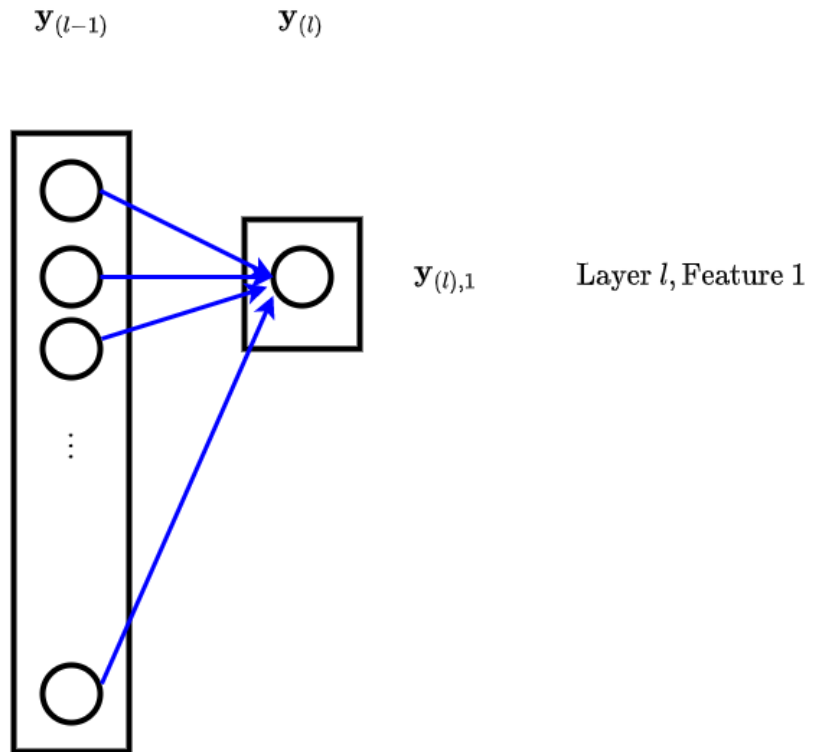# Convolutional Neural Networks

A Fully Connected/Dense Layer with a single unit producing a single feature at layer l computes

$$\mathbf{y}_{(l),1} = a_{(l)}\big(\mathbf{y}_{(l-1)} \cdot \mathbf{W}_{(l),1}\big)$$

Fully connected, single feature

$\mathbf{y}_{(l-1)}$ $\mathbf{y}_{(l)}$

$\mathbf{y}_{(l),1}$ Layer l, Feature 1

That is:

- It recognizes one new synthetic feature
- In the entirety ("fully" connected) of $\mathbf{y}_{(l-1)}$
- Using pattern $\mathbf{W}_{(l),1}$ (same size as $\mathbf{y}_{(l-1)}$)
- To reduce $\mathbf{y}_{(l-1)}$ to a single feature.

The pattern being matched spans the entirety of the input

- Might it be useful to recognize a smaller feature that spanned only *part* of the input ?
- What if this smaller feature could occur *anywhere* in the input rather than at a fixed location ?

For example

- A "spike" in a time series
- The eye in a face

A pattern whose length was that of the entire input could recognize the smaller feature only in a *specific* place

This motivates some of the key ideas behind a Convolutional Layer.

- Recognize smaller features within the whole
- Using small patterns
- That are "slid" over the entire input
- Localizing the specific part of the input containing the smaller feature

Here is the connectivity diagram of a Convolutional Layer producing a **single** feature at layer l

- Using a pattern of length 3
- Eventually we will show how to produce *multiple* features
- Hence the subscript "1" in $\mathbf{y}_{(l),1}$ to denote the first output feature
- The output $\mathbf{y}_{(l),1}$ is called a *feature map* as it attempts to match a feature at each input location

