Introduction to Computer Graphics

– Modeling (1) –

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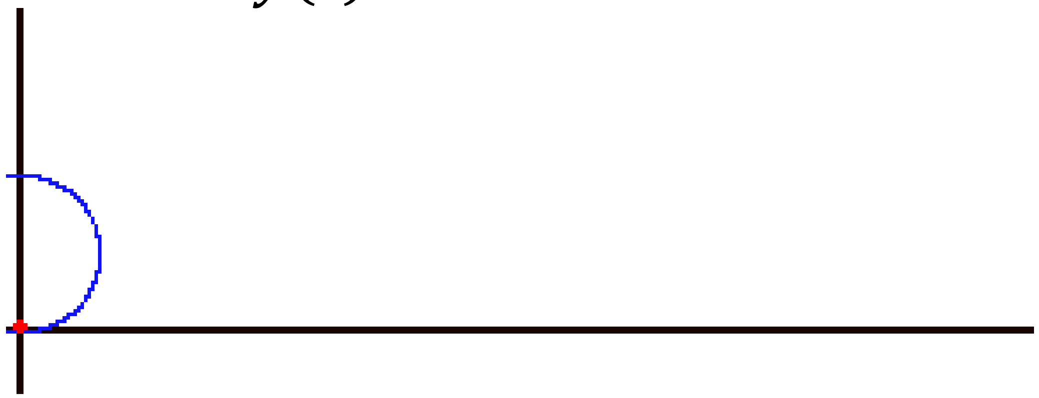
Parametric curves

- X & Y coordinates definted by parameter t
  - Example: Cycloid

$$x(t) = t - \sin t$$
$$y(t) = 1 - \cos t$$



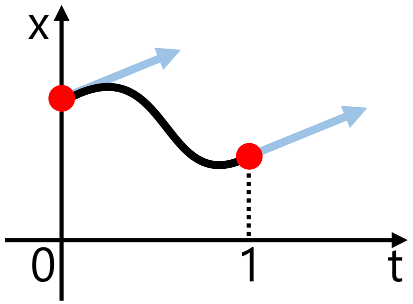
- Tangent (aka. derivative, gradient) vector: $\left(x'(t), y'(t)\right)$

- Polynomial curve: $x(t) = \sum_i a_i t^i$

Cubic Hermite curves

- Cubic polynomial curve interpolating derivative constraints at both ends (Hermite interpolation)

$$\begin{aligned} x(0) &= x_0 \\ x(1) &= x_1 \\ x'(0) &= x'_0 \\ x'(1) &= x'_1 \end{aligned}$$



- 4 constraints ➔ 4 DoF needed ➔ cubic
  - $x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$
  - $x'(t) = a_1 + 2a_2 t + 3a_3 t^2$

$$\begin{aligned} x(0) &= a_0 && = x_0 \\ x(1) &= a_0 + a_1 + a_2 + a_3 && = x_1 \\ x'(0) &= a_1 && = x'_0 \\ x'(1) &= a_1 + 2\,a_2 + 3\,a_3 && = x'_1 \end{aligned}$$

➔

$$\begin{aligned} a_0 &= x_0 \\ a_1 &= x'_0 \\ a_2 &= -3\,x_0 + 3\,x_1 - 2\,x'_0 - x'_1 \\ a_3 &= 2\,x_0 - 2\,x_1 + x'_0 + x'_1 \end{aligned}$$

- Coeffs determined by substituting constrained values & derivatives

Bezier curves

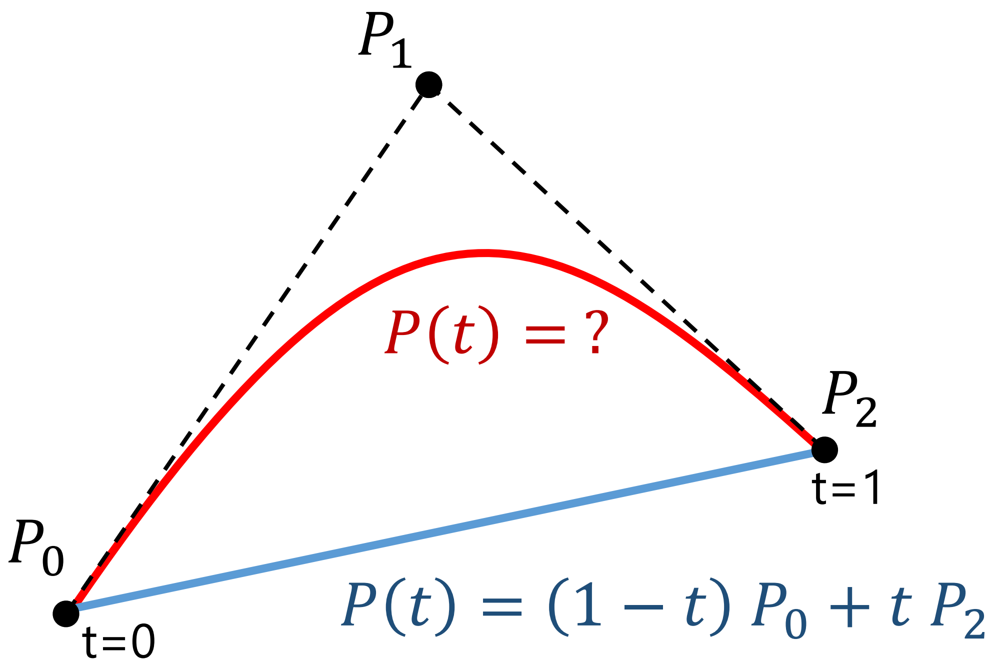
- Input: 3 control points P_0, P_1, P_2
  - Coordinates of points in arbitrary domain (2D, 3D, …)

- Output: Curve $P(t)$ satisfying

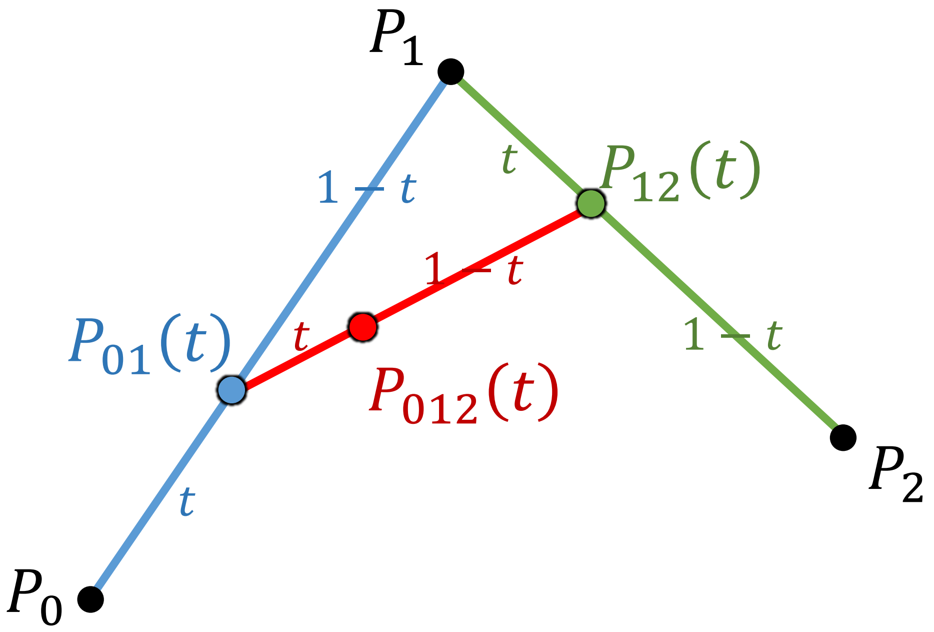
  $P(0) = P_0$

  $P(1) = P_2$

  while being "pulled" by P_1



Bezier curves



- $P_{01}(t) = (1-t)P_0 + t\,P_1$
- $P_{12}(t) = (1-t)P_1 + t\,P_2$
  - $P_{01}(0) = P_0$
  - $P_{12}(1) = P_2$

