The Meaning of *A*

Consider two time series of harmonic signals with angular frequency equals to ω and initial phase angle equals to zero, $a_x(t)$ and $a_y(t)$, the following expression applies:

$$a_x(t) = l_a \, cos(\omega t) \tag{A1}$$

$$a_y(t) = l_b \, sin(\omega t) \tag{A2}$$

The plots for Eqs. (A1) and (A2) are as illustrated in Fig. A1.

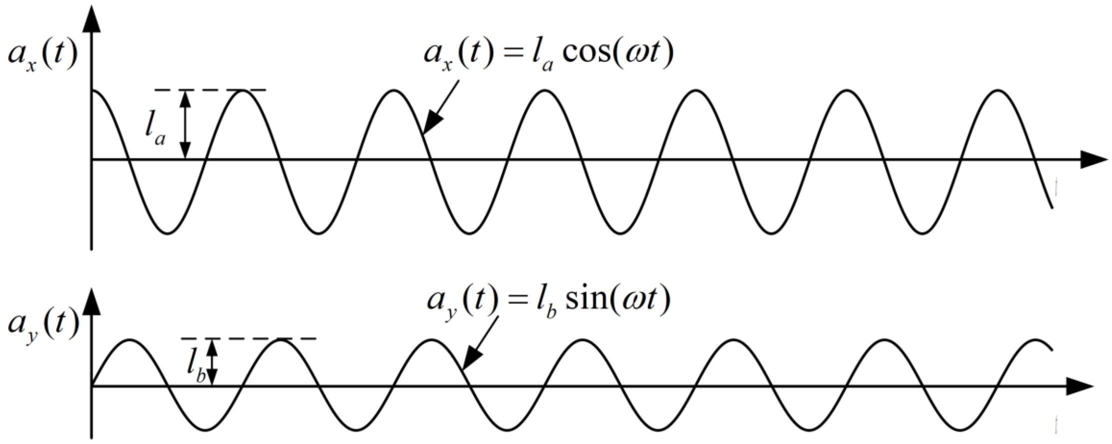

Fig. A1. Plots for Eqs. (A1) and (A2).

And the trajectory of $a_x(t)$ and $a_y(t)$ in the *X*-*Y* plane is in a standard form of ellipse with semi-major axis and semi-minor axis equal to l_a and l_b, respectively, and rotates counterclockwise as is shown in Fig. A2(a). The expression for such ellipse can be written as:

$$\left(\frac{l_a \, cos \, \omega t}{l_a}\right)^2 + \left(\frac{l_b \, sin \, \omega t}{l_b}\right)^2 = 1 \tag{A3}$$

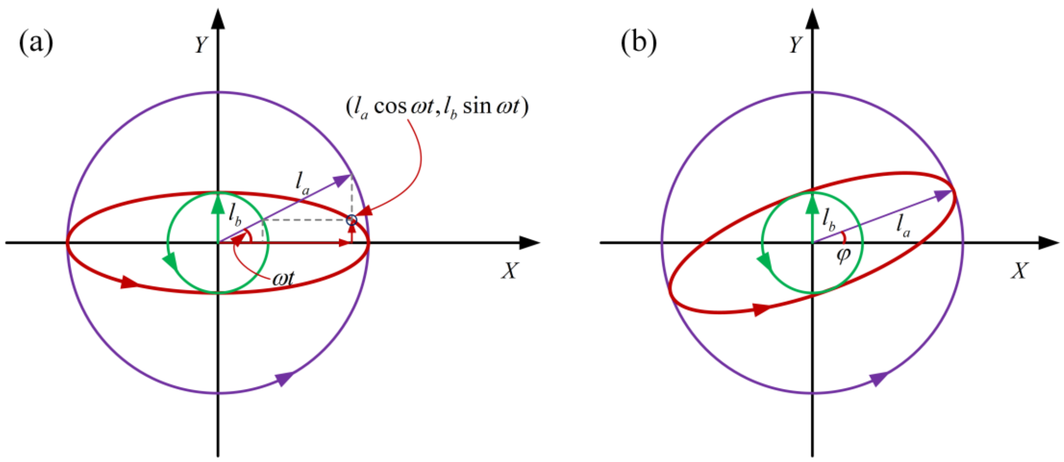

Fig. A2. (a): Trajectory of Eq. (A3); (b): rotated trajectory of Eq. (A3).

