

Applied Probability

1 Continuous time Markov Chains

- right continuous — $\forall t, \exists \epsilon, X_t(\omega) = X_{t+s}(\omega)$ for all $s \in [0, \epsilon]$
- finite dimension marginals $\mathbb{P}(X_{t_1} = x_1, \dots, X_{t_n} = x_n)$

Fact. *process can be determined from the finite dimension marginals*

- Memoryless property $\mathbb{P}(S > t + s | S > s) = \mathbb{P}(S > t)$

Theorem 1.1. *Memoryless iff exponential distribution*

1.1 Poisson process

- Poisson process with intensity λ
 - (i) $N(0) = 0, N(s) \leq N(t)$ for $s < t$
 - (ii) $\mathbb{P}(N(t+h) = n+m | N(t) = n) = \begin{cases} \lambda h + o(h) & \text{if } m = 1 \\ o(h) & \text{if } m > 1 \\ 1 - \lambda h + o(h) & \text{if } m = 0 \end{cases}$
 - (iii) $N(t) - N(s)$ independent of $(N(k))_{k \leq s}$

Theorem 1.2. $N(t) \sim \text{Poi}(\lambda t)$

Proof. derive differential equation, then generating function □

- $p_j(t) = \mathbb{P}(N(t) = j)$
- Generating function $G(s, t) = \sum p_j(t) s^j$
- Arrival time T_n
- interarrival time U_n

Theorem 1.3.

- (i) $U_i \sim \text{Exp}(\lambda)$
- (ii) U_i independent

Proof. use $N(t)$ Poisson

□

Fact. $N(t) \geq j \iff T_j \leq t$

– order statistics

Theorem 1.4. T_1, \dots, T_n conditional on $\{N(t) = n\}$ same as joint distribution of order statistics of n i.i.d. $\text{Uniform}[0, t]$

Proof. U to T , then calculate density

□

Theorem 1.5. (X_n) increasing right-continuous, taking values $\{0, 1, \dots\}$, $X_0 = 0$, then following equivalent:

(i) holding times $S_i \sim \text{Exp}(\lambda)$ i.i.d. jump chain $Y_n = n$,
(Sousi defined X Poisson process in this manner)

(ii) (infinitesimal) X independent increments, $h \downarrow 0$ uniformly in t ,
$$\begin{cases} \mathbb{P}(X_{t+h} - X_t = 1) = \lambda h + o(h) \\ \mathbb{P}(X_{t+h} - X_t = 0) = 1 - \lambda h + o(h) \end{cases}$$

(iii) X has independent, stationary increments, $X_t \sim \text{Poi}(\lambda t)$

Theorem 1.6 (Superposition). X, Y independent Poisson process, with parameters λ, μ , then $Z_t = X_t + Y_t$ Poisson process with parameters $\lambda + \mu$

Proof. infinitesimal

□

Theorem 1.7 (Thining). X Poisson process with parameters λ , $(Z_i) \sim \text{Bernoulli}(p)$ i.i.d., Y jumps $\iff X$ jumps and $Z_{X_t} = 1$, then Y Poisson process of parameter λp , $X - Y$ independent Poisson process of parameter $\lambda(1 - p)$

Proof. infinitesimal for Poisson process, independence follows from expanding $\mathbb{P}(Y_t = n, X_t - Y_t = m)$ (suffice to prove independence using finite dimension marginals)

□

1.2 Birth process

– birth process with birth rates $\lambda_0, \lambda_1, \dots$

(i) $N(0) = 0$, $N(s) \leq N(t)$ for $s < t$

$$(ii) \mathbb{P}(N(t+h) = n+m | N(t) = n) = \begin{cases} \lambda_n h + o(h) & \text{if } m = 1 \\ o(h) & \text{if } m > 1 \\ 1 - \lambda_n h + o(h) & \text{if } m = 0 \end{cases}$$

(iii) $N(t) - N(s)$ independent of $(N(k))_{k \leq s}$

Example.

(i) *Poisson process:* $\lambda_n = \lambda$

(ii) *Simple birth:* $\lambda_n = n\lambda$

(iii) *Simple birth with immigration:* $\lambda_n = n\lambda + \nu$

Proposition 1.8. $T_k \sim \text{Exp}(q_k)$ independent, $0 < q = \sum q_k < \infty$, $T = \inf_k T_k$, then

(i) infimum attained at unique K with probability 1

(ii) T, K independent

(iii) $T \sim \text{Exp}(q)$, $\mathbb{P}(K = k) = \frac{q_k}{q}$

– $T_\infty = \lim T_n = \sum^\infty U_i$

– non-explosive / honest — $\mathbb{P}(T_\infty = \infty) = 1$

Theorem 1.9. birth process N , $\lambda_n > 0$, then non-explosive $\iff \sum_n \frac{1}{\lambda_n} = \infty$

