# 6 - Adjoints, representables and limits

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**Problem 6.1.5.** Interpret the theory of this section in the special case when $\mathbf{I}$ is the discrete category with two objects.

*Solution.* Let $\mathbf{I}$ contain objects $I, J$ with no maps but the identities. First we examine the new definition of a cone as a natural transformation $\eta : \Delta A \to D$ and observe the component maps $(\eta_I)_I$ are simply a pair of maps $A \xrightarrow{\eta_I} DI$ and $A \xrightarrow{\eta_J} DJ$ with no additional commutative properties. So the elements of $Cone(A, D) = [\mathbf{I}, \mathscr{A}](\Delta A, D)$ are all pairs of maps $(A \xrightarrow{\eta_I} DI, A \xrightarrow{\eta_J} DJ)$.

**6.1.1** tells us that each limit cone on $D$ corresponds to a unique bijection between cones on $D$ and maps into the limit cone. We know from Yoneda's Lemma, that $[\mathscr{A}^{op}, \mathbf{Set}](H_{\varprojlim_I D}, Cone(-, D)) \cong Cone(A, D)$ and that each natural transformation is actually just a cone. **6.1.1** tells us that the natural transformations that are also natural isomorphisms are also cones, but they are cones that are also universal elements. The structure of the universal element is another description of a limit cone.

Consider the isomorphism $\alpha \in [\mathscr{A}^{op}, \mathbf{Set}](H_{\varprojlim_I D}, Cone(-, D))$. It places $f \in \mathscr{A}(A, \varprojlim_I D)$ in one-to-one correspondence with $x \in Cone(A, D)$. In particular, we have a universal $u \in Cone(\varprojlim_I D, D)$ where given any cone $x \in Cone(A, D)$, we know there is some unique $f \in \mathscr{A}(A, \varprojlim_I D)$ where $Cone(f, D)(u) = x$.

Let us examine this in the case where $\mathbf{I}$ is binary. Our universal element is some $u = (\varprojlim_I D \xrightarrow{\eta_I} DI, \varprojlim_I D \xrightarrow{\eta_J} DJ)$. Given any $x \in Cone(A, D) = (A \xrightarrow{\eta'_I} DI, A \xrightarrow{\eta'_J} DJ)$ there is a unique $f : A \to \varprojlim_I D$ where $(A \xrightarrow{f\eta_I})_I = x$.

**6.1.3** tells us that natural transformations between diagrams $\alpha \in [\mathbf{I}, \mathscr{A}](D, D')$ induce unique maps between the limit cones. In the case of our binary category, such $\alpha$ are simply a pair of maps with no other properties:

$$\begin{tikzcd} DI \arrow[d, "\alpha_I"] & DJ \arrow[d, "\alpha_J"] \\ D'I & D'J \end{tikzcd}$$

Our induced $\varprojlim_I$ is simply the map that satisfies the commutative squares given in the exercises.

We finally examine the formulation of limits as adjoint functors described in **6.1.4**.

Consider the adjoints (with definitions given in the text):

$$\begin{tikzcd} & {[I, \mathscr{A}]} \arrow[dr, "\varprojlim_I"] & \\ \arrow[ur, "\Delta"] \mathscr{A} & & \end{tikzcd}$$

Where $[I, \mathscr{A}](\Delta A, D) \cong \mathscr{A}(A, \varprojlim_I D)$ naturally in $A, D$. The isomorphism being natural in $A$ is obvious. What is more interesting is that it is natural in $D$. Consider the following commutative diagram, with $lim\alpha$ denoting our induced map between cones.

$$\begin{tikzcd} Cone(A,D) \arrow[r, "Cone(lim\alpha)"] \arrow[d, "\alpha_D"] & Cone(A,D') \arrow[d, "\alpha_{D'}"] \\ \mathscr{A}(A, limD) \arrow[r, "(-)\circ lim\alpha"] & \mathscr{A}(A, limD') \end{tikzcd}$$

A pushback between vertices of limit cones is essentially the same as pushback between the actual limit cones.

