Formalizing double sided auctions in Coq

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Abstract

In this paper we introduce a formal framework for analyzing double sided auction mechanisms in a theorem prover. In double sided auctions multiple buyers and sellers participate for trade. Any mechanism for double sided auctions to match buyers and sellers should satisfy certain properties of matching. For example, fairness, percieved-fairness, individual rationality are some of the important properties. These are critical properties and to verify them we need a formal setting. We formally define all these notions in a theorem prover. This provides us a formal setting in which we prove some useful results on matching in a double sided auction. Finally, we use this framework to analyse properties of two important class of double sided auction mechanism. All the properties that we discuss in this paper are completely formalized in the Coq proof assistant.

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1 Introduction

Trading is a principal component of all modern economy. Over the century more and more complex instruments (for example, index, future, options etc.) are being introduced to trade in the financial markets. With the arrival of computer assisted trading, the volume and liquidity in the markets has improved significantly. Today all big stock exchanges use computer algorithms (matching algorithms) to match buy requests (demands) with sell requests (supplies) of traders. Computer algorithms are also used by many traders to place orders in the markets. This is known as algorithmic trading. As a result of all this the markets has become complex and large. Hence, the analysis of markets is no more feasible without the help of computers.

A potential trader (buyer or seller) places orders in the markets through a broker. These orders are matched by the stock exchange to execute trades. Most stock exchanges divide the trading activity into three main sessions known as pre-markets, continous markets and post markets. While in the pre-markets session an opening price of a product is discovered through double sided auction. In the continous markets session the incoming buyers and sellers are continously matched against each other on a priority basis. In the post-markets session clearing of the remaining orders is done and a closing price is discovered.

A double sided auction mechanism allows multiple buyers and sellers to trade simultaneously [1]. In double sided auctions, auctioneer (e.g. stock exchange) collects buy and sell requests over a period. Each potential trader places the orders with a *limit price*: below which a seller will not sell and above which a buyer will not buy. The exchange at the end of this period matches these orders based on their limit prices. This entire process is completed using a matching algorithm for double sided auctions.

Designing algorithms for double sided auctions is a well studied topic [2, 4, 3, 5]. A major emphasis of many of these algorithms is to maximize the number of matches or maximize the profit of the auctioneer. Note that an increase in the number of matches increases the liquidity in the markets. A matching algorithm can produce a matching with a uniform price or a matching with dynamic prices. While an algorithm which clears each matched bid-ask pair at a single price is referred as uniform price algorithm. An algorithm which may clear each matched bid-ask pair at different prices is refered as dynamic price algorithm. There are other important properties besides the number of macthes which are considered while evaluating the effectiveness of a matching algorithm. For example, fairness, uniform pricing, individual rationality are some of the relevant features used to compare these matching algorithms. However, it is known that no single algorithm can posses all of these properties [4, 2].

In this paper, we describe a formal framework to analyze double sided auctions using a theorem prover. For this work, we assume that each trader wishes to trade a single unit of the product and all the products are indistinguishable as well as indivisible. We have used the Coq proof assistant to formally define the theory of double sided auctions. Furthermore, we use this theory to validate various properties of matching algorithms. We formally prove some important properties of two algorithms; a uniform price algorithm and a dynamic price algorithm.

2 Modeling double sided auctions

To formalize the notion of matching in a double sided auction we use the list data structure. List is also used to define various processes that operate on a matching. However, to conviniently express the properties of these processes we need some relations on lists which are analogous to the relations on multisets. In this section we formally define these relations which are then used for stating important results on matching in a double sided auction.

2.1 Bid, Ask and limit price

An auction is a competetive event, where goods and services are sold to the highest bidders. In any double sided auction multiple buyers and sellers place their orders to buy or sell a unit of the underlying product. The auctioneer matches these buy-sell requests based on their *limit prices*. While the limit price for a buy order (i.e. *bid*), is the price above which the buyer doesn't want to buy one quantity of the item. The limit price of a sell order (i.e. *ask*), is the price below which the seller doesn't want to sell one quantity of the item. In this work we assume that each bid is a buy request for one unit of item. Similarly each ask is a sell request for one unit of item. If a trader wishes to buy or sell multiple units, he can create multiple bids or asks with different *ids*.

