

1 Vectors

D 1.4 (Linear Combination):

- let $v, w \in \mathbb{R}^m, \lambda, \mu \in \mathbb{R}$

$\Rightarrow \sum_{i=1}^n \lambda_i v_i$ are scaled combinations of n vectors v_i .

D 1.7 (Combination types):

- **Affine Combination:** $\sum_{i=1}^n \lambda_i = 1$

Conic Combination: if $\lambda_j \geq 0$ for $j = 1, 2, \dots, n$

Convex Combination: Affine + Conic

D 1.9 (Scalar/dot product):

- $\mathbf{v} \cdot \mathbf{w} := \sum_{i=1}^m v_i w_i$, alternative notation: $[z_i]_{i=1}^m := [v_i + w_i]_{i=1}^m$

D 1.11 (Euclidean norm, squared norm, unit vector):

- $\|\mathbf{v}\| := \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{\mathbf{v}^\top \mathbf{v}}$, **Squared norm:** $\|\mathbf{v}\|^2 := \mathbf{v}^\top \mathbf{v}$,

Unit vector: $\|\mathbf{u}\| = 1 = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\|\mathbf{v}\|} \mathbf{v}$ (for any vector $\mathbf{v} \neq 0$)

L 1.12 (Cauchy-Schwarz inequality):

- $|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\|$ for any two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^m$

D 1.14 (Angle between vectors):

- $\cos(\alpha) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} \in [-1, 1]$

D 1.16 (Hyperplane through origin):

- Let $\mathbf{d} \in \mathbb{R}^m, \mathbf{d} \neq 0, H_{\mathbf{d}} = \{\mathbf{v} \in \mathbb{R}^m : \mathbf{v} \cdot \mathbf{d} = 0\}$

L 1.16 (Triangle inequality):

- $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$

D 1.21 (Linear (in)dependence):

- vectors are linearly dependent if one of them is linear combination of the others: $\mathbf{v}_k = \sum_{j \neq k}^n \lambda_j \mathbf{v}_j$

\Leftrightarrow There are scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ besides $0, 0, \dots, 0$ such that $\sum_{j=1}^n \lambda_j \mathbf{v}_j = \mathbf{0}$. We also say that $\mathbf{0}$ is a nontrivial linear combination of the vectors.

\Leftrightarrow At least one of the vectors is a linear combination of the previous ones.

D 1.25 (Span):

- Span of vectors is a set of all linear combinations of those vectors: $\text{Span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) := \left\{ \sum_{j=1}^n \lambda_j \mathbf{v}_j : \lambda_j \in \mathbb{R} \text{ for all } j \in [n] \right\}$

Construction of vectors with standard unit vectors:

- Every target vector can be written as: $\mathbf{u} = \sum_{i=1}^m u_i \mathbf{e}_i$, where \mathbf{e} is a standard unit vector.

2 Matrices

D 2.1 (Matrix):

- $A = [a_{ij}]_{i=1,j=1}^{m,n}$ - m rows, n columns (Zeilen zuerst, Spalten später)

D 2.2 (Matrix addition, scalar multiplication):

- Addition: $A + B = [a_{ij} + b_{ij}]_{i=1,j=1}^{m,n}$

- Scalar multiplication: $\lambda A = [\lambda a_{ij}]_{i=1,j=1}^{m,n}$

Matrix types:

- Identity matrix ($a_{ii} = 1$ for all i): I
- Diagonal matrix ($a_{ij} = 0$ for all $i \neq j$): $\text{diag}(d_1, \dots, d_n)$
- Upper triangular matrix ($a_{ij} = 0$ for all $i > j$): U
- Lower triangular matrix ($a_{ij} = 0$ for all $i < j$): L
- Symmetric matrix ($a_{ij} = a_{ji}$ for all i, j): $A = A^\top$
- Skew-symmetric matrix ($a_{ij} = -a_{ji}$ for all i, j): $A = -A^\top$

D 2.4 (Matrix-vector product):

- Rows of matrix ($m \times n$) with vector (n elements), i.e.

$u_1 = \sum_{i=1}^m a_{1,i} v_i, Ix = x$; **Trace:** Sum of the diagonal entries.

D 2.9 (Column space):

- The column space $C(A)$ of A is the span (set of all linear combinations) of the columns: $C(A) := \{Ax : x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$

D 2.10 (Rank):

- $\text{rank}(A) :=$ the number of linearly independent column vectors of A .

D 2.11 (Transpose):

- Mirror the matrix along its diagonal. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \leftrightarrow A^\top = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

- $(A^\top)^\top = A$

D 2.13 (Row space):

- $R(A) := C(A^\top)$

D 2.17 (Nullspace):

- Nullspace contains all input vectors that lead to output vector $\mathbf{0}$.

$N(A) = \{\mathbf{x} \in \mathbb{R}^n : Ax = \mathbf{0}\} \subseteq \mathbb{R}^n$

D 2.27 (Kernel & Image):

- **Kernel:** $N(A) = \text{Ker}(T) := \{\mathbf{x} \in \mathbb{R}^n : T(\mathbf{x}) = \mathbf{0}\} \subseteq \mathbb{R}^n$ (If A is the unique $m \times n$ matrix such that $T = TA$)

- **Image:** $C(A) = \text{Im}(T) := \{T(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$ (If A is the unique $m \times n$ matrix such that $T = TA$), the set of all outputs that T can produce.

2.2.2 Working with linear transformations:

A matrix can be understood as a re-mapping of the unit vectors, scaling and re-orienting them. Each column vector can then be understood as the new unit vector e_i , hence essentially adding another coordinate system to the original one, which is moved and rotated a certain way. The rotation matrix under 2 is such an example. To prove that T is a linear transformation, use $T(x+y) = T(x) + T(y)$ and $T(\lambda x) = \lambda T(x)$. Then insert the linear transformation given by the task and replace x (or whatever variable there is) with $x+y$ or λx . $Ax = \sum_{i=1}^n x_i v_i$, where v_i is the i -th column of A .

