Stochastic Financial Models

1 Utility and Mean-Variance analysis

- contingent claims —— r.v. X

- utility function —— non-decreasing

Fact. *Y is preferred to X iff $\mathbb{E}(U(X)) \leq \mathbb{E}(U(Y))$*

- indifferent

- risk neutral

- risk averse

- concave

- strictly concave

Proposition 1.1. *risk averse iff U concave*

- CARA with parameter γ —— $\gamma \in (0, \infty)$, $U(X) = CARA_\gamma(x) = -\exp(-\gamma x)$

- CRRA with parameter R —— $R \in (0,1) \bigcup (1, \infty)$, $U(X) = CRRA_R(x) = \begin{cases} \frac{x^{1-R}}{1-R} & \text{if } x > 0 \\ -\infty & \text{otherwise} \end{cases}$

- CRRA with parameter 1 —— $U(X) = CRRA_1(x) = \begin{cases} \log x & \text{if } x > 0 \\ -\infty & \text{otherwise} \end{cases}$

- constant absolute risk aversion —— CARA

- constant relative risk aversion —— CRRA

Fact (Arrow-Pratt coefficient of absolute risk aversion). *$\omega + X$ preferred to ω iff $\frac{2\mathbb{E}(X)}{\mathbb{E}(X^2)} \geq -\frac{U''(\omega)}{U'(\omega)}$*

Fact (Arrow-Pratt coefficient of relative risk aversion). *$\omega(1+X)$ preferred to ω iff $\frac{2\mathbb{E}(X)}{\mathbb{E}(X^2)} \geq -\frac{\omega U''(\omega)}{U'(\omega)}$*

- available claim $\mathcal{A}$

- reservation bid price $\pi_b(Y)$ —— $\sup \pi$ st $\mathbb{E}(U(X + Y - \pi)) > \mathbb{E}(U(X^*))$

- reservation ask price $\pi_a(Y)$ —— $\inf \pi$ st $\mathbb{E}(U(X - Y + \pi)) > \mathbb{E}(U(X^*))$

Proposition 1.2 (Ask above, bid below). *$\mathcal{A}$ convex, then $\pi_b(Y) \leq \pi_a(Y)$*

Setting 1. *$\mathcal{A}$ affine space, U differentiable, strictly concave*

- marginal price $\pi_m(Y)$ —— $\pi_m(Y) = \frac{\mathbb{E}(U'(X^*)Y)}{\mathbb{E}(U'(X^*))}$

- single-period asset price model

- numeraire

- riskless bond

- interest rate —— $r > -1$

- state-price density ρ —— $S_0^i = \mathbb{E}(S_1^i \rho)$

- wealth ω_0

- portfolio θ

Example. *no bond*

$$
\begin{aligned}
&given & &\mathbb{E}(\theta \cdot S_1) = \theta \cdot \mu, var(\theta \cdot S_1) = \theta^T V \theta \\
&minimize & &var(\theta \cdot S_1) \\
&subject\ to & &\theta \cdot S_0 = \omega_0, \mathbb{E}(\theta \cdot S_1) = \omega_1
\end{aligned}
$$

- mean-variance-efficient frontier —— $\{\theta^*(\omega_1)\}$

- minimum variance portfolio $\theta_{\min^*}$ —— minimise var over ω_1

Example. *with bond*

$$
\begin{aligned}
&minimise & &\theta^T V \theta \\
&subject\ to & &\theta^0 + \theta S_0 = \omega_0, \theta^0(1+r) + \theta\mu = \omega_1
\end{aligned}
$$

Then , $\theta^* = \lambda\theta_m^*$

- market portfolio θ_m^* —— $A(\mu - (1+r)S_0)$, $A = V^{-1}$

Setting 2. S_1 *Gaussian, U CARA*

Example. *no bond*

$$
\begin{aligned}
&maximise & &\mathbb{E}(U(\theta S_1)) \\
&subject\ to & &\theta S_0 = \omega_0
\end{aligned}
$$

Example. *with bond*

$$
\begin{aligned}
&maximise & &\mathbb{E}(U(\bar{\theta}\bar{S}_1)) \\
&subject\ to & &\bar{\theta}\bar{S}_0 = \omega_0
\end{aligned}
$$

then, $\theta^* = \gamma^{-1}\theta_m^*$

- beta/sensitivity β^i —— $\beta^i = \frac{cov(S_1^i, \theta_m^* S_1)}{var(\theta_m^* S_1)}$

- μ^m —— $\theta_m^*\mu$

– S_0^m —— $\theta_m^* S_0$

Proposition 1.3. $\mu^i - (1+r)S_0^i = \beta^i(\mu^m - (1+r)S_0^m)$

- capitalization-weights of the relevant market index

Setting 3.

