A2

1a) Compute the cost at each level, assume $n = 3^k$

i)

level	# nodes	cost per node	total cost
0	1	$n^{3/2}$	$n^{3/2}$
1	5	$(n/3)^{3/2}$	$5(n/3)^{3/2}$
2	25	$(n/9)^{3/2}$	$25(n/9)^{3/2}$
$\vdots$			
i	5^i	$(n/3^i)^{3/2}$	$5^i(n/3^i)^{3/2}$
$\vdots$			
$k-1$	5^{k-1}	$3^{3/2}$	$5^{k-1} \cdot 3^{3/2}$
k	5^k	4	$4 \cdot 5^k$

Total cost is: $$4 \cdot 5^k + \sum_{i=0}^{k-1} 5^i (n/3^i)^{3/2} = 4 \cdot 5^k + n^{3/2} \sum_{i=0}^{k-1} \left(\frac{5}{3\sqrt{3}}\right)^i$$

$$= 4 \cdot 5^k + n^{3/2} \left(\frac{1 - \left(\frac{5}{3\sqrt{3}}\right)^k}{1 - \left(\frac{5}{3\sqrt{3}}\right)} \right)$$

However, $\frac{5}{3\sqrt{3}} \approx 0.962 < 1$ so $\lim_{k \to \infty} \left(\frac{5}{3\sqrt{3}}\right)^k = 0$ ~~and~~

and $\left(\frac{1 - \left(\frac{5}{3\sqrt{3}}\right)^k}{1 - \frac{5}{3\sqrt{3}}} \right) \in \Theta(1)$

Also $5^k = 5^{\log_3 n} = n^{\log_3 5} \approx n^{1.4649}$

$\therefore T(n) \in \Theta(n^{\log_3 5}) + \Theta(n^{3/2}) \in \Theta(n^{3/2})$

a

ii) # levels in recursion tree is $K+1$ where $K = \log_{7/3} n$

level	# nodes	cost per node	total cost
0	1	n^2	n^2
1	6	$\left(\frac{3n}{7}\right)^2$	$6 \cdot \left(\frac{3n}{7}\right)^2$
2	36	$\left(\frac{9n}{49}\right)^2$	$36 \cdot \left(\frac{9n}{49}\right)^2$
$\vdots$			
i	6^i	$\left(\frac{3^i n}{7^i}\right)^2$	$6^i \cdot \left(\frac{3^i n}{7^i}\right)^2$
$\vdots$			
$K-1$	6^{K-1}	$\left(\frac{3^{K-1} n}{7^{K-1}}\right)^2$	$6^{K-1} \left(\frac{3^{K-1} n}{7^{K-1}}\right)^2$
K	6^K	2	$2 \cdot 6^K$

Total cost is $2 \cdot 6^K + \sum_{i=0}^{K-1} 6^i (3^i n / 7^i)^2 = 2 \cdot 6^K + n^2 \sum_{i=0}^{K-1} \left(\frac{54}{49}\right)^i$

$$= 2 \cdot 6^K + n^2 \left(\frac{\left(\frac{54}{49}\right)^K - 1}{\frac{54}{49} - 1} \right)$$

$54/49 > 1$ so $\left(\frac{\left(\frac{54}{49}\right)^K - 1}{\frac{54}{49} - 1} \right) \in \Theta\left(\left(\frac{54}{49}\right)^K \right)$

Also $6^K = 6^{\log_{7/3} n} = n^{\log_{7/3} 6}$

and $\left(\frac{54}{49}\right)^K = \left(\frac{54}{49}\right)^{\log_{7/3} n} = n^{\log_{7/3} (54/49)} = n^{\log_{7/3} 6 - 2}$

$\therefore$ $T(n) \in \Theta(n^{\log_{7/3} 6}) + \Theta(n^2 n^{\log_{7/3} 6 - 2}) \in \Theta(n^{\log_{7/3} 6})$

b) i) $f(n) = n^{3/2}$, $a = 5$, $b = 3$, $\log_b a \approx 1.4649$ $\Rightarrow$ case 3: $T(n) \in \Theta(n^{1.5})$

ii) $f(n) = n^2$, $a = 6$, $b = 7/3$, $\log_b a \approx 2.11$

$\Rightarrow$ case 1: $T(n) \in \Theta(n^{\log_{7/3} 6})$

2. Prove something stronger: $T(n) \le cn - d\sqrt{n}$

Basis: $n = 2$ ① $T(2) = 2T(1) + \sqrt{2} = 6 + \sqrt{2}$

② $c2 - d\sqrt{2}$

so $T(n) \le cn - d\sqrt{n}$ for $n = 2$

if $c \ge 3$, $d \ge 1$ or more generally

for $d > 0$, $c \ge 3 + \frac{(d+1)}{\sqrt{2}}$

Induction Step

n even: $T(n) = 2T(\lfloor \frac{n}{2} \rfloor) + \sqrt{n} \le 2(c\frac{n}{2} - d\sqrt{\tfrac{n}{2}}) + \sqrt{n}$

$\le cn - \frac{2d}{\sqrt{2}}\sqrt{n} + \sqrt{n}$

$\le cn + (1 - \frac{2d}{\sqrt{2}})\sqrt{n}$

we want to show $\le cn - d\sqrt{n}$

True for $d \ge \frac{1}{\sqrt{2}-1}$,

n odd: $T(n) = 2T(\lfloor \frac{n}{2} \rfloor) + \sqrt{n} = 2T(\frac{n-1}{2}) + \sqrt{n} \le 2(c(\frac{n-1}{2}) - d\sqrt{\frac{n-1}{2}}) + \sqrt{n}$

...

$\Rightarrow$ pick a d that meets the restrictions, c accordingly..

3a) Partition $A = (A_1 | A_2 | \cdots | A_k)$ where each A_i is $n \times n$

and $B = \begin{pmatrix} B_1 \\ \hline B_2 \\ \hline \vdots \\ \hline B_k \end{pmatrix}$ where each B_i is $n \times n$

$AB = \sum_{i=1}^{k} A_i B_i$ using Strassen's alg for each product.

Analysis: Multiplications: $O(k n^{\log_2 7})$

Additions : $O(k n^2)$

$\Rightarrow$ Total: $O(k n^{\log_2 7})$

b) Partition A and B as in part a).

$$BA = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1k} \\ C_{21} & C_{22} & \cdots & C_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ C_{k1} & C_{k2} & \cdots & C_{kk} \end{bmatrix}$$

where each $C_{ij} = B_i A_j$, use Strassen's alg for product

- k^2 products

$\Rightarrow O(k^2 n^{\log_2 7})$