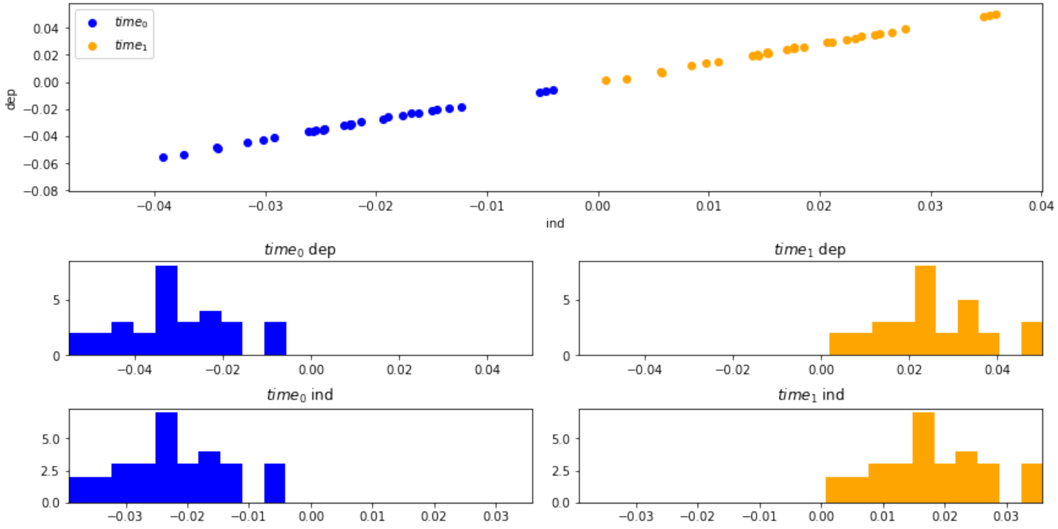
Non-homogeneous data: make it (more) Homogeneous

Normalization via z-score

Let's consider a simple dataset with examples that are drawn from two different groups

In [7]:

```
sph = transform_helper.StockReturn_Pooling_Helper()

means = [ -.02, +.02 ]
s = .16/(252**.5)

df_2means = sph.gen_returns(means, [s, s])

_= sph.plot_data(df_2means)
```



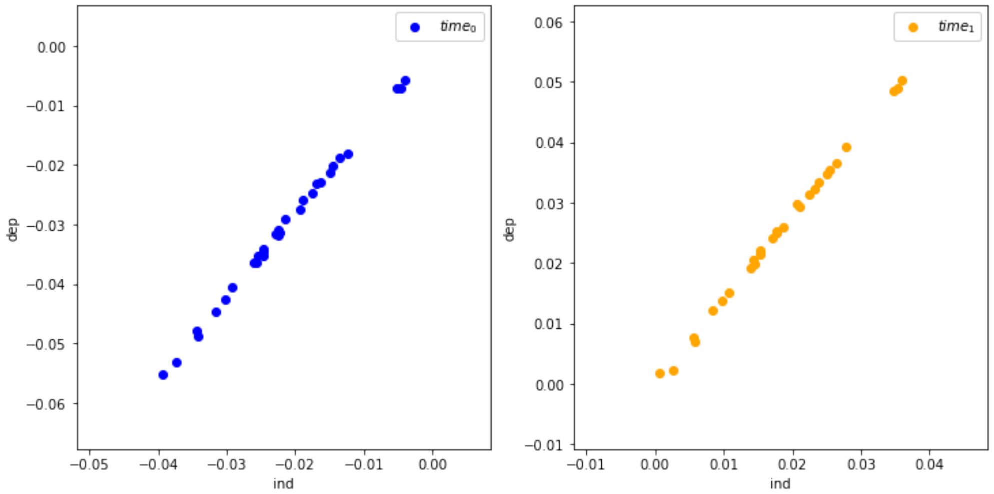
From the top graph: we can see that there is a constant linear relationship

