Hatcher: Algebraic Topology

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1 The Fundamental Group

1.1 Intuition

The Borromean rings are three linked circles where any two circles without a third are unlinked. It is a nice object that elucidates the difference between abelian and nonabelian fundamental groups.

Label the three rings A, B, and C. Consider C as an element of $\mathbb{R}^3 \setminus A \cup B$. C can be represented as $aba^{-1}b^{-1}$, where a and a^{-1} are forward and reverse oriented paths around A (and B). This element is not trivial, so the free group generated by a and b is not abelian.

Modify the ring so that A and B are linked. The same $C = aba^{-1}b^{-1}$ is now trivial. It is easiest to see this visually. Refer to the picture in Hatcher and slide the first loop a all the way around A until it is hanging over the circle. This loop is clearly contractible.

From this, we conclude there are two ways to tell if two circles are linked:

- Show that one of the circles is an element of the fundamental group of the complement of the other circle. (Simple example of loop wrapping around a circle.)
- Show the fundamental group of the complement of both of the circles is the abelian. (As in the Borromean ring example.)

1.2 Basic Constructions

Definition 1.1 (Path)**.** A path in the space X is a continuous map $f : I \to X$ where I is the unit interval $[0, 1]$.

Definition 1.2 (Homotopy of paths)**.** A continuous family of maps represented by the continuous map $f : I \times I \to X$

Example 1.3 (Linear homotopy)**.** Consider two paths f, g that share endpoints ($f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$). Any such paths are always homotopic by the homotopy

$$h_t(s) = (1 - t)f(s) + (t)g(s).$$

Proposition 1.4 (homotopy of paths with fixed endpoints is an equivalence relation)**.**

Definition 1.5 (The product of paths)**.** For paths f, g where $f(1) = g(0)$, we can define the product $h = f \cdot g$ as

$$h(s) = \begin{cases} f(2s) & 0 \leq s \leq 0.5 \\ g(2s-1) & 0.5 \leq s \leq 1 \end{cases}$$

Proposition 1.6 (The fundamental group is in fact a group)**.** *$\pi_1(X, x_0)$ with respect to the operator $[f] \cdot [g] = [f \cdot g]$ is a group.*

Definition 1.7 (Reparamterization of a path)**.** We can precompose any path f with $\phi : I \to I$ such that $f\phi \simeq f$. $f\phi$ is then the reparameterization of f.

Proof. To see our operator is well-defined, $[f \cdot g]$ depends only on $[f]$ and $[g]$. In other words, a homotopy exists between paths in $[f \cdot g]$ if and only if it is the composite of homotopies between paths in $[f]$ and $[g]$.

First, we verify associativity of the operator. Consider $(f \cdot g) \cdot h$. This product composes the composition of f and g with h. Visually, h is the first half and g and f occupy the last quarters of the product path. We can construct a piecewise function to change the proportions of these components - "shrink" h, keep g the same and "expand" f. $[(f \cdot g) \cdot h]\phi \simeq f \cdot (g \cdot h)$ gives us the desired equivalence.

Second, we check the identity property of the constant map c. f can be reparameterized by the piecewise function that speeds it up for the first half of I and holds it constant at $f(1)$ for the second half where $f\phi \simeq f \cdot c$.

Third, we verify the existence of an inverse for each $[f]$. Define $[\bar{f}]$ where $\bar{f}(s) = f(1-s)$. Then $f \cdot \bar{f}$ is homotopic to the constant path. To see this construct $h_t = f_t g_t$ where $f_t = f$ on $[0,t]$ and $f_t = f(t)$ on $[1-t, 1]$ and g_t is the inverse of this function. As t approaches 1, h_t has a larger constant region in its image starting from the middle and growing to the endpoints until $h_1(s) = f(0) = g(1)$ is our constant map at timepoint 1.

□

Proposition 1.8 (The fundamental group is independent of the choice of basepoint up to isomorphism)**.** *If f is a loop with basepoint x_1 and h is a path from x_0 to x_1. The map $\beta_h : \pi_1(X, x_0) \to \pi_1(X, x_1)$ defined as $\beta_h([f]) = [h \cdot f \cdot \hat{h}]$ is an isomorphism.*

Proof. β_h is well-defined. To see this, if $[f] = [f']$ implies that f and f' are homotopic, so certainly $h \cdot f \cdot \bar{h}$ and $h \cdot f' \cdot \bar{h}$ are homotopic and $\beta_h([f]) = \beta([f'])$. β_h is a homomorphism. $\beta_h([f \cdot g]) = [h \cdot f \cdot g \cdot \bar{h}] = [h \cdot f \cdot \cdot h \cdot \bar{h} \cdot g \cdot \bar{h}] = [h \cdot f \cdot \cdot h] \cdot [\bar{h} \cdot g \cdot \bar{h}] = \beta_h([f]) \cdot = \beta_h([g])$. β_h is an isomorphism.

□

Definition 1.9. A space is **simply connected** if it is path connected and has a trivial fundamental group.

Theorem 1.10 ($\mathbb{Z} \cong \pi_1(S^1)$)**.** *The map ϕ sending the integer n to the homotopy class of the loop $\omega_n(x) = (cos 2\pi nx, sin 2\pi nx)$ on S^1 with basepoint $(1,0)$ is an isomorphism.*

Proof. Observe that each loop ω_n can be expressed as the composition $p\tilde{\omega}_n$, where $p(s) = (cos 2\pi s, sin 2\pi s)$ and $\tilde{\omega}_n(s) = ns$ is the path in $\mathbb{R}$ from 0 to n. $\tilde{\omega}_n$ is called the **lift** of ω_n.

We define the image of $n \in Z$ under ϕ as the homotopy class represented by pf_n where f_n is any path in $\mathbb{R}$ from 0 to n. Since any such f_n shares endpoints with $\tilde{\omega}_n$, the paths are homotopic, and pf_n and $p\tilde{\omega}_n$ are also homotopic. Then $\phi(n) = [pf_n] = [w_n]$. This gives an "extended" definition of ϕ that will be useful in showing surfjectivity and injectivity.

We quickly verify ϕ is a homomorphism by considering $\phi(m+n)$ as $p\tilde{\omega}_{m+n} = p(\tilde{\omega_n} \cdot t_n(\tilde{\omega_m})) = \omega_m \cdot \omega_n = \phi(m) \cdot \phi(n)$ where $t_n(s) = s + n$ is just shifts our path in $\mathbb{R}$.

We now define two facts that show ϕ is a bijection:

(a) Given a path f in S^1 starting at x_0, there exists a unique lift of the path f, $\tilde{f}$, in $\mathbb{R}$ starting at $\tilde{x_0}$ for any choice of $\tilde{x_0} \in p^{-1}(x_0)$.

(b) Given a homotopy f_t in S^1 of paths starting at x_0, there exists a unique lifted homotopy $\tilde{f}_t$ of paths in $\mathbb{R}$ starting at $\tilde{x_0}$ for any choice of $\tilde{x_0} \in p^{-1}(x_0)$.

(a) gives injectivity of ϕ. Consider any loop f with basepoint $(1,0)$ reperesenting a homotopy class in $\pi_1(S^1)$. Then there exists $\tilde{f}$ starting at $0 \in p^{-1}((1,0))$ that ends at some $n \in \mathbb{Z} \subset p^{-1}((1,0))$. So exists some n where $\phi(n) = [f]$.

(b) gives surjectivity of ϕ. Consider $\phi(m) = \phi(n)$. Then the representative loops ω_m and ω_n are homotopic. Define this homotopy as f_t. (b) gives us $\tilde{f}_t$ where $\tilde{f}_t(0) = 0$ for all t (Our choice of $\tilde{x_0}$). $\tilde{f}_0(1) = m$ and $\tilde{f}_1(1) = n$, but the endpoint of the homotopy must be invariant with respect to time, so $m = n$.

