3.2 Small and Large Categories

May 3, 2023

Problem 3.2.12 (a).

Solution. We will show $S \subseteq \theta(S)$ and $\theta(S) \subseteq S$ and conclude that $S = \theta(S)$.

First, notice:

$$S = \bigcup_{R \in \mathcal{P}(A): R \subseteq \theta(R)} R \subseteq \bigcup_{R \in \mathcal{P}(A): R \subseteq \theta(R)} \theta(R) = \theta(S)$$

So $S \subseteq \theta(S)$.

To show $\theta(S) \subseteq S$, we show that for any $R \in P(A) : R \subseteq \theta(R)$ that $\theta(R) \subseteq S$. Because θ is order preserving w.r.t. inclusion, $R \subseteq \theta(R) \implies \theta(R) \subseteq \theta(\theta(R))$. So indeed $\theta(R) \subset S$ for every R and $\theta(S) \subseteq S$.

Then $\theta(S) \subseteq S \subseteq \theta(S)$ so $\theta(S) = S$.

□

Problem 3.2.12 (b).

Solution. We claim that $\theta = S \mapsto A \setminus g(B \setminus fS)$ is an order preserving map (with respect to inclusion).

Indeed, $S \subseteq S' \implies fS \subseteq fS' \implies g(B \setminus fS') \subseteq g(B \setminus fS) \implies A \setminus g(B \setminus fS) \subseteq A \setminus g(B \setminus fS')$

Then there exists some $S \subseteq A : S = \theta(S)$ so $A \setminus S = g(B \setminus FS)$ as desired.

□

Problem 3.2.12 (c).

Solution. Still need to finish. □

Problem 3.2.13.

Solution. We will show a contradiction from the assignment $f(a') = \{a | a \notin f(a)\}$ for any a'.

Notice that for any $f(a') = \{a | a \notin f(a)\}$, that $a' \notin \{a | a \notin f(a)\}$ must be true. However, if this is true, then $a' \in \{a | a \notin f(a)\}$ is also true.

Then this assignment is not possible and f cannot be surjective.

□

4.1 Representables Definitions and Examples

May 26, 2023

Problem 4.1.27.

Solution. For each $A \in \mathscr{A}$, let $\alpha_A : H_A \to H_{A'}$ be the component map of the natural isomorphism $H_A \cong H_{A'}$.
Let $f = \alpha_A(1_A)$ and let $g = \alpha_{A'}^{-1}(1_{A'})$.
Consider the following commutative square:

$$\begin{tikzcd} H_A(A) \arrow[r, "H_A(g)"] \arrow[d, "\alpha_A"] & H_A(A') \arrow[d, "\alpha_{A'}"] \\ H_{A'}(A) \arrow[r, "H_{A'}(g)"] & H_{A'}(A') \end{tikzcd}$$

By naturality, $H_{A'}(g)\alpha_A = \alpha_{A'}H_A(g)$ and $H_{A'}(g)\alpha_A(1_A) = \alpha_{A'}H_A(g)(1_A)$. Then $fg = \alpha_{A'}(g)$. Because $g = \alpha_{A'}^{-1}(1'_A)$, $fg = 1_{A'}$.
Now consider the similar commutative square:

$$\begin{tikzcd} H_A(A) & H_A(A') \arrow[l, "H_A(f)"'] \\ H_{A'}(A) \arrow[u, "\alpha_A^{-1}"] & H_{A'}(A') \arrow[l, "H_{A'}(f)"'] \arrow[u, "\alpha_{A'}^{-1}"'] \end{tikzcd}$$

By naturality, $H_A(f)\alpha_{A'}^{-1} = \alpha_A^{-1}H_{A'}(f)$ and $H_A(f)\alpha_{A'}^{-1}(1_{A'}) = \alpha_A^{-1}H_{A'}(f)(1_{A'})$. Then $gf = \alpha_A^{-1}(f)$. Similarly, $gf = 1_A$.

$\square$

Problem 4.1.28.

Solution. Denote $H^{\mathbb{Z}/p\mathbb{Z}}(-)$ as shorthand for $\mathrm{Grp}(\mathbb{Z}/p\mathbb{Z}, -)$. Let G, G' are elements of Grp and $f : G \to G'$ is a homomorphism (a map in Grp). Define $U_p(f)$ as the restriction of f to $U_p(G)$. Note this restriction is also a homomorphism.

We begin by showing the set $H^{\mathbb{Z}/p\mathbb{Z}}(G)$ is in bijective correspondence with $U_p(G)$ for any choice of $G \in \mathrm{Set}$. Define $h : U_p(G) \to H^{\mathbb{Z}/p\mathbb{Z}}(G)$ as $h = g \mapsto \theta : \theta(1) = g$. (Then $\theta(2) = g^2, \theta(3) = g^3$ ... $\theta(p) = 1$ and so forth because θ is a homomorphism).

Because $g \neq g' \implies h(g)(1) \neq h(g')(1)$, h is injective. Because $\theta \neq \theta' \implies \theta(x) \neq \theta'(x)$ for some $x \implies \theta(1)^x \neq \theta'(1)^x \implies \theta(1) \neq \theta'(1) \implies h^{-1}(\theta(1)) \neq h^{-1}(\theta'(1))$, h is also surjective. Then h is a bijection.

We will use this fact to describe elements of $\theta \in H^{\mathbb{Z}/p\mathbb{Z}}(G)$ only by their assignment $\theta(1)$ for the remainder of the proof.

Consider the following commutative diagram, where we introduce the component maps $\alpha_G : H^{\mathbb{Z}/p\mathbb{Z}}(G) \to U_p(G)$ of our natural isomorphism. We define $\alpha_G = \theta \mapsto \theta(1)$.

$$\begin{tikzcd} H^{\mathbb{Z}/p\mathbb{Z}}(G) \arrow[r, "H^{\mathbb{Z}/p\mathbb{Z}}(f)"] \arrow[d, "\alpha_G"] & H^{\mathbb{Z}/p\mathbb{Z}}(G') \arrow[d, "\alpha_{G'}"] \\ U_p(G) \arrow[r, "U_p(f)"] & U_p(G') \end{tikzcd}$$

We first show naturality by proving $U_p(f)\alpha_G = \alpha_{G'} H^{\mathbb{Z}/p\mathbb{Z}}(f)$. Consider $\theta \in H^{\mathbb{Z}/p\mathbb{Z}}(G)$, then $U_p(f)\alpha_G = (1 \mapsto g) \mapsto g \mapsto g'$ and $\alpha_{G'} H^{\mathbb{Z}/p\mathbb{Z}}(f) = (1 \mapsto g) \mapsto (1 \mapsto g') \mapsto g'$. Then $U_p(f)\alpha_G(\theta(1)) = \alpha_{G'} H^{\mathbb{Z}/p\mathbb{Z}}(f)(\theta(1)) = g'$ for any θ so these maps are the same.

