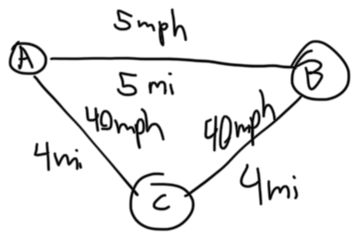
ADM Ch1

1. When $a = -5$, $b = -5$, then $a + b = -5 - 5 = -10$
$-10 < \min(-5, -5)$ $\square$

2. $ab < \min(a, b)$ if $a = -1$, $b = 10$ since
$-1 \cdot 10 = -10 < \min(-1, 10)$

3.


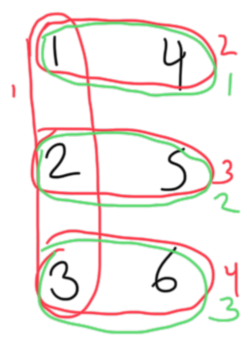
Shortest route takes $\frac{5\text{mi}}{5\text{mi/hr}} = 1$ hr
A-B-C route takes $\frac{8\text{ mi}}{80\text{ mi/hr}} = \frac{1}{10}$ hr

4. Same as ↑ (1 turn is faster than 0 turns)

5. (a) Given $S = \{1, 2, 3\}$, $T = 5$, algo will return $\{1, 2\}$, not $\{2, 3\}$

(b) Same

(c) If $S = \{3, 2, 2\}$, $T = 4$, algo returns $\{3\}$, not $\{2, 2\}$

6.


Greedy algorithm loses in this case for set cover problem.

* 7. Proving base cases for $m(y, z)$

if $y = 0$, $z \in \mathbb{N}$ → then return (0)

failed approach

if ~~$y=1$~~, $z \in \mathbb{N}$ s.t. $z \neq 0$ → $m(c, \lfloor \frac{z}{c} \rfloor) + z \% c$

$\hookrightarrow$ if $z=c \rightarrow m(c,1)+0 = \underbrace{m(c^2, \lfloor \tfrac{1}{c} \rfloor)}_{\because m(c^2,0)=0} + c\underbrace{(1\% c)}_{\because 1\% c = 1}$

$= 0 + c = c = z$

$\hookrightarrow$ if $z<c \rightarrow m(c,0)+z$

$= 0 + z = z$

$\hookrightarrow$ if $z>c \rightarrow$ looks hard to simplify from here...

retry with...

if $z=1$, $y \in \mathbb{N}$ s.t. $y \neq 0$ $\longrightarrow$ $\underbrace{m(cy, \lfloor \tfrac{1}{c} \rfloor)} + y\underbrace{(1\%c)}$

same as

$= 0 + y = y$

Proving inductive case $z=n \in \mathbb{N}$ st $n>1$, $y \in \mathbb{N}$ s.t. $y \neq 0$...

tried y (or z) $= n-1$, $\frac{n}{2}$, $\frac{n}{c}$, $n-c$ for assumption case. all failed?

$m(y,n) = \underbrace{m(cy, \lfloor \tfrac{n}{c} \rfloor)} + y(n\%c)$

$= m\left(c\left\lfloor\frac{n}{c}\right\rfloor, y\right) + y(n \% c)$

$$m\left(\left\lfloor\frac{n}{c}\right\rfloor, cy\right) = \underbrace{m\left(c\cdot\left\lfloor\frac{n}{c}\right\rfloor, \left\lfloor\frac{cy}{c}\right\rfloor\right)}_{} + \underbrace{\left\lfloor\frac{n}{c}\right\rfloor(cy \% c)}_{}$$

$$= m\left(c\left\lfloor\frac{n}{c}\right\rfloor, y\right) + 0$$

$$a - b\left\lfloor\frac{a}{b}\right\rfloor = a \% b$$

if $n \% c = 0$, $\left\lfloor\frac{n}{c}\right\rfloor = \frac{n}{c}$ $\rightarrow$ $m\left(c\left\lfloor\frac{n}{c}\right\rfloor, y\right) + y(n \% c)$

