

A Universal Attractor Decomposition Algorithm for Parity Games

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Abstract

An attractor decomposition meta-algorithm for solving parity games is given that generalizes the classic McNaughton-Zielonka algorithm and its recent quasi-polynomial variants due to Parys (2019), and to Lehtinen, Schewe, and Wojtczak (2019). The central concepts studied and exploited are attractor decompositions of dominia in parity games and the ordered trees that describe the inductive structure of attractor decompositions.

The main technical results include the embeddable decomposition theorem and the dominion separation theorem that together help establish a precise structural condition for the correctness of the universal algorithm: it suffices that the two ordered trees given to the algorithm as inputs embed the trees of some attractor decompositions of the largest dominia for each of the two players, respectively.

The universal algorithm yields McNaughton-Zielonka, Parys's, and Lehtinen-Schewe-Wojtczak algorithms as special cases when suitable universal trees are given to it as inputs. The main technical results provide a unified proof of correctness and deep structural insights into those algorithms.

This paper motivates a research program of developing new efficient algorithms for solving parity games by designing new classes of small trees that embed the largest dominia in relevant classes of parity games. An early success story in this research program is the recent development, by Daviaud, Jurdziński, and Thejaswini (2019), of Strahler-universal trees, which embed dominia in games of bounded register number, introduced by Lehtinen (2018). When run on these trees, the universal algorithm can solve games with bounded register number in polynomial time and in quasi-linear space.

A symbolic implementation of the universal algorithm is also given that improves the symbolic space complexity of solving parity games in quasi-polynomial time from $O(d \lg n)$ —achieved by Chatterjee, Dvořák, Henzinger, and Svozil (2018)—down to $O(\lg d)$, where n is the number of vertices and d is the number of distinct priorities in a parity game. This not only exponentially improves the dependence on d , but it also entirely removes the dependence on n .

Keywords parity games, universal trees, attractor decompositions, separation, embedding, quasi-polynomial, symbolic algorithms

1 Context

1.1 Parity games and their significance

Parity games play a fundamental role in automata theory, logic, and their applications to verification [9], program analysis [1, 19], and synthesis [17, 24]. In particular, parity games are very intimately linked to the problems of emptiness and complementation of non-deterministic automata on trees [9, 29], model checking and satisfiability checking of fixpoint logics [2, 9, 10], evaluation of nested fixpoint expressions [1, 18, 19], or fair simulation relations [11]. It is a long-standing open problem whether parity games can be solved in polynomial time [10].

The impact of parity games goes well beyond their home turf of automata theory, logic, and formal methods. For example, an answer [13] of a question posed originally for parity games [28] has strongly inspired major breakthroughs on the computational complexity of fundamental algorithms in stochastic planning [12] and linear optimization [15, 16], and parity games provide the foundation for the theory of nested fixpoint expressions used in program analysis [1, 19] and coalgebraic model checking [18].

1.2 Related work

The major breakthrough in the study of algorithms for solving parity games occurred in 2017 when Calude, Jain, Khoussainov, Li, and Stephan [4] have discovered the first quasi-polynomial algorithm. Three other—and seemingly distinctly different—techniques for solving parity games in quasi-polynomial time have been proposed in quick succession soon after: by Jurdziński and Lazić [20], by Lehtinen [22], and by Parys [26].

Czerwiński, Daviaud, Fijalkow, Jurdziński, Lazić, and Parys [6] have also uncovered an underlying combinatorial structure of universal trees as provably underlying the techniques of Calude et al., of Jurdziński and Lazić, and of Lehtinen. Czerwiński et al. have also established a quasi-polynomial lower bound for the size of smallest universal trees, providing evidence that the techniques developed in those three papers may be insufficient for leading to further improvements in the complexity of solving parity games. The

work of Parys [26] has not been obviously subject to the quasi-polynomial barrier of Czerwiński et al., making it a focus of current activity. It is worth noting, though, that Lehtinen, Schewe, and Wojtczak [23], who have improved the complexity of Parys’s algorithm somewhat, have made an informal observation that the tree of recursive calls of their algorithm is also universal. The algorithms of Parys and of Lehtinen et al. are modifications of the classic McNaughton-Zielonka algorithm [25, 29], which has exponential running time in the worst case [14], but consistently outperforms most other algorithms in practice [27].

1.3 Our contributions

In this work we provide a meta-algorithm—the *universal attractor decomposition algorithm*—that generalizes McNaughton-Zielonka, Parys’s, and Lehtinen-Schewe-Wojtczak algorithms. There are multiple benefits of considering the universal algorithm.

Firstly, in contrast to Parys’s and Lehtinen-Schewe-Wojtczak algorithms, the universal algorithm has a very simple and transparent structure that minimally departs from the classic McNaughton-Zielonka algorithm. Secondly, we observe that Lehtinen-Schewe-Wojtczak algorithm, as well as non-adaptive versions (see Sections 3.2 and 4.4) of McNaughton-Zielonka and Parys’s algorithms, all arise from the universal algorithm by using specific classes of universal trees, strongly linking the theory of universal trees to the only class of quasi-polynomial algorithms that had no established formal relationship to universal trees so far.

Thirdly, we further develop the theory of dominia and their attractor decompositions in parity games, initiated by Daviaud, Jurdziński, and Łazić [7] and by Daviaud, Jurdziński, and Lehtinen [8], and we prove two new structural theorems (the embeddable decomposition theorem and the dominion separation theorem) about ordered trees of attractor decompositions. Fourthly, we use the structural theorems to provide a unified proof of correctness of various McNaughton-Zielonka-style algorithms, identifying very precise structural conditions on the trees of recursive calls of the universal algorithm that result in it correctly identifying the largest dominia.

Finally, we observe that thanks to its simplicity, the universal algorithm is particularly well-suited for solving parity games efficiently in a symbolic model of computation, when large sizes of input graphs prevent storing them explicitly in memory. Indeed, we argue that already a routine implementation of the universal algorithm improves the state-of-the-art symbolic space complexity of solving parity games in quasi-polynomial time from $O(d \lg n)$ to $O(d)$, but we also show that a more sophisticated symbolic data structure allows to further reduce the symbolic space of the universal algorithm to $O(\lg d)$.

2 Dominia and decompositions

2.1 Strategies, traps, and dominia

A *parity game* \mathcal{G} consists of a finite directed graph (V, E) , a partition $(V_{\text{Even}}, V_{\text{Odd}})$ of the set of vertices V , and a function $\pi : V \rightarrow \{0, 1, \dots, d\}$ that labels every vertex $v \in V$ with a non-negative integer $\pi(v)$ called its *priority*. We say that a cycle is *even* if the highest vertex priority on the cycle is even; otherwise the cycle is *odd*. We say that a parity game is (n, d) -*small* if it has at most n vertices and all vertex priorities are at most d .

For a set S of vertices, we write $\mathcal{G} \cap S$ for the substructure of \mathcal{G} whose graph is the subgraph of (V, E) induced by the sets of vertices S . Sometimes, we also write $\mathcal{G} \setminus S$ to denote $\mathcal{G} \cap (V \setminus S)$. We assume throughout that every vertex has at least one outgoing edge, and we reserve the term *subgame* to substructures $\mathcal{G} \cap S$, such that every vertex in the subgraph of (V, E) induced by S has at least one outgoing edge. For a subgame $\mathcal{G}' = \mathcal{G} \cap S$, we sometimes write $V^{\mathcal{G}'}$ for the set of vertices S that the subgame \mathcal{G}' is induced by. When convenient and if the risk of confusion is contained, we may simply write \mathcal{G}' instead of $V^{\mathcal{G}'}$.

A (positional) *Even strategy* is a set $\sigma \subseteq E$ of edges such that:

- for every $v \in V_{\text{Even}}$, there is an edge $(v, u) \in \sigma$,
- for every $v \in V_{\text{Odd}}$, if $(v, u) \in E$ then $(v, u) \in \sigma$.

We sometimes call all the edges in such an Even strategy σ the *strategy edges*, and the definition of an Even strategy requires that every vertex in V_{Even} has an outgoing strategy edge, and every outgoing edge of a vertex in V_{Odd} is a strategy edge.

For a non-empty set of vertices T , we say that an Even strategy σ *traps Odd in T* if no strategy edge leaves T , that is, $w \in T$ and $(w, u) \in \sigma$ imply $u \in T$. We say that a set of vertices T is a *trap for Odd* if there is an Even strategy that traps Odd in T .

Observe that if T is a trap in a game \mathcal{G} then $\mathcal{G} \cap T$ is a subgame of \mathcal{G} . For brevity, we sometimes say that a subgame \mathcal{G}' is a trap if $\mathcal{G}' = \mathcal{G} \cap T$ and the set T is a trap in \mathcal{G} . Moreover, the following simple “*trap transitivity*” property holds: if T is a trap for Even in game \mathcal{G} and T' is a trap for Even in subgame $\mathcal{G} \cap T$ then T' is a trap in \mathcal{G} .

For a set of vertices $D \subseteq V$, we say that an Even strategy σ is an *Even dominion strategy on D* if:

- σ traps Odd in D ,
- every cycle in the subgraph (D, σ) is even.

Finally, we say that a set D of vertices is an *Even dominion* if there is an Even dominion strategy on it.

Odd strategies, trapping Even, and Odd dominia are defined in an analogous way by swapping the roles of the two players. It is an instructive exercise to prove the following two facts about Even and Odd dominia.

