

Graph theory

1 Basic Definition

Topics covered in Exam.

2016-Paper4-16G **Handshaking**

2016-Paper1-16G **Euler's formula, planar graph inequality with girth**

2017-Paper3-15H **bipartite**

- Graph
- Vertices
- Edges
- Order of a graph
- Size of a graph
- Isomorphic

Example. P_n, K_n, C_n, E_n

- Subgraph
- Induced subgraph
- Connected
- Component
- Forest
- Tree

Theorem 1.1. *Following equiv:*

- (i) *Tree*
- (ii) *Minimal connected*
- (iii) *Maximal acyclic*

Theorem 1.2. *Connected iff has spanning tree*

Lemma 1.3 (Handshaking lemma). $\sum d(v) = 2e(G)$

- Leaf
- Minimum degree $\delta(G)$
- Maximum degree $\Delta(G)$

Theorem 1.4. *Order ≥ 2 , then at least 2 leaves*

Corollary 1.5. *Tree order n has size $n - 1$*

Corollary 1.6. *Following equiv:*

- (i) Tree
- (ii) Connected of order n size $n-1$
- (iii) Acyclic of order n size $n-1$

Theorem 1.7 (Caley). n^{n-2} labelled trees of order n

- r-partite
- Bipartite

Theorem 1.8. *Bipartite iff no odd cycles*

- Eulerian tour
- Eulerian

Theorem 1.9. *Eulerian iff $|G| > 1$, connected, $d(v)$ even*

- Planar
- Plane graph
- Face

Lemma 1.10. $d(v)$ even, then partitioned into cycles

Lemma 1.11. e boundary of two faces iff contained in cycle

Theorem 1.12 (Euler). **Connected** plane graph, order n , size m , faces f

$$n - m + f = 2$$

- Bridge

Fact. f_i number of faces of length i

$$\sum i f_i = 2m$$

- Girth

Theorem 1.13. ~~G connected~~, bridgeless, planar graph

$$e(G) \leq \frac{g}{g-2}(n-2)$$

Especially,

$$e(G) \leq 3(n-2)$$

- complete bipartite graph $K_{p,q}$

Theorem 1.14 (Kuratowski). Planar iff not contain subdivision of $K_{3,3}$ or K_5

- dual graph

Fact. Simple if 3-connected

2 Matching and Connectivity

Topics covered in Exam.

2016-Paper4-16G **Menger**

2017-Paper2-15H **Hall**

- Matching
- Independent
- 1-factor (1 regular spanning subgraph)
- n -regular

Theorem 2.1 (Hall). G bipartite with vertex classes X, Y

Matching from X to Y iff $|\Gamma(A)| \geq |A|$ for all subsets $A \subset X$

Theorem 2.2 (defect form).

$|X| - d$ independent edges iff $|\Gamma(A)| \geq |A| - d$

Theorem 2.3 (polygamous version). *1 to 2 matching iff $|\Gamma(A)| \geq 2|A|$*

- Distinct representatives

Corollary 2.4. *Set of distinct representative iff for all subset $S \in [n], |\bigcup_{i \in S} Y_i| \geq |S|$*

- K -connected (be careful with K_n) — $|G| > k$
- Vertex connectivity $\kappa(G)$
- Local connectivity $\kappa(a, b; G)$ — $ab \notin e(G)$
- Edge connectivity $\lambda(G), \lambda(a, b; G)$

Fact.

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

- Vertex disjoint (set of paths)

Theorem 2.5 (Menger). $ab \notin E(G)$, then

$\exists \kappa(a, b; G)$ vertex disjoint $a - b$ paths

- graph contraction
- contraction

Corollary 2.6. $\kappa(G) \geq k, X, Y$ disjoint subsets $|X|, |Y| \geq k$
Then exists set of k completely vertex disjoint $X - Y$ paths

Theorem 2.7 (edge form of Menger). $a, b \in V(G)$, then

\exists set of $\lambda(a, b; G)$ edge-disjoint $a - b$ paths

- Line graph

Fact. *Menger implies Hall*

3 Extremal Graph Theory

Topics covered in Exam.

