A Numerical Scheme for the Optimal Liquidation Problem Under Jump Diffusion Dynamics on High-Frequency Data

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Abstract

A pairs trade is a portfolio consisting of two historically correlated stocks with a long position on one stock and a short position on the other. Larsson et al. (2013) presented an abstraction for the optimal liquidation problem under jump-diffusion dynamics by presenting evidence of the uniqueness and existence of a numerical solution. This paper builds upon the work of Larsson et al. (2013) and the literature involving empirical strategies involving stop-loss thresholds. The optimal liquidation problem involves evaluating a free-boundary integro partial differential equation. We describe a numerical scheme using finite differences to estimate the non-jump terms and Gauss-Hermite quadrature to estimate the jump term. Additionally, we describe an iterative scheme that uses the previous iteration's estimation as an initial estimation for the integral term to incorporate the integral term into the diffusion terms. We outline a procedure to estimate values for the model parameters using realized variance and bipower variation. Lastly, we apply the framework in this paper in a high-frequency context by using five-minute sampling intervals of the spread between AAPL and NVDA from October 10, 2022 to October 28, 2022 to evaluate the optimal liquidation strategy.

Key words. pairs trading, high-frequency data, jump-diffusion, stopping times, statistical arbitrage

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I Introduction

We discuss a numerical scheme to evaluate the optimal policy for exiting a position in a pair of securities under the assumption that the spread between the two securities is modelled as a jump-diffusion process. According to Vidyamurthy (2004), Pairs trading is a strategy developed by Morgan Stanley in the 1980s. A group of traders at Morgan Stanley decided that a consistent filter of rules would perform better than relying on a trader's intuition and skills. Because of its long history, pairs trading is regarded as the precursor of more complex statistical arbitrage strategies (Vidyamurthy (2004); Avellaneda and Lee (2010)). Pairs trading aims to monitor two historically correlated securities. The idea behind pairs trading involves selling a higher-priced security and buying a lower-priced security under the assumption that both securities are mispriced and will revert to the mean in the future. Vidyamurthy (2004) discussed the connection between the cause of the divergence and the ensuing convergence afterward. The spread between the two securities is defined as a portfolio. Whenever the spread between the two securities is large, both securities are highly mispriced; hence there is an immense profit potential. Elliott et al. (2005) remarked that an advantage of pairs trading is that if the two securities have similar characteristics, then the trader can find a ratio of the two securities such that the portfolio becomes market-neutral in the sense that gains or losses are not heavily affected by the direction of the market. The most cited paper in pairs trading is Gatev et al. (2006), which yielded annualized excess returns of up to 11% and had low exposure to systematic risks. Endres (2019) provided an overview of the various kinds of stochastic differential equations that can be used to model spread dynamics. We refer the reader to Krauss (2017) for a discussion on several other types of pairs trading strategies.

A divergence risk is associated with pairs trading when the two securities cease to be mean-reverting. A practical consideration would be identifying the optimal time when an investor should exit her position in a pairs trade. The standard practice is to set a stop-loss threshold level beforehand and accept the losses should the portfolio fall below the stop-loss level. The stop-loss approach is a favored risk-management strategy because it prevents catastrophic losses in the event of a considerable divergence in security prices. Additionally, the stop-loss approach reduces the need to monitor the securities once the level is placed.

To the authors' knowledge, Ekström et al. (2011) provided one of the earliest works on the optimal liquidation problem in pairs trading. In their setup, Ekström et al. (2011) model the spread between two assets as the solution to an Ornstein-Uhlenbeck process. Larsson et al. (2013) generalized the same optimal liquidation problem as Ekström et al. (2011) under the assumption that the spread follows an Ornstein-Uhlenbeck process with a finite number of jumps driven by a a Lévy process. Larsson et al. (2013) proved a verification theorem for the optimal liquidation problem, designed a conceptual framework for finite-element method to analyze the free-boundary problem and extracted conclusions from numerical simulations.

The optimal liquidation problem considered in this paper is a part of a larger group of stop-loss threshold problems. Literature related to stop-loss thresholds in pairs trading using stochastic control includes the following: Lindberg (2014) considered the optimal liquidation problem under the assumption that a constant rate of growth models opportunity cost. Opportunity cost is defined as the value of the optimal alternative that was not selected. The rationale for modelling opportunity cost is that an investor will always compare her pairs position relative to an opportunity set. Song and Zhang (2013) considered a pairs trading strategy using optimal control by incorporating the stop-loss level as a state constraint and showed that the optimal pairs trading under an Ornstein-Uhlenbeck process could be determined by threshold levels obtained by solving algebraic equations. Tie et al. (2017) generalized the problem posed in Song and Zhang (2013) by determining the optimal strategy to maximize a discounted payoff function with transaction costs under a Geometric Brownian Motion process. The extension to Geometric Brownian Motion was driven by the limitations of a mean-reverting model, such as choosing stocks from the same industrial sector to obtain a mean-reverting spread.

