Linear-time Temporal Logic with Team Semantics: Expressivity and Complexity

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Abstract

We study the expressivity and the model checking problem of linear temporal logic with team semantics (TeamLTL). In contrast to LTL, TeamLTL is capable of defining hyperproperties, i.e., properties which relate multiple execution traces. Logics for hyperproperties have so far been mostly obtained by extending temporal logics like LTL and QPTL with trace quantification, resulting in HyperLTL and HyperQPTL. We study the expressiveness of TeamLTL (and its extensions with downward closed generalised atoms $\mathcal{A}$ and connectives such as Boolean disjunction $\varovee$) in comparison to HyperLTL and HyperQPTL. Thereby, we also obtain a number of model checking results for TeamLTL, a question which is so far an open problem. The two types of logics follow a fundamentally different approach to hyperproperties and are are of incomparable expressiveness. We establish that the universally quantified fragment of HyperLTL subsumes the so-called k-coherent fragment of TeamLTL($\mathcal{A}, \varovee$). This also implies that the model checking problem is decidable for the fragment. We show decidability of model checking of the so-called *left-flat* fragment of TeamLTL($\mathcal{A}, \varovee$) via a translation to a decidable fragment of HyperQPTL. Finally, we show that the model checking problem of TeamLTL with Boolean disjunction and inclusion atom is undecidable.

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1 Introduction

Linear-time temporal logic (LTL) is one of the most prominent logics for the specification and verification of reactive and concurrent systems. Practical model checking tools like SPIN, NuSMV, and many others [17, 3, 7] automatically verify whether a given computer system, such as a hardware circuit or a communication protocol, is correct with respect to its LTL specification. The basic principle, as introduced in 1977 by Amir Pnueli [25], is to specify the correctness of a program as a set of infinite sequences, called *traces*, which define the acceptable executions of the system.

Properties that refer to the *dependence* or *independence* of certain events cannot be specified in LTL. Such properties are of prime interest in information flow security, where dependencies between the secret inputs and the publicly observable outputs of a system are considered potential security violations. Security policies that regulate the permissible dependencies cannot be characterized as properties of individual execution traces. Rather, they are properties of sets of traces, known as *hyperproperties* [5].

One approach to extend LTL to hyperproperties has been to introduce explicit references to traces into the logic. HyperLTL [4] extends LTL with *trace quantifiers* and *trace variables*. For example, the formula

$$\forall\pi.\,\forall\pi'.\,\mathsf{G}(\bigwedge_{i\in pI} i_\pi \leftrightarrow i_{\pi'}) \rightarrow \mathsf{G}(\bigwedge_{o\in pO} o_\pi \leftrightarrow o_{\pi'})$$

states that any two traces which globally agree on the value of the public inputs pI also globally agree on the public outputs pO. Thereby, the value of secret inputs cannot affect the value of the publicly observable outputs. A limitation of HyperLTL is, however, that the set of traces under consideration remains fixed throughout the evaluation of the formula: all quantifiers refer to the full set of system traces. This makes it difficult to reason about properties that refer to subsets, such as that the full set of traces can be decomposed into subsets with different dependencies.

In this paper, we study an attractive, but so far not yet well-studied, alternative to the extension of LTL with trace quantifiers: we switch to the *team semantics* of LTL [21]. Team semantics is a general framework in which hyperproperties, such as dependence and independence, can be expressed directly with atomic statements. Under team semantics, LTL expresses hyperproperties without explicit references to traces. Instead, each subformula is evaluated with respect to a set of traces, called a *team*.

Team semantics is a generalization of Tarski's semantics in which satisfaction of formulae is determined by a set of assignments rather than a single assignment. The development of the area began with the introduction of Dependence Logic in [28] which adds the concept of functional dependence to first-order logic by the means of new atomic dependence formulae

$$\mathrm{dep}(x_1,\ldots,x_n). \tag{1}$$

The dependence atom (1) is satisfied by a set of assignments (aka a *team*) X if any two assignments $s, s' \in X$ assigning the same values to the variables $x_1,\ldots,x_{n-1}$ satisfy $s(x_n) = s'(x_n)$. During the past ten years, team semantics has been generalized to propositional [31], modal [29], temporal [20], and probabilistic [9] frameworks, and fascinating connections to fields such as database theory [12], statistics [8], real valued computation [13], and quantum information theory [18] has been identified.

The team semantics analogues of LTL and CTL were defined in [21, 20]. Our focus in this article is on TeamLTL and its variants, which are natural logics for hyperproperties.

TeamLTL comes with two semantics based on whether the temporal operators are interpreted synchronously or asynchronously. The synchronous semantics can express hyperproperties that reason over subsets of the system. As an example, the TeamLTL formula

$$(\mathsf{G}\,\mathrm{dep}(i_1, i_2, o)) \vee (\mathsf{G}\,\mathrm{dep}(i_2, i_3, o))$$

states that in one part of the system, output o is always determined by inputs i_1 and i_2, while in the other part, it is determined by inputs i_2 and i_3. This type of input-output dependence that relates possibly infinitely many system traces cannot be expressed in HyperLTL. The two logics are, in fact, incomparable in expressive power [21]. Other hyperlogics such as HyperQPTL and HyperQPTL$^+$ can express similar properties [6, 10], but in a much less concise and transparent fashion.

The article [21] settled some questions related to the expressive power and complexity of TeamLTL and its variants. However, most notably, the complexity of model checking under the synchronous semantics was left open. Side stepping the question slightly, it was recently shown that the complexity of satisfiability and model checking of the extension of TeamLTL by the Boolean negation is equivalent to the decision problem of third-order arithmetic [23].

Our contribution. In this article, we explore further the relative expressivity of TeamLTL and temporal hyperlogics like HyperLTL, as well as the decidability frontier of TeamLTL and its variants. We begin by showing that the k-coherent fragment of TeamLTL is expressively weaker than the universal fragment of HyperLTL. Next, we show that the so-called *left-flat* fragment of TeamLTL($\otimes, \overset{1}{\mathsf{A}}$) enjoys decidable model checking via a translation to $\exists^*\forall^*$HyperQPTL. Lastly, we analyse a spectrum of logics between TeamLTL and TeamLTL($\sim$) for the complexity of model checking. We show that already a very restricted access to the Boolean negation $\sim$ leads to high undecidability, whereas $\subseteq$ and $\otimes$ suffice for undecidability. In particular, we show that the model checking problem for TeamLTL($\subseteq, \otimes$) is Σ^0_1-hard, and its extension with one occurrence of A alone is already Σ^1_1-hard.

2 Preliminaries

2.1 HyperLTL, HyperQPTL, and HyperQPTL$^+$

Let us start by recalling the syntax of LTL from the literature. Fix a finite set AP of *atomic propositions*. The set of formulae of LTL is generated by the following grammar:

$$\varphi ::= p \mid \neg p \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid \bigcirc \varphi \mid (\varphi \,\mathcal{U}\, \varphi) \mid (\varphi \,\mathcal{W}\, \varphi), \qquad \text{where } p \in \mathrm{AP}.$$

We adopt, as is common in studies on team logics, the convention that formulae (of LTL and the other logics we consider) are in negation normal form. The logical constants $\top, \bot$ and connectives $\rightarrow, \leftrightarrow$ are defined as usual, and $\mathsf{F}\,\varphi := \top \,\mathcal{U}\, \varphi$ and $\mathsf{G}\,\varphi := \varphi \,\mathcal{W}\, \bot$.

A *trace* t over AP is an infinite sequence from $(2^{\mathrm{AP}})^\omega$. For a natural number $i \in \mathbb{N}$, we denote by $t[i]$ the ith element of t and by $t[i, \infty]$ the postfix $(t[j])_{j \geq i}$ of t. The satisfaction relation $(t, i) \models \varphi$, for LTL-formulae φ, is defined as usual, see e.g., [24]. We use $[\![\varphi]\!]_{(t,i)} \in \{0, 1\}$ to denote the truth value of φ on (t, i).

HyperQPTL$^+$ [10] is a temporal logic for hyperproperties. It extends LTL with explicit trace quantification and quantification of atomic propositions. As such, it subsumes the well studied hyperlogic HyperLTL [4] as well as QPTL and HyperQPTL [26, 6], which can express all ω-regular properties. Fix an infinite set $\mathcal{V}$ of trace variables. The hyperlogic HyperQPTL$^+$ extends LTL with three types of quantifiers, one for traces and two for propositional quantification.

$$\varphi ::= \forall\pi\,\varphi \mid \exists\pi\,\varphi \mid \overset{u}{\forall}p\,\varphi \mid \overset{u}{\exists}p\,\varphi \mid \forall p\,\varphi \mid \exists p\,\varphi \mid \psi$$

$$\psi ::= p_\pi \mid \neg p_\pi \mid (\psi \vee \psi) \mid (\psi \wedge \psi) \mid \bigcirc\psi \mid (\psi\,\mathcal{U}\,\psi) \mid (\psi\,\mathcal{W}\,\psi)$$

Here, $p \in \mathrm{AP}$, $\pi \in \mathcal{V}$, and $\forall\pi$ and $\exists\pi$ stand for universal and existential trace quantifiers, $\forall p$ and $\exists p$ stand for (non-uniform) propositional quantifiers, and $\overset{u}{\forall}p$ and $\overset{u}{\exists}p$ stand for uniform propositional quantifiers.

We also study two syntactic fragments of HyperQPTL$^+$. HyperQPTL is HyperQPTL$^+$ without non-uniform propositional quantifiers, and HyperLTL is HyperQPTL$^+$ without any propositional quantifiers. In fragments of HyperQPTL, when no confusion arises, we also write simply $\forall p$ and $\exists p$ instead of $\overset{u}{\forall}p$ and $\overset{u}{\exists}p$. For an LTL-formula φ and trace variable π, we let φ_π denote the HyperLTL obtained from φ by replacing all proposition symbols p by their indexed versions p_π. We extend this convention to tuples of formulae as well.

The semantics of HyperQPTL$^+$ is defined over a set T of traces. Intuitively, the atomic formula p_π asserts that p holds on trace π. Uniform propositional quantifications $\overset{u}{\forall}p$ and $\overset{u}{\exists}p$ extend the valuations of the traces in T uniformly with respect to p, namely that all traces agree on the valuation of p on any given time step i, whereas nonuniform propositional quantifications $\forall p$ and $\exists p$ color the traces in T in an arbitrary manner.

A *trace assignment* (for T) is a function $\Pi : \mathcal{V} \to T$ that maps each trace variable in $\mathcal{V}$ to some trace in T. The modified trace assignment $\Pi[\pi \mapsto t]$ is one that is otherwise the same as Π except that $\Pi[\pi \mapsto t](\pi) = t$. For any subset $A \subseteq \mathrm{AP}$, we write $t \upharpoonright A$ for the trace over A defined as $(t \upharpoonright A)[i] = t[i] \cap A$ for all $i \in \mathbb{N}$. For any two trace assignments Π and Π', we write $\Pi =_A \Pi'$, if $\big(\Pi(\pi) \upharpoonright A\big) = \big(\Pi'(\pi) \upharpoonright A\big)$ for all $\pi \in \mathcal{V}$. Similarly, for any two sets T and T' of traces, we write $T =_A T'$ whenever $\{t \upharpoonright A \mid t \in T\} = \{t \upharpoonright A \mid t \in T'\}$. For a sequence $s \in (2^{\{p\}})^\omega$ over a single propositional variable p, we write $T[p \mapsto s]$ for the set of traces obtained from T by reinterpreting p on all traces as in s.

