CW complexes

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This note is meant to give a short introduction to CW complexes.

1. Notation and conventions

In the following a space is a topological space and a map $f : X \to Y$ between topological spaces X and Y is a function which is continuous. If X is a space, then a subspace of X is a subset $A \subseteq X$ with the relative (or induced) topology (induced by the topology in X). The n-dimensional disk (just called the n-disk in the following) is the following subspace of $\mathbb{R}^n$:

$$D^n = \{ \ x \in \mathbb{R}^n \ : \ |x| \leq 1 \ \},$$

where $|\cdot| : \mathbb{R}^n \to [0, \infty[$ is the standard norm on $\mathbb{R}^n$. Thus the n-disk is the closed n-disk and is a closed subset of $\mathbb{R}^n$. The open n-disk, denoted $\mathrm{int}(D^n)$, is the interior of D^n in $\mathbb{R}^n$. Thus

$$\mathrm{int}(D^n) = \{ \ x \in \mathbb{R}^n \ : \ |x| < 1 \ \}.$$

The boundary of D^n in $\mathbb{R}^n$ is the standard $(n-1)$-sphere

$$S^{n-1} = \{ \ x \in \mathbb{R}^n \ : \ |x| = 1 \ \}.$$

We note that the 0-disk D^0 is equal to $\mathbb{R}^0 = \{0\}$ by definition. We have $\mathrm{int}(D^0) = D^0 = \{0\}$.

A topological space X is called quasi-compact if every open cover of X has a finite subcover, i.e. whenever $\{U_i\}_{i \in I}$ is a family of open subsets of X s.t. $X = \cup_{i \in I} U_i$ then there exist $i_1, \ldots, i_n \in I$ s.t. $X = \sum_{j=1}^{n} U_{i_j}$. A topological space X is called compact if it is Hausdorff and quasi-compact.

As is standard we will write iff for 'if and only if'.

2. Cell decompositions and CW-complexes

Definition 2.1. An n-cell is a space homeomorphic to the open n-disk $\mathrm{int}(D^n)$. A cell is a space which is an n-cell for some $n \geq 0$.

Note that $\mathrm{int}(D^m)$ and $\mathrm{int}(D^n)$ are homeomorphic if and only if $m = n$. This e.g. follows by noting that $\mathrm{int}(D^n)$ is homeomorphic to $\mathbb{R}^n$ (via the map $x \mapsto \tan(\pi|x|/2)x$), and by the fact that $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ iff $m = n$, cf. [Ha, Theorem 2.26 p. 126]. Thus we can talk about the *dimension* of a cell. An n-cell will be said to have dimension n.

Definition 2.2. A cell-decomposition of a space X is a family $\mathcal{E} = \{e_\alpha \mid \alpha \in I\ \}$ of subspaces of X such that each e_α is a cell and

$$X = \amalg_{\alpha \in I} e_\alpha$$

(disjoint union of sets). The n-skeleton of X is the subspace

$$X^n = \amalg_{\alpha \in I : \dim(e_\alpha) \leq n} e_\alpha.$$

Note that if $\mathcal{E}$ is a cell-decomposition of a space X, then the cells of $\mathcal{E}$ can have many different dimensions. E.g. one cell-decomposition of S^1 is given by $\mathcal{E} = \{e_a, e_b\}$, where e_a is an arbitrary point $p \in S^1$ and $e_b = S^1 \setminus \{p\}$. Here e_a is a 0-cell and e_b is a 1-cell. There are no restrictions on the number of cells in a cell-decomposition. Thus we can have uncountable many cells in such a decomposition. E.g. any space X has a cell-decomposition where each point of X is a 0-cell. A finite cell-decomposition is a cell decomposition consisting of finitely many cells.

Definition 2.3. A pair $(X, \mathcal{E})$ consisting of a Hausdorff space X and a cell-decomposition $\mathcal{E}$ of X is called a CW-complex if the following 3 axioms are satisfied:

Axiom 1: ('Characteristic Maps') For each n-cell $e \in \mathcal{E}$ there is a map $\Phi_e : D^n \to X$ restricting to a homeomorphism $\Phi_e|_{\mathrm{int}(D^n)} : \mathrm{int}(D^n) \to e$ and taking S^{n-1} into X^{n-1}.

