

# Stochastic Financial Models

## 1 Utility and Mean-Variance analysis

- contingent claims — r.v.  $X$
- utility function — non-decreasing

**Fact.**  $Y$  is preferred to  $X$  iff  $\mathbb{E}(U(X)) \leq \mathbb{E}(U(Y))$

- indifferent
- risk neutral
- risk averse
- concave
- strictly concave

**Proposition 1.1.** *risk averse iff  $U$  concave*

- CARA with parameter  $\gamma$  —  $\gamma \in (0, \infty)$ ,  $U(X) = CARA_\gamma(x) = -\exp(-\gamma x)$
- CRRA with parameter  $R$  —  $R \in (0, 1) \cup (1, \infty)$ ,  $U(X) = CRRA_R(x) = \begin{cases} \frac{x^{1-R}}{1-R} & \text{if } x > 0 \\ -\infty & \text{otherwise} \end{cases}$
- CRRA with parameter 1 —  $U(X) = CRRA_1(x) = \begin{cases} \log x & \text{if } x > 0 \\ -\infty & \text{otherwise} \end{cases}$
- constant absolute risk aversion — CARA
- constant relative risk aversion — CRRA

**Fact** (Arrow-Pratt coefficient of absolute risk aversion).  $\omega + X$  preferred to  $\omega$  iff  $\frac{2\mathbb{E}(X)}{\mathbb{E}(X^2)} \geq -\frac{U''(\omega)}{U'(\omega)}$

**Fact** (Arrow-Pratt coefficient of relative risk aversion).  $\omega(1+X)$  preferred to  $\omega$  iff  $\frac{2\mathbb{E}(X)}{\mathbb{E}(X^2)} \geq -\frac{\omega U''(\omega)}{U'(\omega)}$

- available claim  $\mathcal{A}$
- reservation bid price  $\pi_b(Y)$  —  $\sup \pi$  st  $\mathbb{E}(U(X+Y-\pi)) > \mathbb{E}(U(X^*))$
- reservation ask price  $\pi_a(Y)$  —  $\inf \pi$  st  $\mathbb{E}(U(X-Y+\pi)) > \mathbb{E}(U(X^*))$

**Proposition 1.2** (Ask above, bid below).  $\mathcal{A}$  convex, then  $\pi_b(Y) \leq \pi_a(Y)$

**Setting 1.**  $\mathcal{A}$  affine space,  $U$  differentiable, strictly concave

- marginal price  $\pi_m(Y)$  —  $\pi_m(Y) = \frac{\mathbb{E}(U'(X^*)Y)}{\mathbb{E}(U'(X^*))}$
- single-period asset price model
- numeraire
- riskless bond
- interest rate —  $r > -1$
- state-price density  $\rho$  —  $S_0^i = \mathbb{E}(S_1^i \rho)$
- wealth  $\omega_0$
- portfolio  $\theta$

**Example.** *no bond*

$$\begin{array}{ll} \text{given} & \mathbb{E}(\theta \cdot S_1) = \theta \cdot \mu, \text{var}(\theta \cdot S_1) = \theta^T V \theta \\ \text{minimize} & \text{var}(\theta \cdot S_1) \\ \text{subject to} & \theta \cdot S_0 = \omega_0, \mathbb{E}(\theta \cdot S_1) = \omega_1 \end{array}$$

- mean-variance-efficient frontier —  $\{\theta^*(\omega_1)\}$
- minimum variance portfolio  $\theta_{\min^*}$  — minimise var over  $\omega_1$

**Example.** *with bond*

$$\begin{array}{ll} \text{minimise} & \theta^T V \theta \\ \text{subject to} & \theta^0 + \theta S_0 = \omega_0, \theta^0(1+r) + \theta \mu = \omega_1 \end{array}$$

Then ,  $\theta^* = \lambda \theta_m^*$

- market portfolio  $\theta_m^*$  —  $A(\mu - (1+r)S_0)$ ,  $A = V^{-1}$

**Setting 2.**  $S_1$  Gaussian,  $U$  CARA

**Example.** *no bond*

$$\begin{array}{ll} \text{maximise} & \mathbb{E}(U(\theta S_1)) \\ \text{subject to} & \theta S_0 = \omega_0 \end{array}$$

**Example.** *with bond*

$$\begin{array}{ll} \text{maximise} & \mathbb{E}(U(\bar{\theta} \bar{S}_1)) \\ \text{subject to} & \bar{\theta} \bar{S}_0 = \omega_0 \end{array}$$

then,  $\theta^* = \gamma^{-1} \theta_m^*$

- beta/sensitivity  $\beta^i$  —  $\beta^i = \frac{\text{cov}(S_1^i, \theta_m^* S_1)}{\text{var}(\theta_m^* S_1)}$
- $\mu^m$  —  $\theta_m^* \mu$

$$- S_0^m \longrightarrow \theta_m^* S_0$$

**Proposition 1.3.**  $\mu^i - (1+r)S_0^i = \beta^i(\mu^m - (1+r)S_0^m)$

- capitalization-weights of the relevant market index

**Setting 3.**

$$S_1 = (1+R)S_0, S_1^m = (1+R^m)S_0^m, \tilde{\beta}^i = \frac{\text{cov}(R^i, R^m)}{\text{var}(R^m)}$$