Convolutional layer, single feature
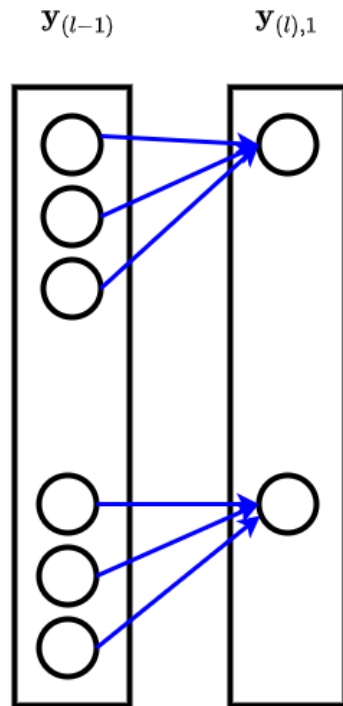
The important differences of a Convolutional Layer from a Fully Connected Layer:

- Produces a new *single* feature *for each location* in $\mathbf{y}_{(l-1)}$

- $\mathbf{y}_{(l),1}$ is thus a *vector* (first feature map) of the same length as $\mathbf{y}_{(l-1)}$

- $\mathbf{y}_{(l)}$ is a vector of $n_{(l)}$ feature maps, one feature map per output feature
- The output feature at location j is **not** fully connected to $\mathbf{y}_{(l-1)}$
    - Only a subsequence of $\mathbf{y}_{(l-1)}$

The lack of full connectivity is significant.

In a Fully Connected network the relationship between

- Feature j and features $(j-1), (j+1)$
- Is no more significant than the relationship between feature j and feature $k \gg j$

That is: spatial locality does not matter.

To see the lack of relationship:

Let perm be a random ordering of the integers in the range $[1 \ldots n]$.

Then

- $\mathbf{x}[\text{perm}]$ is a permutation of input $\mathbf{x}$
- $\Theta[\text{perm}]$ is the corresponding permutation of parameters Θ.

$$\Theta^T \cdot \mathbf{x} = \Theta[\text{perm}]^T \cdot \mathbf{x}[\text{perm}]]$$

But for certain types of inputs (e.g. images) it is easy to imagine that spatial locality is important.

By using a small pattern (and restricting connectivity), we emphasize the importance of neighboring features over far way features.

Mathematically, the One Dimensional Convolutional Layer (Conv1d) we have shown computes $\mathbf{y}_{(l)}$

$$\mathbf{y}_{(l),1} = \begin{pmatrix} a_{(l)} \left(N(\mathbf{y}_{(l-1)}, \mathbf{W}_{(l),1}, 1) \cdot \mathbf{W}_{(l),1} \right) \\ a_{(l)} \left(N(\mathbf{y}_{(l-1)}, \mathbf{W}_{(l),1}, 2) \cdot \mathbf{W}_{(l),1} \right) \\ \vdots \\ a_{(l)} \left(N(\mathbf{y}_{(l-1)}, \mathbf{W}_{(l),1}, n_{(l-1)} \cdot \mathbf{W}_{(l),1} \right) \end{pmatrix}$$

where $N(\ \mathbf{y}_{(l-1)}, \mathbf{W}_{(l),1}, j\ )$

- selects a subsequence of $\mathbf{y}_{(l-1)}$ centered at $\mathbf{y}_{(l-1),j}$

Note that

- The *same* weight matrix $\mathbf{W}_{(l),1}$ is used for the first feature at *all* locations j
- The size of $\mathbf{W}_{(l),1}$ is the same as the size of the subsequence $N(\ \mathbf{y}_{(l-1)}, \mathbf{W}_{(l),1}, j)$
    - Since dot product is element-wise multiplication

So $\mathbf{W}_{(l),1}$

- Is a smaller pattern
- That is applied to *each* location j in $\mathbf{y}_{(l-1)}$
- $\mathbf{y}_{(l),1,j}$ recognizes the match/non-match of the smaller first feature at $\mathbf{y}_{(l-1),j}$

$\mathbf{W}_{(l),1}$ is called a convolutional *filter* or *kernel*

- We will often denote it $\mathbf{k}_{(l),1}$
- But it is just a part of the weights $\mathbf{W}$ of the multi-layer NN.
- We use $f_{(l)}$ to denote the size of the smaller pattern called the *filter size*

Note

The default activation $a_{(l)}$ in Keras is "linear"

- That is: it returns the dot product input unchanged
- Always know what is the default activation for a layer; better yet: always specify !