- between target "dep" and feature "ind"
- both within groups and across groups

From the second and third rows, we see the distribution of features and targets

- has same shape between groups
- with different means

In [8]:
```
_= sph.plot_segments(df_2means)
```



Given the simple linear relationship intra-group

- No harm would come from pooling
- Even though the pooled data comes from distinct groups

However: it the intra-group relationship was more complex (e.g., a curve)

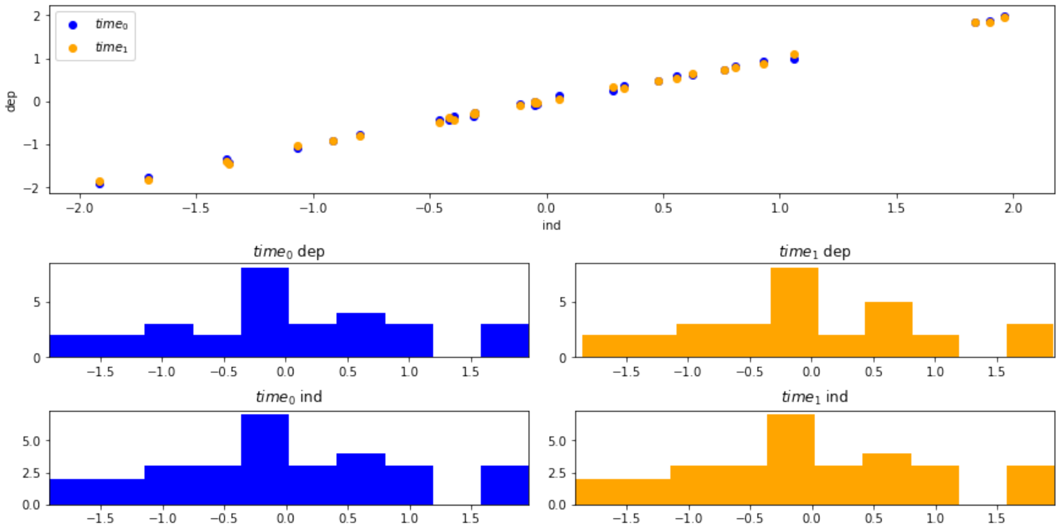
- pooling would be less successful

So although this example may be over-simplified, we still try to make the distinct groups look similar.

Let's normalize each group

- for each variable (target and feature): turn values into z-scores
    - subtract variable mean, divide by variable standard deviation

In [9]:
```
df_2means_norm = sph.normalize_data(df_2means)
_= sph.plot_data(df_2means_norm)
```



You can now see that the two groups are

- congruent in the top joint plot
- have same distributions in the second and third rows

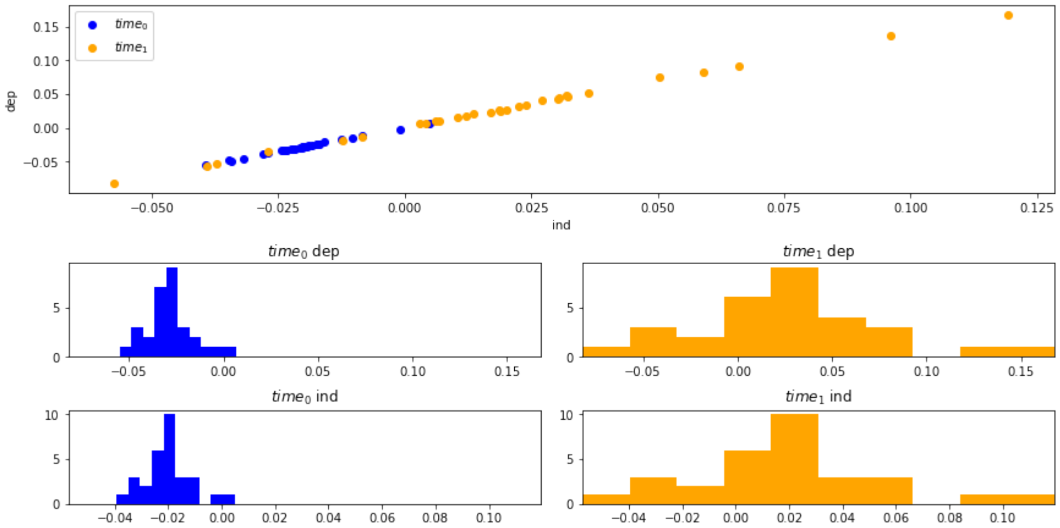
Non-homogeneous groups made homogeneous !

