

The strange world of

PDEs and their discretizations.

Consider the problem

$$Ax = b$$

$A \in \mathbb{R}^{n,n}$ is positive definite and symmetric

\Rightarrow unique, stable solution.

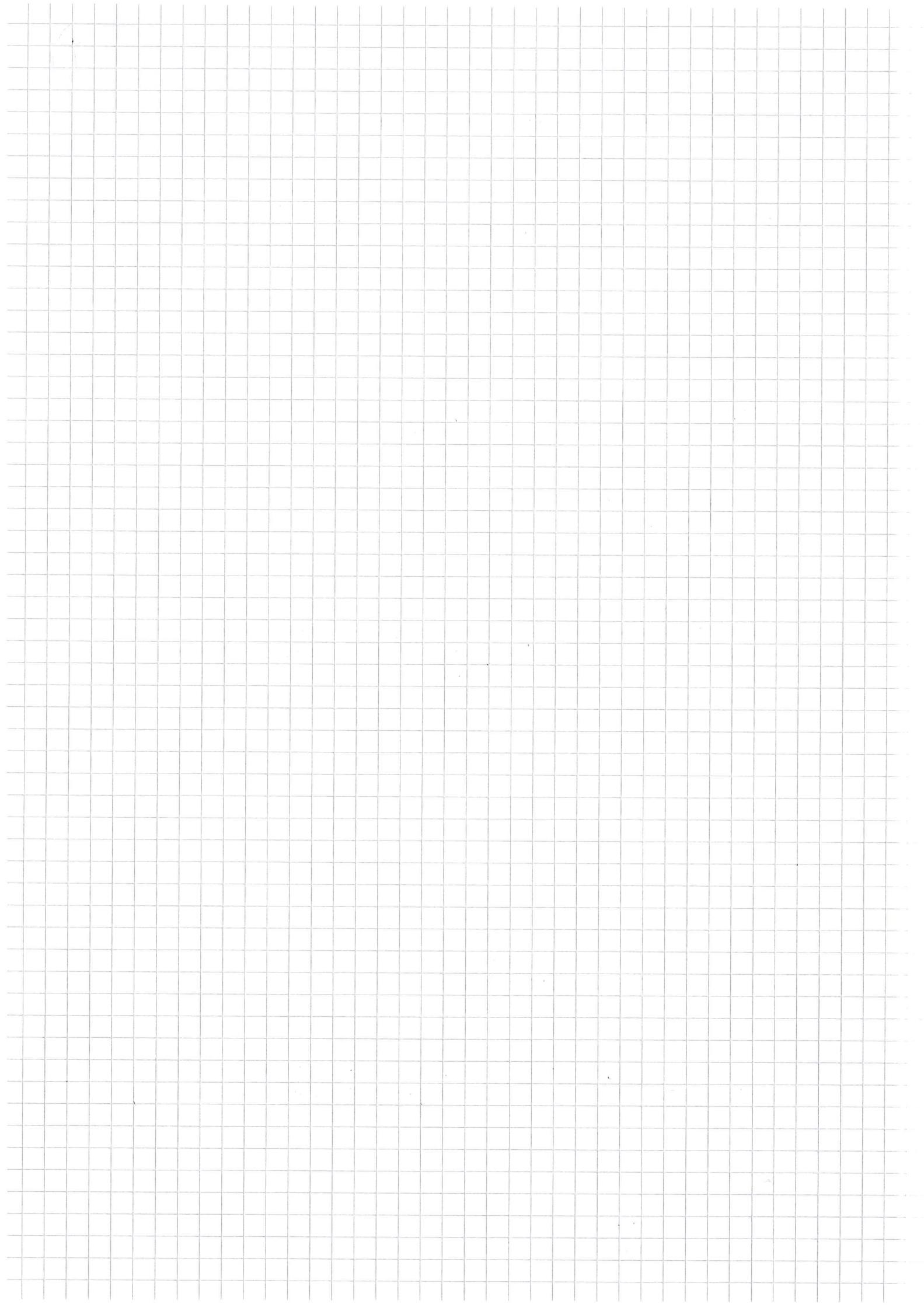
We have

$$x = A^{-1}b$$

$$\|x\| \leq \|A^{-1}b\| \leq \|A^{-1}\| \|b\|$$

and

$$\|Ax\| = \|b\| .$$



2)

For PDEs we do not

have the analog of

$$\|Ax\| \leq \|b\|$$

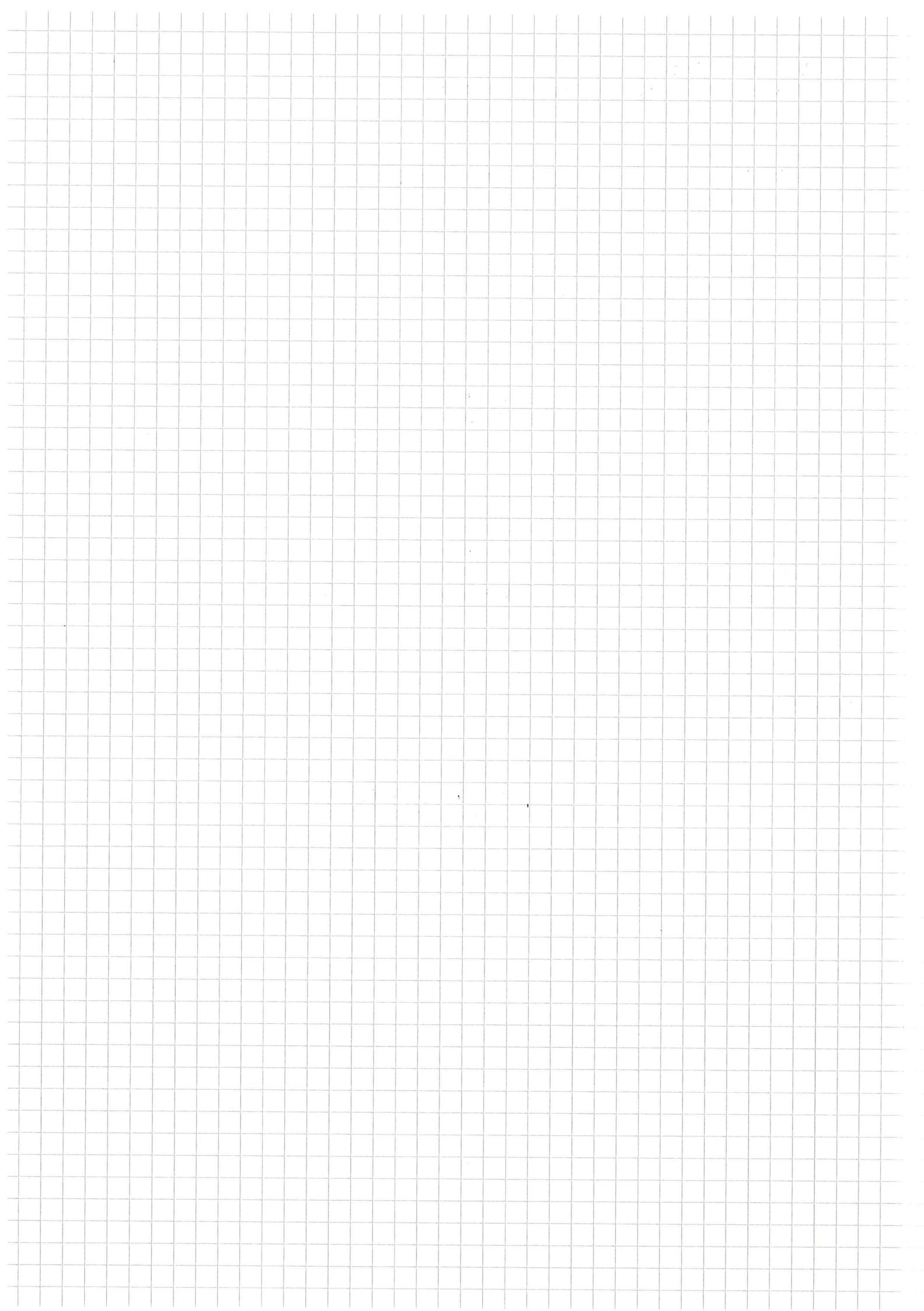
or

$$\|x\| \leq \|A^{-1}\| \|b\|.$$

Instead, we have something

like

$$\|A^{1/2}x\| \leq \|A^{-1/2}b\|$$



3)

