# Stochastic Financial Models

## 1 Utility and Mean-Variance analysis

– contingent claims —— r.v. $X$

– utility function —— non-decreasing

**Fact.** *$Y$ is preferred to $X$ iff* $\mathbb{E}(U(X)) \leq \mathbb{E}(U(Y))$

– indifferent

– risk neutral

– risk averse

– concave

– strictly concave

**Proposition 1.1.** *risk averse iff $U$ concave*

– CARA with parameter $\gamma$ —— $\gamma \in (0, \infty)$, $U(X) = CARA_\gamma(x) = -\exp(-\gamma x)$

– CRRA with parameter $R$ —— $R \in (0,1) \bigcup (1, \infty)$, $U(X) = CRRA_R(x) = \begin{cases} \frac{x^{1-R}}{1-R} & \text{if } x > 0 \\ -\infty & \text{otherwise} \end{cases}$

– CRRA with parameter 1 —— $U(X) = CRRA_1(x) = \begin{cases} \log x & \text{if } x > 0 \\ -\infty & \text{otherwise} \end{cases}$

– constant absolute risk aversion —— CARA

– constant relative risk aversion —— CRRA

**Fact** (Arrow-Pratt coefficient of absolute risk aversion)**.** *$\omega + X$ preferred to $\omega$ iff* $\frac{2\mathbb{E}(X)}{\mathbb{E}(X^2)} \geq -\frac{U''(\omega)}{U'(\omega)}$

**Fact** (Arrow-Pratt coefficient of relative risk aversion)**.** *$\omega(1+X)$ preferred to $\omega$ iff* $\frac{2\mathbb{E}(X)}{\mathbb{E}(X^2)} \geq -\frac{\omega U''(\omega)}{U'(\omega)}$

– available claim $\mathcal{A}$

– reservation bid price $\pi_b(Y)$ —— $\sup \pi$ st $\mathbb{E}(U(X + Y - \pi)) > \mathbb{E}(U(X^*))$

– reservation ask price $\pi_a(Y)$ —— $\inf \pi$ st $\mathbb{E}(U(X - Y + \pi)) > \mathbb{E}(U(X^*))$

**Proposition 1.2** (Ask above, bid below). *$\mathcal{A}$ convex, then $\pi_b(Y) \leq \pi_a(Y)$*

**Setting 1.** *$\mathcal{A}$ affine space, $U$ differentiable, strictly concave*

- marginal price $\pi_m(Y)$ —— $\pi_m(Y) = \frac{\mathbb{E}(U'(X^*)Y)}{\mathbb{E}(U'(X^*))}$
- single-period asset price model
- numeraire
- riskless bond
- interest rate —— $r > -1$
- state-price density $\rho$ —— $S_0^i = \mathbb{E}(S_1^i \rho)$
- wealth $\omega_0$
- portfolio $\theta$

**Example.** *no bond*

$$\begin{aligned} &\textit{given} && \mathbb{E}(\theta \cdot S_1) = \theta \cdot \mu, var(\theta \cdot S_1) = \theta^T V \theta \\ &\textit{minimize} && var(\theta \cdot S_1) \\ &\textit{subject to} && \theta \cdot S_0 = \omega_0, \mathbb{E}(\theta \cdot S_1) = \omega_1 \end{aligned}$$

- mean-variance-efficient frontier —— $\{\theta^*(\omega_1)\}$
- minimum variance portfolio $\theta_{\min^*}$ —— minimise var over $\omega_1$

**Example.** *with bond*

$$\begin{aligned} &\textit{minimise} && \theta^T V \theta \\ &\textit{subject to} && \theta^0 + \theta S_0 = \omega_0, \theta^0(1+r) + \theta\mu = \omega_1 \end{aligned}$$

*Then* , $\theta^* = \lambda \theta_m^*$

- market portfolio $\theta_m^*$ —— $A(\mu - (1+r)S_0)$, $A = V^{-1}$

**Setting 2.** *$S_1$ Gaussian, $U$ CARA*

**Example.** *no bond*

$$\begin{aligned} &\textit{maximise} && \mathbb{E}(U(\theta S_1)) \\ &\textit{subject to} && \theta S_0 = \omega_0 \end{aligned}$$

**Example.** *with bond*

$$\begin{aligned} &\textit{maximise} && \mathbb{E}(U(\bar{\theta}\bar{S}_1)) \\ &\textit{subject to} && \bar{\theta}\bar{S}_0 = \omega_0 \end{aligned}$$

*then*, $\theta^* = \gamma^{-1}\theta_m^*$

- beta/sensitivity $\beta^i$ —— $\beta^i = \frac{cov(S_1^i, \theta_m^* S_1)}{var(\theta_m^* S_1)}$
- $\mu^m$ —— $\theta_m^* \mu$
- $S_0^m$ —— $\theta_m^* S_0$

**Proposition 1.3.** $\mu^i - (1+r)S_0^i = \beta^i(\mu^m - (1+r)S_0^m)$

- capitalization-weights of the relevant market index

**Setting 3.**

$S_1 = (1+R)S_0$, $S_1^m = (1+R^m)S_0^m$, $\tilde{\beta}^i = \frac{cov(R^i, R^m)}{var(R^m)}$