We can express bid as well ask using records containing two fields.

```
Record Bid: Type := Mk_bid { bp:> nat;    idb: nat }.
Record Ask: Type := Mk_ask { sp:> nat;    ida: nat }.
```

For a bid b, ($bp\ b$) is the limit price and ($idb\ b$) is its unique identity. Similarly for an ask a, ($sp\ a$) is the limit price and ($ida\ a$) is the unique identity of a. Note the use of coercion symbol `:>` in the first field of *Bid*. It declares *bp* as a function which is applied automatically to any term of type *Bid* that appears in a context where a term of type `nat` is expected.

Hence, we can use the simple expression b instead of (bp b) to express the limit price of b. Similarly we can use a for the limit price of an ask a.

Since both the fields of *Bid* as well as *Ask* are from domain *nat* in which the equality is decidable (i.e. `nat: eqType`), the equality on *Bid* as well as *Ask* can also be proved decidable. This is achieved by declaring two canonical instances `bid_eqType` and `ask_eqType` which connect *Bid* and *Ask* to the `eqType`.

2.2 Matching in Double Sided Auctions

In a double sided auction (DSA), the auctioneer collects all the buy and sell requests for a fixed duration. All the buy requests can be assumed to be present in a list B. Similarily, all the sell requests can be assumed to be present in a list A. At the time of auction, the auctioneer matches bids in B against asks in A. Furthermore, the auctioneer assigns a trade price to each matched bid-ask pair. This process results in a matching M, which consists of all the matched bid-ask pairs together with their trade prices. We represent matching as a list whose entries are of type `fill_type`.

```
Record fill_type: Type:=  Mk_fill {bid_of: Bid;  ask_of: Ask;  tp: nat}
```

We say a bid-ask pair (b, a) is *matchable* if $b \geq a$ (i.e. $bp\ b \geq sp\ a$). In any matching M, a bid or an ask appears at most once. Note that there might remain some bids in B which are not matched to any ask in a matching M. Similarly there might remain some asks in A which are not matched to any bid in M. The list of bids present in M is denoted as B_M and the list of asks present in M is denoted as A_M. For example, consider Fig. 1 which is a pictorial description of matching M between a list of bids B and a list of asks A. While the asks present in A is shown using left brackets and their limit prices. The bids present in B is shown using right brackets and their limit prices. All the matched bid-ask pair of M is then represented using matched brackets of same colors. For instance, the ask with limit price 53 is matched to the bid with limit price 69 in the matching M. Moreover, we can see that the bid with limit price 37 is not present in B_M since it is not matched to any ask in M.

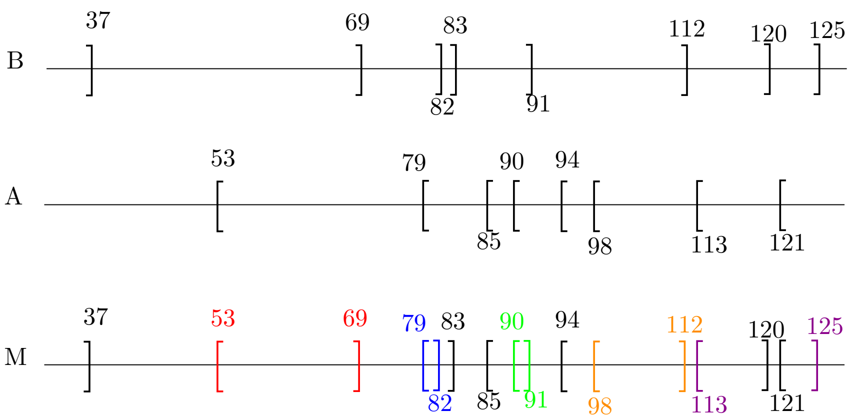


Figure 1 Bids in B and aks in A are represented using right and left brackets respectively. Every matched bid-ask pair in M is shown using the matched brackets of same colors. Note that the bids with limit prices 37, 83 and 120 are not matched to any ask in the matching M.

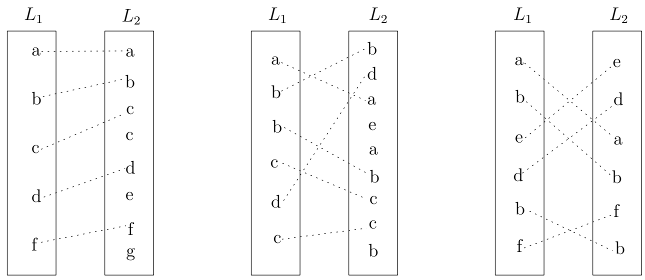
More precisely, for a given list of bids B and list of asks A, M is a matching iff, (1) All the bid-ask pairs in M are matchable, (2) B_M is duplicate-free, (3) A_M is duplicate-free, (4) $B_M \subseteq B$, and (5) $A_M \subseteq A$.

