Convolutional Neural Networks

Our introduction was of a very limited Convolutional Layer

- Recognizing a single feature
- One dimensional

We will relax each restriction in turn.

Multiple features

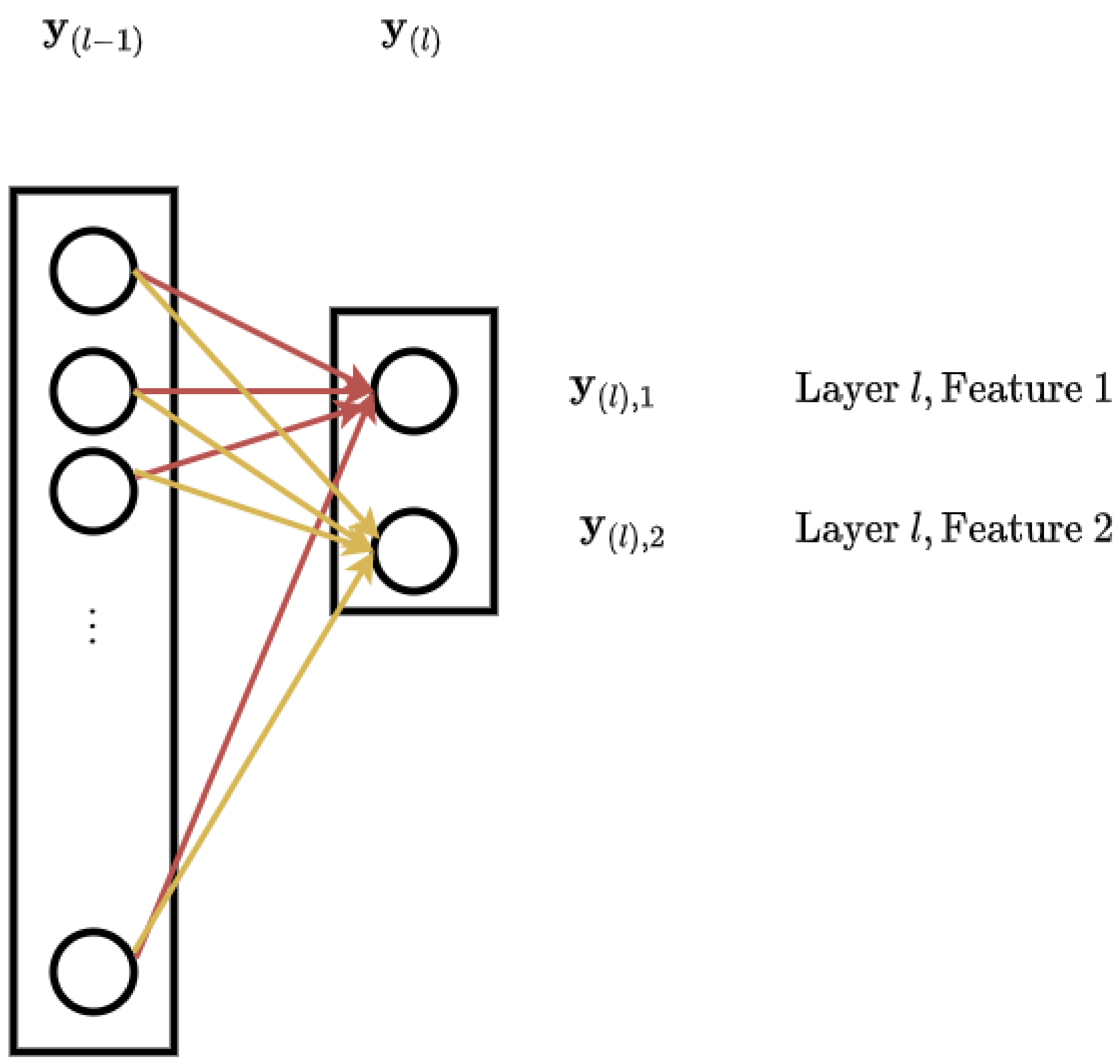
Recall that a Fully Connected layer may have multiple units, so as to compute *multiple* features.