The satisfaction relation $\Pi, i \models_T \varphi$ for HyperQPTL$^+$ formulae φ is defined as follows:

$\Pi, i \models_T p_\pi$	iff	$p \in \Pi(\pi)[i]$
$\Pi, i \models_T \neg p_\pi$	iff	$p \notin \Pi(\pi)[i]$
$\Pi, i \models_T \varphi_1 \vee \varphi_2$	iff	$\Pi, i \models_T \varphi_1$ or $\Pi, i \models_T \varphi_2$
$\Pi, i \models_T \varphi_1 \wedge \varphi_2$	iff	$\Pi, i \models_T \varphi_1$ and $\Pi, i \models_T \varphi_2$
$\Pi, i \models_T \bigcirc\varphi$	iff	$\Pi, i+1 \models_T \varphi$
$\Pi, i \models_T \varphi_1\,\mathcal{U}\,\varphi_2$	iff	$\exists j \geq i$ s.t. $\Pi, j \models_T \varphi_2$ and $\forall k : i \leq k < j \Rightarrow \Pi, k \models_T \varphi_1$
$\Pi, i \models_T \varphi_1\,\mathcal{W}\,\varphi_2$	iff	$\forall j \geq i : \Pi, j \models_T \varphi_1$ or $\exists j' : i \leq j' \leq j : \Pi, j' \models_T \varphi_2$
$\Pi, i \models_T \exists\pi\,\varphi$	iff	$\Pi[\pi \mapsto t], i \models_T \varphi$ for some $t \in T$
$\Pi, i \models_T \forall\pi\,\varphi$	iff	$\Pi[\pi \mapsto t], i \models_T \varphi$ for all $t \in T$
$\Pi, i \models_T \overset{u}{\exists}q\,\varphi$	iff	$\Pi', i \models_{T[q\mapsto s]} \varphi$ for some $s \in (2^{\{q\}})^\omega$ and $\Pi' \colon \mathcal{V} \to T[q \mapsto s]$ s.t $\Pi' =_{\mathrm{AP}\setminus\{q\}} \Pi$
$\Pi, i \models_T \overset{u}{\forall}q\,\varphi$	iff	$\Pi', i \models_{T[q\mapsto s]} \varphi$ for all $s \in (2^{\{q\}})^\omega$ and $\Pi' \colon \mathcal{V} \to T[q \mapsto s]$ s.t $\Pi' =_{\mathrm{AP}\setminus\{q\}} \Pi$
$\Pi, i \models_T \exists q\,\varphi$	iff	$\Pi', i \models_{T'} \varphi$ for some $T' \subseteq (2^{\mathrm{AP}})^\omega$ and $\Pi' \colon \mathcal{V} \to T'$ s.t. $T =_{\mathrm{AP}\setminus\{q\}} T'$ and $\Pi =_{\mathrm{AP}\setminus\{q\}} \Pi'$
$\Pi, i \models_T \forall q\,\varphi$	iff	$\Pi', i \models_{T'} \varphi$ for all $T' \subseteq (2^{\mathrm{AP}})^\omega$ and $\Pi' \colon \mathcal{V} \to T'$ s.t. $T =_{\mathrm{AP}\setminus\{q\}} T'$ and $\Pi =_{\mathrm{AP}\setminus\{q\}} \Pi'$

In the sequel, we will relate extensions of LTL with team semantics to fragments of HyperQPTL$^+$ arising from restricting the quantifier prefixes of formulae. We use $\exists_\pi \,/\, \forall_\pi$ to denote trace quantification, $\overset{u}{\exists}_q \,/\, \overset{u}{\forall}_q$ for uniform propositional quantification, and $\exists_q \,/\, \forall_q$ for nonuniform propositional quantification. We use $\exists$ ($\forall$, resp.) to denote all the relevant existential (universal, resp.) quantifiers. We write Q to refer to both $\exists$ and $\forall$. For a logic L and a regular expression e, we write eL to denote the set of L formulae whose quantifier prefixes are generated by e. E.g., $\forall^*\exists^*$HyperQPTL refers to HyperQPTL formulae with quantifier prefix $\{\forall_q, \forall_\pi\}^*\{\exists_q, \exists_\pi\}^*$.

2.2 TeamLTL

Let us now introduce the logic LTL interpreted in the team semantics setting (denoted TeamLTL). TeamLTL was first studied in [21], where it was called LTL with *synchronous* team semantics. A *(temporal) team* is a pair (T, i) consisting a set of traces $T \subseteq (2^{\mathrm{AP}})^\omega$ and a natural number $i \in \mathbb{N}$ representing the time step.[1] The satisfaction relation $(T, i) \models \varphi$ for TeamLTL-formulae φ is defined inductively as follows:

$$
\begin{array}{lll}
(T,i) \models p & \text{iff} & \forall t \in T : p \in t[i] \\
(T,i) \models \neg p & \text{iff} & \forall t \in T : p \notin t[i] \\
(T,i) \models \varphi \wedge \psi & \text{iff} & (T,i) \models \varphi \text{ and } (T,i) \models \psi \\
(T,i) \models \varphi \vee \psi & \text{iff} & \exists T_1, T_2 \text{ s.t. } T = T_1 \cup T_2,\ (T_1,i) \models \varphi \text{ and } (T_2,i) \models \psi \\
(T,i) \models \bigcirc \varphi & \text{iff} & (T,i+1) \models \varphi \\
(T,i) \models \varphi \,\mathcal{U}\, \psi & \text{iff} & \exists k \geq i \text{ s.t. } (T,k) \models \psi \text{ and } \forall m : i \leq m < k \Rightarrow (T,m) \models \varphi \\
(T,i) \models \varphi \,\mathcal{W}\, \psi & \text{iff} & \forall k \geq i : (T,k) \models \varphi \text{ or } \exists m \text{ s.t. } i \leq m \leq k \text{ and } (T,m) \models \psi
\end{array}
$$

We say that two formulae φ and ψ are *equivalent*, denoted as $\varphi \equiv \psi$, whenever for any (T, i), $(T, i) \models \varphi$ iff $(T, i) \models \psi$. Next we list some important semantical properties or formulae from team semantics literature:

(Downward closure) If $(T, i) \models \varphi$ and $S \subseteq T$, then $(S, i) \models \varphi$.
(Empty team property) $(\emptyset, i) \models \varphi$.
(Flatness) $(T, i) \models \varphi$ iff $(\{t\}, i) \models \varphi$ for all $t \in T$.
(Singleton equivalence) $(\{t\}, i) \models \varphi$ iff $(t, i) \models \varphi$.

It is easy to verify that TeamLTL-formulae satisfy the downward closure, singleton equivalence, and empty team properties. In general, TeamLTL does not satisfy the flatness property; for instance, the formula $\mathsf{F}\, p$ is not flat.

The power of team semantics comes with the ability to enrich the logics with novel atomic statements describing properties of teams. The most prominent examples of such atoms are *dependence atoms* $\mathrm{dep}(\varphi_1, \ldots, \varphi_n, \psi)$ and *inclusion atoms* $\varphi_1, \ldots, \varphi_n \subseteq \psi_1, \ldots, \psi_n$ with $\varphi_1, \ldots, \varphi_n, \psi, \psi_1, \ldots, \psi_n$ being LTL-formulae, whose team semantics is defined as

$$
\begin{aligned}
&(T,i) \models \mathrm{dep}(\varphi_1, \ldots, \varphi_n, \psi) \quad \text{iff} \quad \forall t, t' \in T : \\
&\qquad (\llbracket \varphi_1 \rrbracket_{(t,i)}, \ldots, \llbracket \varphi_n \rrbracket_{(t,i)}) = (\llbracket \varphi_1 \rrbracket_{(t',i)}, \ldots, \llbracket \varphi_n \rrbracket_{(t',i)}) \Rightarrow \llbracket \psi \rrbracket_{(t,i)} = \llbracket \psi \rrbracket_{(t',i)}. \\
&(T,i) \models \varphi_1, \ldots, \varphi_n \subseteq \psi_1, \ldots, \psi_n \quad \text{iff} \quad \forall t \in T\, \exists t' \in T \text{ s.t.} \\
&\qquad (\llbracket \varphi_1 \rrbracket_{(t,i)}, \ldots, \llbracket \varphi_n \rrbracket_{(t,i)}) = (\llbracket \psi_1 \rrbracket_{(t',i)}, \ldots, \llbracket \psi_n \rrbracket_{(t',i)}).
\end{aligned}
$$

[1] Note that in [21] the time step i was not explicitly represented in the team.

More generally, arbitrary properties of teams induce *generalised atoms*. These atoms were first introduced in the first-order team semantics setting by Kuusisto [22] using generalized quantifiers. Propositional generalized atoms essentially encode second-order truth tables.

▶ **Definition 1** (Generalised atoms for LTL). *An n-ary* generalised atom *is an n-ary operator $\#_G(\varphi_1, \dots, \varphi_n)$ with an associated set G of n-ary relations over the Boolean domain $\{0,1\}$ that applies only to* LTL*-formulae $\varphi_1, \dots, \varphi_n$. Its team semantics is defined as:*

$$(T,i) \models \#_G(\varphi_1, \dots, \varphi_n) \qquad \text{iff} \qquad \{([\![\varphi_1]\!]_{(t,i)}, \dots, [\![\varphi_n]\!]_{(t,i)}) \mid t \in T\} \in G.$$

In order to simplify the presentation, we restrict our attention to generalised atoms that have the empty team property. Using this convention mitigates the need to add special cases for our constructions below for the special case of the empty team.

We also consider some other operators known in the team semantics literature: *Boolean disjunction* $\ovee$, *contradictory negation* $\sim$, and *universal subteam quantifiers* A and $\overset{1}{\mathsf{A}}$, with their semantics defined as:

$$\begin{aligned}
&(T,i) \models \varphi \ovee \psi && \text{iff} && (T,i) \models \varphi \text{ or } (T,i) \models \psi\\
&(T,i) \models\, \sim \varphi && \text{iff} && (T,i) \not\models \varphi\\
&(T,i) \models \mathsf{A}\varphi && \text{iff} && \forall S \subseteq T : (S,i) \models \varphi\\
&(T,i) \models \overset{1}{\mathsf{A}}\varphi && \text{iff} && \forall t \in T : (t,i) \models \varphi
\end{aligned}$$

For TeamLTL formulae φ, $\overset{1}{\mathsf{A}}\varphi$ is, in fact, a flat generalised atom and equivalent to $\varphi \subseteq \top$. In the sequel, we treat $\overset{1}{\mathsf{A}}$ as such.

Let $\mathcal{A}$ be a set of generalised atoms and connectives. We denote by TeamLTL($\mathcal{A}$) the extension of TeamLTL with atoms and connectives in $\mathcal{A}$. For any atom or connective $\circ$, we write simply TeamLTL($\mathcal{A}, \circ$) for TeamLTL($\mathcal{A} \cup \{\circ\}$). It is straightforward to verify that for any set $\mathcal{A}$ of downward closed atoms and connectives, all formulae in the logic TeamLTL($\mathcal{A}$) are also downward closed. For instance, TeamLTL(dep, $\ovee, \mathsf{A}, \overset{1}{\mathsf{A}}$) is downward closed.

We say that a logic $\mathcal{L}_2$ is *at most as expressive* than a logic $\mathcal{L}_1$, denoted as $\mathcal{L}_1 \leq \mathcal{L}_2$, if for every $\mathcal{L}_1$-formula φ, there exists an $\mathcal{L}_2$-formula ψ such that $\varphi \equiv \psi$. We write $\mathcal{L}_1 \equiv \mathcal{L}_2$ if both $\mathcal{L}_1 \leq \mathcal{L}_2$ and $\mathcal{L}_2 \leq \mathcal{L}_1$. Clearly, for any downward closed logic TeamLTL($\mathcal{A}$), we have TeamLTL($\mathcal{A}$) $\equiv$ TeamLTL($\mathcal{A}, \mathsf{A}$). TeamLTL and HyperLTL have orthogonal expressiveness [21]. The relation of TeamLTL to HyperQPTL and HyperQPTL$^+$ is still unknown.

Propositional and modal logic with $\ovee$ are known to be expressively complete with respect to nonempty downward closed properties (that are closed under so-called *team bisimulations*)[16, 30]. The next proposition establishes that, in the LTL setting, TeamLTL($\ovee, \overset{1}{\mathsf{A}}$) is very expressive among the downward closed logics; all downward closed atoms that have the empty team property can be expressed in the logic. As TeamLTL formulae are not necessarily flat, the operator $\overset{1}{\mathsf{A}}$ seems to be essential for the result. The proposition follows using an argument similar to an analogous translation given in [30] for propositional team semantics. For details, see Appendix A.