▶ **Definition 1.** *matching_in B A M := All_matchable M* $\wedge$ *NoDup* B_M $\wedge$ *NoDup* A_M $\wedge$ $B_M \subseteq B$ $\wedge$ $A_M \subseteq A$.

While term *NoDup* B_M in the above definition indicates that each bid is a request to trade one unit of item and the items are indivisible. We use the term $B_M \subseteq B$ to represent `Subset` relation between the lists B_M and B. It expresses the fact that each entry in the list B_M is also present in the list B.

Lists, sublist and permutation

While predicate `NoDup` and `Subset` are sufficient to define the notion of a matching. We need more definitions to describe the proerties of matching in a double sided auction. In the following paragraphs we describe three such binary relations on lists namely `sublist`, `included` and `perm` which are then used for stating results on matching in double sided auctions.



(a) sublist (b) included (c) perm

Figure 2 The dotted lines between entries of two lists confirm the presence of same entry in both the lists. (a) If L_1 is `sublist` of L_2 then no two dotted lines can intersect. (b) A list L_1 is `included` in L_2 if every entry in L_1 is also present in L_2. (c) Two lists L_1 and L_2 are permutation of each other if each entry has same number of ocuurences in both the lists L_1 and L_2.

sublist L_1 L_2 **:** The notion of `sublist` is analogous to the subsequence relation on sequences. For the given lists L_1 and L_2 the expression `sublist` L_1 L_2 is `true` if every entry of L_1 is also present in L_2 and they apear in the same succesion. In Fig. 2(a) the list L_1 is a `sublist` of L_2 since there is a line incident on each entry of L_1 and no two lines intersect each other.

Let `T` be an arbitrary `eqType`. Then for any two lists l and s whoes elements are of type `T` we have following lemmas specifying the `sublist` relation.

▶ **Lemma 2.** *sublist_intro1 (a:T): sublist l s-> sublist l (a::s).*

▶ **Lemma 3.** *sublist_elim3a (a e:T): sublist (a::l)(e::s)-> sublist l s.*

▶ **Lemma 4.** *sublist_elim4: sublist l s -> (*$\forall$ *a, count a l* $\leq$ *count a s).*

The term (*count a l*) in Lemma 4 represents the number of occurences of element a in the list l. Note the recursive nature of `sublist` as shown in Lemma 3. It usually makes inductive proofs easier for statements which contains `sublist` in the antecedent. Whereas, this is not true for the other relations (i.e. `included` and `perm`).

included L_1 L_2 **:** A list L_1 is `included` in the list L_2 if every entry of L_1 is also present in L_2. The notion of `included` is analogous to the subset relation in multisets. In Fig 2(b) the list L_1 is `included` in L_2 since there is a line incident on each entry of L_1. More precisely, we have following lemmas specifying the `included` relation.

▶ **Lemma 5.** *included_intro:* *(∀ a, count a l ≤ count a s)-> included l s.*

▶ **Lemma 6.** *included_elim:* *included l s -> (∀ a, count a l ≤ count a s).*

▶ **Lemma 7.** *included_intro3:* *sublist l s -> included l s.*

Note that if l is `sublsit` of s then l is also `included` in s but not the vice versa. However, if both the lists l and s are sorted based on some ordering on type `T` then l is `sublist` of s whenever l is `included` in s.

▶ **Lemma 8.** *sorted_included_sublist:* *Sorted l -> Sorted s -> included l s -> sublist l s.*

perm L_1 L_2 **:** A list L_1 is permutation of list L_2 iff L_1 is included in L_2 and L_2 is included in L_1. The notion of permutation for lists is analogus to the equality in multisets. In Fig 2(c) the list L_1 is perm of list L_2. We have following lemmas specifying the essential properties of the `perm` relation.

