Extended Kalman Filter

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1 Kalamn Filter

We have a system defined as:

$$\mathbf{x}_t = f(\mathbf{x}_{t-1}, \mathbf{u}_{t-1}) - \omega_{t-1} \tag{1}$$
$$\mathbf{z}_t = h(\mathbf{x}_t) + \mathbf{v}_t \tag{2}$$

The goal of the Kalamn Filter is to estimate the Attitude of the IMU. The filter is defined by the following steps.

$$\mathbf{x}_{t|t-1} = f(\mathbf{x}_{t-1|t-1}, \mathbf{u}_{t-1}) \tag{3}$$
$$\mathbf{P}_{t|t-1} = \mathbf{F}_t \mathbf{P}_{t|t-1} \mathbf{F}_t^T + \mathbf{Q}_{t-1} \tag{4}$$

and update step:

$$\boldsymbol{\nu} = \mathbf{z}_t - \mathbf{h}_t(\mathbf{x}_{t|t-1}) \tag{5}$$
$$\mathbf{K}_t = \mathbf{P}_{t|t-1} \mathbf{H}_t^T \left(\mathbf{H}_t \mathbf{P}_{t|t-1} \mathbf{H}_t^T + \mathbf{R}_t \right)^{-1} \tag{6}$$
$$\mathbf{x}_{t|t} = \mathbf{x}_{t|t-1} + \mathbf{K}_t \boldsymbol{\nu} \tag{7}$$
$$\mathbf{P}_{t|t} = (\mathbf{I} - \mathbf{K}_t \mathbf{H}_t) \mathbf{P}_{t|t-1} \tag{8}$$

2 Kinematics

2.1 Quaternion Kinematics

The goal is to change the Quaternion over time through the angular rate vector ω. Typically this is expressed as:

$$\hat{\dot{\mathbf{q}}} = \frac{\partial \hat{\mathbf{q}}}{\partial t} = \frac{1}{2} \omega \hat{\mathbf{q}} \tag{9}$$

This can be expanded to:

$$\frac{1}{2}\omega\hat{\mathbf{q}} = \frac{1}{2}\begin{bmatrix} 0 - \omega_x q_i - \omega_y q_j - \omega_z q_k \\ 0 + \omega_x q_w + \omega_y q_k - \omega_z q_j \\ 0 - \omega_x q_k + \omega_y q_w + \omega_z q_i \\ 0 + \omega_x q_j - \omega_y q_i - \omega_z q_w \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 - \omega_x q_i - \omega_y q_j - \omega_z q_k \\ \omega_x q_w + 0 - \omega_z q_j + \omega_y q_k \\ \omega_y q_w + \omega_z q_i + 0 - \omega_x q_k \\ -\omega_z q_w - \omega_y q_i + \omega_x q_j + 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & -\omega_z & \omega_y \\ \omega_y & \omega_z & 0 & -\omega_x \\ -\omega_z & -\omega_y & \omega_x & 0 \end{bmatrix} \hat{\mathbf{q}} \tag{10}$$

3 Sensor Models

3.1 Gyroscope

The general gyroscope model is:

$$\mathbf{m}_g = \mathbf{m}_B + \omega_g + \mathbf{v}_g \tag{11}$$

Where $\mathbf{m}_B$ is the real angular rate in the body frame, ω_g is the gyroscope bias and $\mathbf{v}_g$ is the noise.

3.2 Accelerometer

$$\mathbf{m}_a = \mathbf{A}(\mathbf{m}_B - \mathbf{g}_B) + \omega_a + \mathbf{v}_a \tag{12}$$

Where $\mathbf{m}_B$ is the real acceleration in the body frame, $\mathbf{g}_B$ is the gravitational vector in the body frame, ω_g is the bias and $\mathbf{v}_g$ is the noise.

