1 Vectors

D 1.4 (Linear Combination):
• let $v, w \in \mathbb{R}^m, \lambda, \mu \in \mathbb{R}$
$\Rightarrow \sum_{i=1}^{n} \lambda_i v_i$ are scaled combinations of n vectors v_i.

D 1.7 (Combination types):
• **Affine Combination:** $\sum_{i=1}^{n} \lambda_i = 1$
Conic Combination: if $\lambda_j \geq 0$ for $j = 1, 2, \ldots, n$
Convex Combination: Affine + Conic

D 1.9 (Scalar/dot product):
• $\mathbf{v} \cdot \mathbf{w} := \sum_{i=1}^{m} v_i w_i$, alternative notation: $[z_i]_{i=1}^{m} := [v_i + w_i]_{i=1}^{m}$

D 1.11 (Euclidean norm, squared norm, unit vector):
• $\|\mathbf{v}\| := \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{\mathbf{v}^\top \mathbf{v}}$, **Squared norm:** $\|\mathbf{v}\|^2 := \mathbf{v}^\top \mathbf{v}$,
Unit vector: $\|\mathbf{u}\| = 1 = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$ (for any vector $\mathbf{v} \neq 0$)

L 1.12 (Cauchy-Schwarz inequality):
• $|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\|\|\mathbf{w}\|$ for any two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^m$

D 1.14 (Angle between vectors):
• $\cos(\alpha) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} \in [-1, 1]$

D 1.16 (Hyperplane through origin):
• Let $\mathbf{d} \in \mathbb{R}^m$, $\mathbf{d} \neq 0$, $H_\mathbf{d} = \{\mathbf{v} \in \mathbb{R}^m : \mathbf{v} \cdot \mathbf{d} = 0\}$

L 1.16 (Triangle inequality):
• $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$

D 1.21 (Linear (in)dependence):
• vectors are linearly dependent if one of them is linear combination of the others: $\mathbf{v}_k = \sum_{\substack{j=1 \\ j \neq k}}^{n} \lambda_j \mathbf{v}_j$
$\Leftrightarrow$ There are scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ besides $0, 0, \ldots, 0$ such that $\sum_{j=1}^{n} \lambda_j \mathbf{v}_j = \mathbf{0}$. We also say that $\mathbf{0}$ is a nontrivial linear combination of the vectors.
$\Leftrightarrow$ At least one of the vectors is a linear combination of the previous ones.

D 1.25 (Span):
• Span of vectors is a set of all linear combinations of those vectors: $\mathbf{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) := \left\{\sum_{j=1}^{n} \lambda_j \mathbf{v}_j : \lambda_j \in \mathbb{R} \text{ for all } j \in [n]\right\}$

Construction of vectors with standard unit vectors:
• Every target vector can be written as: $\mathbf{u} = \sum_{i=1}^{m} u_i \mathbf{e}_i$, where $\mathbf{e}$ is a standard unit vector.

2 Matrices

D 2.1 (Matrix):
• $A = [a_{ij}]_{i=1,j=1}^{m\ \ n}$ - m rows, n columns (*Zeilen zuerst, Spalten später*)

D 2.2 (Matrix addition, scalar multiplication):
• Addition: $A + B = [a_{ij} + b_{ij}]_{i=1,j=1}^{m\ \ n}$
• Scalar multiplication: $\lambda A = [\lambda a_{ij}]_{i=1,j=1}^{m\ \ n}$

Matrix types:
• **Identity matrix** ($a_{ii} = 1$ for all i): I
• **Diagonal matrix** ($a_{ij} = 0$ for all $i \neq j$): $\mathrm{diag}(d_1, \ldots, d_n)$
• **Upper triangular matrix** ($a_{ij} = 0$ for all $i > j$): U
• **Lower triangular matrix** ($a_{ij} = 0$ for all $i < j$): L
• **Symmetric matrix** ($a_{ij} = a_{ji}$ for all i, j): $A = A^\top$
• **Skew-symmetric matrix** ($a_{ij} = -a_{ji}$ for all i, j): $A = -A^\top$

D 2.4 (Matrix-vector product):
• Rows of matrix ($m \times n$) with vector (n elements), i.e.
$u_1 = \sum_{j=1}^{n} a_{1,j}\, v_j$, $Ix = x$; **Trace:** Sum of the diagonal entries.

D 2.9 (Column space):
• The column space $\mathbf{C}(A)$ of A is the span (set of all linear combinations) of the columns: $\mathbf{C}(A) := \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$

D 2.10 (Rank):
• $\mathrm{rank}(A)$:= the number of linearly independent column vectors of A.

D 2.11 (Transpose):
• Mirror the matrix along its diagonal. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \leftrightarrow A^\top \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$
• $(A^\top)^\top = A$

D 2.13 (Row space):
• $\mathbf{R}(A) := \mathbf{C}(A^\top)$

D 2.17 (Nullspace):
• Nullspace contains all input vectors that lead to output vector $\mathbf{0}$.
$\mathbf{N}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{R}^n$

D 2.27 (Kernel & Image):
• **Kernel:** $\mathbf{N}(A) = \mathbf{Ker}(T) := \{\mathbf{x} \in \mathbb{R}^n : T(\mathbf{x}) = \mathbf{0}\} \subseteq \mathbb{R}^n$ (If A is the unique $m \times n$ matrix such that $T = T_A$)
• **Image:** $\mathbf{C}(A) = \mathbf{Im}(T) := \{T(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$ (If A is the unique $m \times n$ matrix such that $T = T_A$), the set of all outputs that T can produce.

2.2.2 Working with linear transformations:
• A matrix can be understood as a re-mapping of the unit vectors, scaling and re-orienting them. Each column vector can then be understood as the new unit vector e_i, hence essentially adding another coordinate system to the original one, which is moved and rotated a certain way. The rotation matrix under 2 is such an example. To prove that T is a linear transformation, use $T(x + y) = T(x) + T(y)$ and $T(\lambda x) = \lambda T(x)$. Then insert the linear transformation given by the task and replace x (or whatever variable there is) with $x + y$ or λx. $Ax = \sum_{i=1}^{n} x_i v_i$, where v_i is the i-th column of A.

O 2.39 (Matrix multiplication):
• $A \times B = C$, $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. Dimension restrictions: A is $m \times n$, B is $n \times p$, result is $m \times p$. For each entry, multiply the i-th row of A with the j-th column of B.
Not commutative, but associative & distributive.

L 2.40 Matrix multiplication with transposition:
• $(AB)^\top = B^\top A^\top$

D 2.44 Outer product:
• $\mathrm{rank}(A) = 1 \iff \exists$ non-zero vectors $v \in \mathbb{R}^m$, $w \in \mathbb{R}^n$ such that A is an outer product, i.e. $A = vw^\mathrm{T}$, thus $\mathrm{rank}(vw^\mathrm{T}) = 1$.

T 2.46 (CR decomposition):
• $A = CR$. Get R from (reduced) row echelon form. C is the columns from A where there is a pivot in R. $C \in \mathbb{R}^{m \times r}$, $R \in \mathbb{R}^{r \times n}$ (in RREF), $r = \mathrm{rank}(A)$. **Row Echelon Form:** To find REF, try to create pivots: $R_0 = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Use Gauss-Jordan elimination to find it (row transformations). **Reduced REF:** RREF has pivots equal to 1 with zeros above and below each pivot and no zero rows (i.e. in R_0, R (in RREF) would be R_0 without the last row).