▶ **Lemma 9.** *perm_intro:* *(∀ a, count a l = count a s) -> perm l s.*

▶ **Lemma 10.** *perm_elim:* *perm l s -> (∀ a, count a l = count a s).*

▶ **Lemma 11.** *perm_sort:* *perm l s -> perm l (sort s).*

The term (*sort s*) in Lemma 11 represents the list s sorted using some ordering relation. Note that any permutation of a matching is also a matching. More precisely, we have the following invariance lemma on matching.

▶ **Lemma 12.** *match_inv:* *perm M M' -> perm B B' -> perm A A' -> matching_in B A M -> matching_in B' A' M'.*

▶ **Lemma 13.**

——> Motivate and explain projection functions and corresponding lemmas.

3 Formal Analysis of Double sided auctions

Usually in a double sided auctions mechanism, the profit of an auctioneer is the difference between the limit prices of matched bid-ask pair. In this work we do not consider analysis of profit for the aiuctioneer. Therefore the buyer of matched bid-ask pair pays the same amout which seller recieves. This price for a matched bid-ask pair is called the trade price for that pair. Since the limit price for a buyer is the price above which she doesn't want to buy, the trade price for this buyer is expected to be below its limit price. Similarly the limit price for a seller is the price below which he doesn't want to sell, hence the trade price for this seller is expected to be be below its limit price. Therefore it is desired that in any matching the trade price of a bid-ask pair lies between their limit prices. A matching which has this property is called an *indivial rational (IR)* matching. Note that any matching can be converted to an IR matching without altering it's bid-ask pair (See Fig 3).

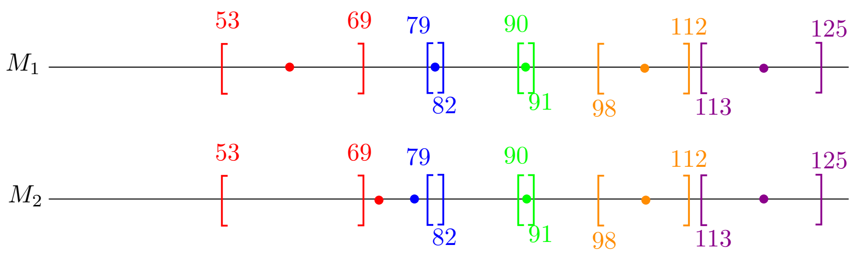


Figure 3 The colored dots represent the trade prices at which the corresponding matched bid-ask pairs are traded. While the matching M_2 is not IR since some dots lie outside the corresponding matched bid ask-pair. The matching M_1 is IR because trade prices for every matched bid-ask pair lie inside the interval. Note that the matching M_1 and M_2 contains exactly same bid-ask pairs.

The number of matched bid-ask pairs produced by any matching algorithm is crucial in the design of a double sided auction mechanism. Increasing the number of matched bid-ask pairs increases the liquidity in market. Therefor, producing a maximum matching is an important aspect of double sided auction mechanism design. For a given list of bids B and list of asks A we say a matching M is a maximum matching if no other matching M' on the same B and A contains more number of matched bid-ask pairs than M. We use predecate `Is_MM` to denote a maximum matching.

▶ **Definition 14.** *Is_MM M B A := (matching_in B A M) ∧ (∀ M', matching_in B A M' → |M'| ≤ |M|).*

In certain situations, to produce a maximum matching, different bid-ask pair must be assigned different trade prices. However, different prices for the same product in the same market simultaneausly leads to dissatisfaction amongst some of the traders. A machanism which clears all the matched bid-ask pairs at same trade price is called a *uniform matching*. It is also known as percived-fairness (cite). In many situation it is not possible to produce an IR matching which is maximum and uniform at the same time. For example in Fig 4 a maximum matching of size two is possible but any uniform matching of size more than one in not possible.

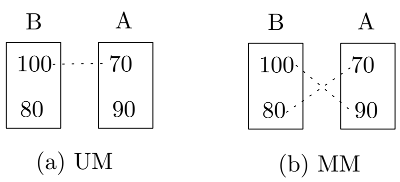


Figure 4 In this figure two bids with limit prices 100 and 80 respectively are matched against two asks of limit price 70 and 90. There is only one matching M_2 of size two possible and it is not uniform.

3.1 Fairness

A bid with higher limit price is more competitive campared to bids with lower limit prices. Similaly an ask with lower limit price is more competitive campared to asks with higher limit prices. In a campetitive market, like double sided auction, it is neccesory to priortise more competitive traders for matching. A matching which priortise competitive traders is a fair matching. Consider the following predicates `fair_on_bids` and `fair_on_asks` which can be used to describe a fair matching.

