Classification: Loss function

It would be natural to expect the Average Loss to be Accuracy (fraction of correct predictions).

On a per example basis, the corresponding loss $\mathcal{L}^{(\mathbf{i})}$ would be either 1 or 0, depending on correctness.

This is not the case.

Recall the mapping of probability to prediction

$$\hat{\mathbf{y}}^{(\mathbf{i})} = \begin{cases} \text{Negative} & \text{if } \hat{p}^{(\mathbf{i})} < 0.5 \\ \text{Positive} & \text{if } \hat{p}^{(\mathbf{i})} \geq 0.5 \end{cases}$$

The prediction for example i changes only when probability $\hat{p}^{(\mathbf{i})}$ crosses the threshold.

Suppose the class for example i is Positive: $\mathbf{y}^{(\mathbf{i})} = \text{Positive}.$

- Is our model "better" when our prediction is "more certain" (extreme probability)

$$
\begin{array}{llll}
\hat{p}^{(\mathbf{i})} & \approx 1 & \text{than when} & \hat{p}^{(\mathbf{i})} = 0.5 \\
\hat{p}^{(\mathbf{i})} & = (.5 - \epsilon) & \text{than when} & \hat{p}^{(\mathbf{i})} \approx 0
\end{array}
$$

- The per-example Accuracy is the same in both comparisons
- But a model with probability $\hat{p}^{(\mathbf{i})}$ closer to 1 for a Positive example i would seem to better

There is no *degree* or magnitude of inaccuracy

- Two models may have the same Accuracy even though the probabilities of one may be closer to perfect than the other
- In our search for the best Θ, Accuracy won't be a guide

In mathematical terms: we want our Loss function be be continuous and differentiable.

Accuracy (and the per-example analog) satisfies neither.

We will introduce Binary Cross Entropy loss to overcome this difficulty.

Think of Binary Cross Entropy as a continuous analog of Accuracy.

Binary Cross Entropy

Let's encode the Positive labels $\mathbf{y}^{(\mathbf{i})}$ with the number 1 and Negative labels with the number 0. The loss for example i will be defined as $$ \loss^\ip_\Theta = \begin{cases}

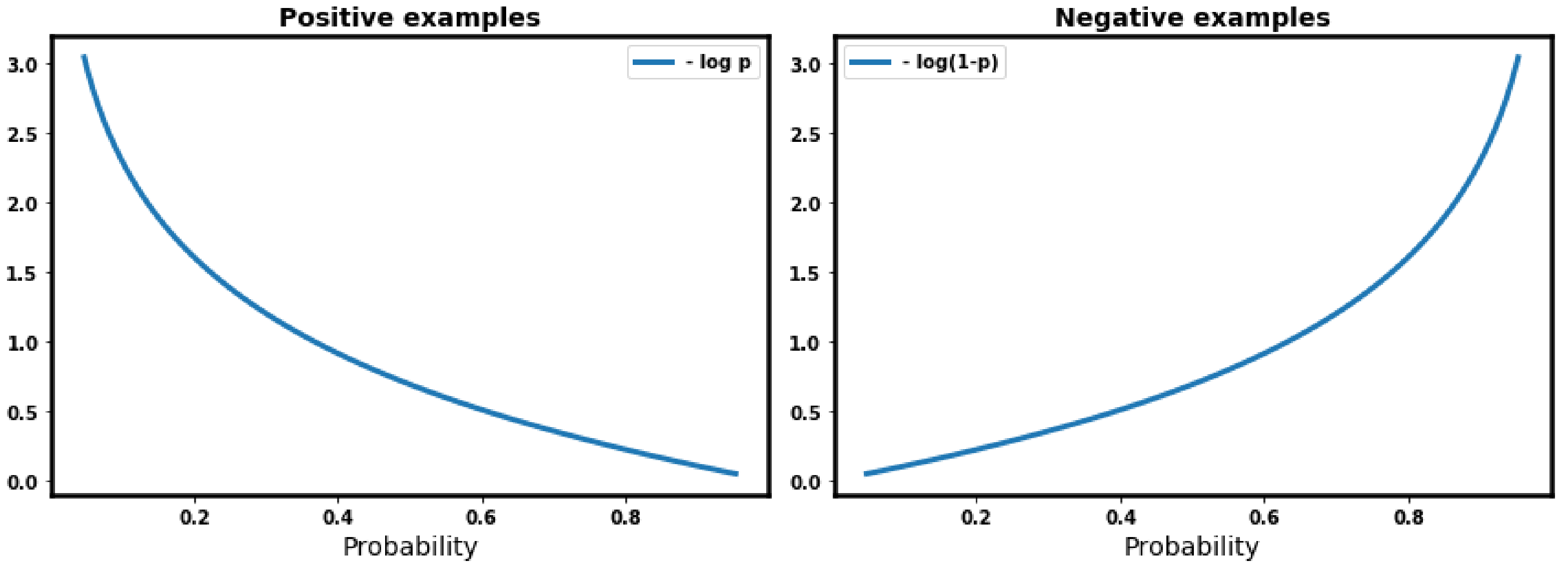
- \log(\hat{p}) & \textrm{if } & \y^\ip = 1 \
- \log(1-\hat{p}) & \textrm{if } & \y^\ip = 0 \ \end{cases} $$