It is widely accepted that financial markets experience significant discontinuities or jumps. Jumps in the financial market occur because of the arrival of new information that has a marginal effect on the prices of the stock. Often, such information is industry-specific and will affect a sector. Since the announcement of important information only occurs during discrete periods, they are modelled by jump processes. There exists a vast amount of literature that considers the theoretical impact and empirical existence of jumps in the financial markets (see Merton (1976), Cont and Tankov (2004), Carr et al. (2002),Bates (1996), Bakshi et al. (1997)). More recently, interest has been in investigating jumps in high-frequency data. Jondeau et al. (2015) performed an empirical analysis on tick-by-tick data of twelve companies and found that jumps occur in high-frequency data. Stübinger and Endres (2018) also performed an empirical analysis on minute-by-minute intervals of the oil sector of the S&P 500 from 1998 to 2015 and remarked that modeling overnight jumps in high-frequency data produced higher returns than the standard Ornstein-Uhlenbeck model by approximately half a standard deviation. Yalaman and Manahov (2022) used five-minute high-frequency data from the Turkish stock exchange during the financial crisis of 2007-2009 and remarked that during a crisis, the proportion of jumps increased while the proportion of Brownian motion decreased in the financial market. Ferriani and Zoi (2022) used five-minute high-frequency data from the S&P 500 and the Euro Stoxx 50 from September 2007 to April 2014 and remarked that significant jump clustering effects occurred during intraday time scales. For our analysis, we use five-minute interval quotes from 9:30 am to 3:55 pm from Yahoo Finance because this sampling frequency reduces micro-structural noise that would affect the estimation of model parameters.

To the authors' knowledge, there does not yet exist a closed-form solution to the optimal liquidation problem under jump-diffusion dynamics; hence we resort to numerical approximations. The non-jump components of the process are approximated using the finite difference method, while the jump components are approximated using Gauss-Hermite quadrature. To incorporate the jump components into the non-jump components, we apply the iterative scheme described by Chiarella et al. (2009) to transform an integro-differential equation into an ordinary differential equation. The iterative scheme uses the best estimate attained from the previous iteration as an initial estimate for the integral in the jump term. The result will be a system of differential equations wherein the solution serves as an estimate of the solution to the optimal liquidation problem. We enhance the existing research in the following aspects:

1. We introduce a procedure to evaluate the associated free-boundary problem for the optimal liquidation problem under the assumption that the spread is modelled as a jump-diffusion process. We develop a numerical scheme using finite differences and Gauss-Hermite quadrature and apply the scheme to the theoretical framework described in Larsson et al. (2013). Further, We present a procedure that can be applied empirically by describing a method to obtain a spread satisfying the model dynamics, obtaining the model parameters, and presenting pseudo code for the numerical scheme. Additionally, whereas Larsson et al. (2013) used computational simulations to draw conclusions from the model, we evaluate the optimal liquidation problem empirically. We consider the optimal liquidation strategy for two cointegrated stocks AAPL and NVDA from October 10, 2022 to October 28, 2022 using a sampling frequency of five-minutes and describing the results.

2. We constructed a pairs trading framework that can capture mean-reversion and jumps in the context of high-frequency data. In our methodology, we consider the effects of jump during the night; however, our model can perform on intraday jumps by using a smaller sampling frequency. To the authors' knowledge, there are limited academic studies that consider statistical arbitrage that allow jumps. Larsson et al. (2013) formulated an abstraction of the optimal stopping problem. Göncü and Akyildirim (2016) introduced a stochastic model for daily commodity pairs trading where a Levy process drives the noise term. Stübinger and Endres (2018) presented a Bollinger Band exit strategy under the assumption that the spread contains overnight jumps. We build upon Stübinger and Endres (2018) by developing a high-frequency pairs trading strategy based on the optimal stopping problem in Larsson et al. (2013). Furthermore, we use five-minute sampling intervals instead of minute-by-minute sampling intervals to reduce microstructural noise and estimate jump intensity using realized variance and realized bipower variation based on Da Fonseca and Ignatieva (2019) rather than using jump thresholds calculated using quantiles.

The rest of this paper is organized as follows. In section 2, we present the model and formulate the optimal liquidation problem. In section 3, we discuss a numerical scheme to evaluate the free-boundary problem. In section 4, we describe a procedure to approximate parameter values for the model. In section 5, we present an empirical example of the optimal liquidation problem. Section 6 concludes this paper.

