

Stochastic Financial Models

1 Utility and Mean-Variance analysis

- contingent claims — r.v. X
- utility function — non-decreasing

Fact. Y is preferred to X iff $\mathbb{E}(U(X)) \leq \mathbb{E}(U(Y))$

- indifferent
- risk neutral
- risk averse
- concave
- strictly concave

Proposition 1.1. *risk averse iff U concave*

- CARA with parameter γ — $\gamma \in (0, \infty)$, $U(X) = CARA_\gamma(x) = -\exp(-\gamma x)$
- CRRA with parameter R — $R \in (0, 1) \cup (1, \infty)$, $U(X) = CRRA_R(x) = \begin{cases} \frac{x^{1-R}}{1-R} & \text{if } x > 0 \\ -\infty & \text{otherwise} \end{cases}$
- CRRA with parameter 1 — $U(X) = CRRA_1(x) = \begin{cases} \log x & \text{if } x > 0 \\ -\infty & \text{otherwise} \end{cases}$
- constant absolute risk aversion — CARA
- constant relative risk aversion — CRRA

Fact (Arrow-Pratt coefficient of absolute risk aversion). $\omega + X$ preferred to ω iff $\frac{2\mathbb{E}(X)}{\mathbb{E}(X^2)} \geq -\frac{U''(\omega)}{U'(\omega)}$

Fact (Arrow-Pratt coefficient of relative risk aversion). $\omega(1+X)$ preferred to ω iff $\frac{2\mathbb{E}(X)}{\mathbb{E}(X^2)} \geq -\frac{\omega U''(\omega)}{U'(\omega)}$

- available claim \mathcal{A}
- reservation bid price $\pi_b(Y)$ — $\sup \pi$ st $\mathbb{E}(U(X+Y-\pi)) > \mathbb{E}(U(X^*))$
- reservation ask price $\pi_a(Y)$ — $\inf \pi$ st $\mathbb{E}(U(X-Y+\pi)) > \mathbb{E}(U(X^*))$

Proposition 1.2 (Ask above, bid below). \mathcal{A} convex, then $\pi_b(Y) \leq \pi_a(Y)$

Setting 1. \mathcal{A} affine space, U differentiable, strictly concave

- marginal price $\pi_m(Y)$ — $\pi_m(Y) = \frac{\mathbb{E}(U'(X^*)Y)}{\mathbb{E}(U'(X^*))}$
- single-period asset price model
- numeraire
- riskless bond
- interest rate — $r > -1$
- state-price density ρ — $S_0^i = \mathbb{E}(S_1^i \rho)$
- wealth ω_0
- portfolio θ

Example. no bond

$$\begin{array}{ll} \text{given} & \mathbb{E}(\theta \cdot S_1) = \theta \cdot \mu, \text{var}(\theta \cdot S_1) = \theta^T V \theta \\ \text{minimize} & \text{var}(\theta \cdot S_1) \\ \text{subject to} & \theta \cdot S_0 = \omega_0, \mathbb{E}(\theta \cdot S_1) = \omega_1 \end{array}$$

- mean-variance-efficient frontier — $\{\theta^*(\omega_1)\}$
- minimum variance portfolio θ_{\min}^* — minimise var over ω_1

Example. with bond

$$\begin{array}{ll} \text{minimise} & \theta^T V \theta \\ \text{subject to} & \theta^0 + \theta S_0 = \omega_0, \theta^0(1+r) + \theta \mu = \omega_1 \end{array}$$

Then, $\theta^* = \lambda \theta_m^*$

- market portfolio θ_m^* — $A(\mu - (1+r)S_0)$, $A = V^{-1}$

Setting 2. S_1 Gaussian, U CARA

Example. no bond

$$\begin{array}{ll} \text{maximise} & \mathbb{E}(U(\theta S_1)) \\ \text{subject to} & \theta S_0 = \omega_0 \end{array}$$

Example. with bond

$$\begin{array}{ll} \text{maximise} & \mathbb{E}(U(\bar{\theta} \bar{S}_1)) \\ \text{subject to} & \bar{\theta} \bar{S}_0 = \omega_0 \end{array}$$

then, $\theta^* = \gamma^{-1} \theta_m^*$

- beta/sensitivity β^i — $\beta^i = \frac{\text{cov}(S_1^i, \theta_m^* S_1)}{\text{var}(\theta_m^* S_1)}$
- μ^m — $\theta_m^* \mu$
- S_0^m — $\theta_m^* S_0$

Proposition 1.3. $\mu^i - (1+r)S_0^i = \beta^i(\mu^m - (1+r)S_0^m)$

- capitalization-weights of the relevant market index

Setting 3.