A *Convolution* is often depicted as

- A filter/kernel
- That is slid over each location in the input
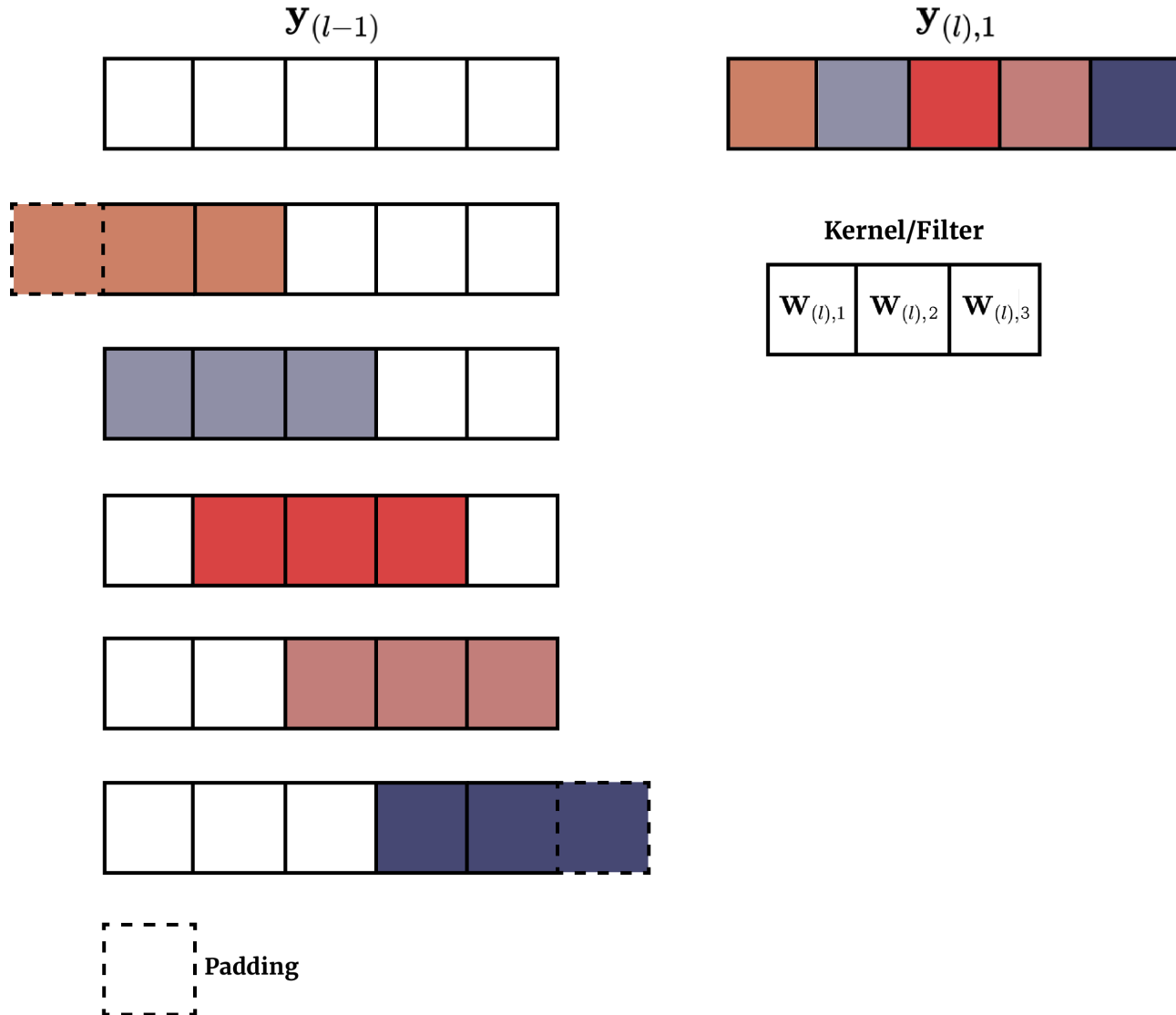- Producing a corresponding output for that location