II The Model

We will consider a Markov process $U = (U_t)_{t\geq 0}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. It is assumed that the sample paths of U are right-continuous and left-continuous over stopping times. It is also assumed that the filtration $\mathcal{F} = (\mathbb{F})_{t\geq 0}$ is right continuous and

$\mathbb{P}$-complete. The term cointegration was first coined by Engle and Granger (1987). The main idea of cointegration is as follows: let y_t and x_t be two nonstationary time series. If there exists a cointegration factor γ such that $y_t - \gamma x_t$ is a stationary time series, then y_t and x_t are said to be cointegrated. Cointegration describes a long-term mean reverting behavior between two stock prices and reflects the similarity of assets in terms of risk-exposure profiles. Since pairs trading is a strategy that attempts to make a profit from deviations from a long-run mean, it follows that cointegrated stocks are a natural candidate for pairs trading. We model the spread between two cointegrated stocks $S^1(t)$ and $S^2(t)$ to have mean-reverting properties. Let $W = \{W_t\}_{t\geq 0}$ be a standard Brownian motion process, $J^\lambda = \{J_t^\lambda\}_{t\geq 0}$ be a Poisson process with intensity $\lambda > 0$, and $\{q_i^\phi\}_{i=1}^\infty$ be a sequence of independent random variables with continuous Gaussian probability density function ϕ. We assume that jumps are unbounded hence ϕ is defined over $\mathbb{R}$. The compound Poisson process $N^{\lambda,\phi} = \{N^{\lambda,\phi}\}$ is defined as:

$$N^{\lambda,\phi} = \sum_{i=1}^{J_t^\lambda} q_i^\phi. \tag{1}$$

The above processes are defined such that they are independent from one another. Define the spread U_t as the stationary series resulting from $S^2 - \gamma S^1$ where S^1 and S^2 are cointegrated. We assume that the spread U_t is the unique solution to the following stochastic differential equation:

$$dU_t = -\mu U_t dt + \sigma dW_t + dN_t, \tag{2}$$

where $\mu > 0$, and $\sigma > 0$.

III Infinite Trading Horizon

In practice, The spread may deviate from the mean-reversion level of zero so an investor would set a stop-loss level in advance as a risk-management strategy. Let τ denote a stopping time with respect to $\mathcal{F}$. Let $\alpha < 0$ denote a stop-loss level. The investor must close her pairs position when $U_t < \alpha$ to prevent further losses. We define the first hitting time on the region $(-\infty, a)$ as

$$\tau_\alpha = \inf\{t \geq 0 : U_t < \alpha\}. \tag{3}$$

The corresponding value function is given by

$$V(u) = \sup_\tau \mathbb{E}_x[U_{\tau_\alpha \wedge \tau}], \tag{4}$$

where the supremum is taken over all stopping times τ and $\mathbb{E}_x$ denotes the expected value with respect to the initial condition $U_0 = x$. The value function can be interpreted as the maximization of the expected value of U where U is a function of the first hitting time of the stop-loss level τ_α and stopping times τ. A key assumption in pairs trading is that the underlying process exhibits mean-reverting behavior. However, a prudent investor should also take note of the drift of the underlying process. Should the drift sink below a certain level, any possible gains from random deviations from the mean would be overshadowed by the losses from the drift. More formally, as long as $\mu \leq 0$, the investor has no reason to liquidate her position since the drift is positive. However, when $\mu > 0$, the drift is working against the investor and there exists a price where an investor would be better off liquidating her position. This indicates that there exists another stopping barrier $\beta > 0$ such that the investor should close her pairs position when $U_t > \beta$. In summary, the investor should maintain her position while U_t is within the open interval (α, β) and close her position otherwise.

The value function is given by $u(x) := V(u)$, where (u, β) is the solution of the following free-boundary problem (see VI):

$$\begin{aligned} \mathcal{L}u(x) &= 0, \quad x \in (\alpha, \beta), \\ u(x) &= x, \quad x \notin (\alpha, \beta), \\ u'(\beta) &= 1, \end{aligned} \tag{5}$$

where, $\mathcal{L}$ is the infinitesimal generator of U, which is defined as:

$$\mathcal{L}f(x) = -\mu x f'(x) + \frac{1}{2}\sigma^2 f''(x) + \lambda \int_{-\infty}^{\infty} [f(x+y) - f(x)]\phi(y)dy, \quad x \in \mathbb{R}. \tag{6}$$

The stopping time when the supremum of the value function (4) is obtained must be

$$\tau_\beta = \inf\{t \geq 0 : U_t > \beta\}. \tag{7}$$

Moreover, Larsson et al. (2013) presented the proof that if (u, β) is a classical solution of (5) with

1. $\mathcal{L}u(x) \leq 0$ for $x > \beta$
2. $u(x) \geq x$ for $x \in \mathbb{R}$

then, $u(x) = V(u) = \mathbb{E}_x[U_{\tau_\alpha \wedge \tau_\beta}]$ for $x \in \mathbb{R}$ where V is given in (4). That is, u is the the expectation with respect to both stopping time thresholds α and β conditional on the initial condition $U_0 = x$.