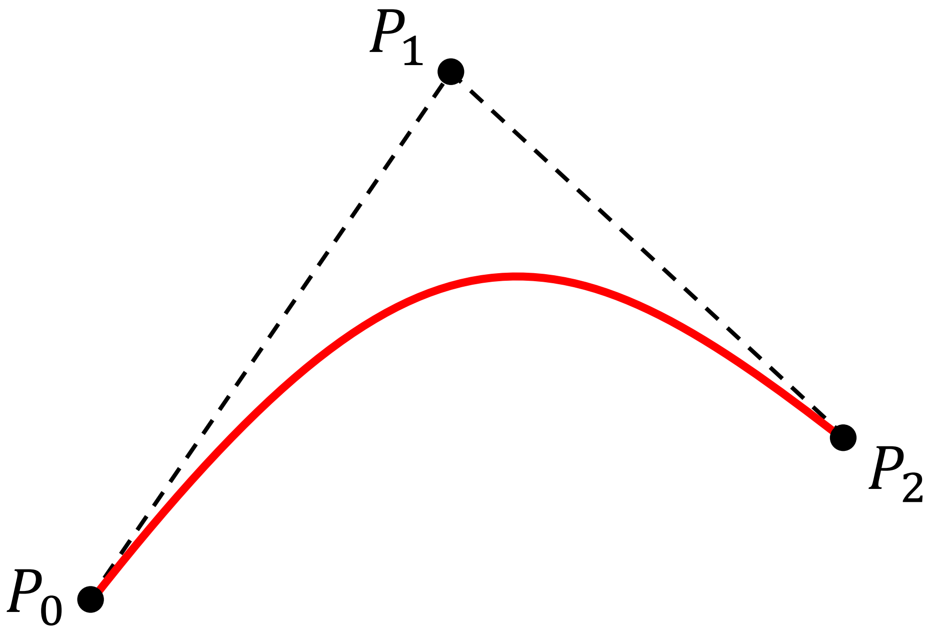
- Idea: "Interpolate the interpolation"
  As t changes $0 \to 1$, smoothly transition from P_{01} to P_{12}

- $P_{012}(t) = (1-t)P_{01}(t) + t\,P_{12}(t)$

$$= (1-t)\{(1-t)P_0 + t\,P_1\} + t\,\{(1-t)P_1 + t\,P_2\}$$

$$= \underbrace{(1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2}_{\text{Quadratic Bezier curve}}$$

Bezier curves



- $P_{01}(t) = (1-t)P_0 + t\,P_1$
- $P_{12}(t) = (1-t)P_1 + t\,P_2$
  - $P_{01}(0) = P_0$
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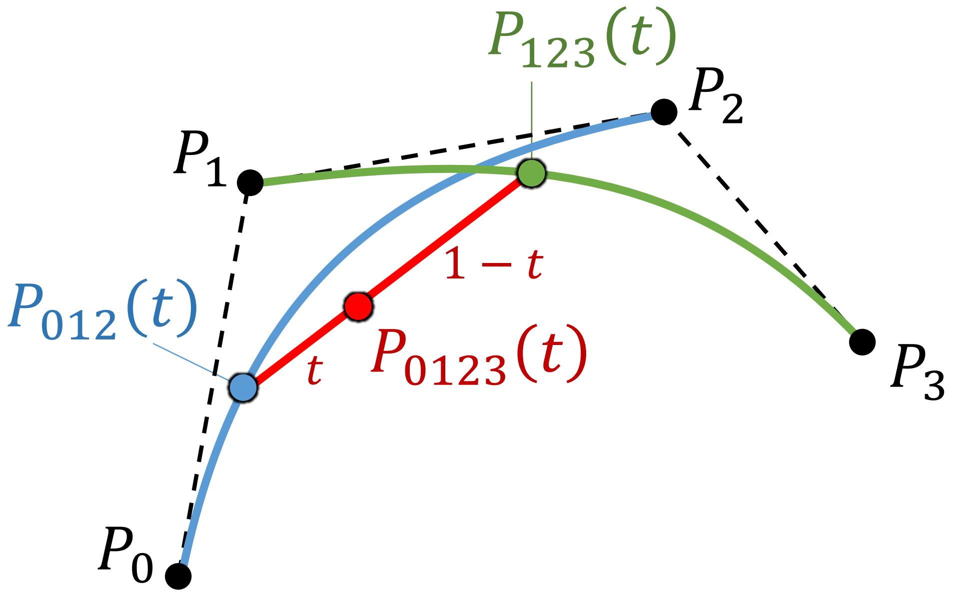
Cubic Bezier curve

- Exact same idea applied to 4 points $P_0, P_1, P_2\ P_3$:
  - As t changes $0 \rightarrow 1$, transition from P_{012} to P_{123}



- $P_{0123}(t) = (1-t)P_{012}(t) + t\ P_{123}(t)$

$$= (1-t)\{(1-t)^2P_0 + 2t(1-t)P_1 + t^2P_2\} + t\ \{(1-t)^2P_1 + 2t(1-t)P_2 + t^2P_3\}$$

$$= \underbrace{(1-t)^3P_0 + 3t(1-t)^2P_1 + 3t^2(1-t)P_2 + t^3P_3}_{\text{Cubic Bezier curve}}$$

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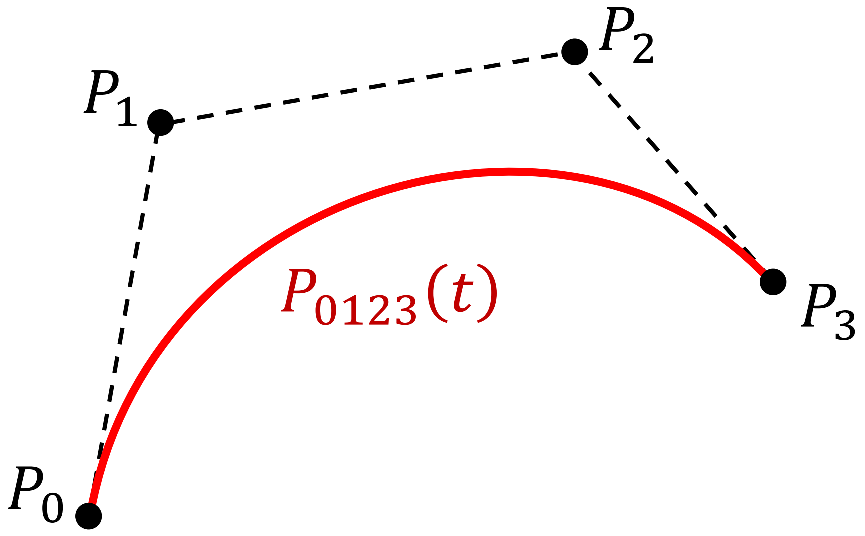

- $P_{0123}(t) = (1-t)P_{012}(t) + t\,P_{123}(t)$

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$$= \underbrace{(1-t)^3P_0 + 3t(1-t)^2P_1 + 3t^2(1-t)P_2 + t^3P_3}_{\text{Cubic Bezier curve}}$$

- Can easily control tangent at endpoints ➔ ubiquitously used in CG

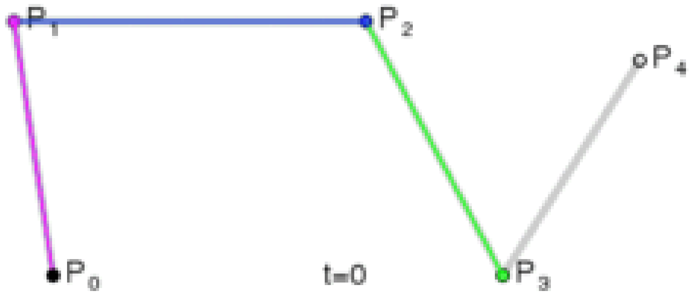
n-th order Bezier curve

- Input: n+1 control points $P_0, \cdots, P_n$

$$P(t) = \sum_{i=0}^{n} \underbrace{{}_n\mathrm{C}_i \, t^i \, (1-t)^{n-i}}_{b_i^n(t)} P_i$$

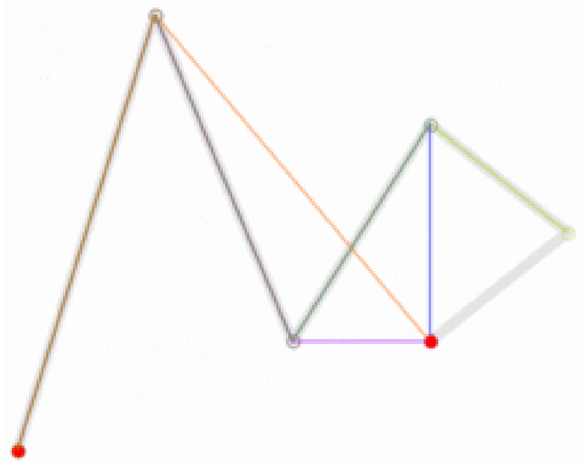
Bernstein basis function

Quartic (4th)



$$\begin{aligned}
&(1-t)^4 P_0 &+\\
&4t(1-t)^3 P_1 &+\\
&6t^2(1-t)^2 P_2 &+\\
&4t^3(1-t) P_3 &+\\
&t^4 P_4
\end{aligned}$$

