# **Comparing Labelled Markov Decision Processes**

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### Abstract

A labelled Markov decision process is a labelled Markov chain with nondeterminism, i.e., together with a strategy a labelled MDP induces a labelled Markov chain. The model is related to interval Markov chains. Motivated by applications of equivalence checking for the verification of anonymity, we study the algorithmic comparison of two labelled MDPs, in particular, whether there exist strategies such that the MDPs become equivalent/inequivalent, both in terms of trace equivalence and in terms of probabilistic bisimilarity. We provide the first polynomial-time algorithms for computing memoryless strategies to make the two labelled MDPs inequivalent if such strategies exist. We also study the computational complexity of qualitative problems about making the total variation distance and the probabilistic bisimilarity distance less than one or equal to one.

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#### 1 Introduction

Given a model of computation (e.g., finite automata), and two instances of it, are they semantically equivalent (i.e., do they accept the same language)? Such equivalence problems can be viewed as a fundamental question for almost any model of computation. As such, they permeate computer science, in particular, theoretical computer science.

In labelled Markov chains (LMCs), which are Markov chains whose states (or, equivalently, transitions) are labelled with an observable letter, there are two natural and very well-studied versions of equivalence, namely trace (or language) equivalence and probabilistic bisimilarity.

The trace equivalence problem has a long history, going back to Schützenberger [36] and Paz [31] who studied weighted and probabilistic automata, respectively. Those models generalize LMCs, but the respective equivalence problems are essentially the same. It can be extracted from [36] that equivalence is decidable in polynomial time, using a technique based on linear algebra. Variants of this technique were developed in [42, 17]. More recently, the efficient decidability of the equivalence problem was exploited, both theoretically and practically, for the verification of probabilistic systems, see, e.g., [23, 24, 32, 30, 28]. In those works, equivalence naturally expresses properties such as obliviousness and anonymity, which are difficult to formalize in temporal logic. In a similar vein, inequivalence can mean detectibility and the lack of anonymity.

Probabilistic bisimilarity is an equivalence that was introduced by Larsen and Skou [27].

It is finer than trace equivalence, i.e., probabilistic bisimilarity implies trace equivalence. A similar notion for Markov chains, called *lumpability*, can be traced back at least to the classical text by Kemeny and Snell [22]. Probabilistic bisimilarity can also be computed in polynomial time [3, 14, 43]. Indeed, in practice, computing the bisimilarity quotient is fast and has become a backbone for highly efficient tools for probabilistic verification such as PRISM [26] and STORM [20].

In this paper, we study equivalence problems for (labelled) Markov decision processes (MDPs), which are LMCs plus nondeterminism, i.e., each state may have several actions (or "moves") one of which is chosen by a controller, potentially randomly. An MDP and a controller strategy together induce an LMC (potentially with infinite state space, depending on the complexity of the strategy). The nondeterminism in MDPs gives rise to a spectrum of equivalence queries: one may ask about the existence of strategies for two given MDPs such that the induced LMCs become trace/bisimulation equivalent, or such that they become trace/bisimulation inequivalent. Another potential dimension of this spectrum is whether to consider general strategies or more restricted ones, such as memoryless or even memoryless deterministic (MD) ones.

In this paper, we focus on memoryless strategies, for several reasons. First, these questions for unrestricted strategies quickly lead to undecidability. For example, in [18, Theorem 3.1] it was shown that whether there exists a general strategy such that a given MDP becomes trace equivalent with a given LMC is undecidable. Second, memoryless strategies are sufficient for a wide range of objectives in MDPs, and their simplicity means that even if it was known that a general strategy exists to accomplish (in)equivalence one might still wonder if there also exists a memoryless strategy. Third, probabilistic bisimilarity is a less natural notion for LMCs induced by general strategies: such LMCs will in general have an infinite state space, even when the MDP is finite. Fourth, applying a memoryless strategy in an MDP is related to choosing an instance of an interval Markov chain (IMC). IMCs are like Markov chains, but the transitions are labelled not with probabilities but with probability intervals. IMCs were introduced by Jonsson and Larsen [21] and have been well studied in verification-related domains [37, 8, 13, 4, 7], but also in areas such as systems biology, security or communication protocols, see, e.g., [12]. Selecting a memoryless strategy in an MDP corresponds to selecting a probability from each interval (one out of generally uncountably many). Parametric Markov chains and parametric MDPs are further related models, see, e.g., [19, 45] and the references therein.

LMCs can also be compared in terms of their distance. We consider two natural distance functions between two LMCs: the total variation distance (between the two trace distributions) and the probabilistic bisimilarity distance [16]. Both distances can be at most 1. The total variation (resp. probabilistic bisimilarity) distance is 0 if and only if the LMCs are trace equivalent (resp. probabilistic bisimilar). Further, the probabilistic bisimilarity distance is an upper bound on the total variation distance [9]. It was shown in [10] (resp. [40]) that whether the total variation (resp. probabilistic bisimilarity) distance of two LMCs equals 1 can be decided in polynomial time. This raises the question whether these results can be extended to MDPs, i.e., what is the complexity of deciding whether there exists a memoryless strategy to make the distance less than 1 or equal to 1, respectively. It turns out that some of these problems are closely related to the corresponding (in)equivalence problem.

Instead of comparing two MDPs with initial distributions/states, one may equivalently compare two initial distributions/states in a single MDP (by taking a disjoint union of the states). In this paper we study the computational complexity of the following problems:

 $\mathbf{T}$   $\mathbf{T}$ 

initial distributions are (not) trace equivalent in the induced labelled Markov chain;

- $\blacksquare$  TV = 1 (TV < 1), which asks whether there is a memoryless strategy such that the two initial distributions (do not) have total variation distance one;
- $\blacksquare$  PB = 0 (PB > 0), which asks whether there is a memoryless strategy such that the two initial states are (not) probabilistic bisimilar;
- PB = 1 (PB < 1), which asks whether there is a memoryless strategy such that the two initial states (do not) have probabilistic bisimilarity distance one.

In Sections 3 and 4 we provide the first polynomial-time algorithms for TV > 0 and PB > 0, respectively. We also show how to compute memoryless strategies that witness trace and probabilistic bisimulation inequivalence, respectively. In Section 5 we discuss TV = 1 and PB = 1, and in Section 6 we establish the complexity of the remaining four problems, which are about making the distance small (= 0 or <1). We conclude in Section 7. Table 1 summarises the results in the paper. Missing proofs can be found in the Appendix.

Problem	Complexity
TV = 0	$\exists \mathbb{R}\text{-complete}$
TV > 0	in P
TV = 1	$NP ext{-}hard$ and in $\exists \mathbb{R}$
TV < 1	$\exists \mathbb{R}\text{-complete}$
PB = 0	NP-complete
PB > 0	in P
PB = 1	NP-complete
PB < 1	NP-complete

**Table 1** Summary of the results. These results also imply results for the problems which state "for all memoryless strategies". For example, TV > 0 is the complement of the decision problem whether for all memoryless strategies the two initial distributions are trace equivalent in the induced labelled Markov chains.

### 2 Preliminaries

We write  $\mathbb{R}$  for the set of real numbers and  $\mathbb{N}$  the set of nonnegative integers. Let S be a finite set. We denote by  $\mathrm{Distr}(S)$  the set of probability distributions on S. By default we view vectors, i.e., elements of  $\mathbb{R}^S$ , as row vectors. For a vector  $\mu \in [0,1]^S$  we write  $|\mu| := \sum_{s \in S} \mu(s)$  for its  $L_1$ -norm. A vector  $\mu \in [0,1]^S$  is a distribution (resp. subdistribution) over S if  $|\mu| = 1$  (resp.  $0 < |\mu| \le 1$ ). We denote column vectors by boldface letters; in particular,  $\mathbf{1} \in \{1\}^S$  and  $\mathbf{0} \in \{0\}^S$  are column vectors all whose entries are 1 and 0, respectively. For  $s \in S$  we write  $\delta_s$  for the (Dirac) distribution over S with  $\delta_s(s) = 1$  and  $\delta_s(r) = 0$  for  $r \in S \setminus \{s\}$ . For a (sub)distribution  $\mu$  we write support  $\mu$  we write support  $\mu$  such that  $\mu$  is a support  $\mu$  such that  $\mu$  is a support  $\mu$  such that  $\mu$  is a subject to  $\mu$  is support.

A labelled Markov chain (LMC) is a quadruple  $\langle S, L, \tau, \ell \rangle$  consisting of a nonempty finite set S of states, a nonempty finite set L of labels, a transition function  $\tau: S \to \mathrm{Distr}(S)$ , and a labelling function  $\ell: S \to L$ .

We denote by  $\tau(s)(t)$  the transition probability from s to t. Similarly, we denote by  $\tau(s)(E) = \sum_{t \in E} \tau(s)(t)$  the transition probability from s to  $E \subseteq S$ . A trace in a LMC  $\mathcal{M}$  is a sequence of labels  $w = a_1 a_2 \cdots a_n$  where  $a_i \in L$ . We denote by  $L^{\leq n}$  the set of traces of length at most n. Let  $M: L \to [0,1]^{S \times S}$  specify the transitions, so that  $\sum_{a \in L} M(a)$  is a stochastic matrix,  $M(a)(s,t) = \tau(s)(t)$  if  $\ell(s) = a$  and M(a)(s,t) = 0 otherwise. We

extend M to the mapping  $M: L^* \to [0,1]^{S \times S}$  with  $M(w) = M(a_1) \cdots M(a_n)$  for a trace  $w = a_1 \cdots a_n$ . If the LMC is in state s, then with probability M(w)(s,s') it emits a trace w and moves to state s' in |w| steps. For a trace  $w \in L^*$ , we define  $Run(w) := \{w\}L^{\omega}$ ; i.e., Run(w) is the set of traces starting with w. To an initial distribution  $\pi$  on S, we associate the probability space  $(L^{\omega}, \mathcal{F}, \Pr_{\mathcal{M}, \pi})$ , where  $\mathcal{F}$  is the  $\sigma$ -field generated by all basic cylinders Run(w) with  $w \in L^*$  and  $\Pr_{\mathcal{M}, \pi} : \mathcal{F} \to [0, 1]$  is the unique probability measure such that  $\Pr_{\mathcal{M}, \pi}(Run(w)) = |\pi M(w)|$ . We generalize the definition of  $\Pr_{\mathcal{M}, \pi}$  to subdistributions  $\pi$  in the obvious way, yielding sub-probability measures. We may drop the subscript  $\mathcal{M}$  if it is clear from the context.

Given two initial distributions  $\mu$  and  $\nu$ , the total variation distance between  $\mu$  and  $\nu$  is defined as follows:

$$d_{tv}(\mu,\nu) = \sup_{E \in \mathcal{F}} |\Pr_{\mu}(E) - \Pr_{\nu}(E)|.$$

We write  $\mu \equiv \nu$  to denote that  $\mu$  and  $\nu$  are trace equivalent, i.e.,  $|\Pr_{\mu}(Run(w))| = |\Pr_{\nu}(Run(w))|$  holds for all  $w \in L^*$ . We have that trace equivalence and the total variation distance being zero are equivalent [10, Proposition 3(a)].

The pseudometric probabilistic bisimilarity distance of Desharnais et al. [15], which we denote by  $d_{pb}$ , is a function from  $S \times S$  to [0, 1], that is, an element of  $[0,1]^{S \times S}$ . It can be defined as the least fixed point of the following function:

$$\Delta(d)(s,t) = \begin{cases} 1 & \text{if } \ell(s) \neq \ell(t) \\ \min_{\omega \in \Omega(\tau(s), \tau(t))} \sum_{u,v \in S} \omega(u,v) \ d(u,v) & \text{otherwise} \end{cases}$$

where the set  $\Omega(\mu, \nu)$  of couplings of  $\mu, \nu \in \mathrm{Distr}(S)$  is defined as  $\Omega(\mu, \nu) = \{ \omega \in \mathrm{Distr}(S \times S) \mid \sum_{t \in S} \omega(s, t) = \mu(s) \land \sum_{s \in S} \omega(s, t) = \nu(t) \}$ . Note that a coupling  $\omega \in \Omega$  is a joint probability distribution with marginals  $\mu$  and  $\nu$  (see, e.g., [5, page 260-262]).

An equivalence relation  $R \subseteq S \times S$  is a probabilistic bisimulation if for all  $(s,t) \in R$ ,  $\ell(s) = \ell(t)$  and  $\tau(s)(E) = \tau(t)(E)$  for each R-equivalence class E. Probabilistic bisimularity, denoted by  $\sim_{\mathcal{M}}$  (or  $\sim$  when  $\mathcal{M}$  is clear), is the largest probabilistic bisimulation. For all  $s, t \in S$ ,  $s \sim t$  if and only if  $d_{pb}(s,t) = 0$  [15, Theorem 1].

A (labelled) Markov decision process (MDP) is a tuple  $\langle S, \mathcal{A}, L, \varphi, \ell \rangle$  consisting of a finite set S of states, a finite set S of actions, a finite set S of labels, a partial function  $\varphi: S \times \mathcal{A} \to \mathrm{Distr}(S)$  denoting the probabilistic transition, and a labelling function  $\ell: S \to L$ . The set of available actions in a state S is S is S is a function, and a labelling function S is defined. A memoryless strategy for an MDP is a function S is a function and that given a state S is returns a probability distribution on all the available actions at that state. Such strategies are also known as positional, as they do not depend on the history of past states. A strategy S is memoryless deterministic (MD) if for all states S there exists an action S is such that S is uch that S is thus view an MD strategy as a function S is a tuple S is a tuple S in the state S is a tuple S in the state S in the state S is a tuple S in the state S is a tuple S in the state S in tuple S is a tuple S in the state S in the state S in the state S is a tuple S

For the remainder of the paper, we fix an MDP  $\mathcal{D} = \langle S, \mathcal{A}, L, \varphi, \ell \rangle$ . Given a memoryless strategy  $\alpha$  for  $\mathcal{D}$ , an LMC  $\mathcal{D}(\alpha) = \langle S, L, \tau, \ell \rangle$  is induced, where  $\tau(s)(t) = \sum_{\mathsf{m} \in \mathcal{A}(s)} \alpha(s)(\mathsf{m}) \cdot \varphi(s, \mathsf{m})(t)$ . The matrix  $M_{\alpha}$  specifies the transitions of the LMC  $\mathcal{D}(\alpha)$  as is defined previously.

We fix two initial distributions  $\mu$  and  $\nu$  on S (resp. two initial states s and t) for problems related to total variation distance (resp. probabilistic bisimilarity distance).

### 3 Trace Inequivalence

In this section we show that one can decide in polynomial time whether there exists a memoryless strategy  $\alpha$  so that  $\mu \not\equiv \nu$  in  $\mathcal{D}(\alpha)$ . In terms of the notation from the introduction,

we show that TV > 0 is in P. Define the following column-vector spaces.

```
\mathcal{V}_1 = \langle M_{\alpha_1}(a_1) M_{\alpha_2}(a_2) \cdots M_{\alpha_m}(a_m) \mathbf{1} : \alpha_i \text{ is a memoryless strategy; } a_i \in L \rangle and
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 $\mathcal{V}_2 = \langle M_\alpha(w) \mathbf{1} : \alpha \text{ is a memoryless strategy; } w \in L^* \rangle \text{ and }$ 

$$\mathcal{V}_3 = \langle M_\alpha(w) \mathbf{1} : \alpha \text{ is an MD strategy}; w \in L^* \rangle.$$

Here and later we use the notation  $\langle \cdot \rangle$  to denote the span of (i.e., the vector space spanned by) a set of vectors. By the definitions, we have that  $\mu \equiv \nu$  in all LMCs induced by all memoryless strategies  $\alpha$  if and only if  $\mu M_{\alpha}(w)\mathbf{1} = \nu M_{\alpha}(w)\mathbf{1}$  holds for all memoryless strategies  $\alpha$  and all  $w \in L^*$ . It follows:

▶ Proposition 1. For all distributions  $\mu, \nu$  over S we have:

 $\exists$  a memoryless strategy  $\alpha$  such that  $\mu \not\equiv \nu$  in  $\mathcal{D}(\alpha) \iff \mu \mathbf{v} \not= \nu \mathbf{v}$  for some  $\mathbf{v} \in \mathcal{V}_2$ .

To decide TV > 0 and to compute the "witness" memoryless strategy such that  $\mu \not\equiv \nu$  in the induced LMC, it suffices to compute a basis for  $\mathcal{V}_2$ ; more precisely, a set of  $\alpha$  and w such that the vectors  $M_{\alpha}(w)\mathbf{1}$  span  $\mathcal{V}_2$ . As the set of memoryless strategies is uncountable, this is not straightforward. From the definitions, we know  $\mathcal{V}_3 \subseteq \mathcal{V}_2 \subseteq \mathcal{V}_1$ . We will show  $\mathcal{V}_1 \subseteq \mathcal{V}_3$  and thus establish the equality of these three vector spaces. It follows from [18, Theorem 5.12] that computing a basis for  $\mathcal{V}_1$  is in P. It follows that our problem TV > 0 is also in P, but this does not explicitly give the witnessing memoryless strategy. Since  $\mathcal{V}_2 = \mathcal{V}_3$ , there must exist an MD strategy that witnesses  $\mu \not\equiv \nu$ . To find this MD strategy, one can go through all MD strategies (potentially exponentially many). In the following, by considering the vector spaces while restricting the word length, we show that a witness MD strategy can also be computed in polynomial time.

We define the following column-vector spaces. For each  $j \in \mathbb{N}$ ,

$$\mathcal{V}_1^j = \langle M_{\alpha_1}(a_1) M_{\alpha_2}(a_2) \cdots M_{\alpha_k}(a_k) \mathbf{1} : \alpha_i \text{ is a memoryless strategy; } a_i \in L; k \leq j \rangle \text{ and}$$

$$\mathcal{V}_2^j = \langle M_{\alpha}(w) \mathbf{1} : \alpha \text{ is a memoryless strategy; } w \in L^{\leq j} \rangle \text{ and}$$

$$\mathcal{V}_3^j = \langle M_{\alpha}(w) \mathbf{1} : \alpha \text{ is an MD strategy; } w \in L^{\leq j} \rangle.$$

Let  $\alpha$  be an MD strategy and  $\mathbf{m}$  be an action available at state i. Recall that an MD strategy can be viewed as a function  $\alpha: S \to \mathcal{A}$ . We define  $\alpha^{i \to \mathbf{m}}$  to be the MD strategy such that  $\alpha^{i \to \mathbf{m}}(i) = \mathbf{m}$  and  $\alpha^{i \to \mathbf{m}}(s) = \alpha(s)$  for all  $s \in S \setminus \{i\}$ . Let  $\mathbf{c}_i \in \{0,1\}^S$  be the column bit vector whose only non-zero entry is the ith one. For a set  $B \subseteq \mathbb{R}^S$ , we define  $\langle B \rangle$  to be the vector space spanned by B.

We call a column vector an MD vector if it is of the form  $M_{\alpha}(w)\mathbf{1}$  for an MD strategy  $\alpha$  and  $w \in L^*$ . Let P be a set of MD strategy and word pairs, i.e.,  $P = \{(\alpha_1, w_1), (\alpha_2, w_2), \cdots, (\alpha_m, w_m)\}$  where  $\alpha_i$  is an MD strategy and  $w_i \in L^*$ . We define a function  $\mathcal{B}$  transforming such a set P to the set of corresponding MD vectors, i.e.,  $\mathcal{B}(P) = \{M_{\alpha_1}(w_1)\mathbf{1}, M_{\alpha_2}(w_2)\mathbf{1}, \cdots, M_{\alpha_n}(w_n)\mathbf{1}\}.$ 

▶ Lemma 2. Let  $j \in \mathbb{N}$ . For all MD strategies  $\alpha_1$  and  $\alpha_2$ ,  $a \in L$  and  $w \in L^{\leq j}$ , we have  $M_{\alpha_1}(a)M_{\alpha_2}(w)\mathbf{1} \in \langle \mathcal{V}_1^j \cup \mathcal{B}(\{(\alpha, aw)\}) \rangle$  where  $\alpha$  is the MD strategy defined by

$$\alpha(i) = \begin{cases} \alpha_1(i) & \text{if } \mathbf{c}_i \notin \mathcal{V}_1^j \\ \alpha_2(i) & \text{otherwise} \end{cases}$$

The next lemma shows that a basis for  $\mathcal{V}_1^j$  for some j < |S| consisting only of MD vectors can be computed in polynomial time.

▶ Lemma 3. Let  $j \in \mathbb{N}$  with j < |S|. One can compute in polynomial time a set  $P_j = \{(\alpha_0, w_0), \cdots, (\alpha_k, w_k)\}$  in which all  $\alpha_i$  are MD strategies and all  $w_i$  are in  $L^{\leq j}$  such that  $\mathcal{B}(P_j)$  is a basis of  $\mathcal{V}_1^j$ .

**Proof sketch.** We prove this lemma by induction on j. The base case where j=0 is vacuously true with  $P_0 = \{(\alpha_0, w_0)\}$  where  $\alpha_0$  is an arbitrary MD strategy,  $w_0 = \varepsilon$  and  $\mathcal{B}(P_0) = \{1\}$ . For the induction step, assume that we can compute in polynomial time a set  $P_j = \{(\alpha_0, w_0), \cdots, (\alpha_k, w_k)\}$  where all the strategies are MD strategies and all the words are in  $L^{\leq j}$  such that  $\mathcal{B}(P_j)$  is a basis for  $\mathcal{V}_1^j$ . We show that the statement holds for j+1. Define

$$\Sigma = \{\alpha_0\} \cup \{\alpha_0^{s \to \mathsf{m}} : s \in S, \ \mathsf{m} \in \mathcal{A}(s)\} \qquad \text{and} \qquad \mathbb{M} = \{M_\alpha(a) \in \mathbb{R}^{S \times S} : \alpha \in \Sigma, \ a \in L\}.$$

Next, we present Algorithm 1 which computes a set  $P_{j+1}$  in polynomial time such that

for all 
$$M \in \mathbb{M}$$
 and all  $\mathbf{b} \in \mathcal{B}(P_j) : M \cdot \mathbf{b} \in \langle \mathcal{B}(P_{j+1}) \rangle$  (1)

**Algorithm 1** Polynomial-time algorithm computing  $P_{j+1}$ .