We observe that both of these facts are specific examples of a more general fact (c)

(c) For an arbitrary space Y, given $F : Y \times I \to S^1$ and a lift $\tilde{F} : Y \times \{0\} \to S^1$ of $F\big|_{Y \times \{0\}}$, there is a unique lift $\tilde{F} : Y \times I \to S^1$ that restricts to the given $\tilde{F}$ on $Y \times \{0\}$.

Observe that (a) follows trivially when we consider Y to be a point. To see that (c) implies (b), observe the homotopy of paths in S^1, f_t given in (b) connects f_0 and f_1. (a) gives us a unique $\tilde{f}_0 : I \times \{0\} \to S^1$, a restriction of the lifted homotopy, and (c) gives us a unique lifted homotopy $\tilde{f}_t$ that restricts to $\tilde{f}_0$. Note that $\tilde{f}_0$ defines the endpoints of homotopic paths in $\mathbb{R}$ and is unique so (b) follows.

□

We can immediately use $\pi_1(S^1)$ to prove important theorems. The big idea is that each full rotation around the circle is a unique element of the group. This suggests, among other things, that a loop once around cannot be continuously deformed to a loop twice around.

Our arguments proceed by contradiction, by assuming that our desired result does not hold and showing that this assumption implies some homotopy between loops on the circle that is not allowed.

Theorem 1.11 (Fundamental Theorem of Algebra)**.** *Every non constant polynomial with coefficients in $\mathbb{C}$ has at least one root in $\mathbb{C}$.*

Proof. Assume our polynomial $p(z) = z^n + a_1 z^{n-1} ... a_n$ has no roots. Define the following $f_r : I \to \mathbb{C}$ for each real number:

$$f_r(s) = \frac{p(re^{2\pi i s})\backslash p(r)}{|p(re^{2\pi i s})|\backslash|p(r)|}$$

We claim this is a loop in the unit circle S^1. To see this, compute some values for fixed r and note for any value of r, $f_r(0) = 1$, $f_r(1) = 1$. Note that f_0 is a constant map with value 1 and is the trivial loop. Then by varying r we obtain a homotopy between any f_r and 0 with (the embedded) basepoint $(1, 0)$.

We now show that p must be constant. Choose a large r such that $r \geq \sum_n |a_n| \geq 1$. Observe then $r^n = rr^{n-1} \geq (\sum_n |a_n|) r^{n-1}$. This expression motivates a new homotopy for each f_r, let $f_{(r,t)}$ be our previous definition with $p_t(z, t) = z^n + t(z^{n-1}a_1 + ... a_n)$ substituted for each p. See that f_t has no zeros on the circle of complex values with radius r satisfying our expression (this is why we constructed it at all). Note that $f_{(r,0)} = e^{2\pi stn}$ (check). Then $f_{(r,t)}$ is a homotopy between f_r and homotopy class $[w_n] \in \pi_1(S^1)$. But f_r is homotopic to 0. Then w_n is also homotopic to 0 and n must be 0. Our $p(z) = a_n$ must be constant. □

Theorem 1.12 (Brouwer fixed point theorem)**.** *Any continuous map $h : D^2 \to D^2$ must have a fixed point $h(x) = x$.*

This theorem was initially proved by a gentleman named L.E.J. Brouwer circa 1910 and seems to be foundational to the rest of algebraic topology and other fields like differential topology.

We now introduce a theorem which proves that there must exist two places on the surface of the earth with both the same temperature and pressure.

Theorem 1.13 (Borsuk-Ulam theorem)**.** *A continuous map $f : S^2 \to \mathbb{R}^2$ must have at least one pair of antipodal points, $f(x) = f(-x)$.*

Lets build intuition for the two dimensional case by proving the theorem in one dimensions. For a map $f : S^1 \to \mathbb{R}$, construct continuous $h = f(x) - f(-x)$. Pick a point x, and evaluate h at points x and $-x$ halfway across the circle from each other. Notice $h(x) = -h(x)$, then there must exist some y where $h(y) = 0 \in [h(x), -h(x)]$ by intermediate value theorem. We proceed with proof in two dimensions:

Proof. Define a path $\eta(s) = (\cos 2\pi s, \sin 2\pi s, 0)$ around the equator of S^2 and a map $g : S^2 \to S^1$ defined as $g(x) = \frac{f(x)-f(-x)}{|f(x)-f(-x)|}$. Let $h = g\eta$. Check that η is nullhomotopic (one can shrink the equator to the origin) and therefore h is also nullhomotopic.

We now examine the lift $\tilde{h}$ of h to see that h cannot be nullhomotopic. Observe $g(x) = -g(-x)$ so $h(s) = -h(s + \frac{1}{2})$ for $s \in [0, \frac{1}{2}]$. Then $\tilde{h}(s + \frac{1}{2}) = \tilde{h}(s) + \frac{q}{2}$ for some odd integer q. (to visualize this, observe $h(s)$ and $-h(s + \frac{1}{2})$ are on opposite sides of S^1, so the distance between them in the lift is a path in $\mathbb{R}$ that is an arbitrary number of full loops and one half loop). Then h lies in the homotopy class represented by the generator of $\pi_1(S^1)$ times a nonzero integer q, so it cannot be nullhomootpic. □

This theorem says a few things. It tells us the surface of a sphere can never be one-to-one with an embedding in R^2, so it cannot be homeomorphic with a subspace of R^2. Think of trying to produce such a map with the surface of a sphere, S^2, and the plane - one would have to introduce a hole somewhere.

Theorem 1.14 (Fundamental group of a product space). *$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$ if X and Y are path connected.*

Example 1.15 (Torus). The fundamental group of the torus can be thought of as a pair of integers. Formally, $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$.

Definition 1.16 (Induced homomorphism). The map $\varphi : X \to Y$ where $\varphi(x_0) = y_0$ induces a homomorphism $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined as $\varphi_*([f]) = [\varphi f]$

We can briefly check some properties of this homomorphism. Let φ, ψ be maps:

- $(\varphi\psi)_* = \varphi_*\psi_*$
- $\mathbb{1}_* = \mathbb{1}$

These properties of the induced homomorphism make the fundamental group a functor.

The induced homomorphism allows us to describe relationships between topological spaces as relationships between fundamental groups.

Proposition 1.17. *For a given retraction $r : X \to A$, the induced homomorphism for the associated inclusion i_* is injective. If r is a deformation retraction, i_* is also an isomorphism.*

Proof. Because $ri = \mathbb{1}_A$, $r_*i_* = \mathbb{1}_{A*}$ so i_* is injective. To see $i_* : \pi(A) \to \pi(X)$ is also surjective, see that any loop $[x] \in \pi_1(X)$ homotopes to a loop in A by r_t if r_t is a deformation retract, so $i_*^{-1}([r_t x]) \in \pi_1(A)$. □

So spaces that are deformation retracts have isomorphic fundamental groups.

Note that deformation retracts are rather strict examples of homotopies as not only do they fix a basepoint between X and A for all timepoints, but they restrict to the identity map over their retracting space for all I ($r_t\big|_{A \times I} = \mathbb{1}$).

Lets work towards an induced isomorphism under the more general condition of homotopy equivalence.

Consider a homotopy $\varphi_t : X \times I \to Y$ that is not a deformation retract but fixes the basepoint in X ($\varphi_t(x_0) = y_0$ for all t). Then $\varphi_{0*} = \varphi_{1*}$ as $[\varphi_0 f] = [\varphi_1 f]$. Any pair of induced homomorphisms under a basepoint preserving homotopy are equivalent.

