Notes on Projective Geometry

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January 3, 2024

1 Basic Definitions

1.1 Linear Subspaces

We begin by proving a result from elementary linear algebra.

Theorem 1.1. *Let W_1 and W_2 be vector spaces. Then* $\dim W_1 + W_2 = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2$

Proof. Let $S = \{u_1 \cdots u_r\}$ be the basis of $W_1 + W_2$. Let $B_1 = \{u_1 \cdots u_r v_1 \cdots v_s\}$ and $B_2 = \{u_1 \cdots u_r w_1 \cdots w_t\}$ be B extended to be the basis of W_1 and W_2 respectively. If we can show B is the basis of $W_1 + W_2$, we have our result, as $\dim B = r + s + t = (r + s) + (r + t) - r = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2$.

First, we show B is linearly independent. Let

$$\sum_i^r a_i u_i + \sum_j^s b_j v_j + \sum_k^t c_k w_k = 0$$

.

Notice if we move terms so

$$\sum_i^r a_i u_i + \sum_j^s b_j v_j = -\sum_k^t c_k w_k$$

,

then the LHS is in W_1 and the RHS is in W_2, so both sides represent the same element in $W_1 + W_2$. Then $\sum_i^r d_i u_i = -\sum_k^t c_k w_k$, where the LHS uses B and the RHS uses B_2. Again moving terms:

$$\sum_i^r d_i u_i + \sum_k^t c_k w_k = 0$$

Where all c_i must be 0 as B_2 is linearly independent. Then

$$\sum_i^r a_i u_i + \sum_j^s b_j v_j = 0$$

But the LHS is described by B_1 which is also linearly independent so all a_i, b_j must also be 0. Then B is linearly independent.

Consider any $w_1 + w_2$.

$$w_1 = \sum_i^r a_i u_i + \sum_j^s b_j v_j$$

$$w_2 = \sum_i^r d_i u_i + \sum_k^t c_k w_k$$

Then

$$w_1 + w_2 = \sum_{i}^{r} (a_i + d_i)u_i + \sum_{j}^{s} b_j v_j + \sum_{k}^{t} c_k w_k \in \operatorname{span} W_1 + W_2$$

□

Theorem 1.2. *In a projective plane $P(V)$, two projective lines, $P(U)$ and $P(U')$, intersect in a unique point.*

Proof. From elementary linear algebra, $\dim V \geq \dim U + U'$. We have shown that $\dim U + U' = \dim U + \dim U' - \dim U \cap U'$. Then $1 \leq \dim U \cap U' \leq 2$. Because $P(U)$ and $P(U')$ are distinct, $\dim U \cap U' = 1$. So $P(U \cap U')$ is a projective point. □