Proposition 2.1 (Closure under union). *If D and D' are Even (resp. Odd) dominia then $D \cup D'$ is also an Even (resp. Odd) dominion.*

Proposition 2.2 (Dominion disjointness). *If D is an Even dominion and D' is an Odd dominion then $D \cap D' = \emptyset$.*

From closure under union it follows that in every parity game, there is the largest Even dominion W_{Even} (which is the union of all Even dominia) and the largest Odd dominion W_{Odd} (which is the union of all Odd dominia), and from dominion disjointness it follows that the two sets are disjoint. The positional determinacy theorem states that, remarkably, the largest Even dominion and the largest Odd dominion form a partition of the set of vertices.

Theorem 2.3 (Positional determinacy [9]). *Every vertex in a parity game is either in the largest Even dominion or in the largest Odd dominion.*

2.2 Reachability strategies and attractors

In a parity game \mathcal{G} , for a target set of vertices B (“bullseye”) and a set of vertices A such that $B \subseteq A$, we say that an Even strategy σ is an *Even reachability strategy to B from A* if every infinite path in the subgraph (V, σ) that starts from a vertex in A contains at least one vertex in B .

For every target set B , there is the largest (with respect to set inclusion) set from which there is an Even reachability strategy to B in \mathcal{G} ; we call this set the *Even attractor to B in \mathcal{G}* and denote it by $\text{Attr}_{\text{Even}}^{\mathcal{G}}(B)$. *Odd reachability strategies and Odd attractors* are defined analogously.

We highlight the simple facts that if A is an attractor for a player in \mathcal{G} then its complement $V \setminus A$ is a trap for her; and that attractors are monotone operators: if $B' \subseteq B$ then the attractor to B' is included in the attractor to B .

2.3 Attractor decompositions

If \mathcal{G} is a parity game in which all priorities do not exceed a non-negative even number d then we say that

$$\mathcal{H} = \langle A, (S_1, \mathcal{H}_1, A_1), \dots, (S_k, \mathcal{H}_k, A_k) \rangle$$

is an *Even d -attractor decomposition* of \mathcal{G} if:

- A is the Even attractor to the (possibly empty) set of vertices of priority d in \mathcal{G} ;

and setting $\mathcal{G}_1 = \mathcal{G}$, for all $i = 1, 2, \dots, k$, we have:

- S_i is a non-empty trap for Odd in \mathcal{G}_i in which every vertex priority is at most $d - 2$;
- \mathcal{H}_i is a $(d - 2)$ -attractor decomposition of subgame $\mathcal{G} \cap S_i$;
- A_i is the Even attractor to S_i in \mathcal{G}_i ;
- $\mathcal{G}_{i+1} = \mathcal{G}_i \setminus A_i$;

and the game \mathcal{G}_{k+1} is empty. If $d = 0$ then we require that $k = 0$.

The following proposition states that if a subgame induced by a trap for Odd has an Even attractor decomposition then

the trap is an Even dominion. Indeed, a routine proof argues that the union of all the reachability strategies, implicit in the attractors listed in the decomposition, is an Even dominion strategy.

Proposition 2.4. *If d is even, T is a trap for Odd in \mathcal{G} , and there is an Even d -attractor decomposition of $\mathcal{G} \cap T$, then T is an Even dominion in \mathcal{G} .*

If \mathcal{G} is a game in which all priorities do not exceed an odd number d , then an *Odd d -attractor decomposition* of \mathcal{G} is defined analogously, with the roles of the two players swapped throughout the definition. As expected and by symmetry, if a trap for Even has an Odd attractor decomposition then it is an Odd dominion.

Proposition 2.5. *If d is odd, T is a trap for Even in \mathcal{G} , and there is an Odd d -attractor decomposition of $\mathcal{G} \cap T$, then T is an Odd dominion in \mathcal{G} .*

In the next subsection we argue that attractor decompositions are witnesses for the largest dominia and that the classic recursive McNaughton-Zielonka algorithm can be amended to produce such witnesses. Since McNaughton-Zielonka algorithm produces Even and Odd attractor decompositions, respectively, of subgames that are induced by sets of vertices that are complements of each other, a by-product of its analysis is a constructive proof of the positional determinacy theorem (Theorem 2.3).

2.4 McNaughton-Zielonka algorithm

The classic recursive McNaughton-Zielonka algorithm (Algorithm 1) computes the largest dominia in a parity game. In order to obtain the largest Even dominion in a parity game \mathcal{G} , it suffices to call $\text{McN-Z}_{\text{Even}}(\mathcal{G}, d)$, where d is even and all vertex priorities in \mathcal{G} are at most d . In order to obtain the largest Odd dominion in a parity game \mathcal{G} , it suffices to call $\text{McN-Z}_{\text{Odd}}(\mathcal{G}, d)$, where d is odd and all vertex priorities in \mathcal{G} are at most d .

The procedures $\text{McN-Z}_{\text{Even}}$ and $\text{McN-Z}_{\text{Odd}}$ are mutually recursive and whenever a recursive call is made, the second argument d decreases by 1. Figure 1 illustrates one iteration of the main loop in a call of procedure $\text{McN-Z}_{\text{Even}}$. The outer rectangle denotes subgame \mathcal{G}_i , the thin horizontal rectangle at the top denotes the set D_i of the vertices in \mathcal{G}_i whose priority is d , and the set below the horizontal wavy line is subgame \mathcal{G}'_i , which is the set of vertices in \mathcal{G}_i that are not in the attractor $\text{Attr}_{\text{Even}}^{\mathcal{G}_i}(D_i)$. The recursive call of $\text{McN-Z}_{\text{Odd}}$ returns the set U_i , and \mathcal{G}_{i+1} is the subgame to the left of the vertical zig-zag line, and it is induced by the set of vertices in \mathcal{G}_i that are not in the attractor $\text{Attr}_{\text{Odd}}^{\mathcal{G}_i}(U_i)$.

A way to prove the correctness of McNaughton-Zielonka algorithm we wish to highlight here is to enhance the algorithm slightly to produce not just a set of vertices but also an Even attractor decomposition of the set and an Odd attractor decomposition of its complement. We explain how

Algorithm 1: McNaughton-Zielonka algorithm

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procedure McN-ZEven( $\mathcal{G}, d$ ):
  if  $d = 0$  then
    return  $V^{\mathcal{G}}$ 
   $i \leftarrow 0$ ;  $\mathcal{G}_1 \leftarrow \mathcal{G}$ 
  repeat
     $i \leftarrow i + 1$ 
     $D_i \leftarrow \pi^{-1}(d) \cap \mathcal{G}_i$ 
     $\mathcal{G}'_i \leftarrow \mathcal{G}_i \setminus \text{Attr}_{\text{Even}}^{\mathcal{G}_i}(D_i)$ 
     $U_i \leftarrow \text{McN-Z}_{\text{Odd}}(\mathcal{G}'_i, d - 1)$ 
     $\mathcal{G}_{i+1} \leftarrow \mathcal{G}_i \setminus \text{Attr}_{\text{Odd}}^{\mathcal{G}_i}(U_i)$ 
  until  $U_i = \emptyset$ 
  return  $V^{\mathcal{G}_i}$ 

procedure McN-ZOdd( $\mathcal{G}, d$ ):
   $i \leftarrow 0$ ;  $\mathcal{G}_1 \leftarrow \mathcal{G}$ 
  repeat
     $i \leftarrow i + 1$ 
     $D_i \leftarrow \pi^{-1}(d) \cap \mathcal{G}_i$ 
     $\mathcal{G}'_i \leftarrow \mathcal{G}_i \setminus \text{Attr}_{\text{Odd}}^{\mathcal{G}_i}(D_i)$ 
     $U_i \leftarrow \text{McN-Z}_{\text{Even}}(\mathcal{G}'_i, d - 1)$ 
     $\mathcal{G}_{i+1} \leftarrow \mathcal{G}_i \setminus \text{Attr}_{\text{Even}}^{\mathcal{G}_i}(U_i)$ 
  until  $U_i = \emptyset$ 
  return  $V^{\mathcal{G}_i}$ 

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to modify procedure McN-Z_{Even} and leave it as an exercise for the reader to analogously modify procedure McN-Z_{Odd}. In procedure McN-Z_{Even}(\mathcal{G}, d), replace the line

$$U_i \leftarrow \text{McN-Z}_{\text{Odd}}(\mathcal{G}'_i, d - 1)$$

by the line

$$U_i, \mathcal{H}_i, \mathcal{H}'_i \leftarrow \text{McN-Z}_{\text{Odd}}(\mathcal{G}'_i, d - 1).$$

Moreover, if upon termination of the **repeat-until** loop we have

$$\mathcal{H}_i = \langle \emptyset, (S_1, \mathcal{I}_1, A_1), \dots, (S_k, \mathcal{I}_k, A_k) \rangle$$

then instead of returning just the set $V^{\mathcal{G}_i}$, let the procedure return both $V^{\mathcal{G}_i}$ and the following two objects:

$$\left\langle \text{Attr}_{\text{Even}}^{\mathcal{G}_i}(D_i), (S_1, \mathcal{I}_1, A_1), \dots, (S_k, \mathcal{I}_k, A_k) \right\rangle \quad (1)$$

and

$$\left\langle \emptyset, (U_1, \mathcal{H}'_1, \text{Attr}_{\text{Odd}}^{\mathcal{G}_i}(U_1)), \dots, (U_i, \mathcal{H}'_i, \text{Attr}_{\text{Odd}}^{\mathcal{G}_i}(U_i)) \right\rangle \quad (2)$$

In an inductive argument by induction on d and i , the inductive hypothesis is that:

- \mathcal{H}'_i is an Odd $(d - 1)$ -attractor decomposition of the subgame $\mathcal{G}'_i \cap U_i$;
- \mathcal{H}_i is an Even d -attractor decomposition of the subgame $\mathcal{G}'_i \setminus U_i$;

and the inductive step is then to show that:

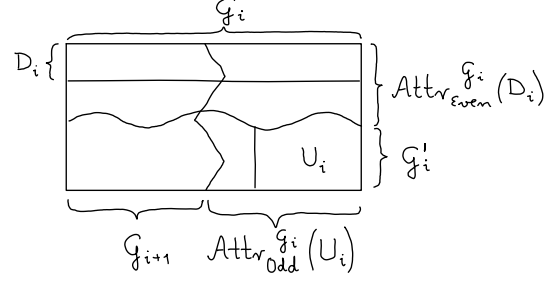


Figure 1. Attractors and subgames in one iteration of the loop in attractor decomposition algorithms.

