Applied Probability

1 Continuous time Markov Chains

– right continuous —— $\forall t, \exists \epsilon, X_t(\omega) = X_{t+s}(\omega)$ for all $s \in [0, \epsilon]$

– finite dimension marginals $\mathbb{P}(X_{t_1} = x_1, \dots, X_{t_n} = x_n)$

Fact. *process can be determined from the finite dimension marginals*

– Memoryless property $\mathbb{P}(S > t + s | S > s) = \mathbb{P}(S > t)$

Theorem 1.1. *Memoryless iff exponential distribution*

1.1 Poisson process

– Poisson process with intensity λ

(i) $N(0) = 0$, $N(s) \leq N(t)$ for $s < t$

(ii) $\mathbb{P}(N(t+h) = n + m | N(t) = n) = \begin{cases} \lambda h + o(h) & \text{if } m = 1 \\ o(h) & \text{if } m > 1 \\ 1 - \lambda h + o(h) & \text{if } m = 0 \end{cases}$

(iii) $N(t) - N(s)$ independent of $(N(k))_{k \leq s}$

Theorem 1.2. $N(t) \sim Poi(\lambda t)$

Proof. derive differential equation, then generating function □

– $p_j(t) = \mathbb{P}(N(t) = j)$

– Generating function $G(s,t) = \sum p_j(t) s^j$

– Arrival time T_n

– interarrival time U_n

Theorem 1.3.

(i) $U_i \sim Exp(\lambda)$

(ii) U_i *independent*

Proof. use $N(t)$ Poisson $\square$

Fact. $N(t) \geq j \iff T_j \leq t$

– order statistics

Theorem 1.4. *$T_1, \dots, T_n$ conditional on $\{N(t) = n\}$ same as joint distribution of order statistics of n i.i.d. $Uniform[0, t]$*

Proof. U to T, then calculate density $\square$

Theorem 1.5. *(X_n) increasing right-continuous, taking values $\{0, 1, \dots\}$, $X_0 = 0$, then following equivalent:*

(i) *holding times $S_i \sim Exp(\lambda)$ i.i.d. ,jump chain $Y_n = n$,*
(Sousi defined X Poisson process in this manner)

(ii) *(**infinitesimal**) X independent increments, $h \downarrow 0$ uniformly in t,*

$$\begin{cases} \mathbb{P}(X_{t+h} - X_t = 1) = \lambda h + o(h) \\ \mathbb{P}(X_{t+h} - X_t = 0) = 1 - \lambda h + o(h) \end{cases}$$

(iii) *X has independent, stationary increments, $X_t \sim Poi(\lambda t)$*

Theorem 1.6 (Superposition)**.** *X, Y independent Poisson process, with parameters λ, μ, then $Z_t = X_t + Y_t$ Poisson process with parameters $\lambda + \mu$*

Proof. infinitesimal $\square$

Theorem 1.7 (Thining)**.** *X Poisson process with parameters λ, $(Z_i) \sim Bernoulli(p)$ i.i.d. , Y jumps $\iff$ X jumps and $Z_{X_t} = 1$, then Y Poisson process of parameter λp, $X - Y$ independent Poisson process of parameter $\lambda(1-p)$*

Proof. infinitesimal for Poisson process, independence follows from expanding $\mathbb{P}(Y_t = n, X_t - Y_t = m)$ (suffice to prove independence using finite dimension marginals) $\square$

1.2 Birth process

– birth process with birth rates $\lambda_0, \lambda_1, \dots$

(i) $N(0) = 0$, $N(s) \leq N(t)$ for $s < t$

(ii) $\mathbb{P}(N(t+h)=n+m|N(t)=n)=\begin{cases}\lambda_n h+o(h) & \text{if } m=1\\ o(h) & \text{if } m>1\\ 1-\lambda_n h+o(h) & \text{if } m=0\end{cases}$

(iii) $N(t)-N(s)$ independent of $(N(k))_{k\leq s}$

Example.