$S_1 = (1+R)S_0$, $S_1^m = (1+R^m)S_0^m$, $\tilde{\beta}^i = \frac{cov(R^i, R^m)}{var(R^m)}$

Fact. $\mathbb{E}(R^i) = r + \tilde{\beta}^i(\mathbb{E}(R^m) - r)$

2   Martingales

- conditional probability

- conditional expectation given event

- conditional expectation given $\mathcal{G}$, $\mathbb{E}(X|\mathcal{G})$

Theorem 2.1. $\mathcal{G} \subset \mathcal{F}$ *sub-σ-algebra, X integrable, then $\exists$ unique Y (up to a.s.) st*

(i) Y *integrable*

(ii) Y $\mathcal{G}$-*measurable*

(iii) $\mathbb{E}(Y\mathbb{1}_A) = \mathbb{E}(X\mathbb{1}_A)$ *for all* $A \in \mathcal{G}$

Fact. *also true if replace integrable by non-negative*

- $\mathbb{E}(X|Z)$ —— $\mathcal{G} = \sigma(Z)$ for r.v. Z

- $\mathbb{P}(A|\mathcal{G})$ —— $X = \mathbb{1}_A$

Fact. $\mathcal{G} = \sigma(B_n)$ *discrete, then* $\mathbb{E}(X|\mathcal{G}) = \sum \mathbb{E}(X|B_n)\mathbb{1}_{B_n}$ *a.s.*

Proposition 2.2. $\mathcal{G} \subset \mathcal{F}$, X, W *integrable, then*

(i) $\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$

(ii) X $\mathcal{G}$-*measurable, then* $\mathbb{E}(X|\mathcal{G}) = X$ *a.s.*

(iii) X *independent of* $\mathcal{G}$*, then* $\mathbb{E}(X|\mathcal{G}) = E(X)$ *a.s.*

(iv) $X \geq 0$ *a.s. , then* $\mathbb{E}(X|\mathcal{G}) \geq 0$ *a.s.*

(v) $\mathbb{E}(\alpha X + \beta W|\mathcal{G}) = \alpha\mathbb{E}(X|\mathcal{G}) + \beta\mathbb{E}(W|\mathcal{G})$ *a.s.*

Proposition 2.3 (Tower property). $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ *sub-σ-algebra, X integrable, then* $\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = \mathbb{E}(X|\mathcal{H})$ *a.s.*

Proposition 2.4 (Taking out what is known). $\mathcal{G} \subset \mathcal{F}$ *sub-σ-algebra, X integrable, Z $\mathcal{G}$-measurable, ZX integrable, then* $\mathbb{E}(ZX|\mathcal{G}) = Z\mathbb{E}(X|\mathcal{G})$ *a.s.*

Proposition 2.5 (Averaging over independent variables). X_1, X_2 *r.v. in* $(E_1, \mathcal{E}_1), (E_2, \mathcal{E}_2), \mathcal{G} \subset \mathcal{F}$, X_1 $\mathcal{G}$*-measurable,* X_2 *independent of* $\mathcal{G}$, F *non-negative, let* $f = \mathbb{E}(F(\cdot, X_2))$, *then* $\mathbb{E}(F(X_1, X_2)|\mathcal{G}) = f(X_1)$ *a.s.*

- filtration $(\mathcal{F}_n)$

- random process

- (X_n) adapted to $(\mathcal{F}_n)$

- natural filtration $(\mathcal{F}_n^X)$ —— $\mathcal{F}_n^X = \sigma(X_k : 0 \le k \le n)$

- martingale ——

  - (Adapted) X_n $\mathcal{F}_n$-measurable
  - (Intergrable) $\mathbb{E}(|X_n|) < \infty$
  - (Martingale property) $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$ a.s.

- supermartingale —— $\mathbb{E}(X_{n+1}|\mathcal{F}_n) \le X_n$

- submartingale —— $\mathbb{E}(X_{n+1}|\mathcal{F}_n) \ge X_n$

Fact. *Any martingale also martingale in natural filtration (natural filtration smalllest)*

- martingale (continuous-time) —— adapted, integrable, $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$ for all $s \le t$

- (X_t) continuous —— $t \mapsto X_t(\omega)$ continuous for all ω

Example. X_n *i.i.d.* , $\mathcal{F}_0 = \{\varnothing, \Omega\}$, $(\mathcal{F}_n)$ *natural filtration*

- *(additive martingale)* X_1 *inegrable,* $E(X_1) = 0$, $S_0 = 0$, $S_n = \sum^n X_k$

- *(multiplicative martingale)* X_1 *non-negative,* $\mathbb{E}(X_1) = 1$, $Z_0 = 1$, $Z_n = \prod^n Z_k$