▶ **Proposition 2.** *For any n-ary downward closed generalised atom $\#_G$ having the empty team property and* LTL*-formulae $\varphi_1, \dots, \varphi_n$, we have that*

$$\#_G(\varphi_1, \dots, \varphi_n) \equiv \bigovee_{R \in G} \bigvee_{(b_1,\dots,b_n) \in R} \overset{1}{\mathsf{A}}(\varphi_1^{b_1} \wedge \dots \wedge \varphi_n^{b_n}),$$

where $\varphi_i^1 := \varphi_i$ and $\varphi_i^0 := \neg\varphi_i$ in negation normal form. In particular, for any set $\mathcal{A}$ of downward closed atoms, TeamLTL($\mathcal{A} \cup \{\ovee, \overset{1}{\mathsf{A}}, \mathsf{A}\}$) $\equiv$ TeamLTL($\ovee, \overset{1}{\mathsf{A}}$)*. The elimination of each atom yields a doubly exponential disjunction over linear sized formulae.*

Similarly one can show that every generalised atom $\#_G$ is expressible in TeamLTL($\sim$), since $\#_G(\vec{\varphi}) \equiv \bigovee_{R\in G} \bigvee_{\vec{b}\in R}(\overset{1}{\mathsf{A}}\vec{\varphi}^{\vec{b}} \wedge \sim \bot)$, where $\ovee$ and $\overset{1}{\mathsf{A}}$ as well as most connectives that have been considered in the literature are shown (e.g., in [14, 23]) to be definable in TeamLTL($\sim$). It is open whether TeamLTL($\sim$) is expressively complete with respect to properties satisfying some natural invariance, as the corresponding logics in the propositional and modal settings are (see [19, 31]).

3 Expressivity of TeamLTL and HyperLTL

In this section, we further study the expressivity of TeamLTL compared to HyperLTL. In Section 4, we utilize this connection to obtain decidable model checking of certain team based logics. The section's main result is that all *k-coherent* properties expressible in extensions of the form TeamLTL($\mathcal{A}$) of TeamLTL are expressible in $\forall^*$HyperLTL. We also conclude that with respect to trace properties, TeamLTL($\mathcal{A}, \overset{1}{\mathsf{A}}$) is equi-expressive to $\forall$HyperLTL. These indicate that the expressive power of TeamLTL($\mathcal{A}$) is mostly manifested with regards to properties that are not k-coherent. Lastly we show that all generalised atoms are expressible in HyperLTL.

3.1 Lower and upper dimensions for TeamLTL

We start by providing compositional estimates for κ-coherence for TeamLTL-formulae.

▶ **Definition 3.** *Let κ be a cardinal number. A formula φ in* TeamLTL($\mathcal{A}$) *is said to be* κ-coherent *if for every team (T, i),*

$$(T, i) \models \varphi \Leftrightarrow (S, i) \models \varphi \text{ for every } S \subseteq T \text{ with } |S| \leq \kappa.$$

The corresponding property in the context of HyperLTL can be formulated naturally as

$$\Pi, i \models_T \varphi \Leftrightarrow \Pi, i \models_S \varphi \text{ for every subset } S \subseteq T \text{ with } |S| \leq \kappa.$$

Sentences $\varphi = \forall \pi_1 \dots \pi_k.\, \psi$ in the fragment $\forall^k$HyperLTL are clearly k-coherent.

It is often useful to give estimates for the cardinal κ, for which a given formula φ is κ-coherent. In the context of modal team logics, compositional estimates for κ-coherence are computed in [16] using the so-called *lower and upper dimensions* of formulae.

▶ **Definition 4.** *The* lower dimension $\dim(\varphi)$ *of a formula φ is defined as the least cardinal κ such that for every team (T, i), for every minimal subteam $S \subseteq T$ with $(S, i) \not\models \varphi$, we have that $|S| \leq \kappa$. The* upper dimension $\mathrm{Dim}(\varphi)$ *of φ is defined as the least cardinal κ such that for every team (T, i), $|M(\varphi, (T, i))| \leq \kappa$, where $M(\varphi, (T, i))$ consists of all maximal subteams $S \subseteq T$ such that $(S, i) \models \varphi$.*

For any downward closed formula φ, it is easy to verify that $\dim(\varphi)$ is the least cardinal κ such that φ is κ-coherent. Clearly, φ if flat iff φ is 1-coherent iff $\dim(\varphi) = 1$. Moreover, by the result of the next proposition, we also have that φ is flat iff $\mathrm{Dim}(\varphi) = 1$. Since lower dimensions often do not admit well-behaved compositional estimates, we will mostly use the upper dimension to estimate the lower dimension, as done in the modal logic setting in [16].

The following two results are proven in the similar fashion as the analogous results for modal team semantics in [16]. See Appendix B for details.

▶ **Proposition 5** (Cf. [16, Lemma 5.6]). *If φ is a formula in some downward closed logic* TeamLTL($\mathcal{A}$)*, then $\dim(\varphi) \leq \mathrm{Dim}(\varphi)$.*

▶ **Lemma 6.** *In any downward closed logic* TeamLTL($\mathcal{A}$)*, we have the following estimates:*

(i) $\mathrm{Dim}(p) = \mathrm{Dim}(\neg p) = 1$
(ii) $\mathrm{Dim}(\#(\varphi_1, \ldots, \varphi_n)) \leq 2^n$
(iii) $\mathrm{Dim}(\varphi \wedge \psi) \leq \mathrm{Dim}(\varphi)\,\mathrm{Dim}(\psi)$
(iv) $\mathrm{Dim}(\varphi \vee \psi) \leq \mathrm{Dim}(\varphi)\,\mathrm{Dim}(\psi)$
(v) $\mathrm{Dim}(\varphi \otimes \psi) \leq \mathrm{Dim}(\varphi) + \mathrm{Dim}(\psi)$
(vi) $\mathrm{Dim}(\bigcirc \varphi) \leq \mathrm{Dim}(\varphi)$
(vii) $\mathrm{Dim}(\mathsf{F}\,\varphi) \leq \omega\,\mathrm{Dim}(\varphi)$
(viii) $\mathrm{Dim}(\mathsf{G}\,\varphi) \leq \mathrm{Dim}(\varphi)^\omega$
(ix) $\mathrm{Dim}(\varphi\,\mathcal{U}\,\psi) \leq \sum_{n\in\mathbb{N}} \mathrm{Dim}(\varphi)^n \mathrm{Dim}(\psi)$
(x) $\mathrm{Dim}(\varphi\,\mathcal{W}\,\psi)$
$\leq \mathrm{Dim}(\varphi)^\omega + \sum_{n\in\mathbb{N}} \mathrm{Dim}(\varphi)^n \mathrm{Dim}(\psi)$

▶ **Corollary 7.** *Let φ be a formula in some downward closed logic* TeamLTL($\mathcal{A}$)*. If every subformula $\psi\,\mathcal{W}\,\theta$ in φ satisfies that* $\mathrm{Dim}(\psi) = 1$ *(or ψ is flat), then φ is ω-coherent.*

The estimates given in Lemma 6, give grounds to look closer to formulae of TeamLTL($\mathcal{A}$) with the left-hand sides of the operators $\mathcal{U}$ and $\mathcal{W}$ restricted to flat formulae. We will prove in Section 4 that such fragments have better computational properties.

3.2 Weak fragments of TeamLTL

We show that, with respect to k-coherent properties ($k \in \mathbb{N}$), TeamLTL($\mathcal{A}, \otimes, \mathsf{A}$) is at most as expressive as $\forall^k$HyperLTL. We define a translation from TeamLTL($\mathcal{A}, \otimes, \mathsf{A}$) to $\forall^*$HyperLTL that preserves the satisfaction relation with respect to teams of bounded size. For any finite set Φ of trace variables, the translation φ^Φ is defined as follows. For $\wedge$, $\otimes$, $\bigcirc$, $\mathcal{U}$, and $\mathcal{W}$, the translation is homomorphic. The remaining cases are defined as follows:

$$p^\Phi := \bigwedge_{\pi\in\Phi} p_\pi, \qquad (\varphi \vee \psi)^\Phi := \bigvee_{\Phi_0\cup\Phi_1=\Phi} \varphi^{\Phi_0} \wedge \psi^{\Phi_1},$$

$$(\neg p)^\Phi := \bigwedge_{\pi\in\Phi} \neg p_\pi, \qquad (\mathsf{A}\varphi)^\Phi := \bigwedge_{\Phi'\subseteq\Phi} \varphi^{\Phi'},$$

$$\big(\#_G(\varphi_1, \cdots, \varphi_n)\big)^\Phi := \bigvee_{R\in G}\Big(\big(\bigwedge_{\pi\in\Phi}\ \bigvee_{\vec{b}\in R} \vec{\varphi}_\pi^{\vec{b}}\big) \wedge \bigwedge_{\vec{b}\in R}\ \bigvee_{\pi\in\Phi} \vec{\varphi}_\pi^{\vec{b}}\Big), \text{ where } \vec{\varphi}^{\vec{b}} = (\varphi_1^{b_1} \wedge \cdots \wedge \varphi_n^{b_n}).$$

Note that the translation is exponential with respect to $|\Phi|$ and doubly exponential with respect to the arity of the generalised atoms. The following Lemma is proved by a straightforward induction, see Appendix C.

▶ **Lemma 8.** *Let $\mathcal{A}$ be a set of generalised atoms and φ a* TeamLTL($\mathcal{A}, \otimes, \mathsf{A}$)*-formula. Let $\Phi = \{\pi_1, \ldots, \pi_k\}$ be a finite set of trace variables. For any team (T, i) with $|T| \leq k$, any set $S \supseteq T$ of traces, and any assignment Π with $\Pi[\Phi] = T$, we have that*

$$(T, i) \models \varphi \quad \textit{iff} \quad \Pi, i \models_S \varphi^\Phi.$$

Furthermore, if the generalised atoms in $\mathcal{A}$ are all downward closed, then

$$(T, i) \models \varphi \quad \textit{iff} \quad \emptyset, i \models_T \forall\pi_1 \ldots \forall\pi_k\, \varphi^\Phi.$$

▶ **Theorem 9.** *Every k-coherent property that is definable in* TeamLTL($\mathcal{A}, \otimes, \mathsf{A}$)*, where $\mathcal{A}$ is any set of generalised atoms, is also definable in* $\forall^k$HyperLTL*.*

Proof. Let φ be a k-coherent TeamLTL($\mathcal{A}, \otimes, \mathsf{A}$)-formula. If $(\emptyset, i) \not\models \varphi$ for some $i \in \mathbb{N}$, then $(T, j) \not\models \varphi$ for every team (T, j). We then translate φ to $\forall\pi\bot$. Assume now $(\emptyset, i) \models \varphi$. For any nonempty team (T, i), we have that

$$\begin{aligned}
(T, i) \models \varphi \ &\Leftrightarrow\ (S, i) \models \varphi \text{ for every } S \subseteq T \text{ with } |S| \leq k && \text{(since } \varphi \text{ is } k\text{-coherent)}\\
&\Leftrightarrow\ \Pi, i \models_T \varphi^{\{\pi_1,\ldots,\pi_k\}} \text{ for every } \Pi \text{ s.t } \Pi[\{\pi_1, \ldots, \pi_k\}] \subseteq T && \text{(by Lemma 8 \& } (\emptyset, i) \models \varphi)\\
&\Leftrightarrow\ \emptyset, i \models_T \forall\pi_1 \ldots \forall\pi_k\, \varphi^{\{\pi_1,\ldots,\pi_k\}}.
\end{aligned}$$

◀

Since, the model checking of $\forall^*$HyperLTL has PSPACE-complete combined complexity and NL-complete data complexity [11], we get the following corollary:

▶ **Corollary 10.** *Let $\mathcal{A}$ be a set of generalised atoms. The model checking problem of* TeamLTL$(\mathcal{A}, \otimes, \mathsf{A})$*, restricted k-coherent properties ($k \in \mathbb{N}$), is decidable for combined complexity, and in* NL *for data complexity.*

Clearly $(T, i) \models \overset{1}{\mathsf{A}}\varphi$ iff $\emptyset, i \models_T \forall\pi.\, \varphi$, for any $\varphi \in$ LTL, and hence we obtain the following:

▶ **Corollary 11.** *Let $\mathcal{A}$ be any collection of generalised atoms and connectives from $\{\otimes, \mathsf{A}\}$. The* 1*-coherent fragment of* TeamLTL$(\mathcal{A}, \overset{1}{\mathsf{A}})$ *is expressively equivalent to* $\forall$HyperLTL*.*

We end this section by showing that all generalised atoms are expressible in HyperLTL.