- for every i , (2) is an Odd $(d + 1)$ -attractor decomposition of subgame $\mathcal{G} \setminus \mathcal{G}_{i+1}$;
- upon termination of the **repeat-until** loop, (1) is an Even d -attractor decomposition of subgame \mathcal{G}_{i+1} .

The general arguments in such a proof are well known [7, 21, 25, 29] and hence we omit the details here.

Theorem 2.6. *McNaughton-Zielonka algorithm can be enhanced to produce both the largest Even and Odd dominia, and an attractor decomposition of each. Every vertex is in one of the two dominia.*

3 Universal trees and algorithms

Every call of McNaughton-Zielonka algorithm takes small polynomial time if the recursive calls it makes are excluded. This is because in every execution of the main **repeat-until** loop, the two attractor computations can be performed in time linear in the size of the game graph, and the loop can only be performed at most linearly many times since the sets U_1, U_2, \dots, U_i are mutually disjoint. Therefore, the running time of McNaughton-Zielonka algorithm is mostly determined by the number of recursive calls it makes overall. While numerous experiments indicate that the algorithm performs very well on some classes of random games and on games arising from applications in model checking, temporal logic synthesis, and equivalence checking [27], it is also well known that there are families of parity games on which McNaughton-Zielonka algorithm performs exponentially many recursive calls [14].

Parys [26] has devised an ingenious modification of McNaughton-Zielonka algorithm that reduced the number of recursive calls of the algorithm to quasi-polynomial number $n^{O(\lg n)}$ in the worst case. Lehtinen, Schewe, and Wojtczak [23] have slightly modified Parys's algorithm in order to improve the running time from $n^{O(\lg n)}$ down to $d^{O(\lg n)}$ for (n, d) -small parity games. They have also made an informal observation that the tree of recursive calls of their recursive procedure is universal.

In this paper, we argue that McNaughton-Zielonka algorithm, Parys's algorithm, and Lehtinen-Schewe-Wojtczak algorithm are special cases of what we call a *universal attractor*

decomposition algorithm. The universal algorithm is parameterized by two ordered trees and we prove a striking structural result that if those trees are capacious enough to embed (in a formal sense explained later) ordered trees that describe the “shape” of some attractor decompositions of the largest Even and Odd dominia in a parity game, then the universal algorithm correctly computes the two dominia. It follows that if the algorithm is run on two universal trees then it is correct, and indeed we reproduce McNaughton-Zielonka, Parys’s, and Lehtinen-Schewe-Wojtczak algorithms by running the universal algorithm on specific classes of universal trees. In particular, Lehtinen-Schewe-Wojtczak algorithm is obtained by using the succinct universal trees of Jurdziński and Lazić [20], whose size nearly matches the quasi-polynomial lower bound on the size of universal trees [6].

3.1 Universal ordered trees

Ordered trees. Ordered trees are defined inductively; an ordered tree is the trivial tree $\langle \rangle$ or a sequence $\langle \mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k \rangle$, where \mathcal{T}_i is an ordered tree for every $i = 1, 2, \dots, k$. The trivial tree has only one node called the root, which is a leaf; and a tree of the form $\langle \mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k \rangle$ has the root with k children, the root is not a leaf, and the i -th child of the root is the root of ordered tree \mathcal{T}_i .

For an ordered tree \mathcal{T} , we write $\text{height}(\mathcal{T})$ for its *height* and $\text{leaves}(\mathcal{T})$ for its *number of leaves*. Both are defined by routine induction: the height of the trivial tree is 0 and it has 1 leaf; the height of tree $\langle \mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k \rangle$ is 1 plus the maximum height of trees $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k$; and the number of leaves of tree $\langle \mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k \rangle$ is the sum of the numbers of leaves of trees $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k$.

Trees of attractor decompositions. The definition of an attractor decomposition is inductive and we define an ordered tree that reflects the hierarchical structure of an attractor decomposition. If d is even and

$$\mathcal{H} = \langle A, (S_1, \mathcal{H}_1, A_1), \dots, (S_k, \mathcal{H}_k, A_k) \rangle$$

is an Even d -attractor decomposition then we define the *tree of attractor decomposition* \mathcal{H} , denoted by $\mathcal{T}_{\mathcal{H}}$, to be the trivial ordered tree $\langle \rangle$ if $k = 0$, and otherwise, to be the ordered tree $\langle \mathcal{T}_{\mathcal{H}_1}, \mathcal{T}_{\mathcal{H}_2}, \dots, \mathcal{T}_{\mathcal{H}_k} \rangle$, where for every $i = 1, 2, \dots, k$, tree $\mathcal{T}_{\mathcal{H}_i}$ is the tree of attractor decomposition \mathcal{H}_i . Trees of Odd attractor decompositions are defined analogously.

Observe that the sets S_1, S_2, \dots, S_k in an attractor decomposition as above are non-empty and pairwise disjoint, which implies that trees of attractor decompositions are small relative to the number of vertices and the number of distinct priorities in a parity game. More precisely, we say that an ordered tree is (n, h) -small if its height is at most h and it has at most n leaves. The following proposition can be proved by routine structural induction.

Proposition 3.1. *If \mathcal{H} is an attractor decomposition of an (n, d) -small parity game then its tree $\mathcal{T}_{\mathcal{H}}$ is $(n, \lceil d/2 \rceil)$ -small.*

Embedding ordered trees. Intuitively, an ordered tree *embeds* another if the latter can be obtained from the former by pruning some subtrees. More formally, every ordered tree embeds the trivial tree $\langle \rangle$, and $\langle \mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k \rangle$ embeds $\langle \mathcal{T}'_1, \mathcal{T}'_2, \dots, \mathcal{T}'_\ell \rangle$ if there are indices i_1, i_2, \dots, i_ℓ , such that $1 \leq i_1 < i_2 < \dots < i_\ell \leq k$ and for every $j = 1, 2, \dots, \ell$, we have that \mathcal{T}_{i_ℓ} embeds \mathcal{T}'_j .

Universal ordered trees. We say that an ordered tree is (n, h) -universal [6] if it embeds every (n, h) -small ordered tree. The complete n -ary tree of height h can be defined by induction on h : if $h = 0$ then $C_{n,0}$ is the trivial tree $\langle \rangle$, and if $h > 0$ then $C_{n,h}$ is the ordered tree $\langle C_{n,h-1} \rangle^n$. The tree $C_{n,h}$ is obviously (n, h) -universal but its size is exponential in h .

We define two further classes $P_{n,h}$ and $S_{n,h}$ of (n, h) -universal trees whose size is only quasipolynomial, and hence they are significantly smaller than the complete n -ary trees of height h . Both classes are defined by induction on $n + h$.

If $h = 0$ then both $P_{n,h}$ and $S_{n,h}$ are defined to be the trivial tree $\langle \rangle$. If $h > 0$ then $P_{n,h}$ is defined to be the ordered tree

$$\langle P_{\lfloor n/2 \rfloor, h-1} \rangle^{\lfloor n/2 \rfloor} \cdot \langle P_{n, h-1} \rangle \cdot \langle P_{\lfloor n/2 \rfloor, h-1} \rangle^{\lfloor n/2 \rfloor},$$

and $S_{n,h}$ is defined to be the ordered tree

$$S_{\lfloor n/2 \rfloor, h} \cdot \langle S_{n, h-1} \rangle \cdot S_{\lfloor n/2 \rfloor, h}.$$

We leave it as an instructive exercise to the reader to prove the following proposition.

Proposition 3.2. *Ordered trees $C_{n,h}$, $P_{n,h}$ and $S_{n,h}$ are (n, h) -universal.*

A proof of universality of $S_{n,h}$ is implicit in the work of Jurdziński and Lazić [20], whose *succinct multi-counters* are merely an alternative presentation of trees $S_{n,h}$. Parys [26] has shown that the number of leaves in trees $P_{n,h}$ is $n^{\lg n + O(1)}$ and Jurdziński and Lazić [20] have proved that the number of leaves in trees $S_{n,h}$ is $n^{\lg h + O(1)}$. Czerwiński et al. [6] have established a quasi-polynomial lower bound on the number of leaves in (n, h) -universal trees, which the size of $S_{n,h}$ exceeds only by a small polynomial factor.