Remark for A

positive def. and symmetric

We may define A^s
for any $s \in \mathbb{R}$

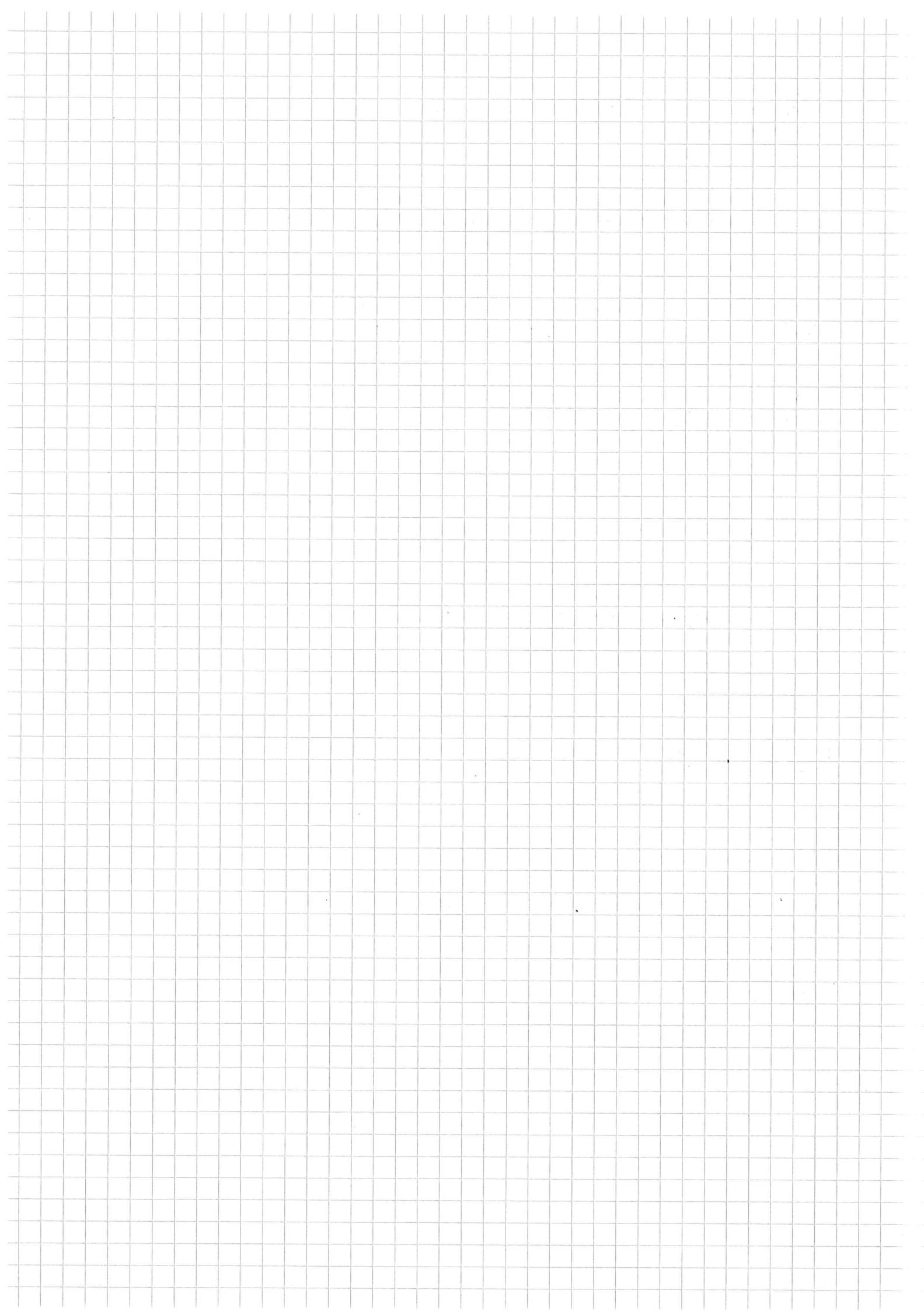
simply α in terms of

the eigenvalue decomposition

Let $A = U \Lambda V$

where Λ are the
eigenvalues.

Then $A^s = U \Lambda^s V$



4)

A^s may not be unique.

Let

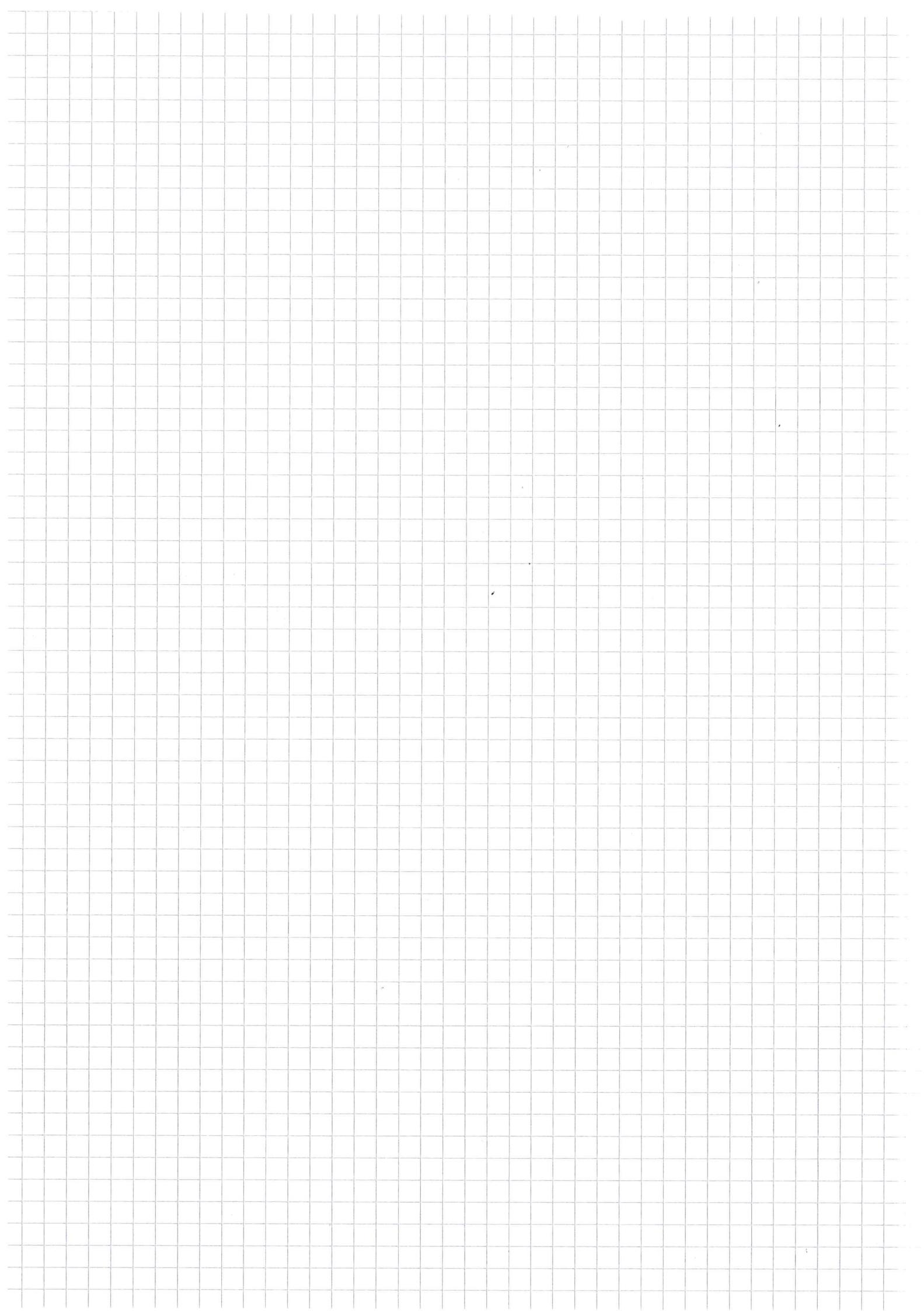
$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \left(\begin{array}{l} \text{- } \Delta \text{ on} \\ \text{a mesh} \\ \text{with two elements} \end{array} \right)$$

