Probability and Measure

1 Boolean Algebras and Finitely Additive Measures

- Boolean algebra $\mathcal{B}$
  - $\varnothing \in \mathcal{B}$
  - stable under finite union
  - stable under complementation

Example.

(i) *trivial Boolean algebra*

(ii) *discrete Boolean algebra*

(iii) *family of constructable sets*

- constructable sets —— finite union of locally closed sets from topological space
- locally closed sets —— $O \cap C$ where O open, C closed
- finitely additive measure, m
  - $m(\varnothing) = 0$
  - $m(E \sqcup F) = m(E) + m(F)$
- sub-additive —— $m(E \cup F) \leq m(E) + m(F)$
- monotone —— $E \subset F \Rightarrow m(E) \leq m(F)$

Fact. *finitely additive measure is sub-additive and monotone*

- counting measure

2 Jordan Measure on $\mathbb{R}^d$

- box $B = I_1 \times \cdots \times I_d$
- elementary subset —— finite union of boxes
- volume of box, $|B|$
- $\mathcal{E}(B)$ —— family of elementary subsets of box B

Proposition 2.1. *Fixed B, then*

(i) *$\mathcal{E}(B)$ Boolean algebra*

(ii) *every $E \in \mathcal{E}(B)$ finite union of disjoint boxes*

(iii) *volume well defined*

– $m(E) = \sum |B_i|$ for $E = \bigsqcup B_i$

Fact. *m finitely additively measure on $(B, \mathcal{E}(B))$*

– Jordan measurable —— For all $\epsilon > 0$, $\exists$ elementary $E \subset A \subset F$ st $m(F \backslash E) < \epsilon$

Fact. *Jordan measurable subsets bounded*

– $m(A)$ for Jordan measurable A ——

$$m(A) = \inf\{m(F) : A \subset F, F \text{ elementary}\} = \sup\{m(F) : A \supset E, E \text{ elementary}\}$$

– $\mathcal{J}(B)$ —— family of Jordan measurable subsets of box B

Proposition 2.2. *Fixed B, then*

(i) *$\mathcal{J}(B)$ Boolean algebra*

(ii) *m finitely additive measure on $(B, \mathcal{J}(B))$*

Fact. *$E \subset$ finite interval $[a, b] \subset \mathbb{R}$, then E Jordan measurable iff $\mathbb{1}_E(x)$ Riemann integrable*

3 Lebesgue measurable setds

– Lebesgue outer-measure —— $E \subset \mathbb{R}^d$,

$$m^*(E) = \inf\{\sum |B_n| : E \subset \bigcup B_n \text{ boxes}\}$$

Fact. *m^* translation invariant*

– Lebesgue measurable —— For $\epsilon > 0$, $\exists C = \bigcup B_n$, $E \subset C$ st

$$m^*(C \backslash E) < \epsilon$$

– $\mathcal{L}$ —— family of Lebesgue measurable sets

Fact. *$\mathcal{L}$ translation invariant, scales naturally*

Fact. *Jordan measurable $\Rightarrow$ Lebesgue measurable*

Proposition 3.1.

(i) *m^* extends m*

(ii) *$\mathcal{L}$ Boolean algebra, stable under countable unions*

(iii) *m^* countably additive on $(\mathbb{R}^d, \mathcal{L})$*

Lemma 3.2. m^*

(i) *monotone* —— $A \subset B \Rightarrow m^*(A) \leq m^*(B)$

(ii) *countably sub-additive* —— $m^*\left(\bigcup A_n\right) \leq \sum m^*(A_n)$

Fact. *Jordan measure countably additive on Jordan measurable set*

– continuity property —— E_n non-increasing, empty intersection $\Rightarrow \lim m(E_n) = 0$

