4.1 Representables Definitions and Examples

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Problem 4.1.27.

Solution. For each $A \in \mathscr{A}$, let $\alpha_A : H_A \to H_{A'}$ be the component map of the natural isomorphism $H_A \cong H_{A'}$.
Let $f = \alpha_A(1_A)$ and let $g = \alpha_{A'}^{-1}(1_{A'})$.
Consider the following commutative square:

$$\begin{tikzcd} H_A(A) \arrow[r, "H_A(g)"] \arrow[d, "\alpha_A"] & H_A(A') \arrow[d, "\alpha_{A'}"] \\ H_{A'}(A) \arrow[r, "H_{A'}(g)"] & H_{A'}(A') \end{tikzcd}$$

By naturality, $H_{A'}(g)\alpha_A = \alpha_{A'}H_A(g)$ and $H_{A'}(g)\alpha_A(1_A) = \alpha_{A'}H_A(g)(1_A)$. Then $fg = \alpha_{A'}(g)$. Because $g = \alpha_{A'}^{-1}(1'_A)$, $fg = 1_{A'}$.
Now consider the similar commutative square:

$$\begin{tikzcd} H_A(A) & H_A(A') \arrow[l, "H_A(f)"'] \\ H_{A'}(A) \arrow[u, "\alpha_A^{-1}"] & H_{A'}(A') \arrow[l, "H_{A'}(f)"'] \arrow[u, "\alpha_{A'}^{-1}"] \end{tikzcd}$$

By naturality, $H_A(f)\alpha_{A'}^{-1} = \alpha_A^{-1}H_{A'}(f)$ and $H_A(f)\alpha_{A'}^{-1}(1_{A'}) = \alpha_A^{-1}H_{A'}(f)(1_{A'})$. Then $gf = \alpha_A^{-1}(f)$. Similarly, $gf = 1_A$.

□

Problem 4.1.28.

Solution. Denote $H^{\mathbb{Z}/p\mathbb{Z}}(-)$ as shorthand for $\text{Grp}(\mathbb{Z}/p\mathbb{Z}, -)$. Let G, G' are elements of Grp and $f : G \to G'$ is a homomorphism (a map in Grp). Define $U_p(f)$ as the restriction of f to $U_p(G)$. Note this restriction is also a homomorphism.

We begin by showing the set $H^{\mathbb{Z}/p\mathbb{Z}}(G)$ is in bijective correspondence with $U_p(G)$ for any choice of $G \in \text{Set}$. Define $h : U_p(G) \to H^{\mathbb{Z}/p\mathbb{Z}}(G)$ as $h = g \mapsto \theta : \theta(1) = g$. (Then $\theta(2) = g^2, \theta(3) = g^3$ ... $\theta(p) = 1$ and so forth because θ is a homomorphism).

Because $g \neq g' \implies h(g)(1) \neq h(g')(1)$, h is injective. Because $\theta \neq \theta' \implies \theta(x) \neq \theta'(x)$ for some $x \implies \theta(1)^x \neq \theta'(1)^x \implies \theta(1) \neq \theta'(1) \implies h^{-1}(\theta(1)) \neq h^{-1}(\theta'(1))$, h is also surjective. Then h is a bijection.

We will use this fact to describe elements of $\theta \in H^{\mathbb{Z}/p\mathbb{Z}}(G)$ only by their assignment $\theta(1)$ for the remainder of the proof.

Consider the following commutative diagram, where we introduce the component maps $\alpha_G : H^{\mathbb{Z}/p\mathbb{Z}}(G) \to U_p(G)$ of our natural isomorphism. We define $\alpha_G = \theta \mapsto \theta(1)$.

$$\begin{tikzcd} H^{\mathbb{Z}/p\mathbb{Z}}(G) \arrow[r, "H^{\mathbb{Z}/p\mathbb{Z}}(f)"] \arrow[d, "\alpha_G"] & H^{\mathbb{Z}/p\mathbb{Z}}(G') \arrow[d, "\alpha_{G'}"] \\ U_p(G) \arrow[r, "U_p(f)"] & U_p(G') \end{tikzcd}$$

We first show naturality by proving $U_p(f)\alpha_G = \alpha_{G'} H^{\mathbb{Z}/p\mathbb{Z}}(f)$. Consider $\theta \in H^{\mathbb{Z}/p\mathbb{Z}}(G)$, then $U_p(f)\alpha_G = (1 \mapsto g) \mapsto g \mapsto g'$ and $\alpha_{G'} H^{\mathbb{Z}/p\mathbb{Z}}(f) = (1 \mapsto g) \mapsto (1 \mapsto g') \mapsto g'$. Then $U_p(f)\alpha_G(\theta(1)) = \alpha_{G'} H^{\mathbb{Z}/p\mathbb{Z}}(f)(\theta(1)) = g'$ for any θ so these maps are the same.

We now show each α_G is an isomorphism. We have already shown there exists a bijective function $h : H^{\mathbb{Z}/p\mathbb{Z}}(G) \to U_p(G)$ for any choice of G. If we define h such that $h\alpha_G = (1 \mapsto g) \mapsto g \mapsto 1 \mapsto g$ it is clear that h is the inverse to each α_G so each α_G is also an isomorphism.

□

Problem 4.1.30.

Solution. For $f \in \mathbf{Top}^{\mathrm{op}}(X, X')$ we define $\mathscr{O}(f) = f^{-1}$. We define a functor $H_S = \mathbf{Top}^{\mathrm{op}}(-, S) : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Set}$ using definition **4.1.16**. Denote $S = \{1, 2\}$ where $\{1\} \subset S$ is the open singleton and $\{2\} \subset S$ is the closed singleton.

Given $X \in \mathbf{Top}^{\mathrm{op}}$, we show a canonical bijection between open sets in X and continuous maps $q : X \to S$ by defining a bijection $\alpha_X : \mathscr{O}(X) \to H_S(X)$ as $\alpha_X = U \mapsto q$ for $U \in \mathscr{O}(X)$, with $q(U) = \{1\}$ and $q(X - U) = \{2\}$. To see that α_X is injective, $U \neq U' \implies \alpha_X(U) \neq \alpha_X(U')$ as $U \subsetneq U' \implies \alpha_X(U)(U' - U) \neq \alpha_X(U')(U' - U)$ while $U' \subsetneq U \implies \alpha_X(U)(U - U') \neq \alpha_X(U')(U - U')$. To see that α_X is surjective, any $q \in H_S(X)$, $\alpha_X^{-1}(q) = q^{-1}(S)$.

We then show $\mathscr{O} \cong H_S$. Consider the following commutative square:

$$\begin{tikzcd}
\mathscr{O}(X) \arrow[r, "\mathscr{O}(f)"] \arrow[d, "\alpha_X"] & \mathscr{O}(X') \arrow[d, "\alpha_{X'}"] \\
H_S(X) \arrow[r, "H_S(f)"] & H_S(X')
\end{tikzcd}$$

We claim $H_S(f)\alpha_X = \alpha_{X'}\mathscr{O}(f)$ for any choice of $f \in \mathbf{Top}^{\mathrm{op}}(X, X')$.

For $U \in \mathscr{O}(X)$, $H_S(f)\alpha_X = U \mapsto q \mapsto qf^{-1}$ and $\alpha_{X'}\mathscr{O}(f) = U \mapsto f^{-1}(U) \mapsto qf^{-1}$, where q is defined as before.

□