Example. (X_n) *Markov chain, countable state space* S, *transition matrix* P, *natural filtration* $(\mathcal{F}_n)$, *bounded/non-negative* f *on* S, *let*

$$Pf(x) = \sum p_{xy} f(y)$$

then if f *subharmonic*

$$\mathbb{E}(f(X_{n+1})|\mathcal{F}_n) = Pf(X_n) \ge f(X_n)$$

then, $(f(X_n))$ *submartingale*

- subharmonic —— $f(x) \le Pf(x)$ for all x

- random time —— $T : \Omega \to \{0, 1, \dots\} \bigcup \{\infty\}$

- stopping time —— $\{T \le n\} \in \mathcal{F}_n$

Theorem 2.6 (Optional stopping). (M_n) *martingale,* T ==bounded==, *stopping time, then* $\mathbb{E}(M_T) = \mathbb{E}(M_0)$

Fact. *Doob's optional sampling theorem basic form*

Theorem 2.7. (M_n) *martingale,* T <mark>*almost surely finite*</mark>*, stopping time, suppose one of the following holds:*

  (i) $|M_n| \le C$ *for all* $n \le T$, C *constant*

  (ii) $\mathbb{E}(T) \le \infty$, $|M_n - M_{n-1}| \le C$ *for all* $n \le T$

Fact. T *stopping time* $\Rightarrow T \wedge n$ *bounded stopping time*

Counter Example. *additive martingale, simple random walk,* $T = \min\{n : S_n = 1\}$, *then almost sure finite as recurrent, but* $\mathbb{E}(T) = \infty$, $\mathbb{E}(S_T) = 1 \ne 0 = S_0$

Counter Example. *multiplicative martingale,* (X_k) *i.i.d.* $,\mathbb{P}(X_1 = 0) = \mathbb{P}(X_1 = 2) = 1/2$, $T = \min\{n : Z_n = 0\}$, *then* $\mathbb{E}(T) = 2$, *but* $\mathbb{E}(Z_T) = 0 \ne 1 = Z_0$

Theorem 2.8. (M_n) *martingale,* T *stopping time, then* $(M_{T \wedge n})$ *martingale*

  – previsible —— H_n $\mathcal{F}_{n-1}$-measurable

  – martingale transform of (M_n) by (H_n) —— $Y_0 = 0$, $Y_n = \sum_1^n H_k(M_k - M_{k-1})$

Theorem 2.9 (Martingale transform). (M_n) *martingale,* (H_n) *bounded, previsible,* (Y_n) *martingale transform, then* (Y_n) *martingale*

Fact. *model stock price as* (M_n), *then*

  (i) *optional stopping* $\Rightarrow$ *expected return* $\mathbb{E}(M_T)$ *the same no matter what stopping time*

  (ii) (H_k) *amount held between time* $k - 1$ *and* k, *no bounded previsible strategy gives expected gain or loss*

3   Pricing contingent claims

  – asset price model $(\bar{S}_n)_{0 \le n \le T}$ with numeraire

  – numeraire (S_n^0) —— $S_n^0 > 0$

  – discounted prices X_n^i —— $X_n^i = S_n^i / S_n^0$

  – $\bar{X}_n = (1, X_n)$

  – interest rate r_n —— $S_n^0 = (1 + r_n)S_{n-1}^0$

  – risky assets (S_n)

  – portfolio $\bar{\theta}_n$

  – self-financing —— $\bar{\theta}_n \bar{S}_n = \bar{\theta}_{n+1} \bar{S}_n$

  – value process (V_n) —— $V_0 = \bar{\theta}_1 \bar{X}_0$, $V_n = \bar{\theta}_n \bar{X}_n$

  – total (discounted) value

  – previsible —— if $\bar{\theta}_n$ $\mathcal{F}_{n-1}$-measurable

Setting 4.

    (i) $(\mathcal{F}_n)$ *filtration generated by* $(\bar{S}_n)$, $\mathcal{F} = \mathcal{F}_T$

    (ii) (S_n) *takes countable values*

    (iii) (S_n^0) *deterministic process*

Proposition 3.1. (θ_n) *preivisible process, then* $\exists \, (\theta_n^0)$ *st*

    (i) (θ_n^0) *previsible*

    (ii) $(\bar{\theta}_n^0)$ *self-financing with initial value* V_0

    (iii) $V_T = V_0 + \sum_1^T \theta_n(X_n - X_{n-1})$

- contingent claim of maturity T —— non-negative $\mathcal{F}_T$-measurable r.v.