Lemma 3.3. *Jordan measure has continuity property on elementary sets*

Lemma 3.4. *Elementary sets E_n decreasing, $A = \bigcap E_n$, then*

(i) *A Lebesgue measurable*

(ii) $m(E_n) \to m^*(A)$

Fact. *countable intersection of elementary sets Lebesgue measurable*

Corollary 3.5. *open and closed subsets Lebesgue measurable*

– null set —— $m^*(E) = 0$

Lemma 3.6. *null set Lebesgue measurable*

Proposition 3.7. *E Lebesgue measurable, then $\exists$ closed C, open O st*

(i) $C \subset E \subset O$

(ii) $m^*(O \backslash C) < \epsilon$

Fact. *E can be written as $\left(\bigcup C_n\right) \sqcup N$ or $\left(\bigcap O_n\right) \backslash N$*

Example. *Vitali's counter example* —— *E set of representatives $E = \{x + \mathbb{Q}\} \subset [0, 1]$*

(i) *m^* not additive on all subsets of $\mathbb{R}^d$*

(ii) *E not Lebesgue measurable*

4 Abstract Measure Theory

– σ-algebra —— Boolean algebra, stable under countable unions

– measurable space, $(X, \mathcal{A})$

– measure μ ——

  (i) $\mu(\varnothing) = 0$

  (ii) countably additive

– measure space, $(X, \mathcal{A}, \mu)$

Example.

(i) $(\mathbb{R}^d, \mathcal{L}, m)$

(ii) $m_0(E) = m(A_0 \bigcap E)$ *for fixed* $A_0 \in \mathcal{L}$

(iii) $(X, 2^X, \#)$, $\#$ *counting measure*

(iv) $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ *where* $\mu(I) = \sum_{i \in I} a_i$ *for fixed* $(a_n)_{n \geq 1}$

Proposition 4.1. $(X, \mathcal{A}, \mu)$ *measure space*

(i) μ *monotone*

(ii) μ *countably sub-additive*

(iii) *upward monotone convergence* —— E_n *increasing, then* $\mu(\bigcup E_n) = \lim \mu(E_n) = \sup \mu(E_n)$

(iv) *downward monotone convergence* —— $\mu(E_1) < \infty$, E_n *decreasing, then* $\mu(\bigcap E_n) = \lim \mu(E_n) = \inf \mu(E_n)$

- finite —— $\mu(X) < \infty$
- σ-finite —— $X = \bigcup E_n$, $\mu(E_n) < \infty$
- probability space
- probability measure
- σ-algebra generated by $\mathcal{F}$, $\sigma(\mathcal{F})$ —— $\mathcal{F}$ family of subsets

Example.

(i) $X = \sqcup X_i$

(ii) X *countable,* $\mathcal{F}$ *singletons*

- Borel σ-algebra, $\mathcal{B}(X)$ —— X topological space, generated by all open subsets
- Borel sets

Fact. $\mathcal{B}(\mathbb{R}^d) \subset \mathcal{L}$

Fact. $\mathcal{B}(\mathbb{R}^d)$ *strictly smaller than* $\mathcal{L}$ —— *every subset of null sets is null*

Fact. $\mathcal{B}(X)$ *(σ-algebra) usually larger than family of constructable sets (Boolean algebra)*

- Boolean algebra generated by $\mathcal{F}$, $\beta(\mathcal{F})$
- explicitly described —— elements of $\beta(\mathcal{F})$ are finite unions of $F_1 \bigcap \cdots \bigcap F_n$, F_i or $\bar{F}_i$ in $\mathcal{F}$

Myth. *Borel hierarchy*

- Borel measure —— measure on $\mathcal{B}(X)$

Setting 1. X *set,* $\mathcal{B}$ *Boolean algebra,* μ *finitely additive measure*

- continuity property —— under setting 1, non-increasing (E_n), $\mu(E_1) < \infty$, empty intersection

$$\lim \mu(E_n) = 0$$

Theorem 4.2 (Caratheodory extension theorem)**.** *Under setting 1, $\mathcal{B}$ continuity property, μ σ-finite, then μ uniquely extends to μ^* on $\sigma(B)$*

- outer-measure μ^* —— $\mu^*(E) = \inf\{\sum \mu(B_i) : E \subset \bigcup B_i, B_i \in \mathcal{B}\}$
- μ^* measurable —— $\exists \bigcup B_n := C$ st $\mu^*(C \backslash E) < \epsilon$
- completion of $\mathcal{B}$, $\mathcal{B}^*$ —— family of μ^* measurable subsets

