Bonds (2) - Sensitivity to time and price

We have so far established a zero rate curve, can we proceed further?

Forward rate

Forward interest rate are the rates of interest implied by current zero rates for periods of time in the future

Suppose the following rate:

Year (n)	Zero rate for an n-year investment [% per annum]	Forward rate for n-th year (% per annum)
1	10.0	
2	10.5	11%
3	10.8	11.4%
4	11.0	
5	11.1	

} to be filled

Suppose an initial investment of \$ 100

1 year:

$$100 \cdot e^{0.1} = \$\,110.52$$

2 years: $100 \cdot e^{0.105 \cdot 2} = \$\,123.37$

Forward 1 year, in 1 year:

It is the rate of interest for year 2 that when combined with 10% per annum for year 1, gives 10.5% overall for the 2 years.

$$100 \cdot e^{0.1} \cdot e^{x} = 100 \cdot e^{0.105 \cdot 2}$$

$$\Rightarrow 0.1 + x = 0.105 \cdot 2 \quad \Rightarrow \quad x = 0.11 = 11\%$$

to do that:
- buy 2 years
- sell 1 year
$\Rightarrow$ forward 1y1

Why is it 11%, and not 10% or 12%? There would otherwise be an arbitrage opportunity.

IF 10% $\Rightarrow$ One sells a 1 year 10% and a forward 1y1 at 10% and buys a 2 year zero rate at 10.5%.

profit after 2 years: $-100 \cdot e^{0.1} \cdot e^{0.1} + 100 \cdot e^{0.105 \cdot 2} = \underbrace{100 \cdot e^{0.01}}_{1\% \text{ arbitrage opportunity}}$

Forward 1 year, in 2 years:

$$100 \cdot e^{0.105 \cdot 2} \cdot e^{x \cdot 1} = 100 \cdot e^{0.108 \cdot 3}$$

$$\Rightarrow x = \frac{0.108 \cdot 3 - 0.105 \cdot 2}{1} = 11.4\%$$

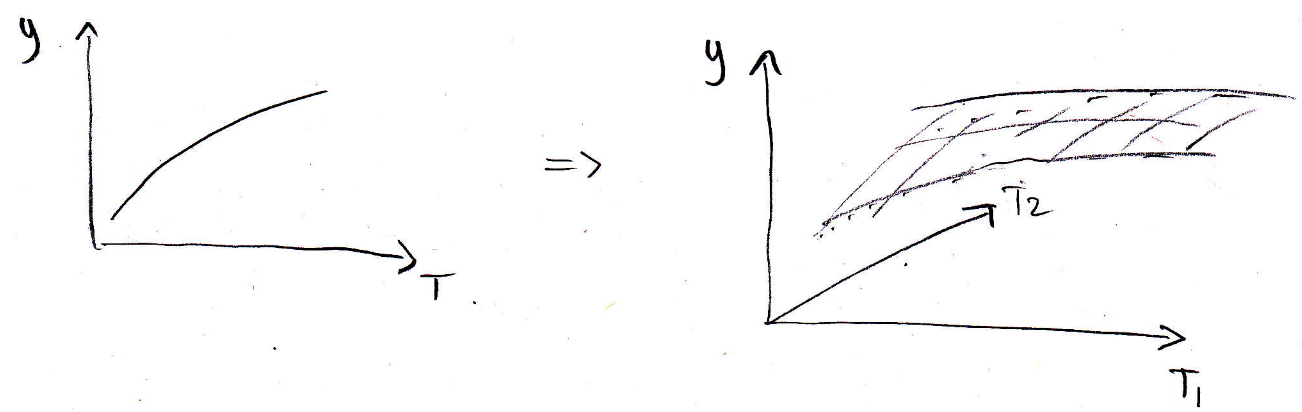
One can the generalise the procedure and write

$$r = \frac{R_2 T_2 - R_1 T_1}{T_2 - T_1}$$

R_1, R_2 rates for maturity T_1 and T_2 respectively.

This relation is slightly more complicated if one considers other compounding convention, e.g. semi-annual.

We now have a surface of rates and not a curve anymore.



