ON PAVLOV'S CONJECTURE ON PRESENTABLY SYMMETRIC MONOIDAL ∞-CATEGORIES

KENSUKE ARAKAWA

Abstract. We prove that the ∞-category of presentably symmetric monoidal ∞-categories is equivalent to that of combinatorial symmetric monoidal model categories and left Quillen monoidal functors. This proves the conjecture by Pavlov.

Warning 0.1. This paper is **not complete** yet, although the proof of the main theorem (Theorem A) is complete. I plan to include the following topics in the near future:

- Stating and proving variations of the main theorem, such as one for simplicial model categories.
- Explain the extent to which our approach extends to the monoidal case, and explain how it relates to Hovey's open problem on monoidal model categories [Hov, Problem 10].
- Localizing the subcategory of $\mathsf{CombSMMC}$ at Quillen equivalences recover the core of $\mathcal{P}\mathrm{r}\mathcal{S}\mathcal{M}$.

Size issues arise inevitably in category theory, and careless handling of them can lead to pathological situations. For example, the theory of small categories with small limits is not categorically interesting, as they are merely preordered sets. However, a careful bookkeeping of size has a substantial payoff: It gives rise to the theory of *locally presentable categories*. These categories contain most large categories of interest, while enjoying good categorical properties (bicomplete, adjoint functor theorems hold, stable under exponential, etc.).

Presentable ∞-categories are an ∞-categorical generalization of locally presentable categories. As in the 1-categorical setting, they include most large ∞-categories of interest (spaces, spectra, derived ∞-categories, sheaves, etc.). At the same time, they retain excellent formal properties. For these reasons, they are now understood as the natural setting in which most homotopy-invariant constructions take place.

Historically, however, the presentable ∞-categories did not arise directly as a response to size issues in homotopy theory. Instead, they were predated by *combinatorial model categories*, which are model categories satisfying some size conditions. Although they are not as flexible as presentable ∞-categories, they have the advantage that homotopy-theoretic constructions can be performed within an *ordinary* category, where many arguments and constructions are significantly more tractable.

This parallel development raises the question of how presentable ∞-categories and combinatorial model categories are related. In this direction, Dugger and Lurie showed that an ∞-category is presentable if and only if it underlies a combinatorial model category [Dug01, Lur09]. More recently, Pavlov [Pav25] upgraded this result by showing that the homotopy theory of combinatorial model categories is equivalent to that of presentable ∞-categories. Informally, this establishes combinatorial model categories provide as concrete models for presentable ∞-categories.

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In many cases, both ∞-categories and model categories come with symmetric monoidal structures. This naturally leads to the notions of *presentably symmetric monoidal ∞-categories* and *combinatorial symmetric monoidal model categories*. Pavlov conjectured that the homotopy theories of these two are likewise equivalent [Pav25, Conjecture 1.9]. Establishing such an equivalence is important not only for conceptual completeness, but also because symmetric monoidal structures play a central role in modern homotopy theory, including operad theory, enrichment, and multiplicative constructions.

The main result of this paper confirms Pavlov's conjecture. More precisely, the main result of this paper proves the following:

Theorem A (Theorem 2.4)**.** *The functor*

$$L\colon \mathsf{CombSMMC}[\mathrm{Quillen.eq}^{-1}] \to \mathcal{P}\mathrm{r}\mathcal{SM}$$

is an equivalence, where:

- $\mathsf{CombSMMC}$ *denotes the category of combinatorial symmetric monoidal model categories and left Quillen symmetric monoidal functors;*
- $\mathrm{Quillen.eq}$ *denotes the class of Quillen equivalences;*
- $\mathcal{P}\mathrm{r}\mathcal{SM}$ *denotes the ∞-category of presentably symmetric monoidal ∞-categories; and*
- *L carries combinatorial symmetric monoidal model categories to their underlying symmetric monoidal ∞-category.*

Remark 0.2. In an earlier work, Nikolaus–Sagave showed that L is essentially surjective [NS17].

Beyond its intrinsic theoretical importance, Theorem A provides an effective way to work with presentably symmetric monoidal ∞-categories. Theorem A allows us to reduce general statements about them to corresponding statements about combinatorial symmetric monoidal model categories, where concrete constructions are often more manageable. In forthcoming work [**?**], we apply this strategy to establish a new equivalence of two models of enriched ∞-operads.

Our method. Our approach to Theorem A is similar in spirit to that of Pavlov, but extending his argument to the symmetric monoidal setting presents new difficulties. His approach is built upon two inputs. The first is the work of Barwick–Kan [BK12b], which says that ∞-categories can be modeled by relative ∞-categories. The second is the work of Dugger and Low on presenting combinatorial model categories [Dug01, Low16]. To prove Theorem A, we must extend both of these works to the symmetric monoidal setting. We will replace the first by the author's earlier work on symmetric monoidal relative categories [Ara], and the second by the new model category of symmetric cubical sets (Subsection 5.1).

Organization of the paper. In Section 1, we recall the definition of monoidal model categories and their variations. In Section 2, we state the main theorem, and then show that it follows from three separate propositions. The proof of these propositions will be given in the next three sections (Section 3, 4, and 5). The appendix contains some miscellaneous results that will be used in the paper.

Notation and convention.

- In addition to the ZFC axioms, we will assume the existence of three nested Grothendieck universes whose elements are called small sets, large sets, and very large sets.[1] All locally presentable categories are assumed to be large

[1]Although Pavlov did not make this assumption in his paper [Pav25], it seems extremely inconvenient to drop this assumption due to the nature of the statements we will prove. For example, the statement of the main theorem itself needs to be adjusted without this.

but not very large, so that the collection of locally presentable categories themselves form a very large set.

- We assume that model categories are at most large.
- We use the term "regular cardinal" to mean "small regular cardinal."
- By ∞-categories, we mean *quasicategories* in the sense of [Joy02, Lur09]. We mostly follow the terminology and notation of [Lur09].
- We let Fin_* denote the category of finite pointed sets $\langle n\rangle = (\{*, 1, \dots, n\}, *)$ for $n \geq 0$ and pointed maps.
- We let $\mathcal{SMC}\mathsf{at}_\infty$ denote the ∞-category of small symmetric monoidal ∞-categories (defined as the localization of the model category described in [Lur17, Variant 2.1.4.13]). We write $\mathcal{SM}\widehat{\mathcal{C}\mathsf{at}}_\infty$ for the ∞-category of large symmetric monoidal ∞-categories.
- We will not notationally distinguish between ordinary categories and their nerves. We will also regard every $(2,1)$-category as an ∞-category by taking its Duskin nerve. (Recall that this converts $(2,1)$-categories into ∞-categories, and it is functorial in strictly unitary pseudofunctors [Lur25, Tag 00AU].)
- If $\mathbf{M}$ is a model category, we write $\mathbf{M}_{\mathrm{cof}} \subset \mathbf{M}$ for the full subcategory of cofibrant objects. We also write $\mathbf{M}_\infty$ for the underlying ∞-category of $\mathbf{M}$ (i.e., localization at weak equivalences).
- Given a pair of symmetric ∞-monoidal categories $\mathcal{C}, \mathcal{D}$, we let $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ denote the ∞-category of symmetric monoidal functors $\mathcal{C} \to \mathcal{D}$. We use similar notation for symmetric monoidal categories.[2]

1. Monoidal model categories and their variations

The use of the term "monoidal model category" and its variations is not entirely standardized in the literature. The goal of this section is to record the the precise definitions we will use.

1.1. **Plain case.** We start with the definition of monoidal model categories. There are several competing definitions in the literature: Most definitions in the literature require the pushout-product axiom, but they often differ in their requirement on how "flat" the unit object should be. In some literature (e.g., [Lur17]), the unit object is required to be cofibrant. This assumption is convenient for theoretical purposes but ends up excluding many interesting examples, such as [Examples]. At the other extreme (e.g., [PS18]), no condition on the unit object is assumed. In this paper, we go for a middle ground by adopting an axiom satisfied by most monoidal model categories we encounter in practice.

Definition 1.1. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be model categories. A functor $F\colon \mathbf{A} \times \mathbf{B} \to \mathbf{C}$ is called a **left Quillen bifunctor** if it satisfies the following pair of conditions:

(1) The functor F preserves small colimits in each variable.
(2) For every pair of cofibrations $f\colon A \to A'$ in $\mathbf{A}$ and $g\colon B \to B'$ in $\mathbf{B}$, the map
$$F(A, B') \amalg_{F(A,B)} F(A', B) \to F(A, B)$$
is a cofibration, and moreover it is a weak equivalence if f or g is one.

Definition 1.2.

[2]When $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal categories regarded as symmetric monoidal ∞-categories, then $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ has potentially two meanings. This is not confusing because the two categories are naturally equivalent, although we interpret $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ as the category of ordinary symmetric monoidal functors unless stated otherwise.

- A **monoidal model category** is a biclosed monoidal category $(\mathbf{M}, \otimes, \mathbf{1})$ equipped with a model structure, satisfying the following axioms:
  (1) (**Pushout-product axiom**) The functor $\otimes\colon \mathbf{M} \times \mathbf{M} \to \mathbf{M}$ is a left Quillen bifunctor (Definition 1.1).
  (2) (**Very strong unit axiom** [Mur15]) There is a weak equivalence $q\colon \widetilde{\mathbf{1}} \to \mathbf{1}$, with $\widetilde{\mathbf{1}}$ cofibrant, such that for every object $X \in \mathbf{M}$, the maps $X \otimes q$ and $q \otimes X$ are weak equivalences.
- We say that a monoidal model category is **combinatorial** if its underlying model category is combinatorial.
- If $\mathbf{M}$ and $\mathbf{N}$ are monoidal model categories, then a **monoidal left Quillen functor** is a monoidal functor[3] $\mathbf{M} \to \mathbf{N}$ whose underlying functor of model categories is left Quillen. We will write $\mathrm{Fun}^{\otimes, LQ}(\mathbf{M}, \mathbf{N})$ for the category of monoidal left Quillen functors $\mathbf{M} \to \mathbf{N}$ and monoidal natural transformations between them.
- We will write $\mathsf{CombMMC}$ for the (very large) category of combinatorial monoidal model categories and monoidal left Quillen functors, and write $\mathsf{CombMMC}^{\mathbf{1}} \subset \mathsf{CombMMC}$ for the full subcategory spanned by the combinatorial monoidal model categories with a cofibrant unit.
- **Symmetric monoidal model categories** and **symmetric monoidal left Quillen functors** are defined similarly. (We will use the notation $\mathrm{Fun}^{\otimes, LQ}(-,-)$ in the symmetric monoidal case, too. This is abusive but is rarely confusing.) The categories $\mathsf{CombSMMC}$ and $\mathsf{CombSMMC}^{\mathbf{1}}$ are also defined similarly, using combinatorial symmetric monoidal model categories.

The reason why we adopt Muro's unit axiom is that it allows us to turn every monoidal model category into one with a cofibrant unit. More precisely, we have the following theorem.

Theorem 1.3. [Mur15, Theorem 1, Proposition 12] *The inclusion* $\mathsf{CombMMC}^{\mathbf{1}} \subset \mathsf{CombMMC}$ *admits a left adjoint. Moreover, the unit of this adjunction preserves and reflects weak equivalences and induces an isomorphism of underlying monoidal categories. A similar claim holds for combinatorial symmetric monoidal model categories.*

The categories $\mathsf{CombMMC}$ can be upgraded to a $(2,1)$-category $\mathsf{CombMMC}_{(2,1)}$, whose mapping groupoid are given by the maximal subgroupoid $\mathrm{Fun}^{\otimes, LQ}(\mathbf{M}, \mathbf{N})^{\cong} \subset \mathrm{Fun}^{\otimes, LQ}(\mathbf{M}, \mathbf{N})$. We will use similar notations for the categories introduced in Definition 1.2. These ∞-categories satisfy the following universal property:

Proposition 1.4. *The functors*

$$\begin{aligned} \mathsf{CombMMC} &\to \mathsf{CombMMC}_{(2,1)}, \\ \mathsf{CombMMC}^{\mathbf{1}} &\to \mathsf{CombMMC}^{\mathbf{1}}_{(2,1)}, \\ \mathsf{CombSMMC} &\to \mathsf{CombSMMC}_{(2,1)}, \\ \mathsf{CombSMMC}^{\mathbf{1}} &\to \mathsf{CombSMMC}^{\mathbf{1}}_{(2,1)} \end{aligned}$$

are ∞-categorical localizations at the left Quillen functors whose underlying functors are equivalences of categories.

Proof. We will prove the assertion for the first functor; the remaining assertions can be proved in a similar way. According to (the dual of) [ACK25, Corollary 3.16], it suffices to show that $\mathsf{CombMMC}_{(2,1)}$ admits weak cotensor by [1]. That

[3] By a monoidal functor, we mean a strong monoidal functor in the sense of [ML98, Chapter XI].

is, it suffices to show that for each $\mathbf{N} \in \mathsf{CombMMC}_{(2,1)}$, there is an object $[1] \pitchfork \mathbf{N}$ admitting an isomorphism of groupoids

$$\mathrm{Fun}^{\otimes,LQ}(\mathbf{M}, [1] \pitchfork \mathbf{N})^{\cong} \cong \mathrm{Fun}\Big([1], \mathrm{Fun}^{\otimes,LQ}(\mathbf{M},\mathbf{N})^{\cong}\Big)$$

natural in $\mathbf{M} \in \mathsf{CombMMC}$. For this, we consider the groupoid J with two objects $0, 1$, and with exactly one morphism between any pair of objects. We can make $\mathbf{N}^J$ into a monoidal model category by using the degreewise tensor product and the model structure coming from the equivalence of categories $\mathbf{N} \xrightarrow{\simeq} \mathbf{N}^J$. Then $\mathbf{N}^J$ is a weak cotensor of $\mathbf{N}$ by $[1]$, and we are done. □

1.2. **Enriched case.** We next turn to the definition of enriched monoidal model categories.

Definition 1.5. Let $\mathbf{V}$ be a combinatorial symmetric monoidal model category.

- A **$\mathbf{V}$-monoidal model category** is a tensored and cotensored $\mathbf{V}$-monoidal category $\mathbf{M}$ (in the sense of [Day70]) equipped with a model structure on its underlying category, such that the tensor bifunctor
$$\mathbf{V} \times \mathbf{M} \to \mathbf{M}$$
is a left Quillen bifunctor.
- We say that a $\mathbf{V}$-monoidal model category is **combinatorial** if its underlying model category is combinatorial.
- If $\mathbf{M}$ and $\mathbf{N}$ are $\mathbf{V}$-monoidal model categories, we write $\mathrm{Fun}_{\mathbf{V}}^{\otimes,LQ}(\mathbf{M},\mathbf{N})$ for the category of $\mathbf{V}$-monoidal functors $\mathbf{M} \to \mathbf{N}$ that are $\mathbf{V}$-cocontinuous and whose underlying functors are left Quillen.
- We write $\mathsf{CombMMC}_{\mathbf{V},(2,1)}$ for the $(2,1)$-category whose objects are the combinatorial $\mathbf{V}$-monoidal model categories, with hom-groupoids given by the maximal subgroupoids of $\mathrm{Fun}_{\mathbf{V}}^{\otimes,LQ}(-,-)$.
- We write $\mathsf{CombMMC}_{\mathbf{V}}$ for the underlying category of $\mathsf{CombMMC}_{\mathbf{V},(2,1)}$.

