3.2 Small and Large Categories

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Problem 3.2.12 (a).

Solution. We will show $S \subseteq \theta(S)$ and $\theta(S) \subseteq S$ and conclude that $S = \theta(S)$.

First, notice:

$$S = \bigcup_{R \in \mathcal{P}(A) : R \subseteq \theta(R)} R \subseteq \bigcup_{R \in \mathcal{P}(A) : R \subseteq \theta(R)} \theta(R) = \theta(S)$$

So $S \subseteq \theta(S)$.

To show $\theta(S) \subseteq S$, we show that for any $R \in P(A) : R \subseteq \theta(R)$ that $\theta(R) \subseteq S$. Because θ is order preserving w.r.t. inclusion, $R \subseteq \theta(R) \implies \theta(R) \subseteq \theta(\theta(R))$. So indeed $\theta(R) \subset S$ for every R and $\theta(S) \subseteq S$.

Then $\theta(S) \subseteq S \subseteq \theta(S)$ so $\theta(S) = S$.

□

Problem 3.2.12 (b).

Solution. We claim that $\theta = S \mapsto A \setminus g(B \setminus fS)$ is an order preserving map (with respect to inclusion).

Indeed, $S \subseteq S' \implies fS \subseteq fS' \implies g(B \setminus fS') \subseteq g(B \setminus fS) \implies A \setminus g(B \setminus fS) \subseteq A \setminus g(B \setminus fS')$

Then there exists some $S \subseteq A : S = \theta(S)$ so $A \setminus S = g(B \setminus FS)$ as desired.

□

Problem 3.2.12 (c).

Solution. Still need to finish. □

Problem 3.2.13.

Solution. We will show a contradiction from the assignment $f(a') = \{a | a \notin f(a)\}$ for any a'.

Notice that for any $f(a') = \{a | a \notin f(a)\}$, that $a' \notin \{a | a \notin f(a)\}$ must be true. However, if this is true, then $a' \in \{a | a \notin f(a)\}$ is also true.

Then this assignment is not possible and f cannot be surjective.

□