$= m(n, y) + 0$

$= yn$

if $n \% c = b$ s.t. $b \neq 0$,

then $n \% c = n - c\left\lfloor\frac{n}{c}\right\rfloor$, so

$$= m\left(c\left\lfloor\frac{n}{c}\right\rfloor, y\right) + y\left(n - c\left\lfloor\frac{n}{c}\right\rfloor\right)$$

$$= \cancel{yc\left\lfloor\frac{n}{c}\right\rfloor} + yn - \cancel{yc\left\lfloor\frac{n}{c}\right\rfloor}$$

$$= yn \quad \blacksquare$$

$$m\left(y, c\left\lfloor\frac{n}{c}\right\rfloor\right) = m\left(cy, \left\lfloor\frac{c\left\lfloor\frac{n}{c}\right\rfloor}{c}\right\rfloor\right) + y\left(c\left\lfloor\frac{n}{c}\right\rfloor \% c\right)$$

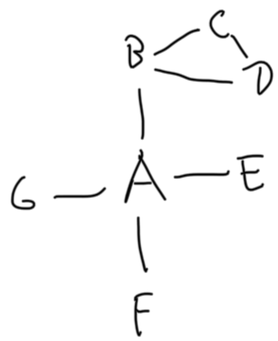
$$= m\left(cy, \left\lfloor\left\lfloor\frac{n}{c}\right\rfloor\right\rfloor\right) + 0$$

$$= m\left(cy, \left\lfloor\frac{n}{c}\right\rfloor\right) = cy\left\lfloor\frac{n}{c}\right\rfloor$$

$\because \left\lfloor\frac{n}{c}\right\rfloor < n$ if $c \geq 2$, so we can use $\left\lfloor\frac{n}{c}\right\rfloor$ as the assumption case.

* (not sure why cy is okay)

7.



vertices sorted by degrees:

A, B, C, D, E, F, G

algo's max clique:

{A, B}

actual max:

{B, C, D}

9. Horner(a, x)

$p = px + a_i$ for $i = n-1$ to 0

Base case: $n = 1$

for i from 0 to 0

So $p = a_1 x + a_0 = \underbrace{a_1 x^1 + \dots + a_0 x^0}_{\text{polynomial}}$

Inductive Case:

$i = 0 \rightarrow p_0 = p_1 x + a_0$ } we assume this based on algo definition.

$= (p_2 x + a_1) x + a_0$

$= ((p_3 x + a_2) x + a_1) x + a_0$

$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \leftarrow$ define p_0 as this

$= \underbrace{(a_n x^{n-1} + a_{n-1} x^{n-2} + \cdots + a_1)}x + a_0$ ← define P_1 as this

$= P_1 x + a_0$

So $\begin{bmatrix} p_i = p_{i+1} x + a_i \\ p_n = a_n \end{bmatrix}$ for $0 \le i < n$ where $H(a, x) = p_0$

Show this is same as $a_n x^n + \cdots a_1 x + a_0 x^0$

aka prove $P_0 = \sum_{j=0}^{n} a_j x^j$

converts to

$$\sum_{j=0}^{n} a_j x^j = a_0 + \sum_{j=1}^{n} a_j x^j$$

→ Prove $P_1 x = \sum_{j=1}^{n} a_j x^j$

if we assume $P_1 = a_n x^{n-1} + \cdots + a_1$

then it reduces to

→ $a_n x^n + \cdots + a_1 x^1 = \sum_{j=1}^{n} a_j x^j$

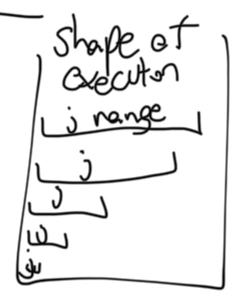
which is self-evident. $\blacksquare$

10. $B(A) = i : n \to 1$

$j : 1 \to i-1$

if $(A[j] > A[j+1]) \to$ swap$(A[j], A[j+1])$

Base case: $A = [\overset{A[1]}{x_1}, \overset{A[2]}{x_2}]$



inner loop only runs once.

Two scenarios: if $x_1 > x_2$, then swap(x_1, x_2)

so that we have $[x_2, x_1]$ which is sorted.

if $x_1 \le x_2$, no op, and we're already sorted

Induction Case.