O 2.5.6 (Invertible matrix):
• Matrix A is invertible if it is square and there is B such that:
$AB = I \Leftrightarrow BA = I \Leftrightarrow AB = BA = I$

D 2.57 (Inverse matrix and its properties):
• If $AB = I$ for invertible A, then B is its inverse and denoted as A^{-1}. • $(A^{-1})^{-1} = A$ • $(AB)^{-1} = B^{-1}A^{-1}$ • $(A^\top)^{-1} = (A^{-1})^\top$

4 Four Fundamental Subspaces

4.1 Vector Spaces

D 4.1 (Vector Space):
• Vector space is a triple $(V, +, \cdot)$ where V is a set (the vectors) with two operations $\oplus$ and $\odot$. They are based on algebras called fields and satisfy axioms: *commutativity, associativity, zero vector, negative vector, identity element, compatibility of multiplications of vectors and scalars ($\in \mathbb{R}$), distributivity over $\oplus$ both for vectors and scalars ($\in \mathbb{R}$).*

D 4.8 (Subspace):
• Let V be a vector space. A nonempty subset $U \subseteq V$ is a subspace of V if following axioms are true $\forall \mathbf{v}, \mathbf{w} \in U$ and $\forall \lambda \mathbf{v} \in U$:
• $\mathbf{v} + \mathbf{w} \in U$ • $\lambda \mathbf{v} \in U$.
They guarantee that vector addition and scalar multiplication "doesn't take us outside of a subspace". For showing U is nonempty it is enough to show that $\mathbf{0} \in U$.

L 4.9 (Subspace always has 0):
• Let $U \subseteq V$ be a subspace of a vector space V. Then $\mathbf{0} \in U$ (at least).

L 4.11 (Column space is a subspace):
• Let $A \in \mathbb{R}^{m \times n}$, then $C(A) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}$ is subspace of $\mathbb{R}^m$.
$\Rightarrow R(A) = C(A^\top)$ is a subspace of $\mathbb{R}^n$.

E 4.13 (The nullspace is a subspace):
• Let $A \in \mathbb{R}^{m \times n}$. Then the nullspace $N(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = 0\}$ is a subspace of $\mathbb{R}^n$

L 4.14 (Subspaces are vector spaces):
• V is a vector space and U is its subspace. Then U is also a vector space with the same $\oplus$ and $\odot$ as V.

4.2 Bases and dimension

D 4.18 (Basis):
• Let V be a vector space. A subset $B \subseteq V$ is called a basis of V if B is linearly independent and it spans V: $\mathrm{Span}(B) = V$.

L 4.19 (Independent columns is a basis):
• Independent columns form basis of column space C(A).

O 4.20 (Non-uniqueness of basis):
• Every set $B = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\} \subseteq \mathbb{R}^m$ of m linearly independent vectors is a basis of $\mathbb{R}^m$.

D 4.21 (Finitely generated vector space):
• There is a finite subset $G \subseteq V$ with $\mathrm{Span}(G) = V$. Then V has a basis $B \subseteq G$.

T 4.22 (Finitely generated VS has a basis):
• If V is finitely generated, then V has a basis $B \subseteq V$.

L 4.23 (Steinitz exchange lemma):
• *"exchanging elements between G and F"*
V is finitely generated vector space, $F \subseteq V$ a finite set of lin. independent vectors, and $G \subseteq V$ a finite set of vectors with $\mathrm{Span}(G) = V$, then:
• $|F| \leq |G|$ and • $\exists E \subseteq G$ of size $|G| - |F|$ such that $\mathrm{Span}(F \cup E) = V$.

T 4.24 (All bases have the same size):
• All bases have the same size: $B, B' \subseteq V \Rightarrow |B| = |B'|$.

D 4.25 (Dimension):
• $\dim(V)$ - the dimension of V. It has a size of arbitrary basis B of V.

D 4.26 (Linear transformation between vector spaces):
• Let V, W be vector spaces. A function $T : V \to W$ is linear if, for all $x_1, x_2 \in V$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, $T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$.

L 4.27 (Bijective lin. transformations preserve basis):
• If $T : V \to W$ is a bijective linear map, then $B \subseteq V$ is a basis of $V \Leftrightarrow T(B)$ is a basis of W, and hence $\dim(V) = \dim(W)$.

D 4.28 (Isomorphic vector spaces):
• $V \cong W \iff \exists T : V \to W$ linear and bijective.

T 4.29 (Basis writes vectors as a unique lin. combination):
• Let V be a finite-dimensional vector space with basis $B = \{v_1, \ldots, v_m\}$. Then every $v \in V$ can be written uniquely as $v = \sum_{j=1}^{m} \lambda_j v_j$, for unique scalars $\lambda_1, \ldots, \lambda_m$.

L 4.30 (Less than $\dim(V)$ vectors do not span V):
• If $|G| < \dim V$, then $\mathrm{span}(G) \neq V$.

4.3 Computing the three fundamental subspaces

T 4.31 (Basis of $C(A)$: Pivots columns of RREF):

• R is RREF of A, then all columns at pivots of R form a basis of $C(A)$: $\dim(C(A)) = \text{rank}(A) = r$

T 4.32 (Basis of $R(A)$: Nonzero rows of RREF(A)):

• Nonzero rows of RREF(A) form a basis of $R(A)$, so, $\dim(R(A)) = r$.

T 4.33 (Row rank equals columns rank):

• $\text{rank}(A) = \text{rank}(A^\top)$

C 4.34 (Rank is at most min of the matrix dimensions):

• A is a $m \times n$ matrix with rank $r \Rightarrow r \leq \min(n, m)$.

L 4.35 (Nullspace isomorphism):

• $R = \text{RREF}(A)$, then $T : N(R) \to \mathbb{R}^{n-r}$ is an isomorphism between $N(R)$ and $\mathbb{R}^{n-r} \Rightarrow \dim(N(R)) = n - r$.

T 4.36 (Basis of $N(A)$: Non-pivot columns of RREF(A)):

• If $\text{rank}(A) = r$, then $\dim(N(A)) = n - r$.

For finding a basis of N(A): First put A into RREF form R. Identify the pivot columns and the non-pivot columns. Write the system Ax = 0. Write the pivot variables in terms of the non-pivot (free) variables. Then, for each non-pivot column, set the corresponding free variable to 1 and all other free variables to 0, and compute the resulting vector. These vectors form a basis of N(A).

4.4 All solutions of $Ax = b$

D 4.37 (Solution space):

• Solution space of $A\mathbf{x} = \mathbf{b}$:

$\mathbf{Sol}(A, \mathbf{b}) := \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\} \subseteq \mathbb{R}^n$

T 4.38 (Solution space from shifting the nullspace):

• Let $\mathbf{s}$ be some solution of $A\mathbf{x} = \mathbf{b}$, then:

$\mathbf{Sol}(A, \mathbf{b}) := \{\mathbf{s} + \mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in N(A)\}$.

We can also compute $\mathbf{Sol}(A, \mathbf{b})$, although it is not a subspace.

T 4.39 (Dimension of a solution space):

• Let $A \in \mathbb{R}^{m\times n}$ with rank r. If $A\mathbf{x} = \mathbf{b}$ is solvable, then: $\dim(\mathbf{Sol}(A, \mathbf{b})) = n - r$, and $\dim(\mathbf{Sol}(A, \mathbf{b})) := \dim(N(A))$.

T 4.40 (Systems of rank m are solvable):

• Let $A \in \mathbb{R}^{m\times n}$ with $\text{rank}(A) = m$, $A\mathbf{x} = \mathbf{b}$ is solvable for all $\mathbf{b} \in \mathbb{R}^m$.

T 4.41 (Systems of rank less than m are typ. unsolvable):

• Systems of rank $r < m$ are typically unsolvable.

D 4.42 (Types of systems):

• Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. The system $A \in \mathbb{R}^{m\times n}$ is called:

• $m = n \Rightarrow$ square (A is a square matrix) $\star$ **typ. solvable**

• $m < n \Rightarrow$ underdetermined (A is a wide matrix) $\star$ **typ. solvable**

• $m > n \Rightarrow$ overdetermined (A is a tall matrix) $\star$ **typ. unsolvable**. **"Typical"** matrices are with $m \leq n$ and have rank $r = m$.