We now show each α_G is an isomorphism. We have already shown there exists a bijective function $h : H^{\mathbb{Z}/p\mathbb{Z}}(G) \to U_p(G)$ for any choice of G. If we define h such that $h\alpha_G = (1 \mapsto g) \mapsto g \mapsto 1 \mapsto g$ it is clear that h is the inverse to each α_G so each α_G is also an isomorphism.

$\square$

Problem 4.1.30.

Solution. For $f \in \mathbf{Top}^{\mathrm{op}}(X, X')$ we define $\mathscr{O}(f) = f^{-1}$. We define a functor $H_S = \mathbf{Top}^{\mathrm{op}}(-, S) : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Set}$ using definition **4.1.16**. Denote $S = \{1, 2\}$ where $\{1\} \subset S$ is the open singleton and $\{2\} \subset S$ is the closed singleton.

Given $X \in \mathbf{Top}^{\mathrm{op}}$, we show a canonical bijection between open sets in X and continuous maps $q : X \to S$ by defining a bijection $\alpha_X : \mathscr{O}(X) \to H_S(X)$ as $\alpha_X = U \mapsto q$ for $U \in \mathscr{O}(X)$, with $q(U) = \{1\}$ and $q(X - U) = \{2\}$. To see that α_X is injective, $U \neq U' \implies \alpha_X(U) \neq \alpha_X(U')$ as $U \subsetneq U' \implies \alpha_X(U)(U' - U) \neq \alpha_X(U')(U' - U)$ while $U' \subsetneq U \implies \alpha_X(U)(U - U') \neq \alpha_X(U')(U - U')$. To see that α_X is surjective, any $q \in H_S(X)$, $\alpha_X^{-1}(q) = q^{-1}(S)$.

We then show $\mathscr{O} \cong H_S$. Consider the following commutative square:

$$\begin{tikzcd}
\mathscr{O}(X) \arrow[r, "\mathscr{O}(f)"] \arrow[d, "\alpha_X"] & \mathscr{O}(X') \arrow[d, "\alpha_{X'}"] \\
H_S(X) \arrow[r, "H_S(f)"] & H_S(X')
\end{tikzcd}$$

We claim $H_S(f)\alpha_X = \alpha_{X'}\mathscr{O}(f)$ for any choice of $f \in \mathbf{Top}^{\mathrm{op}}(X, X')$.

For $U \in \mathscr{O}(X)$, $H_S(f)\alpha_X = U \mapsto q \mapsto qf^{-1}$ and $\alpha_{X'}\mathscr{O}(f) = U \mapsto f^{-1}(U) \mapsto qf^{-1}$, where q is defined as before.

$\square$

4.2 The Yoneda Lemma

Kenny Workman

June 3, 2023

Problem 4.2.3(a).

Solution. We wish to show the (unique) representable $H_M : M^{op} \to Set$ is the same functor as $\underline{M}$. Recall $\underline{M}$ is defined for us as the right M-action of the monoid M on its underlying set defined as $x * m \mapsto xm$ for each $m \in M$.

We first show that H_M is a right M-action. Each $m \in M^{op}(M,M)$ induces a function $H_M(m) : M^{op}(M,M) \to M^{op}(M,M)$, which is a function mapping the morphisms in the single object category (elements of M) to itself. We must show this function satisfies (1) the identity property $H_M(1_M) = x \mapsto x$ and (2) compatibility $H_M(m'm)(x) = H_M(m) \circ H_M(m')(x)$. (1) is true by the definition of $H_M(1_M)$ as $H_M(1_M)(x) = x \mapsto x \circ 1_m = 1$. (2) is true by the composition rules of a contravariant functor.

We conclude by showing $H_M(m)(x) = x \mapsto x \circ m$. This is true by the definition of $H_M(m)$.

$\square$

Problem 4.2.3(b).

Solution. We will refer to $\underline{M}$ as H_M. Both H_M and X are functors in $[M^{op}, Set]$. For each $x \in X(M)$, we define a function $\alpha^x : H_M(M) \to X(M)$ as $m \mapsto X(m)(x)$. Indeed $\alpha^x(1) = X(1)(x) = x$.

We define a bijection $\hat{()} : [H_M, X] \to X$ defined as $(\hat{\alpha}) = \alpha(1)$. We constrain $\hat{()}$ to our set of α_x defined above and claim this function is a bijection. Given $\alpha^x \neq \alpha^{x'}$, $((\hat{\alpha^x}) = x) \neq ((\hat{\alpha^{x'}}) = x')$ so $\hat{()}$ is injective. For each $x \in X$, $(\hat{x})^{-1} = \alpha^x$ so $\hat{()}$ is also surjective.

$\square$

Problem 4.2.3(c).

Solution. We must show $[M^{op}, Set](H_M, X) \cong X(M)$ naturally in $X \in [M^{op}, Set]$ and $M \in M^{op}$. Because M^{op} has one object, we can show naturality in both X and M in one step. Define $X' \in [M^{op}, Set]$ and the natural transformation $f \in [M^{op}, Set](X, X')$. We reuse the definition of $\hat{()}$ from (b). Consider the commutative square:

$$\begin{tikzcd}
{[H_M, X]} \arrow[r, "f(-)"] \arrow[d, "\hat{()}"'] & {[H_M, X']} \arrow[d, "\hat{()}"] \\
X(M) \arrow[r, "f"'] & X'(M)
\end{tikzcd}$$

We claim $f\hat{()} = (\hat{f(-)})$.

