Appendix A: The derivation of Eqs. (10) and (11)

Given two time series of harmonic signals with angular frequency equals to ω and initial phase angle equals to zero, $a_x(t)$ and $a_y(t)$, the following expression applies:

$$a_x(t) = l_a\, cos(\,\omega t) \tag{A.1}$$

$$a_y(t) = l_b\, sin(\,\omega t) \tag{A.2}$$

The plots for Eqs. (A.1) and (A.2) are illustrated in Fig. A.1.

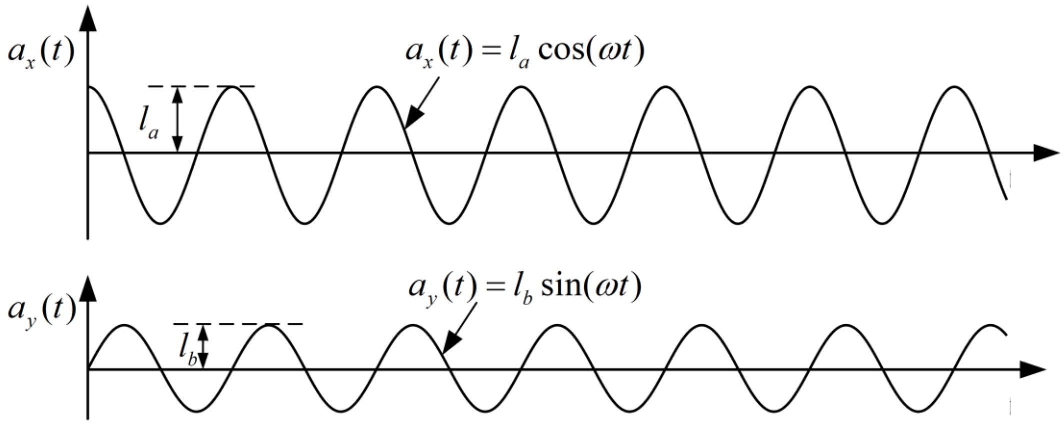

Fig. A.1. Plots for Eqs. (A.1) and (A.2).

The resulting trajectory of $a_x(t)$ and $a_y(t)$ in the *X*-*Y* plane is in a standard form of ellipse with semi-major axis and semi-minor axis equal to l_a and l_b, respectively, and rotates counterclockwise as is shown in Fig. A.2(a). The expression for such ellipse can be written as:

$$\left(\frac{l_a\, cos\, \omega t}{l_a}\right)^2 + \left(\frac{l_b\, sin\, \omega t}{l_b}\right)^2 = 1 \tag{A.3}$$

Further, one can rotate the trajectory counterclockwise by an angle of φ as illustrated in Fig. A.2(b). In doing so, the rotation matrix is introduced:

$$R = \begin{bmatrix} cos\, \varphi & -sin\, \varphi \\ sin\, \varphi & cos\, \varphi \end{bmatrix} \tag{A.4}$$

The coordinates of the new rotated ellipse can be calculated as:

$$\begin{bmatrix} cos\, \varphi & -sin\, \varphi \\ sin\, \varphi & cos\, \varphi \end{bmatrix} \begin{bmatrix} l_a\, cos\, \omega\, t \\ l_b\, sin\, \omega\, t \end{bmatrix} = \begin{bmatrix} l_a\, cos\, \omega\, t\, cos\, \varphi - l_b\, sin\, \omega\, t\, sin\, \varphi \\ l_a\, cos\, \omega\, t\, sin\, \varphi + l_b\, sin\, \omega\, t\, cos\, \varphi \end{bmatrix} \tag{A.5}$$

Now, present the coordinates in the complex plane, that is:

$$(l_a\, cos\, \omega\, t\, cos\, \varphi - l_b\, sin\, \omega\, t\, sin\, \varphi) + i(l_a\, cos\, \omega\, t\, sin\, \varphi + l_b\, sin\, \omega\, t\, cos\, \varphi) \tag{A.6}$$

The ellipse given in Eq. (A.6) can be expressed also in another form as:

$$A_+ e^{i\omega t} + A_- e^{-i\omega t} \tag{A.7}$$

where $A_+ = a_+ + ib_+$, $A_- = a_- + ib_-$, and is the coefficients for the positive and negative frequency, respectively.