III.I Numerical Solution to the Free-Boundary Problem

We derive a numerical method to evaluate the free-boundary problem. We use finite differences to approximate the non-jump components of the integro differential equation and approximate the jump (integral) component using Gauss-Hermite quadrature. Finite differences involve the discretization of the price interval into a finite number of steps and approximating the value of the solution at these discrete points. Gauss-Hermite quadrature is an approximation method used for integrals that are Gaussian in nature. The quadrature method decomposes the integral into a sum of discrete sample points and weights. Further, we apply the iterative scheme from

Chiarella et al. (2009) to incorporate the jump components into the non-jump components to evaluate the free-boundary problem.

We start by transforming the free-boundary problem in (5) to a problem with homogeneous boundary conditions. We apply the transformations $v(x) = u(x) - x$ and $\int_{-\infty}^{\infty} y\phi(y)dy = 0$

$$\begin{aligned} -\frac{1}{2}\sigma^2 v''(x) + \mu x v'(x) \\ -\lambda \int_{-\infty}^{\infty} [v(x+y) - v(x)]\phi(y)dy = -\mu x, \quad & x \in (\alpha, \beta), \\ v(x) = 0, \qquad & x \notin (\alpha, \beta), \\ v'(\beta) = 0. \end{aligned} \tag{8}$$

Our approach is to solve the free-boundary value problem

$$\begin{aligned} \mathcal{L}v = f, \quad & x \in (\alpha, \beta) \\ v(x) = 0 \quad & x \notin (\alpha, \beta), \end{aligned} \tag{9}$$

where $f(x) = -\mu x$ and then fix $\alpha < 0$ to find $\beta > \alpha$ such that $v'(\beta) = 0$. We present a numerical scheme for $\mathcal{L}v$. We begin by decomposing $\mathcal{L}v$ into two parts:

$$\mathcal{L}v = \mathbb{D}v - \lambda \mathbb{J}v, \tag{10}$$

where the non-jump component is defined as:

$$\mathbb{D}v := \mu x v'(x) - \frac{1}{2}\sigma^2 v''(x), \tag{11}$$

and the jump component is defined as:

$$\mathbb{J}v := \int_{-\infty}^{\infty} [v(x+y) - v(x)]\phi(y)dy \tag{12}$$

We consider a spatial discretization of the spread price such that

$$x_{min} := x_0 < x_i < \cdots < x_{m-1} < x_m =: x_{max}, \tag{13}$$

where $\Delta x = x_i - x_{i-1}$ for all $i = 1, \ldots, m$. The first-order derivative of v is approximated using the backward difference formula and the second-order derivative of v is approximated using the central difference formula,

$$\begin{aligned} v'(x_i) &\approx \frac{v(x_i) - v(x_{i-1})}{\Delta x} \\ v''(x_i) &\approx \frac{v(x_{i+1}) - 2v(x_i) + v(x_{i-1})}{(\Delta x)^2}. \end{aligned} \tag{14}$$

A finite-difference approximation of $\mathbb{D}v$ is

$$\mathbb{D}v \approx \mu x \frac{v(x_i) - v(x_{i-1})}{\Delta x} - \frac{1}{2}\sigma^2 \frac{v(x_{i+1}) - 2v(x_i) + v(x_{i-1})}{(\Delta x)^2}. \tag{15}$$

Furthermore, we have the jump term $\mathbb{J}v$ that is governed by the Gaussian probability density function ϕ that is defined over $\mathbb{R}$. We use cubic splines to approximate $v(x_i+y_j)$ for $j = 1, 2, \ldots, l$ from known values of $v(x_i)$ for each $i = 1, 2, \ldots, m$. Here, y_j denotes possible jump values drawn from the probability density function ϕ. An approximation for the integral term is

$$\int_{-\infty}^{\infty} v(x_i + y_j)\phi(y)\,\mathrm{d}y \approx \sum_{j=1}^{L} w_j v(x_i + y_j), \tag{16}$$

where w_i and y_j are the weights and abscissas of the Gauss-Hermite Quadrature scheme with L integration points. Note that $\int_{-\infty}^{\infty} v(x_i)\phi(y)dy = v(x_i)$ and so the the resulting approximation for $\mathbb{J}v$ is

$$\mathbb{J}v \approx \sum_{j=1}^{L} \left[w_j v(x_i + y_j)\right] - v(x_i). \tag{17}$$

