Probability and Measure

1 Boolean Algebras and Finitely Additive Measures

– Boolean algebra $\mathcal{B}$

- $\varnothing \in \mathcal{B}$
- stable under finite union
- stable under complementation

Example.

1. *trivial Boolean algebra*
2. *discrete Boolean algebra*
3. *family of constructable sets*

– constructable sets —— finite union of locally closed sets from topological space

– locally closed sets —— $O \cap C$ where O open, C closed

– finitely additive measure, m

- $m(\varnothing) = 0$
- $m(E \sqcup F) = m(E) + m(F)$

– sub-additive —— $m(E \cup F) \leq m(E) + m(F)$

– monotone —— $E \subset F \Rightarrow m(E) \leq m(F)$

Fact. *finitely additive measure is sub-additive and monotone*

– counting measure

2 Jordan Measure on $\mathbb{R}^d$

– box $B = I_1 \times \cdots \times I_d$

– elementary subset —— finite union of boxes

– volume of box, $|B|$

– $\mathcal{E}(B)$ —— family of elementary subsets of box B

Proposition 2.1. *Fixed B, then*

1. *$\mathcal{E}(B)$ Boolean algebra*
2. *every $E \in \mathcal{E}(B)$ finite union of disjoint boxes*
3. *volume well defined*

– $m(E) = \sum |B_i|$ for $E = \bigsqcup B_i$

Fact. *m finitely additively measure on $(B, \mathcal{E}(B))$*

– Jordan measurable —— For all $\epsilon > 0$, $\exists$ elementary $E \subset A \subset F$ st $m(F \backslash E) < \epsilon$

Fact. *Jordan measurable subsets bounded*

– $m(A)$ for Jordan measurable A ——

$$m(A) = \inf\{m(F) : A \subset F, F \text{ elementary}\} = \sup\{m(F) : A \supset E, E \text{ elementary}\}$$

– $\mathcal{J}(B)$ —— family of Jordan measurable subsets of box B

Proposition 2.2. *Fixed B, then*

1. *$\mathcal{J}(B)$ Boolean algebra*
2. *m finitely additive measure on $(B, \mathcal{J}(B))$*

Fact. *$E \subset$ finite interval $[a, b] \subset \mathbb{R}$, then E Jordan measurable iff $\mathbb{1}_E(x)$ Riemann integrable*

3 Lebesgue measurable setds

– Lebesgue outer-measure —— $E \subset \mathbb{R}^d$,

$$m^*(E) = \inf\{\sum |B_n| : E \subset \bigcup B_n \text{ boxes}\}$$

Fact. *m^* translation invariant*

– Lebesgue measurable —— For $\epsilon > 0$, $\exists C = \bigcup B_n$, $E \subset C$ st

$$m^*(C \backslash E) < \epsilon$$

– $\mathcal{L}$ —— family of Lebesgue measurable sets

Fact. *$\mathcal{L}$ translation invariant, scales naturally*

Fact. *Jordan measurable $\Rightarrow$ Lebesgue measurable*

Proposition 3.1.

1. *m^* extends m*
2. *$\mathcal{L}$ Boolean algebra, stable under countable unions*
3. *m^* countably additive on $(\mathbb{R}^d, \mathcal{L})$*

Lemma 3.2. m^*

1. *monotone* —— $A \subset B \Rightarrow m^*(A) \leq m^*(B)$
2. *countably sub-additive* —— $m^*\left(\bigcup A_n\right) \leq \sum m^*(A_n)$

Fact. *Jordan measure countably additive on Jordan measurable set*

– continuity property —— E_n non-increasing, empty intersection $\Rightarrow \lim m(E_n) = 0$

Lemma 3.3. *Jordan measure has continuity property on elementary sets*

Lemma 3.4. *Elementary sets E_n decreasing, $A = \bigcap E_n$, then*

1. *A Lebesgue measurable*
2. $m(E_n) \to m^*(A)$

Fact. *countable intersection of elementary sets Lebesgue measurable*

Corollary 3.5. *open and closed subsets Lebesgue measurable*

– null set —— $m^*(E) = 0$

Lemma 3.6. *null set Lebesgue measurable*

Proposition 3.7. *E Lebesgue measurable, then $\exists$ closed C, open O st*

1. $C \subset E \subset O$
2. $m^*(O \backslash C) < \epsilon$

Fact. *E can be written as $\left(\bigcup C_n\right) \sqcup N$ or $\left(\bigcap O_n\right) \backslash N$*

Example. *Vitali's counter example —— E set of representatives $E = \{x + \mathbb{Q}\} \subset [0, 1]$*

1. *m^* not additive on all subsets of $\mathbb{R}^d$*
2. *E not Lebesgue measurable*

4 Abstract Measure Theory

– σ-algebra —— Boolean algebra, stable under countable unions

– measurable space, $(X, \mathcal{A})$

– measure μ ——

1. $\mu(\varnothing) = 0$
2. countably additive

– measure space, $(X, \mathcal{A}, \mu)$

Example.