Lemma 1.10. $U_n \sim \text{Exp}(\lambda_n)$, independent, then $\mathbb{P}(T_\infty < \infty) = \begin{cases} 0 & \text{if } \sum_n \frac{1}{\lambda_n} = \infty \\ 1 & \text{if } \sum_n \frac{1}{\lambda_n} < \infty \end{cases}$

– forward system of equations: $p'_{ij}(t) = \lambda_{j-1} p_{i,j-1}(t) - \lambda_j p_{ij}(t)$

– backward system of equations: $p'_{ij}(t) = \lambda_i p_{i+1,j}(t) - \lambda_i p_{ij}(t)$

Theorem 1.11.

(i) forward system has unique solution $\{p_{ij}(t)\}$

(ii) $\{p_{ij}(t)\}$ satisfy backward system

Theorem 1.12. $\{p_{ij}(t)\}$ unique solution of forward equations, $\{\pi_{ij}(t)\}$ any solution of backward equations, then $p_{ij}(t) \leq \pi_{ij}(t)$

Fact. $\sum_j p_{ij}(t) = 1 \iff \mathbb{P}(T_\infty > t) = 1$

- weak Markov property
- stopping time
- strong Markov property
- right continuity
- stationary independent increments
 - (i) $N(t) - N(s)$ only depends on $t - s$
 - (ii) $\{N(t_i) - N(s_i)\}$ independent where $s_1 \leq t_1 \leq \dots \leq s_n \leq t_n$

1.3 Continuous Markov Chain

Setting 1. $(X(t))$ takes values in countable S

- Markov property —
 - $\mathbb{P}(X(t_n) = j | X(t_{n-1}) = i_{n-1}, \dots, X(t_1) = i_1) = \mathbb{P}(X(t_n) = j | X(t_{n-1}) = i_{n-1})$
- continuous-time Markov chain — right-continuous, Markov property
- transition probability $p_{ij}(s, t) = \mathbb{P}(X(t) = j | X(s) = i)$
- homogeneous — $p_{ij}(s, t) = p_{ij}(0, t - s)$
- transition semigroup $(P_t)_{ij} = p_{ij}(t)$
- stochastic semigroup
 - (i) $P_0 = I$
 - (ii) P_t stochastic — non-negative entries, row sum 1
 - (iii) (Chapman-Kolmogorov) $P_{s+t} = P_s P_t$

Setting 2. $(X(t))$ homogeneous Markov chain

Theorem 1.13. P_t stochastic semigroup

- \mathbb{P}_i — probability measure conditional on $X(0) = i$
- \mathbb{E}_i
- t -historical — events given by $\{X(s) : s < t\}$
- t -future — events given by $\{X(s) : s > t\}$
- stopping time T — $\{T \leq t\}$ given by $\{X(s) : s \leq t\}$

Theorem 1.14 (Extended Markov property). H t -historical, F t -future, then $\mathbb{P}(F | X(t) = j, H) = \mathbb{P}(F | X(t) = j)$

Theorem 1.15 (Strong Markov property). T stopping time, conditional on $\{T \leq T_\infty\} \cap \{X(T) = i\}$, then

- (i) $(X_{T+u})_u$ continuous Markov chain start at state i
- (ii) same transition prob
- (iii) independent to $\{X(s) : s < T\}$

Setting 3. $X(0) = i$

– $U_0 = \inf \{t : X(t) \neq i\}$

Fact. right continuous $\Rightarrow U_0 > 0$

Theorem 1.16.

- (i) $U_0 \sim \text{Exp}(g_i)$
- (ii) U_0 stopping time

Proof. Extended Markov and homogeneity to deduce memoryless □

- transition matrix $\mathbf{Y} = (y_{ij})$ — $y_{ij} = \begin{cases} \delta_{ij} & \text{if } g_i = 0 \\ \mathbb{P}_i(X(U_0) = j) & \text{if } g_i > 0 \end{cases}$
- generator $\mathbf{G} = (g_{ij})$ — $g_{ij} = \begin{cases} g_i y_{ij} & \text{if } j \neq i \\ -g_i & \text{if } j = i \end{cases}$

Fact. $\mathbb{P}(X(t+h) = j | X(t) = i) = g_{ij}h + o(h)$

Fact. $g_{ij} = g_i(y_{ij} - \delta_{ij})$

Theorem 1.17. $X(0) = i$, then

- (i) $X(U_0)$ independent of U_0
- (ii) conditional on $X(U_0) = j$, $X^*(s) = X(U_0 + s)$ continuous-time Markov chain, same transition prob, initial state j , independent to the past