2016-Paper2-15G **Turan**

2017-Paper1-16H **Hamiltonian**

- Hamiltonian cycle

Theorem 3.1. $|G| \geq 3$, every pair of **non-adjacent** vertices $d(x) + d(y) \geq k$
If $k < n$, G connected, then exists path of length k
If $k = n$, exist Hamiltonian cycle

Corollary 3.2 (Dirac). $|G| = n$, $\delta \geq n/2$, then exists Hamiltonian cycle

Theorem 3.3. G order n , no length k path. Then $e(G) \leq \frac{k-1}{2}n$.
Equality holds iff $k|n$, G disjoint copies of K_k

- $ex(n, F)$
- r partite Turan graph of order n , $T_r(n)$ with size $t_r(n)$

Theorem 3.4 (Turan). K_{r+1} free, order n , $e(G) \geq t_r(n)$, then $G = T_r(n)$

Proof 1. Induction on n with base case K_r . Remove edges to G' with $t_r(n)$ edges. Delete $\delta(G')$ vertex x . $G - x$ Turan. Put back x . T_{r+1} maximal. \square

Proof 2. Induction on n with base case K_r . Add edges to maximal. $K_n \subset G'$. Count edges. $G - K_n$ Turan. Put back K_n . \square

Theorem 3.5 (problem of Zarankiewicz). Bipartite $n \times n$, no $K_{t,t}$

$$z(n, t) \leq (t-1)^{1/t}(n-t+1)n^{1-1/t} + (t-1)n = O(n^{2-1/t})$$

Theorem 3.6.

$$z(n, 2) \leq \frac{n}{2}(1 + \sqrt{4n-3}), \text{ equality holds for infinitely many } n$$

Theorem 3.7.

$$ex(n, K_{2,2}) \leq \frac{n}{4}(1 + \sqrt{4n-3})$$

Fact.

- (i) $ex(n, K_{r+1}) = t_r(n)$
- (ii) $ex(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$
- (iii) $ex(n, C_5) = t_2(n)$
- (iv) $ex(n, F) = \lfloor n^2/4 \rfloor + 2$
- (v) $ex(n, P) = t_2(n) + n - 2$
- $K_r(t)$ complete r-partite graph, $K_r(t) = T_r(rt)$

Lemma 3.8. *If $n > n_1(r, t, \epsilon)$, $|G| = n$, $\delta(G) \geq (1 - \frac{1}{r} + \epsilon)n$ Then contains $K_{r+1}(t)$*

Theorem 3.9 (Erdos-Stone). *n sufficiently large, $e(G) \geq (1 - \frac{1}{r} + \epsilon)\binom{n}{2}$, then contains $K_{r+1}(t)$*

- chromatic number, $\chi(G)$

Corollary 3.10.

$$\lim_{n \rightarrow \infty} \frac{ex(n, F)}{\binom{n}{2}} = 1 - \frac{1}{\chi(F) - 1}$$

4 Colouring

Topics covered in Exam.

2016-Paper3-15G **chromatic polynomial**

- Vertex colouring
- greedy algorithm

Theorem 4.1.

$$\chi(G) \leq 1 + \max_H \delta(H)$$

Corollary 4.2. $\chi(G) \leq \Delta(G) + 1$

- Block

Fact. *Tree of blocks and bridges*

Theorem 4.3 (Brooks). $\chi(G) = \Delta(G) + 1$, then G complete or odd cycle

- clique number, $\omega(G)$
- independence number, $\alpha(G)$

Fact. $\alpha(G) = \omega(\bar{G})$

Fact. $\max \left\{ \omega(G), \frac{|G|}{\alpha(G)} \right\} \leq \chi(G)$

- chromatic polynomial, $p_G(x)$ — number of ways to colour vertices of G with colours $1, 2, \dots, x$

Example.

- (i) complement of K_n , $p_{\bar{K}_n}(x) = x^n$
- (ii) Tree T , $p_T(x) = x(x-1)^{n-1}$
- (iii) complete graph, $p_{K_n}(x) = x(x-1)(x-2) \cdots (x-n+1)$

Theorem 4.4. Any $e \in E(G)$, $p_G(x) = p_{G-e}(x) - p_{G/e}(x)$

Fact. $p_G(x) = \prod_C p_C(x)$

Corollary 4.5.

$$p_G(x) = x^n - a_{n-1}x^{n-1} + \cdots + (-1)^n a_0$$

where $n = |G|$, $a_{n-1} = e(G)$, $a_j \geq 0$ for all j , $\min\{j : a_j \neq 0\} = k$ the number of components