```
1 P_{j+1} := P_j

2 foreach \alpha_1 \in \Sigma, a \in L and (\alpha_2, w) \in P_j do

3 | if M_{\alpha_1}(a)M_{\alpha_2}(w)1 \not\in \langle \mathcal{B}(P_{j+1}) \rangle then

4 | add (\alpha, aw) to P_{j+1} where \alpha is the MD strategy defined as
\alpha(i) = \begin{cases} \alpha_1(i) & \text{if } \mathbf{c}_i \not\in \mathcal{V}_1^j \\ \alpha_2(i) & \text{otherwise.} \end{cases}
5 | end

6 end
```

All the vectors in  $\mathcal{B}(P_{j+1})$  are linearly independent, as we only add a pair if the corresponding vector is linearly independent to the existing vectors in  $\mathcal{B}(P_{j+1})$  (lines 3-4). Since  $\mathcal{B}(P_j)$  is a basis for  $\mathcal{V}_1^j$ , we can decide whether  $\mathbf{c}_i \in \mathcal{V}_1^j$  for  $i \in S$  in polynomial time, and thus compute a pair  $(\alpha, aw)$  on line 4 in polynomial time. Since  $|\Sigma|$  and |L| are polynomial in the size of the MDP,  $|P_j| < |S|$ , the number of iterations is polynomial in the size of the MDP. The construction of  $P_{j+1}$  is then in polynomial time. It remains to show that after adding  $(\alpha, aw)$  to  $P_{j+1}$  (line 4), we have  $M \cdot \mathbf{b} = M_{\alpha_1}(a)M_{\alpha_2}(w)\mathbf{1} \in \langle \mathcal{B}(P_{j+1})\rangle$ . Since the pair  $(\alpha_2, w)$  is in  $P_j$ , we have  $w \in L^{\leq j}$ . Then,

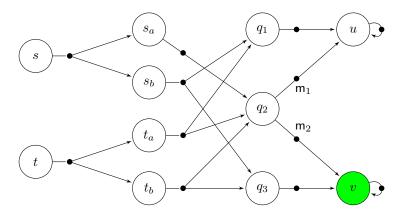
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\begin{array}{ll} M \cdot \mathbf{b} \\ &= M_{\alpha_1}(a) M_{\alpha_2}(w) \mathbf{1} \\ &\in \langle \mathcal{V}_1^j \cup \mathcal{B}(\{(\alpha, aw)\}) \rangle \quad \text{[Lemma 2]} \\ &= \langle \mathcal{B}(P_j) \cup \mathcal{B}(\{(\alpha, aw)\}) \rangle \quad \quad [\mathcal{B}(P_j) \text{ is a basis for } \mathcal{V}_1^j \text{ by induction hypothesis]} \\ &= \langle \mathcal{B}(P_j \cup \{(\alpha, aw)\}) \rangle \end{array}
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Since  $P_j \subseteq P_{j+1}$  (line 1), we have  $\mathcal{B}(P_j) \subseteq \mathcal{B}(P_{j+1})$ . By adding the pair  $(\alpha, aw)$  to  $P_{j+1}$ , we have  $\langle \mathcal{B}(P_j \cup \{(\alpha, aw)\}) \rangle \subseteq \langle \mathcal{B}(P_{j+1}) \rangle$ , and thus  $M \cdot \mathbf{b} \in \langle \mathcal{B}(P_{j+1}) \rangle$ .

Finally, we show that the set  $P_{j+1}$  satisfies  $\mathcal{V}_1^{j+1} = \langle \mathcal{B}(P_{j+1}) \rangle$ . We have

$$\langle \mathcal{B}(P_{j+1}) \rangle \subseteq \mathcal{V}_3^{j+1}$$
 for all  $(\alpha, w) \in P_{j+1} : \alpha$  is an MD strategy and  $w \in L^{\leq j+1}$   $\subseteq \mathcal{V}_1^{j+1}$  from the definitions

We prove the other direction  $\mathcal{V}_1^{j+1} \subseteq \langle \mathcal{B}(P_{j+1}) \rangle$  in Appendix A.



**Figure 1** In this MDP no MD strategy witnesses  $s \nsim t$ . All states have the same label except state v. By default the transition probabilities out of each action are uniformly distributed.

Combining classical linear algebra arguments about equivalence checking (see, e.g., [42]) with Lemma 3, we obtain:

### ▶ Lemma 4.

- 1. For all j < |S| we have  $\mathcal{V}_1^j = \mathcal{V}_2^j = \mathcal{V}_3^j$ . 2. We have  $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{V}_3 = \mathcal{V}_1^{|S|-1} = \mathcal{V}_2^{|S|-1} = \mathcal{V}_3^{|S|-1}$ .

Thus we obtain:

▶ **Proposition 5.** One can compute in polynomial time a set  $P = \{(\alpha_0, w_0), \dots, (\alpha_k, w_k)\}$ of MD strategy and word pairs such that  $\mathcal{B}(P)$  is a basis of  $\mathcal{V}_2$ .

**Proof.** By Lemma 4 it suffices to invoke Lemma 3 for j = |S| - 1.

Now we can prove the main theorem of this section.

**Theorem 6.** The problem TV > 0 is in P. Further, for any positive instance of the problem TV > 0, we can compute in polynomial time an MD strategy  $\alpha$  and a word w that witness  $\mu \not\equiv \nu$ , i.e.,  $\Pr_{\mu,\mathcal{D}(\alpha)}(Run(w)) \not= \Pr_{\nu,\mathcal{D}(\alpha)}(Run(w))$ .

**Proof.** A polynomial algorithm follows naturally from Proposition 5 and Proposition 1. We first compute a set P of MD strategy and word pairs such that  $\mathcal{B}(P)$  is a basis for  $\mathcal{V}_2$ . For each  $\mathbf{b} \in \mathcal{B}(P)$ , we check whether  $\mu \mathbf{b} \neq \nu \mathbf{b}$  and output "yes" indicating a positive instance if the inequality holds. Otherwise, we have  $\mu \mathbf{b} = \nu \mathbf{b}$  for all  $\mathbf{b} \in \mathcal{B}(P)$ , and the algorithm outputs "no" indicating that  $\mu \equiv \nu$  holds for all memoryless strategies.

If the instance is positive, there exists a vector  $\mathbf{b} \in \mathcal{B}(P)$  such that  $\mu \mathbf{b} \neq \nu \mathbf{b}$ . Since  $\mathbf{b}$ is an MD vector which corresponds to a pair  $(\alpha, w) \in P$ , we have  $\mu M_{\alpha}(w) \mathbf{1} \neq \nu M_{\alpha}(w) \mathbf{1}$ , equivalently  $\Pr_{\mu,\mathcal{D}(\alpha)}(Run(w)) \neq \Pr_{\nu,\mathcal{D}(\alpha)}(Run(w))$ .

### Probabilistic Bisimulation Inequivalence

In this section we show that one can decide in polynomial time whether there exists a memoryless strategy  $\alpha$  so that  $s \not\sim t$  in  $\mathcal{D}(\alpha)$ , i.e., we show that PB > 0 is in P.

For some MDPs, there might be memoryless strategies such that  $s \nsim t$  in the induced LMC but no such strategy is MD. The MDP in Figure 1 is such an example. Similar to

### Algorithm 2 Partition Refinement

```
1 i = 0; X_0 := \{S\}
2 repeat
3 | i := i + 1
4 | X_i := S/\equiv_{X_{i-1}}
5 until X_i = X_{i-1}
```

the or-gate construction of [9, Theorem 2], we have  $s \sim t$  if and only if  $q_1 \sim q_2$  or  $q_2 \sim q_3$ . We have  $q_2 \sim q_1$  if the MD strategy maps  $q_2$  to the action that goes to state u, otherwise  $q_2 \sim q_3$  if the MD strategy maps  $q_2$  to the action that goes to state v. This rules out the algorithm that goes through all the MD strategies.

We define an equivalence relation and run the classical polynomial-time partition refinement as shown in Algorithm 2, with an equivalence relation  $\equiv_X$  defined below. At the beginning, all states are in the same equivalence class. In a refinement step, a pair of states is split if there *could* exist a memoryless strategy that makes them not probabilistic bisimilar. Two states s, t remain in the same equivalence class until the end if and only if they are probabilistic bisimilar under all memoryless strategies.

The correctness of this approach is not obvious, as some splits that occurred in different iterations of the algorithm may have been due to different, potentially contradictory, memoryless strategies. Furthermore, the algorithm does not compute a memoryless strategy that witnesses  $s \not\sim t$ . The key to solving both problems will be Lemma 11.

A partition of the states S is a set X consisting of pairwise disjoint subsets E of S with  $\bigcup_{E \in X} = S$ . Recall that  $\varphi(s, \mathsf{m})(s')$  is the transition probability from s to s' when choosing action  $\mathsf{m}$ . Similarly,  $\varphi(s, \mathsf{m})(E)$  is the transition probability from s to  $E \subseteq S$  when choosing action  $\mathsf{m}$ . We write  $\varphi(s, \mathsf{m})(X)$  to denote the vector (probability distribution)  $(\varphi(s, \mathsf{m})(E))_{E \in X}$ . We define  $\varphi(s)(X) = \{\varphi(s, \mathsf{m})(X) : \mathsf{m} \in \mathcal{A}(s)\}$ , which is a set of probabilistic distributions over the partition X when choosing all available actions of s. Each partition is associated with an equivalence relation  $\equiv_X$  on S:  $s \equiv_X s'$  if and only if

```
- \ell(s) = \ell(s');

- s \neq s' \implies |\varphi(s)(X)| = 1 and \varphi(s)(X) = \varphi(s')(X).
```

Let  $S/\equiv_X$  denote the set of equivalence classes with respect to  $\equiv_X$ , which forms a partition of S. We present in Table 2 the partitions of running the algorithm on the MDP in Figure 1. Notice that states s and t are no longer in the same equivalence class at the end. The following lemma is standard, and claims that the partition gets finer.

```
▶ Lemma 7. For all i \in \mathbb{N}, we have \equiv_{X_{i+1}} \subseteq \equiv_{X_i}.
```

If the loop in Algorithm 2 is performed |S| - 1 times then  $X_{|S|-1}$  consists of |S| one-element sets. Hence at most after |S| - 1 refinement steps the partition  $X_i$  cannot be refined.

```
X_{0} = \{S\}
X_{1} = \{\{v\}, S \setminus \{v\}\}\}
X_{2} = \{\{v\}, \{q_{2}\}, \{q_{3}\}, S \setminus \{v, q_{2}, q_{3}\}\}\}
X_{3} = \{\{v\}, \{q_{2}\}, \{q_{3}\}, \{s_{a}\}, \{s_{b}\}, \{t_{a}\}, \{t_{b}\}, \{s, t, q_{1}, u\}\}\}
X_{4} = \{\{v\}, \{q_{2}\}, \{q_{3}\}, \{s_{a}\}, \{s_{b}\}, \{t_{a}\}, \{t_{b}\}, \{s\}, \{t\}, \{q_{1}, u\}\}\}
```

**Table 2** Example of running Algorithm 2 on the MDP in Figure 1.

We aim at proving that  $s \equiv_{X_{|S|-1}} t$  if and only if  $s \sim_{\mathcal{D}(\alpha)} t$  for all memoryless strategies  $\alpha$ . In the following lemma we show the forward direction:

▶ **Lemma 8.** Let X be a partition and  $X = S/\equiv_X$ . We have  $\equiv_X \subseteq \sim_{\mathcal{D}(\alpha)}$  for all memoryless strategies  $\alpha$ .

For the converse, to guarantee  $\equiv_{X_{|S|-1}}$  is not too fine, it suffices to show that there exists a memoryless strategy  $\alpha'$  such that  $\sim_{\mathcal{D}(\alpha')} \subseteq \equiv_X$  where  $X = S/\equiv_X$ . To do that, we define the equivalence relations  $\sim_{\mathcal{D}(\alpha)}^i$  with  $0 \le i \le |S|$  for all memoryless strategies  $\alpha$ .

Let  $\alpha$  be a memoryless strategy. Let  $\tau$  be the transition function for the LMC  $\mathcal{D}(\alpha)$ . Define the equivalence relation  $\sim_{\mathcal{D}(\alpha)}^{i}$  with  $0 \leq i \leq |S|$  on S:  $s \sim_{\mathcal{D}(\alpha)}^{i} s'$  if and only if

```
- \ell(s) = \ell(s');

- i > 0 \implies \tau(s)(E) = \tau(s')(E) for all E \in S/\sim_{\mathcal{D}(\alpha)}^{i-1}.
```

Note that for the LMC  $\mathcal{D}(\alpha)$ , we have  $\sim_{\mathcal{D}(\alpha)}^{i+1} \subseteq \sim_{\mathcal{D}(\alpha)}^{i}$  for all  $i \in \mathbb{N}$  and  $\sim_{\mathcal{D}(\alpha)}^{|S|-1}$  is the probabilistic bisimilarity for the LMC  $\mathcal{D}(\alpha)$  (see, e.g., [3]).

Since the witness strategy might not be MD, we compute a set of prime numbers that can be used to form the weights of the actions. The prime numbers are used to rule out certain "accidental" bisimulations. We denote by  $\operatorname{size}(\mathcal{D})$  the size of the representation of an object  $\mathcal{D}$ . We represent rational numbers as quotients of integers written in binary.

For  $u \in S$ ,  $m \in \mathcal{A}(u)$  and  $E \subseteq S$ , we express  $\varphi(u,m)(E)$  as an irreducible fraction  $\frac{a_{u,m,E}}{b_{u,m,E}}$  where  $a_{u,m,E}$  and  $b_{u,m,E}$  are coprime integers. Similarly, for  $u \in S$ ,  $m_1, m_2 \in \mathcal{A}(u)$  and  $E \subseteq S$ ,  $\varphi(u,m_1)(E) - \varphi(u,m_2)(E)$  is expressed as an irreducible fraction  $\frac{c_{u,m_1,m_2,E}}{d_{u,m_1,m_2,E}}$  that  $c_{u,m_1,m_2,E}$  and  $d_{u,m_1,m_2,E}$  are coprime integers. Let  $N \subseteq \mathbb{N}$  be the following set:

$$\begin{split} N &= \{b_{u,\mathsf{m},E} : u \in S, \ \mathsf{m} \in \mathcal{A}(u) \ \text{and} \ E \in \bigcup_i X_i\} \cup \\ \{c_{u,\mathsf{m}_1,\mathsf{m}_2,E} : u \in S, \ \mathsf{m}_1,\mathsf{m}_2 \in \mathcal{A}(u), \ E \in \bigcup_i X_i \ \text{and} \ c_{u,\mathsf{m}_1,\mathsf{m}_2,E} > 0\}. \end{split}$$

We denote by  $\theta(x)$  the number of different prime factors of a positive integer x, and by  $\theta(N)$  the number of different prime factors in N where N is a set of positive integers.

▶ Lemma 9.  $\theta(N)$  is polynomial in  $size(\mathcal{D})$ .

Using the prime number theorem, we obtain the following lemma which guarantees that one can find |S| extra different prime numbers other than the prime factors in N in time polynomial in  $\operatorname{size}(\mathcal{D})$ .

▶ **Lemma 10.** One can find |S| different prime numbers in time polynomial in  $size(\mathcal{D})$  such that any of them is coprime to all numbers in the set N.

To each  $u \in S$ , we assign a different prime number  $p_u$  that is coprime with all  $b \in N$ . This can be done in polynomial time by Lemma 10. We have

$$p_u \nmid b \text{ for all } b \in N \quad \text{ and } \quad u \neq v \implies p_u \neq p_v \text{ for all } u, v \in S$$
 (2)

We define a partial memoryless strategy for  $\mathcal{D}$  to be a partial function  $\alpha': S \to \mathrm{Distr}(\mathcal{A})$  that, given a state  $s \in S$ , returns  $\alpha'(s) \in \mathrm{Distr}(\mathcal{A}(s))$  if  $\alpha'(s)$  is defined. A memoryless strategy  $\alpha$  is compatible with a partial memoryless strategy  $\alpha'$ , written as  $\alpha \supseteq \alpha'$ , if and only if  $\alpha(s) = \alpha'(s)$  for all s such that  $\alpha'(s)$  is defined. We construct the partial memoryless strategy iteratively.

▶ Lemma 11. Let  $i \in \mathbb{N}$  with  $i \leq |S|$ . One can compute in polynomial time a partial strategy  $\alpha'_i$  such that  $\sim^i_{\mathcal{D}(\alpha)} \subseteq \equiv_{X_i}$  for all  $\alpha \supseteq \alpha'_i$ .

**Proof sketch.** We prove the statement by induction on i. Let  $s, t \in S$ . The base case is i = 0. By definition, we have if  $s \not\equiv_{X_0} t$  then  $\ell(s) \not= \ell(t)$ . We also have if  $\ell(s) \not= \ell(t)$ , then  $s \not\sim_{\mathcal{D}(\alpha)}^0 t$  in  $\mathcal{D}(\alpha)$  for all memoryless strategy  $\alpha$ . We simply let  $\alpha'_0$  be the empty partial function such that  $\alpha \supseteq \alpha'_0$  holds for any memoryless strategy  $\alpha$ .

For the induction step, assume that we can compute in polynomial time a partial strategy  $\alpha'_i$  such that  $\sim^i_{\mathcal{D}(\alpha)} \subseteq \equiv_{X_i}$  for all  $\alpha \supseteq \alpha'_i$ , i.e., if  $s \not\equiv_{X_i} t$  then  $s \not\sim^i_{\mathcal{D}(\alpha)} t$  in  $\mathcal{D}(\alpha)$ . We show the statement holds for i+1.

Algorithm 3 Polynomial-time algorithm constructing  $\alpha'_{i+1}$ .