Our left hand side is defined as $f\hat{()} = \alpha \mapsto \alpha(1) \mapsto f(\alpha(1))$. Our right hand side is defined as $\hat{()}f(-) = \alpha \mapsto f\alpha \mapsto (\hat{f\alpha}) \mapsto f(\alpha(1))$. (Recall $(\hat{\alpha}) = \alpha(1)$). Then $\hat{()}$ is natural and because we have already shown it is a bijective map in Set, it is also an isomorphism. Then $[M^{op}, Set](H_M, X) \cong X(M)$ naturally in $X \in [M^{op}, Set]$ and $M \in M^{op}$.

$\square$

4.3 and 5.1

Kenny Workman

June 16, 2023

Problem 4.3.15.

Solution. Let $J : \mathscr{A} \to \mathscr{B}$ be a full and faithful functor. Consider $A, A' \in \mathscr{A}$.

We first prove (a), that the map $f : A \to B$ in $\mathscr{A}$ is an isomorphism iff the map $J(f) : J(a) \to J(B)$ in $\mathscr{B}$ is an isomorphism. If f is an isomorphism, there exists g where $gf = 1_A$ and $fg = 1_B$. Because J is faithful, $J(f)$ and $J(g)$ are distinct. Furthermore, $J(f)J(g) = J(fg) = J(1_A) = 1_{J(A)}$ and $J(g)J(f) = J(gf) = J(1_B) = 1_{J(B)}$, using the definition of a functor. Similarly, if $J(f)$ is an isomorphism, there exists $J(g)$ where $J(g)J(f) = 1_{J(A)}$ and $J(f)J(g) = 1_{J(B)}$. Because J is full, f, g exist. Because $1_{J(A)} = J(1_A) = J(gf)$, $gf = 1_A$. Similarly $fg = 1_B$.

We then prove (b). Given an isomorphism $g : J(A) \to J(A')$, we showed in (a) that there exists an isomorphism f where $J(f) = g$. Uniqueness of f can be shown using the fact that J is faithful. Constructing some f', where $f' \neq f$, the $J(f') \neq J(f) \neq g$.

To prove (c), we can rephrase the claim as, "There exists an isomorphism $f : A \to A'$ in $\mathscr{A}$ iff there exists an isomorphism $g : J(A) \to J(A')$ in $\mathscr{B}$" (The existence of an isomorphism between two objects is the same as saying those objects are isomorphic). We have already shown the existence of this isomorphism in (a). All that remains is to show the isomorphism is defined on the correct objects in the image of J, but this follows from the definition of a functor ($f : A \to A'$ iff $g = J(f) : J(A) \to J(A')$.

□

Problem 4.3.18(a).

Solution. Because it is not given, we will define the induced functor on natural transformations as $J \circ -(\eta) = J((\eta_C)_{C \in \mathscr{C}})$.

We first show that if J is faithful, then $J \circ -$ is faithful. Fix two functors $G, G' \in [\mathscr{B}, \mathscr{C}]$ and consider $\eta, \eta' \in [\mathscr{B}, \mathscr{C}](G, G')$ where $\eta \neq \eta'$.

To see $J \circ -(\eta) \neq J \circ -(\eta')$ for $J \circ -(\eta), J \circ -(\eta') \in [\mathscr{B}, \mathscr{D}](JG, JG')$, it is sufficient to show that one of the component maps of $J \circ -(\eta)$ and $J \circ -(\eta')$ for some $C \in \mathscr{C}$, are distinct functions. Because $\eta \neq \eta'$, there exists some $B \in \mathscr{B}$ where $\eta_B \neq \eta'_B$. Because $J(\eta_B) \neq J(\eta'_B)$, as J is faithful, certainly $J \circ -(\eta) \neq J \circ -(\eta')$.

To see that if J is full, $J \circ -$ is full, we must show that for each $J(\eta) \in [\mathscr{B}, \mathscr{D}]$ there exists $\eta \in [\mathscr{B}, \mathscr{C}]$. Consider the following commutative diagram:

$$\begin{tikzcd} JG(B) \arrow[r, "Jg"] \arrow[d, "J(\eta_B)"] & JG(B') \arrow[d, "J(\eta_{B'})"] \\ JG'(B) \arrow[r, "Jg'"'] & JG'(B') \end{tikzcd}$$

Indeed $(\eta_B)_{B \in \mathscr{B}}$ exists in the domain of $J \circ -$. To see that it is natural, observe that $Jg'J(\eta_B) = J(\eta_{B'}Jg$ because $J(\eta)$ itself is natural. By functoriality, $J(g'\eta_B) = J(\eta_{B'}g)$. Then $g'\eta_B = \eta_{B'}g$ because J is full so η is natural and we have our result.

□

Problem 4.3.18(b).

Solution. This result follows directly from Lemma 4.3.8

□

Problem 4.3.18(c).

Solution. Because $H_\bullet$ is full and faithful, $H_\bullet \circ -$ is also full and faithful. Then $H_\bullet \circ -(G) \cong H_\bullet \circ -(G')$ implies $G \cong G'$, using the same argument as (b).

□

Problem 5.1.33. Verify that in the category of vector spaces, the product of two vector spaces is their direct sum.

Solution. Consider $X, Y \in \mathbf{Vect}_k$, we claim the direct sum $X \oplus Y$, along with projections $p_1 : X \oplus Y \to X$ and $p_2 : X \oplus Y \to Y$, is the product of X and Y. Denote the elements of $X \oplus Y$ as (x, y). Then our projections are defined as the linear maps $p_1 = (x, y) \mapsto x$ and $p_2 = (x, y) \mapsto y$.

For any $Z \in \mathbf{Vect}_k$ and pairs of maps $f_1 : Z \to X$ and $f_2 : Z \to Y$, there exists a unique function g where $p_1 g = f_1$ and $p_2 g = f_2$. Clearly $f_1 = p_1 g$ ($p_1 g = z \mapsto f_1(z)$). To see uniqueness, define $\hat{g}$ where $p_1 \hat{g} = f_1$ and $p_2 \hat{g} = f_2$, then $\hat{g} = z \mapsto (f_1(z), f_2(z))$, so this is the same function as g.

(As an aside, $X \oplus Y$ can only be the product of X and Y if $X \cap Y = \emptyset$ by the definition of a direct sum.)

□

Problem 5.1.37.

Solution. We must show that the set of all cones on D with vertex 1, defined as $\{(x_I)_{I \in \mathbf{I}} | x_I \in D(I) \text{ and } Du(x_I) = x_J \text{ for all such } u : I \to J\}$, is the same as $\varprojlim_I D$. The projections of this limit, $p_J : \varprojlim_I D \to D(J)$ for each $J \in \mathbf{I}$, are defined as $p_J((x_I)_{I \in \mathbf{I}}) = x_J$.