**Fact.**  $\mathbb{E}(R^i) = r + \tilde{\beta}^i(\mathbb{E}(R^m) - r)$

## 2 Martingales

- conditional probability
- conditional expectation given event
- conditional expectation given  $\mathcal{G}$ ,  $\mathbb{E}(X|\mathcal{G})$

**Theorem 2.1.**  $\mathcal{G} \subset \mathcal{F}$  sub- $\sigma$ -algebra,  $X$  integrable, then  $\exists$  unique  $Y$  (up to a.s.) st

- (i)  $Y$  integrable
- (ii)  $Y$   $\mathcal{G}$ -measurable
- (iii)  $\mathbb{E}(Y\mathbb{1}_A) = \mathbb{E}(X\mathbb{1}_A)$  for all  $A \in \mathcal{G}$

**Fact.** also true if replace integrable by non-negative

- $\mathbb{E}(X|Z) \longrightarrow \mathcal{G} = \sigma(Z)$  for r.v.  $Z$
- $\mathbb{P}(A|\mathcal{G}) \longrightarrow X = \mathbb{1}_A$

**Fact.**  $\mathcal{G} = \sigma(B_n)$  discrete, then  $\mathbb{E}(X|\mathcal{G}) = \sum \mathbb{E}(X|B_n)\mathbb{1}_{B_n}$  a.s.

**Proposition 2.2.**  $\mathcal{G} \subset \mathcal{F}$ ,  $X, W$  integrable, then

- (i)  $\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$
- (ii)  $X$   $\mathcal{G}$ -measurable, then  $\mathbb{E}(X|\mathcal{G}) = X$  a.s.
- (iii)  $X$  independent of  $\mathcal{G}$ , then  $\mathbb{E}(X|\mathcal{G}) = E(X)$  a.s.
- (iv)  $X \geq 0$  a.s. , then  $\mathbb{E}(X|\mathcal{G}) \geq 0$  a.s.
- (v)  $\mathbb{E}(\alpha X + \beta W|\mathcal{G}) = \alpha\mathbb{E}(X|\mathcal{G}) + \beta\mathbb{E}(W|\mathcal{G})$  a.s.

**Proposition 2.3** (Tower property).  $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$  sub- $\sigma$ -algebra,  $X$  integrable, then  $\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = \mathbb{E}(X|\mathcal{H})$  a.s.

**Proposition 2.4** (Taking out what is known).  $\mathcal{G} \subset \mathcal{F}$  sub- $\sigma$ -algebra,  $X$  integrable,  $Z$   $\mathcal{G}$ -measurable,  $ZX$  integrable, then  $\mathbb{E}(ZX|\mathcal{G}) = Z\mathbb{E}(X|\mathcal{G})$  a.s.

**Proposition 2.5** (Averaging over independent variables).  $X_1, X_2$  r.v. in  $(E_1, \mathcal{E}_1), (E_2, \mathcal{E}_2)$ ,  $\mathcal{G} \subset \mathcal{F}$ ,  $X_1$   $\mathcal{G}$ -measurable,  $X_2$  independent of  $\mathcal{G}$ ,  $F$  non-negative, let  $f = \mathbb{E}(F(\cdot, X_2))$ , then  $\mathbb{E}(F(X_1, X_2)|\mathcal{G}) = f(X_1)$  a.s.

- filtration  $(\mathcal{F}_n)$
- random process
- $(X_n)$  adapted to  $(\mathcal{F}_n)$
- natural filtration  $(\mathcal{F}_n^X)$  —  $\mathcal{F}_n^X = \sigma(X_k : 0 \leq k \leq n)$
- martingale —
  - (Adapted)  $X_n$   $\mathcal{F}_n$ -measurable
  - (Integrable)  $\mathbb{E}(|X_n|) < \infty$
  - (Martingale property)  $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$  a.s.
- supermartingale —  $\mathbb{E}(X_{n+1}|\mathcal{F}_n) \leq X_n$
- submartingale —  $\mathbb{E}(X_{n+1}|\mathcal{F}_n) \geq X_n$

**Fact.** Any martingale also martingale in natural filtration (natural filtration smallest)

- martingale (continuous-time) — adapted, integrable,  $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$  for all  $s \leq t$
- $(X_t)$  continuous —  $t \mapsto X_t(\omega)$  continuous for all  $\omega$

**Example.**  $X_n$  i.i.d.,  $\mathcal{F}_0 = \{\emptyset, \Omega\}$ ,  $(\mathcal{F}_n)$  natural filtration

- (additive martingale)  $X_1$  integrable,  $E(X_1) = 0$ ,  $S_0 = 0$ ,  $S_n = \sum^n X_k$
- (multiplicative martingale)  $X_1$  non-negative,  $\mathbb{E}(X_1) = 1$ ,  $Z_0 = 1$ ,  $Z_n = \prod^n Z_k$