O 2.39 (Matrix multiplication):

- $A \times B = C, c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$. Dimension restrictions: A is $m \times n$, B is $n \times p$, result is $m \times p$. For each entry, multiply the i -th row of A with the j -th column of B .

Not commutative, but associative & distributive.

L 2.40 Matrix multiplication with transposition:

- $(AB)^\top = B^\top A^\top$

D 2.44 Outer product:

- $\text{rank}(A) = 1 \iff \exists$ non-zero vectors $v \in \mathbb{R}^m, w \in \mathbb{R}^n$ such that A is an outer product, i.e. $A = vw^\top$, thus $\text{rank}(vw^\top) = 1$.

T 2.46 (CR decomposition):

- $A = CR$. Get R from (reduced) row echelon form. C is the columns from A where there is a pivot in R . $C \in \mathbb{R}^{m \times r}, R \in \mathbb{R}^{r \times n}$ (in RREF), $r = \text{rank}(A)$.

Row Echelon Form: To find REF, try to create pivots: $R_0 = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Use Gauss-Jordan elimination to find it (row formations). **Reduced REF:** RREF is simply REF without any zero rows (i.e. in R_0, R (in RREF) would be R_0 without the last row).

O 2.5.6 (Invertible matrix):

- Matrix A is invertible if it is square and there is B such that:

$$AB = I \Leftrightarrow BA = I \Leftrightarrow AB = BA = I$$

D 2.57 (Inverse matrix and its properties):

- If $AB = I$ for invertible A , then B is its inverse and denoted as A^{-1} . • $(A^{-1})^{-1} = A$ • $(AB)^{-1} = B^{-1}A^{-1}$ • $(A^\top)^{-1} = (A^{-1})^\top$

4 Four Fundamental Subspaces

4.1 Vector Spaces

D 4.1 (Vector Space):

• Vector space is a triple $(V, +, \cdot)$ where V is a set (the vectors) with two operations \oplus and \odot . They are based on algebras called fields and satisfy axioms: *commutativity*, *associativity*, *zero vector*, *negative vector*, *identity element*, *compatibility of multiplications of vectors and scalars* ($\in \mathbb{R}$), *distributivity over \oplus* both for vectors and scalars ($\in \mathbb{R}$).

D 4.8 (Subspace):

- Let V be a vector space. A nonempty subset $U \subseteq V$ is a subspace of V if

following axioms are true $\forall \mathbf{v}, \mathbf{w} \in U$ and $\forall \lambda \mathbf{v} \in U$:

- $\mathbf{v} + \mathbf{w} \in U$
- $\lambda \mathbf{v} \in U$

They guarantee that vector addition and scalar multiplication "doesn't take us outside of a subspace". For showing U is nonempty it is enough to show that $\mathbf{0} \in U$.

L 4.9 (Subspace always has 0):

- Let $U \subseteq V$ be a subspace of a vector space V . Then $\mathbf{0} \in U$ (at least).

L 4.11 (Column space is a subspace):

- Let $A \in \mathbb{R}^{m \times n}$, then $C(A) = \{Ax : x \in \mathbb{R}^n\}$ is subspace of \mathbb{R}^m . $\Rightarrow R(A) = C(A^\top)$ is a subspace of \mathbb{R}^n .

E 4.13 (The nullspace is a subspace):

- Let $A \in \mathbb{R}^{m \times n}$. Then the nullspace $N(A) = \{\mathbf{x} \in \mathbb{R}^n : Ax = \mathbf{0}\}$ is a subspace of \mathbb{R}^n

L 4.14 (Subspaces are vector spaces):

- V is a vector space and U is its subspace. Then U is also a vector space with the same \oplus and \odot as V .

4.2 Bases and dimension

D 4.18 (Basis):

- Let V be a vector space. A subset $B \subseteq V$ is called a basis of V if B is linearly independent and it spans V : $\text{Span}(B) = V$.

L 4.19 (Independent columns is a basis):

- Independent columns form basis of column space $C(A)$.

O 4.20 (Non-uniqueness of basis):

- Every set $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \subseteq \mathbb{R}^m$ of m linearly independent vectors is a basis of \mathbb{R}^m .

D 4.21 (Finitely generated vector space):

- There is a finite subset $G \subseteq V$ with $\text{Span}(G) = V$. Then V has a basis $B \subseteq G$.

T 4.22 (Finitely generated VS has a basis):

- If V is finitely generated, then V has a basis $B \subseteq V$.

L 4.23 (Steinitz exchange lemma):

- "exchanging elements between G and $F"$

V is finitely generated vector space, $F \subseteq V$ a finite set of lin. independent vectors, and $G \subseteq V$ a finite set of vectors with $\text{Span}(G) = V$, then:

- $|F| \leq |G|$ and • $\exists E \subseteq G$ of size $|G| - |F|$ such that $\text{Span}(F \cup E) = V$.

T 4.24 (All bases have the same size):

- All bases have the same size: $B, B' \subseteq V \Rightarrow |B| = |B'|$.

D 4.25 (Dimension):

- $\dim(V)$ - the dimensions of V . It has a size of arbitrary basis B of V .

D 4.26 (Linear transformation between vector spaces):

- Let V, W be vector spaces. A function $T : V \rightarrow W$ is linear if, for all $x_1, x_2 \in V$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, $T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$.

L 4.27 (Bijective lin. transformations preserve basis):

- If $T : V \rightarrow W$ is a bijective linear map, then $B \subseteq V$ is a basis of $V \Leftrightarrow T(B)$ is a basis of W , and hence $\dim(V) = \dim(W)$.

D 4.28 (Isomorphic vector spaces):

- $V \cong W \iff \exists T : V \rightarrow W$ linear and bijective.

T 4.29 (Basis writes vectors as a unique lin. combination):

- Let V be a finite-dimensional vector space with basis $B = \{v_1, \dots, v_m\}$. Then every $v \in V$ can be written uniquely as $v = \sum_{j=1}^m \lambda_j v_j$, for unique scalars $\lambda_1, \dots, \lambda_m$.

L 4.30 (Less than dim(V) vectors do not span V):

- If $|G| < \dim V$, then $\text{span}(G) \neq V$.