Fix an arbitrary cone on D, defined as $(A \xrightarrow{f_I} D(I))_{I \in \mathbf{I}}$. We must show there exists a corresponding $g : A \to \varprojlim_I D$ where $p_J g = f_J$ for each $J \in \mathbf{I}$. Define $g = a \mapsto (f_I(a))_{I \in \mathbf{I}}$. Then $p_J g = a \mapsto (f_I(a))_{I \in \mathbf{I}} \mapsto f_J(a)$ so it is the same function as f_J. In fact, for any cone, we can define a similar unique map, using only the family of maps in the cone, that satisfies the same property. This proves that the set of all cones on D with vertex 1 is the same as the limit of D in *Set*.

□

5 - Limits

Kenny Workman

July 7, 2023

Problem 5.1.34.

Solution. We can prove the converse directly. If the following commutative square is a pullback:

$$\begin{tikzcd} E \arrow[r, "i"] \arrow[d, "i"'] & X \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Then the following diagram commutes for any other object A in the category where $gj = fj$.

$$\begin{tikzcd} A \arrow[dr, dashed, "h"] \arrow[drr, bend left, "j"] \arrow[ddr, bend right, "j"'] & & \\ & E \arrow[r, "i"] \arrow[d, "i"'] & X \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

.

Note there exists a unique $h : A \to E$ where $ih = j$ and $gi = fi$. This diagram collapses to:

$$\begin{tikzcd} A \arrow[dr, dashed, "h"'] \arrow[drr, "j"] & & & \\ & E \arrow[r, "i"'] & X \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & Y \end{tikzcd}$$

Then A is a fork as $gj = fj$ from before. And indeed there exists a unique $h : A \to E$ where $ih = j$. This is true for any such fork in this category, which is a special case of the commutative square describe before, so E is an equalizer.

□

Problem 5.1.38(a).

Solution. We must show that $(L \xrightarrow{p_I} D(I))_{I \in \mathbf{I}}$ is a limit cone of $D : \mathbf{I} \to \mathscr{A}$.

First we show that L is a cone; that for any $u : J \to K$ (where $J, K \in \mathscr{A}$). $Dup_J = p_K$. We know that L is a fork of s and t (in particular it is the equalizer of s and t) so $sp = tp$. Then for any $u : J \to K$, $s_u p = t_u p$. Because $t_u = pr_K$, $t_u p = p_K$. Similarly, because $s_u = D_u pr_J$, $s_u p = D_u p_J$. Then $p_K = D_u p_J$ so L is a cone.

We finish by showing that L is also the limit of D. Notice that any other cone A on D is also a fork of s and t (The family of maps $(A \xrightarrow{f_I} D(I))_{I \in \mathbf{I}}$ can be represented as the single $f : A \to \prod_{I \in \mathbf{I}} D(I)$ with the property $sf = tf$ equivalent to $Duf_J = f_K$ for any $u : J \to K$). Because L is the equalizer of s and t, there exists a unique $g : A \to L$ such that $pg = f$ for any such cone A. Then certainly $p_I g = f_I$ so L is also a limit cone.

□

Problem 5.1.38(b).

Solution. We first that if $\mathscr{A}$ has binary products and a terminal object, $\mathscr{A}$ also has finite products.

Consider $X, Y, Z \in \mathscr{A}$. Because $\mathscr{A}$ has binary products, both the products $X \times Y$ and $(X \times Y) \times Z$ exist.

Define $X \times Y$ as the product with projections p_X and p_Y and P as the product $(X \times Y) \times Z$, with projections $p_{X \times Y}$ and p_Z. We will show that P also has projections $p_X p_{X \times Y}$, $p_Y p_{X \times Y}$, p_Z onto X, Y, and Z respectively that satisfy the necessary properties so that it is also the product $X \times Y \times Z$.

We pick an arbitrary object and set of maps:

$$\begin{tikzcd}
 & X \\
A \arrow[ur, "f_X"] \arrow[r, "f_Y"'] \arrow[dr, "f_Z"'] & Y \\
 & Z
\end{tikzcd}$$

And claim that for any such diagram, there exists a unique $g : A \to P$, where $f_X = p_X p_{X \times Y} g$, $f_Y = p_Y p_{X \times Y} g$ and $f_Z = p_Z g$.

To see $f_X = p_X p_{X \times Y} g$, we introduce a terminal object T and construct the following commutative diagram:

$$\begin{tikzcd}
 & & & X \arrow[r, "t_X"] & T \\
A \arrow[r, "g"] \arrow[urrr, bend left, "f_X"] & P \arrow[r, "p_{X \times Y}"] \arrow[dr, "p_Z"'] & X \times Y \arrow[ur, "p_X"] \arrow[r, "p_Y"] & Y & \\
 & & Z & &
\end{tikzcd}$$

Where g is the map satisfying the unversal property for P as the product $(X \times Y) \times Z$. Then $g p_{X \times Y} p_X t_X = f_X t_X$, because T is a terminal object, so $g p_{X \times Y} p_X = f_X$. An identical argument proves $g p_{X \times Y} p_Y = f_Y$. The result $g p_Z = f_Z$ can be seen considering $(X \times Y) \times Z$ as an ordinary binary product.

We can continue in this way to build products of any finite set of objects in $\mathscr{A}$.

With finite products and equalizers, the argument in **5.1.38(a)** remains the same when D has a finite number of maps $u \in \mathbf{I}$. Then D has finite limits.

(Its actually still unclear to me why we can't make products of infinite objects)

$\square$

Problem 5.2.21.

Solution. We first prove $s = t$ iff there exists an equalizer of s and t the given category and its an isomorphism.

If $s = t$, the equalizer of s and t is X along with the identity 1_X. This equalizer is certainly a fork as $s1_X = t1_X$. For any other fork A with $f : A \to X$ where $sf = tf$, f itself is the unique map such that $s1_X f = t1_X f$. This proves our forward argument.

If there exists an equalizer of s and t that is also an isomorphism, which we shall denote as the object E with map $i : E \to X$, then $si = ti$. If we precompose these maps with the inverse of i denoted j, then $sij = tij$ is the same as $s1_X = t1_X$ and $s = t$, which is our desired result.