Fact. G not specified by $p_G(x)$

- k-edge-colouring
- chromatic index, $\chi'(G)$

Theorem 4.6. Bipartite multigraph, then $\chi'(G) = \Delta(G)$

Fact. fail for non-bipartite graph e.g. K_3

Theorem 4.7 (Vizing). $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$

- list colouring, $\chi_l(G)$

Fact. $\chi_l(G) \geq \chi(G)$

Fact. $\chi_l(G) \leq 1 + \max_H \delta(H)$

Theorem 4.8 (Five Colour Theorem). *G planar, then $\chi(G) \leq 5$*

Theorem 4.9 (Thomasson). *G planar, then $\chi_l(G) \leq 5$*

Fact. *exist graph with $\chi_l(G) = 5$*

Theorem 4.10 (Tait). *Four Colour Theorem holds iff $\chi'(G) = 3$ for every cubic bridgeless planar G*

4.1 Graph on other surfaces

- Euler characteristic, $E \leq 2$

Fact. *simply connected, then $n - m + f = E$*

Example. ***Orientable surface** g handles, $E = 2 - g$*

- $g = 1$, torus
- $g = 2$, double torus

Non-orientable surfaces, one for each $E \leq 1$

- $E = 1$, projective plane
- $E = 0$, Klein bottle

Fact. $m \leq 3(n - E)$

Theorem 4.11 (Heawood). *characteristic $E \leq 1$, then*

$$\chi(G) \leq H(E) = \left\lfloor \frac{7 + \sqrt{49 - 24E}}{2} \right\rfloor$$

5 Ramsey Theory

Topics covered in Exam.

2017-Paper3-15H **Ramsey numbers**

- Ramsey number, $R(s), R(s, t)$ — smallest n st if colour edges of K_n red or blue, then exists red K_s or blue K_t

Theorem 5.1 (Ramsey). $R(s, t) \leq R(s-1, t) + R(s, t-1)$

Fact. $R(s, t) \leq \binom{s+t-2}{s-1} < 2^{s+t}$

Fact. *inequality not exact*

- $R_k(s_1, s_2, \dots, s_k)$

Theorem 5.2. $R_k(s_1, s_2, \dots, s_k)$ exists

Fact. $R_k(s_1, \dots, s_k) \leq R_{k-1}(R(s_1, s_2), s_3, \dots, s_k)$

Fact. $R_k(s_1, \dots, s_k) \leq R_k(s_1 - 1, \dots, s_k) + \dots + R_k(s_1, \dots, s_k - 1) - k + 2$

- r -uniform hypergraph

Theorem 5.3 (Ramsey for r -sets). $R^{(r)}(s, t)$ exists

Theorem 5.4 (Infinite Ramsey). k colours $\mathbb{N}^{(r)}$, then exists infinite $M \subset \mathbb{N}$ s.t. $M^{(r)}$ monochromatic

- big if $|Y| \geq \min Y$

Theorem 5.5. exists $B(s, k, r)$ s.t. if $n \geq B(s, k, r)$, $\{s, s+1, \dots, n\}^{(r)}$ k coloured, then exist big set $Y \subset \{s, s+1, \dots, n\}^{(r)}$ s.t. $Y^{(r)}$ monochromatic

6 Probabilistic Method

Theorem 6.1 (Erdos). $s \geq 3$, then $R(s) \geq 2^{(s-1)/2}$

Theorem 6.2. G has independent set of size at least

$$\sum_{v \in G} \frac{1}{d(v) + 1} \geq \frac{|G|}{d + 1}$$

– $\mathcal{G}(n, p)$ — space of all $2^{\binom{n}{2}}$ labelled graph

Fact. From 6.1, exists graph with $\omega(G)$ and $\alpha(G)$ at most $2 \log_2 n + 1$. Hence, exist G with $\chi(G) \geq \frac{n}{2 \log_2 n + 1}$

Theorem 6.3 (Erdos). $g \geq 3, k \geq 2$, then exists G with $\chi(G) \geq k$ and girth at least g

Theorem 6.4. $n \geq t \geq 2$, then $z(n, t) > \frac{3}{4} n^{2-2/(t+1)}$

Lemma 6.5. $\mathbb{E}(X_n^2)/\mathbb{E}(X_n)^2 \rightarrow 1$ or $\text{Var}(X_n)/\mathbb{E}(X_n)^2 \rightarrow 0$, then any constant c ,

$$\mathbb{P}(|X_n - \mathbb{E}(X_n)| \geq c\mathbb{E}(X_n)) \rightarrow 0$$

let $c = 1$, then

$$\mathbb{P}(X_n = 0) \rightarrow 0$$

Fact. $\mathbb{P}(X = 0) \leq \frac{\text{Var}(X)}{(\mathbb{E}X)^2}$

Fact. let $X = \sum_A I_A$, then

- $\mathbb{E}X^2 = \sum_{A,B} \mathbb{P}(A \cap B) = \sum_{A,B} \mathbb{P}(A)\mathbb{P}(B|A)$
- $\text{Var}(X) = \sum_{A,B} \mathbb{P}(A)[\mathbb{P}(B|A) - \mathbb{P}(B)]$