Let

$$B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \quad \left(\begin{array}{l} \text{gradient on} \\ \text{a mesh with} \\ \text{two elements} \end{array} \right)$$

Then $B^T B = A$

Even though $A \in \mathbb{R}^{2 \times 2}$, $B \in \mathbb{R}^{2 \times 3}$.



Hence, for

5)

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$Ax = b$$

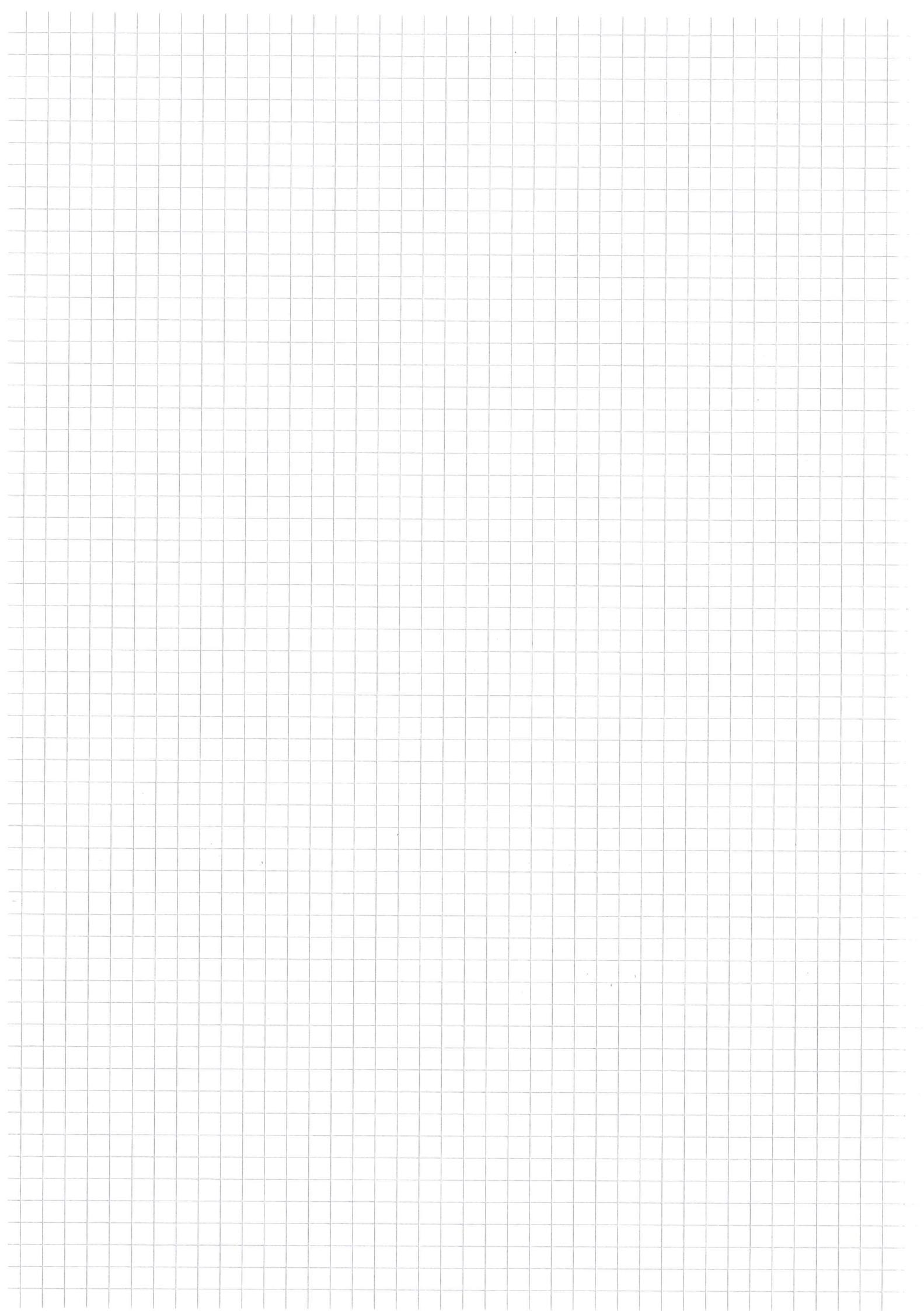
We could hope for something like

$$\|Bx\| \leq \|B^{-T}b\|$$

since

$$B^T B x = Ax = b \quad \text{multiply by } B^{-T}$$

$$\|Bx\| = \|B^{-T}b\|$$



6)

However,

B^{-T} is "difficult" to define since $B \in \mathbb{R}^{2 \times 3}$.

Let us now consider a PDE.