$$S_1 = (1+R)S_0, S_1^m = (1+R^m)S_0^m, \tilde{\beta}^i = \frac{\text{cov}(R^i, R^m)}{\text{var}(R^m)}$$

Fact. $\mathbb{E}(R^i) = r + \tilde{\beta}^i(\mathbb{E}(R^m) - r)$

2 Martingales

- conditional probability
- conditional expectation given event
- conditional expectation given \mathcal{G} , $\mathbb{E}(X|\mathcal{G})$

Theorem 2.1. $\mathcal{G} \subset \mathcal{F}$ sub- σ -algebra, X integrable, then \exists unique Y (up to a.s.) st

- (i) Y integrable
- (ii) Y \mathcal{G} -measurable
- (iii) $\mathbb{E}(Y\mathbb{1}_A) = \mathbb{E}(X\mathbb{1}_A)$ for all $A \in \mathcal{G}$

Fact. also true if replace integrable by non-negative

- $\mathbb{E}(X|Z)$ — $\mathcal{G} = \sigma(Z)$ for r.v. Z
- $\mathbb{P}(A|\mathcal{G})$ — $X = \mathbb{1}_A$

Fact. $\mathcal{G} = \sigma(B_n)$ discrete, then $\mathbb{E}(X|\mathcal{G}) = \sum \mathbb{E}(X|B_n)\mathbb{1}_{B_n}$ a.s.

Proposition 2.2. $\mathcal{G} \subset \mathcal{F}$, X, W integrable, then

- (i) $\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$
- (ii) X \mathcal{G} -measurable, then $\mathbb{E}(X|\mathcal{G}) = X$ a.s.
- (iii) X independent of \mathcal{G} , then $\mathbb{E}(X|\mathcal{G}) = E(X)$ a.s.
- (iv) $X \geq 0$ a.s. , then $\mathbb{E}(X|\mathcal{G}) \geq 0$ a.s.
- (v) $\mathbb{E}(\alpha X + \beta W|\mathcal{G}) = \alpha \mathbb{E}(X|\mathcal{G}) + \beta \mathbb{E}(W|\mathcal{G})$ a.s.

Proposition 2.3 (Tower property). $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ sub- σ -algebra, X integrable, then
 $\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = \mathbb{E}(X|\mathcal{H})$ a.s.

Proposition 2.4 (Taking out what is known). $\mathcal{G} \subset \mathcal{F}$ sub- σ -algebra, X integrable, Z \mathcal{G} -measurable, ZX integrable, then
 $\mathbb{E}(ZX|\mathcal{G}) = Z\mathbb{E}(X|\mathcal{G})$ a.s.

Proposition 2.5 (Averaging over independent variables). X_1, X_2 r.v. in $(E_1, \mathcal{E}_1), (E_2, \mathcal{E}_2)$, $\mathcal{G} \subset \mathcal{F}$, X_1 \mathcal{G} -measurable, X_2 independent of \mathcal{G} , F non-negative, let $f = \mathbb{E}(F(\cdot, X_2))$, then
 $\mathbb{E}(F(X_1, X_2)|\mathcal{G}) = f(X_1)$ a.s.

- filtration (\mathcal{F}_n)
- random process
- (X_n) adapted to (\mathcal{F}_n)
- natural filtration $(\mathcal{F}_n^X) \text{ — } \mathcal{F}_n^X = \sigma(X_k : 0 \leq k \leq n)$
- martingale —
 - (Adapted) X_n \mathcal{F}_n -measurable
 - (Integrable) $\mathbb{E}(|X_n|) < \infty$
 - (Martingale property) $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$ a.s.
- supermartingale — $\mathbb{E}(X_{n+1}|\mathcal{F}_n) \leq X_n$
- submartingale — $\mathbb{E}(X_{n+1}|\mathcal{F}_n) \geq X_n$

Fact. Any martingale also martingale in natural filtration (natural filtration smallest)

- martingale (continuous-time) — adapted, integrable, $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$ for all $s \leq t$
- (X_t) continuous — $t \mapsto X_t(\omega)$ continuous for all ω

Example. X_n i.i.d., $\mathcal{F}_0 = \{\emptyset, \Omega\}$, (\mathcal{F}_n) natural filtration

- (additive martingale) X_1 integrable, $\mathbb{E}(X_1) = 0$, $S_0 = 0$, $S_n = \sum_{k=1}^n X_k$
- (multiplicative martingale) X_1 non-negative, $\mathbb{E}(X_1) = 1$, $Z_0 = 1$, $Z_n = \prod_{k=1}^n X_k$

