Optimal Stopping Problem for Pairs Trading Under Jump Diffusion Dynamics

So, Kenrick Raymond[1] and Garces, Len Patrick[2]

[1]Ateneo de Manila University, Philippines

[2]University of New South Wales, Australia

Abstract

A pairs trade is a portfolio consisting of two historically correlated stocks with a long position on one stock and a short position on the other. Larsson et al. (2013) presented an abstraction for the optimal liquidation problem under jump-diffusion dynamics by presenting evidence of the uniqueness and existence of a numerical solution. This paper builds upon the work of Larsson et al. (2013) and the literature involving empirical strategies involving stop-loss thresholds. The optimal liquidation problem involves evaluating a free-boundary integro partial differential equation. We describe a numerical scheme using finite differences to estimate the non-jump terms and Gauss-Hermite quadrature to estimate the jump term. Additionally, we describe an iterative scheme that uses the previous iteration's estimation as an initial estimation for the integral term to incorporate the integral term into the diffusion terms. We outline a procedure to estimate values for the model parameters using realized variance and bipower variation. Lastly, we apply the framework in this paper in a high-frequency context by using five-minute sampling intervals of the spread between AAPL and NVDA from October 10, 2022 to October 28, 2022 to evaluate the optimal liquidation strategy.

Key words. optimal stopping, numerical quadrature, finite differences, bipower variation

I Introduction

We discuss a numerical scheme to evaluate the optimal policy for exiting a position in a pair of securities under the assumption that the spread between the two securities is modelled as a jump-diffusion process. According to Vidyamurthy (2004), Pairs trading is a strategy developed by Morgan Stanley in the 1980s. A group of traders at Morgan Stanley decided that a consistent filter of rules would perform better than relying on a trader's intuition and skills. Because of its long history, pairs trading is regarded as the precursor of more complex statistical arbitrage strategies (Vidyamurthy 2004; Avellaneda and Lee 2010). Pairs trading aims to monitor two historically correlated securities. The idea behind pairs trading involves selling a higher-priced security and buying a lower-priced security under the assumption that both securities are mispriced and will revert to the mean in the future. Vidyamurthy (2004) discussed the connection between the cause of the divergence and the ensuing convergence afterward. The spread between the two securities is defined as a portfolio. Whenever the spread between the two securities is large, both securities are highly mispriced; hence there is an immense profit potential. Elliott et al. (2005) remarked that an advantage of pairs trading is that if the two securities have similar characteristics, then the trader can find a ratio of the two securities such that the portfolio becomes market-neutral in the sense that gains or losses are not heavily affected by the direction of the market. The most citepd paper in pairs trading is Gatev et al. (2006), which yielded annualized excess returns of up to 11% and had low exposure to systematic risks. Endres (2019) provided an overview of the various kinds of stochastic differential equations that can be used to model spread dynamics. We refer the reader to Krauss (2017) for a discussion on several other types of pairs trading strategies.

A divergence risk is associated with pairs trading when the two securities cease to be mean-reverting. A practical consideration would be identifying the optimal time when an investor should exit her position in a pairs trade. The standard practice is to set a stop-loss threshold level beforehand and accept the losses should the portfolio fall below the stop-loss level. The stop-loss approach is a favored risk-management strategy because it prevents catastrophic losses in the event of a considerable divergence in security prices. Additionally, the stop-loss approach reduces the need to monitor the securities once the level is placed. To the authors' knowledge, Ekström et al. (2011) provided one of the earliest works on the optimal liquidation problem in pairs trading. In their setup, Ekström et al. (2011) model the spread between two assets as the solution to an Ornstein-Uhlenbeck process. Larsson et al. (2013) generalized the same optimal liquidation problem as Ekström et al. (2011) under the assumption that the spread follows an Ornstein-Uhlenbeck process with a finite number of jumps driven by a a Lévy process. Larsson et al. (2013) proved a verification theorem for the optimal liquidation problem, designed a conceptual framework for finite-element method to analyze the free-boundary problem and extracted conclusions from numerical simulations.

The optimal liquidation problem considered in this paper is a part of a larger group of stop-loss threshold problems. Literature related to stop-loss thresholds in pairs trading using stochastic control includes the following: Lindberg (2014) considered the optimal liquidation problem under the assumption that a constant rate of growth models opportunity cost. Opportunity cost is defined as the value of the optimal alternative that was not selected. The rationale for modelling opportunity cost is that an investor will always compare her pairs position relative to an opportunity set. Song and Zhang (2013) considered a pairs trading strategy using optimal control by incorporating the stop-loss level as a state constraint and showed that the optimal pairs trading under an Ornstein-Uhlenbeck process could be determined by threshold levels obtained by solving algebraic equations. Tie et al. (2017) generalized the problem posed in Song and Zhang (2013) by determining the optimal strategy to maximize a discounted payoff function with transaction costs under a Geometric Brownian Motion process. The extension to Geometric Brownian Motion was driven by the limitations of a mean-reverting model, such as choosing stocks from the same industrial sector to obtain a mean-reverting spread.