We then prove there exists an equalizer of s and t and its an isomorphism iff there exists a coequalizer of s and t and its an isomorphism. To see the forward direction, denote our equalizer as E with map $i : E \to X$ and consider the (co?)fork Y with isomoprhism 1_Y. Certainly $1_Y si = 1_Y ti$ and precomposing with the inverse j of i we recover the desired $1_Y s = 1_Y t$. Y is also a coequalizer as any (co?)fork A with $f : Y \to A$ where $fs = ft$ induces $f1_Y s = f1_Y t$.

The proof the reverse direction follows an identical structure and is left as an exercise to the reader.

$\square$

Problem 5.2.22(a).

Solution. The coequalizer is the set of equivalence classes of X generated by the relation $R = \{(f(x), x) | x \in X\}$ denoted by X^*, along with the map $p : X \to X^*$ which sends each $x \in X$ to its respective equivalence class. Indeed $pf = p1$.

We can verify that this coequalizer is universal in this property. Consider any cofork A, with map $h : X \to A$ with $hf = h1$.

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "1"'] & X \arrow[r, "p"] \arrow[rr, bend right, "h"'] & X^* \arrow[r, "g"] & A \end{tikzcd}$$

We define our unique g as $g = x^* \mapsto h(x)$ where x is an arbitrary member of the equivalence class x^*. We can see that h and pg are then the same map. $pg = x \mapsto h(x)$ such that $h(x) = h(f(x))$ by our construction of g. This is an alternative way of stating $h = x \mapsto h(x)$ where $hf = h$.

□

Problem 5.2.22(b).

Solution. Similar to (a), the coequalizer in **Top** is the space whose underlying set is the equivalence classes of X generated by the relation $R = \{(f(x), x) | x \in X\}$ denoted by X^*, along with the map $p : X \to X^*$ which sends each $x \in X$ to its respective equivalence class. The space X^* inherits the topology induced by the quotient map $\{U \subset X^* | p^{-1}(U) \text{ open in } X\}$. Because p is strongly continuous, as $U \subset X^*$ open in X^* iff $p^{-1}(U)$ open in X, p is certainly continuous.

We can verify this coequalizer is universal in this property. Consider any cofork, with continuous map $h : X \to A$ with $hf = h1$.

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "1"'] & X \arrow[r, "p"] \arrow[rr, bend right, "h"'] & X^* \arrow[r, "g"] & A \end{tikzcd}$$

Similar to (a), we define our universal map g as $g = x^* \mapsto h(x)$, where x is any x in the equivalence class x^*. We've already shown that $h(x) = gp(x)$ for any $x \in X$ in (a). Then because h is continuous (open $U \subset A \implies h^{-1}(U)$ open in X as given) so is gp and so is g.

If $X = S^1$, $f = [] \mapsto [0, x]$

□

Problem 5.2.24(a).

Solution. We begin by proving the forward direction. Given isomorphic $e, e' \in \mathbf{Epic}(A)$ we must show they induce the same equivalence relation on A.

Recall a function $h : X \to Y$ induces an equivalence relation on X defined as $\{(x, y) | h(x) = h(y)\}$. Then for two functions $h, h' : X \to Y$, if $(h(x) = h(y) \iff h'(x) = h'(y)$ then e, e' induce the same equivalence relation on A.

Consider the following commutative diagram showing our two objects $e, e' \in \mathbf{Epic}(A)$ with isomorphism f:

$$\begin{tikzcd} A \arrow[d, "e"'] \arrow[dr, "e'"] & \\ X \arrow[r, "f"'] & X' \end{tikzcd}$$

Note that $fe = e'$ and $e = f^{-1}e'$, using the fact that these maps commute in our category and f is an isomorphism.

If $e(x) = e(y)$, then $f^{-1}e'(x) = f^{-1}e'(y)$ and $e'(x) = e'(y)$ (as an isomorphism in **Set**, f is a bijection). An identical argument can be used to show the converse. Then $e(x) = e(y) \iff e'(x) = e'(y)$ and we shown this is an alternative way of stating that e, e' induce the same equivalence relation on A.

To see the reverse argument, that given $e, e' \in \mathbf{Epic}(A)$ that induce the same equivalence relation on A these functions must be isomorphic, we will construct an bijection $f : X \to X'$ such that $fe = e'$ and $e = f^{-1}e'$.

We define $f = e(a) \mapsto e'(a)$ and claim this mapping is a bijection. To see f is injective, recall if $e, e' \in \mathbf{Epic}(A)$ induce the same equivalence relation on A, $e(a) = e(a') \iff e'(a) = e'(a')$. Then $e(a) \neq e(a') \implies e'(a) \neq e'(a')$. To see f is surjective, note that e' is surjective and if $x = e'(a) \in X'$ exists, certainly $e(a)$ exists. To see that f itself is well-defined, note that e is surjective. Since f is a bijection, e, e' are isomorphic.

This proves our result, that each equivalence relation on A corresponds to an isomorphic class of functions out of A.

□

Problem 5.2.24(b).

Solution. Fix some group $G \in \mathbf{Grp}$ and construct the full subcategory $\mathbf{Epic}(G)$ of $G \backslash Grp$ whose objects are epics.

Our "quotient objects" in this subcategory are the isomorphism classes of epics. We will show that each such isomorphism class corresponds to a unique normal subgroup of G.

Consider two such epics ψ, ψ' with isomorphism ϕ:

$$\begin{tikzcd} G \arrow[d, "\psi"'] \arrow[dr, "\psi'"] & \\ X \arrow[r, "\phi"] & X' \end{tikzcd}$$

We claim $ker(\psi) = ker(\psi')$. For any $x \in G$ where $\phi'(x) = 1$, $\psi\phi(x) = 1$ by commutativity and $\psi^{-1}\psi\phi = \phi(x) = 1$ because ψ is an isomorphism. The same argument holds for the converse, so ψ and ψ' share the same kernel.

Then because $ker(\psi) \trianglelefteq G$ for any homormorphism $\psi : G \to X$, the isomorphism class of such epics corresponds to a unique normal subgroup of G.

