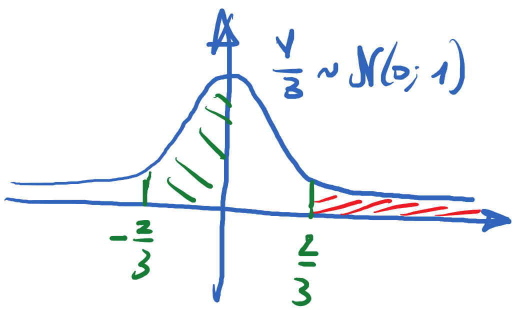
Ex 1

$X \sim \mathcal{N}(0;1) \qquad Y \sim \mathcal{N}(0;3)$

- $P(-2 < Y \leq 1) = P\left(-\frac{2}{3} < \frac{Y}{3} \leq \frac{1}{3}\right)$

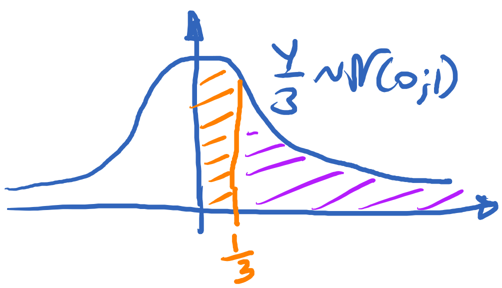


$= \underbrace{P\left(-\frac{2}{3} < \frac{Y}{3} \leq 0\right)} + \underbrace{P\left(0 < \frac{Y}{3} \leq \frac{1}{3}\right)}$

$= \frac{1}{2} - Q\left(\frac{2}{3}\right) + \frac{1}{2} - Q\left(\frac{1}{3}\right)$

$= 1 - Q\left(\frac{2}{3}\right) - Q\left(\frac{1}{3}\right)$

$\approx \boxed{0.38}$



- $P(Y > 5.5) = P\left(\frac{Y}{3} > \frac{5.5}{3}\right) = Q\left(\frac{5.5}{3}\right) \approx \boxed{0.03}$
- $P(-2 < X \leq 2) = 2\left(\frac{1}{2} - Q(2)\right) = 1 - 2Q(2) \approx \boxed{0.95}$
- $P(X > 1.5) = Q(1.5) \approx \boxed{0.07}$

Ex 2

$X \sim \mathcal{N}(30;11)$

- $P(X > 35) = P\left(\frac{X-30}{11} > \frac{5}{11}\right) = Q\left(\frac{5}{11}\right) = \boxed{0{,}32}$
- $P(X \leq 5) = P\left(\frac{X-30}{11} \leq \frac{-25}{11}\right) = 1 - Q\left(\frac{-25}{11}\right) \approx \boxed{0{,}01}$
- $P(20 < X \leq 40) = P\left(-\frac{10}{11} < \frac{X-30}{11} \leq \frac{10}{11}\right) = 2\left(\frac{1}{2} - Q\left(\frac{10}{11}\right)\right)$

$= 1 - 2Q\left(\frac{10}{11}\right)$

$$= 1 - 2Q\left(\frac{10}{11}\right)$$

$$\boxed{\approx 0{,}64}$$

$\boxed{\text{Ex 3}}$ $E(X) = \mu = 0$

$$P(|X| \leq 10) = 0.3$$

$$P(-10 \leq X \leq 10) = 0.3$$

$$P\left(-\frac{10}{\sigma} \leq \frac{X}{\sigma} \leq \frac{10}{\sigma}\right) = 0.3$$

$$2\left(\frac{1}{2} - Q\left(\frac{10}{\sigma}\right)\right) = 0.3$$

$$1 - 2Q\left(\frac{10}{\sigma}\right) = 0.3$$

$$Q\left(\frac{10}{\sigma}\right) = \frac{1 - 0.3}{2}$$

$$\frac{10}{\sigma} = Q^{-1}\left(\frac{0.7}{2}\right) \Rightarrow \boxed{\sigma = \frac{10}{Q^{-1}(0.35)} \approx 26}$$