Example. (X_n) Markov chain, countable state space S , transition matrix P , natural filtration (\mathcal{F}_n) , bounded/non-negative f on S , let

$$Pf(x) = \sum p_{xy}f(y)$$

then if f subharmonic

$$\mathbb{E}(f(X_{n+1})|\mathcal{F}_n) = Pf(X_n) \geq f(X_n)$$

then, $(f(X_n))$ submartingale

- subharmonic — $f(x) \leq Pf(x)$ for all x
- random time — $T : \Omega \rightarrow \{0, 1, \dots\} \cup \{\infty\}$
- stopping time — $\{T \leq n\} \in \mathcal{F}_n$

Theorem 2.6 (Optional stopping). (M_n) martingale, T **bounded**, stopping time, then $\mathbb{E}(M_T) = \mathbb{E}(M_0)$

Fact. Doob's optional sampling theorem basic form

Theorem 2.7. (M_n) martingale, T **almost surely finite**, stopping time, suppose one of the following holds:

- (i) $|M_n| \leq C$ for all $n \leq T$, C constant
- (ii) $\mathbb{E}(T) \leq \infty$, $|M_n - M_{n-1}| \leq C$ for all $n \leq T$

Fact. T stopping time $\Rightarrow T \wedge n$ bounded stopping time

Counter Example. additive martingale, simple random walk, $T = \min\{n : S_n = 1\}$, then almost sure finite as recurrent, but $\mathbb{E}(T) = \infty$, $\mathbb{E}(S_T) = 1 \neq 0 = S_0$

Counter Example. multiplicative martingale, (X_k) i.i.d., $\mathbb{P}(X_1 = 0) = \mathbb{P}(X_1 = 2) = 1/2$, $T = \min\{n : Z_n = 0\}$, then $\mathbb{E}(T) = 2$, but $\mathbb{E}(Z_T) = 0 \neq 1 = Z_0$

Theorem 2.8. (M_n) martingale, T stopping time, then $(M_{T \wedge n})$ martingale

- previsible — H_n \mathcal{F}_{n-1} -measurable
- martingale transform of (M_n) by (H_n) — $Y_0 = 0$, $Y_n = \sum_1^n H_k(M_k - M_{k-1})$

Theorem 2.9 (Martingale transform). (M_n) martingale, (H_n) bounded, previsible, (Y_n) martingale transform, then (Y_n) martingale

Fact. model stock price as (M_n) , then

- (i) optional stopping \Rightarrow expected return $\mathbb{E}(M_T)$ the same no matter what stopping time
- (ii) (H_k) amount held between time $k-1$ and k , no bounded previsible strategy gives expected gain or loss

3 Pricing contingent claims

- asset price model $(\bar{S}_n)_{0 \leq n \leq T}$ with numeraire
- numeraire (S_n^0) — $S_n^0 > 0$
- discounted prices X_n^i — $X_n^i = S_n^i / S_n^0$
- $\bar{X}_n = (1, X_n)$
- interest rate r_n — $S_n^0 = (1 + r_n)S_{n-1}^0$
- risky assets (S_n)
- portfolio $\bar{\theta}_n$
- self-financing — $\bar{\theta}_n \bar{S}_n = \bar{\theta}_{n+1} \bar{S}_n$
- value process (V_n) — $V_0 = \bar{\theta}_1 \bar{X}_0$, $V_n = \bar{\theta}_n \bar{X}_n$
- total (discounted) value
- previsible — if $\bar{\theta}_n$ \mathcal{F}_{n-1} -measurable

Setting 4.

- (i) (\mathcal{F}_n) filtration generated by (\bar{S}_n) , $\mathcal{F} = \mathcal{F}_T$
- (ii) (S_n) takes countable values
- (iii) (S_n^0) deterministic process

Proposition 3.1. (θ_n) previsible process, then $\exists (\theta_n^0)$ st

- (i) (θ_n^0) previsible
- (ii) $(\bar{\theta}_n^0)$ self-financing with initial value V_0
- (iii) $V_T = V_0 + \sum_1^T \theta_n (X_n - X_{n-1})$

- contingent claim of maturity T — non-negative \mathcal{F}_T -measurable r.v.
- European option
- call with strike price K — $(S_T - K)^+$
- put with strike price K — $(S_T - K)^-$
- options
- exotic options — depending on entire path (S_n)
- barrier options —
 - knocked out

- knocked in
- up-and-out call — $C = \begin{cases} (S_T - K)^+ & \text{if } \max S_n < B \\ 0 & \text{otherwise} \end{cases}$
- down-and-in put — $C = \begin{cases} (S_T - K)^- & \text{if } \min S_n \leq B \\ 0 & \text{otherwise} \end{cases}$
- $\tilde{\mathbb{P}}$ equivalent to \mathbb{P} , $\tilde{\mathbb{P}} \sim \mathbb{P}$ — $\exists \rho$, st
 - $\mathbb{P}(\rho > 0) = 1$
 - $\tilde{\mathbb{P}}(A) = \mathbb{E}(\rho \mathbb{1}_A)$
- density $\rho = d\tilde{\mathbb{P}}/d\mathbb{P}$ for $\tilde{\mathbb{P}}$ wrt \mathbb{P}

Fact. $\tilde{\mathbb{E}}(X) = \mathbb{E}(\rho X)$

Fact. $\tilde{\mathbb{P}} \sim \mathbb{P}$ symmetric and transitive

Fact. $d\mathbb{P}/d\tilde{\mathbb{P}} = 1/\rho$ a.s.

