FEniCS Course

Lecture 23: Biot's equations of poroelasticity

Contributors
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Biot's equations of poroelasticity

The Biot's equations of poroelasticity describe the fluid–structure interaction between a porous elastic media saturated by a pressurized fluid

The equations have many forms, one common form is the displacement-pressure formulation:

$$\begin{aligned} \rho u_{tt} - \nabla \cdot 2\mu\epsilon(u) - \nabla\lambda\nabla \cdot uI + \nabla p &= f \\ s\frac{\partial p}{\partial t} + \alpha\frac{\nabla \cdot u}{\partial t} - \nabla \cdot (K\nabla p) &= g \end{aligned}$$

Here

- u and p are the uknown displacement and pressure
- ρ is the density
- μ and λ are Lamé's elastic parameters
- s is the storage coefficent
- K is the hydralic conductivity (permeability / viscosity)

The challenge is to find schemes that are robust to (large) variations in particular in λ and K.

Biot's equations of poroelasticity

The equations are a coupling of linear elasticity and porous Darcy flow. It has a number of intesting and challenging features:

1. Second order derivative in time ρu_{tt}
2. First order derivatives in time $s\frac{\partial p}{\partial t}$ and $\alpha\frac{\nabla\cdot u}{\partial t}$
3. Three elliptic terms $\nabla\cdot 2\mu\epsilon(u)$, $\nabla\lambda\nabla\cdot uI$, and $\nabla\cdot(K\nabla p)$
4. $\nabla\lambda\nabla\cdot uI$ is associated with locking of the displacement as we learned in linear elasticity
5. For $\nabla\cdot(K\nabla p)$, K may be very small or contain large discontinuities resulting in pressure oscillations

Hence, this is a challenging numerical problem which is currently heavily investigated

Biot's equations of poroelasticity

In the quasi-static formulation we ignore the term ρu_{tt}. This can be done as long as the application in focus does not involve shear or compression waves. The equation can the be written:

$$\begin{aligned} -\nabla \cdot 2\mu\epsilon(u) - \nabla\lambda\nabla \cdot uI + \nabla p &= f \\ s\frac{\partial p}{\partial t} + \alpha\frac{\nabla \cdot u}{\partial t} - \nabla \cdot (K\nabla p) &= g \end{aligned}$$

Variational formulation

$$
\begin{aligned}
-\nabla \cdot 2\mu\epsilon(u) - \nabla\lambda\nabla\cdot uI + \nabla p &= f && (1)\\
s\frac{\partial p}{\partial t} + \alpha\frac{\nabla\cdot u}{\partial t} - \nabla\cdot(K\nabla p) &= g && (2)
\end{aligned}
$$

We obtain a variational formulation by multiplying the momentum equation, (1), by v and integrating by parts

$$\int_\Omega 2\mu\epsilon(u) : \epsilon(v)\,\mathrm{d}x + \int_\Omega \lambda\nabla\cdot u\nabla\cdot v\,\mathrm{d}x - \int_\Omega p\nabla\cdot v\,\mathrm{d}x = \int_\Omega f v\,\mathrm{d}x$$

Similarly, multiplying the continuity equation, (2), by q and integrating by parts we obtain and multiplying by -1 to obtain symmetry we get

$$-\int_\Omega s\frac{\partial p}{\partial t}q\,\mathrm{d}x - \int_\Omega \alpha\frac{\nabla\cdot u}{\partial t}q\,\mathrm{d}x - \int_\Omega K\nabla p\cdot\nabla q\,\mathrm{d}x = \int_\Omega gq\,\mathrm{d}x$$