$\boxed{\text{Ex 4}}$

$$\frac{1}{2}\,\mathrm{erfc}\left(\frac{m}{\sqrt{2}}\right) = \frac{1}{2}\,\frac{2}{\sqrt{\pi}}\int_{\frac{m}{\sqrt{2}}}^{+\infty} e^{-x^2}\,dx$$

$u = \sqrt{2}\,x$

$$= \frac{1}{\sqrt{\pi}}\int_{m}^{+\infty} e^{-\frac{u^2}{2}}\,\frac{du}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{m}^{+\infty} e^{-\frac{u^2}{2}}\,du$$

$$\boxed{\frac{1}{2}\,\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right) = Q(x)}$$

$\boxed{\text{Ex } 5}$

$$\begin{cases} H_0 : X = Z \\ H_1 : X = 1 + Z \end{cases} \qquad Z \sim \mathcal{N}(0;1)$$

Décision :

$$\frac{f_{X|H_0}(x)}{f_{X|H_1}(x)} \underset{H_1}{\overset{H_0}{\gtrless}} \frac{p(H_1)}{p(H_0)}$$

let's note $p = p(H_1)$

so $1 - p = p(H_0)$

$$\frac{f_{\mathcal{N}(0;1)}(x)}{f_{\mathcal{N}(1;1)}(x)} \underset{H_1}{\overset{H_0}{\gtrless}} \frac{p}{1-p}$$

$$\frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}{\frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}}} \underset{H_1}{\overset{H_0}{\gtrless}} \frac{p}{1-p}$$

$$e^{\frac{-x^2 + (x-1)^2}{2}} \underset{H_1}{\overset{H_0}{\gtrless}} \frac{p}{1-p}$$

$$\frac{-2x+1}{2} \underset{H_1}{\overset{H_0}{\gtrless}} \ln \frac{p}{1-p}$$

$$x \underset{H_1}{\overset{H_0}{\lessgtr}} \boxed{\frac{1}{2} - \ln\left(\frac{p}{1-p}\right) = \tau}$$

$$x \underset{H_1}{\overset{H_0}{\lessgtr}} \boxed{\frac{1}{2} - \ln\left(\frac{p}{1-p}\right) = \tau}$$

$$p_m = P(A_0 \mid H_1) \quad (\text{decide } A_0 \text{ given } H_1)$$

$$= p\,(x < \tau \mid H_1)$$

$$= P_{\mathcal{N}(1;1)}(x < \tau)$$

$$= P_{\mathcal{N}(0;1)}(x - 1 < \tau - 1)$$

$$= \phi(\tau - 1)$$

$$p_d = 1 - p_m = 1 - \phi(\tau - 1) = Q(\tau - 1)$$

$$\boxed{p_d = Q\left(\frac{1}{2} - \ln\left(\frac{p}{1-p}\right)\right) \quad \text{where } p = p(H_1)}$$

$\boxed{\text{Ex6}}$

$$\begin{cases} H_0 : X = Z \\ H_1 : X = \mu + Z \end{cases} \qquad \begin{matrix} Z \sim \mathcal{N}(0;1) \\ \mu \in \{0;1;2\} \end{matrix}$$

$$p_m = p(A_0 \mid H_1) = p(x < \tau \mid H_1) = P_{\mathcal{N}(\mu,1)}(x < \tau)$$

