Applied Probability

1 Continuous time Markov Chains

– right continuous —— $\forall t, \exists \epsilon, X_t(\omega) = X_{t+s}(\omega)$ for all $s \in [0, \epsilon]$

– finite dimension marginals $\mathbb{P}(X_{t_1} = x_1, \ldots, X_{t_n} = x_n)$

Fact. *process can be determined from the finite dimension marginals*

– Memoryless property $\mathbb{P}(S > t + s | S > s) = \mathbb{P}(S > t)$

Theorem 1.1. *Memoryless iff exponential distribution*

1.1 Poisson process

– Poisson process with intensity λ

(i) $N(0) = 0$, $N(s) \leq N(t)$ for $s < t$

(ii) $$\mathbb{P}(N(t+h) = n + m | N(t) = n) = \begin{cases} \lambda h + o(h) & \text{if } m = 1 \\ o(h) & \text{if } m > 1 \\ 1 - \lambda h + o(h) & \text{if } m = 0 \end{cases}$$

(iii) $N(t) - N(s)$ independent of $(N(k))_{k \leq s}$

Theorem 1.2. $N(t) \sim Poi(\lambda t)$

Proof. derive differential equation, then generating function □

– $p_j(t) = \mathbb{P}(N(t) = j)$

– Generating function $G(s, t) = \sum p_j(t) s^j$

– Arrival time T_n

– interarrival time U_n

Theorem 1.3.

(i) $U_i \sim Exp(\lambda)$

(ii) U_i *independent*

Proof. use $N(t)$ Poisson $\square$

Fact. $N(t) \geq j \iff T_j \leq t$

– order statistics

Theorem 1.4. *$T_1, \ldots, T_n$ conditional on $\{N(t) = n\}$ same as joint distribution of order statistics of n i.i.d. $Uniform[0,t]$*

Proof. U to T, then calculate density $\square$

Theorem 1.5. *(X_n) increasing right-continuous, taking values $\{0, 1, \ldots\}$, $X_0 = 0$, then following equivalent:*

(i) *holding times $S_i \sim Exp(\lambda)$ i.i.d. ,jump chain $Y_n = n$,*
(Sousi defined X Poisson process in this manner)

(ii) *(**infinitesimal**) X independent increments, $h \downarrow 0$ uniformly in t,*
$$\begin{cases} \mathbb{P}(X_{t+h} - X_t = 1) = \lambda h + o(h) \\ \mathbb{P}(X_{t+h} - X_t = 0) = 1 - \lambda h + o(h) \end{cases}$$

(iii) *X has independent, stationary increments, $X_t \sim Poi(\lambda t)$*

Theorem 1.6 (Superposition)**.** *X, Y independent Poisson process, with parameters λ, μ, then $Z_t = X_t + Y_t$ Poisson process with parameters $\lambda + \mu$*

Proof. infinitesimal $\square$

Theorem 1.7 (Thining)**.** *X Poisson process with parameters λ, $(Z_i) \sim Bernoulli(p)$ i.i.d. , Y jumps $\iff$ X jumps and $Z_{X_t} = 1$, then Y Poisson process of parameter λp, $X - Y$ independent Poisson process of parameter $\lambda(1-p)$*

Proof. infinitesimal for Poisson process, independence follows from expanding $\mathbb{P}(Y_t = n, X_t - Y_t = m)$ (suffice to prove independence using finite dimension marginals) $\square$

1.2 Birth process

– birth process with birth rates $\lambda_0, \lambda_1, \ldots$

(i) $N(0) = 0$, $N(s) \leq N(t)$ for $s < t$

$$\textbf{(ii)}\ \mathbb{P}(N(t+h)=n+m|N(t)=n)=\begin{cases}\lambda_n h+o(h) & \text{if } m=1\\ o(h) & \text{if } m>1\\ 1-\lambda_n h+o(h) & \text{if } m=0\end{cases}$$

(iii) $N(t)-N(s)$ independent of $(N(k))_{k\leq s}$

Example.

(i) ***Poisson process****:* $\lambda_n=\lambda$

(ii) ***Simple birth****:* $\lambda_n=n\lambda$

(iii) ***Simple birth with immigration****:* $\lambda_n=n\lambda+\nu$

Proposition 1.8. *$T_k\sim Exp(q_k)$ independent, $0<q=\sum q_k<\infty$, $T=\inf_k T_k$, then*

(i) infimum attained at unique K with probability 1

(ii) T,K independent

(iii) $T\sim Exp(q),\mathbb{P}(K=k)=\frac{q_k}{q}$

- $T_\infty=\lim T_n=\sum^\infty U_i$
- non-explosive / honest —— $\mathbb{P}(T_\infty=\infty)=1$

Theorem 1.9. *birth process N, $\lambda_n>0$, then non-explosive $\iff$ $\sum_n\frac{1}{\lambda_n}=\infty$*

Lemma 1.10. *$U_n\sim Exp(\lambda_n)$, independent, then* $\mathbb{P}(T_\infty<\infty)=\begin{cases}0 & \textit{if } \sum_n\frac{1}{\lambda_n}=\infty\\ 1 & \textit{if } \sum_n\frac{1}{\lambda_n}<\infty\end{cases}$

- forward system of equations: $p'_{ij}(t)=\lambda_{j-1}p_{i,j-1}(t)-\lambda_j p_{ij}(t)$
- backward system of equations: $p'_{ij}(t)=\lambda_i p_{i+1,j}(t)-\lambda_i p_{ij}(t)$

Theorem 1.11.

