Applied Probability

1 Continuous time Markov Chains

1.1 Poisson process

- Poisson process with intensity λ

  (i) $N(0) = 0$, $N(s) \leq N(t)$ for $s < t$

  (ii) $\mathbb{P}(N(t+h) = n+m | N(t) = n) = \begin{cases} \lambda h + o(h) & \text{if } m = 1 \\ o(h) & \text{if } m > 1 \\ 1 - \lambda h + o(h) & \text{if } m = 0 \end{cases}$

  (iii) $N(t) - N(s)$ independent of $(N(k))_{k \leq s}$

Theorem 1.1. $N(t) \sim Poi(\lambda t)$

Proof. derive differential equation, then generating function □

- $p_j(t) = \mathbb{P}(N(t) = j)$
- Generating function $G(s,t) = \sum p_j(t) s^j$
- Arrival time T_n
- interarrival time U_n

Theorem 1.2.

(i) $U_i \sim Exp(\lambda)$

(ii) U_i *independent*

Proof. use $N(t)$ Poisson □

Fact. $N(t) \geq j \iff T_j \leq t$

- order statistics

Theorem 1.3. $T_1, \ldots, T_n$ *conditional on* $\{N(t) = n\}$ *same as joint distribution of order statistics of* n *i.i.d.* $Uniform[0,t]$

Proof. U to T, then calculate density □

1.2 Birth process

- birth process with birth rates $\lambda_0, \lambda_1, \dots$

  (i) $N(0) = 0$, $N(s) \leq N(t)$ for $s < t$

  (ii) $\mathbb{P}(N(t+h) = n+m | N(t) = n) = \begin{cases} \lambda_n h + o(h) & \text{if } m = 1 \\ o(h) & \text{if } m > 1 \\ 1 - \lambda_n h + o(h) & \text{if } m = 0 \end{cases}$

  (iii) $N(t) - N(s)$ independent of $(N(k))_{k \leq s}$

Example.

(i) ***Poisson process****:* $\lambda_n = \lambda$

(ii) ***Simple birth****:* $\lambda_n = n\lambda$

(iii) ***Simple birth with immigration****:* $\lambda_n = n\lambda + \nu$

- $T_\infty = \lim T_n = \sum^\infty U_i$
- non-explosive / honest —— $\mathbb{P}(T_\infty = \infty) = 1$

Theorem 1.4. *birth process N, $\lambda_n > 0$, then non-explosive* $\iff \sum_n \frac{1}{\lambda_n} = \infty$

Lemma 1.5. *$U_n \sim Exp(\lambda_n)$, independent, then* $\mathbb{P}(T_\infty < \infty) = \begin{cases} 0 & \textit{if } \sum_n \frac{1}{\lambda_n} = \infty \\ 1 & \textit{if } \sum_n \frac{1}{\lambda_n} < \infty \end{cases}$

- forward system of equations: $p'_{ij}(t) = \lambda_{j-1} p_{i,j-1}(t) - \lambda_j p_{ij}(t)$
- backward system of equations: $p'_{ij}(t) = \lambda_i p_{i+1,j}(t) - \lambda_i p_{ij}(t)$

Theorem 1.6.

(i) *forward system has unique solution* $\{p_{ij}(t)\}$

(ii) $\{p_{ij}(t)\}$ *satisfy backward system*

Theorem 1.7. $\{p_{ij}(t)\}$ *unique solution of forward equations,* $\{\pi_{ij}(t)\}$ *any solution of backward equations, then* $p_{ij}(t) \leq \pi_{ij}(t)$

Fact. $\sum_j p_{ij}(t) = 1 \iff \mathbb{P}(T_\infty > t) = 1$

- weak Markov property
- stopping time

– strong Markov property

– right continuity

– stationary independent increments

  (i) $N(t) - N(s)$ only depends on $t - s$

  (ii) $\{N(t_i) - N(s_i)\}$ independent where $s_1 \leq t_1 \leq \cdots \leq s_n \leq t_n$