5 Orthogonality and Projections

5.1 Definition

Orthogonality:

• A geometric and algebraic tool in order to be able to decompose a space into subspaces.

D 5.1.1 (Orthogonal subspaces):

• Two vectors are orthogonal if their scalar product is 0: $v^\top w = \sum_{i=1}^n v_i w_i = 0$. Two subspaces are orthogonal if all v and w are orthogonal.

L 5.1.2 (Orthogonality of bases):

• Let $v_1, \ldots, v_k$ and $w_1, \ldots, w_l$ be bases of subspaces W and V. W and V are orthogonal $\Leftrightarrow$ all v_i orthogonal to all w_j

L 5.1.3 (Combinations and interaction of Orthogonal subspaces):

• The set of vectors $\{v_1, \ldots, v_k, w_1, \ldots, w_l\}$ are linearly independent.

• The union of bases of two orthogonal subspaces gives a basis for the new subspace: $\mathcal{B}_V \cup \mathcal{B}_W = \mathcal{B}_{V+W}$ and $V + W = \{\lambda v + \mu w \mid \lambda, \mu \in \mathbb{R},\ v \in V,\ w \in W\}$.

• If V and W are subspaces of $\mathbb{R}^n$, then $V + W$ is a subspace of $\mathbb{R}^n$.

• $V \cap W = \{0\}$ if subspaces are orthogonal.

• $\dim(V) = k$ and $\dim(W) = l$, then $\dim(V + W) = k + l \leq n$.

D 5.1.5 (Orthogonal complement):

• Let V be a subspace of $\mathbb{R}^n$, its **orthogonal complement**: $V^\perp = \{\, w \in \mathbb{R}^n \mid w^T v = 0 \text{ for all } v \in V \,\}$.

T 5.1.6 (Relations between subspaces):

• $N(A) = C(A^\top)^\perp = R(A)^\perp$ and $C(A^\top) = N(A)^\perp$

T 5.1.7 (Vector decomposition by orth. complements):

• $W = V^\perp \Leftrightarrow dim(V) + dim(W) = n \Leftrightarrow$ every $u \in \mathbb{R}^n$ is $u = v + w$, v and w are unique.

L 5.1.10 (Justification of exist. of sol. for normal eq.):

• Let $A \in \mathbb{R}^{m\times n}$. Then $N(A) = N(A^T A)$ and $C(A^T) = C(A^T A)$.

5.2 Projections

D 5.2.1 (Projection):

• **Projection** of $b \in \mathbb{R}^m$ on a subspace S (of $\mathbb{R}^m$) is the point in S that is closest to b: $\text{proj}_S(b) = \arg\min_{p\in S} \|b - p\|$.

L 5.2.2 (One-dimensional Projection Formula):

• Projection of b on $S = \{\, \lambda a \mid \lambda \in \mathbb{R} \,\} = C(a)$: $\text{proj}_S(b) = \frac{aa^T}{a^T a}\, b$.

•"Error vector" ($e = b - p$) is perpendicular projection: $(e = b - \text{proj}_S(b)) \perp \text{proj}_S(b)$.

L 5.2.3 (General Projection Formula):

• Let S be a subspace in $\mathbb{R}^m$ with a basis $a_1, \ldots, a_n$ that span S. Let A be the matrix with column vectors $a_1, \ldots, a_n$.

• The general formula: $\text{proj}_S(b) = A\hat{x}$, where $\hat{x}$ is $A^T A\hat{x} = A^T b$.

L 5.2.4 (Properties of $A^\top A$):

• $A^\top A$ is invertible $\Leftrightarrow$ A has linearly independent columns. $\Rightarrow$ $A^\top A$ is a square matrix, symmetric, invertible.

T 5.2.5 (Projection in terms of projection matrix):

• $\text{proj}_S(b) = Pb$ with projection matrix $P = A(A^\top A)^{-1} A^\top$.

A is matrix given in a task.

6 Applications of Orthogonality and Projections

6.1 Least Squares Approximation

Least Squares:

• Approximate a solution to System of equations: find x for which Ax is as close as possible to b: $\min_{\hat{x}\in\mathbb{R}^n} \|A\hat{x} - b\|^2$. $\hat{x}$ is unique if columns of A are linearly independent. (Fact 6.1.1)

usage:

• find $M = A^\top A$, $b' = A^\top b$, solve $M\hat{x} = b'$

Linear Regression:

• Fitting a parabola, A is obtained from $f(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2$ at given points $t_1, \ldots, t_m$.

$(t_k, b_k) = \{(0, 1), (1, 2), (2, 5)\}$, $b_k \approx \alpha_0 + \alpha_1 t_k + \alpha_2 t_k^2$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, \hat{\alpha} = (A^T A)^{-1} A^T b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \hat{b}(t) = 1 + t^2.$$

L 6.1.2:

• Matrix A ($m \times 2$) has linearly dependent columns $\Leftrightarrow t_i = t_j \forall i \neq j$.

6.2 The set of all solutions to a system of linear equations

L 6.2.1 (Injectivity of A on $C(A^\top)$, uniqueness of sol.):

• $A \in \mathbb{R}^{m\times n}$, $x, y \in C(A^\top)$: $Ax = Ay \Leftrightarrow x = y$

This leads to: $C(A^T) \cap N(A) = \{0\}$

T 6.2.2 (Set of all solution of linear equations):

• Set of all sol. : $\{x \in \mathbb{R}^n | Ax = b\} \neq \emptyset$, then:

$\{x \in \mathbb{R}^n \mid Ax = b\} = x_1 + N(A)$, $\quad x_1 \in R(A)$ is unique s.t. $Ax_1 = b$.

T 6.2.4 (Linear equations with no solution):

• Linear equations has no solution:

$\{x \in \mathbb{R}^n \mid Ax = b\} = \emptyset \iff \{z \in \mathbb{R}^m \mid A^T z = 0,\ b^T z = 1\} \neq \emptyset$.

6.3 Orthonormal Bases and Gram Schmidt

D 6.3.1 (Orthonormal vectors):

• $q_i^\top q_j = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ (orthogonal and have norm 1)

D 6.3.3 (Orthogonal Matrix):

• A square matrix $Q \in \mathbb{R}^{n\times n}$ is an *orthogonal matrix* when $Q^\top Q = I$ If it is square, then, $QQ^\top = I$, $Q^{-1} = Q^\top$, and the columns of Q form an orthonormal basis for $\mathbb{R}^n$.

• Orthogonal (rotation) matrix example: $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

P 6.3.6 (Preserving qualities of orthogonal matrices):

• Orthogonal matrices preserve norm and inner product of vectors: $\|Qx\| = \|x\|$ and $(Qx)^\top (Qy) = x^\top y$

P 6.3.7 (Least square solution to $Qx = b$):

• The least square solution to $Qx = b$, where Q is the matrix whose columns are the vectors forming the orthonomal basis of $S \subseteq \mathbb{R}^m$, is given by $\hat{x} = Q^\top b$ and the projection matrix is given by $QQ^\top$.

D 6.3.8 (Gram-Schmidt algorithm):

• **Gram-Schmidt:** used to construct orthonormal bases.

We have linearly independent vectors $a_1, \ldots, a_n$ that span a subspace S, then we can construct their orthonormal basis $q_1, \ldots, q_n$ by:

• $q_1 = \frac{a_1}{\|a_1\|}$.

• For $k = 2, \ldots, n$ do $q_k' = a_k - \sum_{i=1}^{k-1} (a_k^\top q_i)\, q_i$,

• normalise $q_k = \frac{q_k'}{\|q_k'\|}$.

D 6.3.10 (QR-Decomposition):

• $A = QR$, where $R = Q^\top A$, and Q is a matrix with orthonormal columns produced by Gram-Schmidt.

D 6.3.11 (Well-Defined QR Decomposition):

• R - upper-triangluar and invertible matrix $\Rightarrow QQ^\top A = A$, and hence, $A = QR$ is well-defined.