Note the negative signs:

- The term being negated is a Utility (which we want to maximize)

A plot will give us some intuition.

In [4]:
```
svmh.plot_log_p(x_axis="Probability")
```



- For Positive examples: the loss approaches 0 as the predicted probability approaches the correct value (1).
- For Negative examples: the loss approaches 0 as the predicted probability approaches the correct value (0).

In a Deep Dive (after the introduction of a bit of math) we will gain a greater appreciation it's meaning.

For now: be content that Binary Cross Entropy seems to have the right slope and asymptotic behavior.

Because only one of $\mathbf{y^{(i)}}$ and $(1 - \mathbf{y^{(i)}})$ is non zero, we can re-write the two-case statement into a single expression

$$\mathcal{L}^{(\mathbf{i})}_\Theta = - \left(\mathbf{y^{(i)}} * \log(\hat{p}^{(\mathbf{i})}) + (1 - \mathbf{y^{(i)}}) * \log(1 - \hat{p}^{(\mathbf{i})}) \right)$$

This expression is referred to as *Binary Cross Entropy*; it and the multi-class version will become quite familiar going forward.

The Loss for the entire training set is simply the average (across examples) of the Loss for the example

$$\mathcal{L}_\Theta = \frac{1}{m} \sum_{i=1}^{m} \mathcal{L}_\Theta^{(\mathbf{i})}$$

Loss function for Multinomial Classification: Cross Entropy

A Multinomial Classifier (when categories/classes $||C|| > 2$) can be created from multiple Binary Classifiers

- Create a separate Binary Classifier for each $c \in C$
- The classifier for category c attempts to classify
  - Each example with target category of c as Positive
  - All other examples as Negative
- Combine the $||C||$ classifiers to produce a vector $\hat{p}$ of length $||C||$
  - normalize across $c \in C$ to sum to 1
  - $\hat{p}_c$ denotes the normalized value for category c
    - **Notation abuse**: subscripts should be integers, not categories

Both the target $\mathbf{y}$ and the prediction $\hat{p}$ are represented as vectors of length $||C||$

- We write $\mathbf{y}_c, \hat{p}_c$ to denote the element of the vector corresponding to category c
- Each vector can be interpreted as a probability distribution, e.g.

$$\forall \boldsymbol{c} \in C : \mathbf{y}_c \geq 0$$

$$\sum_{c \in C} \mathbf{y}_c = 1$$

  - $\mathbf{y}$ was created with One Hot Encoding (OHE), so properties satisfied by construction
  - $\hat{p}_c$ satisfies the properties by virtue of the normalization of the predictions of the $||C||$ binary classifiers

With $\mathbf{y}, \hat{p}$ encoded as a vectors, per example Binary Cross Entropy can be generalized to $||C|| \geq 2$ categories:

$$\mathcal{L}_{\Theta}^{(\mathbf{i})} = -\sum_{c=1}^{||C||} \left(\mathbf{y}_c^{(\mathbf{i})} * \log(\hat{\mathbf{p}}_c^{(\mathbf{i})}) \right)$$

This is the multinomial analog of Binary Cross Entropy and is called **Cross Entropy**.

Cross Entropy can be interpreted as a measure of the "distance" between distributions $\mathbf{y}$ and $\hat{p}$

- Minimized when they are identical
- We will use Cross Entropy in the future both as a Loss function and a way of comparing probability distributions

In [5]:

```
print("Done")
```

Done