▶ **Definition 15.** *fair_on_bids M B:= ∀ b b', In b B ∧ In b' B -> b > b' -> In b' B_M -> In b B_M.*

▶ **Definition 16.** *fair_on_asks M A:= ∀ s s', In s A ∧ In s' A -> s < s' -> In s' A_M -> In s A_M.*

▶ **Definition 17.** *Is_fair M B A:= fair_on_asks M A ∧ fair_on_bids M B.*

Here, the predicate `fair_on_bids M B` denotes that the matching M is fair for the list of buyers B. Similarly, the predecate `fair_on_asks M A` assures that the matching M is fair for the list of sellers A. A matching which is fair on both the traders (i.e. B and A) is expessed using the predecate `Is_fair M B A`.

Unlike the uniform matching a fair matching can always be achieved without compromising the the size the matching. We can accopmlish this by converting any matching into a fair matching without changing its size. For example consider the following function `make_FOB`.

```
Fixpoint Make_FOB (M:list fill_type) (B: list Bid):=
match (M,B) with
|(nil,_) => nil
|(m::M',nil) => nil
|(m::M',b::B') => (Mk_fill b (ask_of m) (tp m))::(Make_FOB M' B')
end.
```

The function `make_FOB` produces `fair_on_bids` matching from a given matching M and a list of bids B, both sorted in decresing order bid prices (See Fig 5). The function `make_FOB` is a recursive function and it replaces the largest bid in M with the largest bid in B. Since at any moment the largest bid in B is bigger than the largest bid in M, the new bid-ask pair is still matchable. Note that `make_FOB` doesn't change any of the ask in M and due to recursive nature of `make_FOB` on B, a bid is not repeated in the process of replacement. This ensure that the new B_M is duplicate-free. Once a matching is modified to a fair matching on bids, we use similar function `make_FOA` on this matching to produce a fair on ask matching. Hence the final result is a fair matching.

For the function `make_FOB` we have following lemma prooving it fair on bids.

```
Lemma mfob_fair_on_bid (M: list fill_type) (B:list Bid) (A:list Ask):
  (Sorted m_dbp M) -> (Sorted by_dbp B) -> sublist (bid_prices (bids_of M)) (bid_prices B) ->
  fair_on_bids (Make_FOB M B) B.
```

The proof of above fact is using induction on the size of matching. Note we have not kept matching in the antecedent. For example we get stuck in induction if we try to prove the following claim directly using induction.

–> Insert Claim. Try to signify the role of sublist and how it helps.

—> Explain the final result on fairness

```
Theorem exists_fair_matching (M: list fill_type) (B: list Bid) (A:list Ask) (NDB: NoDup B) (NDA:
  matching_in B A M-> (exists M':list fill_type, matching_in B A M' /\ Is_fair M' B A /\ |M|= |M'
```

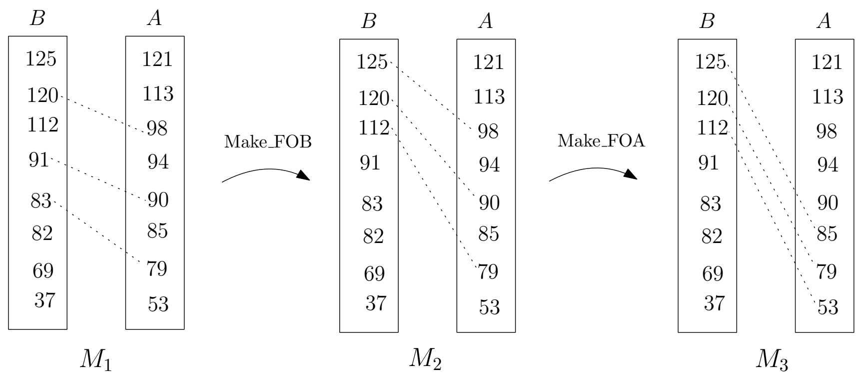


Figure 5 The dotted lines in this figure respresent matched bid-ask pairs in matching M_1, M_2 and M_3. In the first step function `make_FOB` operates on M_1 recursively. At each step it picks the top bid-ask pair, say (b, a) in M_1 and replaces the bid b with most competitive bid available in B. The result of this process is a `fair_on_bids` matching M_2. In a similar way the function `make_FOA` changes M_2 intro a fair on ask matching M_3.