Simplicity of calculation with Q:

• **Projection:** $\text{proj}_{C(A)}(b) = QQ^\top b$, **Least Squares:** $R\hat{x} = Q^\top b$

This is possible because $C(A) = C(Q)$ and R is triangular - we can use back-substitution with it. R is invertible.

6.4 Pseudoinverses

D 6.4.1 (Left pseudoinverse):

• For $A \in \mathbb{R}^{m\times n}$ with full-column $\text{rank}(A) = n$, we get pseudoinverse $A^\dagger \in \mathbb{R}^{n\times m}$ as $A^\dagger = (A^\top A)^{-1} A^\top$. $A^\dagger$ is a left inverse: $A^\dagger A = I$

D 6.4.3 (Right pseudoinverse):

• For $A \in \mathbb{R}^{m\times n}$ with full row rank $\text{rank}(A) = m$ we get $A^\dagger \in \mathbb{R}^{n\times m}$ as $A^\dagger = A^\top (AA^\top)^{-1}$. $A^\dagger$ is a right inverse: $AA^\dagger = I$

D 6.4.7 (CR decomposition with pseudoinverses):

• For $A \in \mathbb{R}^{m\times n}$ with $\text{rank}(A) = r$ and a CR-decomposition $A = CR$, we define $A^\dagger = R^\dagger C^\dagger$. In general, $A^\dagger = R^T (RR^T)^{-1} (C^T C)^{-1} C^T = R^T (C^T C R R^T)^{-1} C^T = R^T (C^T A R^T)^{-1} C^T$.

L 6.4.8 (Unique solution of least sq. with pseudoinverses):

• For any matrix A and vector $b \in \mathcal{C}(A)$, the unique solution of the least squares problem is given by a vector $\hat{x} \in \mathcal{C}(A^T)$ satisfying $A\hat{x} = b$. The solution is $\hat{x} = A^\dagger b$, with $A\hat{x} = b$, and in the general case $A^\dagger = R^\dagger C^\dagger = R^T(C^TAR^T)^{-1}C^T$.

P 6.4.9 (TS decomposition):

• For $A \in \mathbb{R}^{m\times n}$ with $\text{rank}(A) = r$, let $S \in \mathbb{R}^{m\times r}$, $\quad T \in \mathbb{R}^{r\times n}$ such that $A = ST$. Then $A^\dagger = T^\dagger S^\dagger$.

T 6.4.10 (Pseudoinverses properties):

• Let $A \in \mathbb{R}^{m\times n}$. Then $AA^\dagger A = A$, $\quad A^\dagger AA^\dagger = A^\dagger$, $\quad (A^\dagger)^T = (A^T)^\dagger$.
$AA^\dagger$ is symmetric $\Rightarrow$ projection matrix onto $\mathcal{C}(A)$,
$A^\dagger A$ is symmetric $\Rightarrow$ projection matrix onto $\mathcal{C}(A^T)$.
Moreover, $AA^\dagger = CRR^T(RR^T)^{-1}(C^TC)^{-1}C^T = C(C^TC)^{-1}C^T$, which is the projection onto $\mathcal{C}(A)$, and $(AA^\dagger)^T = AA^\dagger$.

7 The Determinant

7.1 2 times 2

D 7.1.1 (2×2 Determinant):

• For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\quad \det(A) = ad - bc$.

L 7.1.2 (Multiplication of determinants):

• $\det(AB) = \det(A)\det(B)$.
Hence, for an LU-decomposition, $\det(A) = \det(L)\det(U)$.

D 7.2.1 (Permutation sign):

• The sign of a permutation is defined as the number of swaps of rows or columns. $\det(\text{permuted matrix}) = (-1)^k \det(\text{original matrix})$, where k is the number of swaps. Even number of swaps $\Rightarrow +1$, odd number $\Rightarrow -1$. $\text{sgn}(\sigma\circ\gamma) = \text{sgn}(\sigma)\,\text{sgn}(\gamma)$. For all $n \geq 2$, half of the permutations have sign $+1$, half have sign -1.

7.2 General case:

D 7.2.3 (Determinant big formula):

• For a square matrix $A \in \mathbb{R}^{n\times n}$,
$\det(A) = \sum_{\sigma\in\Pi_n} \text{sgn}(\sigma)\prod_{i=1}^n A_{i,\sigma(i)}$. *(Number of permutations: n!)*

• **Determinant Properties:**
1. Matrix $T \in \mathbb{R}^{n\times n}$ is triangular, then $\det(T) = \prod_{k=1}^n T_{kk}$, $\quad$ in particular $\det(I) = 1$.
2. Matrix $A \in \mathbb{R}^{n\times n}$, $\det(A) = \det(A^T)$.
3. Matrix $Q \in \mathbb{R}^{n\times n}$ is orthogonal $\Longrightarrow \det(Q) = 1$ or $\det(Q) = -1$.
4. Matrix $A \in \mathbb{R}^{n\times n}$ is invertible $\Longleftrightarrow \det(A) \neq 0$.
5. Matrices $A, B \in \mathbb{R}^{n\times n}$, $\det(AB) = \det(A)\det(B)$, in particular $\det(A^n) = \det(A)^n$.
6. Matrix $A \in \mathbb{R}^{n\times n}$, $\det(A^{-1}) = \frac{1}{\det(A)}$.
7. $\det(\lambda A) = \lambda^n \det(A)$.

P 7.2.4 (Determinant of orthogonal matrices):

• $1 = \det(I) = \det(Q^TQ) = \det(Q^T)\det(Q) = \det(Q)^2$, so $\det(Q) = \pm 1$. If $\det(Q) = 1$, then Q is a rotation matrix. If $\det(Q) = -1$, then Q is a reflection matrix.

P 7.3.2 (Cofactor determinant calculation):

• **Co-factor method**:
$\det(A) = \sum_{j=1}^n A_{ij}C_{ij}$, where cofactors are $C_{ij} = (-1)^{i+j}\det(\mathcal{A}_{ij})$,.

P 7.3.5 (Cramer's Rule):

• **Cramer's Rule**: For $Ax = b$ with $\det(A) \neq 0$, $x_j = \frac{\det(\mathscr{B}_j)}{\det(A)}$, where $\mathscr{B}_j$ is the matrix obtained from A by replacing the j-th column with b.

P 7.3.7 (Linearity of a determinant):

• The determinant is linear in each row (and column). For example, $\det\begin{bmatrix}\alpha_0 a_0^T + \alpha_1 a_1^T \\ a_2^T\end{bmatrix} = \alpha_0 \det\begin{bmatrix}a_0^T \\ a_2^T\end{bmatrix} + \alpha_1\det\begin{bmatrix}a_1^T \\ a_2^T\end{bmatrix}$.

8 Eigenvalues and Eigenvectors

8.1 Complex Numbers

1. Solve $x^2 + 1 = 0 \Rightarrow x = \sqrt{-1} \Rightarrow \mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}$.
2. $(a+ib) + (x+iy) = (a+x) + i(b+y)$,
3. $(a+ib)(x+iy) = (ax - by) + i(ay + bx)$,
4. $(a+ib)(a-ib) = a^2 + b^2$.
5. $\frac{a+ib}{x+iy} = \frac{(a+ib)(x-iy)}{x^2+y^2} = \frac{ax+by}{x^2+y^2} + i\frac{bx-ay}{x^2+y^2}$.
6. $|z| = \sqrt{a^2+b^2}$, $\quad z = a + ib$,
7. $\overline{a+ib} = a - ib$.

R 8.1.1 (Euler's formula):

• For $\theta \in \mathbb{R}$, $e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow e^{i\pi} = -1$

Polar form of a complex number:

• $z = re^{i\theta}$, $\quad z \in \mathbb{C}$, $\quad r > 0$ is the modulus of z, $\quad \theta \in [0, 2\pi)$.