Now consider homotopy equivalences that are also basepoint preserving. Let $\varphi : X \to Y$ and $\psi : Y \to X$ be such equivalences. Then $\psi\varphi \simeq \mathbb{1}$ but we just saw then $(\psi\varphi)_* = \mathbb{1}_*$ so φ_* is injective. φ_* is also surjective as $(\varphi\psi)_* = \mathbb{1}_*$

We can actually see this is true even if our homotopy equivalence does not fix our basepoint.

Lemma 1.18. *Let $\varphi_t : X \to Y$ be a homotopy and $h\big|_{x_0 \times I}$ be a path between $\varphi_0(x_0)$ and $\varphi_1(x_0)$. Then $\varphi_{0*} = \beta_h \varphi_{1*}$*

Note. To see this, for a given loop f in X, we can build a new homotopy $\bar{h_t} \cdot \varphi_t f \cdot h_t$. Then $\beta_h(\varphi_0([f]))$ is homotopic to $\varphi_1([f])$.

Theorem 1.19 (The choice of basepoint between homotopy equivalent spaces is not important). *Consider the homotopy equivalence $\varphi : X \to Y$, then $\pi_1(X, x_0) \cong \pi_1(Y, \varphi(x_0))$.*

Proof. Consider the homotopy inverse $\psi : Y \to X$. Then $\psi\varphi \simeq \mathbb{1}$. We just saw that $(\psi\varphi)_* = \beta_h(\mathbb{1})_*$ for some path h from $\psi\varphi(x_0)$ to x_0. Then $\psi_*\varphi_* = \beta_h$ is an isomorphism and φ_* is injective. Using the same argument, $\varphi\psi \simeq \mathbb{1}$ and φ is also surjective. □

Exercise 1.1.

Proof. □

Exercise 1.2 (1.1.3). $\pi_1(X)$ is abelian iff the basepoint-preserving homomorphism β_h depends only on the choice of endpoints of h

Proof. Consider homotopy classes $[f],[h]$ in X. By definition, $\beta_h([f]) = [hfh^{-1}]$. Consider a new constant path $c = s \mapsto h(0)$ (the basepoint of X). Then if $\beta_c = \beta_h$, $[f] = \beta_c([f]) = \beta_h([f]) = [hfh^{-1}] = [h][f][h^{-1}]$ for any choice of f, h. $\pi_1(X)$ is abelian.

To see the reverse, consider two arbitrary paths g, h (they need not be loops). Because $\pi_1(X)$ is abelian, $\beta_g([f]) = [g^{-1}fg] = [h^{-1}fh] = \beta_h([f])$ for any h.

□

Exercise 1.3 (1.1.5)**.** Let X be some space. The following conditions are equivalent:

- Any map $S^1 \to X$ is homotopic to a constant map
- Any map $S^1 \to X$ extends to some map $D^2 \to X$
- $\pi_1(X, x_0) = 0$ for any $x_0 \in X$

Proof. To see (c) implies (a), consider any loop f. There exists a single homotopy class. Then f homotopic with the constant loop, $\mathbb{1}(I) = x_0$.

Now we show (a) implies (b). Let $f : S^1 \to X$ be our map and $h : S^1 \times I \to X$ be the homotopy relating f to some constant map. Let $g : D^2 \to S^1 \times I$ be a homeomorphism. Then hg is a continuous map where $hg\big|_{S^1} = f$.

Now we show (b) implies (c). Let $f : S^1 \to X$ be an arbitrary loop. Let $h : S^1 \times I \to D^2$ be a homotopy relating the identify on the circle to a constant map. Let g be the extension of f. Then gh homotopes f to a constant map whose image is a point in X. □

Exercise 1.4 (1.1.6)**.** The canonical map $\phi : \pi(X, x_0) \to [S^1, X]$ is onto if X is path connected and places the conjugacy classes of X in one-to-one correspondence with $[S^1, X]$.

Proof. To see ϕ is surjective if X is path connected, consider $[f]$ in $[S^1, X]$. Let h be a path from x_0 to any point on the path f (note that h is constant if x_0 already lies on f). Then $\phi^{-1}([f]) = [\bar{h}fh]$.

A proof of one-to-one correspondence between $[S^1, X]$ and conjugacy classes of π_1 is equivalent to showing $\phi([f]) = \phi([g])$ iff $[f]$ conjugates $[g]$ in π_1.

To see the forward direction, let ψ_t be the basepoint free homotopy relating $\phi([f])$ and $\phi([g])$. Then $\psi_{0*} = \beta_h \psi_{1*}$, where h is the path induced by the image $\psi_t(0)$. Notice that h starts and ends at x_0 because $[f]$ and $[g]$ had basepoints of x_0 under the preimage of ψ, so h is also a loop. Then $\psi_{0*} = \beta_h \psi_{1*}$ is equivalent to $[h \cdot g \cdot \bar{h}] = [f]$ which implies $[h] \cdot [g] \cdot [h]^{-1} = [f]$.

To see the reverse direction, let $[h]$ be the element of π_1 that conjugates $[f]$ and $[g]$.

□

1.3 Van Kampen's Theorem

To understand free products, lets review the direct products.

Definition 1.20 (Direct product of groups)**.** Given an indexed set of subgroups $\{G_\alpha\}$, our direct product group $\prod_\alpha G_\alpha$ is the set of functions $\{f = \alpha \mapsto g_\alpha \in G_\alpha\}$.

Some facts that are useful to verify:

1. $\bigcap G_\alpha = 1$. (Every element is unique in the product, even if two copies of the same group are used in the product.)
2. For each factor group, there exists a surjective "projection homomorphism" $\pi_i : G \to G_\alpha$ where $G_\alpha = \{(1 \dots g_i \dots 1)\}$.
3. $G \backslash G_\alpha \cong \{(g_1, ...g_{\alpha-1}, g_{\alpha+1}, ...g_n)\}$ (To see this, construct a homomorphism that erases α component and invoke the First Isomorphism Theorem).
4. Elements of each factor group commute with each other.

Proposition 1.21 (Proof that direct products are commutative up to isomorphism.)**.** *If $G = H \times K$, then $hk = kh$.*

Proof. Let $\psi : G \to H \times K$ be an isomorphism carrying elements of G to their tuple representation. $\psi(kh) = \psi(k)\psi(h) = (1, k)(h, 1) = (h, k) = \psi(hk)$. □

It is helpful to think of each element of the product as embedded in some latent tuple.

Now we might want to compose a group that is not commutative amongst factors. This will help us distinguish, for example, $\mathbb{Z} \star \mathbb{Z}$ (disjoint circles) from $\mathbb{Z} \times \mathbb{Z}$ (linked circles or torus). We introduce the free group:

To use the language of category theory, the free product is the coproduct in the category of groups.

Formally:

We show the binary operator of free groups is associative.

Associativity of free groups. □

The proof of Van Kampen's also relies on basic facts about homomorphisms and quotient groups. We derive some of these facts here to refresh the dome.

Proposition 1.22. *Consider $\phi : G \to H$ where $ker\phi = K$. If $X = \phi^{-1}(a)$ is an element of G/K, then for any $u \in X$, $X = \{uk \mid k \in K\}$.*

. To see $uK \subseteq X$, observe $\phi(uk) = \phi(u)\phi(k) = \phi(u) \in X$. To see $X \subset uK$, consider $x \in X$ and notice that $\phi(u^{-1}x) = 1$. Then $u^{-1}x = k$ and $x = uk \subset uK$. □

Proposition 1.23. *Consider $\phi : G \to H$ where $ker\phi = K$. The group operation in G/K is well-defined.*

. Consider arbitrary $X, Y \in G/K$ where $X = aK$ and $Y = bK$. Let $Z = XY$. To see multiplication is independent of choice of representative, pick arbitrary representatives $u \in X$ and $v \in Y$. Then $uv \in \phi^{-1}(ab)$. □

In fact, multiplication of quotient group elements is well defined under a more general condition - that our coset is a normal subgroup.