```
1 \alpha'_{i+1} := \alpha'_{i}

2 foreach u \in S such that |\varphi(u)(X_{i})| = 1 and |\varphi(u)(X_{i+1})| \neq 1 do

3 | pick \mathsf{m}_{1}, \mathsf{m}_{2} \in \mathcal{A}(u) such that for a set E \in X_{i+1} : \varphi(u, \mathsf{m}_{1})(E) > \varphi(u, \mathsf{m}_{2})(E)

4 | \alpha'_{i+1}(u)(\mathsf{m}_{1}) := \frac{1}{p_{u}}

5 | \alpha'_{i+1}(u)(\mathsf{m}_{2}) := 1 - \frac{1}{p_{u}}
```

Algorithm 3 computes the partial memoryless strategy  $\alpha'_{i+1}$  in polynomial time. We show that  $\alpha'_j$  does not overwrite  $\alpha'_k$  for all k < j. It follows that for any  $\alpha \supseteq \alpha'_{i+1}$ , it satisfies  $\alpha \supseteq \alpha'_i$ . Let  $\alpha \supseteq \alpha'_{i+1}$ . Assume  $s \not\equiv_{X_{i+1}} t$ . We distinguish the two cases:  $s \not\sim^i_{\mathcal{D}(\alpha)} t$  and  $s \sim^i_{\mathcal{D}(\alpha)} t$ . For both cases we can derive  $s \not\sim^{i+1}_{\mathcal{D}(\alpha)} t$ , i.e.,  $\sim^{i+1}_{\mathcal{D}(\alpha)} \subseteq \equiv_{X_{i+1}}$  as desired. The details can be found in Appendix B.

For example, let  $p_{q_2}$ , the prime number assigned to state  $q_2$  in Figure 1, be 3 which is coprime with numbers in  $N=\{1,2\}$ . We show how the partial strategy  $\alpha_1'$  is constructed. On line 1 of Algorithm 3,  $\alpha_1'$  is equal to  $\alpha_0'$ , the empty partial function. Since  $|\varphi(q_2)(X_0)|=1$  and  $|\varphi(q_2)(X_1)|=2$ , we enter the for loop. We can pick  $\mathsf{m}_1,\mathsf{m}_2\in\mathcal{A}(q_2)$  and  $E=S\setminus\{v\}\in X_1$  on line 3, since  $\varphi(q_2,\mathsf{m}_1)(E)=1>0=\varphi(q_2,\mathsf{m}_2)(E)$ . We then define the strategy for  $q_2$  (line 4):  $\alpha_1'(q_2)(\mathsf{m}_1)=\frac{1}{3}$  and  $\alpha_1'(q_2)(\mathsf{m}_2)=\frac{2}{3}$ . We have completed the construction of  $\alpha_1'$  as  $|\varphi(u)(X_0)|=|\varphi(u)(X_1)|=1$  for all other state u.

▶ Theorem 12. One can compute in polynomial time a memoryless strategy  $\beta$  such that  $\sim_{\mathcal{D}(\beta)} \subseteq \sim_{\mathcal{D}(\alpha)}$  for all memoryless strategies  $\alpha$ .

**Proof.** By invoking Lemma 11 for i = |S| - 1, a partial strategy  $\alpha'_{|S|-1}$  can be computed in polynomial time such that  $\sim_{\mathcal{D}(\alpha)}^{|S|-1} \subseteq \equiv_{X_{|S|-1}}$  for all  $\alpha \supseteq \alpha'_{|S|-1}$ . Since  $\sim_{\mathcal{D}(\alpha)}^{|S|-1} = \sim_{\mathcal{D}(\alpha)}$ , we have  $\sim_{\mathcal{D}(\alpha)} \subseteq \equiv_{X_{|S|-1}}$  for all  $\alpha \supseteq \alpha'_{|S|-1}$ . Let  $\beta$  be a memoryless strategy defined by

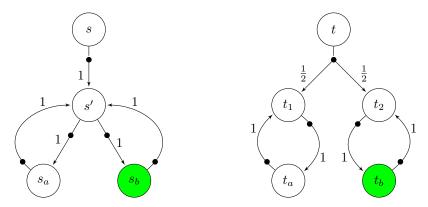
$$\beta(u) = \begin{cases} \alpha'_{|S|-1}(u) & \text{if } \alpha'_{|S|-1}(u) \text{ is defined} \\ \delta_{\mathsf{m}_u} \text{ where } \mathsf{m}_u \in \mathcal{A}(u) & \text{otherwise} \end{cases}$$

By definition the memoryless strategy  $\beta$  is compatible with  $\alpha'_{|S|-1}$ . We have:

$$\begin{array}{c} \sim_{\mathcal{D}(\beta)} \subseteq \equiv_{X_{|S|-1}} & \beta \sqsupseteq \alpha'_{|S|-1} \\ & \subseteq \sim_{\mathcal{D}(\alpha)} \text{ for all strategy } \alpha & X_{|S|-1} = S/\equiv_{X_{|S|-1}} \text{ and Lemma 8} \end{array}$$

▶ Corollary 13. The problem PB > 0 is in P. Further, for any positive instance of the problem PB > 0, we can compute in polynomial time a memoryless strategy that witnesses  $s \nsim t$ .

<sup>&</sup>lt;sup>1</sup> We have  $2 \in N$  since  $\varphi(s, \mathsf{m}_s)(\{s_a\}) = \frac{1}{2}$  where  $\mathsf{m}_s$  is the only available action at state s.



**Figure 2** In this MDP, no MD strategy witnesses  $d_{tv}(\delta_s, \delta_t) = 1$  (nor  $d_{pb}(s, t) = 1$ ). States  $s_b$  and  $t_b$  have label b while all other states have label a.

### 5 The Distance One Problems

In this section, we summarise the results for the two distance one problems, namely TV = 1 and PB = 1. The existential theory of the reals, ETR, is the set of valid formulas of the form

$$\exists x_1 \dots \exists x_n \ R(x_1, \dots, x_n),$$

where R is a boolean combination of comparisons of the form  $p(x_1,\ldots,x_n)\sim 0$ , in which  $p(x_1,\ldots,x_n)$  is a multivariate polynomial (with rational coefficients) and  $\sim \in \{<,>,\leq,\geq,=,\neq\}$ . The complexity class  $\exists \mathbb{R}\ [35]$  consists of those problems that are manyone reducible to ETR in polynomial time. Since ETR is NP-hard and in PSPACE [6, 33], we have NP  $\subseteq \exists \mathbb{R} \subseteq \mathsf{PSPACE}$ .

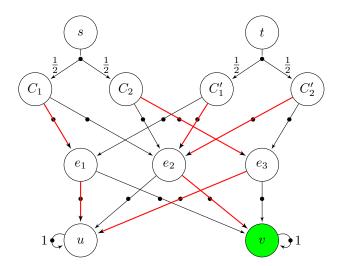
For some MDPs there exist memoryless strategies that make  $d_{tv}(\delta_s, \delta_t) = 1$  but no such strategy is MD. For example, consider the MDP in Figure 2 which has two MD strategies. We have  $d_{tv}(\delta_s, \delta_t) = \frac{1}{2}$  which is less than 1 in the LMC induced by any of the two MD strategies, and  $d_{tv}(\delta_s, \delta_t) = 1$  in the LMC induced by any other strategy. Thus, we cannot simply guess an MD strategy. We show that the problem TV = 1 is in  $\exists \mathbb{R}$ , using the characterization from [10, Theorem 21] of total variation distance 1 in LMCs and some reasoning on convex polyhedra:

### ▶ Theorem 14. The problem TV = 1 is in $\exists \mathbb{R}$ .

The problem TV = 1 is NP-hard, and PB = 1 is NP-complete. The hardness results for both problems are by reductions from the Set Splitting problem. Given a finite set S and a collection C of subsets of S, Set Splitting asks whether there is a partition of S into disjoint sets  $S_1$  and  $S_2$  such that no set in C is a subset of  $S_1$  or  $S_2$ .

Let  $\langle S, \mathcal{C} \rangle$  be an instance of Set Splitting where  $S = \{e_1, \cdots, e_n\}$  and  $\mathcal{C} = \{C_1, \cdots, C_m\}$  is a collection of subsets of S. We construct an MDP  $\mathcal{D}$  consisting of the following states: two states s and t, a state  $e_i$  for each element in S, twin states  $C_j$  and  $C'_j$  for each element in  $\mathcal{C}$ , two sink states u and v. State v has label v while all other states have label v. State v has a single action which goes with uniform probability v to states v and v if v is an action from state v and v is a state v in v

For example, let  $S = \{e_1, e_2, e_3\}$  and  $C = \{C_1, C_2\}$  with  $C_1 = \{e_1, e_2\}$  and  $C_2 = \{e_2, e_3\}$ .



**Figure 3** The MDP in the reduction from Set Splitting for NP-hardness of TV = 1 (or PB = 1).

Figure 3 shows the corresponding MDP. The MD strategy highlighted, corresponding to the partition of  $S_1 = \{e_1, e_3\}$  and  $S_2 = \{e_2\}$ , witnesses  $d_{tv}(\delta_s, \delta_t) = 1$ .

▶ **Theorem 15.** The Set Splitting problem is polynomial-time many-one reducible to TV = 1, hence TV = 1 is NP-hard.

The problem PB = 1 is NP-complete. The MDP in Figure 2 is also an example of no MD strategy witnessing  $d_{pb}(s,t) = 1$ , which rules out the algorithm of simply guessing an MD strategy. By [39], deciding whether  $d_{pb}(s,t) = 1$  in an LMC can be formulated as a reachability problem on a directed graph induced by the LMC. One can nondeterministically guess the graph induced by the LMC and use Algorithm 3 to construct a memoryless strategy that witnesses  $d_{pb}(s,t) = 1$ .

▶ **Theorem 16.** The problem PB = 1 is NP-complete.

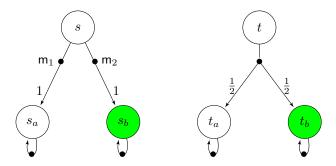
### 6 Making Distances Small

In this section, we summarise the results for the remaining problems, which are all about making the distance small (equal to 0 or less than 1).

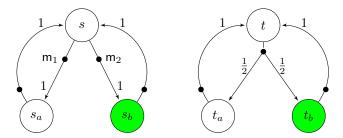
We show that TV = 0 and TV < 1 are  $\exists \mathbb{R}$ -complete. The proof for the membership of TV = 0 in  $\exists \mathbb{R}$  is similar to [18, Theorem 4.3]. For both hardness results we provide reductions from the *Nonnegative Matrix Factorization (NMF)* problem, which asks, given a nonnegative matrix  $J \in \mathbb{Q}^{n \times m}$  and a number  $r \in \mathbb{N}$ , whether there exists a factorization  $J = A \cdot W$  with nonnegative matrices  $A \in \mathbb{R}^{n \times r}$  and  $W \in \mathbb{R}^{r \times m}$ . The NMF problem is  $\exists \mathbb{R}$ -complete by [38, Theorem 2], see also [11, 44, 2] for more details on the NMF problem. The reduction is similar to [18, Theorem 4.5].

- ▶ **Theorem 17.** The problem TV = 0 is  $\exists \mathbb{R}$ -complete.
- ▶ **Theorem 18.** The problem TV < 1 is  $\exists \mathbb{R}$ -complete.

Finally, we show that PB = 0 and PB < 1 are NP-complete. For some MDPs there exist memoryless strategies that make  $d_{pb}(s,t) = 0$  (resp.  $d_{pb}(s,t) < 1$ ) but no such strategy is



**Figure 4** In this MDP, no MD strategy witnesses  $d_{pb}(s,t) = 0$ . States  $s_b$  and  $t_b$  have label b while all other states have label a.



**Figure 5** In this MDP, no MD strategy witnesses  $d_{pb}(s,t) < 1$ . States  $s_b$  and  $t_b$  have label b while all other states have label a.

MD. Indeed, for the MDP in Figure 4 (resp. Figure 5), it is easy to check that the only strategy  $\alpha$  which makes  $d_{pb}(s,t) = 0$  (resp.  $d_{pb}(s,t) < 1$ ), requires randomness, that is,  $\alpha(s)(\mathsf{m}_1) = \alpha(s)(\mathsf{m}_2) = \frac{1}{2}$ , where  $\mathsf{m}_1$  and  $\mathsf{m}_2$  are the two available actions of state s. Thus, to show the NP upper bound, we cannot simply guess an MD strategy. Instead, one can nondeterministically guess a partition of the states and check in polynomial time if the partition is a probabilistic bisimulation.

The hardness results for both problems are by reductions from the Subset Sum problem. The reduction is similar to [18, Theorem 4.1].

▶ **Theorem 19.** The problem PB = 0 is NP-complete.

By [39], deciding whether  $d_{pb}(s,t) < 1$  in an LMC can be formulated as a reachability problem on a directed graph induced by the LMC. In addition to a partition, our NP algorithm also guesses the graph induced by the LMC.

▶ **Theorem 20.** The problem PB < 1 is NP-complete.

### 7 Conclusions

We have studied the computational complexity of qualitative comparison problems in labelled MDPs. Motivated by the connection between obliviousness/anonymity and equivalence, we have devised polynomial-time algorithms to decide the existence of strategies for trace and bisimulation *inequivalence*. In case of trace inequivalence, there always exists an MD witness strategy, and our algorithm computes it. The trace inequivalence algorithm is based on linear-algebra arguments that are considerably more subtle than in the LMC case. For bisimulation inequivalence, MD strategies may not exist, but we have devised a polynomial-time algorithm

to compute a memoryless strategy witnessing inequivalence; here the randomization is based on prime numbers to rule out certain "accidental" bisimulations. The other 6 problems do not have polynomial complexity (unless P = NP), and we have established completeness results for all of them except TV = 1, where a complexity gap between NP and  $\exists \mathbb{R}$  remains.

Concerning the relationship to interval Markov chains and parametric Markov chains mentioned in the introduction, the lower complexity bounds that we have derived in this paper carry over to corresponding problems in these models. Transferring the upper bounds requires additional work, as, e.g., even the consistency problem for IMCs (i.e., whether there exists a Markov chain conforming to an IMC) is not obvious to solve. Nevertheless, the algorithmic insights of this paper will be needed.

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### A Proofs of Section 3

The following lemma is technical and is only used in the proof of Lemma 2.

▶ Lemma 21. Let  $j \in \mathbb{N}$  and  $\alpha$  be an MD strategy. Let  $i \in S$ . If  $\mathbf{c}_i \notin \mathcal{V}_1^j$ , then  $M_{\alpha}(w)\mathbf{1} = M_{\alpha^{i \to m}}(w)\mathbf{1}$  for all  $m \in \mathcal{A}(i)$  and  $w \in L^{\leq j}$ .

**Proof.** Let  $j \in \mathbb{N}$  and  $\alpha$  be an MD strategy. Let  $i \in S$  and  $\mathsf{m} \in \mathcal{A}(i)$ . Assume  $\mathbf{c}_i \notin \mathcal{V}_1^j$ . We prove this lemma by induction on the length of the trace w. The base case where |w| = 0 is vacuously true. For the induction step, assume  $M_{\alpha}(w)\mathbf{1} = M_{\alpha^{i \to \mathsf{m}}}(w)\mathbf{1}$  holds for all w of length less than j. Let  $a \in L$ .

$$\begin{split} &M_{\alpha}(aw)\mathbf{1}\\ &= M_{\alpha}(a)M_{\alpha}(w)\mathbf{1}\\ &= M_{\alpha}(a)M_{\alpha^{i\to m}}(w)\mathbf{1} \qquad [M_{\alpha}(w)\mathbf{1} = M_{\alpha^{i\to m}}(w)\mathbf{1} \text{ by induction hypothesis}]\\ &= (M_{\alpha}(a) - M_{\alpha^{i\to m}}(a))M_{\alpha^{i\to m}}(w)\mathbf{1} + M_{\alpha^{i\to m}}(aw)\mathbf{1}\\ &= x \cdot \mathbf{c}_i + M_{\alpha^{i\to m}}(aw)\mathbf{1} \qquad [\text{for some } x \in \mathbb{R}], \end{split}$$

where the last equality follows from the fact that since the two matrices  $M_{\alpha}(a)$  and  $M_{\alpha^{i\to m}}(a)$  differ only in the *i*th row,  $M_{\alpha}(a) - M_{\alpha^{i\to m}}(a)$  is a matrix all whose rows except possibly the *i*th row are zero vectors. The product of such a matrix with a column vector is then a multiple of  $\mathbf{c}_i$ .

Since both  $M_{\alpha}(aw)\mathbf{1}$  and  $M_{\alpha^{i\to m}}(aw)\mathbf{1}$  are in  $\mathcal{V}_{1}^{j}$ , the difference of them, which is  $x \cdot \mathbf{c}_{i}$ , is in  $\mathcal{V}_{1}^{j}$  as well. However, as  $\mathbf{c}_{i} \notin \mathcal{V}_{1}^{j}$  by assumption, we have x = 0. Hence  $M_{\alpha}(aw)\mathbf{1} = M_{\alpha^{i\to m}}(aw)\mathbf{1}$ .

▶ **Lemma 2.** Let  $j \in \mathbb{N}$ . For all MD strategies  $\alpha_1$  and  $\alpha_2$ ,  $a \in L$  and  $w \in L^{\leq j}$ , we have  $M_{\alpha_1}(a)M_{\alpha_2}(w)\mathbf{1} \in \langle \mathcal{V}_1^j \cup \mathcal{B}(\{(\alpha, aw)\}) \rangle$  where  $\alpha$  is the MD strategy defined by

$$\alpha(i) = \begin{cases} \alpha_1(i) & \text{if } \mathbf{c}_i \notin \mathcal{V}_1^j\\ \alpha_2(i) & \text{otherwise} \end{cases}$$

**Proof.** Let  $j \in \mathbb{N}$ . Let  $\alpha_1$  and  $\alpha_2$  be two MD strategies,  $a \in L$  and  $w \in L^{\leq j}$ . Since  $|w| \leq j$  and for all i with  $\alpha_2(i) \neq \alpha(i)$  we have  $\mathbf{c}_i \notin \mathcal{V}_1^j$ , by Lemma 21, we have

$$M_{\alpha_2}(w)\mathbf{1} = M_{\alpha}(w)\mathbf{1} \tag{3}$$

Then,

$$M_{\alpha_1}(a)M_{\alpha_2}(w)\mathbf{1}$$

$$= M_{\alpha_1}(a)M_{\alpha}(w)\mathbf{1} \qquad [(3)]$$

$$= (M_{\alpha_1}(a) - M_{\alpha}(a))M_{\alpha}(w)\mathbf{1} + M_{\alpha}(aw)\mathbf{1}.$$

The first summand in the previous line is in the vector space  $\mathcal{V}_1^j$ , since  $(M_{\alpha_1}(a) - M_{\alpha}(a))M_{\alpha}(w)\mathbf{1} \in \langle \mathbf{c}_i : \mathbf{c}_i \in \mathcal{V}_1^j \rangle \subseteq \mathcal{V}_1^j$ . Thus,  $M_{\alpha_1}(a)M_{\alpha_2}(w)\mathbf{1} \in \langle \mathcal{V}_1^j \cup \{M_{\alpha}(aw)\mathbf{1}\}\rangle = \langle \mathcal{V}_1^j \cup \mathcal{B}(\{(\alpha, aw)\})\rangle$ .

The next lemma shows that a basis for  $\mathcal{V}_1^j$  for some j < |S| consisting only of MD vectors can be computed in polynomial time.

▶ Lemma 3. Let  $j \in \mathbb{N}$  with j < |S|. One can compute in polynomial time a set  $P_j = \{(\alpha_0, w_0), \cdots, (\alpha_k, w_k)\}$  in which all  $\alpha_i$  are MD strategies and all  $w_i$  are in  $L^{\leq j}$  such that  $\mathcal{B}(P_j)$  is a basis of  $\mathcal{V}_1^j$ .

**Proof.** We have shown  $\langle \mathcal{B}(P_{j+1}) \rangle \subseteq \mathcal{V}_1^{j+1}$  in the proof sketch. To show the other direction,  $\mathcal{V}_1^{j+1} \subseteq \langle \mathcal{B}(P_{j+1}) \rangle$ , it suffices to show for all memoryless strategies  $\alpha'$ ,  $a \in L$  and  $\mathbf{b} \in \langle \mathcal{B}(P_j) \rangle$ , we have  $M_{\alpha'}(a) \cdot \mathbf{b} \in \langle \mathcal{B}(P_{j+1}) \rangle$ . Let  $\alpha'$  be an arbitrary memoryless strategy and  $a \in L$ . The matrix  $M_{\alpha'}(a)$  can be expressed as a linear combination of the matrices from M:

$$M_{\alpha'}(a) = M_{\alpha_0}(a) + \sum_{s \in S} \left( -M_{\alpha_0}(a) + \sum_{\mathsf{m} \in \mathcal{A}(s)} \alpha'(s)(\mathsf{m}) \cdot M_{\alpha_0^{s \to \mathsf{m}}}(a) \right)$$

That is, there are  $y_{\alpha} \in \mathbb{R}$  for all  $\alpha \in \Sigma$  such that

$$M_{\alpha'}(a) = \sum_{\alpha \in \Sigma} y_{\alpha} \cdot M_{\alpha}(a) . \tag{4}$$

Let  $\mathbf{b} \in \mathcal{B}(P_i)$ . Then,

$$M_{\alpha'}(a) \cdot \mathbf{b} = \sum_{\alpha \in \Sigma} y_{\alpha} \cdot M_{\alpha}(a) \cdot \mathbf{b}$$
 [(4)]

Since each term of the summation,  $y_{\alpha} \cdot M_{\alpha}(a) \cdot \mathbf{b}$ , is in  $\langle \mathcal{B}(P_{j+1}) \rangle$  by (1),  $M_{\alpha'}(a) \cdot \mathbf{b}$  is also in  $\langle \mathcal{B}(P_{i+1}) \rangle$ .

For the proof of the following lemma we combine classical linear algebra arguments about equivalence checking (see, e.g., [42]) with Lemma 3.

### ▶ Lemma 4.

- 1. For all j < |S| we have  $\mathcal{V}_1^j = \mathcal{V}_2^j = \mathcal{V}_3^j$ . 2. We have  $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{V}_3 = \mathcal{V}_1^{|S|-1} = \mathcal{V}_2^{|S|-1} = \mathcal{V}_3^{|S|-1}$ .

**Proof.** We prove the items in turn.

- 1. Let j < |S|. From Lemma 3 it follows that  $\mathcal{V}_1^j \subseteq \mathcal{V}_3^j$ . From the definitions of  $\mathcal{V}_1^j, \mathcal{V}_2^j, \mathcal{V}_3^j$ we have  $\mathcal{V}_3^j \subseteq \mathcal{V}_2^j \subseteq \mathcal{V}_1^j$ .
- **2.** We have  $\mathcal{V}_1^j \subseteq \mathcal{V}_1^{j+1}$  for all  $j \in \mathbb{N}$ . Further we have for all  $j \in \mathbb{N}$ :

$$\mathcal{V}_1^{j+1} = \langle \mathbf{v}, M_{\alpha}(a)\mathbf{v} : \alpha \text{ is a memoryless strategy, } a \in L, \mathbf{v} \in \mathcal{V}_1^j \rangle$$
 (5)

It follows that if  $\mathcal{V}_1^j = \mathcal{V}_1^{j+1}$  then  $\mathcal{V}_1^j = \mathcal{V}_1^k$  holds for all  $k \geq j$ . Since dim  $\mathcal{V}_1^j \leq |S|$  for all  $j \in \mathbb{N}$ , it follows that  $\mathcal{V}_1^{|S|-1} = \mathcal{V}_1^k$  holds for all  $k \geq |S|-1$ . By (5) we see that  $\mathcal{V}_1^{|S|-1}$  contains **1** and is closed under pre-multiplication with  $M_{\alpha}(a)$  for any memoryless strategy  $\alpha$  and for any  $a \in L$ . But from the definition of  $\mathcal{V}_1$  we can derive that  $\mathcal{V}_1$  is the smallest vector space that contains 1 and has that closure property. Thus  $\mathcal{V}_1 \subseteq \mathcal{V}_1^{|S|-1}$ We have:

$$\begin{array}{ll} \mathcal{V}_1 \subseteq \mathcal{V}_1^{|S|-1} & \text{as just shown} \\ &= \mathcal{V}_2^{|S|-1} = \mathcal{V}_3^{|S|-1} & \text{by item 1} \\ &\subseteq \mathcal{V}_3 \subseteq \mathcal{V}_2 \subseteq \mathcal{V}_1 & \text{from the definitions} \end{array}$$

Hence these vector spaces are all equal.

### B Proofs of Section 4

The following lemma is quite standard, it shows that the partition gets finer after each iteration of the partition refinement algorithm.

▶ **Lemma 7.** For all  $i \in \mathbb{N}$ , we have  $\equiv_{X_{i+1}} \subseteq \equiv_{X_i}$ .

**Proof.** We prove the statement by induction on i. The base case where i=0 is vacuously true. For the induction step, assume  $\equiv_{X_{i+1}} \subseteq \equiv_{X_i}$ . Then, for each  $\equiv_{X_i}$ -equivalence class E, we have

$$E = \bigcup_{j} E_{j} \text{ where } E_{j} \in S/\equiv_{X_{i+1}}.$$
 (6)

Next, we show  $\equiv_{X_{i+2}} \subseteq \equiv_{X_{i+1}}$ .

Let  $s, t \in S$  and  $s \equiv_{X_{i+2}} t$ . If s = t, then  $s \equiv_{X_{i+1}} t$ . Otherwise, assume  $s \neq t$ . By the definition of  $\equiv_{X_{i+2}}$ , we have  $\ell(s) = \ell(t)$ ,  $|\varphi(s)(X_{i+2})| = 1$  and  $\varphi(s)(X_{i+2}) = \varphi(t)(X_{i+2})$ . For all  $\mathsf{m}_1, \mathsf{m}_2 \in \mathcal{A}(s)$ ,  $\mathsf{m}_1', \mathsf{m}_2' \in \mathcal{A}(t)$  and  $E \in X_{i+2}$ , we have

$$\varphi(s,\mathsf{m}_1)(E) = \varphi(s,\mathsf{m}_2)(E) = \varphi(t,\mathsf{m}_1')(E) = \varphi(t,\mathsf{m}_2')(E).$$

Since  $X_{i+2} = S/\equiv_{X_{i+1}}$  and  $X_{i+1} = S/\equiv_{X_i}$ , by (6), we have for all  $\mathsf{m}_1, \mathsf{m}_2 \in \mathcal{A}(s), \mathsf{m}_1', \mathsf{m}_2' \in \mathcal{A}(t)$  and all  $E' \in X_{i+1}$ ,

$$\varphi(s, \mathsf{m}_1)(E') = \varphi(s, \mathsf{m}_2)(E') = \varphi(t, \mathsf{m}_1')(E') = \varphi(t, \mathsf{m}_2')(E').$$

Thus,  $|\varphi(s)(X_{i+1})| = |\varphi(t)(X_{i+1})| = 1$  and  $\varphi(s)(X_{i+1}) = \varphi(t)(X_{i+1})$ . By the definition of  $\equiv_{X_{i+1}}$ , we have  $s \equiv_{X_{i+1}} t$ .

The next lemma shows that if  $s \equiv_{X_{|S|-1}} t$  then  $s \sim_{\mathcal{D}(\alpha)} t$  for all memoryless strategy  $\alpha$ .

