

# Applied Probability

## 1 Continuous time Markov Chains

- right continuous —  $\forall t, \exists \epsilon, X_t(\omega) = X_{t+s}(\omega)$  for all  $s \in [0, \epsilon]$
- finite dimension marginals  $\mathbb{P}(X_{t_1} = x_1, \dots, X_{t_n} = x_n)$

**Fact.** *process can be determined from the finite dimension marginals*

- Memoryless property  $\mathbb{P}(S > t + s | S > s) = \mathbb{P}(S > t)$

**Theorem 1.1.** *Memoryless iff exponential distribution*

### 1.1 Poisson process

- Poisson process with intensity  $\lambda$ 
  - (i)  $N(0) = 0, N(s) \leq N(t)$  for  $s < t$
  - (ii)  $\mathbb{P}(N(t+h) = n+m | N(t) = n) = \begin{cases} \lambda h + o(h) & \text{if } m = 1 \\ o(h) & \text{if } m > 1 \\ 1 - \lambda h + o(h) & \text{if } m = 0 \end{cases}$
  - (iii)  $N(t) - N(s)$  independent of  $(N(k))_{k \leq s}$

**Theorem 1.2.**  $N(t) \sim \text{Poi}(\lambda t)$

*Proof.* derive differential equation, then generating function □

- $p_j(t) = \mathbb{P}(N(t) = j)$
- Generating function  $G(s, t) = \sum p_j(t) s^j$
- Arrival time  $T_n$
- interarrival time  $U_n$

**Theorem 1.3.**

- (i)  $U_i \sim \text{Exp}(\lambda)$
- (ii)  $U_i$  independent

*Proof.* use  $N(t)$  Poisson

□

**Fact.**  $N(t) \geq j \iff T_j \leq t$

– order statistics

**Theorem 1.4.**  $T_1, \dots, T_n$  conditional on  $\{N(t) = n\}$  same as joint distribution of order statistics of  $n$  i.i.d.  $\text{Uniform}[0, t]$

*Proof.*  $U$  to  $T$ , then calculate density

□

**Theorem 1.5.**  $(X_n)$  increasing right-continuous, taking values  $\{0, 1, \dots\}$ ,  $X_0 = 0$ , then following equivalent:

(i) holding times  $S_i \sim \text{Exp}(\lambda)$  i.i.d. jump chain  $Y_n = n$ ,  
(Sousi defined  $X$  Poisson process in this manner)

(ii) (infinitesimal)  $X$  independent increments,  $h \downarrow 0$  uniformly in  $t$ ,  
$$\begin{cases} \mathbb{P}(X_{t+h} - X_t = 1) = \lambda h + o(h) \\ \mathbb{P}(X_{t+h} - X_t = 0) = 1 - \lambda h + o(h) \end{cases}$$

(iii)  $X$  has independent, stationary increments,  $X_t \sim \text{Poi}(\lambda t)$

**Theorem 1.6** (Superposition).  $X, Y$  independent Poisson process, with parameters  $\lambda, \mu$ , then  $Z_t = X_t + Y_t$  Poisson process with parameters  $\lambda + \mu$

*Proof.* infinitesimal

□

**Theorem 1.7** (Thining).  $X$  Poisson process with parameters  $\lambda$ ,  $(Z_i) \sim \text{Bernoulli}(p)$  i.i.d.,  $Y$  jumps  $\iff X$  jumps and  $Z_{X_t} = 1$ , then  $Y$  Poisson process of parameter  $\lambda p$ ,  $X - Y$  independent Poisson process of parameter  $\lambda(1 - p)$

*Proof.* infinitesimal for Poisson process, independence follows from expanding  $\mathbb{P}(Y_t = n, X_t - Y_t = m)$  (suffice to prove independence using finite dimension marginals)

□

## 1.2 Birth process

– birth process with birth rates  $\lambda_0, \lambda_1, \dots$

(i)  $N(0) = 0$ ,  $N(s) \leq N(t)$  for  $s < t$

- (ii)  $\mathbb{P}(N(t+h) = n+m | N(t) = n) = \begin{cases} \lambda_n h + o(h) & \text{if } m = 1 \\ o(h) & \text{if } m > 1 \\ 1 - \lambda_n h + o(h) & \text{if } m = 0 \end{cases}$
- (iii)  $N(t) - N(s)$  independent of  $(N(k))_{k \leq s}$

**Example.**

- (i) *Poisson process:*  $\lambda_n = \lambda$
- (ii) *Simple birth:*  $\lambda_n = n\lambda$
- (iii) *Simple birth with immigration:*  $\lambda_n = n\lambda + \nu$

**Proposition 1.8.**  $T_k \sim \text{Exp}(q_k)$  independent,  $0 < q = \sum q_k < \infty$ ,  $T = \inf_k T_k$ , then

- (i) infimum attained at unique  $K$  with probability 1
- (ii)  $T, K$  independent
- (iii)  $T \sim \text{Exp}(q)$ ,  $\mathbb{P}(K = k) = \frac{q_k}{q}$

- $T_\infty = \lim T_n = \sum^\infty U_i$
- non-explosive / honest —  $\mathbb{P}(T_\infty = \infty) = 1$