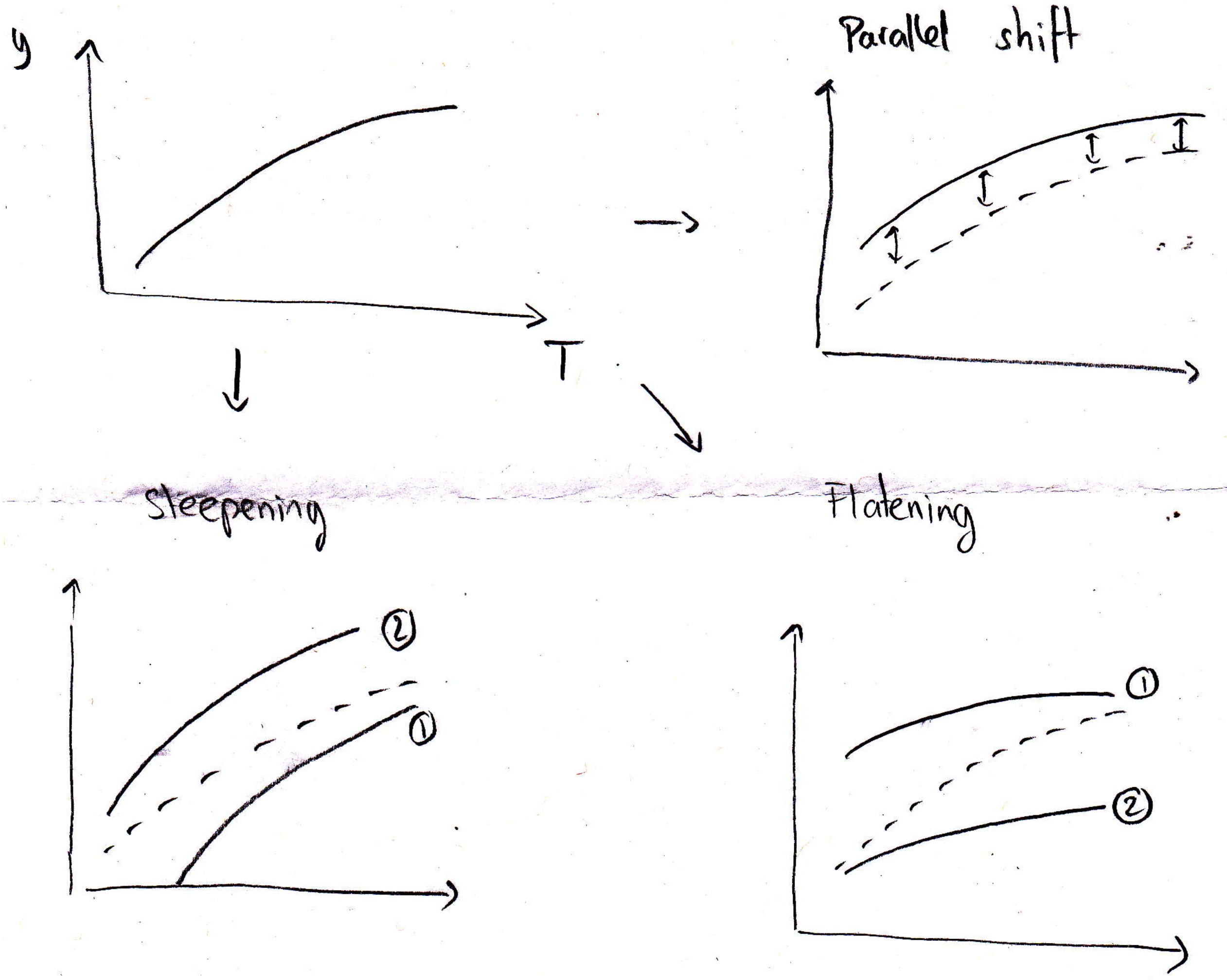
A practical example where forward rates are used:

Mortgage: one wants to buy a house in 1 year, and wants a mortgage for 10 years

The banker uses the 11 year and the 1 year.

Evolution of the curve

The curve evolves in time due to central bank interventions (beginning of the curve) and market sentiments all the curve.



And of course this is theory, in practice it is a bit of a mixture between those 3.

[Show movie]

What is the price sensitivity of a bond?

Duration

The duration of a bond is a measure of how long on average the holder of the bond has to wait before receiving cash payments.