▶ **Lemma 8.** Let X be a partition and  $X = S/\equiv_X$ . We have  $\equiv_X \subseteq \sim_{\mathcal{D}(\alpha)}$  for all memoryless strategies  $\alpha$ .

**Proof.** Let X be a partition and  $X = S/\equiv_X$ . Let  $\alpha$  be a memoryless strategy. We show that  $\equiv_X$  is a probabilistic bisimulation in the induced LMC  $\mathcal{D}(\alpha)$ . By the definition of probabilistic bisimulation, it suffices to show that for all  $(u, v) \in \equiv_X$ , we have  $\ell(u) = \ell(v)$  and  $\tau(u)(E) = \tau(v)(E)$  for each  $\equiv_X$ -equivalence class E.

Since  $X = S/\equiv_X$ , each element  $E \in X$  is an  $\equiv_X$ -equivalence class. Let  $u \equiv_X v$ . We distinguish the following two cases: u = v and  $u \neq v$ . If u = v, then  $u \sim_{\mathcal{D}(\alpha)} v$  is vacuously true. Assume  $u \neq v$ . Let  $\pi$  be a probability distribution over X and  $\pi = \varphi(u, \mathsf{m}_u)(X)$  for some  $\mathsf{m}_u \in \mathcal{A}(u)$ . By definition of  $\equiv_X$ , we have  $\ell(u) = \ell(v)$  and for all  $\mathsf{m}_u \in \mathcal{A}(u)$  and  $\mathsf{m}_v \in \mathcal{A}(v)$ ,  $\varphi(u, \mathsf{m}_u)(X) = \varphi(v, \mathsf{m}_v)(X) = \pi$ .

In the LMC  $\mathcal{D}(\alpha)$ , the transition probability from u to  $E \in X$  is

$$\tau(u)(E) = \sum_{\mathsf{m}_u \in \mathcal{A}(u)} \alpha(u)(\mathsf{m}_u) \cdot \varphi(u, \mathsf{m}_u)(E)$$

$$= \sum_{\mathsf{m}_u \in \mathcal{A}(u)} \alpha(u)(\mathsf{m}_u) \cdot \pi(E) \qquad [\pi(E) = \varphi(u, \mathsf{m}_u)(E)]$$

$$= \pi(E) \qquad [\alpha(u) \text{ is a probability distribution over } \mathcal{A}(u)]$$

Similarly, the transition probability from v to  $E \in X$ ,  $\tau(v)(E)$ , is also equal to  $\pi(E)$ . Thus, for all  $\equiv_X$ -equivalence class E, we have  $\tau(u)(E) = \tau(v)(E)$ . This completes the proof.

▶ Lemma 9.  $\theta(N)$  is polynomial in  $size(\mathcal{D})$ .

**Proof.** Since  $X_i$  for some i is a partition of S, we have  $|X_i| \leq S$ . Together with the fact that Algorithm 2 runs for at most |S| iterations,  $|\bigcup_i X_i|$  is polynomial in |S|. Thus, |N| is polynomial in size( $\mathcal{D}$ ).

Let  $b \in N$ . Since the smallest prime number is 2, we have  $\theta(b) < \log b$ . Furthermore, since  $\log b$  is the bit size of b,  $\theta(b)$  is then polynomial in  $\operatorname{size}(\mathcal{D})$ .

Finally, since 
$$\theta(N) \leq |N| \cdot \max_{b \in N} \theta(b)$$
,  $\theta(N)$  is polynomial in size( $\mathcal{D}$ ).

▶ **Lemma 10.** One can find |S| different prime numbers in time polynomial in size(D) such that any of them is coprime to all numbers in the set N.

**Proof.** We denote by p(x) the number of primes less than or equal to a positive integer x. We show that there exists  $x \in \mathbb{N}$  such that  $p(x) \geq |S| + \theta(N)$  and x is polynomial in  $\operatorname{size}(\mathcal{D})$ . Let x > 55 and  $x \geq (|S| + \theta(N))^2$ . Then,

$$|S| + \theta(N) \leq \sqrt{x}$$

$$= \frac{x}{\sqrt{x}}$$

$$< \frac{x}{\log x + 2} \quad [\sqrt{x} > \log x + 2]$$

$$< p(x) \quad [\frac{x}{\log x + 2} < p(x) \text{ for } x > 55 \text{ by [34]}]$$

It follows that x is polynomial in  $\operatorname{size}(\mathcal{D})$ , as  $\theta(N)$  is polynomial in  $\operatorname{size}(\mathcal{D})$  by Lemma 9. For each positive integer  $i \leq x$ , we can check whether it is prime using the algorithm in [1] and is coprime to all number in N. Each check can be done in polynomial time as shown in [1] and that |N| is polynomial in  $\operatorname{size}(\mathcal{D})$ .

▶ Lemma 11. Let  $i \in \mathbb{N}$  with  $i \leq |S|$ . One can compute in polynomial time a partial strategy  $\alpha'_i$  such that  $\sim^i_{\mathcal{D}(\alpha)} \subseteq \equiv_{X_i}$  for all  $\alpha \supseteq \alpha'_i$ .

**Proof.** Following the proof sketch, we show the rest of the proof in detail.

In Algorithm 2, once a state u satisfies  $|\varphi(u)(X_i)| \neq 1$  for some partition  $X_i$ , it satisfies  $|\varphi(u)(X_j)| \neq 1$  for all  $j \geq i$  since  $\equiv_{X_j} \subseteq \equiv_{X_i}$  for all  $j \geq i$  by Lemma 7. A state u is only added to the domain of the partial strategy once (in Algorithm 3 line 2), which guarantees that  $\alpha'_{i+1}$  does not overwrite  $\alpha'_i$ . It follows that for any  $\alpha \supseteq \alpha'_{i+1}$ , it satisfies  $\alpha \supseteq \alpha'_i$ .

that  $\alpha'_{i+1}$  does not overwrite  $\alpha'_i$ . It follows that for any  $\alpha \supseteq \alpha'_{i+1}$ , it satisfies  $\alpha \supseteq \alpha'_i$ . Let  $s \not\equiv_{X_{i+1}} t$ . Let  $\alpha \supseteq \alpha'_{i+1}$ . If  $s \not\sim^i_{\mathcal{D}(\alpha)} t$ , then  $s \not\sim^{i+1}_{\mathcal{D}(\alpha)} t$ , since  $\sim^{i+1}_{\mathcal{D}(\alpha)} \subseteq \sim^i_{\mathcal{D}(\alpha)}$  for all  $i \in \mathbb{N}$ . Otherwise, assume  $s \sim^i_{\mathcal{D}(\alpha)} t$ . From  $\alpha \supseteq \alpha'_i$  and the induction hypothesis, it follows that  $s \equiv_{X_i} t$ . Since  $s \not\equiv_{X_{i+1}} t$ , by the definition of  $\equiv_{X_i}$  and  $\equiv_{X_{i+1}}$ , we have  $\ell(s) = \ell(t)$ ,  $s \neq t$  and  $|\varphi(s)(X_i)| = |\varphi(t)(X_i)| = 1$ .

Towards a contradiction, assume  $s \sim_{\mathcal{D}(\alpha)}^{i+1} t$ . We show that under this assumption, for all  $E \in X_{i+1}$ ,  $\tau(s)(E) = \tau(t)(E)$  should hold. Let  $E \in X_{i+1}$ . Since  $X_{i+1} = S/\equiv_{X_i}$ , E is an equivalence class with respect to  $\equiv_{X_i}$ . By the induction hypothesis, we have

$$E = \bigcup_{j} E_{j} \text{ where } E_{j} \in S/\sim_{\mathcal{D}(\alpha)}^{i}.$$
 (7)

Then,

$$\tau(s)(E) = \sum_{E_j} \tau(s)(E_j) \qquad [(7)]$$

$$= \sum_{E_j} \tau(t)(E_j) \qquad [s \sim_{\mathcal{D}(\alpha)}^{i+1} t]$$

$$= \tau(t)(E)$$

The two different prime numbers  $p_s$  and  $p_t$  are associated with state s and t, respectively. If  $|\varphi(s)(X_{i+1})| \neq 1$ ,  $\alpha'_{i+1}(s)$  is defined using  $p_s$  on line 4. It follows that  $\tau(s)(E) \neq \tau(t)(E)$ , since  $p_s$  can divide the denominator of  $\tau(s)(E)$  but not the denominator of  $\tau(t)(E)$ . The case when  $|\varphi(t)(X_{i+1})| \neq 1$  is symmetrical. Otherwise,  $|\varphi(s)(X_{i+1})| = |\varphi(t)(X_{i+1})| = 1$ . By the definition of  $\equiv_{X_{i+1}}$ , there must exist a set E that  $\tau(s)(E) \neq \tau(t)(E)$ .

We now show in detail that for all the following cases, we have the contradiction that  $\tau(s)(E) \neq \tau(t)(E)$  for some  $E \in X_{i+1}$ .

- Assume  $|\varphi(s)(X_{i+1})| \neq 1$ . From the construction of  $\alpha'_{i+1}$  on line 3, we pick  $\mathsf{m}_1, \mathsf{m}_2 \in \mathcal{A}(s)$  and  $E \in X_{i+1}$  such that  $\varphi(s, \mathsf{m}_1)(E) > \varphi(s, \mathsf{m}_2)(E)$ . In the LMC  $\mathcal{D}(\alpha)$ , the probability from s to E is

$$\begin{split} \tau(s)(E) &= \sum_{\mathbf{m} \in \mathcal{A}(s)} \alpha(s)(\mathbf{m}) \cdot \varphi(s,\mathbf{m})(E) \\ &= \sum_{\mathbf{m} \in \mathcal{A}(s)} \alpha'_{i+1}(s)(\mathbf{m}) \cdot \varphi(s,\mathbf{m})(E) \qquad [\alpha \sqsupseteq \alpha'_{i+1}] \\ &= \frac{1}{p_s} \cdot \varphi(s,\mathbf{m}_1)(E) + (1 - \frac{1}{p_s}) \cdot \varphi(s,\mathbf{m}_2)(E) \\ &= \varphi(s,\mathbf{m}_2)(E) + \frac{1}{p_s} \cdot \left(\varphi(s,\mathbf{m}_1)(E) - \varphi(s,\mathbf{m}_2)(E)\right) \\ &= \frac{a_{s,\mathbf{m}_2,E}}{b_{s,\mathbf{m}_2,E}} + \frac{1}{p_s} \cdot \frac{c_{s,\mathbf{m}_1,\mathbf{m}_2,E}}{d_{s,\mathbf{m}_1,\mathbf{m}_2,E}} \\ &= \frac{a_{s,\mathbf{m}_2,E}d_{s,\mathbf{m}_1,\mathbf{m}_2,E}p_s + c_{s,\mathbf{m}_1,\mathbf{m}_2,E}b_{s,\mathbf{m}_2,E}}{b_{s,\mathbf{m}_2,E}d_{s,\mathbf{m}_1,\mathbf{m}_2,E}p_s}, \end{split}$$

where  $a_{s,\mathsf{m}_2,E},b_{s,\mathsf{m}_2,E},c_{s,\mathsf{m}_1,\mathsf{m}_2,E}$  and  $d_{s,\mathsf{m}_1,\mathsf{m}_2,E}$  are defined before Lemma 9. The first summand of the numerator in the previous line can be divided by  $p_s$ . By (2), we have  $p_s \nmid b_{s,\mathsf{m}_2,E}$  and  $p_s \nmid c_{s,\mathsf{m}_1,\mathsf{m}_2,E}$ . Thus,  $p_s$  can not divide the second summand, and hence, not the numerator. For  $u \in S$  and  $E \subseteq S$ , we express  $\tau(u)(E)$  as an irreducible fraction  $\frac{x_{\tau(u)(E)}}{y_{\tau(u)(E)}}$  where  $x_{\tau(u)(E)}$  and  $y_{\tau(u)(E)}$  are coprime integers. It follows that  $p_s \mid y_{\tau(s)(E)}$ . For state t, we have either  $|\varphi(t)(X_{i+1})| \neq 1$  or  $|\varphi(t)(X_{i+1})| = 1$ . Assume  $|\varphi(t)(X_{i+1})| \neq 1$ . Similar to s, we have  $\mathsf{m}'_1, \mathsf{m}'_2 \in \mathcal{A}(t)$  and  $E' \in X_{i+1}$  such that  $\varphi(t, \mathsf{m}'_1)(E') > \varphi(t, \mathsf{m}'_2)(E')$ . In the LMC  $\mathcal{D}(\alpha)$ , the probability from t to E is

$$\begin{split} \tau(t)(E) &= \sum_{\mathbf{m} \in \mathcal{A}(t)} \alpha(t)(\mathbf{m}) \cdot \varphi(t,\mathbf{m})(E) \\ &= \sum_{\mathbf{m} \in \mathcal{A}(t)} \alpha'_{i+1}(t)(\mathbf{m}) \cdot \varphi(t,\mathbf{m})(E) \qquad [\alpha \sqsupseteq \alpha'_{i+1}] \\ &= \frac{1}{p_t} \cdot \varphi(t,\mathbf{m}'_1)(E) + (1 - \frac{1}{p_t}) \cdot \varphi(t,\mathbf{m}'_2)(E) \\ &= \frac{1}{p_t} \cdot \frac{a_{t,\mathbf{m}'_1,E}}{b_{t,\mathbf{m}'_1,E}} + \frac{p_t - 1}{p_t} \cdot \frac{a_{t,\mathbf{m}'_2,E}}{b_{t,\mathbf{m}'_2,E}} \\ &= \frac{a_{t,\mathbf{m}'_1,E}b_{t,\mathbf{m}'_2,E} + (p_t - 1) \cdot a_{t,\mathbf{m}'_2,E}b_{t,\mathbf{m}'_1,E}}{p_t b_{t,\mathbf{m}'_1,E}b_{t,\mathbf{m}'_2,E}} \end{split}$$

By (2), the two prime numbers  $p_s$  and  $p_t$  are different and  $p_s \nmid b_{t,\mathsf{m}_1',E}, b_{t,\mathsf{m}_2',E}$ . It follows that  $p_s \nmid y_{\tau(t)(E)}$ , and thus  $\tau(s)(E) \neq \tau(t)(E)$ .

We consider the other case where  $|\varphi(t)(X_{i+1})| = 1$ . It follows that  $\varphi(t, \mathsf{m})(E)$  is the same for all  $\mathsf{m} \in \mathcal{A}(t)$ . Let  $\mathsf{m}' \in \mathcal{A}(t)$ . In the LMC  $\mathcal{D}(\alpha)$ , the probability from t to E is

$$\tau(t)(E) \quad = \quad \sum_{\mathsf{m} \in \mathcal{A}(t)} \alpha(t)(\mathsf{m}) \cdot \varphi(t,\mathsf{m})(E)$$

$$= \varphi(t, \mathbf{m}')(E)$$

$$= \frac{a_{t,\mathbf{m}',E}}{b_{t,\mathbf{m}',E}}$$

By (2),  $p_s \nmid b_{t,m',E}$ . It follows that  $p_s \nmid y_{\tau(t)(E)}$ , and thus  $\tau(s)(E) \neq \tau(t)(E)$ .

- Assume  $|\varphi(s)(X_{i+1})| = 1$  and  $|\varphi(t)(X_{i+1})| \neq 1$ . To avoid redundancy, we do not show the proof as this case is similar to the case  $|\varphi(s)(X_{i+1})| \neq 1$  and  $|\varphi(t)(X_{i+1})| = 1$ .
- Assume  $|\varphi(s)(X_{i+1})| = |\varphi(t)(X_{i+1})| = 1$ . Since  $s \not\equiv_{X_{i+1}} t$ , by definition of  $\equiv_{X_{i+1}}$ , we have  $\varphi(s)(X_{i+1}) \neq \varphi(t)(X_{i+1})$ . Let  $\mathsf{m}_s \in \mathcal{A}(s)$  and  $\mathsf{m}_t \in \mathcal{A}(t)$ . There exists a set  $E \in X_{i+1}$  such that  $\varphi(s, \mathsf{m}_s)(E) \neq \varphi(t, \mathsf{m}_t)(E)$ . In the LMC  $\mathcal{D}(\alpha)$ , we have

$$\begin{split} \tau(s)(E) &= \sum_{\mathbf{m} \in \mathcal{A}(s)} \alpha(s)(\mathbf{m}) \cdot \varphi(s,\mathbf{m})(E) \\ &= \varphi(s,\mathbf{m}_s)(E) \\ &\neq \varphi(t,\mathbf{m}_t)(E) \\ &= \sum_{\mathbf{m} \in (t)} \alpha(t)(\mathbf{m}) \cdot \varphi(t,\mathbf{m})(E) \\ &= \tau(t)(E). \end{split}$$

This completes the proof.

## C Proofs of Section 5

### **C.1** Proofs of TV = 1

In this section, we show that the problem TV = 1 is in  $\exists \mathbb{R}$  and is NP-hard. Recall that TV = 1 is the problem asking whether there is a memoryless strategy  $\alpha$  for  $\mathcal{D}$  such that the total variation distance of the two initial distributions is one in the induced labelled Markov chain  $\mathcal{D}(\alpha)$ , i.e.,  $d_{tv}(\mu, \nu) = 1$ .

Define the set  $R^{\mu,\nu}:=\{(r_1,r_2)\in S\times S:\exists w\in L^*:r_1\in \text{support}(\mu M(w))\text{ and }r_2\in \text{support}(\nu M(w))\}$ , which can be computed in polynomial time as shown in [10, Lemma 20]. For each  $r_1\in S$ , define the projection  $R^{\mu,\nu}_{r_1}:=\{r_2\in S:(r_1,r_2)\in R^{\mu,\nu}\}$ . According to [10, Theorem 21], the following proposition holds.

▶ Proposition 22. We have  $d_{tv}(\mu, \nu) < 1$  if and only if there are  $r_1 \in S$  and subdistributions  $\mu_1$  and  $\mu_2$  such that

$$\mu_1 \equiv \mu_2 \quad and \quad r_1 \in \text{support}(\mu_1) \quad and \quad \text{support}(\mu_2) \subseteq R_{r_1}^{\mu,\nu}$$
 (8)

It is known that  $\exists \mathbb{R}$  is closed under NP-reductions [41] which is needed for showing the membership of TV = 1 in  $\exists \mathbb{R}$ , and later the membership of TV < 1 in  $\exists \mathbb{R}$ .

▶ Theorem 14. The problem TV = 1 is in  $\exists \mathbb{R}$ .

**Proof.** Let  $B_{\alpha} \in \mathbb{R}^{S \times r}$  be a matrix consisting of  $r \leq |S|$  linearly independent columns which we denote by  $\mathbf{b}_0, \dots, \mathbf{b}_{r-1}$ . Furthermore, we have

- $\mathbf{b}_0 = \mathbf{1}$ ;
- $\mathbf{b}_i = M_{\alpha}(w_i)\mathbf{1}$  where  $w_i \in L^{\leq |S|}$  for all  $1 \leq i < r$ .

The columns of  $B_{\alpha}$  are linearly independent, i.e.,  $B_{\alpha}$  has full rank r, if and only if there exists a reduced QR factorization of  $B_{\alpha}$ , i.e., there exist a matrix  $Q \in \mathbb{R}^{S \times r}$  with orthonormal

columns and an upper triangular matrix  $R \in \mathbb{R}^{r \times r}$  with all diagonal entries being nonzero such that  $B_{\alpha} = QR$ .

The matrix  $B_{\alpha}$  is a basis for the vector space

$$\langle M_{\alpha}(w) \cdot \mathbf{1} : w \in L^* \rangle$$

if and only if  $B_{\alpha}$  is closed under pre-multiplication with  $M_{\alpha}(a)$  for any  $a \in L$ , i.e., for each label  $a \in L$ , there exists a matrix  $F(a) \in \mathbb{R}^{r \times r}$  such that

$$M_{\alpha}(a) \cdot B_{\alpha} = B_{\alpha}F(a).$$

Let  $I_n \in \mathbb{R}^{n \times n}$  denote the identity matrix of size n. Let  $H_{\alpha} \in \mathbb{R}^{S \times r'}$  be a matrix consisting of r' columns which are denoted by  $\mathbf{h}_0, \dots, \mathbf{h}_{r'-1}$ . Furthermore, we require that all of the columns have length one and they are mutually orthogonal, i.e.,  $H_{\alpha}^T H_{\alpha} = I_{r'}$ . It is an orthonormal basis for the vector space  $\langle \mathbf{x} : B_{\alpha}^T \mathbf{x} = \mathbf{0} \rangle$ , i.e., the orthogonal complement of  $\langle B_{\alpha} \rangle$ , if and only if  $B_{\alpha}^T \mathbf{h}_i = \mathbf{0}$  for all  $0 \le i < r'$  and  $rank(B_{\alpha}) + rank(H_{\alpha}) = r + r' = |S|$ .

Recall that  $\mathbf{c}_i \in \{0,1\}^S$  is the column bit vector whose only non-zero entry is the *i*th one. For each  $s \in S$ , define a convex polyhedron

$$\mathcal{P}_s = \Big\{ \mathbf{c}_s + \sum_{t \in S} \lambda_t \mathbf{c}_t + \sum_{t \in R^{\mu, \nu}} \lambda'_t (-\mathbf{c}_t) \mid \lambda_t \ge 0, \ \lambda'_t \ge 0 \Big\}.$$

We call  $\mathbf{c}_t$  for  $t \in S$  and  $-\mathbf{c}_t$  for  $t \in R_s^{\mu,\nu}$  the spanning vectors of  $\mathcal{P}_s$ .

Assume the matrix  $B_{\alpha}$  is a basis for  $\langle M_{\alpha}(w) \cdot \mathbf{1} : w \in L^* \rangle$  and  $H_{\alpha}$  is an orthonormal basis for the orthogonal complement of  $\langle B_{\alpha} \rangle$ . We show that the two convex polyhedra  $\langle H_{\alpha} \rangle$  and  $\mathcal{P}_s$  intersect if and only if  $d_{tv}(\mu, \nu) < 1$  in  $\mathcal{D}(\alpha)$ . We distinguish the following two cases:

- Assume  $s \in R_s^{\mu,\nu}$ . It is easy to check that  $\mathbf{0} \in \langle H_\alpha \rangle \cap \mathcal{P}_s$ . Define the two subdistributions  $\mu_1$  and  $\mu_2$  as  $\mu_1 = \mu_2 = \delta_s$ . By Proposition 22,  $d_{tv}(\mu, \nu) < 1$  holds since  $\mu_1$  and  $\mu_2$  satisfy (8).
- Assume  $s \notin R_s^{\mu,\nu}$ . We first show the backward implication. From Proposition 22, there exist subdistributions  $\mu_1$  and  $\mu_2$  satisfying (8). Let  $N = \mu_1(s) \mu_2(s)$ . Since  $s \in \text{support}(\mu_1)$  and  $s \notin R_s^{\mu,\nu}$ , we have  $N = \mu_1(s) > 0$ . Define the vector  $\mathbf{v} = \frac{(\mu_1 \mu_2)^T}{N}$ . We can easily verify that it is in both  $\langle H_\alpha \rangle$  and  $\mathcal{P}_s$ , and hence,  $\langle H_\alpha \rangle \cap \mathcal{P}_s \neq \emptyset$ . For the converse, assume  $\langle H_\alpha \rangle \cap \mathcal{P}_s \neq \emptyset$ . Let  $\mathbf{v}$  be a column vector such that  $\mathbf{v} \in \langle H_\alpha \rangle$  and  $\mathbf{v} \in \mathcal{P}_s$ . Since  $\mathbf{v} \in \mathcal{P}_s$  and  $s \notin R_s^{\mu,\nu}$ , we have  $\mathbf{v}(s) \geq 1$  and  $\mathbf{v}(t) \geq 0$  for all  $t \in S \setminus R_s^{\mu,\nu}$ . Since  $B_\alpha^T \mathbf{v} = \mathbf{0}$  and  $\mathbf{b}_0 = \mathbf{1}$ , we have  $\mathbf{1}^T \mathbf{v} = 0$ . It follows that  $\{t : \mathbf{v}(t) < 0\} \subseteq R_s^{\mu,\nu}$  and  $\{t : \mathbf{v}(t) < 0\} \neq \emptyset$ . Let  $N = \sum_{u \in S} |\mathbf{v}(u)|$ . Define the two subdistributions  $\mu_1$  and  $\mu_2$  as

follows: 
$$\mu_1(u) = \begin{cases} \frac{\mathbf{v}(u)}{N} & \text{if } \mathbf{v}(u) > 0, \\ 0 & \text{otherwise;} \end{cases} \text{ and } \mu_2(u) = \begin{cases} -\frac{\mathbf{v}(u)}{N} & \text{if } \mathbf{v}(u) < 0, \\ 0 & \text{otherwise.} \end{cases}$$
Since  $\mu_1 - \mu_2 = \frac{\mathbf{v}}{N}$  and  $\mathbf{v}$  is orthogonal with  $\langle B_2 \rangle$ ,  $\mu_1 - \mu_2$  is also only the solution of the solution

Since  $\mu_1 - \mu_2 = \frac{\mathbf{v}}{N}$  and  $\mathbf{v}$  is orthogonal with  $\langle B_{\alpha} \rangle$ ,  $\mu_1 - \mu_2$  is also orthogonal with  $\langle B_{\alpha} \rangle$ , and thus  $\mu_1 \equiv \mu_2$ . Furthermore, we have  $\mu_1(s) = \frac{\mathbf{v}(s)}{N} \geq \frac{1}{N} > 0$ , and support $(\mu_2) = \{t : \mathbf{v}(t) < 0\} \subseteq R_s^{\mu,\nu}$ . From Proposition 22, it follows that  $d_{tv}(\mu,\nu) < 1$  in the LMC  $\mathcal{D}(\alpha)$  since  $\mu_1$  and  $\mu_2$  satisfy (8).

From the analysis above, to show that there exists a memoryless strategy  $\alpha$  such that  $d_{tv}(\mu,\nu)=1$  in the LMC  $\mathcal{D}(\alpha)$ , it suffices to show that there exists a memoryless strategy  $\alpha$  such that  $\langle H_{\alpha} \rangle \cap \mathcal{P}_s = \emptyset$  for all  $s \in S$ . By [29, Theorem 5.5.1], if the two convex polyhedra  $\langle H_{\alpha} \rangle$  and  $\mathcal{P}_s$  are disjoint, then there exists a hyperplane that strictly separates them, i.e., there exist  $a_s, b_s \in \mathbb{R}$  and a row vector  $v_s \in \mathbb{R}^S$  such that  $v_s \cdot \mathbf{x} \leq a_s$  for all  $\mathbf{x} \in \langle H_{\alpha} \rangle$ ,

 $v_s \cdot \mathbf{x} \ge b_s$  for all  $\mathbf{x} \in \mathcal{P}_s$  and  $a_s < b_s$ . Since  $\mathbf{0} \in \langle H_{\alpha} \rangle$ , we have  $a_s \ge 0$  and  $0 \le a_s < b_s$ . For any column vector  $\mathbf{h}$  of  $H_{\alpha}$ ,  $a \cdot \mathbf{h}$  is also in  $\langle H_{\alpha} \rangle$  for any  $a \in \mathbb{R}$ . It follows that  $v_s \cdot \mathbf{h} \le \frac{a_s}{a}$  and  $v_s \cdot \mathbf{h} \ge -\frac{a_s}{a}$  for all a > 0, and hence,

$$-\lim_{a\to\infty} \frac{a_s}{a} \le v_s \cdot \mathbf{h} \le \lim_{a\to\infty} \frac{a_s}{a}.$$

Since both the left and right limits exist and are equal to zero, we have  $v_s \cdot \mathbf{h} = 0$  for all column vectors  $\mathbf{h}$  of  $H_{\alpha}$ . It follows that  $v_s \cdot \mathbf{x} = 0$  for all  $\mathbf{x} \in \langle H_{\alpha} \rangle$ ,  $v_s \cdot \mathbf{x} \geq b_s$  for all  $\mathbf{x} \in \mathcal{P}_s$  and  $b_s > 0$ .

A memoryless strategy  $\alpha$  for  $\mathcal{D}$  can be characterised by numbers  $x_{s,m} \in [0,1]$  where  $s \in S$  and  $m \in \mathcal{A}$  such that  $x_{s,m} = \alpha(s)(m)$ . We write  $\bar{x}$  for the collection  $(x_{s,m})_{s \in S, m \in \mathcal{A}}$ . Thus, to decide if there exists a memoryless strategy such that  $d_{tv}(\mu, \nu) = 1$ , we nondeterministically guess a set of r-1 words  $w_i \in L^{\leq |S|}$  where  $1 \leq i < r$  and a nonnegative integer r', then check the following decision problem, which is a closed formula in the existential theory of the reals:

 $\exists \bar{x}$ , a matrix  $B_{\alpha} \in \mathbb{R}^{S \times r}$  the columns of which are denoted by  $\mathbf{b}_0, \dots, \mathbf{b}_{r-1}$ , a matrix  $Q \in \mathbb{R}^{S \times r}$ , an upper triangular matrix  $R \in \mathbb{R}^{r \times r}$ , matrices  $F(a) \in \mathbb{R}^{r \times r}$  for all  $a \in L$ , a matrix  $H_{\alpha} \in \mathbb{R}^{S \times r'}$  the columns of which are denoted by  $\mathbf{h}_0, \dots, \mathbf{h}_{r'-1}$ , row vectors  $v_s \in \mathbb{R}^S$  and  $b_s \in \mathbb{R}$  for all  $s \in S$  such that