▶ **Proposition 12.** *There is an* 2EXPTIME*-computable function mapping each n-ary generalised atom $\#_G$ to a* HyperLTL*-formula ψ_G in the $\exists^*_\pi\forall_\pi$ fragment such that for any non-empty set T of traces, $i \in \mathbb{N}$, assignment $\Pi\colon \mathcal{V} \to T$, and* LTL*-formulae $\varphi_1, \ldots, \varphi_n$,*

$$(T, i) \models \#_G(\varphi_1, \ldots, \varphi_n) \quad \Leftrightarrow \quad \Pi, i \models_T \psi_G(\varphi_1, \ldots, \varphi_n).$$

Moreover, if $\#_G$ is downward closed, ψ_G can be defined in the $\forall^*_\pi$ fragment.

Proof. The idea is to describe in the HyperLTL-formula ψ_G the second-order truth table encoded by $\#_G$. For downward closed $\#_G$ the translation is similar to the one in Proposition 2:

$$\psi_G := \forall\pi_1 \ldots \forall\pi_{2^n} \bigvee_{R \in G} \bigwedge_{i=1}^{2^n} \bigvee_{\vec{b} \in R} \vec{\varphi}^{\vec{b}}_{\pi_i}, \text{ where } \vec{\varphi}^{\vec{b}}_{\pi_i} := (\varphi_1)^{b_1}_{\pi_i} \wedge \cdots \wedge (\varphi_n)^{b_n}_{\pi_i}.$$

For an arbitrary generalised atom $\#_G$, we define

$$\psi_G := \exists\pi_1 \ldots \exists\pi_{2^n}\forall\pi \left(\bigvee_{R \in G} \left(\Big(\bigwedge_{i=1}^{2^n} \bigvee_{\vec{b} \in R} \vec{\varphi}^{\vec{b}}_{\pi_i} \Big) \wedge \bigwedge_{\vec{b} \in R} \bigvee_{i=1}^{2^n} \vec{\varphi}^{\vec{b}}_{\pi_i} \right) \wedge \bigvee_{i=1}^{2^n} \bigwedge_{j=1}^{n} \big((\varphi_j)_\pi \leftrightarrow (\varphi_j)_{\pi_i} \big) \right).$$

It is not hard to prove that the above formulae ψ_G are as required. ◀

4 HyperQPTL$^+$ and decidable fragments of TeamLTL

In this section, we compare the expressivity of extensions of TeamLTL to that of HyperQPTL and HyperQPTL$^+$. By doing so, we provide a partial answer to the open problem posed in [21] concerning the complexity of the model checking problem of TeamLTL and its extensions, which is known to be decidable (in fact in PSPACE) for the fragment of TeamLTL without $\vee$ [21]. However, for TeamLTL with $\vee$, no meaningful upper bound for the problem was known. The best previous upper bound could be obtained from TeamLTL($\sim$), for which the problem is highly undecidable [23]. The main source of difficulties comes from the fact that the semantical definition of $\vee$ does not yield any reasonable compositional brute force algorithm: The verification of $(T, i) \models \varphi \vee \psi$ with T generated by a finite Kripke structure proceeds by checking that $(T_1, i) \models \varphi$ and $(T_2, i) \models \psi$ for some $T_1 \cup T_2 = T$, but it can well be that T_1 and T_2 cannot be generated from any finite Kripke structure whatsoever. The main result of the section is the decidability of model checking of the so-called *left-flat fragment* of TeamLTL$(\otimes, \overset{1}{\mathsf{A}})$. We obtain inclusion to EXPTIME by a translation to $\overset{u}{\exists}{}^*_p\forall_\pi$HyperQPTL.

We first give translations from TeamLTL$(\otimes, \mathcal{A})$ to weak fragments of HyperQPTL$^+$ in the following theorem, whose proof can be found in Appendix D. The theorem gives insight on the limits of the expressivity of different extensions of TeamLTL.

▶ **Theorem 13.** *For every* TeamLTL$(\mathcal{A}, \varovee)$*-formula* φ*, with* $\mathcal{A}$ *a set of generalised atoms, there exists an equivalent* 2EXPTIME*-computable HyperQPTL*$^+$*-formula* φ^* *in the* $\exists_p \overset{u}{Q}{}^*_p \exists^*_\pi \forall_\pi$ *fragment. If the atoms in* $\mathcal{A}$ *are downward closed,* φ^* *can be defined in the* $\exists_p \overset{u}{Q}{}^*_p \forall_\pi$ *fragment. For formulae without generalised atoms, the translation can be done in linear time.*

We adopt a variant of the model checking problem where one asks whether the team of traces generated by a given Kripke structure satisfies a given formula. We consider Kripke structures of the form $\mathrm{K} = (W, R, \eta, w_0)$, where W is a finite set of states, $R \subseteq W^2$ a left-total transition relation, $\eta \colon W \to 2^{\mathrm{AP}}$ a labeling function, and $w_0 \in W$ an initial state of W. A path σ through K is an infinite sequence $\sigma \in W^\omega$ such that $\sigma[0] = w_0$ and $(\sigma[i], \sigma[i+1]) \in R$ for every $i \geq 0$. The trace of σ is defined as $t(\sigma) := \eta(\sigma[0])\eta(\sigma[1]) \cdots \in (2^{\mathrm{AP}})^\omega$. A Kripke structure K induces a set of traces $\mathrm{Traces}(\mathrm{K}) = \{t(\sigma) \mid \sigma \text{ is a path through K}\}$.

▶ **Definition 14.** *The model checking problem of* TeamLTL *(HyperQPTL*$^+$*) is, given a formula* φ *and a Kripke structure* K *over* AP*, to determine whether* $(\mathit{Traces}(K), 0) \models \varphi$*.*

In the remainder of this section, we show that formulae φ from the left-flat fragment of TeamLTL can be translated to formulae HyperQPTL that are linear in the size of φ. The known model checking algorithm of HyperQPTL [26] then immediately yields a model checking algorithm for the left-flat fragment of TeamLTL$(\varovee, \overset{1}{\mathsf{A}})$.

▶ **Definition 15** (The left-flat fragment)**.** *Let* $\mathcal{A}$ *be a collection of atoms and connectives. A* TeamLTL$(\mathcal{A})$ *formula belongs to the left-flat fragment if in each of its subformulae of the form* $\varphi \mathcal{U} \psi$ *or* $\varphi \mathcal{W} \psi$*,* φ *is a flat formula.*

Such defined fragment allows for arbitrary use of the F operator, and therefore remains incomparable to HyperLTL [21]. For instance, $\mathsf{F}\,\mathrm{dep}(a, b) \vee \mathsf{F}\,\mathrm{dep}(c, d)$ is a nontrivial formula in this fragment. It states that the set of traces can be partitioned into two parts, one where eventually a determines the value of b, and another where eventually c determines the value of d. A system satisfying this property can have some unobserved input that influences the dependency of some of its variables. The property is not expressible in HyperLTL, because HyperLTL cannot state the property "there is a point in time at which p holds on all (or infinitely many) traces" [2].

Next we translate formulae from left-flat TeamLTL$(\varovee, \overset{1}{\mathsf{A}})$ into $\overset{u}{\exists}{}^*_q \forall_\pi$HyperQPTL. In this translation, we essentially make use of the fact that satisfaction of flat formulae φ can be determined with the usual (single-traced) LTL semantics. In the evaluation of φ, it is thus sufficient to consider only finitely many subteams, whose behaviour (i.e., synchronization points) can be simulated by existentially quantified q-sequences.

A left-flat TeamLTL$(\varovee, \overset{1}{\mathsf{A}})$-formula φ will be translated into a $\overset{u}{\exists}{}^*_q \forall_\pi$HyperQPTL-formula of the form $\exists r_1 \ldots r_n \forall \pi.\, [\varphi, r]$, where $[\varphi, r]$ is quantifier-free. For any flat formula ψ, let $\forall \pi.\, \hat{\psi}$ be the HyperLTL formula given by Theorem 9, which is of linear size in the size of ψ. We now define inductively the quantifier free formula $[\varphi, r]$ as follows:

$$[p, r] \coloneqq \mathsf{G}(r_\pi \to p_\pi) \qquad [\bigcirc \varphi, r] \coloneqq r \prec r^\varphi \wedge [\varphi, r^\varphi]$$

$$[\neg p, r] \coloneqq \mathsf{G}(r_\pi \to \neg p_\pi) \qquad [\varphi \wedge \psi, r] \coloneqq [\varphi, r^\varphi] \wedge [\psi, r^\psi]$$

$$[\overset{1}{\mathsf{A}}\varphi, r] \coloneqq \mathsf{G}(r_\pi \to \hat{\varphi}) \qquad [\varphi \vee \psi, r] \coloneqq [\varphi, r^\varphi] \vee [\psi, r^\psi]$$

$$[\varphi \varovee \psi, r] \coloneqq (d^{\varphi \varovee \psi}_\pi \to [\varphi, r]) \wedge (\neg d^{\varphi \varovee \psi}_\pi \to [\psi, r])$$

$$[\varphi \mathcal{W} \psi, r] \coloneqq \mathit{once}(r^\varphi) \wedge \mathit{once}(r^\psi) \wedge \mathsf{G}(r_\pi \to r^\varphi_\pi \mathcal{W} r^\psi_\pi) \wedge \mathsf{G}(r^\varphi_\pi \to \hat{\varphi}) \wedge [\psi, r^\psi]$$

$$[\varphi \mathcal{U} \psi, r] \coloneqq \mathit{once}(r^\varphi) \wedge \mathit{once}(r^\psi) \wedge \mathsf{G}(r_\pi \to r^\varphi_\pi \mathcal{U} r^\psi_\pi) \wedge \mathsf{G}(r^\varphi_\pi \to \hat{\varphi}) \wedge [\psi, r^\psi]$$

where $r \prec r' := \mathsf{G}(r_\pi \leftrightarrow \bigcirc r'_\pi)$ and $once(r) := \mathsf{G}(r_\pi \rightarrow \bigcirc \mathsf{G} \neg r_\pi)$.

Now, let $r^1, \dots r^n$ be the free propositions occurring in $[\varphi, r^0]$ (except r^0) and π the free trace variable. Define the following $\overset{u}{\exists}{}^*_q \forall_\pi$ HyperQPTL formula:

$$\overset{u}{\exists} r^0 \overset{u}{\exists} r^1 \dots \overset{u}{\exists} r^n .\, \forall \pi .\, r^0_\pi \wedge \bigcirc \mathsf{G} \neg r^0_\pi \wedge [\varphi, r^0].$$

All time points at which the teams have to synchronize are existentially quantified in the formula. The formula works essentially because when evaluating the left-flat formula φ, only finitely many teams will be generated, and thus there are only finitely many points of synchronization. Every trace then has to fit into one of the teams described by the quantified propositional variables. We verify that the translation is indeed correct in Appendix E.