**Theorem 1.9.** birth process  $N$ ,  $\lambda_n > 0$ , then non-explosive  $\iff \sum_n \frac{1}{\lambda_n} = \infty$

**Lemma 1.10.**  $U_n \sim \text{Exp}(\lambda_n)$ , independent, then  $\mathbb{P}(T_\infty < \infty) = \begin{cases} 0 & \text{if } \sum_n \frac{1}{\lambda_n} = \infty \\ 1 & \text{if } \sum_n \frac{1}{\lambda_n} < \infty \end{cases}$

- forward system of equations:  $p'_{ij}(t) = \lambda_{j-1} p_{i,j-1}(t) - \lambda_j p_{ij}(t)$
- backward system of equations:  $p'_{ij}(t) = \lambda_i p_{i+1,j}(t) - \lambda_i p_{ij}(t)$

**Theorem 1.11.**

- (i) forward system has unique solution  $\{p_{ij}(t)\}$
- (ii)  $\{p_{ij}(t)\}$  satisfy backward system

**Theorem 1.12.**  $\{p_{ij}(t)\}$  unique solution of forward equations,  $\{\pi_{ij}(t)\}$  any solution of backward equations, then  $p_{ij}(t) \leq \pi_{ij}(t)$

**Fact.**  $\sum_j p_{ij}(t) = 1 \iff \mathbb{P}(T_\infty > t) = 1$

- weak Markov property
- stopping time
- strong Markov property
- right continuity
- stationary independent increments
  - (i)  $N(t) - N(s)$  only depends on  $t - s$
  - (ii)  $\{N(t_i) - N(s_i)\}$  independent where  $s_1 \leq t_1 \leq \dots \leq s_n \leq t_n$

### 1.3 Continuous Markov Chain

**Setting 1.**  $(X(t))$  takes values in countable  $S$

- Markov property —
  - $\mathbb{P}(X(t_n) = j | X(t_{n-1}) = i_{n-1}, \dots, X(t_1) = i_1) = \mathbb{P}(X(t_n) = j | X(t_{n-1}) = i_{n-1})$
- continuous-time Markov chain — right-continuous, Markov property
- transition probability  $p_{ij}(s, t) = \mathbb{P}(X(t) = j | X(s) = i)$
- homogeneous —  $p_{ij}(s, t) = p_{ij}(0, t - s)$
- transition semigroup  $(P_t)_{ij} = p_{ij}(t)$
- stochastic semigroup
  - (i)  $P_0 = I$
  - (ii)  $P_t$  stochastic — non-negative entries, row sum 1
  - (iii) (Chapman-Kolmogorov)  $P_{s+t} = P_s P_t$

**Setting 2.**  $(X(t))$  homogeneous Markov chain

**Theorem 1.13.**  $P_t$  stochastic semigroup

- $\mathbb{P}_i$  — probability measure conditional on  $X(0) = i$
- $\mathbb{E}_i$
- $t$ -historical — events given by  $\{X(s) : s < t\}$
- $t$ -future — events given by  $\{X(s) : s > t\}$
- stopping time  $T$  —  $\{T \leq t\}$  given by  $\{X(s) : s \leq t\}$

**Theorem 1.14** (Extended Markov property).  $H$   $t$ -historical,  $F$   $t$ -future, then  $\mathbb{P}(F | X(t) = j, H) = \mathbb{P}(F | X(t) = j)$

**Theorem 1.15** (Strong Markov property).  $T$  stopping time, conditional on  $\{T \leq T_\infty\} \cap \{X(T) = i\}$ , then

- (i)  $(X_{T+u})_u$  continuous Markov chain start at state  $i$
- (ii) same transition prob
- (iii) independent to  $\{X(s) : s < T\}$

**Setting 3.**  $X(0) = i$

–  $U_0 = \inf \{t : X(t) \neq i\}$

**Fact.** right continuous  $\Rightarrow U_0 > 0$

**Theorem 1.16.**

- (i)  $U_0 \sim \text{Exp}(g_i)$
- (ii)  $U_0$  stopping time

*Proof.* Extended Markov and homogeneity to deduce memoryless □

- transition matrix  $\mathbf{Y} = (y_{ij})$  —  $y_{ij} = \begin{cases} \delta_{ij} & \text{if } g_i = 0 \\ \mathbb{P}_i(X(U_0) = j) & \text{if } g_i > 0 \end{cases}$
- generator  $\mathbf{G} = (g_{ij})$  —  $g_{ij} = \begin{cases} g_i y_{ij} & \text{if } j \neq i \\ -g_i & \text{if } j = i \end{cases}$

**Fact.**  $\mathbb{P}(X(t+h) = j | X(t) = i) = g_{ij}h + o(h)$

**Fact.**  $g_{ij} = g_i(y_{ij} - \delta_{ij})$

**Theorem 1.17.**  $X(0) = i$ , then

- (i)  $X(U_0)$  independent of  $U_0$
- (ii) conditional on  $X(U_0) = j$ ,  $X^*(s) = X(U_0 + s)$  continuous-time Markov chain, same transition prob, initial state  $j$ , independent to the past