Here's a picture with a kernel of size $f_{(l)} = 3$

Conv 1D, single feature: sliding the filter

$\mathbf{y}_{(l-1)}$

Kernel/Filter

$\mathbf{W}_{(l),1}$ $\mathbf{W}_{(l),2}$ $\mathbf{W}_{(l),3}$

Kernel/Filter

$\mathbf{W}_{(l),1}$ $\mathbf{W}_{(l),2}$ $\mathbf{W}_{(l),3}$

$\mathbf{y}_{(l),1}$

Kernel/Filter

$\mathbf{W}_{(l),1}$ $\mathbf{W}_{(l),2}$ $\mathbf{W}_{(l),3}$

$\mathbf{y}_{(l),1}$
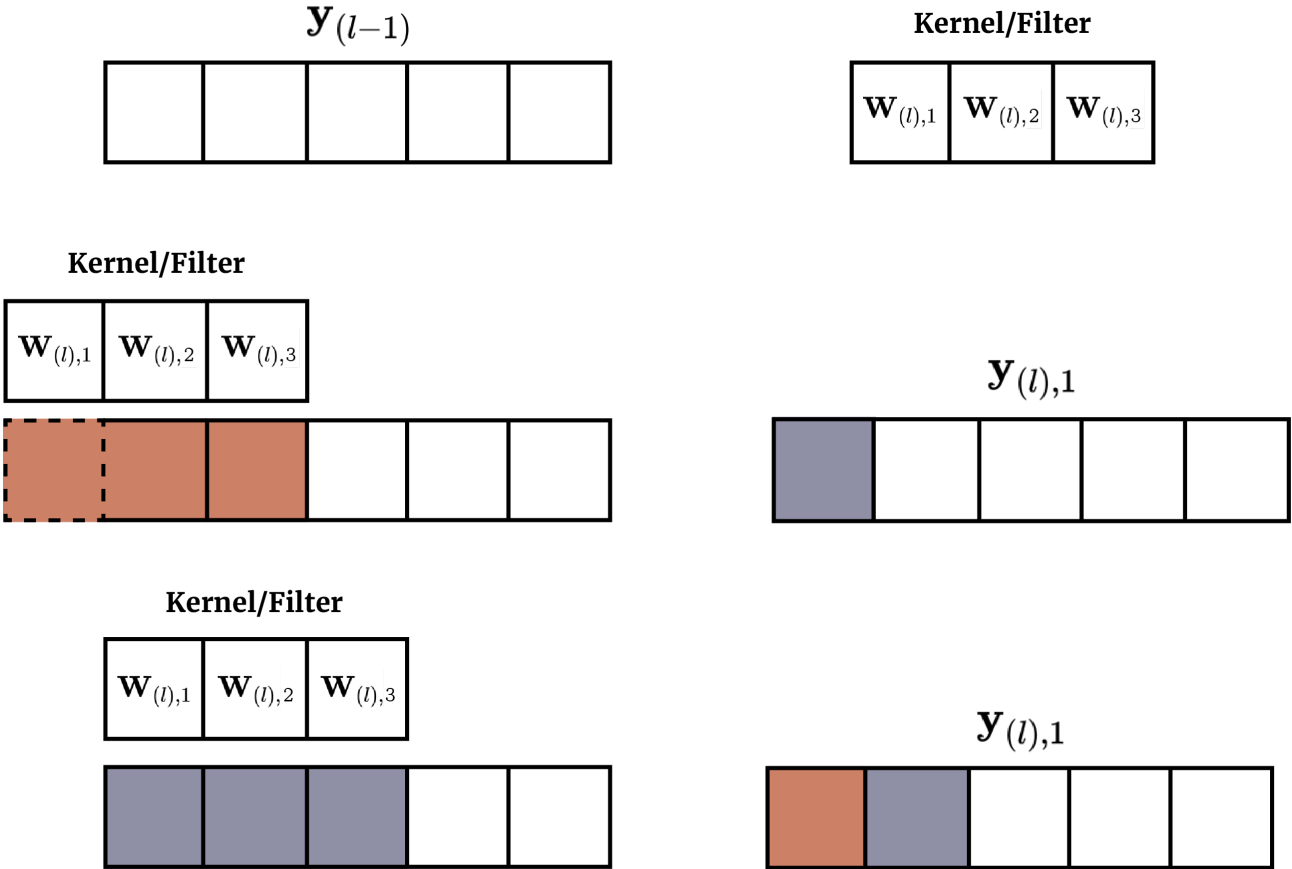
After sliding the Kernel over the whole $\mathbf{y}_{(l-1)}$ we get:

Conv 1D, single feature

$\mathbf{y}_{(l-1)}$

$\mathbf{y}_{(l),1}$

Kernel/Filter

$\mathbf{W}_{(l),1}$	$\mathbf{W}_{(l),2}$	$\mathbf{W}_{(l),3}$

Padding

Element j of output $\mathbf{y}_{(l).1}$ (i.e., $\mathbf{y}_{(l),1,j}$)

- Is colored (e.g., $j = 1$ is colored Red)
- Is computed by applying the *same* $\mathbf{W}_{(l),1}$ to
    - The $f_{(l)}$ elements of $\mathbf{y}_{(l-1)}$, centered at $\mathbf{y}_{(l-1),j}$
    - Which have the same color as the output

Note however that, at the "ends" of $\mathbf{y}_{(l-1)}$ the kernel may extend beyond the input vector.

In that case $\mathbf{y}_{(l-1)}$ may be extended with *padding* (elements with 0 value typically)
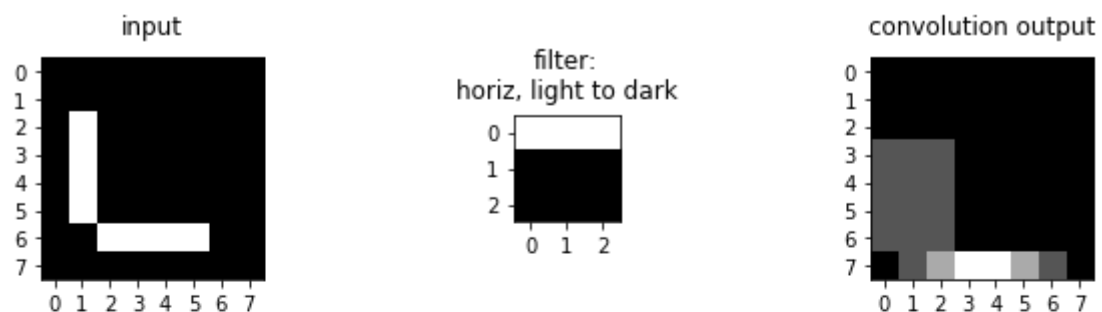
Conv2d in action

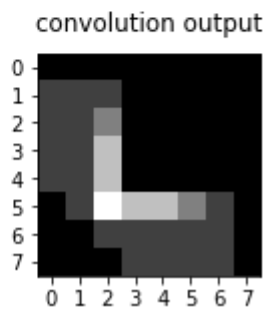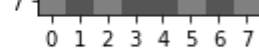Pre-Deep Learning: manually specified filters have a rich history for image recognition.

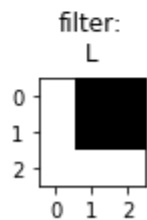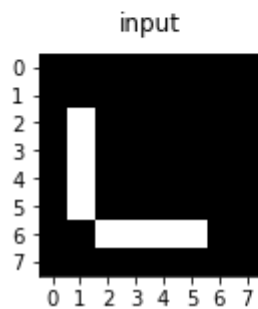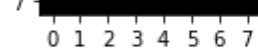Here is a list of manually constructed kernels (templates) that have proven useful

- list of filter matrices (https://en.wikipedia.org/wiki/Kernel_(image_processing))

Let's see some in action to get a better intuition.