$$= P_{\mathcal{N}(0,1)}(x - \mu < \tau - \mu) = \boxed{\phi(\tau - \mu)}$$

$$p_d = 1 - p_m = 1 - \phi(\tau - \mu) = \boxed{Q(\tau - \mu)}$$

$$p_d = 1 - p_m = 1 - \phi(\tau - \mu) = \boxed{Q(\tau - \mu)}$$

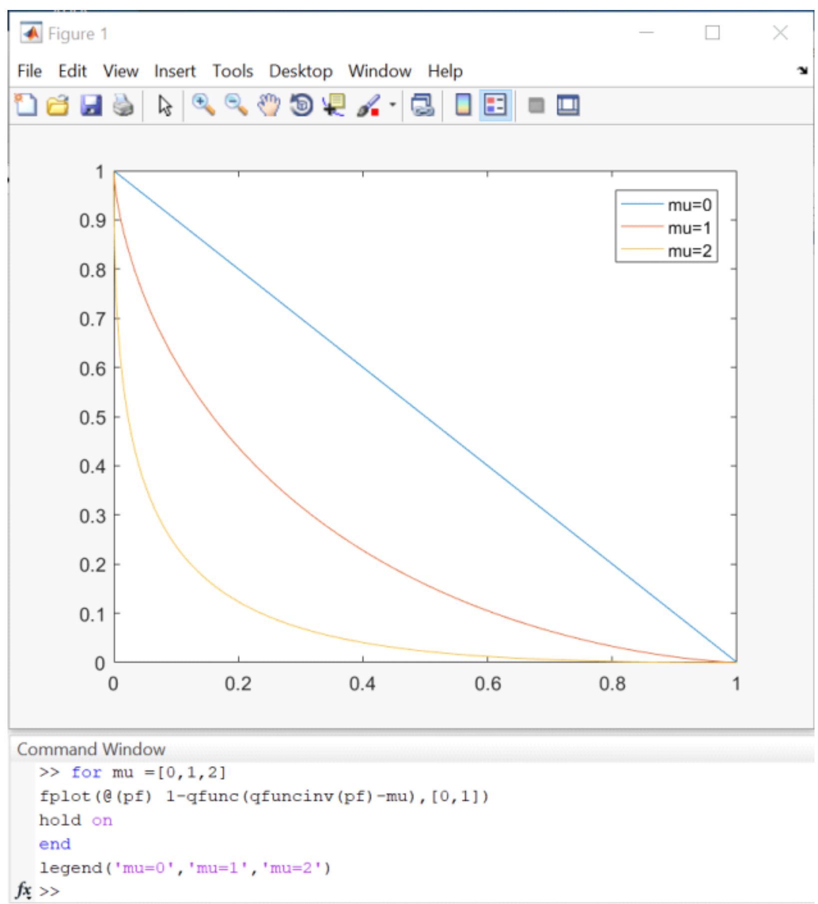
$$p_f = p(A_1 | H_0) = p(x > \tau | H_0) = P_{N(0,1)}(x > \tau)$$

$$\boxed{p_f = Q(\tau)}$$

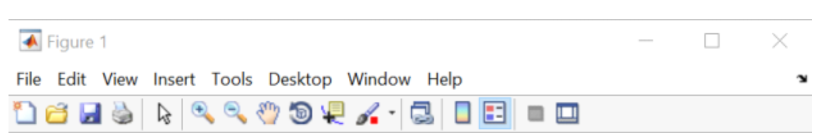
$$p_m = \phi(\tau - \mu) = \phi(Q^{-1}(p_f) - \mu)$$

$$\boxed{p_m = 1 - Q(Q^{-1}(p_f) - \mu)} \quad (\text{Roc})$$

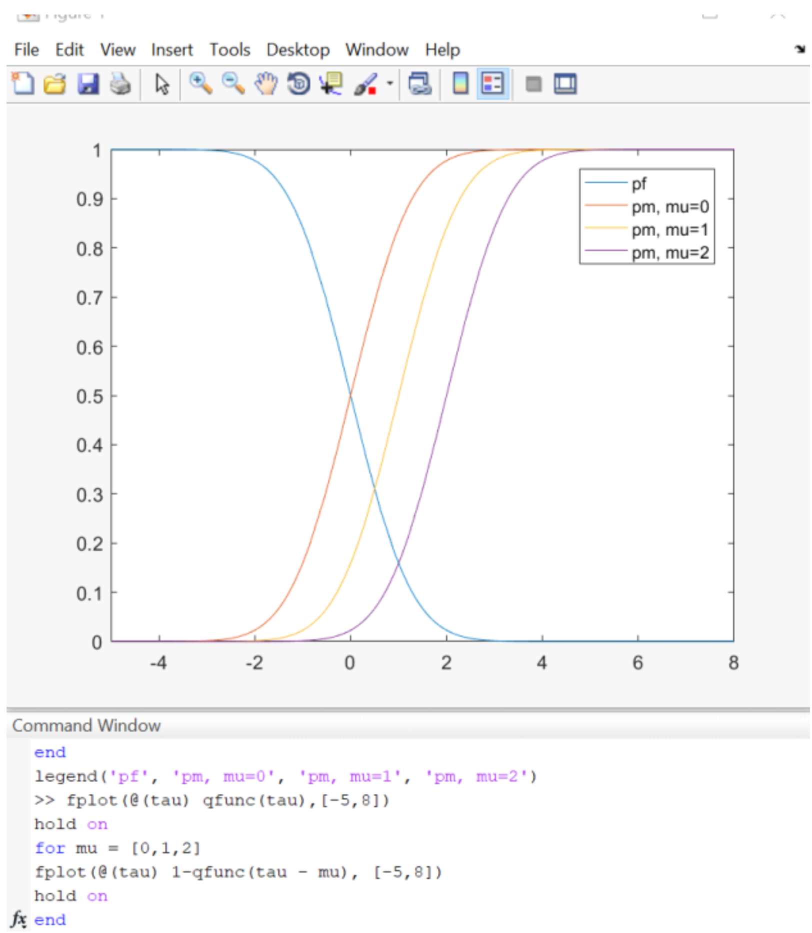
```
for mu =[0,1,2]
  fplot(@(pf) 1-qfunc(qfuncinv(pf)-mu),[0,1])
  hold on
end
legend('mu=0','mu=1','mu=2')
```