**Fact.** $\mathbb{E}(R^i) = r + \tilde{\beta}^i(\mathbb{E}(R^m) - r)$

## 2 Martingales

- conditional probability
- conditional expectation given event
- conditional expectation given $\mathcal{G}$, $\mathbb{E}(X|\mathcal{G})$

**Theorem 2.1.** *$\mathcal{G} \subset \mathcal{F}$ sub-$\sigma$-algebra, $X$ integrable, then $\exists$ unique $Y$ (up to a.s.) st*

***(i)*** *$Y$ integrable*

***(ii)*** *$Y$ $\mathcal{G}$-measurable*

***(iii)*** *$\mathbb{E}(Y\mathbb{1}_A) = \mathbb{E}(X\mathbb{1}_A)$ for all $A \in \mathcal{G}$*

**Fact.** *also true if replace integrable by non-negative*

- $\mathbb{E}(X|Z)$ —— $\mathcal{G} = \sigma(Z)$ for r.v. $Z$
- $\mathbb{P}(A|\mathcal{G})$ —— $X = \mathbb{1}_A$

**Fact.** *$\mathcal{G} = \sigma(B_n)$ discrete, then $\mathbb{E}(X|\mathcal{G}) = \sum \mathbb{E}(X|B_n)\mathbb{1}_{B_n}$ a.s.*

**Proposition 2.2.** *$\mathcal{G} \subset \mathcal{F}$, $X, W$ integrable, then*

***(i)*** *$\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$*

***(ii)*** *$X$ $\mathcal{G}$-measurable, then $\mathbb{E}(X|\mathcal{G}) = X$ a.s.*

***(iii)*** *$X$ independent of $\mathcal{G}$, then $\mathbb{E}(X|\mathcal{G}) = E(X)$ a.s.*

***(iv)*** *$X \geq 0$ a.s. , then $\mathbb{E}(X|\mathcal{G}) \geq 0$ a.s.*

***(v)*** *$\mathbb{E}(\alpha X + \beta W|\mathcal{G}) = \alpha\mathbb{E}(X|\mathcal{G}) + \beta\mathbb{E}(W|\mathcal{G})$ a.s.*

**Proposition 2.3** (Tower property). *$\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ sub-$\sigma$-algebra, $X$ integrable, then* $\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = \mathbb{E}(X|\mathcal{H})$ *a.s.*

**Proposition 2.4** (Taking out what is known). *$\mathcal{G} \subset \mathcal{F}$ sub-$\sigma$-algebra, $X$ integrable, $Z$ $\mathcal{G}$-measurable, $ZX$ integrable, then*
$\mathbb{E}(ZX|\mathcal{G}) = Z\mathbb{E}(X|\mathcal{G})$ *a.s.*

**Proposition 2.5** (Averaging over independent variables). *$X_1, X_2$ r.v. in $(E_1, \mathcal{E}_1), (E_2, \mathcal{E}_2)$, $\mathcal{G} \subset \mathcal{F}$, $X_1$ $\mathcal{G}$-measurable, $X_2$ independent of $\mathcal{G}$, $F$ non-negative, let $f = \mathbb{E}(F(\cdot, X_2))$, then*
$\mathbb{E}(F(X_1, X_2)|\mathcal{G}) = f(X_1)$ *a.s.*

– filtration $(\mathcal{F}_n)$

– random process

– $(X_n)$ adapted to $(\mathcal{F}_n)$

– natural filtration $(\mathcal{F}_n^X)$ —— $\mathcal{F}_n^X = \sigma(X_k : 0 \le k \le n)$

– martingale ——

- (Adapted) $X_n$ $F_n$-measurable
- (Intergrable) $\mathbb{E}(|X_n|) < \infty$
- (Martingale property) $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$ a.s.

– supermartingale —— $\mathbb{E}(X_{n+1}|\mathcal{F}_n) \le X_n$

– submartingale —— $\mathbb{E}(X_{n+1}|\mathcal{F}_n) \ge X_n$

**Fact.** *Any martingale also martingale in natural filtration (natural filtration smalllest)*

– martingale (continuous-time) —— adapted, integrable, $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$ for all $s \le t$

– $(X_t)$ continuous —— $t \mapsto X_t(\omega)$ continuous for all $\omega$

**Example.** *$X_n$ i.i.d. , $\mathcal{F}_0 = \{\varnothing, \Omega\}$, $(\mathcal{F}_n)$ natural filtration*

– *(additive martingale) $X_1$ inegrable, $E(X_1) = 0$, $S_0 = 0$, $S_n = \sum^n X_k$*

– *(multiplicative martingale) $X_1$ non-negative, $\mathbb{E}(X_1) = 1$, $Z_0 = 1$, $Z_n = \prod^n Z_k$*

**Example.** *$(X_n)$ Markov chain, countable state space $S$, transition matrix $P$, natural filtration $(\mathcal{F}_n)$, bounded/non-negative $f$ on $S$, let*

$$Pf(x) = \sum p_{xy} f(y)$$

*then if $f$ subharmonic*

$$\mathbb{E}(f(X_{n+1})|\mathcal{F}_n) = Pf(X_n) \ge f(X_n)$$

*then, $(f(X_n))$ submartingale*

- subharmonic —— $f(x) \leq Pf(x)$ for all $x$
- random time —— $T : \Omega \to \{0, 1, \dots\} \bigcup \{\infty\}$
- stopping time —— $\{T \leq n\} \in \mathcal{F}_n$