We define $\mathbf{V}$-symmetric monoidal model categories similarly, and define a $(2,1)$-category $\mathsf{CombSMMC}_{\mathbf{V},(2,1)}$ and an ordinary category $\mathsf{CombSMMC}_{\mathbf{V}}$ similarly.

In ordinary algebra, a commutative algebra A over a commutative ring k can be defined as a commutative ring equipped with a ring homomorphism $k \to A$. Extending an analogy of this to the model-categorical setting, we arrive at the notion of *symmetric algebras*:

Definition 1.6. Let $\mathbf{V}$ be a combinatorial symmetric monoidal model category.

- A **symmetric $\mathbf{V}$-algebra** is a monoidal model category $\mathbf{M}$ equipped with a monoidal left Quillen functor $\mathbf{V} \to \mathbf{M}$.
- We say that a symmetric $\mathbf{V}$-algebra is **combinatorial** if its underlying model category is combinatorial.
- We write $\mathsf{CombSym}\mathbf{V}\text{-}\mathsf{Alg}_{(2,1)} = \int^{\mathbf{M} \in \mathsf{CombSMMC}_{(2,1)}} \mathrm{Fun}^{\otimes,LQ}(\mathbf{V},\mathbf{M})$ for the $(2,1)$-category of combinatorial symmetric $\mathbf{V}$-algebras, where $\int$ denotes the 2-Grothendieck construction [Str80, 1.10]. Explicitly:
  - Objects are combinatorial symmetric $\mathbf{V}$-algebras.
  - A 1-morphism $(F\colon \mathbf{V} \to \mathbf{M}) \to (G\colon \mathbf{V} \to \mathbf{N})$ is a pair (H, α), where $H\colon \mathbf{M} \to \mathbf{N}$ is a symmetric monoidal functor and $\alpha\colon HF \Rightarrow G$ is a symmetric monoidal natural transformation.
  - A 2-morphism $(H,\alpha) \Rightarrow (H',\alpha')\colon (F\colon \mathbf{V} \to \mathbf{M}) \to (G\colon \mathbf{V} \to \mathbf{N})$ is a symmetric monoidal natural isomorphism $\beta\colon H \overset{\cong}{\Rightarrow} H'$ satisfying

$\alpha' \circ \beta F = \alpha$:

$$
\begin{tikzcd}
 & \mathbf{V} \arrow[dl, "F"'] \arrow[dr, "G"] \arrow[d, phantom, "=\alpha\Rightarrow"] & \\
\mathbf{M} \arrow[rr, "H"'] & {} & \mathbf{N}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & \mathbf{V} \arrow[dl, "F"'] \arrow[dr, "G"] \arrow[d, phantom, "=\alpha'\Rightarrow"] & \\
\mathbf{M} \arrow[rr, bend left, "H'"] \arrow[rr, bend right, "H"'] \arrow[rr, phantom, "\Uparrow\beta"] & {} & \mathbf{N}.
\end{tikzcd}
$$

- We write $\mathsf{CombSym}\mathbf{V}\text{-}\mathsf{Alg}$ for the underlying category of $\mathsf{CombSym}\mathbf{V}\text{-}\mathsf{Alg}_{(2,1)}$.

Remark 1.7. There is a related notion of *central algebras* [Hov99, Definition 4.1.10], but they are not amenable to the techniques we use in this paper.

As in Proposition 1.4, we have:

Proposition 1.8. *Let* $\mathbf{V}$ *be a combinatorial symmetric monoidal model category. The functors*

$$
\begin{aligned}
\mathsf{CombMMC}_{\mathbf{V}} &\to \mathsf{CombMMC}_{\mathbf{V},(2,1)} \\
\mathsf{CombSMMC}_{\mathbf{V}} &\to \mathsf{CombSMMC}_{\mathbf{V},(2,1)} \\
\mathsf{CombSym}\mathbf{V}\text{-}\mathsf{Alg} &\to \mathsf{CombSym}\mathbf{V}\text{-}\mathsf{Alg}_{(2,1)}
\end{aligned}
$$

are all localizations.

1.3. **Semi case (draft).** A (left) **semi-model category** is a weakening of a Quillen model category in which the axioms of lifting and factorization are only required to hold for cofibrations with *cofibrant* source [BW24, Definition 2.1]. While this leads to a less attractive set of axioms, semi-model categories are practically indistinguishable from model categories. The main advantages of semi-model categories over ordinary model categories is that they are much easier to construct than a full model structure, and that they are fully compatible with (left) Bousfield localizations [BW24, Theorem A]. For these reasons, semi-model categories have become increasingly popular, and we have no reason not to include them in this paper.

Definition 1.9. The definition of monoidal model categories carries over to the semi-model categorical case. We define categories $\mathsf{CombMMC}_{\mathrm{semi}}$, $\mathsf{CombMMC}^{\mathbf{1}}_{\mathrm{semi}}$, $\mathsf{CombSMMC}_{\mathrm{semi}}$, and $\mathsf{CombSMMC}^{\mathbf{1}}_{\mathrm{semi}}$ exactly as in Definition 1.2. These categories have associated $(2,1)$-categories, which we denote by $\mathsf{CombMMC}_{\mathrm{semi},(2,1)}$, etc.

Given a combinatorial symmetric monoidal semi-model category $\mathbf{V}$, we also define $\mathsf{CombSym}\mathbf{V}\text{-}\mathsf{Alg}_{\mathrm{semi}}$ exactly as in

Remark 1.10. Theorem 1.3 and Proposition 1.4 remains valid for semi-model categories, with the same proof. [Check this!]

2. Main Result

In this section, we state the main result of this paper and explain how we will prove it.

To state our main result, we need a bit of terminology and notation.

Definition 2.1. [Ara, Definition 0.1] A **symmetric monoidal relative category** is a symmetric monoidal category $\mathbf{C}$ equipped with a subcategory $\mathbf{W} \subset \mathbf{C}$ of **weak equivalences**, which contains all isomorphisms and is stable under tensor products.

We write $\mathsf{SMRelCat}$ for the category of small symmetric monoidal relative categories and symmetric monoidal relative functors that preserve weak equivalences.

We define the category $\mathsf{SMRel}\widehat{\mathsf{Cat}}$ of *large* symmetric monoidal relative categories similarly.

Example 2.2. Let $\mathbf{M}$ be a symmetric monoidal model category. Weak equivalences of $\mathbf{M}$ are generally not stable under tensor products, so $\mathbf{M}$ is generally *not* a symmetric monoidal relative category. Nonetheless, its full subcategory $\mathbf{M}^\flat \subset \mathbf{M}$ of cofibrant objects and the objects isomorphic to the unit objects is a symmetric monoidal relative category.

For the next definition, we recall that localization of symmetric monoidal relative categories determines a functor $L\colon \mathsf{SMRelCat} \to \mathcal{SMC}\mathsf{at}_\infty$ [Ara, Notation 2.1].

Definition 2.3. We define a functor

$$(-)_\infty\colon \mathsf{CombSMMC} \to \mathcal{P}\mathrm{r}\mathcal{SM}$$

as the codomain restriction of the composite

$$\mathsf{CombSMMC} \xrightarrow{(-)^\flat} \mathsf{SMRel}\widehat{\mathsf{Cat}} \xrightarrow{L} \mathcal{SM}\widehat{\mathcal{C}\mathsf{at}}_\infty.$$

(Note that this is well-defined by [Lur17, Proposition 1.3.4.22 and Corollary 1.3.4.26].) If $\mathbf{M}$ is a combinatorial symmetric monoidal model category, we refer to $\mathbf{M}_\infty$ as its **underlying symmetric monoidal ∞-category**.

We now arrive at the statement of the main result.

Theorem 2.4. *The functor $(-)_\infty$ of Definition 2.3 is a localization at Quillen equivalences, i.e., it induces a categorical equivalence*

$$\mathsf{CombSMMC}[\mathrm{Quill.eq}^{-1}] \xrightarrow{\simeq} \mathcal{P}\mathrm{r}\mathcal{SM}.$$

We will prove Theorem 2.4 as follows: Fix a left proper tractable symmetric monoidal model category $\mathbf{S}$ with a cofibrant unit, whose underlying symmetric monoidal ∞-category is equivalent to the cartesian monoidal ∞-category $\mathcal{S}$ of ∞-groupoids.[4] (Examples include the cartesian model category of simplicial sets with the Kan–Quillen model structure.) We also define a category $\mathsf{SMRel}\widehat{\mathsf{Cat}}_{\mathcal{P}\mathrm{r}\mathcal{SM}}$ by the pullback

$$\begin{tikzcd}
\mathsf{SMRel}\widehat{\mathsf{Cat}}_{\mathcal{P}\mathrm{r}\mathcal{SM}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{SMRel}\widehat{\mathsf{Cat}} \arrow[d, "L"] \\
\mathcal{P}\mathrm{r}\mathcal{SM} \arrow[r] & \mathcal{SM}\widehat{\mathcal{C}\mathsf{at}}_\infty.
\end{tikzcd}$$

We then contemplate the following commutative diagram

$$\begin{tikzcd}
\mathsf{CombSym}\mathbf{S}\text{-}\mathsf{Alg} \arrow[r, "U"] \arrow[d, "(-)_\flat"'] & \mathsf{CombSMMC}^{\mathbf{1}} \arrow[r, hook, "\iota"] & \mathsf{CombSMMC} \arrow[d, "(-)_\infty"] \\
\mathsf{SMRel}\widehat{\mathsf{Cat}}_{\mathcal{P}\mathrm{r}\mathcal{SM}} \arrow[rr, "L"'] & & \mathcal{P}\mathrm{r}\mathcal{SM}\widehat{\mathcal{C}\mathsf{at}}_\infty,
\end{tikzcd}$$

where U is the forgetful functor and ι is the inclusion. We will then prove that:

Proposition 2.5. *The functor $L\colon \mathsf{SMRel}\widehat{\mathsf{Cat}}_{\mathcal{P}\mathrm{r}\mathcal{SM}} \to \mathcal{P}\mathrm{r}\mathcal{SM}\widehat{\mathcal{C}\mathsf{at}}_\infty$ induces a categorical equivalence*

$$\mathsf{SMRel}\widehat{\mathsf{Cat}}_{\mathcal{P}\mathrm{r}}[\mathrm{loc.eq}^{-1}] \to \mathcal{P}\mathrm{r}\mathcal{SM}\widehat{\mathcal{C}\mathsf{at}}_\infty,$$

where loc.eq *denotes the subcategory of* ***local equivalences****, i.e., morphisms inverted by L.*

[4]Since $\mathcal{S}$ is the initial presentably symmetric monoidal ∞-category [Lur17, Corollary 3.2.1.9, Example 4.8.1.20], such an equivalence is unique up to a contractible space of choices if it exists.

Proposition 2.6. *The functor $(-)_\flat$ induces a categorical equivalence*

$$\mathsf{CombSym}\mathbf{S}\text{-}\mathsf{Alg}[\mathrm{Quill.eq}^{-1}] \xrightarrow{\simeq} \mathsf{SMRel}\widehat{\mathsf{Cat}}_{\mathcal{P}\mathsf{r}\mathcal{SM}}[\mathrm{loc.eq}^{-1}].$$

Proposition 2.7. *The functor U induces a categorical equivalence*

$$\mathsf{CombSym}\mathbf{S}\text{-}\mathsf{Alg}[\mathrm{Quill.eq}^{-1}] \xrightarrow{\simeq} \mathsf{CombSMMC}^{\mathbf{1}}[\mathrm{Quill.eq}^{-1}].$$

The proof of these propositions will be given in the next three sections (Section 3, 4, and 5). Since we know that ι induces a categorical equivalence upon localizing at Quillen equivalences (Theorem 1.3), these proposition will prove Theorem 2.4.

3. Proof of Proposition 2.5

We first prove Proposition 2.5, which asserts that the functor

$$\mathsf{SMRel}\widehat{\mathsf{Cat}}_{\mathcal{P}\mathsf{r}\mathcal{SM}}[\mathrm{loc.eq}^{-1}] \to \mathcal{P}\mathsf{r}\mathcal{SM}$$

is an equivalence. In fact, we will prove the following more general assertion:

Lemma 3.1. *For every conservative functor $\mathcal{X} \to \mathcal{SMC}\mathsf{at}_\infty$ of ∞-categories, the functor*

$$\mathsf{SMRelCat}_{\mathcal{X}} = \mathcal{X} \times_{\mathcal{SMC}\mathsf{at}_\infty} \mathsf{SMRelCat} \to \mathcal{X}$$

is a localization.

Replacing $\mathcal{SMC}\mathsf{at}_\infty$ by $\mathcal{SM}\widehat{\mathcal{C}\mathsf{at}}_\infty$ and then substituting $\mathcal{P}\mathsf{r}\mathcal{SM}$ for $\mathcal{X}$, we obtain Proposition 2.5.

For the proof of Lemma 3.1, we need some terminology and notation.

- A **relative ∞-category** is an ∞-category equipped with a subcategory $\mathcal{W} \subset \mathcal{C}$ containing all equivalences. Morphisms in $\mathcal{W}$ are typically called **weak equivalences**. When $\mathcal{C}$ is (the nerve of) an ordinary category, we call such pairs **relative categories**. A relative functor is a functor This should not be confused with the definition of Barwick–Kan [BK12b], which they only require that $\mathcal{W}$ contains all objects.
- If $\mathcal{C}$ and $\mathcal{D}$ are relative ∞-categories, a **relative functor** $f\colon \mathcal{C} \to \mathcal{D}$ is a functor between the underlying ∞-categories that preserve weak equivalences. We say that such an f is a **homotopy equivalence** if there is a relative functor $g\colon \mathcal{D} \to \mathcal{C}$ and zig-zags of natural weak equivalences connecting $f \circ g$ and $g \circ f$ to the identity functors.
- We write RelCat for the category of small relative categories and functors preserving weak equivalences. For disambiguation, we will write write $\mathsf{RelCat}_{\mathrm{BK}}$ for the slightly larger category of Barwick–Kan's relative categories that are small.
- We call a morphsim $f\colon \mathcal{C} \to \mathcal{D}$ of $\mathsf{RelCat}_{\mathrm{BK}}$ a **local equivalences** if it induces an equivalence between the (∞-categorical) localizations.
- Recall that a symmetric monoidal category is called a **permutative category** if its underlying monoidal category is strict (i.e., its associators and the unitors are the identity maps). We write $\mathsf{PermRelCat} \subset \mathsf{SMRelCat}$ for the subcategory of objects whose underlying symmetric monoidal categories are permutative categories, and morphisms whose underlying symmetric monoidal functors are strict.
- We will regard the categories $\mathsf{RelCat}, \mathsf{RelCat}_{\mathrm{BK}}, \mathsf{PermRelCat}, \mathsf{SMRelCat}$ as relative categories whose weak equivalences are the local equivalences.
- We write Fin_* for the category of finite pointed sets and pointed maps. We write $\mathrm{Fun}^{\mathrm{Seg}}(\mathsf{Fin}_*, \mathsf{RelCat}) \subset \mathrm{Fun}^{\mathrm{Seg}}(\mathsf{Fin}_*, \mathsf{RelCat})$ for the full subcategory

spanned by the functors $F\colon \mathsf{Fin}_* \to \mathsf{RelCat}$ satisfying the following **Segal condition**: For every $n \geq 0$, the map $\rho_i\colon \langle n\rangle \to \langle 1\rangle$ defined by

$$\rho_i(j) = \begin{cases} * & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases}$$

induces a local equivalence

$$F\langle n\rangle \xrightarrow{\simeq} \prod_{1\leq i\leq n} F\langle 1\rangle.$$

We define $\mathrm{Fun}^{\mathrm{Seg}}(\mathsf{Fin}_*, \mathsf{RelCat}_{\mathrm{BK}}) \subset \mathrm{Fun}^{\mathrm{Seg}}(\mathsf{Fin}_*, \mathsf{RelCat}_{\mathrm{BK}})$ similarly.