We now show the reverse, that each normal subgroup of G corresponds to a unique quotient object. Define $N \trianglelefteq G$, and construct an arbitrary surjective homomorphism $\psi : G \to X$ where $ker(\psi) = N$. We claim that any additional homormophism ψ' that is surjective and shares this kernel is isomorphic to ψ. The isomorphism theorems tell us that $\psi(G) \cong G \backslash N$ and $\psi'(G) \cong G \backslash N$. Then $\psi(G) \cong \psi'(G)$ and this proves our result.

□

Problem 5.2.25(a).

Solution. Let $m : A \to B$ be a split monic. We will show this is also a regular monic. Define any $e : B \to A$ such that $em = fm$. We claim that m is the equalizer of e and f. To see this, consider any other fork C, such that $eh = fh$, in the following diagram:

$$\begin{tikzcd} C \arrow[dr, "g"'] \arrow[drr, "h"] & & & \\ & A \arrow[r, "m"] & B \arrow[r, shift left, "e"] \arrow[r, shift right, "f"'] & A \end{tikzcd}$$

By construction, $eh = emg$. Then $eh = 1_A g$, so our universal map is defined as $g = eh$. Indeed to see, $emg = fmg$, observe $emeh = fmeh$ and by substitution $1_A eh = 1_A eh$, giving us our result.

We now show that m is also a monic. For any $x, x' : X \to A$, where $mx = mx'$, we show $x = x'$. Notice that since m is an equalizer for some maps s, t, the object X and map mx must be a fork of these maps ($tmx = smx$). If we construct the diagram of this fork along with its projection g on our equalizer:

$$\begin{tikzcd} X \arrow[dr, "g"'] \arrow[drr, "mx"] & & & \\ & A \arrow[r, "m"] & B \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & A \end{tikzcd}$$

We notice that our universal g for mx is exactly x and because it is unique, $mx = mx' \implies x = x'$.

$\square$

Problem 5.3.8.

Solution. We define $F : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ as $F = (X, Y) \mapsto X \times Y$ where $X \times Y$ is the binary product of (X, Y). Because $\mathscr{A}$ has binary products, this assignment is straightforward.

To define assignment of morphisms, consider an additional $(X', Y') \in \mathscr{A} \times \mathscr{A}$ with morphism $(f, g) : (X, Y) \to (X', Y')$ induced by $f : X \to X', g : Y \to Y'$. When we treat $X \times Y$ as any other object satisfying the properties of product $X' \times Y'$, the following diagram emerges:

$$\begin{tikzcd} & X \times Y \arrow[dl, "p_x"'] \arrow[d, "\bar{g}"] \arrow[dr, "p_y"] & \\ X \arrow[d, "f"'] & X' \times Y' \arrow[dl] \arrow[dr] & Y \arrow[d, "g"] \\ X' & & Y' \end{tikzcd}$$

$F((f, g)) = \bar{g}$, where $\bar{g}$ is universal map associated with $fp_x : X \times Y \to X'$ and $gp_y : X \times Y \to Y'$.

We can verify F satisfies the necessary axioms. For any $(A, A) \in \mathscr{A} \times \mathscr{A}$. $F(1_{(A,A)}) = 1_{A \times A}$. We can examine the commutative diagram to verify that the universal map assigned under our definition is the same as the desired identity morphism:

$$\begin{tikzcd} & X \times Y \arrow[d, "1_{X \times Y}"] \arrow[ddl, "p_X"'] \arrow[ddr, "p_Y"] & \\ & X \times Y \arrow[dl, "p_X"] \arrow[dr, "p_Y"'] & \\ X & & Y \end{tikzcd}$$

To see composition, we define an additional $(X'', Y'') \in \mathscr{A} \times \mathscr{A}$ with morphism $(f', g') : (X', Y') \to (X'', Y'')$ induced by $f : X' \to X'', g : Y' \to Y''$. We wish to show that $F((f', g') \circ (f, g)) = F((f', g')) \circ F((f, g))$

$\square$

$$\begin{tikzcd} & X \times Y \arrow[dl, "p_X"] \arrow[d, "\bar{g}"] \arrow[dr, "p_Y"'] \arrow[dddl, bend right=60, "f'fp_X"'] \arrow[dddr, bend left=60, "g'gp_Y"] & \\ X \arrow[d, "f"'] & X' \times Y' \arrow[dl, "p_{X'}"] \arrow[d, "\bar{\bar{g}}"] \arrow[dr, "p_{Y'}"'] & Y \arrow[d, "g"] \\ X' \arrow[d, "f'"'] & X'' \times Y'' \arrow[dl, "p_{X''}"] \arrow[dr, "p_{Y''}"'] & Y' \arrow[d, "g'"] \\ X'' & & Y'' \end{tikzcd}$$

$F((f', g')) \circ F((f, g)) = \bar{\bar{g}}\bar{g}$. $F((f', g') \circ (f, g))$ is equivalent to the universal map induced by $f'fp_X$ and $g'gp_Y$. But it is clear from our diagram that is the same as $\bar{\bar{g}}\bar{g}$, giving us our result.

Problem 5.3.11(a).

Solution. We will show that the forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$ creates arbitrary limits.

To see this we define a diagram $D : \mathbf{I} \to \mathbf{Grp}$ where $\mathbf{I}$ is any small category. For any limit cone $(B \xrightarrow{q_I} UDI)_{I \in \mathbf{I}}$ on FD, we must show there exists a corresponding limit cone $(A \xrightarrow{p_I} DI)_{I \in \mathbf{I}}$ on D where $U(A) = B$ and $U(p_I) = q_I$ for each $I \in \mathbf{I}$.

We know that B is the set $\{(x_I)_{I \in \mathbf{I}} \in \prod_{I \in \mathbf{I}} UDI | UDu(x_J) = x_K$ for each $UDu : UDJ \to UDK\}$ and each $(B \xrightarrow{q_I} UDI)$ is the projection map.

Then there is a unique group structure we can impose on A that satisfies the desired properties. We know $p_I : \prod_{I \in \mathbf{I}} DI \to DI$ is defined as $(x_I)_I \mapsto x_I$. Then for each $a, a' \in A$, $p_I(a \circ a') = p_I(a) \circ p_I(a') = x_I \circ x'_I$ for each $I \in \mathbf{I}$. So $a \circ a' = (x_I \circ x'_I)_{I \in \mathbf{I}}$. A similar argument recovers inverses for each element and the identity for our group and we have our result.

$\square$

Problem 5.3.11(b).