We can make the separation between groups less trivial by also having different standard deviations per group.

Here's what the data looks like

In [10]:

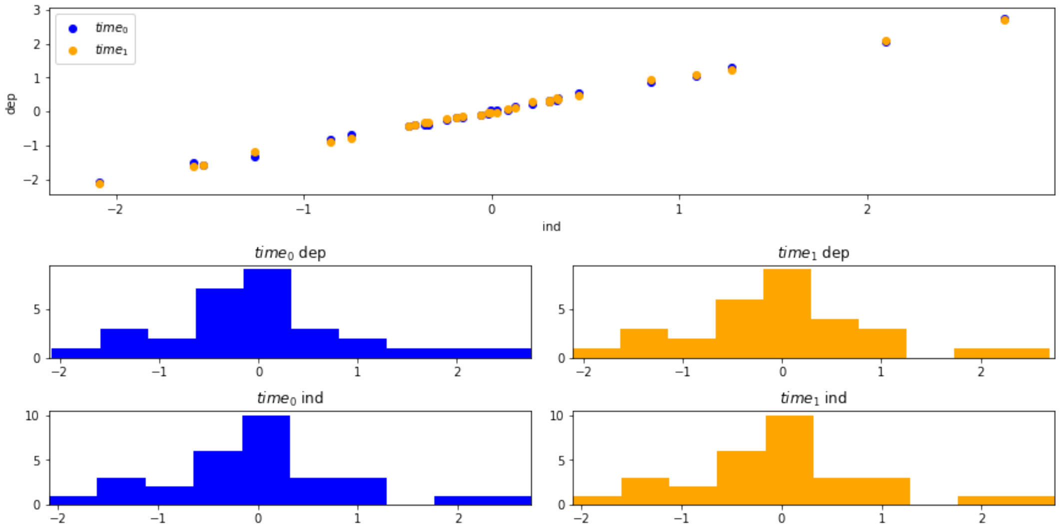
```
df_2means_2sdevs = sph.gen_returns(means, [s, 4*s])

_= sph.plot_data(df_2means_2sdevs)
```



Again: normalization does the trick

In [11]:

```
df_2means_2sdevs_norm = sph.normalize_data(df_2means_2sdevs)
_= sph.plot_data(df_2means_2sdevs_norm)
```