Proposition 1.24. *Let $uH, vH \in G/H$. Then $(uH)(vH) = uvH$ is well defined iff $H \trianglelefteq G$.*

. For the forward argument, choose arbitrary $x, x^{-1} \in G$. Then $(xH)(x^{-1}H) = H$, so $(xh)(x^{-1}1) \in H$. Then H is normal as desired. For the reverse argument, consider alternative representatives $u' \in uH$ and $v' \in vH$. We must show $u'v' \in uvH$. $u' = um$ and $v' = vn$ for some $m, n \in H$. Then $(um)(vn) = uvv^{-1}mvn$. But $v^{-1}mv \in H$ by assumption (denote this n'). Then $uvv^{-1}mvn = uvn'n \in uvH$. □

Theorem 1.25 (Van Kampen's Theorem)**.** *Let $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ be the homomorphism induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ and $j_\alpha : \pi_1(A_\alpha) \to \pi_1(X)$ be the homomorphism induced by $A_\alpha \hookrightarrow X$.*

Observe $\phi(i_{\alpha\beta}(\omega)i_{\beta\alpha}(\omega)^{-1}) = j_\alpha i_{\alpha\beta}(\omega) j_\beta i_{\beta\alpha}(\omega)^{-1} = 1$ because $j_\alpha i_{\alpha\beta} = j_\beta i_{\beta\alpha}$. (Recall ϕ is the inclusion homomorphism applied to each letter in the word). Clearly all such elements of the free group are in the kernel of ϕ. We will show the normal group generated by such elements are exactly the kernel of ϕ.

Proof. To see N is the kernel of ϕ, we show that $\star_\alpha \pi_1(A_\alpha) \backslash N \cong \pi_1(X)$. Observe $[f_i]_\alpha N = [f_i]_\beta N$ by the definition of N ($[f_i]_\alpha [f_i]_\beta^{-1} \in N$). Furthermore $[f_1][f_i]_\alpha[f_2]N = [f_1][f_i]_\beta[f_2]N$ as N is normal so the group operation is well defined on cosets.

□

Note. Van Kampen's allows us to treat the fundamental group of wedge sums as the free group of summands.

It gives us the isomorphism $\star_\alpha \pi_1(X_\alpha) \cong \pi_1(A_\alpha)$

$\pi_1(X_\alpha) \cong \pi_1(A_\alpha)$

where $A_\alpha = U_\alpha \bigvee_{\beta \neq \alpha} X_\beta$.

Note. Group descriptions of basic objects

- $\pi_1(S^1) \cong \mathbb{Z}$

- $\pi_1(S^2) \cong 0$
- The fundamental group of a torus is $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \star \mathbb{Z}$

Van Kampen's can be used to derive a powerful result about 2-dimensional complexes. If we consider the decomposition of such a complex as a 1-skeleton (wedge sum of circles) with a collection of attached 2-cells, then the fundamental group of this complex is the free group on some number of generators corresponding to the number of circles in the 1-skeleton modulo a normal group generated by loops where the 2-cells are attached. Recall 2-dimensional complexes includes quite a broad group of objects we are familiar with, such as the orientable surfaces of genus 2 (M_2 being the torus) and the non-orientable surfaces like the mobius strip and klein bottle.

Theorem 1.26 (van Kampen's applied to 2-dimensional cell complex)**.** *Let X be some path connected space and Y be the result of attaching a collection of 2-cells $\{e_\alpha\}$. Then the isomomorphism induced by the inclusion $\pi(X) \to \pi(Y)$ has a kernel that is exactly $N =< \bar{\gamma_\alpha} \varphi_\alpha \gamma_\alpha >$. Then, (by van Kampen's), $\pi(X)/N \cong \pi(Y)$*

We can use this theorem to compute the fundamental group of the torus from the abelianization of the free group in two generators. Let $X = S^1 \vee S^1$ and $Y = S^1 \times S^1$ and observe that X is the one-skeleton of the torus and Y is obtained by attaching a 2-cell to this skeleton along the loop $a^{-1}b^{-1}ab$ (trace the edges of a rectangle).

We know $\mathbb{Z}\star\mathbb{Z}/[\mathbb{Z},\mathbb{Z}] \cong \mathbb{Z}\times\mathbb{Z}$. Because $\pi(X) \cong \mathbb{Z}\star\mathbb{Z}$ and $\pi(Y) = \mathbb{Z}\times\mathbb{Z}$, it remains to see that $N \cong [\mathbb{Z},\mathbb{Z}]$ to have our result.

Theorem 1.27. *If $g \neq h$, then $\pi_1(M_g)$ and $pi_1(M_h)$ are not isomoprhic. Nor are the spaces homotopy equivalent.*

Proof. If $M_g \simeq M_h$, then their fundamental groups are isomorphic. These groups are the abelianizations of free groups in $2g$ and $2h$ generators respectively (this is immediate from the previous result). Because these groups are isomorphic, g and h must be the same. □

We will used the fact that a punctured n-sphere is homeomorphic to the

Proposition 1.28 (Proof of generalized stereographic projection)**.** *There exists a homeomorphism between $S^n - 1$ and*

Exercise 1.5 (1.2.4)**.** Compute $\pi_1(\mathbb{R}^3 - X)$ where X is the union of n lines through the origin.

Proof. Observe the complement deformation retracts to S^2 with $2n$ points removed (the poles of our n lines). This object is homoemorphic to $\mathbb{R}^2$ with $2n-1$ points removed. (maybe construct the explicit stereographic projection). We can use van kampens to show the fundamental group of this space is the free group of $2n-1$ generators. □

Exercise 1.6 (1.2.6)**.** Supppose a space Y is obtained from a path-connected space X by attaching n-cells for a fixed $n \geq 3$. Then show the inclusion $X \hookrightarrow Y$ induces an isomorphism on π_1. Use this result to then show that the complement of a discrete subspace of R^n is simply-connected if $n \geq 3$.

Proof. We proceed using the structure of the proof of 1.26. Let A be Y with a hole in each of the attached n-cells. Let B be the union of the attached n-cells. See that A and B are both open and $Y = A \cup B$, so we invoke van Kampen's theorem: $\pi_1(Y) = \pi_1(A) \star \pi_1(B)/N$ where N is defined in the usual way.

As before, A deformation retracts onto X, so $\pi_1(A) \cong \pi_1(X)$, and B is contractible, so $\pi_1(B) = 0$. Then it remains to show that N generated by $\{i_{AB}(\omega)i_{BA}(\omega)^{-1} | \omega \in \pi_1(A \cap B)\}$ is trivial (where the map i_{AB} is defined as $i_{AB} : \pi_1(A \cap B) \to \pi_1(A)$). See that any loop $i_{AB}(\omega)$ is contractible, because despite introducing a hole, the loop can homotope around it within the interior of the n-cell (if $n \geq 3$). Any $i_{BA}(\omega)$ is trivially contractible, being included into an open subset of an n-cell with no holes. Then $\pi_1(Y) \cong \pi_1(X)$.

To prove our last result, let X be our space of interest (the complement of some discrete subspace of R^n), we can obtain a covering $\{A_\alpha\}_\alpha$ where each $A_\alpha = X - x_\alpha$ is the complement of each point in the discrete subspace. See each A_α is open. Then $\pi_1(Z) = \star\pi_1(A_\alpha)/N$.

□

Exercise 1.7 (1.2.7)**.** Construct a cell complex for the two-sphere with north and south poles identified and use this to compute the fundamental group of this space.

Proof. We first construct the cell complex. Begin with a 0-cell x and attach a 1-cell e to x to form a circle. Then take a 2-disk σ and attach its boundary circle $\delta\sigma$ to e in two pieces, $\delta_1\sigma$ traces one endpoint to the other and attaches to e, while $\delta_2\sigma$ completes the circle in the same orientation but attaches to $\bar{e}$.