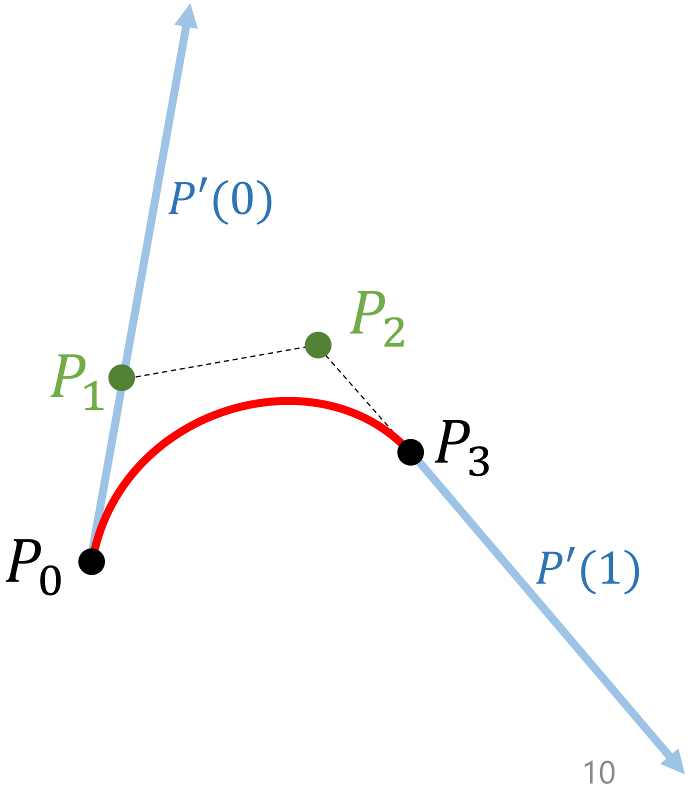
Quintic (5th)



$$\begin{aligned}
&(1-t)^5 P_0 &+\\
&5t(1-t)^4 P_1 &+\\
&10t^2(1-t)^3 P_2 &+\\
&10t^3(1-t)^2 P_3 &+\\
&5t^4(1-t) P_4 &+\\
&t^5 P_5
\end{aligned}$$

Cubic Bezier curves & cubic Hermite curves

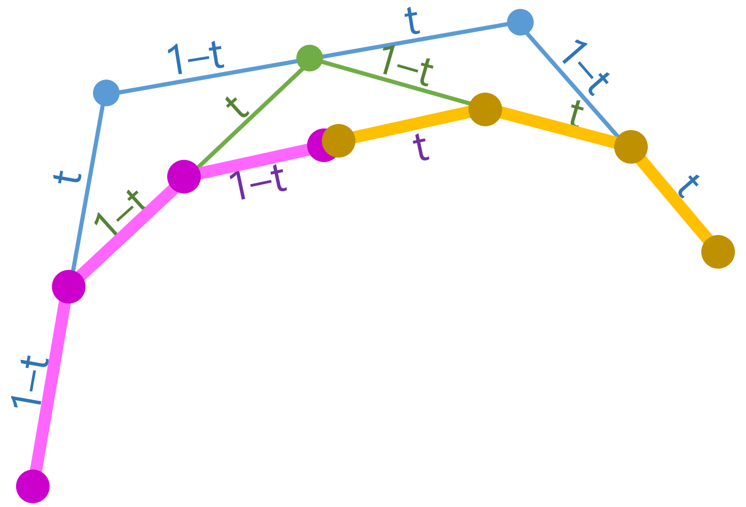
- Cubic Bezier curve & its derivative:
  - $P(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t)P_2 + t^3 P_3$
  - $P'(t) = -3(1-t)^2 P_0 + 3\{(1-t)^2 - 2t(1-t)\}P_1 + 3\{2t(1-t) - t^2\}P_2 + 3t^2 P_3$
- Derivatives at endpoints:
  - $P'(0) = -3P_0 + 3P_1 \quad \Rightarrow \quad P_1 = P_0 + \frac{1}{3}P'(0)$
  - $P'(1) = -3P_2 + 3P_3 \quad \Rightarrow \quad P_2 = P_3 - \frac{1}{3}P'(1)$
- Different ways of looking at cubic curves



Evaluating Bezier curves

- Method 1: Direct evaluation of polynomials
  - Simple & fast☺, could be numerically unstable☹

- Method 2: de Casteljau's algorithm
  - Directly after the recursive definition of Bezier curves
  - More computation steps☹, numerically stable☺
  - Also useful for splitting Bezier curves

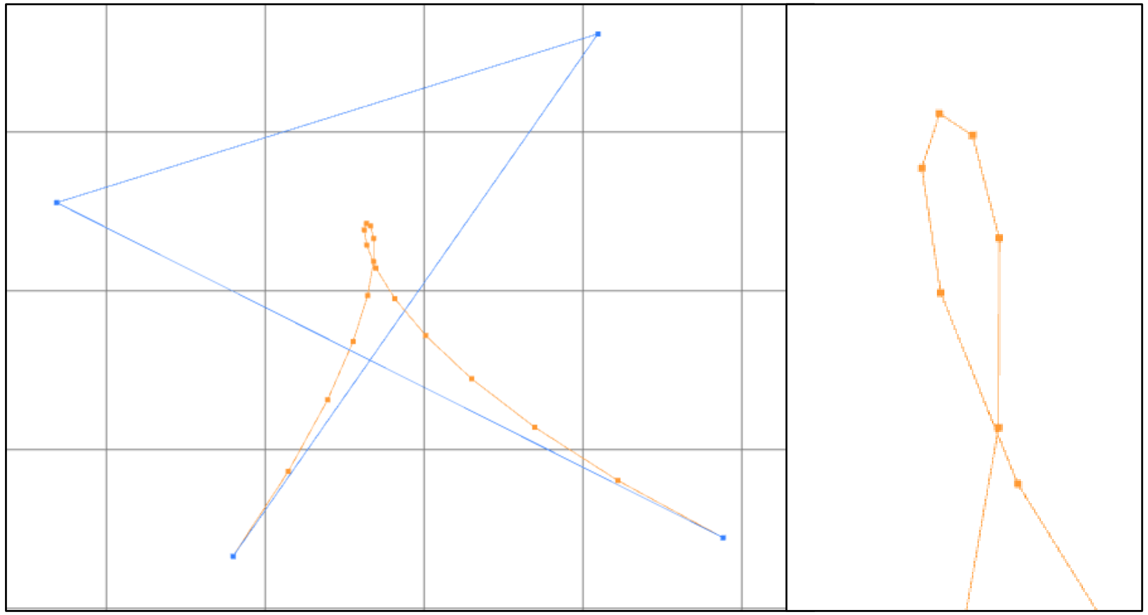


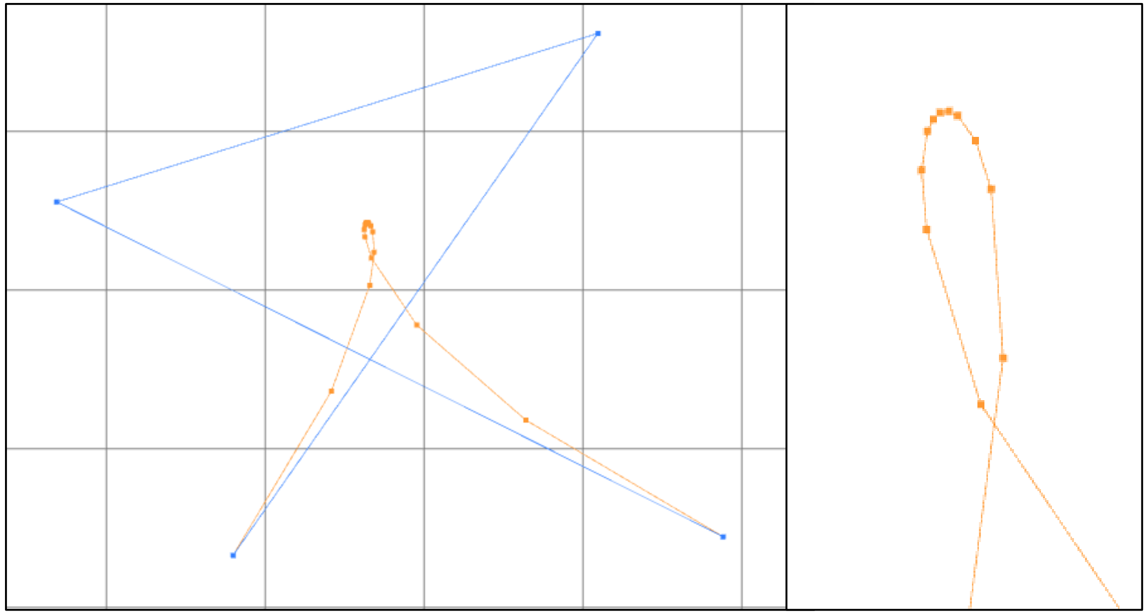
Drawing Bezier curves

- In the end, everything is drawn as polyline
  - Main question: How to sample paramter t?