It is widely accepted that financial markets experience significant discontinuities or jumps. Jumps in the financial market occur because of the arrival of new information that has a marginal effect on the prices of the stock. Often, such information is industry-specific and will affect a sector. Since the announcement of important information only occurs during discrete periods, they are modelled by jump processes. There exists a vast amount of literature that considers the theoretical impact and empirical existence of jumps in the financial markets (Bakshi et al. 1997; Bates 1996; Carr et al. 2002; Cont and Tankov 2004; Merton 1976). More recently, interest has been in investigating jumps in high-frequency data. Jondeau et al. (2015) performed an empirical analysis on tick-by-tick data of twelve companies and found that jumps occur in high-frequency data. Stübinger and Endres (2018) also performed an empirical analysis on minute-by-minute intervals of the oil sector of the S&P 500 from 1998 to 2015 and remarked that modeling overnight jumps in high-frequency data produced higher returns than the standard Ornstein-Uhlenbeck model by approximately half a standard deviation. Yalaman and Manahov (2022) used five-minute high-frequency data from the Turkish stock exchange during the financial crisis of 2007-2009 and remarked that during a crisis, the proportion of jumps increased while the proportion of Brownian motion decreased in the financial market. Ferriani and Zoi (2022) used five-minute high-frequency data from the S&P 500 and the Euro Stoxx 50 from September 2007 to April 2014 and remarked that significant jump clustering effects occurred during intraday time scales. For our analysis, we use five-minute interval quotes from 9:30 am to 3:55 pm from Yahoo Finance because this sampling frequency reduces micro-structural noise that would affect the estimation of model parameters.

There does not yet exist a closed-form solution to the optimal liquidation problem under jump-diffusion dynamics; hence we resort to numerical approximations. The non-jump components of the process are approximated using the finite difference method, while the jump components are approximated using Gauss-Hermite quadrature. To incorporate the jump components into the non-jump components, we apply the iterative scheme described by Chiarella et al. (2009) to transform an integro-differential equation into an ordinary differential equation. The iterative scheme uses the best estimate attained from the previous iteration as an initial estimate for the integral in the jump term. The result will be a system of differential equations wherein the solution serves as an estimate of the solution to the optimal liquidation problem. We enhance the existing research in the following aspects:

1. We introduce a procedure to evaluate the associated free-boundary problem for the optimal liquidation problem under the assumption that the spread is modelled as a jump-diffusion process. We develop a numerical scheme using finite differences and Gauss-Hermite quadrature and apply the scheme to the theoretical framework described in Larsson et al. (2013). Further, We present a procedure that can be applied empirically

by describing a method to obtain a spread satisfying the model dynamics, obtaining the model parameters, and presenting pseudo code for the numerical scheme. Additionally, whereas Larsson et al. (2013) used computational simulations to draw conclusions from the model, we evaluate the optimal liquidation problem empirically. We consider the optimal liquidation strategy for two cointegrated stocks AAPL and NVDA from October 10, 2022 to October 28, 2022 using a sampling frequency of five-minutes and describing the results.

2. We constructed a pairs trading framework that can capture mean-reversion and jumps in the context of high-frequency data. In our methodology, we consider the effects of jump during the night; however, our model can perform on intraday jumps by using a smaller sampling frequency. To the authors' knowledge, there are limited academic studies that consider statistical arbitrage that allow jumps. Larsson et al. (2013) formulated an abstraction of the optimal stopping problem. Göncü and Akyildirim (2016) introduced a stochastic model for daily commodity pairs trading where a Levy process drives the noise term. Stübinger and Endres (2018) presented a Bollinger Band exit strategy under the assumption that the spread contains overnight jumps. We build upon Stübinger and Endres (2018) by developing a high-frequency pairs trading strategy based on the optimal stopping problem in Larsson et al. (2013). Furthermore, we use five-minute sampling intervals instead of minute-by-minute sampling intervals to reduce microstructural noise and estimate jump intensity using realized variance and realized bipower variation based on Da Fonseca and Ignatieva (2019) rather than using jump thresholds calculated using quantiles.

The rest of this paper is organized as follows. In section 2, we present the model and formulate the optimal liquidation problem. In section 3, we discuss a numerical scheme to evaluate the free-boundary problem. In section 4, we describe a procedure to approximate parameter values for the model. In section 5, we present an empirical example of the optimal liquidation problem. Section 6 concludes this paper.

II The Model

We consider a Markov process $U = (U_t)_{t\geq 0}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. It is assumed that the sample paths of U are right-continuous and left-continuous over stopping times. It is also assumed that the filtration $\mathcal{F} = (\mathbb{F})_{t\geq 0}$ is right continuous and $\mathbb{P}$-complete. The term cointegration was first coined by Engle and Granger (1987). The main idea of cointegration is as follows: let y_t and x_t be two nonstationary time series. If there exists a cointegration factor γ such that $y_t - \gamma x_t$ is a stationary time series, then y_t and x_t are said to be cointegrated. Cointegration describes a long-term mean reverting behavior between two stock prices and reflects the similarity of assets in terms of risk-exposure profiles. Since pairs trading is a strategy that attempts to make a profit from deviations from a long-run mean, it follows that cointegrated stocks are a natural candidate for pairs trading. We model the spread between two cointegrated stocks $S^1(t)$ and $S^2(t)$ to have mean-reverting properties. Let $W = \{W_t\}_{t\geq 0}$ be a standard Brownian motion process, $J^\lambda = \{J_t^\lambda\}_{t\geq 0}$ be a Poisson process with intensity $\lambda > 0$, and $\{q_i^\phi\}_{i=1}^\infty$ be a sequence of independent random variables with continuous Gaussian probability density function ϕ. We assume that jumps are unbounded hence ϕ is defined over $\mathbb{R}$. The compound Poisson process $N^{\lambda,\phi} = \{N^{\lambda,\phi}\}$ is defined as:

$$N^{\lambda,\phi} = \sum_{i=1}^{J_t^\lambda} q_i^\phi. \tag{1}$$

The above processes are defined such that they are independent from one another. Define the spread U_t as the stationary series resulting from $S^2 - \gamma S^1$ where S^1 and S^2 are cointegrated. We assume that the spread U_t is the unique solution to the following stochastic differential equation:

$$dU_t = -\mu U_t dt + \sigma dW_t + dN_t, \tag{2}$$

where $\mu > 0$, and $\sigma > 0$. In practice, The spread may deviate from the mean-reversion level of zero so an investor would set a stop-loss level in advance as a risk-management strategy. Let τ denote a stopping time with respect to $\mathcal{F}$. Let $\alpha < 0$ denote a stop-loss level. The investor must close her pairs position when $U_t < \alpha$ to prevent further losses. We define the first hitting time on the region $(-\infty, a)$ as

$$\tau_\alpha = \inf\{t \geq 0 : U_t < \alpha\}. \tag{3}$$

The corresponding value function is given by

$$V(u) = \sup_\tau \mathbb{E}_x[U_{\tau_\alpha \wedge \tau}], \tag{4}$$

where the supremum is taken over all stopping times τ and $\mathbb{E}_x$ denotes the expected value with respect to the initial condition $U_0 = x$. The value function can be interpreted as the maximization of the expected value of U where U is a function of the first hitting time of the stop-loss level τ_α and stopping times τ. A key assumption in pairs trading is that the underlying process exhibits mean-reverting behavior. However, a prudent investor should also take note of the drift of the underlying process. Should the drift sink below a certain level, any possible gains from random deviations from the mean would be overshadowed by the losses from the drift. More formally, as long as $\mu \leq 0$, the investor has no reason to liquidate her position since the drift is positive. However, when $\mu > 0$, the drift is working against the investor and there exists a price where an investor would be better off liquidating her position. This indicates that there exists another stopping barrier $\beta > 0$ such that the investor should close her pairs position when $U_t > \beta$. In summary, the investor should maintain her position while U_t is within the open interval (α, β) and close her position otherwise.

The value function is given by $u(x) := V(u)$, where (u, β) is the solution of the following free-boundary problem:

$$\begin{aligned} \mathcal{L}u(x) &= 0, \quad x \in (\alpha, \beta), \\ u(x) &= x, \quad x \notin (\alpha, \beta), \\ u'(\beta) &= 1, \end{aligned} \tag{5}$$

where, $\mathcal{L}$ is the infinitesimal generator of U, which is defined as:

$$\mathcal{L}f(x) = -\mu x f'(x) + \frac{1}{2}\sigma^2 f''(x) + \lambda \int_{-\infty}^{\infty} [f(x+y) - f(x)]\phi(y)dy, \quad x \in \mathbb{R}. \tag{6}$$

The stopping time when the supremum of the value function (4) is obtained must be

$$\tau_\beta = \inf\{t \geq 0 : U_t > \beta\}. \tag{7}$$

Moreover, Larsson et al. (2013) presented the proof that if (u, β) is a classical solution of (5) with

1. $\mathcal{L}u(x) \leq 0$ for $x > \beta$
2. $u(x) \geq x$ for $x \in \mathbb{R}$

then, $u(x) = V(u) = \mathbb{E}_x[U_{\tau_\alpha \wedge \tau_\beta}]$ for $x \in \mathbb{R}$ where V is given in (4). That is, u is the the expectation with respect to both stopping time thresholds α and β conditional on the initial condition $U_0 = x$.

III Numerical Solution

We derive a numerical method to evaluate the free-boundary problem. We use finite differences to approximate the non-jump components of the integro differential equation and approximate the jump (integral) component using Gauss-Hermite quadrature. Finite differences involve the discretization of the price interval into a finite number of steps and approximating the value of the solution at these discrete points. Gauss-Hermite quadrature is an approximation method used for integrals that are Gaussian in nature. The quadrature method decomposes the integral into a sum of discrete sample points and weights. Further, we apply the iterative scheme from Chiarella et al. (2009) to incorporate the jump components into the non-jump components to evaluate the free-boundary problem.

We start by transforming the free-boundary problem in (5) to a problem with homogeneous boundary conditions. We apply the transformations $v(x) = u(x) - x$ and $\int_{-\infty}^{\infty} y\phi(y)dy = 0$

$$\begin{aligned} -\frac{1}{2}\sigma^2 v''(x) &+ \mu x v'(x) \\ -\lambda \int_{-\infty}^{\infty} [v(x+y) - v(x))]\phi(y)dy &= -\mu x, \quad x \in (\alpha, \beta), \\ v(x) &= 0, \qquad x \notin (\alpha, \beta), \\ v'(\beta) &= 0. \end{aligned} \tag{8}$$

Our approach is to solve the free-boundary value problem

$$\begin{aligned} \mathcal{L}v &= f, \quad x \in (\alpha, \beta) \\ v(x) &= 0 \quad\ \ x \notin (\alpha, \beta), \end{aligned} \tag{9}$$

where $f(x) = -\mu x$ and then fix $\alpha < 0$ to find $\beta > \alpha$ such that $v'(\beta) = 0$. We present a numerical scheme for $\mathcal{L}v$. We begin by decomposing $\mathcal{L}v$ into two parts:

$$\mathcal{L}v = \mathbb{D}v - \lambda \mathbb{J}v, \tag{10}$$

where the non-jump component is defined as:

$$\mathbb{D}v := \mu x v'(x) - \frac{1}{2}\sigma^2 v''(x), \tag{11}$$

and the jump component is defined as:

$$\mathbb{J}v := \int_{-\infty}^{\infty} [v(x+y) - v(x))]\phi(y)dy \tag{12}$$

We consider a spatial discretization of the spread price such that

$$x_{min} := x_0 < x_i < \cdots < x_{m-1} < x_m =: x_{max}, \tag{13}$$

where $\Delta x = x_i - x_{i-1}$ for all $i = 1, \ldots, m$. The first-order derivative of v is approximated using the backward difference formula and the second-order derivative of v is approximated using the central difference formula,

$$\begin{aligned} v'(x_i) &\approx \frac{v(x_i) - v(x_{i-1})}{\Delta x} \\ v''(x_i) &\approx \frac{v(x_{i+1}) - 2v(x_i) + v(x_{i-1})}{(\Delta x)^2}. \end{aligned} \tag{14}$$