A Fully Connected/Dense Layer producing multiple features at layer l computes

$$\mathbf{y}_{(l),j} = a_{(l)}(\mathbf{y}_{(l-1)} \cdot \mathbf{W}_{(l),j})$$

using separate weights to recognize each feature

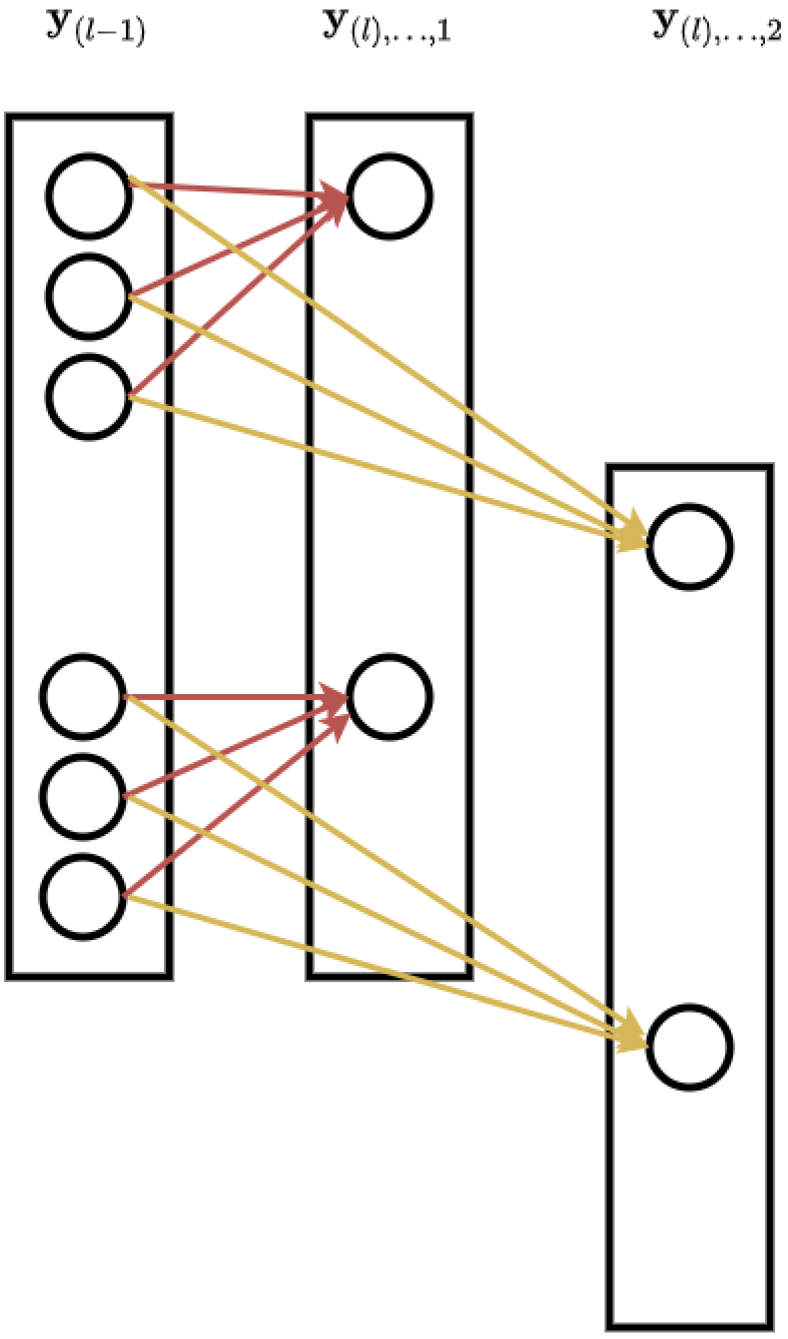
Fully connected, two features



Similary, a Convolutional layer may compute *multiple* features:

- Using separate kernels to recognize each output feature map
- Indicated via separate colors

CNN layer, multiple features


$\mathbf{y}_{(l-1)}$
$\mathbf{y}_{(l),\dots,1}$
$\mathbf{y}_{(l),\dots,2}$

Remember: each output feature is a **vector** !

A *feature map* is

- the same feature
- evaluated at *each spatial location* of the input

So the output of Convolutional Layer l is $n_{(l)}$ feature maps.

- Different feature maps $\mathbf{y}_{(l),j}$ use *different* kernels
    - e.g., $\mathbf{k}_{(l),1}, \mathbf{k}_{(l),2}, \ldots$
- But are applied over the *same* input locations
- Recognizing *different* features at the same location
- e.g., $\mathbf{y}_{(l),1}, \mathbf{y}_{(l),2}, \ldots$

As a preview of concepts to be introduced:

Consider an example where

- the input layer $(l-1)$ is a two-dimensional grid of pixels
- layer l is a Convolutional Layer identifying 3 features

Convolution: 1 input feature to 3 output features

Layer $(l-1)$ is three-dimensional tensor: $8 \times 8 \times 1$

- Spatial dimension 8×8
- 1 feature map (channel dimension $= 1$)

The idea behind a convolution is that *each* kernel

- is slid over the spatial dimension of the layer l input
- to create *one* feature map of the layer l output
    - the spatial dimension of the output is the same as the input
    - indicators for the locations in the input that match the kernel pattern

We do this for each kernel, resulting in the output having one feature map per kernel.

Layer l implements a convolution

- Kernel size $k = 2$ (shape 2×2)
- 3 Kernels, each matching a simple pattern
    - "eye"
    - "left smile"
    - "right smile"

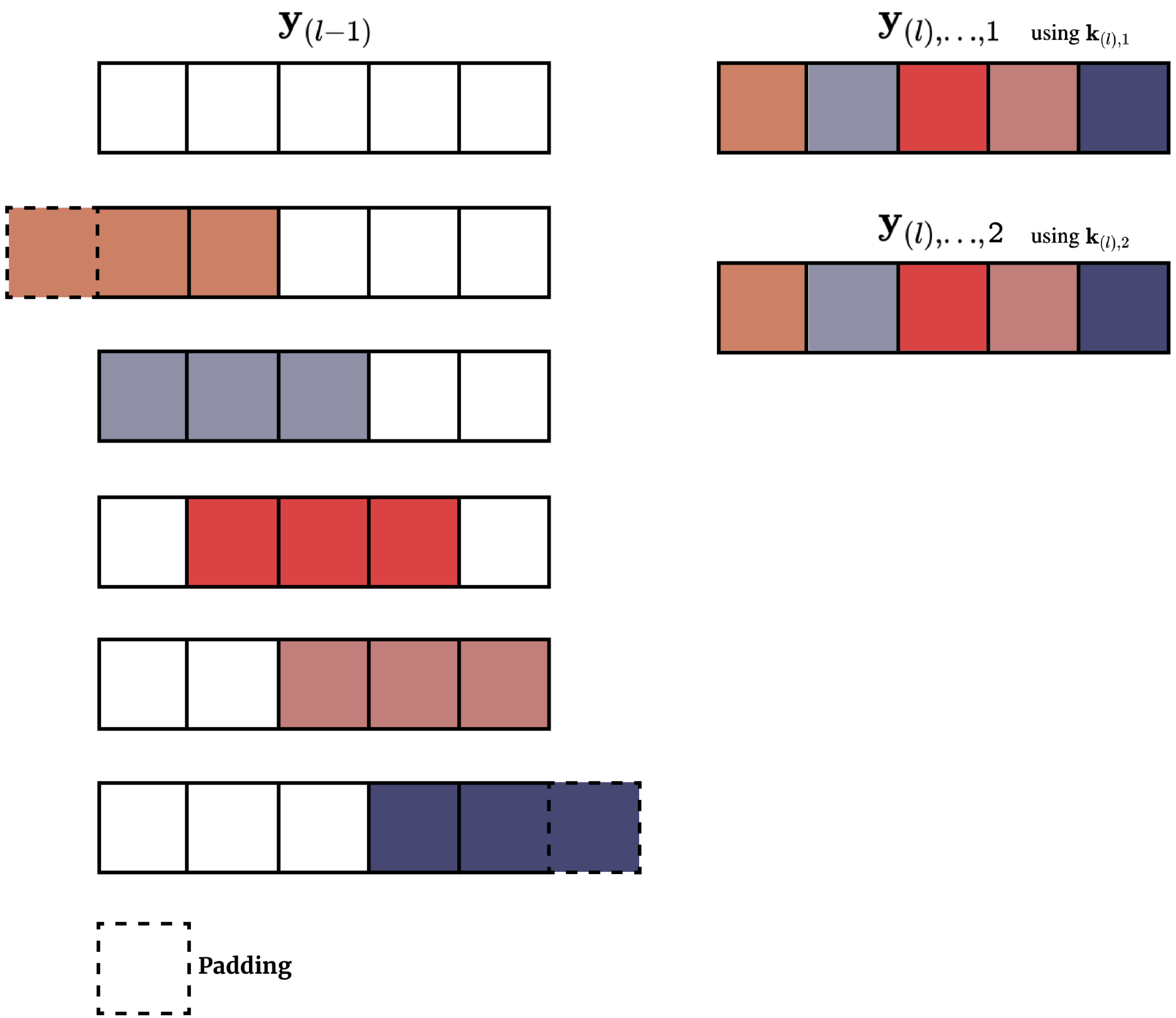
Layer l output is three-dimensional: $8 \times 8 \times 3$