4.3 Computing the three fundamental subspaces

T 4.31 (Basis of $C(A)$: Pivots columns of RREF):

- R is RREF of A , then all columns at pivots of R form a basis of $C(A)$:

P 6.4.9 (TS decomposition):

- For $A \in \mathbb{R}^{m \times n}$ with $\text{rank}(A) = r$, let $S \in \mathbb{R}^{m \times r}$, $T \in \mathbb{R}^{r \times n}$ such that $A = ST$. Then $A^\dagger = T^\dagger S^\dagger$.

T 6.4.10 (Pseudoinverses properties):

- Let $A \in \mathbb{R}^{m \times n}$. Then $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, $(A^\dagger)^T = (A^T)^\dagger$. AA^\dagger is symmetric \Rightarrow projection matrix onto $\mathcal{C}(A)$, $A^\dagger A$ is symmetric \Rightarrow projection matrix onto $\mathcal{C}(A^T)$. Moreover, $AA^\dagger = CRR^T(RR^T)^{-1}(C^T)^{-1}C^T = C(C^TC)^{-1}C^T$, which is the projection onto $\mathcal{C}(A)$, and $(AA^\dagger)^T = AA^\dagger$.

7 The Determinant

7.1 2 times 2

D 7.1.1 (2 × 2 Determinant):

- For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A) = ad - bc$.

L 7.1.2 (Multiplication of determinants):

- $\det(AB) = \det(A)\det(B)$.

Hence, for an LU -decomposition, $\det(A) = \det(L)\det(U)$.

D 7.2.1 (Permutation sign):

- The sign of a permutation is defined as the number of swaps of rows or columns. $\det(\text{permuted matrix}) = (-1)^k \det(\text{original matrix})$, where k is the number of swaps. Even number of swaps $\Rightarrow +1$, odd number $\Rightarrow -1$. $\text{sgn}(\sigma \circ \gamma) = \text{sgn}(\sigma) \text{sgn}(\gamma)$. For all $n \geq 2$, half of the permutations have sign $+1$, half have sign -1 .

7.2 General case:

D 7.2.3 (Determinant big formula):

- For a square matrix $A \in \mathbb{R}^{n \times n}$, $\det(A) = \sum_{\sigma \in \Pi_n} \text{sgn}(\sigma) \prod_{i=1}^n A_{i,\sigma(i)}$. (Number of permutations: $n!$)

Determinant Properties:

- Matrix $T \in \mathbb{R}^{n \times n}$ is triangular, then $\det(T) = \prod_{k=1}^n T_{kk}$, in particular $\det(I) = 1$.
- Matrix $A \in \mathbb{R}^{n \times n}$, $\det(A) = \det(A^T)$.
- Matrix $Q \in \mathbb{R}^{n \times n}$ is orthogonal $\iff \det(Q) = 1$ or $\det(Q) = -1$.
- Matrix $A \in \mathbb{R}^{n \times n}$ is invertible $\iff \det(A) \neq 0$.
- Matrices $A, B \in \mathbb{R}^{n \times n}$, $\det(AB) = \det(A)\det(B)$, in particular $\det(A^n) = \det(A)^n$.
- Matrix $A \in \mathbb{R}^{n \times n}$, $\det(A^{-1}) = \frac{1}{\det(A)}$.
- $\det(\lambda A) = \lambda^n \det(A)$.

P 7.2.4 (Determinant of orthogonal matrices):

- $1 = \det(I) = \det(Q^T Q) = \det(Q^T) \det(Q) = \det(Q)^2$, so $\det(Q) = \pm 1$. If $\det(Q) = 1$, then Q is a rotation matrix. If $\det(Q) = -1$, then Q is a reflection matrix.

P 7.3.2 (Cofactor determinant calculation):

Co-factor method:

$\det(A) = \sum_{j=1}^n A_{ij} C_{ij}$, where cofactors are $C_{ij} = (-1)^{i+j} \det(A_{ij})$.

P 7.3.5 (Cramer's Rule):

- Cramer's Rule:** For $Ax = b$ with $\det(A) \neq 0$, $x_j = \frac{\det(\mathcal{B}_j)}{\det(A)}$, where \mathcal{B}_j is the matrix obtained from A by replacing the j -th column with b .

P 7.3.7 (Linearity of a determinant):

- The determinant is linear in each row (and column). For example, $\det \begin{bmatrix} \alpha_0 a_0^T + \alpha_1 a_1^T \\ a_2^T \end{bmatrix} = \alpha_0 \det \begin{bmatrix} a_0^T \\ a_2^T \end{bmatrix} + \alpha_1 \det \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}$.

8 Eigenvalues and Eigenvectors

8.1 Complex Numbers

- Solve $x^2 + 1 = 0 \Rightarrow x = \sqrt{-1} \Rightarrow \mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}$.
- $(a + ib) + (x + iy) = (a + x) + i(b + y)$,
- $(a + ib)(x + iy) = (ax - by) + i(ay + bx)$,
- $(a + ib)(a - ib) = a^2 + b^2$.
- $\frac{a+ib}{x+iy} = \frac{(a+ib)(x-iy)}{x^2+y^2} = \frac{ax+by}{x^2+y^2} + i \frac{bx-ay}{x^2+y^2}$.
- $|z| = \sqrt{a^2 + b^2}$, $z = a + ib$,
- $a + ib = a - ib$.

R 8.1.1 (Euler's formula):

- For $\theta \in \mathbb{R}$, $e^{i\theta} = \cos \theta + i \sin \theta \Rightarrow e^{i\pi} = -1$

Polar form of a complex number:

- $z = re^{i\theta}$, $z \in \mathbb{C}$, $r > 0$ is the modulus of z , $\theta \in [0, 2\pi)$.

T 8.1.2 (Fundamental Theorem of Algebra):

- Any degree n non-constant ($n \geq 1$) polynomial $P(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + \dots + \alpha_1 z + \alpha_0$, ($\alpha_n \neq 0$) has a zero: there exists $\lambda \in \mathbb{C}$ such that $P(\lambda) = 0$. \Rightarrow A degree- n polynomial has at most n distinct zeros (roots).