(i) forward system has unique solution $\{p_{ij}(t)\}$

(ii) $\{p_{ij}(t)\}$ satisfy backward system

Theorem 1.12. *$\{p_{ij}(t)\}$ unique solution of forward equations, $\{\pi_{ij}(t)\}$ any solution of backward equations, then $p_{ij}(t)\leq\pi_{ij}(t)$*

Fact. $\sum_j p_{ij}(t)=1\iff\mathbb{P}(T_\infty>t)=1$

– weak Markov property

– stopping time

– strong Markov property

– right continuity

– stationary independent increments

(i) $N(t) - N(s)$ only depends on $t - s$

(ii) $\{N(t_i) - N(s_i)\}$ independent where $s_1 \leq t_1 \leq \cdots \leq s_n \leq t_n$

1.3 Continuous Markov Chain

Setting 1. $(X(t))$ *takes values in countable* S

– Markov property ——
$\mathbb{P}(X(t_n) = j | X(t_{n-1}) = i_{n-1}, \ldots, X(t_1) = i_1) = \mathbb{P}(X(t_n) = j | X(t_{n-1}) = i_{n-1})$

– continuous-time Markov chain —— right-continuous, Markov property

– transition probability $p_{ij}(s,t) = \mathbb{P}(X(t) = j | X(s) = i)$

– homogeneous —— $p_{ij}(s,t) = p_{ij}(0, t-s)$

– transition semigroup $(P_t)_{ij} = p_{ij}(t)$

– stochastic semigroup

(i) $P_0 = I$

(ii) P_t stochastic —— non-negative entries, row sum 1

(iii) (Chapman-Kolmogorov) $P_{s+t} = P_s P_t$

Setting 2. $(X(t))$ *homogeneous Markov chain*

Theorem 1.13. P_t *stochastic semigroup*

– $\mathbb{P}_i$ —— probability measure conditional on $X(0) = i$

– $\mathbb{E}_i$

– t-historical —— events given by $\{X(s) : s < t\}$

– t-future —— events given by $\{X(s) : s > t\}$

– stopping time T —— $\{T \leq t\}$ given by $\{X(s) : s \leq t\}$

Theorem 1.14 (Extended Markov property)**.** H *t-historical,* F *t-future, then*
$\mathbb{P}(F | X(t) = j, H) = \mathbb{P}(F | X(t) = j)$

Theorem 1.15 (Strong Markov property). *T stopping time, conditional on $\{T \leq T_\infty\} \bigcap \{X(T) = i\}$, then*

(i) *$(X_{T+u})_u$ continuous Markov chain start at state i*

(ii) *same transition prob*

(iii) *independent to $\{X(s) : s < T\}$*

Setting 3. $X(0) = i$

- $U_0 = \inf\{t : X(t) \neq i\}$

Fact. *right continuous* $\Rightarrow U_0 > 0$

Theorem 1.16.

(i) *$U_0 \sim Exp(g_i)$*

(ii) *U_0 stopping time*

Proof. Extended Markov and homogeneity to deduce memoryless □

- transition matrix $\mathbf{Y} = (y_{ij})$ —— $y_{ij} = \begin{cases} \delta_{ij} & \text{if } g_i = 0 \\ \mathbb{P}_i(X(U_0) = j) & \text{if } g_i > 0 \end{cases}$
- generator $\mathbf{G} = (g_{ij})$ —— $g_{ij} = \begin{cases} g_i y_{ij} & \text{if } j \neq i \\ -g_i & \text{if } j = i \end{cases}$

Fact. $\mathbb{P}(X(t+h) = j | X(t) = i) = g_{ij}h + o(h)$

Fact. $g_{ij} = g_i(y_{ij} - \delta_{ij})$

Theorem 1.17. *$X(0) = i$, then*

(i) *$X(U_0)$ independent of U_0*

(ii) *conditional on $X(U_0) = j$, $X^*(s) = X(U_0 + s)$ continuous-time Markov chain, same transition prob, initial state j, independent to the past*

- T_m
- holding time $U_m = T_{m+1} - T_m$
- jump chain $Y = \{Y_n\}$
- $T_\infty = \lim T_n$
- minimal process
- explode from state i —— $\mathbb{P}_i(T_\infty < \infty) > 0$

Proposition 1.18. *X minimal process, then* $\boldsymbol{P}_{s+t} = \boldsymbol{P}_s \boldsymbol{P}_t$

Proof. may go to $\{\infty\}$ □

Theorem 1.19. *$i \in S$, non-explosive from i if any of the following holds:*

(i) *S finite*

(ii) $\sup_j g_j < \infty$

(iii) *i recurrent in jump chain Y*

Proof. be dominated by Poisson process which is non-explosive □

- irreducible —— $\forall i, j, \exists t > 0, p_{ij}(t) > 0$

Theorem 1.20.

(i) *(Levy dichotomy) X irreducible, then $\forall t > 0$, $p_{ij}(t) > 0$*

(ii) *X irreducible $\iff$ Y irreducible*

Proof. look at jump chain, $g_{i_0} \cdots g_{i_n} > 0$, $p_{i_k, i_{k+1}}(t) > 0$ □

Fact. *birth process not irreducible*

- $T_A = \inf\{t > 0 : X_t \in A\}$
- $H_A = \inf\{n \geq 0 : Y_n \in A\}$
- hitting probability $h_A(x) = \mathbb{P}_x(T_A < \infty)$
- expected hitting time $k_A(x) = \mathbb{E}_x(T_A)$

Theorem 1.21. *$(h_A(x))_x$ minimal non-negative solution to*

$$\begin{cases} h_A(x) = 1 & \forall x \in A \\ Qh_A(x) = \sum_y q_{xy} h_A(y) = 0 & \forall x \notin A \end{cases}$$

Theorem 1.22. *$q_x > 0 \forall x \notin A$, then $k_A(x)$ minimal non-negative solution to*

$$\begin{cases} k_A(x) = 0 & \forall x \in A \\ Qk_A(x) = \sum_y q_{xy} k_A(y) = -1 & \forall x \notin A \end{cases}$$