- Spatial dimension 8×8
- 3 feature maps (channel dimension $= 3$)
    - One for each kernel
    - Each feature map has a green square at the spatial location in layer $(l-1)$ where the kernel matches

We can show the process of sliding two kernels over an input layer $(l-1)$ that contains a single feature

- "One dimensional" input
- More accurately: Two dimensional with a singleton dimension corresponding to the feature/channel dimension

Conv 1D, single input, multiple output features



Notation

Input dimensions: Spatial, channel

Our examples thus far have input layers that are one dimensional (having a single feature).

This will not always be the case:

- When Convolutional Layer l creates *multiple* features, as above
- Layer l output is 2 dimensional

We will soon deal with even higher dimensional inputs (e.g, 3 dimensional).

First, some common terminology.

Suppose the input $\mathbf{y}_{(l-1)}$ is $(N+1)$ dimensional of shape

$$||\mathbf{y}_{(l-1)}|| = (d_{(l-1),1} \times d_{(l-1),2} \times \ldots d_{(l-1),N} \quad \times n_{(l-1)})$$

(Thus far: $N = 1$ and $n_{(l-1)} = 1$ but that will soon change)

The first N dimensions $(d_{(l-1),1} \times d_{(l-1),2} \times \ldots d_{(l-1),N})$

- Are called the *spatial* dimensions

The last dimension (of size $n_{(l-1)}$)

- Indexes the features i.e., varies over the number of features
- Called the *feature* or *channel* dimension

Notation

- N denotes the *number* of spatial dimensions
- $n_{(l)}$ denotes the *number of features* in layer l
- Thus far: $N = n_{(l)} = 1$

Rather than treating the single feature input as a special case

- The shape of $\mathbf{y}_{(l-1)}$ would be better written with an extra dimension of length 1:
$$||\mathbf{y}_{(l-1)}|| = (d_{(l-1),1} \times d_{(l-1),2} \times \ldots d_{(l-1),N} \times \mathbf{1})$$
- More clearly indicating that layer $l-1$ has just one feature

With this terminology we can say that a Convolution

- Uses a different kernel $\mathbf{k}_{(l),j}$ for each output feature/channel $1 \le j \le n_{(l)}$
- Applies this kernel to *each* element in the the *spatial* dimensions
- Feature map for feature number $1 \le j \le n_{(l)}$
    - Is of same shape as the spatial dimension
    - Recognizing a single feature at each location within the spatial dimension

Channel Last/First

As we have seem: we are dealing with objects of $(N + 1)$ dimensions

- Have identified the first N dimensions as "spatial"
- The last ($(N+1)^{th}$) as the feature/channel dimension

This is known as *channel last* because the feature/channel dimension is the last.

Some toolkits

- Identify the *first* dimension as the feature/channel dimension
- The remaining N dimensions as the spatial dimensions

This is called *channel first* because the feature/channel dimension is first.

You may arrange the data in Keras according to *either* convention, but it defaults to channel last so we will use that as well.

That's why we write the output of layer l at feature j as

$$\mathbf{y}_{(l),\ldots,j}$$

where the dots $(\ldots)$ indicate the (variable number of) spatial dimensions

Conv1d when input layer has multiple features: $n_{(l-1)} > 1$

Our examples thus far have input layer $(l-1)$ with a single feature

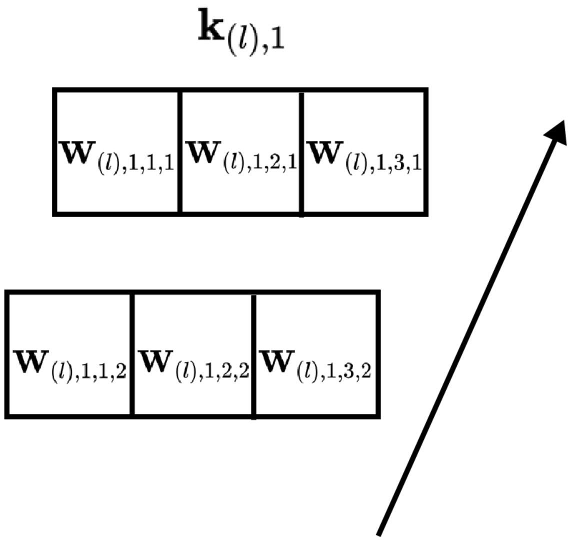
How does a convolution work when the input layer has *more than one* feature ?

- As would be the case of layer l which is the *result* of applying a Convolutional Layer to layer $l-1$
  - Face example: Combine the two half smiles into a single smile

The answer is that we again slide a kernel over each location in the spatial dimension

- **but** each spatial location is now a *vector* of all $n_{(l-1)}$ input features
- Hence the kernel has an extra dimension of length $n_{(l-1)}$
    - That is, of shape $(f_{(l)} \times n_{(l-1)})$

Conv 1D: 2 input features: kernel 1



Note: Weights notation

- $\mathbf{w}_{(l),k,j,f}$
  - layer l
  - output feature k
  - spatial location j
  - input feature f

