Formalization of double sided auctions

Abhishek Kr Singh
Tata Institute of Fundamental Research, India
abhishek.uor@gmail.com

Suneel Sarswat
Tata Institute of Fundamental Research, India
suneel.sarswat@gmail.com

Abstract

In this paper we introduce a formal framework for analyzing double sided auction mechanisms in a theorem prover. In double sided auctions multiple buyers and sellers participate for trade. Any mechanism for double sided auctions to match buyers and sellers should satisfy certain properties of matching. For example, fairness, percieved-fairness, individual rationality are some of the important properties. These are critical properties and to verify them we need a formal setting. We formally define all these notions in a theorem prover. This provides us a formal setting in which we prove some useful results on matching in a double sided auction. Finally, we use this framework to analyse properties of two important class of double sided auction mechanism. All the properties that we discuss in this paper are completely formalized in the Coq proof assistant.

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1 Introduction

Trading is a principal component of all modern economy. Over the century more and more complex instruments (for example, index, future, options etc.) are being introduced to trade in the financial markets. With the arrival of computer assisted trading, the volume and liquidity in the markets has improved significantly. Today all big stock exchanges use computer algorithms (matching algorithms) to match buy requests (demands) with sell requests (supplies) of traders. Computer algorithms are also used by many traders to place orders in the markets. This is known as algorithmic trading. As a result of all this the markets has become complex and large. Hence, the analysis of markets is no more feasible without the help of computers.

A potential trader (buyer or seller) places orders in the markets through a broker. These orders are matched by the stock exchange to execute trades. Most stock exchanges divide the trading activity into three main sessions known as pre-markets, continous markets and post markets. While in the pre-markets session an opening price of a product is discovered through double sided auction. In the continous markets session the incoming buyers and sellers are continously matched against each other on a priority basis. In the post-markets session clearing of the remaining orders is done and a closing price is discovered.

A double sided auction mechanism allows multiple buyers and sellers to trade simultaneously [1]. In double sided auctions, auctioneer (e.g. stock exchange) collects buy and sell requests over a period. Each potential trader places the orders with a *limit price*: below which a seller will not sell and above which a buyer will not buy. The exchange at the end of this period matches these orders based on their limit prices. This entire process is completed using a matching algorithm for double sided auctions.

Designing algorithms for double sided auctions is a well studied topic [2, 4, 3, 5]. A major emphasis of many of these algorithms is to maximize the number of matches or maximize the profit of the auctioneer. Note that an increase in the number of matches increases the liquidity in the markets. A matching algorithm can produce a matching with a uniform price or a matching with dynamic prices. While an algorithm which clears each matched bid-ask pair at a single price is referred as uniform price algorithm. An algorithm which may clear each matched bid-ask pair at different prices is refered as dynamic price algorithm. There are other important properties besides the number of macthes which are considered while evaluating the effectiveness of a matching algorithm. For example, fairness, uniform pricing, individual rationality are some of the relevant features used to compare these matching algorithms. However, it is known that no single algorithm can posses all of these properties [4, 2].

In this paper, we describe a formal framework to analyze double sided auctions using a theorem prover. For this work, we assume that each trader wishes to trade a single unit of the product and all the products are indistinguishable as well as indivisible. We have used the Coq proof assistant to formally define the theory of double sided auctions. Furthermore, we use this theory to validate various properties of matching algorithms. We formally prove some important properties of two algorithms; a uniform price algorithm and a dynamic price algorithm.

2 Modeling double sided auctions

Bid, Ask and limit price

An auction is a competetive event, where goods/services are sold to highest bidders. In a double sided auction, multiple buyers and multiple sellers places their orders of buy/sell to an agent. The agent, known as auctioneer, matches these buy-sell requests against each others based on their *limit prices*. The limit price for a bid (buy order), is the price above which buyer doesn't want to buy one quantity of the item. Similarily, the limit price of an ask (sell order), is the price below which seller doesn't want to sell one quantity of the item. We defined the notions of the bid as well ask as record in Coq.

```
Record Bid: Type := Mk_bid { bp:> nat;    idb: nat }.
Record Ask: Type := Mk_ask { sp:> nat;    ida: nat }.
```

In the above definition of bid `b`, `bp` (`b`) is the limit price of `b` and `idb` (`b`) is the unique identity of bid `b`. Similarily for the ask `a`, `sp` (`a`) is the limit price of `a` and `ida` (`a`) is the unique identity of ask `a`. In our work, each bid (ask) is a buy (sell) request for one unit of the item. If a trader wish to buy (sell) multiple units, he can create multiple bids (asks) with different ids.

