boldsymbol{P}_t$ minimal non-negative solution of backward equation $\boldsymbol{P}_t' = \boldsymbol{G}\boldsymbol{P}_t$ with boundary condition $\boldsymbol{P}_0 = 1$*

(ii) *$\boldsymbol{P}_t$ minimal non-negative solution of forward equation $\boldsymbol{P}_t' = \boldsymbol{P}_t\boldsymbol{G}$*

Proof. **Solution:** condition on $T_1 > t$ or $T_1 \le t$.
Minimal: reverse argument and induction. ∎

Fact. *any solution to both equations sub-stochastic*

Fact. *non-explosive $\Rightarrow$ $\boldsymbol{P}_t$ unique solution to both equations*

- measure
- stationary measure —— $\boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{P}_t$
- stationary distribution
- unique measure —— unique up to scalar multiplication
- first return time R_i
- $m_i = \mathbb{E}_i(R_i)$

Theorem 1.18. *X irreducible, $|S| \ge 2$*

(i) *some state k positive recurrent, then*

(a) $\exists$ unique stationary distribution $\boldsymbol{\pi}$

(b) unique distribution st $\boldsymbol{\pi}\boldsymbol{G} = 0$

(c) all states positive recurrent

(ii) *X non-explosive, $\exists\ \boldsymbol{\pi}$ st $\boldsymbol{\pi}\boldsymbol{G} = 0$, then*

(a) all states positive recurrent

(b) $\boldsymbol{\pi}$ stationary

(c) $\pi_k = \frac{1}{m_k g_k}$

Proof. **(i)** use 1.19(**iv**) $\boldsymbol{\pi} = \mu(k)/m_k$, then uniqueness of measure $\Rightarrow$ all state non-null

(ii) $\boldsymbol{\nu}' = \frac{\pi_i g_i}{\pi_k g_k}$, then $\boldsymbol{\rho}(k) \leq \boldsymbol{\nu}'$ from discrete MC $\square$

- $\nu_i = x_i g_i$
- $\boldsymbol{\mu}(k) = (\mu_j(k))_j$ —— $\mu_j(k) = \mathbb{E}_k \left(\int_0^{R_k} \mathbb{1}\{X(s) = j\}\, ds \right)$
- $\boldsymbol{\rho}(k) = (\rho_j(k))$ —— mean visit to j starting from k in jump chain Y

Lemma 1.19. *X irreducible Markov chain, $|S| \geq 2$*

(i) *measure $\boldsymbol{x}$, then $\boldsymbol{xG} = 0 \iff \boldsymbol{\nu Y} = \boldsymbol{\nu}$*

(ii) *X recurrent, $\boldsymbol{xG} = 0$ unique measure*

(iii) *$\boldsymbol{x}$ measure, $\boldsymbol{xG} = 0$, then $x_j > 0$*

(iv) *X recurrent, $k \in S$, then $\boldsymbol{\mu}(k)\boldsymbol{G} = 0$ and stationary*

Proof. **(i)** expand

(ii) $\boldsymbol{\nu Y} = \boldsymbol{\nu}$, then uniqueness from discrete MC

(iii) $\mu_j(k) = \frac{1}{g_j}\rho_j(k)$, then $\boldsymbol{\rho}(k)\boldsymbol{Y} = \boldsymbol{\rho}(k)$ from discrete MC

(iv) strong Markov to shift time t $\square$

Fact. *X non-explosive, then $R_k = \sum_j \int_0^{R_k} \mathbb{1}\{X(s) = j\}\, ds$*

Fact. *X irreducible, $\exists$ more than one stationary distribution, then X explosive*

Theorem 1.20 (Markov chain limit theorem)**.** *X irreducible, non-explosive*

(i) *if $\exists$ stationary distribution $\boldsymbol{\pi}$, then*

(a) *$\boldsymbol{\pi}$ unique*

(b) *$p_{ij}(t) \to \pi_j$*

(ii) *if no stationary distribution, then $p_{ij}(t) \to 0$*

Proof. skeleton $Z_n = X(nh)$ $\square$

Lemma 1.21. *X minimal, then $|p_{ij}(t+u) - p_{ij}(t)| \leq 1 - e^{-g_i u}$*

1.4 Birth-death process and imbedding

– birth rate $\lambda_0, \lambda_1, \dots$

– death rate $\mu_1, \mu_2, \dots$

– birth-death process

Theorem 1.22. *X birth-death process, generator $\boldsymbol{G}$*

(i) *measure $x_n = \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}x_0$ satisfies $\boldsymbol{xG} = 0$*

(ii) *$\exists$ distribution $\boldsymbol{\pi}$ satisfies $\boldsymbol{\pi G} = 0 \iff \sum \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n} < \infty$*

(iii) *if $\sum \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}(\lambda_n + \mu_n) < \infty$, then $\boldsymbol{\pi}$ stationary*

Proof. **(i)** solve $\boldsymbol{xG} = 0$

(ii) trivial

(iii) condition for jump chain Y recurrent, then non-explosive

□

Example.

– ***Pure birth*** $\mu_n = 0$

– ***Simple death with immigration*** $\lambda_n = \lambda, \mu_n = n\mu$

Theorem 1.23. *$X(t)$ asymptotically Poi(ρ) = Poi $\left(\frac{\lambda}{\mu}\right)$*

– ***Simple birth-death*** $\lambda_n = n\lambda, \mu_n = n\mu,\ X(0) = I$

Fact. *state 0 absorbing*

Theorem 1.24. $G(s,t) = \mathbb{E}(s^{X(t)}) = \begin{cases} \left(\frac{\lambda t(1-s)+s}{\lambda t(1-s)+1}\right)^I & \text{if } \mu = \lambda \\ \left(\frac{\mu(1-s)-(\mu-\lambda s)e^{-t(\lambda-\mu)}}{\lambda(1-s)-(\mu-\lambda s)e^{-t(\lambda-\mu)}}\right)^I & \text{if } \mu \neq \lambda \end{cases}$

Proof. Forward equation □

Fact. *non-explosive as $\sum p_j(t) = G(1,t) = 1$*

Fact. $\mathbb{E}_I(X(t)) \to \begin{cases} 0 & \text{if } \rho < 1 \\ \infty & \text{if } \rho > 1 \end{cases}$

- *extinction probability $\eta(t) = \mathbb{P}_I(X(t) = 0)$*

Corollary 1.25. $\eta(t) \to \begin{cases} 1 & \textit{if } \rho \leq 1 \\ \rho^{-I} & \textit{if } \rho > 1 \end{cases}$

- imbedded random walk —— jump chain Y with parameter $\frac{\lambda}{\lambda+\mu}$, absorbing at 0
- imbedded branching process —— lives $Exp(\lambda+\mu)$, then born n individuals where $\begin{cases} p_0 = \mathbb{P}(n=0) = \frac{\mu}{\lambda+\mu} \\ p_2 = \mathbb{P}(n=2) = \frac{\lambda}{\lambda+\mu} \end{cases}$
- age-dependent branching process
- age density function $f_T(u) = (\lambda+\mu)e^{-(\lambda+\mu)u}$
- family-size generating function $G(s) = \frac{\mu+\lambda s^2}{\mu+\lambda} = p_0 + p_2 s^2$