(Physical Review Letters, to be published)

Newton's cradle *vs* non-binary collisions

Ken Sekimoto[*1,*2]
[*1] *Matières et Systèmes Complexes, CNRS-UMR7057, Université Paris 7, France*
[*2] *Gulliver, CNRS-UMR7083, ESPCI, Paris, France*

(Dated: Received on December 23, 2009; accepted on March 3, 2010)

Newtons cradle is a classical example of a one-dimensional impact problem. In the early 1980s the naïve perception of its behavior was corrected: For example, the impact of a particle does *not* exactly cause the release of the farthest particle of the target particle train, if the target particles have been just in contact with their own neighbors. It is also known that the nave picture would be correct if the whole process consisted of purely binary collisions. Our systematic study of particle systems with truncated power-law repulsive force shows that the quasi binary collision is recovered in the limit of hard core repulsion, or a very large exponent. Contrastingly, a discontinuous step-like repulsive force mimicking a hard contact, or a very small exponent, leads to a completely different process: the impacting cluster and the targeted cluster act respectively as if they were non-deformable blocks.

PACS numbers: 45.50.Tn, 01.50.Wg , 05.45.-a, 02.30.Hq

Impacts of more than two bodies in one dimension arouse many interesting phenomena. A classical example is the Newton's cradle: Before the collision, n_1 identical particles are just in contact with their neighbor(s) and moving together with a velocity v_0, while other n_2 particles being also identical to the former ones are just in contact with their neighbor(s) and are staying at rest ahead of the aforementioned n_1 particles. We shall call this setup the $(n_1 \triangleright n_2)$-collision. Fig. 1 shows the case of $(1 \triangleright n-1)$-collision.

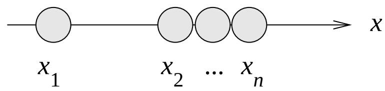

FIG. 1: Setup of $(1 \triangleright n-1)$-collision (see the text). Before the collision, all the latter $(n-1)$ particles are at rest and just in contact with its neighbor(s).

Except for the $(1 \triangleright 1)$-collision, i.e. binary collision, the conservation laws of energy and momentum are not sufficient to determine the final velocities of all particles [1]. Nevertheless, outside the physicists community, it has been widely believed that the $(n_1 \triangleright n_2)$-collision ends up with ejecting the farthest n_1 particles with the initial velocity v_0. In fact it is not exactly what occurs [1–4]. Using the high speed camera, Donahue *et al.* [5] observed directly that the $(1 \triangleright 2)$-collision of steel balls causes the bouncing back of the impacting particle. Experimentally, the simple mass and spring model has been shown to describe well the observations [5–7] if the springs obey the Hertz' force law [8], i.e. repulsive force with 3/2-th power of the positive "overlap" between the particles and zero otherwise. Theoretically, the case of Hertz type force as well as that of the truncated harmonic force (i.e. Hooke's law for the positive overlap and zero otherwise) have been studied in detail; the relation with the dispersion law was discussed [1, 2], and the propagation of solitary waves have been also studied both theoretically and experimentally [3, 6, 7, 9, 10]. Numerical and analytical approximation studies also explored the behaviors of large n_2 cases s[11, 12].

While these studies have elucidated the phenomenon of multi-body collisions, they focused rather on particular force laws, such as of Hertz's force or of truncated harmonic force; the systematic study about different force laws is still lacking to the author's knowledge (cf. [11]). Such study may shed light on these realistic cases from different point of view, and may also give an insight about the *contact between hard spheres*, which involves some non-determinism [13]. This non-determinism is related to the notion of hard sphere, a discontinuous infinite potential barrier at a critical distance requires a proper definition as a limiting procedure. One way would be to use a power-law repulsive potential and let its exponent to infinity ($\alpha \to \infty$, see below). Hereafter we shall call this limit the hard core limit or hard core repulsion. The other way would be to use a linear potential slope and let its gradient to infinity ($\alpha \to 0$, see below). We shall call this limit the step-like force. We will show that these alternatives are not equivalent with each other in the context of the contact between hard spheres and, therefore, in the study of $(2 \triangleright 3)$-collision, for example. While a simple picture of the network of quasi binary collisions for the Newton's cradle is compatible with the hard core repulsion [2, 14], the model with the step-like force leads to completely different and somehow complementary phenomenon.