- European option

- call with strike price K —— $(S_T - K)^+$

- put with strike price K —— $(S_T - K)^-$

- options

- exotic options —— depending on entire path (S_n)

- barrier options ——

    • knocked out

    • knocked in

- up-and-out call —— $C = \begin{cases} (S_T - K)^+ & \text{if } \max S_n < B \\ 0 & \text{otherwise} \end{cases}$

- down-and-in put —— $C = \begin{cases} (S_T - K)^- & \text{if } \min S_n \leq B \\ 0 & \text{ptherwise} \end{cases}$

- $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$, $\tilde{\mathbb{P}} \sim \mathbb{P}$ —— $\exists \rho$, st

    • $\mathbb{P}(\rho > 0) = 1$

    • $\tilde{\mathbb{P}}(A) = \mathbb{E}(\rho \mathbb{1}_A)$

- density $\rho = d\tilde{\mathbb{P}}/d\mathbb{P}$ for $\tilde{\mathbb{P}}$ wrt $\mathbb{P}$

Fact. $\tilde{\mathbb{E}}(X) = \mathbb{E}(\rho X)$

Fact. $\tilde{\mathbb{P}} \sim \mathbb{P}$ *symmetric and transitive*

Fact. $d\mathbb{P}/d\tilde{\mathbb{P}} = 1/\rho$ *a.s.*

- arbitrage for $(\bar{S}_n)_{0 \leq n \leq T}$ ——

    (i) $(\bar{\theta}_n)_{1 \leq n \leq T}$ previsible, self-financing

     (ii) $V_0 = 0$

     (iii) $V_T \geq 0$ a.s.

     (iv) $V_T > 0$ with positive probability

  – $(\bar{S}_n)$ arbitrage free

Proposition 3.2. *(X_n) martingale $\Rightarrow$ (X_n) arbitrage free*

Fact. *proof can be simpler when (θ_n) bounded*

Setting 5. *single period model, $\mathcal{F}_0 = \varnothing, \Omega$*

Proposition 3.3. *Y r.v. , following equivalent:*

  (i) *arbitrage free (i.e. no θ st $\theta Y \geq 0$ a.s. with $\theta Y > 0$ with positive prob)*

  (ii) *$\exists$ equivalent probability measure $\tilde{\mathbb{P}}$ st Y integrable with $\mathbb{E}(Y) = 0$*

  – equivalent martingale measure $\tilde{\mathbb{P}}$ (risk neutral measure) —— $\tilde{\mathbb{P}} \sim \mathbb{P}$, (X_n) martingale under $\tilde{\mathbb{P}}$

Theorem 3.4 ((1st) Fundamental theorem of asset pricing)**.** *following equivalent:*

  (i) *$(\bar{S}_n)$ arbitrage free*

  (ii) *$(\bar{S}_n)$ has equivalent martingale measure*

Setting 6. *C time-T contingent claim, $D = C/S_T^0$ discounted value*

  – attainable/replicable —— $\exists$ previsible, self-financing $\overline{(\theta_n)}$ st $C = \bar{\theta}_n \bar{S}_T$

Fact. *Alternative def: $\exists V_0$ $\mathcal{F}_0$-measurable, θ_n previsible st $D = V_0 + \sum^T \theta_n (X_n - X_{n-1})$*

  – fair price —— V_0

  – replicating portfolio/hedging portfolio —— $\bar{\theta}_n$

  – $(\bar{S}_n)$ complete —— all contingent claims attainable

Proposition 3.5. *$\mathcal{F}_0 = \{\varnothing, \Omega\}$, $\mathcal{F}_T = \sigma(\bar{S}_1)$*

  (i) *C non-negative, attainable, time-T contingent claim, $\tilde{\mathbb{P}}$ equivalent martingale measure, then fair price $V_0 = \tilde{\mathbb{E}}(D)$, $D = C/S_T^0$*

  (ii) *$(\bar{S}_n)$ complete, numeraire non-random, then at most one equivalent martingale measure*

  – binomial model (Cox-Ross-Rubinstein model) —— interest rate r, parameters $a < b$, R_i i.i.d. with parameter p

    • $S_n^0 = (1 + r)^n$

    • $S_n = S_0 \prod (1 + R_k)$

    • $\begin{cases} \mathbb{P}(R_1 = a) = 1 - p \\ \mathbb{P}(R_1 = b) = p \end{cases}$

Proposition 3.6. *binomial model has arbitrage unless* $r \in (a, b)$

Proposition 3.7. $(\bar{S}_n)$ *binamial model,* $r \in (a, b)$*, define*

(i) $p^* = \frac{r-a}{b-a}$

(ii) *equivalent prob measure* $\frac{d\mathbb{P}^*}{d\mathbb{P}} = \left(\frac{p^*}{p}\right)^{U_T} \left(\frac{1-p^*}{1-p}\right)^{D_T}$ *where* $U_T = (T + S_T)/2, D_T = (T - S_T)/2,$ S_T *number of* b

Then under $\mathbb{P}^*$,

(i) $R_1, \ldots, R_T$ *i.i.d.* , $\begin{cases} \mathbb{P}^*(R_1 = a) = 1 - p^* \\ \mathbb{P}^*(R_1 = b) = p^* \end{cases}$

(ii) (X_n) *martingale under* $\mathbb{P}^*$

Fact. $r = \mathbb{E}^*(R_1)$

Fact. *Binomial model arbitrage free when* $r \in (a, b)$

Setting 7. *if* $C = f(S_0, \ldots, S_T)$*, then*

$$V(C) = \frac{\mathbb{E}^*(C)}{(1+r)^T} = (1+r)^{-T} \sum f(s_0, s_1, \ldots, s_T)\mathbb{P}^*(S_1 = s_1, \ldots, S_T = s_T)$$