C 8.1.3 (Algebraic multiplicity, num. of 0 in polynomial):

- Any degree n non-constant ($n \geq 1$) polynomial has n zeros $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, and $P(z) = \alpha_n(z - \lambda_1)(z - \lambda_2) \dots (z - \lambda_n)$. The number of times $\lambda \in \mathbb{C}$ appears in the expression is called the *algebraic multiplicity* of the zero.

Inner product on \mathbb{C}^n :

- The inner product on \mathbb{C}^n is given by $\langle v, w \rangle = w^* v$.

Conjugate transpose:

- $A^* = \overline{A^T}$.

8.2 Introduction to Eigenvalues and Eigenvectors

D 8.2.1 (EW/EV pair):

- Given $A \in \mathbb{R}^{n \times n}$, we say $\lambda \in \mathbb{C}$ is an *eigenvalue* of A and $v \in \mathbb{C}^n \setminus \{0\}$ is an *eigenvector* of A associated with λ when $Av = \lambda v$. (λ, v) is an eigenvalue–eigenvector pair. If $\lambda \in \mathbb{R}$, then we have a real eigenvalue–eigenvector pair.

L 8.2.3 (Real EW/EV):

- Let $A \in \mathbb{R}^{n \times n}$. Then $\lambda \in \mathbb{R}$ is a real eigenvalue of A if and only if $\det(A - \lambda I) = 0$. A vector $v \in \mathbb{R}^n \setminus \{0\}$ is an eigenvector associated with λ if and only if $v \in \mathcal{N}(A - \lambda I)$.

D 8.3.4 (Characteristic Polynomial):

- The characteristic polynomial: $(-1)^n \det(A - zI) = \det(zI - A) = (z - \lambda_1)(z - \lambda_2) \dots (z - \lambda_n)$. The coefficient of z^n is $(-1)^n$.

T 8.2.5 (Existence of EW):

- Every matrix $A \in \mathbb{R}^{n \times n}$ has an eigenvalue (possibly complex-valued).

P 8.2.7 (EW of orthogonal matrix):

- If $Q \in \mathbb{R}^{n \times n}$ is orthogonal and $\lambda \in \mathbb{C}$ is an eigenvalue of Q , then $|\lambda| = 1$.

L 8.2.8 (Complex EW exist in conjugate pairs for real A):

- Let $A \in \mathbb{R}^{n \times n}$. If (λ, v) is an eigenvalue–eigenvector pair, then $(\bar{\lambda}, \bar{v})$ is also an eigenvalue–eigenvector pair.

8.3 Properties of Eigenvalues and Eigenvectors

P 8.3.1 (EW modifications based on types of a matrix):

- If (λ, v) is an eigenvalue–eigenvector pair of A , then (λ^k, v) is an eigenvalue–eigenvector pair of A^k for $k \geq 1$.

- If (λ, v) is an eigenvalue–eigenvector pair of A with $\lambda \neq 0$, then $(\frac{1}{\lambda}, v)$ is an eigenvalue–eigenvector pair of A^{-1} .

L 8.3.2 (Linear independence):

- If $\lambda_1, \dots, \lambda_n$ are all distinct, the corresponding eigenvectors v_1, \dots, v_n are linearly independent.

T 8.3.3 (Existence of a basis from EV):

- Let $A \in \mathbb{R}^{n \times n}$ with n distinct real eigenvalues. Then there exists a basis of \mathbb{R}^n , v_1, \dots, v_n , made of eigenvectors of A .

D 8.3.4 (Trace of a matrix):

- The trace of A is defined by $\text{Tr}(A) = \sum_{i=1}^n A_{ii}$.

L 8.3.5 (Transposition equality of EW):

- The eigenvalues of $A \in \mathbb{R}^{n \times n}$ are the same as those of A^T . But the eigenvectors can be different.

L 8.3.6 (Determinant and Trace via EW):

- Let $A \in \mathbb{R}^{n \times n}$ and let $\lambda_1, \dots, \lambda_n$ be its eigenvalues as they appear in the characteristic polynomial. Then $\det(A) = \prod_{i=1}^n \lambda_i$, $\text{Tr}(A) = \sum_{i=1}^n \lambda_i$.

L 8.3.7 (Cyclic invariance of the trace):

- For $A, B, C \in \mathbb{R}^{n \times n}$:
 $\text{Tr}(AB) = \text{Tr}(BA)$, and $\text{Tr}(ABC) = \text{Tr}(BCA) = \text{Tr}(CAB)$.

9 Diagonalizable Matrices, Singular Value Decomposition

9.1 Diagonalization

T 9.1.1 (Diagonalization Theorem, ability changing basis):

- $A = V \Lambda V^{-1}$, where V 's columns are its eigenvectors and Λ is a diagonal matrix with $\Lambda_{ii} = \lambda_i$ and all other entries 0. $A \in \mathbb{R}^{n \times n}$ and has to have a complete set of real eigenvectors (eigenbasis). Equivalently, $\Lambda = V^{-1}AV$, since V is invertible.

$$\text{Std. coord.} \xrightarrow{V^{-1}} \text{EV. coord.} \xrightarrow{\Lambda} \text{EV. coord.} \xrightarrow{V} \text{Std. coord.}$$

D 9.1.2 (Diagonalizable matrix):

- A matrix $A \in \mathbb{R}^{n \times n}$ is called *diagonalizable* if there exists an invertible matrix V such that $V^{-1}AV = \Lambda$, where Λ is a diagonal matrix.

D 9.1.3 (Complete set of EV):

- If we can find eigenvectors forming a basis of \mathbb{R}^n for A , we say that A has a *complete set of real eigenvectors*.

P 9.1.6 (Projection and EW/EV):

- Let P be a projection matrix onto a subspace $U \subset \mathbb{R}^n$. Then P has two eigenvalues, 0 and 1, and a complete set of real eigenvectors.

D 9.1.7 (Similar matrices):

- Matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$ are called *similar* if there exists an invertible matrix S such that $B = S^{-1}AS$. **P 9.1.8:** Similar matrices have the same eigenvalues.