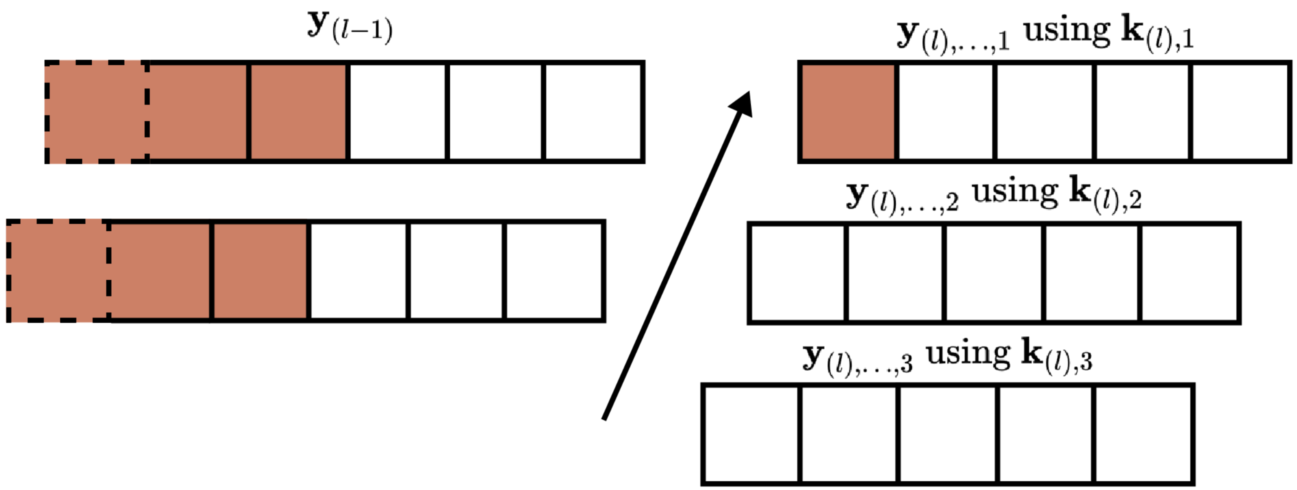
Note

- Dot product is only defined over one dimensional vectors
- When we use "dot product" on two higher dimensional objects of the same shape:
    - Element-wise product
    - Reduced to a scalar by summing the products
- Consider it to be the dot product of the flattened versions of the two objects

Let's illustrate how this works.