Proposition 4.3. *Under setting 1,*

(i) *$\mathcal{B}^*$ σ-algebra containing $\mathcal{B}$*

(ii) *μ^* countably additive on $\mathcal{B}^*$*

(iii) *μ^* extends μ*

Myth. *X compact metric space, μ probability measure on Borel σ-algebra $\mathcal{B}$, no atom, then $\exists$ measure preseving measurable isomorphism between $(X, \mathcal{B}^*, \mu)$ and $([0,1], \mathcal{L}, m)$*

5 Uniqueness of Measures

- π-system —— family $\mathcal{F}$
  - **(i)** contains $\varnothing$
  - **(ii)** stable under finite intersection

Proposition 5.1 (measure uniqueness)**.** *$(X, \mathcal{A})$ measurable space, μ_1, μ_2 finite measures st*

(i) *$\mu_1 = \mu_2$ on $\mathcal{F} \bigcup \{X\}$*

(ii) *$\mathcal{F}$ π-system st $\sigma(\mathcal{F}) = \mathcal{A}$*

then $\mu_1 = \mu_2$ on $\mathcal{A}$

Fact. *For general measures, if $\exists F_n \subset \mathcal{F}$ st μ_1, μ_2 finite on F_n, $X = \bigcup F_n$, then uniqueness also holds*

Lemma 5.2 (Dynkin's lemma)**.**

(i) *$\mathcal{F}$ π-system*

(ii) *$\mathcal{F} \subset \mathcal{C}$*

(iii) *$\mathcal{C}$ stable under complementation, disjoint countable union*

then $\sigma(\mathcal{F}) \subset \mathcal{C}$

- translation invariant —— $m(A + x) = m(A)$ for all A, x

Proposition 5.3. *Lebesgue measure unique measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ st*

(i) *translation invariant*

(ii) *$m([0,1]^d) = 1$*

6 Measurable Functions

Setting 2. $(X,\mathcal{A}),(Y,\mathcal{B})$ *measurable space*

- $f : X \to \mathbb{R}$ measurable function
- $f : X \to Y$ measurable map

Fact. *can extend to* $\{\infty\}$ *or* $\{-\infty\}$

Fact. *continuous function measurable*

Fact. $E \in \mathcal{A}$ *iff* $\mathbb{1}_E$ *measurable*

- $\mathbb{R}$-algebra

Proposition 6.1. $(f_n)_{n\geq 1}$ *measurable functions*

(i) f, g *measurable* $\Rightarrow g \circ f$ *measurable*

(ii) *Family of measurable functions form* $\mathbb{R}$*-algebra*

(iii) $\limsup f_n, \liminf f_n, \sup f_n, \inf f_n$ *measurable functions*

Proposition 6.2. $f = (f_1, f_2, \ldots, f_d)^T$*, then* f *measurable iff* f_i *measurable*

- Borel measurable (or simply Borel)

Fact. f *measurable*

(i) $f^{-1}(L)$ *need not measurable for* $L \in \mathcal{L}$

(ii) $f(X)$ *need not measurable even for* f *continuous*

Example. ***(i)*** f *sends to trivial* σ*-algebra*

7 Integration

- simple function —— $\sum^N a_i \mathbb{1}_{A_i}$ with $a_i \geq 0$

Lemma 7.1. f *simple,* $f = \sum a_i \mathbb{1}_{A_i} = \sum b_j \mathbb{1} B_j$*, then* $\sum a_i \mu(A_i) = \sum b_j \mu(B_j)$

- integral $\mu(f)$ for simple f —— $\mu(f) = \sum a_i \mu(A_i) = \int f d\mu$
- integral $\mu(f)$ for non-negative f —— $\mu(f) = \sup\{\mu(g) : g \leq f, g \text{ simple}\}$

Proposition 7.2 (positivity)**.** f, g *non-negative measurable, then*

- $f \geq g \Rightarrow \mu(f) \geq \mu(g)$
- $f \geq g,\ \mu(f) = \mu(g) \Rightarrow f = g$ *a.e.*
- $f = g$ almost everywhere