- recurrent —— $\mathbb{P}(\{t : X(t) = i\} \text{ unbounded}) = 1$
- transient —— $\mathbb{P}(\{t : X(t) = i\} \text{ unbounded}) = 0$
- $R_i = \inf\{t > U_0 : X(t) = i\}$
- mean return time $m_i = \mathbb{E}(R_i)$
- positive recurrent / non-null recurrent —— $m_i < \infty$

Theorem 1.23. *continuous-time chain X, jump chain Y*

(i) *$g_i = 0$, then i recurrent for X*

(ii) *$g_i > 0$, then i recurrent for $X \iff$ recurrent for Y*

(iii) *i recurrent $\iff \int p_{ii}(t)dt = \infty$*

(iv) *i transient $\iff \int p_{ii}(t)dt < \infty$*

(v) *X irreducible, then every state recurrent or every state transient*

Proof. main point is no explosion. Interchange summation, then old result. □

- **Forward equation:** $\boldsymbol{P}_t' = \boldsymbol{P}_t\boldsymbol{G}$ with boundary condition $\boldsymbol{P}_0 = 1$
- **Backward equation:** $\boldsymbol{P}_t' = \boldsymbol{G}\boldsymbol{P}_t$ with boundary condition $\boldsymbol{P}_0 = 1$

Fact. *If states S finite, then $\boldsymbol{P}_t = e^{t\boldsymbol{G}}$*

- minimal solution —— $p_{ij}(t) \leq \pi_{ij}(t)$
- sub-stochastic —— $\sum_j p_{ij}(t) < 1$

Theorem 1.24. *S countable, X minimal Markov chain with generator $\boldsymbol{G}$, then*

(i) *$\boldsymbol{P}_t$ minimal non-negative solution of backward equation $\boldsymbol{P}_t' = \boldsymbol{G}\boldsymbol{P}_t$ with boundary condition $\boldsymbol{P}_0 = 1$*

(ii) *$\boldsymbol{P}_t$ minimal non-negative solution of forward equation $\boldsymbol{P}_t' = \boldsymbol{P}_t\boldsymbol{G}$*

Proof. **Solution:** condition on $T_1 > t$ or $T_1 \leq t$.
Minimal: reverse argument and induction. □

Fact. *any solution to both equations sub-stochastic*

Fact. *non-explosive $\Rightarrow$ $\boldsymbol{P}_t$ unique solution to both equations*

- measure

- stationary measure —— $\boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{P}_t$
- stationary distribution
- unique measure —— unique up to scalar multiplication
- first return time R_i
- $m_i = \mathbb{E}_i(R_i)$

Theorem 1.25. *X irreducible, $|S| \geq 2$*

(i) *some state k positive recurrent, then*

(a) $\exists$ unique stationary distribution $\boldsymbol{\pi}$

(b) unique distribution st $\boldsymbol{\pi}\boldsymbol{G} = 0$

(c) all states positive recurrent

(ii) *X non-explosive, $\exists\ \boldsymbol{\pi}$ st $\boldsymbol{\pi}\boldsymbol{G} = 0$, then*

(a) all states positive recurrent

(b) $\boldsymbol{\pi}$ stationary

(c) $\pi_k = \frac{1}{m_k g_k}$

Proof. **(i)** use 1.26(**iv**) $\boldsymbol{\pi} = \mu(k)/m_k$, then uniqueness of measure $\Rightarrow$ all state non-null

(ii) $\boldsymbol{\nu}' = \frac{\pi_i g_i}{\pi_k g_k}$, then $\boldsymbol{\rho}(k) \leq \boldsymbol{\nu}'$ from discrete MC

□

- $\nu_i = x_i g_i$
- $\boldsymbol{\mu}(k) = (\mu_j(k))_j$ —— $\mu_j(k) = \mathbb{E}_k\left(\int_0^{R_k} \mathbb{1}\left\{X(s) = j\right\} ds\right)$
- $\boldsymbol{\rho}(k) = (\rho_j(k))$ —— mean visit to j starting from k in jump chain Y

Lemma 1.26. *X irreducible Markov chain, $|S| \geq 2$*

(i) *measure $\boldsymbol{x}$, then $\boldsymbol{x}\boldsymbol{G} = 0 \iff \boldsymbol{\nu}\boldsymbol{Y} = \boldsymbol{\nu}$*

(ii) *X recurrent, $\boldsymbol{x}\boldsymbol{G} = 0$ unique measure*

(iii) *$\boldsymbol{x}$ measure, $\boldsymbol{x}\boldsymbol{G} = 0$, then $x_j > 0$*

(iv) *X recurrent, $k \in S$, then $\boldsymbol{\mu}(k)\boldsymbol{G} = 0$ and stationary*

Proof. **(i)** expand

(ii) $\boldsymbol{\nu}\boldsymbol{Y} = \boldsymbol{\nu}$, then uniqueness from discrete MC

(iii) $\mu_j(k) = \frac{1}{g_j}\rho_j(k)$, then $\boldsymbol{\rho}(k)\boldsymbol{Y} = \boldsymbol{\rho}(k)$ from discrete MC

(iv) strong Markov to shift time t

□

Fact. *X non-explosive, then $R_k = \sum_j \int_0^{R_k} \mathbb{1}\{X(s) = j\}\, ds$*

Fact. *X irreducible, $\exists$ more than one stationary distribution, then X explosive*

Theorem 1.27 (Markov chain limit theorem)**.** *X irreducible, non-explosive*

(i) *if $\exists$ stationary distribution $\boldsymbol{\pi}$, then*

(a) *$\boldsymbol{\pi}$ unique*

(b) *$p_{ij}(t) \to \pi_j$*

(ii) *if no stationary distribution, then $p_{ij}(t) \to 0$*

Proof. skeleton $Z_n = X(nh)$ □

Lemma 1.28. *X minimal, then $|p_{ij}(t+u) - p_{ij}(t)| \leq 1 - e^{-g_i u}$*

1.4 Reversibility

Theorem 1.29. *X irreducible, non-explosive, with invariant distribution $\boldsymbol{\pi}$, let $X_0 \sim \boldsymbol{\pi}$, fix T, $\hat{X}_t = X_{T-t}$, then*