Then, by van Kampen's, our fundamental group is isomorphic to $< a \mid aa^{-1} >$, but this is just the free group on one generator, so $\pi_1(X) \cong \mathbb{Z}$.

□

Seeing that the cell complex is the same as the sphere with north and south poles identified was not immediately obvious.

This post was helpful (notation was taken from Lee Mosher's solution), viewing our space as the quotient map $f : D_2 \to X$ performing exactly two identifications. 2-disks can take different forms equivalent up to homeomorphism and their boundaries can simply be circles. Seeing that S^2 is simply D_2 attached along e and $\bar{e}$, where e is an arc connecting the poles, was crucial.

Exercise 1.8 (1.2.8)**.** Consider a surface M_g of genus g that is split into two closed surfaces M_h and M_k by a circle C. M_h' and M_k' are constructed by deleting an open disk from each. Show that neither M_h' or M_k' retract to their boundary circle, but both (and all of M_g) retract to a lateral C'.

More familiarity with the two torus (really genus g surfaces in general) and practice with abelianization is needed to tackle this problem.

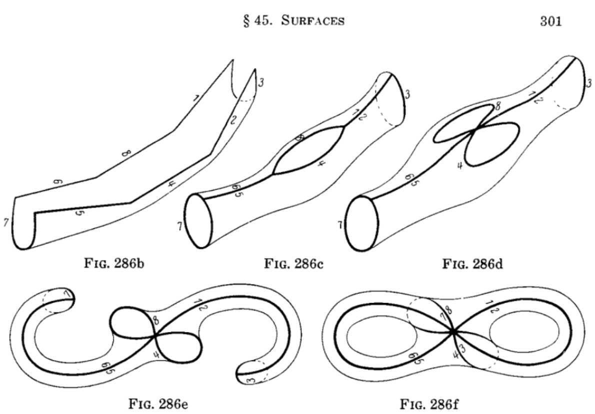

Figure 1: The geometry of the two torus. Hilbert's *Geometry and the Imagination*

Note (Abelianization of $\pi_1(M_1)$)**.** Consider $\pi_1(M_1) =< a, b \mid [a, b]$ as the familiar free group on two generators modulo the commutator (where $[a, b] = aba^{-1}b^{-1}$).

The abelianization of $\pi_1(M_1)$ is then $\pi_1(M_1)$ $[\pi_1(M_1), \pi_1(M_1)]$

1.4 Covering Spaces

Covering spaces let us (1) calculate fundamental groups of structures and (2) think about algebraic properties using geometric intuition. (I do not really understand how yet, so revisit this explanation after some exercise.)

Definition 1.29. Given some space X, the space $\tilde{X}$, along with map $p : \tilde{X} \to X$, is a **covering space** if each $x \in X$ *has some* open neighborhood U of x, $p^{-1}(U)$ is a disjoint union of open sets where each one is mapped homeomorphically onto U by p. These open sets are called **sheets**.

$p^{-1}(U)$ is allowed to be empty, so p need not be surjective.

The helicoid (denoted S) is a good example that is easy to visualize. The helicoid can be parameterized as:

$$x = s\cos 2\pi t$$
$$y = s\sin 2\pi t$$
$$z = t$$

for $s \in (0, \inf), t \in \mathbb{R}$

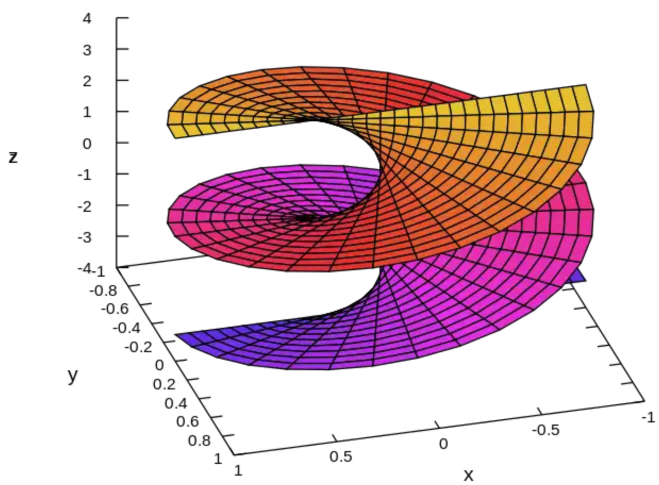

Figure 2:

Then $p : S \to \mathbb{R} \setminus \{0\}$ given by $(x, y, z) \mapsto (x, y)$ defines a covering space.

I found this is best seen by looking at single line segments extending radially from the origin. Consider $X = (0.5, 1.5) \times \{0\}$. $p^{-1}(X)$ is a collection of disjoint segments. Our parameterization allows this segment of the x axis to exist whenever we make one full rotation around the circle. There are an infinite number of these segments and each maps to X homeomorphically.

Once place to see the utility of covering spaces is with oriented graphs.

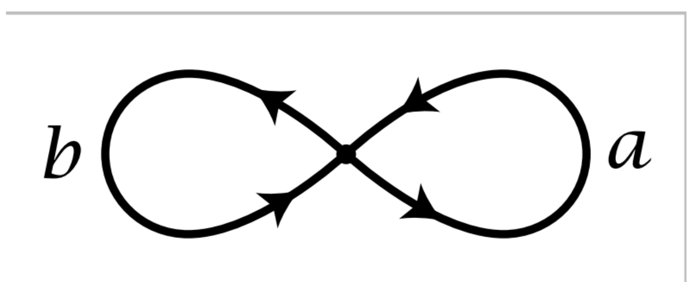

Figure 3:

Let X (Fig. 3) be basepoint with two loops labeled a and b. Now consider any graph, $\tilde{X}$, where edge vertex has four edges. If we label, with a or b, and orient each edge we obtain a structure that looks like X (respecting orientation!) as we zoom into each vertex. We can call this an **2-oriented graph**.

If we construct a map $p : \tilde{X} \to X$, where the interior of each edge in $\tilde{X}$ maps homeomorphically to corresponding labeled edge in X following the orientation of both edges, we satisfy the properties for a covering space. It follows both that every oriented graph is a covering space of X and every covering space of X can be 2-oriented (every vertex has 4 edges that can be labeled so the local picture looks like X).

We are given a table of some of these $\tilde{X}$.

Theorem 1.30. $\pi(X) \cong < a, b^2, bab^{-1} >$

Proof. Hatcher instructs us that we can use Van Kampen's. $\square$

When we consider the fundamental group of the covering space as the "image subgroup" of $\pi_1(X)$, $p_*(\pi_1(\tilde{X})) \leq \pi_1(X)$, we recover yet another beautiful correspondence between algebra and geometry - there is a one to one correspondence between the subgroups $\pi_1(X)$ and covering spaces of X.

A few more algebraic interpretations of covering spaces:

- If we change our basepoint in a covering space that is otherwise the same, $p_*(\pi_1(\tilde{X}, x_1))$ is a conjugacy subgroup of $p_*(\pi_1(\tilde{X}, x_0))$ in $\pi(X)$ where the conjugating element is the loop connecting the basepoints.
- If a **symmetry** is an automorphism on a graph that preserves labels and orientations, then a graph can be "more" symmetric if it has more automorphisms and "the most" symmetric if every possible permutation of the graph vertexes have automorphisms with the desired properties. We will show that "the most symmetric" graphs are exactly those induced subgroups that are normal in $\pi_1(X)$.

We now define three **lifting properties** of covering spaces and show some applications. Given some covering space and a map into X, these properties tell us when a lift of a homotopy exists, when a lift of a regular map exists and when lifts are unique.