Assuming $A = [X_1, \dots, X_{n-1}]$ & $B(A)$ sorts A

Prove $A' = [X_1, \dots, X_{n-1}, X_n]$ is sorted by $B(A')$

$\downarrow$ 3 cases:

$X_n < X_{left}$	$X_n > X_{right}$	$X_{left} \le X_n \le X_{right}$

where X_{left} is $B(A)_1$, X_{right} is $B(A)_{n-1}$

each iteration moves it left

$[\dots, X_n, X_{right}]$

$\downarrow$

$[\dots, X_n, X_{right-1}, X_{right}]$

$\downarrow$

$[X_n, X_{left}, \dots, X_{right}]$

$[\underbrace{X_{left}, \dots, X_{right}}_{\text{sorted}}, X_n]$

$[X_{left}, \dots, X_n, \dots, X_{right}]$

In any case, its sorted.

11. Struggled

12. Prove $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for $n \geq 0$

Base cases, $n=0$

$$\sum_{i=1}^{0} i = 0 = \frac{0(0+1)}{2}$$

$n=1$

$$\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}$$

Inductive case, n, given $\sum_{i=1}^{n-1} i = \frac{(n-1)[(n-1)+1]}{2} = \frac{n(n-1)}{2}$

$$\sum_{i=1}^{n} i = n + \sum_{i=1}^{n-1} i$$

$n(n-1)$

$$= n + \frac{n(n-1)}{2}$$

$$= \frac{n^2+2n-n}{2} = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$$

13. Prove $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for $n \geq 0$ $\dashrightarrow \frac{(n^2+n)(2n+1)}{6} = \frac{2n^3+2n^2+n^2+n}{6}$

$$= \frac{2n^3+3n^2+n}{6}$$

Base case, $n=0$

$$\sum_{i=1}^{0} i^2 = 1^2 = 1 = \frac{1(1+1)(2+1)}{6} = \frac{6}{6} = 1$$

Inductive case, n given $\sum_{i=1}^{n-1} i^2 = \frac{(n-1)[(n-1)+1][2(n-1)+1]}{6}$

$$= \frac{(n-1)n(2n-1)}{6}$$

$$\sum_{i=1}^{n} i^2 = n^2 + \sum_{i=1}^{n-1} i^2 = n^2 + \frac{(n-1)n(2n-1)}{6}$$

$$= \frac{6n^2+(n^2-n)(2n-1)}{6} = \frac{6n^2+2n^3-n^2-2n^2+n}{6}$$

$$= \frac{2n^3+3n^2+n}{6} = \frac{n(2n^2+3n+1)}{6}$$

$$= \frac{n(n+1)(2n+1)}{6}$$

14. Prove $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ for $n \geq 0$

Base case, $n=0$

$$\sum_{i=1}^{0} i^3 = 0^3 = 0 = \frac{0^2(0+1)^2}{4}$$

Inductive case n, given $\sum_{i=1}^{n-1} i^3 = \frac{(n-1)^2[(n-1)+1]^2}{4}$

$$\sum_{i=1}^{n} i^3 = n^3 + \sum_{i=1}^{n-1} i^3 = n^3 + \frac{(n-1)^2[(n-1)+1]^2}{4} = \frac{4n^3+(n^2-2n+1)n^2}{4}$$

$$= \frac{4n^3+n^4-2n^3+n^2}{4} = \frac{n^4+2n^3+n^2}{4} = \frac{n^2(n^2+2n+1)}{4}$$

4 4 4

$$= \frac{n^2(n+1)^2}{4}$$ $\blacksquare$

15. Base case, n=1

$$\sum_{i=1}^{1} i(i+1)(i+2) = 1\cdot2\cdot3 = \frac{1\cdot2\cdot3\cdot\cancel{4}}{\cancel{4}}$$

