Extended Kalman Filter

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1 Kalamn Filter

The goal of the Kalamn Filter is to estimate the Attitude of the IMU. The filter is defined by the following steps.

$$\mathbf{x}_{t|t-1} = \mathbf{F}_t \mathbf{x}_{t-1|t-1} \tag{1}$$
$$\mathbf{P}_{t|t-1} = \mathbf{F}_t \mathbf{P}_{t|t-1} \mathbf{F}_t^T + \mathbf{Q}_{t-1} \tag{2}$$

and update step:

$$\boldsymbol{\nu} = \mathbf{z}_t - \mathbf{h}_t(\mathbf{x}_{t|t-1}) \tag{3}$$
$$\mathbf{K}_t = \mathbf{P}_{t|t-1} \mathbf{H}_t^T \left(\mathbf{H}_t \mathbf{P}_{t|t-1} \mathbf{H}_t^T + \mathbf{R}_t\right)^{-1} \tag{4}$$
$$\mathbf{x}_{t|t} = \mathbf{x}_{t|t-1} + \mathbf{K}_t \boldsymbol{\nu} \tag{5}$$
$$\mathbf{P}_{t|t} = \left(\mathbf{I} - \mathbf{K}_t \mathbf{H}_t\right) \mathbf{P}_{t|t-1} \tag{6}$$

2 Kinematics

2.1 Quaternion Kinematics

The goal is to change the Quaternion over time through the angular rate vector ω. Typically this is expressed as:

$$\dot{\hat{\mathbf{q}}} = \frac{\partial \hat{\mathbf{q}}}{\partial t} = \frac{1}{2}\omega \hat{\mathbf{q}} \tag{7}$$

This can be expanded to:

$$\frac{1}{2}\omega\hat{\mathbf{q}} = \frac{1}{2}\begin{bmatrix} 0 - \omega_x q_i - \omega_y q_j - \omega_z q_k \\ 0 + \omega_x q_w + \omega_y q_k - \omega_z q_j \\ 0 - \omega_x q_k + \omega_y q_w + \omega_z q_i \\ 0 + \omega_x q_j - \omega_y q_i - \omega_z q_w \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 - \omega_x q_i - \omega_y q_j - \omega_z q_k \\ \omega_x q_w + 0 - \omega_z q_j + \omega_y q_k \\ \omega_y q_w + \omega_z q_i + 0 - \omega_x q_k \\ -\omega_z q_w - \omega_y q_i + \omega_x q_j + 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & -\omega_z & \omega_y \\ \omega_y & \omega_z & 0 & -\omega_x \\ -\omega_z & -\omega_y & \omega_x & 0 \end{bmatrix} \hat{\mathbf{q}} \tag{8}$$

3 Sensor Models

3.1 Gyroscope

The general gyroscope model is:

$$\mathbf{m}_g = \mathbf{m}_B + \omega_g + \mathbf{v}_g \tag{9}$$

Where $\mathbf{m}_B$ is the real angular rate in the body frame, ω_g is the gyroscope bias and $\mathbf{v}_g$ is the noise.

3.2 Accelerometer

$$\mathbf{m}_a = \mathbf{A}(\mathbf{m}_B - \mathbf{g}_B) + \omega_a + \mathbf{v}_a \tag{10}$$

Where $\mathbf{m}_B$ is the real acceleration in the body frame, $\mathbf{g}_B$ is the gravitational vector in the body frame, ω_g is the bias and $\mathbf{v}_g$ is the noise.

3.3 Magnetometer

$$\mathbf{m}_m = \mathbf{m}_B + \omega_m + \mathbf{v}_m \tag{11}$$

Where $\mathbf{m}_m$ is the real magnetic filed in the body frame, ω_m is the bias and $\mathbf{v}_m$ is the noise.
In actuality the Magnetometer measurement is reduced to a Yaw pseudo measurement with the aid of the Accelerometer measurements. First the roll ϕ and pitch θ will be determined with the help of the corrected Magnetometer measurement $\mathbf{m}_{a,c}$.

$$\phi = atan2(m_{a,c,y}, -m_{a,c,z}) \tag{12}$$

$$\theta = atan\left(\frac{-m_{a,c,x}}{\sqrt{m_{a,c,y}^2 + m_{a,c,z}^2}}\right) \tag{13}$$

$$\tag{14}$$

These are then used to calculate the compensated magnetic field vector $\mathbf{m}_{m,c}$ (z is omitted)

$$m_{m,c,x} = m_{m,x} \cdot \cos\theta + m_{m,z} \cdot \sin\theta \tag{15}$$

$$m_{m,c,y} = m_{m,x} \cdot \sin\phi \sin\theta + m_{m,y} \cdot \sin\phi - m_{m,z} \cdot \sin\phi \cos\theta \tag{16}$$

$$\tag{17}$$

The compensated Vector is then used to calculate the yaw.

$$\psi = atan2(-m_{m,c,y}, m_{m,c,x}) \tag{18}$$

4 Measurement Prediction

4.1 Accelerometer

Generally the prediction is that the gravitational vector points downwards. Hence,

$$\mathbf{h_a} = \hat{\mathbf{q}}^* \cdot \hat{\mathbf{g}}_g \cdot \hat{\mathbf{q}} = \hat{\mathbf{q}}^* \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \hat{\mathbf{q}} = \mathbf{R}_q^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{19}$$

Here $\mathbf{R_q}$ is the rotor from the current attitude. Since the gravitational vector is mostly 0 and that $\mathbf{R}^{-1} = \mathbf{R}^\mathsf{T}$ for orthogonal matrices the computation can be simplified:

$$\mathbf{h_a} = \begin{bmatrix} 2(q_i \cdot q_k - q_w \cdot q_j) \\ 2(q_j \cdot q_k + q_w \cdot q_i) \\ 1 - 2(q_i^2 + q_j^2) \end{bmatrix} \tag{20}$$