D 9.1.10 (Geometric multiplicity):

- Let $A \in \mathbb{R}^{n \times n}$ and let λ be an eigenvalue of A . Then $\dim \mathcal{N}(A - \lambda I)$ is called the *geometric multiplicity* of λ .

L 9.1.11 (Complete set of real EV):

- A matrix has a complete set of real eigenvectors if and only if all its eigenvalues are real and the geometric multiplicities equal the algebraic multiplicities for all eigenvalues.

9.2 Symmetric Matrices, Spectral Theorem

T 9.2.1 (Spectral Theorem):

- Any symmetric matrix $A \in \mathbb{R}^{n \times n}$ has n real eigenvalues and an orthonormal basis of \mathbb{R}^n consisting of eigenvectors of A .

C 9.2.2 (Eigendecomposition):

- For any symmetric matrix $A \in \mathbb{R}^{n \times n}$, there exists an orthogonal matrix $V \in \mathbb{R}^{n \times n}$ (whose columns are eigenvectors of A) such that $A = V \Lambda V^T$, where $\Lambda \in \mathbb{R}^{n \times n}$ is diagonal with diagonal entries equal to the eigenvalues

of A , and $V^T V = I$. This decomposition is called the *eigendecomposition*.

C 9.2.4 (Rank of real symmetric matrix):

- If A is a real symmetric matrix, then $\text{rank}(A)$ is the number of nonzero eigenvalues of A (counting repetitions).
- For a general $n \times n$ matrix, $\text{rank}(A) = n - \dim \mathcal{N}(A)$, so the geometric multiplicity of the eigenvalue $\lambda = 0$ equals $\dim \mathcal{N}(A)$.

P 9.2.6 (Rank-One Spectral Decomposition):

- Let $A \in \mathbb{R}^{n \times n}$ be symmetric, and let v_1, \dots, v_n be an orthonormal basis of eigenvectors of A (the columns of V), with associated eigenvalues $\lambda_1, \dots, \lambda_n$. Then $A = \sum_{k=1}^n \lambda_k v_k v_k^T$.

A *real symmetric matrix is a weighted sum of orthogonal projections onto its eigenvector directions, with weights given by the eigenvalues.*

L 9.2.7 (Orthogonality of EV):

- Let $A \in \mathbb{R}^{n \times n}$ be symmetric and let $\lambda_1 \neq \lambda_2 \in \mathbb{R}$ be two distinct eigenvalues of A with corresponding eigenvectors v_1, v_2 . Then v_1 and v_2 are orthogonal.

L 9.2.8 (Symmetric matrix has real EW):

- A symmetric matrix $A \in \mathbb{R}^{n \times n}$ has only real eigenvalues: $\lambda \in \mathbb{C} \Rightarrow \lambda \in \mathbb{R}$. Indeed, if $Av = \lambda v$:
 $\lambda \|v\|^2 = \bar{\lambda} v^* v = (\lambda v)^* v = (Av)^* v = v^* A^* v = v^* Av = v^* \lambda v = \lambda \|v\|^2$. \Rightarrow every symmetric matrix $A \in \mathbb{R}^{n \times n}$ has a real eigenvalue. (C 9.2.9)

P 9.2.10 (Rayleigh Quotient):

- $A \in \mathbb{R}^{n \times n}$ is symmetric. For $x \in \mathbb{R}^n \setminus \{0\}$, the Rayleigh quotient $R(x) = \frac{x^T Ax}{x^T x}$.

The minimum of $R = R(v_{\min}) = \lambda_{\min}$, and the maximum $R(v_{\max}) = \lambda_{\max}$. Here $\lambda_{\max}/\lambda_{\min}$ are the largest/smallest eigenvalues of A , and v_{\max}/v_{\min} their associated eigenvectors.

D 9.2.11 (PSD and PD matrices):

- $A = A^T \bullet A \succeq 0$ (PSD) $\Leftrightarrow \lambda_i(A) \geq 0$ • $A > 0$ (PD) $\Leftrightarrow \lambda_i(A) > 0$.

P 9.2.12 (Positivity of the quadratic form):

- Let $A \in \mathbb{R}^{n \times n}$ be symmetric. Then $A \succeq 0 \iff x^T Ax \geq 0 \quad \forall x \in \mathbb{R}^n$, and $A > 0 \iff x^T Ax > 0 \quad \forall x \neq 0$.

D 9.2.13 (Gram Matrix):

- Given vectors $v_1, \dots, v_n \in \mathbb{R}^m$, their *Gram matrix* is $G \in \mathbb{R}^{n \times n}$ defined by $G_{ij} = v_i^T v_j$. If $V = [v_1 \dots v_n] \in \mathbb{R}^{m \times n}$, then $G = V^T V$.
- If $A = [a_1 \dots a_n] \in \mathbb{R}^{m \times n}$, one also calls AA^T a Gram matrix; note that $AA^T = \sum_{i=1}^n a_i a_i^T$. It is $m \times m$ matrix.

P 9.2.15 (Same EV of transposed matrices):

- For a real matrix $A \in \mathbb{R}^{m \times n}$, the non-zero eigenvalues of $A^T A \in \mathbb{R}^{n \times n}$ and $AA^T \in \mathbb{R}^{m \times m}$ are the same. Also both are symmetric and PSD.

P 9.2.16 (Cholesky Decomposition):

- Every symmetric PSD matrix M is a Gram matrix of upper-triangular matrix C : $M = C^T C$.

9.3 Singular Value Decomposition

D 9.3.1 (Singular Value Decomposition):

- Let $A \in \mathbb{R}^{m \times n}$. There exist orthogonal matrices $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ and a diagonal matrix $\Sigma \in \mathbb{R}^{m \times n}$ with nonnegative diagonal entries $\sigma_1 \geq \dots \geq \sigma_{\min(m,n)}$ such that

$$A = U \Sigma V^T.$$

The columns of U and V are called the left and right singular vectors of A , and the diagonal entries of Σ are the singular values of A . Columns of U are the eigenvectors of AA^T and columns of V^T are the eigenvectors of $A^T A$. Σ has the square root of the eigenvalues of $AA^T / A^T A$ (they are the same). U has the dimensions $m \times m$ and V has the dimensions $n \times n$. U, V are orthogonal matrices.