- T_m
- holding time $U_m = T_{m+1} - T_m$
- jump chain $Y = \{Y_n\}$
- $T_\infty = \lim T_n$
- minimal process
- explode from state i — $\mathbb{P}_i(T_\infty < \infty) > 0$

Proposition 1.18. X minimal process, then $P_{s+t} = P_s$

Proof. may go to $\{\infty\}$ □

Theorem 1.19. $i \in S$, non-explosive from i if any of the following holds:

- (i) S finite
- (ii) $\sup_j g_j < \infty$
- (iii) i recurrent in jump chain Y

Proof. be dominated by Poisson process which is non-explosive □

- irreducible — $\forall i, j, \exists t > 0, p_{ij}(t) > 0$

Theorem 1.20.

- (i) (Levy dichotomy) X irreducible, then $\forall t > 0, p_{ij}(t) > 0$
- (ii) X irreducible $\iff Y$ irreducible

Proof. look at jump chain, $g_{i_0} \cdots g_{i_n} > 0, p_{i_k, i_{k+1}}(t) > 0$ □

Fact. birth process not irreducible

- $T_A = \inf \{t > 0 : X_t \in A\}$
- $H_A = \inf \{n \geq 0 : Y_n \in A\}$
- hitting probability $h_A(x) = \mathbb{P}_x(T_A < \infty)$
- expected hitting time $k_A(x) = \mathbb{E}_x(T_A)$

Theorem 1.21. $(h_A(x))_x$ minimal non-negative solution to

$$\begin{cases} h_A(x) = 1 & \forall x \in A \\ Qh_A(x) = \sum_y q_{xy} h_A(y) = 0 & \forall x \notin A \end{cases}$$

Theorem 1.22. $q_x > 0 \forall x \notin A$, then $k_A(x)$ minimal non-negative solution to

$$\begin{cases} k_A(x) = 0 & \forall x \in A \\ Qk_A(x) = \sum_y q_{xy} k_A(y) = -1 & \forall x \notin A \end{cases}$$

- recurrent — $\mathbb{P}(\{t : X(t) = i\} \text{ unbounded}) = 1$
- transient — $\mathbb{P}(\{t : X(t) = i\} \text{ unbounded}) = 0$
- $R_i = \inf \{t > U_0 : X(t) = i\}$
- mean return time $m_i = \mathbb{E}(R_i)$
- positive recurrent / non-null recurrent — $m_i < \infty$

Theorem 1.23. *continuous-time chain X , jump chain Y*

- (i) $g_i = 0$, then i recurrent for X
- (ii) $g_i > 0$, then i recurrent for $X \iff$ recurrent for Y
- (iii) i recurrent $\iff \int p_{ii}(t)dt = \infty$
- (iv) i transient $\iff \int p_{ii}(t)dt < \infty$
- (v) X irreducible, then every state recurrent or every state transient

Proof. main point is no explosion. Interchange summation, then old result. \square

- **Forward equation:** $P'_t = P_t G$ with boundary condition $P_0 = 1$
- **Backward equation:** $P'_t = G P_t$ with boundary condition $P_0 = 1$

Fact. *If states S finite, then $P_t = e^{tG}$*

- minimal solution — $p_{ij}(t) \leq \pi_{ij}(t)$
- sub-stochastic — $\sum_j p_{ij}(t) < 1$

Theorem 1.24. *S countable, X minimal Markov chain with generator G , then*

- (i) P_t minimal non-negative solution of backward equation $P'_t = G P_t$ with boundary condition $P_0 = 1$
- (ii) P_t minimal non-negative solution of forward equation $P'_t = P_t G$

Proof. Solution: condition on $T_1 > t$ or $T_1 \leq t$.
Minimal: reverse argument and induction. \square

Fact. *any solution to both equations sub-stochastic*

Fact. *non-explosive $\Rightarrow P_t$ unique solution to both equations*

- measure

- stationary measure — $\pi = \pi P_t$
- stationary distribution
- unique measure — unique up to scalar multiplication
- first return time R_i
- $m_i = \mathbb{E}_i(R_i)$

Theorem 1.25. X irreducible, $|S| \geq 2$

- (i) some state k positive recurrent, then
 - (a) \exists unique stationary distribution π
 - (b) unique distribution st $\pi G = 0$
 - (c) all states positive recurrent
- (ii) X non-explosive, $\exists \pi$ st $\pi G = 0$, then
 - (a) all states positive recurrent
 - (b) π stationary
 - (c) $\pi_k = \frac{1}{m_k g_k}$