(i) *$\hat{X}$ Markov with generator $\hat{Q}$ and invariant distribution $\boldsymbol{\pi}$, $\pi(x)\hat{q}_{xy} = \pi(y)q_{yx}$*

(ii) *$\hat{X}$ irreducible, non-explosive*

Proof. expand $\mathbb{P}(\hat{X}_{t_0} = x_0, \ldots, \hat{X}_{t_n} = x_n)$, then $\hat{P}$ satisfies Komogorov backward with $\hat{Q}$, then minimal, easy to show irreducible, finally $\hat{p}_{xy}(t) = \mathbb{P}_x(\hat{X}_t = y, t < \hat{\zeta})$ where ζ explosion time □

- reversible —— (X_t), (X_{T-t}) same distribution
- detailed balanced —— $\lambda(x)q_{xy} = \lambda(y)q_{yx}$

Lemma 1.30. *detail balanced $\Rightarrow$ λ invariant measure*

Theorem 1.31. *X irreducible, non-explosive, $X_0 \sim \boldsymbol{\pi}$, then*
detail balanced $\iff$ (X_t) reversible

Lemma 1.32. *$\boldsymbol{\pi}$ invariant for birth-death chain $\iff$ detail balanced*

1.5 Ergodic theorem

- long run proportion of time spends at x —— $\frac{1}{t}\int_0^t \mathbb{1}(X_s = x)ds$

Theorem 1.33. *X irreducible, then*

(i) $\frac{1}{t}\int_0^t \mathbb{1}(X_s = x)ds \xrightarrow{a.s.} \frac{1}{m_x g_x}$

(ii) *if π invariant, f bounded, then* $\frac{1}{t}\int_0^t f(X_s)ds \xrightarrow{a.s.} \sum_x f(x)\pi(x)$

1.6 Birth-death process and imbedding

- birth rate $\lambda_0, \lambda_1, \ldots$
- death rate $\mu_1, \mu_2, \ldots$
- birth-death process

Theorem 1.34. *X birth-death process, generator $\boldsymbol{G}$*

(i) *measure* $x_n = \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}x_0$ *satisfies* $\boldsymbol{x}\boldsymbol{G} = 0$

(ii) $\exists$ *distribution $\boldsymbol{\pi}$ satisfies* $\boldsymbol{\pi}\boldsymbol{G} = 0 \iff \sum \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n} < \infty$

(iii) *if* $\sum \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}(\lambda_n + \mu_n) < \infty$, *then $\boldsymbol{\pi}$ stationary*

Proof. **(i)** solve $\boldsymbol{x}\boldsymbol{G} = 0$

(ii) trivial

(iii) condition for jump chain Y recurrent, then non-explosive

□

Example.

- ***Pure birth*** $\mu_n = 0$
- ***Simple death with immigration*** $\lambda_n = \lambda, \mu_n = n\mu$

  Theorem 1.35. *$X(t)$ asymptotically* $Poi(\rho) = Poi\left(\frac{\lambda}{\mu}\right)$

- ***Simple birth-death*** $\lambda_n = n\lambda, \mu_n = n\mu,\ X(0) = I$

  Fact. *state 0 absorbing*

  Theorem 1.36. $G(s,t) = \mathbb{E}(s^{X(t)}) = \begin{cases} \left(\frac{\lambda t(1-s)+s}{\lambda t(1-s)+1}\right)^I & \text{if } \mu = \lambda \\ \left(\frac{\mu(1-s)-(\mu-\lambda s)e^{-t(\lambda-\mu)}}{\lambda(1-s)-(\mu-\lambda s)e^{-t(\lambda-\mu)}}\right)^I & \text{if } \mu \neq \lambda \end{cases}$

Proof. Forward equation □

Fact. *non-explosive as* $\sum p_j(t) = G(1,t) = 1$

Fact. $\mathbb{E}_I(X(t)) \to \begin{cases} 0 & \text{if } \rho < 1 \\ \infty & \text{if } \rho > 1 \end{cases}$

- *extinction probability* $\eta(t) = \mathbb{P}_I(X(t) = 0)$

Corollary 1.37. $\eta(t) \to \begin{cases} 1 & \text{if } \rho \leq 1 \\ \rho^{-I} & \text{if } \rho > 1 \end{cases}$

– imbedded random walk —— jump chain Y with parameter $\frac{\lambda}{\lambda+\mu}$, absorbing at 0

– imbedded branching process —— lives $Exp(\lambda+\mu)$, then born n individuals where $\begin{cases} p_0 = \mathbb{P}(n=0) = \frac{\mu}{\lambda+\mu} \\ p_2 = \mathbb{P}(n=2) = \frac{\lambda}{\lambda+\mu} \end{cases}$

– age-dependent branching process

– age density function $f_T(u) = (\lambda+\mu)e^{-(\lambda+\mu)u}$

– family-size generating function $G(s) = \frac{\mu+\lambda s^2}{\mu+\lambda} = p_0 + p_2 s^2$

2 Queues

– interarrival time X_n with common distribution F_X

– service time S_n with common distribution F_S

– n-th customer arrival time $T_n = \sum X_i$

– length of queue $Q(t)$

– $A/B/s$ —— $F_X/F_S/$#servers

Example.

– $D(d)$ —— *deterministic*

– $M(\lambda)$ —— $Exp(\lambda)$ *(Markovian)*

– $\Gamma(\lambda, k)$

– G —— *general*

Example.