We are, therefore, motivated to study systematically different force laws, $F_\alpha(\delta)$, between the two neighboring particles:

$$F_\alpha(\delta) = \begin{cases} a\,\delta^\alpha & (\delta \geq 0) \\ 0 & (\delta < 0) \end{cases}, \tag{1}$$

where a is a positive constant and the "overlap" δ is the distance approached with respect to the point of "just in contact" ($\delta = 0$). Depending of the value of α, the force represents, for example,

$$\begin{cases} \alpha \gg 1 & \text{(hard core repulsion)} \\ \alpha = \frac{3}{2} & \text{(Hertz' contact force)} \\ \alpha = 1 & \text{(truncated harmonic force)} \\ \alpha \ll 1 & \text{(step-like force)} \end{cases} . \quad (2)$$

The truncated harmonic case is analytically solvable and serves to check the numerical calculations. (We ignore the effect of the internal degrees of freedom of each particle [15, 16] or adhesive interaction among particles [17], with the light of fairly good reproduction of experimental results by the Hertz force [5–7].) As mentioned above the hard core limit corresponds to $\alpha \gg 1$. In the opposite limit, the step-like force ($\alpha \ll 1$), the repulsive force is discontinuously switched across the contact point ($\delta = 0$).

Our system is defined as follows. We denote by x_i and p_i ($i = 1, \ldots, n$) the position and momentum of i-th movable particle of a common mass M. We assume the spatial ordering, $x_1 \le x_2 \le \ldots \le x_n$. The "radius" of the particles, R, is also assumed to be common. Then the overlap δ between the neighboring particle pair $(i, i+1)$ is $\delta = 2R - x_i + x_{i+1}$. With a given force law (1) the Newton's equations of motion are identical to those studied before [2–7]:

$$\frac{dx_i}{dt} = \frac{p_i}{M}, \quad \frac{dp_i}{dt} = F_\alpha(2R - x_i + x_{i-1}) - F_\alpha(2R - x_{i+1} + x_i), \quad (3)$$

for $i = 1, \ldots, n$. We define $x_0 = -\infty$ and $x_{n+1} = +\infty$ with $n = n_1 + n_2$ in the setup of $(n_1 \triangleright n_2)$-collision (Fig. 1). In this setup a simple dimensional analysis tells that the values of the mass M, force parameter a, injecting velocity v_0, and the radius R can be set equal to unity without loosing generality [3]. (Especially R can be eliminated by using the displacements, $u_i(t) = x_i(t) - 2Ri$+const., with respect to the positions at the moment of initial impact. In other words, the ratio of the ultimate velocities v_i of the i-th perticle after the whole impact process to the impacting velocity v_0 is a function of the exponent α and the pair (n_1, n_2) alone. Hereafter, we therefore use the units, $M = a = v_0 = R = 1$.

Fig. 2 shows the $(1 \triangleright 2)$-collisions (left) and $(2 \triangleright 3)$-collisions (right), with the force exponents, $\alpha = 10$ (top), $3/2$ (middle) and $1/10$ (bottom), respectively. We integrated the equations (3) using Mathematica™. The errors in the total momentum (=1) and the total energy (1/2) are, respectively, of order of 10^{-12} and 10^{-6}. On the top row in Fig. 2, the force with $\alpha = 10$ mimics the hard core repulsion. The outcomes are what we expect from the naïve picture of the Newton's cradle. At the same time, it is clearly visible that the collision consists of a network of (quasi) binary collisions [2, 14]. Such simple explanation may have been given somewhere, but we did not find it. In this case the particles behave as if they had enough spaces between their neighbors although the target particles were just in contact at the moment of collision. Such somewhat paradoxical result is ascribed to the hard core repulsion, which rises steeply only at the overlap $\delta \sim 1$ but remains very soft for $0 \le \delta < 1$. In fact the results for $\alpha = 10$ are robust against the addition of small (true) gaps between the particles before the impact.