Matching

In double sided auctions, auctioneer collects all the bids (asks) for a duration. We can assume, all the bids are present in a list B. Similarily, all the asks are present in a list A. At the end of the duration, the auctioneer matches bids in B against asks in A. Furthermore, auctioneer assign a trade price to each matched bid-ask pair. The result of process is a matching M, which is also represented using list.

In any matching M, a bid (ask) appears at most once in M. A bid `b` can be matched against an ask `a` if `bp (b)` $\geq$ `sp (a)`. We say, a bid-ask pair (b,a) is matchable if `bp (b)` $\geq$ `sp (a)`. Note that, there can be bids (asks) that are not matched in M. The collection of bids present in M is denoted as B_M and collection of asks present in M is denoted as A_M. More precisely, for a given list of bids B and list of asks A, M is a matching iff, (1) All the bid-ask pairs in M are matchable, (2) B_M is duplicate-free, (3) A_M is duplicate-free, (4) $B_M \subseteq B$, and (5) $A_M \subseteq A$. Formaaly, Matching is,

```
Definition matching (M: list fill_type):=
  (All_matchable M) /\ (NoDup (bids_of M)) /\ (NoDup (asks_of M)).

Definition matching_in (B:list Bid) (A:list Ask) (M:list fill_type):=
(matching M) /\ ((bids_of M) [<=] B) /\ ((asks_of M) [<=] A).
```

▶ **Lemma 1** (Lorem ipsum). *Vestibulum sodales dolor et dui cursus iaculis. Nullam ullamcorper purus vel turpis lobortis eu tempus lorem semper. Proin facilisis gravida rutrum. Etiam sed sollicitudin lorem. Proin pellentesque risus at elit hendrerit pharetra. Integer at turpis varius libero rhoncus fermentum vitae vitae metus.*

Proof. Cras purus lorem, pulvinar et fermentum sagittis, suscipit quis magna.

▷ Claim 2. content...

Proof. content... ◁

◀

▶ **Corollary 3** (Curabitur pulvinar,). *Nam liber tempor cum soluta nobis eleifend option congue nihil imperdiet doming id quod mazim placerat facer possim assum. Lorem ipsum dolor sit amet, consectetuer adipiscing elit, sed diam nonummy nibh euismod tincidunt ut laoreet dolore magna aliquam erat volutpat.*

▶ **Proposition 4.** *This is a proposition*

Proposition 4 and Proposition 4 . . .

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▶ Remark 5. content...

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▶ **Lemma 6** (Quisque blandit tempus nunc). *Sed interdum nisl pretium non. Mauris sodales consequat risus vel consectetur. Aliquam erat volutpat. Nunc sed sapien ligula. Proin faucibus sapien luctus nisl feugiat convallis faucibus elit cursus. Nunc vestibulum nunc ac massa pretium pharetra. Nulla facilisis turpis id augue venenatis blandit. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus.*

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A Styles of lists, enumerations, and descriptions

List of different predefined enumeration styles:

- `\begin{itemize}...\end{itemize}`
- ...
- ...

1. `\begin{enumerate}...\end{enumerate}`
2. ...
3. ...

(a) `\begin{alphaenumerate}...\end{alphaenumerate}`
(b) ...
(c) ...

(i) `\begin{romanenumerate}...\end{romanenumerate}`

(ii) ...
(iii) ...

(1) \begin{bracketenumerate}...\end{bracketenumerate}
(2) ...
(3) ...

Description 1 \begin{description} \item[Description 1] ...\end{description}
Description 2 Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.
Description 3 ...

B Theorem-like environments

List of different predefined enumeration styles:

▶ **Theorem 7.** *Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.*

▶ **Lemma 8.** *Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.*

▶ **Corollary 9.** *Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.*

▶ **Proposition 10.** *Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.*

▶ **Exercise 11.** *Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.*

▶ **Definition 12.** *Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.*

▶ **Example 13.** Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.

▶ Note 14. Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.

▶ Note. Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.

▶ Remark 15. Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.

▶ Remark. Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.

▷ Claim 16. Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.

▷ Claim. Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.

Proof. Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque. ◀

Proof. Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque. ◁