- arbitrage for $(\bar{S}_n)_{0 \leq n \leq T}$ —
 - (i) $(\bar{\theta}_n)_{1 \leq n \leq T}$ previsible, self-financing
 - (ii) $V_0 = 0$
 - (iii) $V_T \geq 0$ a.s.
 - (iv) $V_T > 0$ with positive probability
- (\bar{S}_n) arbitrage free

Proposition 3.2. (X_n) martingale $\Rightarrow (X_n)$ arbitrage free

Fact. proof can be simpler when (θ_n) bounded

Setting 5. single period model, $\mathcal{F}_0 = \emptyset, \Omega$

Proposition 3.3. Y r.v. , following equivalent:

- (i) arbitrage free (i.e. no θ st $\theta Y \geq 0$ a.s. with $\theta Y > 0$ with positive prob)
- (ii) \exists equivalent probability measure $\tilde{\mathbb{P}}$ st Y integrable with $\mathbb{E}(Y) = 0$

- equivalent martingale measure $\tilde{\mathbb{P}}$ (risk neutral measure) — $\tilde{\mathbb{P}} \sim \mathbb{P}$, (X_n) martingale under $\tilde{\mathbb{P}}$

Theorem 3.4 ((1st) Fundamental theorem of asset pricing). *following equivalent:*

- (i) (\bar{S}_n) arbitrage free
- (ii) (\bar{S}_n) has equivalent martingale measure

Setting 6. C time- T contingent claim, $D = C/S_T^0$ discounted value

- attainable/replicable — \exists previsible, self-financing $\bar{\theta}_n$ st $C = \bar{\theta}_n \bar{S}_T$

Fact. *Alternative def:* $\exists V_0$ \mathcal{F}_0 -measurable, θ_n previsible st $D = V_0 + \sum^T \theta_n (X_n - X_{n-1})$

- fair price — V_0
- replicating portfolio/hedging portfolio — $\bar{\theta}_n$
- (\bar{S}_n) complete — all contingent claims attainable

Proposition 3.5. $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \sigma(\bar{S}_1)$

- (i) C non-negative, attainable, time- T contingent claim, $\tilde{\mathbb{P}}$ equivalent martingale measure, then fair price $V_0 = \tilde{\mathbb{E}}(D)$, $D = C/S_T^0$
- (ii) (\bar{S}_n) complete, numeraire non-random, then at most one equivalent martingale measure

- binomial model (Cox-Ross-Rubinstein model) — interest rate r , parameters $a < b$, R_i i.i.d. with parameter p
 - $S_n^0 = (1+r)^n$
 - $S_n = S_0 \prod (1+R_k)$
 - $\begin{cases} \mathbb{P}(R_1 = a) = 1-p \\ \mathbb{P}(R_1 = b) = p \end{cases}$

Proposition 3.6. *binomial model has arbitrage unless $r \in (a, b)$*

Proposition 3.7. (\bar{S}_n) binomial model, $r \in (a, b)$, define

(i) $p^* = \frac{r-a}{b-a}$

- (ii) equivalent prob measure $\frac{d\mathbb{P}^*}{d\mathbb{P}} = \left(\frac{p^*}{p}\right)^{U_T} \left(\frac{1-p^*}{1-p}\right)^{D_T}$ where $U_T = (T + S_T)/2$, $D_T = (T - S_T)/2$, S_T number of b

Then under \mathbb{P}^* ,

(i) R_1, \dots, R_T i.i.d., $\begin{cases} \mathbb{P}^*(R_1 = a) = 1-p^* \\ \mathbb{P}^*(R_1 = b) = p^* \end{cases}$

- (ii) (X_n) martingale under \mathbb{P}^*

Fact. $r = \mathbb{E}^*(R_1)$

Fact. Binomial model arbitrage free when $r \in (a, b)$

Setting 7. if $C = f(S_0, \dots, S_T)$, then

$$V(C) = \frac{\mathbb{E}^*(C)}{(1+r)^T} = (1+r)^{-T} \sum f(s_0, s_1, \dots, s_T) \mathbb{P}^*(S_1 = s_1, \dots, S_T = s_T)$$