**Theorem 2.6** (Optional stopping)**.** *$(M_n)$ martingale, $T$ bounded, stopping time, then $\mathbb{E}(M_T) = \mathbb{E}(M_0)$*

**Fact.** *Doob's optional sampling theorem basic form*

**Theorem 2.7.** *$(M_n)$ martingale, $T$ almost surely finite, stopping time, suppose one of the following holds:*

***(i)*** *$|M_n| \leq C$ for all $n \leq T$, $C$ constant*

***(ii)*** *$\mathbb{E}(T) < \infty$, $|M_n - M_{n-1}| \leq C$ for all $n \leq T$*

**Fact.** *$T$ stopping time $\Rightarrow$ $T \wedge n$ bounded stopping time*

**Counter Example.** *additive martingale, simple random walk, $T = \min\{n : S_n = 1\}$, then almost sure finite as recurrent, but $\mathbb{E}(T) = \infty$, $\mathbb{E}(S_T) = 1 \neq 0 = S_0$*

**Counter Example.** *multiplicative martingale, $(X_k)$ i.i.d. ,$\mathbb{P}(X_1 = 0) = \mathbb{P}(X_1 = 2) = 1/2$, $T = \min\{n : Z_n = 0\}$, then $\mathbb{E}(T) = 2$, but $\mathbb{E}(Z_T) = 0 \neq 1 = Z_0$*

**Theorem 2.8.** *$(M_n)$ martingale, $T$ stopping time, then $(M_{T \wedge n})$ martingale*

- previsible —— $H_n$ $\mathcal{F}_{n-1}$-measurable
- martingale transform of $(M_n)$ by $(H_n)$ —— $Y_0 = 0$, $Y_n = \sum_1^n H_k(M_k - M_{k-1})$

**Theorem 2.9** (Martingale transform)**.** *$(M_n)$ martingale, $(H_n)$ bounded, previsible, $(Y_n)$ martingale transform, then $(Y_n)$ martingale*

**Fact.** *model stock price as $(M_n)$, then*

***(i)*** *optional stopping $\Rightarrow$ expected return $\mathbb{E}(M_T)$ the same no matter what stopping time*

***(ii)*** *$(H_k)$ amount held between time $k-1$ and $k$, no bounded previsible strategy gives expected gain or loss*

## 3 Pricing contingent claims

– asset price model $(\bar{S}_n)_{0\leq n\leq T}$ with numeraire

– numeraire $(S_n^0)$ —— $S_n^0 > 0$

– discounted prices $X_n^i$ —— $X_n^i = S_n^i/S_n^0$

– $\bar{X}_n = (1, X_n)$

– interest rate $r_n$ —— $S_n^0 = (1+r_n)S_{n-1}^0$

– risky assets $(S_n)$

– portfolio $\bar{\theta}_n$

– self-financing —— $\bar{\theta}_n\bar{S}_n = \bar{\theta}_{n+1}\bar{S}_n$

– value process $(V_n)$ —— $V_0 = \bar{\theta}_1\bar{X}_0$, $V_n = \bar{\theta}_n\bar{X}_n$

– total (discounted) value

– previsible —— if $\bar{\theta}_n$ $\mathcal{F}_{n-1}$-measurable

**Setting 4.**

***(i)*** *$(\mathcal{F}_n)$ filtration generated by $(\bar{S}_n)$, $\mathcal{F} = \mathcal{F}_T$*

***(ii)*** *$(S_n)$ takes countable values*

***(iii)*** *$(S_n^0)$ deterministic process*

**Proposition 3.1.** *$(\theta_n)$ previsible process, then $\exists$ $(\theta_n^0)$ st*

***(i)*** *$(\theta_n^0)$ previsible*

***(ii)*** *$(\bar{\theta}_n^0)$ self-financing with initial value $V_0$*

***(iii)*** *$V_T = V_0 + \sum_1^T \theta_n(X_n - X_{n-1})$*

– contingent claim of maturity $T$ —— non-negative $\mathcal{F}_T$-measurable r.v.

– European option

– call with strike price $K$ —— $(S_T - K)^+$

– put with strike price $K$ —— $(S_T - K)^-$

– options

– exotic options —— depending on entire path $(S_n)$

– barrier options ——

- knocked out

- knocked in

– up-and-out call —— $C = \begin{cases} (S_T - K)^+ & \text{if } \max S_n < B \\ 0 & \text{otherwise} \end{cases}$

– down-and-in put —— $C = \begin{cases} (S_T - K)^- & \text{if } \min S_n \leq B \\ 0 & \text{ptherwise} \end{cases}$

– $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$, $\tilde{\mathbb{P}} \sim \mathbb{P}$ —— $\exists \rho$, st

- $\mathbb{P}(\rho > 0) = 1$
- $\tilde{\mathbb{P}}(A) = \mathbb{E}(\rho \mathbb{1}_A)$