▶ **Theorem 16.** *For every* TeamLTL$(\circledvee, \overset{1}{\mathsf{A}})$*-formula φ, we can compute an equivalent $\overset{u}{\exists}{}^*_q \forall_\pi$ HyperQPTL-formula of size linear in the size of φ.*

Recall that the model checking problem of HyperLTL formulae with one quantifier alternation is EXPSPACE-complete [11] in the size of the formula, and PSPACE-complete in the size of the Kripke structure [11]. These results directly transfer to HyperQPTL [26] (in which HyperQPTL was called *HyperLTL with extended quantification* instead): for model checking a HyperQPTL-formula, the Kripke structure can be extended by two states generating all possible q-sequences. Since the translation from TeamLTL$(\circledvee, \overset{1}{\mathsf{A}})$ to HyperQPTL yields a formula in the $\overset{u}{\exists}{}^*_q \forall_\pi$ fragment with a single quantifier alternation and preserves the size of the formula, we obtain the following theorem.

▶ **Theorem 17.** *The model checking problem for* TeamLTL$(\circledvee, \overset{1}{\mathsf{A}})$ *is in* EXPSPACE *w.r.t. combined complexity, and in* PSPACE *w.r.t. data complexity.*

Together with Proposition 2, we obtain the following corollary.

▶ **Corollary 18.** *Let $\mathcal{A}$ be a set of downward closed generalised atoms. The model checking problem for* TeamLTL$(\mathcal{A}, \circledvee, \overset{1}{\mathsf{A}})$ *is in* 3EXPSPACE *w.r.t. combined complexity, and in* PSPACE *w.r.t. data complexity.*

5 Undecidable extensions of TeamLTL

Having established the decidability of the model checking problem of the left-flat fragment of TeamLTL$(\mathcal{A}, \circledvee)$ with $\mathcal{A}$ a set of downward closed generalised atoms, we now turn to the side of undecidability and explore what suffices to make TeamLTL model checking undecidable. In [23] Lück established that the model checking problem for TeamLTL$(\sim)$ is complete for third-order arithmetics and thus highly undecidable. The proof heavily utilizes the interplay between the Boolean negation $\sim$ and the disjunction $\vee$; it was left as an open problem whether some sensible restrictions of the use of the Boolean negation would lead toward discovering decidable logics. We show by a reduction from (lossy) counter machines that already very restricted access to $\sim$ leads to high undecidability, whereas $\subseteq$ and $\circledvee$ suffice for undecidability.

5.1 Counter machines and lossy counter machines

A *non-deterministic 3-counter machine* M consists of a finite list of instructions I that manipulate the content of three counters C_l, C_m, and C_r taking natural numbers as values. An *instruction set* I of cardinality n consists of labelled instructions of the following form:

- $i\colon C_a^+$ goto $\{j_1, j_2\}$,
- $i\colon C_a^-$ goto $\{j_1, j_2\}$,
- $i\colon$ if $C_a = 0$ goto j_1, else goto j_2,

where $a \in \{l, m, r\}$, $0 \leq i, j_1, j_2 < n$. A *configuration* is a tuple (i, j, k, t), where $0 \leq i < n$ is the current instruction, and $j, k, t \in \mathbb{N}$ the values of the counters C_l, C_m, and C_r, respectively. The execution of the instruction $i\colon C_a^+$ goto $\{j_1, j_2\}$ ($i\colon C_a^-$ goto $\{j_1, j_2\}$, resp.) increments (decrements, resp.) the value of the counter C_a by 1. The next instruction is selected nondeterministically from the set $\{j_1, j_2\}$. The instruction $i\colon$ if $C_a = 0$ goto j_1, else goto j_2 checks whether the value of the counter C_a is 0 and proceeds to the next instruction accordingly. The *consecution relation* $\rightsquigarrow_{\mathrm{c}}$ of configurations is defined in the obvious way. The *lossy consecution relation* $(i_1, i_2, i_3, i_4) \rightsquigarrow_{\mathrm{lc}} (j_1, j_2, j_3, j_4)$ of configurations holds if $(i_1, i_2', i_3', i_4') \rightsquigarrow_{\mathrm{c}} (j_1, j_2', j_3', j_4')$ holds for some $i_2', i_3', i_4', j_2', j_3', j_4'$ with $i_2' \leq i_2$, $i_3' \leq i_3$, $i_4' \leq i_4$, $j_2 \leq j_2'$, $j_3 \leq j_3'$, and $j_4 \leq j_4'$. A *(lossy) computation* is an infinite sequence of (lossy) consecutive configurations starting from the initial configuration $(0, 0, 0, 0)$. A (lossy) computation is *b-recurring* if the instruction labelled b occurs infinitely often in it.

▶ **Theorem 19** ([1, 27]). *Deciding whether a given non-deterministic* 3*-counter machine has a b-recurring (b-recurring lossy, resp.) computation for a given b is Σ_1^1-complete (Σ_1^0-complete).*

5.2 Model checking is (highly) undecidable

We reduce the existence of a b-recurring lossy computation of a given 3-counter machine M and an instruction label b to the model checking problem of TeamLTL$(\subseteq, \varovee)$. We also illustrate that with a single instance of A we can enforce non-lossy computation instead.

▶ **Theorem 20.** *The model checking problem for* TeamLTL$(\subseteq, \varovee)$ *is Σ_1^0-hard.*

Proof. For any set I of instructions of a 3-counter machine M, and any instruction label b, we construct a TeamLTL$(\subseteq, \varovee)$-formula $\varphi_{I,b}$ and a Kripke structure K_I such that

$$\big(\mathrm{Traces}(\mathrm{K}_I), 0\big) \models \varphi_{I,b} \Leftrightarrow M \text{ has a } b\text{-recurring lossy computation.} \tag{2}$$

The Σ_1^0-hardness then follows from Theorem 19, since our construction is clearly computable.

The idea is as follows. Put $n = |I|$. A sequence $(\vec{c}_i)_{i \in \mathbb{N}}$ of configurations is encoded by a set T of traces over the set $\{c_l, c_m, c_r, 0, \dots, n-1\}$ of propositions if, for each $j \in \mathbb{N}$ and $\vec{c}_j = (i, l, m, r)$,

- $t[j] \cap \{0, \dots, n-1\} = \{i\}$ for each $t \in T$,
- $|\{t[j, \infty] \mid c_s \in t[j] \text{ and } t \in T\}| = s$ for each $s \in \{l, m, r\}$.

We then use the formula $\varphi_{I,b}$ to enforce that in such a computation the rules of the counter machine are followed.

Now define $\varphi_{I,b} := (\theta_{\mathrm{comp}} \wedge \theta_{b-\mathrm{rec}}) \vee_{\mathrm{L}} \top$, where $\vee_{\mathrm{L}}$ is a new disjunction with the semantics: $(T, i) \models \varphi \vee_{\mathrm{L}} \psi$ iff $\exists T_1, T_2 \subseteq T$ s.t. $T_1 \neq \emptyset$, $T = T_1 \cup T_2$, $(T_1, i) \models \varphi$ and $(T_2, i) \models \psi$. The disjunction $\vee_{\mathrm{L}}$ can be defined using $\subseteq$, $\vee$, and a built-in trace $\{p\}^\omega$, where p is a fresh proposition (see e.g., [15, Lemma 3.4]). The formula $\theta_{b-\mathrm{rec}} := \mathsf{G}\,\mathsf{F}\, b$ describes the b-recurrence condition of the computation. The other formula θ_{comp}, which we define below in steps, states that the encoded computation is a legal one.

First, define

$$\mathit{singleton} := \mathsf{G} \bigwedge_{a \in \mathrm{AP}} (a \varovee \neg a), \quad \text{and} \quad c_s\!-\!\mathit{preserve} := c_s \vee (\neg c_s \wedge \bigcirc \neg c_s) \text{ for } s \in \{l, m, r\}.$$

Next, for each label i of an instruction, we define a formula θ_i describing the result of the execution of the instruction as follows:

- For the instruction $i\colon C_l^+$ goto $\{j,j'\}$, define
  $\theta_i := \bigcirc(j \varovee j') \wedge \big((\mathit{singleton} \wedge \neg c_l \wedge \bigcirc c_l) \vee_{\mathrm{L}} c_l{-}\mathit{preserve}\big) \wedge c_r{-}\mathit{preserve} \wedge c_m{-}\mathit{preserve}$
- For the instruction $i\colon C_l^-$ goto $\{j,j'\}$, define
  $\theta_i := \bigcirc(j \varovee j') \wedge \big((\mathit{singleton} \wedge c_l \wedge \bigcirc \neg c_l) \vee_{\mathrm{L}} c_l{-}\mathit{preserve}\big) \wedge c_r{-}\mathit{preserve} \wedge c_m{-}\mathit{preserve}$
- For instructions $i\colon C_s^+$ goto $\{j,j'\}$ and $i\colon C_s^-$ goto $\{j,j'\}$ with $s \in \{m,r\}$, the formulae θ_i are defined analogously with the indices l, m, and r permuted.
- For the instruction $i\colon$ if $C_s = 0$ goto j, else goto j', define
  $\theta_i := \big(\bigcirc(\neg c_s \wedge j) \varovee (\top \subseteq c_s \wedge \bigcirc j')\big) \wedge c_l{-}\mathit{preserve} \wedge c_m{-}\mathit{preserve} \wedge c_r{-}\mathit{preserve}.$

Finally, define $\theta_{\mathrm{comp}} := \mathsf{G} \bigovee_{i<n}(i \wedge \theta_i)$.

Next, we define the Kripke structure $\mathrm{K}_I = (W, R, \eta, w_0)$ over $\{c_l, c_m, c_r, 0, \dots, n-1\}$. Let $W := \{(i,j,k,t) \mid 0 \leq i < n$ and $j,k,t \in \{0,1\}\}$, $w_0 = (0,0,0,0)$, R is defined to be the complete binary relation on W, and η is a valuation such that $\eta\big((i,j,k,t)\big) \cap \{0,\dots,n-1\} = i$, $c_l \in \eta\big((i,j,k,t)\big)$ if $j = 1$, $c_m \in \eta\big((i,j,k,t)\big)$ if $k = 1$, and $c_r \in \eta\big((i,j,k,t)\big)$ if $t = 1$.

Finally, to see that (2) holds, note that $\big(\mathrm{Traces}(\mathrm{K}_I), 0\big) \models \theta_{I,b}$ iff there exists a non-empty team $T \subseteq \big(\mathrm{Traces}(\mathrm{K}_I), 0\big)$ such that $(T, 0) \models \theta_{\mathrm{comp}} \wedge \theta_{b-\mathrm{rec}}$. A team T that potentially encodes a computation is guessed. The formulae θ_{comp} and $\theta_{b-\mathrm{rec}}$ then enforce that the guess indeed encodes a b-recurrent lossy computation. The detailed proof is omitted. ◀

The formula *singleton* in the above proof only expresses that from now on all traces in the team in question coincide, which does not necessarily imply that the team is actually a singleton itself. For this reason, the above proof employs a lossy computation, instead of a non-lossy one. Nevertheless, the formula $\theta_{\mathrm{diff}} := \mathsf{A}\big(\,\mathsf{G}(q \varovee \neg q) \varovee \mathsf{G}\,\mathsf{F}(\top \subseteq q \wedge \bot \subseteq q)\big)$ with an A operator (which states that each pair of traces that differ, differ infinitely often) aids the formula *singleton* to define exactly the property that the team in question is a singleton. Now, the theorem below can be obtained by redefining the formulae in the proof of Theorem 20 as $\varphi_{I,b} := (\theta_{\mathrm{diff}} \wedge \theta_{\mathrm{comp}} \wedge \theta_{b-\mathrm{rec}}) \vee_{\mathrm{L}} \top$, $c_s{-}\mathit{preserve} := \big((c_s \wedge \bigcirc c_s) \vee (\neg c_s \wedge \bigcirc \neg c_s)\big)$ for $s \in \{l,m,r\}$, and for the instruction $i\colon$ if $C_s = 0$ goto j, else goto j', define

$$\theta_i := \big((\neg c_s \wedge \bigcirc j) \varovee (\top \subseteq c_s \wedge \bigcirc j')\big) \wedge c_l{-}\mathit{preserve} \wedge c_m{-}\mathit{preserve} \wedge c_r{-}\mathit{preserve}.$$

▶ **Theorem 21.** *The model checking problem for* TeamLTL$(\subseteq, \varovee, \mathsf{A})$ *is* Σ_1^1*-hard. The result holds also for the fragment with only one occurrence of* A*.*

6 Conclusion

We studied TeamLTL under the synchronous semantics, a logic that can concisely express a collection of hyperproperties that reasons about sub-parts of systems. We related the expressiveness of TeamLTL to the hyperlogics HyperLTL, HyperQPTL, and HyperQPTL$^+$, which are obtained by extending traditional temporal logics with trace quantifiers. We showed that for k-coherent properties, TeamLTL$(\mathcal{A}, \varovee, \mathsf{A})$ is subsumed by $\forall^*$HyperLTL. We also proved that all generalised atoms can be expressed in $\exists^*\forall$HyperLTL. For a more expressive logic, we studied the left-flat fragment of TeamLTL, which contains interesting properties not expressible in HyperLTL. We showed that this fragment is contained in $\overset{u}{\exists}{}^*_q\forall_\pi$HyperQPTL, from which we obtained that model checking of the left-flat fragment is decidable, with different complexities depending on the included atoms and connectives. We also related full TeamLTL$(\mathcal{A}, \varovee)$ to a fragment of HyperQPTL$^+$, which underlines the expressiveness of TeamLTL when extended with additional connectives and atoms. As another indicator, we showed that TeamLTL$(\subseteq, \varovee)$ is already undecidable.

