Prior independent scheduler as Onsager correction for AMP

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Spike

$$\mathbf{Y}_\lambda = \sqrt{\frac{\lambda}{N}}\mathbf{X}^*\mathbf{X}^{*\mathsf{T}} + \mathbf{W}$$

Signal: $\mathbf{X}^* \in \mathbb{R}^N$, $\mathbb{N}_+ \ni N \to \infty$
Signal-to-Noise ratio (SNR): $\lambda \geq 0$
Noise: $W_{ij} = W_{ji} \sim\perp \mathcal{N}(0,1)$

Spike

$$\mathbf{Y}_\lambda = \sqrt{\frac{\lambda}{N}}\mathbf{X}^*\mathbf{X}^{*\intercal} + \mathbf{W}$$

Signal: $\mathbf{X}^* \in \mathbb{R}^N$, $\mathbb{N}_+ \ni N \to \infty$
Signal-to-Noise ratio (SNR): $\lambda \geq 0$
Noise: $W_{ij} = W_{ji} \sim\perp \mathcal{N}(0,1)$

$$X^*_{i\leq N} \sim P_X = \begin{cases} \text{Ber}(\rho) := \rho\delta_1 + (1-\rho)\delta_0 \\ \mathcal{N}(\mu,\sigma^2) := \dfrac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\dfrac{1}{2}\left(\dfrac{x-\mu}{\sigma}\right)^2\right] \\ \text{Rad}(\rho) := \dfrac{\rho}{2}(\delta_1 + \delta_{-1}) + (1-\rho)\delta_0 \\ \mathcal{U}\{0, r-1\} := \dfrac{1}{r} \end{cases}$$

Introduction

Structured noise

Spike

$$\mathbf{Y}_\lambda = \frac{\lambda}{N}\mathbf{X}^*\mathbf{X}^{*\intercal} + \mathbf{W}$$

$$dP_W(\mathbf{W}) = C_V \exp\left(-\frac{N}{2}\mathrm{Tr}V(\mathbf{W})\right)\prod_{i\le j} dW_{ij}$$

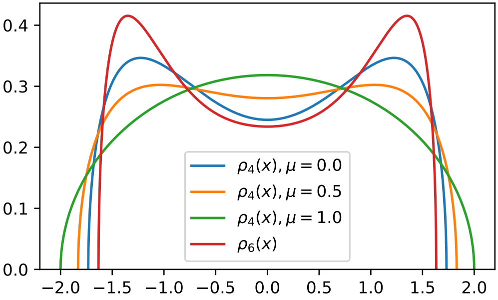
- $V_4(x) = \frac{\mu}{2}x^2 + \frac{\gamma}{4}x^4$
- $V_6(x) = \frac{\mu}{2}x^2 + \frac{\gamma}{4}x^4 + \frac{\xi}{6}x^6$

$$\mu, \gamma, \xi \geq 0$$

- $\rho_4(x) = \frac{(\mu+2a^2\gamma+\gamma x^2)\sqrt{4a^2-x^2}}{2\pi}$
- $\rho_6(x) = \frac{(\mu+2a^2\gamma+6a^4\xi+(\gamma+2a^2\xi)x^2+\xi x^4)\sqrt{4a^2-x^2}}{2\pi}$

Spike

$$\mathbf{Y}_\lambda = \frac{\lambda}{N}\mathbf{X}^*\mathbf{X}^{*\intercal} + \mathbf{W}$$



- $V_4(x) = \frac{\mu}{2}x^2 + \frac{\gamma}{4}x^4$
- $V_6(x) = \frac{\mu}{2}x^2 + \frac{\gamma}{4}x^4 + \frac{\xi}{6}x^6$

$$\int x^2 d\rho(x) = 1$$

- $\rho_4(x) = \frac{(\mu+2a^2\gamma+\gamma x^2)\sqrt{4a^2-x^2}}{2\pi}$
  $$\mu \in [0,1]$$
  $$\gamma(\mu) = \frac{8-9\mu+\sqrt{64-144\mu+108\mu^2-27\mu^3}}{27}$$
  $$a^2 = (\sqrt{\mu^2+12\gamma}-\mu)/(6\gamma)$$
- $\rho_6(x) = \frac{\xi(6a^4+2a^2x^2+x^4)\sqrt{4a^2-x^2}}{2\pi}$
  $$\xi = 27/80, \quad a = \sqrt{2/3}$$

Introduction

Approximate Message Passing (AMP) algorithm

Low complexity, *computationally efficient*, iterative algorithm

Marginals:

$$\mathbf{p}^{t+1} := \eta(\mathbf{P}^t, \mathbf{Q}^t), \quad \mathbf{q}^{t+1} := \partial_{\mathbf{P}^t}\eta(\mathbf{P}^t, \mathbf{Q}^t)$$