3.2 Universal algorithm

Every call of McNaughton-Zielonka algorithm (Algorithm 1) repeats the main loop until the set U_i (returned by a recursive call) is empty. If the number of iterations for each value of d is large then the overall number of recursive calls may be exponential in d in the worst case, and that is indeed what happens for some families of hard parity games [14].

In our *universal attractor decomposition algorithm* (Algorithm 2), every iteration of the main loop performs exactly the same actions as in McNaughton-Zielonka algorithm (see Algorithm 1 and Figure 1), but the algorithm uses a different mechanism to determine how many iterations of the main loop are performed in each recursive call. In the mutually recursive procedures Univ_{Odd} and $\text{Univ}_{\text{Even}}$, this is determined

Algorithm 2: The universal algorithm

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procedure UnivEven( $\mathcal{G}, d, \mathcal{T}^{\text{Even}}, \mathcal{T}^{\text{Odd}}$ ):
  let  $\mathcal{T}^{\text{Odd}} = \langle \mathcal{T}_1^{\text{Odd}}, \mathcal{T}_2^{\text{Odd}}, \dots, \mathcal{T}_k^{\text{Odd}} \rangle$ 
   $\mathcal{G}_1 \leftarrow \mathcal{G}$ 
  for  $i \leftarrow 1$  to  $k$  do
     $D_i \leftarrow \pi^{-1}(d) \cap \mathcal{G}_i$ 
     $\mathcal{G}'_i \leftarrow \mathcal{G}_i \setminus \text{Attr}_{\text{Even}}^{\mathcal{G}_i}(D_i)$ 
     $U_i \leftarrow \text{UnivOdd}(\mathcal{G}'_i, d-1, \mathcal{T}^{\text{Even}}, \mathcal{T}_i^{\text{Odd}})$ 
     $\mathcal{G}_{i+1} \leftarrow \mathcal{G}_i \setminus \text{Attr}_{\text{Odd}}^{\mathcal{G}_i}(U_i)$ 
  return  $V^{\mathcal{G}_{k+1}}$ 