– $M/M/1$

– $M/D/1$

– $G/G/1$

– traffic intensity $\rho = \frac{\mathbb{E}(S)}{\mathbb{E}(X)}$

2.1 M/M/1

Setting 4. $M(\lambda)/M(\mu)/1$, $\lambda_n = \lambda$, $\mu_n = \mu$

Fact. $\rho = \frac{\lambda}{\mu}$

Theorem 2.1.

(i) *if* $\rho < 1$, *then* $\mathbb{P}(Q(t) = n) \to (1-\rho)\rho^n = \pi_n$

(ii) *if* $\rho \geq 1$, *then* $\mathbb{P}(Q(t) = n) \to 0$

Fact. *can define underlying discrete random walk* $Q_{n+1} = \begin{cases} Q_n + 1 & \textit{with probability } \frac{\lambda}{\lambda+\mu} = \frac{\rho}{1+\rho} \\ Q_n - 1 & \textit{with probability } \frac{\mu}{\lambda+\mu} = \frac{1}{1+\rho} \end{cases}$ *for* $n \geq 1$, *and* $\mathbb{P}(Q_{n+1} = 1 | Q_n = 0) = 1$

Fact. Q_n *is* $\begin{cases} \textit{positive recurrent} & \textit{if } \rho < 1 \\ \textit{null recurrent} & \textit{if } \rho = 1 \\ \textit{transient} & \textit{if } \rho > 1 \end{cases}$

- waiting time of customer arrived at time t, W

Theorem 2.2. $\rho < 1$, *queue in equilibrium, then* $W \sim Exp(\mu - \lambda)$

Fact. *expected queue length at equilibrium* $= \frac{\lambda}{\lambda+\mu}$

2.2 M/M/∞

Setting 5. $\begin{cases} q_{i,i+1} = \lambda \\ q_{i,i-1} = i\mu \end{cases}$

Theorem 2.3.

(i) $Q(t)$ *positive recurrent*

(ii) *invariant distribution* $\pi \sim Poi(\rho)$

Proof. solve detail balanced for invariant, coupling to prove non-explosive $\square$

Setting 6. $M/M/1$ *queue,* $\rho < 1$

- D_t —— number of customers have departed queue up to time t

Theorem 2.4 (Burke's theorem)**.**

(i) *At equalibrium,* $D_t \sim Poi(\lambda)$

(ii) X_t *independent from* $(D_s : s \leq t)$

Proof. **(i)** fix T, time reversal, then Poisson process for all T, use independent increment criterion.

(ii) X_0 independent to $[0, T]$, then reverse □

2.3 Queues in tandem

Setting 7. *two* $M/M/1$ *with* λ, μ_1, μ_2

Theorem 2.5. X_t*,* Y_t *queue length of first, second queue, then* (X, Y) *positive recurrent Markov chain* $\iff$ $\lambda < \mu_1, \mu_2$ *In this case,* $\pi(m, n) = (1 - \rho_1)\rho_1^m(1 - \rho_2)\rho_n$*, so* X_t, Y_t *indepedent, geometric distributed*

Proof. **(i)** (**Proof 1:**) $(m, n) \to \begin{cases} (m+1, n) & \text{with rate } \lambda \\ (m, n+1) & \text{with rate } \mu_1 \text{ if } m \geq 1 \\ (m, n-1) & \text{with rate } \mu_2 \text{ if } n \geq 1 \end{cases}$, then check directly.

Rate bounded so non-explosive

(ii) (**Proof 2:**) Burke's □

Fact. *r.v. independent while process not independent*

2.4 Jackson network

Setting 8. *finite set* $S = \{s_1, \ldots, s_c\}$*,* $Q_i(t)$ *individual in station* i*,* $\mathcal{N} = \{\boldsymbol{n} = (n_1, \ldots, n_c)\}$ *countable,* $g_{\boldsymbol{m},\boldsymbol{n}} = \begin{cases} \lambda_{ij}\phi_i(m_i) & \textit{if } \boldsymbol{n} = \boldsymbol{m} - \boldsymbol{e}_i + \boldsymbol{e}_j \ \textbf{\textit{(transfer)}} \\ \nu_j & \textit{if } \boldsymbol{n} = \boldsymbol{m} + \boldsymbol{e}_j \ \textbf{\textit{(arrival)}} \\ \mu_i\phi_i(m_i) & \textit{if } \boldsymbol{n} = \boldsymbol{m} - \boldsymbol{e}_i \ \textbf{\textit{(departure)}} \\ 0 & \textit{other } \boldsymbol{m} \neq \boldsymbol{n} \end{cases}$, $\begin{cases} \phi_i(0) = 0 \\ \phi_i(m) > 0 \end{cases}$, $\lambda_{ii} = 0$

- closed migration process —— $\nu_j = \mu_j = 0$
- irreducible

Setting 9. *closed migration process,* N *number of customers*

Example (Base case)**.** $N = 1$*,* $\phi_j(1) = 1$*, generator* $h_{ij} = \begin{cases} \lambda_{ij} & \textit{if } i \neq j \\ -\sum_k \lambda_{ik} & \textit{if } i = j \end{cases}$*, irreducible*

Fact (Traffic equations)**.** $\sum_j \alpha_j \lambda_{ji} = \alpha_i \sum_j \lambda_{ij}$

- α —— stationary distribution for base case $N = 1$

Theorem 2.6. *irreducible closed migration process, N customers, then unique stationary distribution* $\pi(\boldsymbol{n}) = B_N \prod_{i=1}^{c} \left\{ \frac{\alpha_i^{n_i}}{\prod_{r=1}^{n_i} \phi_i(r)} \right\}$