Denoiser:

$$\begin{aligned}\eta(b, v) :=& \mathbb{E}\left[X \middle| Xv + W\sqrt{v} = b\right] \\ =& \frac{\int dP_X(x) x \exp(xb - x^2 v/2)}{\int dP_X(x) \exp(xb - x^2 v/2)}\end{aligned}$$

Cavity Fields:

$$\mathbf{P}^{t+1} := \sqrt{\frac{\lambda}{N}}\mathbf{Y}\mathbf{p}^t - \mathbf{p}^{t-1} \circ \left[\frac{\lambda}{N}\mathbf{Y}^2\mathbf{q}^t\right] \quad \mathbf{Q}^{t+1} := \frac{\lambda}{N}\left(\mathbf{q}^t + \|\mathbf{p}^t\|^2\mathbf{1}\right) - \left[\frac{\lambda}{N}\mathbf{Y}^2\mathbf{q}^t\right]$$

State evolution (SE)

Mean Square Error (MSE)

$$\begin{aligned}\mathrm{MSE}^t_{\mathrm{AMP}} &\equiv \frac{1}{N^2}\left\|\mathbf{x}^* \otimes \mathbf{x}^* - \mathbf{p}^t \otimes \mathbf{p}^t\right\|_F^2 \\ &= \frac{1}{N^2}\left(\|\mathbf{x}^*\|_2^4 + \|\mathbf{b}^t\|_2^4 - 2(\mathbf{x}^* \cdot \mathbf{b}^t)^2\right)\end{aligned}$$

Methodology

Quartic

Algorithm 1: Spike Wigner Model

Input : λ, ρ, μ

$\mathbf{x}^* \leftarrow \mathbf{X}^*_{i \leq N} \sim P_X(\rho)$

$\gamma \leftarrow \frac{8-9\mu+\sqrt{64-144\mu+108\mu^2-27\mu^3}}{27}$

$a^2 \leftarrow \frac{\sqrt{\mu^2+12\gamma}-\mu}{6\gamma}$

$\mathbf{V} \leftarrow$ Orthogonal matrix

$\mathbf{\Lambda} \leftarrow \text{diag}\left(\mathbf{W}_i \sim \rho_4(x) = \frac{(\mu+2a^2\gamma+\gamma x^2)\sqrt{4a^2-x^2}}{2\pi}\right)$

$\mathbf{Y} \leftarrow \frac{\lambda}{N}\mathbf{x}^*\mathbf{x}^{*\intercal} + \mathbf{V}\mathbf{\Lambda}\mathbf{V}^\intercal$

Output: $\mathbf{Y}$

Algorithm 2: Spike Wigner Model

Input : λ, ρ

1 $\mathbf{x}^* \leftarrow \mathbf{X}^*_{i \leq N} \sim P_X(\rho)$

2 $\xi \leftarrow 27/80$

3 $a^2 \leftarrow 2/3$

4 $\mathbf{V} \leftarrow$ Orthogonal matrix

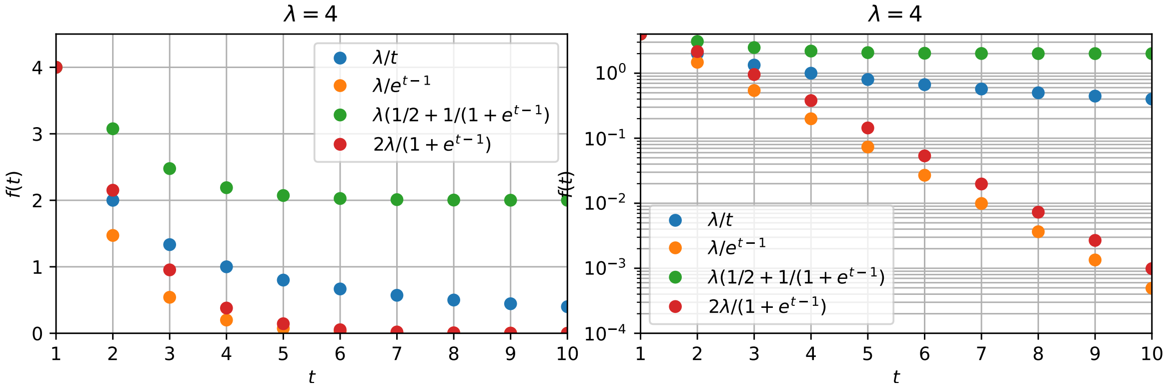
5 $\mathbf{\Lambda} \leftarrow \text{diag}\left(\mathbf{W}_i \sim \rho_6(x) = \frac{\xi(6a^4+2a^2x^2+x^4)\sqrt{4a^2-x^2}}{2\pi}\right)$

6 $\mathbf{Y} \leftarrow \frac{\lambda}{N}\mathbf{x}^*\mathbf{x}^{*\intercal} + \mathbf{V}\mathbf{\Lambda}\mathbf{V}^\intercal$

Output: $\mathbf{Y}$

Methodology

Prior independent scheduler



Results

Conclusions & Discussions