4.3 and 5.1

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Problem 4.3.15.

Solution. Let $J : \mathscr{A} \to \mathscr{B}$ be a full and faithful functor. Consider $A, A' \in \mathscr{A}$.

We first prove (a), that the map $f : A \to B$ in $\mathscr{A}$ is an isomorphism iff the map $J(f) : J(a) \to J(B)$ in $\mathscr{B}$ is an isomorphism. If f is an isomorphism, there exists g where $gf = 1_A$ and $fg = 1_B$. Because J is faithful, $J(f)$ and $J(g)$ are distinct. Furthermore, $J(f)J(g) = J(fg) = J(1_A) = 1_{J(A)}$ and $J(g)J(f) = J(gf) = J(1_B) = 1_{J(B)}$, using the definition of a functor. Similarly, if $J(f)$ is an isomorphism, there exists $J(g)$ where $J(g)J(f) = 1_{J(A)}$ and $J(f)J(g) = 1_{J(B)}$. Because J is full, f, g exist. Because $1_{J(A)} = J(1_A) = J(gf)$, $gf = 1_A$. Similarly $fg = 1_B$.

We then prove (b). Given an isomorphism $g : J(A) \to J(A')$, we showed in (a) that there exists an isomorphism f where $J(f) = g$. Uniqueness of f can be shown using the fact that J is faithful. Constructing some f', where $f' \neq f$, the $J(f') \neq J(f) \neq g$.

To prove (c), we can rephrase the claim as, "There exists an isomorphism $f : A \to A'$ in $\mathscr{A}$ iff there exists an isomorphism $g : J(A) \to J(A')$ in $\mathscr{B}$" (The existence of an isomorphism between two objects is the same as saying those objects are isomorphic). We have already shown the existence of this isomorphism in (a). All that remains is to show the isomorphism is defined on the correct objects in the image of J, but this follows from the definition of a functor ($f : A \to A'$ iff $g = J(f) : J(A) \to J(A')$.

□

Problem 4.3.18(a).

Solution. Because it is not given, we will define the induced functor on natural transformations as $J \circ -(\eta) = J((\eta_C)_{C \in \mathscr{C}})$.

We first show that if J is faithful, then $J \circ -$ is faithful. Fix two functors $G, G' \in [\mathscr{B}, \mathscr{C}]$ and consider $\eta, \eta' \in [\mathscr{B}, \mathscr{C}](G, G')$ where $\eta \neq \eta'$.

To see $J \circ -(\eta) \neq J \circ -(\eta')$ for $J \circ -(\eta), J \circ -(\eta') \in [\mathscr{B}, \mathscr{D}](JG, JG')$, it is sufficient to show that one of the component maps of $J \circ -(\eta)$ and $J \circ -(\eta')$ for some $C \in \mathscr{C}$, are distinct functions. Because $\eta \neq \eta'$, there exists some $B \in \mathscr{B}$ where $\eta_B \neq \eta'_B$. Because $J(\eta_B) \neq J(\eta'_B)$, as J is faithful, certainly $J \circ -(\eta) \neq J \circ -(\eta')$.

To see that if J is full, $J \circ -$ is full, we must show that for each $J(\eta) \in [\mathscr{B}, \mathscr{D}]$ there exists $\eta \in [\mathscr{B}, \mathscr{C}]$. Consider the following commutative diagram:

$$\begin{array}{ccc} JG(B) & \xrightarrow{Jg} & JG(B') \\ \big\downarrow{\scriptstyle J(\eta_B)} & & \big\downarrow{\scriptstyle J(\eta_{B'})} \\ JG'(B) & \xrightarrow[Jg']{} & JG'(B') \end{array}$$

Indeed $(\eta_B)_{B \in \mathscr{B}}$ exists in the domain of $J \circ -$. To see that it is natural, observe that $Jg' J(\eta_B) = J(\eta_{B'} Jg$ because $J(\eta)$ itself is natural. By functoriality, $J(g'\eta_B) = J(\eta_{B'} g)$. Then $g'\eta_B = \eta_{B'} g$ because J is full so η is natural and we have our result.

□

Problem 4.3.18(b).

Solution. This result follows directly from Lemma 4.3.8

□

Problem 4.3.18(c).

Solution. Because $H_\bullet$ is full and faithful, $H_\bullet \circ -$ is also full and faithful. Then $H_\bullet \circ -(G) \cong H_\bullet \circ -(G')$ implies $G \cong G'$, using the same argument as (b).

$\square$

Problem 5.1.33. Verify that in the category of vector spaces, the product of two vector spaces is their direct sum.

Solution. Consider $X, Y \in \mathbf{Vect}_k$, we claim the direct sum $X \oplus Y$, along with projections $p_1 : X \oplus Y \to X$ and $p_2 : X \oplus Y \to Y$, is the product of X and Y. Denote the elements of $X \oplus Y$ as (x, y). Then our projections are defined as the linear maps $p_1 = (x, y) \mapsto x$ and $p_2 = (x, y) \mapsto y$.

For any $Z \in \mathbf{Vect}_k$ and pairs of maps $f_1 : Z \to X$ and $f_2 : Z \to Y$, there exists a unique function g where $p_1 g = f_1$ and $p_2 g = f_2$. Clearly $f_1 = p_1 g$ ($p_1 g = z \mapsto f_1(z)$). To see uniqueness, define $\hat{g}$ where $p_1 \hat{g} = f_1$ and $p_2 \hat{g} = f_2$, then $\hat{g} = z \mapsto (f_1(z), f_2(z))$, so this is the same function as g.

(As an aside, $X \oplus Y$ can only be the product of X and Y if $X \cap Y = \emptyset$ by the definition of a direct sum.)

$\square$

Problem 5.1.37.

Solution. We must show that the set of all cones on D with vertex 1, defined as $\{(x_I)_{I \in \mathbf{I}} | x_I \in D(I) \text{ and } Du(x_I) = x_J \text{ for all such } u : I \to J\}$, is the same as $\varprojlim_I D$. The projections of this limit, $p_J : \varprojlim_I D \to D(J)$ for each $J \in \mathbf{I}$, are defined as $p_J((x_I)_{I \in \mathbf{I}}) = x_J$.

Fix an arbitrary cone on D, defined as $(A \xrightarrow{f_I} D(I))_{I \in \mathbf{I}}$. We must show there exists a corresponding $g : A \to \varprojlim_I D$ where $p_J g = f_J$ for each $J \in \mathbf{I}$. Define $g = a \mapsto (f_I(a))_{I \in \mathbf{I}}$. Then $p_J g = a \mapsto (f_I(a))_{I \in \mathbf{I}} \mapsto f_J(a)$ so it is the same function as f_J. In fact, for any cone, we can define a similar unique map, using only the family of maps in the cone, that satisfies the same property. This proves that the set of all cones on D with vertex 1 is the same as the limit of D in *Set*.

$\square$