R 9.3.2 (Compact form of SVD):

- If $\text{rank}(A) = r$, then the SVD can be written as

$$A = U_r \Sigma_r V_r^T,$$

where $U_r \in \mathbb{R}^{m \times r}$ and $V_r \in \mathbb{R}^{n \times r}$ have orthonormal columns, and $\Sigma_r = \text{diag}(\sigma_1, \dots, \sigma_r)$. This representation stores $r(m+n+1)$ real numbers instead of mn . For small r , this yields substantial savings and motivates low-rank approximations.

T 9.3.3 (Every matrix has SVD):

- Every matrix $A \in \mathbb{R}^{m \times n}$ has SVD: $A = U \Sigma V^T$. Equivalently, every linear transformation is diagonal in orthonormal bases of singular vectors.

P 9.3.4 (SVD as a sum of rank-one matrices):

- Let $A \in \mathbb{R}^{m \times n}$ have rank r , with singular values $\sigma_1, \dots, \sigma_r$ and corresponding singular vectors u_1, \dots, u_r and v_1, \dots, v_r . Then

$$A = \sum_{k=1}^r \sigma_k u_k v_k^T.$$

Main idea: We can write any rank- r matrix $A \in \mathbb{R}^{m \times n}$ as a sum of r rank-1 matrices.

SVD of the Inverse A^{-1} :

- If $A = U \Sigma V^T$, $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_n)$, then the SVD of the inverse can be written as

$$A^{-1} = V \Sigma^{-1} U^T.$$

where

$$\Sigma^{-1} = \text{diag}\left(\frac{1}{\sigma_1}, \dots, \frac{1}{\sigma_n}\right).$$

Algorithms

Gaussian Elimination.

Given $Ax = b$, form the augmented matrix $[A | b]$ and apply elementary row operations to reach row echelon form (REF): pivot \rightarrow swap \rightarrow eliminate below \rightarrow repeat. If a row $(0 \dots 0 | c)$ with $c \neq 0$ appears, the system is inconsistent; otherwise solve by back-substitution.

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 1 & 4 \end{array} \right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -1 & -2 \end{array} \right] \Rightarrow (x, y) = (1, 2).$$

Gauss-Jordan Elimination.

Starting from $[A | b]$, apply Gaussian elimination, then normalize each pivot to 1 and eliminate all other entries in the pivot columns (pivot columns must be basis vectors). The resulting reduced row echelon form (RREF) gives the solution directly. Pivot positions move strictly to the right as you go down the rows. Solve:

$$\left\{ \begin{array}{l} x + y = 3, \\ 2x + y = 4. \end{array} \right. \iff \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 1 & 4 \end{array} \right]$$

Row-reduce to RREF:

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 1 & 4 \end{array} \right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -1 & -2 \end{array} \right] \xrightarrow{R_2 + R_1} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 2 \end{array} \right] \xrightarrow{R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array} \right].$$

$$x = 1, \quad y = 2.$$

Inverse via Gauss-Jordan.

To compute A^{-1} , form the augmented matrix $[A | I]$ and apply Gauss-Jordan elimination. If

$$[A | I] \xrightarrow{} [I | B],$$

then $B = A^{-1}$. If I cannot be obtained on the left, A is not invertible.

$$[A | I] \xrightarrow{} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array} \right] \xrightarrow{R_2 - \frac{1}{2}R_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -\frac{1}{2} & 1 \end{array} \right] \xrightarrow{R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & -\frac{1}{2} & 1 \end{array} \right].$$

$$\hat{A} = \arg \min_{\alpha \in \mathbb{R}^2} \|A\alpha - b\|^2 = (A^T A)^{-1} A^T b, \quad A = \begin{pmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_m \end{pmatrix}$$

Forming orthonormal basis via Gram-Schmidt.

Gram-Schmidt used to construct orthonormal bases.

We have linearly independent vectors a_1, \dots, a_n that span a subspace S , then we can construct their orthonormal basis q_1, \dots, q_n by:

- $q_1 = \frac{a_1}{\|a_1\|}$.
- For $k = 2, \dots, n$ do $q'_k = a_k - \sum_{i=1}^{k-1} (a_i^T q_i) q_i$,
- normalise $q_k = \frac{q'_k}{\|q'_k\|}$.

Rule of Sarrus (Computing 3x3 Determinants).

The green arrows represent $\text{sum1} = aei + bfg + cdh$ and the red arrows represent $\text{sum2} = gec + hfa + idb$. The determinant is given by $\det(A) = \text{sum1} - \text{sum2}$.

$$\left(\begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array} \right)$$

Solving Linear Recurrences via Matrix Diagonalization

We are given the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2}, \quad \text{for } n \geq 2.$$

Using the given formula, we can derive a matrix M such that

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ a_{n-2} \end{pmatrix}, \quad M = \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{g}_n = \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix},$$

With initial vector $\mathbf{g}_0 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, we have $\mathbf{g}_n = M^n \mathbf{g}_0$.

Eigenvalues of M

We compute $\det(M - \lambda I) = (5 - \lambda)(-\lambda) + 6 = \lambda^2 - 5\lambda + 6$.

Solving, $\lambda^2 - 5\lambda + 6 = 0 \Rightarrow (\lambda - 3)(\lambda - 2) = 0$.

Hence, $\lambda_1 = 3, \lambda_2 = 2$.