A finite-difference approximation of $\mathbb{D}v$ is

$$\mathbb{D}v \approx \mu x \frac{v(x_i) - v(x_{i-1})}{\Delta x} - \frac{1}{2}\sigma^2 \frac{v(x_{i+1}) - 2v(x_i) + v(x_{i-1})}{(\Delta x)^2}. \tag{15}$$

Furthermore, we have the jump term $\mathbb{J}v$ that is governed by the Gaussian probability density function ϕ that is defined over $\mathbb{R}$. We use cubic splines to approximate $v(x_i + y_j)$ for $j = 1, 2, \ldots, l$ from known values of $v(x_i)$ for each $i = 1, 2, \ldots, m$. Here, y_j denotes possible jump values drawn from the probability density function ϕ. An approximation for the integral term is

$$\int_{-\infty}^{\infty} v(x_i + y_j)\phi(y)\,\mathrm{d}y \approx \sum_{j=1}^{L} w_j v(x_i + y_j), \tag{16}$$

where w_i and y_j are the weights and abscissas of the Gauss-Hermite Quadrature scheme with L integration points. Note that $\int_{-\infty}^{\infty} v(x_i)\phi(y)dy = v(x_i)$ and so the the resulting approximation for $\mathbb{J}v$ is

$$\mathbb{J}v \approx \sum_{j=1}^{L} \left[w_j v(x_i + y_j) \right] - v(x_i). \tag{17}$$

The differential equation $\mathcal{L}v = f$ becomes the discrete version

$$\begin{aligned} \mu x_i \frac{v(x_i) - v(x_{i-1})}{\Delta x} - \frac{1}{2}\sigma^2 \frac{v(x_{i+1}) - 2v(x_i) + v(x_{i-1})}{(\Delta x)^2} \\ - \lambda \left[\sum_{j=1}^{L} \left[w_j v(x_i + y_j) \right] - v(x_i) \right] = -\mu x_i. \end{aligned} \tag{18}$$

Our approach involves solving a system of integro-differential equations at each price step i defined as:

$$Av(x_i) = f \tag{19}$$

where $A \in \mathbb{R}^{(m-1)\times(m-1)}$ is a constant sparse tridiagonal martix. Since the non-jump and the jump components need to be evaluated simultaneously, we need to incorporate the jump components into the non-jump components. Let k denote the index of the current iteration and $v^{(k)}$ be the corresponding solution to the system. Following Chiarella et al. (2009), we treat the integro-differential equations as ordinary differential equations by using $v^{(k-1)}$ as an initial approximation for $v^{(k)}$ in the integral term $\mathbb{J}^{(k)}v$. We then solve the ordinary differential equation for increasing iterations until v converges to the desired level of accuracy. We begin by decomposing $\mathbb{D}v$ as follows:

$$\mathbb{D}v \approx v(x_{i-1}) \left[-\frac{\mu x_i}{\Delta x} - \frac{\sigma^2}{2(\Delta x)^2} \right] + v(x_i) \left[\frac{\mu x_i}{\Delta x} + \frac{\sigma^2}{(\Delta x)^2} \right] + v(x_{i+1}) \left[\frac{-\sigma^2}{2(\Delta x)^2} \right]. \tag{20}$$

Define the abbreviations ω_i and ψ as follows,

$$\omega_i := \frac{\mu x_i}{\Delta x}, \quad \psi := \frac{\sigma^2}{2(\Delta x)^2}. \tag{21}$$

Define the matrix A:

$$A := \begin{bmatrix} \omega_i + 2\psi & -\psi & & 0 \\ -\omega_i - \psi & \ddots & \ddots & \\ & \ddots & \ddots & -\psi \\ 0 & & -\omega_{M-1} - \psi & \omega_{M-1} + 2\psi \end{bmatrix} \in \mathbb{R}^{(M-1)\times(M-1)}. \tag{22}$$

We begin the procedure by initializing $v^{(0)} = [0, 0, \ldots, 0]' \in \mathbb{R}^{(M-1)}$, where $'$ denotes the transpose of a matrix and solving the first iteration of the system:

$$Av^{(0)} = f^{(0)}. \tag{23}$$

Next, we apply the Gauss-Hermite quadrature using values from the $(k-1)$'st iteration to obtain an estimate for $v^{(k-1)}(x_i)$. The resulting value for $v^{(k-1)}(x_i)$ serves as an initial estimate for the integral term in the k'th iteration. Define an estimate for the integral term at the k'th iteration $\mathbb{J}^{(k)}v$ as

$$\mathbb{J}_{\mathbb{E}}^{(k)}v := \sum_{j=1}^{L} \left[w_j^{(k-1)} v^{(k-1)}(x_i + y_j) \right] - v(x_i). \tag{24}$$

where $w_j^{(k-1)} v^{(k-1)}(x_i+y_j)$ denote Gauss-Hermite approximations for the integral term $\mathbb{J}v$ using values from the $(k-1)$'st iteration. Afterward, we incorporate the integral term into f and denote this as $f^{(k)}$,

$$f^{(k)} = f^{(k-1)} + \lambda \mathbb{J}_{\mathbb{E}}^{(k)} v. \tag{25}$$

Lastly, we solve the system

$$Av^{(k)} = f^{(k)}, \tag{26}$$

for an increasing number of iterations until v converges to a level of accuracy $\epsilon < 1e^{-5}$. We present a core algorithm below:

Algorithm 1: Prototype Core Algorithm

Result: A vector of the unobserved process v from the differential equation $\mathcal{L}v = f$