```
 -\text{for all } s \in S : \sum_{\mathsf{m} \in \mathcal{A}(s)} x_{s,\mathsf{m}} = 1; \qquad [\bar{x} \text{ characterising a memoryless strategy}] 
 -\mathbf{b}_0 = \mathbf{1}; \\ -\text{for all } 1 \leq i < r : \mathbf{b}_i = M_{\alpha}(w_i) \mathbf{1}; \\ -Q^T Q = I_r; \\ -R[i,i] \neq 0 \text{ for all } i; \\ -B_{\alpha} = QR; \\ -\text{for all labels } a \in L : M_{\alpha}(a) \cdot B_{\alpha} = B_{\alpha}F(a); 
 -H_{\alpha}^T H_{\alpha} = I_{r'}; \\ -\text{for all } 0 \leq i < r' : B_{\alpha}^T \mathbf{h}_i = \mathbf{0}; \\ -r + r' = |S|; 
 -\text{for all } s \in S : v_s \cdot \mathbf{h}_i = 0 \text{ for all } 0 \leq i < r'; \\ -\text{for all } s \in S : v_s \cdot \mathbf{x} \geq b_s \text{ for all } \mathbf{x} \in \mathcal{P}_s, \text{ i.e.,} \\ v_s \cdot \mathbf{c}_s = b_s \text{ and } v_s \cdot \mathbf{c} \geq 0 \text{ for all spanning vectors } \mathbf{c} \text{ of } \mathcal{P}_s; \\ -\text{for all } s \in S : b_s > 0. 
 -\text{for all } s \in S : b_s > 0. 
 -\text{for all } s \in S : b_s > 0. 
 -\text{for all } s \in S : b_s > 0.
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Let  $\mu_1, \mu_2$  be two subdistributions on S. We write  $\mu_1 \leq \mu_2$  to say that  $\mu_1(u) \leq \mu_2(u)$  for all  $u \in S$ . According to [10, Proposition 17], the following proposition holds.