3.3 Magnetometer

$$\mathbf{m}_m = \mathbf{m}_B + \omega_m + \mathbf{v}_m \tag{13}$$

Where $\mathbf{m}_m$ is the real magnetic filed in the body frame, ω_m is the bias and $\mathbf{v}_m$ is the noise.
In actuality the Magnetometer measurement is reduced to a Yaw pseudo measurement with the aid of the Accelerometer measurements. First the roll ϕ and pitch θ will be determined with the help of the corrected Magnetometer measurement $\mathbf{m}_{a,c}$.

$$\phi = atan2(m_{a,c,y}, m_{a,c,z}) \tag{14}$$

$$\theta = atan\left(\frac{-m_{a,c,x}}{\sqrt{m_{a,c,y}^2 + m_{a,c,z}^2}}\right) \tag{15}$$

$$\tag{16}$$

These are then used to calculate the compensated magnetic field vector $\mathbf{m}_{m,c}$ (z is omitted)

$$m_{m,c,x} = m_{m,x} \cdot \cos\theta + m_{m,z} \cdot \sin\theta \tag{17}$$

$$m_{m,c,y} = m_{m,x} \cdot \sin\phi \sin\theta + m_{m,y} \cdot \cos\phi - m_{m,z} \cdot \sin\phi \cos\theta \tag{18}$$

$$\tag{19}$$

The compensated Vector is then used to calculate the yaw.

$$\psi = atan2(-m_{m,c,y}, m_{m,c,x}) \tag{20}$$

4 Measurement Prediction

4.1 Accelerometer

Generally the prediction is that the gravitational vector points downwards. Hence,

$$\mathbf{h_a} = \hat{\mathbf{q}}^* \cdot \hat{\mathbf{g}}_g \cdot \hat{\mathbf{q}} = \hat{\mathbf{q}}^* \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \hat{\mathbf{q}} = \mathbf{R}_q^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{21}$$

Here $\mathbf{R_q}$ is the rotor from the current attitude. Since the gravitational vector is mostly 0 and that $\mathbf{R}^{-1} = \mathbf{R}^\top$ for orthogonal matrices the computation can be simplified:

$$\mathbf{h_a} = \begin{bmatrix} 2(q_i \cdot q_k - q_w \cdot q_j) \\ 2(q_j \cdot q_k + q_w \cdot q_i) \\ q_w^2 - q_i^2 - q_j^2 + q_k^2 \end{bmatrix} \tag{22}$$

For the observation matrix the following then follows:

$$\mathbf{H}_a = \frac{\partial \mathbf{h_a}}{\partial \mathbf{x}} = \begin{bmatrix} -2q_j & 2q_k & -2q_w & 2q_i & 0 & 0 & 0 \\ 2q_i & 2q_w & 2q_k & 2q_j & 0 & 0 & 0 \\ 2q_w & -2q_i & -2q_j & 2q_k & 0 & 0 & 0 \end{bmatrix} \tag{23}$$

4.2 Magnetometer

Since the magnetometer is only used to compute the yaw of the sensor a projection to the x-y plane is necessary to eliminate negative influence the z axis.

$$\mathbf{m}_B = \mathbf{R}^{-1}\left((\mathbf{R}\cdot\mathbf{m}_B)\cdot\begin{bmatrix}1\\1\\0\end{bmatrix}\right) \tag{24}$$

The yaw axis is then:

$$\Psi = atan2(-m_y, m_x) \tag{25}$$

The prediction of Ψ can be calculated from the current attitude:

$$h_a = atan2(2(q_w \cdot q_k + q_i \cdot q_j), 1 - 2q_j^2 - 2q_k^2) \tag{26}$$

We can rewrite this as:

$$\begin{aligned} h_a &= atan2(A, B) &(27)\\ A &= 2(q_w \cdot q_k + q_i \cdot q_j) &(28)\\ B &= 1 - 2q_j^2 - 2q_k^2 &(29)\\ & &(30) \end{aligned}$$