Inductive case, n given $\sum_{i=1}^{n-1} i(i+1)(i+2) = \frac{(n-1)[(n-1)+1][(n-1)+2][(n-1)+3]}{4}$

$$\sum_{i=1}^{n} i(i+1)(i+2) = n(n+1)(n+2) + \sum_{i=1}^{n-1} i(i+1)(i+2) = n(n+1)(n+2) + \frac{(n-1)(n)(n+1)(n+2)}{4}$$

$$= \frac{4n(n+1)(n+2) + (n^2-n)(n^2+3n+2)}{4} = \frac{(4n^2+4n)(n+2) + n^4+3n^3+2n^2-n^3-3n^2-2n}{4}$$

$$= \frac{4n^3+4n^2+8n^2+8n+n^4+3n^3+2n^2-n^3-3n^2-2n}{4} = \frac{n^4+6n^3+11n^2+6n}{4}$$

Since $(n+1)(n+2)(n+3) = (n^2+3n+2)(n+3)$
$= n^3+3n^2+3n^2+9n+2n+6$
$= n^3+6n^2+11n+6$

reduces to

$$= \frac{(n+1)(n+2)(n+3)}{4}$$ $\blacksquare$

16. Prove $\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}$ for $n \geq 1$, every $a \neq 1$

Base case, n=1

$$\sum_{i=0}^{1} a^i = a^0 + a^1 = 1 + a = \frac{a^{1+1}-1}{a-1} = \frac{a^2-1}{a-1} = \frac{(a+1)\cancel{(a-1)}}{\cancel{a-1}}$$

Inductive case n, given $\sum_{i=0}^{n-1} a^i = \frac{a^{(n-1)+1}-1}{a-1} = \frac{a^n-1}{a-1}$

$$\sum_{i=0} a^i = a^n + \sum_{i=0} a^i = a^n + \frac{a^n - 1}{a-1} = \frac{a^n(a-1) + a^n - 1}{a-1}$$

$$= \frac{a^{n+1} \cancel{-a^n} \cancel{+a^n} - 1}{a-1}$$ ■

17. Prove $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ for $n \geq 1$

Base case, $n=1$

$$\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1 \cdot 2} = \frac{1}{2} = \frac{(1)}{(1)+1}$$

Inductive case n, given $\sum_{i=1}^{n-1} \frac{1}{i(i+1)} = \frac{n-1}{n-1+1} = \frac{n-1}{n}$

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{n(n+1)} + \sum_{i=1}^{n-1} \frac{1}{i(i+1)} = \frac{1}{n(n+1)} + \frac{n-1}{n} = \frac{1 + (n-1)(n+1)}{n(n+1)}$$

$$= \frac{\cancel{1} + n^2 \cancel{-1}}{n(n+1)} = \frac{n^2}{n(n+1)} = \frac{n}{n+1}$$ ■

18. Prove $3 \mid (n^3 + 2n)$ for all $n \geq 0$

is divisible

Base case, $n=0$

$0^3 + 2(0) = 0$

$0 \bmod 3 = 0$

$n^3 - 3n^2 + 5n - 3$

Inductive case n, given $3 \mid [(n-1)^3 + (n-1)2]$

$n^3 + 2n = n(n^2 + 2)$

$\boxed{0 = a \bmod b \text{ means } \exists \text{ some } f \in \mathbb{Z} \ni fb = a}$

So if $(n^3 - 3n^2 + 5n - 3) \bmod 3 = 0$,
then

— — ; $n^3-3n^2+5n-3=3f$

which means $3(f+k)=m \Rightarrow f=\frac{m}{3}-k$

an adjustment to f's value

~~any~~ other number divisible by 3

$\to f=\frac{n^3-3n^2+5n-3}{3}$

$\Rightarrow f=\frac{n^3}{3}-n^2+\frac{5}{3}n-1$ } f's relationship w.r.t n

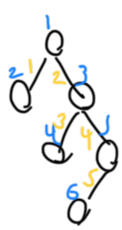
If we can show that there's a k such that

m —— $\frac{n(n^2+2)}{3}-k=\underbrace{\frac{n^3-3n^2+5n-3}{3}}_{f}$

then we would have proved $3\backslash(n^3+2n)$

$k=\frac{n^3+2n-n^3+3n^2-5n+3}{3}$

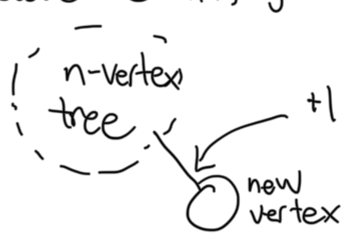
$=\frac{3n^2-3n+3}{3}=n^2-n+1$ $\boxed{?}$

19. 