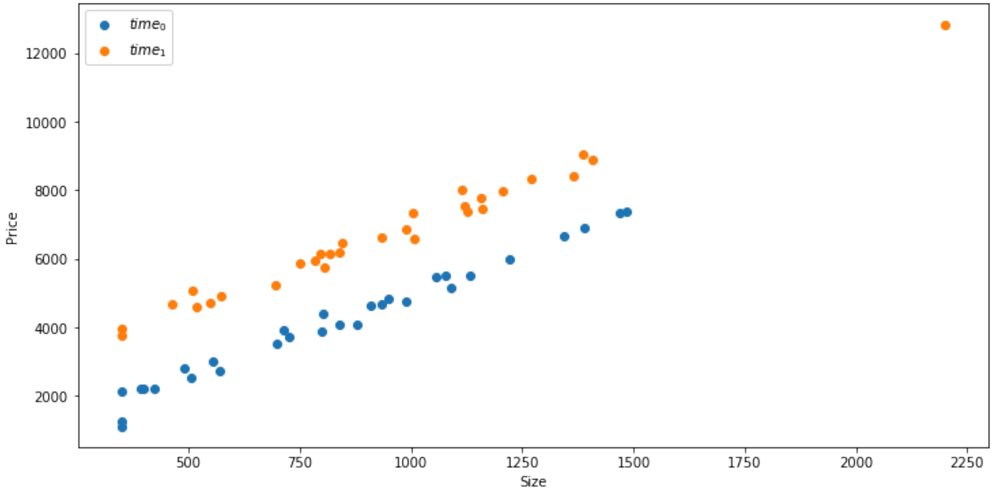


Pooled over time alternate method: normalization

Let's revisit our "pooled over time" dataset.

In [12]:

```
sph = transform_helper.ShiftedPrice_Helper()
series_over_time = sph.gen_data(m=30)

fig, ax = plt.subplots(1,1, figsize=(12,6) )
_= sph.plot_data(series_over_time, ax=ax)
```



We observed that the two groups have the same slope (Θ_1) but different intercepts (Θ_0)

$$
\begin{aligned}
\mathbf{y}_{(\text{time}_0)} &= \Theta_{(\text{time}_0)} + \Theta_1 * \mathbf{x} \\
\mathbf{y}_{(\text{time}_1)} &= \Theta_{(\text{time}_1)} + \Theta_1 * \mathbf{x}
\end{aligned}
$$

We had previously addressed this by adding a missing feature

- distinct intercept per group
- by adding a "group indicator" feature

$$
\mathbf{y} = \Theta_{(\text{time}_0)} * \text{Is}_0 + \Theta_{(\text{time}_1)} * \text{Is}_1 + \Theta_1 * \mathbf{x}
$$

Which, after dropping one group indicator to avoid the Dummy Variable Trap gave us

$$
\mathbf{y} = \Theta_0 + \Theta'_{(\text{time}_1)} * \text{Is}_1 + \Theta_1 * \mathbf{x}
$$

We show an alternate solution using a (trivial) standardization

Some simple algebra allows us to derive the intercept for Linear Regression in terms of the other features

$$
\begin{array}{lll}
\mathbf{y^{(i)}} & = & \Theta_0 + \Theta * \mathbf{x^{(i)}} & \text{hypothesize linear relationship} \\
 & & & \Theta \text{ is a vector of non-intercept features} \\
\frac{1}{m} \sum_i \mathbf{y^{(i)}} & = & \frac{1}{m} \sum_i \left(\Theta_0 + \Theta * \mathbf{x^{(i)}} \right) & \text{sum over all examples, divide by no. o} \\
\bar{\mathbf{y}} & = & \Theta_0 + \Theta * \bar{\mathbf{x}} & \text{definition of average} \\
\Theta_0 & = & \bar{\mathbf{y}} - \Theta * \bar{\mathbf{x}} & \text{re-arrange terms}
\end{array}
$$

That is, the intercept

- is the average target
- less "average prediction"
    - the prediction (excluding intercept) at the average value of all features

Let's standardize the features $\mathbf{x}$ and target $\mathbf{y}$ in our original equations

$$
\begin{aligned}
\mathbf{y}_{(\text{time}_0)} &= \Theta_{(\text{time}_0)} + \Theta_1 * \mathbf{x} \\
\mathbf{y}_{(\text{time}_1)} &= \Theta_{(\text{time}_1)} + \Theta_1 * \mathbf{x}
\end{aligned}
$$

giving us the equations

$$
\begin{aligned}
\tilde{\mathbf{y}}_{(\text{time}_0)} &= \tilde{\Theta}_{(\text{time}_0)} + \tilde{\Theta}_1 * \tilde{\mathbf{x}} \\
\tilde{\mathbf{y}}_{(\text{time}_1)} &= \tilde{\Theta}_{(\text{time}_1)} + \tilde{\Theta}_1 * \tilde{\mathbf{x}}
\end{aligned}
$$

where the $\tilde{\mathbf{y}}$ and $\tilde{\mathbf{x}}$ variables are the standardized forms of $\mathbf{y}$ and $\mathbf{x}$

According to our algebra, when

- the average target
- and the average feature

are both 0: the intercept is 0.