Variational formulation, cont'd

The equations

$$\int_\Omega 2\mu\epsilon(u) : \epsilon(v)\,\mathrm{d}x + \int_\Omega \lambda\nabla\cdot u\nabla\cdot v\,\mathrm{d}x - \int_\Omega p\nabla\cdot v\,\mathrm{d}x = \int_\Omega fv\,\mathrm{d}x$$

$$-\int_\Omega s\frac{\partial p}{\partial t}q\,\mathrm{d}x - \int_\Omega \alpha\frac{\nabla\cdot u}{\partial t}q\,\mathrm{d}x - \int_\Omega K\nabla p\cdot\nabla q\,\mathrm{d}x = \int_\Omega gq\,\mathrm{d}x$$

with an implicit Euler discretization (where we have multiplied the continuity equation with Δt) reads: The equations

$$\begin{aligned}
\int_\Omega 2\mu\epsilon(u^n) : \epsilon(v) + \lambda\nabla\cdot u^n\nabla\cdot v - p^n\nabla\cdot v\,\mathrm{d}x &= \int_\Omega fv\,\mathrm{d}x \\
\int_\Omega -sp^nq - \alpha\nabla\cdot u^nq - \Delta tK\nabla p^n\cdot\nabla q\,\mathrm{d}x &= \int_\Omega \ldots\,\mathrm{d}x
\end{aligned}$$

Variational formulation

This equation may be written as a saddle point problem

$$\begin{bmatrix} A & B \\ B^T & -C \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \tag{3}$$

Where, if N_i are the basis functions for the displacement and L_i are the basis functions of the pressure

- $A_{ij} = \int_\Omega 2\mu\epsilon(N_i) : \epsilon(N_j)\,\mathrm{d}x + \lambda\nabla\cdot N_i \nabla\cdot N_j\,\mathrm{d}x$
- $B_{ij} = \int_\Omega \nabla\cdot N_i L_j\,\mathrm{d}x$
- $C_{ij} = \int_\Omega sL_iL_j\,\mathrm{d}x + K\nabla L_i\cdot\nabla L_j\,\mathrm{d}x$

Notice that the C matrix (as A) is positive, but in (3) there is a '-' sign in front of C. As for the previously mentioned mixed form of elasticity (with "solid pressure"), this negative term is stabilizing the saddle point problem.

Comparison with Stokes

We remember Stokes problem This equation may be written as a saddle point problem

$$\begin{bmatrix} -\nabla^2 & -\nabla \\ \nabla\cdot & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

In our case A and C are different. This equation may be written as a saddle point problem

$$\begin{bmatrix} -\nabla \cdot 2\mu\epsilon - \nabla\lambda\nabla\cdot & -\nabla \\ \nabla\cdot & -s + \nabla \cdot K\nabla \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

- We already addressed the difference between ∇^2 and $\nabla \cdot \epsilon$
- We therefore replace $\nabla \cdot \epsilon$ with ∇^2 to simplify the discussion
- We known that large λ is associated with locking of displacement
- Pressure oscillations are associated with K small or with large jumps

An attempt to avoid displacement locking: introduce solid pressure

We remember that for linear elasticity (when $\lambda \gg \mu$) we introduced the solid pressure. The displacement formulation of linear elasticity was:

$$-\mu \nabla^2 u - \nabla \lambda \nabla \cdot u = f$$

introducing the solid pressure as $p_S = \lambda \nabla \cdot u$ we obtain

$$\begin{aligned} -\mu \nabla^2 u - \nabla p_S &= f \\ \nabla \cdot u - \frac{1}{\lambda} p_S &= g \end{aligned}$$

This formulation did not suffer from locking when using Brezzi stable elements such as, e.g., Taylor–Hood.

Biot with solid pressure

Using the same trick we obtain an alternative formulation of the Biot's equations with two pressures (solid p_S and fluid p_F)

$$\begin{bmatrix} -\nabla^2 & -\nabla & -\nabla \\ \nabla\cdot & -\frac{1}{\lambda} & 0 \\ \nabla\cdot & 0 & -s+\nabla\cdot K\nabla \end{bmatrix} \begin{bmatrix} u \\ p_S \\ p_F \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ g \end{bmatrix}$$

Is this formulation ok?

Biot with solid pressure

Using the same trick we obtain an alternative formulation of the Biot's equations with two pressures (solid p_S and fluid p_F)

$$\begin{bmatrix} -\nabla^2 & -\nabla & -\nabla \\ \nabla\cdot & -\frac{1}{\lambda} & 0 \\ \nabla\cdot & 0 & -s + \nabla \cdot K\nabla \end{bmatrix} \begin{bmatrix} u \\ p_S \\ p_F \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ g \end{bmatrix}$$

As $\lambda \to \infty$ and $K, s \to 0$ we obtain

$$\begin{bmatrix} -\nabla^2 & -\nabla & -\nabla \\ \nabla\cdot & 0 & 0 \\ \nabla\cdot & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ p_S \\ p_F \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ g \end{bmatrix}$$

This formulation is not stable in the parameters λ, s and K!