Prove tree with n vertices has n-1 edges

Base case, 1 vertex → no edges.

Inductive case n+1, given n vertex → n-1 edges



+1 edge, so $n-1+1=n$ edges for n+1 tree ■

20. $\sum_{i=1}^{n} i^3=\left(\sum_{i=1}^{n} i\right)^2$を証明。

ベースケース n=1

$$\sum_{i=1}^{1} i^3 = 1^3 = 1 = \left(\sum_{i=1}^{1} i\right)^2 = 1^2$$

誘導ケース n+1 given $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$

$$= \left[\frac{n(n+1)}{2}\right]^2 = \left(\frac{n^2+n}{2}\right)^2 = \frac{n^4+2n^3+n^2}{4}$$

$$\sum_{i=1}^{n+1} i^3 = (n+1)^3 + \sum_{i=1}^{n} i^3$$

$$= (n+1)^3 + \frac{n^4+2n^3+n^2}{4} = n^3+3n^2+3n+1 + \frac{n^4+2n^3+n^2}{4}$$

$$= \frac{4n^3+12n^2+12n+4+n^4+2n^3+n^2}{4} = \frac{n^4+6n^3+13n^2+12n+4}{4}$$

$$\left(\sum_{i=1}^{n+1} i\right)^2 = \left(\frac{(n+1)[(n+1)+1]}{2}\right)^2 = \left(\frac{(n+1)(n+2)}{2}\right)^2 = \frac{(n^2+2n+1)(n^2+4n+4)}{4}$$

$$= \frac{n^4+4n^3+4n^2 + 2n^3+8n^2+8n + n^2+4n+4}{4}$$

$$= \frac{n^4+6n^3+13n^2+12n+4}{4}$$ ■

21. Books owned: ~50
Pages ea: ~300 ×
15,000 pgs

Library books: 300 pages per book
20 books per row
8 rows per shelf
20 shelves per floor
× 5 floors
4,800,000 pgs

22. ~800 pages
~45 lines per page
× ~10 word per line
360,000 words

23. 60 is approx $\frac{100}{2}$, 24 is approx $\frac{100}{4}$

So 1 M s $\approx$ 10K m $\approx$ 100 h $\approx$ 4 days ← should be more than actual because we divided by 50 twice (and 50 < 60), even though we used 25 in place of 24 (since it's pretty close)

Actually, there's 277.8 hours

or 11.6 days

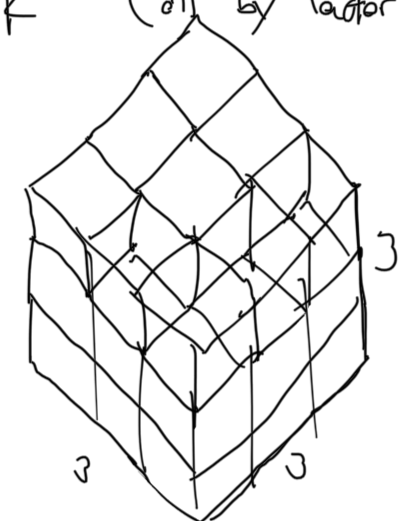
(off by factor of ~3)

24. 5 towns per county ← biggest deviance from reality
40 counties per state
× 50 states
1000 towns

actual: 19.5K (off by factor of 20)

25. mi^3 output from Mqi river?

- 10 mph
- 1 mile wide mouth } $\frac{1}{6} mi^2$ mouth
- $\frac{1}{6}$ mile deep mouth



if 1 mph stream means in an hour
$1 mi^3$ moves thru a $1 mi^2$ opening every hr

then 10 mph → $10 mi^3$ moves thru $1 mi^2$ opening every hr

since we have $\frac{1}{6} mi^2$ opening, then
flow is $\frac{10}{6} mi^3/hr$ = 1.67 mi^3/hr
× 24 hr/day
40 mi^3/day

actual: 593K ft^3/s
= .35 mi^3/day
(off by factor of ~114)

26. Starbucks:

0.1 avg per small town
4000 towns per state
× 50 states

20,000 slaves (actual: 15,444) within same factor!