Define recursive relation:

$$f_T(s_0, \ldots, s_T) = f(s_0, \ldots, s_T)$$
$$f_n(s_0, \ldots, s_n) = (1 - p^*)f_{n+1}(s_0, \ldots, s_n, (1 + a)s_n) + p^* f_{n+1}(s_0, \ldots, s_n, (1 + b)s_n)$$

Proposition 3.8. $\mathbb{E}^*(f(S_0, \ldots, S_T)|\mathcal{F}_n) = f_n(S_0, \ldots, S_n)$, $\mathbb{E}^*(C) = f_0(S_0)$

Proposition 3.9. *Define*

$$\Delta_n(s_0, \ldots, s_{n-1}) = \frac{f_n(s_0, \ldots, s_{n-1}, (1 + b)s_{n-1}) - f_n(s_0, \ldots, s_{n-1}, (1 + a)s_{n-1})}{(1+r)^{T-n}(b - a)s_{n-1}}$$

then $\theta_n = \Delta_n(S_0, \ldots, S_{n-1})$ *replicating portfolio for* C

Fact. *Binomial model complete when* $r \in (a, b)$

Proposition 3.10. (W_n) *simple random walk with* $\mathbb{P}(W_1 = 1) = p$*, let* $M_T = \max W_n$*, then with* $k \leq T$, $2k - T \leq m \leq k$

$$\mathbb{P}(M_T = m, W_T = 2k - T) = \left(\binom{T}{k - m} - \binom{T}{k - m - 1}\right) p^k(1 - p)^{T-k}$$

Example. *if* $(1+a)(1+b) = 1$*, then* $S_n = S_0(1+b)^{W_n}$*, so can give fair price of* $C = F(S_T, \max S_n)$

4   Dynamic Programming

– state-space E

– action-space A

Setting 8. $F : \{0, \ldots, T-1\} \times E \times A \times [0,1] \to E$, (ϵ_n) *i.i.d.* , $\mathcal{F}_n = \sigma(\epsilon_1, \ldots, \epsilon_n)$

– adapted control $u = (u_n)_{k \leq n \leq T-1}$ —— initial time k, u_n $\mathcal{F}_n$-measurable

Setting 9. *initial state $x \in E$, adapted control u, define $X_k = x$, $X_{n+1} = F(n, X_n, u_n, \epsilon_{n+1})$*
Write $X_n = X_n^u(k, x)$

– expected reward —— $V^u(k, x) = \mathbb{E}\left(\left(\sum_k^{T-1} r(n, X_n^u(k, x), u_n)\right) + R(X_T^u(k, x))\right)$

– reward function r, R —— non-negative measurable function

– value function V —— $V(k, x) = \sup_u V^u(k, x)$

– u optimal control from (k, x) —— $V(k, x) = V^u(k, x)$

Proposition 4.1 (Bellman equation). *Let $Pv(n, x, a) = \mathbb{E}\left(v(n+1, F(n, x, a, \epsilon_{n+1}))\right)$*

$$v(T, x) = R(x)$$
$$v(n, x) = \sup_{a \in A}\{r(n, x, a) + Pv(n, x, a)\} \qquad\qquad n = 0, \ldots, T-1$$

Suppose $\exists a$ st

$$v(n, x) = r(n, x, a(n, x)) + Pv(n, x, a(n, x)) \qquad\qquad n = 0, \ldots, T-1$$

Then,

   (i) $V = v$

  (ii) *optimal control $u_n^* = a(n, X_n^{u*}(k, x))$*

Fact. *Possible variations:*

   (i) r, R *as costs*

  (ii) *mixture of costs and rewards*

 (iii) *time-dependent state-space E_n*

  (iv) *time-and-state-dependent action-space $A_{n,x}$*

– American call —— family of time-T contingent claim $(1+r)^{T-\tau}(S_\tau - K)^+$

– American put —— family of time-T contingent claim $(1+r)^{T-\tau}(S_\tau - K)^-$

Setting 10. (S_n) *binomial model, $r \in (a, b)$*

Fact. *complete $\Rightarrow$ can hedge all C with $\mathbb{E}^*(C) = 0$*

Example (American call). *$\tau = T$ always optimal, American and European calls equivalent*

Fact. *fair price can be founded using Bellman equation*

5 Brownian motion

- Brownian motion ——

  - $B_0 = 0$
  - $(B_{s+t} - B_s) \sim N(0, t)$, independent of $\sigma(B_r : r \leq s)$
  - $t \mapsto B_t(\omega)$ continuous

- Brownian motion starting from x —— $B_0 = x$

- Gaussian process —— $\forall (t_1, \ldots, t_n), (X_{t_1}, \ldots, X_{t_n})$ multivariate normal