`_= cnnh.plot_convs()`

input

filter:
horiz, light to dark

convolution output

input

filter:
vert, light to dark

convolution output

input

filter:
horiz, light band

convolution output

input

filter:
vert, light band

convolution output

input

filter:

L

convolution output
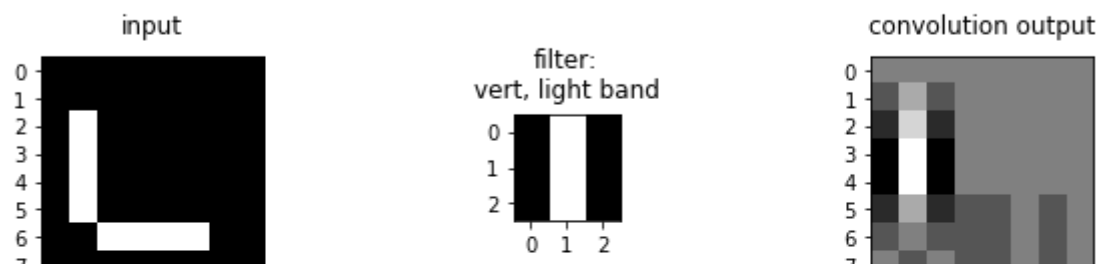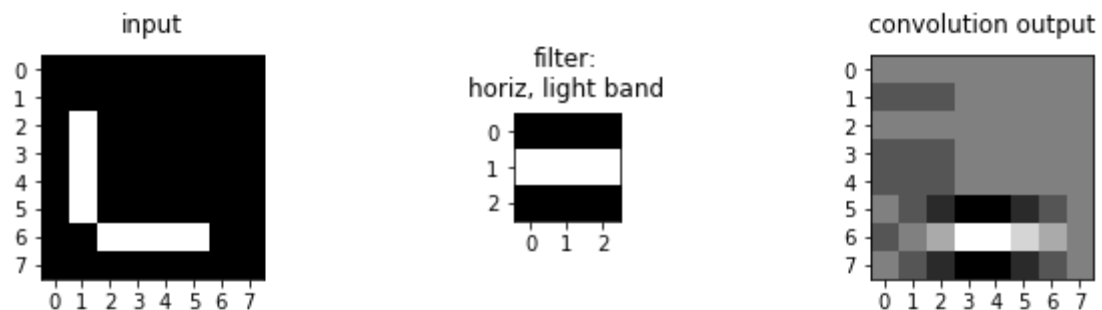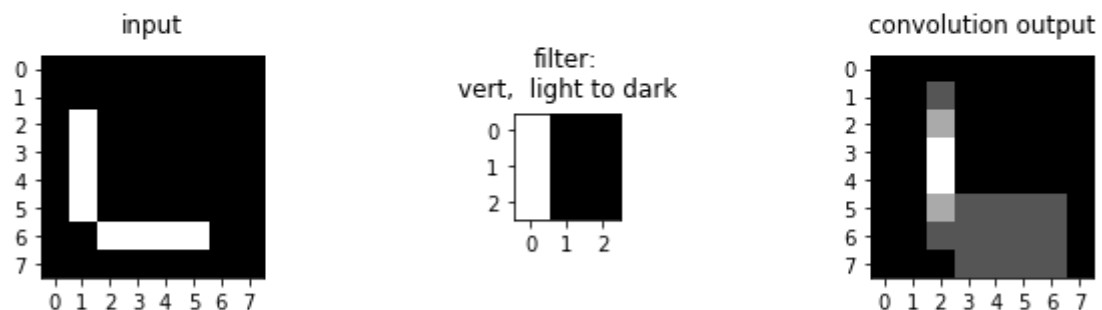
- A bright element in the output indicates a high, positive dot product
- A dark element in the output indicates a low (or highly negative) dot product

In our example

- $N = 2$: Two spatial dimensions
- One input feature: $n_{(l-1)} = 1$
- One output feature $n_{(l)} = 1$
- $f_{(l)} = 3$
  - Kernel is $(3 \times 3 \times 1)$.

The template match will be maximized when

- high values in the input correspond to high values in the matching location of the template
- low values in the input correspond to low values in the matching locations of the template

```
In [5]: print("Done")
```

Done