- $T_m$
- holding time  $U_m = T_{m+1} - T_m$
- jump chain  $Y = \{Y_n\}$
- $T_\infty = \lim T_n$
- minimal process
- explode from state  $i$  —  $\mathbb{P}_i(T_\infty < \infty) > 0$

**Proposition 1.18.**  $X$  minimal process, then  $P_{s+t} = P_s$

*Proof.* may go to  $\{\infty\}$  □

**Theorem 1.19.**  $i \in S$ , non-explosive from  $i$  if any of the following holds:

- (i)  $S$  finite
- (ii)  $\sup_j g_j < \infty$
- (iii)  $i$  recurrent in jump chain  $Y$

*Proof.* be dominated by Poisson process which is non-explosive □

- irreducible —  $\forall i, j, \exists t > 0, p_{ij}(t) > 0$

**Theorem 1.20.**

- (i) (Levy dichotomy)  $X$  irreducible, then  $\forall t > 0, p_{ij}(t) > 0$
- (ii)  $X$  irreducible  $\iff Y$  irreducible

*Proof.* look at jump chain,  $g_{i_0} \cdots g_{i_n} > 0, p_{i_k, i_{k+1}}(t) > 0$  □

**Fact.** birth process not irreducible

- $T_A = \inf \{t > 0 : X_t \in A\}$
- $H_A = \inf \{n \geq 0 : Y_n \in A\}$
- hitting probability  $h_A(x) = \mathbb{P}_x(T_A < \infty)$
- expected hitting time  $k_A(x) = \mathbb{E}_x(T_A)$

**Theorem 1.21.**  $(h_A(x))_x$  minimal non-negative solution to

$$\begin{cases} h_A(x) = 1 & \forall x \in A \\ Qh_A(x) = \sum_y q_{xy} h_A(y) = 0 & \forall x \notin A \end{cases}$$

**Theorem 1.22.**  $q_x > 0 \forall x \notin A$ , then  $k_A(x)$  minimal non-negative solution to

$$\begin{cases} k_A(x) = 0 & \forall x \in A \\ Qk_A(x) = \sum_y q_{xy} k_A(y) = -1 & \forall x \notin A \end{cases}$$

- recurrent —  $\mathbb{P}(\{t : X(t) = i\} \text{ unbounded}) = 1$
- transient —  $\mathbb{P}(\{t : X(t) = i\} \text{ unbounded}) = 0$
- $R_i = \inf \{t > U_0 : X(t) = i\}$
- mean return time  $m_i = \mathbb{E}(R_i)$
- positive recurrent / non-null recurrent —  $m_i < \infty$

**Theorem 1.23.** *continuous-time chain  $X$ , jump chain  $Y$*

- (i)  $g_i = 0$ , then  $i$  recurrent for  $X$
- (ii)  $g_i > 0$ , then  $i$  recurrent for  $X \iff$  recurrent for  $Y$
- (iii)  $i$  recurrent  $\iff \int p_{ii}(t)dt = \infty$
- (iv)  $i$  transient  $\iff \int p_{ii}(t)dt < \infty$
- (v)  $X$  irreducible, then every state recurrent or every state transient

*Proof.* main point is no explosion. Interchange summation, then old result.  $\square$

- **Forward equation:**  $P'_t = P_t G$  with boundary condition  $P_0 = 1$
- **Backward equation:**  $P'_t = G P_t$  with boundary condition  $P_0 = 1$

**Fact.** *If states  $S$  finite, then  $P_t = e^{tG}$*

- minimal solution —  $p_{ij}(t) \leq \pi_{ij}(t)$
- sub-stochastic —  $\sum_j p_{ij}(t) < 1$

**Theorem 1.24.**  *$S$  countable,  $X$  minimal Markov chain with generator  $G$ , then*

- (i)  $P_t$  minimal non-negative solution of backward equation  $P'_t = G P_t$  with boundary condition  $P_0 = 1$
- (ii)  $P_t$  minimal non-negative solution of forward equation  $P'_t = P_t G$

*Proof. Solution:* condition on  $T_1 > t$  or  $T_1 \leq t$ .  
**Minimal:** reverse argument and induction.  $\square$

**Fact.** *any solution to both equations sub-stochastic*

**Fact.** *non-explosive  $\Rightarrow P_t$  unique solution to both equations*

- measure

- stationary measure —  $\pi = \pi P_t$
- stationary distribution
- unique measure — unique up to scalar multiplication
- first return time  $R_i$
- $m_i = \mathbb{E}_i(R_i)$

**Theorem 1.25.**  $X$  irreducible,  $|S| \geq 2$

- (i) some state  $k$  positive recurrent, then
  - (a)  $\exists$  unique stationary distribution  $\pi$
  - (b) unique distribution st  $\pi G = 0$
  - (c) all states positive recurrent
- (ii)  $X$  non-explosive,  $\exists \pi$  st  $\pi G = 0$ , then
  - (a) all states positive recurrent
  - (b)  $\pi$  stationary
  - (c)  $\pi_k = \frac{1}{m_k g_k}$