procedure UnivOdd( $\mathcal{G}, d, \mathcal{T}^{\text{Even}}, \mathcal{T}^{\text{Odd}}$ ):
  let  $\mathcal{T}^{\text{Even}} = \langle \mathcal{T}_1^{\text{Even}}, \mathcal{T}_2^{\text{Even}}, \dots, \mathcal{T}_\ell^{\text{Even}} \rangle$ 
   $\mathcal{G}_1 \leftarrow \mathcal{G}$ 
  for  $i \leftarrow 1$  to  $\ell$  do
     $D_i \leftarrow \pi^{-1}(d) \cap \mathcal{G}_i$ 
     $\mathcal{G}'_i \leftarrow \mathcal{G}_i \setminus \text{Attr}_{\text{Odd}}^{\mathcal{G}_i}(D_i)$ 
     $U_i \leftarrow \text{UnivEven}(\mathcal{G}'_i, d-1, \mathcal{T}_i^{\text{Even}}, \mathcal{T}^{\text{Odd}})$ 
     $\mathcal{G}_{i+1} \leftarrow \mathcal{G}_i \setminus \text{Attr}_{\text{Even}}^{\mathcal{G}_i}(U_i)$ 
  return  $V^{\mathcal{G}_{\ell+1}}$ 

```

by the numbers of children of the root in the input trees $\mathcal{T}^{\text{Even}}$ (the third argument) and \mathcal{T}^{Odd} (the fourth argument), respectively. Note that the sole recursive call of UnivOdd in the i -th iteration of the main loop in a call of UnivEven is given subtree $\mathcal{T}_i^{\text{Odd}}$ as its fourth argument and, analogously, the sole recursive call of UnivEven in the j -th iteration of the main loop in a call of UnivOdd is given subtree $\mathcal{T}_j^{\text{Even}}$ as its third argument.

Define the *interleaving* operation on two ordered trees inductively as follows: $\langle \rangle \bowtie \mathcal{T} = \langle \rangle$ and $\langle \mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k \rangle \bowtie \mathcal{T} = \langle \mathcal{T} \bowtie \mathcal{T}_1, \mathcal{T} \bowtie \mathcal{T}_2, \dots, \mathcal{T} \bowtie \mathcal{T}_k \rangle$. Then the following simple proposition provides an explicit description of the tree of recursive calls of our universal algorithm.

Proposition 3.3. *If d is even then the tree of recursive calls of a call $\text{UnivEven}(\mathcal{G}, d, \mathcal{T}^{\text{Even}}, \mathcal{T}^{\text{Odd}})$ is the interleaving $\mathcal{T}^{\text{Odd}} \bowtie \mathcal{T}^{\text{Even}}$ of trees \mathcal{T}^{Odd} and $\mathcal{T}^{\text{Even}}$.*

Proposition 3.4. *If d is odd then the tree of recursive calls of a call $\text{UnivOdd}(\mathcal{G}, d, \mathcal{T}^{\text{Even}}, \mathcal{T}^{\text{Odd}})$ is the interleaving $\mathcal{T}^{\text{Even}} \bowtie \mathcal{T}^{\text{Odd}}$ of trees $\mathcal{T}^{\text{Even}}$ and \mathcal{T}^{Odd} .*

The following elementary proposition helps estimate the size of an interleaving of two ordered trees and hence the running time of a call of the universal algorithm that is given two ordered trees as inputs.

Proposition 3.5. *If \mathcal{T} and \mathcal{T}' are ordered trees then:*

- $\text{height}(\mathcal{T} \bowtie \mathcal{T}') \leq \text{height}(\mathcal{T}) + \text{height}(\mathcal{T}')$;
- $\text{leaves}(\mathcal{T} \bowtie \mathcal{T}') \leq \text{leaves}(\mathcal{T}) \cdot \text{leaves}(\mathcal{T}')$.

In contrast to the universal algorithm, the tree of recursive calls of McNaughton-Zielonka algorithm is not pre-determined by a structure separate from the game graph, such as the pair of trees $\mathcal{T}^{\text{Even}}$ and \mathcal{T}^{Odd} . Instead, McNaughton-Zielonka algorithm determines the number of iterations of its main loop adaptively, using the adaptive *empty-set early termination rule*: terminate the main loop as soon as $U_i = \emptyset$. We argue that if we add the empty-set early termination rule to the universal algorithm in which both trees $\mathcal{T}^{\text{Even}}$ and \mathcal{T}^{Odd} are the tree $C_{n,d/2}$ then its behaviour coincides with McNaughton-Zielonka algorithm.

Proposition 3.6. *The universal algorithm performs the same actions and produces the same output as McNaughton-Zielonka algorithm if it is run on an (n, d) -small parity game and with both trees $\mathcal{T}^{\text{Even}}$ and \mathcal{T}^{Odd} equal to $C_{n,d/2}$, and if it uses the adaptive empty-set early termination rule.*

The idea of using rules for implicitly pruning the tree of recursive calls of a McNaughton-Zielonka-style algorithm that are significantly different from the adaptive empty-set early termination rule is due to Parys [26]. In this way, he has designed the first McNaughton-Zielonka-style algorithm that works in quasi-polynomial time $n^{O(\lg n)}$ in the worst case, and Lehtinen, Schewe, and Wojtczak [23] have refined Parys's algorithm, improving the worst-case running time down to $n^{O(\lg d)}$. Both algorithms use two numerical arguments (one for Even and one for Odd) and “halving tricks” on those parameters, which results in pruning the tree of recursive calls down to quasi-polynomial size in the worst case. We note that our universal algorithm yields the algorithms of Parys and of Lehtinen et al., respectively, if, when run on an (n, d) -small parity game and if both trees $\mathcal{T}^{\text{Even}}$ and \mathcal{T}^{Odd} set to be the $(n, d/2)$ -universal trees $P_{n,d/2}$ and $S_{n,d/2}$, respectively.

Proposition 3.7. *The universal algorithm performs the same actions and produces the same output as Lehtinen-Schewe-Wojtczak algorithm if it is run on an (n, d) -small parity game with both trees $\mathcal{T}^{\text{Even}}$ and \mathcal{T}^{Odd} equal to $S_{n,d/2}$.*

The correspondence between the universal algorithm run on $(n, d/2)$ -universal trees $P_{n,d/2}$ and Parys's algorithm is a bit more subtle. While both run in quasi-polynomial time in the worst case, the former may perform more recursive calls than the latter. The two coincide, however, if the former is enhanced with a simple adaptive tree-pruning rule similar to the empty-set early termination rule. The discussion of this and other adaptive tree-pruning rules will be better informed once we have discussed sufficient conditions for the correctness of our universal algorithm. Therefore, we will return to elaborating the full meaning of the following proposition in Section 4.4.

Proposition 3.8. *The universal algorithm performs the same actions and produces the same output as a non-adaptive version*

of Parys's algorithm if it is run on an (n, d) -small parity games with both trees $\mathcal{T}^{\text{Even}}$ and \mathcal{T}^{Odd} equal to $P_{n, d/2}$.

4 Correctness via structural theorems

The proof of correctness of McNaughton-Zielonka algorithm that is based on the algorithm recursively producing attractor decompositions of largest Even and Odd dominia (as discussed in Section 2.4) critically relies on the $U_i = \emptyset$ termination condition of the main loop in McNaughton-Zielonka algorithm. The argument breaks down if the loop terminates before that empty-set condition obtains. Instead, Parys [26] has developed a novel *dominion separation technique* to prove correctness of his algorithm and Lehtinen et al. [23] use the same technique to justify theirs.

In this paper, we significantly generalize the dominion separation technique of Parys, which allows us to intimately link the correctness of our meta-algorithm to shapes (modelled as ordered trees) of attractor decompositions of largest Even and Odd dominia. We say that the universal algorithm is correct on a parity game if $\text{Univ}^{\text{Even}}$ returns the largest Even dominion and Univ^{Odd} returns the largest Odd dominion. We also say that an ordered tree \mathcal{T} embeds a dominion D in a parity game \mathcal{G} if it embeds the tree of some attractor decomposition of $\mathcal{G} \cap D$. The main technical result we aim to prove in this section is the sufficiency of the following condition for the universal algorithm to be correct.

Theorem 4.1 (Correctness of universal algorithm). *The universal algorithm is correct on a parity game \mathcal{G} if it is run on ordered trees $\mathcal{T}^{\text{Even}}$ and \mathcal{T}^{Odd} , such that $\mathcal{T}^{\text{Even}}$ embeds the largest Even dominion in \mathcal{G} and \mathcal{T}^{Odd} embeds the largest Odd dominion in \mathcal{G} .*

4.1 Embeddable decomposition theorem

Before we prove Theorem 4.1 in Section 4.2, in this section we establish another technical result—the embeddable decomposition theorem—that enables our generalization of Parys's dominion separation technique. Its statement is intuitive: a subgame induced by a trap has a simpler attractor decomposition structure than the whole game itself; its proof, however, seems to require some careful surgery.

Theorem 4.2 (Embeddable decomposition). *If T is a trap for Even in a parity game \mathcal{G} and $\mathcal{G}' = \mathcal{G} \cap T$ is the subgame induced by T , then for every Even attractor decomposition \mathcal{H} of \mathcal{G} , there is an Even attractor decomposition \mathcal{H}' of \mathcal{G}' , such that $\mathcal{T}_{\mathcal{H}}$ embeds $\mathcal{T}_{\mathcal{H}'}$.*

In order to streamline the proof of the embeddable decomposition theorem, we state the following two propositions, which synthesize or generalize some of the arguments that were also used by Parys [26] and Lehtinen et al. [23]. Proofs are included in the Appendix.

Proposition 4.3. *Suppose that R is a trap for Even in game \mathcal{G} . Then if T is a trap for Odd in \mathcal{G} then $T \cap R$ is a trap for Odd in*

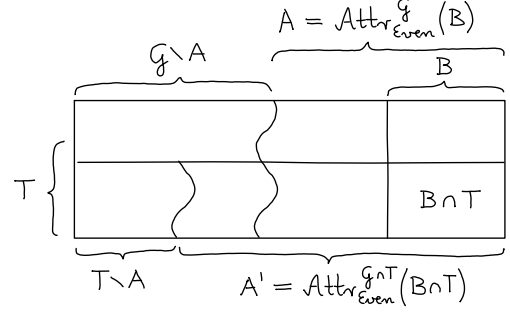


Figure 2. Traps and attractors in Proposition 4.4.

subgame $\mathcal{G} \cap R$, and if T is an Even dominion in \mathcal{G} then $T \cap R$ is an Even dominion in $\mathcal{G} \cap R$.