Then the standard derivitive is:

$$\frac{\partial h_a}{\partial q_n} = \frac{B}{A^2+B^2}\frac{\partial A}{\partial x} - \frac{A}{A^2+B^2}\frac{\partial B}{\partial x} \tag{31}$$

So in total:

$$\begin{aligned} \mathbf{H}_m = \frac{\partial \mathbf{h_m}}{\partial \mathbf{x}} &= \begin{bmatrix}\frac{\partial \mathbf{h_m}}{\partial \mathbf{q}_w} & \frac{\partial \mathbf{h_m}}{\partial \mathbf{q}_i} & \frac{\partial \mathbf{h_m}}{\partial \mathbf{q}_j} & \frac{\partial \mathbf{h_m}}{\partial \mathbf{q}_k} & 0 & 0 & 0\end{bmatrix} &(32)\\ \frac{\partial \mathbf{h_m}}{\partial \mathbf{q}_w} &= \frac{B}{A^2+B^2}2q_k &(33)\\ \frac{\partial \mathbf{h_m}}{\partial \mathbf{q}_i} &= \frac{B}{A^2+B^2}2q_j &(34)\\ \frac{\partial \mathbf{h_m}}{\partial \mathbf{q}_j} &= \frac{2Bq_i + 4Aq_j}{A^2+B^2} &(35)\\ \frac{\partial \mathbf{h_m}}{\partial \mathbf{q}_k} &= \frac{2Bq_w + 4q_k}{A^2+B^2} &(36) \end{aligned}$$

5 State Model

The EKF should predict the attitude and gyroscope bias. Bias terms for the other sensors and non orthogonality are to be ignored. The noise is assumed to be $\sim \mathcal{N}(0, \sigma^2)$. Therefore:

$$\mathbf{x_k} = \begin{bmatrix}\hat{\mathbf{q}}\\ \mathbf{b}\end{bmatrix} \tag{37}$$

Where $\hat{\mathbf{q}}$ is the attitude Quaternion and $\mathbf{b}$ is the bias.
Notice that:

$$\begin{aligned} \mathbf{q_{k+1}} &= \mathbf{q_k} + \frac{1}{2}\begin{bmatrix}0 & -\omega_x & -\omega_y & -\omega_z\\ \omega_x & 0 & -\omega_z & \omega_y\\ \omega_y & \omega_z & 0 & -\omega_x\\ -\omega_z & -\omega_y & \omega_x 0 & \end{bmatrix}\cdot \mathbf{q_k}\Delta t &(38)\\ \mathbf{b}_{k+1} &= \mathbf{b}_k &(39) \end{aligned}$$

Here $\omega = \mathbf{m}_g - \mathbf{b}_k$
Hence,

$$\mathbf{x_{k+1}} = \mathbf{f}(k,x) = \begin{bmatrix} \mathbf{q_{k+1}} \\ \mathbf{b} \end{bmatrix} \tag{40}$$

The state transition matrix is then the Jacobain of $\mathbf{f}$:

$$\mathbf{F_k} = \frac{\partial \mathbf{f}(k,x)}{\partial x} = \begin{bmatrix} \frac{\partial \mathbf{q_{k+1}}}{\partial q} & \frac{\partial \mathbf{q_{k+1}}}{\partial b} \\ 0 & I \end{bmatrix} \tag{41}$$

Where:

$$\frac{\partial \mathbf{q_{k+1}}}{\partial q} = \mathbf{q_{k+1}} = I + \frac{1}{2} \begin{bmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & -\omega_z & \omega_y \\ \omega_y & \omega_z & 0 & -\omega_x \\ -\omega_z & -\omega_y & \omega_x 0 & \end{bmatrix} \Delta t \tag{42}$$

$$\frac{\partial \mathbf{q_{k+1}}}{\partial b} = \begin{bmatrix} \frac{1}{2} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} q_k \Delta t & \frac{1}{2} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} q_k \Delta t & \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} q_k \Delta t \end{bmatrix} \tag{43}$$

$$= \begin{bmatrix} \frac{1}{2}\Delta t \begin{bmatrix} -q_i \\ q_w \\ -q_k \\ q_j \end{bmatrix} & \frac{1}{2}\Delta t \begin{bmatrix} -q_j \\ q_k \\ q_w \\ -q_i \end{bmatrix} & \frac{1}{2}\Delta t \begin{bmatrix} -q_k \\ -q_j \\ q_i \\ -q_w \end{bmatrix} \end{bmatrix} \tag{44}$$