- ▶ Proposition 23. We have  $d_{tv}(\mu, \nu) < 1$  if and only if there are  $w \in L^*$  and  $\mu_1$  and  $\mu_2$  with  $\mu_1 \leq \mu M(w)$  and  $\mu_2 \leq \nu M(w)$  and  $\mu_1 \equiv \mu_2$  and  $|\mu_1| = |\mu_2| > 0$ .
- ▶ **Theorem 15.** The Set Splitting problem is polynomial-time many-one reducible to TV = 1, hence TV = 1 is NP-hard.

**Proof.** Let  $\langle S, \mathcal{C} \rangle$  be an instance of Set Splitting where  $S = \{e_1, \dots, e_n\}$  and  $\mathcal{C} = \{C_1, \dots, C_m\}$  is a collection of subsets of S. We construct an MDP  $\mathcal{D}$ , see Figure 3 for example, consisting of the following states: two states s and t, a state  $e_i$  for each element in S, twin states  $C_j$  and  $C_j'$  for each element in C, two sink states u and v. State v has label

b while all other states have label a. State s (t) has a single action which goes with uniform probability  $\frac{1}{m}$  to states  $C_i$  ( $C_i'$ ) for  $1 \le i \le m$ . For each  $e_i \in C_j$ , there is an action from state  $C_j$  and  $C_j'$  leading to state  $e_i$  with probability one. Each state  $e_i$  has two actions going to the sink states u and v with probability one, respectively. We show that

 $\langle S, \mathcal{C} \rangle \in \text{Set Splitting} \iff \exists \text{ memoryless strategy } \alpha \text{ such that } d_{tv}(\mu, \nu) = 1 \text{ in } \mathcal{D}(\alpha).$ 

Intuitively, making  $C_i$  (resp.  $C'_i$ ) select the transition to  $e_j$  simulates the membership of  $e_j$  in  $S_1$  (resp.  $S_2$ ).

( $\Longrightarrow$ ) Let  $S_1$  and  $S_2$  be the two disjoint sets that partition S and split the elements in C. For the MDP D, we define an MD strategy  $\alpha$  as follows: let state  $e_i \in S_1$  select the action transitioning to u and state  $e_i \in S_2$  the action to v; let state  $C_i$  select an available action that goes to a state in  $S_1$  and  $C'_i$  an available action that goes to a state in  $S_2$ .

We show that  $d_{tv}(\mu, \nu) = 1$  in the LMC  $\mathcal{D}(\alpha)$ . Let  $\mu_1$  and  $\mu_2$  be subdistributions over the states that are reachable from s and t, respectively. Let  $E' \in \mathcal{F}$  be a set of words always ending with infinite number of b's. Since a word emitted by running  $\mathcal{D}(\alpha)$  from an arbitrary state in support( $\mu_2$ ) always ends with infinitely many b's, we have  $\Pr_{\mu_2}(E') > 0$ . On the other hand, a word emitted by running  $\mathcal{D}(\alpha)$  from an arbitrary state in support( $\mu_1$ ) always ends with infinitely many a's, we have  $\Pr_{\mu_1}(E') = 0$ . Then,

$$d_{tv}(\mu_1, \mu_2) = \sup_{E \in \mathcal{F}} |\Pr_{\mu_1}(E) - \Pr_{\mu_2}(E)|$$
  
 
$$\ge |\Pr_{\mu_1}(E') - \Pr_{\mu_2}(E')| > 0$$

By Proposition 23, we have  $d_{tv}(\mu, \nu) = 1$  in the LMC  $\mathcal{D}(\alpha)$ .

( $\iff$ ) Let  $\alpha$  be a memoryless strategy for  $\mathcal{D}$  such that  $d_{tv}(\mu,\nu)=1$ . Let  $\tau$  be the transition function for the LMC  $\mathcal{D}(\alpha)$ . Let  $S_1=\bigcup_{C_i} \operatorname{support}(\tau(C_i))$  and  $S_2=S\setminus S_1$ . Let  $S_2'=\bigcup_{C_i'} \operatorname{support}(\tau(C_i'))$ . It suffices to show that  $S_2'\subseteq S_2$  and  $S_1$  and  $S_2'$  split the elements of  $\mathcal{C}$ .

Since  $S_2' \cap S_1 = \emptyset$ , otherwise  $d_{tv}(\mu, \nu) < 1$  by Proposition 23. We have  $S_2' \subseteq S_2$ . We prove by contradiction that  $S_1$  and  $S_2'$  split the elements of  $\mathcal{C}$ . Assume there is a set  $C_i \in \mathcal{C}$  which is not split by  $S_1$  and  $S_2'$ . Furthermore, without loss of generality, assume  $C_i \subseteq S_1$ , that is, for all states  $e \in C_i : e \in \text{support}(\tau(C_i))$ . Since state  $C_i$  and  $C_i'$  have the same successors in the MDP  $\mathcal{D}$ , there must exist a state  $e' \in C_i$  such that  $e' \in \text{support}(\tau(C_i'))$ . Let  $\mu_1 = \mu_2 = \delta_{e'}$ . We have  $d_{tv}(\mu_1, \mu_2) = 0$ , which leads to the desired contradiction  $d_{tv}(\mu, \nu) < 1$  in  $\mathcal{D}(\alpha)$  by Proposition 23.

### **C.2** Proofs of PB = 1

Next, we show that the problem PB = 1 is NP-complete. Recall that PB = 1 is the problem asking whether there is a memoryless strategy  $\alpha$  for  $\mathcal{D}$  such that the probabilistic bisimilarity distance of the two initial states is one in the induced labelled Markov chain  $\mathcal{D}(\alpha)$ , i.e.,  $d_{pb}(s,t)=1$ .

▶ **Definition 24.** The directed graph G = (V, E) is defined by

$$V = \{(u, v) : \ell(u) = \ell(v)\}$$
  
 
$$E = \{ \langle (u, v), (s', t') \rangle : \tau(s')(u) > 0 \land \tau(t')(v) > 0 \}$$

By [39, Theorem 4, Proposition 5], the following proposition holds.

▶ Proposition 25. We have  $d_{pb}(s,t) < 1$  if and only if in the graph G = (V, E) the vertex (s,t) is reachable from some (u,v) with  $u \sim v$ .

▶ **Theorem 26.** The problem PB = 1 is in NP.

**Proof.** Suppose there exists a memoryless strategy  $\beta$  such that  $d_{pb}(s,t) = 1$  in  $\mathcal{D}(\beta)$ . Let G be the graph of Definition 24 induced by the LMC  $\mathcal{D}(\beta)$ . Consider an MDP  $\mathcal{D}' = \langle S, \mathcal{A}', L, \varphi', \ell \rangle$ , which is over the same state space as  $\mathcal{D}$  but is restricted to choose actions that conform to the graph G. Thus,  $\beta$  is also a strategy of  $\mathcal{D}'$ . Furthermore, we have  $\mathcal{D}'(\alpha) = \mathcal{D}(\alpha)$  for all memoryless strategy  $\alpha$  of  $\mathcal{D}'$ .

According to Theorem 12, a memoryless strategy  $\alpha'$  of  $\mathcal{D}'$  such that  $\sim_{\mathcal{D}'(\alpha')} \subseteq \sim_{\mathcal{D}'(\alpha)}$  for all memoryless strategy  $\alpha$  can be computed in polynomial time. Thus, we have  $\sim_{\mathcal{D}'(\alpha')} \subseteq \sim_{\mathcal{D}'(\beta)}$ , that is, if  $u \not\sim_{\mathcal{D}'(\beta)} v$  then  $u \not\sim_{\mathcal{D}'(\alpha')} v$  for  $u, v \in S$ . Let G' be the graph of Definition 24 for the LMC  $\mathcal{D}'(\alpha')$ . Since  $\mathcal{D}'$  conforms to G, G' is a subgraph of G. Let G and G' be the set of state pairs that can reach (s,t) in G and G', respectively. We have  $G' \subseteq G$ .

According to Proposition 25, since  $d_{pb}(s,t)=1$  in the LMC  $\mathcal{D}'(\beta)$ , we have  $u \not\sim_{\mathcal{D}'(\beta)} v$  for all  $(u,v) \in R$ . By  $\sim_{\mathcal{D}'(\alpha')} \subseteq \sim_{\mathcal{D}'(\beta)}$  and  $R' \subseteq R$ , we have  $u \not\sim_{\mathcal{D}'(\alpha')} v$  for all  $(u,v) \in R'$ . By Proposition 25, we have  $d_{pb}(s,t)=1$  in the LMC  $\mathcal{D}'(\alpha')$ , and hence,  $\alpha'$  is a memoryless strategy that witnesses  $d_{pb}(s,t)=1$ .

This induces the following nondeterministic algorithm: we guess the graph G and check whether  $d_{pb}(s,t)=1$  holds in  $\mathcal{D}(\alpha')$ , where both the construction of the memoryless strategy  $\alpha'$  (using Algorithm 3) and the checking of  $d_{pb}(s,t)=1$  are in polynomial time.

▶ **Theorem 27.** The Set Splitting problem is polynomial-time many-one reducible to PB = 1, hence PB = 1 is NP-hard.

**Proof.** Given an instance of Set Splitting  $\langle S, \mathcal{C} \rangle$  where  $S = \{e_1, \dots, e_n\}$  and  $\mathcal{C} = \{C_1, \dots, C_m\}$  is a collection of subsets of S, we construct the same MDP  $\mathcal{D}$  as shown in Theorem 15, see Figure 3 for example. We show that

```
\langle S, C \rangle \in \text{Set Splitting} \iff \exists \alpha \text{ for } \mathcal{D} \text{ such that } d_{pb}(s,t) = 1 \text{ in } \mathcal{D}(\alpha).
```

( $\Longrightarrow$ ) Let  $S_1$  and  $S_2$  be the two disjoint sets that partition S and split the elements of C. According to Theorem 15, there exists a memoryless strategy  $\alpha$  such that  $d_{tv}(\delta_s, \delta_t) = 1$  in the induced LMC  $\mathcal{D}(\alpha)$ . Since probabilistic bisimilarity distance is an upper bound of the total variational distance [9], we have that  $d_{pb}(s,t) = 1$  in  $\mathcal{D}(\alpha)$ .

 $(\Leftarrow)$  Let  $\alpha$  be a memoryless strategy for  $\mathcal{D}$  such that  $d_{pb}(s,t)=1$  in the LMC  $\mathcal{D}(\alpha)$ . Let  $\tau$  be the transition function for the LMC  $\mathcal{D}(\alpha)$ . Let  $S_1=\bigcup_{C_i} \operatorname{support}(\tau(C_i))$  and  $S_2=S\setminus S_1$ . Let  $S_2'=\bigcup_{C_i'} \operatorname{support}(\tau(C_i'))$ . It suffices to show that  $S_1$  and  $S_2'$  split the elements of  $\mathcal{C}$  and  $S_2'\subseteq S_2$ .

Since  $d_{pb}(s,t)=1$ , by definition of probabilistic bisimilarity distance,  $d_{pb}(C_i,C'_j)=1$  for any choice of  $C_i$  and  $C'_j$ . We can obtain, by the same argument,  $d_{pb}(e_k,e_l)=1$  for any  $e_k \in \text{support}(\tau(C_i))$  and  $e_l \in \text{support}(\tau(C'_j))$ . Thus, we have  $\text{support}(\tau(C_i)) \cap \text{support}(\tau(C'_j))=\emptyset$  for any choice of  $C_i$  and  $C'_j$ . It follows that  $S_1 \cap S'_2 = \emptyset$ , that is,  $S'_2 \subseteq S_2$ . Furthermore, for any set  $C_i \in \mathcal{C}$ , there are two states  $e_k, e_l \in C_i$  such that  $e_k \in \text{support}(\tau(C_i))$  and  $e_l \in \text{support}(\tau(C'_i))$ , that is,  $e_k$  and  $e_l$  split the set  $C_i$ .

▶ **Theorem 16.** The problem PB = 1 is NP-complete.

**Proof.** It follows from Theorem 26 and Theorem 27.

### D Proofs of Section 6

### **D.1** Proofs of TV = 0

In this section we show that the problem TV = 0 is  $\exists \mathbb{R}$ -complete. Recall that TV = 0 is the problem asking whether there is a memoryless strategy  $\alpha$  for  $\mathcal{D}$  such that the total variation distance of the two initial distributions is zero in the induced labelled Markov chain  $\mathcal{D}(\alpha)$ , i.e.,  $d_{tv}(\mu, \nu) = 0$ .

The following proposition is adapted from [25, Proposition 10], which will be used to prove Theorem 29.

- ▶ Proposition 28. Let  $\mathcal{M} = \langle S, L, \tau, \ell \rangle$  be an LMC and  $\mu$  and  $\nu$  be two (sub)distributions. We have that  $\mu \equiv \nu$  if and only if there exists  $F \in \mathbb{R}^{S \times S}$  such that
  - the first row of F is  $\mu \nu$ ;
  - $F\mathbf{1} = \mathbf{0}$  and for each label  $a \in L$  there exists a matrix  $B(a) \in \mathbb{R}^{S \times S}$  such that

$$FM(a) = B(a)F.$$

▶ Theorem 29. The problem TV = 0 is in  $\exists \mathbb{R}$ .

**Proof.** The proof is very similar to the one of [18, Theorem 4.3].

A memoryless strategy  $\alpha$  for  $\mathcal{D}$  can be characterised by numbers  $x_{s,m} \in [0,1]$  where  $s \in S$  and  $m \in \mathcal{A}$  such that  $x_{s,m} = \alpha(s)(m)$ . We write  $\bar{x}$  for the collection  $(x_{s,m})_{s \in S, m \in \mathcal{A}}$ .

According to Proposition 28, in the LMC  $\mathcal{D}(\alpha)$ , we have  $\mu \equiv \nu$  if and only if the following decision problem, which is a closed formula in the existential theory of the reals, has answer "ves":

 $\exists \bar{x}$ , matrices  $B(a) \in \mathbb{R}^{S \times S}$  for all  $a \in L$  and a matrix  $F \in \mathbb{R}^{S \times S}$  such that

- $\sum_{m \in \mathcal{A}(s)} x_{s,m} = 1$  for all  $s \in S$ ;
- the first row of F is  $\mu \nu$ ;
- F1 = 0:
- $FM_{\alpha}(a) = B(a)F$  for all  $a \in L$ .