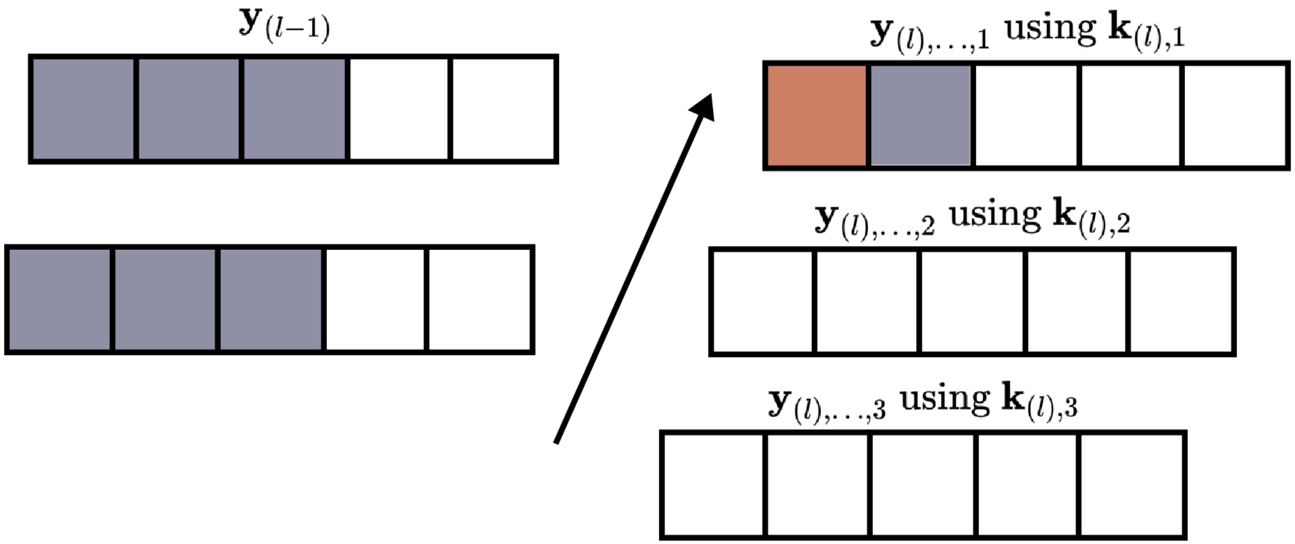
- Output feature 1
- Spatial location 1

Conv 2D: 2 features to 3 features: kernel 1



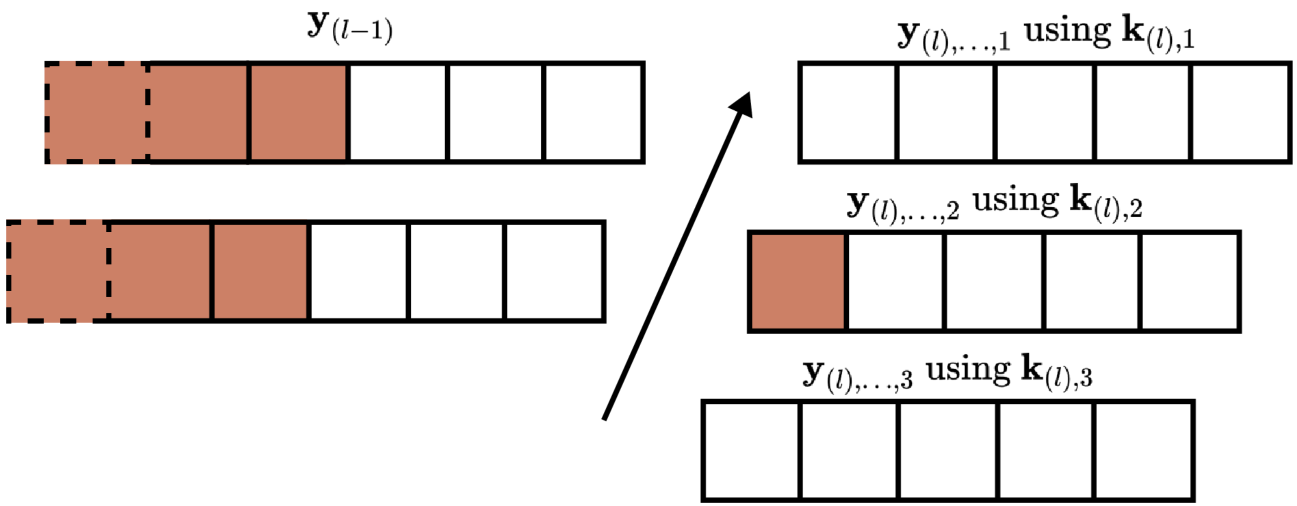
- Output feature 1
- Spatial location 2

Conv 2D: 2 features to 3 features: kernel 1



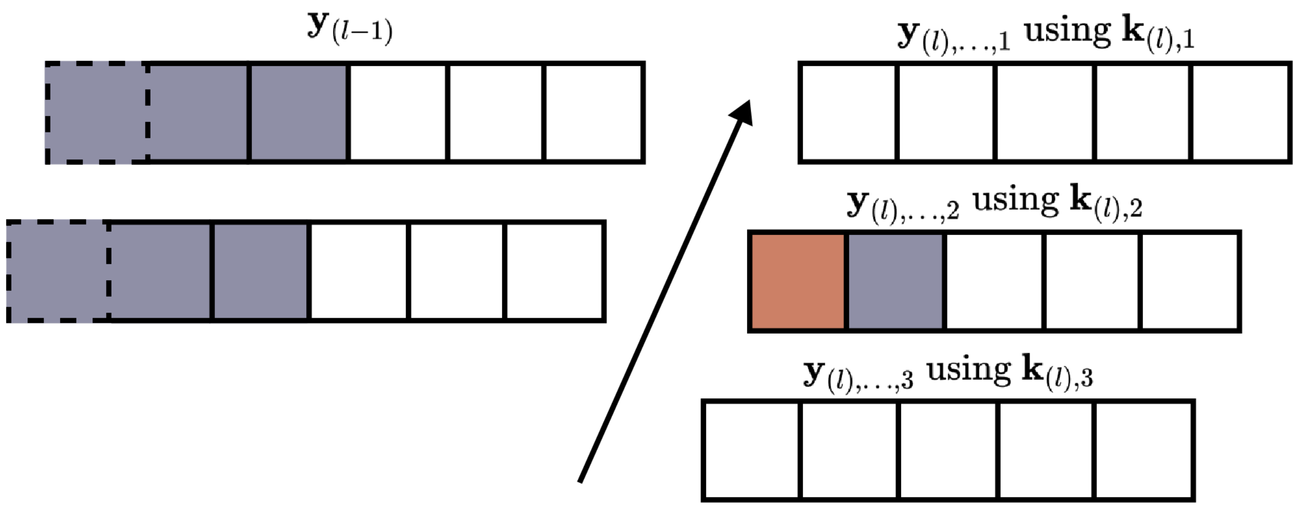
- Output feature 2
- Spatial location 1

Conv 2D: 2 features to 3 features: kernel 2



- Output feature 2
- Spatial location 2

Conv 2D: 2 features to 3 features: kernel 2



With an input layer having N spatial dimensions, a Convolutional Layer l producing $n_{(l)}$ features

- Preserves the "spatial" dimensions of the input
- Replaces the channel/feature dimensions

That is\

$$\begin{aligned}
||\mathbf{y}_{(l-1)}|| &= (n_{(l-1),1} \times n_{(l-1),2} \times \ldots n_{(l-1),N}, \quad \mathbf{n_{(l-1)}}) \\
||\mathbf{y}_{(l)}|| &= (n_{(l-1),1} \times n_{(l-1),2} \times \ldots n_{(l-1),N}, \quad \mathbf{n_{(l)}})
\end{aligned}$$

Conv2d: Two dimensional convolution ($N = 2$)

Thus far, the spatial dimension has been of length $N = 1$.

Generalizing to $N = 2$ is straightforward.

For example, here is a two dimensional convolution with a single input and output feature ($n_{(l-1)} = n_{(l)} = 1$)

- Kernel
    - Two spatial dimensions of size $f_{(l)}$ each
    - A single input feature dimension of size $n_{(l-1)} = 1$
    - Dimension $\left(f_{(l)} \times f_{(l)} \times n_{(l-1)}\right)$
- Is "slid" over 2 dimensional segments of the input
- The "dot product" of the kernel and a two dimensional region of $\mathbf{y}_{(l-1)}$ is performed
- There are $n_{(l)} = 1$ kernels and output features