Set $v = 0$ to be a zero vector that has the same length as the spread;

while $\epsilon > 1e^{-5}$ **do**

- **for** $k > 1$ **do**
  - Interpolate a polynomial for $v^{(k-1)}(x_i + y_j)$ using cubic splines;
  - Apply Gauss-Hermite quadrature on the interpolated polynomial from the previous iteration $v^{(k-1)}(x_i + y_j)$ to obtain an approximation for the integral term $\mathbb{J}_{\mathbb{E}}^{(k)} v$;
  - Incorporate the jumps into f through $f^{(k)} = f^{(k-1)} + \lambda \mathbb{J}_{\mathbb{E}}^{(k)} v$;
  - Solve for $v^{(k)}$ in the system $Av^{(k)} = f^{(k)}$;
  - $k = k + 1$;
- $\epsilon = max(v^{(k)} - v^{(k-1)})$;

Now that we have v, we solve for β using the terminal condition $v'(\beta) = 0$. We can identify when $v'(x_i)$ is zero and fit a piecewise-third-order polynomial through the three points v'_1, v'_2, v'_3 around where $v'(x_i)$ is machine precision error level to zero. Afterward, we evaluate the root of the polynomial using the *uniroot* function in R (see (Brent 1972)) and the root serves as the index for an approximation of β.

IV Parameter Estimation for Model

Since empirical evidence points to the existence of jumps in financial markets (Stübinger and Endres 2018; Göncü and Akyildirim 2016; Jondeau et al. 2015), the normality assumption in the classic Ornstein-Uhlenbeck process is a deficiency. To proceed with our analysis, we need to obtain estimate parameter values for our model. We start by describing the procedure to obtain parameters related to the jump term then proceed to the procedure to obtain the non-jump term parameters. The procedure for the jump detection scheme is based on Barndorff-Nielsen and Shephard (2004) and involves decomposing the realized variance into a jump component and a continuous component through the the difference between the realized variance and realized bipower variation. Afterward, we use Maximum Likelihood Estimation for the non-jump component parameters. The first step is to identify the days when a jump occurs and the corresponding magnitude of the jump. We follow the daily jump detection scheme outlined by Da Fonseca and Ignatieva (2019). Let y_t denote the spread process at day t. Define

$$r_{t,i} = y_{t,i\Delta} - y_{t,(i-1)\Delta}, \tag{27}$$

where $r_{t,i}$ is the i'th intra-day return on day t and Δ is the sampling frequency for each day. Let $m = \frac{1}{\Delta}$ be the number of observations each day. The jump volatility JV_t is calculated as:

$$JV_t = QV_t - IV_t, \tag{28}$$

where QV_t is the quadratic variation and IV_t is the integrated variance. Barndorff-Nielsen and Shephard (2004) demonstrated that as the sampling frequency gets larger ($m \to \infty$), the realized bipower variation estimates the integrated variance and that the difference between the realized variance and realized bipower variation estimates the quadratic variation. The realized variance RV_t is calculated as the sum of the squared returns at the selected frequency.

$$RV_t = \sum_{i=1}^{m} r_{t,i}^2, \tag{29}$$

The next estimator is the realized bipower variation BV_t defined as

$$BV_t = \frac{\pi}{2}\frac{m}{m-1}\sum_{i=2}^{m} \left|r_{t,i}\right| \left|r_{t,i-1}\right|, \tag{30}$$

We follow Andersen et al. (2012); Da Fonseca and Ignatieva (2019); Huang and Tauchen (2005) and use the test statistic:

$$RJ_t = \frac{RV_t - BV_t}{RV_t}, \tag{31}$$

measures the contributions of jumps to the total intra-day variance of the spread. The test statistic RJ_t converges to the standard normal distribution when the following scaling is applied:

$$ZJ_t = \frac{RJ_t}{\sqrt{\left\{\left(\frac{\pi}{2}\right)^2 + \pi - 5\right\}\Delta \max\left(1, \frac{TP_t}{BV_t^2}\right)}}, \tag{32}$$

where the tripower quarticity TP_t that is robust to jumps is defined as in Barndorff-Nielsen and Shephard (2004):

$$TP_t = m\mu_{4/3}^{-3}\frac{m}{m-2}\sum_{i=3}^{m} \left|r_{t,i-2}\right|^{4/3} \left|r_{t,i-1}\right|^{4/3} \left|r_{t,i}\right|^{4/3}, \tag{33}$$

and

$$\mu_{4/3} = 2^{2/3}\frac{\Gamma((7/6)))}{\Gamma(1/2)}. \tag{34}$$

We assume that jumps are infrequent and only occur overnight. That is, either a jump occured overnight affecting the opening price the next day or that no jump occured overnight. However, the model can be extended to include intra-day jumps by using a more granular sampling frequency. The magnitude of the jumps can be obtained using the formula

$$J_t = sign(r_t) \times \sqrt{(RV - BV) \times I_{ZJ \geq \Phi_\alpha^{-1}}} \tag{35}$$

where $\Phi^{-1}(.)$ is the inverse normal cumulative distribution function, α is the level of significance and $I_{ZJ \geq \Phi_\alpha^{-1}}$ is an indicator function that takes on a value of 1 if the jump detected is statistically significant at the level α and 0 otherwise. The jump intensity parameter λ can be calculated using

$$\lambda = \frac{\text{days when jumps occured}}{\text{total number of days}}. \tag{36}$$

Now that we have the magnitude of the jumps and the days when they occur, we remove the jumps from the spread and the process is reduced to a standard Ornstein-Uhlenbeck process. We use maximum likelihood estimation to get the values of μ, and σ. This was done using the language R and applying the function mle to the model.