*Proof.* (i) use 1.26(iv)  $\pi = \mu(k)/m_k$ , then uniqueness of measure  $\Rightarrow$  all state non-null

(ii)  $\nu' = \frac{\pi_i g_i}{\pi_k g_k}$ , then  $\rho(k) \leq \nu'$  from discrete MC

□

- $\nu_i = x_i g_i$
- $\mu(k) = (\mu_j(k))_j$  —  $\mu_j(k) = \mathbb{E}_k \left( \int_0^{R_k} \mathbb{1} \{X(s) = j\} ds \right)$
- $\rho(k) = (\rho_j(k))$  — mean visit to  $j$  starting from  $k$  in jump chain  $Y$

**Lemma 1.26.**  $X$  irreducible Markov chain,  $|S| \geq 2$

- (i) measure  $x$ , then  $xG = 0 \iff \nu Y = \nu$
- (ii)  $X$  recurrent,  $xG = 0$  unique measure
- (iii)  $x$  measure,  $xG = 0$ , then  $x_j > 0$
- (iv)  $X$  recurrent,  $k \in S$ , then  $\mu(k)G = 0$  and stationary



*Proof.* (i) expand

(ii)  $\nu Y = \nu$ , then uniqueness from discrete MC

(iii)  $\mu_j(k) = \frac{1}{g_j} \rho_j(k)$ , then  $\rho(k)Y = \rho(k)$  from discrete MC

(iv) strong Markov to shift time  $t$

□

**Fact.**  $X$  non-explosive, then  $R_k = \sum_j \int_0^{R_k} \mathbb{1}\{X(s) = j\} ds$

**Fact.**  $X$  irreducible,  $\exists$  more than one stationary distribution, then  $X$  explosive

**Theorem 1.27** (Markov chain limit theorem).  $X$  irreducible, non-explosive

(i) if  $\exists$  stationary distribution  $\pi$ , then

(a)  $\pi$  unique

(b)  $p_{ij}(t) \rightarrow \pi_j$

(ii) if no stationary distribution, then  $p_{ij}(t) \rightarrow 0$

*Proof.* skeleton  $Z_n = X(nh)$

□

**Lemma 1.28.**  $X$  minimal, then  $|p_{ij}(t+u) - p_{ij}(t)| \leq 1 - e^{-g_i u}$

## 1.4 Reversibility

**Theorem 1.29.**  $X$  irreducible, non-explosive, with invariant distribution  $\pi$ , let  $X_0 \sim \pi$ , fix  $T$ ,  $\hat{X}_t = X_{T-t}$ , then

(i)  $\hat{X}$  Markov with generator  $\hat{Q}$  and invariant distribution  $\pi$ ,  $\pi(x)\hat{q}_{xy} = \pi(y)q_{yx}$

(ii)  $\hat{X}$  irreducible, non-explosive

*Proof.* expand  $\mathbb{P}(\hat{X}_{t_0} = x_0, \dots, \hat{X}_{t_n} = x_n)$ , then  $\hat{P}$  satisfies Komogorov backward with  $\hat{Q}$ , then minimal, easy to show irreducible, finally  $\hat{p}_{xy}(t) = \mathbb{P}_x(\hat{X}_t = y, t < \hat{\zeta})$  where  $\zeta$  explosion time □

– reversible —  $(X_t), (X_{T-t})$  same distribution

– detailed balanced —  $\lambda(x)q_{xy} = \lambda(y)q_{yx}$

**Lemma 1.30.** detail balanced  $\Rightarrow \lambda$  invariant measure

**Theorem 1.31.**  $X$  irreducible, non-explosive,  $X_0 \sim \pi$ , then  
detail balanced  $\iff (X_t)$  reversible

**Lemma 1.32.**  $\pi$  invariant for birth-death chain  $\iff$  detail balanced

## 1.5 Ergodic theorem

- long run proportion of time spends at  $x$  —  $\frac{1}{t} \int_0^t \mathbb{1}(X_s = x) ds$

**Theorem 1.33.**  $X$  irreducible, then

(i)  $\frac{1}{t} \int_0^t \mathbb{1}(X_s = x) ds \xrightarrow{a.s.} \frac{1}{m_x g_x}$

(ii) if  $\pi$  invariant,  $f$  bounded, then  $\frac{1}{t} \int_0^t f(X_s) ds \xrightarrow{a.s.} \sum_x f(x) \pi(x)$

## 1.6 Birth-death process and imbedding

- birth rate  $\lambda_0, \lambda_1, \dots$
- death rate  $\mu_1, \mu_2, \dots$
- birth-death process

**Theorem 1.34.**  $X$  birth-death process, generator  $G$

(i) measure  $x_n = \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n} x_0$  satisfies  $\mathbf{x}G = 0$

(ii)  $\exists$  distribution  $\pi$  satisfies  $\pi G = 0 \iff \sum \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n} < \infty$

(iii) if  $\sum \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n} (\lambda_n + \mu_n) < \infty$ , then  $\pi$  stationary