Conv 2D: single input feature: kernel 1

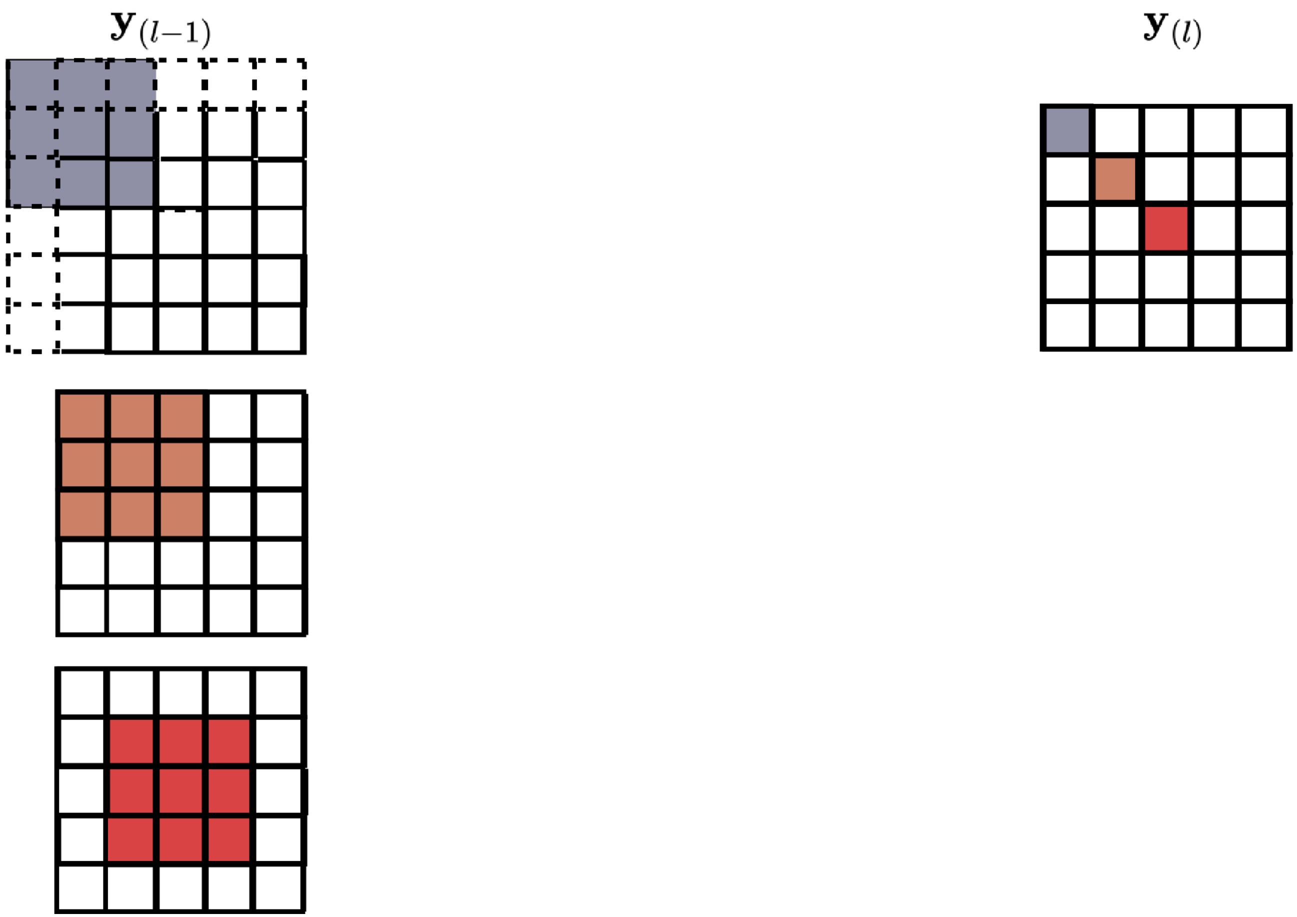
$$\mathbf{k}_{(l),1,1}$$

$$\begin{array}{|c|c|c|}
\hline
\mathbf{w}_{(l),1,1,1} & \mathbf{w}_{(l),1,2,1} & \mathbf{w}_{(l),1,3,1} \\
\hline
\mathbf{w}_{(l),2,1,1} & \mathbf{w}_{(l),2,2,1} & \mathbf{w}_{(l),2,3,1} \\
\hline
\mathbf{w}_{(l),3,1,1} & \mathbf{w}_{(l),3,2,1} & \mathbf{w}_{(l),3,3,1} \\
\hline
\end{array}$$

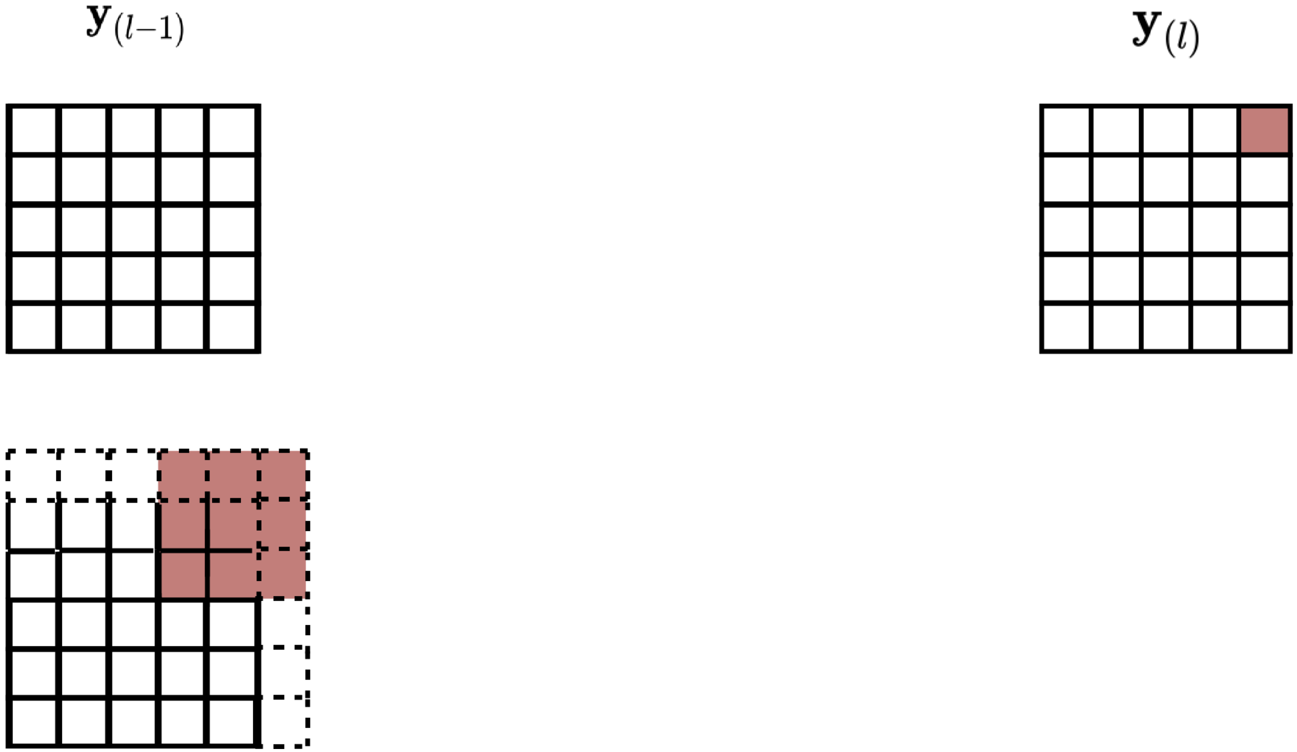
$\mathbf{k}_{(l),j,j'}$

- layer l
- output feature j
- input feature j'