Proof of Lemma 3.1. Since symmetric categories are functorially equivalent to permutative categories [May78, Proposition 4.2], using Proposition B.3, we may replace $\mathsf{SMRelCat}$ by $\mathsf{PermRelCat}$.

According to [Ara, Theorem 1.1], there is a homotopy equivalence $\mathrm{Fact}\colon \mathsf{PermRelCat} \to \mathrm{Fun}^{\mathrm{Seg}}(\mathsf{Fin}_*, \mathsf{RelCat})$ of relative categories. The proof of loc. cit. further shows that the localization of $\mathrm{Fun}^{\mathrm{Seg}}(\mathsf{Fin}_*, \mathsf{RelCat})$ is equivalent to $\mathcal{SMC}\mathsf{at}_\infty$, and the diagram

$$\begin{array}{ccc} \mathsf{PermRelCat} & \xrightarrow{\mathrm{Fact}} & \mathrm{Fun}^{\mathrm{Seg}}(\mathsf{Fin}_*, \mathsf{RelCat}) \\ \big\downarrow & & \big\downarrow{\scriptstyle \text{localization}} \\ \mathsf{SMRelCat} & \xrightarrow[L]{} & \mathcal{SMC}\mathsf{at}_\infty. \end{array}$$

commutes up to natural equivalence. Thus, by Proposition B.3, it will suffice to show that the functor

$$\mathrm{Fun}^{\mathrm{Seg}}(\mathsf{Fin}_*, \mathsf{RelCat})_{\mathfrak{X}} \to \mathfrak{X}$$

is a localization. Since the inclusion $\mathsf{RelCat} \hookrightarrow \mathsf{RelCat}_{\mathrm{BK}}$ is a homotopy equivalence of relative categories, we are free to replace RelCat by $\mathsf{RelCat}_{\mathrm{BK}}$.

We now recall from [BK12b, Theorem 6.1] and [BK12a, Theorem 1.8] that $\mathsf{RelCat}_{\mathrm{BK}}$ and its local equivalences is part of a combinatorial model structure. Therefore, Theorem B.1 and [Hir03, Theorem 11.6.1] show that the localization

$$\mathrm{Fun}(\mathsf{Fin}_*, \mathsf{RelCat}_{\mathrm{BK}}) \to \mathrm{Fun}(\mathsf{Fin}_*, \mathsf{RelCat}_{\mathrm{BK}})[\mathrm{weq}^{-1}]$$

is stable under pullback. In particular, the functor

$$\mathrm{Fun}^{\mathrm{Seg}}(\mathsf{Fin}_*, \mathsf{RelCat}_{\mathrm{BK}}) \to \mathfrak{X}$$

is a localization, which was to be proved. □

4. Proof of Proposition 2.6

Notation 4.1. Throughout this section, we fix a left proper tractable symmetric monoidal model category $\mathbf{S}$ with a cofibrant unit, whose underlying symmetric monoidal ∞-category is equivalent to the cartesian monoidal ∞-category $\mathcal{S}$ of ∞-groupoids.

In this section, we give a proof of Proposition 2.6, which asserts that the functor

$$\mathsf{CombSym}\mathbf{S}\text{-}\mathsf{Alg}[\mathrm{Quill.eq}^{-1}] \xrightarrow{\simeq} \mathsf{SMRel}\widehat{\mathsf{Cat}}_{\mathcal{P}r\mathcal{SM}}[\mathrm{loc.eq}^{-1}]$$

is an equivalence. This will need two ingredients: A relative-categorical version of the multiplicative Gabriel–Ulmer duality, and some formal cardinality argument of model categories. We will tackle each of these in the next two subsections, and then prove Proposition 2.6 at the end of this subsection.

4.1. **Relative categorical multiplicative Gabriel–Ulmer duality.** In this subsection, we state and prove a relative categorical multiplicative Gabriel–Ulmer duality (Proposition 4.6). We refer the reader to Section A for a review of Gabriel–Ulmer duality.

To state the main result, we need a bit of preliminaries.

Notation 4.2. Let κ be a regular cardinal. We write $\mathcal{SMC}\mathsf{at}_\infty(\kappa)$ for the subcategory of $\mathcal{SMC}\mathsf{at}_\infty$ spanned by the symmetric monoidal ∞-categories that compatible with κ-small colimits (i.e., those ∞-categories that have κ-small colimits and whose tensor product preserves those colimits in each variable).

We write $\mathcal{SMC}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa) \subset \mathcal{SMC}\mathsf{at}_\infty(\kappa)$ for the full subcategory spanned by the objects that are idempotent complete.

Construction 4.3. Let κ be a regular cardinal. For each $\mathcal{C} \in \mathsf{SMRelCat}_{\mathcal{SMC}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa)}$, we define an $\mathbf{S}$-algebra $\mathbf{Ind}_\kappa^{\mathbf{S}}(L(\mathcal{C}))$ as follows:

- Its underlying category is $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{S})$.
- The symmetric monoidal structure comes from Day's convolution product.
- The symmetric monoidal functor $\mathbf{S} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{S})$ is given by $X \mapsto \mathcal{C}(-, \mathbf{1}) \cdot X$, where $\mathbf{1} \in \mathcal{C}$ denote the monoidal unit and the dot "$\cdot$" denotes copowering by set.
- The model structure is the left Bousfield localization of the projective model structure on $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{S})$ whose weak equivalences are the maps inverted by the composite
$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{S}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Fun}(L(\mathcal{C})^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Ind}_\kappa(L(\mathcal{C})).$$
Here, the first functor is the postcomposes the localization $\mathbf{S} \to \mathcal{S}$, the second functor is the left adjoint Kan extension along the localization functor $\mathcal{C}^{\mathrm{op}} \to L(\mathcal{C})^{\mathrm{op}}$, and the third functor is the left adjoint to the inclusion $\mathrm{Ind}_\kappa(L(\mathcal{C})) \hookrightarrow \mathrm{Fun}(L(\mathcal{C}), \mathcal{S})$.

We sometimes use the notation $\mathbf{Ind}_\kappa^{\mathbf{S}}(L(\mathcal{C}))$ for essentially small symmetric monoidal relative categories whose symmetric monoidal localization is idempotent complete and is compatible with κ-small colimits.

Lemma 4.4. *Construction 4.3 is well-defined. More precisely, in the situation of Construction 4.3, the following holds:*

(1) The model category $\mathbf{Ind}_\kappa^{\mathbf{S}}(L(\mathcal{C}))$ exists.
(2) The model category $\mathbf{Ind}_\kappa^{\mathbf{S}}(L(\mathcal{C}))$ satisfies the pushout-product axiom, and the monoidal unit is cofibrant.
(3) The symmetric monoidal functor $\mathbf{S} \to \mathbf{Ind}_\kappa^{\mathbf{S}}(L(\mathcal{C}))$ is left Quillen.

Proof. Part (1) is a consequence of Proposition B.4.

Next, we prove (2). Choose generating sets I, J of cofibrations and trivial cofibrations of $\mathbf{S}$. According to [Hir03, Theorem 11.6.1], the sets $I_\mathcal{C} = \{\mathcal{C}(-, C) \cdot i\}_{i \in I, C \in \mathcal{C}}$ and $J_\mathcal{C} = \{\mathcal{C}(-, C) \cdot j\}_{j \in J, C \in \mathcal{C}}$ generate cofibrations and trivial cofibrations of the projective model structure on $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{S})$. It follows immediately that:

- The monoidal unit of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{S})$ is cofibrant (because $\mathbf{S}$ has a cofibrant unit).
- The Day convolution product satisfies the pushout-product axiom for the projective model structure.
- The projective model structure is tractable.

Therefore, to prove (2), it will suffice to prove the following: For every cofibrant object $X \in \mathbf{Ind}_\kappa^{\mathbf{S}}(L(\mathcal{C}))$, the functor
$$X \otimes -\colon \mathbf{Ind}_\kappa^{\mathbf{S}}(L(\mathcal{C}))_{\mathrm{cof}} \to \mathbf{Ind}_\kappa^{\mathbf{S}}(L(\mathcal{C}))_{\mathrm{cof}}$$

preserves weak equivalences. But this is clear, because the composite

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{S})_{\mathrm{cof}} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Fun}(L(\mathcal{C})^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Ind}_\kappa(L(\mathcal{C}))$$

is symmetric monoidal.

Claim (3) follows from the fact that its right adjoint is the evaluation at the unit object, which is evidently right Quillen. The claim follows. □

Notation 4.5. For each subcategory $\mathcal{X} \subset \mathcal{SMC}\mathsf{at}_\infty$ containing all equivalences, we will write $\mathsf{SMRelCat}_{\mathcal{X}} = \mathcal{X} \times_{\mathcal{SMC}\mathsf{at}_\infty} \mathsf{SMRelCat}$, and define $\mathsf{SMRelCat}_{(2,1),\mathcal{X}} \subset \mathsf{SMRelCat}_{(2,1)}$ for the sub $(2,1)$-category whose mapping groupoids are the components corresponding to the morphisms in $\mathsf{SMRelCat}_{\mathcal{X}}$.

We can now state the main result of this subsection.

Proposition 4.6. *The strictly unitary pseudofunctor*

$$\mathsf{SMRelCat}_{\mathcal{SMC}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa)} \to \mathsf{SMRel}\widehat{\mathsf{Cat}}_{\kappa\text{-}\mathcal{P}\mathsf{r}\mathcal{SM},(2,1)}$$
$$\mathcal{C} \mapsto \mathbf{Ind}^{\mathbf{S}}_\kappa(L(\mathcal{C}))_\flat$$

induces a categorical equivalence

$$\mathsf{SMRelCat}_{\mathcal{SMC}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa)}[\mathrm{loc.eq}^{-1}] \xrightarrow{\simeq} \mathsf{SMRel}\widehat{\mathsf{Cat}}_{\kappa\text{-}\mathcal{P}\mathsf{r}\mathcal{SM},(2,1)}[\mathrm{loc.eq}^{-1}].$$

We need a lemma for the proof of Proposition 4.6. We will use the following notation: For each subcategory $\mathcal{X} \subset \mathcal{SMC}\mathsf{at}_\infty$ containing all equivalences, we will write $\mathsf{SMRelCat}_{(2,1),\mathcal{X}} \subset \mathsf{SMRelCat}_{(2,1)}$ for the sub $(2,1)$-category whose mapping groupoids are the components corresponding to the morphisms in $\mathsf{SMRelCat}_{\mathcal{X}}$. We then have:

Lemma 4.7. *For every subcategory $\mathcal{X} \subset \mathcal{SMC}\mathsf{at}_\infty$ containing all equivalences, the inclusion $\mathsf{SMRelCat}_{\mathcal{X}} \hookrightarrow \mathsf{SMRelCat}_{(2,1),\mathcal{X}}$ is a localization.*

Proof. According to [ACK25, Corollary 3.15], it suffices to show that the $(2,1)$-category $\mathsf{SMRelCat}_{(2,1),\mathcal{X}}$ admits a weak cotensor by $[1]$. Given an object $\mathcal{C} \in \mathsf{SMRelCat}_{(2,1)}$, a weak cotensor by $[1]$ is given by $\mathcal{C}^J$, where the tensor product is given by the degreewise tensor product, and the weak equivalences the natural transformations whose components are weak equivalences. □

We can now prove Proposition 4.6.

Proof of Proposition 4.6. The Yoneda embedding determines a pseudonatural transformation depicted as

$$\begin{tikzcd}
\mathsf{SMRelCat}_{\mathcal{SMC}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa)} \arrow[rr, "\mathbf{Ind}^{\mathbf{S}}_\kappa(L(-))_\flat"] \arrow[dr, hook] & \arrow[d, phantom, "\Longrightarrow"] & \mathsf{SMRel}\widehat{\mathsf{Cat}}_{\kappa\text{-}\mathcal{P}\mathsf{r}\mathcal{SM},(2,1)} \arrow[dl, hook] \\
& \mathsf{SMRel}\widehat{\mathsf{Cat}}_{(2,1)} &
\end{tikzcd}$$

Using Lemmas 3.1 and 4.7 to identify the localization of each category at local equivalences, we obtain a functor $I\colon \mathcal{SMC}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa) \to \kappa\text{-}\mathcal{P}\mathsf{r}\mathcal{SM}$ and a natural transformation depicted as

$$\begin{tikzcd}
\mathcal{SMC}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa) \arrow[rr, "I"] \arrow[dr, hook] & \arrow[d, phantom, "\overset{\alpha}{\Longrightarrow}"] & \kappa\text{-}\mathcal{P}\mathsf{r}\mathcal{SM} \arrow[dl, hook] \\
& \mathcal{SM}\widehat{\mathcal{C}\mathsf{at}}_\infty(\kappa). &
\end{tikzcd}$$

By construction, the pair (I, α) satisfies the hypothesis of Corollary A.9. It follows from this corollary that I is an equivalence, and we are done. □

4.2. **Strongly κ-combinatorial model categories.** Combinatorial model categories enjoy the following curious property, first articulated by Dugger in [Dug01] and later expanded by Low [Low16]: Beyond sufficiently large regular cardinals, the distinction between ordinary-categorical notion and ∞-categorical notions gets blurry. In this subsection, we record several results that embody this principle.

The following definition is essentially due to Low [Low16, Definition 5.1]:

Definition 4.8. Let κ be a regular cardinal. We say that a model category $\mathbf{M}$ is **strongly κ-combinatorial** if there is a regular cardinal $\kappa_0 \lhd \kappa$ with the following properties:

(1) $\mathbf{M}$ is locally κ_0-presentable.
(2) $\mathbf{M}_\kappa$ is closed under finite limits in $\mathbf{M}$.
(3) Each hom-set in $\mathbf{M}_{\kappa_0}$ is κ-small.
(4) There are κ-small sets of morphisms in $\mathbf{M}_{\kappa_0}$ that cofibrantly generate the model structure on $\mathbf{M}$.

The following results summarize the basic properties of strongly κ-combinatorial model categories.