(i) ***Poisson process****:* $\lambda_n=\lambda$

(ii) ***Simple birth****:* $\lambda_n=n\lambda$

(iii) ***Simple birth with immigration****:* $\lambda_n=n\lambda+\nu$

Proposition 1.8. *$T_k\sim Exp(q_k)$ independent, $0<q=\sum q_k<\infty$, $T=\inf_k T_k$, then*

(i) *infimum attained at unique K with probability 1*

(ii) *T,K independent*

(iii) *$T\sim Exp(q),\mathbb{P}(K=k)=\frac{q_k}{q}$*

- $T_\infty=\lim T_n=\sum^\infty U_i$
- non-explosive / honest —— $\mathbb{P}(T_\infty=\infty)=1$

Theorem 1.9. *birth process N, $\lambda_n>0$, then non-explosive $\iff$ $\sum_n\frac{1}{\lambda_n}=\infty$*

Lemma 1.10. *$U_n\sim Exp(\lambda_n)$, independent, then* $\mathbb{P}(T_\infty<\infty)=\begin{cases}0 & \textit{if } \sum_n\frac{1}{\lambda_n}=\infty\\ 1 & \textit{if } \sum_n\frac{1}{\lambda_n}<\infty\end{cases}$

- forward system of equations: $p'_{ij}(t)=\lambda_{j-1}p_{i,j-1}(t)-\lambda_j p_{ij}(t)$
- backward system of equations: $p'_{ij}(t)=\lambda_i p_{i+1,j}(t)-\lambda_i p_{ij}(t)$

Theorem 1.11.

(i) *forward system has unique solution $\{p_{ij}(t)\}$*

(ii) *$\{p_{ij}(t)\}$ satisfy backward system*

Theorem 1.12. *$\{p_{ij}(t)\}$ unique solution of forward equations, $\{\pi_{ij}(t)\}$ any solution of backward equations, then $p_{ij}(t)\leq\pi_{ij}(t)$*

Fact. $\sum_j p_{ij}(t)=1\iff\mathbb{P}(T_\infty>t)=1$

– weak Markov property

– stopping time

– strong Markov property

– right continuity

– stationary independent increments

  (i) $N(t) - N(s)$ only depends on $t - s$

  (ii) $\{N(t_i) - N(s_i)\}$ independent where $s_1 \leq t_1 \leq \cdots \leq s_n \leq t_n$

1.3 Continuous Markov Chain

Setting 1. $(X(t))$ *takes values in countable* S

– Markov property —— $\mathbb{P}(X(t_n) = j | X(t_{n-1}) = i_{n-1}, \ldots, X(t_1) = i_1) = \mathbb{P}(X(t_n) = j | X(t_{n-1}) = i_{n-1})$

– continuous-time Markov chain —— right-continuous, Markov property

– transition probability $p_{ij}(s,t) = \mathbb{P}(X(t) = j | X(s) = i)$

– homogeneous —— $p_{ij}(s,t) = p_{ij}(0, t-s)$

– transition semigroup $(P_t)_{ij} = p_{ij}(t)$

– stochastic semigroup

  (i) $P_0 = I$

  (ii) P_t stochastic —— non-negative entries, row sum 1

  (iii) (Chapman-Kolmogorov) $P_{s+t} = P_s P_t$

Setting 2. $(X(t))$ *homogeneous Markov chain*

Theorem 1.13. P_t *stochastic semigroup*

– $\mathbb{P}_i$ —— probability measure conditional on $X(0) = i$

– $\mathbb{E}_i$

– t-historical —— events given by $\{X(s) : s < t\}$

– t-future —— events given by $\{X(s) : s > t\}$

– stopping time T —— $\{T \leq t\}$ given by $\{X(s) : s \leq t\}$

Theorem 1.14 (Extended Markov property)**.** *H t-historical, F t-future, then* $\mathbb{P}(F | X(t) = j, H) = \mathbb{P}(F | X(t) = j)$

Theorem 1.15 (Strong Markov property). *T stopping time, conditional on $\{T \le T_\infty\} \bigcap \{X(T) = i\}$, then*

(i) $(X_{T+u})_u$ *continuous Markov chain start at state i*

(ii) *same transition prob*

(iii) *independent to $\{X(s) : s < T\}$*

Setting 3. $X(0) = i$

- $U_0 = \inf \{t : X(t) \neq i\}$

Fact. *right continuous* $\Rightarrow U_0 > 0$

Theorem 1.16.

(i) $U_0 \sim Exp(g_i)$

(ii) U_0 *stopping time*

Proof. Extended Markov and homogeneity to deduce memoryless □

- transition matrix $\mathbf{Y} = (y_{ij})$ —— $y_{ij} = \begin{cases} \delta_{ij} & \text{if } g_i = 0 \\ \mathbb{P}_i(X(U_0) = j) & \text{if } g_i > 0 \end{cases}$

- generator $\mathbf{G} = (g_{ij})$ —— $g_{ij} = \begin{cases} g_i y_{ij} & \text{if } j \neq i \\ -g_i & \text{if } j = i \end{cases}$