T 8.1.2 (Fundamental Theorem of Algebra):

• Any degree n non-constant $(n \geq 1)$ polynomial $P(z) = \alpha_n z^n + \alpha_{n-1}z^{n-1} + \cdots + \alpha_1 z + \alpha_0$, $\quad (\alpha_n \neq 0)$ has a zero: there exists $\lambda \in \mathbb{C}$ such that $P(\lambda) = 0$.
$\Rightarrow$ A degree-n polynomial has at most n distinct zeros (roots).

C 8.1.3 (Algebraic multiplicity, num. of 0 in polynomial):

• Any degree n non-constant $(n \geq 1)$ polynomial has n zeros $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, and $P(z) = \alpha_n(z - \lambda_1)(z - \lambda_2)\cdots(z - \lambda_n)$. The number of times $\lambda \in \mathbb{C}$ appears in the expression is called the *algebraic multiplicity* of the zero.

Inner product on $\mathbb{C}^n$:

• The inner product on $\mathbb{C}^n$ is given by $\langle v, w\rangle = w^*v$.

Conjugate transpose:

• $A^* = \overline{A}^T$.

8.2 Introduction to Eigenvalues and Eigenvectors

D 8.2.1 (EW/EV pair):

• Given $A \in \mathbb{R}^{n\times n}$, we say $\lambda \in \mathbb{C}$ is an *eigenvalue* of A and $v \in \mathbb{C}^n \setminus \{0\}$ is an *eigenvector* of A associated with λ when $Av = \lambda v$. (λ, v) is an eigenvalue–eigenvector pair. If $\lambda \in \mathbb{R}$, then we have a real eigenvalue–eigenvector pair.

L 8.2.3 (Real EW/EV):

• Let $A \in \mathbb{R}^{n\times n}$. Then $\lambda \in \mathbb{R}$ is a real eigenvalue of A if and only if $\det(A - \lambda I) = 0$. A vector $v \in \mathbb{R}^n \setminus \{0\}$ is an eigenvector associated with λ if and only if $v \in \mathcal{N}(A - \lambda I)$.

D 8.3.4 (Characteristic Polynomial):

• The characteristic polynomial: $(-1)^n\det(A - zI) = \det(zI - A) = (z - \lambda_1)(z - \lambda_2)\cdots(z - \lambda_n)$. The coefficient of z^n is $(-1)^n$.

T 8.2.5 (Existence of EW):

• Every matrix $A \in \mathbb{R}^{n\times n}$ has an eigenvalue (possibly complex-valued).

P 8.2.7 (EW of orthogonal matrix):

• If $Q \in \mathbb{R}^{n\times n}$ is orthogonal and $\lambda \in \mathbb{C}$ is an eigenvalue of Q, then $|\lambda| = 1$.

L 8.2.8 (Complex EW exist in conjugate pairs for real A):

• Let $A \in \mathbb{R}^{n\times n}$. If (λ, v) is an eigenvalue–eigenvector pair, then $(\overline{\lambda}, \overline{v})$ is also an eigenvalue–eigenvector pair.

8.3 Properties of Eigenvalues and Eigenvectors

P 8.3.1 (EW modifications based on types of a matrix):

• If (λ, v) is an eigenvalue–eigenvector pair of A, then (λ^k, v) is an eigenvalue–eigenvector pair of A^k for $k \geq 1$.
• If (λ, v) is an eigenvalue–eigenvector pair of A with $\lambda \neq 0$, then $\left(\frac{1}{\lambda}, v\right)$ is an eigenvalue–eigenvector pair of A^{-1}.

L 8.3.2 (Linear independence):

• If $\lambda_1, \ldots, \lambda_n$ are all distinct, the corresponding eigenvectors $v_1, \ldots, v_n$ are linearly independent.

T 8.3.3 (Existence of a basis from EV):

• Let $A \in \mathbb{R}^{n\times n}$ with n distinct real eigenvalues. Then there exists a basis of $\mathbb{R}^n$, $v_1, \ldots, v_n$, made of eigenvectors of A.

D 8.3.4 (Trace of a matrix):

• The trace of A is defined by $\text{Tr}(A) = \sum_{i=1}^n A_{ii}$.

L 8.3.5 (Transposition equality of EW):

• The eigenvalues of $A \in \mathbb{R}^{n\times n}$ are the same as those of A^T But the eigenvectors can be different.

L 8.3.6 (Determinant and Trace via EW):

• Let $A \in \mathbb{R}^{n\times n}$ and let $\lambda_1, \ldots, \lambda_n$ be its eigenvalues as they appear in the characteristic polynomial. Then
$\det(A) = \prod_{i=1}^n \lambda_i$, $\text{Tr}(A) = \sum_{i=1}^n \lambda_i$.

L 8.3.7 (Cyclic invariance of the trace):

• For $A, B, C \in \mathbb{R}^{n\times n}$:
$\text{Tr}(AB) = \text{Tr}(BA)$, and $\text{Tr}(ABC) = \text{Tr}(BCA) = \text{Tr}(CAB)$.

9 Diagonizable Matrices, Singular Value Decomposition

9.1 Diagonalization

T 9.1.1 (Diagonalization Theorem, ability changing basis):

• $A = V\Lambda V^{-1}$, where V's columns are its eigenvectors and Λ is a diagonal matrix with $\Lambda_{ii} = \lambda_i$ and all other entries 0. $A \in \mathbb{R}^{n\times n}$ and has to have a complete set of real eigenvectors (eigenbasis).
Equivalently, $\Lambda = V^{-1}AV$, since V is invertible.

$$\text{Std. coord.} \xrightarrow{V^{-1}} \text{EV. coord.} \xrightarrow{\Lambda} \text{EV. coord.} \xrightarrow{V} \text{Std. coord.}$$

D 9.1.2 (Diagonalizable matrix):

• A matrix $A \in \mathbb{R}^{n\times n}$ is called *diagonalizable* if there exists an invertible matrix V such that $V^{-1}AV = \Lambda$, where Λ is a diagonal matrix.

D 9.1.3 (Complete set of EV):

• If we can find eigenvectors forming a basis of $\mathbb{R}^n$ for A, we say that A has a *complete set of real eigenvectors*.

P 9.1.6 (Projection and EW/EV):

• Let P be a projection matrix onto a subspace $U \subset \mathbb{R}^n$. Then P has two eigenvalues, 0 and 1, and a complete set of real eigenvectors.

D 9.1.7 (Similar matrices):

• Matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ are called *similar* if there exists an invertible matrix S such that $B = S^{-1}AS$. **P 9.1.8**: Similar matrices have the same eigenvalues.

D 9.1.10 (Geometric multiplicity):

• Let $A \in \mathbb{R}^{n\times n}$ and let λ be an eigenvalue of A. Then $\dim\mathcal{N}(A - \lambda I)$ is called the *geometric multiplicity* of λ.

L 9.1.11 (Complete set of real EV):

• A matrix has a complete set of real eigenvectors if and only if all its eigenvalues are real and the geometric multiplicities equal the algebraic multiplicities for all eigenvalues.

9.2 Symmetric Matrices, Spectral Theorem

T 9.2.1 (Spectral Theorem):

• Any symmetric matrix $A \in \mathbb{R}^{n\times n}$ has n real eigenvalues and an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

C 9.2.2 (Eigendecomposition):

• For any symmetric matrix $A \in \mathbb{R}^{n\times n}$, there exists an orthogonal matrix $V \in \mathbb{R}^{n\times n}$ (whose columns are eigenvectors of A) such that $A = V\Lambda V^T$, where $\Lambda \in \mathbb{R}^{n\times n}$ is diagonal with diagonal entries equal to the eigenvalues of A, and $V^TV = I$. This decomposition is called the *eigendecomposition*.

