

Appendix A: The derivation of Eqs. (12) and (13)

Given two time series of harmonic signals with angular frequency equals to ω and initial phase angle equals to zero, $a_x(t)$ and $a_y(t)$, the following expression applies:

$$a_x(t) = l_a \cos(\omega t) \quad (\text{A.1})$$

$$a_y(t) = l_b \sin(\omega t) \quad (\text{A.2})$$

The plots for Eqs. (A.1) and (A.2) are illustrated in Fig. A.1.

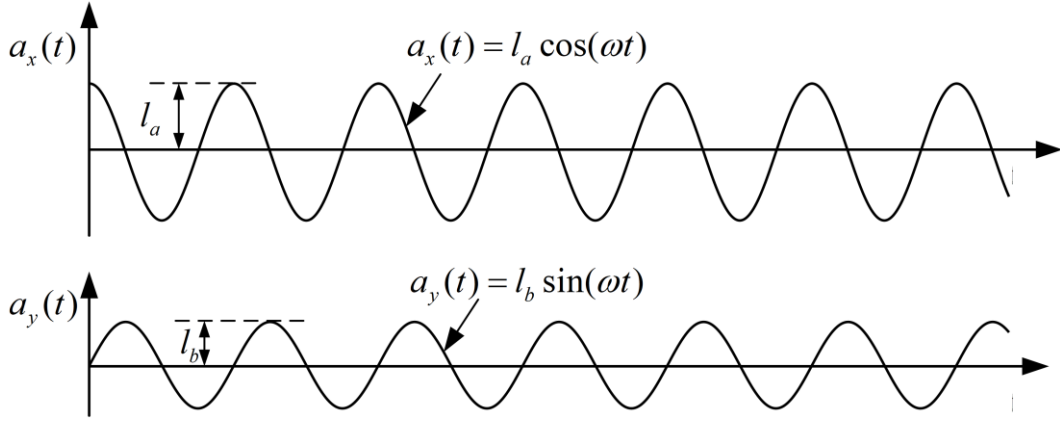


Fig. A.1. Plots for Eqs. (A.1) and (A.2).

The resulting trajectory of $a_x(t)$ and $a_y(t)$ in the X - Y plane is in a standard form of ellipse with semi-major axis and semi-minor axis equal to l_a and l_b , respectively, and rotates counterclockwise as is shown in Fig. A.2(a). The expression for such ellipse can be written as:

$$\left(\frac{l_a \cos \omega t}{l_a}\right)^2 + \left(\frac{l_b \sin \omega t}{l_b}\right)^2 = 1 \quad (\text{A.3})$$

Further, one can rotate the trajectory counterclockwise by an angle of φ as illustrated in Fig. A.2(b). In doing so, the rotation matrix is introduced:

$$R = \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix} \quad (\text{A.4})$$

The coordinates of the new rotated ellipse can be calculated as:

$$\begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix} \begin{bmatrix} l_a \cos \omega t \\ l_b \sin \omega t \end{bmatrix} = \begin{bmatrix} l_a \cos \omega t \cos \varphi - l_b \sin \omega t \sin \varphi \\ l_a \cos \omega t \sin \varphi + l_b \sin \omega t \cos \varphi \end{bmatrix} \quad (\text{A.5})$$

Now, present the coordinates in the complex plane, that is:

$$(l_a \cos \omega t \cos \varphi - l_b \sin \omega t \sin \varphi) + i(l_a \cos \omega t \sin \varphi + l_b \sin \omega t \cos \varphi) \quad (\text{A.6})$$

The ellipse given in Eq. (A.6) can be expressed also in another form as:

$$A_+ e^{i\omega t} + A_- e^{-i\omega t} \quad (\text{A.7})$$

where $A_+ = a_+ + ib_+$, $A_- = a_- + ib_-$, and is the coefficients for the positive and negative frequency, respectively.