3.2 Maximum Matching

The liquidity in any market is a meausre of how quickly one can trade in the market without much cost. A highly liquid market boosts the investor's confidence in the market. One way to increase the liquidity in a double sided auction is to maximize the number of matched bid-ask pair. In the previous section we have seen that any matching can be changed to a fair matching without altering its size. Therefore, we can have a maximum matching without compromising on the fairness of the matching. In this section we describe a matching which fair as well as maximum. For a given bid B and ask A, a maximum and fair matching can achieved in two steps. In first we have function `produce_MM` which produce a matching which is maximum and fair on bids. In the next step we apply `make_FOA` to this maximum matching to produce a fair on ask matching (See Fig 6).

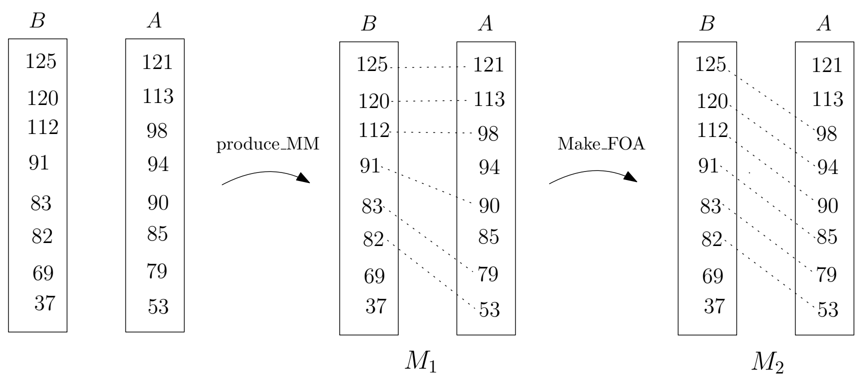


Figure 6 In the first step, the function `produce_MM` operates reccursively on the list of bids B and list of asks A. At each step the function `produce_MM` selects the most competitive available bid and then pairs it with the largest mathchable ask. Note that the output of this function is fair on bid since it doesn't leave any bid from top. In the second step, the function `make_FOA` converts the M_1 into fair matching M_2.

```
Fixpoint produce_MM (B:list Bid) (A: list Ask): (list fill_type) :=
  match (B, A) with
```

```
  |(nil, _) => nil
  |(b::B', nil) => nil
  |(b::B', a::A') => match (Nat.leb (sp a) (bp b)) with
                     |true => ({|bid_of:= b ; ask_of:= a ; tp:=(bp b) |})::(produce_MM B' A')
                     |false => produce_MM B A'
                     end
  end.
```

–> edit the above definition.

The function `produce_MM` produces a maximum matching from a given lists of bids B and a list of asks A, both sorted in decresing order by limit prices (See Fig **??**). At each iteration it generates a matchable bid-ask pair. Due to the recursive nature of function `produce_MM` on both B and A, it never pair any bid with more than two asks. This ensures that the list of bids in matching B_M is duplicate-free. Note that the function `produce_MM` tries to match a bid until it finds a matchable ask before pairing the next bid. The function terminates when either all the bids are matched or it encounters a bid for which no matchable ask is available. Therefore, it produces a matching which is fair on bid.

```
 Lemma produce_MM_fob (B: list Bid)(A: list Ask):
   Sorted by_dbp B -> Sorted by_dsp A -> fair_on_bids (produce_MM B A) B.
```

Talk about the maximality proof.

```
Lemma produce_MM_is_MM (B: list Bid)(A: list Ask)(no_dup_B: NoDup B)(no_dup_A: NoDup A):
   Sorted by_dbp B -> Sorted by_dsp A-> Is_MM (produce_MM B A) B A.
```

—> Insert the proof diagram —> Proof Idea.

—> Final lemmas stating that there exists a maximal and fair matching.

```
Theorem exists_fair_maximum (B: list Bid)(A: list Ask): exists M, (Is_fair M B A /\ Is_MM M B A).
```

4 Matching in financial markets

▶ **Lemma 18** (Lorem ipsum). *Vestibulum sodales dolor et dui cursus iaculis. Nullam ullamcorper purus vel turpis lobortis eu tempus lorem semper. Proin facilisis gravida rutrum. Etiam sed sollicitudin lorem. Proin pellentesque risus at elit hendrerit pharetra. Integer at turpis varius libero rhoncus fermentum vitae vitae metus.*

Proof. Cras purus lorem, pulvinar et fermentum sagittis, suscipit quis magna.