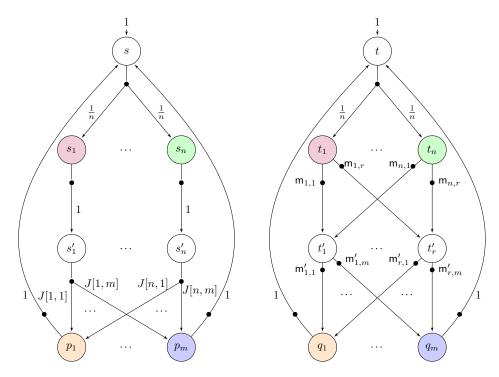
To show that the problem TV = 0 is hard for  $\exists \mathbb{R}$ , we present the reduction from the nonnegative matrix factorization (NMF) problem. Given the instance of the NMF, a nonnegative matrix  $J \in \mathbb{Q}^{n \times m}$  and a number  $r \in \mathbb{N}$ , we construct an MDP  $\mathcal{D}$ ; see Figure 6. Similar to [18, Theorem 4.5], we assume, without loss of generality, that J is a stochastic matrix. The left part is an LMC. The transition probability from  $s_i'$  to  $p_j$  in the LMC encodes the entry J[i,j].

The other part is an MDP; see the right of Figure 6. The initial state t transitions to the successors  $t_1, \dots, t_n$  with equal probabilities. In each  $t_i$  where  $1 \le i \le n$ , there are r actions  $\mathsf{m}_{i,1}, \mathsf{m}_{i,2}, \dots, \mathsf{m}_{i,r}$  where  $\varphi(t_i, \mathsf{m}_{i,k}) = \delta_{t_k'}$  for  $1 \le k \le r$ . In each  $t_k'$ , there are m actions  $\mathsf{m}'_{k,1}, \mathsf{m}'_{k,2}, \dots, \mathsf{m}'_{k,m}$  where  $\varphi(t_k', \mathsf{m}'_{k,j}) = \delta_{q_j}$  where  $1 \le j \le m$ . In state  $q_j$ , there is only one action which transitions back to state t with probability one.

The probabilities of choosing the action  $\mathsf{m}_{i,k}$  in  $s_i$  and choosing  $\mathsf{m}'_{k,j}$  in  $s'_k$  simulate the entries of A[i,k] and W[k,j].

The distribution  $\mu$  and  $\nu$  are the Dirac distribution on s and t, respectively. The labels of the states are as follows:  $\ell(s_i) = \ell(t_i) = a_i$  for  $1 \le i \le n$ ,  $\ell(p_j) = \ell(q_j) = b_j$  for  $1 \le j \le m$  and all remaining states have label c. The construction is very similar to the one in [18, Theorem 4.5].

The following proposition is technical and is used in proving Theorem 31 and Theorem 34.



**Figure 6** The MDP  $\mathcal{D}$  in the reduction for  $\exists \mathbb{R}$ -hardness of TV = 0 (or TV < 1). The labels of the states are as follows:  $\ell(s_i) = \ell(t_i) = a_i$  for  $1 \le i \le n$ ,  $\ell(p_j) = \ell(q_j) = b_j$  for  $1 \le j \le m$  and all remaining states have label c.

▶ **Proposition 30.** The NMF instance is a yes-instance if and only if there is a memoryless strategy  $\alpha$  such that  $d_{tv}(\mu, \nu) = 0$  in  $\mathcal{D}(\alpha)$ .

**Proof.** ( $\iff$ ) Assume there is a memoryless strategy  $\alpha$  such that in the induced Markov chain  $d_{tv}(\mu,\nu)=0$ , that is, we have  $\Pr_{\mu}(Run(w))=\Pr_{\nu}(Run(w))$  for all words  $w\in L^*$ . For all  $1\leq i\leq n,\ 1\leq k\leq r$  and  $1\leq j\leq m$ , let

$$A[i,k] = \alpha(t_i)(\mathsf{m}_{i,k})$$
 and  $W[k,j] = \alpha(t_k')(\mathsf{m}_{k,j}')$ .

In the LMC  $\mathcal{D}(\alpha)$ , for all  $1 \leq i \leq n$  and all  $1 \leq j \leq m$ , we have

$$\begin{split} &\Pr_{\mu}(Run(ca_icb_j)) = \frac{1}{n}J[i,j] \quad \text{ and} \\ &\Pr_{\nu}(Run(ca_icb_j)) = \frac{1}{n}\sum_{k=1}^{r}\alpha(t_i)(\mathsf{m}_{i,k}) \cdot \alpha(t_k')(\mathsf{m}_{k,j}') = \frac{1}{n}\sum_{k=1}^{r}A[i,k] \cdot W[k,j]. \end{split}$$

For all i, j we have  $\Pr_{\mu}(Run(ca_icb_j)) = \Pr_{\nu}(Run(ca_icb_j))$ . Thus, we have  $\sum_{k=1}^r A[i, k] \cdot W[k, j] = J[i, j]$  for all i, j.

 $(\Longrightarrow)$  Assume the NMF instance is a yes-instance, that is,  $\sum_{k=1}^r A[i,k] \cdot W[k,j] = J[i,j]$  for all i,j. We construct a memoryless strategy  $\alpha$  such that  $d_{tv}(\mu,\nu) = 0$  in  $\mathcal{D}(\alpha)$ . For all

state  $s' \in S$  and  $m \in A$ , the strategy  $\alpha$  is defined by

$$\alpha(s')(\mathsf{m}) = \left\{ \begin{array}{ll} A[i,k] & \text{if } s' = t_i \text{ and } \mathsf{m} = \mathsf{m}_{i,k} \text{ where } 1 \leq i \leq n \text{ and } 1 \leq k \leq r \\ W[k,j] & \text{if } s' = t'_k \text{ and } \mathsf{m} = \mathsf{m}'_{k,j} \text{ where } 1 \leq k \leq r \text{ and } 1 \leq j \leq m \\ \\ 1 & \text{if } \mathsf{m} \text{ is the only action available to } s' \\ 0 & \text{otherwise.} \end{array} \right.$$

Let  $k \in \mathbb{N}$ . Define  $w_k$  to be the word  $ca_{i_k}cb_{j_k}$  where  $1 \leq i_k \leq n$  and  $1 \leq j_k \leq m$ . To show that  $\Pr_{\mu}(Run(w)) = \Pr_{\nu}(Run(w))$  for all  $w \in L^*$ , it suffices to show for all  $k \in \mathbb{N}$ , we have:

- $\Pr_{\mu}(Run(w_1 \cdots w_k)) = \Pr_{\nu}(Run(w_1 \cdots w_k)) = \frac{1}{n^k} \prod_{k'=1}^k J[i_{k'}, j_{k'}];$
- $\Pr_{\mu}(Run(w_1\cdots w_kc)) = \Pr_{\nu}(Run(w_1\cdots w_kc));$
- $\Pr_{\mu}(Run(w_1\cdots w_kca_{i_{k+1}})) = \Pr_{\nu}(Run(w_1\cdots w_kca_{i_{k+1}}));$
- $\Pr_{\mu}(Run(w_1 \cdots w_k ca_{i_{k+1}} c)) = \Pr_{\nu}(Run(w_1 \cdots w_k ca_{i_{k+1}} c)).$

We prove the statement by induction on k. The base case is k=0. We have  $\Pr_{\mu}(Run(\varepsilon))=$  $\Pr_{\nu}(Run(\varepsilon)) = \Pr_{\mu}(Run(c)) = \Pr_{\nu}(Run(c)) = 1 \text{ and } \Pr_{\mu}(Run(ca_{i_1})) = \Pr_{\nu}(Run(ca_{i_1})) = \Pr_{\nu}(R$  $\Pr_{\mu}(Run(ca_{i_1}c)) = \Pr_{\nu}(Run(ca_{i_1}c)) = \frac{1}{n}.$ 

For the induction step, assume the statement holds for all  $k' \leq k$ . By the induction hypothesis, we have:

$$\mu M_{\alpha}(w_{1} \cdots w_{k}) = \left(\frac{1}{n^{k}} \prod_{k'=1}^{k} J[i_{k'}, j_{k'}]\right) \delta_{s} = \left(\frac{1}{n^{k}} \prod_{k'=1}^{k} J[i_{k'}, j_{k'}]\right) \mu \text{ and}$$

$$\nu M_{\alpha}(w_{1} \cdots w_{k}) = \left(\frac{1}{n^{k}} \prod_{k'=1}^{k} J[i_{k'}, j_{k'}]\right) \delta_{t} = \left(\frac{1}{n^{k}} \prod_{k'=1}^{k} J[i_{k'}, j_{k'}]\right) \nu$$

$$(10)$$

$$\nu M_{\alpha}(w_1 \cdots w_k) = \left(\frac{1}{n^k} \prod_{k'=1}^k J[i_{k'}, j_{k'}]\right) \delta_t = \left(\frac{1}{n^k} \prod_{k'=1}^k J[i_{k'}, j_{k'}]\right) \nu \tag{10}$$

First, we show that

$$\Pr_{\mu}(Run(w_1 \cdots w_{k+1})) = \Pr_{\nu}(Run(w_1 \cdots w_{k+1})) = \frac{1}{n^{k+1}} \prod_{k'=1}^{k+1} J[i_{k'}, j_{k'}]. \tag{11}$$

We have

$$\begin{split} &\Pr_{\mu}(Run(w_{1}\cdots w_{k+1})) \\ = &|\mu M_{\alpha}(w_{1}\cdots w_{k+1})| \\ = &|\mu M_{\alpha}(w_{1}\cdots w_{k})M_{\alpha}(w_{k+1})| \\ = &|(\frac{1}{n^{k}}\prod_{k'=1}^{k}J[i_{k'},j_{k'}])\mu M_{\alpha}(w_{k+1})| \\ = &(\frac{1}{n^{k}}\prod_{k'=1}^{k}J[i_{k'},j_{k'}])|\mu M_{\alpha}(w_{k+1})| \\ = &(\frac{1}{n^{k}}\prod_{k'=1}^{k}J[i_{k'},j_{k'}])\frac{1}{n}J[i_{k+1},j_{k+1}] \qquad \text{[induction hypothesis]} \\ = &\frac{1}{n^{k+1}}\prod_{k'=1}^{k+1}J[i_{k'},j_{k'}] \end{split}$$

Similarly,

$$\Pr_{\nu}(Run(w_1 \cdots w_{k+1}))$$

$$= |\nu M_{\alpha}(w_1 \cdots w_{k+1})|$$

$$= |\nu M_{\alpha}(w_1 \cdots w_k) M_{\alpha}(w_{k+1})|$$

$$= \left| \left( \frac{1}{n^k} \prod_{k'=1}^k J[i_{k'}, j_{k'}] \right) \nu M_{\alpha}(w_{k+1}) \right| \qquad [(10)]$$

$$= \left( \frac{1}{n^k} \prod_{k'=1}^k J[i_{k'}, j_{k'}] \right) \left| \nu M_{\alpha}(w_{k+1}) \right|$$

$$= \left( \frac{1}{n^k} \prod_{k'=1}^k J[i_{k'}, j_{k'}] \right) \frac{1}{n} J[i_{k+1}, j_{k+1}] \qquad [\text{induction hypothesis}]$$

$$= \frac{1}{n^{k+1}} \prod_{k'=1}^{k+1} J[i_{k'}, j_{k'}].$$

By equation (11), we have

$$\mu M_{\alpha}(w_1 \cdots w_{k+1}) = \left(\frac{1}{n^{k+1}} \prod_{k'=1}^{k+1} J[i_{k'}, j_{k'}]\right) \delta_s = \left(\frac{1}{n^{k+1}} \prod_{k'=1}^{k+1} J[i_{k'}, j_{k'}]\right) \mu \text{ and}$$
 (12)

$$\nu M_{\alpha}(w_1 \cdots w_{k+1}) = \left(\frac{1}{n^{k+1}} \prod_{k'=1}^{k+1} J[i_{k'}, j_{k'}]\right) \delta_t = \left(\frac{1}{n^k} \prod_{k'=1}^k J[i_{k'}, j_{k'}]\right) \nu \tag{13}$$

Thus,

$$\Pr_{\mu}(Run(w_{1}\cdots w_{k+1}c))$$

$$=|\mu M_{\alpha}(w_{1}\cdots w_{k+1}c)|$$

$$=|\mu M_{\alpha}(w_{1}\cdots w_{k+1})M_{\alpha}(c)|$$

$$=|\left(\frac{1}{n^{k+1}}\prod_{k'=1}^{k+1}J[i_{k'},j_{k'}]\right)\mu M_{\alpha}(c)| \qquad [(12)]$$

$$=\left(\frac{1}{n^{k+1}}\prod_{k'=1}^{k+1}J[i_{k'},j_{k'}]\right)|\mu M_{\alpha}(c)|$$

$$=\frac{1}{n^{k+1}}\prod_{k'=1}^{k+1}J[i_{k'},j_{k'}]$$

Similarly,  $\Pr_{\nu}(Run(w_1\cdots w_{k+1}c)) = \frac{1}{n^{k+1}}\prod_{k'=1}^{k+1}J[i_{k'},j_{k'}]$ . We also have

$$\Pr_{\mu}(Run(w_{1}\cdots w_{k+1}ca_{i_{k+2}}))$$

$$=|\mu M_{\alpha}(w_{1}\cdots w_{k+1}ca_{i_{k+2}})|$$

$$=|\mu M_{\alpha}(w_{1}\cdots w_{k+1})M_{\alpha}(ca_{i_{k+2}})|$$

$$=|(\frac{1}{n^{k+1}}\prod_{k'=1}^{k+1}J[i_{k'},j_{k'}])\mu M_{\alpha}(ca_{i_{k+2}})|$$

$$=(\frac{1}{n^{k+1}}\prod_{k'=1}^{k+1}J[i_{k'},j_{k'}])|\mu M_{\alpha}(ca_{i_{k+2}})|$$

$$=\frac{1}{n^{k+2}}\prod_{k'=1}^{k+1}J[i_{k'},j_{k'}]$$
[(12)]

Similarly, 
$$\Pr_{\nu}(Run(w_1 \cdots w_{k+1}ca_{i_{k+2}})) = \Pr_{\mu}(Run(w_1 \cdots w_{k+1}ca_{i_{k+2}}c)) = \Pr_{\nu}(Run(w_1 \cdots w_{k+1}ca_{i_{k+2}}c)) = \frac{1}{n^{k+2}} \prod_{k'=1}^{k+1} J[i_{k'}, j_{k'}].$$

▶ Theorem 31. The NMF problem is polynomial-time reducible to the problem TV = 0, hence TV = 0 is  $\exists \mathbb{R}$ -hard.

**Proof.** Proposition 30 shows that the NMF problem is polynomial-reducible to the problem TV = 0. Since the NMF problem is  $\exists \mathbb{R}$ -complete [38], we have that the problem TV = 0 is  $\exists \mathbb{R}$ -hard.

▶ **Theorem 17.** The problem TV = 0 is  $\exists \mathbb{R}$ -complete.

**Proof.** It follows from Theorem 29 and Theorem 31.

#### **D.2** Proofs of TV < 1

Next, we show that the problem TV < 1 is  $\exists \mathbb{R}$ -complete. Recall that TV < 1 is the problem asking whether there is a memoryless strategy  $\alpha$  for  $\mathcal{D}$  such that the total variation distance of the two initial distributions is less than one in the induced labelled Markov chain  $\mathcal{D}(\alpha)$ , i.e.,  $d_{tv}(\mu, \nu) < 1$ .

With Proposition 22 at hand, we obtain:

▶ Theorem 32. The problem TV < 1 is in  $\exists \mathbb{R}$ .

**Proof.** A memoryless strategy  $\alpha$  for  $\mathcal{D}$  can be characterised by numbers  $x_{s,m} \in [0,1]$  where  $s \in S$  and  $m \in A$  such that  $x_{s,m} = \alpha(s)(m)$ . We write  $\bar{x}$  for the collection  $(x_{s,m})_{s \in S, m \in A}$ .

From Proposition 22, to check whether there is a memoryless strategy  $\alpha$  such that  $d_{tv}(\mu,\nu)<1$ , it suffices to check if there are subdistributions  $\mu_1$  and  $\mu_2$  that satisfy Equation (8). Thus, we can nondeterministically guess  $r_1$  and support of  $\mu_2$  such that support  $(\mu_2) \subseteq R_{r_1}^{\mu,\nu}$ , and then check the following decision problem, which is a closed formula in the existential theory of the reals:

 $\exists \bar{x}$ , matrices  $B(a) \in \mathbb{R}^{S \times S}$  for all  $a \in L$ , a matrix  $F \in \mathbb{R}^{S \times S}$ , subdistributions  $\mu_1$  and  $\mu_2$ such that

```
- \sum_{\mathsf{m}\in\mathcal{A}(s)} x_{s,\mathsf{m}} = 1 for all s\in S;
- the first row of F is \mu_1 - \mu_2;
```

- F1 = 0;
- FM(a) = B(a)F for all  $a \in L$ ;
- $r_1 \in \operatorname{support}(\mu_1)$ ;
- support $(\mu_2) \subseteq R_{r_1}^{\mu,\nu}$ .

It follows that the problem is in  $\exists \mathbb{R}$  since  $\exists \mathbb{R}$  is closed under NP-reductions [41].

To show that the problem TV < 1 is hard for  $\exists \mathbb{R}$ , we present the reduction from the nonnegative maitrx factorization (NMF) problem. We construct the same MDP  $\mathcal{D}$  as shown in Figure 6. The reduction is similar to [18, Theorem 4.5].

The proposition below is technical and is only used in the proof of Theorem 34.

▶ Proposition 33. If the NMF instance is a no-instance then for all memoryless strategy  $\alpha$ and all (sub)distributions  $\mu_1$  over the left part of  $\mathcal{D}$  and all (sub)distributions  $\mu_2$  over the right part, we have  $d_{tv}(\mu_1, \mu_2) > 0$  in the LMC  $\mathcal{D}(\alpha)$ .

**Proof.** Let  $\mu_1$  and  $\mu_2$  be two (sub)distributions where  $\mu_1$  is over the left part of  $\mathcal{D}$  and  $\mu_2$ is over the right part. Let the NMF instance be a no-instance. Let  $\alpha$  be any memoryless strategy.

By the construction of the MDP  $\mathcal{D}$  (see Figure 6), there must exist a word  $w' \in L^*$  such that  $\mu_1 M_{\alpha}(w')$  is a Dirac distribution on state s. Let  $\mu'_1 = \mu_1 M_{\alpha}(w')$  and  $\mu'_2 = \mu_2 M_{\alpha}(w')$ . We have that  $\mu'_1 = \mu'_1(s)\delta_s$ . We distinguish the following three cases.

