Github Document

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These are Wolfram Mathematica codes for systematically constructing Lyapunov functions for dynamical systems corresponding to optimization methods and deriving their convergence rates. These codes were used to yield the Lyapunov functions and convergence rates in our article "Computer-Assisted Search for Differential Equations Corresponding to Optimization Methods and Their Convergence Rates" (`https://doi.org/10.48550/arXiv.2512.09712`). See this article for the technical details of the numerical method.

Each source file in this folder corresponds to a class of dynamical systems with certain kinds of objective functions and convergence assumptions.

- "Theorem 6" corresponds to the dynamical system $\nabla^2 f\dot{x} + \nabla f = 0$ with a convex objective function and an exponential convergence assumption.
- "Theorem 7,8" corresponds to the dynamical system $\dot{x} + b\nabla^2 f\dot{x} + \nabla f = 0$ with a strongly convex and smooth objective function and an exponential convergence assumption.
- "Theorem 9,10" corresponds to the dynamical system $\ddot{x} + a\dot{x} + b\nabla^2 f\dot{x} + \nabla f = 0$ with a strongly convex objective function and an exponential convergence assumption.
- "Theorem 11" corresponds to the dynamical system $\ddot{x} + \frac{r}{t}\dot{x} + \nabla f = 0$ with a convex objective function and a polynomial convergence assumption.
- "Theorem 12" corresponds to the dynamical system $\ddot{x} + \frac{r}{t}\dot{x} + \nabla f = 0$ with a strongly convex objective function and a polynomial convergence assumption.
- "Theorem 13" corresponds to the dynamical system $\ddot{x} + \frac{r}{t}\dot{x} + \nabla f = 0$ with a strongly convex objective function and an exponential convergence assumption.
- "Theorem 14" corresponds to the dynamical system $\ddot{x} + \frac{r}{t^\alpha}\dot{x} + \nabla f = 0$ with a strongly convex objective function and an unique convergence assumption.

For each source file, each block does the following calculation;

1. Block 1 defines each operations in Appendix A.
2. Block 2 defines sets of operations in Appendix B.1.
3. Block 3 combines the sets of operations as in Appendix B.2 and B.3.
4. Block 4 sets the function that calculates the convergence rate from matrices P, Q. It sets the convergence rate assumption by $\gamma[t]$.
5. Block 5 conducts the matrices calculations and rate calculations. The ODE is represented by Q_0, and the traits of objective functions are given by μ, L. The time limit for calculating each convergence rate is given here by "TimeConstrained".

By executing these five blocks in order, you can perform the matrix computations and derive the convergence rates. The output is provided as a pair consisting of the matrices P, Q and the corresponding convergence rate. Outputs that share the same characteristics are grouped together into a single "group". The matrix P obtained in this process corresponds to the Lyapunov function.

If the convergence rate includes case distinctions with respect to t, you only need to check the rate for $t > 0$.

If you want to carry out an experiment with our proposed algorithm for another ODE and/or objective function and/or convergence assumption and/or time limit, you should change the corresponding blocks mentioned above.