Proof. stationary condition gives $\sum_{i,j} \gamma(\boldsymbol{n} - \boldsymbol{e}_i + \boldsymbol{e}_j)\lambda_{ji}\phi_j(n_j + 1) = \gamma(\boldsymbol{n}) \sum_{i,j} \lambda_{ij}\phi_i(n_i)$ partial balance equation: $\sum_j \gamma(\boldsymbol{n} - \boldsymbol{e}_i + \boldsymbol{e}_j)\lambda_{ji}\phi_j(n_j + 1) = \gamma(\boldsymbol{n}) \sum_j \lambda_{ij}\phi_i(n_i)$, then plug in π □

Fact. *In equilibrium, not independence as constraint of* $\sum n_i = N$

Example (Open migration process)**.**

- *auxiliary process* —— *add station* $\{\infty\}$
- $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_c, \beta_\infty)$ —— *stationary distribution of auxiliary process*
- $\alpha_i = \frac{\beta_i}{\beta_\infty}$

Fact. $\nu_i + \sum_j \alpha_j \lambda_{ji} = \alpha_i \left(\mu_i + \sum_j \lambda_{ij} \right)$

- $D_i = \sum_{n=0}^{\infty} \frac{\alpha_i^n}{\prod_{r=1}^{n} \phi_j(r)}$
- $\pi_i(n_i) = D^{-1} \frac{\alpha_i^{n_i}}{\prod_{r=1}^{n_i} \phi_i(r)}$

Theorem 2.7. *auxiliary process irreducible, $D_i < \infty$, then stationary distribution* $\pi(\boldsymbol{n}) = \prod_i^c \pi_i(n_i)$

Fact. *In equilibrium, queue length of different stations independent r.v.*

- $g_i = \mu_i + \sum_{j \neq i} \lambda_{ij}$ —— total rate of departure of a customer at station i
- $\gamma_i = \alpha_i g_i$ ——aggregate/effective arrival rate at i

Fact (Traffic equation)**.** $\gamma_i = \nu_i + \sum_j \gamma_j y_{ij}$ *where* (y_{ij}) *jump chain of single-customer system*

Theorem 2.8. $\boldsymbol{Q} = \{\boldsymbol{Q}(t) : -\infty < t < \infty\}$ *irreducible open migration process,* $\exists$ *stationary distribution* π*, let* $\boldsymbol{Q}'(t) = \boldsymbol{Q}(-t)$*, then* $\boldsymbol{Q}'$ *open migration network with parameters* $\begin{cases} \lambda'_{ij} = \frac{\alpha_j \lambda_{ji}}{\alpha_i} \\ \nu'_j = \alpha_j \mu_j \\ \mu'_i = \frac{\nu_i}{\alpha_i} \\ \phi'_i(\cdot) = \phi_i(\cdot) \end{cases}$

Proof. similar to reversal □

Fact. *arrival processes of different station independent Poisson process, so by reversal, departure process independent Poisson process*

2.5 M/G/1

– $Q(D_n)$ —— $Q(D_n+)$ number of customers left after n-th customer left

Fact. $Q(D_{n+1}) = \begin{cases} U_n + Q(D_n) - 1 & \text{if } Q(D_n) > 0 \\ U_n & \text{if } Q(D_n) = 0 \end{cases}$

Theorem 2.9. *S typical service time, then $Q(D)$ Markov chain with transition matrix*

$$P_D = \begin{pmatrix} \delta_0 & \delta_1 & \delta_2 & \cdots \\ \delta_0 & \delta_1 & \delta_2 & \cdots \\ 0 & \delta_0 & \delta_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \ \delta_j = \mathbb{E}\left(\frac{(\lambda S)^j}{j!} e^{-\lambda S}\right)$$

– traffic intensity $\rho = \lambda \mathbb{E}(S)$

– M_S —— moment generating function of service time

Theorem 2.10.

(i) *$\rho < 1$, then $Q(D)$ ergodic, unique stationary distribution π with generating function* $G(s) = \sum \pi_j s^j = (1-\rho)(s-1)\frac{M_S(\lambda(s-1))}{s - M_S(\lambda(s-1))}$

(ii) *$\rho > 1$, $Q(D)$ transient*

(iii) *$\rho = 1$, $Q(D)$ null recurrent*

Proof. stationary $\iff$ $\lim_{s\uparrow 1} G(s) = 1$ □

– busy period B —— time during which server continuously occupied

Fact. $\begin{cases} \mathbb{E}(B) < \infty & \text{if } \rho < 1 \\ \mathbb{E}(B) = \infty, \mathbb{P}(B = \infty) = 0 & \text{if } \rho = 1 \\ \mathbb{P}(B = \infty) > 0 & \text{if } \rho > 1 \end{cases}$

– imbedded branching process —— C_2 offspring of C_1 if C_2 join when C_1 being served

Theorem 2.11. $\mathbb{P}(B < \infty) \begin{cases} = 1 & \text{if } \rho \leq 1 \\ < 1 & \text{if } \rho > 1 \end{cases}$

Proof. branching process dies out $\iff$ mean number of offsprings $= \rho < 1$ □

3 Renewals

- X_i i.i.d. —— non-negative, $\mathbb{P}(X_i > 0) = 1$
- $T_n = X_1 + \cdots + X_n$
- renewal process N

Theorem 3.1. $\mathbb{P}(N(t) < \infty) = 1 \iff \mathbb{E}(X_1) > 0$

- F distribution of X
- F_k distribution of T_k

Lemma 3.2. $F_1 = F$, $F_{k+1}(x) = \int_0^x F_k(x-y)dF(y)$

Lemma 3.3. $\mathbb{P}(N(t) = k) = F_k(t) - F_{k+1}(t)$

- renewal function $m(t) = \mathbb{E}(N(t))$

Lemma 3.4. $m(t) = \sum F_k(t)$

Lemma 3.5. $m(t) = F(t) + \int_0^t m(t-x)dF(x)$

Proof. condition on X_1 □

- renewal-type equation $\mu(t) = H(t) + \int_0^t \mu(t-x)dF(x)$, H uniformly bounded

Theorem 3.6.