Proof. (i) use 1.26(iv) $\pi = \mu(k)/m_k$, then uniqueness of measure \Rightarrow all state non-null

(ii) $\nu' = \frac{\pi_i g_i}{\pi_k g_k}$, then $\rho(k) \leq \nu'$ from discrete MC

□

- $\nu_i = x_i g_i$
- $\mu(k) = (\mu_j(k))_j$ — $\mu_j(k) = \mathbb{E}_k \left(\int_0^{R_k} \mathbb{1} \{X(s) = j\} ds \right)$
- $\rho(k) = (\rho_j(k))$ — mean visit to j starting from k in jump chain Y

Lemma 1.26. X irreducible Markov chain, $|S| \geq 2$

- (i) measure x , then $xG = 0 \iff \nu Y = \nu$
- (ii) X recurrent, $xG = 0$ unique measure
- (iii) x measure, $xG = 0$, then $x_j > 0$
- (iv) X recurrent, $k \in S$, then $\mu(k)G = 0$ and stationary

Proof. (i) expand

(ii) $\nu Y = \nu$, then uniqueness from discrete MC

(iii) $\mu_j(k) = \frac{1}{g_j} \rho_j(k)$, then $\rho(k)Y = \rho(k)$ from discrete MC

(iv) strong Markov to shift time t

□

Fact. X non-explosive, then $R_k = \sum_j \int_0^{R_k} \mathbb{1}\{X(s) = j\} ds$

Fact. X irreducible, \exists more than one stationary distribution, then X explosive

Theorem 1.27 (Markov chain limit theorem). X irreducible, non-explosive

(i) if \exists stationary distribution π , then

(a) π unique

(b) $p_{ij}(t) \rightarrow \pi_j$

(ii) if no stationary distribution, then $p_{ij}(t) \rightarrow 0$

Proof. skeleton $Z_n = X(nh)$

□

Lemma 1.28. X minimal, then $|p_{ij}(t+u) - p_{ij}(t)| \leq 1 - e^{-g_i u}$

1.4 Reversibility

Theorem 1.29. X irreducible, non-explosive, with invariant distribution π , let $X_0 \sim \pi$, fix T , $\hat{X}_t = X_{T-t}$, then

(i) \hat{X} Markov with generator \hat{Q} and invariant distribution π , $\pi(x)\hat{q}_{xy} = \pi(y)q_{yx}$

(ii) \hat{X} irreducible, non-explosive

Proof. expand $\mathbb{P}(\hat{X}_{t_0} = x_0, \dots, \hat{X}_{t_n} = x_n)$, then \hat{P} satisfies Komogorov backward with \hat{Q} , then minimal, easy to show irreducible, finally $\hat{p}_{xy}(t) = \mathbb{P}_x(\hat{X}_t = y, t < \hat{\zeta})$ where ζ explosion time □

– reversible — $(X_t), (X_{T-t})$ same distribution

– detailed balanced — $\lambda(x)q_{xy} = \lambda(y)q_{yx}$

Lemma 1.30. detail balanced $\Rightarrow \lambda$ invariant measure

Theorem 1.31. X irreducible, non-explosive, $X_0 \sim \pi$, then
detail balanced $\iff (X_t)$ reversible

Lemma 1.32. π invariant for birth-death chain \iff detail balanced

1.5 Ergodic theorem

- long run proportion of time spends at x — $\frac{1}{t} \int_0^t \mathbb{1}(X_s = x) ds$

Theorem 1.33. X irreducible, then

(i) $\frac{1}{t} \int_0^t \mathbb{1}(X_s = x) ds \xrightarrow{a.s.} \frac{1}{m_x g_x}$

(ii) if π invariant, f bounded, then $\frac{1}{t} \int_0^t f(X_s) ds \xrightarrow{a.s.} \sum_x f(x) \pi(x)$

1.6 Birth-death process and imbedding

- birth rate $\lambda_0, \lambda_1, \dots$
- death rate μ_1, μ_2, \dots
- birth-death process

Theorem 1.34. X birth-death process, generator G

(i) measure $x_n = \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n} x_0$ satisfies $\mathbf{x}G = 0$

(ii) \exists distribution π satisfies $\pi G = 0 \iff \sum \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n} < \infty$

(iii) if $\sum \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n} (\lambda_n + \mu_n) < \infty$, then π stationary

Proof. (i) solve $\mathbf{x}G = 0$

(ii) trivial

(iii) condition for jump chain Y recurrent, then non-explosive

□

Example.