V An Empirical Example

We implemented the numerical scheme outlined in this paper to solve an optimal liquidation problem formulated as a free boundary problem. In this setting, the spread between two assets is modeled as a mean-reverting jump-diffusion process, capturing both continuous fluctuations and discrete jumps. We first describe a data normalization procedure using z-scores to ensure that the mean reversion level is set to zero. Subsequently, we present the parameter estimates obtained through the methodology outlined in Section 4, and finally, we evaluate the solution to the optimal stopping problem as detailed in Section 3.

In this study, we focus on two American technology companies, Apple Inc. (AAPL) and Nvidia Corporation (NVDA), considering data from the period October 10, 2022, to October 28, 2022, comprising 15 trading days. Intraday stock prices were retrieved from Yahoo Finance at five-minute intervals between 9:30 a.m. and 3:55 p.m., yielding 78 observations per trading day and a total of 1170 observations across the sample period. Letting S^1 represent the stock price of AAPL and S^2 represent that of NVDA, we apply the augmented Engle-Granger two-step cointegration test, obtaining a p-value of approximately 0.029. Since this p-value is less than the standard threshold of 0.05, we conclude that AAPL and NVDA are cointegrated, validating the spread-trading framework. To model the spread, we fit the cointegration factor γ via ordinary least squares regression and transform the spread using

$$s = S^2 - \gamma S^1.$$

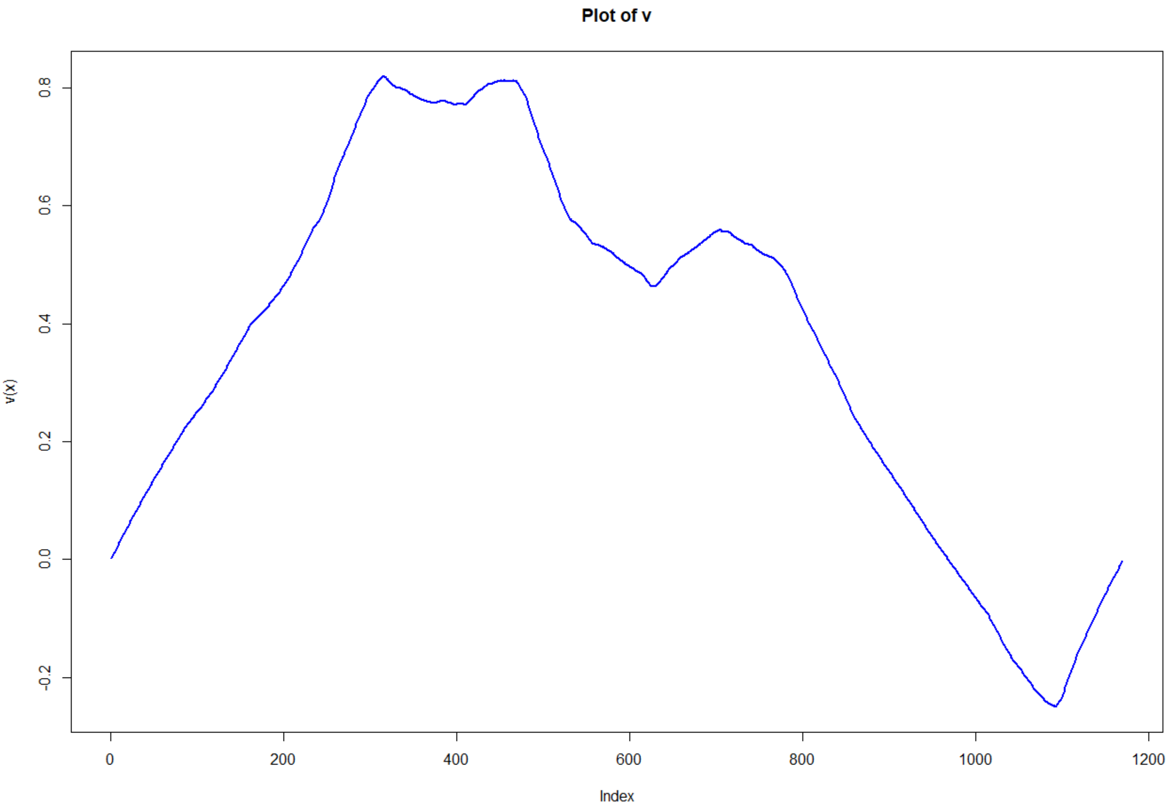

Figure 1: Plot of value function v against the number of time steps.

To ensure the mean reversion level is exactly zero, we apply z-score normalization to the spread process, using the transformation

$$x_i = \frac{s_i - \mu}{\sigma},$$

where μ and σ denote the mean and standard deviation of the spread, respectively, and $i = 1, 2, \dots, 1170$. The resulting process x exhibits mean-reverting behavior around zero, which is crucial for the stability and interpretability of the subsequent numerical analysis. The normalized spread observations are then partitioned by trading day to facilitate the calculation of realized variance (RV_t) and bipower variation (BV_t), enabling the detection of jumps within each day. Jumps are identified using a significance level of 5%, determined through the threshold $\Phi^{-1}_{0.05}$ applied to the normalized returns. The jump intensity λ is measured as the fraction of days during which at least one jump occurred, yielding an estimate of $\lambda \approx 0.2667$. After identifying and removing the jumps from x, the residual continuous component is modeled as an Ornstein-Uhlenbeck process. Using maximum likelihood estimation, we obtain parameter estimates of approximately $\mu = 0.2323$ and $\sigma = 0.6818$, leading to the following summary:

$$\mu \approx 0.2323, \quad \sigma \approx 0.6818, \quad \lambda \approx 0.2667.$$

We then proceed to evaluate the value function $v(x)$ by applying the numerical procedure described in Section 3. We set the lower stop-loss boundary at $\alpha = -2$, representing a conservative