For the observation matrix the following then follows:

$$\mathbf{H}_a = \frac{\partial \mathbf{h_a}}{\partial \mathbf{x}} = \begin{bmatrix} -2q_j & 2q_k & -2q_w & 2q_i & 0 & 0 & 0 & 0 & 0 & 0 \\ 2q_i & 2q_w & 2q_k & 2q_j & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4q_i & -4q_j & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{21}$$

4.2 Magnetometer

Since the magnetometer is only used to compute the yaw of the sensor a projection to the x-y plane is necessary to eliminate negative influence the z axis.

$$\mathbf{m}_B = \mathbf{R}^{-1}\left((\mathbf{R}\cdot\mathbf{m}_B)\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}\right) \tag{22}$$

The yaw axis is then:

$$\Psi = atan2(-m_y, m_x) \tag{23}$$

The prediction of Ψ can be calculated from the current attitude:

$$h_a = atan2(2(q_i\cdot q_k + q_w\cdot q_j), 1-2q_j^2-2q_k^2) \tag{24}$$

For the observation matrix the following then follows if I use my immense mathematical talents (and Wolfram Alpha):

$$\mathbf{H}_m = \frac{\partial\mathbf{h_m}}{\partial\mathbf{x}} = \begin{bmatrix}\frac{\partial\mathbf{h_m}}{\partial\mathbf{q}_w} & \frac{\partial\mathbf{h_m}}{\partial\mathbf{q}_i} & \frac{\partial\mathbf{h_m}}{\partial\mathbf{q}_j} & \frac{\partial\mathbf{h_m}}{\partial\mathbf{q}_k} & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} \tag{25}$$

$$\frac{\partial\mathbf{h_m}}{\partial\mathbf{q}_w} = \frac{2q_z(-2q_j^2-2q_z^2+1)}{4(q_wq_z+q_iq_j)^2+(-2q_j^2-2q_z^2+1)^2} \tag{26}$$

$$\frac{\partial\mathbf{h_m}}{\partial\mathbf{q}_i} = \frac{2q_j(-2q_j^2-2q_z^2+1)}{4(q_wq_z+q_iq_j)^2+(-2q_j^2-2q_z^2+1)^2} \tag{27}$$

$$\frac{\partial\mathbf{h_m}}{\partial\mathbf{q}_j} = \frac{2(4q_wq_jq_z+2q_iq_j^2-2q_iq_z^2+q_i)}{4(q_w^2-1)q_z^2+8q_wq_iq_jq_z+4q_j^2(q_i^2+2q_z^2-1)+4q_j^4+4q_z^4+1} \tag{28}$$

$$\frac{\partial\mathbf{h_m}}{\partial\mathbf{q}_k} = \frac{q_w(-4q_j^2+4q_z^2+2)+8q_iq_jq_z}{4(q_w^2-1)q_z^2+8q_wq_iq_jq_z+4q_j^2(q_i^2+2q_z^2-1)+4q_j^4+4q_z^4+1} \tag{29}$$

5 State Model

The EKF should predict the attitude and gyroscope bias. Bias terms for the other sensors and non orthogonality are to be ignored. The noise is assumed to be $\sim\mathcal{N}(0,\sigma^2)$. Therefore:

$$\mathbf{x_k} = \begin{bmatrix}\hat{\mathbf{q}}\\ \omega\\ \mathbf{b}\end{bmatrix} \tag{30}$$

Where $\hat{\mathbf{q}}$ is the attitude Quaternion and $\mathbf{b}$ is the bias.
Notice that:

$$\mathbf{q_{k+1}} = \mathbf{q_k} + \frac{1}{2}\begin{bmatrix}0 & -\omega_x & -\omega_y & -\omega_z\\ \omega_x & 0 & -\omega_z & \omega_y\\ \omega_y & \omega_z & 0 & -\omega_x\\ -\omega_z & -\omega_y & \omega_x 0 & \end{bmatrix}\cdot\mathbf{q_k} \tag{31}$$

$$\omega = \mathbf{m}_B - \omega_g \tag{32}$$

$$\mathbf{b} = \omega_g \tag{33}$$

Hence,

$$\mathbf{x_{k+1}} = \mathbf{f}(k,x) = \begin{bmatrix}\mathbf{q_{k+1}}\\ \omega\\ \mathbf{b}\end{bmatrix} \tag{34}$$

The state transition matrix is then the Jacobain of $\mathbf{f}$:

$$\mathbf{F_k} = \frac{\partial\mathbf{f}(k,x)}{\partial x} = \begin{bmatrix}\frac{\partial\mathbf{q_{k+1}}}{\partial q} & \frac{\partial\mathbf{q_{k+1}}}{\partial\omega} & 0\\ 0 & 0 & \frac{\partial\omega}{\partial\omega}\\ 0 & 0 & \frac{\partial\mathbf{b}}{\partial\omega}\end{bmatrix} \tag{35}$$

6 Noise

The noise of the Accelerometer and the Magnetometer are trivial:

$$\mathbf{R}_m = \mathbf{I}_3 \sigma_m^2 \tag{36}$$
$$\mathbf{R}_\Psi = \sigma_A^2 \tag{37}$$
$$\tag{38}$$

The process Noise is less trivial. The noise response of the system

$$\mathbf{F}_R = \frac{\partial f_k}{\partial v} = \begin{bmatrix} 0_{4,3} & 0_{4,3} \\ I_3 & 0_3 \\ 0_3 & I_3 \end{bmatrix} \tag{39}$$

So,

$$\mathbf{Q} = F_R U F_R^T \tag{40}$$

7 Initial Values

Dont know yet, eh