66) Let $(a_n), (b_n) \in V$ and (a). Then

a)

$$\varphi(a) + \varphi(b) = (0, a_1 + b_1, a_2 + b_2, ...) = \varphi(a + b) \tag{1}$$
$$\alpha \cdot \varphi(a) = (0, \alpha \cdot a_1, \alpha \cdot a_2, ...) = \varphi(\alpha \cdot a) \tag{2}$$
$$\psi(a) + \psi(b) = (a_2 + b_1, ...) = \psi(a + b) \tag{3}$$
$$\alpha \cdot \psi(a) = (\alpha \cdot a_2, \alpha \cdot a_3 ...) = \psi(\alpha \cdot a) \tag{4}$$
$$\varphi(0) = 0, \quad \psi(0) = 0 \tag{5}$$

b)

$$\psi(\varphi(a)) = a \tag{6}$$

So ψ is left-inverse to φ but $(1, 0, 0, ...)$ for instance is not in the image of φ.

c) $\psi(\varphi(a)) = (a_1, a_2, ...)$ but $\phi((x, a_1, a_2, ..) = a$ for $x \in \mathbb{R}$, so preimages are not unique.

67) a) $(\gamma \circ \varphi)(a) = 0$ if $\gamma((a_1, a_2, ...) = (a_1, 0, 0, ...)$ but

$$(\varphi \circ \gamma)(a) = 0 \iff \gamma = 0 \in R$$

b) $(\psi \circ \gamma)(a) = 0$ for any γ whose image is contained in $\{(x, 0, 0,), x \in \mathbb{R}\}$, for instance $\gamma((a_n)) = (a_1, 0, 0, ..)$, but

$$(\gamma \circ \psi)(a) = 0 \iff \gamma(a) = 0, \quad \forall a \in V$$

68) a) Let $R = \{a/p^n | a, n \in \mathbb{Z}\}$. Then $a/p^n + b/p^m = \frac{(ap^m + bp^n)}{p^{n+m}} \in R$ and $a/p^n \cdot b/p^m = \frac{ab}{p^{n+m}} \in R$
It is an not an ideal. Take $\frac{1}{2} \in R = \{\frac{a}{2^n}\}$. Then $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{2 \cdot 3} \notin R$

b) If $p \nmid a$, $p \nmid b$, then $p \nmid ab$
Let $R = \{a/b \,|\, a, b \in \mathbb{Z},\, p \nmid b\}, \quad \alpha, \beta \in R$
Then $\alpha + \beta = \frac{a}{b} + \frac{d}{c} = \frac{ac+db}{bc}$ and $\alpha \cdot \beta = \frac{ad}{bc}$, which are both elements of R. It is not an ideal.

69) a) Let $a = (a_1, ..., a_n), b = (b_1, ..., b_n) \in S = S_1 \times \cdots \times S_n$.
Then $a + b = (a_1, ..., a_n) + (b_1, ..., b_n) = (a_1 + b_1, ..., a_n + b_n)$ in S and $ab = (a_1 b_1, ..., a_n b_n)$ in S

b) We only need to prove $ab \in I = I_1 \times \cdots \times I_n$, where $a \in R, b \in S$
$ab = (a_1 b_1, ..., a_n b_n)$ and we know that $a_i b_i \in I_i$. Thus I is an ideal

70) a) Let $a, b \in Z(R)$. Then $x(a + b) = xa + xb = ax + bx = (a + b)x$ and $abx = axb = xab$

b) Let $a = (a_1, ..., a_n) \in Z(R_1 \times \cdots \times R_n)$. By definition, each a_i commutes with every element of R_i, so

$$Z(R_1 \times \cdots \times R_n) \in Z(R_1) \times \cdots \times Z(R_n)$$

For the other direction let $a \in Z(R_1) \times \cdots \times Z(R_n)$. Since each a_i commutes with elements from R_i, a commutes with every element from $R_1 \times \cdots \times R_n$