Define recursive relation:

$$\begin{aligned} f_T(s_0, \dots, s_T) &= f(s_0, \dots, s_T) \\ f_n(s_0, \dots, s_n) &= (1-p^*)f_{n+1}(s_0, \dots, s_n, (1+a)s_n) + p^*f_{n+1}(s_0, \dots, s_n, (1+b)s_n) \end{aligned}$$

Proposition 3.8. $\mathbb{E}^*(f(S_0, \dots, S_T) | \mathcal{F}_n) = f_n(S_0, \dots, S_n)$, $\mathbb{E}^*(C) = f_0(S_0)$

Proposition 3.9. Define

$$\Delta_n(s_0, \dots, s_{n-1}) = \frac{f_n(s_0, \dots, s_{n-1}, (1+b)s_{n-1}) - f_n(s_0, \dots, s_{n-1}, (1+a)s_{n-1})}{(1+r)^{T-n}(b-a)s_{n-1}}$$

then $\theta_n = \Delta_n(S_0, \dots, S_{n-1})$ replicating portfolio for C

Fact. Binomial model complete when $r \in (a, b)$

Proposition 3.10. (W_n) simple random walk with $\mathbb{P}(W_1 = 1) = p$, let $M_T = \max W_n$, then with $k \leq T$, $2k - T \leq m \leq k$

$$\mathbb{P}(M_T = m, W_T = 2k - T) = \left(\binom{T}{k-m} - \binom{T}{k-m-1} \right) p^k (1-p)^{T-k}$$

Example. if $(1+a)(1+b) = 1$, then $S_n = S_0(1+b)^{W_n}$, so can give fair price of $C = F(S_T, \max S_n)$

4 Dynamic Programming

- state-space E
- action-space A

Setting 8. $F : \{0, \dots, T-1\} \times E \times A \times [0, 1] \rightarrow E$, (ϵ_n) i.i.d., $\mathcal{F}_n = \sigma(\epsilon_1, \dots, \epsilon_n)$

- adapted control $u = (u_n)_{k \leq n \leq T-1}$ — initial time k , u_n \mathcal{F}_n -measurable

Setting 9. initial state $x \in E$, adapted control u , define $X_k = x$, $X_{n+1} = F(n, X_n, u_n, \epsilon_{n+1})$
Write $X_n = X_n^u(k, x)$

- expected reward — $V^u(k, x) = \mathbb{E} \left(\left(\sum_{k=0}^{T-1} r(n, X_n^u(k, x), u_n) \right) + R(X_T^u(k, x)) \right)$

- reward function r, R — non-negative measurable function
- value function V — $V(k, x) = \sup_u V^u(k, x)$
- u optimal control from (k, x) — $V(k, x) = V^u(k, x)$

Proposition 4.1 (Bellman equation). *Let $Pv(n, x, a) = \mathbb{E}(v(n+1, F(n, x, a, \epsilon_{n+1})))$*

$$\begin{aligned} v(T, x) &= R(x) \\ v(n, x) &= \sup_{a \in A} \{r(n, x, a) + Pv(n, x, a)\} \end{aligned} \quad n = 0, \dots, T-1$$

Suppose $\exists a$ st

$$v(n, x) = r(n, x, a(n, x)) + Pv(n, x, a(n, x)) \quad n = 0, \dots, T-1$$

Then,

(i) $V = v$

(ii) optimal control $u_n^* = a(n, X_n^{u^*}(k, x))$

Fact. Possible variations:

- (i) r, R as costs
- (ii) mixture of costs and rewards
- (iii) time-dependent state-space E_n
- (iv) time-and-state-dependent action-space $A_{n,x}$
 - American call — family of time- T contingent claim $(1+r)^{T-\tau}(S_\tau - K)^+$
 - American put — family of time- T contingent claim $(1+r)^{T-\tau}(S_\tau - K)^-$

Setting 10. (S_n) binomial model, $r \in (a, b)$

Fact. complete \Rightarrow can hedge all C with $\mathbb{E}^*(C) = 0$

Example (American call). $\tau = T$ always optimal, American and European calls equivalent

Fact. fair price can be founded using Bellman equation

5 Brownian motion

- Brownian motion —
 - $B_0 = 0$
 - $(B_{s+t} - B_s) \sim N(0, t)$, independent of $\sigma(B_r : r \leq s)$
 - $t \mapsto B_t(\omega)$ continuous
- Brownian motion starting from x — $B_0 = x$
- Gaussian process — $\forall(t_1, \dots, t_n), (X_{t_1}, \dots, X_{t_n})$ multivariate normal

Proposition 5.1. (B_t) continuous process starting from 0, then following equivalent:

- (i) (B_t) Brownian motion
- (ii) (B_t) zero mean Gaussian process, $\mathbb{E}(B_s B_t) = s \wedge t$

Proposition 5.2 (Scaling property). (B_t) Brownian motion, set $\tilde{B}_t = c^{-1} B_{c^2 t}$, then (\tilde{B}_t) Brownian motion

Proposition 5.3. (B_t) Brownian motion, (B_t) exit every finite interval a.s.