Fact. $\mathbb{P}(X(t+h) = j | X(t) = i) = g_{ij}h + o(h)$

Fact. $g_{ij} = g_i(y_{ij} - \delta_{ij})$

Theorem 1.17. *$X(0) = i$, then*

(i) *$X(U_0)$ independent of U_0*

(ii) *conditional on $X(U_0) = j$, $X^*(s) = X(U_0 + s)$ continuous-time Markov chain, same transition prob, initial state j, independent to the past*

- T_m
- holding time $U_m = T_{m+1} - T_m$
- jump chain $Y = \{Y_n\}$
- $T_\infty = \lim T_n$
- minimal process
- explode from state i —— $\mathbb{P}_i(T_\infty < \infty) > 0$

Proposition 1.18. *X minimal process, then $\boldsymbol{P}_{s+t} = \boldsymbol{P}_s$*

Proof. may go to $\{\infty\}$ □

Theorem 1.19. *$i \in S$, non-explosive from i if any of the following holds:*

(i) *S finite*

(ii) $\sup_j g_j < \infty$

(iii) *i recurrent in jump chain Y*

Proof. be dominated by Poisson process which is non-explosive □

- irreducible —— $\forall i, j, \exists t > 0, p_{ij}(t) > 0$

Theorem 1.20.

(i) *(Levy dichotomy) X irreducible, then $\forall t > 0$, $p_{ij}(t) > 0$*

(ii) *X irreducible $\iff$ Y irreducible*

Proof. look at jump chain, $g_{i_0} \cdots g_{i_n} > 0$, $p_{i_k, i_{k+1}}(t) > 0$ □

Fact. *birth process not irreducible*

- $T_A = \inf\{t > 0 : X_t \in A\}$
- $H_A = \inf\{n \geq 0 : Y_n \in A\}$
- hitting probability $h_A(x) = \mathbb{P}_x(T_A < \infty)$
- expected hitting time $k_A(x) = \mathbb{E}_x(T_A)$

Theorem 1.21. *$(h_A(x))_x$ minimal non-negative solution to*
$$\begin{cases} h_A(x) = 1 & \forall x \in A \\ Qh_A(x) = \sum_y q_{xy} h_A(y) = 0 & \forall x \notin A \end{cases}$$

Theorem 1.22. *$q_x > 0 \forall x \notin A$, then $k_A(x)$ minimal non-negative solution to*
$$\begin{cases} k_A(x) = 0 & \forall x \in A \\ Qk_A(x) = \sum_y q_{xy} k_A(y) = -1 & \forall x \notin A \end{cases}$$

- recurrent —— $\mathbb{P}(\{t : X(t) = i\} \text{ unbounded}) = 1$
- transient —— $\mathbb{P}(\{t : X(t) = i\} \text{ unbounded}) = 0$
- $R_i = \inf\{t > U_0 : X(t) = i\}$
- mean return time $m_i = \mathbb{E}(R_i)$
- positive recurrent / non-null recurrent —— $m_i < \infty$

Theorem 1.23. *continuous-time chain X, jump chain Y*

(i) *$g_i = 0$, then i recurrent for X*

(ii) *$g_i > 0$, then i recurrent for $X \iff$ recurrent for Y*

(iii) *i recurrent $\iff \int p_{ii}(t)dt = \infty$*

(iv) *i transient $\iff \int p_{ii}(t)dt < \infty$*

(v) *X irreducible, then every state recurrent or every state transient*

Proof. main point is no explosion. Interchange summation, then old result. □

- **Forward equation:** $\boldsymbol{P}_t' = \boldsymbol{P}_t\boldsymbol{G}$ with boundary condition $\boldsymbol{P}_0 = 1$
- **Backward equation:** $\boldsymbol{P}_t' = \boldsymbol{G}\boldsymbol{P}_t$ with boundary condition $\boldsymbol{P}_0 = 1$

Fact. *If states S finite, then $\boldsymbol{P}_t = e^{t\boldsymbol{G}}$*

- minimal solution —— $p_{ij}(t) \le \pi_{ij}(t)$
- sub-stochastic —— $\sum_j p_{ij}(t) < 1$

Theorem 1.24. *S countable, X minimal Markov chain with generator $\boldsymbol{G}$, then*

(i) *$\boldsymbol{P}_t$ minimal non-negative solution of backward equation $\boldsymbol{P}_t' = \boldsymbol{G}\boldsymbol{P}_t$ with boundary condition $\boldsymbol{P}_0 = 1$*

(ii) *$\boldsymbol{P}_t$ minimal non-negative solution of forward equation $\boldsymbol{P}_t' = \boldsymbol{P}_t\boldsymbol{G}$*