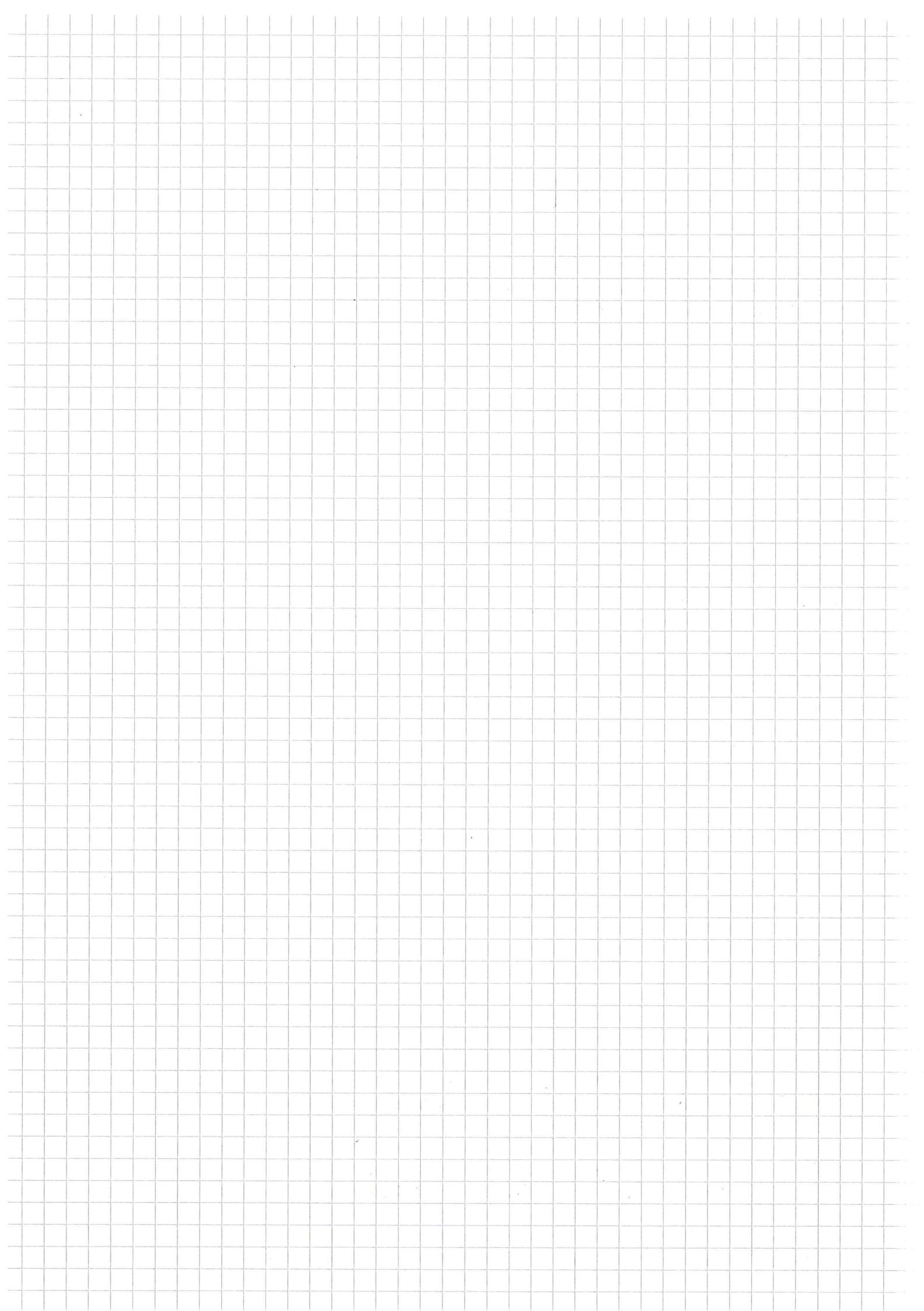
$$-\Delta u = f \quad \text{in } \Omega \quad (1)$$

$$u = 0 \quad \text{on } \partial\Omega.$$

We know that formally

(1) is not well defined ~~as~~

gets as (1) is its strong formulation.



We are not allowed to

7)

say do

$$u = -\Delta^{-1} f$$

and $\|u\| \leq \|-\Delta^{-1} f\|$

①

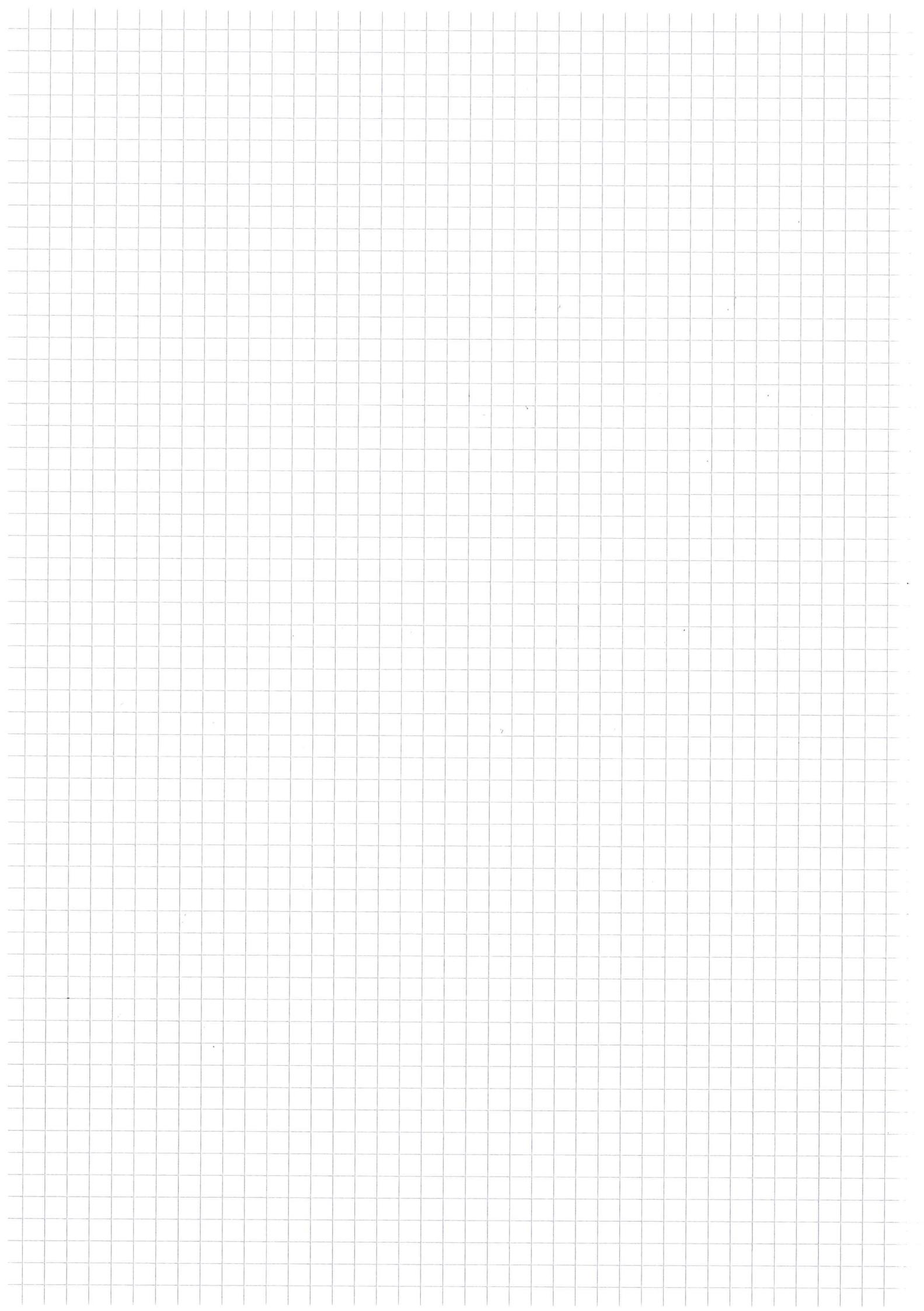
What we have is the

weak form : Find $u \in H_0^1(\Omega)$

$$\int_{\Omega} \nabla u : \nabla v = \int_{\Omega} f v$$

A

$$V \in H_0^1(\Omega)$$



From the weak form

8)