Axiom 2: ('**C**losure Finiteness') For any cell $e \in \mathcal{E}$ the closure $\bar{e}$ intersects only a finite number of other cells in $\mathcal{E}$.

Axiom 3: ('**W**eak Topology') A subset $A \subseteq X$ is closed iff $A \cap \bar{e}$ is closed in X for each $e \in \mathcal{E}$.

Here $\bar{e}$ of course is the closure of e in X. Note that the Axioms 2 and 3 are only needed in case $\mathcal{E}$ is infinite (i.e. they are automatically satisfied if $\mathcal{E}$ is finite). It is not difficult to give examples of pairs $(X, \mathcal{E})$ with X a Hausdorff space and $\mathcal{E}$ an infinite cell-decomposition of X such that Axiom 1 is satisfied and either Axiom 2 or Axiom 3 is satisfied, see e.g. [J, p. 97]. Thus Axiom 2 and 3 are independent of each other. Note that the characteristic map for a 0-cell $e \subseteq X$ is simply the map mapping 0 to the one-point space e.

Lemma 2.4. *Let $(X, \mathcal{E})$ be a Hausdorff space X together with a cell-decomposition $\mathcal{E}$. If $(X, \mathcal{E})$ satisfies Axiom 1 in Definition 2.3 then we have*

$$\bar{e} = \Phi_e(D^n)$$

for any cell $e \in \mathcal{E}$. In particular $\bar{e}$ is a compact subspace of X and the 'cell boundary' $\bar{e} \setminus e = \Phi_e(S^{n-1})$ lies in X^{n-1}.

Proof. For any map $f : Y \to Z$ between topological spaces Y and Z and any subset $B \subseteq Y$ we have $f(\bar{B}) \subseteq \overline{f(B)}$, see e.g. [D, Theorem III.8.3 pp. 79-80] or [A, Theorem 2.9 p. 33]. Thus

$$\bar{e} = \overline{\Phi_e(\mathrm{int}(D^n))} \supseteq \Phi_e(D^n) \supseteq e.$$

But $\Phi_e(D^n)$ is compact hence closed in X since X is Hausdorff. Thus $\Phi_e(D^n) = \bar{e}$. By Axiom 1 we have $\Phi_e(\mathrm{int}(D^n)) = e$ and $\Phi_e(S^{n-1}) \cap e = \emptyset$ so $\Phi_e(S^{n-1}) = \bar{e} \setminus e$. □

Note, if X is not Hausdorff we still have $\Phi_e(D^n) \subseteq \bar{e}$ but we don't necessarily have equality. We have no garantee that $\Phi_e(D^n)$ is closed in X.

3. Subcomplexes

Let $(X, \mathcal{E})$ be a CW-complex, $\mathcal{E}' \subseteq \mathcal{E}$ a set of cells in it and

$$X' = \cup_{e \in \mathcal{E}'} e.$$

Lemma 3.1. *The following 3 conditions are equivalent:*

(a) *The pair $(X', \mathcal{E}')$ is a CW-complex.*
(b) *The subset X' is closed in X.*
(c) *The closure $\bar{e} \subseteq X'$ for each $e \in \mathcal{E}'$, where $\bar{e}$ is the closure of e in X.*

For a proof, see [J, p. 98].

Definition 3.2. Let $(X, \mathcal{E})$ be a CW-complex and let $(X', \mathcal{E}')$ be as above. Then $(X', \mathcal{E}')$ is called a subcomplex (of $(X, \mathcal{E})$) if the 3 equivalent conditions (a), (b) and (c) in the above lemma are satisfied.