The other proposition is illustrated in Figure 2. Its statement is more complex than that of the first proposition. The statement and the proof describe the relationship between the Even attractor of a set B of vertices in a game \mathcal{G} and the Even attractor of the set $B \cap T$ in subgame $\mathcal{G} \cap T$, where T is a trap for Even in \mathcal{G} .

Proposition 4.4. *Let $B \subseteq V^{\mathcal{G}}$ and let T be a trap for Even in game \mathcal{G} . Define $A = \text{Attr}_{\text{Even}}^{\mathcal{G}}(B)$ and $A' = \text{Attr}_{\text{Even}}^{\mathcal{G} \cap T}(B \cap T)$. Then $T \setminus A'$ is a trap for Even in subgame $\mathcal{G} \setminus A$.*

Finally, we are ready to prove the embeddable decomposition theorem by induction on the number of leaves of the tree of attractor decomposition \mathcal{H} .

Proof of Theorem 4.2. Without loss of generality, assume that d is even and

$$\mathcal{H} = \langle A, (S_1, \mathcal{H}_1, A_1), \dots, (S_k, \mathcal{H}_k, A_k) \rangle$$

is an Even d -attractor decomposition of \mathcal{G} , where A is the Even attractor to the set D of vertices of priority d in \mathcal{G} . In Figure 3, set T and the subgame \mathcal{G}' it induces form the pentagon obtained from the largest rectangle by removing the triangle above the diagonal line in the top-left corner. Sets A, S_1 , and A_1 are also illustrated, together with sets A', S'_1, A'_1 and subgames $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}'_1$, and \mathcal{G}'_2 , which are defined as follows.

Let $\mathcal{G}_1 = \mathcal{G} \setminus A$, and $\mathcal{G}_2 = \mathcal{G}_1 \setminus A_1$. We will define sets $A', S'_1, A'_1, \dots, S'_\ell, A'_\ell$, and Even $(d-2)$ -attractor decompositions $\mathcal{H}'_1, \dots, \mathcal{H}'_\ell$ of subgames $\mathcal{G} \cap S'_1, \dots, \mathcal{G} \cap S'_\ell$, respectively, such that

$$\mathcal{H}' = \langle A', (S'_1, \mathcal{H}'_1, A'_1), \dots, (S'_k, \mathcal{H}'_\ell, A'_\ell) \rangle$$

is an Even d -attractor decomposition of subgame \mathcal{G}' and $\mathcal{T}_{\mathcal{H}}$ embeds $\mathcal{T}_{\mathcal{H}'}$.

Let A' be the Even attractor to $D \cap T$ in \mathcal{G}' and let $\mathcal{G}'_1 = \mathcal{G}' \setminus A'$. Set $S'_1 = S_1 \cap \mathcal{G}'_1$, let A'_1 be the Even attractor to S'_1 in \mathcal{G}'_1 , and let $\mathcal{G}'_2 = \mathcal{G}'_1 \setminus A'_1$.

Firstly, since $D \subseteq V^{\mathcal{G}}$ and T is a trap for Even in \mathcal{G} , by Proposition 4.4, we have that \mathcal{G}'_1 is a trap for Even in subgame \mathcal{G}_1 . Since $S_1 \subseteq V^{\mathcal{G}_1}$ and subgame \mathcal{G}'_1 is a trap for Even

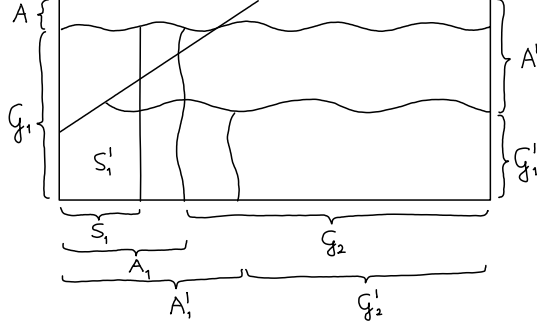


Figure 3. Attractors, subgames, and dominia in the proof of the embeddable decomposition theorem.

in subgame \mathcal{G}_1 , again by Proposition 4.4, we conclude that \mathcal{G}'_2 is a trap for Even in subgame \mathcal{G}_2 .

Secondly, we argue that S'_1 is an Even dominion in subgame \mathcal{G}'_1 . This follows by recalling that S_1 is a dominion for Even in \mathcal{G}_1 and \mathcal{G}'_1 is a trap for Even in \mathcal{G}_1 , and then applying Proposition 4.3.

Thirdly, we argue that S'_1 is a trap for Even in subgame $\mathcal{G} \cap S_1$. This follows by recalling that S_1 is a trap for Odd in \mathcal{G}_1 and that \mathcal{G}'_1 is a trap for Even in \mathcal{G}_1 , and then applying Proposition 4.3.

We are now in a position to apply the inductive hypothesis twice in order to complete the definition of the attractor decomposition \mathcal{H}' . Firstly, recall that S'_1 is a trap for Even in subgame $\mathcal{G} \cap S_1$ and that \mathcal{H}_1 is a $(d-2)$ -attractor decomposition of $\mathcal{G} \cap S_1$, so we can apply the inductive hypothesis to obtain a $(d-2)$ -attractor decomposition \mathcal{H}'_1 of subgame $\mathcal{G} \cap S'_1$, such that $\mathcal{T}_{\mathcal{H}'_1}$ embeds $\mathcal{T}_{\mathcal{H}_1}$. Secondly, note that

$$\mathcal{I} = \langle \emptyset, (S_2, \mathcal{H}_2, A_2), \dots, (S_k, \mathcal{H}_k, A_k) \rangle$$

is a d -attractor decomposition of \mathcal{G}_2 . We find a d -attractor decomposition \mathcal{I}' of subgame \mathcal{G}'_2 , such that $\mathcal{T}_{\mathcal{I}'}$ embeds $\mathcal{T}_{\mathcal{I}}$. Recalling that \mathcal{G}'_2 is a trap for Even in subgame \mathcal{G}_2 , it suffices to use the inductive hypothesis for subgame \mathcal{G}'_2 of game \mathcal{G}_2 and the d -attractor decomposition \mathcal{I} of \mathcal{G}_2 .

Verifying that \mathcal{H}' is a d -attractor decomposition of \mathcal{G}' is routine. That $\mathcal{T}_{\mathcal{H}}$ embeds $\mathcal{T}_{\mathcal{H}'}$ also follows routinely from $\mathcal{T}_{\mathcal{H}_1}$ embedding $\mathcal{T}_{\mathcal{H}'_1}$ and $\mathcal{T}_{\mathcal{I}}$ embedding $\mathcal{T}_{\mathcal{I}'}$. \square

4.2 Dominion separation theorem

The simple dominion disjointness property (Proposition 2.2) states that every Even dominion is disjoint from every Odd dominion. For two sets A and B , we say that another set X separates A from B if $A \subseteq X$ and $X \cap B = \emptyset$. In this section we establish a very general dominion separation property for subgames that occur in iterations of the universal algorithm. This allows us to prove one of the main technical results of this paper (Theorem 4.1) that describes a detailed structural sufficient condition for the correctness of the universal algorithm.

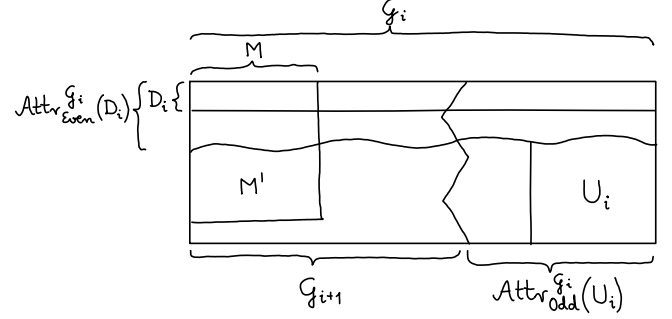


Figure 4. Attractors, subgames, and dominia in the first part of the proof of the dominion separation theorem.

Theorem 4.5 (Dominion separation). *Let \mathcal{G} be an (n, d) -small parity game and let $\mathcal{T}^{\text{Even}} = \langle \mathcal{T}_1^{\text{Even}}, \dots, \mathcal{T}_\ell^{\text{Even}} \rangle$ and $\mathcal{T}^{\text{Odd}} = \langle \mathcal{T}_1^{\text{Odd}}, \dots, \mathcal{T}_k^{\text{Odd}} \rangle$ be trees of height at most $\lceil d/2 \rceil$ and $\lfloor d/2 \rfloor$, respectively.*

- (a) *If d is even and $\mathcal{G}_1, \dots, \mathcal{G}_{k+1}$ are the games that are computed in the successive iterations of the loop in the call $\text{UnivEven}(\mathcal{G}, d, \mathcal{T}^{\text{Even}}, \mathcal{T}^{\text{Odd}})$, then for every $i = 0, 1, \dots, k$, we have that \mathcal{G}_{i+1} separates every Even dominion in \mathcal{G} that tree $\mathcal{T}^{\text{Even}}$ embeds from every Odd dominion in \mathcal{G} that tree $\langle \mathcal{T}_1^{\text{Odd}}, \dots, \mathcal{T}_i^{\text{Odd}} \rangle$ embeds.*
- (b) *If d is odd and $\mathcal{G}_1, \dots, \mathcal{G}_{\ell+1}$ are the games that are computed in the successive iterations of the loop in the call $\text{UnivOdd}(\mathcal{G}, d, \mathcal{T}^{\text{Even}}, \mathcal{T}^{\text{Odd}})$, then for every $i = 0, 1, \dots, \ell$, we have that \mathcal{G}_{i+1} separates every Odd dominion in \mathcal{G} that tree \mathcal{T}^{Odd} embeds from every Even dominion in \mathcal{G} that tree $\langle \mathcal{T}_1^{\text{Even}}, \dots, \mathcal{T}_i^{\text{Even}} \rangle$ embeds.*

Before we prove the dominion separation theorem, we recall a simple proposition from Parys [26], also stated explicitly by Lehtinen et al. [23]. Note that it is a straightforward corollary of the dual of Proposition 4.4 (in case $B \cap T = \emptyset$).

Proposition 4.6. *If T is a trap for Odd in \mathcal{G} and $T \cap B = \emptyset$ then we also have that $T \cap \text{Attr}_{\text{Odd}}^{\mathcal{G}}(B) = \emptyset$.*

Proof of Theorem 4.5. We prove the statement of part (a); the proof of part (b) is analogous.

The proof is by induction on the height of tree $\mathcal{T}^{\text{Odd}} \bowtie \mathcal{T}^{\text{Even}}$ (the “outer” induction). If the height is 0 then tree \mathcal{T}^{Odd} is the trivial tree $\langle \rangle$; hence $k = 0$, the algorithm returns the set $V^{\mathcal{G}_1} = V^{\mathcal{G}}$, which contains the largest Even dominion, and which is trivially disjoint from the largest Odd dominion (because the latter is empty).

If the height of $\mathcal{T}^{\text{Odd}} \bowtie \mathcal{T}^{\text{Even}}$ is positive, then we split the proof of the separation property into two parts.

Even dominia embedded by $\mathcal{T}^{\text{Even}}$ are included in \mathcal{G}_{i+1} . We prove by induction on i (the “inner” induction) that for $i = 0, 1, 2, \dots, k$, if M is an Even dominion in \mathcal{G} that $\mathcal{T}^{\text{Even}}$ embeds, then $M \subseteq \mathcal{G}_{i+1}$.

For $i = 0$, this is moot because $\mathcal{G}_1 = \mathcal{G}$.

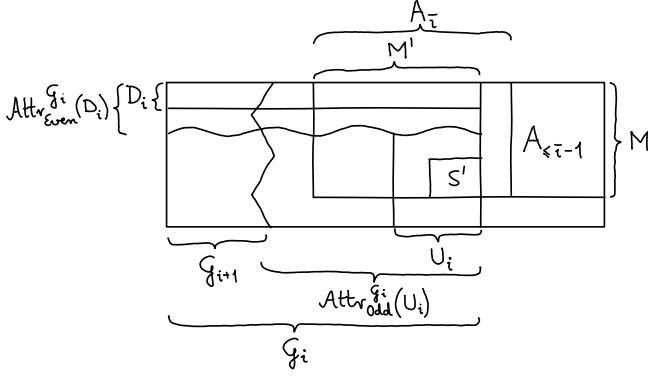


Figure 5. Attractors, subgames, and dominia in the second part of the proof of the dominion separation theorem.

For $i > 0$, let M be an Even dominion that has an Even d -attractor decomposition \mathcal{H} such that $\mathcal{T}^{\text{Even}}$ embeds $\mathcal{T}_{\mathcal{H}}$. The inner inductive hypothesis (for $i - 1$) implies that $M \subseteq \mathcal{G}_i$.

The reader is encouraged to systematically refer to Figure 4 to better follow the rest of this part of the proof.

Let $M' = M \setminus \text{Attr}_{\text{Even}}^{G_i}(D_i)$. Because $\mathcal{G}_i \setminus \text{Attr}_{\text{Even}}^{G_i}(D_i)$ is a trap for Even in \mathcal{G}_i and M is a trap for Odd in \mathcal{G}_i , the dual of Proposition 4.3 yields that M' is a trap for Even in $\mathcal{G}_i \cap M$.