27.

volume: $36\ ft^2$

straw diameter $= \frac{1}{3}$ in $\rightarrow \frac{\pi^2}{36}\ in^2$ area

suction speed: $\frac{1}{8}$ in/s

insight: flow rate in X m/s means an object in the water moves X m over a second. To get the m³/s flow rate, multiply X m/s by cutaway surface area/m²

$\frac{1}{8}$ in/s $\rightarrow$ 1 inch / 8 sec $\rightarrow \frac{\pi^2 in^2}{36} \cdot 1\ in/s = \frac{\pi^2}{36}\ in^3/s$

12 in / ft $\rightarrow$

$12^3\ in^3/ft^3$
$= 1{,}728\ in^3\ H_2O$ in tub

$\frac{\pi^2}{36} = \frac{3.14^2}{36} \approx \frac{10}{36} \approx \frac{1}{4}\ in^3/s$

$\frac{1728\ in^3}{\frac{1}{4}\ in^3/s} = 6917 \approx 7000\ s \approx 120$ min
≈ 2 hrs

28. disk access is microseconds (1,000,000 μs = 1 s
→ 1000 ms = 1 s
→ 1,000 μs = 1 ms)

wrong

✓ ram should be less than 1 μs

Actual:
70s HDD seek time: 600 ms
modern HDD: ~10 ms × 1/60
SSD: ~100 μs × 1/100
RAM: ~10 ns × 1/10,000

Conclusion:
SSD 100x faster than HDD
RAM 1,000,000 x faster than HDD

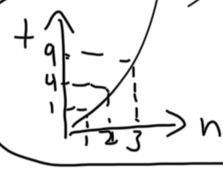
3 GHz = 3 billion instructions per second

$3 \times 10^9 \times \frac{60\,s}{m} \times \frac{60\,m}{h} \times \frac{24\,h}{d} \times \frac{365\,d}{y}$
$= 9.46 \times 10^{16}$ (~95 peta instructions per year)

29. 1000 items => 10000 items if algo is O(n)

1s 10s

(a) if it's $O(n^2)$, then time would be



$$1s\left(\frac{10,000}{1000}\right)^2 = 100s$$

$$\log_2 x = \frac{\log_{whatever} x}{\log_{whatever} 2}$$

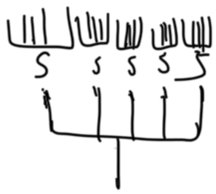
(b) $1s\,(10\,\underbrace{\log_2 10}) = 3.32 \times 10 = 33.2s$

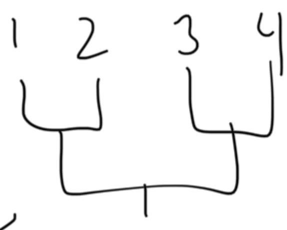
$$x = \log_2 10 \Leftrightarrow 2^x = 10$$

33. $\{1, 2, 3 \cdots \cdots 24, 25\}$ $\{2, 5, 6, 10, \}$

somewhere in here is 1st, 2nd, 3rd

25 馬/race ⇒ 1 race needed (to determine 1st 2nd 3rd)





6 races to determine 1st
remove 1st, repeat for 2nd, 3rd...

∴ 6×3=18 races

6 races
@5 per race ⟶ 1st only

thus

3 races → 1st only
@2 per race

optimizes to

5 races, then rem slowest 2 horses (15 rem)

3 races " (9 rem)

2 races, " (6 rem)