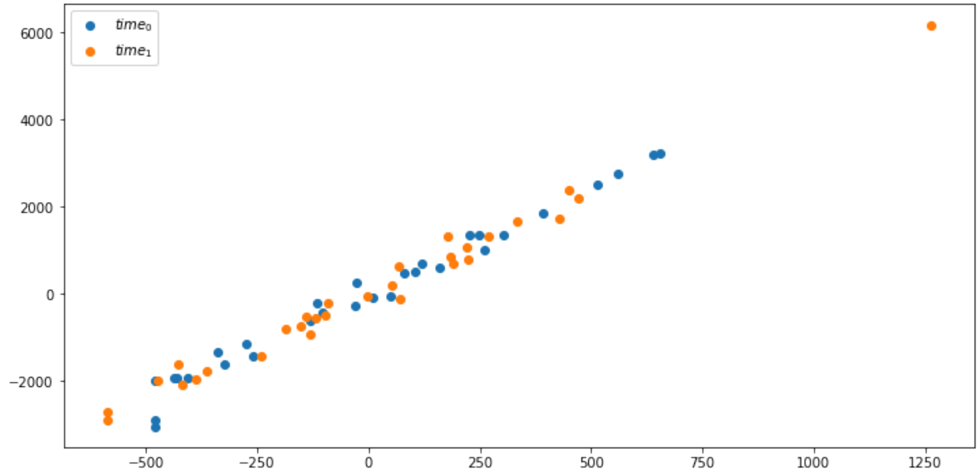
Hence are two equations simplify to

$$
\begin{array}{lll}
\tilde{\mathbf{y}}_{(\text{time}_0)} & = & 0 + \tilde{\Theta}_1 * \tilde{\mathbf{x}} \\
\tilde{\mathbf{y}}_{(\text{time}_1)} & = & 0 + \tilde{\Theta}_1 * \tilde{\mathbf{x}}
\end{array}
$$

That is, a single equation describes both groups

$$
\tilde{\mathbf{y}} \quad = \quad 0 + \tilde{\Theta}_1 * \tilde{\mathbf{x}}
$$

In [13]:
```
fig, ax = plt.subplots(1,1, figsize=(12,6) )

demean_x0 = sph.x0 - sph.x0.mean()
demean_x1 = sph.x1 - sph.x1.mean()

_= ax.scatter(demean_x0, sph.y0 - sph.y0.mean(), label="$time_0$")
_= ax.scatter(demean_x1, sph.y1 - sph.y1.mean(), label="$time_1$")
_= ax.legend()
```



Now it looks like each group comes from the same distribution.