We have some immediate consequences:

Corollary 3.3. *Let $(X, \mathcal{E})$ be a CW-complex. Then*

(1) *Let $\{A_i \mid i \in I\}$ be any family of subcomplexes of $(X, \mathcal{E})$. Then $\cup_{i \in I} A_i$ and $\cap_{i \in I} A_i$ are subcomplex of $(X, \mathcal{E})$.*
(2) *The n-skeleton X^n is a subcomplex of $(X, \mathcal{E})$ for each $n \geq 0$.*

(3) *Let $\{e_i | i \in I\}$ be an arbitrary family of n-cells in $\mathcal{E}$. Then $X^{n-1} \cup (\cup_{i \in I} e_i)$ is a subcomplex.*

Proof. For (1) note that $\cup_{i \in I} A_i$ is a subcomplex by characterization (c) in Lemma 3.1 and $\cap_{i \in I} A_i$ is a subcomplex by (b) in that lemma. Both (2) and (3) follow by using characterization (c) in Lemma 3.1 together with Lemma 2.4. $\square$

Note that (2) is a special case of (3). By the above we have that the n-skeleton X^n of a CW-complex $(X, \mathcal{E})$ is a closed subset of X.

4. Identification Topology and Quotient Spaces

In the next section we need a general proceedure for constructing new spaces from old spaces by gluing spaces together via maps. Let us in this section describe the general concepts from point set topology needed.

4.1. **Identification Topology.** Let X be a topological space and let Y be an arbitrary set and let $p : X \to Y$ be a surjection. Then we can define a topology in Y by: a subset $U \subseteq Y$ is open iff $p^{-1}(U)$ is open in X. This topology is the largest topology in Y for which $p : X \to Y$ is continuous. We call it the identification topology in Y determined by p, and $p : X \to Y$ is called an identification map.

If X and Y are two spaces and $p : X \to Y$ a surjective map, then p is called an identification map if the topology in Y is the identification topology determined by p.

Lemma 4.1. *Let X be a compact space and Y a Hausdorff space and let $p : X \to Y$ be a surjective map. Then p is an identification map.*

Proof. It is enough to prove that $C \subseteq Y$ is closed iff $p^{-1}(C)$ is closed. Since p is continuous we only have to prove that C is closed in Y if $p^{-1}(C)$ is closed in X. But if $p^{-1}(C)$ is closed in X then it is compact, since X is compact. Thus $C = p(p^{-1}(C))$ is compact in the Hausdorff space Y, hence C is closed in Y. $\square$

Lemma 4.2. *Let $p : X \to Y$ be an identification map and let Z be a space. Then $f : Y \to Z$ is continuous iff $f \circ p : X \to Z$ is continuous.*

Proof left to the reader.

4.2. **Quotient spaces.** Let X be a set and let $\sim$ be an equivalence relation on X. Let $X/_\sim$ be the set of equivalence classes and let $\pi : X \to X/_\sim$ be the canonical projection, i.e. the function mapping x to the equivalence class containing x. Recall here that the equivalence classes are mutually disjoint subsets of X and that X is the disjoint union of these equivalence classes. Oppositely, if we have given a disjoint family $\{A_i\}_{i \in I}$ of subsets of X covering X, i.e. $X = \cup_{i \in I} A_i$, then we can define an equivalence relation on X by $x \sim y$ if and only if $\exists i \in I$ s.t. $x, y \in A_i$. The equivalence classes for that equivalence relation $\sim$ are nothing but our subsets A_i. Thus an equivalence relation in X is nothing but a partition of X into subsets.

Now, given a space X and an equivalence relation $\sim$ we equip $X/_\sim$ with the identification topology determined by the canonical projection $\pi : X \to X/_\sim$. This topology is normally called the quotient topology and $X/_\sim$ is called a quotient space of X (the quotient of X by $\sim$).

If $p : X \to Y$ is an identification map, then we can identify Y with a quotient space. Namely, the subsets $p^{-1}(y)$, $y \in Y$, gives a partition of X. Thus the equivalence relation $\sim$ in X induced by this partition is $x \sim x'$ iff $p(x) = p(x')$. We thus have a bijection $q : X/_\sim \to Y$ given by

$q(\pi(x)) = p(x)$, where $\pi : X \to X/_\sim$ is the canonical projection. By Lemma 4.2 both q and q^{-1} are continuous since p and π are both identification maps and $q \circ \pi = p$ and $q^{-1} \circ p = \pi$ are continuous. Thus $q : X/_\sim \to Y$ is a homeomorphism so from a topological point of view we consider Y and $X/_\sim$ to be the same space. Thus quotient spaces and identification spaces are one and the same thing.