Proposition 4.9. *For every combinatorial model category* $\mathbf{M}$*, there is a regular cardinal* κ *such that* $\mathbf{M}$ *is strongly* κ*-combinatorial. Moreover, in this situation,* $\mathbf{M}_\kappa$ *is a model category whose cofibrations, fibrations, and weak equivalences are precisely those of* $\mathbf{M}$ *of* κ*-compact objects.*

Proof. This is [Low16, Propositions 5.6, 5.12]. □

Proposition 4.10. *Let* $\kappa' \rhd \kappa$ *be regular cardinals. Every strongly* κ*-combinatorial model category is strongly* κ'*-combinatorial.*

Proof. This is [Low16, Remark 5.2]. □

Proposition 4.11. *Let* κ *be a regular cardinal, and let* $\mathbf{M}$ *be a strongly* κ*-combinatorial model category. Then the* ∞*-category* $\mathbf{M}_{\kappa,\infty} = (\mathbf{M}_\kappa)_\infty$ *admits* κ*-small colimits, and the functor*

$$\iota\colon \mathbf{M}_{\kappa,\infty} \to \mathbf{M}_\infty$$

exhibits $\mathbf{M}_\infty$ *as an* Ind_κ*-completion of* $\mathbf{M}_{\kappa,\infty}$*.*

Proof. The fact that $\mathbf{M}_{\kappa,\infty}$ admits κ-small colimits follows from the argument of [Cis19a, Proposition 7.7.4]. To show that ι induces a categorical equivalence $\mathrm{Ind}_\kappa(\mathbf{M}_{\kappa,\infty}) \xrightarrow{\simeq} \mathbf{M}_\infty$, it suffices to prove the following:

(1) The functor ι is fully faithful.
(2) The essential image of ι generates $\mathbf{M}_\infty$ under κ-filtered colimits.
(3) The image of ι lies in $(\mathbf{M}_\infty)_\kappa$.

Claim (1) follows from Proposition 4.9 and [ACK25, Corrollary 3.1], which asserts that the classical derived mapping spaces (computed using simplicial resolutions objects) computes the mapping spaces of the localization of model categories.

For (2), we observe that weak equivalences of $\mathbf{M}$ are stable under κ-filtered colimits [RR15, Proposition 4.1]. This implies that κ-filtered colimits in $\mathbf{M}$ are already homotopy colimits, so we only have to show that $\mathbf{M}_\kappa$ generates $\mathbf{M}$ under κ-filtered colimits. This is clear, because $\mathbf{M}$ is locally κ-presentable.

For (3), suppose we are given a κ-filtered ∞-category $\mathcal{I}$ and a colimit diagram $\overline{F}\colon \mathcal{I}^\rhd \to \mathbf{M}_\infty$. We must show that, for every object $X \in \mathbf{M}_\kappa$, the diagram $\mathbf{M}_\infty(X,-) \circ \overline{F}\colon \mathcal{I}^\rhd \to \mathcal{S}$ is a colimit diagram. Using [Lur25, Tag 02QA], we may assume that $\mathcal{I}$ is (the nerve of) a poset. In this case, the functor $\mathrm{Fun}(\mathcal{I}^\rhd, \mathbf{M}) \to \mathrm{Fun}(\mathcal{I}^\rhd, \mathbf{M}_\infty)$ is a localization [Cis19a, Theorem 7.9.8], so we may assume that $\overline{F}$ lifts to a diagram $\overline{G}\colon \mathcal{I}^\rhd \to \mathbf{M}$. Without loss of generality, we may assume that $\overline{G}$

takes values in the full subcategory of fibrant objects. Set $G = \overline{G}|\mathcal{J}$. Since weak equivalences of $\mathbf{M}$ are stable under κ-filtered colimits, the map

$$\operatorname{colim}_{\mathcal{J}} G \to \overline{G}(\infty)$$

is a weak equivalence. Moreover, since fibrations of $\mathbf{M}$ are stable under κ-filtered colimits, the object $\operatorname{colim}_{\mathcal{J}} G \in \mathbf{M}$ is fibrant. Thus, we may assume that $\overline{G}$ is a strict colimit diagram.

Now choose a cosimplicial resolution $X^\bullet$ of X in $\mathbf{M}_\kappa$. By [ACK25, Corollary 3.1], the functor $\mathbf{M}(X^\bullet, -)\colon \mathbf{M}_{\mathrm{fib}} \to \mathsf{sSet}$ descends to the functor $\mathbf{M}_\infty \to \mathcal{S}$ corepresented by X. Therefore, we are reduced to showing that the diagram $\mathbf{M}\big(X^\bullet, \overline{G}\big)\colon \mathcal{J}^\triangleright \to \mathsf{sSet}$ is a homotopy colimit diagram. This is clear, since each $X^\bullet$ is κ-compact and κ-filtered colimits are homotopy colimits in sSet. The proof is now complete. $\square$

Corollary 4.12. *Let κ be a regular cardinal, and let $\mathbf{M}$ be a symmetric $\mathbf{S}$-algebra whose underlying model category is strongly κ-combinatorial. The inclusion $\mathbf{M}_{\kappa,\mathrm{cof}} \hookrightarrow \mathbf{M}$ induces a left Quillen equivalence of $\mathbf{S}$-algebras*

$$\theta\colon \mathbf{Ind}^{\mathbf{S}}_\kappa(L(\mathbf{M}_{\kappa,\mathrm{cof}})) \xrightarrow{\simeq} \mathbf{M}$$
$$X \mapsto \int^{M \in \mathbf{M}_{\kappa,\mathrm{cof}}} M \otimes X(M).$$

Proof. We first show that θ induces a left Quillen functor

$$\theta'\colon \operatorname{Fun}\Big(\mathbf{M}^{\mathrm{op}}_{\kappa,\mathrm{cof}}, \mathbf{S}\Big) \to \mathbf{M},$$

where $\operatorname{Fun}\Big(\mathbf{M}^{\mathrm{op}}_{\kappa,\mathrm{cof}}, \mathbf{S}\Big)$ carries the projective model structure. Since $\mathbf{M}$ is strongly κ-combinatorial, it has a generating set I of cofibrations consisting of maps in $\mathbf{M}_\kappa$. The maps $\{\mathbf{M}(-, M) \otimes i\}_{M \in \mathbf{M}_{\kappa,\mathrm{cof}},\, i \in I}$ generate projective cofibrations of $\operatorname{Fun}\Big(\mathbf{M}^{\mathrm{op}}_{\kappa,\mathrm{cof}}, \mathbf{S}\Big)$ [Hir03, Theorem 11.6.1]. The images of these maps under θ are simply the maps $\{M \otimes i\}_{M \in \mathbf{M}_{\kappa,\mathrm{cof}},\, i \in I}$, which are all cofibrations. Hence θ carries projective cofibrations to cofibrations. A similar argument shows that θ carries projective trivial cofibrations to trivial cofibrations. Hence θ' is left Quillen, as claimed.

We now consider the following diagram, which commutes up to natural equivalence:

$$\begin{tikzcd}
\mathbf{M}_{\kappa,\mathrm{cof}} \arrow[r] \arrow[d] & \mathbf{M}_{\kappa,\infty} \arrow[d, "i"] \\
\operatorname{Fun}(\mathbf{M}^{\mathrm{op}}_{\kappa,\mathrm{cof}}, \mathbf{S})_\infty \arrow[r, "\theta'_\infty"] \arrow[d] & \mathbf{M}_\infty \\
\mathbf{Ind}^{\mathbf{S}}_\kappa(L(\mathbf{M}_{\kappa,\mathrm{cof}}))_\infty. &
\end{tikzcd}$$

According to Proposition 4.11, the functor ι exhibits $\mathbf{M}_\infty$ as an Ind_κ-completion of $\mathbf{M}_{\kappa,\infty}$. Also, by what we have just shown in the previous paragraph, the functor θ'_∞ preserves small colimits. It follows that θ'_∞ can be identified with the functor

$$\operatorname{Fun}\Big(\mathbf{M}^{\mathrm{op}}_{\kappa,\mathrm{cof}}, \mathcal{S}\Big) \to \mathrm{Ind}_\kappa(\mathbf{M}_{\kappa,\mathrm{cof}})$$

described in Construction 4.3. By the definition of weak equivalences of $\mathbf{Ind}^{\mathbf{S}}_\kappa(L(\mathbf{M}_{\kappa,\mathrm{cof}}))$, this means that θ preserves weak equivalences of cofibrant objects of $\mathbf{Ind}^{\mathbf{S}}_\kappa(L(\mathbf{M}_{\kappa,\mathrm{cof}}))$. Since we already know that θ' is a left Quillen functor, and since $\mathbf{Ind}^{\mathbf{S}}_\kappa(L(\mathbf{M}_{\kappa,\mathrm{cof}}))$

is tractable, this is enough to conclude that θ is a left Quillen functor. The resulting functor

$$\theta_\infty\colon \operatorname{Ind}^{\mathbf{S}}_\kappa(L(\mathbf{M}_{\kappa,\mathrm{cof}}))_\infty \to \mathbf{M}_\infty$$

is an equivalence, because both sides are the $\operatorname{Ind}_\kappa$-completion of $\mathbf{M}_{\kappa,\infty}$. Hence θ is a left Quillen equivalence, and we are done. □

We conclude this subsection by taking monoidal structures into account.

Definition 4.13. Let κ be a regular cardinal. A **monoidally κ-combinatorial model category** is a monoidal model category $\mathbf{M}$ satisfying the following conditions:

- As a model category, $\mathbf{M}$ is strongly κ-combinatorial.
- κ-compact objects are stable under finite tensor products. (In particular, the unit object is κ-compact.)

We write $\mathsf{CombMMC}(\kappa) \subset \mathsf{CombMMC}$ for the subcategory spanned by the monoidally κ-combinatorial model category and those maps that preserve κ-compact objects.

Lemma 4.14.

(1) For every pair of regular cardinals $\kappa \lhd \lambda$, we have $\mathsf{CombMMC}(\kappa) \subset \mathsf{CombMMC}(\lambda)$.
(2) Every morphism of $\mathsf{CombMMC}$ belongs to $\mathsf{CombMMC}(\kappa)$ for some regular cardinal κ.

Proof. We first show that every combinatorial monoidal model category $\mathbf{M}$ is monoidally κ-combinatorial for some κ, and if this is true, then $\mathbf{M}$ is monoidally λ-combinatorial for every $\lambda \rhd \kappa$. By Proposition 4.9, there is some regular cardinal κ_0 such that $\mathbf{M}$ is strongly κ_0-combinatorial as a model category. Find a regular cardinal $\kappa_0 \lhd \kappa$ such that, if $X, Y \in \mathbf{M}_{\kappa_0}$, then $X \otimes Y \in \mathbf{M}_\kappa$. Since every κ-compact object is a κ-small colimit of κ_0-compact object [MP89, Theorem 2.3.11], this ensures that κ-compact objects is stable under tensor product. Hence $\mathbf{M}$ is monoidally κ-combinatorial. A similar argument, using Proposition 4.10, shows that every monoidally κ-combinatorial model category is monoidally λ-combinatorial.

We now prove (1) and (2). Part (1) follows from the result in the previous paragraph and [AR94, Remark 2.20]. For (2), let $F\colon \mathbf{M} \to \mathbf{N}$ be a morphism of $\mathsf{CombMMC}$. By the result in the previous paragraph and part (1), we can find some κ_0 such that $\mathbf{M}$ and $\mathbf{N}$ are monoidally κ_0-combinatorial. Choose $\kappa \rhd \kappa_0$ such that $F(\mathbf{M}_{\kappa_0}) \subset \mathbf{N}_\kappa$. We claim that F belongs to $\mathsf{CombMMC}(\kappa)$.

Since every κ-compact object is a κ-small colimit of κ_0-compact object [MP89, Theorem 2.3.11], our choice of κ and κ_0 ensure that $F(\mathbf{M}_\kappa) \subset \mathbf{N}_\kappa$. We also know from (1) that $\mathbf{M}$ and $\mathbf{N}$ are monoidally κ-combinatorial from. Thus F belongs to $\mathsf{CombMMC}(\kappa)$, as desired. □

4.3. **Proof of Proposition 2.6.** We now arrive at the proof of Proposition 2.6.

Notation 4.15. We write urCard for the (large) poset of (small) uncountable regular cardinals, ordered by $\lhd$. Given a functor $F\colon \mathsf{urCard} \to \widehat{\mathsf{Cat}}$, we write $\int^{\mathsf{urCard}} F \to \mathsf{urCard}$ for the Grothendieck construction of F. Likewise, given a functor $G\colon \mathsf{urCard}^{\mathrm{op}} \to \widehat{\mathsf{Cat}}$, its Grothendieck construction will be denoted by $\int_{\mathsf{urCard}} G \to \mathsf{urCard}$.

Notation 4.16. We let $\mathsf{CombSym}\mathbf{S}\text{-}\mathsf{Alg}^{\mathrm{sk}} \subset \mathsf{CombSym}\mathbf{S}\text{-}\mathsf{Alg}$ denote the full subcategory of $\mathbf{S}$-algebras $\mathbf{M}$ whose underlying category is skeletal (i.e., isomorphic objects are equal). Note that for such an $\mathbf{M}$, the full subcategory $\mathbf{M}_\kappa$ of κ-compact objects is *literally* small for every κ.

We define a full subcategory $\mathsf{CombSym}\mathbf{S}\text{-}\mathsf{Alg}^{\mathrm{sk}}_{(2,1)} \subset \mathsf{CombSym}\mathbf{S}\text{-}\mathsf{Alg}_{(2,1)}$ similarly.

Proof of Proposition 2.6. By Proposition 1.8 and Lemma 4.7, it suffices to show that the functor

$$\mathsf{CombSym}\mathbf{S}\text{-}\mathsf{Alg}^{\mathrm{sk}}_{(2,1)}[\mathrm{Quill.eq}^{-1}] \to \mathsf{SMRel}\widehat{\mathsf{Cat}}_{(2,1)}[\mathrm{loc.eq}^{-1}]$$

is an equivalence.

For each uncountable regular cardinal κ, let $\mathsf{CombSym}\mathbf{S}\text{-}\mathsf{Alg}(\kappa)^{\mathrm{sk}}$ denote the fiber product

$$\mathsf{CombSym}\mathbf{S}\text{-}\mathsf{Alg}^{\mathrm{sk}} \times_{\mathsf{CombMMC}} \mathsf{CombMMC}(\kappa).$$

We now consider the following diagram:

$$\begin{tikzcd}
\int^{\mathsf{urCard}} \mathsf{CombSym}\mathbf{S}\text{-}\mathsf{Alg}(-)^{\mathrm{sk}} \arrow[r, "F"] \arrow[d, "U"'] & \int_{\mathsf{urCard}} \mathsf{SMRelCat}_{\mathcal{SMC}\mathsf{at}_\infty(-)} \arrow[d, "(-)_\flat \circ \mathbf{Ind}^{\mathbf{S}}"] \arrow[dl, "\mathbf{Ind}^{\mathbf{S}}"'] \\
\mathsf{CombSym}\mathbf{S}\text{-}\mathsf{Alg}^{\mathrm{sk}}_{(2,1)} \arrow[r, "(-)_\flat"'] & \mathsf{SMRel}\widehat{\mathsf{Cat}}_{\mathcal{P}\mathrm{r}\mathcal{SM},(2,1)}.
\end{tikzcd}$$

Here U and F are functors and $\mathbf{Ind}^{\mathbf{S}}$ is a strictly unitary pseudofunctor, defined by the formulas

$$\begin{aligned}
F(\kappa, \mathbf{M}) &= (\kappa, \mathbf{M}_{\kappa,\mathrm{cof}}),\\
U(\kappa, \mathbf{M}) &= \mathbf{M},\\
\mathbf{Ind}^{\mathbf{S}}(\kappa, \mathcal{C}) &= \mathbf{Ind}^{\mathbf{S}}_{\kappa}(L(\mathcal{C})).
\end{aligned}$$

The lower triangle commutes by construction; the upper triangle does not commute on the nose, but Corollary 4.12 gives a pseudonatural Quillen equivalence $\mathbf{Ind}^{\mathbf{S}} \circ F \xrightarrow{\simeq} U$. Thus, to prove the claim, it suffices to show that U and $(-)_\flat \circ \mathbf{Ind}^{\mathbf{S}}$ induce equivalences of ∞-categories when localized at the maps whose images in $\mathcal{P}\mathrm{r}\mathcal{SM}$ are equivalences.