- Method 1: Uniform sampling
  - Simple
  - Potentially insufficient sampling density

- Method 2: Adaptive sampling
  - If control points deviate too much from straight line, split by de Casteljau's algorithm





Further control: Rational Bezier curve

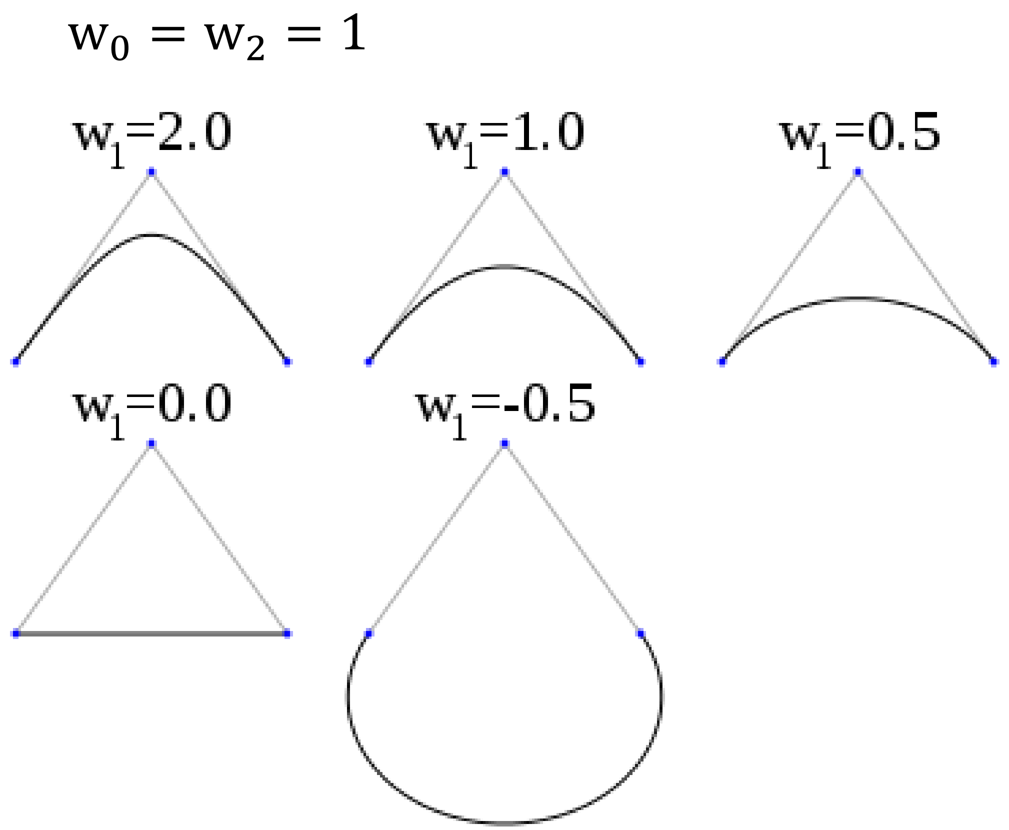
- Another view on Bezier curve:
  "Weighted average" of control points
  - $P_{012}(t) = (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 \quad P_2$
    $= \lambda_0(t) \quad P_0 + \lambda_1(t) \quad P_1 + \lambda_2(t)P_2$
  - Important property: $\lambda_0(t) + \lambda_1(t) + \lambda_2(t) = 1 \quad \forall t$

- Multiply each $\lambda_i(t)$ by arbitrary coeff w_i:

$$\xi_i(t) = w_i \, \lambda_i(t)$$

- Normalize to obtain new weights:

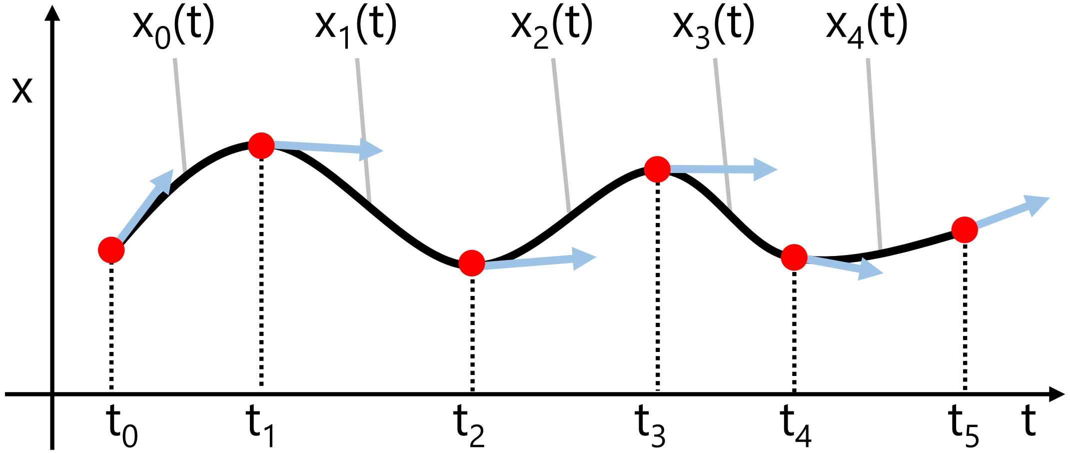
$$\lambda_i'(t) = \frac{\xi_i(t)}{\sum_j \xi_j(t)}$$

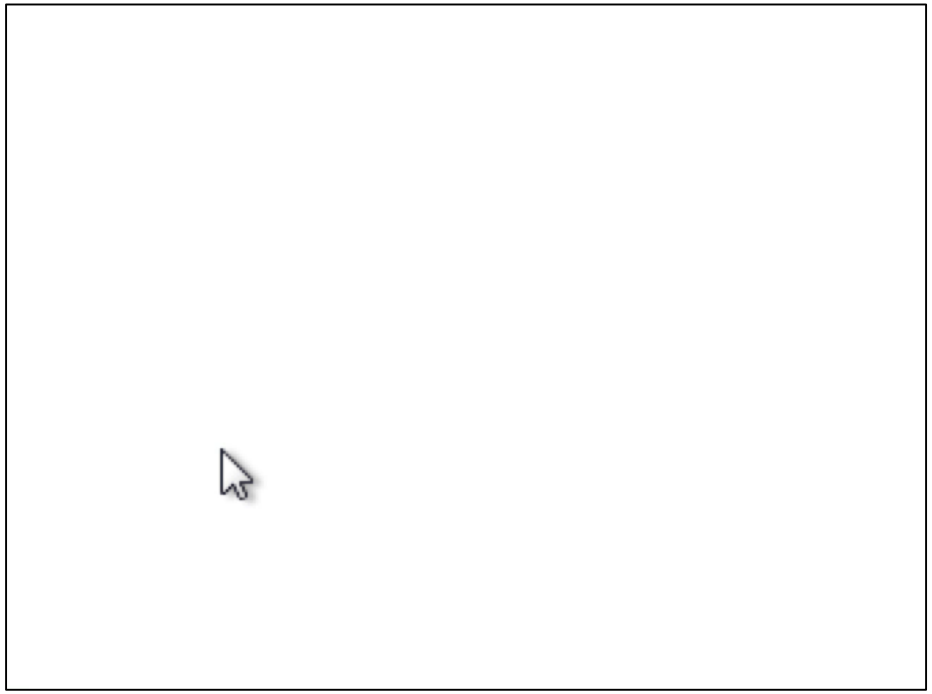


Non-polynomial curve ➔ can represent arcs etc.