Proposition 1.31 (Homotopy lifting property)**.** *Given a covering space $p : \tilde{X} \to X$, a homotopy $f_t : Y \to X$ and a lift of timepoint 0 of the homotopy $\tilde{f}_0 : Y \to \tilde{X}$, there exists a unique lift of the entire homotopy $\tilde{f}_t : Y \to X$.*

When Y is a point, this tells us that paths in X induce unique paths in the covering space.

When Y is I, this tells us that homotopies in X induce unique homotopies in the covering space.

The key idea here is that these homotopies are unique. If we consider the lift of a constant path, this implies that the lifted path is also constant

Note. Check this. If $p\tilde{f} = f$ and f is constant, if $\tilde{f}$ had anything other than $\tilde{f}(0)$ in its image, what does that imply about the set of allowed $\tilde{f}$s? If $\tilde{f}$ is itself constant, then $p\tilde{f} = f$ obvioulsy holds.

We will use this to show p_* is injective.

Proposition 1.32. *The induced homomorphism $p_* : \pi_1(\tilde{X}, \tilde{x_0}) \to \pi_1(X, x_0)$ is injective. The image subgroup $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ consists of loops with baspoint x_0 that lift to loops in $\tilde{X}$ with basepoint $\tilde{x_0}$.*

Proof. To prove injectivity, consider two distinct homotopy classes, $[\tilde{f_0'}]$ and $[\tilde{f_0''}]$, that have the same image under p_*, so $p_*([\tilde{f_0'}]) = p_*([\tilde{f_0'}])$. Indeed $p\tilde{f_0'} \simeq p\tilde{f_0''}$, given by some homotopy f_t, so our homotopy lifting property gives us a lifted homotopy $\tilde{f}_t$ where $\tilde{f_0'} \simeq \tilde{f_0''}$. Then $[\tilde{f_0'}] = [\tilde{f_0''}]$, so p_* is injective.

(This is a bit more explicit than Hatcher's proof using the kernel of p_*)

A loop in X with basepoint x_0 that lifts to a loop in $\tilde{X}$ with basepoint $\tilde{x_0}$ is by definition an element of $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$.

The second statement is a bit bizarre, because should every element in the image subgroup, represented as $p_*\tilde{f}$, admit a straightforward lift $\tilde{f}$. I do not see where I have to invoke homotopy lifting, unless you consider some other representative of each homotopy class $g \simeq p\tilde{f}$, then $\tilde{g} \simeq \tilde{f}$, so the representative admits the desired lift. $\square$

Proposition 1.33. *The number of sheets in a covering space p is equal to the index of $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ in $\pi_1(X, x_0)$*

Proof. Let g be a loop in $\pi_1(X, x_0)$. $\tilde{g}$ is its lift by the homotopy lifting property. Note that $\tilde{g}$ need not be a loop! Consider the lift of a single wrap around the circle in $p : \mathbb{R} \to S^1$.

Each representative of our coset, where $H = p_*(\pi_1(\tilde{X}, \tilde{x}))$, $H[g] = h \cdot g$ has a lift $\tilde{h} \cdot \tilde{g}$. Notice that this lift is a path that starts at $\tilde{x_0}$ and ends at $\tilde{g}(1)$.

We construct a bijection $\phi : H[g] \to p^{-1}(x_0)$ defined as $H[g] \to \tilde{g}(1)$. Because $\tilde{X}$ is path connected, for each point in $p^{-1}(x_0)$, we can draw a path to it from $\tilde{x_0}$. Notice that the projection of this path is always a loop at x_0 by definition of p. Then ϕ is surjective. If $\theta(H[g]) = \theta(H[g'])$, then $g(1) = g'(1)$ and $g \cdot \bar{g'}$ lifts to a loop in $\tilde{X}$ with basepoint $\tilde{x_0}$. Then $[g \cdot \bar{g'}] \in H$ and $H[g] = H[g']$ (θ is injective). $\square$

We have two more properties. One will show when a lift exists and another will show when lifts are unique.

Proposition 1.34 (Lifting criterion)**.** *Given a covering space $p : \tilde{X} \to X$ and some map $f : Y \to X$, there exists a lift of this map, denoted $\tilde{f} : Y \to \tilde{X}$, iff $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x_0}))$.*

Proof. The necessary condition is straightforward. The existence of a lift implies that $f = p\tilde{f}$, then $f_* = p_*\tilde{f}_*$, so certainly $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x_0}))$

The sufficient condition is less straighforward. We proceed by constucting our lift $\tilde{f}$ explicitly. Let $\tilde{f} = y \mapsto \tilde{f\gamma}$, where γ is any path from y_0 to y. We will show that this function is well-defined - that it does not matter what path we draw to y.

Consider another path γ'. We build a loop $\gamma \cdot \bar{\gamma}'$. Consider $f(\gamma\bar{\gamma}')$. By our assumption, this is some element of $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$, call it $[x]$, so $f(\gamma\bar{\gamma}')$ homotopes to x. Because x lifts to a loop with basepoint $\tilde{x_0}$, by the homotopy lifting property, $f(\gamma\bar{\gamma}')$ itself lifts to a loop with basepoint $\tilde{x_0}$.

We also know that this lifted path must be unique (by the same homotopy lifting property when considering a path as a homotopy of points, see the note above). Then $\widetilde{f(\gamma\bar{\gamma}')} = \widetilde{f(\gamma)} \cdot \widetilde{f(\gamma')}^{-1}$. The equivalence comes from properties of the induced homomorphism and the lifts of the "half paths" each exist by the homotopy lifting property. The LHS is a loop so certainly the half paths must meet at their endpoints and $\tilde{f\gamma}(1) = \tilde{f\gamma}'(1)$ as desired.

We must also check for continuity. Consider $y \in Y$, and let U be a neighborhood about $f(y)$. Let $\tilde{U}$ be the sheet that covers $\tilde{f}(y)$ with a homeomorphism $p : \tilde{U} \to U$.

Let N be a neighborhood about y such that $f(N) \subseteq U$ by continuity. For any $y' \in N$, let $\gamma \cdot \eta$ be a path from y_0 to y', where γ goes from y_0 to y and η goes from y to y'. $\tilde{f}$ is defined by the endpoint of the lifted path $\gamma\eta$. But $\tilde{f\eta}$ is given by $p^{-1}f$ as it agrees with at least $\tilde{fy}$, the junction of γ and η, so it cannot lie in some other sheet.

Then restricted to N, $\tilde{f} = p^{-1}f$. This is true for arbitrary N so $\tilde{f}$ is continuous. □

Proposition 1.35 (Unique lifting property)**.** *Given a covering space $p : \tilde{X} \to X$ and some map $f : Y \to X$, if $\tilde{f}_1$ and $\tilde{f}_2$ agree on one point of Y and Y is connected, they agree on all of Y.*

Proof. For a point $y \in Y$ let U be an evenly-covered neighborhood of $f(y)$ (recall this means that $p^{-1}(U)$ is a set of open sets in $\tilde{X}$ each mapping homeomorphically to U). Let $\tilde{U_1}$ and $\tilde{U_2}$ be the open sets that $\tilde{f}_1(y)$ and $\tilde{f}_2(y)$ reside in respectively.

If $\tilde{f}_1(y) \neq \tilde{f}_2(y)$, then $\tilde{U_1} \neq \tilde{U_2}$ so these sheets are disjoint.

Consider an open N about y that maps into $\tilde{U_1}$ and $\tilde{U_2}$ by each lift. Such an N exists because of continuity. If $\tilde{f}_1(y) = \tilde{f}_2(y)$, then N must map into the same $\tilde{U}$. Because $p\tilde{f}_1 = p\tilde{f}_2$ and p is injective over over $\tilde{U}$ (it is homeomorphic with U), these lifts agree over N. □

Definition 1.36 (semilocal simple connectedness)**.** For each point $x \in X$, there exists U, where $\pi_1(U) \hookrightarrow \pi_1(X)$ is trivial.