I would like to
write

$$\|\nabla u\| \leq \|\nabla^{-1} f\|$$

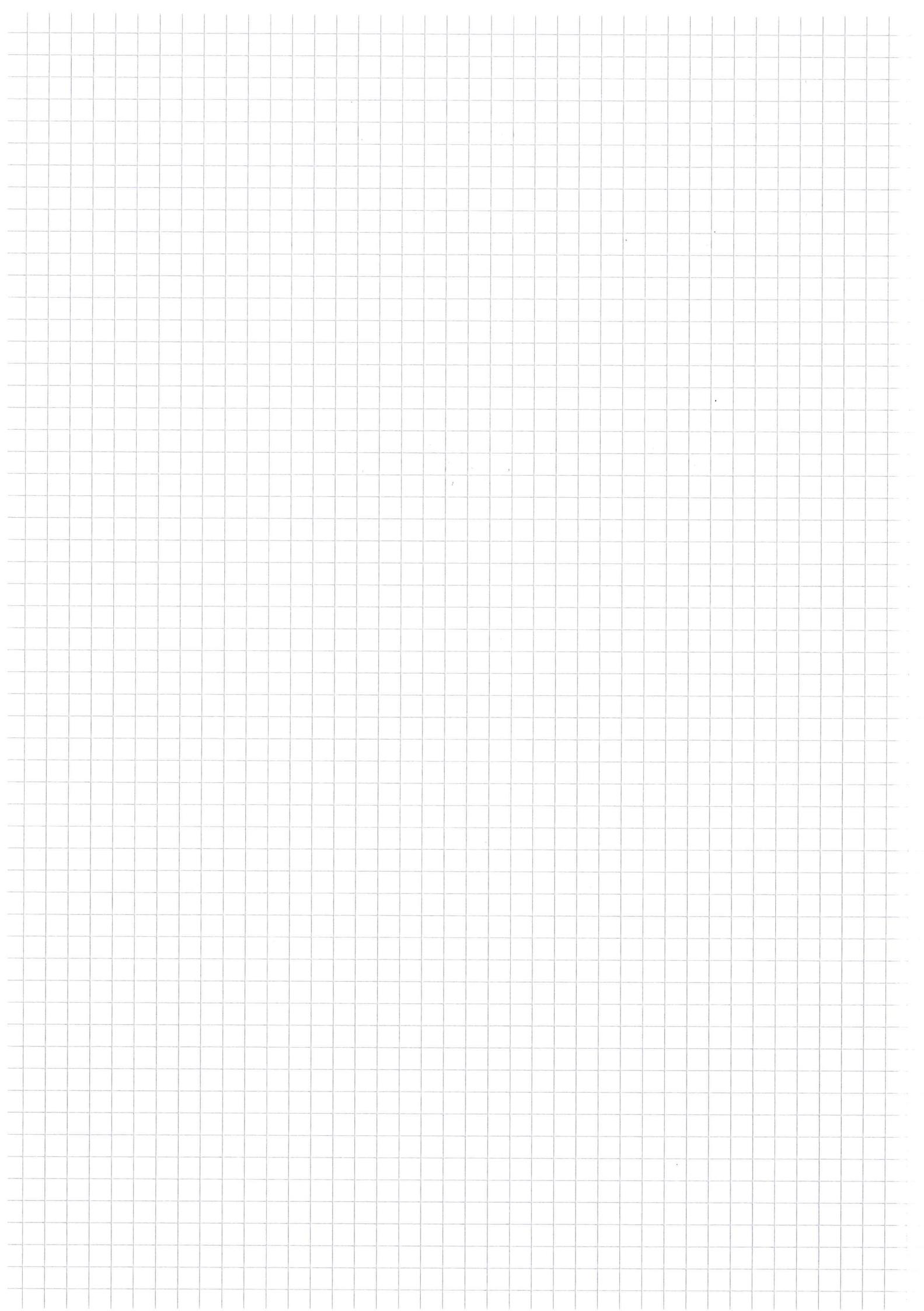


Problems here

again

o o





9)

The correct thing

to write here is

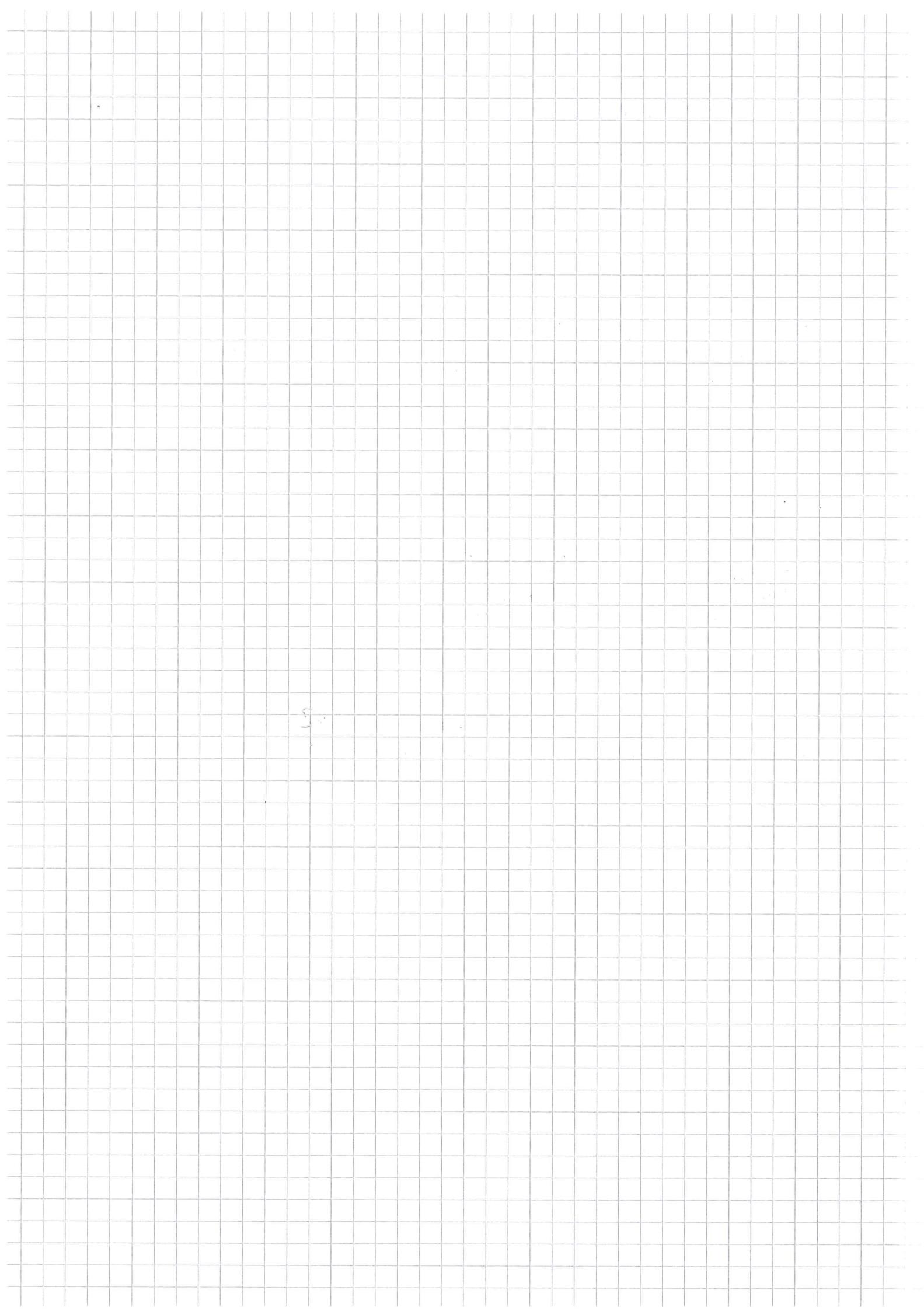
$$\|u\|_1 \leq \|f\|_{-1}$$

should
be semi-norms.

here

$$\|u\|_1^2 = \int (\nabla u)^2$$

what about $\|f\|_{-1}$?