**Example.**  $(X_n)$  Markov chain, countable state space  $S$ , transition matrix  $P$ , natural filtration  $(\mathcal{F}_n)$ , bounded/non-negative  $f$  on  $S$ , let

$$Pf(x) = \sum p_{xy}f(y)$$

then if  $f$  subharmonic

$$\mathbb{E}(f(X_{n+1})|\mathcal{F}_n) = Pf(X_n) \geq f(X_n)$$

then,  $(f(X_n))$  submartingale

- subharmonic —  $f(x) \leq Pf(x)$  for all  $x$
- random time —  $T : \Omega \rightarrow \{0, 1, \dots\} \cup \{\infty\}$
- stopping time —  $\{T \leq n\} \in \mathcal{F}_n$

**Theorem 2.6** (Optional stopping).  $(M_n)$  martingale,  $T$  **bounded**, stopping time, then  $\mathbb{E}(M_T) = \mathbb{E}(M_0)$

**Fact.** Doob's optional sampling theorem basic form

**Theorem 2.7.**  $(M_n)$  martingale,  $T$  **almost surely finite**, stopping time, suppose one of the following holds:

- (i)  $|M_n| \leq C$  for all  $n \leq T$ ,  $C$  constant
- (ii)  $\mathbb{E}(T) \leq \infty$ ,  $|M_n - M_{n-1}| \leq C$  for all  $n \leq T$

**Fact.**  $T$  stopping time  $\Rightarrow T \wedge n$  bounded stopping time

**Counter Example.** additive martingale, simple random walk,  $T = \min\{n : S_n = 1\}$ , then almost sure finite as recurrent, but  $\mathbb{E}(T) = \infty$ ,  $\mathbb{E}(S_T) = 1 \neq 0 = S_0$

**Counter Example.** multiplicative martingale,  $(X_k)$  i.i.d.,  $\mathbb{P}(X_1 = 0) = \mathbb{P}(X_1 = 2) = 1/2$ ,  $T = \min\{n : Z_n = 0\}$ , then  $\mathbb{E}(T) = 2$ , but  $\mathbb{E}(Z_T) = 0 \neq 1 = Z_0$

**Theorem 2.8.**  $(M_n)$  martingale,  $T$  stopping time, then  $(M_{T \wedge n})$  martingale

- previsible —  $H_n \mathcal{F}_{n-1}$ -measurable
- martingale transform of  $(M_n)$  by  $(H_n)$  —  $Y_0 = 0$ ,  $Y_n = \sum_1^n H_k(M_k - M_{k-1})$

**Theorem 2.9** (Martingale transform).  $(M_n)$  martingale,  $(H_n)$  bounded, previsible,  $(Y_n)$  martingale transform, then  $(Y_n)$  martingale

**Fact.** model stock price as  $(M_n)$ , then

- (i) optional stopping  $\Rightarrow$  expected return  $\mathbb{E}(M_T)$  the same no matter what stopping time
- (ii)  $(H_k)$  amount held between time  $k-1$  and  $k$ , no bounded previsible strategy gives expected gain or loss

### 3 Pricing contingent claims

- asset price model  $(\bar{S}_n)_{0 \leq n \leq T}$  with numeraire
- numeraire  $(S_n^0)$  —  $S_n^0 > 0$
- discounted prices  $X_n^i$  —  $X_n^i = S_n^i / S_n^0$
- $\bar{X}_n = (1, X_n)$
- interest rate  $r_n$  —  $S_n^0 = (1 + r_n)S_{n-1}^0$
- risky assets  $(S_n)$
- portfolio  $\bar{\theta}_n$
- self-financing —  $\bar{\theta}_n \bar{S}_n = \bar{\theta}_{n+1} \bar{S}_n$
- value process  $(V_n)$  —  $V_0 = \bar{\theta}_1 \bar{X}_0$ ,  $V_n = \bar{\theta}_n \bar{X}_n$
- total (discounted) value
- previsible — if  $\bar{\theta}_n \mathcal{F}_{n-1}$ -measurable

**Setting 4.**

- (i)  $(\mathcal{F}_n)$  filtration generated by  $(\bar{S}_n)$ ,  $\mathcal{F} = \mathcal{F}_T$
- (ii)  $(S_n)$  takes countable values
- (iii)  $(S_n^0)$  deterministic process

**Proposition 3.1.**  $(\theta_n)$  previsible process, then  $\exists (\theta_n^0)$  st

- (i)  $(\theta_n^0)$  previsible
- (ii)  $(\bar{\theta}_n^0)$  self-financing with initial value  $V_0$
- (iii)  $V_T = V_0 + \sum_1^T \theta_n(X_n - X_{n-1})$ 
  - contingent claim of maturity  $T$  — non-negative  $\mathcal{F}_T$ -measurable r.v.
  - European option
  - call with strike price  $K$  —  $(S_T - K)^+$
  - put with strike price  $K$  —  $(S_T - K)^-$
  - options
  - exotic options — depending on entire path  $(S_n)$
  - barrier options —
    - knocked out
    - knocked in
  - up-and-out call —  $C = \begin{cases} (S_T - K)^+ & \text{if } \max S_n < B \\ 0 & \text{otherwise} \end{cases}$
  - down-and-in put —  $C = \begin{cases} (S_T - K)^- & \text{if } \min S_n \leq B \\ 0 & \text{otherwise} \end{cases}$
  - $\tilde{\mathbb{P}}$  equivalent to  $\mathbb{P}$ ,  $\tilde{\mathbb{P}} \sim \mathbb{P}$  —  $\exists \rho$ , st
    - $\mathbb{P}(\rho > 0) = 1$
    - $\tilde{\mathbb{P}}(A) = \mathbb{E}(\rho \mathbb{1}_A)$
  - density  $\rho = d\tilde{\mathbb{P}}/d\mathbb{P}$  for  $\tilde{\mathbb{P}}$  wrt  $\mathbb{P}$