6 Noise

In an Extended Kalman Filter (EKF), two primary sources of noise are modeled:

- **Process noise**, representing uncertainty in the system dynamics.
- **Measurement noise**, representing uncertainty in sensor observations.

6.1 Measurement Noise

The measurement noise arises from the accelerometer and magnetometer. Assuming white, zero-mean, isotropic noise, the corresponding covariance matrices are

$$\mathbf{R}_A = \mathbf{I}_3 \sigma_A^2, \tag{45}$$

$$\mathbf{R}_M = \mathbf{I}_3 \sigma_M^2, \tag{46}$$

where σ_A and σ_M are the standard deviations of the accelerometer and magnetometer noise, respectively. The combined measurement noise covariance used in the EKF update step is

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_M & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{R}_A \end{bmatrix} = \begin{bmatrix} \mathbf{I}_3\sigma_M^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_3\sigma_A^2 \end{bmatrix}. \tag{47}$$

6.2 Process Noise

The system state includes both the quaternion and the gyroscope bias:

$$\mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \mathbf{b}_g \end{bmatrix} \in \mathbb{R}^7, \tag{48}$$

The quaternion propagation is modeled as

$$\dot{\mathbf{q}} = \frac{1}{2}\,\mathbf{\Omega}(\mathbf{q})\,(\mathbf{m}_g - \mathbf{b}_g - \mathbf{v}_g), \tag{49}$$

where $\mathbf{v}_g$ is the gyroscope noise and

$$\boldsymbol{\Omega}(\mathbf{q}) = \begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{bmatrix}. \tag{50}$$

The gyroscope bias is modeled as a random walk:

$$\dot{\mathbf{b}}_g = \mathbf{w}_b, \tag{51}$$

where $\mathbf{w}_b$ is zero-mean Gaussian noise.

6.3 Discrete Process Noise Covariance

After discretization over a time step Δt, the Jacobian of the quaternion with respect to gyro noise is

$$\frac{\partial \mathbf{q}_{k+1}}{\partial \mathbf{v}_g} = -\frac{1}{2}\,\boldsymbol{\Omega}(\mathbf{q}_k)\,\Delta t. \tag{52}$$

Assuming the gyroscope and bias noise covariances are

$$\mathbf{Q}_g = \mathbf{I}_3\sigma_g^2, \qquad \mathbf{Q}_b = \mathbf{I}_3\sigma_b^2\Delta t^2, \tag{53}$$

Then:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_q & \mathbf{0}_{4\times 3} \\ \mathbf{0}_{3\times 4} & \mathbf{Q}_b \end{bmatrix}, \quad \mathbf{Q}_q = \frac{1}{4}\boldsymbol{\Omega}(\mathbf{q}_k)\,\mathbf{Q}_g\,\boldsymbol{\Omega}(\mathbf{q}_k)^T\,\Delta t^2. \tag{54}$$

7 Initial Values

We generate Initial values for the roll, pitch and yaw using a method similar to the complementary filter. For the variances we use Welford's online algorithm:

$$\bar{x}_n = \bar{x}_{n-1} + \frac{x_n - \bar{x}_{n-1}}{n} \tag{55}$$

$$M_n = M_{n-1} + (x_n - \bar{x}_{n-1})(x_n - \bar{x}_n) \tag{56}$$

$$\sigma_n^2 = \frac{M_n}{n-1} \tag{57}$$