- $\mathcal{F}_t = \sigma(B_s : s \in [0, t])$
- stopping time — $\{T \leq t\} \in \mathcal{F}_t$ for all t
- \mathcal{F}_T — $A \in \mathcal{F}_\infty$ st $A \cap \{T \leq t\} \in \mathcal{F}_t$

Proposition 5.4 (Strong Markov property). (B_t) Brownian motion, T a.s. finite stopping time. Define $\tilde{B}_t = B_{T+t} - B_T$, then

- (i) (\tilde{B}_t) Brownian motion
- (ii) independent of \mathcal{F}_T

Proposition 5.5. (B_t) Brownian motion, define $T_a = \inf \{t \geq 0 : B_t = a\}$, then

- (i) T_a stopping time
- (ii) T_a almost surely finite

Theorem 5.6. $(\Omega, \mathcal{F}, \mathbb{P})$ not discrete, m prob measure on \mathbb{R} , mean 0, variance 1, then $\exists (B_t), (W_t^{(k)})$ for all $k \in \mathbb{N}$ st

- (i) (B_t) Brownian motion
- (ii) $(W_{\frac{n}{k}}^{(k)})$ random walk with distribution m , $(W_t^{(k)})$ linear interpolation of values $\{\frac{n}{k}\}$
- (iii) $\frac{W_t^{(k)}}{\sqrt{k}} \rightarrow B_t$ uniformly on compacts in t a.s.

Fact. combination of Wiener's Theorem and Donsker's Invariance Principle

- Wiener measure

Proposition 5.7. let $T \geq 0$, $c \in \mathbb{R}$, $B = (B_t)_{\{0 \leq t \leq T\}}$ brownian motion, $\tilde{B}_t = B_t + ct$, then \forall measurable set $A \subset C[0, T]$, $\mathbb{P}(\tilde{B} \in A) = \mathbb{E}(\mathbb{1}_{\{B \in A\}} e^{cB_T - \frac{c^2 T}{2}})$

Fact. special case of Cameron-Martin theorem

Proposition 5.8 (Reflection principle). (B_t) Brownian motion, $a \geq 0$, set $T_a = \inf \{t \geq 0 : B_t = a\}$, define $\tilde{B}_t = \begin{cases} B_t & \text{if } t \leq T_a \\ 2a - B_t & \text{if } t > T_a \end{cases}$, then (\tilde{B}_t) Brownian motion

– maximum process — $M_t = \sup_{\{0 \leq s \leq t\}} B_s$

Fact. M_t same distribution as $|B_t|$

Proposition 5.9. T_a has density $h_a(t) = \frac{a}{\sqrt{2\pi t^3}} e^{-\frac{a^2}{2t}}$

- $p_t(x, y)$ — density of B_t starting at x
- $p_t^a(x, y) = p_t(x, y) - p_t(x, 2a - y)$

Proposition 5.10. $x \leq a$, (B_t) Brownian motion with density starting from x , then \forall non-negative measurable f , $\mathbb{E}_x(f(B_t) \mathbb{1}_{\{T_a > t\}}) = \int_{-\infty}^a f(y) p_t^a(x, y) dy$

6 Black-Scholes model

- Black-Scholes model — $S_t^0 = e^{rt}$, $S_t = S_0 e^{\sigma B_t + \mu t}$
- price of riskless bond S_t^0
- interest rate r
- price of risky asset S_t
- drift μ
- volatility σ

Fact. $(e^{\sigma B_t - \sigma^2 t/2})$ martingale

Fact. with $\mu^* = r - \sigma^2/2$, discounted asset price $(e^{rt} S_t)$ martingale

Proposition 6.1. (S_t^0, S_t) Black-Scholes, fix T , consider $\frac{d\mathbb{P}^*}{d\mathbb{P}} = e^{\lambda B_T - \lambda^2 T/2}$ where $\sigma\lambda = \mu^* - \mu$, then under \mathbb{P}^* , discounted stock price $(e^{-rt}S_t)$ martingale

Fact. abuse notation in writing \mathbb{P}^* instead of \mathbb{P}

- time- T contingent claim C — \mathcal{F}_T -measurable r.v.
- Black-Scholes price V_0 — $V_0 = e^{-rT}\mathbb{E}^*(C)$

Fact. fair price unique

Example.