Conv 2D, single input, single output feature: padding at border



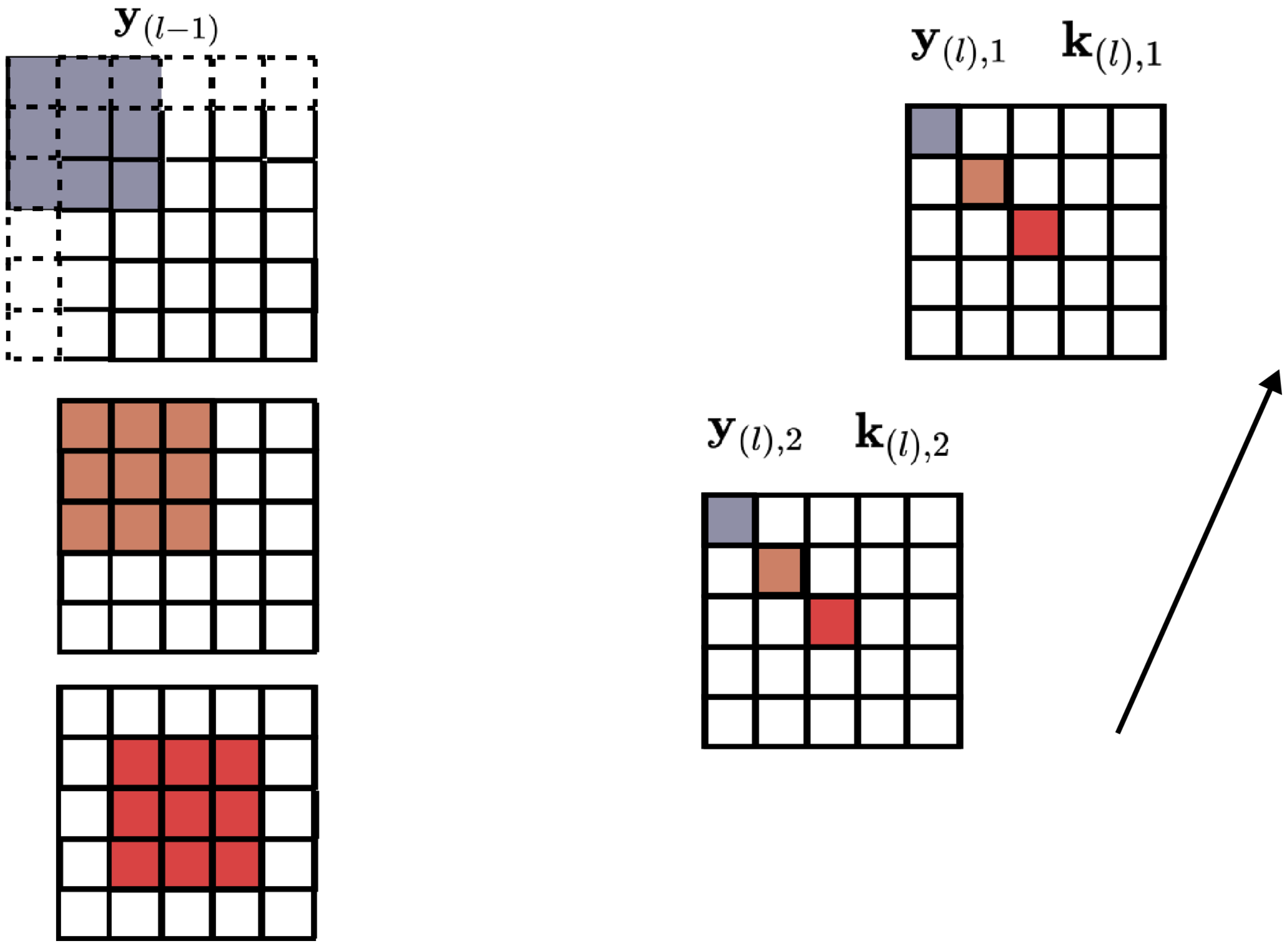
Conv 2D, single input, single output feature: padding at borderpadding at border



The above example was for a single feature.

Of course, we can (and it's common) to recognize multiple features ($n_{(l)} > 1$)

Conv 2D, single input, multiple output feature: padding at border



Dealing with multiple input features works similarly as for $N = 1$:

- The dot product
- Is over a spatial region that now has a "depth" $n_{(l-1)}$ equal to the number of input features
- Which means the kernel has a depth $n_{(l-1)}$

Conv 2D: multiple input features: kernel 1

$\mathbf{k}_{(l),1,1}$

$\mathbf{w}_{(l),1,1,1}$	$\mathbf{w}_{(l),1,2,1}$	$\mathbf{w}_{(l),1,3,1}$
$\mathbf{w}_{(l),2,1,1}$	$\mathbf{w}_{(l),2,2,1}$	$\mathbf{w}_{(l),2,3,1}$
$\mathbf{w}_{(l),3,1,1}$	$\mathbf{w}_{(l),3,2,1}$	$\mathbf{w}_{(l),3,3,1}$