We start from the claim on U. According to Lemma 4.14, the category $\mathsf{CombSym}\mathbf{S}\text{-}\mathsf{Alg}$ is the colimit of $\mathsf{CombSym}\mathbf{S}\text{-}\mathsf{Alg}(\kappa)$ as κ ranges over urCard. It follows from [Lur25, Tag 02UU] that U is already a localization.

Next, for $(-)_\flat \circ \mathbf{Ind}^{\mathbf{S}}$, we factor it as

$$\int_{\mathsf{urCard}} \mathsf{SMRelCat}_{\mathcal{SMC}\mathsf{at}_\infty(-)} \xrightarrow{\Phi} \int_{\mathsf{urCard}} \mathsf{SMRel}\widehat{\mathsf{Cat}}_{(-)\text{-}\mathcal{P}\mathrm{r}\mathcal{SM},(2,1)} \xrightarrow{\Psi} \mathsf{SMRel}\widehat{\mathsf{Cat}}_{\mathcal{P}\mathrm{r}\mathcal{SM}},$$

where $\int^{\mathsf{urCard}} \mathsf{SMRel}\widehat{\mathsf{Cat}}_{(-)\text{-}\mathcal{P}\mathrm{r}\mathcal{SM},(2,1)}$ denotes the cartesian fibration associated with the functor $\kappa \mapsto \mathsf{SMRel}\widehat{\mathsf{Cat}}_{\kappa\text{-}\mathcal{P}\mathrm{r}\mathcal{SM},(2,1)}$, concretely realized as (the dual of) Lurie's relative nerve [Lur09, Definition 3.2.5.2]. The functor Φ is induced by the functors $\{\mathbf{Ind}^{\mathbf{S}}_{\kappa}(L(-))_\flat\}_\kappa$, and Ψ is the forgetful functor. As in the previous paragraph, the functor Ψ is already a localization, so it suffices to verify that Φ induces a localization upon localizing at the maps whose images in $\mathcal{P}\mathrm{r}\mathcal{SM}$ are equivalences. This follows from (the dual of) [Ara, Corollary B.6] and Proposition 4.6, which show more strongly that it induces a localization when localizing at fiberwise local equivalences. The proof is now complete. ☐

5. Proof of Proposition 2.7

We finally turn to the proof of Proposition 2.7. In light of Propositions 2.5 and 2.6, it will suffice to prove this for one *specific* choice of $\mathbf{S}$. To this end, we will construct a new model category of *symmetric cubical sets* in Subsection 5.1. We then use this to prove Proposition 2.7 in Subsection 5.2.

5.1. **Symmetric cubical sets.** In this subsection, we construct the symmetric monoidal model category of symmetric cubical sets (Theorem 5.10). It models the homotopy theory of spaces, in the sense that its underlying symmetric monoidal ∞-category is equivalent to the cartesian monoidal ∞-category of spaces. We will see that every combinatorial symmetric monoidal model category with cofibrant unit can be enriched over symmetric cubical sets in an essentially unique manner (Corollary 5.12).

Remark 5.1. Part of the contents of this subsection is similar in spirit to Isaacson's papers [Isa11, Isa09]. In these papers, he considered symmetric cubical sets by using a slightly bigger category of symmetric cubes. He then showed that every combinatorial symmetric monoidal model category with cofibrant unit and satisfying the monoid axiom can be enriched over symmetric cubical sets [Isa09, Theorem 10.1]. However, his approach does not seem to give an essential uniqueness of this enrichment.

We start by recalling the classical cube category and the Grothendieck model structure.

Definition 5.2. The **box category** $\square$ has the following descriptions:

- Objects are the posets $[1]^n$, where $n \geq 0$.
- A morphism $f\colon [1]^n \to [1]^m$ is a poset map that erases coordinates and inserts 0 and 1 without changing the order of the coordinates. Equivalently, they are the generated under composition by the following maps:
  1. The **face map** $\delta^{i,\varepsilon} = \delta^{i,\varepsilon}_n\colon [1]^n \to [1]^{n+1}$ for $n \geq 0$ and $1 \leq i \leq n+1$, which inserts $\varepsilon \in \{0,1\}$ in the ith coordinate.
  2. The **degeneracy map** $\sigma^i_n\colon [1]^{n+1} \to [1]^n$ for $n \geq 0$ and $1 \leq i \leq n+1$, which deletes the ith coordinate.

The cartesian product of posets makes $\square$ into a monoidal category. It is also naturally a Reedy category. The category $\mathsf{Set}^{\square^{\mathrm{op}}}$ of **cubical sets** of admits an induced monoidal structure, given by Day's convolution. We will write $\square_{\leq 1} \subset \square$ for the full subcategory spanned by the objects $[1]^n$ with $n \leq 1$. The presheaf represented by $[1]^n$ will be denote by $\square^n$.

The category $\mathsf{Set}^{\square^{\mathrm{op}}}$ can be endowed with a model structure which models the homotopy theory of spaces. To state it more precisely, let $\partial\square^n = \bigcup_{i,\varepsilon} \delta^{i,\varepsilon}\big(\square^{n-1}\big)$ and $\sqcap^n_{i,\varepsilon} = \bigcup_{(j,\eta)\neq(i,\varepsilon)} \delta^{i,\varepsilon}\big(\square^{n-1}\big)$. We consider the following sets of morphisms of cubical sets:

$$I = \{\partial\square^n \to \square^n \mid n \geq 0\},\ J = \{\sqcap^n_{i,\varepsilon} \to \square^n \mid n \geq 1,\ 1 \leq i \leq n,\ \varepsilon \in \{0,1\}\}.$$

Theorem 5.3. [Cis06], [Jar06, Theorem 6.2, Theorem 8.6, Theorem 8.8] *The category* $\mathsf{Set}^{\square^{\mathrm{op}}}$ *of* ***cubical sets*** *has a combinatorial monoidal model structure called the* ***Grothendieck model structure****, whose:*

- *cofibrations are the monomorphisms.*
- *weak equivalences are preserved and detected by the triangulation functor* $\mathsf{Set}^{\square^{\mathrm{op}}} \to \mathsf{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$*, which is the left adjoint carrying* $\square^n \mapsto \big(\Delta^1\big)^n$*.*

Moreover, I *and* J *cofibrantly generate this model structure, and* T *is a left Quillen equivalence with respect to the Kan–Quillen model structure on simplicial sets.*

Our goal for now is to consider the symmetric monoidal version of this.

Definition 5.4. We define the symmetric monoidal category of **symmetric box category** $\square_\Sigma$ by adjoining to the box category the permutation map $\pi_p\colon [1]^n \to [1]^n$ for each $n \geq 0$ and $p \in \Sigma_n$, defined by $\pi_p(x_1,\dots,x_n) = \big(x_{p^{-1}(1)},\dots,x_{p^{-1}(n)}\big)$.

There is an inclusion $i\colon \square \to \square_\Sigma$. We denote by $\square^n_\Sigma$ the presheaves on $\square_\Sigma$ represented by $[1]^n$. Note that $\square_{\leq 1}$ is a full subcategory of $\square_\Sigma$.

Notation 5.5. The following proposition is immediate from the definitions:

Proposition 5.6. *We have the following **cocubical relations** for maps in $\square_\Sigma$:*

- $\delta^{j,\eta}\delta^{i,\varepsilon} = \delta^{i,\varepsilon}\delta^{j-1,\eta}$ *for $j \leq i$.*
- $\sigma^j\delta^{i,\varepsilon} = \begin{cases} \delta^{i,\varepsilon}\sigma^{j-1} & \text{if } i < j, \\ \mathrm{id} & \text{if } i = j, \\ \delta^{i-1}\sigma^j & \text{if } i > j. \end{cases}$
- $\sigma^j\sigma^i = \sigma^i\sigma^{j+1}$ *if $i \leq j$.*
- $\pi_p\pi_q = \pi_{pq}$.
- $\pi_p\delta^{i,\varepsilon} = \delta^{p(i)}\pi_q$, *where q denotes the composite*
$$\{1,\dots,n\} \cong \{1,\dots,n+1\} \setminus \{i\} \xrightarrow{p} \{1,\dots,n+1\} \setminus \{p(i)\} \cong \{1,\dots,n\}.$$
- $\sigma^i\pi_p = \pi_q\sigma^{p^{-1}(i)}$, *where q denotes the composite*
$$\{1,\dots,n-1\} \cong \{1,\dots n\} \setminus \{p^{-1}(i)\} \xrightarrow{p} \{1,\dots,n\} \setminus \{i\} \cong \{1,\dots,n-1\}.$$

Moreover, every map in $\square_\Sigma$ factors uniquely as
$$\delta^{i_1,\varepsilon_1}\cdots\delta^{i_n\varepsilon_n}\pi_p\sigma^{j_1}_l\cdots\sigma^{j_m},$$
with $i_1 > \cdots i_n$, $j_1 < \cdots < j_m$, and $p \in \Sigma_l$.

Corollary 5.7. *The symmetric cube category $\square_\Sigma$ is generated by the face maps, degeneracy maps, and permutation maps under the cocubical relations. A similar claim holds for the classical cube category.*

Corollary 5.8. *Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal category. The evaluation at the unit object $[1] \in \square_\Sigma$ determines a categorical equivalence*
$$\theta\colon\ \mathrm{Fun}^{\otimes}(\square_\Sigma, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}(\square_{\leq 1}, \mathcal{C}) \times_{\mathrm{Fun}(\{[1]^0\},\mathcal{C})} \{\mathbf{1}\}.$$
A similar claim holds for monoidal categories and the plain cube category.

Proof. The functor θ needs a bit of explanation. If $F\colon \square_\Sigma \to \mathcal{C}$ is a symmetric monoidal functor, then its image in $\mathrm{Fun}(\square_{\leq 1}, \mathcal{C}) \times_{\mathrm{Fun}(\{[1]^0\},\mathcal{C})} \{\mathbf{1}\}$ is given by the functor $\square_{\leq 1} \to \mathcal{C}$ obtained by modifying the value of F at $[1]^0$ to $\mathbf{1}$ using the structure map $\mathbf{1} \xrightarrow{\cong} F\big([1]^0\big)$ of F.

With this in mind, we will give an explicit inverse equivalence to θ. Given a functor $F\colon \square_{\leq 1} \to \mathcal{C}$ carrying $[1]^0$ to the unit object, we can define a new functor $\widetilde{F}\colon \square_\Sigma \to \mathcal{C}$ by $\widetilde{F}([1]^n) = F([1])^{\otimes n}$, with structure maps given by that of F and the braiding of $\mathcal{C}$ (Corollary 5.7). The coherence maps of $\mathcal{C}$ makes $\widetilde{F}$ into a symmetric monoidal functor. The functor $F \mapsto \widetilde{F}$ is an inverse equivalence of θ. $\square$

Corollary 5.9. *Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a cocomplete symmetric monoidal category whose tensor product preserves small colimits in each variable. The evaluation at the unit object $\square^0_\Sigma \in \mathsf{Set}^{\square^{\mathrm{op}}_\Sigma}$ induces a categorical equivalence*
$$\mathrm{Fun}^{\otimes,cc}\Big(\mathsf{Set}^{\square^{\mathrm{op}}_\Sigma}, \mathcal{C}\Big) \xrightarrow{\simeq} \mathrm{Fun}(\square_{\leq 1}, \mathcal{C}) \times_{\mathrm{Fun}(\{[1]^0\},\mathcal{C})} \{\mathbf{1}\},$$
where $\mathrm{Fun}^{\otimes,cc}\Big(\mathsf{Set}^{\square^{\mathrm{op}}_\Sigma}, \mathcal{C}\Big)$ denotes the category of symmetric monoidal categories whose underlying functors preserve small colimits, and symmetric monoidal natural transformations between them.

Proof. This follows from Corollary 5.8 and the universal property of Day convolution product monoidal structure. $\square$

For the following theorem, we will write $\partial\square^n_\Sigma \subset \square^n_\Sigma$ for the subpresheaf consisting of the maps $[1]^k \to [1]^n$ that inserts at least one 0 or 1. Equivalently, we have $i_!(\partial\square^n) = \partial\square^n_\Sigma$.

Theorem 5.10. *The category $\mathsf{Set}^{\square^{\mathrm{op}}_\Sigma}$ has a tractable model structure whose weak equivalences and fibrations are preserved and detected by the forgetful functor $i^*\colon \mathsf{Set}^{\square^{\mathrm{op}}_\Sigma} \to \mathsf{Set}^{\square^{\mathrm{op}}}$. Moreover:*

(I) The adjunction

$$i_!\colon \mathsf{Set}^{\square^{\mathrm{op}}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathsf{Set}^{\square^{\mathrm{op}}_\Sigma} : i^*$$

is a Quillen equivalence.

(II) The model structure on $\mathsf{Set}^{\square^{\mathrm{op}}_\Sigma}$ is symmetric monoidal with respect to the Day convolution.

(III) The model structure is left proper.

(IV) If $\mathbf{M}$ is a symmetric monoidal model category and $F\colon \mathsf{Set}^{\square^{\mathrm{op}}_\Sigma} \to \mathbf{M}$ is a symmetric monoidal functor which is also a left adjoint, then F is left Quillen precisely when the map $F\big(\partial\square^1_\Sigma \to \square^1_\Sigma\big)$ is a cofibration, the map $F\big(\square^1_\Sigma\big) \to F\big(\square^0_\Sigma\big)$ is a weak equivalence, and the monoidal unit of $\mathbf{M}$ is cofibrant.

(V) The assignment $\square^n_\Sigma \mapsto \big(\Delta^1\big)^n$ determines a symmetric monoidal left Quillen equivalence $\mathsf{Set}^{\square^{\mathrm{op}}_\Sigma} \to \mathsf{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$.

The proof of Theorem 5.10 relies on the following lemma.

Lemma 5.11. *The diagram*

$$\begin{tikzcd}
\mathsf{Set}^{\square^{\mathrm{op}}} \arrow[r, hook] \arrow[d] & \mathcal{S}^{\square^{\mathrm{op}}} \arrow[d, "\operatorname{colim}_{\square^{\mathrm{op}}}"] \\
\mathsf{Set}^{\square^{\mathrm{op}}}[\mathrm{weq}^{-1}] \arrow[r, "\simeq"] & \mathcal{S}
\end{tikzcd}$$

commutes up to natural equivalence.

Proof. We first give a model for $\operatorname{colim}_{\square^{\mathrm{op}}}$. Consider the functor

$$\Phi\colon\ \operatorname{Fun}\Big(\square^{\mathrm{op}}, \mathsf{Set}^{\square^{\mathrm{op}}}\Big) \to \mathsf{Set}^{\square^{\mathrm{op}}},\ Y \mapsto \int^{[1]^n\in\square} Y_n \otimes \square^n.$$

Since the cocubical object $\square^\bullet \in \operatorname{Fun}\Big(\square, \mathsf{Set}^{\square^{\mathrm{op}}}\Big)$ is Reedy cofibrant, the functor Φ is left Quillen for the Reedy model structure [Lur09, Proposition A.2.9.26]. The right adjoint of this functor, given by $K \mapsto K^{\square^\bullet}$, is weakly equivalent to that of the diagonal functor on the full subcategory of fibrant objects. So Φ is a model of the homotopy colimit, in the sense that the induced functor functor

$$\mathcal{S}^{\square^{\mathrm{op}}} \to \mathcal{S}$$

is equivalent to $\operatorname{colim}_{\square^{\mathrm{op}}}$.