Then, because \mathcal{H} is an Even d -attractor decomposition of $\mathcal{G} \cap M$, it follows by Theorem 4.2 that there is an Even d -attractor decomposition \mathcal{H}' of $\mathcal{G}_i \cap M'$ such that $\mathcal{T}_{\mathcal{H}'}$ embeds $\mathcal{T}_{\mathcal{H}}$, and hence also $\mathcal{T}^{\text{Even}}$ embeds $\mathcal{T}_{\mathcal{H}'}$.

Therefore, because M' is an Even dominion in the game $\mathcal{G}_i \setminus \text{Attr}_{\text{Even}}^{G_i}(D_i)$, part (b) of the outer inductive hypothesis yields $M' \cap U_i = \emptyset$.

Finally, because $M \setminus M' \subseteq \text{Attr}_{\text{Even}}^{G_i}(D_i)$ and $(M' \setminus M) \cap U_i = \emptyset$, it follows that $M \cap U_i = \emptyset$. By Proposition 4.6, we obtain $M \cap \text{Attr}_{\text{Odd}}^{G_i}(U_i) = \emptyset$ and hence $M \subseteq \mathcal{G}_{i+1}$.

Odd dominia embedded by $\langle \mathcal{T}_1^{\text{Odd}}, \dots, \mathcal{T}_i^{\text{Odd}} \rangle$ are disjoint from \mathcal{G}_{i+1} . We prove by induction on i (another “inner” induction) that for $i = 0, 1, \dots, k$, if M is an Odd dominion in \mathcal{G} that $\langle \mathcal{T}_1^{\text{Odd}}, \dots, \mathcal{T}_i^{\text{Odd}} \rangle$ embeds, then $\mathcal{G}_{i+1} \cap M = \emptyset$.

For $i = 0$, note that $\langle \mathcal{T}_1^{\text{Odd}}, \dots, \mathcal{T}_i^{\text{Odd}} \rangle = \langle \rangle$ and the only Odd dominion M in \mathcal{G} that has an Odd $(d + 1)$ -attractor decomposition whose tree is the trivial tree $\langle \rangle$ is the empty set, and hence $\mathcal{G}_1 \cap M = \emptyset$, because $\mathcal{G}_1 = \mathcal{G}$.

The reader is encouraged to systematically refer to Figure 5 to better follow the rest of this part of the proof.

For $i > 0$, let

$$\mathcal{H} = \langle \emptyset, (S_1, \mathcal{H}_1, A_1), \dots, (S_i, \mathcal{H}_i, A_i) \rangle$$

be an Odd $(d + 1)$ -attractor decomposition of $\mathcal{G} \cap M$ such that $\langle \mathcal{T}_1^{\text{Odd}}, \dots, \mathcal{T}_i^{\text{Odd}} \rangle$ embeds $\mathcal{T}_{\mathcal{H}}$. Note that the embedding implies that $\bar{i} \leq i$.

If $\langle \mathcal{T}_1^{\text{Odd}}, \dots, \mathcal{T}_{i-1}^{\text{Odd}} \rangle$ embeds $\mathcal{T}_{\mathcal{H}}$ then the inner inductive hypothesis (for $i - 1$) implies that $\mathcal{G}_i \cap M = \emptyset$ and thus $\mathcal{G}_{i+1} \cap M = \emptyset$ since $\mathcal{G}_{i+1} \subseteq \mathcal{G}_i$.

Otherwise, it must be the case that

$$\mathcal{T}_i^{\text{Odd}} \text{ embeds } \mathcal{T}_{\mathcal{H}_i}. \quad (3)$$

Observe that the set $A_{\leq \bar{i}-1} = A_1 \cup A_2 \cup \dots \cup A_{\bar{i}-1}$ is a trap for Even in $\mathcal{G} \cap M$, and hence by trap transitivity it is a trap for Even in \mathcal{G} because M is a trap for Even in \mathcal{G} . Moreover, subgame $\mathcal{G} \cap A_{\leq \bar{i}-1}$ has an Odd $(d + 1)$ -attractor decomposition

$$\mathcal{I} = \langle \emptyset, (S_1, \mathcal{H}_1, A_1), \dots, (S_{\bar{i}-1}, \mathcal{H}_{\bar{i}-1}, A_{\bar{i}-1}) \rangle$$

in \mathcal{G} and hence—by Proposition 2.5—it is an Odd dominion in \mathcal{G} , and ordered tree $\langle \mathcal{T}_1^{\text{Odd}}, \dots, \mathcal{T}_{\bar{i}-1}^{\text{Odd}} \rangle$ embeds $\mathcal{T}_{\mathcal{I}}$. Hence, the inner inductive hypothesis (for $i - 1$) yields

$$\mathcal{G}_i \cap A_{\leq \bar{i}-1} = \emptyset. \quad (4)$$

Set $M' = \mathcal{G}_i \cap M$ and note that not only $M' \subseteq A_{\bar{i}}$, but also M' is a trap for Odd in $A_{\bar{i}}$, because \mathcal{G}_i is a trap for Odd in \mathcal{G} . Moreover—by Proposition 4.3— M' is an Odd dominion in \mathcal{G}_i because \mathcal{G}_i is a trap for Odd in \mathcal{G} and M is a dominion for Odd in \mathcal{G} .

Observe that $\mathcal{J} = \langle \emptyset, (S_i, \mathcal{H}_i, A_i) \rangle$ is an Odd $(d + 1)$ -attractor decomposition of $\mathcal{G} \cap A_{\bar{i}}$. By the embeddable decomposition theorem (Theorem 4.2), it follows that there is an Odd $(d + 1)$ -attractor decomposition \mathcal{K} of $\mathcal{G} \cap M'$ such that $\mathcal{T}_{\mathcal{J}}$ embeds $\mathcal{T}_{\mathcal{K}}$. Because of this embedding, \mathcal{K} must have the form $\mathcal{K} = \langle \emptyset, (S', \mathcal{K}', M') \rangle$. Since $\mathcal{T}_{\mathcal{J}}$ embeds $\mathcal{T}_{\mathcal{K}}$, we also have that $\mathcal{T}_{\mathcal{H}_i}$ embeds $\mathcal{T}_{\mathcal{K}'}$, and hence—by (3)— $\mathcal{T}_i^{\text{Odd}}$ embeds $\mathcal{T}_{\mathcal{K}'}$.

Note that S' is a trap for Odd in $\mathcal{G} \cap M'$ in which every vertex priority is at most $d - 1$, because \mathcal{K} is an Odd $(d + 1)$ -attractor decomposition of $\mathcal{G} \cap M'$. It follows that S' is also an Odd dominion in $\mathcal{G}_i \setminus \text{Attr}_{\text{Even}}^{G_i}(D_i)$.

The outer inductive hypothesis then yields $S' \subseteq U_i$. It follows that

$$M' = \text{Attr}_{\text{Odd}}^{G_i \cap M'}(S') \subseteq \text{Attr}_{\text{Odd}}^{G_i}(S') \subseteq \text{Attr}_{\text{Odd}}^{G_i}(U_i),$$

where the first inclusion holds because M' is a trap for Even in \mathcal{G}_i , and the second follows from monotonicity of the attractor operator. When combined with (4), this implies $\mathcal{G}_{i+1} \cap M = \emptyset$. \square

4.3 Correctness and complexity

The dominion separation theorem (Theorem 4.5) allows us to conclude the proof of the main universal algorithm correctness theorem (Theorem 4.1). Indeed, if trees $\mathcal{T}^{\text{Even}}$ and \mathcal{T}^{Odd} satisfy the conditions of Theorem 4.1 then, by the dominion separation theorem, the set returned by the call $\text{UnivEven}(\mathcal{G}, d, \mathcal{T}^{\text{Even}}, \mathcal{T}^{\text{Odd}})$ separates the largest Even dominion from the largest Odd dominion, and hence—by the positional determinacy theorem (Theorem 2.3)—it is the largest Even dominion. The argument for procedure UnivOdd is analogous.

We note that the universal algorithm correctness theorem, together with Propositions 3.8 and 3.7, imply correctness of the non-adaptive version of Parys’s algorithm [26] and of Lehtinen-Schewe-Wojtczak algorithm [23], because trees of

attractor decompositions are $(n, d/2)$ -small (Proposition 3.1) and trees $P_{n,d/2}$ and $S_{n,d/2}$ are $(n, d/2)$ -universal.

The following fact, an alternative restatement of the conclusion of Lehtinen et al. [23], is a simple corollary of the precise asymptotic upper bounds on the size of the universal trees $S_{n,d/2}$ established by Jurdziński and Lazić [20], and of Propositions 3.7, 3.3, and 3.5.

Proposition 4.7 (Complexity). *The universal algorithm that uses universal trees $S_{n,d/2}$ (aka. Lehtinen-Schewe-Wojtczak algorithm) solves (n, d) -small parity games in polynomial time if $d = O(\log n)$, and in time $n^{2 \lg(d/\lg n) + O(1)}$ if $d = \omega(\log n)$.*

4.4 Acceleration by tree pruning

As we have discussed in Section 3.2, Parys [26] has achieved a breakthrough of developing the first quasi-polynomial McNaughton-Zielonka-style algorithm for parity games by pruning the tree of recursive calls down to quasi-polynomial size. Proposition 3.8 clarifies that Parys’s scheme can be reproduced by letting the universal algorithm run on universal trees $P_{n,d/2}$, but as it also mentions, just doing so results in a “non-adaptive” version of Parys’s algorithm. What is the “adaptive” version actually proposed by Parys?

Recall that the root of tree $P_{n,h}$ has $n + 1$ children, the first $n/2$ and the last $n/2$ children are the roots of copies of tree $P_{n/2,h-1}$, and the middle child is the root of a copy of tree $P_{n,h-1}$. The adaptive version of Parys’s algorithm also uses another tree-pruning rule, which is adaptive and a slight generalization of the empty-set rule: whenever the algorithm is processing the block of the first $n/2$ children of the root or the last $n/2$ children of the root, if one of the recursive calls in this block returns an empty set then the rest of the block is omitted.

We expect that our structural results (such as Theorems 4.1 and 4.5) will provide insights to inspire development and proving correctness of further and more sophisticated adaptive tree-pruning rules, but we leave it to future work. This may be critical for making quasi-polynomial versions of McNaughton-Zielonka competitive in practice with its basic version that is exponential in the worst case, but remains very hard to beat in practice [26, 27].

5 Symbolic algorithms

Parity games that arise in applications, for example from the automata-theoretic model checking approaches to verification and automated synthesis, often suffer from the *state-space explosion problem*: the sizes of models are exponential (or worse) in the sizes of natural descriptions of the modelled objects, and hence the models obtained may be too large to store them explicitly in memory. One method of overcoming this problem that has been successful in the practice of algorithmic formal methods is to represent the models symbolically rather than explicitly, and to develop algorithms

for solving the models that work directly on such succinct symbolic representations [3].

We adopt the *set-based symbolic model of computation* that was already considered for parity games by Chatterjee, Dvořák, Henzinger, and Svozil [5]. In this model, any standard computational operations on any standard data structures are allowed, but there are also the following symbolic resources available: *symbolic set variables* can be used to store sets of vertices in the graph of a parity game; basic set-theoretic operations on symbolic set variables are available as *primitive symbolic operations*; the *controllable predecessors* operations are available as primitive symbolic operations: the Even (resp. Odd) controllable predecessor, when applied to a symbolic set variable X , returns the set of vertices from which Even (resp. Odd) can force to move into the set X , by taking just one outgoing edge. Since symbolic set variables can represent possibly very large and complex objects, they should be treated as a costly resource.