Proposition 5.1. (B_t) *continuous process starting from 0, then following equivalent:*

(i) (B_t) *Brownian motion*

(ii) (B_t) *zero mean Gaussian process,* $\mathbb{E}(B_s B_t) = s \wedge t$

Proposition 5.2 (Scaling property)**.** (B_t) *Brownian motion, set* $\tilde{B}_t = c^{-1} B_{c^2 t}$, *then* $(\tilde{B}_t)$ *Brownian motion*

Proposition 5.3. (B_t) *Brownian motion,* (B_t) *exit every finite interval a.s.*

  - $\mathcal{F}_t = \sigma(B_s : s \in [0, t])$

  - stopping time —— $\{T \leq t\} \in \mathcal{F}_t$ for all t

  - $\mathcal{F}_T$ —— $A \in \mathcal{F}_\infty$ st $A \bigcap \{T \leq t\} \in \mathcal{F}_t$

Proposition 5.4 (Strong Markov property)**.** (B_t) *Brownian motion,* T *a.s. finite stopping time. Define* $\tilde{B}_t = B_{T+t} - B_T$, *then*

(i) $(\tilde{B}_t)$ *Brownian motion*

(ii) *independent of* $\mathcal{F}_T$

Proposition 5.5. (B_t) *Brownian motion, define* $T_a = \inf \{t \geq 0 : B_t = a\}$, *then*

(i) T_a *stopping time*

(ii) T_a *almost surely finite*

Theorem 5.6. $(\Omega, \mathcal{F}, \mathbb{P})$ *not discrete,* m *prob measure on* $\mathbb{R}$, *mean 0, variance 1, then* $\exists (B_t), (W_t^{(k)})$ *for all* $k \in \mathbb{N}$ *st*

(i) (B_t) *Brownian motion*

(ii) $(W_{\frac{n}{k}}^{(k)})$ *random walk with distribution* m, $(W_t^{(k)})$ *linear interpolation of values* $\{\frac{n}{k}\}$

(iii) $\frac{W_t^{(k)}}{\sqrt{k}} \to B_t$ *uniformly on compacts in* t *a.s.*

Fact. *combination of Wiener's Theorem and Donsker's Invariance Principle*

  - Wiener measure

Proposition 5.7. *let $T \geq 0$, $c \in \mathbb{R}$, $B = (B_t)_{\{0 \leq t \leq T\}}$ brownian motion, $\tilde{B}_t = B_t + ct$, then*

$\forall$ *measurable set* $A \subset C[0,T]$, $\mathbb{P}(\tilde{B} \in A) = \mathbb{E}(\mathbb{1}_{\{B \in A\}} e^{cB_T - \frac{c^2 T}{2}})$

Fact. *special case of Cameron-Martin theorem*

Proposition 5.8 (Reflection principle). *(B_t) Brownian motion, $a \geq 0$, set $T_a = \inf\{t \geq 0 : B_t = a\}$,*

define $\tilde{B}_t = \begin{cases} B_t & \text{if } t \leq T_a \\ 2a - B_t & \text{if } t > T_a \end{cases}$, *then $(\tilde{B}_t)$ Brownian motion*

  – maximum process —— $M_t = \sup_{\{0 \leq s \leq t\}} B_s$

Fact. *M_t same distribution as $|B_t|$*

Proposition 5.9. *T_a has density $h_a(t) = \frac{a}{\sqrt{2\pi t^3}} e^{-\frac{a^2}{2t}}$*

  – $p_t(x,y)$ —— density of B_t starting at x

  – $p_t^a(x,y) = p_t(x,y) - p_t(x, 2a - y)$

Proposition 5.10. *$x \leq a$, (B_t) Brownian motion with density starting from x, then $\forall$ non-negative measurable f, $\mathbb{E}_x(f(B_t)\mathbb{1}_{\{T_a > t\}}) = \int_\infty^a f(y) p_t^a(x,y) dy$*

6   Black-Scholes model

  – Black-Scholes model —— $S_t^0 = e^{rt}, S_t = S_0 e^{\sigma B_t + \mu t}$

  – price of riskless bond S_t^0

  – interest rate r

  – price of risky asset S_t

  – drift μ

  – volatility σ

Fact. *$(e^{\sigma B_t - \sigma^2 t/2})$ martingale*

Fact. *with $\mu^* = r - \sigma/2$, discounted asset price $(e^{rt} S_t)$ martingale*

Proposition 6.1. *(S_t^0, S_t) Black-Scholes, fix T, consider $\frac{d\mathbb{P}*}{d\mathbb{P}} = e^{\lambda B_T - \lambda^2 T/2}$ where $\sigma\lambda = \mu^* - \mu$, then under $\mathbb{P}^*$, discounted seet price $(e^{-rt} S_t)$ martingale*

Fact. *abuse notation in writing $\mathbb{P}^*$ instead of $\mathbb{P}$*

  – time-T contingent claim C —— $\mathcal{F}_T$-measurable r.v.