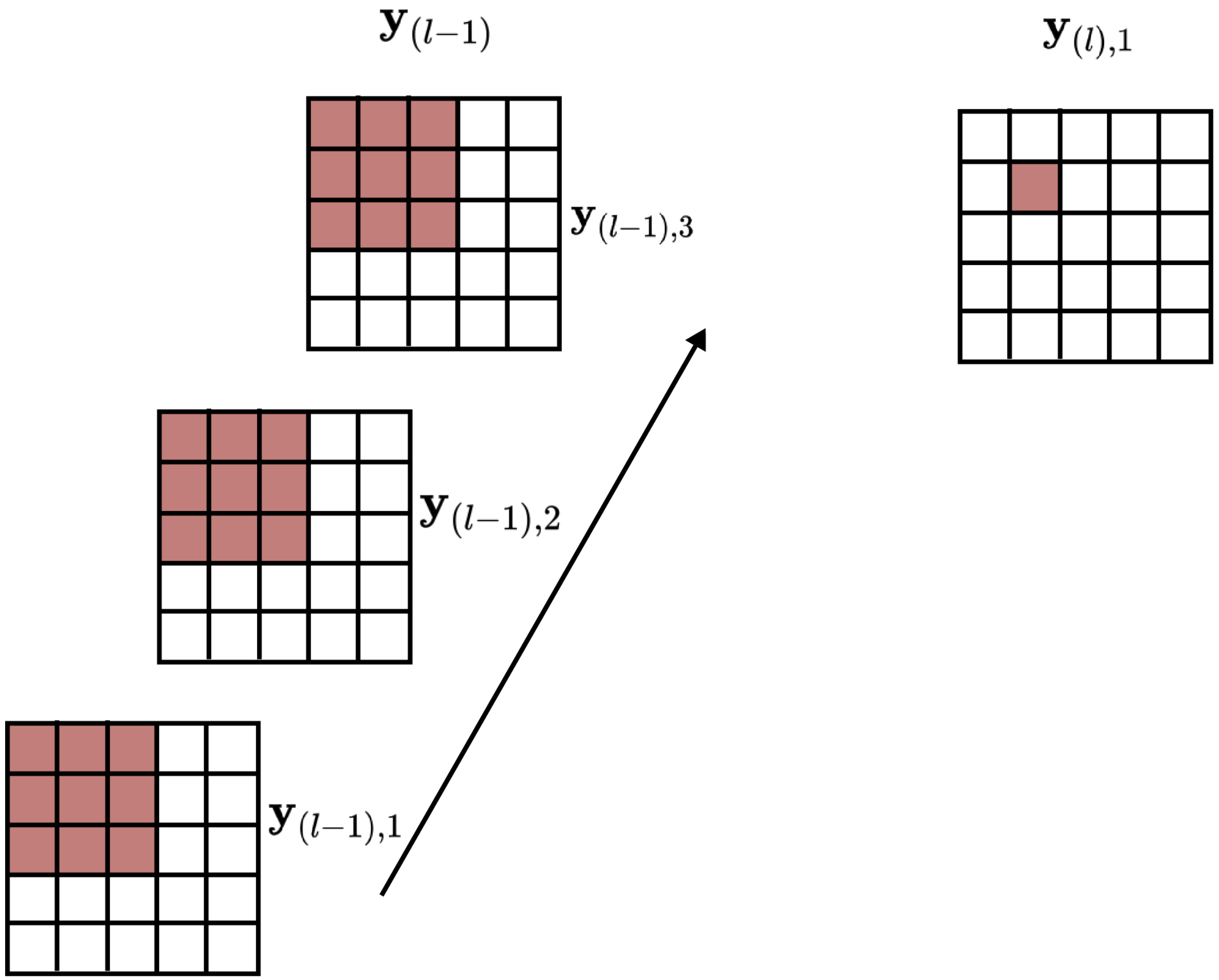
$\mathbf{k}_{(l),1,2}$

$\mathbf{w}_{(l),1,1,2}$	$\mathbf{w}_{(l),1,2,2}$	$\mathbf{w}_{(l),1,3,2}$
$\mathbf{w}_{(l),2,1,2}$	$\mathbf{w}_{(l),2,2,2}$	$\mathbf{w}_{(l),2,3,2}$
$\mathbf{w}_{(l),3,1,2}$	$\mathbf{w}_{(l),3,2,3}$	$\mathbf{w}_{(l),3,3,2}$

$\mathbf{k}_{(l),j,j'}$

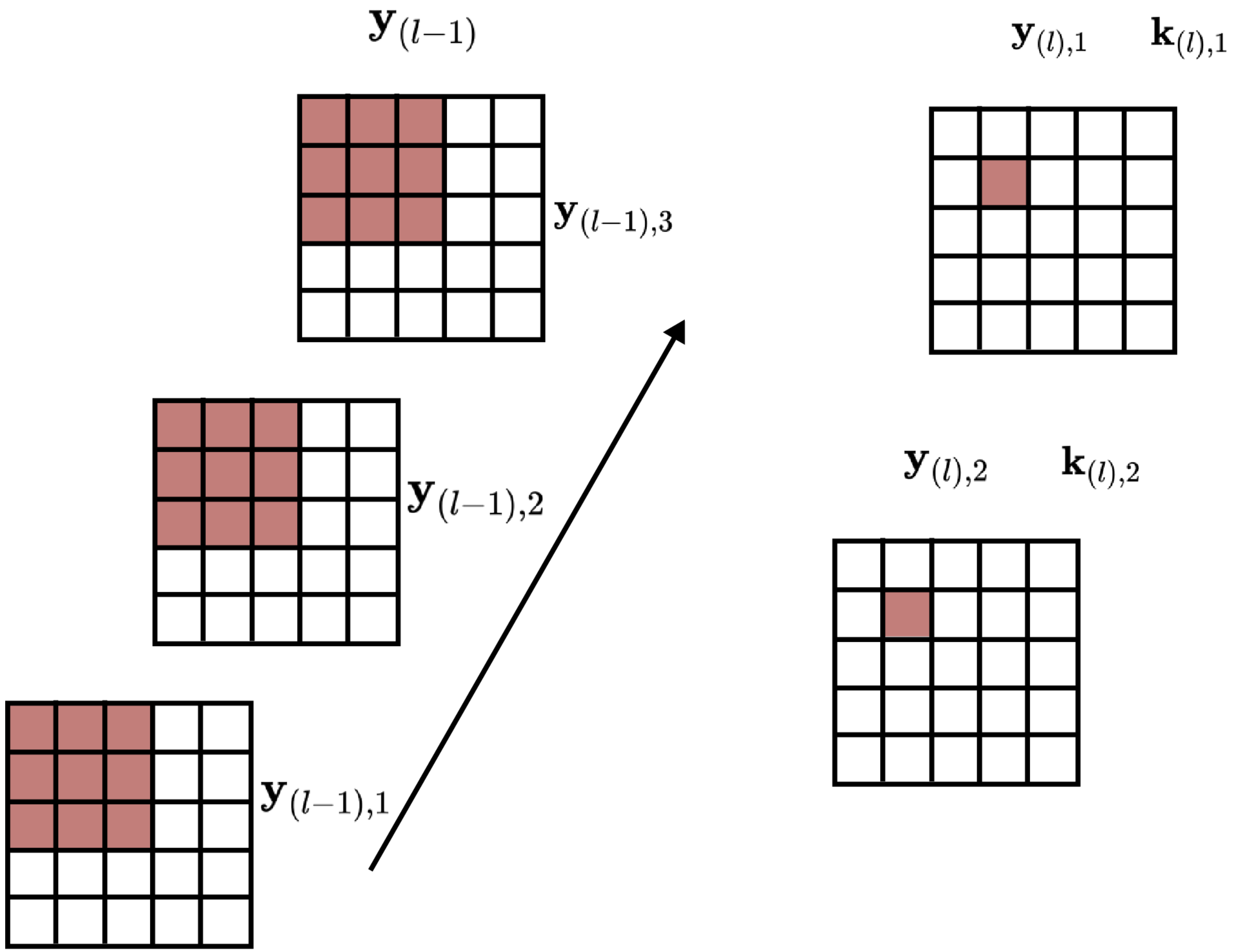
- layer l
- output feature j
- input feature j'

Conv 2D, multiple input, single output feature: padding at border



When we compute *multiple* feature maps, we get

Conv 2D, multiple input, multiple output features



Training a CNN

Hopefully you understand how kernels are "feature recognizers".

But you may be wondering: how do we determine the weights in each kernel ?

Answer: a Convolutional Layer is "just another" layer in a multi-layer network

- The kernels are just weights (like the weights in Fully Connected layers)
- We solve for all the weights $\mathbf{W}$ in the multi-layer network in the same way

The answer is: exactly as we did in Classical Machine Learning

- Define a loss function that is parameterized by $\mathbf{W}$:
$$\mathcal{L} = L(\hat{\mathbf{y}}, \mathbf{y}; \mathbf{W})$$
- The kernel weights are just part of $\mathbf{W}$
- Our goal is to find $\mathbf{W}^*$ the "best" set of weights
$$\mathbf{W}^* = \operatorname*{argmin}_W L(\hat{\mathbf{y}}, \mathbf{y}; \mathbf{W})$$
- Using Gradient Descent !

In other words: their is nothing special about finding the "best" kernels.

In [5]:

```
print("Done")
```

```
Done
```