(i) *$\mu(t) = H(t) + \int_0^t H(t-x)dm(x)$ solution of renewal-type equation*

(ii) *H bounded on finite interval, then μ bounded on finite intervals, unique sol to equation*

- Laplace-Stieltjes transform —— $g^*(\theta) = \int_0^\infty e^{-\theta x}dg(x)$

Fact. *after transform, $\mu^*(\theta) = \frac{H^*(\theta)}{1-F^*(\theta)}$*
if $H = F$, $m^*(\theta) = \frac{F^*(\theta)}{1-F^*(\theta)}$

Theorem 3.7. $\frac{1}{t}N(t) \xrightarrow{a.s.} \frac{1}{\mu}$

Proof. Strong law on $T_{N(t)}$ and $T_{N(t)+1}$ $\square$

Theorem 3.8. $Var(X_1) = \sigma^2$, $\frac{N(t)-\frac{t}{\mu}}{\sqrt{\frac{t\sigma^2}{\mu^3}}} \xrightarrow{d} \mathcal{N}(0,1)$

Theorem 3.9 (Elementary renewal theorem)**.** $\frac{1}{t}m(t) \to \frac{1}{\mu}$

Proof. $M = N(t) + 1$ stopping time for X_i, then $t < T_{N(t)+1}$ with Wald giving lower bound. $t \geq T_{N(t)}$ and truncation for upper bound. $\square$

Lemma 3.10 (Wald's equation)**.** *X_i i.i.d. , finite mean, M stopping time wrt (X_i), $\mathbb{E}(M) < \infty$, then $\mathbb{E}\left(\sum_{i=1}^{M} X_i\right) = \mathbb{E}(X_1)\mathbb{E}(M)$*

Fact. *$N(t)$ not stopping time*

Fact. *$\mathbb{E}_{N(t)+1} \neq \mu$ in general*

- F_X arithmetic with span λ —— X take values $\{m\lambda\}$ a.s.

Theorem 3.11 (Blackwell's renewal theorem)**.**

(i) *X_1 not arithmetic, then $m(t+h) - m(t) \xrightarrow{t\to\infty} \frac{h}{\mu}$ for all h*

(ii) *If X_1 arithmetic, still holds when h multiple of λ*

Theorem 3.12 (Key renewal theorem)**.** *X_1 not arithmetic, g such that*

(i) *$g(t) \geq 0$ for all t*

(ii) $\int_0^\infty g(t)dt < \infty$

(iii) *g non-increasing*

then $\int_0^t g(t-x)dm(x) \to \frac{1}{\mu}\int_0^\infty g(x)dx$

3.1 Excess Life

- excess lifetime $E(t) = T_{N(t)+1} - t$
- current lifetime/age $C(t) = t - T_{N(t)}$
- total lifetime $D(t) = E(t) + C(t) = X_{N(t)+1}$

Theorem 3.13. $\mathbb{P}(E(t) \leq y) = F(t+y) - \int_0^t [1 - F(t+y-x)] dm(x)$

Proof. consider $\mathbb{P}(E(t) > y)$ and condition on X_1 to get renewal-type equation □

Corollary 3.14. $\mathbb{P}(C(t) \geq y) = \begin{cases} 0 & \text{if } y > t \\ 1 - F(t) + \int_0^{t-y} (1 - F(t-x)) dm(x) & \text{if } y \leq t \end{cases}$

Proof. $\mathbb{P}(C(t) \geq y) = \mathbb{P}(E(t-y) > y)$ □

Setting 10. $X = X_1$

Theorem 3.15. *X not arithmetic, $\mu = \mathbb{E}(X) < \infty$, then* $\mathbb{P}(E(t) \leq y) \xrightarrow{t\to\infty} \frac{1}{\mu} \int_0^y (1 - F(x)) dx$

Proof. key renewal and $\mu = \int F(x) dx$ □

Theorem 3.16. *X takes values in $\mathbb{N}$ with span 1, $\mu = \mathbb{E}(X) < \infty$, then $\mathbb{P}(E(n) = k) \to \frac{1}{\mu}\mathbb{P}(X \geq k)$ for $k \in \mathbb{N}$*

Proof. $p_{k,k-1} = 1, p_{1,j} = \mathbb{P}(X = j)$, invariant distribution $\pi_k = \frac{1}{\mu}\mathbb{P}(X \geq k)$ □

3.2 Renewal-reward processes

Setting 11. *$\{(X_i, R_i)\}$ i.i.d.*

- reward process $C(t) = \sum_{i=1}^{N(t)} R_i$
- reward function $c(t) = \mathbb{E}(C(t))$

Theorem 3.17 (Renewal-reward theorem). $0 < \mathbb{E}X < \infty$, $\mathbb{E}|R| < \infty$, *then*

(i) $\frac{C(t)}{t} \xrightarrow{a.s.} \frac{\mathbb{E}R}{\mathbb{E}X}$

(ii) $\frac{c(t)}{t} \to \frac{\mathbb{E}R}{\mathbb{E}X}$

Proof. **(i)** $\frac{C(t)}{t} = \frac{C(t)}{N(t)} \frac{N(t)}{t}$

(ii) **1)** Wald, suffice $t^{-1}\mathbb{E}(R_{N(t)+1}) \to 0$, then condition on X_1, renewal, then bound

2) uniform integrablity, then cvg in prob $\Rightarrow L^1$ convergence

□

Fact. *also work for accumulative reward as long as monotone*

Setting 12. *Continuous Markov Chain, irreducible, mean return time* μ_i

Theorem 3.18. $X(0) = i$, $\mu_i < \infty$, *then*

(i) $\frac{1}{t}\int_0^t \mathbb{1}\{X(s) = i\}\, ds \xrightarrow{a.s.} \frac{1}{\mu_i g_i}$

(ii) $\frac{1}{t}\int_0^t p_{ii}(s) ds \to \frac{1}{\mu_i g_i}$

Proof. process (P_r, Q_r) where P_r r-th passage time, Q_r r-th holding time □

Fact. *proportion of time spent in* i*, and corresponding expectation*

Assumption 1.