The differential equation $\mathcal{L}v = f$ becomes the discrete version

$$\begin{aligned} \mu x_i \frac{v(x_i) - v(x_{i-1})}{\Delta x} - \frac{1}{2}\sigma^2 \frac{v(x_{i+1}) - 2v(x_i) + v(x_{i-1})}{(\Delta x)^2} \\ - \lambda \left[\sum_{j=1}^{L} \left[w_j v(x_i + y_j)\right] - v(x_i) \right] = -\mu x_i. \end{aligned} \tag{18}$$

Our approach involves solving a system of integro-differential equations at each price step i defined as:

$$Av(x_i) = f \tag{19}$$

where $A \in \mathbb{R}^{(m-1)\times(m-1)}$ is a constant sparse tridiagonal martix. Since the non-jump and the jump components need to be evaluated simultaneously, we need to incorporate the jump components into the non-jump components. Let k denote the index of the current iteration and $v^{(k)}$ be the corresponding solution to the system. Following Chiarella et al. (2009), we treat the integro-differential equations as ordinary differential equations by using $v^{(k-1)}$ as an initial approximation for $v^{(k)}$ in the integral term $\mathbb{J}^{(k)}v$. We then solve the ordinary differential equation for increasing iterations until v converges to the desired level of accuracy. We begin by decomposing $\mathbb{D}v$ as follows:

$$\mathbb{D}v \approx v(x_{i-1}) \left[-\frac{\mu x_i}{\Delta x} - \frac{\sigma^2}{2(\Delta x)^2} \right] + v(x_i) \left[\frac{\mu x_i}{\Delta x} + \frac{\sigma^2}{(\Delta x)^2} \right] + v(x_{i+1}) \left[\frac{-\sigma^2}{2(\Delta x)^2} \right]. \tag{20}$$

Define the abbreviations ω_i and ψ as follows,

$$\omega_i := \frac{\mu x_i}{\Delta x}, \quad \psi := \frac{\sigma^2}{2(\Delta x)^2}. \tag{21}$$

Define the matrix A:

$$A := \begin{bmatrix} \omega_i + 2\psi & -\psi & & 0 \\ -\omega_i - \psi & \ddots & \ddots & \\ & \ddots & \ddots & -\psi \\ 0 & & -\omega_{M-1} - \psi & \omega_{M-1} + 2\psi \end{bmatrix} \in \mathbb{R}^{(M-1)\times(M-1)}. \tag{22}$$

We begin the procedure by initializing $v^{(0)} = [0, 0, \dots, 0]' \in \mathbb{R}^{(M-1)}$, where $'$ denotes the transpose of a matrix and solving the first iteration of the system:

$$Av^{(0)} = f^{(0)}. \tag{23}$$

Next, we apply the Gauss-Hermite quadrature using values from the $(k-1)$'st iteration to obtain an estimate for $v^{(k-1)}(x_i)$. The resulting value for $v^{(k-1)}(x_i)$ serves as an initial estimate for the integral term in the k'th iteration. Define an estimate for the integral term at the k'th iteration $\mathbb{J}^{(k)}v$ as

$$\mathbb{J}^{(k)}_{\mathbb{E}} v := \sum_{j=1}^{L} \left[w_j^{(k-1)} v^{(k-1)}(x_i + y_j) \right] - v(x_i). \tag{24}$$

where $w_j^{(k-1)} v^{(k-1)}(x_i + y_j)$ denote Gauss-Hermite approximations for the integral term $\mathbb{J}v$ using values from the $(k-1)$'st iteration. Afterward, we incorporate the integral term into f and denote this as $f^{(k)}$,

$$f^{(k)} = f^{(k-1)} + \lambda \mathbb{J}^{(k)}_{\mathbb{E}} v. \tag{25}$$

Lastly, we solve the system

$$Av^{(k)} = f^{(k)}, \tag{26}$$

for an increasing number of iterations until v converges to a level of accuracy $\epsilon < 1e^{-5}$. We present a core algorithm below:

Algorithm 1: Prototype Core Algorithm (Infinite Horizon)

Result: A vector of the unobserved process v from the differential equation $\mathcal{L}v = f$

Set $v = 0$ to be a zero vector that has the same length as the spread;

while $\epsilon > 1e^{-5}$ **do**

- **for** $k > 1$ **do**
  - Interpolate a polynomial for $v^{(k-1)}(x_i + y_j)$ using cubic splines;
  - Apply Gauss-Hermite quadrature on the interpolated polynomial from the previous iteration $v^{(k-1)}(x_i + y_j)$ to obtain an approximation for the integral term $\mathbb{J}^{(k)}_{\mathbb{E}} v$;
  - Incorporate the jumps into f through $f^{(k)} = f^{(k-1)} + \lambda \mathbb{J}^{(k)}_{\mathbb{E}} v$;
  - Solve for $v^{(k)}$ in the system $Av^{(k)} = f^{(k)}$;
  - $k = k + 1$;
- $\epsilon = max(v^{(k)} - v^{(k-1)})$;

Now that we have v, we solve for β using the terminal condition $v'(\beta) = 0$. We can identify when $v'(x_i)$ is zero and fit a piecewise-third-order polynomial through the three points v'_1, v'_2, v'_3 around where $v'(x_i)$ is machine precision error level to zero. Afterward, we evaluate the root of the polynomial using the *uniroot* function in R (see Brent (1972)) and the root serves as the index for an approximation of β.