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A Proof of Proposition 2

▶ **Proposition 2.** *For any n-ary downward closed generalised atom $\#_G$ having the empty team property and* LTL*-formulae $\varphi_1, \ldots, \varphi_n$, we have that*

$$\#_G(\varphi_1, \ldots, \varphi_n) \equiv \bigovee_{R \in G} \bigvee_{(b_1,\ldots,b_n) \in R} \overset{1}{\mathsf{A}}(\varphi_1^{b_1} \wedge \cdots \wedge \varphi_n^{b_n}),$$

where $\varphi_i^1 := \varphi_i$ and $\varphi_i^0 := \neg\varphi_i$ in negation normal form. In particular, for any set $\mathcal{A}$ of downward closed atoms, $\mathrm{TeamLTL}(\mathcal{A} \cup \{\varovee, \overset{1}{\mathsf{A}}, \mathsf{A}\}) \equiv \mathrm{TeamLTL}(\varovee, \overset{1}{\mathsf{A}})$*. The elimination of each atom yields a doubly exponential disjunction over linear sized formulae.*

Proof. The proposition follows essentially from a similar argument to that of a similar translation given in [30]. For any $\vec{b} = (b_1, \ldots, b_n) \in R \in G$, put $\vec{\varphi}^{\vec{b}} = \varphi_1^{b_1} \wedge \cdots \wedge \varphi_n^{b_n}$. By the empty team property TeamLTL, we have that

$$\begin{aligned}
(S, i) \models \overset{1}{\mathsf{A}}\vec{\varphi}^{\vec{b}} &\Leftrightarrow S = \emptyset \text{ or } \forall t \in S : ([\![\varphi_1]\!]_{(t,i)}, \ldots, [\![\varphi_n]\!]_{(t,i)}) = (b_1, \ldots, b_n) \\
&\Leftrightarrow \{([\![\varphi_1]\!]_{(t,i)}, \ldots, [\![\varphi_n]\!]_{(t,i)}) \mid t \in S\} \subseteq \{(b_1, \ldots, b_n)\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
(T, i) \models \bigvee_{\vec{b} \in R} \overset{1}{\mathsf{A}}\vec{\varphi}^{\vec{b}} &\Leftrightarrow \forall \vec{b} \in R, \exists T_{\vec{b}} \text{ s.t. } T = \bigcup_{\vec{b} \in R} T_{\vec{b}},\ \{([\![\varphi_1]\!]_{(t,i)}, \ldots, [\![\varphi_n]\!]_{(t,i)}) \mid t \in T_{\vec{b}}\} \subseteq \{(b_1, \ldots, b_n)\} \\
&\Leftrightarrow \{([\![\varphi_1]\!]_{(t,i)}, \ldots, [\![\varphi_n]\!]_{(t,i)}) \mid t \in T\} \subseteq R.
\end{aligned}$$

Finally,

$$\begin{aligned}
(T, i) \models \bigovee_{R \in G} \bigvee_{\vec{b} \in R} \overset{1}{\mathsf{A}}\vec{\varphi}^{\vec{b}} &\Leftrightarrow (T, i) \models \bigvee_{\vec{b} \in R} \overset{1}{\mathsf{A}}\vec{\varphi}^{\vec{b}} \text{ for some } R \in G \\
&\Leftrightarrow \{([\![\varphi_1]\!]_{(t,i)}, \ldots, [\![\varphi_n]\!]_{(t,i)}) \mid t \in T\} \subseteq R \text{ for some } R \in G \\
&\Leftrightarrow \{([\![\varphi_1]\!]_{(t,i)}, \ldots, [\![\varphi_n]\!]_{(t,i)}) \mid t \in T\} \in G \\
&\qquad\qquad (\because \#_G \text{ is downward closed}) \\
&\Leftrightarrow (T, i) \models \#_G(\varphi_1, \ldots, \varphi_n).
\end{aligned}$$

◀

B Proof of Proposition 5 and Lemma 6

▶ **Proposition 5** (Cf. [16, Lemma 5.6])**.** *If φ is a formula in some downward closed logic* $\mathrm{TeamLTL}(\mathcal{A})$*, then* $\dim(\varphi) \leq \mathrm{Dim}(\varphi)$*.*

Proof. Let (T, i) be a team, and $S \subseteq T$ a minimal subteam such that $(S, i) \not\models \varphi$. It suffices to show that $|S| \leq \mathrm{Dim}(\varphi)$. If $S = \emptyset$, then obviously $|S| = 0 \leq \mathrm{Dim}(\varphi)$ and we are done. Now assume that $S \neq \emptyset$. For every $R \in M(\varphi, (T, i))$, observe that $S \setminus R \neq \emptyset$, since otherwise $S \subseteq R$, which contradicts the downward closure of φ. Choose $t_R \in S \setminus R$, and consider the team $S_0 = \{t_R \mid R \in M(\varphi, (T, i))\}$.

Now, since $S_0 \not\supseteq R$ for every $R \in M(\varphi, (T, i))$ and φ is downward closed, we must have that $(S_0, i) \not\models \varphi$. On the other hand, since $S_0 \subseteq S$, by minimality of S we are forced to conclude that $S_0 = S$. Hence, $|S| = |S_0| \leq |M(\varphi, (T, i))| \leq \mathrm{Dim}(\varphi)$. ◀

▶ **Lemma 6.** *In any downward closed logic* $\mathrm{TeamLTL}(\mathcal{A})$*, we have the following estimates:*

(i) $\mathrm{Dim}(p) = \mathrm{Dim}(\neg p) = 1$
(ii) $\mathrm{Dim}(\#(\varphi_1, \dots, \varphi_n)) \leq 2^n$
(iii) $\mathrm{Dim}(\varphi \wedge \psi) \leq \mathrm{Dim}(\varphi)\,\mathrm{Dim}(\psi)$
(iv) $\mathrm{Dim}(\varphi \vee \psi) \leq \mathrm{Dim}(\varphi)\,\mathrm{Dim}(\psi)$
(v) $\mathrm{Dim}(\varphi \otimes \psi) \leq \mathrm{Dim}(\varphi) + \mathrm{Dim}(\psi)$
(vi) $\mathrm{Dim}(\bigcirc \varphi) \leq \mathrm{Dim}(\varphi)$
(vii) $\mathrm{Dim}(\mathsf{F}\,\varphi) \leq \omega\,\mathrm{Dim}(\varphi)$
(viii) $\mathrm{Dim}(\mathsf{G}\,\varphi) \leq \mathrm{Dim}(\varphi)^\omega$
(ix) $\mathrm{Dim}(\varphi\,\mathcal{U}\,\psi) \leq \sum_{n\in\mathbb{N}} \mathrm{Dim}(\varphi)^n \mathrm{Dim}(\psi)$
(x) $\mathrm{Dim}(\varphi\,\mathcal{W}\,\psi)$
$\leq \mathrm{Dim}(\varphi)^\omega + \sum_{n\in\mathbb{N}} \mathrm{Dim}(\varphi)^n \mathrm{Dim}(\psi)$

Proof. Items (i), (ii), (v) are easy to see. Item (iii) is analogous to (iv). We only show the remaining items.

For (iv), for any team (T, i), it is not hard to see that $M(\varphi \vee \psi, (T, i)) \subseteq \{R \cup S \mid R \in M(\varphi, (T, i))$ and $S \in M(\psi, (T, i))\}$. Thus, $|M(\varphi\vee\psi, (T, i)| \leq |M(\varphi, (T, i))|\times|M(\psi, (T, i))| \leq \mathrm{Dim}(\varphi)\mathrm{Dim}(\psi)$, thereby $\mathrm{Dim}(\varphi \vee \psi) \leq \mathrm{Dim}(\varphi)\mathrm{Dim}(\psi)$.

For (vi), for any team (T, i), it is easy to see that $|M(\bigcirc \varphi, (T, i))| = |M(\varphi, (T, i + 1))| \leq \mathrm{Dim}(\varphi)$. Thus, $\mathrm{Dim}(\bigcirc \varphi) \leq \mathrm{Dim}(\varphi)$.

For (vii), for any team (T, i), it is easy to see that $M(\mathsf{F}\,\varphi, (T, i)) \subseteq \bigcup_{k\geq i} M(\varphi, (T, k))$. Thus, $|M(\mathsf{F}\,\varphi, (T, i))| \leq \sum_{k\geq i} |M(\varphi, (T, k))| \leq \sum_{n\in\mathbb{N}} \mathrm{Dim}(\varphi) \leq \omega\mathrm{Dim}(\varphi)$, thereby $\mathrm{Dim}(\mathsf{F}\,\varphi \leq \omega\mathrm{Dim}(\varphi)$.

For (viii), for any team (T, i), since $\mathsf{G}\,\varphi$ is downward closed, it is not hard to see that $M(\mathsf{G}\,\varphi, (T, i)) \subseteq \{\bigcap_{k\geq i} S_k \mid S_k \in M(\varphi, (T, k))$ for all $k \geq i\}$. Thus, $|M(\mathsf{G}\,\varphi, (T, i))| \leq \prod_{k\geq i} |M(\varphi, (T, k))| \leq \prod_{n\in\mathbb{N}} \mathrm{Dim}(\varphi) \leq \mathrm{Dim}(\varphi)^\omega$, thereby $\mathrm{Dim}(\mathsf{G}\,\varphi) \leq \mathrm{Dim}(\varphi)^\omega$.

