Notes on Projective Geometry

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1 Basic Definitions

Definition 1.1. Let V be a vector space. The projective space $P(V)$ is the set of 1-dimensional vector subspaces of V.

It can be useful to think of projective spaces, at least in the real case, as bundles of lines that pass through the origin. This reduces our space by a dimension, motivating the common shorthand for the reals $P^n(R) = P(R^{n+1})$.

Many sources encourage thinking about the sphere S^n for a projective space $P^n(\mathbb{R})$ to ease the visualization of a space of lines. Antipodal points on the sphere are identified and every such antipodal pair is injective with the actual elements of the projective space.

I have found it easier to just think of the vector subspaces the projective elements represent (projective points and lines are lines and planes through the origin in $\mathbb{R}^3$) for more natural geometric intuition.

1.1 Decomposition

The following decomposition is useful to understand the structure of theses spaces:

$$P(R^n) = R^{n-1} + P(R^{n-1})$$

Essentially our goal is to take R^n and partition the set of points into 1-dim vector subspaces such that each partition has a nice representation. Recall:

Definition 1.2. A representative vector is any of the non-zero vectors from the 1-dimensional subspace corresponding to a point $[v] \in P(V)$.

Then if $[x] = [\lambda x] = [a]$, x and a are both representatives for the same projective point.

We also want to define the notion of the homogoenous coordinates for each projective point, which are just the real points that exist in the corresponding vector subspace.

Definition 1.3. The homogoenous coordinates for $[v] \in P(V)$ are the set $[(x_0 \cdots x_n)]$ equivalent under scalar multiplication by λ.

If we construct a subset of homogoenous coordinates U_0 where $x_0 \neq 1$, notice that each $[(x_0 \cdots x_n)] = [1 \cdots x_n/x_0]$, so $U_0 \cong R^n - 1$. We are left to "partition" the coordinates where $x_0 = 0$, but this is exactly the set of 1-dimensional subspaces of $V^n - 1$, so $P(R^{n-1})$.

1.2 Applications

1.3 Linear Subspaces

We begin by proving a result from elementary linear algebra.

Theorem 1.4. *Let W_1 and W_2 be vector spaces. Then* $\dim W_1 + W_2 = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2$

Theorem 1.5. *In a projective plane $P(V)$, two projective lines, $P(U)$ and $P(U')$, intersect in a unique point.*

Proof. From elementary linear algebra, $\dim V \geq \dim U + U'$. We have shown that $\dim U + U' = \dim U + \dim U' - \dim U \cap U'$. Then $1 \leq \dim U \cap U' \leq 2$. Because $P(U)$ and $P(U')$ are distinct, $\dim U \cap U' = 1$. So $P(U \cap U')$ is a projective point. □

It is useful to think about this result using our model of the projective plane as a sphere and using our decomposition.

We can think of projective lines as planes in $\mathbb{R}^3$ that intersect the sphere in two great circles. These great circles intersect in a pair of antipodal points, which is a projective line.

Alternatively

1.4 Projective Transformations

Given a linear transformation $T : V \to W$, we might want to recover a bijective map $\tau : P(V) \to P(W)$.

Definition 1.6. If T is invertible, τ is a projective transformation between projective spaces.

This is a map of lines to lines.

It seems natural to define τ as $[x] \mapsto [T(x)]$ for any $x \in P(V)$. But notice that because there is no 0 in $P(W)$ (as the collection of 1-dimensional subspaces), $\dim T([x]) = 1$ if τ is to be well-defined over its codomain. Then T must be invertible.

Note. It is not immediately obvious that projective transformations describe bijections (or isomorphisms between underlying vector spaces). However, a T that takes any U in V to 0 induces an ill-defined τ (no 0 in $P(W)$). Furthermore, if $T(U) = T(U')$ for distinct U, U', T cannot be invertible so there will exist some U'' where $T(U'')$ goes to 0. So the requirement that T be invertible is really a requirment that τ needs to be a well-defined map with only 1-dimensional subspaces in its codomain.

Note. Projective transformations are also called *homographies* and have roots in the non abstract origins of projective geometry as a tool to study perspective. In broad strokes, a homography just describes a transformation of perpsectives of the same underlying object.

In fact, projective transformations descibe a collection of linear transformations T that are equivalent up to scalar multiplication.

Proposition 1.7. *If $T, T' : V \to W$ define the same projective transformation, $T = \lambda T'$.*

Proof. If V is generated by basis $\{v_1 \cdots v_n\}$, then $[Tv_i] = [T'v_i]$ by assumption. Certainly for each basis element, $Tv_i = \lambda_i T' v_i$. For an arbitrary element, $\sum T v_i = \sum \lambda_i T' v_i$, our assumption tells us $T(\sum v_i) = \lambda T'(\sum v_i)$. Then, by linearity:

$$\lambda T'(\sum v_i) = T(\sum v_i) = \sum T v_i = \sum \lambda_i T' v_i$$

. So $\lambda = \lambda_i$. Because $\lambda x = \lambda \sum v_i$, $T = \lambda T'$. □

In the real projective plane, there is very a natural geometric picture one can construct to see these transformations are bundles of lines related by a disjoint "observing point".

Example 1.8. Consider two projective lines $P(U)$ and $P(U')$ in the projective plane $P(V)$. If there exists a point $K \in P(V)$ disjoint from both lines, this point induces a natural projective transformation τ. For each point (line) in $P(U)$, draw a line through K, and where it meets $P(U')$ is its image under τ.