Further, we can rotate the trajectory counterclockwise by an angle of φ as illustrated in Fig. A2(b). In doing so, the rotation matrix is introduced:

$$R = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \tag{A4}$$

The coordinates of the new rotated ellipse can be calculated as:

$$\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} l_a \cos\omega t \\ l_b \sin\omega t \end{bmatrix} = \begin{bmatrix} l_a \cos\omega t \cos\varphi - l_b \sin\omega t \sin\varphi \\ l_a \cos\omega t \sin\varphi + l_b \sin\omega t \cos\varphi \end{bmatrix} \tag{A5}$$

Now, let the coordinates be presented in the complex plane, that is:

$$(l_a \cos\omega t \cos\varphi - l_b \sin\omega t \sin\varphi) + i(l_a \cos\omega t \sin\varphi + l_b \sin\omega t \cos\varphi) \tag{A6}$$

The ellipse given in Eq. (A6) can be expressed also in another form as:

$$A_+ e^{i\omega t} + A_- e^{-i\omega t} \tag{A7}$$

where $A_+ = a_+ + ib_+$, $A_- = a_- + ib_-$, and is the coefficients for the positive and negative frequency, respectively. Further, by using the Euler's relation, Eq. (A7) can be expanded as:

$$\begin{aligned} A_+ e^{i\omega t} + A_- e^{-i\omega t} &= (a_+ + ib_+)(\cos\omega t + i\sin\omega t) \\ &\quad + (a_- + ib_-)(\cos\omega t - i\sin\omega t) \\ &= [(a_+ + a_-)\cos\omega t + (b_- - b_+)\sin\omega t] \\ &\quad + i[(b_+ + b_-)\cos\omega t + (a_+ - a_-)\sin\omega t] \end{aligned} \tag{A8}$$

Comparing Eq. (A8) with Eq. (A6), the following relationship applies:

$$\begin{cases} a_+ + a_- = l_a \cos\varphi \\ a_+ - a_- = l_b \cos\varphi \end{cases} \qquad \begin{cases} b_- - b_+ = -l_b \sin\varphi \\ b_+ + b_- = l_a \sin\varphi \end{cases} \tag{A9}$$

Furthermore, one can determine the expression for a_+, a_-, b_+ and b_- is:

$$\begin{cases} a_+ = \frac{l_a + l_b}{2} \cos\varphi \\ a_- = \frac{l_a - l_b}{2} \cos\varphi \end{cases} \qquad \begin{cases} b_+ = \frac{l_a + l_b}{2} \sin\varphi \\ b_- = \frac{l_a - l_b}{2} \sin\varphi \end{cases} \tag{A10}$$

Thus, the expression for A_+ and A_- is:

$$A_+ = \frac{l_a + l_b}{2} \cos\varphi + i\frac{l_a + l_b}{2} \sin\varphi \qquad A_- = \frac{l_a - l_b}{2} \cos\varphi + i\frac{l_a - l_b}{2} \sin\varphi \tag{A11}$$

Using the Euler's relation, Eq. (A11) can also be expressed:

$$A_+ = \frac{l_a + l_b}{2} e^{i\varphi} \qquad A_- = \frac{l_a - l_b}{2} e^{i\varphi} \tag{A12}$$

Hence, the length of the semi-major axis and semi-minor axis of this ellipse, l_a and l_b, can be determined as:

$$l_a = |A_+| + |A_-| \qquad\qquad l_b = \big||A_+| - |A_-|\big| \tag{A13}$$

Further, the orientation azimuth φ, which is the angle that the semi-major axis of the ellipse makes with the X(real) axis, can be calculated using Eq. (A14):

$$\varphi = \frac{\angle(A_+) + \angle(A_-)}{2} \tag{A14}$$