IV Parameter Estimation for Model

Since empirical evidence points to the existence of jumps in financial markets (see Göncü and Akyildirim (2016), Stübinger and Endres (2018), Jondeau et al. (2015)), the normality assumption in the classic Ornstein-Uhlenbeck process is a deficiency. To proceed with our analysis, we need to obtain estimate parameter values for our model. We start by describing the procedure to obtain parameters related to the jump term then proceed to the procedure to obtain the non-jump term parameters. The procedure for the jump detection scheme is based on Barndorff-Nielsen and Shephard (2004) and involves decomposing the realized variance into a jump component and a continuous component through the the difference between the realized variance and realized bipower variation. Afterward, we use Maximum Likelihood Estimation for the non-jump component parameters. The first step is to identify the days when a jump occurs and the corresponding magnitude of the jump. We follow the daily jump detection scheme outlined by Da Fonseca and Ignatieva (2019). Let y_t denote the spread process at day t. Define

$$r_{t,i} = y_{t,i\Delta} - y_{t,(i-1)\Delta}, \tag{27}$$

where $r_{t,i}$ is the i'th intra-day return on day t and Δ is the sampling frequency for each day. Let $m = \frac{1}{\Delta}$ be the number of observations each day. The jump volatility JV_t is calculated as:

$$JV_t = QV_t - IV_t, \tag{28}$$

where QV_t is the quadratic variation and IV_t is the integrated variance. Barndorff-Nielsen and Shephard (2004) demonstrated that as the sampling frequency gets larger ($m \to \infty$), the realized bipower variation estimates the integrated variance and that the difference between the realized variance and realized bipower variation estimates the quadratic variation. The realized variance RV_t is calculated as the sum of the squared returns at the selected frequency.

$$RV_t = \sum_{i=1}^{m} r_{t,i}^2, \tag{29}$$

The next estimator is the realized bipower variation BV_t defined as

$$BV_t = \frac{\pi}{2} \frac{m}{m-1} \sum_{i=2}^{m} \left| r_{t,i} \right| \left| r_{t,i-1} \right|, \tag{30}$$

We follow Da Fonseca and Ignatieva (2019), Andersen et al. (2012), Huang and Tauchen (2005) and use the test statistic:

$$RJ_t = \frac{RV_t - BV_t}{RV_t}, \tag{31}$$

measures the contributions of jumps to the total intra-day variance of the spread. The test statistic RJ_t converges to the standard normal distribution when the following scaling is applied:

$$ZJ_t = \frac{RJ_t}{\sqrt{\left\{\left(\frac{\pi}{2}\right)^2 + \pi - 5\right\} \Delta \max\left(1, \frac{TP_t}{BV_t^2}\right)}}, \tag{32}$$

where the tripower quarticity TP_t that is robust to jumps is defined as in Barndorff-Nielsen and Shephard (2004):

$$TP_t = m\mu_{4/3}^{-3} \frac{m}{m-2} \sum_{i=3}^{m} \left|r_{t,i-2}\right|^{4/3} \left|r_{t,i-1}\right|^{4/3} \left|r_{t,i}\right|^{4/3}, \tag{33}$$

and

$$\mu_{4/3} = 2^{2/3} \frac{\Gamma((7/6)))}{\Gamma(1/2)}. \tag{34}$$

We assume that jumps are infrequent and only occur overnight. That is, either a jump occured overnight affecting the opening price the next day or that no jump occured overnight. However, the model can be extended to include intra-day jumps by using a more granular sampling frequency. The magnitude of the jumps can be obtained using the formula

$$J_t = sign(r_t) \times \sqrt{(RV - BV) \times I_{ZJ \geq \Phi_\alpha^{-1}}} \tag{35}$$

where $\Phi^{-1}(.)$ is the inverse normal cumulative distribution function, α is the level of significance and $I_{ZJ \geq \Phi_\alpha^{-1}}$ is an indicator function that takes on a value of 1 if the jump detected is statistically significant at the level α and 0 otherwise. The jump intensity parameter λ can be calculated using

$$\lambda = \frac{\text{days when jumps occured}}{\text{total number of days}}. \tag{36}$$

Now that we have the magnitude of the jumps and the days when they occur, we remove the jumps from the spread and the process is reduced to a standard Ornstein-Uhlenbeck process. We use maximum likelihood estimation to get the values of μ, and σ. This was done using the language R and applying the function mle to the model.