(i) $C = S_T$

(ii) $C = K$

- simple replicable claim — constant C_0 , $0 = t_0 \leq \dots \leq t_n = T$, θ_k bounded $\mathcal{F}_{t_{k-1}}$ -measurable

$$e^{rT}C = C_0 + \sum_1^n \theta_k (X_{t_k} - X_{t_{k-1}})$$

(Replicating strategy) at time t_{k-1} , buy θ_k , then sell θ_k at time t_k , then buy θ_k bond

Fact. any simple replicable claim can be replicated for cost C_0 at time 0, $V_0 = C_0$

Fact (Brownian martingale representation theorem). every integrable \mathcal{F}_T -measurable contingent claim is limit in probability of simple replicable claims

Setting 11. $S_0 = s$

Fact. $\log S_t = \log s + \sigma B_t + \mu t$

- $p(t, x, z) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{|x-z|^2}{2t}}$
- $p(\sigma^2 t, x(t), \cdot)$ — density of $\log S_t$, $x(t) = \log s + \mu t$
- $\rho(t, s, \cdot)$ — density of S_t
- $\dot{\rho}$ — derivative in first argument
- ρ' — derivative in second argument

Fact. $y\rho(t, s, y) = p(\sigma^2 t, x(t), z)$, $z = \log y$

Fact. $\dot{\rho} = \frac{1}{2}\sigma^2 s^2 \rho'' + rs\rho'$

Proposition 6.2. F on $(0, \infty)$ continuous, polynomial growth, $t \in [0, T], s \in (0, \infty)$
 $V(t, s)$ time- t value of time- T contingent claim $F(S_T)$, conditional on $S_t = s$
 $V(t, s) = e^{-r(T-t)} \mathbb{E}^*(F(S_T) | S_t = s) = e^{-r(T-t)} \mathbb{E}(F(se^{\sigma B_{T-t} + \mu^*(T-t)}))$, then

- (i) V continuous on $(0, T) \times (0, \infty)$
- (ii) $V \in C^{1,2}$ on $(0, T) \times (0, \infty)$
- (iii) (Black Scholes PDE) $\mathcal{L}V = \dot{V} + \frac{1}{2}\sigma^2 s^2 V'' + rsV' - rV$ with $V(\cdot, T) = F$

Setting 12 (Binomial approximation to BS). Consider convergence of random walk to Brownian motion, special case $(W_{n/k}^{(k)})$ simple symmetric random walk on $\{-1, 1\}$

- $1 + a_k = \exp\left(-\frac{\sigma}{\sqrt{k}} + \frac{\mu}{k}\right)$
- $1 + b_k = \exp\left(\frac{\sigma}{\sqrt{k}} + \frac{\mu}{k}\right)$
- $1 + r_k = \exp\left(\frac{r}{k}\right)$
- $S_t^{(k)} = S_0 \exp\left(\frac{\sigma W_t^{(k)}}{\sqrt{k}} + \mu t\right)$
- $S_t = S_0 \exp(\sigma B_t + \mu t)$
- $S_t^{(k)0} = S_t^0 = \exp(rt)$

Fact.

- (i) (S_t^0, S_t) Black-Scholes of drift μ , volatility σ , interest rate r
- (ii) $(S_{n/k}^{(k)0}, S_{n/k}^{(k)})$ binomial model of parameters $a_k < r_k < b_k$, $p = \frac{1}{2}$
- (iii) $S_t^{(k)} \rightarrow S_t$ uniformly on compacts in t a.s.
- $\mathbb{P}^{(k)*}$ — martingale measures for binomial model
- \mathbb{P}^* — martingale measures for Black-Scholes model

Fact. $\mathbb{E}^{(k)*}(G(S^{(k)})) \rightarrow \mathbb{E}^*(G(S))$, so can approximate fair price using Binomial model

Setting 13. $\mu = \mu^* = r - \frac{\sigma^2}{2}$ for convenience

- expression $C = F(B)$ — F on $C[0, T]$
- terminal-value option $C = f(B_T)$

Fact. for $C = f(B_T)$, $V_0 = e^{-rT} \int f\left(\sqrt{T}y\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$

Fact. May not be efficient way:

- (i) multiple assets — exponentially growth computational cost

(ii) want to compute pricing surface

- pricing surface — $V(t, s) = e^{-r(T-t)} \mathbb{E}^*(C|S_t = s)$

Setting 14. terminal-value option $C = g(S_T)$, g continuous, no more than linear growth

