Hatcher: Algebraic Topology

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1 The Fundamental Group

1.1 Basic Constructions

Definition 1.1 (Path)**.** A path in the space X is a continuous map $f : I \to X$ where I is the unit interval $[0, 1]$.

Definition 1.2 (Homotopy of paths)**.** A continuous family of maps represented $f : I \times I \to X$
A path in the space X is a continuous map $f : I \to X$ where I is the unit interval $[0, 1]$.

Example 1.3 (Linear homotopy)**.** Consider two paths f, g that share endpoints ($f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$). Any such paths are always homotopic by the homotopy

$$h_t(s) = (1 - t)f(s) + (t)g(s).$$

Proposition 1.4 (homotopy of paths with fixed endpoints is an equivalence relation)**.**

Definition 1.5 (The product of paths)**.** For paths f, g where $f(1) = g(0)$, we can define the product $h = f \cdot g$ as

$$h(s) = \begin{cases} f(2s) & 0 \leq s \leq 0.5 \\ g(2s-1) & 0.5 \leq s \leq 1 \end{cases}$$

Proposition 1.6 (The fundamental group is in fact a group)**.** *$\pi_1(X, x_0)$ with respect to the operator $[f] \cdot [g] = [f \cdot g]$ is a group.*

Definition 1.7 (Reparamterization of a path)**.** We can precompose any path f with $\phi : I \to I$ such that $f\phi \simeq f$. $f\phi$ is then the reparameterization of f.

Proof. To see our operator is well-defined, $[f \cdot g]$ depends only on $[f]$ and $[g]$. In other words, a homotopy exists between paths in $[f \cdot g]$ if and only if it is the composite of homotopies between paths in $[f]$ and $[g]$.

First, we verify associativity of the operator. Consider $(f \cdot g) \cdot h$. This product composes the composition of f and g with h. Visually, h is the first half and g and f occupy the last quarters of the product path. We can construct a piecewise function to change the proportions of these components - "shrink" h, keep g the same and "expand" f. $[(f \cdot g) \cdot h]\phi \simeq f \cdot (g \cdot h)$ gives us the desired equivalence.

Second, we check the identity property of the constant map c. f can be reparameterized by the piecewise function that speeds it up for the first half of I and holds it constant at $f(1)$ for the second half where $f\phi \simeq f \cdot c$.

Third, we verify the existence of an inverse for each $[f]$. Define $[\bar{f}]$ where $\bar{f}(s) = f(1 - s)$. Then $f \cdot \bar{f}$ is homotopic to the constant path. To see this construct $h_t = f_t g_t$ where $f_t = f$ on $[0, t]$ and $f_t = f(t)$ on $[1 - t, 1]$ and g_t is the inverse of this function. As t approaches 1, h_t has a larger constant region in its image starting from the middle and growing to the endpoints until $h_1(s) = f(0) = g(1)$ is our constant map at timepoint 1.

□

Proposition 1.8 (The fundamental group is independent of the choice of basepoint up to isomorphism)**.** *If f is a loop with basepoint x_1 and h is a path from x_0 to x_1. The map $\beta_h : \pi_1(X, x_0) \to \pi_1(X, x_1)$ defined as $\beta_h([f]) = [h \cdot f \cdot \hat{h}]$ is an isomorphism.*

Proof. β_h is well-defined. To see this, if $[f] = [f']$ implies that f and f' are homotopic, so certainly $h \cdot f \cdot \bar{h}$ and $h \cdot f' \cdot \bar{h}$ are homotopic and $\beta_h([f]) = \beta([f'])$. β_h is a homomorphism. $\beta_h([f \cdot g]) = [h \cdot f \cdot g \cdot \bar{h}] = [h \cdot f \cdot \cdot h \cdot \bar{h} \cdot g \cdot \bar{h}] = [h \cdot f \cdot \cdot h] \cdot [\bar{h} \cdot g \cdot \bar{h}] = \beta_h([f]) \cdot = \beta_h([g])$. β_h is an isomorphism. □

Definition 1.9. A space is **simply connected** if it is path connected and has a trivial fundamental group.

Theorem 1.10 ($\mathbb{Z} \cong \pi(S1)$)**.** *The map ϕ sending the integer n to the homotopy class of the loop $\omega_n(x) = (cos2\pi nx, sin2\pi nx)$ on S^1 with basepoint $(1, 0)$ is an isomorphism.*

Proof. Observe that each loop ω_n can be expressed as the composition $p\tilde{\omega}_n$, where $p(s) = (cos2\pi s, sin2\pi s)$ and $\tilde{\omega}_n(s) = ns$ is the path in $\mathbb{R}$ from 0 to n. $\tilde{\omega}_n$ is called the lift of ω_n.

We define the image $n \in Z$ under ϕ as the homotopy class represented by pf_n where f_n is any path in $\mathbb{R}$ from 0 to n. Since any such f_n shares endpoints with $\tilde{\omega}_n$, the paths are homotopic, and pf_n and $p\tilde{\omega}_n$ are also homotopic. Then $\phi(n) = [pf_n] = [w_n]$.

