Extended Kalman Filter

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1 Kalamn Filter

The filter to estimate $\mathbf{x}$ is a Kalman-Filter with the prediction:

$$\mathbf{x}_{t|t-1} = \mathbf{F}_t \mathbf{x}_{t-1|t-1} \tag{1}$$
$$\mathbf{P}_{t|t-1} = \mathbf{F}_t \mathbf{P}_{t|t-1} \mathbf{F}_t^T + \mathbf{Q}_{t-1} \tag{2}$$

and update step:

$$\boldsymbol{\nu} = \mathbf{z}_t - \mathbf{h}_t(\mathbf{x}_{t|t-1}) \tag{3}$$
$$\mathbf{K}_t = \mathbf{P}_{t|t-1} \mathbf{H}_t^T \left(\mathbf{H}_t \mathbf{P}_{t|t-1} \mathbf{H}_t^T + \mathbf{R}_t \right)^{-1} \tag{4}$$
$$\mathbf{x}_{t|t} = \mathbf{x}_{t|t-1} + \mathbf{K}_t \boldsymbol{\nu} \tag{5}$$
$$\mathbf{P}_{t|t} = \left(\mathbf{I} - \mathbf{K}_t \mathbf{H}_t \right) \mathbf{P}_{t|t-1} \tag{6}$$

2 Kinematics

2.1 Quaternion Kinematics

The goal is to change the Quaternion over time through the angular rate vector ω. Typically this is expressed as:

$$\dot{\hat{\mathbf{q}}} = \frac{\partial \hat{\mathbf{q}}}{\partial t} = \frac{1}{2} \omega \hat{\mathbf{q}} \tag{7}$$

This can be expanded to:

$$\frac{1}{2}\omega\hat{\mathbf{q}} = \frac{1}{2} \begin{bmatrix} 0 - \omega_x q_i - \omega_y q_j - \omega_z q_k \\ 0 + \omega_x q_w + \omega_y q_k - \omega_z q_j \\ 0 - \omega_x q_k + \omega_y q_w + \omega_z q_i \\ 0 + \omega_x q_j - \omega_y q_i - \omega_z q_w \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 - \omega_x q_i - \omega_y q_j - \omega_z q_k \\ \omega_x q_w + 0 - \omega_z q_j + \omega_y q_k \\ \omega_y q_w + \omega_z q_i + 0 - \omega_x q_k \\ -\omega_z q_w - \omega_y q_i + \omega_x q_j + 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & -\omega_z & \omega_y \\ \omega_y & \omega_z & 0 & -\omega_x \\ -\omega_z & -\omega_y & \omega_x & 0 \end{bmatrix} \hat{\mathbf{q}} \tag{8}$$

3 Sensor Models

3.1 Gyroscope

The general gyroscope model is:

$$\mathbf{m}_g = \mathbf{m}_B + \omega_g + \mathbf{v}_g \tag{9}$$

Where $\mathbf{m}_B$ is the real angular rate in the body frame, ω_g is the gyroscope bias and $\mathbf{v}_g$ is the noise.

3.2 Accelerometer

$$\mathbf{m}_a = \mathbf{m}_B - \mathbf{g}_B + \omega_a + \mathbf{v}_a \tag{10}$$

Where $\mathbf{m}_B$ is the real acceleration in the body frame, $\mathbf{g}_B$ is the gravitational vector in the body frame, ω_g is the bias and $\mathbf{v}_g$ is the noise.

3.3 Magnetometer

$$\mathbf{m}_m = \mathbf{m}_B + \omega_m + \mathbf{v}_m \tag{11}$$

Where $\mathbf{m}_m$ is the real magnetic filed in the body frame, ω_m is the bias and $\mathbf{v}_m$ is the noise.

4 Measurement Prediction

4.1 Accelerometer

The prediction of the Accelerometer presumes that the IMU is static in space. This does not hold in reality and a revision of the Filter will be necessary in the future. Generally the prediction is that the gravitational vector points downwards. Hence,

$$\mathbf{h_a} = \mathbf{R_q}\hat{\mathbf{g_g}} = \mathbf{R_q}\begin{bmatrix}0\\0\\-1\end{bmatrix} \tag{12}$$

Here $\mathbf{R_q}$ is the rotor from the current attitude. Since the gravitational vector is mostly 0 the computation can be simplified:

$$\mathbf{h_a} = -\begin{bmatrix}2(q_i \cdot q_k + q_w \cdot q_j)\\ 2(q_j \cdot q_k - q_w \cdot q_i)\\ q_w^2 - q_i^2 - q_j^2 + q_k^2\end{bmatrix} \tag{13}$$

For the observation matrix the following then follows:

$$\mathbf{H}_a = \frac{\partial \mathbf{h_a}}{\partial \mathbf{x}} = \begin{bmatrix}-2q_j & -2q_k & -2q_w & -2q_i & 0 & 0 & 0 & 0 & 0 & 0\\ 2q_i & 2q_w & -2q_k & -2q_j & 0 & 0 & 0 & 0 & 0 & 0\\ -2q_w & 2q_i & 2q_j & -2q_k & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} \tag{14}$$