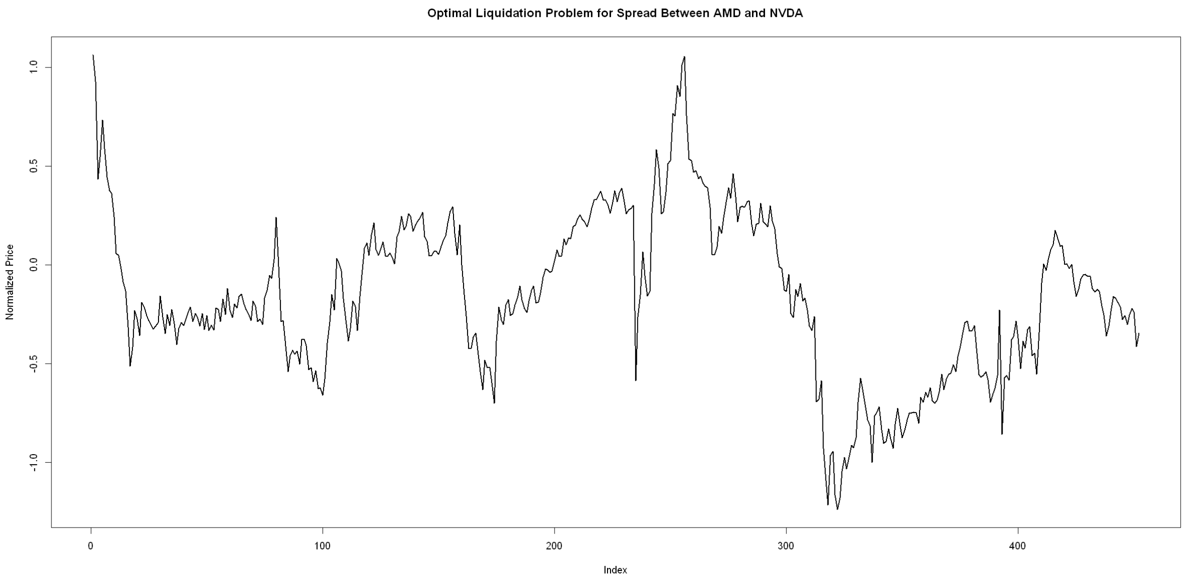

Figure 2: Path taken by the cointegrated pair up to optimal stopping price.

level beyond which losses would be considered unacceptable. Notably, in our case study, the stop-loss boundary α was never hit prior to reaching the optimal liquidation boundary β, affirming the robustness of the strategy under the given market conditions. A plot of the function is seen in Figure 1. The solution $v(x)$ exhibits a structure: initially, it grows from the origin, rising to approximately 0.4 at $x = 200$ and further to around 0.8 at $x = 300$. From $x = 300$ to $x = 400$, $v(x)$ plateaus, suggesting that the marginal benefit of continuing the trade diminishes in this region. This behavior signals proximity to the optimal stopping boundary, where the decision to liquidate becomes increasingly favorable. Beyond $x = 400$, the value function begins to decline, dropping to around 0.5 at $x = 600$ and eventually turning negative at $x = 1100$, reaching approximately -0.2. The negative values of $v(x)$ indicate that continuation beyond this point results in expected losses, affirming the presence of a critical free boundary earlier in the domain.

To determine the precise location of the optimal liquidation boundary β, we identify the index l at which the derivative $v'(x)$ approaches zero within machine precision. A local cubic spline interpolation is then fitted using the values of v' at $l-1$, l, and $l+1$, and the root of the interpolated cubic polynomial provides a refined estimate for the free boundary. Through this approach, we estimate

$$\beta \approx -0.3448.$$

This result implies that optimal liquidation should occur once the normalized spread reaches approximately -0.3448. Furthermore, we locate the index corresponding to the optimal liquidation point within the original dataset, finding it at the 452^{nd} observation. This observation corresponds to a price point on the sixth trading day within the sample period. The optimal liquidation point is visually represented as the final point in the plot of the spread process shown below: In conclusion, the behavior of the solution $v(x)$ reflects the fundamental nature of the free boundary problem: an initial region where continuation is valuable, a critical region where the incentive to continue diminishes, and a final region where immediate liquidation becomes optimal. The identified free boundary encapsulates the optimal decision rule for the trading strategy based on the underlying mean-reverting jump-diffusion dynamics of the spread.

VI Conclusion

In this paper, we developed a comprehensive approach to the optimal liquidation problem for pairs trading when the spread exhibits both mean-reversion and jump behavior. By introducing a finite difference–Gauss-Hermite quadrature scheme, we provided a practical and flexible method to solve the resulting free-boundary problem numerically. Beyond theoretical development, we proposed a fully implementable procedure, bridging the gap between model assumptions and empirical application. Our empirical study, which applied the method to high-frequency data of AAPL and NVDA, demonstrated that optimal liquidation strategies derived from jump-diffusion models can offer practical advantages over classical approaches that neglect jump risks. Furthermore, by explicitly accounting for overnight and intraday jumps in the spread dynamics, our framework extends existing pairs trading methodologies to more realistic high-frequency settings. This work shows that incorporating jump components and careful sampling frequency selection can substantially improve the robustness of statistical arbitrage strategies. Future research may explore further refinements, such as incorporating stochastic jump intensities or transaction costs, and evaluating the performance of the proposed framework across different market regimes and asset classes.

References

Andersen, T. G., Dobrev, D., and Schaumburg, E. (2012). Jump-robust volatility estimation using nearest neighbor truncation. *Journal of Econometrics*, 169(1):75–93. https://doi.org/10.1016/j.jeconom.2012.01.011.

Avellaneda, M. and Lee, J.-H. (2010). Statistical arbitrage in the us equities market. *Quantitative Finance*, 10(7):761–782. https://doi.org/10.1080/14697680903124632.