V An Empirical Example

We implemented the numerical scheme outlined in this paper. We model the spread as a mean-reverting jump diffusion process. First, we describe a procedure using z-score normalization to obtain a zero mean reversion level. Next, we present the resulting values from the parameter estimation procedure outlined in section 4. Lastly, we evaluate the optimal liquidation problem using the procedure in section 3.

In this paper, we consider two American techonology companies for the period October 10, 2022 to October 28, 2022, which includes 15 trading days. The first stock is for Apple Inc. (AAPL) and the second stock is for Nvidia Corporation (NVDA). We use Yahoo Finance to obtain the stock prices from 9:30 am to 3:55 pm at five-minute intervals so each day consists of 78 observations. In total, we have 1170 observations of the spread process. Let $S^1 = AAPL$ and $S^2 = NVDA$, applying the augmented Engle-Granger two-step cointegration test results in a p-value of approximately $0.029 < 0.05$ hence we consider AAPL and NVDA to be cointegrated. We apply ordinary least squares regression to fit the cointegration factor γ and apply the following transformation to obtain a mean-reverting spread.

$$s = S^2 - \gamma * S^1. \tag{37}$$

Next, we apply z-score normalization on the spread to get a mean-reverting spread with mean zero. The following formula is used on each observation of the spread:

$$x_i = \frac{s_i - \mu}{\sigma}, \tag{38}$$

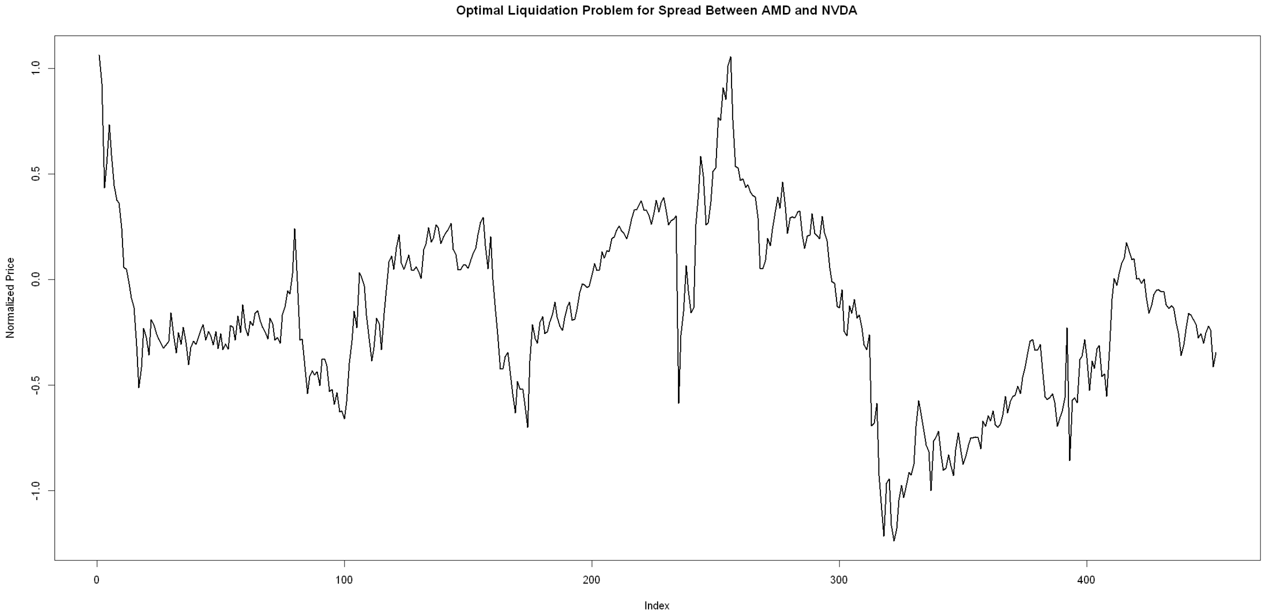
for $i = 1, 2, \ldots, 1170$. The resulting process x now has a mean-reversion level of zero. Afterward, we partition the observations of x by the day of occurrence. The partitioning was done so that we could calculate RV_t and BV_t to measure the magnitude and frequency of the jumps. For the indicator function in J_t, we use $\Phi^{-1}_{0.05}$; that is, we detect jumps at the level of significance of 5%. The jump frequency λ is measured as the fraction of the number of days when a jump occured over the total number of days considered, we measured that $\lambda \approx 0.2667$. Now that we have the magnitude and the index of the days when a jump occurs, we can remove them from x resulting in an Ornstein-Uhlenbeck process that can be estimated using maximum likelihood estimation. The resulting parameter estimates are $\mu \approx 0.2323$ and $\sigma \approx 0.6818$. In summary the parameter estimates are:

$$\begin{aligned} \mu &\approx 0.2323 \\ \sigma &\approx 0.6818 \\ \lambda &\approx 0.2667. \end{aligned} \tag{39}$$