4.2 Magnetometer

Since the magnetometer is only used to compute the yaw of the sensor a projection to the x-y plane is necessary to eliminate negative influence the z axis.

$$\mathbf{m}_B = \mathbf{R}^{-1}\left((\mathbf{R}\cdot\mathbf{m}_B)\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}\right) \tag{15}$$

The yaw axis is then:

$$\Psi = atan2(-m_y, m_x) \tag{16}$$

The prediction of Ψ can be calculated from the current attitude:

$$h_a = atan2(2(q_i \cdot q_k + q_w \cdot q_j), 1 - 2q_j^2 - 2q_k^2) \tag{17}$$

For the observation matrix the following then follows if I use my immense mathematical talents (and Wolfram Alpha):

$$\mathbf{H}_m = \frac{\partial \mathbf{h_m}}{\partial \mathbf{x}} = \begin{bmatrix}\frac{\partial \mathbf{h_m}}{\partial \mathbf{q}_w} & \frac{\partial \mathbf{h_m}}{\partial \mathbf{q}_i} & \frac{\partial \mathbf{h_m}}{\partial \mathbf{q}_j} & \frac{\partial \mathbf{h_m}}{\partial \mathbf{q}_k} & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} \tag{18}$$

$$\frac{\partial \mathbf{h_m}}{\partial \mathbf{q}_w} = \frac{2q_z(-2q_j^2 - 2q_z^2 + 1)}{4(q_w q_z + q_i q_j)^2 + (-2q_j^2 - 2q_z^2 + 1)^2} \tag{19}$$

$$\frac{\partial \mathbf{h_m}}{\partial \mathbf{q}_i} = \frac{2q_j(-2q_j^2 - 2q_z^2 + 1)}{4(q_w q_z + q_i q_j)^2 + (-2q_j^2 - 2q_z^2 + 1)^2} \tag{20}$$

$$\frac{\partial \mathbf{h_m}}{\partial \mathbf{q}_j} = \frac{2(4q_w q_j q_z + 2q_i q_j^2 - 2q_i q_z^2 + q_i)}{4(q_w^2 - 1)q_z^2 + 8q_w q_i q_j q_z + 4q_j^2(q_i^2 + 2q_z^2 - 1) + 4q_j^4 + 4q_z^4 + 1} \tag{21}$$

$$\frac{\partial \mathbf{h_m}}{\partial \mathbf{q}_k} = \frac{q_w(-4q_j^2 + 4q_z^2 + 2) + 8q_i q_j q_z}{4(q_w^2 - 1)q_z^2 + 8q_w q_i q_j q_z + 4q_j^2(q_i^2 + 2q_z^2 - 1) + 4q_j^4 + 4q_z^4 + 1} \tag{22}$$

5 State Model

The EKF should predict the attitude and gyroscope bias. Bias terms for the other sensors and non orthogonality are to be ignored. The noise is assumed to be $\sim \mathcal{N}(0, \sigma^2)$. Therefore:

$$\mathbf{x_k} = \begin{bmatrix} \hat{\mathbf{q}} \\ \omega \\ \mathbf{b} \end{bmatrix} \tag{23}$$

Where $\hat{\mathbf{q}}$ is the attitude Quaternion and $\mathbf{b}$ is the bias.
Notice that:

$$\mathbf{q_{k+1}} = \mathbf{q_k} + \frac{1}{2} \begin{bmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & -\omega_z & \omega_y \\ \omega_y & \omega_z & 0 & -\omega_x \\ -\omega_z & -\omega_y & \omega_x 0 & \end{bmatrix} \cdot \mathbf{q_k} \tag{24}$$

$$\omega = \mathbf{m}_B - \omega_g \tag{25}$$

$$\mathbf{b} = \omega_g \tag{26}$$

Hence,

$$\mathbf{x_{k+1}} = \mathbf{f}(k, x) = \begin{bmatrix} \mathbf{q_{k+1}} \\ \omega \\ \mathbf{b} \end{bmatrix} \tag{27}$$

The state transition matrix is then the Jacobain of $\mathbf{f}$:

$$\mathbf{F_k} = \frac{\partial \mathbf{f}(k, x)}{\partial x} = \begin{bmatrix} \frac{\partial \mathbf{q_{k+1}}}{\partial q} & \frac{\partial \mathbf{q_{k+1}}}{\partial \omega} & 0 \\ 0 & 0 & \frac{\partial \omega}{\partial \omega} \\ 0 & 0 & \frac{\partial \mathbf{b}}{\partial \omega} \end{bmatrix} \tag{28}$$

6 Noise

The noise of the Accelerometer and the Magnetometer are trivial:

$$\mathbf{R}_m = \mathbf{I}_3 \sigma_m^2 \tag{29}$$

$$\mathbf{R}_\Psi = \sigma_A^2 \tag{30}$$

$$\tag{31}$$

The process Noise is less trivial. The noise response of the system

$$\mathbf{F}_R = \frac{\partial f_k}{\partial v} = \begin{bmatrix} 0_{4,3} & 0_{4,3} \\ I_3 & 0_3 \\ 0_3 & I_3 \end{bmatrix} \tag{32}$$

So,

$$\mathbf{Q} = F_R U F_R^T \tag{33}$$

7 Initial Values

Dont know yet, eh