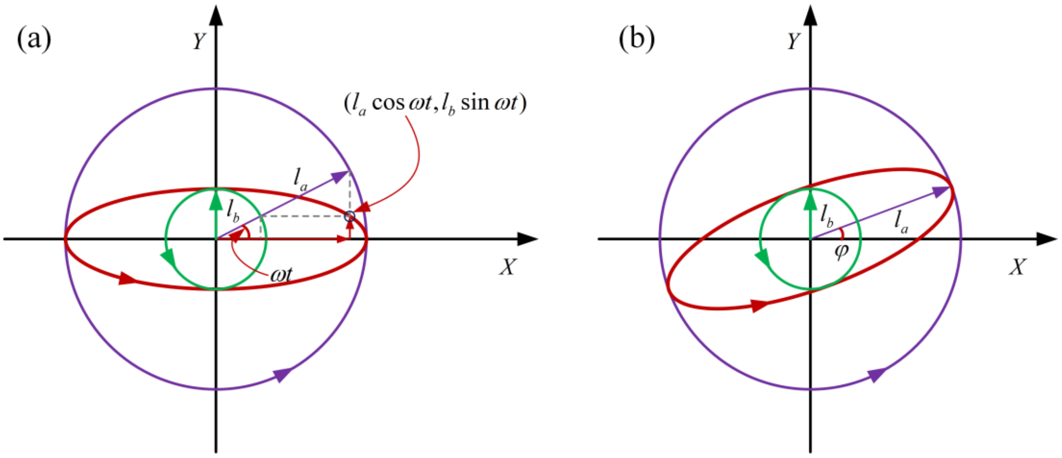

Fig. A.2. (a): Trajectory of Eq. (A.3); (b): rotated trajectory of Eq. (A.3).

Further, by using the Euler's relation, Eq. (A.7) can be expanded as:

$$\begin{aligned} A_+ e^{i\omega t} + A_- e^{-i\omega t} &= (a_+ + ib_+)(\cos\omega t + i\sin\omega t) \\ &\quad + (a_- + ib_-)(\cos\omega t - i\sin\omega t) \\ &= [(a_+ + a_-)\cos\omega t + (b_- - b_+)\sin\omega t] \\ &\quad + i[(b_+ + b_-)\cos\omega t + (a_+ - a_-)\sin\omega t] \end{aligned} \tag{A.8}$$

Comparing Eq. (A.8) with Eq. (A.6), the following relationship applies:

$$\begin{cases} a_+ + a_- = l_a \cos\varphi \\ a_+ - a_- = l_b \cos\varphi \end{cases} \qquad \begin{cases} b_+ + b_- = l_a \sin\varphi \\ b_+ - b_- = l_b \sin\varphi \end{cases} \tag{A.9}$$

Furthermore, the expression for a_+, a_-, b_+, and b_- can be determined as:

$$\begin{cases} a_+ = \frac{l_a + l_b}{2}\cos\varphi \\ a_- = \frac{l_a - l_b}{2}\cos\varphi \end{cases} \qquad \begin{cases} b_+ = \frac{l_a + l_b}{2}\sin\varphi \\ b_- = \frac{l_a - l_b}{2}\sin\varphi \end{cases} \tag{A.10}$$

Thus, the expression for A_+ and A_- is:

$$A_+ = \frac{l_a + l_b}{2}\cos\varphi + i\frac{l_a + l_b}{2}\sin\varphi \qquad A_- = \frac{l_a - l_b}{2}\cos\varphi + i\frac{l_a - l_b}{2}\sin\varphi \tag{A.11}$$

Using the Euler's relation, Eq. (A.11) can also be expressed:

$$A_+ = \frac{l_a + l_b}{2}e^{i\varphi} \qquad A_- = \frac{l_a - l_b}{2}e^{i\varphi} \tag{A.12}$$

Hence, the length of the semi-major axis and semi-minor axis of this ellipse, l_a and l_b, can be determined as:

$$l_a = |A_+| + |A_-| \qquad l_b = \big||A_+| - |A_-|\big| \tag{A.13}$$

Further, the orientation azimuth φ, which is the angle that the semi-major axis of the ellipse makes with the X(real) axis, can be calculated using Eq. (A.14):

$$\varphi = \frac{\angle(A_+) + \angle(A_-)}{2} \tag{A.14}$$