Hatcher: Algebraic Topology

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1 The Fundamental Group

1.1 Basic Constructions

Definition 1.1 (Path)**.** A path in the space X is a continuous map $f : I \to X$ where I is the unit interval $[0, 1]$.

Definition 1.2 (Homotopy of paths)**.** A continuous family of maps represented $f : I \times I \to X$

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Example 1.3 (Linear homotopy)**.** Consider two paths f, g that share endpoints ($f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$). Any such paths are always homotopic by the homotopy

$$h_t(s) = (1 - t)f(s) + (t)g(s).$$

Proposition 1.4 (The relation of homotopy of paths with fixed endpoints is an equivalence relation)**.**

Theorem 1.5 ($\mathbb{Z} \cong \pi(S1)$)**.** *The map ϕ sending the integer n to the homotopy class of the loop $\omega_n(x) = (cos2\pi nx, sin2\pi nx)$ on S^1 with basepoint $(1, 0)$ is an isomorphism.*

Proof. Observe that each loop ω_n can be expressed as the composition $p\tilde{\omega}_n$, where $p(s) = (cos2\pi s, sin2\pi s)$ and $\tilde{\omega}_n(s) = ns$ is the path in $\mathbb{R}$ from 0 to n. $\tilde{\omega}_n$ is called the lift of ω_n.

We define the image $n \in Z$ under ϕ as the homotopy class represented by pf_n where f_n is any path in $\mathbb{R}$ from 0 to n. Since any such f_n shares endpoints with $\tilde{\omega}_n$, the paths are homotopic, and pf_n and $p\tilde{\omega}_n$ are also homotopic. Then $\phi(n) = [pf_n] = [w_n]$.

We quickly verify ϕ is a homomorphism by considering $\phi(m + n)$ as $p\tilde{\omega}_{m+n}$

We now define two facts that show ϕ is a bijection:

(a) Given a path f in S^1 starting at x_0, there exists a unique lift of the path f, $\tilde{f}$, in $\mathbb{R}$ starting at $\tilde{x_0}$ for any choice of $\tilde{x_0} \in p^{-1}(x_0)$.

(b) Given a homotopy f_t in S^1 of paths starting at x_0, there exists a unique lifted homotopy $\tilde{f}_t$ of paths in $\mathbb{R}$ starting at $\tilde{x_0}$ for any choice of $\tilde{x_0} \in p^{-1}(x_0)$.

(a) gives injectivity of ϕ. Consider any loop f with basepoint $(1, 0)$ reperesenting a homotopy class in $\pi_1(S^1)$. Then there exists $\tilde{f}$ starting at $0 \in p^{-1}((1, 0))$ that ends at $n \in \mathbb{Z} \subset p^{-1}((1, 0))$. So $\phi(n) = [f]$

(b) gives surjectivity of ϕ. Consider $\phi(m) = \phi(n)$. Then the representative loops ω_m and ω_n are homotopic. Define this homotopy as f_t. (b) gives us $\tilde{f}_t$ where $\tilde{f}_t(0) = 0$ for all t (Our choice of $\tilde{x_0}$). $\tilde{f}_0(1) = m$ and $\tilde{f}_1(1) = n$, but the endpoint of the homotopy must be invariant with respect to time, so $m = n$.

We observe that both of these facts are specific examples of a more general fact (c)

(c) For an arbitrary space Y, given $F : Y \times I \to S^1$ and a lift $\tilde{F} : Y \times \{0\} \to S^1$ of $F\big|_{Y \times \{0\}}$, there is a unique lift $\tilde{F} : Y \times I \to S^1$ that restricts to the given $\tilde{F}$ on $Y \times \{0\}$.

Observe that (a) follows trivially when we consider Y to be a point. To see that (c) implies (b), observe the homotopy of paths in S^1, f_t given in (b) connects f_0 and f_1. (a) gives us a unique $\tilde{f}_0 : I \times \{0\} \to S^1$, a restriction of the lifted homotopy, and (c) gives us a unique lifted homotopy $\tilde{f}_t$ that restricts to $\tilde{f}_0$. Note that $\tilde{f}_0$ defines the endpoints of homotopic paths in $\mathbb{R}$ and is unique so (b) follows.

□

Theorem 1.6 (Fundamental Theorem of Algebra). *Every non constant polynomial with coefficients in $\mathbb{C}$ has at least one root in $\mathbb{C}$.*

Proof. Assume our polynomial $p(z) = z^n + a_1 z^{n-1} ... a_n$ has no roots. Define the following $f_r : I \to \mathbb{C}$ for each real number:

$$f_r(s) = \frac{p(re^{2\pi is}) \backslash p(r)}{|p(re^{2\pi is})| \backslash |p(r)|}$$

We claim this is a loop in the unit circle S^1. To see this, compute some values for fixed r and note for any value of r, $f_r(0) = 1$, $f_r(1) = 1$. Note that f_0 is a constant map with value 1 and is the trivial loop. Then by varying r we obtain a homotopy between any f_r and 0 with (the embedded) basepoint $(1, 0)$.

We now show that p must be constant. Choose a large r such that $r \geq \sum_n |a_n| \geq 1$. Observe then $r^n = rr^{n-1} \geq (\sum_n |a_n|) r^{n-1}$. This expression motivates a new homotopy for each f_r, let $f_{(r,t)}$ be our previous definition with $p_t(z,t) = z^n + t(z^{n-1}a_1 + ...a_n)$ substituted for each p. See that f_t has no zeros on the circle of complex values with radius r satisfying our expression (this is why we constructed it at all). Note that $f_{(r,0)} = e^{2\pi stn}$ (check). Then $f_{(r,t)}$ is a homotopy between f_r and homotopy class $[w_n] \in \pi_1(S^1)$. But f_r is homotopic to 0. Then w_n is also homotopic to 0 and n must be 0. Our $p(z) = a_n$ must be constant.

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