– density $\rho = d\tilde{\mathbb{P}}/d\mathbb{P}$ for $\tilde{\mathbb{P}}$ wrt $\mathbb{P}$

**Fact.** $\tilde{\mathbb{E}}(X) = \mathbb{E}(\rho X)$

**Fact.** $\tilde{\mathbb{P}} \sim \mathbb{P}$ *symmetric and transitive*

**Fact.** $d\mathbb{P}/d\tilde{\mathbb{P}} = 1/\rho$ *a.s.*

– arbitrage for $(\bar{S}_n)_{0 \leq n \leq T}$ ——

**(i)** $(\bar{\theta}_n)_{1 \leq n \leq T}$ previsible, self-financing

**(ii)** $V_0 = 0$

**(iii)** $V_T \geq 0$ a.s.

**(iv)** $V_T > 0$ with positive probability

– $(\bar{S}_n)$ arbitrage free

**Proposition 3.2.** $(X_n)$ *martingale* $\Rightarrow$ $(X_n)$ *arbitrage free*

**Fact.** *proof can be simpler when* $(\theta_n)$ *bounded*

**Setting 5.** *single period model,* $\mathcal{F}_0 = \{\varnothing, \Omega\}$

**Proposition 3.3.** $Y$ *r.v. , following equivalent:*

***(i)*** *arbitrage free (i.e. no* $\theta$ *st* $\theta Y \geq 0$ *a.s. with* $\theta Y > 0$ *with positive prob)*

***(ii)*** $\exists$ *equivalent probability measure* $\tilde{\mathbb{P}}$ *st* $Y$ *integrable with* $\mathbb{E}(Y) = 0$

– equivalent martingale measure $\tilde{\mathbb{P}}$ (risk neutral measure) —— $\tilde{\mathbb{P}} \sim \mathbb{P}$, $(X_n)$ martingale under $\tilde{\mathbb{P}}$

**Theorem 3.4** ((1st) Fundamental theorem of asset pricing). *following equivalent:*

**(i)** $(\bar{S}_n)$ *arbitrage free*

**(ii)** $(\bar{S}_n)$ *has equivalent martingale measure*

**Setting 6.** $C$ *time-$T$ contigent claim,* $D = C/S^0_T$ *discounted value*

- attainable/replicable —— $\exists$ previsible, self-financing $(\bar{\theta}_n)$ st $C = \bar{\theta}_n \bar{S}_T$

**Fact.** *Alternative def:* $\exists V_0$ $\mathcal{F}_0$*-measurable,* $\theta_n$ *previsible st* $D = V_0 + \sum^T \theta_n (X_n - X_{n-1})$

- fair price —— $V_0$
- replicating portfolio/hedging portfolio —— $\bar{\theta}_n$
- $(\bar{S}_n)$ complete —— all contingent claims attainable

**Proposition 3.5.** $\mathcal{F}_0 = \{\varnothing, \Omega\}$, $\mathcal{F}_T = \sigma(\bar{S}_1)$

**(i)** $C$ *non-negative, attainable, time-$T$ contingent claim,* $\tilde{\mathbb{P}}$ *equivalent martingale measure, then fair price* $V_0 = \tilde{\mathbb{E}}(D)$, $D = C/S^0_T$

**(ii)** $(\bar{S}_n)$ *complete, numeraire non-random, then at most one equivalent martingale measure*

- binomial model (Cox-Ross-Rubinstein model) —— interest rate $r$, parameters $a < b$, $R_i$ i.i.d. with parameter $p$
  - $S^0_n = (1+r)^n$
  - $S_n = S_0 \prod (1 + R_k)$
  - $\begin{cases} \mathbb{P}(R_1 = a) = 1 - p \\ \mathbb{P}(R_1 = b) = p \end{cases}$

**Proposition 3.6.** *binomial model has arbitrage unless* $r \in (a, b)$

**Proposition 3.7.** $(\bar{S}_n)$ *binomial model,* $r \in (a, b)$*, define*

**(i)** $p^* = \frac{r-a}{b-a}$

**(ii)** *equivalent prob measure* $\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} = \left(\frac{p^*}{p}\right)^{U_T} \left(\frac{1-p^*}{1-p}\right)^{D_T}$ *where* $\begin{cases} U_T = (T + S_T)/2 \\ D_T = (T - S_T)/2 \end{cases}$,

$S_T$ *number of* $b$

*Then under* $\mathbb{P}^*$,

**(i)** $R_1, \ldots, R_T$ *i.i.d.* , $\begin{cases} \mathbb{P}^*(R_1 = a) = 1 - p^* \\ \mathbb{P}^*(R_1 = b) = p^* \end{cases}$

**(ii)** $(X_n)$ *martingale under* $\mathbb{P}^*$

**Fact.** $r = \mathbb{E}^*(R_1)$

**Fact.** *Binomial model arbitrage free when $r \in (a, b)$*

**Setting 7.** *if $C = f(S_0, \ldots, S_T)$, then*

$$V(C) = \frac{\mathbb{E}^*(C)}{(1+r)^T} = (1+r)^{-T} \sum f(s_0, s_1, \ldots, s_T) \mathbb{P}^*(S_1 = s_1, \ldots, S_T = s_T)$$