We evaluate $v(x)$ using the procedure in section 3. We set $\alpha = -2$ signifying the lower stop-loss level. We remark that α had not been hit before β had been hit in our example. Afterward, Let l be the index where the v' is within machine precision to zero. We used local cubic splines interpolation to fit a piecewise-third-order polynomial on $v'(l-1), v'(l), v'(l+1)$. Lastly, we evaluate the root of the cubic spline interpolation to get the index of the root and obtain an estimate for beta.

$$\beta \approx -0.3448. \tag{40}$$

We remark that the index of the optimal liquidation points is at the 452'nd observation corresponding to a price on the 6th trading day considered in this example. The optimal liquidation point is the last observation in the plot below:



VI Conclusion

In this paper, we introduce a procedure to evaluate the free-boundary problem for the optimal liquidation problem under the assumption that the spread is modelled as a jump-diffusion process. We resort to numerical methods using the finite-difference method for the non-jump components and Gauss-Hermite quadrature for the jump-components. We apply an iterative scheme to incorporate the jump component into the non-jump components using the previous iterations as an initial estimate for the integral in the jump term. Additionally, we outline an estimation procedure for the model parameters based on the realized variance and the realized bipower variation. Lastly, we deploy the procedure outlined in this paper on five-minute intervals of the spread between two cointegrated American technology companies.

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Supplemental Materials: A Numerical Scheme for the Optimal Liquidation Problem Under Jump Diffusion Dynamics on High-Frequency Data

VII Derivation of the solution to the Value function

This appendix shows a derivation of the solution to the value function. Consider a strong Markov process $U = (U_t)_{t\geq 0}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{P})$ taking values on $E = \mathbb{R}^d$ for some $d \geq 1$. Assume that U is continuous and takes on values in $\mathbb{R}$. Assume we are given an open bounded set $C = (\alpha, \beta) \subseteq E$, consider:

$$\tau_D = \inf\{t \geq 0 : U_t \in D\}, \tag{S1}$$

where $D = C^c = (\alpha, \beta)^c$. The goal is to determine an integro-differential equation solved by

$$V(u) = \sup_{\tau} \mathbb{E}_u \left[U_{\tau_\alpha \wedge \tau}\right] \tag{S2}$$

for $u \in E$. The value function V solves the follow Neumann Problem:

$$\mathcal{L}V(u) = 0, \quad u \in (\alpha, \beta) \tag{S3}$$

$$V(u) = u, \quad u \notin (\alpha, \beta) \tag{S4}$$

$$V'(\beta) = 1. \tag{S5}$$

As in Peskir and Shiryaev (2006) Chapter 3, we prove (S3). Given $u \in C$ choose an open bounded set I such that $u \in I \subseteq C$. Let the shift operator $\theta_t : \Omega \to \Omega$ be defined by $\theta_t(\omega)(s) = w(t+s)$ for $\omega \in \Omega$ with $t, s \geq 0$. Let τ denote a stopping time and let $V = V(\omega)$ be any $\mathcal{F}$-measurable bounded functional. By the strong Markov property,

$$\begin{aligned}
\mathbb{E}_u V(U_{\tau_I^c}) &= \mathbb{E}_u \left[\mathbb{E}_{U_{\tau_I^c}} V(U_{\tau_D})\right] \\
&= \mathbb{E}_u \left[\mathbb{E}_u (V(U_{\tau_D}) \circ \theta_{U_{\tau_I^c}} | \mathcal{F}_{\tau_I^c})\right] \\
&= \mathbb{E}_u \left[V\left(U_{\tau_I^c + \tau_D + \theta_{U_{\tau_I^c}}}\right)\right] \\
&= \mathbb{E}_u \left[V\left(U_{\tau_D}\right)\right] \\
&= V(u).
\end{aligned} \tag{S6}$$

Hence it follows that

$$\mathcal{L}V(u) = \lim_{I \downarrow u} \frac{\mathbb{E}_u V(U_{\tau_I^c}) - V(u)}{\mathbb{E}_u \tau_I^c} = 0. \tag{S7}$$

The condition (S4) is evident and we emphasize the critical boundary points $V(\alpha) = \alpha$ and $V(\beta) = \beta$. Further, (S5) follows from (S4).