Theorem 6.6. $\omega(n) \rightarrow \infty, G \in \mathcal{G}(n, p)$

If $p = \frac{\log n - \omega(n)}{n}$, G has isolated vertices a.s.

If $p = \frac{\log n + \omega(n)}{n}$, G has no isolated vertices a.s.

- almost surely — $\mathbb{P}(A) \rightarrow 1$ as $n \rightarrow \infty$
- threshold phenomenon

Theorem 6.7. let $\mu(d) = \binom{n}{d} p^{\binom{d}{2}}$, fixed p , then

If $\mu(d) \rightarrow 0$, $G \in \mathcal{G}(n, p)$ almost certainly not contain K_d

If $\mu(d) \rightarrow \infty$, $G \in \mathcal{G}(n, p)$ almost certainly contain K_d

Corollary 6.8. $0 < p < 1$ fixed, then clique number is $(2 + o(1)) \log_{1/p} n$ almost surely

7 Eigenvalue Methods

Topics covered in Exam.

2017-Paper4-16H **strongly regular, Petersen graph**

- adjacency matrix, A

Fact. A real symmetric

Fact. $(A^d)_{ij}$ number of walks length d from i to j

- diameter, $\text{diam}(G)$

Fact. $\{I, A, A^2, \dots, A^{\text{diam}(G)}\}$ linearly independent, so at least $\text{diam}(G) + 1$ distinct eigenvalues

Fact. $\lambda_{\min} = \min_{|x|=1} x^T A x$, $\lambda_{\max} = \max_{|x|=1} x^T A x$

Fact. H induced subgraph, $\lambda_{\min}(G) \leq \lambda_{\min}(H) \leq \lambda_{\max}(H) \leq \lambda_{\max}(G)$

Fact. bipartite G , then $A = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$. Also, characteristic polynomial is in t^2

Theorem 7.1. G graph

(i) $\delta(G) \leq \lambda_{\max} \leq \Delta(G)$

(ii) $|\lambda| \leq \Delta(G)$ for all λ

If G connected, then

(i) $\lambda_{\max} = \Delta(G)$ iff G regular, in this case multiplicity of $\lambda_{\max} = 1$

(ii) $\lambda_{\min} = -\Delta(G)$ iff G regular and bipartite, in this case multiplicity of $\lambda_{\min} = 1$

- orientation
- incidence matrix, B
- combinatorial Laplacian, $L = BB^T$

Fact. let $D = \text{diag}(d(1), \dots, d(n))$

$$L = D - A = \begin{cases} d(i) & \text{if } i = j \\ -1 & i \rightarrow j \text{ or } j \rightarrow i \\ 0 & \text{otherwise} \end{cases}$$

Fact. L positive semi-definite

Fact. let eigenvalue be $\mu_1 \leq \mu_2 \leq \dots \leq \mu_n$
 $x^T L x = \sum_{i,j \in E(G)} (x_i - x_j)^2$, hence eigenvalue $\mu_1 = 0$

Proposition 7.2. $\mu_2 = \min\{x^T Lx / \|x\|^2 : x \neq 0, \sum x_i = 0\}$

Theorem 7.3. G graph, $U \subset V(G)$, then at least $\frac{\mu_2 |U| |V-U|}{|G|}$ edges between U and $V - U$

- Moore graph, d -regular, diameter 2, order $d^2 + 1$
- strongly regular (d, a, b) :
 - (i) regular of degree d
 - (ii) every pair of adjacent vertices exactly a common neighbours
 - (iii) every pair of non-adjacent vertices exactly b common neighbours

Fact. Moore graph is $(d, 0, 1)$

Theorem 7.4. G strongly regular (d, a, b) , order n , then

$$\frac{1}{2} \left(n - 1 \pm \frac{(n-1)(b-a) - 2d}{\sqrt{(a-b)^2 + 4(d-b)}} \right) \in \mathbb{N}$$

Theorem 7.5. exist Moore graph only if $d \in \{1, 2, 3, 7, 57\}$