There are many standard constructions in algebraic topology using the above ideas. In particular we mention:

Collapsing a subspace to a point. Let X be a topological space and let A be some non-empty subspace. Then we let $X/A = X/_\sim$, where the equivalence classes w.r.t. $\sim$ are A and the singletons $\{x\}$, $x \in X \setminus A$.

The Wedge of Spaces. Given two spaces X and Y and chosen points $x_0 \in X$ and $y_0 \in Y$ we let

$$X \vee Y = (X \amalg Y)/\{x_0, y_0\}.$$

That is, we collapse the subset $\{x_0, y_0\}$ to a point. Here $X \amalg Y$ is the topological disjoint union of X and Y (sometimes called the topological sum of X and Y), see below. We note that the construction depends on the points x_0 and y_0. However, different choices can lead to homeomorphic results. Thus e.g. $S^1 \vee S^1$ does not depend on the choice of the points x_0 and y_0.

More generally we can talk about the wedge of a family of spaces $\{X_i\}_{i\in I}$, namely if $x_i \in X_i$ we let

$$\vee_{i\in I} X_i = \amalg_{i\in I} X_i/A,$$

where $A = \{x_i\}_{i\in I}$, and where $\amalg_{i\in I} X_i$ is the topological disjoint union of the spaces X_i.

Here and in the following we use the following construction. Let $\{X_i\}_{i\in I}$ be a family of topological spaces. Then the topological disjoint union $X = \amalg X_i$ is the following topological space. As a set it is equal to $\cup_{i\in I}\{i\} \times X_i$. We identify $\{i\} \times X_i$ with X_i. The topology on X is given by: U is open in X iff $U \cap X_i$ is open in X_i for all $i \in I$. Note that if $f_i : X_i \to Y$, $i \in I$ is a family of maps then we get a unique map $f = \amalg_{i\in I} f_i : X \to Y$ s.t. $f|_{X_i} = f_i$ for each $i \in I$. A function $f : X \to Y$ is continuous iff $f|_{X_i} : X_i \to Y$ is continuous for each $i \in I$.

Attaching a space to another via map. Let X and Y be two spaces and let A be a subspace of X and $f : A \to Y$ a map. Then

$$X \amalg_f Y := X \amalg Y/_\sim,$$

where $X \amalg Y$ is given the disjoint union topology, see above, and the equivalence classes w.r.t. $\sim$ are the singletons $\{p\}$, $p \in X \setminus A$ and $p \in Y \setminus f(A)$, and the subsets $\{y\} \cup f^{-1}(y)$, $y \in f(A)$. We will only need this in situations where A is a closed subset of X.

Lemma 4.3. *Let the situation be as above with A a closed subset of X, let $\pi : X \amalg Y \to X \amalg_f Y$ be the canonical projection and let $i : Y \to X \amalg Y$ and $j : X \setminus A \to X \amalg Y$ be the inclusion maps. Then $\pi \circ i : Y \to X \amalg_f Y$ is an embedding with image a closed subset of $X \amalg_f Y$ and $\pi \circ j : X \setminus A \to X \amalg_f Y$ is an embedding with image an open subset of $X \amalg_f Y$.*

Exercise 4.4. Prove the above lemma.

5. CW-complexes considered as spaces obtained by attaching cells to each other

We start by a small

Lemma 5.1. *Let $(X, \mathcal{E})$ be a CW-complex and let Y be any space and let $f : X \to Y$ be a function. Then the following are equivalent.*

(1) *$f : X \to Y$ is continuous.*
(2) *The restriction $f|_{\bar{e}} : \bar{e} \to Y$ is continuous for all $e \in \mathcal{E}$.*

(3) *The restriction* $f|_{X^n} : X^n \to Y$ *is continuous for all* $n \geq 0$.

Exercise 5.2. Prove the above lemma.

Proposition 5.3. *Let* $(X, \mathcal{E})$ *be a CW-complex. Then* X^n *is obtained from* X^{n-1} *by attaching of the* n*-cells in* X.