- **Pure birth** $\mu_n = 0$
- **Simple death with immigration** $\lambda_n = \lambda, \mu_n = n\mu$

Theorem 1.35. $X(t)$ asymptotically $\text{Poi}(\rho) = \text{Poi}\left(\frac{\lambda}{\mu}\right)$

- **Simple birth-death** $\lambda_n = n\lambda, \mu_n = n\mu, X(0) = I$

Fact. state 0 absorbing

Theorem 1.36. $G(s, t) = \mathbb{E}(s^{X(t)}) = \begin{cases} \left(\frac{\lambda t(1-s) + s}{\lambda t(1-s) + 1} \right)^I & \text{if } \mu = \lambda \\ \left(\frac{\mu(1-s) - (\mu - \lambda s)e^{-t(\lambda - \mu)}}{\lambda(1-s) - (\mu - \lambda s)e^{-t(\lambda - \mu)}} \right)^I & \text{if } \mu \neq \lambda \end{cases}$

Proof. Forward equation □

Fact. *non-explosive as $\sum p_j(t) = G(1, t) = 1$*

Fact. $\mathbb{E}_I(X(t)) \rightarrow \begin{cases} 0 & \text{if } \rho < 1 \\ \infty & \text{if } \rho > 1 \end{cases}$

- *extinction probability $\eta(t) = \mathbb{P}_I(X(t) = 0)$*

Corollary 1.37. $\eta(t) \rightarrow \begin{cases} 1 & \text{if } \rho \leq 1 \\ \rho^{-I} & \text{if } \rho > 1 \end{cases}$

- imbedded random walk — jump chain Y with parameter $\frac{\lambda}{\lambda+\mu}$, absorbing at 0
- imbedded branching process — lives $Exp(\lambda+\mu)$, then born n individuals where $\begin{cases} p_0 = \mathbb{P}(n=0) = \frac{\mu}{\lambda+\mu} \\ p_2 = \mathbb{P}(n=2) = \frac{\lambda}{\lambda+\mu} \end{cases}$
- age-dependent branching process
- age density function $f_T(u) = (\lambda + \mu)e^{-(\lambda+\mu)u}$
- family-size generating function $G(s) = \frac{\mu+\lambda s^2}{\mu+\lambda} = p_0 + p_2 s^2$

2 Queues

- interarrival time X_n with common distribution F_X
- service time S_n with common distribution F_S
- n -th customer arrival time $T_n = \sum X_i$
- length of queue $Q(t)$
- $A/B/s$ — $F_X/F_S/\text{\#servers}$

Example.

- $D(d)$ — *deterministic*
- $M(\lambda)$ — $Exp(\lambda)$ (*Markovian*)
- $\Gamma(\lambda, k)$
- G — *general*

Example.

- $M/M/1$
- $M/D/1$
- $G/G/1$
- traffic intensity $\rho = \frac{\mathbb{E}(S)}{\mathbb{E}(X)}$

2.1 M/M/1

Setting 4. $M(\lambda)/M(\mu)/1$, $\lambda_n = \lambda$, $\mu_n = \mu$

Fact. $\rho = \frac{\lambda}{\mu}$

Theorem 2.1.

(i) if $\rho < 1$, then $\mathbb{P}(Q(t) = n) \rightarrow (1 - \rho)\rho^n = \pi_n$

(ii) if $\rho \geq 1$, then $\mathbb{P}(Q(t) = n) \rightarrow 0$

Fact. can define underlying discrete random walk $Q_{n+1} = \begin{cases} Q_n + 1 & \text{with probability } \frac{\lambda}{\lambda + \mu} = \frac{\rho}{1 + \rho} \\ Q_n - 1 & \text{with probability } \frac{\mu}{\lambda + \mu} = \frac{1}{1 + \rho} \end{cases}$
for $n \geq 1$, and $\mathbb{P}(Q_{n+1} = 1 | Q_n = 0) = 1$

Fact. Q_n is $\begin{cases} \text{positive recurrent} & \text{if } \rho < 1 \\ \text{null recurrent} & \text{if } \rho = 1 \\ \text{transient} & \text{if } \rho > 1 \end{cases}$

– waiting time of customer arrived at time t , W

Theorem 2.2. $\rho < 1$, queue in equilibrium, then $W \sim \text{Exp}(\mu - \lambda)$

Fact. expected queue length at equilibrium = $\frac{\lambda}{\lambda + \mu}$

2.2 M/M/ ∞

Setting 5. $\begin{cases} q_{i,i+1} = \lambda \\ q_{i,i-1} = i\mu \end{cases}$

Theorem 2.3.

(i) $Q(t)$ positive recurrent

(ii) invariant distribution $\pi \sim \text{Poi}(\rho)$

Proof. solve detail balanced for invariant, coupling to prove non-explosive □

Setting 6. $M/M/1$ queue, $\rho < 1$

– D_t — number of customers have departed queue up to time t

Theorem 2.4 (Burke's theorem).