- We can pool the observations from the two groups

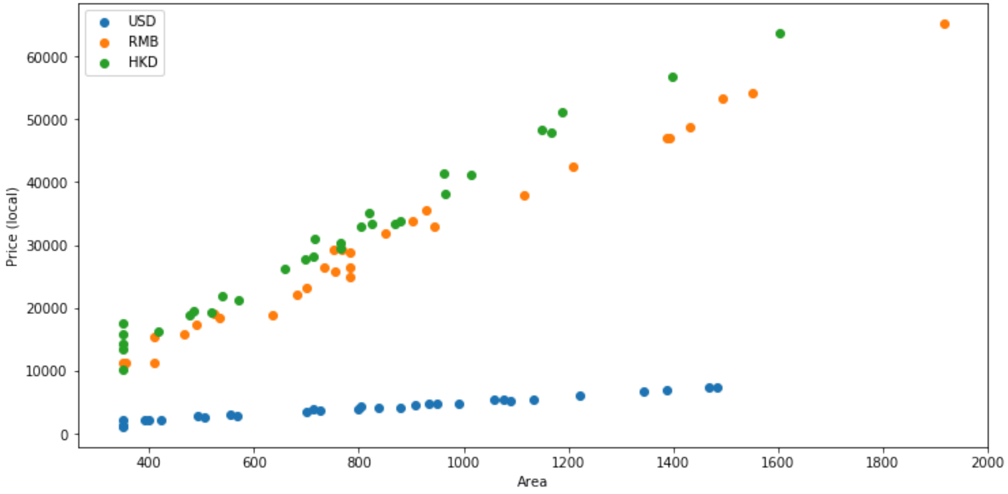
Normalization by uncovering the hidden relationship

Consider the following multi-group data (our multiple geography pooling of data)

- house price as a function of size
- in different geographies

In [14]:
```
fig_rp
```

Out[14]:



There is clearly a linear relationship intra-group, but the slope differs between groups (local currencies).

$$\begin{aligned} \mathbf{y}_g &= \Theta_{0,g} + \Theta_{1,g} * \mathbf{x} & \text{Equation for group } g \\ & & g \in \{\text{USD}, \text{HKD}, \text{RMB}\} \\ & & \text{Separate parameter vectors } \Theta_g \text{ per group} \end{aligned}$$

The apparent diversity in the target may obscure a simple relationship that is common to all groups

Notice that the "units" of the target $\mathbf{y}$ differ for each group

- different currencies

Let's transform the targets to a common unit

- by applying an exchange rate β_g to convert currency g into a common currency (USD)
$$\tilde{\mathbf{y}}_g = \frac{\mathbf{y}_g}{\beta_g}$$

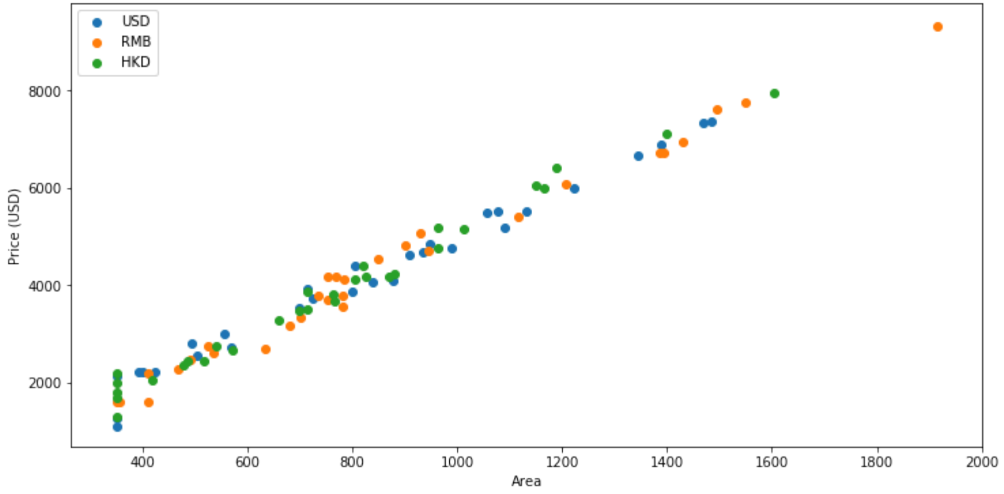
Let's re-denominate the target in a common unit.

