CW Complexes

Kenny Workman

August 23, 2023

Problem 4.4.

Solution. By construction of the disjoint union topology, $j(X \setminus A)$ is open in $X \sqcup Y$ because $X \setminus A$ is open in X. Because π preserves open sets, $\pi j(X \setminus A)$ is open in $X \sqcup_f Y$.

Similarly, $i(Y)$ is clopen in $X \sqcup Y$ (Y as a space itself is both open and closed). Then $\pi i(Y)$ is clopen in $X \sqcup_f Y$. In particular, it is closed.

□

Problem 5.2.

Solution. To show (1) $\implies$ (2), we argue that the restriction of a continuous function to a closed subspace is also continuous.

Because f is continous, if $U \subseteq Y$ is closed then $f^{-1}(U)$ is also closed. Any closed subset of the image of restricted $f\big|_{\bar{e}}$ can be written as $U \cap f(\bar{e})$. Then $f^{-1}(U \cap f(\bar{e}) = f^{-1}(U) \cap \bar{e}$ is also closed. This holds for any closed U so our restriction must also be continuous.

An identical argument holds for (1) $\implies$ (3) ($X^n \subseteq X$ is closed for any n because it is a subcomplex).

To show (2) $\implies$ (1), again consider any closed $U \subseteq Y$ and note it is partitioned by a subset of $\mathcal{E}$, rather $U = \sqcup_{e \in \mathcal{E}} U \cup f(\bar{e})$. Because $f\big|_{\bar{e}}$ is continuous, $f\big|_{\bar{e}}^{-1}(U \cup f(\bar{e}))$ is closed for each e by assumption. But then $f^{-1}(U) = \sqcup_{e \in \mathcal{E}} f\big|_{\bar{e}}^{-1}(U \cup f(\bar{e}))$ is also closed. (TODO: need to show this union is finite). This holds for any closed U so f is continuous as desired.

To see (3) $\implies$ (1), it is sufficient to see there exists some $n >= 0$ where $X^n = X$. Then $f\big|_{X^n} = f$ and we are done.

□

Problem 5.4.1.

Solution. We construct a bijection $\alpha : Z \sqcup_\phi X^{n-1} \to X^n$. $\alpha\big|_{X^{n-1}} = 1$. $\alpha\big|_{int(D_e^n)} = \Phi_e\big|_{int(D^n)}$.

We can verify the union of our restrictions is exactly the domain of α, $\sqcup_{\mathcal{E}_n} int(D_e^n) \sqcup X^{n-1} = Z \sqcup_\phi X^{n-1}$. The image of α is $\sqcup_{\mathcal{E}_n} e_n \sqcup X^{n-1} = X^n$. Each restriction is a bijection ($\Phi_e\big|_{int(D^n)}$ is a homeomorphism by definition, $\alpha\big|_{X^{n-1}}$ is the identity map). So α is also a bijection.

□