*Proof.* (i) solve  $\mathbf{x}G = 0$

(ii) trivial

(iii) condition for jump chain  $Y$  recurrent, then non-explosive

□

**Example.**

- **Pure birth**  $\mu_n = 0$
- **Simple death with immigration**  $\lambda_n = \lambda, \mu_n = n\mu$

**Theorem 1.35.**  $X(t)$  asymptotically  $\text{Poi}(\rho) = \text{Poi}\left(\frac{\lambda}{\mu}\right)$

- **Simple birth-death**  $\lambda_n = n\lambda, \mu_n = n\mu, X(0) = I$

**Fact.** state 0 absorbing

**Theorem 1.36.**  $G(s, t) = \mathbb{E}(s^{X(t)}) = \begin{cases} \left( \frac{\lambda t(1-s) + s}{\lambda t(1-s) + 1} \right)^I & \text{if } \mu = \lambda \\ \left( \frac{\mu(1-s) - (\mu - \lambda s)e^{-t(\lambda - \mu)}}{\lambda(1-s) - (\mu - \lambda s)e^{-t(\lambda - \mu)}} \right)^I & \text{if } \mu \neq \lambda \end{cases}$

*Proof.* Forward equation □

**Fact.** *non-explosive as*  $\sum p_j(t) = G(1, t) = 1$

**Fact.**  $\mathbb{E}_I(X(t)) \rightarrow \begin{cases} 0 & \text{if } \rho < 1 \\ \infty & \text{if } \rho > 1 \end{cases}$

- *extinction probability*  $\eta(t) = \mathbb{P}_I(X(t) = 0)$

**Corollary 1.37.**  $\eta(t) \rightarrow \begin{cases} 1 & \text{if } \rho \leq 1 \\ \rho^{-I} & \text{if } \rho > 1 \end{cases}$

- imbedded random walk — jump chain  $Y$  with parameter  $\frac{\lambda}{\lambda+\mu}$ , absorbing at 0
- imbedded branching process — lives  $Exp(\lambda+\mu)$ , then born  $n$  individuals where  $\begin{cases} p_0 = \mathbb{P}(n=0) = \frac{\mu}{\lambda+\mu} \\ p_2 = \mathbb{P}(n=2) = \frac{\lambda}{\lambda+\mu} \end{cases}$
- age-dependent branching process
- age density function  $f_T(u) = (\lambda + \mu)e^{-(\lambda+\mu)u}$
- family-size generating function  $G(s) = \frac{\mu+\lambda s^2}{\mu+\lambda} = p_0 + p_2 s^2$

## 2 Queues

- interarrival time  $X_n$  with common distribution  $F_X$
- service time  $S_n$  with common distribution  $F_S$
- $n$ -th customer arrival time  $T_n = \sum X_i$
- length of queue  $Q(t)$
- $A/B/s$  —  $F_X/F_S/\text{\#servers}$

**Example.**

- $D(d)$  — *deterministic*
- $M(\lambda)$  —  $Exp(\lambda)$  (*Markovian*)
- $\Gamma(\lambda, k)$
- $G$  — *general*

**Example.**

- $M/M/1$
- $M/D/1$
- $G/G/1$
- traffic intensity  $\rho = \frac{\mathbb{E}(S)}{\mathbb{E}(X)}$

## 2.1 M/M/1

**Setting 4.**  $M(\lambda)/M(\mu)/1$ ,  $\lambda_n = \lambda$ ,  $\mu_n = \mu$

**Fact.**  $\rho = \frac{\lambda}{\mu}$

**Theorem 2.1.**

(i) if  $\rho < 1$ , then  $\mathbb{P}(Q(t) = n) \rightarrow (1 - \rho)\rho^n = \pi_n$

(ii) if  $\rho \geq 1$ , then  $\mathbb{P}(Q(t) = n) \rightarrow 0$

**Fact.** can define underlying discrete random walk  $Q_{n+1} = \begin{cases} Q_n + 1 & \text{with probability } \frac{\lambda}{\lambda + \mu} = \frac{\rho}{1 + \rho} \\ Q_n - 1 & \text{with probability } \frac{\mu}{\lambda + \mu} = \frac{1}{1 + \rho} \end{cases}$   
for  $n \geq 1$ , and  $\mathbb{P}(Q_{n+1} = 1 | Q_n = 0) = 1$

**Fact.**  $Q_n$  is  $\begin{cases} \text{positive recurrent} & \text{if } \rho < 1 \\ \text{null recurrent} & \text{if } \rho = 1 \\ \text{transient} & \text{if } \rho > 1 \end{cases}$

– waiting time of customer arrived at time  $t$ ,  $W$

**Theorem 2.2.**  $\rho < 1$ , queue in equilibrium, then  $W \sim \text{Exp}(\mu - \lambda)$

**Fact.** expected queue length at equilibrium =  $\frac{\lambda}{\lambda + \mu}$

## 2.2 M/M/ $\infty$

**Setting 5.**  $\begin{cases} q_{i,i+1} = \lambda \\ q_{i,i-1} = i\mu \end{cases}$

**Theorem 2.3.**

(i)  $Q(t)$  positive recurrent

(ii) invariant distribution  $\pi \sim \text{Poi}(\rho)$

*Proof.* solve detail balanced for invariant, coupling to prove non-explosive □

**Setting 6.**  $M/M/1$  queue,  $\rho < 1$

–  $D_t$  — number of customers have departed queue up to time  $t$

**Theorem 2.4** (Burke's theorem).

(i) At equilibrium,  $D_t \sim \text{Poi}(\lambda)$

(ii)  $X_t$  independent from  $(D_s : s \leq t)$

*Proof.* (i) fix  $T$ , time reversal, then Poisson process for all  $T$ , use independent increment criterion.

