MEK2200

Kap. 2

1) a) $\underline{\underline{P}} = P_{xx}\,\hat{i}\hat{i} + P_{xy}(\hat{i}\hat{j} + \hat{j}\hat{i}) + P_{yy}\,\hat{j}\hat{j}$

Matriseform: $\underline{\underline{P}} = \begin{bmatrix} P_{xx} & P_{xy} \\ P_{xy} & P_{yy} \end{bmatrix}$

b) Flaten $y = \eta(x)$. Konstruerer en konstant flate ved:

$$\beta(x,y) = y - \eta(x) = 0$$

Siden $\beta(x,y)$ er konstant, så er normalen til flaten gitt ved $\hat{n} = \dfrac{\nabla\beta}{|\nabla\beta|}$

$$\nabla\beta = \frac{\partial\beta}{\partial x}\hat{i} + \frac{\partial\beta}{\partial y}\hat{j} = -\frac{\partial\eta}{\partial x}\hat{i} + 1\hat{j}$$

$$|\nabla\beta| = \sqrt{1 + \left(\frac{\partial\eta}{\partial x}\right)^2}$$

$$\Rightarrow \hat{n} = \frac{1}{\sqrt{1+\left(\frac{\partial\eta}{\partial x}\right)^2}}\left(-\frac{\partial\eta}{\partial x}\hat{i} + \hat{j}\right)$$

c) Spenningen er gitt ved $\underline{P_n} = \hat{n}\,\underline{\underline{P}}$

$$\Rightarrow \underline{P_n} = \frac{1}{|\nabla\beta|}\begin{bmatrix} -\frac{\partial\eta}{\partial x} & 1 \end{bmatrix}\begin{bmatrix} P_{xx} & P_{xy} \\ P_{xy} & P_{yy} \end{bmatrix}$$

$$= \frac{1}{|\nabla\beta|}\left[\left(-\frac{\partial\eta}{\partial x}P_{xx} + P_{xy}\right)\hat{i} + \left(-\frac{\partial\eta}{\partial x}P_{xy} + P_{yy}\right)\hat{j}\right]$$

– Ved indeksnotasjon:

$$\underline{P}_j = n_i\,\underline{\underline{P}}_{ij} \qquad \text{der} \quad i,j \in [x,y]$$

For $j=x$: $\quad \underline{P}_x = n_x\,\underline{\underline{P}}_{xx} + n_y\,\underline{\underline{P}}_{yx} = \frac{1}{|\nabla\beta|}\left(-\frac{\partial\eta}{\partial x}P_{xx} + P_{xy}\right)$

For $j=y$: $\quad \underline{P}_y = n_x\,\underline{\underline{P}}_{xy} + n_y\,\underline{\underline{P}}_{yy} = \frac{1}{|\nabla\beta|}\left(-\frac{\partial\eta}{\partial x}P_{xy} + P_{yy}\right)$

der $n_x = \frac{-1}{|\nabla\beta|}\frac{\partial\eta}{\partial x}$ og $n_y = \frac{1}{|\nabla\beta|}$.

skalarproduktet

d) Normalspenningen: $P_{nn} = \underline{P_n}\cdot\hat{n}$

$$\Rightarrow P_{nn} = \frac{1}{|\nabla\beta|}\begin{bmatrix}\left(-\frac{\partial\eta}{\partial x}P_{xx} + P_{xy}\right) & \left(-\frac{\partial\eta}{\partial x}P_{xy} + P_{yy}\right)\end{bmatrix}\begin{bmatrix} -\frac{1}{|\nabla\beta|}\frac{\partial\eta}{\partial x} \\ \frac{1}{|\nabla\beta|}\end{bmatrix}$$

$$= \frac{1}{|\nabla\beta|^2}\left(\left(\frac{\partial\eta}{\partial x}\right)^2 P_{xx} - 2\frac{\partial\eta}{\partial x}P_{xy} + P_{yy}\right) \quad (\star\star)$$

Tangensialspenningen $P_{nt} = \| \underline{P}_n \times \hat{n} \| = \left\| \begin{vmatrix} i & j & k \\ \frac{1}{|\nabla\beta|}\left(-\frac{\partial\eta}{\partial x} P_{xx} + P_{xy}\right) & \frac{1}{|\nabla\beta|}\left(-\frac{\partial\eta}{\partial x} P_{xy} + P_{yy}\right) & 0 \\ -\frac{1}{|\nabla\beta|}\frac{\partial\eta}{\partial x} & \frac{1}{|\nabla\beta|} & 0 \end{vmatrix} \right\|$

$$\Rightarrow P_{nt} = \frac{1}{|\nabla\beta|^2}\left(-\frac{\partial\eta}{\partial x} P_{xx} + P_{xy} - \left(\frac{\partial\eta}{\partial x}\right)^2 P_{xy} + \frac{\partial\eta}{\partial x} P_{yy} \right)$$

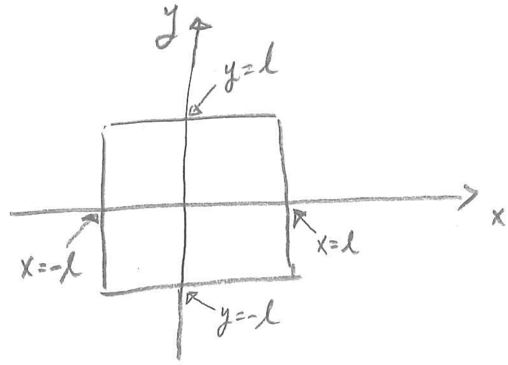
$$= \frac{1}{|\nabla\beta|^2}\left(\frac{\partial\eta}{\partial x}\left(P_{yy} - P_{xx}\right) + \left(1 - \left(\frac{\partial\eta}{\partial x}\right)^2\right) P_{xy} \right) \qquad (*)$$

e) Hvis $P_{xx} = P_{yy}$ og $P_{xy} = 0$; ser vi fra $(*)$ at $P_{nt} = 0$.