*Define recursive relation:*

$$\begin{aligned} f_T(s_0, \ldots, s_T) &= f(s_0, \ldots, s_T) \\ f_n(s_0, \ldots, s_n) &= (1-p^*) f_{n+1}(s_0, \ldots, s_n, (1+a)s_n) + p^* f_{n+1}(s_0, \ldots, s_n, (1+b)s_n) \end{aligned}$$

**Proposition 3.8.** $\mathbb{E}^*(f(S_0, \ldots, S_T)|\mathcal{F}_n) = f_n(S_0, \ldots, S_n)$, $\mathbb{E}^*(C) = f_0(S_0)$

**Proposition 3.9.** *Define*

$$\Delta_n(s_0, \ldots, s_{n-1}) = \frac{f_n(s_0, \ldots, s_{n-1}, (1+b)s_{n-1}) - f_n(s_0, \ldots, s_{n-1}, (1+a)s_{n-1})}{(1+r)^{T-n}(b-a)s_{n-1}}$$

*then $\theta_n = \Delta_n(S_0, \ldots, S_{n-1})$ replicating portfolio for $C$*

**Fact.** *Binomial model complete when $r \in (a, b)$*

**Proposition 3.10.** *$(W_n)$ simple random walk with $\mathbb{P}(W_1 = 1) = p$, let $M_T = \max W_n$, then with $k \leq T$, $2k - T \leq m \leq k$*

$$\mathbb{P}(M_T = m, W_T = 2k - T) = \left( \binom{T}{k-m} - \binom{T}{k-m-1} \right) p^k (1-p)^{T-k}$$

**Example.** *if $(1+a)(1+b) = 1$, then $S_n = S_0(1+b)^{W_n}$, so can give fair price of $C = F(S_T, \max S_n)$*

# 4 Dynamic Programming

- state-space $E$
- action-space $A$

**Setting 8.** $F : \{0, \ldots, T-1\} \times E \times A \times [0,1] \to E$, *$(\epsilon_n)$ i.i.d.* , $\mathcal{F}_n = \sigma(\epsilon_1, \ldots, \epsilon_n)$

- adapted control $u = (u_n)_{k \leq n \leq T-1}$ —— initial time $k$, $u_n$ $\mathcal{F}_n$-measurable

**Setting 9.** *initial state $x \in E$, adapted control $u$, define $X_k = x, X_{n+1} = F(n, X_n, u_n, \epsilon_{n+1})$*
*Write $X_n = X_n^u(k, x)$*

- expected reward —— $V^u(k, x) = \mathbb{E}\left( \left( \sum_k^{T-1} r(n, X_n^u(k, x), u_n) \right) + R(X_T^u(k, x)) \right)$

– reward function $r, R$ —— non-negative measurable function

– value function $V$ —— $V(k,x) = \sup_u V^u(k,x)$

– $u$ optimal control from $(k,x)$ —— $V(k,x) = V^u(k,x)$

**Proposition 4.1** (Bellman equation)**.** *Let* $Pv(n,x,a) = \mathbb{E}\left(v(n+1, F(n,x,a,\epsilon_{n+1}))\right)$

$$
\begin{aligned}
v(T,x) &= R(x) \\
v(n,x) &= \sup_{a \in A} \{r(n,x,a) + Pv(n,x,a)\} \qquad\qquad n = 0, \dots, T-1
\end{aligned}
$$

*Suppose* $\exists a$ *st*

$$v(n,x) = r(n,x,a(n,x)) + Pv(n,x,a(n,x)) \qquad\qquad n = 0, \dots, T-1$$

*Then,*

***(i)*** $V = v$

***(ii)*** *optimal control* $u_n^* = a(n, X_n^{u*}(k,x))$

**Fact.** *Possible variations:*

***(i)*** $r, R$ *as costs*

***(ii)*** *mixture of costs and rewards*

***(iii)*** *time-dependent state-space* $E_n$

***(iv)*** *time-and-state-dependent action-space* $A_{n,x}$

– American call —— family of time-$T$ contingent claim $(1+r)^{T-\tau}(S_\tau - K)^+$

– American put —— family of time-$T$ contingent claim $(1+r)^{T-\tau}(S_\tau - K)^-$

**Setting 10.** $(S_n)$ *binomial model,* $r \in (a,b)$

**Fact.** *complete* $\Rightarrow$ *can hedge all* $C$ *with* $\mathbb{E}^*(C) = 0$

**Example** (American call)**.** $\tau = T$ *always optimal, American and European calls equivalent*

**Fact.** *fair price can be founded using Bellman equation*

# 5 Brownian motion

– Brownian motion ——

- $B_0 = 0$
- $(B_{s+t} - B_s) \sim N(0,t)$, independent of $\sigma(B_r : r \leq s)$
- $t \mapsto B_t(\omega)$ continuous

– Brownian motion starting from $x$ —— $B_0 = x$

– Gaussian process —— $\forall (t_1, \dots, t_n), (X_{t_1}, \dots, X_{t_n})$ multivariate normal