For (ix), for any team (T, i), since $\varphi\,\mathcal{U}\,\psi$ is downward closed, it is not hard to see that $M(\varphi\,\mathcal{U}\,\psi, (T, i)) \subseteq \bigcup_{k\geq i}\{R \cap \bigcap_{i=m}^{k} S_m \mid R \in M(\psi, (T, k)), S_m \in M(\varphi, (T, m))$ for all $i \leq m < k\}$. Thus, $|M(\varphi\,\mathcal{U}\,\psi, (T, i)| \leq \sum_{k\geq i} |M(\varphi, (T, i))|\times\cdots\times|M(\varphi, (T, k-1))|\times|M(\psi, (T, k))| \leq \sum_{k\geq i} \mathrm{Dim}(\varphi)^k\mathrm{Dim}(\psi) \leq \sum_{n\in\mathbb{N}} \mathrm{Dim}(\varphi)^n\mathrm{Dim}(\psi)$, thereby $\mathrm{Dim}(\varphi\,\mathcal{U}\,\psi) \leq \sum_{n\in\mathbb{N}} \mathrm{Dim}(\varphi)^n\mathrm{Dim}(\psi)$

For (x), for any team (T, i), we have $|M(\varphi\,\mathcal{W}\,\psi, (T, i))| \leq |M(\mathsf{G}\,\varphi, (T, i))\cup M(\varphi\,\mathcal{U}\,\psi, (T, i))| = |M(\mathsf{G}\,\varphi, (T, i))| + |M(\varphi\,\mathcal{U}\,\psi, (T, i))| = \mathrm{Dim}(\varphi)^\omega + \sum_{n\in\mathbb{N}} \mathrm{Dim}(\varphi)^n\mathrm{Dim}(\psi)$. ◀

C Proof of Lemma 8

▶ **Lemma 8.** *Let $\mathcal{A}$ be a set of generalised atoms and φ a* $\mathrm{TeamLTL}(\mathcal{A}, \otimes, \mathsf{A})$*-formula. Let $\Phi = \{\pi_1, \dots, \pi_k\}$ be a finite set of trace variables. For any team (T, i) with $|T| \leq k$, any set $S \supseteq T$ of traces, and any assignment Π with $\Pi[\Phi] = T$, we have that*

$$(T, i) \models \varphi \quad \textit{iff} \quad \Pi, i \models_S \varphi^\Phi.$$

Furthermore, if the generalised atoms in $\mathcal{A}$ are all downward closed, then

$$(T, i) \models \varphi \quad \textit{iff} \quad \emptyset, i \models_T \forall\pi_1 \dots \forall\pi_k\, \varphi^\Phi.$$

Proof. The second claim of the lemma follows easily from the first claim and the downward closure of the logic $\mathrm{TeamLTL}(\mathcal{A}, \otimes, \mathsf{A})$. We only give the detailed proof for the first claim. We proceed by induction on φ. We only give the details for three nontrivial inductive cases.

The case for $\vee$:

$$\begin{aligned}(T, i) \models \varphi \vee \psi \quad &\Leftrightarrow \quad (\Pi[\Phi_0], i) \models \varphi \text{ and } (\Pi[\Phi_1], i) \models \psi \text{ for some } \Phi_0, \Phi_1 \text{ with } \Phi_0 \cup \Phi_1 = \Phi\\ &\qquad\qquad (\text{since } T = \Pi[\Phi])\\ &\Leftrightarrow \quad \Pi, i \models_S \varphi^{\Phi_0} \text{ and } \Pi, i \models_S \psi^{\Phi_1} \text{ for some } \Phi_0, \Phi_1 \text{ with } \Phi_0 \cup \Phi_1 = \Phi\\ &\qquad\qquad (\text{by induction hypothesis, since } S \supseteq T = \Pi[\Phi] \supseteq \Pi[\Phi_0], \Pi[\Phi_1])\end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \quad \Pi, i \models_S \bigvee_{\Phi_0 \cup \Phi_1 = \Phi} (\varphi^{\Phi_0} \wedge \psi^{\Phi_1})\\
&\Leftrightarrow \quad \Pi, i \models_S (\varphi \vee \psi)^{\Phi}.
\end{aligned}$$

The case for $\mathcal{U}$:

$$\begin{aligned}
(T,i) \models \psi \,\mathcal{U}\, \varphi \quad &\Leftrightarrow \quad \exists n \geq i : (T,n) \models \varphi, \text{ and } \forall m : i \leq m < n \Rightarrow (T,m) \models \psi\\
&\Leftrightarrow \quad \exists n \geq i : \Pi, n \models_S \varphi^{\Phi}, \text{ and } \forall m : i \leq m < n \Rightarrow \Pi, m \models_S \psi^{\Phi}\\
&\Leftrightarrow \quad \Pi, i \models_S \psi^{\Phi} \,\mathcal{U}\, \varphi^{\Phi}\\
&\Leftrightarrow \quad \Pi, i \models_S (\psi \,\mathcal{U}\, \varphi)^{\Phi}.
\end{aligned}$$

The case for A:

$$\begin{aligned}
(T,i) \models \mathsf{A}\varphi \quad &\Leftrightarrow \quad (\Pi[\Psi], i) \models \varphi \text{ for all } \Psi \subseteq \Phi && \text{(since } T = \Pi[\Phi])\\
&\Leftrightarrow \quad \Pi, i \models_S \varphi^{\Psi} \text{ for all } \Psi \subseteq \Phi && \text{(by IH, since } S \supseteq T \supseteq \Phi[\Psi])\\
&\Leftrightarrow \quad \Pi, i \models_S \bigwedge_{\Psi \subseteq \Phi} \varphi^{\Psi}\\
&\Leftrightarrow \quad \Pi, i \models_S (\mathsf{A}\varphi)^{\Phi}.
\end{aligned}$$

◀

D Proof of Theorem 13

For a given team (T,i) and TeamLTL$(\mathcal{A}, \otimes)$-formula φ, the verification of $(T,i) \models \varphi$, using the semantical rules of the connectives and atoms, boils down to checking statements of the form $(S,j) \models \psi$, where $S \in \mathcal{S}_T \subseteq 2^T$, $j \in \mathbb{N}$, and ψ is an atomic formula, together with expressions of the form $S_1 = S_2 \cup S_3$, where $S_1, S_2, S_3 \in \mathcal{S}_T$. The following lemma implies that the set $\mathcal{S}_T$ can be fixed as a countable set that depends only on T.

▶ **Lemma 22.** *For every set T of traces over* AP*, there exists a countable set $\mathcal{S}_T \subseteq 2^T$ such that for every* TeamLTL$(\mathcal{A}, \otimes)$*-formula φ and $i \in \mathbb{N}$*

$$(T,i) \models \varphi \quad \Leftrightarrow \quad (T,i) \models^* \varphi, \tag{3}$$

where the satisfaction relation $\models^*$ is defined the same way as $\models$ except that in the semantic clause for $\vee$ we require additionally that the two subteams $T_1, T_2 \in \mathcal{S}_T$.

Proof. We first define inductively and nondeterministically a function Sub: $(2^T \times \mathbb{N}) \times$ TeamLTL$(\otimes, \mathcal{A}) \to 2^T$ as follows:

- $\mathrm{Sub}\big((S,j), \varphi\big) := \{S\}$, if φ is an atomic formula or a generalised atom.
- $\mathrm{Sub}\big((S,j), \bigcirc\varphi\big) := \mathrm{Sub}\big((S, i+1), \varphi\big)$.
- $\mathrm{Sub}\big((S,j), \varphi \wedge \psi\big) := \mathrm{Sub}\big((S,j), \varphi \otimes \psi\big) := \mathrm{Sub}\big((S,j), \varphi\big) \cup \mathrm{Sub}\big((S,j), \psi\big)$.
- Nondeterministically guess subsets S_1 and S_2 of S such that $S_1 \cup S_2 = S$, and define $\mathrm{Sub}\big((S,j), \varphi \vee \psi\big) := \mathrm{Sub}\big((S_1,j), \varphi\big) \cup \mathrm{Sub}\big((S_2,j), \psi\big) \cup \{S\}$.
- $\mathrm{Sub}\big((S,j), \psi \,\mathcal{U}\, \varphi\big) := \mathrm{Sub}\big((S,j), \psi \,\mathcal{W}\, \varphi\big) := \bigcup_{i \geq j} \big(\mathrm{Sub}\big((S,i), \psi\big) \cup \mathrm{Sub}\big((S,i), \varphi\big)\big)$.

Clearly $\mathrm{Sub}\big((S,j), \psi\big)$ is a countable set. Now, define

$$\mathcal{S}_T := \bigcup_{\substack{j \in \mathbb{N}\\ \psi \in \mathrm{TeamLTL}(\otimes, \mathcal{A})}} \mathrm{Sub}\big((T,j), \psi\big).$$

The set $\mathcal{S}_T$ is a countable union of countable sets, and thus itself countable. For each team (S, j) and formula ψ, assuming $\mathcal{S}_T \supseteq \mathrm{Sub}\big((S, j), \psi\big)$, it is straightforward to show by induction that (3) holds. ◀

▶ **Theorem 13.** *For every* TeamLTL$(\mathcal{A}, \mathbin{\otimes})$*-formula φ, with $\mathcal{A}$ a set of generalised atoms, there exists an equivalent* 2EXPTIME*-computable HyperQPTL^{+}-formula φ^* in the $\exists_p \overset{u}{Q}{}^*_p \exists^*_\pi \forall_\pi$ fragment. If the atoms in $\mathcal{A}$ are downward closed, φ^* can be defined in the $\exists_p \overset{u}{Q}{}^*_p \forall_\pi$ fragment. For formulae without generalised atoms, the translation can be done in linear time.*

Proof. For any two distinct propositional variables q and r, we define a compositional translation $\mathrm{TR}_{(q,r)}\colon \mathrm{TeamLTL}(\mathcal{A}, \mathbin{\otimes}) \to \mathrm{HyperQPTL}$ such that for every team (T, i) and TeamLTL$(\mathcal{A}, \mathbin{\otimes})$-formula φ,

$$(T, i) \models \varphi \quad \Leftrightarrow \quad \Pi, i \models_T \exists q^{\mathcal{S}_T} \overset{u}{\exists} q \overset{u}{\exists} r \big(\mathrm{TR}_{(q,r)}(\varphi) \wedge \varphi_{\mathrm{aux}}\big), \tag{4}$$

where $\varphi_{\mathrm{aux}} := \forall \pi\, (q^{\mathcal{S}_T}_\pi \wedge q_\pi \wedge \bigcirc^i r)$.

We first fix some conventions. All quantified variables in the translation below are assumed to be fresh and distinct. We also assume that the uniformly quantified propositional variables $q, r, \ldots$ are true in exactly one level, that is, they satisfy the formula $\forall \pi\, \mathsf{F}\, q_\pi \wedge \mathsf{G}(q_\pi \to \bigcirc \mathsf{G}\, \neg q_\pi)$, which we will omit in the presentation of the translation for simplicity.

The idea behind the translation is the following. Let (T', i) denote the team obtained from (T, i) by evaluating the quantifier $\exists q^{\mathcal{S}_T}$.

- The variable $q^{\mathcal{S}_T}$ is used to encode the countable set $\mathcal{S}_T$ of sets of traces given by Lemma 22. To be precise, for each $i \in \mathbb{N}$, $q^{\mathcal{S}_T}$ encodes the set $(\{t \in T' \mid q^{\mathcal{S}_T} \in t[i]\} \upharpoonright \mathrm{AP}) \in \mathcal{S}_T$.
- The uniformly quantified variable q in $\mathrm{TR}_{(q,r)}$ is used to encode an element of $\mathcal{S}_T$ using $q^{\mathcal{S}_T}$: If $q \in t[i]$, then q encodes the set $\{t \in T' \mid q^{\mathcal{S}_T} \in t[i]\} \upharpoonright \mathrm{AP}$.
- The uniformly quantified variable r in $\mathrm{TR}_{(q,r)}$ is used to encode the time step i of a team (T, i): If $r \in t[i]$, then r encodes the time step i.

After fixing a suitable interpretation for $q^{\mathcal{S}_T}$, teams (S, i) can be encoded with pairs of uniformly quantified variables (q, r), whenever $S \in \mathcal{S}_T$. The formula φ_{aux} expresses that the pair (q, r) encodes the team (T, i) in question.