(i) At equilibrium, $D_t \sim \text{Poi}(\lambda)$

(ii) X_t independent from $(D_s : s \leq t)$

Proof. (i) fix T , time reversal, then Poisson process for all T , use independent increment criterion.

(ii) X_0 independent to $[0, T]$, then reverse

□

2.3 Queues in tandem

Setting 7. two $M/M/1$ with λ, μ_1, μ_2

Theorem 2.5. X_t, Y_t queue length of first, second queue, then (X, Y) positive recurrent Markov chain $\iff \lambda < \mu_1, \mu_2$ In this case, $\pi(m, n) = (1 - \rho_1)\rho_1^m(1 - \rho_2)\rho_2^n$, so X_t, Y_t independent, geometric distributed

Proof. (i) (**Proof 1:**) $(m, n) \rightarrow \begin{cases} (m+1, n) & \text{with rate } \lambda \\ (m, n+1) & \text{with rate } \mu_1 \text{ if } m \geq 1, \text{ then check directly.} \\ (m, n-1) & \text{with rate } \mu_2 \text{ if } n \geq 1 \end{cases}$

Rate bounded so non-explosive

(ii) (**Proof 2:**) Burke's

□

Fact. r.v. independent while process not independent

2.4 Jackson network

Setting 8. finite set $S = \{s_1, \dots, s_c\}$, $Q_i(t)$ individual in station i , $\mathcal{N} = \{\mathbf{n} = (n_1, \dots, n_c)\}$ count-

$$\text{able, } g_{\mathbf{m}, \mathbf{n}} = \begin{cases} \lambda_{ij}\phi_i(m_i) & \text{if } \mathbf{n} = \mathbf{m} - \mathbf{e}_i + \mathbf{e}_j \text{ (transfer)} \\ \nu_j & \text{if } \mathbf{n} = \mathbf{m} + \mathbf{e}_j \text{ (arrival)} \\ \mu_i\phi_i(m_i) & \text{if } \mathbf{n} = \mathbf{m} - \mathbf{e}_i \text{ (departure)} \\ 0 & \text{other } \mathbf{m} \neq \mathbf{n} \end{cases}, \begin{cases} \phi_i(0) = 0 \\ \phi_i(m) > 0 \end{cases}, \lambda_{ii} = 0$$

– closed migration process — $\nu_j = \mu_j = 0$

– irreducible

Setting 9. closed migration process, N number of customers

Example (Base case). $N = 1$, $\phi_j(1) = 1$, generator $h_{ij} = \begin{cases} \lambda_{ij} & \text{if } i \neq j \\ -\sum_k \lambda_{ik} & \text{if } i = j \end{cases}$, irreducible

Fact (Traffic equations). $\sum_j \alpha_j \lambda_{ji} = \alpha_i \sum_j \lambda_{ij}$

- α — stationary distribution for base case $N = 1$

Theorem 2.6. *irreducible closed migration process, N customers, then unique stationary distribution $\pi(\mathbf{n}) = B_N \prod_{i=1}^c \left\{ \frac{\alpha_i^{n_i}}{\prod_{r=1}^{n_i} \phi_i(r)} \right\}$*

Proof. stationary condition gives $\sum_{i,j} \gamma(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) \lambda_{ji} \phi_j(n_j + 1) = \gamma(\mathbf{n}) \sum_{i,j} \lambda_{ij} \phi_i(n_i)$ partial balance equation: $\sum_j \gamma(\mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) \lambda_{ji} \phi_j(n_j + 1) = \gamma(\mathbf{n}) \sum_j \lambda_{ij} \phi_i(n_i)$, then plug in π \square

Fact. *In equilibrium, not independence as constraint of $\sum n_i = N$*

Example (Open migration process).

- auxiliary process — add station $\{\infty\}$
- $\beta = (1, \dots, \beta_c, \beta_\infty)$ — stationary distribution of auxiliary process
- $\alpha_i = \frac{\beta_i}{\beta_\infty}$

Fact. $\nu_i + \sum_j \alpha_j \lambda_{ji} = \alpha_i \left(\mu_i + \sum_j \lambda_{ij} \right)$

- $D_i = \sum_{n=0}^{\infty} \frac{\alpha_i^n}{\prod_{r=1}^n \phi_i(r)}$
- $\pi_i(n_i) = D_i^{-1} \frac{\alpha_i^{n_i}}{\prod_{r=1}^{n_i} \phi_i(r)}$

Theorem 2.7. *auxiliary process irreducible, $D_i < \infty$, then stationary distribution $\pi(\mathbf{n}) = \prod_i \pi_i(n_i)$*