Now since every cubical set X is Reedy cofibrant when regarded as a level-wise discrete cubical object in $\mathsf{Set}^{\square^{\mathrm{op}}}$, the above argument shows that the functor $\Phi'\colon \mathsf{Set}^{\square^{\mathrm{op}}} \to \mathsf{Set}^{\square^{\mathrm{op}}}$, $X \mapsto \Phi(X)$ descends to the composite

$$\mathsf{Set}^{\square^{\mathrm{op}}} \to \mathcal{S}^{\square^{\mathrm{op}}} \to \mathcal{S}.$$

The coYoneda lemma shows that Φ' is naturally isomorphic to the identity functor, and the claim follows. $\square$

Proof of Theorem 5.10. We start by showing that the model structure on $\mathsf{Set}^{\square^{\mathrm{op}}}$ transfers to a model structure on $\mathsf{Set}^{\square_\Sigma^{\mathrm{op}}}$, using [Hir03, Theorem 11.3.2]. Let $I = \{\partial\square^n \to \square^n \mid n \geq 0\}$ and $J = \{\sqcap^n_{i,\varepsilon} \to \square^n \mid n \geq 1,\, 1 \leq i \leq n,\, \varepsilon \in \{0,1\}\}$ denote the generating cofibrations and trivial cofibrations of $\mathsf{Set}^{\square^{\mathrm{op}}}$. We wish to show that i^* takes $i_!J$-cell complexes to weak equivalences. For this, it will suffice to show that each of the map in $i^*i_!J$ is a trivial cofibration. The cofibration part is clear, because $i_!$ and i^* preserve monomorphisms. Consequently, it will suffice to show that each of the maps in $i^*i_!J$ is a weak equivalence. We will prove this by showing that the unit $\eta\colon \mathrm{id} \to i^*i_!$ is a natural weak equivalence.

By the standard argument using the skeletal filtration, it will suffice to show that $\eta_{\square^n}\colon \square^n \to i^*i_!\square^n$ is a weak equivalence for all n. In other words, our task it to prove that each $i^*i_!\square^n$ is weakly contractible. By Lemma 5.11 and [Lur09, Corollary 3.3.4.6], this is equivalent to the condition that the category $\square_{/i^*i_!\square^n} \cong \square \times_{\square_\Sigma} \square_{\Sigma/\square^n_\Sigma}$ be weakly contractible for all n. Thus, it suffices to show that the functor $i\colon \square \to \square_\Sigma$ is homotopy initial. We prove this as follows: Consider the commutative diagram

$$\begin{tikzcd} & \square_+ \arrow[dl, "j"'] \arrow[dr, "k"] & \\ \square \arrow[rr, "i"'] & & \square_\Sigma \end{tikzcd}$$

where $\square_+ \subset \square$ denotes the subcategory spanned by the face maps. Since homotopy initial functors have the right cancellation property [Cis19b, Corollary 4.1.9], it will suffice to show that j and k are homotopy initial. We will show that j is homotopy initial; the proof for k is similar. Our task is to show that the category $\square_+ \times_\square \square_{/\square^n}$ is weakly contractible for every $n \geq 0$. But the inclusion $\square_+ \times_\square \square_{/\square^n} \hookrightarrow \square_{/\square^n}$ admits a homotopy inverse, which factors a map as a composition of degeneracy maps followed by a composition of face maps. So we are reduced to showing that $\square_{/\square^n}$ is weakly contractible, which is clear.

We now prove (I) through (V). For (I), observe that we have just shown that the derived unit is an isomorphism. Since $\mathbb{R}i^*$ is conservative by construction, the triangle identities show that the derived counit is also an isomorphism. Hence the adjunction is a Quillen equivalence. Part (II) follows from the fact that the model structure on $\mathsf{Set}^{\square^{\mathrm{op}}}$ is monoidal for the Day convolution. Part (III) follows from the left properness of $\mathsf{Set}^{\square^{\mathrm{op}}}$, since i^* preserves cofibrations and small colimits and detects weak equivalences. Part (IV) is proved exactly as in [Law17, Corollary 1.5]. Finally, part (V) follows from parts (I, IV) and Theorem 5.3. $\square$

Corollary 5.12. *Every cofibrantly generated symmetric monoidal model category* $\mathbf{M}$ *with a cofibrant unit is a symmetric* $\mathsf{Set}^{\square_\Sigma^{\mathrm{op}}}$*-algebra in an essentially unique way. More precisely, the category* $\mathrm{Fun}^{\otimes,LQ}\left(\mathsf{Set}^{\square_\Sigma^{\mathrm{op}}}, \mathbf{M}\right)$ *is weakly contractible.*

Proof. Define a full subcategory $\mathcal{X}$ of

$$\mathrm{Fun}(\square_{\leq 1}, \mathbf{M})_{\mathbf{1}} = \mathrm{Fun}(\square_{\leq 1}, \mathbf{M}) \times_{\mathrm{Fun}(\{[1]^0\}, \mathbf{M})} \{\mathbf{1}\}$$

as follows: An object of $F \in \mathrm{Fun}(\square_{\leq 1}, \mathbf{M})_{\mathbf{1}}$ can be identified with a diagram of the form

$$\begin{tikzcd} \mathbf{1} \arrow[dr] \arrow[drr, bend left, "\mathrm{id}"] & & \\ & F([1]) \arrow[r] & \mathbf{1}. \\ \mathbf{1} \arrow[ur] \arrow[urr, bend right, "\mathrm{id}"'] & & \end{tikzcd}$$

We declare that such a diagram belongs to $\mathcal{X}$ if and only if the map $\mathbf{1} \amalg \mathbf{1} \to F([1])$ is a cofibration, and the map $F([1]) \to \mathbf{1}$ is a weak equivalence.

According to Theorem 5.10 and Corollary 5.9, there is an equivalence of categories

$$\mathrm{Fun}^{\otimes, LQ}\left(\mathsf{Set}^{\square_\Sigma^{\mathrm{op}}}, \mathbf{M}\right) \xrightarrow{\simeq} \mathcal{X}.$$

Therefore, it suffices to show that $\mathcal{X}$ is weakly contractible. To this end, we define a functor $\Phi\colon \mathrm{Fun}(\square_{\leq 1}, \mathbf{M})_{\mathbf{1}} \to \mathrm{Fun}(\square_{\leq 1}, \mathbf{M})_{\mathbf{1}}$ as follows: It carries an object $(\mathbf{1} \amalg \mathbf{1} \to I \to \mathbf{1})$ to an object $(\mathbf{1} \amalg \mathbf{1} \to I' \to \mathbf{1})$, where I' is obtained by (functorially) factoring the map $\mathbf{1} \amalg \mathbf{1} \to I$ as a cofibration $\mathbf{1} \amalg \mathbf{1} \rightarrowtail I'$ followed by a weak equivalence $I' \xrightarrow{\simeq} I$. Then Φ restricts to a functor $\Phi_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X}$. The functor $\Phi_{\mathcal{X}}$ admits natural transformations to the identity functor and the constant functor at $\Phi(\mathbf{1} \amalg \mathbf{1} \to \mathbf{1} \to \mathbf{1})$. Therefore, the identity functor of $\mathcal{X}$ can be connected by a zig-zag of natural transformation to a constant functor. Hence $\mathcal{X}$ is weakly contractible, as claimed. □

5.2. **Proof of Proposition 2.7.** We can now give a proof of Proposition 2.7.

Proof of Proposition 2.7. As discussed at the beginning of the section, it will suffice to prove the claim for a specific choice of $\mathbf{S}$.

Take $\mathbf{S} = \mathsf{Set}^{\square_\Sigma^{\mathrm{op}}}$. The functor $U\colon \mathsf{CombSym}\mathbf{S}\text{-}\mathsf{Alg} \to \mathsf{CombSMMC}^{\mathbf{1}}$ is a cocartesian fibration, and its fibers are weakly contractible by Corollary 5.12. It follows from [Lur25, Tag 02LY] that U is already a localization, so in particular it induces an equivalence

$$\mathsf{CombSym}\mathbf{S}\text{-}\mathsf{Alg}[\mathrm{Quill.eq}^{-1}] \xrightarrow{\simeq} \mathsf{CombSMMC}^{\mathbf{1}}[\mathrm{Quill.eq}^{-1}].$$

□

Appendix A. Multiplicative Gabriel–Ulmer Duality

The Gabriel–Ulmer duality [GU71] is one precise formulation of the slogan that "locally presentable categories are controlled by small data." In the ∞-categorical setting, it asserts that for every regular cardinal κ, the Ind_κ-completion functor

$$\mathrm{Ind}_\kappa\colon \mathcal{C}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa) \to \mathcal{P}\mathsf{r}^L(\kappa)$$

is an equivalence. Here $\mathcal{C}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa)$ denotes the ∞-category of small idempotent cocomplete ∞-categories with κ-small colimits and functors preserving κ-small colimits, and $\mathcal{P}\mathsf{r}^L(\kappa)$ denotes the ∞-category of κ-presentable ∞-categories.[5]

The goal of this section is to enhance this to an equivalence of symmetric monoidal ∞-categories (Proposition A.5 and Corollary A.6). As a corollary to this, we obtain a Gabriel–Ulmer duality for κ-presentably monoidal ∞-categories (Corollary A.8).

For the following definition, we recall that the ∞-category $\mathcal{P}\mathsf{r}^L$ admits a natural symmetric monoidal structure [Lur17, Proposition 4.8.1.15].

Definition A.1. Let κ be a regular cardinal. We write $\mathcal{P}\mathsf{r}^L(\kappa)^\otimes \subset \left(\mathcal{P}\mathsf{r}^L\right)^\otimes$ for the suboperad spanned by the κ-presentable ∞-categories and those maps $(\mathcal{C}_1, \dots, \mathcal{C}_n) \to \mathcal{C}$ whose corresponding functors $F\colon \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \to \mathcal{C}$ have the following property:

- Given an object $(X_1, \dots, X_n) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_n$, if each X_i is κ-presentable, then so is $F(X_1, \dots, X_n)$.

Given another object $\mathcal{D} \in \mathcal{P}\mathsf{r}^L(\kappa)$, we write $\mathrm{Fun}_{\mathcal{P}\mathsf{r}^L(\kappa)}((\mathcal{C}_1, \dots, \mathcal{C}_n), \mathcal{D}) \subset \mathrm{Fun}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n, \mathcal{D})$ for the full subcategory spanned by the maps $(\mathcal{C}_1, \dots, \mathcal{C}_n) \to \mathcal{D}$ in $\mathcal{P}\mathsf{r}^L(\kappa)^\otimes$.

[5]An ∞-category is **κ-presentable** if it is κ-accessible and presentable.

Our goal for the moment is to show that $\mathcal{P}\mathsf{r}^L(\kappa)^{\otimes}$ is a symmetric monoidal ∞-category, and upgrade the Gabriel–Ulmar duality to a symmetric monoidal equivalence $\mathcal{P}\mathsf{r}^L(\kappa)^{\otimes}$ (Proposition A.5).

Definition A.2. Let κ be a regular cardinal. We write $\mathcal{C}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa)$ for the ∞-category of idempotent complete small ∞-categories with κ-small colimits. We also write $\overline{\mathcal{C}\mathsf{at}}_\infty^{\mathrm{idem}}(\kappa)$ for the ∞-category of large but essentially small, idempotent complete ∞-categories with κ-small colimits. Each of these ∞-categories have a natural symmetric moniodal structure, as explained in [Lur17, Corollary 4.8.1.4]. (Briefly, if $\mathcal{A},\mathcal{B},\mathcal{C} \in \mathcal{C}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa)$, then a giving a map $\mathcal{A}\otimes\mathcal{B}\to\mathcal{C}$ is equivalent to giving a functor $\mathcal{A}\times\mathcal{B}\to\mathcal{C}$ that preserves κ-small colimits in each variable.) We denote these symmetric monoidal ∞-categories by $\mathcal{C}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa)^{\otimes}$ and $\overline{\mathcal{C}\mathsf{at}}_\infty^{\mathrm{idem}}(\kappa)^{\otimes}$. (When κ is uncountable, we drop idem from the notation because idempotent completeness is automatic.)

Our goal for the moment is to show that $\mathcal{P}\mathsf{r}^L(\kappa)^{\otimes}$ is a symmetric monoidal ∞-category, and upgrade the Gabriel–Ulmar duality to a symmetric monoidal equivalence (Proposition A.5).

Construction A.3. For each $\mathcal{C}\in\mathcal{C}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa)$, the **$\kappa$-ind completion** $\mathrm{Ind}_\kappa(\mathcal{C})$ is a κ-presentable ∞-category and is characterized by the following universal property: For every $\mathcal{D}\in\mathcal{P}\mathsf{r}^L(\kappa)$, the map

$$\mathrm{Fun}_{\mathcal{P}\mathsf{r}^L(\kappa)}(\mathrm{Ind}_\kappa(\mathcal{C}),\mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}_{\overline{\mathcal{C}\mathsf{at}}_\infty^{\mathrm{idem}}(\kappa)}(\mathcal{C},\mathcal{D}_\kappa)$$

is a categorical equivalence ([Lur09, Propositions 5.3.5.10 and 5.5.1.9]). It follows that the assignment $\mathcal{C}\mapsto\mathrm{Ind}_\kappa(\mathcal{C})$ can be assembled into a left adjoint $\mathrm{Ind}_\kappa\colon \mathcal{C}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa)\to\mathcal{P}\mathsf{r}^L(\kappa)$. By inspecting the unit and counit, we deduce that this functor is in fact an equivalence of ∞-categories. Using [Lur17, Remark 4.8.1.8], we find that the κ-ind completion functor $\mathrm{Ind}_\kappa\colon\mathcal{C}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa)\to\mathcal{P}\mathsf{r}^L$ can be enhanced to a symmetric monoidal functor $\mathrm{Ind}_\kappa\colon\mathcal{C}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa)^{\otimes}\to\left(\mathcal{P}\mathsf{r}^L\right)^{\otimes}$.

Remark A.4. The proof of [Lur09, Proposition 5.5.1.9] and [Lur09, Propositions 5.3.5.10] give a stronger universal property of $\mathrm{Ind}_\kappa(\mathcal{C})$, where $\mathcal{C}\in\mathcal{C}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa)$. Namely, if $\mathcal{D}$ is an ∞-category with small colimits, the functor

$$\mathrm{Fun}^L(\mathrm{Ind}_\kappa(\mathcal{C}),\mathcal{D})\xrightarrow{\simeq}\mathrm{Fun}_\kappa(\mathcal{C},\mathcal{D})$$

is an equivalence. Here $\mathrm{Fun}^L(\mathrm{Ind}_\kappa(\mathcal{C}),\mathcal{D})\subset\mathrm{Fun}(\mathrm{Ind}_\kappa(\mathcal{C}),\mathcal{D})$denotes the full subcategory of functors preserving small colimits, and $\mathrm{Fun}_\kappa(\mathcal{C},\mathcal{D})\subset\mathrm{Fun}(\mathcal{C},\mathcal{D})$ denotes the full subcategory of functors preserving κ-small colimits.