C 9.2.4 (Rank of real symmetric matrix):

- If A is a real symmetric matrix, then $\operatorname{rank}(A)$ is the number of nonzero eigenvalues of A (counting repetitions).
- For a general $n \times n$ matrix, $\operatorname{rank}(A) = n - \dim \mathcal{N}(A)$, so the geometric multiplicity of the eigenvalue $\lambda = 0$ equals $\dim \mathcal{N}(A)$.

P 9.2.6 (Rank-One Spectral Decomposition):

- Let $A \in \mathbb{R}^{n\times n}$ be symmetric, and let $v_1, \dots, v_n$ be an orthonormal basis of eigenvectors of A (the columns of V), with associated eigenvalues $\lambda_1, \dots, \lambda_n$. Then $A = \sum_{k=1}^{n} \lambda_k\, v_k v_k^T$.

A real symmetric matrix is a weighted sum of orthogonal projections onto its eigenvector directions, with weights given by the eigenvalues.

L 9.2.7 (Orthogonality of EV):

- Let $A \in \mathbb{R}^{n\times n}$ be symmetric and let $\lambda_1 \neq \lambda_2 \in \mathbb{R}$ be two distinct eigenvalues of A with corresponding eigenvectors v_1, v_2. Then v_1 and v_2 are orthogonal.

L 9.2.8 (Symmetric matrix has real EW):

- A symmetric matrix $A \in \mathbb{R}^{n\times n}$ has only real eigenvalues: $\lambda \in \mathbb{C} \quad \Rightarrow \lambda \in \mathbb{R}$. Indeed, if $Av = \lambda v$:
$\overline{\lambda}\|v\|^2 = \overline{\lambda}\, v^* v = (\lambda v)^* v = (Av)^* v = v^* A^T v = v^* A v = v^* \lambda v = \lambda \|v\|^2$. Thus $\lambda = \overline{\lambda}$, so $\lambda \in \mathbb{R}$. $\Rightarrow$ every symmetric matrix $A \in \mathbb{R}^{n\times n}$ has a real eigenvalue. (C 9.2.9)

P 9.2.10 (Rayleigh Quotient):

- $A \in \mathbb{R}^{n\times n}$ is symmetric. For $x \in \mathbb{R}^n \setminus \{0\}$, the Rayleigh quotient $R(x) = \frac{x^T A x}{x^T x}$.

The minimum of $R = R(v_{\min}) = \lambda_{\min}$, and the maximum $R(v_{\max}) = \lambda_{\max}$. Here $\lambda_{\max}/\lambda_{\min}$ are the largest/smallest eigenvalues of A, and $v_{\max}/v_{\min}$ their associated eigenvectors.

D 9.2.11 (PSD and PD matrices):

- $A = A^\top$ • $A \succeq 0$ **(PSD)** $\Leftrightarrow \lambda_i(A) \geq 0$ • $A \succ 0$ **(PD)** $\Leftrightarrow \lambda_i(A) > 0$.

P 9.2.12 (Positivity of the quadratic form):

- Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Then $A \succeq 0 \iff x^\top A x \geq 0 \quad \forall x \in \mathbb{R}^n$, and $A \succ 0 \iff x^\top A x > 0 \quad \forall x \neq 0$.

D 9.2.13 (Gram Matrix):

- Given vectors $v_1, \dots, v_n \in \mathbb{R}^m$, their *Gram matrix* is $G \in \mathbb{R}^{n\times n}$ defined by $G_{ij} = v_i^\top v_j$. If $V = [v_1 \cdots v_n] \in \mathbb{R}^{m\times n}$, then $G = V^\top V$.
- If $A = [a_1 \cdots a_n] \in \mathbb{R}^{m\times n}$, one also calls $AA^\top$ a Gram matrix; note that $AA^\top = \sum_{i=1}^{n} a_i a_i^\top$. It is $m \times m$ matrix.

P 9.2.15 (Same EV of transposed matrices):

- For a real matrix $A \in \mathbb{R}^{m\times n}$, the non-zero eigenvalues of $A^\top A \in \mathbb{R}^{n\times n}$ and $AA^\top \in \mathbb{R}^{m\times m}$ are the same. Also both are symmetric and PSD.

P 9.2.16 (Cholesky Decomposition):

- Every symmetric PSD matrix M is a Gram matrix of upper-triangular matrix C: $M = C^\top C$.

9.3 Singular Value Decomposition

D 9.3.1 (Singular Value Decomposition):

- Let $A \in \mathbb{R}^{m\times n}$. There exist orthogonal matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ and a diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ with nonnegative diagonal entries $\sigma_1 \geq \cdots \geq \sigma_{\min(m,n)}$ such that

$$A = U\Sigma V^\top.$$

The columns of U and V are called the left and right singular vectors of A, and the diagonal entries of Σ are the singular values of A. Columns of U are the eigenvectors of $AA^\top$ and columns of V are the eigenvectors of $A^\top A$. Σ has the square root of the eigenvalues of $AA^\top$ / $A^\top A$ (they are the same). U has the dimensions mxm and V has the dimensions nxn. U, V are orthogonal matrices.

R 9.3.2 (Compact form of SVD):

- If $\operatorname{rank}(A) = r$, then the SVD can be written as

$$A = U_r \Sigma_r V_r^\top,$$

where $U_r \in \mathbb{R}^{m\times r}$ and $V_r \in \mathbb{R}^{n\times r}$ have orthonormal columns, and $\Sigma_r = \operatorname{diag}(\sigma_1, \dots, \sigma_r)$. This representation stores $r(m+n+1)$ real numbers instead of mn. For small r, this yields substantial savings and motivates low-rank approximations.

T 9.3.3 (Every matrix has SVD):

- Every matrix $A \in \mathbb{R}^{m\times n}$ has SVD: $A = U\Sigma V^\top$. Equivalently, every linear transformation is diagonal in orthonormal bases of singular vectors.

P 9.3.4 (SVD as a sum of rank-one matrices):

- Let $A \in \mathbb{R}^{m\times n}$ have rank r, with singular values $\sigma_1, \dots, \sigma_r$ and corresponding singular vectors $u_1, \dots, u_r$ and $v_1, \dots, v_r$. Then

$$A = \sum_{k=1}^{r} \sigma_k\, u_k v_k^\top.$$

Main idea: *We can write any rank-r matrix $A \in \mathbb{R}^{m\times n}$ as a sum of r rank-1 matrices.*

SVD of the Inverse A^{-1}:

- If $A = U\Sigma V^\top$, $\quad \Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_n)$., then the SVD of the inverse can be written as

$$A^{-1} = V\Sigma^{-1}U^\top.$$

where

$$\Sigma^{-1} = \operatorname{diag}\Big(\frac{1}{\sigma_1}, \dots, \frac{1}{\sigma_n}\Big).$$

Algorithms

Gaussian Elimination.

Given $Ax = b$, form the augmented matrix $[A \,|\, b]$ and apply elementary row operations to reach row echelon form (REF): pivot $\to$ swap $\to$ eliminate below $\to$ repeat. If a row $(0 \cdots 0 \,|\, c)$ with $c \neq 0$ appears, the system is inconsistent; otherwise solve by back-substitution.

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 1 & 4 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -1 & -2 \end{array}\right] \Rightarrow (x, y) = (1, 2).$$

Gauss-Jordan Elimination.

Starting from $[A \,|\, b]$, apply Gaussian elimination, then normalize each pivot to 1 and eliminate all other entries in the pivot columns (pivot columns must be basis vectors). The resulting reduced row echelon form (RREF) gives the solution directly. Pivot positions move strictly to the right as you go down the rows. Solve:

$$\begin{cases} x + y = 3, \\ 2x + y = 4. \end{cases} \iff \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 1 & 4 \end{array}\right]$$

Row-reduce to RREF:

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 1 & 4 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow -R_2} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right].$$

$$x = 1, \qquad y = 2.$$

Inverse via Gauss–Jordan.