10)

Let's play !

Say $-\Delta u = f$ ← solutions
 $-\Delta v = g$ ← defined weakly

We have

$$(u, v)_1 = (-\Delta u, v) = (\nabla u, \nabla v) = (u, -\Delta v)$$

Analogously, let's say that

$$(f, g)_{-1} = (-\Delta^{-1}f, g) = (f, -\Delta^{-1}g)$$

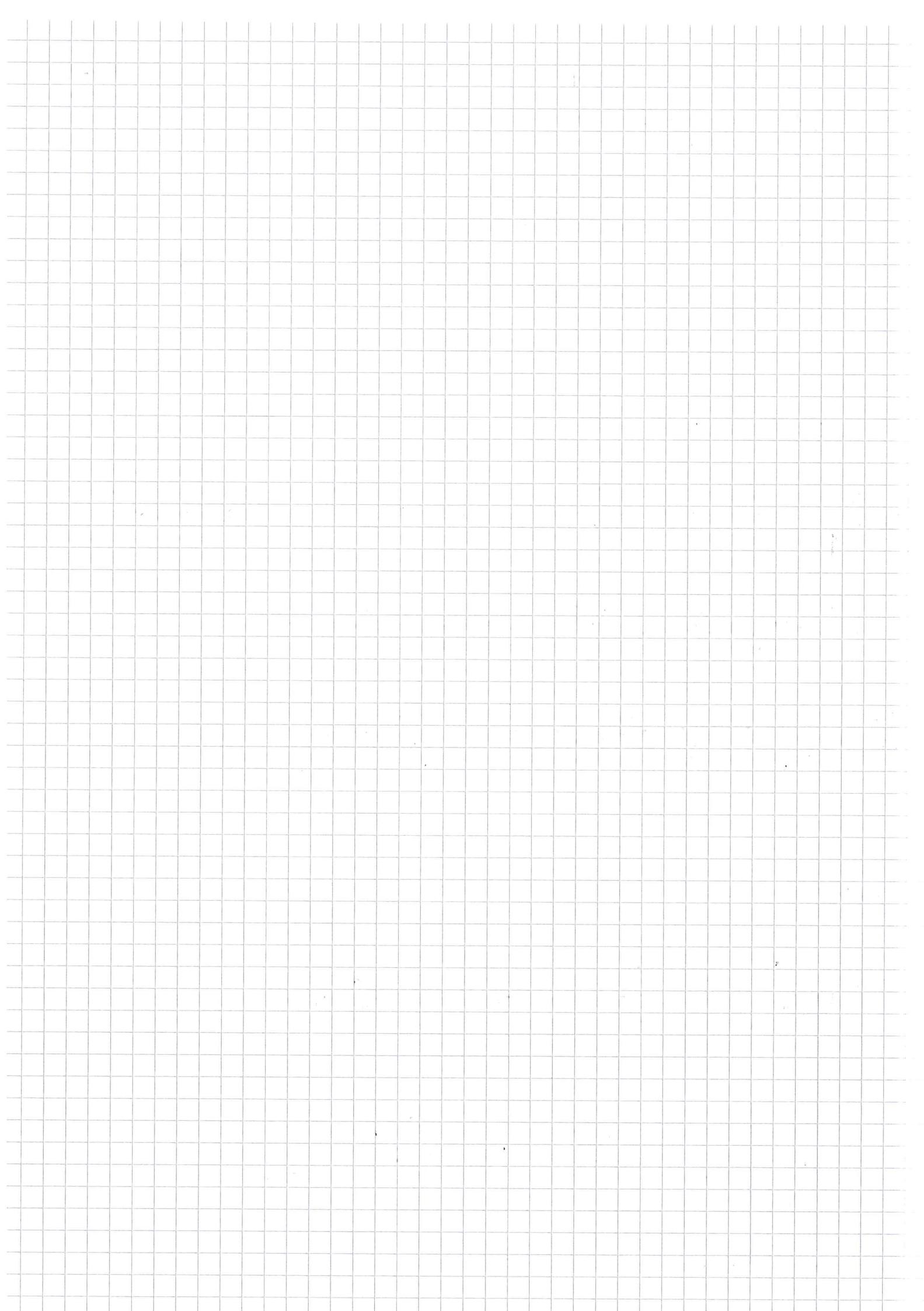
Now

$$(u, v)_1 = (-\Delta u, v) = (f, v)$$

$$\Downarrow \quad (f, -\Delta^{-1}g) = (f, g)_{-1}$$

↑

$$v = -\Delta^{-1}g$$



Let $u = \sin(k\pi x)$

11)