Cubic splines

- Series of connected cubic curves
  - Piecewise-polynomial
  - Share value & derivative at every transition of intervals (C^1 continuity)
- Parameter range can be other than $[0, 1]$
  - Assumption: $t_k < t_{k+1}$
- Given values as only input,
  we want to automatically set derivatives

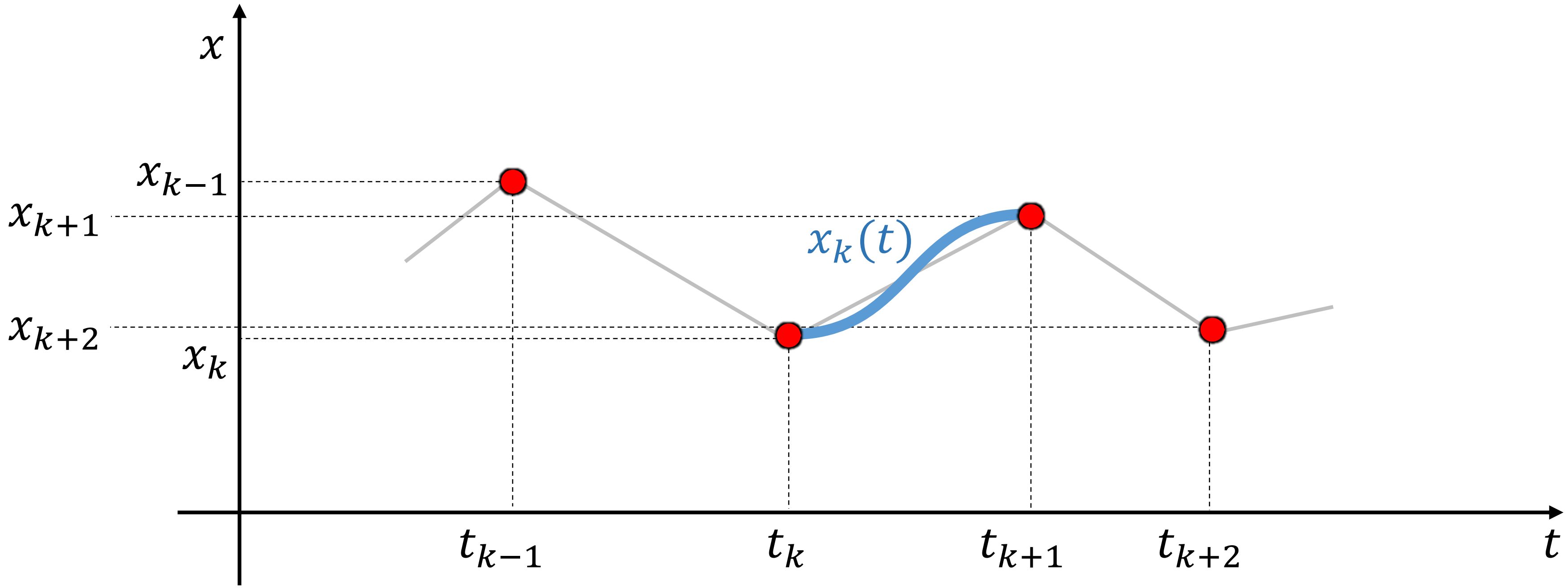




Curve tool in PowerPoint

Cubic Catmull-Rom spline

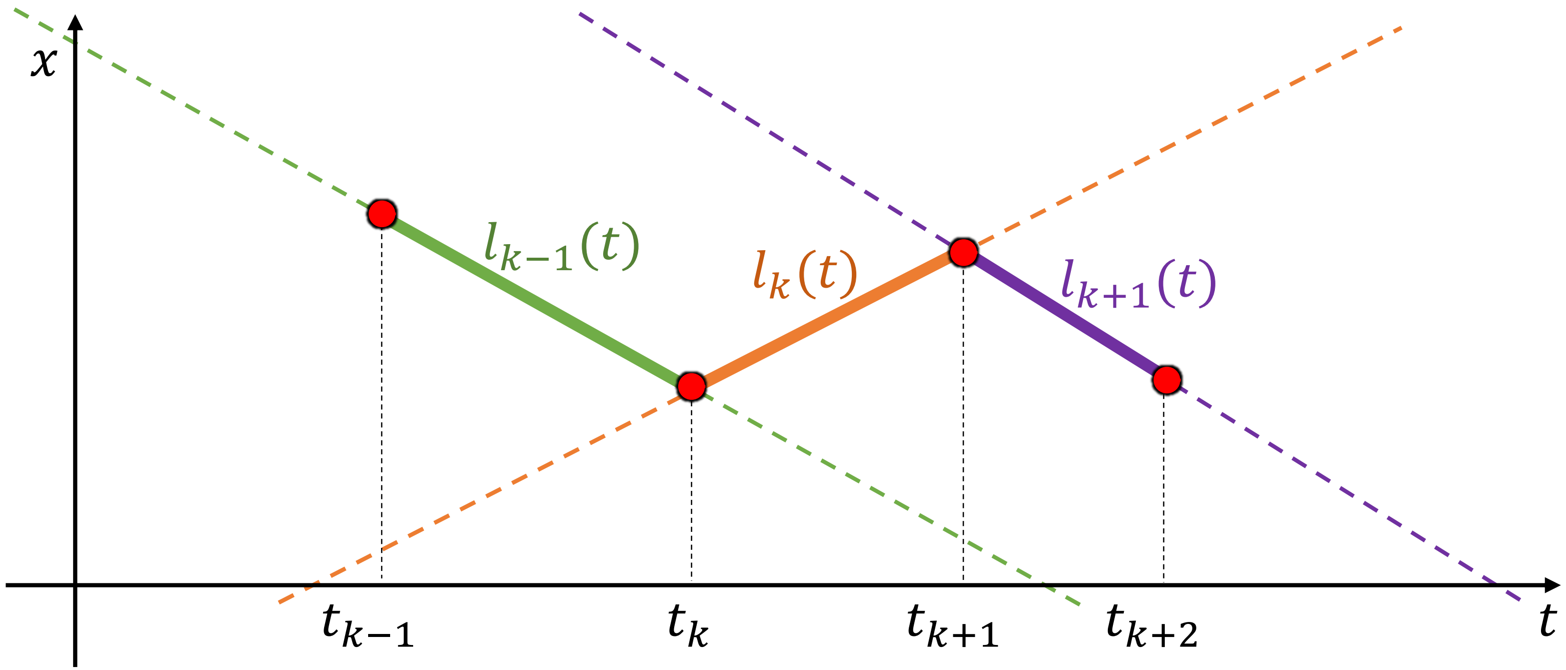
- Cubic function $x_k(t)$ for range $t_k \le t \le t_{k+1}$ is defined by adjacent constrained values $x_{k-1}, x_k, x_{k+1}, x_{k+2}$



Cubic Catmull-Rom spline: Step 1

- As $t_k \to t_{k+1}$, interpolate such that $x_k \to x_{k+1}$ ➔ Line

$$l_k(t) = \left(1 - \frac{t - t_k}{t_{k+1} - t_k}\right) x_k + \frac{t - t_k}{t_{k+1} - t_k} x_{k+1}$$

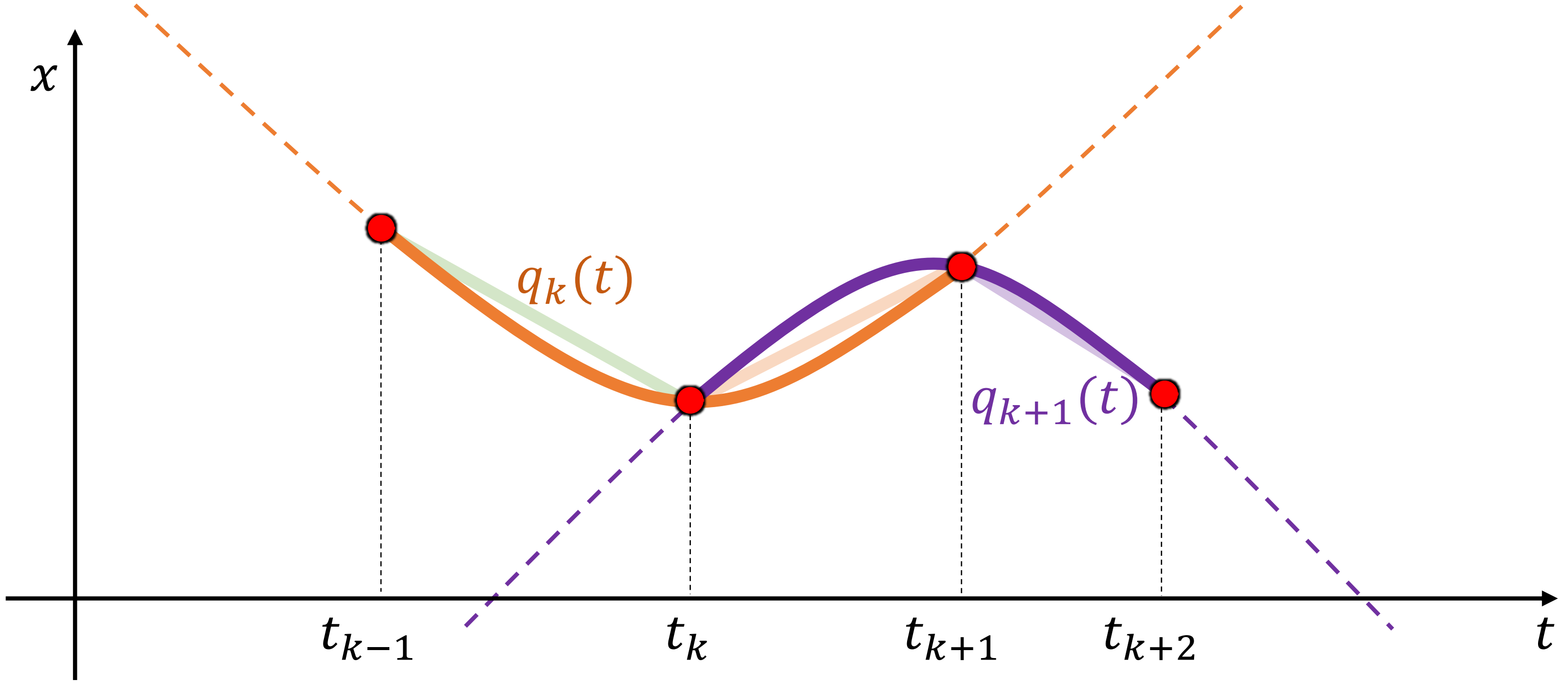


Cubic Catmull-Rom spline: Step 2

- As $t_{k-1} \rightarrow t_{k+1}$, interpolate such that $l_{k-1} \rightarrow l_k$ ➔ Quadratic curve

$$q_k(t) = \left(1 - \frac{t - t_{k-1}}{t_{k+1} - t_{k-1}}\right) l_{k-1}(t) + \frac{t - t_{k-1}}{t_{k+1} - t_{k-1}} l_k(t)$$