Fact. *In equilibrium, queue length of different stations independent r.v.*

- $g_i = \mu_i + \sum_{j \neq i} \lambda_{ij}$ — total rate of departure of a customer at station i
- $\gamma_i = \alpha_i g_i$ — aggregate/effective arrival rate at i

Fact (Traffic equation). $\gamma_i = \nu_i + \sum_j \gamma_j y_{ij}$ where (y_{ij}) jump chain of single-customer system

Theorem 2.8. $\mathbf{Q} = \{\mathbf{Q}(t) : -\infty < t < \infty\}$ irreducible open migration process, \exists stationary distribution π , let $\mathbf{Q}'(t) = \mathbf{Q}(-t)$, then \mathbf{Q}' open migration network with parameters $\begin{cases} \lambda'_{ij} = \frac{\alpha_j \lambda_{ji}}{\alpha_i} \\ \nu'_j = \alpha_j \mu_j \\ \mu'_i = \frac{\nu_i}{\alpha_i} \\ \phi'_i(\cdot) = \phi_i(\cdot) \end{cases}$

Proof. similar to reversal \square

Fact. *arrival processes of different station independent Poisson process, so by reversal, departure process independent Poisson process*

2.5 M/G/1

- $Q(D_n)$ — $Q(D_{n+})$ number of customers left after n -th customer left

Fact. $Q(D_{n+1}) = \begin{cases} U_n + Q(D_n) - 1 & \text{if } Q(D_n) > 0 \\ U_n & \text{if } Q(D_n) = 0 \end{cases}$

Theorem 2.9. S typical service time, then $Q(D)$ Markov chain with transition matrix

$$P_D = \begin{pmatrix} \delta_0 & \delta_1 & \delta_2 & \cdots \\ \delta_0 & \delta_1 & \delta_2 & \cdots \\ 0 & \delta_0 & \delta_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \delta_j = \mathbb{E} \left(\frac{(\lambda S)^j}{j!} e^{-\lambda S} \right)$$

- traffic intensity $\rho = \lambda \mathbb{E}(S)$
- M_S — moment generating function of service time

Theorem 2.10.

- (i) $\rho < 1$, then $Q(D)$ ergodic, unique stationary distribution π with generating function $G(s) = \sum \pi_j s^j = (1 - \rho)(s - 1) \frac{M_S(\lambda(s-1))}{s - M_S(\lambda(s-1))}$
- (ii) $\rho > 1$, $Q(D)$ transient
- (iii) $\rho = 1$, $Q(D)$ null recurrent

Proof. stationary $\iff \lim_{s \uparrow 1} G(s) = 1$

□

- busy period B — time during which server continuously occupied

Fact. $\begin{cases} E(B) < \infty & \text{if } \rho < 1 \\ E(B) = \infty, \mathbb{P}(B = \infty) = 0 & \text{if } \rho = 1 \\ \mathbb{P}(B = \infty) > 0 & \text{if } \rho > 1 \end{cases}$

3 Renewals

- X_i i.i.d. — non-negative, $\mathbb{P}(X_i > 0) = 1$
- $T_n = X_1, \dots, X_n$
- renewal process N

Theorem 3.1. $\mathbb{P}(N(t) < \infty) = 1 \iff \mathbb{E}(X_1) > 0$

- F distribution of X
- F_k distribution of T_k

Lemma 3.2. $F_1 = F$, $F_{k+1}(x) = \int_0^x F_k(x-y)dF(y)$

Lemma 3.3. $\mathbb{P}(N(t) = k) = F_k(t) - F_{k+1}(t)$

– renewal function $m(t) = \mathbb{E}(N(t))$

Lemma 3.4. $m(t) = \sum F_k(t)$

Lemma 3.5. $m(t) = F(t) + \int_0^t m(t-x)dF(x)$

Proof. condition on X_1

□

– renewal-type equation $\mu(t) = H(t) + \int_0^t \mu(t-x)dF(x)$, H uniformly bounded

Theorem 3.6.

(i) $\mu(t) = H(t) + \int_0^t H(t-x)dm(x)$ solution of renewal-type equation

(ii) H bounded on finite interval, then μ bounded on finite intervals, unique sol to equation

– Laplace-Stieltjes transform — $g^*(\theta) = \int_0^\infty e^{-\theta x} dg(x)$

Fact. after transform, $\mu^*(\theta) = \frac{H^*(\theta)}{1-F^*(\theta)}$

if $H = F$, $m^*(\theta) = \frac{F^*(\theta)}{1-F^*(\theta)}$

Theorem 3.7. $\frac{1}{t}N(t) \xrightarrow{a.s.} \frac{1}{\mu}$

Proof. Strong law on $T_{N(t)}$ and $T_{N(t)+1}$

□

Theorem 3.8. $Var(X_1) = \sigma^2$, $\frac{N(t) - \frac{t}{\mu}}{\sqrt{\frac{t\sigma^2}{\mu^3}}} \xrightarrow{d} \mathcal{N}(0, 1)$