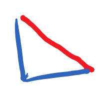

```
fplot(@(tau) qfunc(tau),[-5,8])
hold on
for mu = [0,1,2]
fplot(@(tau) 1-qfunc(tau - mu), [-5,8])
hold on
```

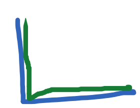


```
hold on
for mu = [0,1,2]
fplot(@(tau) 1-qfunc(tau - mu), [-5,8])
hold on
end
legend('pf', 'pm, mu=0', 'pm, mu=1', 'pm, mu=2')
```



When $\mu = 0$, there is no separability ($p_{X|H_0} = p_{X|H_1}$)

ROC is the worst 

When $\mu \nearrow$, $p_d \nearrow$ $p_m \searrow$ $p_f \searrow$, there is better separability, ROC tends to the perfect detector 

$\boxed{\text{Ex 7}}$

$$\begin{cases} H_0 : Y = W & X=0 \\ H_1 : Y = V + W & X=1 \end{cases} \qquad p(H_0) = p(H_1) = 0.5 \qquad V \perp W \sim \mathcal{E}(1)$$

Decision rule: $\dfrac{f_{Y|H_0}(x)}{\quad} \underset{<}{\overset{H_0}{>}} \dfrac{p(H_1)}{\quad}$

Decision Rule: $\dfrac{f_{Y|H_0}(x)}{f_{Y|H_1}(x)} \underset{H_1}{\overset{H_0}{\gtrless}} \dfrac{p(H_1)}{p(H_0)}$

$x \geq 0$ & $\dfrac{\cancel{e^{-x}}}{x\cancel{e^{-x}}} \underset{H_1}{\overset{H_0}{\gtrless}} \dfrac{0.5}{0.5}$

$$\boxed{x \geq 0 \ \& \quad x \underset{H_1}{\overset{H_0}{\lessgtr}} 1 = \tau}$$

Decision Rule minimizing perr

$$p_{err} = p_m \times p(H_1) + p_f \times p(H_0) = \frac{p_m + p_f}{2}$$

$$p_m = p(A_0|H_1) = p_{Y|H_1}(0 \leq x < \tau)$$

$$= \int_0^\tau x e^{-x}\,dx = \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx$$

$$= -\frac{1}{e} + \left[-e^{-x}\right]_0^1 = -\frac{1}{e} + 1 - \frac{1}{e} = 1 - \frac{2}{e} = \frac{e-2}{e}$$

$$p_f = p(A_1|H_0) = p_{Y|H_0}(x > \tau)$$

$$= \int_\tau^{+\infty} e^{-x}\,dx = \left[-e^{-x}\right]_1^{+\infty} = \frac{1}{e}$$

So $$\boxed{p_{err} = \frac{\frac{e-2}{e} + \frac{1}{e}}{2} = \frac{e-1}{2e}}$$

So $\boxed{p_{err} = \dfrac{\frac{\cdots}{e} + \frac{\cdots}{e}}{2} = \dfrac{e-1}{2e}}$