  – Black-Scholes price V_0 —— $V_0 = e^{-rT} \mathbb{E}^*(C)$

Fact. *fair price unique*

Example.

(i) $C = S_T$

(ii) $C = K$

   – simple replicable claim —— constant C_0, $0 = t_0 \leq \cdots \leq t_n = T$, θ_k bounded $\mathcal{F}_{t_{k-1}}$-measurable

$$e^{rT}C = C_0 + \sum_1^n \theta_k(X_{t_k} - X_{t_{k-1}})$$

   (Replicating strategy) at time t_{k-1}, buy θ_k, then sell θ_k at time t_k, then buy θ_k bond

Fact. *any simple replicable claim can be replicated for cost C_0 at time 0, $V_0 = C_0$*

Fact (Brownian martingale representation theorem). *every integrable $\mathcal{F}_T$-measurable contingent claim is limit in probability of simple replicable claims*

Setting 11. $S_0 = s$

Fact. $\log S_t = \log s + \sigma B_t + \mu t$

   – $p(t, x, z) = \frac{1}{\sqrt{2\pi t}} e^{\frac{|x-z|^2}{2t}}$

   – $p(\sigma^2 t, x(t), \cdot)$ —— density of $\log S_t$, $x(t) = \log s + \mu t$

   – $\rho(t, s, \cdot)$ —— density of S_t

   – $\dot{\rho}$ —— derivative in first argument

   – ρ' —— derivative in second argument

Fact. $y\rho(t, s, y) = p(\sigma^2 t, x(t), z)$, $z = \log y$

Fact. $\dot{\rho} = \frac{1}{2}\sigma^2 s^2 \rho'' + rs\rho'$

Proposition 6.2. *F on $(0, \infty)$ continuous, polynomial growth, $t \in [0, T], s \in (0, \infty)$*
$V(t, s)$ time-t value of time-T contingent claim $F(S_T)$, conditional on $S_t = s$
$V(t, s) = e^{-r(T-t)}\mathbb{E}^*(F(S_T)|S_t = s) = e^{-r(T-t)}\mathbb{E}(F(se^{\sigma B_{T-t} + \mu^*(T-t)}))$, then

(i) *V continous on $(0, T) \times (0, \infty)$*

(ii) *$V \in C^{1,2}$ on $(0, T) \times (0, \infty)$*

(iii) *(Black Scholes PDE) $\mathcal{L}V = \dot{V} + \frac{1}{2}\sigma^2 s^2 V'' + rsV' - rV$ with $V(\cdot, T) = F$*

Setting 12 (Binomial approximation to BS). *Consider convergence of random walk to Brownian motion, special case $(W_{n/k}^{(k)})$ simple symmetric random walk on $\{-1, 1\}$*

   – $1 + a_k = \exp\left(-\frac{\sigma}{\sqrt{k}} + \frac{\mu}{k}\right)$

   – $1 + b_k = \exp\left(\frac{\sigma}{\sqrt{k}} + \frac{\mu}{k}\right)$

   – $1 + r_k = \exp\left(\frac{r}{k}\right)$

- $S_t^{(k)} = S_0 \exp\left(\frac{\sigma W_t^{(k)}}{\sqrt{k}} + \mu t\right)$

- $S_t = S_0 \exp\left(\sigma B_t + \mu t\right)$

- $S_t^{(k)0} = S_t^0 = \exp(rt)$

Fact.

  (i) (S_t^0, S_t) *Black-Scholes of drift μ, volatility σ, interest rate r*

 (ii) $(S_{n/k}^{(k)0}, S_{n/k}^{(k)})$ *binomial model of parameters $a_k < r_k < b_k$, $p = \frac{1}{2}$*

(iii) $S_t^{(k)} \to S_t$ *uniformly on compacts in t a.s.*

  - $\mathbb{P}^{(k)*}$ —— *martingale measures for binomial model*

  - $\mathbb{P}^*$ —— *martingale measures for Black-Scholes model*

Fact. $\mathbb{E}^{(k)*}(G(S^{(k)})) \to \mathbb{E}^*(G(S))$, *so can approximate fair price using Binomial model*

Setting 13. $\mu = \mu^* = r - \frac{\sigma^2}{2}$ *for convenience*

  - *expression* $C = F(B)$ —— F *on* $C[0, T]$

  - *terminal-value option* $C = f(B_T)$

Fact. *for* $C = f(B_T)$, $V_0 = e^{-rT} \int f\left(\sqrt{T}y\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$

Fact. *May not be efficient way:*

  (i) *multiple assets* —— *exponentially growth computational cost*

 (ii) *want to compute pricing surface*

  - *pricing surface* —— $V(t, s) = e^{-r(T-t)} \mathbb{E}^*(C | S_t = s)$

Setting 14. *terminal-value option* $C = g(S_T)$, g *continuous, no more than linear growth*