□

**Problem 6.2.20.**

*Solution.* We will show that $\alpha$ is monic iff $\alpha_A$ is monic for each $A \in \mathbf{A}$.

The forward direction is trivial. If $\alpha$ is a monic in $[\mathbf{A}, \mathscr{P}]$, each of its component maps are certainly monic.

The reverse direction follows from **6.2.5**. The fact that $\alpha_A$ is monic in $\mathscr{P}$ means that the following pullback exists in $\mathscr{P}$ for each $A$ in $\mathbf{A}$:

$$\begin{tikzcd} X(A) \arrow[r, "1"] \arrow[d, "1"'] & X(A) \arrow[d, "\alpha_A"] \\ X(A) \arrow[r, "\alpha_A"'] & Y(A) \end{tikzcd}$$

Each such pullback can be described as a limit on the diagram $ev_A \circ D$. Where the functor $D : \mathbf{I} \to [\mathbf{A}, \mathscr{P}]$ assigns $DI = X$ and $DJ = Y$ and it domain is the small category $\mathbf{I}$:

$$\begin{tikzcd} & I \arrow[d, "u"] \\ I \arrow[r, "u"'] & J \end{tikzcd}$$

Our limit is the family $(L \xrightarrow{1} DI(A)_{I,J}$ where $L = XA = DIA$ for each $A \in \mathbf{A}$.

**6.2.5** tells us that there is a limit cone on $D$ itself and its image under $ev_A$ is exactly $L = XA$ for each such $A \in \mathbf{A}$. Then this limit cone is exactly $X$ which is another way of saying $[\mathbf{A}, \mathscr{P}]$ has the pullback:

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, "1"'] & X \arrow[d, "\alpha"] \\ X \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

So $\alpha$ is monic.

□

**Problem 6.2.22.** Show how a category of elements can be described as a comma category.

*Solution.* Define an arbitrary presheaf $X : \mathscr{A}^{op} \to \mathbf{Set}$ and consider the comma category denoted $(X \Rightarrow 1)$ as in the diagram:

$$\begin{tikzcd} & \{1\} \arrow[d, "1"] \\ \mathscr{A}^{op} \arrow[r, "X"'] & Set \end{tikzcd}$$

Where the functor 1 is defined on the singleton set as $1 : \{1\} \to Z$ where $Z = \coprod_{A \in \mathscr{A}^{op}} XA$.

Then the elements of $(X \Rightarrow 1)$ are $(A, g_A, x$ where $g_A : XA \to Z$ maps $x \mapsto x$.

□

**Problem 6.3.21(a).**

*Solution.* It is enough to show that $U : \mathbf{Grp} \to \mathbf{Set}$ does not preserve colimits to see it has no right adjoint.

Consider the "empty sum" in any category $X$, the colimit of the diagram $D : \mathbf{I} \to X$ where $I$ has no objects or maps, and observe this is the same as the initial object in this category. In **Grp** the colimit of this diagram is then the trivial group and in **Set** it is the empty set. Then $U$ does not preserve colimits. □

**Problem 6.3.21(b).**

*Solution.* We begin with the right side of the adjoint chain. Recall $I : \mathbf{Set} \to \mathbf{Cat}$ right adjoint to $C$ turns any set $X$ into a category $IX$ with all possible morphisms between objects. Any function $f : X \to Y$ becomes a functor $If = x \mapsto fx$ for each object $x \in IX$ that carries all maps in $IX(x, y)$ to $IY(Ifx, Ify)$. It is enough to show that this functor does not preserve colimits.

As we continue to the left side of the adjoint chain $C \dashv D$, recall $D : \mathbf{Set} \to \mathbf{C}$ turns sets into "discrete" categories (with only the identity maps) while $C : \mathbf{Cat} \to \mathbf{Set}$ maps categories to sets where each function $Cf$ inherits □

**Problem 6.3.24(a).**

*Solution.* If $A$ is finite, the subgroup in question is finitely generated and is itself finite. Its cardinal number is then a natural number.