A zero coupon that matures in n years has a duration of n years. A coupon-bearing bond has a duration smaller than n.

Recall that $$B = \sum_{i=1}^{n} c_i e^{-y t_i}$$ (c_n includes the principal) (*)

The duration D of a bond is defined as

$$D = \frac{\sum_{i=1}^{n} t_i c_i e^{-y t_i}}{B} \qquad (D)$$

that can be written as

$$D = \sum_{i=1}^{n} t_i \left[\frac{c_i e^{-y t_i}}{B} \right]$$

≥ 0

≤ 1

$\Sigma = 1$

The duration is therefore a weighted average of the times when payments are made.

From (*) :

$$\frac{\partial B}{\partial y} = -\sum c_i t_i e^{-yt_i}$$

From (Δ)

$$\Rightarrow \quad \frac{\partial B}{\partial y} = -BD$$

Hence if we make a small parallel shift to the yield curve δy, one can approximate a δB change of price

as

$$\frac{\delta B}{\delta y} = -B \cdot D$$

or

$$\underbrace{\frac{\delta B}{B}}_{\text{price change in \%}} = -\underbrace{D}_{\text{duration}} \cdot \underbrace{\delta y}_{\text{parallel shift}}$$

Example: A 3 year 10% capon bond, yield is 12%.

Time (years)	Cash flow ($)	Present value ($)	Weight	Time×weight
0.5	5	$5e^{-0.12\cdot 0.5} = 4.709$	$4.709/94.213 = 0.050$	0.025
1.0	5	4.435	0.047	0.047
1.5	5	4.176	0.044	0.066
2.0	5	3.933	0.042	0.083
2.5	5	3.704	0.039	0.098
3.0	105	73.256	0.778	2.333
		94.213	1.000	2.653

D = 2.653

Suppose $\delta y = 0.10\%$ (small parallel shift)

$$5e^{-0.121\cdot 0.5} + 5e^{-0.121} + 5\cdot e^{-0.121\cdot 1.5} + 5\cdot e^{-0.121\cdot 2} +$$

$$5\cdot e^{-0.121\cdot 2.5} + 105\cdot e^{-0.121\cdot 3} = 93.963$$

Does that match ?

$$\delta B = 93.963 - 94.213 = -0.250 \stackrel{?}{=} -B\cdot D\cdot \delta y = -94.213\cdot 2.653\cdot 10^{-3}$$

$$= -0.250 \quad ✓$$

Suppose a larger parallel shift, $\delta y = 1\%$.

$$5 \cdot e^{-0.13 \cdot 0.5} + 5 e^{-0.13} + \cdots = 91.749$$

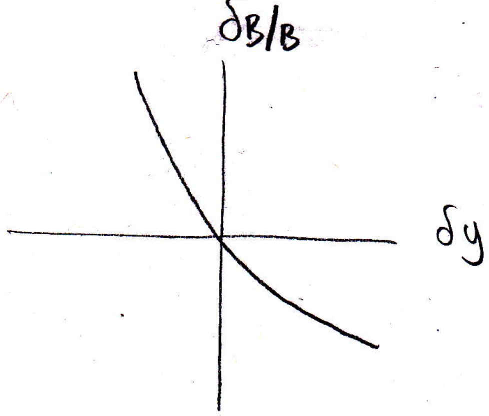
Does that match?

$$\delta B = 91.749 - 94.213 = -2.464$$

$$\overset{?}{=} -B \cdot D \cdot \delta y = -94.213 \cdot 2.653 \cdot 10^{-2}$$

$$= -2.4991$$

There is a significant Δ due to the non-linearity of the instrument. $-2.464 + 2.499 = 0.035$



A measure of convexity is $C = \frac{1}{B} \frac{\partial^2 B}{\partial y^2} = \sum_{i=1}^{n} c_i t_i^2 e^{-y t_i}$

And we show that $\frac{\delta B}{B} = -D \delta y + \frac{1}{2} C (\delta y^2)$

$C = 7.570$, $-B \cdot D \cdot \delta y + \frac{1}{2} (B \cdot) C (\delta y)^2 = -2.464$

$\Rightarrow \Delta_2 = 0.000$ (the 4^{th} decimal is wrong)