then

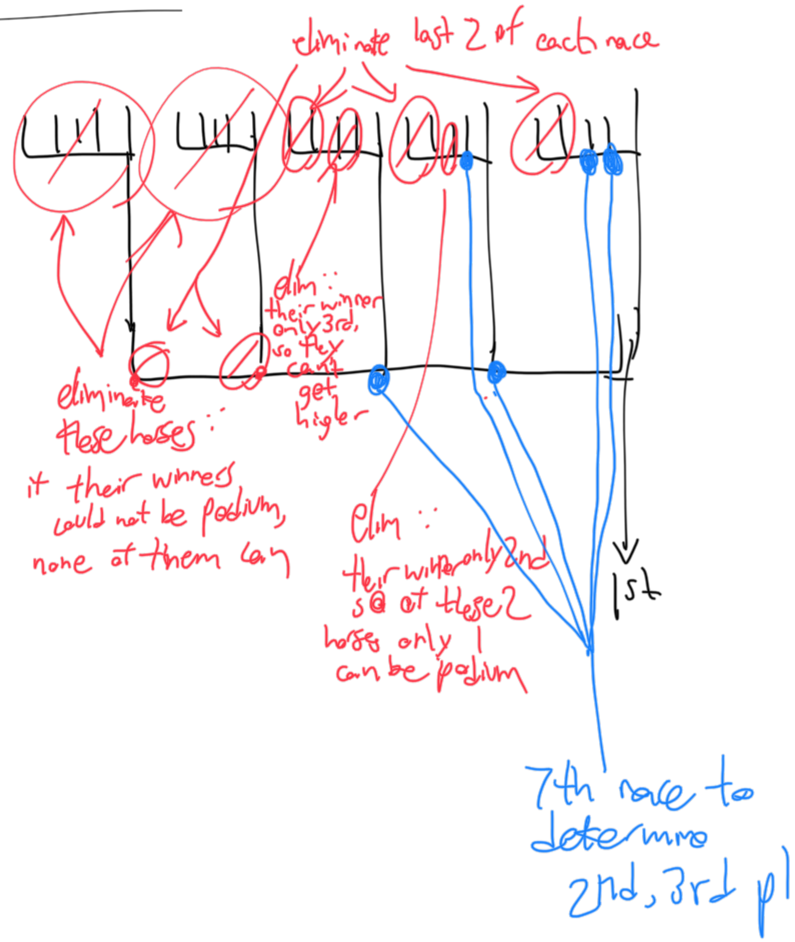
2 more

+ diff pattern

5 races, then remove slowest horse (20 rem.)
4 " (16 rem)
4 " (12 rem)
+ 3 " (9 rem)
2 " (7 rem)
2 " (5 rem)
1 " (done)

21 races

12 races



34. Piano tuners in World

8B × .01 play piano
.50 play non-electronic
.25 play enough to need tuning
.05 tuners for every one of people

500K

35. US gas station ct

4000 towns/st
50 states
8 stations/town

1.6 M   Actual: 115K (off by ~14x)

36. Hockey rink ice weight

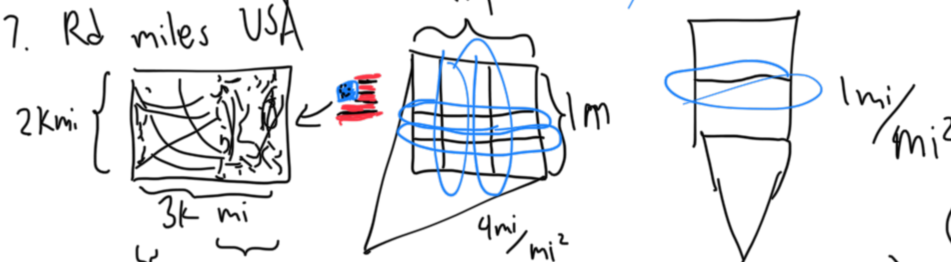
	Actual
.2 m deep	1"
100 m long	200 ft
50 m wide	85 ft

$1417\ ft^3 \times 57.3\ lbs/ft^3 = 80K\ lbs \approx 36000K$

off by 26x

$1000\,m^3$ ice × ice weigh (919 kg/m³) = 919,000 kg ≈ 1000 tons

37. Rd miles USA



60% well-populated

40% Sparse

⇒ $(.6 \times 4) + (.4 \times 1) \times 3000 \times 2000$

= 14.4M mi

Actual: 4M (off by ~3.5x)

38. Manhattan phone book avg rand opening ct to find specific name

5 M pop

800 pg book

↓

6,250 name/pg → $\frac{1}{6250}$ chance to find a specific person

→ 3125 times (on avg) to find "

(based on if you have $\frac{1}{4}$ odds to win, then avg winner did it in 2 tries)