Proof. **Solution:** condition on $T_1 > t$ or $T_1 \le t$.
Minimal: reverse argument and induction. □

Fact. *any solution to both equations sub-stochastic*

Fact. *non-explosive $\Rightarrow \boldsymbol{P}_t$ unique solution to both equations*

- measure

- stationary measure —— $\boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{P}_t$
- stationary distribution
- unique measure —— unique up to scalar multiplication
- first return time R_i
- $m_i = \mathbb{E}_i(R_i)$

Theorem 1.25. *X irreducible, $|S| \geq 2$*

(i) *some state k positive recurrent, then*

(a) *$\exists$ unique stationary distribution $\boldsymbol{\pi}$*

(b) *unique distribution st $\boldsymbol{\pi}\boldsymbol{G} = 0$*

(c) *all states positive recurrent*

(ii) *X non-explosive, $\exists$ $\boldsymbol{\pi}$ st $\boldsymbol{\pi}\boldsymbol{G} = 0$, then*

(a) *all states positive recurrent*

(b) *$\boldsymbol{\pi}$ stationary*

(c) *$\pi_k = \frac{1}{m_k g_k}$*

Proof. **(i)** use 1.26**(iv)** $\boldsymbol{\pi} = \mu(k)/m_k$, then uniqueness of measure $\Rightarrow$ all state non-null

(ii) $\boldsymbol{\nu}' = \frac{\pi_i g_i}{\pi_k g_k}$, then $\boldsymbol{\rho}(k) \leq \boldsymbol{\nu}'$ from discrete MC $\square$

- $\nu_i = x_i g_i$
- $\boldsymbol{\mu}(k) = (\mu_j(k))_j$ —— $\mu_j(k) = \mathbb{E}_k\left(\int_0^{R_k} \mathbb{1}\left\{X(s) = j\right\} ds\right)$
- $\boldsymbol{\rho}(k) = (\rho_j(k))$ —— mean visit to j starting from k in jump chain Y

Lemma 1.26. *X irreducible Markov chain, $|S| \geq 2$*

(i) *measure $\boldsymbol{x}$, then $\boldsymbol{x}\boldsymbol{G} = 0 \iff \boldsymbol{\nu}\boldsymbol{Y} = \boldsymbol{\nu}$*

(ii) *X recurrent, $\boldsymbol{x}\boldsymbol{G} = 0$ unique measure*

(iii) *$\boldsymbol{x}$ measure, $\boldsymbol{x}\boldsymbol{G} = 0$, then $x_j > 0$*

(iv) *X recurrent, $k \in S$, then $\boldsymbol{\mu}(k)\boldsymbol{G} = 0$ and stationary*

Proof. **(i)** expand

(ii) $\boldsymbol{\nu Y} = \boldsymbol{\nu}$, then uniqueness from discrete MC

(iii) $\mu_j(k) = \frac{1}{g_j}\rho_j(k)$, then $\boldsymbol{\rho}(k)\boldsymbol{Y} = \boldsymbol{\rho}(k)$ from discrete MC

(iv) strong Markov to shift time t

□

Fact. *X non-explosive, then $R_k = \sum_j \int_0^{R_k} \mathbb{1}\{X(s) = j\}\, ds$*

Fact. *X irreducible, $\exists$ more than one stationary distribution, then X explosive*

Theorem 1.27 (Markov chain limit theorem)**.** *X irreducible, non-explosive*

(i) if $\exists$ stationary distribution $\boldsymbol{\pi}$, then

(a) $\boldsymbol{\pi}$ unique

(b) $p_{ij}(t) \to \pi_j$

(ii) if no stationary distribution, then $p_{ij}(t) \to 0$

Proof. skeleton $Z_n = X(nh)$ □

Lemma 1.28. *X minimal, then $|p_{ij}(t+u) - p_{ij}(t)| \le 1 - e^{-g_i u}$*

1.4 Reversibility

Theorem 1.29. *X irreducible, non-explosive, with invariant distribution $\boldsymbol{\pi}$, let $X_0 \sim \boldsymbol{\pi}$, fix T, $\hat{X}_t = X_{T-t}$, then*

(i) $\hat{X}$ Markov with generator $\hat{Q}$ and invariant distribution $\boldsymbol{\pi}$, $\pi(x)\hat{q}_{xy} = \pi(y)q_{yx}$