**Proposition 5.1.** $(B_t)$ *continuous process starting from 0, then following equivalent:*

***(i)*** $(B_t)$ *Brownian motion*

***(ii)*** $(B_t)$ *zero mean Gaussian process,* $\mathbb{E}(B_s B_t) = s \wedge t$

**Proposition 5.2** (Scaling property)**.** $(B_t)$ *Brownian motion, set* $\tilde{B}_t = c^{-1} B_{c^2 t}$*, then* $(\tilde{B}_t)$ *Brownian motion*

**Proposition 5.3.** $(B_t)$ *Brownian motion,* $(B_t)$ *exit every finite interval a.s.*

- $\mathcal{F}_t = \sigma(B_s : s \in [0,t])$
- stopping time —— $\{T \leq t\} \in \mathcal{F}_t$ for all $t$
- $\mathcal{F}_T$ —— $A \in \mathcal{F}_\infty$ st $A \bigcap \{T \leq t\} \in \mathcal{F}_t$

**Proposition 5.4** (Strong Markov property)**.** $(B_t)$ *Brownian motion,* $T$ *a.s. finite stopping time. Define* $\tilde{B}_t = B_{T+t} - B_T$*, then*

***(i)*** $(\tilde{B}_t)$ *Brownian motion*

***(ii)*** *independent of* $\mathcal{F}_T$

**Proposition 5.5.** $(B_t)$ *Brownian motion, define* $T_a = \inf\{t \geq 0 : B_t = a\}$*, then*

***(i)*** $T_a$ *stopping time*

***(ii)*** $T_a$ *almost surely finite*

**Theorem 5.6.** $(\Omega, \mathcal{F}, \mathbb{P})$ *not discrete,* $m$ *prob measure on* $\mathbb{R}$*, mean 0, variance 1, then* $\exists (B_t), (W_t^{(k)})$ *for all* $k \in \mathbb{N}$ *st*

***(i)*** $(B_t)$ *Brownian motion*

***(ii)*** $(W_{\frac{n}{k}}^{(k)})$ *random walk with distribution* $m$*,* $(W_t^{(k)})$ *linear interpolation of values* $\left\{\frac{n}{k}\right\}$

***(iii)*** $\frac{W_t^{(k)}}{\sqrt{k}} \to B_t$ *uniformly on compacts in* $t$ *a.s.*

**Fact.** *combination of Wiener's Theorem and Donsker's Invariance Principle*

- Wiener measure

**Proposition 5.7.** *let $T \geq 0$, $c \in \mathbb{R}$, $B = (B_t)_{\{0 \leq t \leq T\}}$ brownian motion, $\tilde{B}_t = B_t + ct$, then $\forall$ measurable set $A \subset C[0,T]$, $\mathbb{P}(\tilde{B} \in A) = \mathbb{E}(\mathbb{1}_{\{B \in A\}} e^{cB_T - \frac{c^2 T}{2}})$*

**Fact.** *special case of Cameron-Martin theorem*

**Proposition 5.8** (Reflection principle)**.** *$(B_t)$ Brownian motion, $a \geq 0$, set $T_a = \inf\{t \geq 0 : B_t = a\}$, define $\tilde{B}_t = \begin{cases} B_t & \text{if } t \leq T_a \\ 2a - B_t & \text{if } t > T_a \end{cases}$, then $(\tilde{B}_t)$ Brownian motion*

– maximum process —— $M_t = \sup_{\{0 \leq s \leq t\}} B_s$

**Fact.** *$M_t$ same distribution as $|B_t|$*

**Proposition 5.9.** *$T_a$ has density $h_a(t) = \frac{a}{\sqrt{2\pi t^3}} e^{-\frac{a^2}{2t}}$*

– $p_t(x,y)$ —— density of $B_t$ starting at $x$

– $p_t^a(x,y) = p_t(x,y) - p_t(x, 2a - y)$

**Proposition 5.10.** *$x \leq a$, $(B_t)$ Brownian motion with density starting from $x$, then $\forall$ non-negative measurable $f$, $\mathbb{E}_x(f(B_t)\mathbb{1}_{\{T_a > t\}}) = \int_\infty^a f(y) p_t^a(x,y) dy$*

# 6 Black-Scholes model

– Black-Scholes model —— $S_t^0 = e^{rt}, S_t = S_0 e^{\sigma B_t + \mu t}$

- price of riskless bond $S_t^0$
- interest rate $r$
- price of risky asset $S_t$
- drift $\mu$
- volatility $\sigma$

**Fact.** *$(e^{\sigma B_t - \sigma^2 t/2})$ martingale*

**Fact.** *with $\mu^* = r - \sigma/2$, discounted asset price $(e^{rt} S_t)$ martingale*

**Proposition 6.1.** *$(S_t^0, S_t)$ Black-Scholes, fix $T$, consider $\frac{\mathrm{d}\mathbb{P}*}{\mathrm{d}\mathbb{P}} = e^{\lambda B_T - \lambda^2 T/2}$ where $\sigma\lambda = \mu^* - \mu$, then under $\mathbb{P}^*$, discounted seet price $(e^{-rt} S_t)$ martingale*

**Fact.** *abuse notation in writing $\mathbb{P}^*$ instead of $\mathbb{P}$*

– time-$T$ contingent claim $C$ —— $\mathcal{F}_T$-measurable r.v.