Solution. We will show that the forgetful functor $U : \mathbf{Ab} \to \mathbf{Set}$ creates arbitrary limits.

We can reuse much of the previous argument but must verify that the group structure we define is abelian. As before, $a, a' \in A$, $p_I(a \circ a') = p_I(a) \circ p_I(a') = x_I \circ x'_I$ for each $I \in \mathbf{I}$. So $a \circ a' = (x_I \circ x'_I)_{I \in \mathbf{I}}$. But each DI is abelian, so $(x'_I \circ x_I)_{I \in \mathbf{I}}$ for each $I \in \mathbf{I}$ so certainly $(x'_I \circ x_I)_{I \in \mathbf{I}}$. Then $a \circ a' = a' \circ a$.

$\square$

6 - Adjoints, representables and limits

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Problem 6.1.5. Interpret the theory of this section in the special case when $\mathbf{I}$ is the discrete category with two objects.

Solution. Let $\mathbf{I}$ contain objects I, J with no maps but the identities. First we examine the new definition of a cone as a natural transformation $\eta : \Delta A \to D$ and observe the component maps $(\eta_I)_I$ are simply a pair of maps $A \xrightarrow{\eta_I} DI$ and $A \xrightarrow{\eta_J} DJ$ with no additional commutative properties. So the elements of $Cone(A, D) = [\mathbf{I}, \mathscr{A}](\Delta A, D)$ are all pairs of maps $(A \xrightarrow{\eta_I} DI, A \xrightarrow{\eta_J} DJ)$.

6.1.1 tells us that each limit cone on D corresponds to a unique bijection between cones on D and maps into the limit cone. We know from Yoneda's Lemma, that $[\mathscr{A}^{op}, \mathbf{Set}](H_{\lim_{\leftarrow I} D}, Cone(-, D)) \cong Cone(A, D)$ and that each natural transformation is actually just a cone. **6.1.1** tells us that the natural transformations that are also natural isomorphisms are also cones, but they are cones that are also universal elements. The structure of the universal element is another description of a limit cone.

Consider the isomorphism $\alpha \in [\mathscr{A}^{op}, \mathbf{Set}](H_{\lim_{\leftarrow I} D}, Cone(-, D))$. It places $f \in \mathscr{A}(A, \lim_{\leftarrow I} D)$ in one-to-one correspondence with $x \in Cone(A, D)$. In particular, we have a universal $u \in Cone(\lim_{\leftarrow I} D, D)$ where given any cone $x \in Cone(A, D)$, we know there is some unique $f \in \mathscr{A}(A, \lim_{\leftarrow I} D)$ where $Cone(f, D)(u) = x$.

Let us examine this in the case where $\mathbf{I}$ is binary. Our universal element is some $u = (\lim_{\leftarrow I} D \xrightarrow{\eta_I} DI, \lim_{\leftarrow I} D \xrightarrow{\eta_J} DJ)$. Given any $x \in Cone(A, D) = (A \xrightarrow{\eta'_I} DI, A \xrightarrow{\eta'_J} DJ)$ there is a unique $f : A \to \lim_{\leftarrow I} D$ where $(A \xrightarrow{f\eta_I})_I = x$.

6.1.3 tells us that natural transformations between diagrams $\alpha \in [\mathbf{I}, \mathscr{A}](D, D')$ induce unique maps between the limit cones. In the case of our binary category, such α are simply a pair of maps with no other properties:

$$\begin{tikzcd} DI \arrow[d, "\alpha_I"] & DJ \arrow[d, "\alpha_J"] \\ D'I & D'J \end{tikzcd}$$

Our induced $\lim_{\leftarrow I}$ is simply the map that satisfies the commutative squares given in the exercises.

We finally examine the formulation of limits as adjoint functors described in **6.1.4**.

Consider the adjoints (with definitions given in the text):

$$\begin{tikzcd} {[I, \mathscr{A}]} \arrow[d, shift left, "\lim_{\leftarrow I}"] \\ \mathscr{A} \arrow[u, shift left, "\Delta"] \end{tikzcd}$$

Where $[I, \mathscr{A}](\Delta A, D) \cong \mathscr{A}(A, \lim_{\leftarrow I} D)$ naturally in A, D. The isomorphism being natural in A is obvious. What is more interesting is that it is natural in D. Consider the following commutative diagram, with $lim\alpha$ denoting our induced map between cones.

$$\begin{tikzcd}
Cone(A,D) \arrow[r, "Cone(lim\alpha)"] \arrow[d, "\alpha_D"] & Cone(A,D') \arrow[d, "\alpha_{D'}"] \\
\mathscr{A}(A, limD) \arrow[r, "(-)\circ lim\alpha"] & \mathscr{A}(A, limD')
\end{tikzcd}$$

A pushback between vertices of limit cones is essentially the same as pushback between the actual limit cones.

□

Problem 6.2.20.

Solution. We will show that α is monic iff α_A is monic for each $A \in \mathbf{A}$.

The forward direction is trivial. If α is a monic in $[\mathbf{A}, \mathscr{P}]$, each of its component maps are certainly monic.

The reverse direction follows from **6.2.5**. The fact that α_A is monic in $\mathscr{P}$ means that the following pullback exists in $\mathscr{P}$ for each A in $\mathbf{A}$:

$$\begin{tikzcd}
X(A) \arrow[r, "1"] \arrow[d, "1"'] & X(A) \arrow[d, "\alpha_A"] \\
X(A) \arrow[r, "\alpha_A"'] & Y(A)
\end{tikzcd}$$

Each such pullback can be described as a limit on the diagram $ev_A \circ D$. Where the functor $D : \mathbf{I} \to [\mathbf{A}, \mathscr{P}]$ assigns $DI = X$ and $DJ = Y$ and it domain is the small category $\mathbf{I}$:

$$\begin{tikzcd}
 & I \arrow[d, "u"] \\
I \arrow[r, "u"'] & J
\end{tikzcd}$$

Our limit is the family $(L \xrightarrow{1} DI(A)_{I,J}$ where $L = XA = DIA$ for each $A \in \mathbf{A}$.

6.2.5 tells us that there is a limit cone on D itself and its image under ev_A is exactly $L = XA$ for each such $A \in \mathbf{A}$. Then this limit cone is exactly X which is another way of saying $[\mathbf{A}, \mathscr{P}]$ has the pullback:

$$\begin{tikzcd}
X \arrow[r, "1"] \arrow[d, "1"'] & X \arrow[d, "\alpha"] \\
X \arrow[r, "\alpha"'] & Y
\end{tikzcd}$$

So α is monic.

