Griffiths: Introduction to Electrodynamics

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1 Mathematical Foundations

2 Electrostatics

In electrodynamics, we are often interested in the force a collection of source charges $q_1 \dots q_n$ exerts on a test charge Q.

This problem becomes difficult if our source charge(s) are moving, so we constrain our cases to those where these charges are fixed. Electrostatics is the study of these cases.

2.1 The Electric Field

Definition 2.1. The principle of superposition tells us the force exerted from different charges are independent from each other. If F_1 is the force of q_1 acting on Q, then the force from the collection $\{q_i\}$ is simply $F = \sum_i q_i$.

Some notational convention:

- r' is a position vector of a source charge
- r is a position vector of a test charge
- $\mathscr{r} = r' - r$, the distance vector between our source and test

Definition 2.2. Coulomb's law gives us the force on our test charge Q from a single source charge q:

$$F = \frac{1}{4\pi\epsilon_0} \frac{qQ}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}}$$

Definition 2.3. We can start to think of the force from $\{q_i\}$ in aggregate by factoring a common Q from a sum of forces:

$$F = Q\boldsymbol{E}$$

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0} \sum_i \frac{q_i}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}}$$

$\boldsymbol{E}$ is called the **electric field** of the source charges $\{q_i\}$.

Notice that E is a vector function dependent on the location of our test charge. This function is a mathematical description of our "field" and we are encouraged to think of it as some jelly-like substances that permeates space.

This gives machinery if we have a discrete collection of points. However, often our charge is more like a continuous schmear over some line, surface or volume, motivating new expansions of Coulomb's law:

Definition 2.4 (Coulomb's for continuous line charge)**.** $E = \frac{1}{4\pi\epsilon_0} \int_l \frac{\lambda(r')}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}} \, \mathrm{d}\ell$

Here we are integrating over the contribution of each infinitessimal section of the continuous charge to the field. Notice that multiple things are "moving" - the displacement vector $\boldsymbol{\mathscr{r}}$ as well as the source charge (if it is not constant) $\lambda(r')$.

This extends to surfaces and volumes:

Definition 2.5 (Coulomb's for continuous surface charge)**.** $E = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(r')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\ \mathrm{d}a$

Definition 2.6 (Coulomb's for continuous volume charge)**.** $E = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(r')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\ \mathrm{d}\tau$

Example. 2.2 Calculation for the field distance z from the midpoint of a line of length $2L$ with constant charge λ.

We use Coulomb's law:

$$\int_{-L}^{L} \frac{1}{4\pi\epsilon_o}\frac{\lambda}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\ \mathrm{d}\ell$$

where:

$$\mathscr{r}^2 = z^2 + x^2$$

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{x\hat{\mathbf{x}} + z\hat{\mathbf{z}}}{\sqrt{z^2+x^2}}$$

Skipping integration steps we are left with our field $\boldsymbol{E}$. Note our field is in general a function of our field point but what we provide here is an evaluation of this function.

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_o}\frac{2\lambda L}{z\sqrt{z^2+L^2}}\hat{\mathbf{z}}$$

Exercise (2.4)**.** Calculate the electric field distance z away from the midpoint of a square loop of constant charge λ and side a.

Solution (2.4)**.** The field is symmetric with respect to each side of the square, so we can compute a value for each side and add them up using the principal of superposition.

To find the $\boldsymbol{E}'$ contributing from a single side, proceed as in (2.2), but set $z = \sqrt{z^2 + (\frac{a}{2})^2}$. We are interested in the vertical component of this vector, as horizontal components will cancel from opposing sides of the square, so our result is $4\cos(\theta)\boldsymbol{E}'$ where $\cos(\theta) = \frac{z}{\sqrt{z^2+(\frac{a}{2})^2}}$ and 4 comes from each side.

Exercise (2.7)**.** Calculate the electric field distance z from the surface of a sphere centered at the origin with radius R and constant charge λ.

Solution.

$$\mathbf{r} = z\hat{\mathbf{z}}$$

$$\mathbf{r}' = R(\sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}})$$

$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'$$

$$E = \frac{1}{4\pi\epsilon_0}\int_\theta\int_\phi \frac{\lambda}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\ \mathrm{d}a$$

Really do not want to calculate this and will use as motivation for using Gauss's law.

3 Divergence and curl of electric fields

4 Appendix