Biot with Total pressure

Let us instead introduce the total pressure: $p_T = p_S + p_F$. The problem then reads

$$\begin{bmatrix} -\nabla^2 & -\nabla & 0 \\ \nabla\cdot & -\frac{1}{\lambda} & \frac{1}{\lambda} \\ 0 & \frac{1}{\lambda} & -s - \frac{1}{\lambda} + \nabla \cdot K\nabla \end{bmatrix} \begin{bmatrix} u \\ p_T \\ p_F \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ g \end{bmatrix}$$

This formulation is perfectly stable in the parameters λ, s and K!

Standard Stokes elements can be used!

Lee JJ, Mardal KA, Winther R. Parameter-robust discretization and preconditioning of Biot's consolidation model. SIAM Journal on Scientific Computing. 2017 Jan 3;39(1):A1-24.

The Galerkin method and oscillations

Consider the following simplified problem in 1D

$$s\frac{\partial p}{\partial t} - \nabla \cdot (K \nabla p) = f$$

Given a small K, does a usual finite element scheme yield a good approximation?

The Galerkin method and oscillations

An implicit Euler leads to

$$sp^n - \Delta t \nabla \cdot (K \nabla p^n) = p^{n-1} + \Delta t f$$

Given a small K, does a usual finite element scheme yield a good approximation?

The Galerkin method and oscillations, cont'd

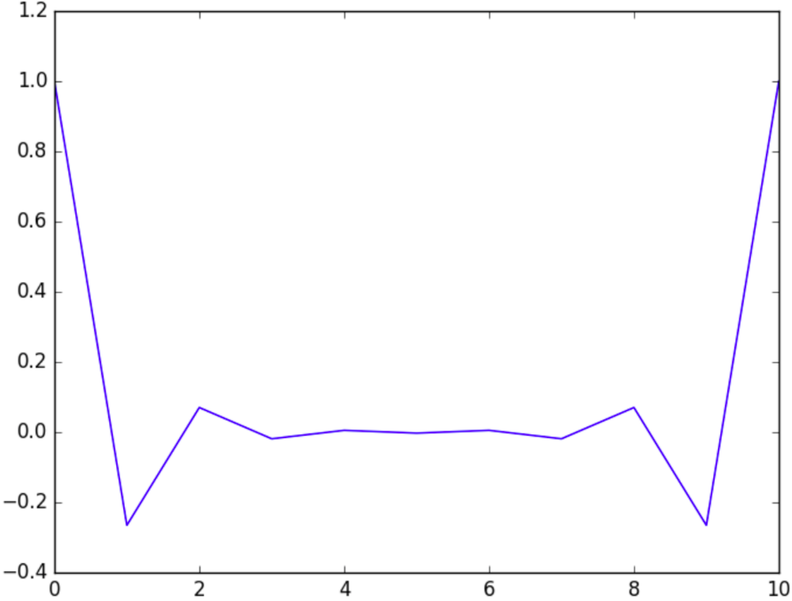
```
from dolfin import *
mesh = UnitIntervalMesh(10)
V = FunctionSpace(mesh, "Lagrange", 1)
u = TrialFunction(V)
v = TestFunction(V)

K = Constant(0.00001)
a = u*v*dx + K*inner(grad(u), grad(v))*dx
L = Constant(0)*v*dx

def boundary(x): return near(x[0], 0) or
    near(x[0], 1)
bc = DirichletBC(V, Constant(1), boundary)

u = Function(V)
solve(a == L, u, bc)
```

The Galerkin method and oscillations, cont'd



Is the best approximation the best approximation?

- The finite element method actually find the best approximation (in the inner product defined by our weak form)
- Clearly, we saw *un-physical oscillations*
- This illustrates that the best approximation may be treacherous
- To guarantee oscillation-free solutions we would need monotom schemes or schemes that conserve special properties
- In this case it is natural to employ more advanced (mixed) schemes for the Darcy problem