The translation $\mathrm{TR}_{(q,r)}$ is defined inductively as follows:

$$
\begin{aligned}
\mathrm{TR}_{(q,r)}(p) &:= \forall \pi \big(\mathsf{F}(q_\pi \wedge q^{\mathcal{S}_T}_\pi) \to \mathsf{F}(r_\pi \wedge p_\pi)\big),\\
\mathrm{TR}_{(q,r)}(\neg p) &:= \forall \pi \big(\mathsf{F}(q_\pi \wedge q^{\mathcal{S}_T}_\pi) \to \mathsf{F}(r_\pi \wedge \neg p_\pi)\big),\\
\mathrm{TR}_{(q,r)}(\varphi \vee \psi) &:= \overset{u}{\exists} q_1 q_2 \,\big(\varphi_{\cup}(q, q_1, q_2) \wedge \mathrm{TR}_{(q_1,r)}(\varphi) \wedge \mathrm{TR}_{(q_2,r)}(\psi)\big),\\
\mathrm{TR}_{(q,r)}(\varphi \mathbin{\otimes} \psi) &:= \mathrm{TR}_{(q,r)}(\varphi) \vee \mathrm{TR}_{(q,r)}(\psi),\\
\mathrm{TR}_{(q,r)}(\varphi \wedge \psi) &:= \mathrm{TR}_{(q,r)}(\varphi) \wedge \mathrm{TR}_{(q,r)}(\psi),\\
\mathrm{TR}_{(q,r)}(\bigcirc \varphi) &:= \overset{u}{\exists} r' \big(r \prec r' \wedge \mathrm{TR}_{(q,r')}(\varphi)\big),\\
\mathrm{TR}_{(q,r)}(\varphi\, \mathcal{U}\, \psi) &:= \overset{u}{\exists} r' \Big(r \preceq r' \wedge \mathrm{TR}_{(q,r')}(\psi) \wedge \overset{u}{\forall} r'' \big((r \preceq r'' \wedge r'' \prec r') \to \mathrm{TR}_{(q,r'')}(\varphi)\big)\Big),\\
\mathrm{TR}_{(q,r)}(\varphi\, \mathcal{W}\, \psi) &:= \forall r' \Big(r \leq r' \to \Big(\mathrm{TR}_{(q,r')}(\varphi) \vee \exists r'' \big(r'' \preceq r' \wedge \mathrm{TR}_{q,r''}(\psi)\big)\Big)\Big),
\end{aligned}
$$

$\mathrm{TR}_{(q,r)}(\#_G(\varphi_1, \ldots, \varphi_n)) :=$

$$
\begin{cases}
\exists \pi_1 \ldots \exists \pi_{2^n} \forall \pi \Big(\mathsf{F}\left(q_\pi \wedge q^{\mathcal{S}_T}_\pi\right) \to \mathsf{F}\left(q_\pi \wedge \bigwedge_{1 \leq i \leq 2^n} q^{\mathcal{S}_T}_{\pi_i}\right) \wedge \mathsf{F}\left(r_\pi \wedge \theta_G\right)\Big), & \text{if } \emptyset \in G,\\
\exists \pi_1 \ldots \exists \pi_{2^n} \forall \pi \Big(\mathsf{F}\left(q_\pi \wedge \bigwedge_{1 \leq i \leq 2^n} q^{\mathcal{S}_T}_{\pi_i}\right) \wedge \Big(\mathsf{F}\left(q_\pi \wedge q^{\mathcal{S}_T}_\pi\right) \to \mathsf{F}\left(r_\pi \wedge \theta_G\right)\Big)\Big), & \text{if } \emptyset \notin G,
\end{cases}
$$

where $r \prec r' := \mathsf{G}(r_\pi \leftrightarrow \bigcirc r'_\pi)$, $r \preceq r' := \mathsf{G}(r_\pi \to \mathsf{G}\, r'_\pi)$,

$$\varphi_\cup(q, q', q'') := \mathsf{F}(q_\pi \wedge q_\pi^{\mathcal{S}_T}) \leftrightarrow \mathsf{F}\left((q'_\pi \vee q''_\pi) \wedge q_\pi^{\mathcal{S}_T}\right),$$

and θ_G is the quantifier-free subformula of the formula $\psi_G = \exists\pi_1 \ldots \exists\pi_{2^n}\forall\pi\, \theta_G$ given by Proposition 12.

It is easy to transform $\mathrm{TR}_{(q,r)}(\varphi) \wedge \varphi_{\mathrm{aux}}$ to an equivalent prenex formula in the fragment $\overset{u}{Q}{}^*_p \exists^*_\pi \forall_\pi$, and the translation is at most doubly exponential. The equivalence (4) is proved by a straightforward inductive argument. We omit the detailed proof.

For any formula $\varphi \in \mathrm{TeamLTL}(\otimes)$ without the generalised atoms, the above translation yields a formula $\mathrm{TR}_{(q,r)}(\varphi) \wedge \varphi_{\mathrm{aux}}$ that is linear in the size of φ, and the equivalent formula in the $\overset{u}{Q}{}^*_p \forall_\pi$ fragment of HyperQPTL is of the same size.

If $\mathcal{A}$ is a set of downward closed generalised atoms, the translation $\mathrm{TR}_{(q,r)}$ can be defined in the same way as above except that the case for generalised atoms can alternatively be defined using the doubly exponential translation of Proposition 2 to $\mathrm{TeamLTL}(\otimes, \overset{1}{\mathsf{A}})$. To be precise, if $\#_G$ is downward closed, define $\mathrm{TR}_{(q,r)}(\#_G(\varphi_1, \ldots, \varphi_n)) :=$

$$\begin{cases} \mathrm{TR}_{(q,r)}\Big(\bigotimes_{R \in G} \bigvee_{(b_1,\ldots,b_n)\in R} \overset{1}{\mathsf{A}}(\varphi_1^{b_1} \wedge \cdots \wedge \varphi_n^{b_n})\Big), & \text{if } \emptyset \in G, \\ \bot, & \text{if } \emptyset \notin G \text{ (i.e., } G = \emptyset), \end{cases}$$

where $\mathrm{TR}_{(q,r)}(\overset{1}{\mathsf{A}}\varphi) := \forall\pi\big(\mathsf{F}(q_\pi \wedge q_\pi^{\mathcal{S}_T}) \to \mathsf{F}(r_\pi \wedge \varphi_\pi)\big)$. ◀

E Proof of Theorem 16

▶ **Theorem 16.** *For every* $\mathrm{TeamLTL}(\otimes, \overset{1}{\mathsf{A}})$*-formula* φ*, we can compute an equivalent* $\overset{u}{\exists}{}^*_q \forall_\pi$ *HyperQPTL-formula of size linear in the size of* φ*.*

The theorem follows from the following lemma.

▶ **Lemma 23.** *Let* φ *be a* $\mathrm{TeamLTL}(\otimes, \overset{1}{\mathsf{A}})$ *formula, and* $r_1, \ldots, r_n$ *free variables occurring in* $[\varphi, r]$ *but not in* φ *(except* r*). Let* $i \in \mathbb{N}$*, and* $s \in (2^{\{r\}})^\omega$ *a sequence that has* r *set exactly at position* i*. For every team* (T, i)*,*

$$(T, i) \models \varphi \Leftrightarrow \emptyset, 0 \models_{T[r \mapsto s]} \exists r_1 \ldots r_n \forall\pi.\, [\varphi, r].$$

Proof. We proceed by induction on φ.
Case for p:

$$\begin{aligned}(T, i) \models p &\equiv \forall t \in T : p \in t[i] \\ &\equiv \emptyset, 0 \models_{T[r \mapsto s]} \forall\pi.\, \mathsf{G}(r_\pi \to p_\pi)\end{aligned}$$

Case for $\neg p$: Similar to the above.
Case for $\overset{1}{\mathsf{A}}\varphi$:

$$\begin{aligned}(T, i) \models \overset{1}{\mathsf{A}}\varphi &\equiv \forall t \in T : (t, i) \models \varphi \\ &\equiv \emptyset, 0 \models_T \forall\pi.\, \mathsf{G}(r_\pi \to \hat{\varphi}) && \text{(by Theorem 9, as } \overset{1}{\mathsf{A}}\varphi \text{ is flat or 1-coherent)}\end{aligned}$$

Case for $\bigcirc\varphi$:

$$(T, i) \models \bigcirc\varphi \equiv (T, i+1) \models \varphi$$

$$\overset{\text{IH}}{\equiv} \emptyset, 0 \models_{T[r^\varphi \mapsto s^\varphi]} \exists r_1 \dots r_n. \forall \pi. [\varphi, r^\varphi]$$
$$\equiv \emptyset, 0 \models_{T[r \mapsto s]} \exists r^\varphi. \exists r_1 \dots r_n. \forall \pi. r \prec r^\varphi \wedge [\varphi, r^\varphi]$$

Case for $\varphi \wedge \psi$: Easy, by induction hypothesis.
Case for $\varphi \vee \psi$:

$$\begin{aligned}(T, i) \models \varphi \vee \psi &\equiv \exists T_1, T_2 \text{ s.t. } T = T_1 \cup T_2 \text{ and } (T_1, i) \models \varphi \text{ and } (T_2, i) \models \psi \\ &\overset{\text{IH}}{\equiv} \emptyset, 0 \models_{T_1[r \mapsto s]} \exists r^1_1 \dots r^1_n. \forall \pi. [\varphi, r] \text{ and } \emptyset, 0 \models_{T_2[r \mapsto s]} \exists r^2_1 \dots r^2_m. \forall \pi. [\psi, r] \\ &\equiv \emptyset, 0 \models_{T[r \mapsto s]} \exists r^1_1 \dots r^1_n r^2_1 \dots r^2_m. \forall \pi. [\varphi, r] \vee [\psi, r] \\ &\qquad\qquad (\text{since } T_1[r \mapsto s] \cup T_2[r \mapsto s] = T[r \mapsto s])\end{aligned}$$

Case for $\varphi \otimes \psi$:

$$\begin{aligned}(T, i) \models \varphi \otimes \psi &\equiv (T, i) \models \varphi \text{ or } (T, i) \models \psi \\ &\overset{\text{IH}}{\equiv} \emptyset, 0 \models_{T[r \mapsto s]} \exists r^1_1 \dots r^1_n. \forall \pi. [\varphi, r] \text{ or } \emptyset, 0 \models_{T[r \mapsto s]} \exists r^2_1 \dots r^2_m. \forall \pi. [\psi, r] \\ &\equiv \emptyset, 0 \models_{T[r \mapsto s]} \exists d^{\varphi \otimes \psi}. \exists r^1_1 \dots r^1_n r^2_1 \dots r^2_m. \forall \pi. \\ &\qquad (d^{\varphi \otimes \psi}_\pi \to [\varphi, r]) \wedge (\neg d^{\varphi \otimes \psi}_\pi \to [\psi, r])\end{aligned}$$

Case for $\varphi \mathcal{U} \psi$:

$$\begin{aligned}(T, i) \models \varphi \mathcal{U} \psi &\equiv \exists i' \geq i. (T, i') \models \psi \text{ and } \forall i \leq i'' < i'. (T, i'') \models \varphi \\ &\equiv \exists i' \geq i. (T, i') \models \psi \text{ and } \forall i \leq i'' < i'. \emptyset, i'' \models_T \forall \pi. \hat{\varphi} \\ &\qquad\qquad (\text{by Theorem 9, since } \varphi \text{ is 1-coherent}) \\ &\overset{\text{IH}}{\equiv} \exists i' \geq i. \emptyset, 0 \models_{T[r^\psi \mapsto s^\psi]} \exists r_1 \dots r_n. \forall \pi. [\psi, r^\psi] \text{ and } \forall i \leq i'' < i'. \\ &\qquad \emptyset, i'' \models_T \forall \pi. \hat{\varphi} \text{ where } r^\psi \text{ is set in } s^\psi \text{ exactly at position } i' \\ &\equiv \exists i' \geq i. \emptyset, 0 \models_{T[r \mapsto s, r^\psi \mapsto s^\psi, r^\varphi \mapsto s^\varphi]} \exists r_1 \dots r_n. \forall \pi. \mathsf{G}(r^\varphi_\pi \to \hat{\varphi}) \wedge [\psi, r^\psi] \\ &\qquad \text{where } r^\psi \text{ is set in } s^\psi \text{ exactly at position } i' \text{ and} \\ &\qquad r^\varphi \text{ is set in } s^\varphi \text{ exactly at all positions between } i \text{ and } i' \\ &\equiv \emptyset, 0 \models_{T[r \mapsto s]} \models \exists r^\psi. \exists r^\varphi. \exists r_1 \dots r_n. \forall \pi. \mathit{once}(r^\psi) \wedge \\ &\qquad\qquad \mathsf{G}(r_\pi \to r^\varphi_\pi \mathcal{U} r^\psi_\pi) \wedge \mathsf{G}(r^\varphi_\pi \to \hat{\varphi}) \wedge [\psi, r^\psi]\end{aligned}$$

Case for $\varphi \mathcal{W} \psi$. Similar to $\varphi \mathcal{U} \psi$. ◀