Notes on Projective Geometry

Kenny Workman

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1 Basic Definitions

Definition 1.1. Let V be a vector space. The projective space $P(V)$ is the set of 1-dimensional vector subspaces of V.

It can be useful to think of projective spaces, at least in the real case, as bundles of lines that pass through the origin. This reduces our space by a dimension, motivating the common shorthand for the reals $P^n(R) = P(R^{n+1})$.

Many sources encourage thinking about the sphere S^n for a projective space $P^n(\mathbb{R})$ to ease the visualization of a space of lines. Antipodal points on the sphere are identified and every such antipodal pair is injective with the actual elements of the projective space.

I have found it easier to just think of the vector subspaces the projective elements represent (projective points and lines are lines and planes through the origin in $\mathbb{R}^3$) for more natural geometric intuition.

1.1 Decomposition

The following decomposition is useful to understand the structure of theses spaces:

$$P(R^n) = R^n - 1 + P(R^{n-1})$$

Essentially our goal is to take R^n and partition the set of points into 1-dim vector subspaces such that each partition has a nice representation. Recall:

Definition 1.2. A representative vector is any of the non-zero vectors from the 1-dimensional subspace corresponding to a point $[v] \in P(V)$.

Then if $[x] = [x] = [a]$, x and a are both representatives for the same projective point.

We also want to define the notion of the homogoenous coordinates for each projective point, which are just the real points that exist in the corresponding vector subspace.

Definition 1.3. The homogoenous coordinates for $[v] \in P(V)$ are the set $[(x_0 \cdots x_n)]$ equivalent under scalar multiplication by λ.

If we construct a subset of homogoenous coordinates U_0 where $x_0 \neq 1$, notice that each $[(x_0 \cdots x_n)] = [1 \cdots x_n/x_0]$, so $U_0 \cong R^n - 1$. We are left to "partition" the coordinates where $x_0 = 0$, but this is exactly the set of 1-dimensional subspaces of $V^n - 1$, so $P(R^{n-1})$.

1.2 Applications

1.3 Linear Subspaces

We begin by proving a result from elementary linear algebra.

Theorem 1.4. *Let W_1 and W_2 be vector spaces. Then* $\dim W_1 + W_2 = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2$

Proof. Let $S = \{u_1 \cdots u_r\}$ be the basis of $W_1 + W_2$. Let $B_1 = \{u_1 \cdots u_r v_1 \cdots v_s\}$ and $B_2 = \{u_1 \cdots u_r w_1 \cdots w_t\}$ be B extended to be the basis of W_1 and W_2 respectively. If we can show B is the basis of $W_1 + W_2$, we have our result, as $\dim B = r + s + t = (r + s) + (r + t) - r = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2$.

First, we show B is linearly independent. Let

$$\sum_i^r a_i u_i + \sum_j^s b_j v_j + \sum_k^t c_k w_k = 0$$

.

Notice if we move terms so

$$\sum_i^r a_i u_i + \sum_j^s b_j v_j = -\sum_k^t c_k w_k$$

,

then the LHS is in W_1 and the RHS is in W_2, so both sides represent the same element in $W_1 + W_2$. Then $\sum_i^r d_i u_i = -\sum_k^t c_k w_k$, where the LHS uses B and the RHS uses B_2. Again moving terms:

$$\sum_i^r d_i u_i + \sum_k^t c_k w_k = 0$$

Where all c_i must be 0 as B_2 is linearly independent. Then

$$\sum_i^r a_i u_i + \sum_j^s b_j v_j = 0$$

But the LHS is described by B_1 which is also linearly independent so all a_i, b_j must also be 0. Then B is linearly independent.

Consider any $w_1 + w_2$.

$$w_1 = \sum_i^r a_i u_i + \sum_j^s b_j v_j$$

$$w_2 = \sum_i^r d_i u_i + \sum_k^t c_k w_k$$

Then

$$w_1 + w_2 = \sum_i^r (a_i + d_i) u_i + \sum_j^s b_j v_j + \sum_k^t c_k w_k \in \operatorname{span} W_1 + W_2$$

$\square$

Theorem 1.5. *In a projective plane $P(V)$, two projective lines, $P(U)$ and $P(U')$, intersect in a unique point.*

Proof. From elementary linear algebra, $\dim V \geq \dim U + U'$. We have shown that $\dim U + U' = \dim U + \dim U' - \dim U \cap U'$. Then $1 \leq \dim U \cap U' \leq 2$. Because $P(U)$ and $P(U')$ are distinct, $\dim U \cap U' = 1$. So $P(U \cap U')$ is a projective point. $\square$

It is useful to think about this result using our model of the projective plane as a sphere and using our decomposition.

We can think of projective lines as planes in $\mathbb{R}^3$ that intersect the sphere in two great circles. These great circles intersect in a pair of antipodal points, which is a projective line.

Alternatively