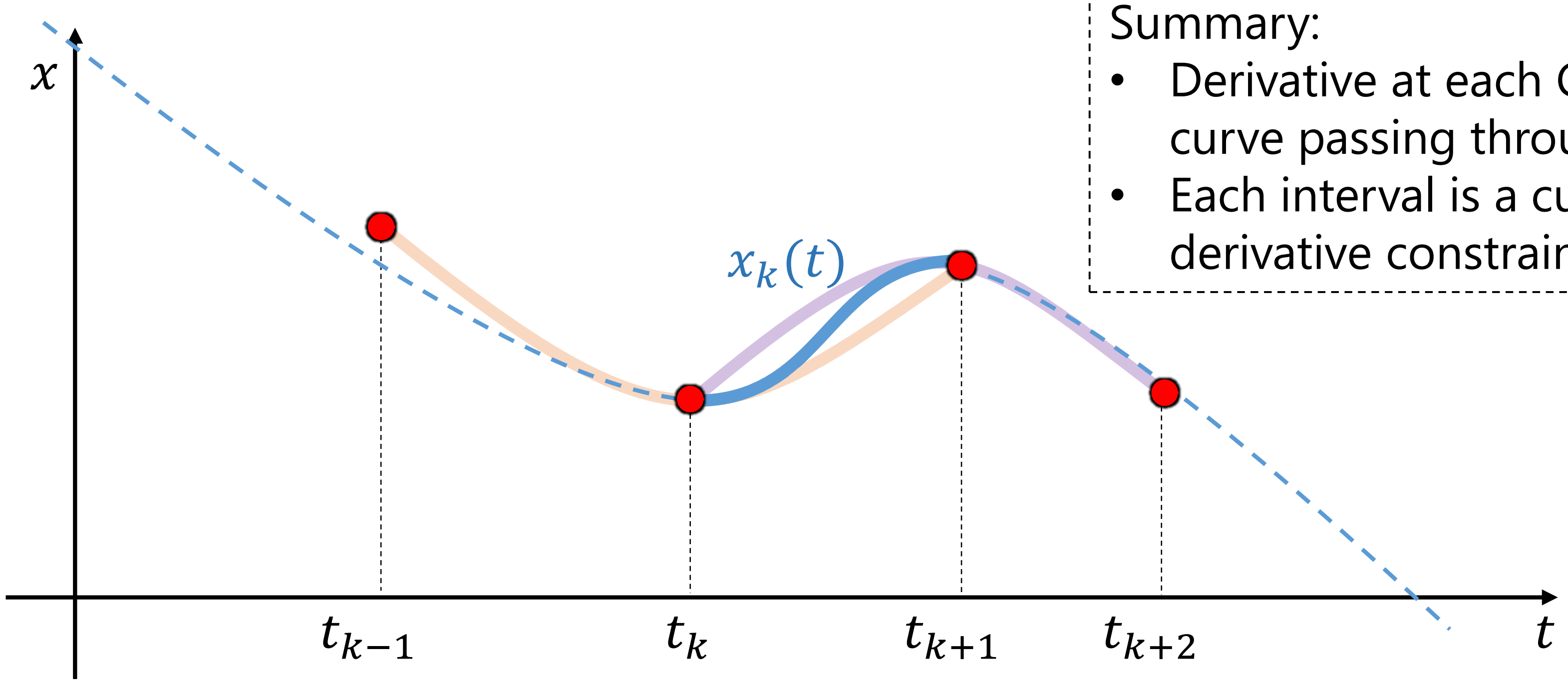
  - Passes through 3 points $(t_{k-1}, x_{k-1}), (t_k, x_k), (t_{k+1}, x_{k+1})$



Cubic Catmull-Rom spline: Step 3

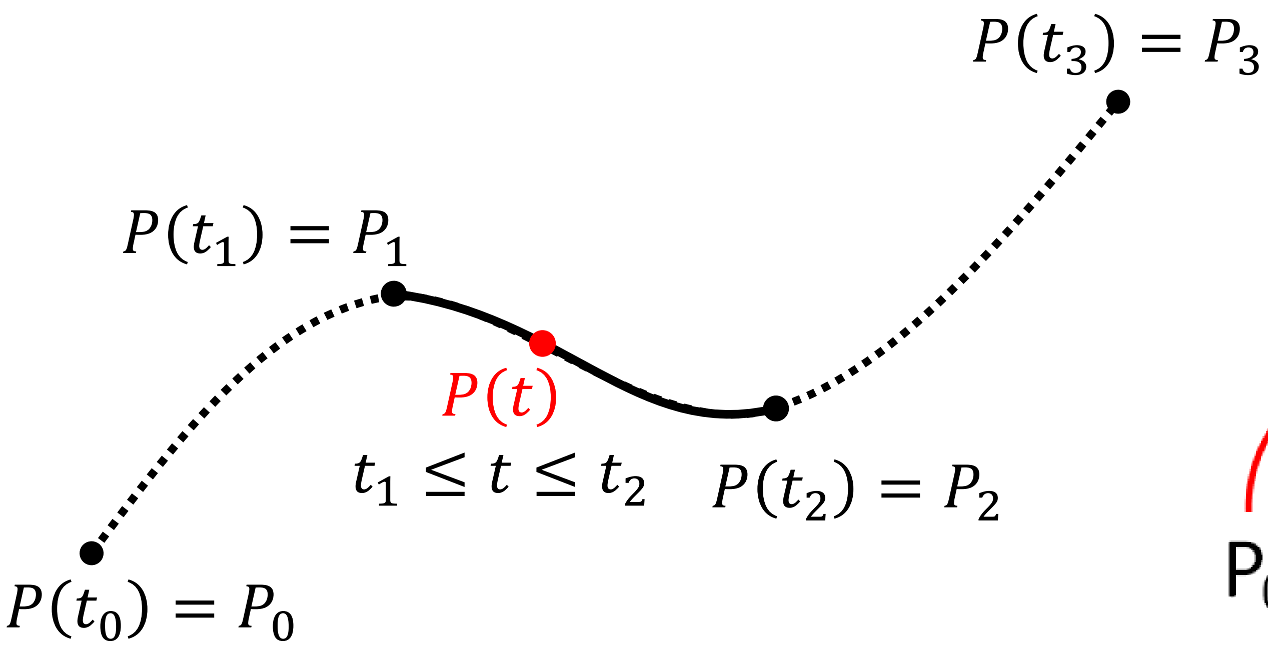
- As $t_k \rightarrow t_{k+1}$, interpolate such that $q_k \rightarrow q_{k+1}$ ➔ Cubic curve