To compute A^{-1}, form the augmented matrix $[A \,|\, I]$ and apply Gauss–Jordan elimination. If

$$[A \,|\, I] \longrightarrow [I \,|\, B],$$

then $B = A^{-1}$. If I cannot be obtained on the left, A is not invertible.

$$[A \,|\, I] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 3R_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow -\frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right].$$

Fitting a line with least squares.

$$\hat{\alpha} \;=\; \arg\min_{\alpha \in \mathbb{R}^2} \|A\alpha - b\|^2 \;=\; (A^TA)^{-1}A^Tb, \quad A \;=\; \begin{pmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_m \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \hat{\alpha} = (A^TA)^{-1}A^Tb = \begin{pmatrix} \frac{7}{6} \\ \frac{1}{2} \end{pmatrix}, \hat{b}(t) = \tfrac{7}{6} + \tfrac{1}{2}t.$$

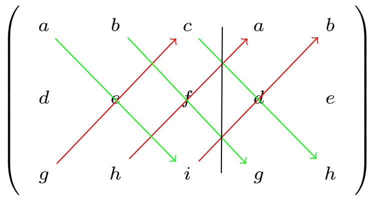
Forming orthonormal basis via Gram-Schmidt.

Gram-Schmidt used to construct orthonormal bases.
We have linearly independent vectors $a_1, \dots, a_n$ that span a subspace S, then we can construct their orthonormal basis $q_1, \dots, q_n$ by:

- $q_1 = \frac{a_1}{\|a_1\|}$.
- For $k = 2, \dots, n$ do $q'_k = a_k - \sum_{i=1}^{k-1} (a_k^\top q_i)\, q_i$,
- normalise $q_k = \frac{q'_k}{\|q'_k\|}$.

Rule of Sarrus (Computing 3x3 Determinants).

The green arrows represent sum1 = aei + bfg + cdh and the red arrows represent sum2 = gec + hfa + idb. The determinant is given by $\det(A) = \text{sum1} - \text{sum2}$.



Solving Linear Recurrences via Matrix Diagonalization

We are given the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2}, \quad \text{for } n \geq 2.$$

Using the given formula, we can derive a matrix M such that

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ a_{n-2} \end{pmatrix}, \quad M = \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix}, \ \mathbf{g}_n = \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix},$$

With initial vector $\mathbf{g}_0 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, we have $\mathbf{g}_n = M^n \mathbf{g}_0$.

Eigenvalues of M

We compute $\det(M - \lambda I) = (5 - \lambda)(-\lambda) + 6 = \lambda^2 - 5\lambda + 6$.
Solving, $\lambda^2 - 5\lambda + 6 = 0 \Rightarrow (\lambda - 3)(\lambda - 2) = 0$.
Hence, $\lambda_1 = 3, \quad \lambda_2 = 2$.

Eigenvectors
For $\lambda = 3$:

$$(M-3I) = \begin{pmatrix} 2 & -6 \\ 1 & -3 \end{pmatrix}.$$

Solving $(M-3I)\mathbf{v} = 0$ gives $2x - 6y = 0 \;\Rightarrow\; x = 3y$, so EV $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.
For $\lambda = 2$:

$$(M-2I) = \begin{pmatrix} 3 & -6 \\ 1 & -2 \end{pmatrix}.$$

Solving $(M-2I)\mathbf{v} = 0$ gives $x - 2y = 0 \;\Rightarrow\; x = 2y$, so EV $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Closed Form
Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, we can write

$$\mathbf{g}_0 = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2.$$

That is,

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \alpha_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Solving,

$$\alpha_1 = 2, \quad \alpha_2 = -2.$$

Therefore, $\mathbf{g}_n = \alpha_1 3^n \mathbf{v}_1 + \alpha_2 2^n \mathbf{v}_2 = 2 \cdot 3^n \begin{pmatrix} 3 \\ 1 \end{pmatrix} - 2^{n+1} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.
Thus, $a_n = 2 \cdot 3^n - 2^{n+1}$.

Quizzes - Computations

Basis of a plane and related subspaces:

- $$\text{Plane } P = \{x \in \mathbb{R}^3 : 6x_1 - x_2 + 5x_3 = 0\}$$

1. Basis for P
Solve for x_2:

$$x_2 = 6x_1 + 5x_3$$

Let $x_1 = s,\ x_3 = t$:

$$(x_1, x_2, x_3) = (s, 6s + 5t, t) = s(1, 6, 0) + t(0, 5, 1)$$

$$\boxed{\mathcal{B}_P = \{(1, 6, 0), (0, 5, 1)\}}$$

2. Intersection with $\operatorname{span}\{e_1, e_2\}$

$$x_3 = 0 \Rightarrow 6x_1 - x_2 = 0 \Rightarrow x_2 = 6x_1$$

$$(x_1, x_2, x_3) = s(1, 6, 0)$$

$$\boxed{\mathcal{B}_{P \cap \operatorname{span}\{e_1, e_2\}} = \{(1, 6, 0)\}}$$

3. Perpendicular vectors to P
Normal vector from plane equation:

$$\mathbf{n} = (6, -1, 5)$$

$$P^\perp = \operatorname{span}\{(6, -1, 5)\}$$

$$\boxed{\mathcal{B}_{P^\perp} = \{(6, -1, 5)\}}$$

Compute bases for orthogonal spaces:

- **Finding** $S^\perp \subset \mathbb{R}^3$ **when** $S = \operatorname{span}\{v\}$

Let $v = (a, b, c) \neq 0$ and $S = \operatorname{span}\{v\}$. Then

$$S^\perp = \{u \in \mathbb{R}^3 : u \cdot v = 0\}.$$

Let $u = (x, y, z)$. Orthogonality gives

$$ax + by + cz = 0.$$

Solve for one variable and parametrize. Choose convenient values for the free variables to obtain two linearly independent solutions.
These two vectors form a basis for $S^\perp$.
Example: v=(-6,-9,7)

$$-6x - 9y + 7z = 0$$

Choose $(y, z) = (2, 0)$ and $(0, 6)$:

$$u_1 = (-3, 2, 0), \quad u_2 = (7, 0, 6).$$

Calculating four fundamental subspaces:

- **Four Fundamental Subspaces**

Let $A \in \mathbb{R}^{m \times n}$ with $\operatorname{rank}(A) = r$.

$$\dim \mathcal{C}(A) = r$$
$$\dim \mathcal{C}(A^T) = r$$
$$\dim \mathcal{N}(A) = n - r$$
$$\dim \mathcal{N}(A^T) = m - r$$

$$\mathcal{C}(A) \perp \mathcal{N}(A^T), \quad \mathcal{C}(A^T) \perp \mathcal{N}(A)$$

Pseudoinverse of diagonal matrix:

- Diagonal matrix $\Rightarrow$ invert nonzero diagonals, keep zeros.

Diagonalization of a symmetric matrix:

- **Diagonalization of a symmetric matrix (example)** Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Characteristic polynomial:

$$p_A(\lambda) = \det(A - \lambda I) = (2 - \lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1).$$

Eigenvalues:

$$\lambda_1 = 3, \quad \lambda_2 = 1.$$

Eigenvectors:

$$\lambda_1 = 3 : \; v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \lambda_2 = 1 : \; v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The eigenvectors are orthogonal:

$$v_1 \cdot v_2 = 0.$$

Define

$$V = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$A = V \Lambda V^{-1}, \qquad V^{-1} = (V^T V)^{-1} V^T = \tfrac{1}{2} V^T.$$

Remark: The order of eigenvalues on the diagonal of Λ is arbitrary. Reordering eigenvalues requires the same reordering of eigenvectors.