- $f(x) = g(e^{\sigma x})$
- $u(t, x) = \mathbb{E}_x(f(B_t))$

Fact. $V(t, s) = e^{-r(T-t)} \mathbb{E}(g(se^{\sigma B_{T-t} + \mu(T-t)})) = e^{-r(T-t)} u\left(T-t, \frac{\log s + \mu(T-t)}{\sigma}\right)$

- $p_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}}$

Fact. $u(t, x) = \int p_t(x, y) f(y) dy$

Fact. $\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2 p}{\partial x^2} \Rightarrow \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}$

Setting 15. assume can accurately approximate $u(t, \pm L)$ for all t

- grid $\{(ik, jh)\} \subset [0, T] \times [-L, L]$ — $k = \frac{T}{N}, h = \frac{L}{M}$
- grid points U_j^i — idea $U_j^i \approx u(ik, jh)$
- FTCS (forward-in-time, central-in-space) — $\frac{U_j^{i+1} - U_j^i}{k} = \frac{U_{j-1}^i - 2U_j^i + U_{j+1}^i}{2h^2}$
 - explicit
 - first-order in time
- BTCS (backward-in-time, central-in-space) — $\frac{U_j^{i+1} - U_j^i}{k} = \frac{U_{j-1}^{i+1} - 2U_j^{i+1} + U_{j+1}^{i+1}}{2h^2}$
 - require to solve $2M \times 2M$ matrix inversion
 - better stability
 - first-order in time
- Crank-Nicolson — $\frac{U_j^{i+1} - U_j^i}{k} = \frac{1}{2} \left(\frac{U_{j-1}^i - 2U_j^i + U_{j+1}^i}{2h^2} + \frac{U_{j-1}^{i+1} - 2U_j^{i+1} + U_{j+1}^{i+1}}{2h^2} \right)$
 - require to solve $2M \times 2M$ matrix inversion
 - better stability
 - second-order in time
- Monte Carlo —
 - time-step $k = \frac{T}{N}$
 - (i) $(B_t^{(N)} : t = ik)$ linear interpolation of random walk with step distribution $N(0, k)$
 - (ii) $(X_t : t = ik)$ simple symmetric random walk with step-size $h = \sqrt{k}$
 - generate sample $(B^{(N), i})$
 - (Idea) $\frac{1}{n} \sum F(B^{(N), i}) \approx \mathbb{E}(F(B^{(N)})) \rightarrow \mathbb{E}(F(B))$

- fair price for European call $EC(x, K, \sigma, r, T) = \mathbb{E}^*(e^{-rT}(S_T - K)^+)$
- $\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$
- $\Phi(a) = \int_{-\infty}^a \phi(y) dy$
- $\bar{\Phi}(a) = 1 - \Phi(a)$
- $d^\pm = \frac{\log(\frac{x}{K}) + rT}{\sigma\sqrt{T}} \pm \frac{\sigma\sqrt{T}}{2}$

Proposition 6.3 (Black-Scholes formula). $EC(x, K, \sigma, r, T) = x\Phi(d^+) - e^{-rT}K\Phi(d^-)$

- put-call parity — $(S_T - K)^+ - (S_T - K)^- = S_T - K$
- fair price for forward contract — $\mathbb{E}^*(e^{-rT}(S_T - K)) = x - e^{-rT}K$
- fair price for European put $EP(x, K, \sigma, r, T) = e^{-rT}K\bar{\Phi}(d^-) - x\bar{\Phi}(d^+)$

Setting 16. $v(x) = v(x, C, \sigma, r, T) = \mathbb{E}^*(e^{-rT}C)$

- **Sensitivities**
- Delta $\Delta = \frac{\partial v}{\partial x}$
- Gamma $\Gamma = \frac{\partial^2 v}{\partial x^2}$
- Vega $\mathcal{V} = \frac{\partial v}{\partial \sigma}$
- Rho $\rho = \frac{\partial v}{\partial r}$

Example. C European call, $\Delta = \Phi(d^+)$, $\mathcal{V} = x\phi(d^+)\sqrt{T}$

Proposition 6.4.

- (i) $\sigma \mapsto EC(x, K, \sigma, r, T)$ increasing bijection
- (ii) $\lim_{\sigma \rightarrow 0} EC(x, K, \sigma, r, T) = (x - e^{-rT})^+$
- (iii) $\lim_{\sigma \rightarrow \infty} EC(x, K, \sigma, r, T) = x$

- implied volatility $\sigma_{implied}(K, T)$ — $EC(S_0, K, \sigma_{implied}(K, T), r, T) = EC_{market}(K, T)$

Example. up-and-out call $C = h(S_T) \mathbb{1}_{\{\sup_{0 \leq t \leq T} S_t < A\}}$, $h(s) = (s - K)^+$, $A \geq \max\{S_0, K\}$