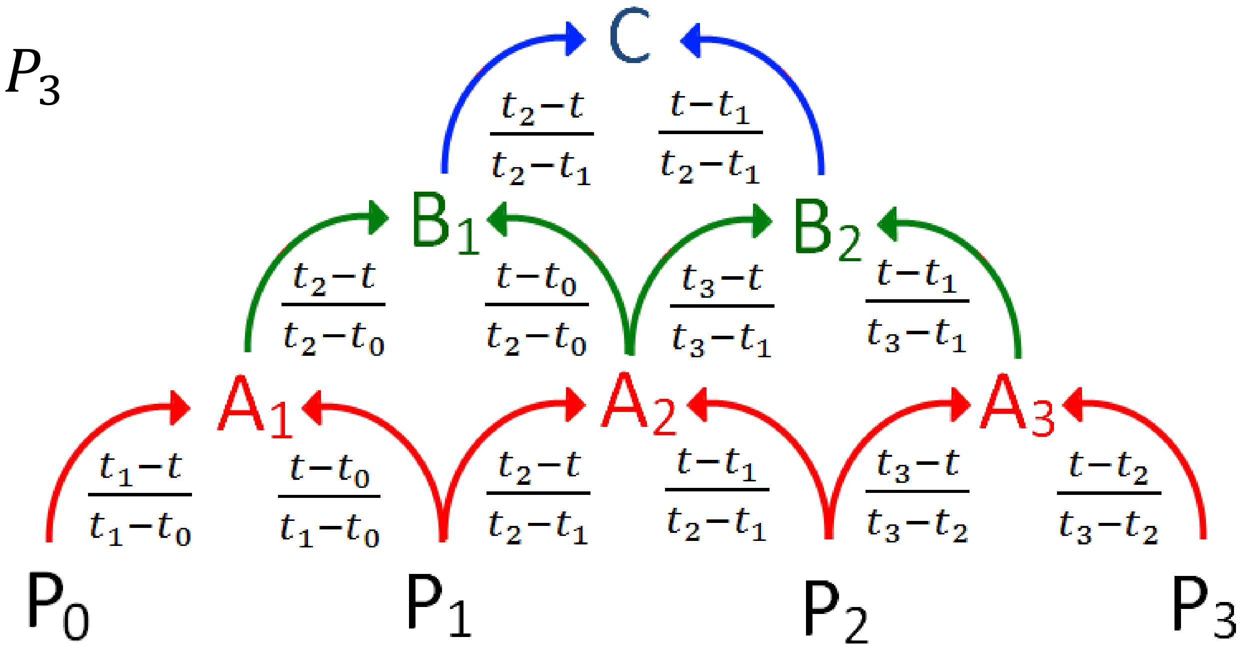
$$x_k(t) = \left(1 - \frac{t - t_k}{t_{k+1} - t_k}\right) q_k(t) + \frac{t - t_k}{t_{k+1} - t_k} q_{k+1}(t)$$



Summary:
- Derivative at each CP is defined by a quadratic curve passing through its adjacent CPs
- Each interval is a cubic curve satisfying derivative constraints at both ends

Evaluating cubic Catmull-Rom spline

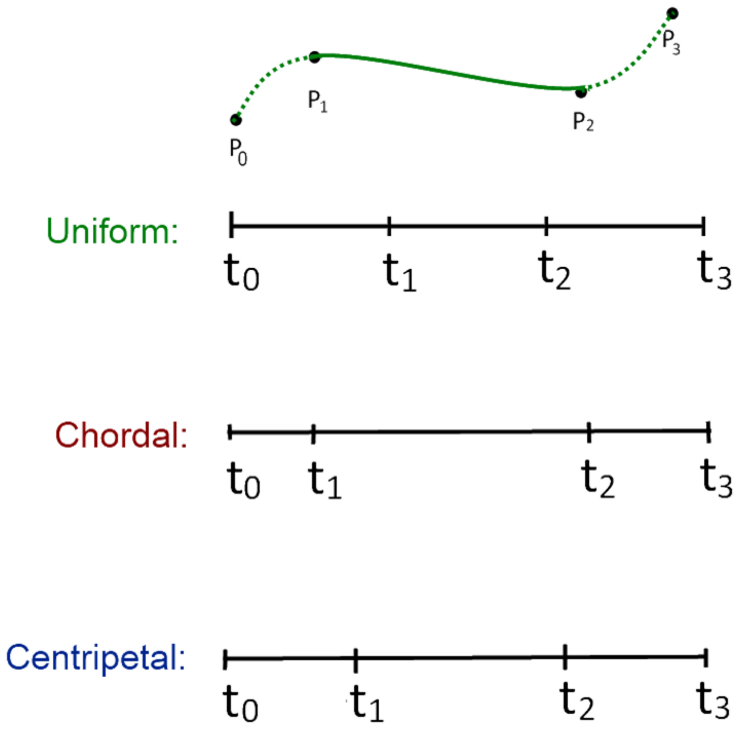


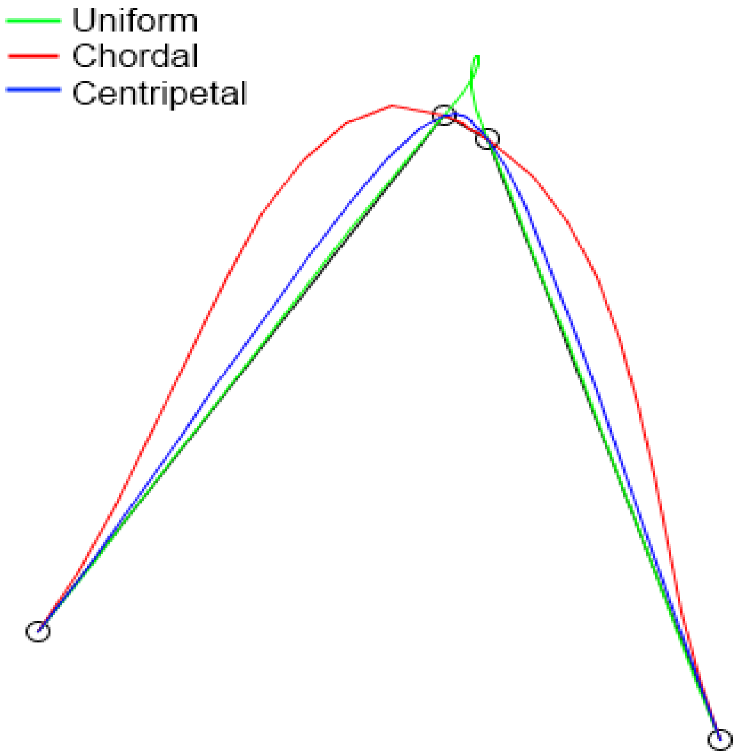


A recursive evaluation algorithm for a class of Catmull-Rom splines [Barry,Colgman,SIGGRAPH88]

Ways of setting parameter values t_k (aka. knot sequence)

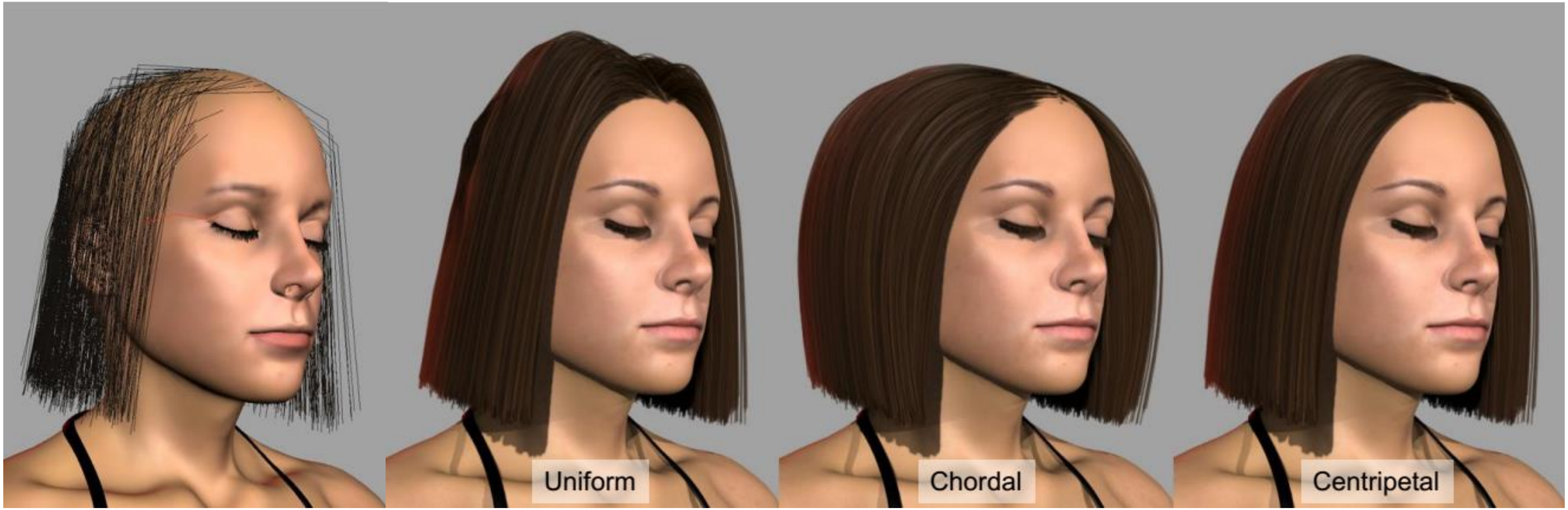
- $t_0 = 0$
- Uniform
  $$t_k = t_{k-1} + 1$$
- Chordal
  $$t_k = t_{k-1} + |P_{k-1} - P_k|$$
- Centripetal
  $$t_k = t_{k-1} + \sqrt{|P_{k-1} - P_k|}$$





Parameterization of Catmull-Rom Curves [Yuksel,Schaefer,Keyser,CAD11]