We quickly verify ϕ is a homomorphism by considering $\phi(m + n)$ as $p\tilde{\omega}_{m+n}$

We now define two facts that show ϕ is a bijection:

(a) Given a path f in S^1 starting at x_0, there exists a unique lift of the path f, $\tilde{f}$, in $\mathbb{R}$ starting at $\tilde{x_0}$ for any choice of $\tilde{x_0} \in p^{-1}(x_0)$.

(b) Given a homotopy f_t in S^1 of paths starting at x_0, there exists a unique lifted homotopy $\tilde{f}_t$ of paths in $\mathbb{R}$ starting at $\tilde{x_0}$ for any choice of $\tilde{x_0} \in p^{-1}(x_0)$.

(a) gives injectivity of ϕ. Consider any loop f with basepoint $(1, 0)$ reperesenting a homotopy class in $\pi_1(S^1)$. Then there exists $\tilde{f}$ starting at $0 \in p^{-1}((1, 0))$ that ends at $n \in \mathbb{Z} \subset p^{-1}((1, 0))$. So $\phi(n) = [f]$

(b) gives surjectivity of ϕ. Consider $\phi(m) = \phi(n)$. Then the representative loops ω_m and ω_n are homotopic. Define this homotopy as f_t. (b) gives us $\tilde{f}_t$ where $\tilde{f}_t(0) = 0$ for all t (Our choice of $\tilde{x_0}$). $\tilde{f}_0(1) = m$ and $\tilde{f}_1(1) = n$, but the endpoint of the homotopy must be invariant with respect to time, so $m = n$.

We observe that both of these facts are specific examples of a more general fact (c)

(c) For an arbitrary space Y, given $F : Y \times I \to S^1$ and a lift $\tilde{F} : Y \times \{0\} \to S^1$ of $F\big|_{Y\times\{0\}}$, there is a unique lift $\tilde{F} : Y \times I \to S^1$ that restricts to the given $\tilde{F}$ on $Y \times \{0\}$.

Observe that (a) follows trivially when we consider Y to be a point. To see that (c) implies (b), observe the homotopy of paths in S^1, f_t given in (b) connects f_0 and f_1. (a) gives us a unique $\tilde{f}_0 : I \times \{0\} \to S^1$, a restriction of the lifted homotopy, and (c) gives us a unique lifted homotopy $\tilde{f}_t$ that restricts to $\tilde{f}_0$. Note that $\tilde{f}_0$ defines the endpoints of homotopic paths in $\mathbb{R}$ and is unique so (b) follows. □

We can immediately use $\pi_1(S^1)$ to prove important theorems. The big idea is that each full rotation around the circle is a unique element of the group. This suggests, among other things, that a loop once around cannot be continuously deformed to a loop twice around.

Our arguments proceed by contradiction, by assuming that our desired result does not hold and showing that this assumption implies some homotopy between loops on the circle that is not allowed.

Theorem 1.11 (Fundamental Theorem of Algebra)**.** *Every non constant polynomial with coefficients in $\mathbb{C}$ has at least one root in $\mathbb{C}$.*

Proof. Assume our polynomial $p(z) = z^n + a_1 z^{n-1} ... a_n$ has no roots. Define the following $f_r : I \to \mathbb{C}$ for each real number:

$$f_r(s) = \frac{p(re^{2\pi is})\backslash p(r)}{|p(re^{2\pi is})|\backslash|p(r)|}$$

We claim this is a loop in the unit circle S^1. To see this, compute some values for fixed r and note for any value of r, $f_r(0) = 1$, $f_r(1) = 1$. Note that f_0 is a constant map with value 1 and is the trivial loop. Then by varying r we obtain a homotopy between any f_r and 0 with (the embedded) basepoint $(1, 0)$.

We now show that p must be constant. Choose a large r such that $r \geq \sum_n |a_n| \geq 1$. Observe then $r^n = rr^{n-1} \geq (\sum_n |a_n|)r^{n-1}$. This expression motivates a new homotopy for each f_r, let $f_{(r,t)}$ be our previous definition with $p_t(z,t) = z^n + t(z^{n-1}a_1 + ...a_n)$ substituted for each p. See that f_t has no zeros on the circle of complex values with radius r satisfying our expression (this is why we constructed it at all). Note that $f_{(r,0)} = e^{2\pi stn}$ (check). Then $f_{(r,t)}$ is a homotopy between f_r and homotopy class $[w_n] \in \pi_1(S^1)$. But f_r is homotopic to 0. Then w_n is also homotopic to 0 and n must be 0. Our $p(z) = a_n$ must be constant. □

Theorem 1.12 (Brouwer fixed point theorem). *Any continuous map $h : D^2 \to D^2$ must have a fixed point $h(x) = x$.*

This theorem was initially proved by a gentleman named L.E.J. Brouwer circa 1910 and seems to be foundational to the rest of algebraic topology and other fields like differential topology.

Exercise 1.1 (1.3).

Proof. □