EigenvectorsFor $\lambda = 3$:

$$(M - 3I) = \begin{pmatrix} 2 & -6 \\ 1 & -3 \end{pmatrix}.$$

Solving $(M - 3I)\mathbf{v} = 0$ gives $2x - 6y = 0 \Rightarrow x = 3y$, so EV $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

For $\lambda = 2$:

$$(M - 2I) = \begin{pmatrix} 3 & -6 \\ 1 & -2 \end{pmatrix}.$$

Solving $(M - 2I)\mathbf{v} = 0$ gives $x - 2y = 0 \Rightarrow x = 2y$, so EV $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Closed FormSince \mathbf{v}_1 and \mathbf{v}_2 are linearly independent, we can write

$$\mathbf{g}_0 = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2.$$

That is,

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \alpha_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Solving,

$$\alpha_1 = 2, \quad \alpha_2 = -2.$$

Therefore, $\mathbf{g}_n = \alpha_1 3^n \mathbf{v}_1 + \alpha_2 2^n \mathbf{v}_2 = 2 \cdot 3^n \begin{pmatrix} 3 \\ 1 \end{pmatrix} - 2^{n+1} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Thus, $a_n = 2 \cdot 3^n - 2^{n+1}$.

Quizzes - Computations

Basis of a plane and related subspaces:

- Plane $P = \{x \in \mathbb{R}^3 : 6x_1 - x_2 + 5x_3 = 0\}$

1. Basis for P

Solve for x_2 :

$$x_2 = 6x_1 + 5x_3$$

Let $x_1 = s$, $x_3 = t$:

$$(x_1, x_2, x_3) = (s, 6s + 5t, t) = s(1, 6, 0) + t(0, 5, 1)$$

$$\mathcal{B}_P = \{(1, 6, 0), (0, 5, 1)\}$$

2. Intersection with $\text{span}\{e_1, e_2\}$

$$\begin{aligned} x_3 = 0 &\Rightarrow 6x_1 - x_2 = 0 \Rightarrow x_2 = 6x_1 \\ (x_1, x_2, x_3) &= s(1, 6, 0) \end{aligned}$$

$$\mathcal{B}_{P \cap \text{span}\{e_1, e_2\}} = \{(1, 6, 0)\}$$

3. Perpendicular vectors to P

Normal vector from plane equation:

$$\mathbf{n} = (6, -1, 5)$$

$$P^\perp = \text{span}\{(6, -1, 5)\}$$

$$\mathcal{B}_{P^\perp} = \{(6, -1, 5)\}$$

Compute bases for orthogonal spaces:

- Finding $S^\perp \subset \mathbb{R}^3$ when $S = \text{span}\{v\}$

Let $v = (a, b, c) \neq 0$ and $S = \text{span}\{v\}$. Then

$$S^\perp = \{u \in \mathbb{R}^3 : u \cdot v = 0\}.$$

Let $u = (x, y, z)$. Orthogonality gives

$$ax + by + cz = 0.$$

Solve for one variable and parametrize. Choose convenient values for the free variables to obtain two linearly independent solutions.

These two vectors form a basis for S^\perp .

Example: $v = (-6, -9, 7)$

$$-6x - 9y + 7z = 0$$

Choose $(y, z) = (2, 0)$ and $(0, 6)$:

$$u_1 = (-3, 2, 0), \quad u_2 = (7, 0, 6).$$

Calculating four fundamental subspaces:

- Four Fundamental Subspaces

Let $A \in \mathbb{R}^{m \times n}$ with $\text{rank}(A) = r$.

$$\dim \mathcal{C}(A) = r$$

$$\dim \mathcal{C}(A^T) = r$$

$$\dim \mathcal{N}(A) = n - r$$

$$\dim \mathcal{N}(A^T) = m - r$$

$$\mathcal{C}(A) \perp \mathcal{N}(A^T), \quad \mathcal{C}(A^T) \perp \mathcal{N}(A)$$

Pseudoinverse of diagonal matrix:

- Diagonal matrix \Rightarrow invert nonzero diagonals, keep zeros.

Diagonalization of a symmetric matrix:

- Diagonalization of a symmetric matrix (example) Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Characteristic polynomial:

$$p_A(\lambda) = \det(A - \lambda I) = (2 - \lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1).$$

Eigenvalues:

$$\lambda_1 = 3, \quad \lambda_2 = 1.$$

Eigenvectors:

$$\lambda_1 = 3 : \quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda_2 = 1 : \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The eigenvectors are orthogonal:

$$v_1 \cdot v_2 = 0.$$

Define

$$V = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$A = V\Lambda V^{-1}, \quad V^{-1} = (V^T V)^{-1} V^T = \frac{1}{2} V^T.$$

Remark: The order of eigenvalues on the diagonal of Λ is arbitrary. Reordering eigenvalues requires the same reordering of eigenvectors.

Computing singular values:

- Computing singular values

For a matrix $A \in \mathbb{R}^{m \times n}$, the singular values are

$$\sigma_i = \sqrt{\lambda_i},$$

where λ_i are the eigenvalues of $A^T A$.

Example:

$$A = \begin{pmatrix} 3 & 0 \\ 4 & 0 \end{pmatrix}.$$

Compute

$$A^T A = \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 25 & 0 \\ 0 & 0 \end{pmatrix}.$$

Eigenvalues of $A^T A$:

$$\lambda_1 = 25, \quad \lambda_2 = 0.$$

Singular values:

$$\sigma_1 = \sqrt{25} = 5, \quad \sigma_2 = \sqrt{0} = 0.$$

$$\text{Singular values of } A = \sqrt{\text{eigenvalues of } A^T A}.$$

Why determinant expansion along a row/column works:

- Why determinant expansion along a row/column works

For any $n \times n$ matrix $M = (m_{ij})$, the determinant can be expanded along any row i or any column j (Laplace expansion):

$$\det(M) = \sum_{j=1}^n m_{ij} C_{ij} \quad \text{or} \quad \det(M) = \sum_{i=1}^n m_{ij} C_{ij},$$

where

$$C_{ij} = (-1)^{i+j} \det(M_{ij}).$$

is the cofactor, and M_{ij} is obtained by deleting row i and column j .