Notice that this does not mean that U itself is simply connected, rather that loops within U can homotope throughout the whole space to X. Local simple connectedness is a much stronger condition.

Definition 1.37 (local simple connectedness)**.** For each point $x \in X$ the

Example 1.38 (Constructing the universal cover)**.** We're going to build a topological space in quite an abstract way.

Consider a semi-locally simply connected, path connected and semilocally path connected space X (fucking mouth full - let us make sure we actually understand why each of these properties are needed throughout the proof and are not just listing them). We want to build a covering space, $p : \tilde{X} \to X$, that is simply connected.

We first define the points of $\tilde{X}$ as $\{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\}$. To see why we are able to do this, we start with $p : \tilde{X} \to X$ that has already been constructed and is simply connected. For a basepoint $\tilde{x_0}$, each x has exacly one homotopy class of paths because it is simply connected, call it $[\gamma]$. These paths also form a homotopy class $[p\gamma]$ under the image of p, and by the homotopy lifting property, any member of

$[p\gamma]$ must itself lift to or is homotopic to a path that lifts to $[\gamma]$. So the homotopy classes of paths in X and the points in $\tilde{X}$ are in one to one correspondence and we may use them interchangeably.

We now must define a topology on $\tilde{X}$. Observe first that the collection $\mathscr{U} = \{U \mid \pi_1(U) \hookrightarrow \pi_1(X) \text{ is trivial}\}$ is a basis for X. Clearly it covers X, as X is semilocal simply connected (esoteric property checked off), but observe also any $W \subseteq V \cap U \subset U$ for $U, V \in \mathscr{U}$ also lies in $\mathscr{U}$ ($\pi_1(W) \hookrightarrow \pi_1(U) \hookrightarrow \pi_1(X)$ is trivial).

We introduce some new notation. $U_{[\gamma]} = \{[\gamma \cdot \eta] \mid \eta \text{ is any path in } U \text{ where } \eta(0) = \gamma(1)\}$. Observe that $U_{[\gamma]} = U_{[\gamma']}$ iff $\gamma' \in [U_{[\gamma]}$, but **not** necessary if the endpoint just ends up in U (see Hatcher for this quick proof).

Then we build a basis $\mathscr{B} = \{U_{[\gamma]} \mid U \in \mathscr{U}\}$. Again $\mathscr{B}$ covers $\tilde{X}$ for the same reason, and for $\gamma'' \in U_{[\gamma]} \cap V_{[\gamma']}$ where $U_{[\gamma]}, V_{[\gamma']} \in \mathscr{B}$, there exists $W \subseteq U \cap V$ where $W_{[\gamma'']} \subseteq U_{[\gamma'']} \cap V_{[\gamma'']}$. Our topology on $\tilde{X}$ is then generated by $\mathscr{B}$.

Let $p : \tilde{X} \to X$ be defined as $\gamma \mapsto \gamma(1)$. p is surjective as X is path connected (another esoteric property checked off). p is also injective as $p\gamma = p\gamma' \implies \gamma(1) = \gamma'(1) \implies \gamma \simeq \gamma'$.

For each sheet, $p : U_{[\gamma]} \to U$ is also a homeomorphism. $p(V_{[\gamma]}) = V$ and $p^{-1}(U) \cap U_{[\gamma]} = V_{[\gamma'}$ (recall that the inverse images can be mapped into disjoint sets for each distinct homotopy class so the intersection is crucial).

To see that $p^{-1}(U)$ is a collection of disjoint sheets. Notice that various γ partition $U_{[\gamma]}$. Either $[\gamma] \neq [\gamma']$, so $U_{[\gamma]} \cap U_{[\gamma']} = \emptyset$. Else $[\gamma''] \in U_{[\gamma]} \cap U_{[\gamma']}$ then $U_{[\gamma]} = U_{[\gamma']} = U_{[\gamma'']}$. Notice that we get different sheets when the homotopy classes are different and we can start to get a feel for how eg. holes in X lead to different sheets in $\tilde{X}$.

Our last task is to show that $\tilde{X}$ is simply connected. Because p is injective it is sufficient to show $p_*(\pi_1(\tilde{X}, [\tilde{x}_0]))$ is trivial (check - why do we need p to be injective?).

Example 1.39. Let $X_{m,n}$ be the quotient space of a cylinder $S^1 \times I$ with the identifications $(z, 0) \simeq (e^{\frac{2\pi i}{m}} z, 0)$ and $(z, 1) \simeq (e^{\frac{2\pi i}{n}} z, 1)$. Here z is a complex representation of a point on the circle, so the transformation $e^{\frac{2\pi i}{m}} z$ is rotating it by $\frac{2\pi}{m}$.

A and B are subspaces of X and the quotients of $S^1 \times [0, \frac{1}{2}]$ and $S^1 \times [\frac{1}{2}, 1]$ respectively. One end of each half cylinder is just the circle ($A \cap B = S^1$), but they "twist" to the identification.

See that if $m = n = 2$, then A and B are the mobius band.

Exercise 1.9 (1.3.1)**.** This is an exercise is basic definitions from point-set topology.

For arbitrary $x \in A \subseteq X$, let U be a neighborhood satisfying the covering space property when x is considered a member of X. Namely, U is both open and $p^{-1}(U)$ is a collection of disjoint open subsets of $\tilde{X}$ that map homeomoprhically to X by p.

Then U is open in the subspace A by the definition of a subspace. If under the unrestricted map, $p^{-1}(U)$ is a collection of open and disjoint sets in $tildeX$ where each is homemorphic to U. It is easy to see that $p^{-1}(U \cap A)$ each is disjoint and has the desired homeomorphic correspondence.

Exercise 1.10 (1.3.3)**.**

2 Appendix

2.1 Point Set Topology

Definition 2.1 (Basis)**.** A basis for a topology $\mathscr{T}$ on X is a collection $\mathscr{B} \subseteq \mathscr{T}$ such that every open set of the topology can be represented as a union of elements in $\mathscr{B}$.

Equivalently, for each $U \in \mathscr{T}$ and $x \in U$, $\mathscr{B}$ is the collection of B where $x \in B \subseteq U$.

It is straightforward to show these definitions are equivalent (check).

Bases then admit two properties:

- $\mathscr{B}$ covers X (X is certainly an open set that can be represented as a union of basis elements)
- For each $x \in X$ and neighborhoods $B_1, B_2 \in \mathscr{B}$, there exists $B_3 \in \mathscr{B}$ where $x \in B_3 \in B_1 \cap B_2$ (As finite intersections of open sets are open, we need basis elements to fit inside each intersection)

2.2 Presentations

After more exposure to presentations to describe the group structure of some of these geometric objects, it became clear that my understanding lacked rigor.

$G =< X \mid R >$ can be considered as the largest possible group with elements that are words composed of the letters in X **but** also subject to the relations described in R. "Subject to the relations" means that we can swap subwords for equivalent subwords or cancel subwords towards reducing different words to the same underlying group element.

(For $< a \mid a^3 >$, such reductions look like $aaaaaa = aaa = 1$, $(a^{-1}a^{-1})a = (a)a$)

I have found this hard to reason about the equivalence of words for presentations (in abstract, not for particular groups) and it turns out that this problem is very hard. Deciding that words are equivalent in G just from the generators and relations is unsolvable in general.

Note. This problem is called the **word problem for groups**. In 1911, Max Dehn proposed that this problem was an important area of study for group theory, whereas prior most mathematicians used normal forms (irreducable representations) for group computations which made the word problem less relevant.

This does not mean that there do not exist groups where equivalences between words are obvious. Consider again $< a \mid a^3 >$, any string / word, eg. $a^4a^{-2}a^3\cdots$, can be reduced monotonically to one of a, aa, aaa.