(i) *customers arrive one by one, spend waiting time* V_n *time before departure*

(ii) $\exists$ *a.s. finite* T *st* $Q(T) = Q(0) = 0$*, and the distribution the same after time* T

- $X_i = T_i - T_{i-1}$
- cycles $[T_{i-1}, T_i)$
- $P_i = \{Q(t) : t \in [T_{i-1}, T_i]\}$
- N_i number of arrival in $[T_{i-1}, T_i]$
- $\begin{cases} N = N_1 \\ X = T_1 \end{cases}$
- waiting time V_n

Fact. *let* $R = \int_0^X Q(u)du$, $S = \sum^N V_i$

(i) ***long run average queue length*** L***:*** $\frac{1}{t}\int_0^t \xrightarrow{a.s.} \frac{\mathbb{E}R}{\mathbb{E}X}$

(ii) ***long run rate of arrival*** λ***:*** $\frac{N(t)}{t} \xrightarrow{a.s.} \frac{\mathbb{E}N}{\mathbb{E}X}$

(iii) ***long run average waiting time*** W***:*** $\frac{\sum^n V_i}{n} \xrightarrow{a.s.} \frac{\mathbb{E}S}{\mathbb{E}N}$

Theorem 3.19 (Little's theorem)**.** *Under assumption,* $L = \lambda W$

Proof. use above, then suffice $\sum^N V_i = \int_0^X Q(u)du$ $\square$

4 Spatial Poisson processes

- Poisson process with (constant) intensity λ, Π —— $\Pi \subset \mathbb{R}^d$ random countable, $N(A) = |\Pi \bigcap A|$, $A \in \mathcal{B}^d$
  - **(i)** $N(A) \sim Poi(\lambda|A|)$
  - **(ii)** $A_1, \ldots, A_n$ disjoint, then $N(A_1), \ldots, N(A_n)$ independent
- density λ —— non-negative measurable
- mean measure $\Lambda(A) = \int_A \lambda(x)dx$
- non-homogeneous Poisson process with intensity function λ, Π —— $\Pi \subset \mathbb{R}^d$ random countable, $N(A) = |\Pi \bigcap A|$, $A \in \mathcal{B}^d$, $\Lambda(A) < \infty$ for all bounded A
  - **(i)** $N(A) \sim Poi(\Lambda(A))$
  - **(ii)** $A_1, \ldots, A_n$ disjoint, then $N(A_1), \ldots, N(A_n)$ independent

Fact. *measures* Λ *no atoms*

Theorem 4.1 (Superposition theorem)**.** Π', Π'' *independent (non-homogeneous) Poisson process wrt* λ', λ'', *then* $\Pi = \Pi' \bigcup \Pi''$ *(non-homogeneous) Poisson process wrt* $\lambda = \lambda' + \lambda''$

Proof. hard part: show $\Pi' \bigcap \Pi'' = \varnothing$, suffice to consider n-boxes $\square$

Setting 13. $f : \mathbb{R}^d \to \mathbb{R}^s$, Π *non-homogeneous Poisson process,* $\Lambda(f^{-1}\{y\}) = 0$

Theorem 4.2 (Mapping theorem)**.** *let* $\mu(B) = \Lambda(f^{-1}B) = \int_{f^{-1}B} \lambda(x)dx$, $B \in \mathcal{B}^s$, $\mu(B) < \infty$ *for all* B, *then* $f(\Pi)$ *non-homogeneous Poisson process on* $\mathbb{R}^s$ *with mean measure* μ

Proof. hard part: show $f(\Pi)$ a.s. distinct, $\square$

Theorem 4.3 (Conditional property)**.** Π *non-homogeneous Poisson process with* λ, $A \subset \mathbb{R}^d$, $0 < \Lambda(A) < \infty$,
Conditional on $|\Pi \bigcap A| = n$, *then* n *points in* A *have same distribution as* n *points chosen independently according* $\mathbb{Q}(B) = \frac{\Lambda(B)}{\Lambda(A)}$ *where* $B \subset A$, *i.e. density* $\frac{\lambda(x)}{\Lambda(A)}$

Proof. $A_1, \dots, A_k$ partition of A, then calculate $\mathbb{P}(N(A_i) = n_i \mid N(A) = n)$ where $\sum n_i = n$ $\square$

Fact. *used as prood of existence of Poisson process: A_i partition of $\mathbb{R}^d$, then sample n_i from $Poi(\Lambda(A_i))$, then sample according to $\mathbb{Q}$*

Theorem 4.4 (Colouring theorem)**.** *Π non-homogeneous Poisson process with λ, colour $x \in \Pi$ $\begin{cases} \text{green with probability } \gamma(x) \\ \text{scalet with probability } \sigma(x) = 1 - \gamma(x) \end{cases}$, $\begin{cases} \Gamma \text{ set of green points} \\ \Sigma \text{ set of scalet points} \end{cases}$, then Γ, Σ independent Poisson processes with density $\lambda(x)\gamma(x), \lambda(x)\sigma(x)$*

Proof. conditional on $|\Pi \bigcap A| = n$, work with $\mathbb{Q}$, $\begin{cases} \overline{\gamma} = \int_A \gamma(x) d\mathbb{Q} \\ \overline{\sigma} = \int_A \sigma(x) d\mathbb{Q} \end{cases}$ $\square$

Theorem 4.5 (Renyi's theorem)**.** *Π random countable, λ non-negative integrable, $\Lambda(A) < \infty$ for bounded A, if $\mathbb{P}(\Pi \in A = \varnothing) = e^{-\lambda(A)}$ for any finite union of boxes A, then Π Poisson process with λ*