Chatterjee et al. [5] have given a symbolic set-based algorithm that on (n, d) -small parity games uses $O(d \log n)$ of symbolic set variables and runs in quasi-polynomial time. While the dependence on n is only logarithmic, a natural question is whether this dependence is inherent. Given that n can be prohibitively large in applications, reducing dependence on n is desirable. In this section we argue that it is not only possible to eliminate the dependence on n entirely, but it is also possible to exponentially improve the dependence on d , resulting in a quasi-polynomial symbolic algorithm for solving parity games that uses only $O(\lg d)$ symbolic set variables.

5.1 Universal algorithm symbolically

In the set-based symbolic model of computation, it is routine to compute the attractors efficiently: it is sufficient to iterate the controllable predecessor operations. Using the results of Jurdziński and Lazić [20], one can also represent a path of nodes from the root to a leaf in the tree $S_{n,d/2}$ in $O(\lg n \cdot \lg d)$ bits, and for every node on such a path, to compute its number of children in $O(\lg n \cdot \lg d)$ standard primitive operations. This allows to run the whole universal algorithm (Algorithm 2) on an (n, d) -small parity game and two copies of trees $S_{n,d/2}$, using only $O(\lg n \cdot \lg d)$ bits to represent the relevant nodes in the trees $\mathcal{T}^{\text{Even}}$ and \mathcal{T}^{Odd} throughout the execution.

The depth of the tree of recursive calls of the universal algorithm on an (n, d) -small parity game is at most d . Moreover, in every recursive call, only a small constant number of set variables is needed because only the latest sets $V^{\mathcal{G}_i}$, D_i , $V^{\mathcal{G}'_i}$, and U_i are needed at any time. It follows that the overall number of symbolic set variables needed to run the universal algorithm is $O(d)$. Also note that every recursive call can be implemented symbolically using a constant number of

primitive symbolic operations and two symbolic attractor computations.

This improves the symbolic space from Chatterjee et al.'s $O(d \lg n)$ to $O(d)$, while keeping the running time quasipolynomial. Moreover, this symbolic algorithm is very simple and straightforward to implement, which makes it particularly promising and attractive for empirical evaluation and deployment in applications.

5.2 Succinct symbolic partitions

In this section we describe how the number of symbolic set variables in the symbolic implementation of the universal algorithm can be further reduced from $O(d)$ to $O(\log d)$. We use letters G , D , G' , and U to denote the sets $V^{\mathcal{G}_i}$, D_i , $V^{\mathcal{G}'_i}$, and U_i for some i -th iteration of any of the recursive calls of the universal algorithm. Observe that we do not need to keep the symbolic variables that store the sets D , G' , and U on the stack of recursive calls because on any return from a recursive call, their values are not needed to proceed. How can we store the sets denoted by all the symbolic set variables G on the stack using only $O(\log d)$ symbolic set variables, while the height of the stack may be as large as d ?

Firstly, we argue that we can symbolically represent any sequence $\langle G_{d-1}, \dots, G_i \rangle$ of set variables that would normally occur on the stack of recursive calls of the universal algorithm, by another sequence $\langle H_{d-1}, \dots, H_0 \rangle$, in which the sets form a partition of the set of vertices in the parity game. Indeed, a sequence $\langle G_d, \dots, G_i \rangle$ on the stack of recursive calls at any time forms a descending chain w.r.t. inclusion, and G_d is the set of all vertices, so it suffices to consider the sequence $\langle G_d \setminus G_{d-1}, \dots, G_{i+1} \setminus G_i, G_i, \emptyset, \dots, \emptyset \rangle$.

Secondly, we argue that the above family of d mutually disjoint sets can be succinctly represented and maintained using $O(\log d)$ set variables. W.l.o.g., assume that d is a power of 2. For every $k = 1, 2, \dots, \lg d$, and for every $i = 1, 2, \dots, d$, let $\text{bit}_k(i)$ be the k -th digit in the binary representation of i (and zero if there are less than k digits). We now define the following sequence of sets $\langle S_1, S_2, \dots, S_{\lg d} \rangle$ that provides a succinct representation of the sequence $\langle H_{d-1}, \dots, H_0 \rangle$. For every $k = 1, 2, \dots, \lg d$, we set:

$$S_k = \bigcup \{H_i : 0 \leq i \leq d-1 \text{ and } \text{bit}_k(i) = 1\}.$$

By sets $\langle H_{d-1}, \dots, H_0 \rangle$ forming a partition of the set of all vertices, it follows that for every $i = 0, 1, \dots, d-1$, we have:

$$H_i = \bigcap \{S_k : 1 \leq k \leq \lg d \text{ and } \text{bit}_k(i) = 1\} \\ \cap \bigcap \{\overline{S_k} : 1 \leq k \leq \lg d \text{ and } \text{bit}_k(i) = 0\},$$

where \overline{X} is the complement of set X .

What remains to be shown is that the operations on the sequence of sets $\langle G_{d-1}, \dots, G_i \rangle$ that reflect changes on the stack of recursive calls of the universal algorithm can indeed

be implemented using small numbers of symbolic set operations on the succinct representation $\langle S_1, \dots, S_{\lg d} \rangle$ of the sequence $\langle H_{d-1}, \dots, H_0 \rangle$. We note that there are two types of changes to the sequence $\langle G_{d-1}, \dots, G_i \rangle$ that the universal algorithm makes:

- (a) all components are as before, except for G_i that is replaced by $G_i \setminus B$, for some set $B \subseteq G_i$;
- (b) all components are as before, except that a new entry G_{i-1} is added equal to $G_i \setminus B$, for some set $B \subseteq G_i$.

The corresponding changes to the sequence $\langle H_{d-1}, \dots, H_0 \rangle$ are then:

- (a) all components are as before, except that set H_{i+1} is replaced by $H_{i+1} \cup B$, and set H_i is replaced by $H_i \setminus B$;
- (b) all components are as before, except that set H_i is replaced by B , and set H_{i-1} is replaced by $H_i \setminus B$.

To implement the update of type (a), it suffices to perform the following update to the succinct representation:

$$S'_k = \begin{cases} S_k & \text{if } \text{bit}_k(i+1) = \text{bit}_k(i), \\ S_k \cup B & \text{if } \text{bit}_k(i+1) = 1 \text{ and } \text{bit}_k(i) = 0, \\ S_k \setminus B & \text{if } \text{bit}_k(i+1) = 0 \text{ and } \text{bit}_k(i) = 1. \end{cases}$$

and to implement the update of type (b), it suffices to perform the following:

$$S'_k = \begin{cases} S_k & \text{if } \text{bit}_k(i) = \text{bit}_k(i-1), \\ S_k \setminus (H_i \setminus B) & \text{if } \text{bit}_k(i) = 1 \text{ and } \text{bit}_k(i-1) = 0, \\ S_k \cup (H_i \setminus B) & \text{if } \text{bit}_k(i) = 0 \text{ and } \text{bit}_k(i-1) = 1. \end{cases}$$

This completes the proof sketch of the main technical result in this section.

Theorem 5.1. *There is a symbolic algorithm that solves (n, d) -small parity games using $O(\lg d)$ symbolic set variables, $O(\log d \cdot \log n)$ bits of conventional space, and whose running time is polynomial if $d = O(\log n)$, and $n^{2 \lg(d/\lg n) + O(1)}$ if $d = \omega(\log n)$.*

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A Appendix

Proof of Proposition 4.3. Firstly, we argue that $T \cap R$ is a trap for Odd in subgame $\mathcal{G} \cap R$. Let $v \in (T \cap R) \cap V_{\text{Odd}}$. We need to argue that if $(v, w) \in E$ and $w \in R$ then $w \in T$, which follows directly from the assumption that T is a trap for Odd in \mathcal{G} . Let $v \in (T \cap R) \cap V_{\text{Even}}$. We need to argue that there is $(v, w) \in E$, such that $w \in T \cap R$. Observe that there is an edge $(v, w) \in E$, such that $w \in T$, because T is a trap for Odd in \mathcal{G} . Note, however, that we also have $w \in R$ because R is a trap for Even in \mathcal{G} .

Now suppose that T is not only a trap for Odd in \mathcal{G} but it is also an Even dominion in \mathcal{G} and let σ be an Even dominion strategy on T . Note that σ is also an Even dominion strategy on $T \cap R$ in subgame $\mathcal{G} \cap R$ because R is a trap for Even. \square

Proof of Proposition 4.4. Firstly, we argue that $A \cap T \subseteq A'$. Let σ be an Even reachability strategy from A to B in \mathcal{G} . Note that σ is also an Even reachability strategy from $A \cap T$ to $B \cap T$ in $\mathcal{G} \cap T$ because T is a trap for Even in \mathcal{G} . It then follows that $A \cap T \subseteq A'$ because A' is the Even attractor to $B \cap T$ in $\mathcal{G} \cap T$.

Secondly, we argue that $T \setminus A'$ is a trap for Even in $\mathcal{G} \setminus A$. Note that $T \setminus A'$ is a trap for Even in \mathcal{G} by trap transitivity: T is a trap for Even in \mathcal{G} , and $T \setminus A'$ is a trap for Even in $\mathcal{G} \cap T$ because it is a complement of an Even attractor. From $A \cap T \subseteq A'$ it follows that $T \setminus A'$ is disjoint from A , and hence it is also a trap for Even in $\mathcal{G} \setminus A$. \square