**Fact.**  $\tilde{\mathbb{E}}(X) = \mathbb{E}(\rho X)$

**Fact.**  $\tilde{\mathbb{P}} \sim \mathbb{P}$  symmetric and transitive

**Fact.**  $d\mathbb{P}/d\tilde{\mathbb{P}} = 1/\rho$  a.s.

- arbitrage for  $(\bar{S}_n)_{0 \leq n \leq T}$  —
- (i)  $(\bar{\theta}_n)_{1 \leq n \leq T}$  previsible, self-financing

- (ii)  $V_0 = 0$
- (iii)  $V_T \geq 0$  a.s.
- (iv)  $V_T > 0$  with positive probability

–  $(\bar{S}_n)$  arbitrage free

**Proposition 3.2.**  $(X_n)$  martingale  $\Rightarrow (X_n)$  arbitrage free

**Fact.** proof can be simpler when  $(\theta_n)$  bounded

**Setting 5.** single period model,  $\mathcal{F}_0 = \emptyset, \Omega$

**Proposition 3.3.**  $Y$  r.v. , following equivalent:

(i) arbitrage free (i.e. no  $\theta$  st  $\theta Y \geq 0$  a.s. with  $\theta Y > 0$  with positive prob)

(ii)  $\exists$  equivalent probability measure  $\tilde{\mathbb{P}}$  st  $Y$  integrable with  $\mathbb{E}(Y) = 0$

– equivalent martingale measure  $\tilde{\mathbb{P}}$  (risk neutral measure) —  $\tilde{\mathbb{P}} \sim \mathbb{P}$ ,  $(X_n)$  martingale under  $\tilde{\mathbb{P}}$

**Theorem 3.4** ((1st) Fundamental theorem of asset pricing). following equivalent:

(i)  $(\bar{S}_n)$  arbitrage free

(ii)  $(\bar{S}_n)$  has equivalent martingale measure

**Setting 6.**  $C$  time- $T$  contingent claim,  $D = C/S_T^0$  discounted value

– attainable/replicable —  $\exists$  previsible, self-financing  $(\bar{\theta}_n)$  st  $C = \bar{\theta}_n \bar{S}_T$

**Fact.** Alternative def:  $\exists V_0$   $\mathcal{F}_0$ -measurable,  $\theta_n$  previsible st  $D = V_0 + \sum^T \theta_n (X_n - X_{n-1})$

– fair price —  $V_0$

– replicating portfolio/hedging portfolio —  $\bar{\theta}_n$

–  $(\bar{S}_n)$  complete — all contingent claims attainable

**Proposition 3.5.**  $\mathcal{F}_0 = \{\emptyset, \Omega\}$ ,  $\mathcal{F}_T = \sigma(\bar{S}_1)$

(i)  $C$  non-negative, attainable, time- $T$  contingent claim,  $\tilde{\mathbb{P}}$  equivalent martingale measure, then fair price  $V_0 = \tilde{\mathbb{E}}(D)$ ,  $D = C/S_T^0$

(ii)  $(\bar{S}_n)$  complete, numeraire non-random, then at most one equivalent martingale measure

– binomial model (Cox-Ross-Rubinstein model) — interest rate  $r$ , parameters  $a < b$ ,  $R_i$  i.i.d. with parameter  $p$

- $S_n^0 = (1 + r)^n$
- $S_n = S_0 \prod (1 + R_k)$
- $\begin{cases} \mathbb{P}(R_1 = a) = 1 - p \\ \mathbb{P}(R_1 = b) = p \end{cases}$

**Proposition 3.6.** *binomial model has arbitrage unless  $r \in (a, b)$*

**Proposition 3.7.**  *$(\bar{S}_n)$  binomial model,  $r \in (a, b)$ , define*

(i)  $p^* = \frac{r-a}{b-a}$

(ii) *equivalent prob measure  $\frac{d\mathbb{P}^*}{d\mathbb{P}} = \left(\frac{p^*}{p}\right)^{U_T} \left(\frac{1-p^*}{1-p}\right)^{D_T}$  where  $U_T = (T + S_T)/2, D_T = (T - S_T)/2$ ,  $S_T$  number of  $b$*

*Then under  $\mathbb{P}^*$ ,*

(i)  $R_1, \dots, R_T$  i.i.d. ,  $\begin{cases} \mathbb{P}^*(R_1 = a) = 1 - p^* \\ \mathbb{P}^*(R_1 = b) = p^* \end{cases}$