Setting 1. $(X(t))$ *takes values in countable* S

– Markov property —— $\mathbb{P}(X(t_n) = j | X(t_{n-1}) = i_{n-1}, \ldots, X(t_1) = i_1) = \mathbb{P}(X(t_n) = j | X(t_{n-1}) = i_{n-1})$

– continuous-time Markov chain —— right-continuous, Markov property

– transition probability $p_{ij}(s,t) = \mathbb{P}(X(t) = j | X(s) = i)$

– homogeneous —— $p_{ij}(s,t) = p_{ij}(0, t-s)$

– transition semigroup $(P_t)_{ij} = p_{ij}(t)$

– stochastic semigroup

  (i) $P_0 = I$

  (ii) P_t stochastic —— non-negative entries, row sum 1

  (iii) (Chapman-Kolmogorov) $P_{s+t} = P_s P_t$

Setting 2. $(X(t))$ *homogeneous Markov chain*

Theorem 1.8. P_t *stochastic semigroup*

– $\mathbb{P}_i$ —— probability measure conditional on $X(0) = i$

– $\mathbb{E}_i$

– t-historical —— events given by $\{X(s) : s < t\}$

– t-future —— events given by $\{X(s) : s > t\}$

– stopping time T —— $\{T \leq t\}$ given by $\{X(s) : s \leq t\}$

Theorem 1.9 (Extended Markov property)**.** *H t-historical, F t-future, then* $\mathbb{P}(F | X(t) = j, H) = \mathbb{P}(F | X(t) = j)$

Theorem 1.10 (Strong Markov property)**.** *T stopping time, conditional on $\{T \leq T_\infty\} \bigcap \{X(T) = i\}$, then*

(i) *$(X_{T+u})_u$ continuous Markov chain start at state i*

(ii) *same transition prob*

(iii) *independent to $\{X(s) : s < T\}$*

Setting 3. $X(0) = i$

– $U_0 = \inf \{t : X(t) \neq i\}$

Fact. *right continuous $\Rightarrow U_0 > 0$*

Theorem 1.11.

(i) *$U_0 \sim Exp(g_i)$*

(ii) *U_0 stopping time*

Proof. Extended Markov and homogeneity to deduce memoryless □

– transition matrix $\mathbf{Y} = (y_{ij})$ —— $y_{ij} = \begin{cases} \delta_{ij} & \text{if } g_i = 0 \\ \mathbb{P}_i(X(U_0) = j) & \text{if } g_i > 0 \end{cases}$

– generator $\mathbf{G} = (g_{ij})$ —— $g_{ij} = \begin{cases} g_i y_{ij} & \text{if } j \neq i \\ -g_i & \text{if } j = i \end{cases}$

Fact. $\mathbb{P}(X(t+h) = j | X(t) = i) = g_{ij}h + o(h)$

Fact. $g_{ij} = g_i(y_{ij} - \delta_{ij})$

Theorem 1.12. *$X(0) = i$, then*

(i) *$X(U_0)$ independent of U_0*

(ii) *conditional on $X(U_0) = j$, $X^*(s) = X(U_0 + s)$ continuous-time Markov chain, same transition prob, initial state j, independent to the past*

– T_m

– holding time $U_m = T_{m+1} - T_m$

– jump chain $Y = \{Y_n\}$

– $T_\infty = \lim T_n$

– minimal process

– explode from state i —— $\mathbb{P}_i(T_\infty < \infty) > 0$

Proposition 1.13. *X minimal process, then $\boldsymbol{P}_{s+t} = \boldsymbol{P}_s$*

Proof. may go to $\{\infty\}$ □

Theorem 1.14. *$i \in S$, non-explosive from i if any of the following holds:*

(i) *S finite*

(ii) $\sup_j g_j < \infty$

(iii) *i recurrent in jump chain Y*

Proof. be dominated by Poisson process which is non-explosive □

- irreducible —— $\forall i, j, \exists t > 0, p_{ij}(t) > 0$

Theorem 1.15.