1. $(\mathbb{R}^d, \mathcal{L}, m)$

2. *$m_0(E) = m\left(A_0 \bigcap E\right)$ for fixed $A_0 \in \mathcal{L}$*

3. *$(X, 2^X, \#)$, $\#$ counting measure*

4. *$(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ where $\mu(I) = \sum_{i \in I} a_i$ for fixed $(a_n)_{n \geq 1}$*

Proposition 4.1. *$(X, \mathcal{A}, \mu)$ measure space*

1. *μ monotone*

2. *μ countably sub-additive*

3. *upward monotone convergence —— E_n increasing, then $\mu(\bigcup E_n) = \lim \mu(E_n) = \sup \mu(E_n)$*

4. *downward monotone convergence —— $\mu(E_1) < \infty$, E_n decreasing, then $\mu(\bigcap E_n) = \lim \mu(E_n) = \inf \mu(E_n)$*

– finite —— $\mu(X) < \infty$

– σ-finite —— $X = \bigcup E_n$, $\mu(E_n) < \infty$

– probability space

– probability measure

– σ-algebra generated by $\mathcal{F}$, $\sigma(\mathcal{F})$ —— $\mathcal{F}$ family of subsets

Example.

1. *$X = \sqcup X_i$*

2. *X countable, $\mathcal{F}$ singletons*

– Borel σ-algebra, $\mathcal{B}(X)$ —— X topological space, generated by all open subsets

– Borel sets

Fact. $\mathcal{B}(\mathbb{R}^d) \subset \mathcal{L}$

Fact. *$\mathcal{B}(\mathbb{R}^d)$ strictly smaller than $\mathcal{L}$ —— every subset of null sets is null*

Fact. *$\mathcal{B}(X)$ (σ-algebra) usually larger than family of constructable sets (Boolean algebra)*

– Boolean algebra generated by $\mathcal{F}$, $\beta(\mathcal{F})$

– explicitly described —— elements of $\beta(\mathcal{F})$ are finite unions of $F_1 \bigcap \cdots \bigcap F_n$, F_i or $\bar{F}_i$ in $\mathcal{F}$

Myth. *Borel hierarchy*

– Borel measure —— measure on $\mathcal{B}(X)$

Setting 1. *X set, $\mathcal{B}$ Boolean algebra, μ finitely additive measure*

– continuity property —— under setting 1, non-increasing (E_n), $\mu(E_1) < \infty$, empty intersection

$$\lim \mu(E_n) = 0$$

Theorem 4.2 (Caratheodory extension theorem)**.** *Under setting 1, $\mathcal{B}$ continuity property, μ σ-finite, then μ uniquely extends to μ^* on $\sigma(B)$*

- outer-measure μ^* —— $\mu^*(E) = \inf\left\{\sum \mu(B_i) : E \subset \bigcup B_i, B_i \in \mathcal{B}\right\}$
- μ^* measurable —— $\exists \bigcup B_n := C$ st $\mu^*(C \backslash E) < \epsilon$
- completion of $\mathcal{B}$, $\mathcal{B}^*$ —— family of μ^* measurable subsets

Proposition 4.3. *Under setting 1,*

1. *$\mathcal{B}^*$ σ-algebra containing $\mathcal{B}$*
2. *μ^* countably additive on $\mathcal{B}^*$*
3. *μ^* extends μ*

Myth. *X compact metric space, μ probability measure on Borel σ-algebra $\mathcal{B}$, no atom, then $\exists$ measure preseving measurable isomorphism between $(X, \mathcal{B}^*, \mu)$ and $([0,1], \mathcal{L}, m)$*

5 Uniqueness of Measures

- π-system —— family $\mathcal{F}$
  1. contains $\varnothing$
  2. stable under finite intersection

Proposition 5.1 (measure uniqueness)**.** *$(X, \mathcal{A})$ measurable space, μ_1, μ_2 finite measures st*

1. *$\mu_1 = \mu_2$ on $\mathcal{F} \bigcup \{X\}$*
2. *$\mathcal{F}$ π-system st $\sigma(\mathcal{F}) = \mathcal{A}$*

then $\mu_1 = \mu_2$ on $\mathcal{A}$

Fact. *For general measures, if $\exists F_n \subset \mathcal{F}$ st μ_1, μ_2 finite on F_n, $X = \bigcup F_n$, then uniqueness also holds*

Lemma 5.2 (Dynkin's lemma)**.**

1. *$\mathcal{F}$ π-system*
2. *$\mathcal{F} \subset \mathcal{C}$*
3. *$\mathcal{C}$ stable under complementation, disjoint countable union*

then $\sigma(\mathcal{F}) \subset \mathcal{C}$

- translation invariant —— $m(A + x) = m(A)$ for all A, x

Proposition 5.3. *Lebesgue measure unique measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ st*

1. *translation invariant*
2. *$m([0,1]^d) = 1$*

6 Measurable Functions

Setting 2. $(X, \mathcal{A}), (Y, \mathcal{B})$ *measurable space*

- $f : X \to \mathbb{R}$ measurable function
- $f : X \to Y$ measurable map

Fact. *can extend to* $\{\infty\}$ *or* $\{-\infty\}$

Fact. *continuous function measurable*

Fact. $E \in \mathcal{A}$ *iff* $\mathbb{1}_E$ *measurable*

- $\mathbb{R}$-algebra

Proposition 6.1. $(f_n)_{n\geq 1}$ *measurable functions*

1. f, g *measurable* $\Rightarrow g \circ f$ *measurable*
2. *Family of measurable functions form* $\mathbb{R}$*-algebra*
3. $\limsup f_n, \liminf f_n, \sup f_n, \inf f_n$ *measurable functions*

Proposition 6.2. $f = (f_1, f_2, \ldots, f_d)^T$, *then* f *measurable iff* f_i *measurable*

- Borel measurable (or simply Borel)

Fact. f *measurable*

1. $f^{-1}(L)$ *need not measurable for* $L \in \mathcal{L}$
2. $f(X)$ *need not measurable even for* f *continuous*

Example. 1. f *sends to trivial* σ*-algebra*

7 Integration

- simple function