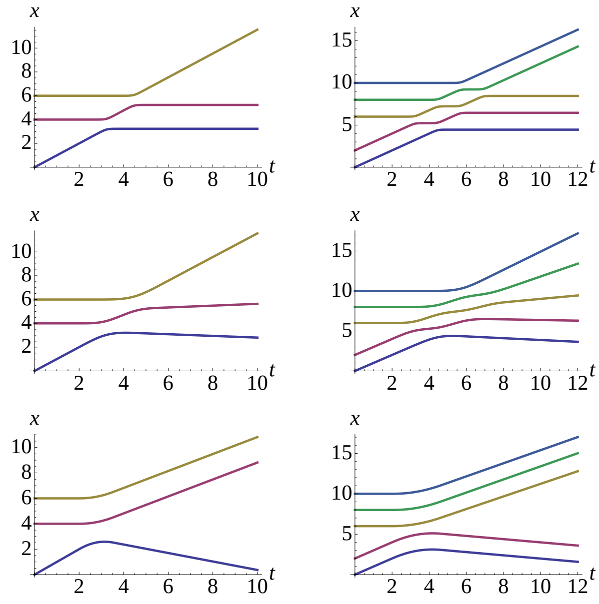

FIG. 2: (Color online) Time courses of the $(1 \triangleright 2)$-collisions (left column) and the $(2 \triangleright 3)$-collisions (right column) on the x vs t plane. The exponent α of the truncated repulsive force (1) is 10 (top row), 3/2 (middle row) and 1/10 (bottom row).

For the Hertz's repulsive interaction (middle row in Fig. 2), the impacting particles ($x_1(t)$ (left) or $x_1(t)$ and $x_2(t)$ (right)) undergo the bouncing back, as reported previously [3–7]. This is a generic result for any finite positive values of α (see below). For $\alpha = 1$ we can compare the numerical result with the analytic result of e.g. the $(1 \triangleright 2)$-collision. Mathematically, for any $(1 \triangleright n-1)$-collision with $n \le 3$, the ultimate momentum p_1 *must* be negative unless all the momentum is transmitted to the farthest particle, p_n. [**?**]

Finally the force with $\alpha = 1/10$ (bottom row in Fig. 2) mimics the step-like force; the force raises abruptly upon the overlapping, $\delta > 0$. It is a surprise that the groups of particles, impacting group on the one hand and target group on the other hand, keeps their identity and moves as if they were non-deformable blocks. Except for the interface between these two groups, the overlap between the neighboring particles is kept almost zero throughout the process. (For the $(1 \triangleright 2)$-collision, δ for the pair (x_2, x_3) is 4×10^{-4} at the maximum, while for the $(2 \triangleright 3)$-collision, δ for the pairs (x_1, x_2), (x_3, x_4) and (x_4, x_5) are up to 10^{-3}, 1.5×10^{-2} and 1.5×10^{-5}, respectively.) We will give later a physical argument for such non-deformable blocks in the limit of $\alpha \to 0$.

In order to see systematically the momentum redistribution through the $(1 \triangleright 2)$-collision, we plotted the ultimate momenta (p_1, p_2, p_3) on the plane of the conserved momentum, $p_1 + p_2 + p_3 = 1$. Fig. 3 views this plane perpendicularly, i.e., from the direction, $p_1 = p_2 = p_3$. On this plane the energy conservation, ${p_1}^2 + {p_2}^2 + {p_3}^2 = 1$, defines the circle that passes through the three vertices $(p_1, p_2, p_3) = (1, 0, 0)$ (=the initial condition: left bottom corner), $(0, 1, 0)$ (right bottom corner) and $(0, 0, 1)$ (top corner) of the regular triangle. For each values of α the ultimate momenta are represented as a thick dot. As α increases, the ultimate momenta approach the vertex $(0, 0, 1)$ (cf. top-left figure of Fig. 2). By contrast in the limit of small α the momenta looks to converge at the mid-point of the arc, $(-1/3, 2/3, 2/3)$ (cf. bottom-left figure of Fig. 2).

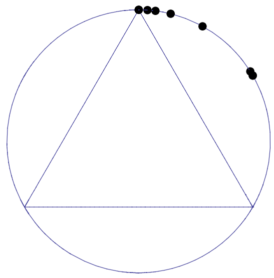

FIG. 3: Ultimate momenta of three particles (thick dots) with different values of the force exponent, $\alpha = 10, 5, 2, 3/2, 1, 1/2, 1/5$ and $1/10$ in the increasing order of the distances from the top vertex of the triangle. See the text as for the reading out of the momenta. The first two points for $\alpha = 10$ and 5 are not distinguishable in the figure, while the last two points ($\alpha = 1/5$ and $1/10$) are very close with each other and to the mid-point, $(-1/3, 2/3, 2/3)$.