– Black-Scholes price $V_0$ —— $V_0 = e^{-rT}\mathbb{E}^*(C)$

**Fact.** *fair price unique*

**Example.**

***(i)*** $C = S_T$

***(ii)*** $C = K$

– simple replicable claim —— constant $C_0$, $0 = t_0 \leq \cdots \leq t_n = T$, $\theta_k$ bounded $\mathcal{F}_{t_{k-1}}$-measurable

$$e^{rT}C = C_0 + \sum_1^n \theta_k (X_{t_k} - X_{t_{k-1}})$$

(Replicating strategy) at time $t_{k-1}$, buy $\theta_k$, then sell $\theta_k$ at time $t_k$, then buy $\theta_k$ bond

**Fact.** *any simple replicable claim can be replicated for cost $C_0$ at time 0, $V_0 = C_0$*

**Fact** (Brownian martingale representation theorem)**.** *every integrable $\mathcal{F}_T$-measurable contingent claim is limit in probability of simple replicable claims*

**Setting 11.** $S_0 = s$

**Fact.** $\log S_t = \log s + \sigma B_t + \mu t$

– $p(t, x, z) = \frac{1}{\sqrt{2\pi t}} e^{\frac{|x-z|^2}{2t}}$

– $p(\sigma^2 t, x(t), \cdot)$ —— density of $\log S_t$, $x(t) = \log s + \mu t$

– $\rho(t, s, \cdot)$ —— density of $S_t$

– $\dot{\rho}$ —— derivative in first argument

– $\rho'$ —— derivative in second argument

**Fact.** $y\rho(t, s, y) = p(\sigma^2 t, x(t), z)$, $z = \log y$

**Fact.** $\dot{\rho} = \frac{1}{2}\sigma^2 s^2 \rho'' + rs\rho'$

**Proposition 6.2.** *$F$ on $(0, \infty)$ continuous, polynomial growth, $t \in [0, T], s \in (0, \infty)$*
*$V(t, s)$ time-$t$ value of time-$T$ contingent claim $F(S_T)$, conditional on $S_t = s$*
*$V(t, s) = e^{-r(T-t)}\mathbb{E}^*(F(S_T)|S_t = s) = e^{-r(T-t)}\mathbb{E}(F(se^{\sigma B_{T-t} + \mu^*(T-t)}))$, then*

***(i)*** *$V$ continous on $(0, T) \times (0, \infty)$*

***(ii)*** *$V \in C^{1,2}$ on $(0, T) \times (0, \infty)$*

***(iii)*** *(Black Scholes PDE) $\mathcal{L}V = \dot{V} + \frac{1}{2}\sigma^2 s^2 V'' + rsV' - rV$ with $V(\cdot, T) = F$*

**Setting 12** (Binomial approximation to BS)**.** *Consider convergence of random walk to Brownian motion, special case* $(W^{(k)}_{n/k})$ *simple symmetric random walk on* $\{-1, 1\}$

- $1 + a_k = \exp\left(-\frac{\sigma}{\sqrt{k}} + \frac{\mu}{k}\right)$
- $1 + b_k = \exp\left(\frac{\sigma}{\sqrt{k}} + \frac{\mu}{k}\right)$
- $1 + r_k = \exp\left(\frac{r}{k}\right)$
- $S^{(k)}_t = S_0 \exp\left(\frac{\sigma W^{(k)}_t}{\sqrt{k}} + \mu t\right)$
- $S_t = S_0 \exp\left(\sigma B_t + \mu t\right)$
- $S^{(k)0}_t = S^0_t = \exp(rt)$

**Fact.**

***(i)*** $(S^0_t, S_t)$ *Black-Scholes of drift* $\mu$*, volatility* $\sigma$*, interest rate* $r$

***(ii)*** $(S^{(k)0}_{n/k}, S^{(k)}_{n/k})$ *binomial model of parameters* $a_k < r_k < b_k$*,* $p = \frac{1}{2}$

***(iii)*** $S^{(k)}_t \to S_t$ *uniformly on compacts in* $t$ *a.s.*

- $\mathbb{P}^{(k)*}$ —— martingale measures for binomial model
- $\mathbb{P}^{*}$ —— martingale measures for Black-Scholes model

**Fact.** $\mathbb{E}^{(k)*}(G(S^{(k)})) \to \mathbb{E}^{*}(G(S))$*, so can approximate fair price using Binomial model*

**Setting 13.** $\mu = \mu^* = r - \frac{\sigma^2}{2}$ *for convenience*

- expression $C = F(B)$ —— $F$ on $C[0, T]$
- terminal-value option $C = f(B_T)$

**Fact.** *for* $C = f(B_T)$*,* $V_0 = e^{-rT} \int f\left(\sqrt{T} y\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$