Lemma 7.3. $f \geq 0$*, then* $\exists$ *increasing simple functions* g_n *st* $g_n \to f$ *pointwise*

Proof. $g_n(x) = 2^{-n}\lfloor 2^n(f(x) \wedge n)\rfloor$ □

Theorem 7.4 (Monotone Convergence Theorem)**.**

- ***(i)*** *(f_n) non-negative, non-decresing*
- ***(ii)*** *let $f(x) = \lim f_n(x)$, the pointwise limit*

Then, $\mu(f) = \lim \mu(f_n)$

Lemma 7.5. *Fixed g simple, then $m_g(E) := \mu(\mathbb{1}_E g)$ is a measure*

Lemma 7.6 (Fatou)**.** *$f_n \geq 0$, then $\mu(\liminf f_n) \leq \liminf \mu(f_n)$*

- f^+, f^-
- μ-integrable —— $\mu(|f|) < \infty$
- intergral $\mu(f)$ for integrable f —— $\mu(f) = \mu(f^+) - \mu(f^-)$

Proposition 7.7 (Linearity of integral)**.** *f, g integrable*

- ***(i)*** *$\alpha f + \beta g$ integrable*
- ***(ii)*** *$\mu(\alpha f + \beta g) = \alpha\mu(f) + \beta\mu(g)$*

Fact. *Also holds for nonegative f, g, α, β*

Theorem 7.8 (Dominated Convergence Theorem)**.** *f, f_n measurable, g integrable*

- ***(i)*** *$|f_n(x)| \leq g(x)$*
- ***(ii)*** *$\lim f_n(x) = f(x)$*

Then,

- ***(i)*** *$\lim \mu(f_n) = \mu(f)$*
- ***(ii)*** *f integrable*

Fact. *condition for MCT, Fatou, DCT only need to hold μ-almost everywhere*

Corollary 7.9 (Exchange of $\int$ and $\sum$)**.**

- ***(i)*** *$f_n \geq 0$, then $\mu\left(\sum^\infty f_n\right) = \sum^\infty \mu(f_n)$*
- ***(ii)*** *$\sum |f_n|$ μ-integrable, then*
  - *$\sum f_n$ integrable*
  - *$\mu\left(\sum f_n\right) = \sum \mu(f_n)$*

Corollary 7.10 (Differentiation under $\int$ sign)**.** *U open set, $f : U \times X \to \mathbb{R}$ st*

- ***(i)*** *$f(t, \cdot)$ μ-integrable*
- ***(ii)*** *$f(\cdot, x)$ differentiable*
- ***(iii)*** *(domination) $\exists$ integrable g st $\sup_t |\frac{\partial f}{\partial t}(t, x)| \leq g(x)$*

Then,

- ***(i)*** $\frac{\partial f}{\partial t}(t, \cdot)$ *μ-integrable*
- ***(ii)*** *let* $F(t) = \int_X f(t,x)d\mu(x)$*, then*
  - *(a)* F *differentiable*
  - *(b)* $F' = \int_X \frac{\partial f}{\partial t}(t,x)d\mu(x)$

Fact. f *bounded, then* f *Riemann integrable iff* $\{x : f(x) \text{ not continuous}\}$ *has Lebsegue measure 0*

Fact (invariance under affine map)**.** $g \in GL_d(\mathbb{R}), f$ *integrable, then* $m(f \circ g) = \frac{1}{|detg|}m(f)$

Fact. $\phi \in C^1$*, then* $\int f(\phi(x))J_\phi(x)dx = \int f(x)dx$

- Radon measure —— Borel measure, finite on every compact subset

Fact (Riesz Representation for locally compact spaces)**.**

- ***(i)*** μ *Radon measure, let* $\Lambda(f) = \mu(f)$*, then* $\Lambda \in C_c(X)'$
- ***(ii)*** *let* $\Lambda \in C_c(X)'$*,* Λ *non-negative, then* $\exists$ *Radon measure* μ *st* $\Lambda(f) = \mu(f)$