How can the step-like force among particles ($\alpha \to 0$) lead to the behavior of non-deformable blocks ? A semi-quantitative explanation is as follows. For $\alpha \to 0$ the truncated force is the Heviside step-function, $F_{0+}(\delta) = \theta(\delta)$. Let us take a $(1 \triangleright n-1)$-collision, as example. As soon as the impacting particle (x_1) has an overlap $\delta > 0$ with the closest target particle (x_2), the force accelerates the latter particle, and leads to the overlap between this particle and the next particle x_3, and so on. Shortly after the first overlap, the second equation of (3) reads, therefore, $dp_1/dt = -1$, $dp_2/dt = \ldots = dp_{n-1}/dt = 0$ and $dp_n/dt = 1$. Since the propagation of the overlap takes little time for small $\alpha(>0)$, the overlap between the neighboring particles are all small. In this stage, the farthest particle (x_n) being accelerated should soon detach from the $(n-1)$-th particle. It is then the latter particle x_{n-1} that is accelerated, and detaches from x_{n-2}, and so on. The events of acceleration and gap creation will, therefore, back-propagate up to the pair (x_2, x_3). Now the story will restart again, and it will continue until the impacting particle x_1 definitively leaves its neighbor x_2. Throughout the process, all the neighboring particle pairs except for the interface pair (x_1, x_2) remain almost just in contact. Despite such pathological process [**?**], we can calculate the ultimate momenta of the non-deformable blocks, merely by the assuming the non-deformability, i.e. $\delta = 0$ for all neighboring pairs except at the interface of the clusters: After some time-coarse graining of the momenta, $p_2, \ldots, p_n$, which we denote with the over-bar, the equations of motion *during* the collision become,

$$\bar{p}_2 = \ldots = \bar{p}_n \equiv p', \tag{4}$$

and

$$\frac{dp_1}{dt} = -1, \qquad (n-1)\frac{dp'}{dt} = 1. \tag{5}$$

The problem is thus reduced to the binary collision between the masses 1 and $(n-1)$. The ultimate momenta are then given as $p_1 = -(n-2)/n$ and $p' = 2/n$ just by the laws of conservation. For the $(n_1 \triangleright n_2)$-collision, the same argument leads to $p_1 = \ldots = p_{n_1} = (n_1 - n_2)/(n_1 + n_2)$ and $p_{n_1+1} = \ldots = p_{n_1+n_2} = 2n_2/(n_1 + n_2)$. Our numerical results with $\alpha = 1/10$ approximately reproduces the above formula of ultimate momenta for $(1\triangleright2)$-, $(1\triangleright3)$-, $(1\triangleright4)$-, $(2\triangleright2)$- and $(2\triangleright3)$-collisions within 1% of deviations.

We should note that the system with step-like force is not robust against the insertion of gaps between the neighboring particle pairs. For $\alpha = 1/10$ the initial inter-particle gap of 0.1 within each cluster is sufficient to render all process as a network of binary collisions. For general α, a dimensionless number characterizing the importance of a gap ϵ for a $(1 \triangleright n-1)$-collision is the ratio between the injected kinetic energy ${p_0}^2/(2M)$ and the characteristic potential energy $a\epsilon^{\alpha+1}/(\alpha + 1)$. If a is very large, the injected momentum p_0 should be very large in order for the non-binary collisions come into play. This explains the observation [5] that the smallest gaps between the particles before the collision eliminated the rebounding of the impacting particle.

It is among future problems to study inhomogeneous systems. A typical example is the system with a hard wall, which transmits the momentum but not the energy. In [18] the authors studied experimentally the impact of n hard spheres in contact and moving at the same velocity v_0 against a wall at rest. They observed that, when n is large enough, the furthermost particles from the wall bounce successively whereas the last 5 particles nearest to the wall bounce in block. For very large n and very small α, the numerical calculation is very delicate and efficient schemes are under exploration [**?**]. Instead, we did preliminary studies of the $(2 \triangleright 2)$-collisions in the presence of a rigid wall just in contact with the farthest target particle (x_4). For $\alpha = 10$ the collisions ended up with bouncing back of the two injected particles just in contact, as expected by the picture of quasi binary collision. For $\alpha = 1/10$, however, all the four particles left the wall at comparable velocities (data not shown). The wall or inhomogeneity, therefore, prevented the target particles from behaving as non-deformable cluster.

In conclusion, our systematic study about different laws of inter-particle force clarified several things: The notions of contact between hard spheres became clearer, and we understood qualitatively the prevailing naïve notion of Newton's cradle in terms of a network of quasi binary collisions. The solitary wave and its propagation studied for the Hertz' force