(a) Assume  $|\mu'_1| \neq |\mu'_2|$ . Let  $E' = L^{\omega}$ . We have

$$\begin{array}{rcl} d_{tv}(\mu_1', \mu_2') & = & \sup_{E \in \mathcal{F}} |\Pr_{\mu_1'}(E) - \Pr_{\mu_2'}(E)| \\ & \geq & |\Pr_{\mu_1'}(E') - \Pr_{\mu_2'}(E')| & [E' \in \mathcal{F}] \\ & = & ||\mu_1'| - |\mu_2'|| \\ & > & 0 & [|\mu_1'| \neq |\mu_2'|] \end{array}$$

(b) Assume  $|\mu'_1| = |\mu'_2|$  and  $\mu'_1(s) \neq \mu'_2(t)$ . Let  $E' = Run(ca_i) \in \mathcal{F}$ . We have

$$\begin{array}{lcl} d_{tv}(\mu_1',\mu_2') & = & \sup_{E \in \mathcal{F}} |\operatorname{Pr}_{\mu_1'}(E) - \operatorname{Pr}_{\mu_2'}(E)| \\ & \geq & |\operatorname{Pr}_{\mu_1'}(E') - \operatorname{Pr}_{\mu_2'}(E')| & [E' \in \mathcal{F}] \\ & = & |\operatorname{Pr}_{\mu_1'}(Run(ca_i)) - \operatorname{Pr}_{\mu_2'}(Run(ca_i))| & [E' = Run(ca_i)] \\ & = & ||\mu_1' M_{\alpha}(ca_i)| - |\mu_2' M_{\alpha}(ca_i)|| \\ & = & |\frac{1}{n} \mu_1'(s) - \frac{1}{n} \mu_2'(t)| \\ & > & 0 & [\mu_1'(s) \neq \mu_2'(t)] \end{array}$$

(c) Assume  $|\mu'_1| = |\mu'_2|$  and  $\mu'_1(s) = \mu'_2(t) > 0$ . Since  $\mu'_1 = \mu'_1(s)\delta_s$ ,  $|\mu'_1| = |\mu'_2|$  and  $\mu'_2(t) = \mu'_2(s) > 0$ , we have that  $\mu'_2 = \mu'_2(t)\delta_t$ . By Proposition 30, we have that if the NMF instance is a no-instance, then  $d_{tv}(\mu, \nu) > 0$ , that is, there exists a word  $w \in L^*$  such that  $\Pr_{\mu}(Run(w)) \neq \Pr_{\nu}(Run(w))$ . This word w is of the form  $ca_{i_1}cb_{j_1}ca_{i_2}\cdots$ , since it is emitted by running the MDP  $\mathcal{D}$  from state s. Let  $E' = Run(w) \in \mathcal{F}$ . We have

```
\begin{array}{lll} d_{tv}(\mu'_1,\mu'_2) & = & \sup_{E\in\mathcal{F}}|\Pr_{\mu'_1}(E)-\Pr_{\mu'_2}(E)| \\ & \geq & |\Pr_{\mu'_1}(E')-\Pr_{\mu'_2}(E')| \quad [E'\in\mathcal{F}] \\ & = & |\Pr_{\mu'_1}(Run(w))-\Pr_{\mu'_2}(Run(w))| \quad [E'=Run(w)] \\ & = & ||\mu'_1M_\alpha(w)|-|\mu'_2M_\alpha(w)|| \\ & = & ||\mu'_1(s)\delta_sM_\alpha(w)|-|\mu'_2(t)\delta_tM_\alpha(w)|| \quad [\mu'_1=\mu'_1(s)\delta_s \text{ and } \mu'_2=\mu'_2(t)\delta_t] \\ & = & |\Pr_{\mu'_1(s)\delta_s}(Run(w))-\Pr_{\mu'_2(t)\delta_t}(Run(w))| \\ & = & |\Pr_{\mu'_1(s)\mu}(Run(w))-\Pr_{\mu'_2(t)\nu}(Run(w))| \quad [\mu=\delta_s \text{ and } \nu=\delta_t] \\ & = & |\mu'_1(s)\Pr_\mu(Run(w))-\mu'_2(t)\Pr_\nu(Run(w))| \\ & = & \mu'_1(s)|\Pr_\mu(Run(w))-\Pr_\nu(Run(w))| \quad [\mu'_1(s)=\mu'_2(t)] \\ & > & 0 \quad [\mu'_1(s)>0 \text{ and } \Pr_\mu(Run(w))\neq\Pr_\nu(Run(w))] \end{array}
```

Following the three cases, we have  $d_{tv}(\mu'_1, \mu'_2) > 0$ , that is, there exists a word  $w \in L^*$  such that  $\Pr_{\mu'_1}(Run(w)) \neq \Pr_{\mu'_2}(Run(w))$ . Consider the word w'w, we have

```
\begin{array}{lll} \Pr_{\mu_1}(Run(w'w)) & = & |\mu_1 M_{\alpha}(w'w)| \\ & = & |\mu_1 M_{\alpha}(w') M_{\alpha}(w)| \\ & = & \Pr_{\mu_1 M_{\alpha}(w')}(Run(w) \\ & = & \Pr_{\mu_1'}(Run(w)) \quad [\mu_1' = \mu_1 M_{\alpha}(w')] \\ & \neq & \Pr_{\mu_2'}(Run(w)) \\ & = & \Pr_{\mu_2 M_{\alpha}(w')}(Run(w)) \quad [\mu_2' = \mu_2 M_{\alpha}(w')] \\ & = & |\mu_2 M_{\alpha}(w') M_{\alpha}(w)| \\ & = & |\mu_2 M_{\alpha}(w'w)| \\ & = & \Pr_{\mu_2}(Run(w'w)). \end{array}
```

Thus, we have  $d_{tv}(\mu_1, \mu_2) > 0$ .

▶ Theorem 34. The NMF problem is polynomial-time reducible to the problem TV < 1, hence TV < 1 is  $\exists \mathbb{R}$ -hard.

**Proof.** According to Proposition 30, we have that if the NMF instance is a yes-instance then there is a memoryless strategy such that  $d_{tv}(\mu,\nu) = 0$  in the induced LMC, which implies  $d_{tv}(\mu,\nu) < 1$ .

It remains to show that if there is a memoryless strategy such that  $d_{tv}(\mu,\nu) < 1$ , then the NMF instance is a yes-instance. We show the contrapositive, that is, if the NMF instance is a no-instance, then for all memoryless strategy  $d_{tv}(\mu,\nu) = 1$  in the induced LMC. For all

 $w \in L^*$  and memoryless strategy  $\alpha$ , we have that if  $|\mu M_{\alpha}(w)| > 0$  and  $|\nu M_{\alpha}(w)| > 0$  then  $\mu M_{\alpha}(w)$  and  $\nu M_{\alpha}(w)$  are subdistributions over the left and right part of  $\mathcal{D}$ , respectively. It follows from Proposition 33 that  $d_{tv}(\mu_1, \mu_2) > 0$  for all subdistributions  $\mu_1, \mu_2$  over the left and right part of  $\mathcal{D}$ , respectively. By Proposition 23, we have that  $d_{tv}(\mu, \nu) = 1$  in all LMC  $\mathcal{D}(\alpha)$ .

▶ Theorem 18. The problem TV < 1 is  $\exists \mathbb{R}$ -complete.

**Proof.** It follows from Theorem 32 and Theorem 34.

### **D.3** Proofs of PB = 0

Next, we show that the problem PB = 0 is NP-complete. Recall that PB = 0 is the problem asking whether there is a memoryless strategy  $\alpha$  for  $\mathcal{D}$  such that the probabilistic bisimilarity distance of the two initial states is zero in the induced labelled Markov chain  $\mathcal{D}(\alpha)$ , i.e.,  $d_{pb}(s,t)=0$ .

- ▶ **Theorem 35.** [15, Theorem 1] For all  $s, t \in S$ ,  $s \sim t$  if and only if  $d_{nb}(s, t) = 0$ .
- ▶ **Theorem 36.** The problem PB = 0 is in NP.

**Proof.** According to Theorem 35 and the definition of probabilistic bisimilarity, there exists a memoryless strategy  $\alpha$  such that  $d_{pb}(s,t)=0$  in the induced LMC  $\mathcal{D}(\alpha)$ , if and only if the initial states s and t are probabilistic bisimilar, i.e., s and t are in the same probabilistic bisimulation induced equivalence class.

We can nondeterministically guess a partition  $E_1, \dots, E_n$  of S such that each subset  $E_i$  is a probabilistic bisimulation induced equivalence class and state s, t are in the same equivalence class, that is,  $\bigcup_{E_i} = S$ ,  $E_i \cap E_j = \emptyset$  for any  $i \neq j$ , and  $s, t \in E_i$  for some i.

A memoryless strategy  $\alpha$  for  $\mathcal{D}$  can be characterised by numbers  $x_{s,m} \in [0,1]$  where  $s \in S$  and  $m \in \mathcal{A}$  such that  $x_{s,m} = \alpha(s)(m)$ . We write  $\bar{x}$  for the collection  $(x_{s,m})_{s \in S, m \in \mathcal{A}}$ . To check  $d_{pb}(s,t) = 0$  in the induced LMC  $\mathcal{D}(\alpha)$  amounts to a feasibility test of the following linear program:

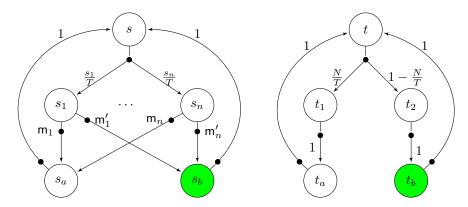
$$\exists \bar{x} \text{ such that } \sum_{\mathsf{m} \in \mathcal{A}(s)} x_{s,\mathsf{m}} = 1 \text{ for all } s \in S \text{ and}$$
 
$$\tau(s')(E_j) = \tau(t')(E_j) \text{ for all } E_i, E_j \text{ and all } s',t' \in E_i,$$

and hence can be decided in polynomial time.

Given a set  $S = \{s_1, \dots, s_n\}$  and  $N \in \mathbb{N}$ , Subset Sum asks whether there exists a set  $P \subseteq S$  such that  $\sum_{s_i \in P} s_i = N$ .

▶ **Theorem 37.** The Subset Sum problem is polynomial-time many-one reduction to PB = 0, hence PB = 0 is NP-hard.

**Proof.** Given an instance of Subset Sum  $\langle S, N \rangle$  where  $S = \{s_1, \cdots, s_n\}$  and  $N \in \mathbb{N}$ , we construct an MDP  $\mathcal{D}$ ; see Figure 7. Let  $T = \sum_{s_i \in S} s_i$ . In the MDP  $\mathcal{D}$ , state s transitions to state  $s_i$  with probability  $s_i/T$  for all  $1 \leq i \leq n$ . Each state  $s_i$  has two available actions, each transitions to  $s_a$  and  $s_b$  by taking the action  $\mathsf{m}_i$  and  $\mathsf{m}'_i$ , respectively. State t transitions to  $t_1$  and  $t_2$  with probability N/T and  $t_2$  with probability  $t_1$  and  $t_3$  with probability  $t_4$  and  $t_4$  with probability  $t_4$  and  $t_5$  have label  $t_6$  and all other states have label  $t_6$ .



**Figure 7** The MDP  $\mathcal{D}$  in the reduction for NP-hardness of PB = 0 (or PB < 1). All states have the same label a except  $s_b$  and  $t_b$  which have label b.

Next, we show that

$$\langle S, N \rangle \in \text{Subset Sum} \iff \exists \text{ memoryless strategy } \alpha \text{ such that } d_{pb}(s, t) = 0 \text{ in } \mathcal{D}(\alpha).$$

Intuitively, making  $s_i$  probabilistic bisimilar with  $t_1$  simulates the membership of  $s_i$  in P. Conversely, making  $s_i$  probabilistic bisimilar with  $t_2$  simulates the membership of  $s_i$  in  $S \setminus P$ .

( $\Longrightarrow$ ) Let  $P \subseteq S$  be the set such that  $\sum_{s_i \in P} s_i = N$ . Let  $\alpha$  be an MD strategy such that if  $s_i \in P$  then  $\alpha(s_i) = \mathsf{m}_i$  and  $\alpha(s_i) = \mathsf{m}_i'$  otherwise. Consider the following partition of states of  $\mathcal{D}$ ,

$$E_1 = \{s, t\}, E_2 = P \cup \{t_1\}, E_3 = (S \setminus P) \cup \{t_2\}, E_4 = \{s_a, t_a\} \text{ and } E_5 = \{s_b, t_b\}.$$

Then,

$$\tau(s)(E_2) = \tau(s)(P) = \sum_{s_i \in P} \frac{s_i}{T}$$

$$= \frac{N}{T}$$

$$= \tau(t)(t_1) = \tau(t)(E_2) \text{ and}$$

$$[\sum_{s_i \in P} s_i = N]$$

$$\tau(s)(E_3) = \tau(s)(S \setminus P) = \sum_{s_i \in S \setminus P} \frac{s_i}{T} = 1 - \sum_{s_i \in P} \frac{s_i}{T}$$

$$= 1 - \frac{N}{T}$$

$$= \tau(t)(t_2) = \tau(t)(E_3).$$

$$[\sum_{s_i \in P} s_i = N]$$

Similarly, we can verify that for all  $E_i, E_j$  and all  $s', t' \in E_i : \tau(s')(E_j) = \tau(t')(E_j)$ .

By the definition of probabilistic bisimulation, each set  $E_i$  is a probabilistic bisimulation induced equivalence class. Since s and t are in the same equivalence class, we have  $s \sim t$ , and hence  $d_{pb}(s,t) = 0$  by Theorem 35.

( $\iff$ ) Assume there is a memoryless strategy  $\alpha$  such that  $d_{pb}(s,t) = 0$  in the LMC  $\mathcal{D}(\alpha)$ . By Theorem 35, s and t are probabilistic bisimilar in  $\mathcal{D}(\alpha)$ . Let P be the set of successor states of s that are probabilistic bisimilar to  $t_1$ . Then,

$$\tau(s)(P) = \sum_{s_i \in P} \frac{s_i}{T}$$
 and  $\tau(t)(t_1) = \frac{N}{T}$ .

Since  $\ell(t_a) \neq \ell(t_b)$ , by definition of probabilistic bisimilarity, we have  $t_a \not\sim t_b$ , and hence  $t_a$  and  $t_b$  are not in the same  $\sim$ -equivalence class. Since  $\tau(t_1)(t_a) = \tau(t_2)(t_b) = 1$  in the LMC  $\mathcal{D}(\alpha)$ , again by definition of probabilistic bisimilarity, we have  $t_1 \not\sim t_2$  and  $t_1$  and  $t_2$  are not in the same  $\sim$ -equivalence class, and thus  $t_2$  is not in the same  $\sim$ -equivalence class as the states in P. Since  $s \sim t$ , we have  $\sum_{s_i \in P} \frac{s_i}{T} = \frac{N}{T}$ , and hence,  $\sum_{s_i \in P} = N$ .

▶ **Theorem 19.** The problem PB = 0 is NP-complete.

**Proof.** It follows from Theorem 36 and Theorem 37.

### **D.4** Proofs of PB < 1

We show in this section that the problem PB < 1 is NP-complete. Recall that PB < 1 is the problem asking whether there is a memoryless strategy  $\alpha$  for  $\mathcal{D}$  such that the probabilistic bisimilarity distance of the two initial states is less than one in the induced labelled Markov chain  $\mathcal{D}(\alpha)$ , i.e.,  $d_{vb}(s,t) < 1$ .

▶ **Theorem 20.** The problem PB < 1 is NP-complete.

**Proof.** We first show that this problem is in NP. We nondeterministically guess a partition  $E_1, \dots, E_n$  of the states of  $\mathcal{D}$  and two states u, v in the same subset  $E_i$  for some i. We also nondeterministically guess the graph G of Definition 24 for the LMC  $\mathcal{D}(\alpha)$  induced by some strategy  $\alpha$ . By Proposition 25, if (s,t) is reachable from some state pair (u,v) in the graph G and  $u \sim v$  then  $d_{pb}(s,t) < 1$ . The condition that (s,t) is reachable from (u,v) in the graph G can be checked in polynomial time using e.g. breadth-first search. To check  $u \sim v$ , it suffices to check each subset  $E_i$  is a probabilistic bisimulation induced equivalence class, which amounts to a feasibility test of the linear program:

$$\exists \bar{x} \text{ such that } \sum_{\mathsf{m} \in \mathcal{A}(s)} x_{s,\mathsf{m}} = 1 \text{ for all } s \in S \text{ and}$$

$$\tau(s')(E_j) = \tau(t')(E_j) \text{ for all } E_i, E_j \text{ and all } s', t' \in E_i,$$

and hence can be decided in polynomial time.

Next, we establish NP-hardness of the problem. Similar to Theorem 37, we provide a polynomial-time many-one reduction from Subset Sum. Given an instance of  $\langle S, N \rangle$  of Subset Sum, we construct the same MDP  $\mathcal{D}$  as shown in Figure 7.

Next, we show that

- $\langle S, N \rangle \in \text{Subset Sum} \iff \exists \text{ memoryless strategy } \alpha \text{ such that } d_{pb}(s, t) < 1 \text{ in } \mathcal{D}(\alpha).$
- ( $\Longrightarrow$ ) Let  $P\subseteq S$  be the set such that  $\sum_{s_i\in P} s_i=N$ . By Theorem 37, there exists a memoryless strategy  $\alpha$  such that  $d_{pb}(s,t)=0$ , and hence  $d_{pb}(s,t)<1$ .
- ( $\Leftarrow$ ) We prove its contrapositive, that is, if the instance is a no-instance then for all memoryless strategy  $\alpha$  we have  $d_{pb}(s,t) = 1$  in  $\mathcal{D}(\alpha)$ .

Assume the instance is a no-instance. By Theorem 37, we have  $d_{pb}(s,t) > 0$  in  $\mathcal{D}(\alpha)$  for all memoryless strategy  $\alpha$ , i.e.  $s \not\sim t$ . Let  $\alpha$  be an arbitrary memoryless strategy. By the construction of the MDP, we have  $s_a \not\sim t_a$  and  $s_b \not\sim t_b$ . Since  $\ell(s_a) \neq \ell(t_b)$  and  $\ell(s_b) \neq \ell(t_a)$ , we also have  $s_a \not\sim t_b$  and  $s_b \not\sim t_a$ . Thus, in the LMC  $\mathcal{D}(\alpha)$ ,  $s_i$  for all  $1 \le i \le n$  is not probabilistic bisimilar to  $t_1$  or  $t_2$ . In the graph of Definition 24, the following vertices could reach (s,t):  $(s_i,t_1)$  or  $(s_i,t_2)$  for all  $1 \le i \le n$ ,  $(s_a,t_a)$  and  $(s_b,t_b)$ . However, none of them are probabilistic bisimilar. By Proposition 25, we have  $d_{pb}(s,t) = 1$  in the LMC  $\mathcal{D}(\alpha)$ .