- Let the target of example i in group g be
$$\mathbf{y}_g^{(\mathbf{i})}$$
- Change the units in which $\mathbf{y}_g^{(\mathbf{i})}$ is expressed
- Into a common unit
- Via an "exchange rate" equal to the slope of group g
$$\beta_g$$
- yielding
$$\tilde{\mathbf{y}}_g^{(\mathbf{i})} = \frac{\mathbf{y}_g^{(\mathbf{i})}}{\beta_g}$$

Here's the re-denominated plot

In [15]:

```
# Relative price levels
rel_price = rph.relative_price()

# Normalize the price of each series by the relative price
series_normalized = [ series[i]/(1,rel_price[i]) for i in range(len(series))]

fig_rp_norm, ax_rp_norm = plt.subplots(1,1, figsize=(12,6))
_= rph.plot_data(series_normalized, ax=ax_rp_norm, labels=labels, xlabel="Area",
ylabel="Price (USD)")

# plt.close(fig_rp_norm)
```



The three groups are now homogeneous !

$$
\begin{aligned}
\mathbf{y}_g &= \Theta_{0,g} + \Theta_{1,g} * \mathbf{x} && \text{Equation for group } g \\
& && g \in \{\mathrm{USD}, \mathrm{HKD}, \mathrm{RMB}\} \\
& && \text{Separate parameter vectors } \Theta_g \text{ per group} \\
\frac{\mathbf{y}_g}{\beta_g} &= \frac{\Theta_{0,g}}{\beta_g} + \frac{\Theta_{1,g}}{\beta_g} * \mathbf{x} && \text{divide by exchange rate} \\
\mathbf{y} &= \Theta_0 + \Theta_1 * \mathbf{x}
\end{aligned}
$$

Apparantly:

$$\frac{\Theta_{j,g}}{\beta_g} = \Theta_j$$

We have argued that Transformations should be well motivated.

In this case

- the "buying power" of one unit of each currency is different
- by re-denominating in a common currency
- we discover that the relationship is in "buying power" and not "local currency"

We have used "common currency" as a proxy for "buying power".

But there may be better alternative

- "number of months of salary"
    - Interpretation: each additional increment is Size is worth some number of "months of salary"
    - Compensates for differences in salary levels across geographies
- "number of MacDonald's hamburgers"
    - Compensates for differences in price level of a common commodity

It is up to you, the Data Scientist, to propose (and verify) which units reveal the true relationship.

This conversion into common units is a type of *scaling* transformation.

- the common relationship only becomes apparent when the target (or some features) are placed on a common scale
- often see this when target/features are scaled by their standard deviation
    - re-denominate in terms of *number of standard deviations*
    - e.g., returns of two equities are both normal but with different volatilities

Normalization: creating the correct units

There is a similar need for "re-denomination" that arises in a different context

- when the raw feature
- does not express the key semantics as well as a re-denominated feature

The Geron book has a more sophisticated example of predicting house Price from features (external/handson-ml2/02_end_to_end_machine_learning_project.ipynb#Experimenting-with-Attribute-Combinations)

- a lot more features

In [16]:
```
housing.info()
```

```
<class 'pandas.core.frame.DataFrame'>
RangeIndex: 20640 entries, 0 to 20639
Data columns (total 10 columns):
longitude             20640 non-null float64
latitude              20640 non-null float64
housing_median_age    20640 non-null float64
total_rooms           20640 non-null float64
total_bedrooms        20433 non-null float64
population            20640 non-null float64
households            20640 non-null float64
median_income         20640 non-null float64
median_house_value    20640 non-null float64
ocean_proximity       20640 non-null object
dtypes: float64(9), object(1)
memory usage: 1.6+ MB
```

In terms of predictive value, there are some features

- `total_rooms`, `total_bedrooms` that are not predictive because their units are not informative
- both features will have greater magnitude in a multi-family house than a single family house

A more meaningful feature can be synthesized by normalizing by the number of families

```
housing["rooms_per_household"] = housing["total_rooms"]/housing["households"]
housing["bedrooms_per_room"] = housing["total_bedrooms"]/housing["total_rooms"]
```

That is:

- the normalized variable has units "per household"
- that is more predictive of price than the raw feature

In [17]:

```
print("Done")
```

Done