(ii)  $X_0$  independent to  $[0, T]$ , then reverse

□

## 2.3 Queues in tandem

**Setting 7.** two  $M/M/1$  with  $\lambda, \mu_1, \mu_2$

**Theorem 2.5.**  $X_t, Y_t$  queue length of first, second queue, then  $(X, Y)$  positive recurrent Markov chain  $\iff \lambda < \mu_1, \mu_2$  In this case,  $\pi(m, n) = (1 - \rho_1)\rho_1^m(1 - \rho_2)\rho_2^n$ , so  $X_t, Y_t$  independent, geometric distributed

*Proof.* (i) (**Proof 1:**)  $(m, n) \rightarrow \begin{cases} (m+1, n) & \text{with rate } \lambda \\ (m, n+1) & \text{with rate } \mu_1 \text{ if } m \geq 1, \text{ then check directly.} \\ (m, n-1) & \text{with rate } \mu_2 \text{ if } n \geq 1 \end{cases}$

Rate bounded so non-explosive

(ii) (**Proof 2:**) Burke's

□

**Fact.** r.v. independent while process not independent

## 2.4 Jackson network

**Setting 8.** finite set  $S = \{s_1, \dots, s_c\}$ ,  $Q_i(t)$  individual in station  $i$ ,  $\mathcal{N} = \{\mathbf{n} = (n_1, \dots, n_c)\}$  count-

$$\text{able, } g_{\mathbf{m}, \mathbf{n}} = \begin{cases} \lambda_{ij}\phi_i(m_i) & \text{if } \mathbf{n} = \mathbf{m} - \mathbf{e}_i + \mathbf{e}_j \text{ (transfer)} \\ \nu_j & \text{if } \mathbf{n} = \mathbf{m} + \mathbf{e}_j \text{ (arrival)} \\ \mu_i\phi_i(m_i) & \text{if } \mathbf{n} = \mathbf{m} - \mathbf{e}_i \text{ (departure)} \\ 0 & \text{other } \mathbf{m} \neq \mathbf{n} \end{cases}, \begin{cases} \phi_i(0) = 0 \\ \phi_i(m) > 0 \end{cases}, \lambda_{ii} = 0$$

– closed migration process —  $\nu_j = \mu_j = 0$

– irreducible

**Setting 9.** closed migration process,  $N$  number of customers

**Example** (Base case).  $N = 1$ ,  $\phi_j(1) = 1$ , generator  $h_{ij} = \begin{cases} \lambda_{ij} & \text{if } i \neq j \\ -\sum_k \lambda_{ik} & \text{if } i = j \end{cases}$ , irreducible

**Fact** (Traffic equations).  $\sum_j \alpha_j \lambda_{ji} = \alpha_i \sum_j \lambda_{ij}$

- $\alpha$  — stationary distribution for base case  $N = 1$

**Theorem 2.6.** *irreducible closed migration process,  $N$  customers, then unique stationary distribution  $\pi(\mathbf{n}) = B_N \prod_{i=1}^c \left\{ \frac{\alpha_i^{n_i}}{\prod_{r=1}^{n_i} \phi_i(r)} \right\}$*

*Proof.* stationary condition gives  $\sum_{i,j} \gamma(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) \lambda_{ji} \phi_j(n_j + 1) = \gamma(\mathbf{n}) \sum_{i,j} \lambda_{ij} \phi_i(n_i)$  partial balance equation:  $\sum_j \gamma(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) \lambda_{ji} \phi_j(n_j + 1) = \gamma(\mathbf{n}) \sum_j \lambda_{ij} \phi_i(n_i)$ , then plug in  $\pi$   $\square$

**Fact.** *In equilibrium, not independence as constraint of  $\sum n_i = N$*

**Example** (Open migration process).

- auxiliary process — add station  $\{\infty\}$
- $\beta = (1, \dots, \beta_c, \beta_\infty)$  — stationary distribution of auxiliary process
- $\alpha_i = \frac{\beta_i}{\beta_\infty}$

**Fact.**  $\nu_i + \sum_j \alpha_j \lambda_{ji} = \alpha_i \left( \mu_i + \sum_j \lambda_{ij} \right)$

- $D_i = \sum_{n=0}^{\infty} \frac{\alpha_i^n}{\prod_{r=1}^n \phi_i(r)}$
- $\pi_i(n_i) = D_i^{-1} \frac{\alpha_i^{n_i}}{\prod_{r=1}^{n_i} \phi_i(r)}$

**Theorem 2.7.** *auxiliary process irreducible,  $D_i < \infty$ , then stationary distribution  $\pi(\mathbf{n}) = \prod_i \pi_i(n_i)$*