Example:

$$\lambda I - A = \begin{pmatrix} \lambda - \frac{9}{2} & 0 & -\frac{1}{2} \\ 0 & \lambda & 0 \\ -\frac{1}{2} & 0 & \lambda - \frac{9}{2} \end{pmatrix}.$$

Expanding along row 2:

$$\det(\lambda I - A) = 0 \cdot C_{21} + \lambda \cdot C_{22} + 0 \cdot C_{23} = \lambda C_{22}.$$

Cofactor computation:

$$C_{22} = (-1)^{2+2} \det \begin{pmatrix} \lambda - \frac{9}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda - \frac{9}{2} \end{pmatrix}.$$

Since $(-1)^{2+2} = (-1)^4 = 1$, we obtain

$$C_{22} = \det \begin{pmatrix} \lambda - \frac{9}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda - \frac{9}{2} \end{pmatrix}.$$

Therefore,

$$\det(\lambda I - A) = \lambda \det \begin{pmatrix} \lambda - \frac{9}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda - \frac{9}{2} \end{pmatrix}.$$

Conclusion: Expanding along rows or columns with many zeros is always valid and simplifies determinant computations.

Fast computation of singular values (symmetric case):

- Fast computation of singular values (symmetric case) Let $A \in \mathbb{R}^{n \times n}$.
- Key fact: If $A = A^T$ (i.e. A is symmetric), then

$$\sigma_i(A) = |\lambda_i(A)|,$$

where $\lambda_i(A)$ are the eigenvalues of A . **Reason:** Singular values are defined by

$$\sigma_i(A) = \sqrt{\lambda_i(A^T A)}.$$

If $A = A^T$, then

$$A^T A = A^2,$$

and eigenvalues of A^2 are $\lambda_i(A)^2$. Hence

$$\sigma_i(A) = \sqrt{\lambda_i(A)^2} = |\lambda_i(A)|.$$

Example:

$$A = \begin{pmatrix} \frac{9}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{9}{2} \end{pmatrix} \quad (\text{symmetric}).$$

Eigenvalues:

$$\lambda(A) = \{5, 4, 0\}.$$

Singular values:

$$\sigma(A) = \{5, 4, 0\}.$$

General fallback (always works): If A is not symmetric,

compute eigenvalues of $A^T A$ and take square roots.

Remember:

$$A = A^T \Rightarrow \sigma_i = |\lambda_i|.$$

Similar (2x2) matrices:

- Similar (2x2) matrices always have the same eigenvalues. It is the fast way to check if matrices are similar.

Norm of a vector:

- Norm (2) of a vector:

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Characteristic polynomial & eigenvalues & eigenvectors:

- We can find Characteristic polynomial via:

$$\det(\lambda I - A) = (-1)^n \det(A - \lambda I)$$

- We can find corresponding eigenvectors \mathbf{v} to eigenvalues λ via:

$$(A - \lambda I)\mathbf{v} = 0$$

Distance between vector and its projection:

- Distance between vector and its projection is:

$$\|\text{vector} - \text{projection}\| = \|\mathbf{v} - \text{proj}_S(\mathbf{v})\|$$

Angle between vectors:

- Angle between vectors:

$$\cos(\theta) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right)$$

When T is linear transformation:

- When T is linear transformation, then:

$$2T(\mathbf{x}) + 3T(\mathbf{y}) = T(2\mathbf{x} + 3\mathbf{y})$$

This might simplify some calculations a lot.

Inverse of 2×2 matrix:

- Inverse of 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det(A) \neq 0$:

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Determining when matrix returns to identity:

- If $A^m = I$ and $A^n = I$, then

$$A^{\gcd(m,n)} = I.$$

Proof sketch to template:

Let $d = \gcd(m, n)$. Then there exist integers x, y with $d = xm + yn$ (Bézout). Assuming A is invertible (true in any group; for matrices this means $A \in GL$),

$$A^d = A^{xm+yn} = (A^m)^x (A^n)^y = I^x I^y = I.$$

When can you conclude $A = I$? You can conclude $A = I$ if $\gcd(m, n) = 1$, because then $A^1 = A = I$. Otherwise, you can only conclude that the order of A divides $\gcd(m, n)$ (i.e., A is a root of unity of that exponent).

Pairwise linear independence:

- Pairwise independence is weaker than (joint) linear independence. \Rightarrow Linear independence is a global property: checking vectors two at a time is not enough. Collectively, they might still fail independence.

Geometrically in \mathbb{R}^2 : you can have infinitely many vectors that are pairwise non-collinear, but at most two vectors can be linearly independent.

Complex expression and geometric Interpretation:

- Using $z\bar{z} = |z|^2$, the condition reduces to $x^2 + y^2 = 1$, which describes the unit circle.

SVD of rank-1 matrix:

- SVD of rank-1 matrix:

$$A = \sigma u v^\top$$

Where:

- σ is only on non-zero singular value

- u is a unit column vector (left singular vector)

- v is a unit column vector (right singular vector)

More specifically, if

$$A = xy^\top$$

with $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, then

$$A = \sigma u v^\top$$

where

$$\sigma = \|x\| \|y\|, \quad u = \frac{x}{\|x\|}, \quad v = \frac{y}{\|y\|}$$

Example:

$$A = \begin{bmatrix} \sqrt{6} \\ 3 \\ -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\|x\| = \sqrt{6 + 9 + 1} = 4, \quad \|y\| = 1$$

$$A = \begin{bmatrix} \frac{\sqrt{6}}{4} \\ \frac{3}{4} \\ -\frac{1}{4} \end{bmatrix} [4] \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Unique non-zero singular value $\sigma = 4$

SVD validity check:

- To verify whether a proposed factorization $A = U\Sigma V^\top$ is a valid SVD, it suffices to check the following:

- **Orthogonality:** U (and V) has orthonormal columns, i.e. $U^\top U = I$ and $V^\top V = I$;
- **Singular values:** Σ is diagonal with non-negative entries;
- **Dimensions:** if $A \in \mathbb{R}^{m \times n}$, then $\Sigma \in \mathbb{R}^{m \times n}$.

Invertible matrix and EW:

- A matrix is invertible iff 0 is not an eigenvalue.

- $A = A^\top \Rightarrow A$ has real eigenvalues $\Rightarrow \lambda + i \neq 0 \Rightarrow \det(A + iI) \neq 0 \Rightarrow A + iI$ is invertible