(ii) $\hat{X}$ irreducible, non-explosive

Proof. expand $\mathbb{P}(\hat{X}_{t_0} = x_0, \ldots, \hat{X}_{t_n} = x_n)$, then $\hat{P}$ satisfies Komogorov backward with $\hat{Q}$, then minimal, easy to show irreducible, finally $\hat{p}_{xy}(t) = \mathbb{P}_x(\hat{X}_t = y, t < \hat{\zeta})$ where ζ explosion time □

- reversible —— (X_t), (X_{T-t}) same distribution
- detailed balanced —— $\lambda(x)q_{xy} = \lambda(y)q_{yx}$

Lemma 1.30. *detail balanced $\Rightarrow$ λ invariant measure*

Theorem 1.31. *X irreducible, non-explosive, $X_0 \sim \boldsymbol{\pi}$, then*
detail balanced $\iff$ (X_t) reversible

Lemma 1.32. *$\boldsymbol{\pi}$ invariant for birth-death chain $\iff$ detail balanced*

1.5 Ergodic theorem

– long run proportion of time spends at x —— $\frac{1}{t}\int_0^t \mathbb{1}(X_s = x)ds$

Theorem 1.33. *X irreducible,*

(i) $\frac{1}{t}\int_0^t \mathbb{1}(X_s = x)ds \xrightarrow{a.s.} \frac{1}{m_x g_x}$

(ii) *if π invariant, f bounded, then* $\frac{1}{t}\int_0^t f(X_s)ds \xrightarrow{a.s.} \sum_x f(x)\pi(x)$

1.6 Birth-death process and imbedding

– birth rate $\lambda_0, \lambda_1, \ldots$

– death rate $\mu_1, \mu_2, \ldots$

– birth-death process

Theorem 1.34. *X birth-death process, generator $\boldsymbol{G}$*

(i) *measure* $x_n = \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}x_0$ *satisfies* $\boldsymbol{x}\boldsymbol{G} = 0$

(ii) $\exists$ *distribution* $\boldsymbol{\pi}$ *satisfies* $\boldsymbol{\pi}\boldsymbol{G} = 0 \iff \sum \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n} < \infty$

(iii) *if* $\sum \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}(\lambda_n + \mu_n) < \infty$, *then* $\boldsymbol{\pi}$ *stationary*

Proof. **(i)** solve $\boldsymbol{x}\boldsymbol{G} = 0$

(ii) trivial

(iii) condition for jump chain Y recurrent, then non-explosive

□

Example.

– ***Pure birth*** $\mu_n = 0$

– ***Simple death with immigration*** $\lambda_n = \lambda, \mu_n = n\mu$

Theorem 1.35. *$X(t)$ asymptotically* $Poi(\rho) = Poi\left(\frac{\lambda}{\mu}\right)$

– ***Simple birth-death*** $\lambda_n = n\lambda, \mu_n = n\mu,\ X(0) = I$

Fact. *state 0 absorbing*

Theorem 1.36. $G(s,t) = \mathbb{E}(s^{X(t)}) = \begin{cases} \left(\frac{\lambda t(1-s)+s}{\lambda t(1-s)+1}\right)^I & \text{if } \mu = \lambda \\ \left(\frac{\mu(1-s)-(\mu-\lambda s)e^{-t(\lambda-\mu)}}{\lambda(1-s)-(\mu-\lambda s)e^{-t(\lambda-\mu)}}\right)^I & \text{if } \mu \neq \lambda \end{cases}$

Proof. Forward equation □

Fact. *non-explosive as* $\sum p_j(t) = G(1,t) = 1$

Fact. $\mathbb{E}_I(X(t)) \to \begin{cases} 0 & \textit{if } \rho < 1 \\ \infty & \textit{if } \rho > 1 \end{cases}$

- *extinction probability* $\eta(t) = \mathbb{P}_I(X(t) = 0)$

Corollary 1.37. $\eta(t) \to \begin{cases} 1 & \textit{if } \rho \leq 1 \\ \rho^{-I} & \textit{if } \rho > 1 \end{cases}$

– imbedded random walk —— jump chain Y with parameter $\frac{\lambda}{\lambda+\mu}$, absorbing at 0

– imbedded branching process —— lives $Exp(\lambda+\mu)$, then born n individuals where $\begin{cases} p_0 = \mathbb{P}(n=0) = \frac{\mu}{\lambda+\mu} \\ p_2 = \mathbb{P}(n=2) = \frac{\lambda}{\lambda+\mu} \end{cases}$

– age-dependent branching process

– age density function $f_T(u) = (\lambda + \mu)e^{-(\lambda+\mu)u}$

– family-size generating function $G(s) = \frac{\mu+\lambda s^2}{\mu+\lambda} = p_0 + p_2 s^2$