(ii)  $(X_n)$  martingale under  $\mathbb{P}^*$

**Fact.**  $r = \mathbb{E}^*(R_1)$

**Fact.** *Binomial model arbitrage free when  $r \in (a, b)$*

**Setting 7.** *if  $C = f(S_0, \dots, S_T)$ , then*

$$V(C) = \frac{\mathbb{E}^*(C)}{(1+r)^T} = (1+r)^{-T} \sum f(s_0, s_1, \dots, s_T) \mathbb{P}^*(S_1 = s_1, \dots, S_T = s_T)$$

*Define recursive relation:*

$$\begin{aligned} f_T(s_0, \dots, s_T) &= f(s_0, \dots, s_T) \\ f_n(s_0, \dots, s_n) &= (1-p^*)f_{n+1}(s_0, \dots, s_n, (1+a)s_n) + p^*f_{n+1}(s_0, \dots, s_n, (1+b)s_n) \end{aligned}$$

**Proposition 3.8.**  $\mathbb{E}^*(f(S_0, \dots, S_T) | \mathcal{F}_n) = f_n(S_0, \dots, S_n), \mathbb{E}^*(C) = f_0(S_0)$

**Proposition 3.9.** *Define*

$$\Delta_n(s_0, \dots, s_{n-1}) = \frac{f_n(s_0, \dots, s_{n-1}, (1+b)s_{n-1}) - f_n(s_0, \dots, s_{n-1}, (1+a)s_{n-1})}{(1+r)^{T-n}(b-a)s_{n-1}}$$

*then  $\theta_n = \Delta_n(S_0, \dots, S_{n-1})$  replicating portfolio for  $C$*

**Fact.** *Binomial model complete when  $r \in (a, b)$*

**Proposition 3.10.**  $(W_n)$  simple random walk with  $\mathbb{P}(W_1 = 1) = p$ , let  $M_T = \max W_n$ , then with  $k \leq T, 2k - T \leq m \leq k$

$$\mathbb{P}(M_T = m, W_T = 2k - T) = \left( \binom{T}{k-m} - \binom{T}{k-m-1} \right) p^k (1-p)^{T-k}$$

**Example.** *if  $(1+a)(1+b) = 1$ , then  $S_n = S_0(1+b)^{W_n}$ , so can give fair price of  $C = F(S_T, \max S_n)$*



## 4 Dynamic Programming

- state-space  $E$
- action-space  $A$

**Setting 8.**  $F : \{0, \dots, T-1\} \times E \times A \times [0, 1] \rightarrow E$ ,  $(\epsilon_n)$  i.i.d.,  $\mathcal{F}_n = \sigma(\epsilon_1, \dots, \epsilon_n)$

- adapted control  $u = (u_n)_{k \leq n \leq T-1}$  — initial time  $k$ ,  $u_n$   $\mathcal{F}_n$ -measurable

**Setting 9.** initial state  $x \in E$ , adapted control  $u$ , define  $X_k = x$ ,  $X_{n+1} = F(n, X_n, u_n, \epsilon_{n+1})$   
Write  $X_n = X_n^u(k, x)$

- expected reward —  $V^u(k, x) = \mathbb{E} \left( \left( \sum_k^{T-1} r(n, X_n^u(k, x), u_n) \right) + R(X_T^u(k, x)) \right)$
- reward function  $r, R$  — non-negative measurable function
- value function  $V$  —  $V(k, x) = \sup_u V^u(k, x)$
- $u$  optimal control from  $(k, x)$  —  $V(k, x) = V^u(k, x)$

**Proposition 4.1** (Bellman equation). Let  $Pv(n, x, a) = \mathbb{E}(v(n+1, F(n, x, a, \epsilon_{n+1})))$

$$\begin{aligned} v(T, x) &= R(x) \\ v(n, x) &= \sup_{a \in A} \{r(n, x, a) + Pv(n, x, a)\} \end{aligned} \quad n = 0, \dots, T-1$$

Suppose  $\exists a$  st

$$v(n, x) = r(n, x, a(n, x)) + Pv(n, x, a(n, x)) \quad n = 0, \dots, T-1$$

Then,

- (i)  $V = v$
- (ii) optimal control  $u_n^* = a(n, X_n^{u^*}(k, x))$

**Fact.** Possible variations:

- (i)  $r, R$  as costs
- (ii) mixture of costs and rewards
- (iii) time-dependent state-space  $E_n$
- (iv) time-and-state-dependent action-space  $A_{n,x}$

- American call — family of time- $T$  contingent claim  $(1+r)^{T-\tau}(S_\tau - K)^+$
- American put — family of time- $T$  contingent claim  $(1+r)^{T-\tau}(S_\tau - K)^-$

**Setting 10.**  $(S_n)$  binomial model,  $r \in (a, b)$

**Fact.** complete  $\Rightarrow$  can hedge all  $C$  with  $\mathbb{E}^*(C) = 0$

**Example** (American call).  $\tau = T$  always optimal, American and European calls equivalent

**Fact.** fair price can be founded using Bellman equation

## 5 Brownian motion

– Brownian motion —

- $B_0 = 0$
- $(B_{s+t} - B_s) \sim N(0, t)$ , independent of  $\sigma(B_r : r \leq s)$
- $t \mapsto B_t(\omega)$  continuous