Then

$$(u, u)_1 = (-\Delta u, u)$$

$$= (\pi k)^2 (u, u) = (\pi k)^2 \|u\|_0^2$$

Further

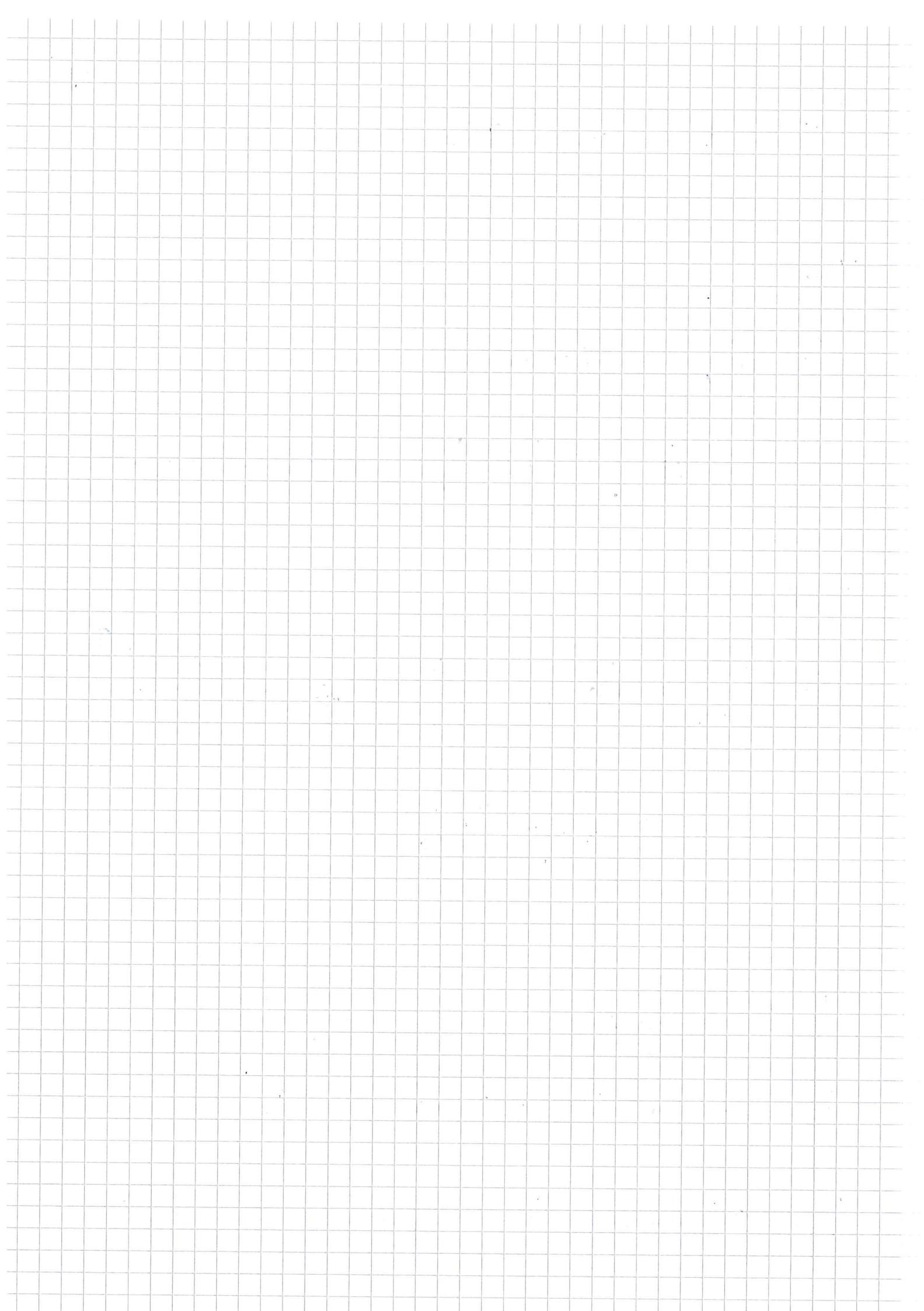
$$(u, u)_{-1} = ((-\Delta)^{-1} u, u) = \frac{1}{(\pi k)^2} (u, u)$$

$$= \frac{1}{(\pi k)^2} \|u\|_0^2$$

High frequencies are important

in H^1 , with weight $(k\pi)^2$

but inversely less important in H^{-1}

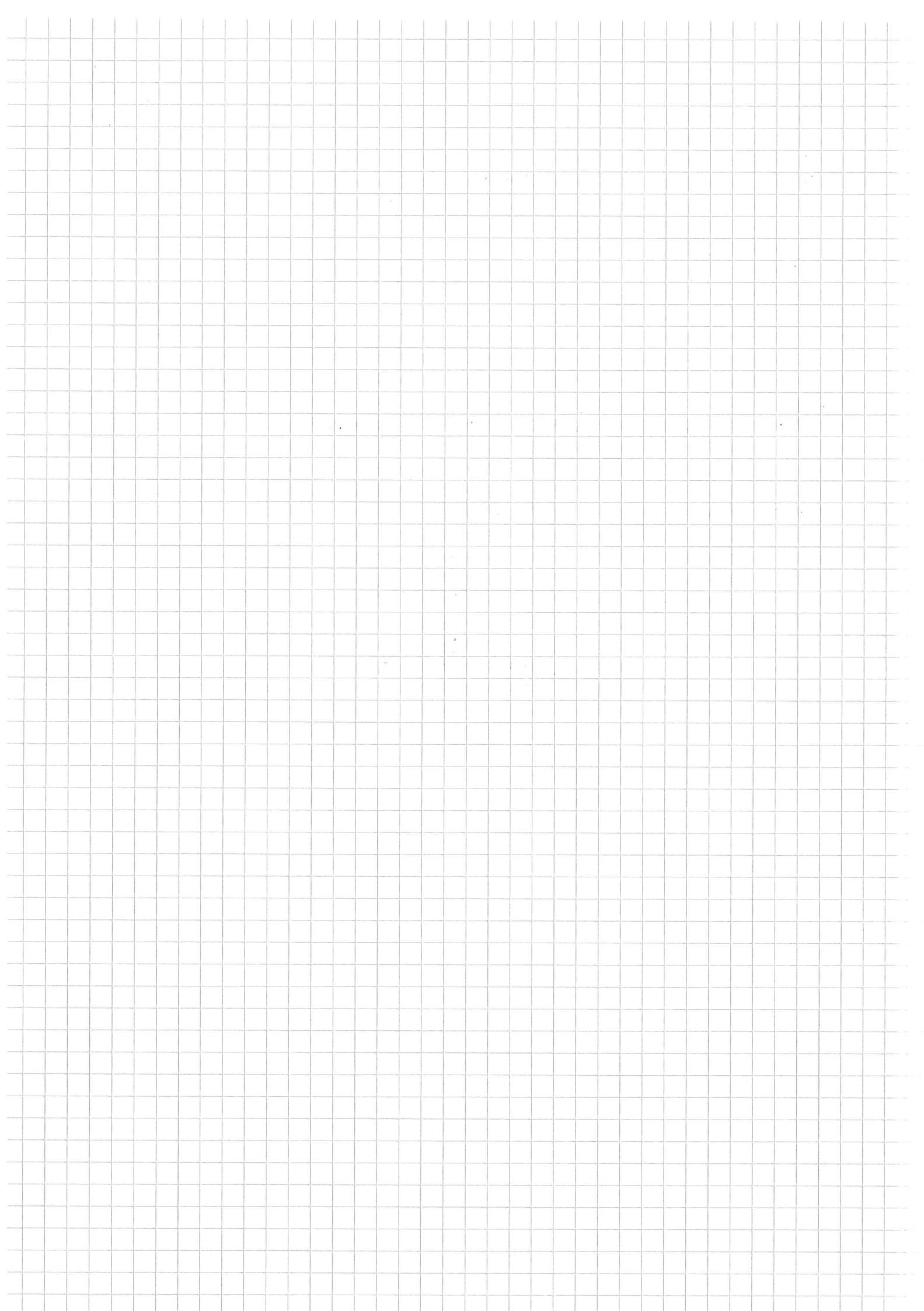


12)

H^{-1} is the dual space
of H_0' .

