Wasserman: All of Statistics

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1 Probability

1.1 Basics

Example 1.1. It's actually tricky to show $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ using only the three axioms:

$$\begin{aligned}
\mathbb{P}(A \cup B) &= \mathbb{P}(AB^c \cap AB \cap A^cB) \\
&= \mathbb{P}(AB^c) + \mathbb{P}(AB) + \mathbb{P}(A^cB) \\
&= \mathbb{P}(AB^c) + \mathbb{P}(AB) + \mathbb{P}(A^cB) + P(AB) - P(AB) \\
&= \mathbb{P}(AB^c \cup AB) + \mathbb{P}(A^cB \cup AB) - P(AB) \\
&= \mathbb{P}(A) + \mathbb{P}(B) - P(AB)
\end{aligned}$$

Another simple idea is that events that are identical at the limit should have identical probabilities.

Theorem 1.2 (Continuity of Events)**.** *If $A_n \to A$ then $\mathbb{P}(A_n) \to \mathbb{P}(A)$.*

Proof. Let A_n be monotone increasing: $A_1 \subset A_2 \subset \dots$. Let $A = \lim_{n\to\infty} A_n = \bigcup_{i=1}^{\infty} A_i$.

Construct disjoint sets B_i from each A_i where $B_1 = A_1$ and $B_n = \{\omega \in \Omega : \omega \in A_n, \omega \notin \bigcup_{i=1}^{i-1} A_i\}$. It will be shown that (1) each pair of B_i are disjoint, (2) $\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i$ and (3) $A = \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$ (Exercise 1.1).

From Axiom 3: $\mathbb{P}(A_n) = \mathbb{P}(\bigcup\limits_{i=1}^{n} A_i) = \mathbb{P}(\bigcup\limits_{i=1}^{n} B_i) = \sum\limits_{i=1}^{n} \mathbb{P}(B_i)$.

Then $\lim_{n\to\infty} \mathbb{P}(A_n) = \lim_{n\to\infty} \sum\limits^{n} \mathbb{P}(B_n) = \sum\limits^{\infty} \mathbb{P}(B_n) = \mathbb{P}(\bigcup\limits^{\infty} B_n) = \mathbb{P}(A)$ □

Exercise 1.1. Fill in the details for Theorem 1.2 and extend to the case where A_n is monotone decreasing.

Proof. For any pair B_{n+1} and B_n, because $B_n \subset A_n$ and $B_{n+1} \cap A_n = \emptyset$, it follows that $B_{n+1} \cap B_n = \emptyset$.
Let $\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i$. Then $\bigcup_{i=1}^{n+1} B_i = (A_{n+1} \setminus \bigcup_{i=1}^{n} A_i) \bigcup (\bigcup_{i=1}^{n} A_i) = \bigcup_{i=1}^{n+1} A_i$.
For the monotone decreasing case, let A_n be a sequence where $A_1 \supset A_2 \supset A_3 \dots$.
Observe $A_1^c \subset A_2^c \dots$ and $\lim_{n\to\infty} A_n = \Omega \setminus \bigcup^{\infty} A_i^c$. Construct disjoint B_n^c from A^c in the same way.
Then $\lim_{n\to\infty} \mathbb{P}(A_n) = 1 - \sum\limits^{\infty} \mathbb{P}(B_i^c) = 1 - \mathbb{P}(A^c) = \mathbb{P}(A)$ □

Exercise 1.3. Let Ω be a sample space and $A_1, A_2, \dots$ be events. Define $B_n = \cup_{i=n}^{\infty} A_i$ and $C_n = \cap_{i=n}^{\infty} A_i$.

(a) Show $B_1 \supset B_2 \supset B_3 \dots$ and $C_1 \subset C_2 \subset C_3 \dots$

(b) Show $\omega \in \cap_{n=1}^{\infty} B_n$ iff ω is in an infinite number of the events

(c) Show $\omega \in \cup n = 1^{\infty} C_n$ iff ω belongs to all of the events, except possibly a finite number of those events.

Proof. (a) Certainly $\cup_{i=1}^{\infty} A_i \supset \cup_{i=2}^{\infty} A_i \dots$ and $\cap_{i=1}^{\infty} A_i \subset \cap_{i=2}^{\infty} \dots$.

(b) Forward. Assume $\omega \in \cap_{n=1}^{\infty} B_n$. If ω does not belong to an infinite number of events A_i, there exists some index j past which $\omega \notin B_j$. Then certainly $\omega \notin \cap_{n=1}^{\infty} B_n$. Reverse. ω belonging to infinite events means there cannot exist such a j described previously so $\omega \in B_n$ for all n. Indeed $\omega \in \cap_{n=1}^{\infty} B_n$

(c) Forward. Assume $\omega \in \cup_{n=1}^{\infty} C_n$. Then $\omega \in C_j = \cap_{i=j}^{\infty} A_i$ for some j. This is another way of saying ω is in every single event except for perhaps a finite number in $A_{i<j}$. Reverse. Let j be the index of the largest event that ω is not in. Then $\omega \in C_{n>j}$ and certainly $\omega \in \cup^{\infty} C_n$.

□

Note. The key idea above is this notion of "infinitely often" (i.o.) and "all but finitely often" (eventually) which are two distinct structures of infinite occurence in sequences. Consider an ω that exists in every other event (eg. just the odd indices) for infinite events and revisit its inclusion in $\cap^{\infty} B_i$ and $\cup^{\infty} C_i$.

Note. $\lim \cap \cup A_n$ is also referred to as the limit infimum of A_n. Similarly, $\lim \cup \cap A_n$ is referred to as the limit supremum of A_n.

Exercise 1.7. Let $\mathbb{P}(\bigcup^{n} A_i) \leq \sum^{n} \mathbb{P}(A_i)$. Then $\mathbb{P}(A_{n+1} \cup (\bigcup^{n} A_i)) \leq \mathbb{P}(A_{n+1}) + (\sum^{n} \mathbb{P}(A_i)) - \mathbb{P}(A_{n+1} \cap (\bigcup^{n} A_i)) \leq \sum^{n+1} \mathbb{P}(A_i)$

Note. Expand a bit on the Boole inequality.