– Brownian motion starting from  $x$  —  $B_0 = x$

– Gaussian process —  $\forall(t_1, \dots, t_n), (X_{t_1}, \dots, X_{t_n})$  multivariate normal

**Proposition 5.1.**  $(B_t)$  continuous process starting from 0, then following equivalent:

(i)  $(B_t)$  Brownian motion

(ii)  $(B_t)$  zero mean Gaussian process,  $\mathbb{E}(B_s B_t) = s \wedge t$

**Proposition 5.2** (Scaling property).  $(B_t)$  Brownian motion, set  $\tilde{B}_t = c^{-1} B_{c^2 t}$ , then  $(\tilde{B}_t)$  Brownian motion

**Proposition 5.3.**  $(B_t)$  Brownian motion,  $(B_t)$  exit every finite interval a.s.

- $\mathcal{F}_t = \sigma(B_s : s \in [0, t])$
- stopping time —  $\{T \leq t\} \in \mathcal{F}_t$  for all  $t$
- $\mathcal{F}_T$  —  $A \in \mathcal{F}_\infty$  st  $A \cap \{T \leq t\} \in \mathcal{F}_t$

**Proposition 5.4** (Strong Markov property).  $(B_t)$  Brownian motion,  $T$  a.s. finite stopping time. Define  $\tilde{B}_t = B_{T+t} - B_T$ , then

(i)  $(\tilde{B}_t)$  Brownian motion

(ii) independent of  $\mathcal{F}_T$

**Proposition 5.5.**  $(B_t)$  Brownian motion, define  $T_a = \inf \{t \geq 0 : B_t = a\}$ , then

- (i)  $T_a$  stopping time
- (ii)  $T_a$  almost surely finite

**Theorem 5.6.**  $(\Omega, \mathcal{F}, \mathbb{P})$  not discrete,  $m$  prob measure on  $\mathbb{R}$ , mean 0, variance 1, then  $\exists(B_t), (W_t^{(k)})$  for all  $k \in \mathbb{N}$  st

- (i)  $(B_t)$  Brownian motion
- (ii)  $(W_{\frac{n}{k}}^{(k)})$  random walk with distribution  $m$ ,  $(W_t^{(k)})$  linear interpolation of values  $\{\frac{n}{k}\}$
- (iii)  $\frac{W_t^{(k)}}{\sqrt{k}} \rightarrow B_t$  uniformly on compacts in  $t$  a.s.

**Fact.** combination of Wiener's Theorem and Donsker's Invariance Principle

- Wiener measure

**Proposition 5.7.** let  $T \geq 0$ ,  $c \in \mathbb{R}$ ,  $B = (B_t)_{\{0 \leq t \leq T\}}$  brownian motion,  $\tilde{B}_t = B_t + ct$ , then  $\forall$  measurable set  $A \subset C[0, T]$ ,  $\mathbb{P}(\tilde{B} \in A) = \mathbb{E}(\mathbb{1}_{\{B \in A\}} e^{cB_T - \frac{c^2 T}{2}})$

**Fact.** special case of Cameron-Martin theorem

**Proposition 5.8** (Reflection principle).  $(B_t)$  Brownian motion,  $a \geq 0$ , set  $T_a = \inf \{t \geq 0 : B_t = a\}$ , define  $\tilde{B}_t = \begin{cases} B_t & \text{if } t \leq T_a \\ 2a - B_t & \text{if } t > T_a \end{cases}$ , then  $(\tilde{B}_t)$  Brownian motion

- maximum process —  $M_t = \sup_{\{0 \leq s \leq t\}} B_s$

**Fact.**  $M_t$  same distribution as  $|B_t|$

**Proposition 5.9.**  $T_a$  has density  $h_a(t) = \frac{a}{\sqrt{2\pi t^3}} e^{-\frac{a^2}{2t}}$

- $p_t(x, y)$  — density of  $B_t$  starting at  $x$
- $p_t^a(x, y) = p_t(x, y) - p_t(x, 2a - y)$

**Proposition 5.10.**  $x \leq a$ ,  $(B_t)$  Brownian motion with density starting from  $x$ , then  $\forall$  non-negative measurable  $f$ ,  $\mathbb{E}_x(f(B_t) \mathbb{1}_{\{T_a > t\}}) = \int_{-\infty}^a f(y) p_t^a(x, y) dy$

## 6 Black-Scholes model

- Black-Scholes model —  $S_t^0 = e^{rt}$ ,  $S_t = S_0 e^{\sigma B_t + \mu t}$
- price of riskless bond  $S_t^0$
- interest rate  $r$
- price of risky asset  $S_t$
- drift  $\mu$
- volatility  $\sigma$

**Fact.**  $(e^{\sigma B_t - \sigma^2 t/2})$  martingale

**Fact.** with  $\mu^* = r - \sigma^2/2$ , discounted asset price  $(e^{rt} S_t)$  martingale

**Proposition 6.1.**  $(S_t^0, S_t)$  Black-Scholes, fix  $T$ , consider  $\frac{d\mathbb{P}^*}{d\mathbb{P}} = e^{\lambda B_T - \lambda^2 T/2}$  where  $\sigma \lambda = \mu^* - \mu$ , then under  $\mathbb{P}^*$ , discounted asset price  $(e^{-rt} S_t)$  martingale