However, while this idea is intuitive, it is not satisfying, and a more formal definition of the group describe by a presentation is desired:

$$G =< X \mid R >= F(X)/ < R >^{F(X)}$$

Definition 2.2. The normal closure of H in G is the smallest normal subgroup containing H. It is computed by $< \{xHx^{-1} | x \in G\} >$

We will prove three properties of the presentation using this formal definition motivated by this post.

Proposition 2.3. *G is generated by the images of X in the quotient group.*

Proof. Let $N =< R >^{F(X)}$. Clearly the generating set $\{xN | x \in X\} \subseteq F(X)/N$. Then the generated set $< \{xN | x \in X\} >\subseteq F(X)/N$ as any word $xNyN \cdots zN$ can be reduced to $(xy \cdots z)N$ and this word is in $F(X)/N$.

To see the reverse, consider any $gN \in F(X)/N$ and observe g is some word of elements of X. Expand g to the equivalent representation in letters of X, $(xy \cdots z)N$ and see this element lies in the desired generated set.

We finish this proof by constructing an isomorphism by mapping each coset of $< \{xN | x \in X\}$ to the coset in that shares its representative $\subseteq F(X)/N$. ☐

Proposition 2.4. *G is generated by the images of X in the quotient group.*

Note (Free Groups)**.** See the **myasnikov.1.free.groups.pdf** for a category theory flavored exposition on free groups, their universal property and role in presentations.

Note (Universal Property of Free Groups)**.** Let S be the generating set of a group $F(S)$. For any group G and associated map $f : S \to G$ (note f is not a homomorphism and is just mapping symbols into a group), f extends to a unique (homomorphism) $f' : F(S) \to G$ that commutes in the following diagram.

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "i"'] & G \\ F(S) \arrow[ur, "f'"'] & \end{tikzcd}$$

We proceed with a brief proof and then an example.

If we consider $F(S)$ as reduced words in letters of S, let $f' = f(s_1) \cdots f(s_n)$ and claim this is our extended homomorphism.

Uniqueness can be seen becauase f' must agree with f for each $s \in S$.

Lets construct two isomomorphisms between presentations and groups to understand what a quotient of this enormous free group and almost just as enormous normal subgroup actually mean in the context of groups we already understand.

- $\mathbb{Z} \times \mathbb{Z} \cong < a, b \mid [a, b] >$
- $\S_3 \cong < a, b \mid a^2 = b^3 = aba^{-1}b^2 >$

Proposition 2.5. $D_3 \cong < x, y \mid x^3 = y^2 = xyx^{-2}y >$

Proof. Lets begin by invoking the universal property:

$$\begin{tikzcd}
\{x, y\} \arrow[r, "i"] \arrow[dr, "f"'] & F(x, y) \arrow[d, "\phi"] \\
& D_3
\end{tikzcd}$$

Where $F(x, y)$ is our free group on two generator, i is the inclusion map, f is a map from our generators to letters in the dihedral group. The diagram commutes ($f = \phi i$).

By definition, our presentation is $F(x, y)/ << x^2, y^3, xyx^{-2}y >>$ (where the $<< - >>$ denotes the normal closure). I claim that our universal θ gives us the desired isomorphism when restricted to the quotient group.

Let us expand our commutative diagram by factoring ϕ through the quotient group:

$$\begin{tikzcd}
\{x, y\} \arrow[r, "i"] \arrow[dr, "f"'] & F(x, y) \arrow[d, "\pi"] \\
& D_3/N \arrow[d, "\phi'"] \\
& D_3
\end{tikzcd}$$

Where $pi = w \to w << x^2, y^3, xyx^{-2}y >>$ projects words onto their cosets and $\phi' = wN \to \phi(w)$. Notice ϕ' is well defined as the image of every element of the normal closure is the identity ("ϕ respects the relations").

Then ϕ' is surjective as ϕ is surjective (every element of D_3 is word of letters in the image of f). To see ϕ' is injective, consider distinct cosets $aN \neq bN$. Then $\phi'(aN) = \phi'(a)\phi'(N) = \phi'(a) = \phi(a)$. Similarly, $\phi'(bN) = \phi(b)$. $\phi(a) \neq \phi(b)$

□

Note. Tietze transforms define operations on presentations that do not change the generated group. Adding and removing both generators and relations are allowed if the generator or relation can be derived from other information in the presentation. $< x, y \mid x^2 = y = 1 >$

2.3 Derived Subgroups

Commutators and derived subgroups appear frequently to describe the structure of fundamental groups of cell complexes. We will review some definitions and proofs from elementary algebra.

Definition 2.6. The **commutator** of two elements $a, b \in G$ is $a^{-1}b^{-1}ab$ and is denoted $[a, b]$.

Definition 2.7. a^x is the conjugate of a by x (xax^{-1}).

Clearly the commutator is the identity element if a and b commute in G (hence its name). We derive some pedagogical identities that will be useful.

Definition 2.8. $[a, b]^x = [a^x, b^x]$

Proof. $xa^{-1}b^{-1}abx^{-1} = xa^{-1}x^{-1}xb^{-1}x^{-1}xax^{-1}xbx^{-1} = [a^x, b^x]$ ☐

Note. The set of commutators need not be closed over the group operation.

Consider the free group on four generators. Then $[a,b][c,d] = a^{-1}b^{-1}abc^{-1}d^{-1}cd$. This is not a commutator in this group (there are no two elements x, y such that $[x,y] = [a,b][c,d]$.

Even though products of commutators are not themselves commutators in general, the subgroup *generated* by all commutators in a group is certainly closed over the group operation. This leads naturally to the derived subgroup.

Definition 2.9. The group generated by all of the commutators in G is the **derived subgroup** of G, denoted $[G,G]$ or G'.

Theorem 2.10. $[G,G] \trianglelefteq G$

Proof. $([a,b]\cdots[e,f])^x = ([a^x,b^x]\cdots[e^x,f^x])$ ☐

Theorem 2.11. *If the quotient group G/N is abelian, then $[G,G] \subseteq N$. In other words, the derived subgroup the smallest subgroup that abelianizes G.*

Proof. Let N be some normal subgroup of G, then N abelianizes G iff for each $x, y \in G$, $xyN = yxN$. Then $N = y^{-1}x^{-1}yxN$ (the group operation is well-defined as N is normal) so $y^{-1}x^{-1}yx \in N$.

Certainly the set of commutators must be included in N. So any minimal N is the minimal group generated by these commutators. But this is exactly $[G,G]$. ☐

Definition 2.12. $G/[G,G]$ is the abelianization of G and sometimes denoted G^{ab}.

2.4 Continuity

I found myself confused about the equivalence between the $\delta\epsilon$ definition of continuity from calculus and the topological definition using open sets. The direction of the implications is what is not obvious.

Recall the conceptual meat of continuity is that small perturbations in the range can always be .

More specifically the preimage of some small interval in the range always exists in the domain if we

Definition 2.13. A function is continuous at x if for all ϵ there exists an δ where

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$$

The structure of this implication is crucial. We can construct arbitrarily small intervals of the range and always recover some δ about x_0 that entirely maps within this interval. Simple piecewise discontinuous funcions from analysis illustrate how clearly discontinuous maps (they have big holes) map onto this delta epsilon definition. We will walk through one for the sake of pedagogy.

Example 2.14. Consider

$$\begin{cases} x^2 - 3 & x \leq 0 \\ \sin x & x > 0 \end{cases}$$

Consider continuity at $x = 0$. For $\epsilon = 2$, there exists no value of δ where $|x - 0| \implies |f(x) - f(0)| < 2$.

Continuity is defined differently in topology:

Definition 2.15. Let $f : X \to Y$ be a continuous map. Then for each open $U \in Y$, $f^{-1}(U)$ is open in X.