**Fact.** *In equilibrium, queue length of different stations independent r.v.*

- $g_i = \mu_i + \sum_{j \neq i} \lambda_{ij}$  — total rate of departure of a customer at station  $i$
- $\gamma_i = \alpha_i g_i$  — aggregate/effective arrival rate at  $i$

**Fact** (Traffic equation).  $\gamma_i = \nu_i + \sum_j \gamma_j y_{ij}$  where  $(y_{ij})$  jump chain of single-customer system

**Theorem 2.8.**  $Q = \{Q(t) : -\infty < t < \infty\}$  irreducible open migration process,  $\exists$  stationary distribution  $\pi$ , let  $Q'(t) = Q(-t)$ , then  $Q'$  open migration network with parameters  $\begin{cases} \lambda'_{ij} = \frac{\alpha_j \lambda_{ji}}{\alpha_i} \\ \nu'_j = \alpha_j \mu_j \\ \mu'_i = \frac{\nu_i}{\alpha_i} \\ \phi'_i(\cdot) = \phi_i(\cdot) \end{cases}$

*Proof.* similar to reversal  $\square$

**Fact.** *arrival processes of different station independent Poisson process, so by reversal, departure process independent Poisson process*

## 2.5 M/G/1

- $Q(D_n)$  —  $Q(D_{n+})$  number of customers left after  $n$ -th customer left

**Fact.**  $Q(D_{n+1}) = \begin{cases} U_n + Q(D_n) - 1 & \text{if } Q(D_n) > 0 \\ U_n & \text{if } Q(D_n) = 0 \end{cases}$

**Theorem 2.9.**  $S$  typical service time, then  $Q(D)$  Markov chain with transition matrix

$$P_D = \begin{pmatrix} \delta_0 & \delta_1 & \delta_2 & \cdots \\ \delta_0 & \delta_1 & \delta_2 & \cdots \\ 0 & \delta_0 & \delta_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \delta_j = \mathbb{E} \left( \frac{(\lambda S)^j}{j!} e^{-\lambda S} \right)$$

- traffic intensity  $\rho = \lambda \mathbb{E}(S)$
- $M_S$  — moment generating function of service time

**Theorem 2.10.**

- (i)  $\rho < 1$ , then  $Q(D)$  ergodic, unique stationary distribution  $\pi$  with generating function  $G(s) = \sum \pi_j s^j = (1 - \rho)(s - 1) \frac{M_S(\lambda(s-1))}{s - M_S(\lambda(s-1))}$
- (ii)  $\rho > 1$ ,  $Q(D)$  transient
- (iii)  $\rho = 1$ ,  $Q(D)$  null recurrent

*Proof.* stationary  $\iff \lim_{s \uparrow 1} G(s) = 1$

□

- busy period  $B$  — time during which server continuously occupied

**Fact.**  $\begin{cases} E(B) < \infty & \text{if } \rho < 1 \\ E(B) = \infty, \mathbb{P}(B = \infty) = 0 & \text{if } \rho = 1 \\ \mathbb{P}(B = \infty) > 0 & \text{if } \rho > 1 \end{cases}$

## 3 Renewals

- $X_i$  i.i.d. — non-negative,  $\mathbb{P}(X_i > 0) = 1$
- $T_n = X_1, \dots, X_n$
- renewal process  $N$

**Theorem 3.1.**  $\mathbb{P}(N(t) < \infty) = 1 \iff \mathbb{E}(X_1) > 0$

- $F$  distribution of  $X$
- $F_k$  distribution of  $T_k$

**Lemma 3.2.**  $F_1 = F$ ,  $F_{k+1}(x) = \int_0^x F_k(x-y)dF(y)$

**Lemma 3.3.**  $\mathbb{P}(N(t) = k) = F_k(t) - F_{k+1}(t)$

– renewal function  $m(t) = \mathbb{E}(N(t))$

**Lemma 3.4.**  $m(t) = \sum F_k(t)$

**Lemma 3.5.**  $m(t) = F(t) + \int_0^t m(t-x)dF(x)$

*Proof.* condition on  $X_1$  □

– renewal-type equation  $\mu(t) = H(t) + \int_0^t \mu(t-x)dF(x)$ ,  $H$  uniformly bounded

**Theorem 3.6.**

(i)  $\mu(t) = H(t) + \int_0^t H(t-x)dm(x)$  solution of renewal-type equation

(ii)  $H$  bounded on finite interval, then  $\mu$  bounded on finite intervals, unique sol to equation

– Laplace-Stieltjes transform —  $g^*(\theta) = \int_0^\infty e^{-\theta x} dg(x)$

**Fact.** after transform,  $\mu^*(\theta) = \frac{H^*(\theta)}{1-F^*(\theta)}$

if  $H = F$ ,  $m^*(\theta) = \frac{F^*(\theta)}{1-F^*(\theta)}$

**Theorem 3.7.**  $\frac{1}{t}N(t) \xrightarrow{a.s.} \frac{1}{\mu}$

*Proof.* Strong law on  $T_{N(t)}$  and  $T_{N(t)+1}$  □

**Theorem 3.8.**  $Var(X_1) = \sigma^2$ ,  $\frac{N(t) - \frac{t}{\mu}}{\sqrt{\frac{t\sigma^2}{\mu^3}}} \xrightarrow{d} \mathcal{N}(0, 1)$