□

Problem 6.2.22. Show how a category of elements can be described as a comma category.

Solution. Define an arbitrary presheaf $X : \mathscr{A}^{op} \to \mathbf{Set}$ and consider the comma category denoted $(X \Rightarrow 1)$ as in the diagram:

$$\begin{tikzcd}
 & \{1\} \arrow[d, "1"] \\
\mathscr{A}^{op} \arrow[r, "X"'] & Set
\end{tikzcd}$$

Where the functor 1 is defined on the singleton set as $1 : \{1\} \to Z$ where $Z = \coprod_{A \in \mathscr{A}^{op}} XA$.

Then the elements of $(X \Rightarrow 1)$ are $(A, g_A, x$ where $g_A : XA \to Z$ maps $x \mapsto x$.

□

Problem 6.3.21(a).

Solution. It is enough to show that $U : \mathbf{Grp} \to \mathbf{Set}$ does not preserve colimits to see it has no right adjoint.

Consider the "empty sum" in any category X, the colimit of the diagram $D : \mathbf{I} \to X$ where I has no objects or maps, and observe this is the same as the initial object in this category. In **Grp** the colimit of this diagram is then the trivial group and in **Set** it is the empty set. Then U does not preserve colimits.

□

Problem 6.3.21(b).

Solution. We begin with the right side of the adjoint chain. Recall $I : \mathbf{Set} \to \mathbf{Cat}$ right adjoint to C turns any set X into a category IX with all possible morphisms between objects. Any function $f : X \to Y$ becomes a functor $If = x \mapsto fx$ for each object $x \in IX$ that carries all maps in $IX(x, y)$ to $IY(Ifx, Ify)$. It is enough to show that this functor does not preserve colimits.

As we continue to the left side of the adjoint chain $C \dashv D$, recall $D : \mathbf{Set} \to \mathbf{C}$ turns sets into "discrete" categories (with only the identity maps) while $C : \mathbf{Cat} \to \mathbf{Set}$ maps categories to sets where each function Cf inherits

□

Problem 6.3.24(a).

Solution. If A is finite, the subgroup in question is finitely generated and is itself finite. Its cardinal number is then a natural number.

If A is infinite, the subgroup will have

□

Problem 6.3.24(b).

Solution. The number of different group structures (isomorphic classes) ¡ S is small.

□

Problem 6.3.24(c).

Solution. For each $A \in Set$, we show that the category $(A \Rightarrow U)$ has a weakly initial set. For any object $A \xrightarrow{f} U(X)$, we claim there is some group $B \cong < (x_a)_{a \in A} >$, where $(x_a)_{a \in A}$ is the subgroup generated by the elements of X defined in the function f. Then there exists object $A \xrightarrow{g} U(B)$, where

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, "f"'] & UB \arrow[d, "U\psi"] \\ & UX \end{tikzcd}$$

commutes, if ψ is any isomorphism between these groups.

For any A, we know that any group $< (x_a)_{a \in A} >$ has cardinality $\max \mathcal{N}, A$ and the collection of isomorphism classes of groups at most $\max \mathcal{N}, A$ is small. Our argument shows that we need at most one member of each isomorphism class of groups at $\max \mathcal{N}, A$ to and that this set is our weakly initial set of $A \Rightarrow U$ for each $A \in Set$

□

Problem 6.3.26(a).

Solution. For any $f : A' \to A$, consider any monic $m \in Sub(A)$. Observe the limit of the following diagram:

$$\begin{tikzcd} & X \arrow[d, "m"] \\ A' \arrow[r, "f"'] & A \end{tikzcd}$$

is the pullback:

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "m'"'] & X \arrow[d, "m"] \\ A' \arrow[r, "f"'] & A \end{tikzcd}$$

which defines a unique m'. We know that m' is also monic from **5.1.42** and can be considered an element of $Sub(A')$. Then any $f : A' \to A$ induces a map $Sub(f) : Sub(A) \to Sub(A')$.

□

Problem 6.3.26(b).

Solution. We must show that the map $Sub(f)$ we constructed is functorial

Define $g : A'' \to A'$ and pick some $m \in Sub(A), m' \in Sub(A'), m'' \in Sub(A'')$. Then $Sub(g)Sub(f)$ is the diagram:

$$\begin{tikzcd} X'' \arrow[r, "g'"] \arrow[d, "m''"'] & X' \arrow[r, "f'"] \arrow[d, "m'"'] & X \arrow[d, "m"] \\ A'' \arrow[r, "g"'] & A' \arrow[r, "f"'] & A \end{tikzcd}$$

Using the result **5.1.35**, because the left and right squares are both pullbacks, the outer rectangle is a pullback. So the correspondence between m and m'' is again unique.

It is easy to see that for the same $m \in Sub(A), m'' \in Sub(A'')$, the pullback $Sub(fg)$ is the same as the outer rectangle above:

$$\begin{tikzcd} X'' \arrow[r, "f'g'"] \arrow[d, "m''"] & X \arrow[d, "m"] \\ A'' \arrow[r, "fg"] & A \end{tikzcd}$$

This gives our result $Sub(fg) = Sub(g)Sub(f)$.

□

Problem 6.3.26(c).

Solution. For any $A \in$ **Set**, an isomorphism class of monics into A, eg. element of $Sub(A)$, is a subset of A.

To see this, recall monics in **Set** are injective functions and two monics into A are in the same isomorphism class iff the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "m"'] & X' \arrow[d, "m'"] \\ & A \end{tikzcd}$$

Where h is a bijection (an isomorphism in **Set**). This diagram only commutes if m and m' share the same range $m'h(X) = mh^{-1}(X')$. Then the isomorphism class is this range, a subset of A.

That fact that $H_2 \cong Sub$ naturally in $A \in \mathscr{A}$ is then a formal description that functions into 2 and isomorphism classes of monics into A are both ways of describing subsets of A

The isomorphism $\alpha_A : H_2(A) \to Sub(A)$, defined as $\alpha_A = (f : A \to 2) \mapsto *x$, where $*x \in Sub(A)$ and each $X \xrightarrow{m} A \in *x$ has the property $m(X) = f^{-1}(1)$.

□