4.2 The Yoneda Lemma

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Problem 4.2.3(a).

Solution. We wish to show the (unique) representable $H_M : M^{op} \to Set$ is the same functor as $\underline{M}$. Recall $\underline{M}$ is defined for us as the right M-action of the monoid M on its underlying set defined as $x * m \mapsto xm$ for each $m \in M$.

We first show that H_M is a right M-action. Each $m \in M^{op}(M, M)$ induces a function $H_M(m) : M^{op}(M, M) \to M^{op}(M, M)$, which is a function mapping the morphisms in the single object category (elements of M) to itself. We must show this function satisfies (1) the identity property $H_M(1_M) = x \mapsto x$ and (2) compatibility $H_M(m'm)(x) = H_M(m) \circ H_M(m')(x)$. (1) is true by the definition of $H_M(1_M)$ as $H_M(1_M)(x) = x \mapsto x \circ 1_m = 1$. (2) is true by the composition rules of a contravariant functor.

We conclude by showing $H_M(m)(x) = x \mapsto x \circ m$. This is true by the definition of $H_M(m)$.

□

Problem 4.2.3(b).

Solution. We will refer to $\underline{M}$ as H_M. Both H_M and X are functors in $[M^{op}, Set]$. For each $x \in X(M)$, we define a function $\alpha^x : H_M(M) \to X(M)$ as $m \mapsto X(m)(x)$. Indeed $\alpha^x(1) = X(1)(x) = x$.

We define a bijection $\hat{()} : [H_M, X] \to X$ defined as $(\hat{\alpha}) = \alpha(1)$. We constrain $\hat{()}$ to our set of α_x defined above and claim this function is a bijection. Given $\alpha^x \neq \alpha^{x'}$, $((\hat{\alpha^x}) = x) \neq ((\hat{\alpha^{x'}}) = x')$ so $\hat{()}$ is injective. For each $x \in X$, $(\hat{x})^{-1} = \alpha^x$ so $\hat{()}$ is also surjective.

□

Problem 4.2.3(c).

Solution. We must show $[M^{op}, Set](H_M, X) \cong X(M)$ naturally in $X \in [M^{op}, Set]$ and $M \in M^{op}$. Because M^{op} has one object, we can show naturality in both X and M in one step. Define $X' \in [M^{op}, Set]$ and the natural transformation $f \in [M^{op}, Set](X, X')$. We reuse the definition of $\hat{()}$ from (b). Consider the commutative square:

$$\begin{array}{ccc} [H_M, X] & \xrightarrow{f(-)} & [H_M, X'] \\ \hat{()}\big\downarrow & & \big\downarrow\hat{()} \\ X(M) & \xrightarrow[f]{} & X'(M) \end{array}$$

We claim $f\hat{()} = (\hat{f(-)})$.

Our left hand side is defined as $f\hat{()} = \alpha \mapsto \alpha(1) \mapsto f(\alpha(1))$. Our right hand side is defined as $\hat{()}f(-) = \alpha \mapsto f\alpha \mapsto (\hat{f\alpha}) \mapsto f(\alpha(1))$. (Recall $(\hat{\alpha}) = \alpha(1)$). Then $\hat{()}$ is natural and because we have already shown it is a bijective map in Set, it is also an isomorphism. Then $[M^{op}, Set](H_M, X) \cong X(M)$ naturally in $X \in [M^{op}, Set]$ and $M \in M^{op}$.

□