**Theorem 3.9** (Elementary renewal theorem).  $\frac{1}{t}m(t) \rightarrow \frac{1}{\mu}$



*Proof.*  $M = N(t) + 1$  stopping time for  $X_i$ , then  $t < T_{N(t)+1}$  with Wald giving lower bound.  $t \geq T_{N(t)}$  and truncation for upper bound.  $\square$

**Lemma 3.10** (Wald's equation).  $X_i$  i.i.d., finite mean,  $M$  stopping time wrt  $(X_i)$ ,  $\mathbb{E}(M) < \infty$ , then  $\mathbb{E}\left(\sum_{i=1}^M X_i\right) = \mathbb{E}(X_1)\mathbb{E}(M)$

**Fact.**  $N(t)$  not stopping time

**Fact.**  $\mathbb{E}_{N(t)+1} \neq \mu$  in general

- $F_X$  arithmetic with span  $\lambda$  —  $X$  take values  $\{m\lambda\}$  a.s.

**Theorem 3.11** (Blackwell's renewal theorem).

(i)  $X_1$  not arithmetic, then  $m(t+h) - m(t) \xrightarrow{t \rightarrow \infty} \frac{h}{\mu}$  for all  $h$

(ii) If  $X_1$  arithmetic, still holds when  $h$  multiple of  $\lambda$

**Theorem 3.12** (Key renewal theorem).  $X_1$  not arithmetic,  $g$  such that

(i)  $g(t) \geq 0$  for all  $t$

(ii)  $\int_0^\infty g(t)dt < \infty$

(iii)  $g$  non-increasing

then  $\int_0^t g(t-x)dm(x) \rightarrow \frac{1}{\mu} \int_0^\infty g(x)dx$

### 3.1 Excess Life

- excess lifetime  $E(t) = T_{N(t)+1} - t$
- current lifetime/age  $C(t) = t - T_{N(t)}$
- total lifetime  $D(t) = E(t) + C(t) = X_{N(t)+1}$

**Theorem 3.13.**  $\mathbb{P}(E(t) \leq y) = F(t+y) - \int_0^t [1 - F(t+y-x)]dm(x)$

*Proof.* consider  $\mathbb{P}(E(t) > y)$  and condition on  $X_1$  to get renewal-type equation  $\square$

**Corollary 3.14.**  $\mathbb{P}(C(t) \geq y) = \begin{cases} 0 & \text{if } y > t \\ 1 - F(t) + \int_0^{t-y} (1 - F(t-x))dm(x) & \text{if } y \leq t \end{cases}$

*Proof.*  $\mathbb{P}(C(t) \geq y) = \mathbb{P}(E(t - y) > y)$

□

**Setting 10.**  $X = X_1$

**Theorem 3.15.**  $X$  not arithmetic,  $\mu = \mathbb{E}(X) < \infty$ , then  $\mathbb{P}(E(t) \leq y) \xrightarrow{t \rightarrow \infty} \frac{1}{\mu} \int_0^y (1 - F(x)) dx$

*Proof.* key renewal and  $\mu = \int F(x) dx$

□

**Theorem 3.16.**  $X$  takes values in  $\mathbb{N}$  with span 1,  $\mu = \mathbb{E}(X) < \infty$ , then  $\mathbb{P}(E(n) = k) \rightarrow \frac{1}{\mu} \mathbb{P}(X \geq k)$  for  $k \in \mathbb{N}$

*Proof.*  $p_{k,k-1} = 1, p_{1,j} = \mathbb{P}(X = j)$ , invariant distribution  $\pi_k = \frac{1}{\mu} \mathbb{P}(X \geq k)$

□

### 3.2 Renewal-reward processes

**Setting 11.**  $\{(X_i, R_i)\}$  i.i.d.

- reward process  $C(t) = \sum_{i=1}^{N(t)} R_i$
- reward function  $c(t) = \mathbb{E}(C(t))$

**Theorem 3.17** (Renewal-reward theorem).  $0 < \mathbb{E}X < \infty$ ,  $\mathbb{E}|R| < \infty$ , then

(i)  $\frac{C(t)}{t} \xrightarrow{a.s.} \frac{\mathbb{E}R}{\mathbb{E}X}$

(ii)  $\frac{c(t)}{t} \rightarrow \frac{\mathbb{E}R}{\mathbb{E}X}$

*Proof.* (i)  $\frac{C(t)}{t} = \frac{C(t)}{N(t)} \frac{N(t)}{t}$

- (ii) 1) Wald, suffice  $t^{-1} \mathbb{E}(R_{N(t)+1}) \rightarrow 0$ , then condition on  $X_1$ , renewal, then bound
- 2) uniform integrability, then cvg in prob  $\Rightarrow L^1$  convergence

□

**Fact.** also work for accumulative reward as long as monotone