Bakshi, G., Cao, C., and Chen, Z. (1997). Empirical performance of alternative option pricing models. *The Journal of Finance*, 52(5):2003–2049. https://doi.org/10.1111/j.1540-6261.1997.tb02749.x.

Barndorff-Nielsen, O. E. and Shephard, N. (2004). Power and Bipower Variation with Stochastic Volatility and Jumps. *Journal of Financial Econometrics*, 2(1):1–37. https://doi.org/10.1093/jjfinec/nbh001.

Bates, D. S. (1996). Jumps and stochastic volatility: Exchange rate processes implicit in deutsche mark options. *The Review of Financial Studies*, 9(1):69–107. http://doi.org/10.3386/w4596.

Brent, R. P. (1972). *Algorithms for minimization without derivatives 78039*. Prentice-Hall series in automatic computation. Prentice-Hall.

Carr, P., Geman, H., Madan, D., and Yor, M. (2002). The fine structure of asset returns: An empirical investigation. *The Journal of Business*, 75(2):305–332. https://doi.org/10.1086/338705.

Chiarella, C., Kang, B., Meyer, G. H., and Ziogas, A. (2009). The evaluation of american option prices under stochastic volatility and jump-diffusion dynamics using the method of lines. *International Journal of Theoretical and Applied Finance*, 12(03):393–425. http://doi.org/10.1142/S0219024909005270.

Cont, R. and Tankov, P. (2004). *Financial modelling with jump processes*. Chapman & Hall/CRC financial mathematics series. Chapman & Hall/CRC. https://doi.org/10.1201/9780203485217.

Da Fonseca, J. and Ignatieva, K. (2019). Jump activity analysis for affine jump-diffusion models: Evidence from the commodity market. *Journal of Banking & Finance*, 99:45–62. https://doi.org/10.1016/j.jbankfin.2018.11.014.

Ekström, E., Lindberg, C., and Tysk, J. (2011). *Optimal Liquidation of a Pairs Trade*, pages 247–255. Springer Berlin Heidelberg, Berlin, Heidelberg. https://doi.org/10.1007/978-3-642-18412-3_9.

Elliott, R. J., *, J. V. D. H., and Malcolm, W. P. (2005). Pairs trading. *Quantitative Finance*, 5(3):271–276. https://doi.org/10.1080/14697680500149370.

Endres, S. (2019). Review of stochastic differential equations in statistical arbitrage pairs trading. *Managerial Economics*, 20(2):71–118. https://doi.org/10.7494/manage.2019.20.2.71.

Engle, R. F. and Granger, C. W. J. (1987). Co-integration and error correction: Representation, estimation, and testing. *Econometrica*, 55(2):251–276. https://doi.org/10.2307/1913236.

Ferriani, F. and Zoi, P. (2022). The dynamics of price jumps in the stock market: an empirical study on europe and u.s. *The European Journal of Finance*, 28(7):718–742. https://doi.org/10.1080/1351847X.2020.1740288.

Gatev, E., Goetzmann, W. N., and Rouwenhorst, K. G. (2006). Pairs trading: Performance of a relative-value arbitrage rule. *The Review of Financial Studies*, 19(3):797 – 827. http://dx.doi.org/10.2139/ssrn.141615.

Göncü, A. and Akyildirim, E. (2016). A stochastic model for commodity pairs trading. *Quantitative Finance*, 16(12):1843–1857. https://doi.org/10.1080/14697688.2016.1211793.

Huang, X. and Tauchen, G. (2005). The Relative Contribution of Jumps to Total Price Variance. *Journal of Financial Econometrics*, 3(4):456–499. https://doi.org/10.1093/jjfinec/nbi025.

Jondeau, E., Lahaye, J., and Rockinger, M. (2015). Estimating the price impact of trades in a high-frequency microstructure model with jumps. *Journal of Banking & Finance*, 61:S205–S224. https://doi.org/10.1016/j.jbankfin.2015.09.005.

Krauss, C. (2017). Statistical arbitrage pairs trading strategies: Review and outlook. *Journal of Economic Surveys*, 31(2):513–545. https://doi.org/10.1111/joes.12153.

Larsson, S., Lindberg, C., and Warfheimer, M. (2013). Optimal closing of a pair trade with a model containing jumps. *Applications of Mathematics*, 58(3):249–268. https://doi.org/10.1007/s10492-013-0012-8.

Lindberg, C. (2014). Pairs trading with opportunity cost. *Journal of Applied Probability*, 51(1):282–286. https://doi.org/10.1239/jap/1395771429.

Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 3(1):125–144. https://doi.org/10.1016/0304-405X(76)90022-2.

Song, Q. and Zhang, Q. (2013). An optimal pairs-trading rule. *Automatica*, 49(10):3007–3014. https://doi.org/10.1016/j.automatica.2013.07.012.

Stübinger, J. and Endres, S. (2018). Pairs trading with a mean-reverting jump–diffusion model on high-frequency data. *Quantitative Finance*, 18(10):1735–1751. https://doi.org/10.1080/14697688.2017.1417624.

Tie, J., Zhang, H., and Zhang, Q. (2017). An optimal strategy for pairs trading under geometric brownian motions. *Journal of Optimization Theory and Applications*, 179(2):654–675. https://doi.org/10.1007/s10957-017-1065-8.

Vidyamurthy, G. (2004). *Pairs trading: Quantitative methods and analysis*. J. Wiley.

Yalaman, A. and Manahov, V. (2022). Analysing emerging market returns with high-frequency data during the global financial crisis of 2007–2009. *The European Journal of Finance*, 28(10):1019–1051. https://doi.org/10.1080/1351847X.2021.1957698.