Proof. Let $\mathcal{E}_n$ be the n-cells in X and let for each $e \in \mathcal{E}_n$, $\Phi_e : D^n \to X$ be the characteristic map of e. Since $\Phi_e(S^{n-1}) \subseteq X^{n-1}$ we can consider $\varphi_e = \Phi_e|_{S^{n-1}} : S^{n-1} \to X^{n-1}$. For each $e \in \mathcal{E}_n$, let $D^n_e = \{e\} \times D^n$ and identify this with D^n. (Here e is just some index.) We let $Z = \amalg_{e\in\mathcal{E}_n} D^n_e$, $A = \amalg_{e\in\mathcal{E}_n} \partial D^n_e$ and $\varphi = \amalg_{e\in\mathcal{E}_n} \varphi_e : A \to X^{n-1}$. Thus $\varphi|_{\partial D^n_e} = \varphi_e$. (Here $\partial D^n_e = \{e\} \times S^{n-1} \cong S^{n-1}$.) We let

$$Y = Z \amalg_\varphi X^{n-1}$$

and let $\pi_n : Z \amalg X^{n-1} \to Y$ be the canonical projection. We have a surjective map

$$f = (\amalg_{e\in\mathcal{E}_n} \Phi_e) \amalg j : Z \amalg X^{n-1} \to X^n,$$

where $j : X^{n-1} \to X^n$ is the inclusion. The proof is finalized by the following exercise. □

Exercise 5.4. Let the situation be as in the above proof. Prove the following facts:

(1) There is a unique bijection $\alpha : Y \to X^n$ such that $\alpha \circ \pi_n = f$.
(2) The bijection α is a homeomorphism. (Hint: To prove that α^{-1} is continuous use Lemma 5.1 and prove that $\alpha^{-1}|_{\bar{e}} = \pi_n|_{\bar{e}}$ if $e \subseteq X^{n-1}$ and $\alpha^{-1}|_{\bar{e}} \circ \Phi_e = \pi_n|_{D^n_e} : D^n_e \to Y$ for each $e \in \mathcal{E}_n$. Prove and use that $\Phi_e : D^n_e \to \bar{e}$ is an identification map.)
(3) The composition $\alpha \circ \pi_n|_{X^{n-1}} : X^{n-1} \to X^n$ is the inclusion.

We can now formulate an alterantive definition of CW-complexes.

Proposition/Definition 5.5. *A CW-complex is a space together with a filtration of subspaces*

$$\emptyset = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq \ldots \subseteq X^n \subseteq \ldots \subseteq X$$

such that

(1) X^n *is obtained by attaching of* n*-cells to* X^{n-1}. *That is, we have maps* $\varphi_\alpha : \partial D^n_\alpha \to X^{n-1}$, $\alpha \in I_n$, $n \geq 0$, *and homeomorphisms* $\alpha_n : Y^n \to X^n$ *such that* $\alpha_n \circ \pi_n|_{X^{n-1}} : X^{n-1} \to X^n$ *is the inclusion, where*

$$Y^n = Z^n \amalg_{\varphi_n} X^{n-1},$$

where $Z^n = \amalg_{\alpha\in I_n} D^n_\alpha$ *and* $\varphi_n = \amalg_{\alpha\in I_n} \varphi_\alpha : \amalg_{\alpha\in I_n} \partial D^n_\alpha \to X^{n-1}$, *and where* $\pi_n : Z^n \amalg X^{n-1} \to Y^n$ *is the canonical projection.*
(2) $X = \cup_{n\geq 0} X^n$.
(3) X *carries the weak topology w.r.t. the family of spaces* $\{X^n\}_{n\geq 0}$. *That is, a subset* $A \subseteq X$ *is closed iff* $A \cap X^n$ *is closed in* X^n *for all* $n \geq 0$.

Thus by (1) we have (inductively) a unique topology on X^n for each $n \geq 0$ starting by noting that X^0 necessarily have the discrete topology (since $X^0 = \amalg_{\alpha\in I_0} D^0_\alpha$), and by (3) the total space X then has the weak topology w.r.t. these topological spaces.

Note that we have not included in the definition above that X is Hausdorff. That follows automatically as the proof will reveal.

Proof. Exercise. See later version. □

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