**Fact.** abuse notation in writing  $\mathbb{P}^*$  instead of  $\mathbb{P}$

- time- $T$  contingent claim  $C$  —  $\mathcal{F}_T$ -measurable r.v.
- Black-Scholes price  $V_0$  —  $V_0 = e^{-rT} \mathbb{E}^*(C)$

**Fact.** fair price unique

**Example.**

(i)  $C = S_T$

(ii)  $C = K$

– simple replicable claim — constant  $C_0$ ,  $0 = t_0 \leq \dots \leq t_n = T$ ,  $\theta_k$  bounded  $\mathcal{F}_{t_{k-1}}$ -measurable

$$e^{rT}C = C_0 + \sum_1^n \theta_k (X_{t_k} - X_{t_{k-1}})$$

(Replicating strategy) at time  $t_{k-1}$ , buy  $\theta_k$ , then sell  $\theta_k$  at time  $t_k$ , then buy  $\theta_k$  bond

**Fact.** any simple replicable claim can be replicated for cost  $C_0$  at time 0,  $V_0 = C_0$

**Fact** (Brownian martingale representation theorem). every integrable  $\mathcal{F}_T$ -measurable contingent claim is limit in probability of simple replicable claims

**Setting 11.**  $S_0 = s$

**Fact.**  $\log S_t = \log s + \sigma B_t + \mu t$

- $p(t, x, z) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{|x-z|^2}{2t}}$
- $p(\sigma^2 t, x(t), \cdot)$  — density of  $\log S_t$ ,  $x(t) = \log s + \mu t$
- $\rho(t, s, \cdot)$  — density of  $S_t$
- $\dot{\rho}$  — derivative in first argument
- $\rho'$  — derivative in second argument

**Fact.**  $y\rho(t, s, y) = p(\sigma^2 t, x(t), z)$ ,  $z = \log y$

**Fact.**  $\dot{\rho} = \frac{1}{2}\sigma^2 s^2 \rho'' + rs\rho'$

**Proposition 6.2.**  $F$  on  $(0, \infty)$  continuous, polynomial growth,  $t \in [0, T]$ ,  $s \in (0, \infty)$

$V(t, s)$  time- $t$  value of time- $T$  contingent claim  $F(S_T)$ , conditional on  $S_t = s$

$V(t, s) = e^{-r(T-t)} \mathbb{E}^*(F(S_T) | S_t = s) = e^{-r(T-t)} \mathbb{E}(F(se^{\sigma B_{T-t} + \mu^*(T-t)}))$ , then

(i)  $V$  continuous on  $(0, T) \times (0, \infty)$

(ii)  $V \in C^{1,2}$  on  $(0, T) \times (0, \infty)$

(iii) (Black Scholes PDE)  $\mathcal{L}V = \dot{V} + \frac{1}{2}\sigma^2 s^2 V'' + rsV' - rV$  with  $V(\cdot, T) = F$

**Setting 12** (Binomial approximation to BS). Consider convergence of random walk to Brownian motion, special case  $(W_{n/k}^{(k)})$  simple symmetric random walk on  $\{-1, 1\}$

- $1 + a_k = \exp\left(-\frac{\sigma}{\sqrt{k}} + \frac{\mu}{k}\right)$
- $1 + b_k = \exp\left(\frac{\sigma}{\sqrt{k}} + \frac{\mu}{k}\right)$
- $1 + r_k = \exp\left(\frac{r}{k}\right)$

$$- S_t^{(k)} = S_0 \exp \left( \frac{\sigma W_t^{(k)}}{\sqrt{k}} + \mu t \right)$$

$$- S_t = S_0 \exp (\sigma B_t + \mu t)$$

$$- S_t^{(k)0} = S_t^0 = \exp(rt)$$

**Fact.**

(i)  $(S_t^0, S_t)$  Black-Scholes of drift  $\mu$ , volatility  $\sigma$ , interest rate  $r$

(ii)  $(S_{n/k}^{(k)0}, S_{n/k}^{(k)})$  binomial model of parameters  $a_k < r_k < b_k$ ,  $p = \frac{1}{2}$

(iii)  $S_t^{(k)} \rightarrow S_t$  uniformly on compacts in  $t$  a.s.

-  $\mathbb{P}^{(k)*}$  — martingale measures for binomial model

-  $\mathbb{P}^*$  — martingale measures for Black-Scholes model

**Fact.**  $\mathbb{E}^{(k)*}(G(S^{(k)})) \rightarrow \mathbb{E}^*(G(S))$ , so can approximate fair price using Binomial model

**Setting 13.**  $\mu = \mu^* = r - \frac{\sigma^2}{2}$  for convenience

- expression  $C = F(B)$  —  $F$  on  $C[0, T]$

- terminal-value option  $C = f(B_T)$

**Fact.** for  $C = f(B_T)$ ,  $V_0 = e^{-rT} \int f(\sqrt{T}y) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$

**Fact.** May not be efficient way:

(i) multiple assets — exponentially growth computational cost

(ii) want to compute pricing surface

- pricing surface —  $V(t, s) = e^{-r(T-t)} \mathbb{E}^*(C | S_t = s)$

**Setting 14.** terminal-value option  $C = g(S_T)$ ,  $g$  continuous, no more than linear growth

-  $f(x) = g(e^{\sigma x})$

-  $u(t, x) = \mathbb{E}_x(f(B_t))$

**Fact.**  $V(t, s) = e^{-r(T-t)} \mathbb{E}(g(se^{\sigma B_{T-t} + \mu(T-t)})) = e^{-r(T-t)} u\left(T-t, \frac{\log s + \mu(T-t)}{\sigma}\right)$