(i) *(Levy dichotomy) X irreducible, then $\forall t > 0$, $p_{ij}(t) > 0$*

(ii) *X irreducible $\iff$ Y irreducible*

Proof. look at jump chain, $g_{i_0} \cdots g_{i_n} > 0$, $p_{i_k, i_{k+1}}(t) > 0$ □

Fact. *birth process not irreducible*

- recurrent —— $\mathbb{P}(\{t : X(t) = i\} \text{ unbounded}) = 1$
- transient —— $\mathbb{P}(\{t : X(t) = i\} \text{ unbounded}) = 0$
- $R_i = \inf \{t > U_0 : X(t) = i\}$
- mean return time $m_i = \mathbb{E}(R_i)$
- positive recurrent / non-null recurrent —— $m_i < \infty$

Theorem 1.16. *continuous-time chain X, jump chain Y*

(i) *$g_i = 0$, then i recurrent for X*

(ii) *$g_i > 0$, then i recurrent for X $\iff$ recurrent for Y*

(iii) *i recurrent $\iff$ $\int p_{ii}(t)dt = \infty$*

(iv) *i transient $\iff$ $\int p_{ii}(t)dt < \infty$*

(v) *X irreducible, then every state recurrent or every state transient*

Proof. main point is no explosion. Interchange summation, then old result. □

- **Forward equation:** $\mathbf{P}_t' = \mathbf{P}_t\mathbf{G}$ with boundary condition $\mathbf{P}_0 = 1$
- **Backward equation:** $\mathbf{P}_t' = \mathbf{G}\mathbf{P}_t$ with boundary condition $\mathbf{P}_0 = 1$

Fact. *If states S finite, then $\boldsymbol{P}_t = e^{t\boldsymbol{G}}$*

- minimal solution —— $p_{ij}(t) \leq \pi_{ij}(t)$
- sub-stochastic —— $\sum_j p_{ij}(t) < 1$

Theorem 1.17. *S countable, X minimal Markov chain with generator $\boldsymbol{G}$, then*

(i) *$\boldsymbol{P}_t$ minimal non-negative solution of backward equation $\boldsymbol{P}_t' = \boldsymbol{G}\boldsymbol{P}_t$ with boundary condition $\boldsymbol{P}_0 = 1$*

(ii) *$\boldsymbol{P}_t$ minimal non-negative solution of forward equation $\boldsymbol{P}_t' = \boldsymbol{P}_t\boldsymbol{G}$*

Proof. **Solution:** condition on $T_1 > t$ or $T_1 \leq t$.
Minimal: reverse argument and induction. □

Fact. *any solution to both equations sub-stochastic*

Fact. *non-explosive $\Rightarrow$ $\boldsymbol{P}_t$ unique solution to both equations*

- measure
- stationary measure —— $\pi = \pi P_t$
- stationary distribution
- unique measure —— unique up to scalar multiplication
- first return time R_i
- $m_i = \mathbb{E}_i(R_i)$

Theorem 1.18. *X irreducible, $|S| \geq 2$*

(i) *some state k positive recurrent, then*

(a) $\exists$ unique stationary distribution π

(b) unique distribution st $\pi\boldsymbol{G} = 0$

(c) all states positive recurrent

(ii) *X non-explosive, $\exists\ \pi$ st $\pi\boldsymbol{G} = 0$, then*

(a) all states positive recurrent

(b) π stationary

(c) $\pi_k = \frac{1}{m_k g_k}$

- $\nu_i = x_i g_i$
- $\mu(k) = (\mu_j(k))_j$ —— $\mu_j(k) = \mathbb{E}_k \left(\int_0^{R_k} \mathbb{1} \left\{ X(s) = j \right\} ds \right)$

Lemma 1.19. *X irreducible, $|S| \geq 2$*

(i) *measure x, then $x\boldsymbol{G} = 0 \iff \nu \boldsymbol{Y} = \nu$*

(ii) *X recurrent, $x\boldsymbol{G} = 0$ unique measure*

(iii) *x measure, $x\boldsymbol{G} = 0$, then $x_j > 0$*

(iv) *X recurrent, $k \in S$, then $\mu(k)\boldsymbol{G} = 0$ and stationary*