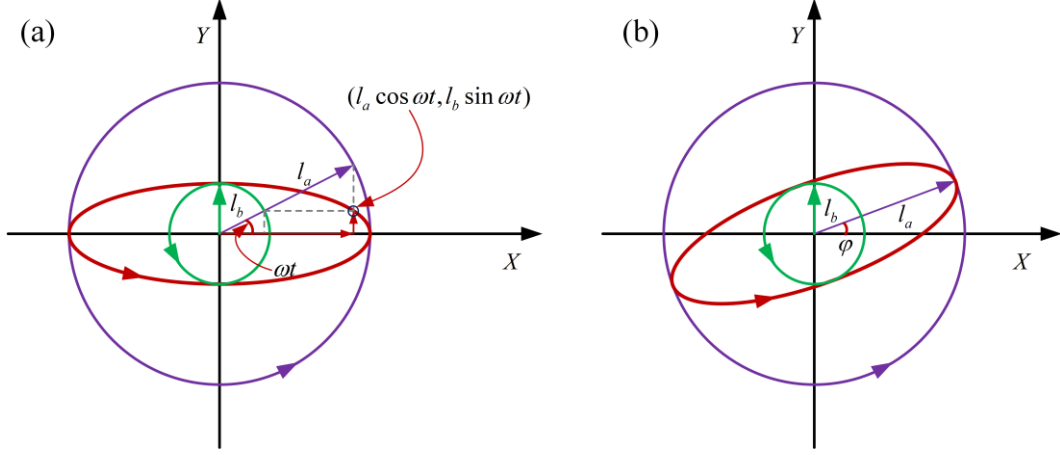


Fig. A.2. (a): Trajectory of Eq. (A.3); (b): rotated trajectory of Eq. (A.3).

Further, by using the Euler's relation, Eq. (A.7) can be expanded as:

$$\begin{aligned}
 A_+ e^{i\omega t} + A_- e^{-i\omega t} &= (a_+ + ib_+)(\cos \omega t + i \sin \omega t) \\
 &\quad + (a_- + ib_-)(\cos \omega t - i \sin \omega t) \\
 &= [(a_+ + a_-) \cos \omega t + (b_- - b_+) \sin \omega t] \\
 &\quad + i[(b_+ + b_-) \cos \omega t + (a_+ - a_-) \sin \omega t]
 \end{aligned} \tag{A.8}$$

Comparing Eq. (A.8) with Eq. (A.6), the following relationship applies:

$$\begin{cases} a_+ + a_- = l_a \cos \varphi \\ a_+ - a_- = l_b \cos \varphi \end{cases} \quad \begin{cases} b_+ + b_- = l_a \sin \varphi \\ b_+ - b_- = l_b \sin \varphi \end{cases} \tag{A.9}$$

Furthermore, the expression for a_+ , a_- , b_+ , and b_- can be determined as:

$$\begin{cases} a_+ = \frac{l_a + l_b}{2} \cos \varphi \\ a_- = \frac{l_a - l_b}{2} \cos \varphi \end{cases} \quad \begin{cases} b_+ = \frac{l_a + l_b}{2} \sin \varphi \\ b_- = \frac{l_a - l_b}{2} \sin \varphi \end{cases} \tag{A.10}$$

Thus, the expression for A_+ and A_- is:

$$A_+ = \frac{l_a + l_b}{2} \cos \varphi + i \frac{l_a + l_b}{2} \sin \varphi \quad A_- = \frac{l_a - l_b}{2} \cos \varphi + i \frac{l_a - l_b}{2} \sin \varphi \tag{A.11}$$

Using the Euler's relation, Eq. (A.11) can also be expressed:

$$A_+ = \frac{l_a + l_b}{2} e^{i\varphi} \quad A_- = \frac{l_a - l_b}{2} e^{i\varphi} \tag{A.12}$$

Hence, the length of the semi-major axis and semi-minor axis of this ellipse, l_a and l_b , can be determined as:

$$l_a = |A_+| + |A_-| \quad l_b = ||A_+| - |A_-|| \tag{A.13}$$

Further, the orientation azimuth φ , which is the angle that the semi-major axis of the ellipse makes with the X(real) axis, can be calculated using Eq. (A.14):

$$\varphi = \frac{\angle(A_+) + \angle(A_-)}{2} \tag{A.14}$$