-  $p_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}}$

**Fact.**  $u(t, x) = \int p_t(x, y) f(y) dy$

**Fact.**  $\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2 p}{\partial x^2} \Rightarrow \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}$

**Setting 15.** assume can accurately approximate  $u(t, \pm L)$  for all  $t$

- grid  $\{(ik, jh)\} \subset [0, T] \times [-L, L]$  —  $k = \frac{T}{N}, h = \frac{L}{M}$
- grid points  $U_j^i$  — idea  $U_j^i \approx u(ik, jh)$
- FTCS (forward-in-time, central-in-space) —  $\frac{U_j^{i+1} - U_j^i}{k} = \frac{U_{j-1}^i - 2U_j^i + U_{j+1}^i}{2h^2}$ 
  - explicit
  - first-order in time
- BTCS (backward-in-time, central-in-space) —  $\frac{U_j^{i+1} - U_j^i}{k} = \frac{U_{j-1}^{i+1} - 2U_j^{i+1} + U_{j+1}^{i+1}}{2h^2}$ 
  - require to solve  $2M \times 2M$  matrix inversion
  - better stability
  - first-order in time
- Crank-Nicolson —  $\frac{U_j^{i+1} - U_j^i}{k} = \frac{1}{2} \left( \frac{U_{j-1}^i - 2U_j^i + U_{j+1}^i}{2h^2} + \frac{U_{j-1}^{i+1} - 2U_j^{i+1} + U_{j+1}^{i+1}}{2h^2} \right)$ 
  - require to solve  $2M \times 2M$  matrix inversion
  - better stability
  - second-order in time
- Monte Carlo —
  - time-step  $k = \frac{T}{N}$
  - (i)  $(B_t^{(N)} : t = ik)$  linear interpolation of random walk with step distribution  $N(0, k)$
  - (ii)  $(X_t : t = ik)$  simple symmetric random walk with step-size  $h = \sqrt{k}$
  - generate sample  $(B^{(N), i})$
  - (Idea)  $\frac{1}{n} \sum F(B^{(N), i}) \approx \mathbb{E}(F(B^{(N)})) \rightarrow \mathbb{E}(F(B))$
- fair price for European call  $EC(x, K, \sigma, r, T) = \mathbb{E}^*(e^{-rT}(S_T - K)^+)$
- $\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$
- $\Phi(a) = \int_{-\infty}^a \phi(y) dy$
- $\bar{\Phi}(a) = 1 - \Phi(a)$
- $d^\pm = \frac{\log(\frac{x}{K}) + rT}{\sigma\sqrt{T}} \pm \frac{\sigma\sqrt{T}}{2}$

**Proposition 6.3** (Black-Scholes formula).  $EC(x, K, \sigma, r, T) = x\Phi(d^+) - e^{-rT}K\Phi(d^-)$

- put-call parity —  $(S_T - K)^+ - (S_T - K)^- = S_T - K$
- fair price for forward contract —  $\mathbb{E}^*(e^{-rT}(S_T - K)) = x - e^{-rT}K$
- fair price for European put  $EP(x, K, \sigma, r, T) = e^{-rT}K\bar{\Phi}(d^-) - x\bar{\Phi}(d^+)$

**Setting 16.**  $v(x) = v(x, C, \sigma, r, T) = \mathbb{E}^*(e^{-rT}C)$

– **Sensitivities**

– Delta  $\Delta = \frac{\partial v}{\partial x}$

– Gamma  $\Gamma = \frac{\partial^2 v}{\partial x^2}$

– Vega  $\mathcal{V} = \frac{\partial v}{\partial \sigma}$

– Rho  $\rho = \frac{\partial v}{\partial r}$

**Example.** *C* European call,  $\Delta = \Phi(d^+)$ ,  $\mathcal{V} = x\phi(d^+)\sqrt{T}$

**Proposition 6.4.**

(i)  $\sigma \mapsto EC(x, K, \sigma, r, T)$  increasing bijection

(ii)  $\lim_{\sigma \rightarrow 0} EC(x, K, \sigma, r, T) = (x - e^{-rT})^+$

(iii)  $\lim_{\sigma \rightarrow \infty} EC(x, K, \sigma, r, T) = x$

– implied volatility  $\sigma_{implied}(K, T) \text{ ——— } EC(S_0, K, \sigma_{implied}(K, T), r, T) = EC_{market}(K, T)$

**Example.** *up-and-out call*  $C = h(S_T) \mathbb{1}_{\{\sup_{0 \leq t \leq T} S_t < A\}}$ ,  $h(s) = (s - K)^+$ ,  $A \geq \max\{S_0, K\}$