Computing singular values:

- **Computing singular values**

For a matrix $A \in \mathbb{R}^{m \times n}$, the singular values are

$$\sigma_i = \sqrt{\lambda_i},$$

where λ_i are the eigenvalues of $A^T A$.
Example:

$$A = \begin{pmatrix} 3 & 0 \\ 4 & 0 \end{pmatrix}.$$

Compute

$$A^T A = \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 25 & 0 \\ 0 & 0 \end{pmatrix}.$$

Eigenvalues of $A^T A$:

$$\lambda_1 = 25, \quad \lambda_2 = 0.$$

Singular values:

$$\sigma_1 = \sqrt{25} = 5, \quad \sigma_2 = \sqrt{0} = 0.$$

$$\boxed{\text{Singular values of } A = \sqrt{\text{eigenvalues of } A^T A}.}$$

Why determinant expansion along a row/column works:

- **Why determinant expansion along a row/column works**

For any $n \times n$ matrix $M = (m_{ij})$, the determinant can be expanded along any row i or any column j (Laplace expansion):

$$\det(M) = \sum_{j=1}^{n} m_{ij} C_{ij} \quad \text{or} \quad \det(M) = \sum_{i=1}^{n} m_{ij} C_{ij},$$

where

$$C_{ij} = (-1)^{i+j} \det(M_{ij})$$

is the cofactor, and M_{ij} is obtained by deleting row i and column j.

Example:

$$\lambda I - A = \begin{pmatrix} \lambda - \frac{9}{2} & 0 & -\frac{1}{2} \\ 0 & \lambda & 0 \\ -\frac{1}{2} & 0 & \lambda - \frac{9}{2} \end{pmatrix}.$$

Expanding along row 2:

$$\det(\lambda I - A) = 0 \cdot C_{21} + \lambda \cdot C_{22} + 0 \cdot C_{23} = \lambda\, C_{22}.$$

Cofactor computation:

$$C_{22} = (-1)^{2+2} \det \begin{pmatrix} \lambda - \frac{9}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda - \frac{9}{2} \end{pmatrix}.$$

Since $(-1)^{2+2} = (-1)^4 = 1$, we obtain

$$C_{22} = \det \begin{pmatrix} \lambda - \frac{9}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda - \frac{9}{2} \end{pmatrix}.$$

Therefore,

$$\det(\lambda I - A) = \lambda \det \begin{pmatrix} \lambda - \frac{9}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda - \frac{9}{2} \end{pmatrix}.$$

Conclusion: Expanding along rows or columns with many zeros is always valid and simplifies determinant computations.

Fast computation of singular values (symmetric case):

- **Fast computation of singular values (symmetric case)** Let $A \in \mathbb{R}^{n \times n}$.

Key fact: If $A = A^T$ (i.e. A is symmetric), then

$$\boxed{\sigma_i(A) = |\lambda_i(A)|},$$

where $\lambda_i(A)$ are the eigenvalues of A. **Reason:** Singular values are defined by

$$\sigma_i(A) = \sqrt{\lambda_i(A^T A)}.$$

If $A = A^T$, then

$$A^T A = A^2,$$

and eigenvalues of A^2 are $\lambda_i(A)^2$. Hence

$$\sigma_i(A) = \sqrt{\lambda_i(A)^2} = |\lambda_i(A)|.$$

Example:

$$A = \begin{pmatrix} \frac{9}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{9}{2} \end{pmatrix} \quad \text{(symmetric)}.$$

Eigenvalues:

$$\lambda(A) = \{5,\ 4,\ 0\}.$$

Singular values:

$$\boxed{\sigma(A) = \{5,\ 4,\ 0\}}.$$

General fallback (always works): If A is not symmetric,

compute eigenvalues of $A^T A$ and take square roots.

Remember:

$$\boxed{A = A^T \ \Rightarrow\ \sigma_i = |\lambda_i|.}$$

Similar (2x2) matrices:

- Similar (2x2) matrices always have the same eigenvalues. It is the fast way to check if matrices are similar.

Norm of a vector:

- Norm (2) of a vector:

$$\boxed{\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}}$$

Characteristic polynomial & eigenvalues & eigenvectors:

- We can find Characteristic polynomial via:

$$\boxed{\det(\lambda I - A) = (-1)^n \det(A - \lambda I)}$$

- We can find corresponding eigenvectors $\mathbf{v}$ to eigenvalues λ via:

$$\boxed{(A - \lambda I)\mathbf{v} = 0}$$

Distance between vector and its projection:

- Distance between vector and its projection is:

$$\boxed{\|\text{vector} - \text{projection}\| = \|\mathbf{v} - \text{proj}_S(\mathbf{v})\|}$$

Angle between vectors:

- Angle between vectors:

$$\boxed{\cos(\theta) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)}$$

When T is linear transformation:

- When T is linear transformation, then:

$$\boxed{2\,T(\mathbf{x}) + 3\,T(\mathbf{y}) = T(2\mathbf{x} + 3\mathbf{y})}$$

This might simplify some calculations a lot.

Inverse of 2×2 matrix:

- Inverse of 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det(A) \neq 0$:

$$\boxed{A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

Determining when matrix returns to identity:

- If $A^m = I$ and $A^n = I$, then

$$A^{\gcd(m,n)} = I.$$

Proof sketch to template:
Let $d = \gcd(m,n)$. Then there exist integers x, y with $d = xm + yn$ (Bézout). Assuming A is invertible (true in any group; for matrices this means $A \in GL$),

$$A^d = A^{xm+yn} = (A^m)^x (A^n)^y = I^x I^y = I.$$

When can you conclude $A = I$? You can conclude $A = I$ if $\gcd(m,n) = 1$, because then $A^1 = A = I$. Otherwise, you can only conclude that the order of A divides $\gcd(m,n)$ (i.e., A is a root of unity of that exponent).

Pairwise linear independence:

- Pairwise independence is weaker than (joint) linear independence. $\Rightarrow$ Linear independence is a global property: checking vectors two at a time is not enough. Collectively, they might still fail independence.

Geometrically in $\mathbb{R}^2$: you can have infinitely many vectors that are pairwise non-collinear, but at most two vectors can be linearly independent.

Complex expression and geometric Interpretation:

- Using $z\bar{z} = |z|^2$, the condition reduces to $x^2 + y^2 = 1$, which describes the unit circle.

SVD of rank-1 matrix:

- SVD of rank-1 matrix:

$$A = \sigma u v^\top$$

Where:

1. σ is only on non-zero singular value
2. u is a unit column vector (left singular vector)
3. v is a unit column vector (right singular vector)

More *specifically*, if

$$A = x\,y^T$$

with $x \in \mathbb{R}^m,\ y \in \mathbb{R}^n$, then

$$A = \sigma\,u\,v^T$$

where

$$\sigma = \|x\|\,\|y\|, \quad u = \frac{x}{\|x\|}, \quad v = \frac{y}{\|y\|}$$

Example:

$$A = \begin{bmatrix} \sqrt{6} \\ 3 \\ -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\|x\| = \sqrt{6 + 9 + 1} = 4, \quad \|y\| = 1$$

$$A = \begin{bmatrix} \frac{\sqrt{6}}{4} \\ \frac{3}{4} \\ -\frac{1}{4} \end{bmatrix} [4] \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\boxed{\text{Unique non-zero singular value } \sigma = 4}$$

SVD validity check:

- To verify whether a proposed factorization $A = U\Sigma V^\top$ is a valid SVD, it suffices to check the following:
  - **Orthogonality:** U (and V) has orthonormal columns, i.e. $U^\top U = I$ and $V^\top V = I$;
  - **Singular values:** Σ is diagonal with non-negative entries;
  - **Dimensions:** if $A \in \mathbb{R}^{m\times n}$, then $\Sigma \in \mathbb{R}^{m\times n}$.

Invertible matrix and EW:

- A matrix is invertible iff 0 is not an eigenvalue.
- $A = A^T \Rightarrow A$ has real eigenvalues $\Rightarrow \lambda + i \neq 0 \Rightarrow \det(A + iI) \neq 0 \Rightarrow A + iI$ is invertible