The formal, but less instructive
definition is that

$$\|f\|_{-1} = \sup_{v \in H_0'} \frac{(f, v)}{\|v\|_1}$$

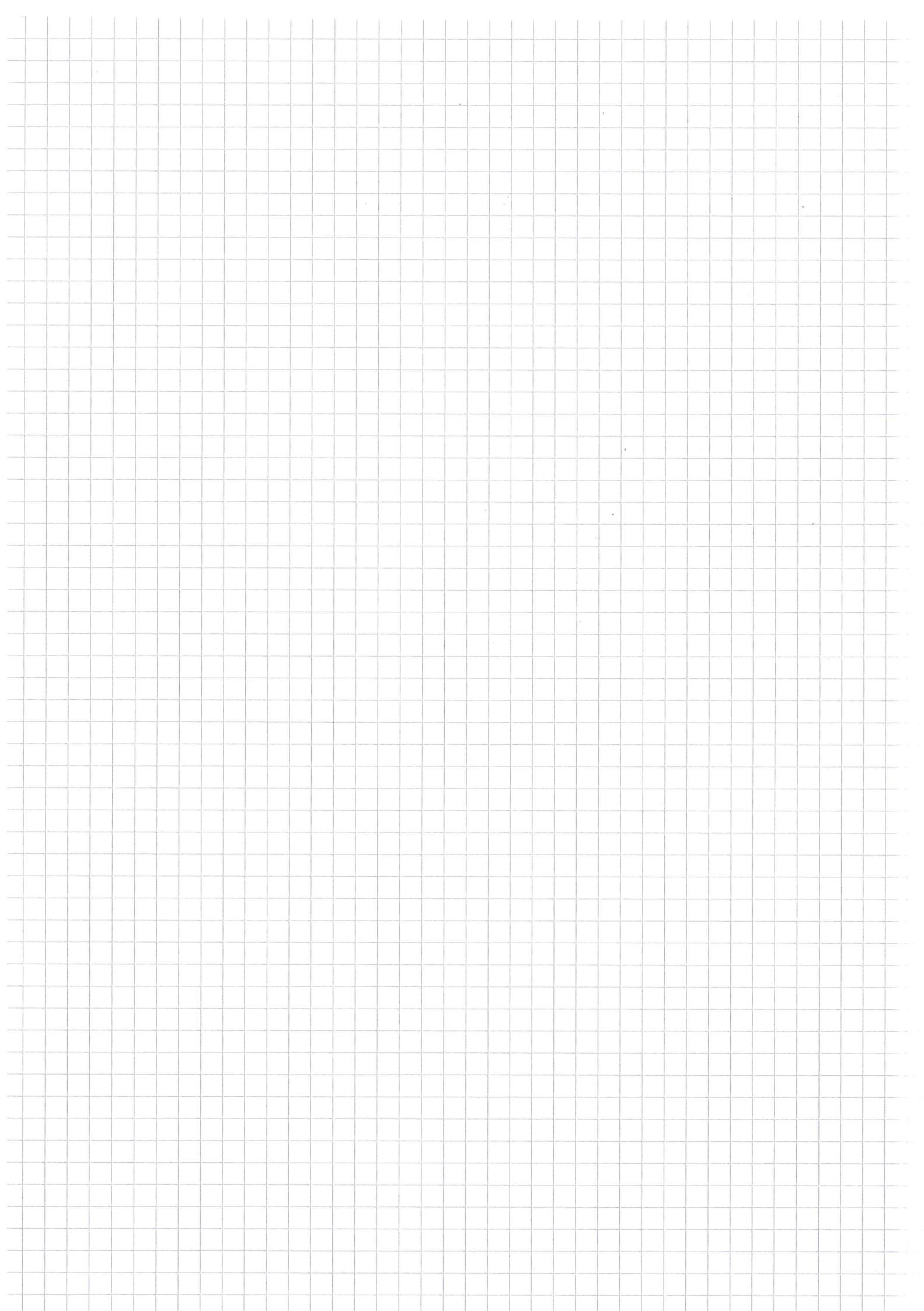


13.)

There is a distinction
between functions and
functionals.

A functional is
something that takes a
function and produce a number,

~~function~~
~~functional~~



14)

Any function can together

with the L^2 inner product

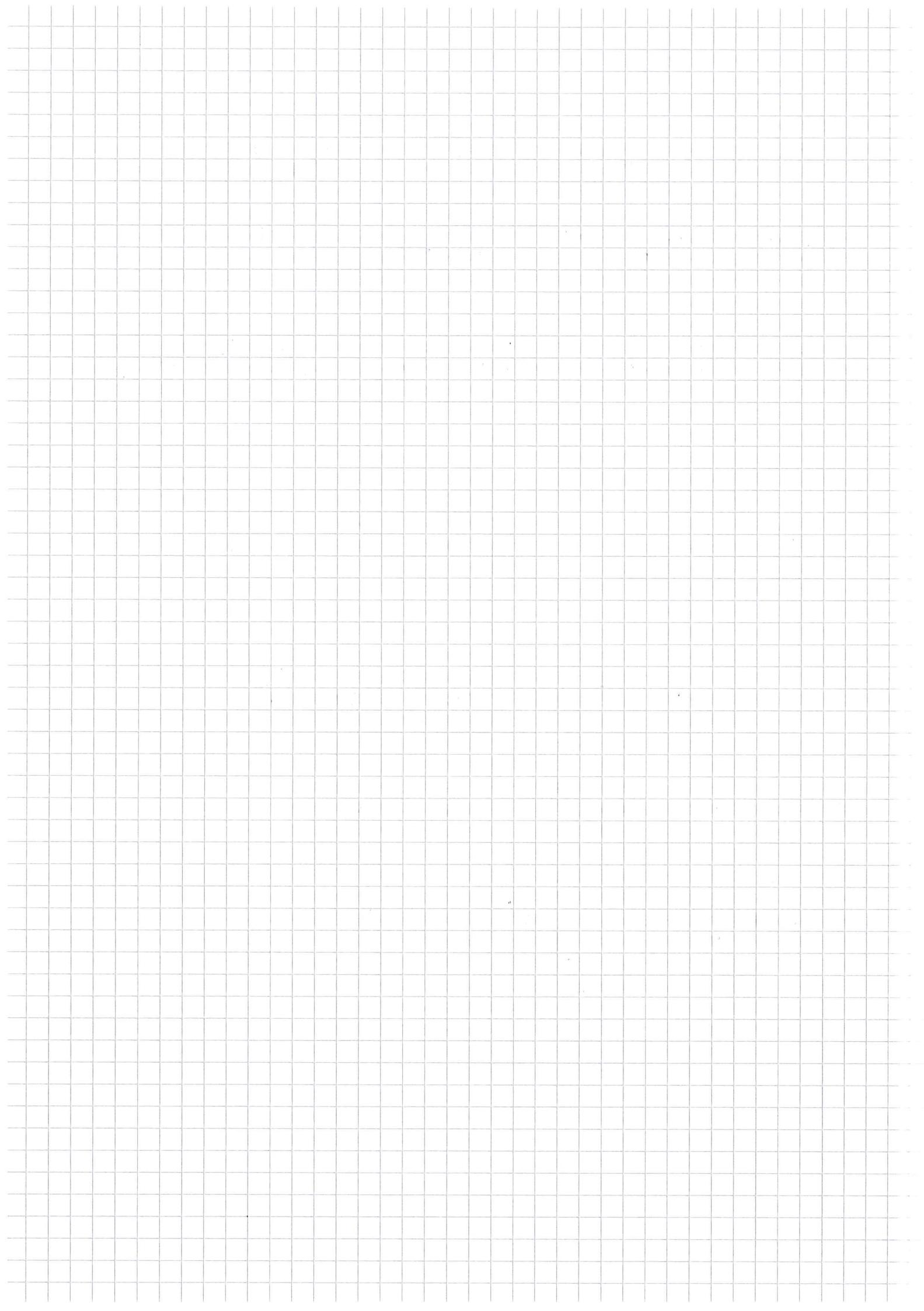
be turned into a functional \mathcal{F}

Let f be some function,

Then the corresponding functional

is when applied to v

$$\mathcal{F}(v) = \int_{\Omega} f v \, dx.$$



15)

Hence, in some sense

the ~~discrete~~ distinction

between functions and

functionals are not that

important in practice.

But functionals may be

defined for "generalized functions"

such as Dirac's delta.