▷ Claim 19. content...

Proof. content... ◁

◀

▶ **Corollary 20** (Curabitur pulvinar,). *Nam liber tempor cum soluta nobis eleifend option congue nihil imperdiet doming id quod mazim placerat facer possim assum. Lorem ipsum dolor sit amet, consectetuer adipiscing elit, sed diam nonummy nibh euismod tincidunt ut laoreet dolore magna aliquam erat volutpat.*

▶ **Proposition 21.** *This is a proposition*

Proposition 21 and Proposition 21 . . .

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▶ Remark 22. content...

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Nec urna malesuada sollicitudin. Nulla facilisi. Vivamus aliquam tempus ligula eget ornare. Praesent eget magna ut turpis mattis cursus. Aliquam vel condimentum orci. Nunc congue, libero in gravida convallis , orci nibh sodales quam, id egestas felis mi nec nisi. Suspendisse tincidunt, est ac vestibulum posuere, justo odio bibendum urna, rutrum bibendum dolor sem nec tellus.

▶ **Lemma 23** (Quisque blandit tempus nunc). *Sed interdum nisl pretium non. Mauris sodales consequat risus vel consectetur. Aliquam erat volutpat. Nunc sed sapien ligula. Proin faucibus sapien luctus nisl feugiat convallis faucibus elit cursus. Nunc vestibulum nunc ac massa pretium pharetra. Nulla facilisis turpis id augue venenatis blandit. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus.*

Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.

References

1. Daniel Friedman. The double auction market institution: A survey. 01 1993.
2. R Preston McAfee. A dominant strategy double auction. *Journal of economic Theory*, 56(2):434–450, 1992.
3. Jinzhong Niu and Simon Parsons. Maximizing matching in double-sided auctions. In Maria L. Gini, Onn Shehory, Takayuki Ito, and Catholijn M. Jonker, editors, *International conference on Autonomous Agents and Multi-Agent Systems, AAMAS '13, Saint Paul, MN, USA, May 6-10, 2013*, pages 1283–1284. IFAAMAS, 2013. URL: `http://dl.acm.org/citation.cfm?id=2484920`.
4. Peter R. Wurman, William E. Walsh, and Michael P. Wellman. Flexible double auctions for electronic commerce: theory and implementation. *Decision Support Systems*, 24(1):17–27, 1998.
5. Dengji Zhao, Dongmo Zhang, Md Khan, and Laurent Perrussel. Maximal matching for double auction. In Jiuyong Li, editor, *Australasian Conference on Artificial Intelligence*, volume 6464 of *Lecture Notes in Computer Science*, pages 516–525. Springer, 2010.

A Styles of lists, enumerations, and descriptions

List of different predefined enumeration styles:

- `\begin{itemize}...\end{itemize}`
- ...
- ...

1. `\begin{enumerate}...\end{enumerate}`
2. ...
3. ...

(a) `\begin{alphaenumerate}...\end{alphaenumerate}`
(b) ...
(c) ...

(i) `\begin{romanenumerate}...\end{romanenumerate}`

(ii) ...
(iii) ...

(1) `\begin{bracketenumerate}...\end{bracketenumerate}`
(2) ...
(3) ...

Description 1 `\begin{description} \item[Description 1] ...\end{description}`
Description 2 Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.
Description 3 ...

B Theorem-like environments

List of different predefined enumeration styles:

▶ **Theorem 24.** *Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.*

▶ **Lemma 25.** *Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.*

▶ **Corollary 26.** *Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.*

▶ **Proposition 27.** *Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.*

▶ **Exercise 28.** *Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.*

▶ **Definition 29.** *Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.*

▶ **Example 30.** Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.

▶ Note 31. Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.

▶ Note. Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.

▶ Remark 32. Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.

▶ Remark. Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.

▷ Claim 33. Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.

▷ Claim. Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque.

Proof. Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque. ◀

Proof. Fusce eu leo nisi. Cras eget orci neque, eleifend dapibus felis. Duis et leo dui. Nam vulputate, velit et laoreet porttitor, quam arcu facilisis dui, sed malesuada risus massa sit amet neque. ◁