

Why function spaces ?)

Properties :

- if u, v are functions in V
then $w = u + v$ is also in V
- if $u \in V$ and $c \in \mathbb{R}$ then
 $w = cu \in V$.
- if V is spanned by
 $\{N_i\}_{i=0}^N$ then $u = \sum_{i=0}^N u_i N_i$
is a typical function in V .

Function spaces have nice properties 2)

- Cauchy - Schwarz inequality

If V is a Hilbert space it involves an inner product $\langle \cdot, \cdot \rangle$

Then

$$\langle u, v \rangle \leq \|u\|_V \|v\|$$

- We use function spaces that are complete (Hilbert, Banach)

Hence, if ~~$v_n \rightarrow v$~~ $v_n \xrightarrow{n \rightarrow \infty} v$

and $u_n \in V$ then

$$u \in V.$$

Our function spaces 3)

typically have approximation properties.

Given a family of meshes \mathcal{R}_h

that approximate \mathcal{R} .

Let V be the continuous function space while $\{\check{V}_h\}$ is the discrete spaces.

Typically we can then say that for any $u \in V$ we have

$$\|u - u_h\|_V \leq ch^\alpha \|u\|_V$$

Fundamental lemma (Gauss-Green) 4)

(integration by parts)

$$\int_{\Omega} -\nabla \cdot (K \nabla u) v \, dx = \int_{\partial\Omega} -K \frac{\partial u}{\partial n} v \, ds$$

$$+ \int_{\Omega} K \nabla u \cdot \nabla v \, dx$$

As already mentioned, this

is incredibly useful

and the basis for a lot
of results.

5)

Deriving a variational formulation

3 Step

6)

1. multiply by a test function
2. integrate over the domain
3. integrate by parts.

Poisson problem :

(6)

$$-\nabla \cdot (K \nabla u) = f \quad \text{in } \Omega \quad (1)$$

$$u = u_0 \quad \text{on } \partial\Omega_D \quad (2)$$

$$-K \nabla u \cdot n = g \quad \text{on } \partial\Omega_N \quad (3)$$

Step 1.

$$-\nabla \cdot (K \nabla u) \cdot v = f \cdot v$$

[You may ask whether
you should hit (2) and (3)
with a test function as
well. Sometimes this is useful.]

8)

7)

Step 2.

$$\int_{\Omega} -\nabla \cdot (K \nabla u) v \, dx = \int_{\Omega} f v \, dx$$

||

Step 3.

$$\int_{\Omega} K \nabla u \cdot \nabla v - \int_{\partial\Omega} K \frac{\partial u}{\partial n} v \, ds$$

Here, we know

 $K \frac{\partial u}{\partial n}$ on $\partial\Omega_N$ by (3)

What about
 $\partial\Omega_D$ and $u=u_0$?

14)

Result that we

8)

On $\partial\Omega_D$ we

set $u = u_0$ and let

$v = 0$. This happens in

bc. apply () in FEniKS

in the sense that

$u = u_0$ is put in pointwise

at every point on the boundary.

Summarizing we have that

$$\int_{\partial\Omega} K \frac{\partial u}{\partial n} v \, ds = \int_{\partial\Omega_D} K \frac{\partial u}{\partial n} v \, ds + \int_{\partial\Omega_N} K \frac{\partial u}{\partial n} v \, ds$$

\uparrow

\uparrow

$= 0$

$$= \int_{\partial\Omega_N} g v \, ds$$

Result that we

14)

The variational formulation

9)

Find $u \in V_g$ such that

$$a(u, v) = L(v) \quad \forall v \in V_0$$

where

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx$$

$$L(v) = \int_{\Omega} fv \, dx + \int_{\partial\Omega} gv \, ds$$

want to prove.

Comments

10)

$$u \in V_g$$

determines the number of degrees of freedom / unknowns

$$v \in V_0$$

determine the number of equations.

$$\text{If } V_g = V_0$$

except on the Dirichlet boundary

we call it the Galerkin method.

We then have that

$$\dim(V_g) = \dim(V_0)$$

→ "well defined" problem
in terms of a rectangular matrix

Result that we
want to prove.

II)

FEM version

Find $u_h \in V_{g,h}$ such that

(*)

$$a(u_h, v_h) = L(v_h) \quad \forall v \in V_{0,h}$$

The specific form of (*)
that we use is

$$u_h = \sum u_i N_i$$

↑

FEniCS
Function

FEniCS
Trial Function

$$v_h = \sum N_j$$

12)

This is then written
as a linear system.

$$Ax = b$$

$$A_{ij} = a(N_i, N_j)$$

$$b_j = L(N_j)$$

$$x_i = u_i$$

Notice that

$$Ax = \sum a(N_i, N_j) u_i$$

$$= \textcircled{Q} a\left(\sum u_i N_i, N_j\right)$$

$$= a(u_n, N_j) = L(N_j)$$

want to prove.

Useful function spaces

13

L^2 : consist of functions such that

$$\int_{\Omega} u^2 dx < \infty$$

: Inner product between u, v

$$(u, v)_{L^2} = \int_{\Omega} uv dx$$

H^1 : consists of functions such that

$$\int_{\Omega} (u^2 + (\nabla u)^2) dx < \infty$$

Inner product ~~between~~ between u, v

$$(u, v)_{H^1} = \int_{\Omega} (uv + \nabla u \cdot \nabla v) dx$$

Result that we
want to prove.

14)

u is the continuous solution.

u_h is the family of
solutions parameterized by h .

Then

$$\|u - u_h\|_1 \leq c h^\alpha \|u\|_\infty$$

Question:

What is the cool thing
with an inner product?