Proposition A.5. *Let κ be a regular cardinal.*

(1) *The ∞-operad $\mathcal{P}\mathsf{r}^L(\kappa)^{\otimes}$ is a symmetric monoidal ∞-category, and the inclusion $\mathcal{P}\mathsf{r}^L(\kappa)^{\otimes}\hookrightarrow\left(\mathcal{P}\mathsf{r}^L\right)^{\otimes}$ is symmetric monoidal.*
(2) *The symmetric monoidal functor $\mathrm{Ind}_\kappa\colon\mathcal{C}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa)^{\otimes}\to\left(\mathcal{P}\mathsf{r}^L\right)^{\otimes}$ restricts to an equivalence of symmetric monoidal ∞-categories $\mathcal{C}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa)^{\otimes}\xrightarrow{\simeq}\mathcal{P}\mathsf{r}^L(\kappa)^{\otimes}$.*

Proof. We start with (1). Let $\mathcal{C}_1,\dots,\mathcal{C}_n$ be κ-presentable ∞-categories. We wish to show that there is a κ-presentable ∞-category $\mathcal{C}$ and a morphism $(\mathcal{C}_1,\dots,\mathcal{C}_n)\to\mathcal{C}$ in $\mathcal{P}\mathsf{r}^L(\kappa)^{\otimes}$ with the following properties:

I For every $\mathcal{D}\in\mathcal{P}\mathsf{r}^L$, the functor

$$\theta_I\colon\ \mathrm{Fun}_{\mathcal{P}\mathsf{r}^L}(\mathcal{C},\mathcal{D})\to\mathrm{Fun}_{\mathcal{P}\mathsf{r}^L}((\mathcal{C}_1,\dots,\mathcal{C}_n),\mathcal{D})$$

is an equivalence.

II For every $\mathcal{D} \in \mathcal{P}\mathsf{r}^L(\kappa)$, the functor

$$\theta_{II} \colon \operatorname{Fun}_{\mathcal{P}\mathsf{r}^L(\kappa)}(\mathcal{C}, \mathcal{D}) \to \operatorname{Fun}_{\mathcal{P}\mathsf{r}^L(\kappa)}((\mathcal{C}_1, \dots, \mathcal{C}_n), \mathcal{D})$$

is an equivalence.

To this end, choose $\mathcal{A}_i \in \mathcal{C}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa)$ and equivalences $\operatorname{Ind}_\kappa(\mathcal{A}_i) \xrightarrow{\simeq} \mathcal{C}_i$, and set $\mathcal{C} = \operatorname{Ind}_\kappa(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_k)$. Using Remark A.4 repeatedly, we obtain a functor $\alpha \colon \mathcal{C}_1 \times \cdots \times \mathcal{C}_k \to \mathcal{C}$ that preserves colimits in each variable separately and which makes the diagram

$$\begin{tikzcd}
\mathcal{A}_1 \times \cdots \times \mathcal{A}_n \arrow[r] \arrow[d, "\alpha'"'] & \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \arrow[d, "\alpha"] \\
\mathcal{A} \arrow[r] & \mathcal{C}
\end{tikzcd}$$

commutative up to equivalence. The commutativity of the diagram implies that α determines a morphism $\alpha \colon (\mathcal{C}_1, \dots, \mathcal{C}_n) \to \mathcal{C}$ in $\mathcal{P}\mathsf{r}^L(\kappa)^\otimes$. Remark A.4 shows that α satisfies condition (I). It follows that the functor θ_{II} in condition (II) is fully faithful. To show that it is essentially surjective, it will suffice to show that the essential image of α' generates $\mathcal{A}$ under κ-small colimits. This follows from the proof of [Lur17, Proposition 4.8.1.3], proving (1).

Next, we prove (2). Since $\operatorname{Ind}_\kappa \colon \mathcal{C}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa) \to \mathcal{P}\mathsf{r}^L(\kappa)$ is an equivalence of ∞-categories, it suffices to show that $\operatorname{Ind}_\kappa$ carries $\mathcal{C}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa)^\otimes$ into $\mathcal{P}\mathsf{r}^L(\kappa)^\otimes$. (Because this and part (1) will prove that the resulting functor $\mathcal{C}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa)^\otimes \to \mathcal{P}\mathsf{r}^L(\kappa)^\otimes$ is symmetric monoidal.) This is equivalent to saying that $\operatorname{Ind}_\kappa \colon \mathcal{C}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa)^\otimes \to \left(\mathcal{P}\mathsf{r}^L\right)^\otimes$ carries active cocartesian morphisms into $\left(\mathcal{P}\mathsf{r}^L(\kappa)\right)^\otimes$, which follows from the proof of (1). The proof is now complete. □

We conclude this subsection with three corollaries of Proposition A.5.

Corollary A.6. *Let κ be a regular cardinal. The forgetful functor*

$$(-)_\kappa \colon \mathcal{P}\mathsf{r}^L(\kappa)^\otimes \to \overline{\mathcal{C}\mathsf{at}}_\infty^{\mathrm{idem}}(\kappa)^\otimes$$

is an equivalence of symmetric monoidal ∞-categories.

Proof. The composite $\operatorname{Ind}_\kappa \colon \mathcal{C}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa)^\otimes \xrightarrow{\simeq} \mathcal{P}\mathsf{r}^L(\kappa)^\otimes \to \overline{\mathcal{C}\mathsf{at}}_\infty^{\mathrm{idem}}(\kappa)^\otimes$ is equivalent to the inclusion, which is an equivalence of symmetric monoidal ∞-categories. □

Definition A.7. Let $\mathcal{C}$ be a monoidal ∞-category, and let κ be a regular cardinal. We say that $\mathcal{C}$ is **κ-presentably monoidal** if it satisfies the following conditions:

(1) The underlying ∞-category of $\mathcal{C}$ is κ-presentable.
(2) For every $n \geq 0$ and κ-presentable objects $X_1, \dots, X_n \in \mathcal{C}$, the object $X_1 \otimes \cdots \otimes X_n$ is κ-presentable. (In particular, the unit object is κ-presentable).

We write $\kappa\text{-}\mathcal{P}\mathsf{r}\mathcal{S}\mathcal{M} \subset \mathcal{S}\mathcal{M}\widehat{\mathcal{C}\mathsf{at}}_\infty$ for the subcategory of κ-presentably monoidal ∞-categories and monoidal functors that preserves small colimits and κ-presentable objects.

Corollary A.8. *Let κ be a regular cardinal. The forgetful functor*

$$\kappa\text{-}\mathcal{P}\mathsf{r}\mathcal{S}\mathcal{M} \to \mathcal{S}\mathcal{M}\overline{\mathcal{C}\mathsf{at}}^{\mathrm{idem}}(\kappa),\ \mathcal{C}^\otimes \mapsto \mathcal{C}^\otimes_\kappa$$

is a categorical equivalence, where $\mathcal{S}\mathcal{M}\overline{\mathcal{C}\mathsf{at}}_\infty^{\mathrm{idem}}(\kappa) \subset \mathcal{S}\mathcal{M}\widehat{\mathcal{C}\mathsf{at}}_\infty$ denotes the subcategory of essentially small monoidal ∞-categories compatible with κ-small colimits, and those monoidal functors preserving κ-small colimits.

Proof. This follows from Corollary A.6 and the straightening–unstraightening equivalence. □

Corollary A.9. *Let κ be a regular cardinal, and let $I\colon \mathcal{SMC}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa) \to \kappa\text{-}\mathcal{P}\mathsf{r}\mathcal{SM}$ be a functor equipped with a natural transformation depicted as*

$$\begin{tikzcd}
\mathcal{SMC}\mathsf{at}_\infty^{\mathrm{idem}}(\kappa) \arrow[rr, "I"] \arrow[dr, hook] & \arrow[d, phantom, "\overset{\alpha}{\Longrightarrow}"] & \kappa\text{-}\mathcal{P}\mathsf{r}\mathcal{SM} \arrow[dl, hook'] \\
& \mathcal{SM}\widehat{\mathcal{C}\mathsf{at}}_\infty. &
\end{tikzcd}$$

Suppose that, for each $\mathcal{C} \in \mathcal{SMC}\mathsf{at}_\infty(\kappa)$, the map $\alpha_{\mathcal{C}}\colon \mathcal{C} \to I(\mathcal{C})$ exhibits $I(\mathcal{C})$ as an Ind_κ-completion of $\mathcal{C}$. Then I is a categorical equivalence.

Proof. The natural transformation α determines a natural equivalence

$$\mathcal{C} \xrightarrow{\simeq} I(\mathcal{C})_\kappa$$

from the inclusion $\iota\colon \mathcal{SMC}\mathsf{at}_\infty(\kappa) \hookrightarrow \mathcal{SM}\overline{\mathcal{C}\mathsf{at}}_\infty(\kappa)$ to the composite $\mathcal{SMC}\mathsf{at}_\infty(\kappa) \xrightarrow{I} \kappa\text{-}\mathcal{P}\mathsf{r}\mathcal{SM} \xrightarrow{(-)_\kappa} \mathcal{SM}\overline{\mathcal{C}\mathsf{at}}_\infty(\kappa)$. The functors ι and $(-)_\kappa$ are equivalences (Corollary A.8), so we find that I is also an equivalence. □

Appendix B. Miscellaneous results on localization

In this appendix, we record some results on localization of ∞-categories.

B.1. **Base change of localization.** Let $L\colon \mathcal{C} \to \mathcal{D}$ be a localization of ∞-categories, and suppose we are given a pullback square in $\mathcal{C}\mathsf{at}_\infty$

$$\begin{tikzcd}
\mathcal{C}' \arrow[r] \arrow[d, "L'"'] & \mathcal{D} \arrow[d, "L"] \\
\mathcal{D}' \arrow[r] & \mathcal{D}.
\end{tikzcd}$$

The functor L' is not generally a localization. This naturally raises the question of when L' is actually a localization. The goal of this subsection is to record a few results related to this question.

We start with a sufficient condition for a localization functor to be stable under base change; such a functor is called a **universal localization**. (A general criteria for universal localization is discussed in [Hin24].) The following theorem says that localization of ∞-categories with weak equivalences and fibrations [Cis19a, Definition 7.4.12] (such as that of model categories) enjoy this property:

Theorem B.1. *Let $\mathcal{C}$ be an ∞-category with weak equivalences and fibrations. The localization functor*

$$\mathcal{C} \to \mathcal{C}[\mathrm{weq}^{-1}]$$

is a universal localization.

The proof of Theorem B.1 requires the following lemma:

Lemma B.2. *Let $\mathcal{C}$ be an ∞-category, let $\mathcal{W} \subset \mathcal{C}$ be a subcategory containing all objects, and let $L\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which carries every morphism in $\mathcal{W}$ to an equivalence. If for each $n \geq 0$, the functor*

$$\theta_{L,n}\colon \mathrm{Fun}([n], \mathcal{C}) \times_{\mathcal{C}^{n+1}} \mathcal{W}^{n+1} \to \mathrm{Fun}([n], \mathcal{D})^{\simeq}$$

is a weak homotopy equivalence, then L is a universal localization.

Proof. In the case where $\mathcal{W}$ is the subcategory of morphisms mapped to equivalences in $\mathcal{D}$, this is more or less immediate from Mazel–Gee's localization theorem (see [MG19, Theorem 3.8] or [AC25, Theorem 1.1]) and can be deduced from [Hin24, Lemma 1.3 and 1.5]. The proof of the general case we present below is very similar. (In fact, we do not need this generality for this paper, but we record it for completeness.)

Let $\mathcal{D}' \to \mathcal{D}$ be a categorical fibration of ∞-categories, and set $\mathcal{C}' = \mathcal{D}' \times_{\mathcal{D}} \mathcal{C}$. We must show that the functor $L' \colon \mathcal{C}' \to \mathcal{D}'$ is a localization. For this, let $\mathcal{W}' = (\mathcal{D}')^{\simeq} \times_{\mathcal{D}} \mathcal{W} \subset \mathcal{C}'$. We claim that L' is a localization at $\mathcal{W}'$. According to the generalized Mazel-Gee's localization theorem [Ara23, Theorem 1.7], it suffices to show that for each $n \geq 0$, the map $\theta_{L',n}$ is a weak homotopy equivalence. But θ'_n is a pullback of the map θ_n along the the Kan fibration $\mathrm{Fun}([n], \mathcal{D}')^{\simeq} \to \mathrm{Fun}([n], \mathcal{D})^{\simeq}$, so the claim follows from the right properness of the Kan–Quillen model structure. $\square$

Proof of Theorem B.1. Let $\mathcal{W} \subset \mathcal{C}$ denote the subcategory of weak equivalences. By [Cis19a, Remark 7.5.22], we may assume that $\mathcal{W}$ is saturated, i.e., that it consists of the morphisms whose images in $L(\mathcal{C}) = \mathcal{C}[\mathrm{weq}^{-1}]$ are equivalences. By Lemma B.2, it will suffice to show that the functor

$$\theta_{\mathcal{C},n} \colon \mathrm{Fun}([n], \mathcal{C}) \times_{\mathcal{C}^{n+1}} \mathcal{W}^{n+1} \to \mathrm{Fun}([n], L(\mathcal{C}))$$

is a weak homotopy equivalence for every $n \geq 0$. By [Cis19a, Theorem 7.4.20], the ∞-category $\mathrm{Fun}([n], \mathcal{C})$ has the structure of an ∞-category with weak equivalences and fibrations whose subcategory of weak equivalences is given by $\mathrm{Fun}([n], \mathcal{C}) \times_{\mathcal{C}^{n+1}} \mathcal{W}^{n+1}$. Moreover, by [Cis19a, Theorem 7.6.17], the functor

$$L(\mathrm{Fun}([n], \mathcal{C})) \to \mathrm{Fun}([n], L(\mathcal{C}))$$

is a categorical equivalence. Therefore, the map $\theta_{L,n}$ can be identified with $\theta_{\mathrm{Fun}([n],\mathcal{C}),0}$. Thus, replacing $\mathcal{C}$ by $\mathrm{Fun}([n], \mathcal{C})$, we are reduced to showing that $\theta_{\mathcal{C},0} \colon \mathcal{W} \to L(\mathcal{C})$ is a weak homotopy equivalence. This is the content of [Cis19a, Lemma 7.6.9], and the proof is complete. $\square$

We next consider the interaction of pullback and homotopy equivalences of relative ∞-categories.

Proposition B.3. *Consider a diagram* $[2] \times [1] \to \mathsf{Cat}_\infty$ *depicted as*

$$\begin{tikzcd}
 & & & \mathcal{A} \arrow[rr, "f"] \arrow[ddr, "p"] & & \mathcal{B} \arrow[ddl, "q"'] \\
\mathcal{A}' \arrow[urrr] \arrow[rr, "f'"] \arrow[ddr, "p'"'] & & \mathcal{B}' \arrow[urrr] \arrow[ddl, "q'"] & & & \\
 & & & & \mathcal{C}, & \\
 & \mathcal{C}' \arrow[urrr, "u"'] & & & &
\end{tikzcd}$$

where the squares are all cartesian. Regard $\mathcal{A}$ and $\mathcal{B}$ (resp. $\mathcal{A}'$ and $\mathcal{B}'$) as relative ∞-categories whose weak equivalences are the maps whose images in $\mathcal{C}$ (resp. $\mathcal{C}'$) are equivalences. Suppose that the following conditions are satisfied:

1. *The functor $f \colon \mathcal{A} \to \mathcal{B}$ is a homotopy equivalence of relative ∞-categories.*
2. *The functor $u \colon \mathcal{C}' \to \mathcal{C}$ is conservative (i.e., reflects equivalences).*

Then f' is a homotopy equivalence of relative ∞-categories.