We can see this is indeed a projective transformation by proving the underlying linear transformation $T : U \to U'$ is invertible. If W is the subspace corresponding to K, any $a \in U$ can be uniquely expressed as $w + a'$ from $W \bigoplus U'$ (W is disjoint from both U and U'). Then $a' = a - w$ where the w component guarantees that $\ker T = 0$.

Note. As outlined here, visualizing the above example in $\mathbb{R}^2$ leads to a natural image of an observer (our extra point!) connecting two planes together using their perspective. In fact, this transformation is also called a perspectivity in computer graphics for this reason.

In vanilla linear algebra, we can fully characterize a linear transformation from an n dimensional space by observing what it does to n linearly independent vectors.

Definition 1.9. Points $X_1 \cdots X_{n+1} \in P(V)$ are in general position if any subset of n points have representative vectors that are linearly independent.

Theorem 1.10. *If $X_1 \cdots X_{n+2} \in P(V)$ (in n dimensional $P(V)$) are in general position in $P(V)$ and $Y_1 \cdots Y_{n+2}$ are in general position in $P(W)$, then there is a unique projective transformation such that $\tau(X_i) = Y_i$.*

Proof. We can choose representatives, $v_i \in V$, such that $n+1$ representatives form a basis of V. We can choose representatives such that $v_{n+2} = \sum_{i=0}^{n+1} v_i$. Note that the sum of vectors is unique by linear indpendence and must exist because $(v_i)_i$ form a basis for V.

Similarly, we can choose representatives from W such that $w_{n+2} = \sum_{i=0}^{n+1} w_i$. Again this sum of elements is unique.

Then there exists a unique and invertible $T : V \to W$ described by the mapping of basis elements (where $T(v_i) = w_i$) that induces a projective transform with the desired properties

To see uniqueness, consider an alternative $T' : V \to W$ such that $T'(v_i) = \mu_i w_i$, taking our basis elements to a different representative of a point in $P(W)$.

Then $T'(v_{n+2}) = \mu_{n+2} w_{n+2} = \sum_{i=0}^{n+2} \mu_i w_i = \sum_{i=0}^{n+2} T'(v_i)$. Because w_{n+2}, is the unique sum of representatives expressed earlier, $\frac{\mu_i}{\mu_{n+2}} = 1$. So $\mu_i = \mu_{n+2}$ and $T = \mu_{n+2} T'$. $\square$

1.5 Duality

We first review definitions of duality from elementary linear algebra.

Definition 1.11. For a vector space V defined over field F, the dual of V is the vector space V' with elements that are linear transformations $f : V \to F$.

Definition 1.12. If the basis of V is $\{v_1 \cdots v_n\}$, V' has a corresponding basis where for each v_i, $f_i(v_i) = 1$ and $f_i(v_j) = 0$ for all $j \neq i$.

Given a linear transformation $T : V \to W$ over vector spaces, there is a canonical linear transformation induced over its duals $T' : W' \to V'$, given by $T'(f) = f(T)$. Indeed $T'f = fT : V \to F$ and $T'f = fT = v_i \mapsto f(T(v_i))$.

Note. We can use the language of contravariant functors to illustrate the correspondance between linear transformations in their vector and dual spaces.

The correspondence of a category of functions to another category with the domain and codmain swapped is ubituiquitous in category theory and is used to introduce covariant and contravariant functors in Leinster.

1.6 The Fundamental Group of the Projective Plane

1.7 Appendix

Theorem 1.13. $\dim W_1 + W_2 = \dim W_1 + \dim W_2$

Proof. Let $S = \{u_1 \cdots u_r\}$ be the basis of $W_1 + W_2$. Let $B_1 = \{u_1 \cdots u_r v_1 \cdots v_s\}$ and $B_2 = \{u_1 \cdots u_r w_1 \cdots w_t\}$ be B extended to be the basis of W_1 and W_2 respectively. If we can show B is the basis of $W_1 + W_2$, we have our result, as $\dim B = r + s + t = (r + s) + (r + t) - r = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2$.

First, we show B is linearly independent. Let

$$\sum_i^r a_i u_i + \sum_j^s b_j v_j + \sum_k^t c_k w_k = 0$$

.

Notice if we move terms so

$$\sum_i^r a_i u_i + \sum_j^s b_j v_j = -\sum_k^t c_k w_k$$

,

then the LHS is in W_1 and the RHS is in W_2, so both sides represent the same element in $W_1 + W_2$. Then $\sum_i^r d_i u_i = -\sum_k^t c_k w_k$, where the LHS uses B and the RHS uses B_2. Again moving terms:

$$\sum_i^r d_i u_i + \sum_k^t c_k w_k = 0$$

Where all c_i must be 0 as B_2 is linearly independent. Then

$$\sum_i^r a_i u_i + \sum_j^s b_j v_j = 0$$

But the LHS is described by B_1 which is also linearly independent so all a_i, b_j must also be 0. Then B is linearly independent.

Consider any $w_1 + w_2$.

$$w_1 = \sum_i^r a_i u_i + \sum_j^s b_j v_j$$

$$w_2 = \sum_i^r d_i u_i + \sum_k^t c_k w_k$$

Then

$$w_1 + w_2 = \sum_i^r (a_i + d_i) u_i + \sum_j^s b_j v_j + \sum_k^t c_k w_k \in \operatorname{span} W_1 + W_2$$

$\square$