Fra likning $(**)$ får vi

$$P_{nn} = \frac{1}{|\nabla\beta|^2} \underbrace{\left(\left(\frac{\partial\eta}{\partial x}\right)^2 + 1 \right)}_{= |\nabla\beta|^2} P_{xx} = P_{xx}.$$

(2) $$\underline{\underline{P}} = \begin{bmatrix} \beta x & \varkappa y \\ \varkappa y & \beta x \end{bmatrix}$$



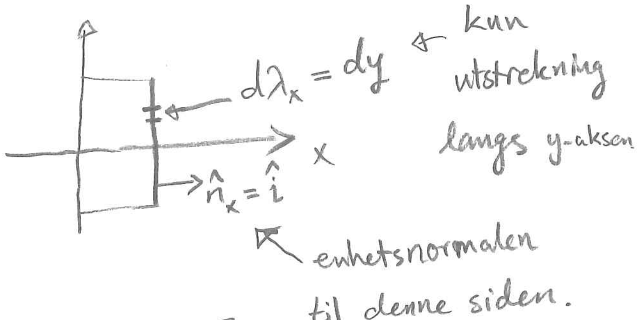
Spenningskraften er gitt ved:

$$\underline{F}_n = \int \underline{P}_n \, d\lambda_n$$

der $\underline{P}_n$ er spenningen og $d\lambda_n$ linjeelementet på kvadratet.

$$\underline{F}_n = \underbrace{\int_{-l}^{l} \underline{P}_x \, d\lambda_x}_{I_1} + \underbrace{\int_{-l}^{l} \underline{P}_y \, d\lambda_y}_{I_2} + \underbrace{\int_{-l}^{l} \underline{P}_{(-x)} \, d\lambda_{(-x)}}_{I_3} + \underbrace{\int_{-l}^{l} \underline{P}_{(-y)} \, d\lambda_{(-y)}}_{I_4}$$

Först ser på I_1:



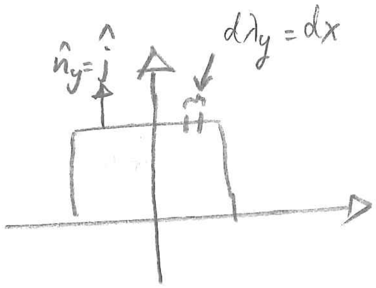
$$\underline{P}_x = \hat{n}_x \cdot \underline{\underline{P}}$$

$$= \hat{i} \cdot \underline{\underline{P}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \beta x & \varkappa y \\ \varkappa y & \beta x \end{bmatrix} = \begin{bmatrix} \beta x & \varkappa y \end{bmatrix}$$

$$\Rightarrow I_1 = \int_{-l}^{l} \left(\beta x \, \hat{i} + \varkappa y \, \hat{j}\right)\Big|_{x=l} dy = \int_{-l}^{l} \beta x \Big|_{x=l} dy \, \hat{i} + \int_{-l}^{l} \varkappa y \, dy \, \hat{j}$$

$$= \beta l \left[y\right]_{-l}^{l} \hat{i} + \varkappa \left[\tfrac{1}{2} y^2\right]_{-l}^{l} \hat{j} = 2l^2 \beta \, \hat{i} + 0 \, \hat{j} = \underline{2l^2 \beta \, \hat{i}}$$

$$I_2 = \int_{-l}^{l} P_y \, d\lambda_y$$



$$\underline{P}_y = \hat{j} \cdot \underline{\underline{P}} = [0 \quad 1] \begin{bmatrix} \beta x & \varkappa y \\ \varkappa y & \beta x \end{bmatrix} = [\varkappa y \quad \beta x] = \varkappa y \, \hat{i} + \beta x \, \hat{j}$$

$$I_2 = \int_{-l}^{l} (\varkappa y \, \hat{i} + \beta x \, \hat{j})\Big|_{y=l} dx = \int_{-l}^{l} \varkappa y\Big|_{y=l} dx \, \hat{i} + \int_{-l}^{l} \beta x \, dx \, \hat{j}$$

$$= \underline{2 l^2 \varkappa \, \hat{i}}$$

Tilsvarende for de to resterende sidene får vi

$$I_3 = \int_{-l}^{l} \underline{P}_{(-x)} \, d\lambda_{(-x)} = -\int_{-l}^{l} (\beta x \, \hat{i} + \varkappa y \, \hat{j})\Big|_{x=-l} dy = \underline{2 l^2 \beta \, \hat{i}}$$

$$I_4 = \int_{-l}^{l} \underline{P}_{(-y)} \, d\lambda_{(-y)} = -\int_{-l}^{l} (\varkappa y \, \hat{i} + \beta x \, \hat{j})\Big|_{y=-l} dx = \underline{2 l^2 \varkappa \, \hat{i}}$$

$$\Rightarrow \underline{F}_n = I_1 + I_2 + I_3 + I_4 = 4 l^2 (\beta + \varkappa) \, \hat{i}$$

Hvis $\underline{F}_n = 0 \Rightarrow \beta + \varkappa = 0 \Rightarrow \underline{\beta = -\varkappa}$