If $A$ is infinite, the subgroup will have □

**Problem 6.3.24(b).**

*Solution.* The number of different group structures (isomorphic classes) ¡ S is small. □

**Problem 6.3.24(c).**

*Solution.* For each $A \in Set$, we show that the category $(A \Rightarrow U)$ has a weakly initial set. For any object $A \xrightarrow{f} U(X)$, we claim there is some group $B \cong < (x_a)_{a\in A} >$, where $(x_a)_{a\in A}$ is the subgroup generated by the elements of $X$ defined in the function $f$. Then there exists object $A \xrightarrow{g} U(B)$, where

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, "f"'] & UB \arrow[d, "U\psi"] \\ & UX \end{tikzcd}$$

commutes, if $\psi$ is any isomorphism between these groups.

For any $A$, we know that any group $< (x_a)_{a\in A} >$ has cardinality $\max \mathcal{N}, A$ and the collection of isomorphism classes of groups at most $\max \mathcal{N}, A$ is small. Our argument shows that we need at most one member of each isomorphism class of groups at $\max \mathcal{N}, A$ to and that this set is our weakly initial set of $A \Rightarrow U$ for each $A \in Set$ □

**Problem 6.3.26(a).**

*Solution.* For any $f : A' \to A$, consider any monic $m \in Sub(A)$. Observe the limit of the following diagram:

$$\begin{tikzcd} & X \arrow[d, "m"] \\ A' \arrow[r, "f"'] & A \end{tikzcd}$$

is the pullback:

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "m'"'] & X \arrow[d, "m"] \\ A' \arrow[r, "f"'] & A \end{tikzcd}$$

which defines a unique $m'$. We know that $m'$ is also monic from **5.1.42** and can be considered an element of $Sub(A')$. Then any $f : A' \to A$ induces a map $Sub(f) : Sub(A) \to Sub(A')$.

□

**Problem 6.3.26(b).**

*Solution.* We must show that the map $Sub(f)$ we constructed is functorial

Define $g : A'' \to A'$ and pick some $m \in Sub(A), m' \in Sub(A'), m'' \in Sub(A'')$. Then $Sub(g)Sub(f)$ is the diagram:

$$\begin{tikzcd} X'' \arrow[r, "g'"] \arrow[d, "m''"'] & X' \arrow[r, "f'"] \arrow[d, "m'"'] & X \arrow[d, "m"] \\ A'' \arrow[r, "g"'] & A' \arrow[r, "f"'] & A \end{tikzcd}$$

Using the result **5.1.35**, because the left and right squares are both pullbacks, the outer rectangle is a pullback. So the correspondence between $m$ and $m''$ is again unique.

It is easy to see that for the same $m \in Sub(A), m'' \in Sub(A'')$, the pullback $Sub(fg)$ is the same as the outer rectangle above:

$$\begin{tikzcd} X'' \arrow[r, "f'g'"] \arrow[d, "m''"] & X \arrow[d, "m"] \\ A'' \arrow[r, "fg"] & A \end{tikzcd}$$

This gives our result $Sub(fg) = Sub(g)Sub(f)$.

□

**Problem 6.3.26(c).**

*Solution.* For any $A \in$ **Set**, an isomorphism class of monics into $A$, eg. element of $Sub(A)$, is a subset of $A$.

To see this, recall monics in **Set** are injective functions and two monics into $A$ are in the same isomorphism class iff the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "m"'] & X' \arrow[d, "m'"] \\ & A \end{tikzcd}$$

Where $h$ is a bijection (an isomorphism in **Set**). This diagram only commutes if $m$ and $m'$ share the same range $m'h(X) = mh^{-1}(X')$. Then the isomorphism class is this range, a subset of $A$.

That fact that $H_2 \cong Sub$ naturally in $A \in \mathscr{A}$ is then a formal description that functions into 2 and isomorphism classes of monics into $A$ are both ways of describing subsets of $A$

The isomorphism $\alpha_A : H_2(A) \to Sub(A)$, defined as $\alpha_A = (f : A \to 2) \mapsto *x$, where $*x \in Sub(A)$ and each $X \xrightarrow{m} A \in *x$ has the property $m(X) = f^{-1}(1)$.

□