**Fact.** *May not be efficient way:*

***(i)*** *multiple assets —— exponentially growth computational cost*

***(ii)*** *want to compute pricing surface*

- pricing surface —— $V(t, s) = e^{-r(T-t)} \mathbb{E}^{*}(C | S_t = s)$

**Setting 14.** *terminal-value option* $C = g(S_T)$*,* $g$ *continuous, no more than linear growth*

- $f(x) = g(e^{\sigma x})$
- $u(t, x) = \mathbb{E}_x(f(B_t))$

**Fact.** $V(t,s) = e^{-r(T-t)}\mathbb{E}(g(se^{\sigma B_{T-t}+\mu(T-t)})) = e^{-r(T-t)}u\left(T-t, \frac{\log s+\mu(T-t)}{\sigma}\right)$

– $p_t(x,y) = \frac{1}{\sqrt{2\pi t}}e^{\frac{-(y-x)^2}{2t}}$

**Fact.** $u(t,x) = \int p_t(x,y)f(y)dy$

**Fact.** $\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial x^2} \Rightarrow \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}$

**Setting 15.** *assume can accurately approximate* $u(t, \pm L)$ *for all* $t$

– grid $\{(ik, jh)\} \subset [0,T]\times[-L,L]$ —— $k = \frac{T}{N}, h = \frac{L}{M}$

– grid points $U^i_j$ —— idea $U^i_j \approx u(ik, jh)$

– FTCS (forward-in-time, central-in-space) —— $\frac{U^{i+1}_j - U^i_j}{k} = \frac{U^i_{j-1} - 2U^i_j + U^i_{j+1}}{2h^2}$

- explicit
- first-order in time

– BTCS (backward-in-time, central-in-space) —— $\frac{U^{i+1}_j - U^i_j}{k} = \frac{U^{i+1}_{j-1} - 2U^{i+1}_j + U^{i+1}_{j+1}}{2h^2}$

- require to solve $2M \times 2M$ matrix inversion
- better stability
- first-order in time

– Crack-Nicolson —— $\frac{U^{i+1}_j - U^i_j}{k} = \frac{1}{2}\left(\frac{U^i_{j-1} - 2U^i_j + U^i_{j+1}}{2h^2} + \frac{U^{i+1}_{j-1} - 2U^{i+1}_j + U^{i+1}_{j+1}}{2h^2}\right)$

- require to solve $2M \times 2M$ matrix inversion
- better stability
- second-order in time

– Monte Carlo ——

- time-step $k = \frac{T}{N}$
- **(i)** $(B^{(N)}_t : t = ik)$ linear interpolation of random walk with step distribution $N(0,k)$
  **(ii)** $(X_t : t = ik)$ simple symmetric random walk with step-size $h = \sqrt{k}$
- generate sample $(B^{(N),i})$
- **(Idea)** $\frac{1}{n}\sum F(B^{(N),i}) \approx \mathbb{E}(F(B^{(N)})) \to \mathbb{E}(F(B))$

– fair price for European call $EC(x, K, \sigma, r, T) = \mathbb{E}^*(e^{-rT}(S_T - K)^+)$

– $\phi(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$

– $\Phi(a) = \int_{\infty}^{a}\phi(y)dy$

– $\bar{\Phi}(a) = 1 - \Phi(a)$

– $d^{\pm} = \frac{\log\left(\frac{x}{K}\right)+rT}{\sigma\sqrt{T}} \pm \frac{\sigma\sqrt{T}}{2}$

**Proposition 6.3** (Black-Scholes formula). $EC(x, K, \sigma, r, T) = x\Phi(d^+) - e^{-rT}K\Phi(d^-)$

- put-call parity —— $(S_T - K)^+ - (S_T - K)^- = S_T - K$
- fair price for forward contract —— $\mathbb{E}^*(e^{-rT}(S_T - K)) = x - e^{-rT}K$
- fair price for European put $EP(x, K, \sigma, r, T) = e^{-rT}K\bar{\Phi}(d^-) - x\bar{\Phi}(d^+)$

**Setting 16.** $v(x) = v(x, C, \sigma, r, T) = \mathbb{E}^*(e^{-rT}C)$

- **Sensitivities**
- Delta $\Delta = \frac{\partial v}{\partial x}$
- Gamma $\Gamma = \frac{\partial^2 v}{\partial x^2}$
- Vega $\mathcal{V} = \frac{\partial v}{\partial \sigma}$
- Rho $\rho = \frac{\partial v}{\partial r}$

**Example.** *$C$ European call,* $\Delta = \Phi(d^+), \mathcal{V} = x\phi(d^+)\sqrt{T}$

**Proposition 6.4.**

***(i)*** $\sigma \mapsto EC(x, K, \sigma, r, T)$ *increasing bijection*

***(ii)*** $\lim_{\sigma\to 0} EC(x, K, \sigma, r, T) = (x - e^{-rT}K)^+$

***(iii)*** $\lim_{\sigma\to\infty} EC(x, K, \sigma, r, T) = x$

- implied volatility $\sigma_{implied}(K, T)$ —— $EC(S_0, K, \sigma_{implied}(K, T), r, T) = EC_{market}(K, T)$

**Example.** *up-and-out call* $C = h(S_T)\mathbb{1}_{\{\sup_{0\le t\le T} S_t < A\}}$, $h(s) = (s - K)^+$, $A \ge \max\{S_0, K\}$

# 7 Fun fact

**Fact.** $(B_t)$ *Brownian motion, then* $tB_{\frac{1}{t}}$ *Brownian motion*