Theorem 3.9 (Elementary renewal theorem). $\frac{1}{t}m(t) \rightarrow \frac{1}{\mu}$

Proof. $M = N(t) + 1$ stopping time for X_i , then $t < T_{N(t)+1}$ with Wald giving lower bound. $t \geq T_{N(t)}$ and truncation for upper bound. \square

Lemma 3.10 (Wald's equation). X_i i.i.d., finite mean, M stopping time wrt (X_i) , $\mathbb{E}(M) < \infty$, then $\mathbb{E}\left(\sum_{i=1}^M X_i\right) = \mathbb{E}(X_1)\mathbb{E}(M)$

Fact. $N(t)$ not stopping time

Fact. $\mathbb{E}_{N(t)+1} \neq \mu$ in general

- F_X arithmetic with span λ — X take values $\{m\lambda\}$ a.s.

Theorem 3.11 (Blackwell's renewal theorem).

(i) X_1 not arithmetic, then $m(t+h) - m(t) \xrightarrow{t \rightarrow \infty} \frac{h}{\mu}$ for all h

(ii) If X_1 arithmetic, still holds when h multiple of λ

Theorem 3.12 (Key renewal theorem). X_1 not arithmetic, g such that

(i) $g(t) \geq 0$ for all t

(ii) $\int_0^\infty g(t)dt < \infty$

(iii) g non-increasing

then $\int_0^t g(t-x)dm(x) \rightarrow \frac{1}{\mu} \int_0^\infty g(x)dx$

3.1 Excess Life

- excess lifetime $E(t) = T_{N(t)+1} - t$
- current lifetime/age $C(t) = t - T_{N(t)}$
- total lifetime $D(t) = E(t) + C(t) = X_{N(t)+1}$

Theorem 3.13. $\mathbb{P}(E(t) \leq y) = F(t+y) - \int_0^t [1 - F(t+y-x)]dm(x)$

Proof. consider $\mathbb{P}(E(t) > y)$ and condition on X_1 to get renewal-type equation \square

Corollary 3.14. $\mathbb{P}(C(t) \geq y) = \begin{cases} 0 & \text{if } y > t \\ 1 - F(t) + \int_0^{t-y} (1 - F(t-x))dm(x) & \text{if } y \leq t \end{cases}$

Proof. $\mathbb{P}(C(t) \geq y) = \mathbb{P}(E(t - y) > y)$

□

Setting 10. $X = X_1$

Theorem 3.15. X not arithmetic, $\mu = \mathbb{E}(X) < \infty$, then $\mathbb{P}(E(t) \leq y) \xrightarrow{t \rightarrow \infty} \frac{1}{\mu} \int_0^y (1 - F(x)) dx$

Proof. key renewal and $\mu = \int F(x) dx$

□

Theorem 3.16. X takes values in \mathbb{N} with span 1, $\mu = \mathbb{E}(X) < \infty$, then $\mathbb{P}(E(n) = k) \rightarrow \frac{1}{\mu} \mathbb{P}(X \geq k)$ for $k \in \mathbb{N}$

Proof. $p_{k,k-1} = 1, p_{1,j} = \mathbb{P}(X = j)$, invariant distribution $\pi_k = \frac{1}{\mu} \mathbb{P}(X \geq k)$

□

3.2 Renewal-reward processes

Setting 11. $\{(X_i, R_i)\}$ i.i.d.

- reward process $C(t) = \sum_{i=1}^{N(t)} R_i$
- reward function $c(t) = \mathbb{E}(C(t))$

Theorem 3.17 (Renewal-reward theorem). $0 < \mathbb{E}X < \infty$, $\mathbb{E}|R| < \infty$, then

(i) $\frac{C(t)}{t} \xrightarrow{a.s.} \frac{\mathbb{E}R}{\mathbb{E}X}$

(ii) $\frac{c(t)}{t} \rightarrow \frac{\mathbb{E}R}{\mathbb{E}X}$

Proof. (i) $\frac{C(t)}{t} = \frac{C(t)}{N(t)} \frac{N(t)}{t}$

- (ii) 1) Wald, suffice $t^{-1} \mathbb{E}(R_{N(t)+1}) \rightarrow 0$, then condition on X_1 , renewal, then bound
- 2) uniform integrability, then cvg in prob $\Rightarrow L^1$ convergence

□

Fact. also work for accumulative reward as long as monotone