Proof. We claim that there are maps $g\colon \mathcal{B} \to \mathcal{A}$ and $i\colon \mathcal{B} \to \mathcal{B}$ in $\mathsf{Cat}_{\infty/\mathcal{C}}$ and a diagram of the form

(B.1)
$$\begin{tikzcd}
\mathcal{B}\times\{0\} \arrow[dr, hook] \arrow[drr, bend left, "fg"] & & \\
& \mathcal{B}\times\mathcal{I} \arrow[r, "\Phi"] & \mathcal{B} \\
\mathcal{B}\times\{1\} \arrow[ur, hook] \arrow[urr, bend right, "i"'] & &
\end{tikzcd}$$

in $\mathsf{Cat}_{\infty/\mathcal{C}}$, where:

(I) $\mathcal{I}$ is a weakly contractible ∞-category equipped with two distinguished objects $0, 1 \in \mathcal{I}$.
(II) For each $B \in \mathcal{B}$, the functor $\Phi|\{B\} \times \mathcal{I}$ carries each morphism to a weak equivalence.
(III) The functor i is an equivalence of ∞-categories.
(IV) $\mathcal{B} \times \mathcal{I}$ lies over $\mathcal{C}$ via the composite $\mathcal{B} \times \mathcal{I} \xrightarrow{\mathrm{pr}} \mathcal{B} \xrightarrow{q} \mathcal{C}$.

Changing base along the map $\mathcal{C}' \to \mathcal{C}$, we obtain functors $g'\colon \mathcal{B}' \to \mathcal{A}'$, $i'\colon \mathcal{B}' \to \mathcal{B}'$, and $\Phi'\colon \mathcal{B}' \times \mathcal{I} \to \mathcal{B}'$. Using condition (2), we find that Φ' witnesses that f' has a right homotopy inverse. Applying the same argument to g, we find that g' also has a right homotopy inverse. It then follows from the two out of six property of homotopy equivalences that f' is a homotopy equivalence of relative categories.

To prove the above claim, use condition (1) to find a relative functor $g\colon \mathcal{B} \to \mathcal{A}$ and diagrams of the form (B.1) in Cat_∞ (not $\mathsf{Cat}_{\infty/\mathcal{C}}$ yet!) that satisfy conditions (I), (II), and (III). Let S denote the set of morphisms of $\mathcal{B} \times \mathcal{I}$ of the form $(B, x) \to (B, y)$, where $B \in \mathcal{B}$ and $x \to y$ is a moprhism in $\mathcal{I}$. By condition (I), $q\Phi$ factors through the localization $(\mathcal{B} \times \mathcal{I})[S^{-1}] \simeq \mathcal{B} \times \mathcal{I}[\mathcal{I}^{-1}]$. Since $\mathcal{I}$ is weakly contractible, this localization can be identified with $\mathcal{B}$, with localizing functor given by the projection $\mathcal{B} \times \mathcal{I} \to \mathcal{B}$. Thus, there is a diagram $[1] \times [1] \to \mathsf{Cat}_\infty$ of the form

$$\begin{tikzcd}
\mathcal{B}\times\mathcal{I} \arrow[r, "\Phi"] \arrow[d, "\mathrm{pr}"'] & \mathcal{B} \arrow[d, "q"] \\
\mathcal{B} \arrow[r, "r"'] & \mathcal{C}
\end{tikzcd}$$

Since $r \simeq (q\Phi)|\mathcal{B} \times \{1\} = q$, we may assume that $r = q$. Thus we obtain a diagram $\sigma\colon [2] \to \mathsf{Cat}_\infty$ whose boundary is depicted as

$$\begin{tikzcd}
\mathcal{B}\times\mathcal{I} \arrow[r, "\Phi"] \arrow[dr, "q\circ\mathrm{pr}"'] & \mathcal{B} \arrow[d, "q"] \\
& \mathcal{C}.
\end{tikzcd}$$

We now contemplate the diagrams $([1] \times [1])^{\triangleright} \to \mathsf{Cat}_\infty$ and $[3] \to \mathsf{Cat}_\infty$ depicted as

$$\begin{tikzcd}
\mathcal{B}\times\{0\} \arrow[rr, "g"] \arrow[d, hook] \arrow[drrr, "fg"] \arrow[ddrr, "q"'] & & \mathcal{A} \arrow[dr, "f"] \arrow[dd, dashed, "p"] & \\
\mathcal{B}\times\mathcal{I} \arrow[rrr, "\Phi"] \arrow[drr, "q\circ\mathrm{pr}"'] & & & \mathcal{B} \arrow[dl, "q"] \\
& & \mathcal{C} &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathcal{B}\times\mathcal{I} \arrow[rr, "\Phi"] \arrow[ddr, "q\circ\mathrm{pr}"'] & & \mathcal{B} \arrow[ddl, "q"] \\
\mathcal{B}\times\{1\} \arrow[ur, hook] \arrow[urrr, "i"] \arrow[drr, "q"'] & & & \\
& & \mathcal{C} &
\end{tikzcd}$$

The left-hand diagram is constructed as follows: Think of this as an amalgamation of two 3-simplices $\Delta^3 \to \mathsf{Cat}_\infty$. The front 3-simplex is obtained by filling the horn

$\Lambda^3_1 \subset \Delta^3$, using σ, the natural equivalence $fg \simeq \Phi|\mathcal{B} \times \{0\}$, and the 2-simplex corresponding to the equality $q \circ \mathrm{pr}|\mathcal{B} \times \{0\} = q$. We then fill the 3-simplex in the back by filling the horn $\Lambda^3_2 \subset \Delta^3$. Likewise, the right-hand 3-simplex is obtained by filling the inner horn $\Lambda^3_1 \subset \Delta^3$. These diagrams lifts the maps g, i, and Φ to those in $\mathcal{C}\mathsf{at}_{\infty/\mathcal{C}}$, and they give rise to a diagram in $\mathcal{C}\mathsf{at}_{\infty/\mathcal{C}}$ satisfying conditions (I) through (IV). The proof is now complete. $\square$

B.2. **Bousfield localization.** In this subsection, we record a proof of the following well-known result on Bousfield localization. It has been certainly floating in the literature for quite some time, but we could not find a proof.

Proposition B.4. *Let* $\mathbf{M}$ *be a combinatorial left proper model category with underlying* ∞*-category* $\mathcal{M}$*, and let* $F\colon \mathcal{M} \to \mathcal{N}$ *be a functor admitting a fully faithful right adjoint, where* $\mathcal{N}$ *is presentable. There is a left Bousfield localization* $\mathbf{N}$ *of* $\mathbf{M}$ *whose weak equivalences are the maps inverted by the composite* $\mathbf{M} \to \mathcal{M} \xrightarrow{F} \mathcal{N}$*. Moreover, the induced functor* $\mathbf{N} \to \mathcal{N}$ *is a localization at weak equivalences.*

Proof. Recall from [Lur09, Proposition 5.5.4.2] that there is a small set S of morphisms of $\mathcal{M}$ with the following property: An object $M \in \mathcal{M}$ lies in the essential image of the right adjoint $G\colon \mathcal{N} \to \mathcal{M}$ if and only if it is S**-local**, i.e., $\mathcal{M}(f, M)$ is an equivalence for every $f \in S$. By [Bar10, Theorem 4.7], there is a left Bousfield localization $L_S\mathbf{M}$ of $\mathbf{M}$ at S, which has the following description:

- Fibrant objects are the fibrant objects of $\mathbf{M}$ whose image in $\mathcal{M}$ is S-local.
- Weak equivalences are the maps inverted by the composite $\mathbf{M} \to \mathcal{M} \xrightarrow{F} \mathcal{N}$.

Using the universal property of localization, we obtain functors $\phi\colon \mathcal{M} \to (L_S\mathbf{M})[\mathrm{weq}^{-1}]$ and $\psi\colon (L_S\mathbf{M})[\mathrm{weq}^{-1}] \to \mathcal{N}$, and the diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[rr, "F"] \arrow[dr, "\phi"'] & & \mathcal{N} \\
& (L_S\mathbf{M})[\mathrm{weq}^{-1}] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

commutes up to natural equivalence. The functors F and ϕ are localizations because it has a fully faithful right adjoint [Lur25, 04JL]. Moreover, a morphism of $\mathcal{M}$ is inverted by F if and only if it is inverted by ϕ. This means that F and ϕ are localization at the same class of maps, so ψ is an equivalence. It follows that the functor $L_S\mathbf{M} \to \mathcal{N}$ is a localization. Therefore, the left Bousfield localization $L_S\mathbf{M}$ has the desired properties. $\square$

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References

[AC25] Kensuke Arakawa and Bastiaan Cnossen, *A short proof of the universality of the relative rezk nerve*, `https://arxiv.org/abs/2505.14123`, 2025, To appear in *Proc. Amer. Math. Soc.*

[ACK25] Kensuke Arakawa, Daniel Carranza, and Chris Kapulkin, *Derived mapping spaces of* ∞*-categories*, 2025.

[AR94] J. Adámek and J. Rosický, *Locally presentable and accessible categories*, Lecture note series / London mathematical society, Cambridge University Press, 1994.

[Ara] Kensuke Arakawa, *Monoidal relative categories model monoidal* ∞*-categories*, To appear in *J. Pure Appl. Algebra*, preprint available at `https://arxiv.org/abs/2504.20606`.

[Ara23] ———, *Classification diagrams of marked simplicial sets*, `https://arxiv.org/abs/2311.01101`, 2023.

[Bar10] Clark Barwick, *On left and right model categories and left and right Bousfield localizations*, Homology Homotopy Appl. **12** (2010), no. 2, 245–320. MR 2771591

[BK12a] C. Barwick and D. M. Kan, *A characterization of simplicial localization functors and a discussion of DK equivalences*, Indag. Math. (N.S.) **23** (2012), no. 1-2, 69–79. MR 2877402

[BK12b] ———, *Relative categories: another model for the homotopy theory of homotopy theories*, Indag. Math. (N.S.) **23** (2012), no. 1-2, 42–68. MR 2877401

[BW24] Michael Batanin and David White, *Left Bousfield localization without left properness*, J. Pure Appl. Algebra **228** (2024), no. 6, Paper No. 107570, 23. MR 4670515

[Cis06] Denis-Charles Cisinski, *Les préfaisceaux comme modèles des types d'homotopie*, Astérisque (2006), no. 308, xxiv+390. MR 2294028

[Cis19a] ———, *Higher categories and homotopical algebra*, Cambridge Studies in Advanced Mathematics, vol. 180, Cambridge University Press, Cambridge, 2019. MR 3931682

[Cis19b] ———, *Higher categories and homotopical algebra*, Cambridge Studies in Advanced Mathematics, vol. 180, Cambridge University Press, Cambridge, 2019. MR 3931682

[Day70] Brian Day, *On closed categories of functors*, Reports of the Midwest Category Seminar, IV, Lecture Notes in Math., vol. Vol. 137, Springer, Berlin-New York, 1970, pp. 1–38. MR 272852

[Dug01] Daniel Dugger, *Combinatorial model categories have presentations*, Adv. Math. **164** (2001), no. 1, 177–201. MR 1870516

[GU71] Peter Gabriel and Friedrich Ulmer, *Lokal präsentierbare Kategorien*, Lecture Notes in Mathematics, vol. Vol. 221, Springer-Verlag, Berlin-New York, 1971. MR 327863

[Hin24] Vladimir Hinich, *Key lemma and universal localization*, 2024.

[Hir03] Philip S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003, errata available at `https://math.mit.edu/~psh/MCATL-errata-2018-08-01.pdf`. MR 1944041

[Hov] Mark Hovey, *Mark hovey's algebraic topology problem list (Model categories)*, https://www-users.cse.umn.edu/ tlawson/hovey/model.html.

[Hov99] Mark Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999. MR 1650134

[Isa09] Samuel B. Isaacson, *Symmetric cubical sets*, 2009.

[Isa11] Samuel B. Isaacson, *Symmetric cubical sets*, J. Pure Appl. Algebra **215** (2011), no. 6, 1146–1173. MR 2769224

[Jar06] J. F. Jardine, *Categorical homotopy theory*, Homology Homotopy Appl. **8** (2006), no. 1, 71–144. MR 2205215

[Joy02] A. Joyal, *Quasi-categories and Kan complexes*, J. Pure Appl. Algebra **175** (2002), no. 1-3, 207–222, Special volume celebrating the 70th birthday of Professor Max Kelly. MR 1935979

[Law17] Tyler Lawson, *Localization of enriched categories and cubical sets*, Theory Appl. Categ. **32** (2017), Paper No. 35, 1213–1221. MR 3695491

[Low16] Zhen Lin Low, *The heart of a combinatorial model category*, Theory Appl. Categ. **31** (2016), Paper No. 2, 31–62. MR 3448618

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659

[Lur17] J. Lurie, *Higher algebra*, `https://www.math.ias.edu/~lurie/papers/HA.pdf`, 2017.

[Lur25] Jacob Lurie, *Kerodon*, `https://kerodon.net`, 2025.

[May78] J. P. May, *The spectra associated to permutative categories*, Topology **17** (1978), no. 3, 225–228. MR 508886

[MG19] Aaron Mazel-Gee, *The universality of the Rezk nerve*, Algebr. Geom. Topol. **19** (2019), no. 7, 3217–3260. MR 4045352

[ML98] Saunders Mac Lane, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998. MR 1712872

[MP89] Michael Makkai and Robert Paré, *Accessible categories: the foundations of categorical model theory*, Contemporary Mathematics, vol. 104, American Mathematical Society, Providence, RI, 1989. MR 1031717

[Mur15] Fernando Muro, *On the unit of a monoidal model category*, Topology Appl. **191** (2015), 37–47. MR 3361052

[NS17] Thomas Nikolaus and Steffen Sagave, *Presentably symmetric monoidal ∞-categories are represented by symmetric monoidal model categories*, Algebr. Geom. Topol. **17** (2017), no. 5, 3189–3212. MR 3704256

[Pav25] Dmitri Pavlov, *Combinatorial model categories are equivalent to presentable quasicategories*, J. Pure Appl. Algebra **229** (2025), no. 2, Paper No. 107860, 39. MR 4851928

[PS18] Dmitri Pavlov and Jakob Scholbach, *Admissibility and rectification of colored symmetric operads*, J. Topol. **11** (2018), no. 3, 559–601, Revised proofs of 7.5 and 7.9 are available at `https://arxiv.org/abs/1410.5675`. MR 3830876

[RR15] G. Raptis and J. Rosický, *The accessibility rank of weak equivalences*, Theory Appl. Categ. **30** (2015), Paper No. 19, 687–703. MR 3356340

[Str80] Ross Street, *Fibrations in bicategories*, Cahiers Topologie Géom. Différentielle **21** (1980), no. 2, 111–160. MR 574662

Email address: `arakawa.kensuke.22c@st.kyoto-u.ac.jp`

Department of Mathematics, Kyoto University, Kyoto, 606-8502, Japan