[3, 6, 7, 9] can be regarded as an interpolation to $\alpha < \infty$ of the above mentioned quasi binary collisions, as suggested by [3]. The hitherto unexplored case of step-like force lead to a complementary feature of collisions, where the group of impacting particles and that of target particles behave, respectively, as non-deformable blocks. It will be noteworthy that, in continuum approximation, an estimation [11] suggested the divergence of the width of their mechanical pulse for $\alpha \to 0$ (cf.[19]) although the authors of [11] have not considered the case with $\alpha < 1$. Are interactions with $\alpha < 1$ realizable? The step-like force ($\alpha = 0$) is reminiscent of the interface energy of two-fluid interface [**?**]. Also a uniform long-range force (gravity, electrostatic force between plates, etc.) could be devised to work in a macroscopic setup. Unlike Hertz' force, the restoring force of rubber balloon obeys $\alpha = 1$ for spherical balloons (data not shown) and should obey $\alpha = 1/2$ for cylindrical balloons contacting side by side. Impact problem of soft materials may show different aspects from the hard ones. For example, simple dimensional analysis shows that, for $\alpha < 1$, the negligence of the rag time due to intra-particle elastic wave becomes a better approximation for small impact velocities while the opposite is the case for $\alpha > 1$. Even outside the pure scientific interests such as granular materials and soft matters, the redistribution of injected momentum may play important roles in both macroscopic and microscopic phenomena, e.g. in the martial arts [20], biomechanics [21], robotics [22] or composite materials [23].

The author deeply thanks Claude Gignoux for the enlightening question that ignited the present study, and also Alain Laverne, Élie Raphaël and Michael Schindler for the discussion and encouragements.

[1] F. Herrmann and P. Schmälzle, Am. J. Phys. **49**, 761 (1981).
[2] F. Herrmann and M. Seitz, Am. J. Phys. **50**, 977 (1982).
[3] E. J. Hinch and S. Saint-Jean, Proc. R. Soc. Lond. A **455**, 3201 (1999).
[4] S. Hutzler, G. Delaney, D. Weaire, and F. MacLeod, Am. J. Phys. **72**, 1508 (2004).
[5] C. M. Donahue, C. M. Hrenya, A. P. Zelinskaya, and K. J. Nakagawa, Phys. Fluids **20**, 113301 (2008).
[6] C. Coste, E. Falcon, and S. Fauve, Phys. Rev. E **56**, 6104 (1997).
[7] C. Coste and B. Gilles, Eur. Phys. J. B **7**, 155 (1999).
[8] K. L. Johnson, *Contact Mechanics* (Cambridge Univ. Press, 1987).
[9] V. F. Nesterenko, J. Appl. Mech. Tech. Phys. (USSR) **5**, 733 (1983).
[10] G. Friesecke and K. Matthies, Physica D **171**, 211 (2002).
[11] A. Rosas and K. Lindenberg, Phys. Rev. E **69**, 037601 (2004).
[12] U. Harbola, A. Rosas, M. Esposito, and K. Lindenberg, Phys. Rev. E **80**, 031303 (2009).
[13] V. Ceanga and Y. Hurmuzlu, J. Appl. Mech. **68**, 575 (2001).
[14] D. T. Wu, Physica A **315**, 194 (2002).
[15] S.I. Nagahiro and Y. Hayakawa, Phys. Rev. E **67**, 036609 (2003).
[16] Y. Tanaka, Phys. Rev. E **73**, 031403 (2006).
[17] N. V. Brilliantov, N. Albers, F. Spahn, and T. Poschel, Phys. Rev. E **76**, 051302 (2007).
[18] E. Falcon, C. Laroche, S. Fauve, and C. Coste, Eur. Phys. J. B **5**, 111 (1998).
[19] V. Nesterenko, in *MRS Symposium Proceedings 627*, edited by M. H. E. S. Sen (2001), p. BB3.1.
[20] M. Sprague, *Fighting Science: the laws of physics for martial artists* (Turtle Press, 2002).
[21] M. Alam, M. Wahab, and C. Jenkins, Mechanics of Materials **39**, 145 (2007).
[22] T. Umedachi, T. Kitamura, K. Takeda, T. Nakagaki, R. Kobayashi, and A. Ishiguro, in *Distributed Autonomous Robotic Systems 8* (Springer Berlin Heidelberg, 2009), pp. 193–202.
[23] S. Abrate, Composite Structures **51**, 129 (2001).
[24] The momentum constraint, $\sum_{i=1}^{n} p_i = 1$ and the conditions $p_i \geq 0$ for all i defines a $(n-1)$-dimensional simplex. With this simplex, $(n-1)$-sphere defined by the energy constraint, $\sum_{i=1}^{n} {p_i}^2 = 1$, shears only the corner points of the former.
[25] For small but finite values of α, smoothed trajectories replace the pathological process.
[26] Work under way by 'Physical Numerics Collaboration'.
[27] M. Schindler, private communication.