  - $f(x) = g(e^{\sigma x})$

  - $u(t, x) = \mathbb{E}_x(f(B_t))$

Fact. $V(t, s) = e^{-r(T-t)} \mathbb{E}(g(se^{\sigma B_{T-t} + \mu(T-t)})) = e^{-r(T-t)} u\left(T - t, \frac{\log s + \mu(T-t)}{\sigma}\right)$

  - $p_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{\frac{-(y-x)^2}{2t}}$

Fact. $u(t, x) = \int p_t(x, y) f(y) dy$

Fact. $\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2 p}{\partial x^2} \Rightarrow \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}$

Setting 15. *assume can accurately approximate $u(t, \pm L)$ for all t*

- grid $\{(ik, jh)\} \subset [0, T] \times [-L, L]$ —— $k = \frac{T}{N}, h = \frac{L}{M}$

- grid points U_j^i —— idea $U_j^i \approx u(ik, jh)$

- FTCS (forward-in-time, central-in-space) —— $\frac{U_j^{i+1} - U_j^i}{k} = \frac{U_{j-1}^i - 2U_j^i + U_{j+1}^i}{2h^2}$

  - explicit
  - first-order in time

- BTCS (backward-in-time, central-in-space) —— $\frac{U_j^{i+1} - U_j^i}{k} = \frac{U_{j-1}^{i+1} - 2U_j^{i+1} + U_{j+1}^{i+1}}{2h^2}$

  - require to solve $2M \times 2M$ matrix inversion
  - better stability
  - first-order in time

- Crack-Nicolson —— $\frac{U_j^{i+1} - U_j^i}{k} = \frac{1}{2} \left(\frac{U_{j-1}^i - 2U_j^i + U_{j+1}^i}{2h^2} + \frac{U_{j-1}^{i+1} - 2U_j^{i+1} + U_{j+1}^{i+1}}{2h^2} \right)$

  - require to solve $2M \times 2M$ matrix inversion
  - better stability
  - second-order in time

- Monte Carlo ——

  - time-step $k = \frac{T}{N}$
  - (i) $(B_t^{(N)} : t = ik)$ linear interpolation of random walk with step distribution $N(0, k)$
    (ii) $(X_t : t = ik)$ simple symmetric random walk with step-size $h = \sqrt{k}$
  - generate sample $(B^{(N),i})$
  - (Idea) $\frac{1}{n} \sum F(B^{(N),i}) \approx \mathbb{E}(F(B^{(N)})) \to \mathbb{E}(F(B))$

- fair price for European call $EC(x, K, \sigma, r, T) = \mathbb{E}^*(e^{-rT}(S_T - K)^+)$

- $\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$

- $\Phi(a) = \int_\infty^a \phi(y) dy$

- $\bar{\Phi}(a) = 1 - \Phi(a)$

- $d^\pm = \frac{\log\left(\frac{x}{K}\right) + rT}{\sigma\sqrt{T}} \pm \frac{\sigma\sqrt{T}}{2}$

Proposition 6.3 (Black-Scholes formula). $EC(x, K, \sigma, r, T) = x\Phi(d^+) - e^{-rT} K\Phi(d^-)$

- put-call parity —— $(S_T - K)^+ - (S_T - K)^- = S_T - K$

- fair price for forward contract —— $\mathbb{E}^*(e^{-rT}(S_T - K)) = x - e^{-rT} K$

- fair price for European put $EP(x, K, \sigma, r, T) = e^{-rT} K\bar{\Phi}(d^-) - x\bar{\Phi}(d^+)$

Setting 16. $v(x) = v(x, C, \sigma, r, T) = \mathbb{E}^*(e^{-rT} C)$

– **Sensitivities**

 – Delta $\Delta = \frac{\partial v}{\partial x}$

 – Gamma $\Gamma = \frac{\partial^2 v}{\partial x^2}$

 – Vega $\mathcal{V} = \frac{\partial v}{\partial \sigma}$

 – Rho $\rho = \frac{\partial v}{\partial r}$

Example. *C European call,* $\Delta = \Phi(d^+), \mathcal{V} = x\phi(d^+)\sqrt{T}$

Proposition 6.4.

(*i*) $\sigma \mapsto EC(x, K, \sigma, r, T)$ *increasing bijection*

(*ii*) $\lim_{\sigma \to 0} EC(x, K, \sigma, r, T) = (x - e^{-rT})^+$

(*iii*) $\lim_{\sigma \to \infty} EC(x, K, \sigma, r, T) = x$

 – implied volatility $\sigma_{implied}(K, T)$ —— $EC(S_0, K, \sigma_{implied}(K, T), r, T) = EC_{market}(K, T)$

Example. *up-and-out call* $C = h(S_T)\mathbb{1}_{\left\{\sup_{0 \le t \le T} S_t < A\right\}}, h(s) = (s - K)^+, A \ge \max\{S_0, K\}$