Application of cubic Catmull-Rom spline: Hair modeling



Parameterization of Catmull-Rom Curves [Yuksel,Schaefer,Keyser,CAD11]

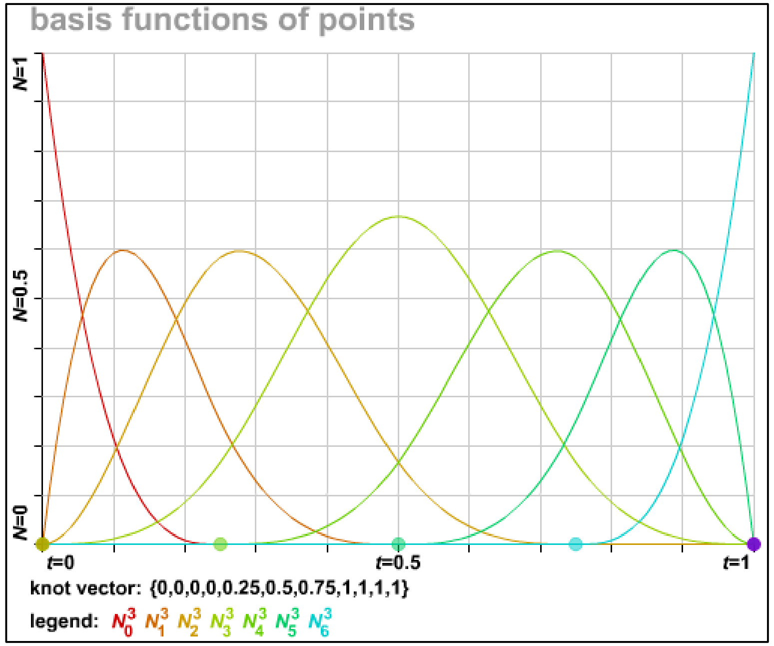
B-spline & NURBS

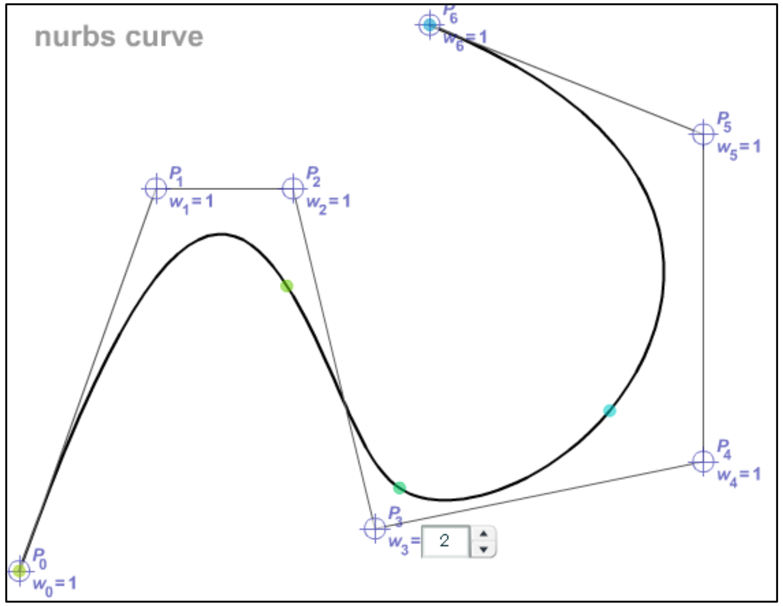
- B (**B**asis) Spline:
  - Another way of defining polynomial spline
  - Represent curve as sum of basis functions
  - Cubic basis is mostly used in practice

- **N**on-**U**niform **R**ational **B**-**S**pline
  - Non-uniform spacing of knots (t_k)
  - Rational: arbitrary weights for CPs

- Deeply related to subdivision surfaces
  ➔ Next lecture

- Cool Flash demo:
  http://geometrie.foretnik.net/files/NURBS-en.swf

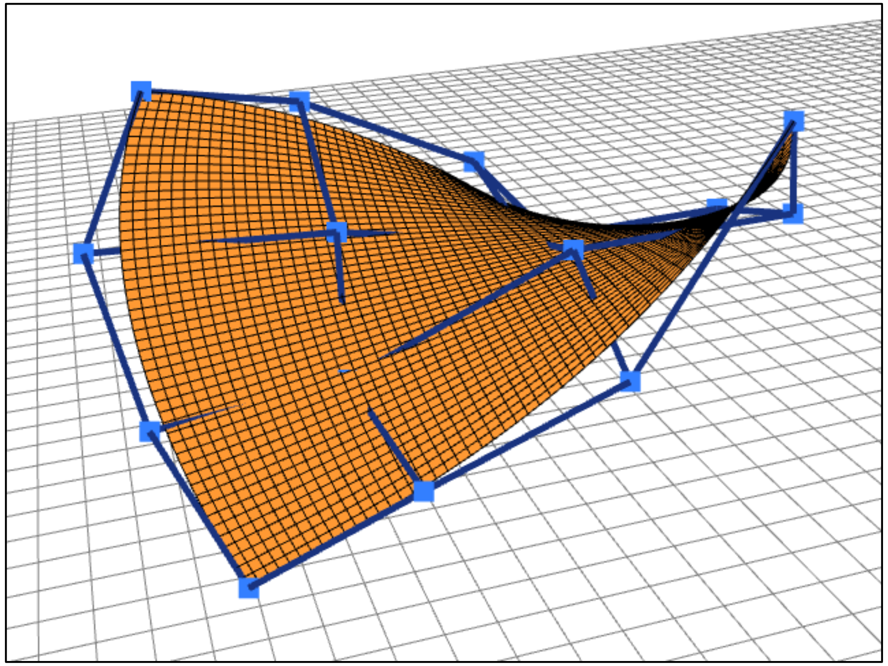




Parametric surfaces

- One parameter ➔ Curve $P(t)$
- Two parameters ➔ Surface $P(s,t)$

- Cubic Bezier surface:
  - Input: 4×4=16 control points P_{ij}



$$P(s,t) = \sum_{i=0}^{3}\sum_{j=0}^{3} b_i^3(s) b_j^3(t) P_{ij}$$

Bernstein basis functions

$b_0^3(t) = (1-t)^3$

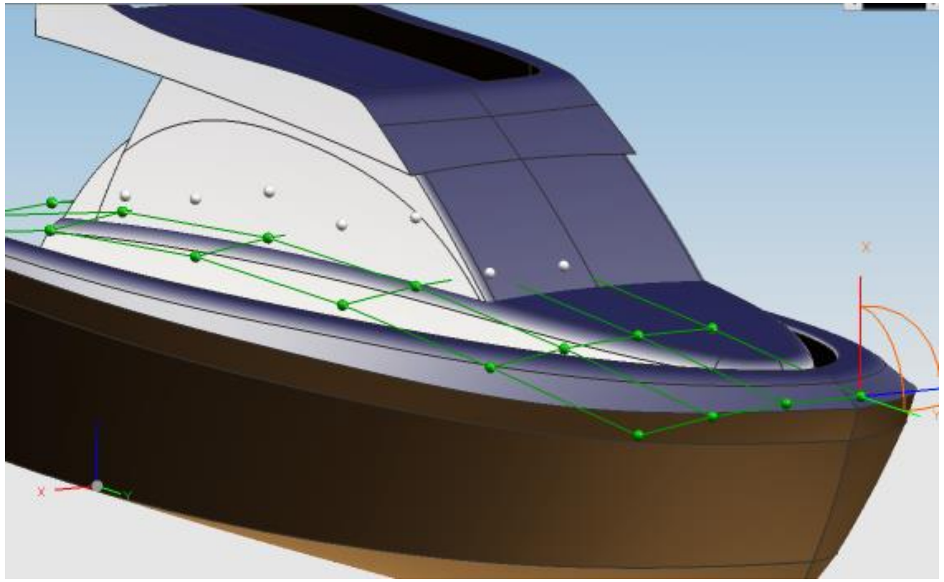
$b_1^3(t) = 3t(1-t)^2$

$b_2^3(t) = 3t^2(1-t)$

$b_3^3(t) = t^3$

3D modeling using parametric surface patches

- Pros
  - Can compactly represent smooth surfaces
  - Can accurately represent spheres, cones, etc



- Cons
  - Hard to design nice layout of patches
  - Hard to maintain continuity across patches
- Often used for designing man-made objects consisting of simple parts

Pointers

- http://en.wikipedia.org/wiki/Bezier_curve
- http://antigrain.com/research/adaptive_bezier/
- https://groups.google.com/forum/#!topic/comp.graphics.algorithms/2FypAv29dG4
- http://en.wikipedia.org/wiki/Cubic_Hermite_spline
- http://en.wikipedia.org/wiki/Centripetal_Catmull%E2%80%93Rom_spline