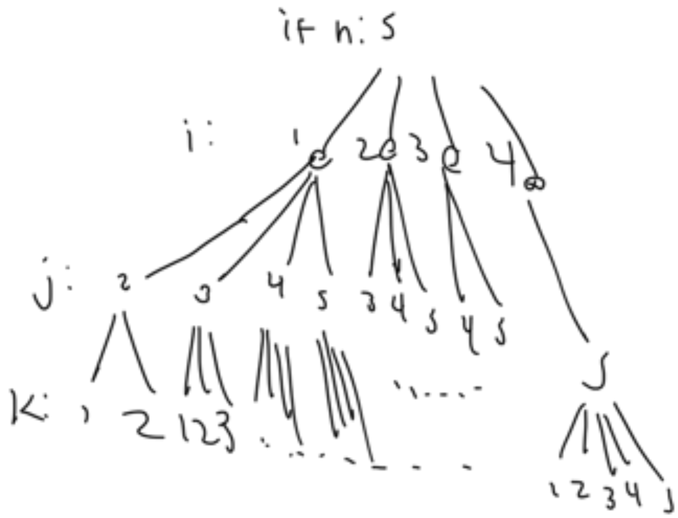
ADM ch2

1. i loops n-1 times
j loops n-i times
k loops j times



$$\text{Mystery}(n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{k=1}^{j} 1$$

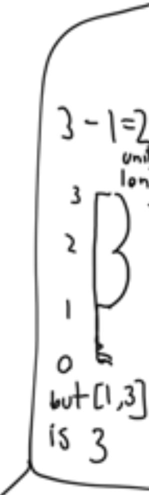
$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} j$$

$$= \sum_{i=1}^{n-1} \frac{(n+i+1)(n-(i\cancel{+1})\cancel{+1})}{2}$$

$$= \sum_{i=1}^{n-1} \frac{(n+i+1)(n-i)}{2}$$

Side note

It seems whenever you have an inclusive, discrete range, then the no. of elements of that range is $(b-a+1)$ if the range is $[a,b]$.



x units long $\Rightarrow$ spans x+1 members

E.g. $[1,3]$'s length is

$$3-1+1$$
$$=3$$

$$\sum_{i=x}^{n} i$$

$$= x+(x+1)+(x+2)+\dots+(n-1)+n$$

n-x+1 because range is $[x, n]$ ← $\frac{n-x+1}{2}$ pairs, each $(n+x)$

$$= \frac{(n+x)(n-x+1)}{2}$$

$$\sum^{n} i = \frac{n(n+1)}{}$$

$$= \sum_{i=1}^{n-1} \frac{n^2 + \cancel{in} + n - \cancel{in} - i^2 - i}{2} = \sum_{i=1}^{n-1} \frac{n^2+n-i^2-i}{2}$$

$$= \frac{1}{2} \sum_{i=1}^{n-1} \underbrace{n(n+1)}_{\text{notice there are } (n-1) \text{ number of these terms}} - i(i+1)$$

$$= \frac{1}{2}\left[n(n+1)(n-1) - 1(1+1) - 2(2+1) - 3(3+1) - \cdots - (n-2)(n-1) - (n-1)n \right]$$

$$- (1\cdot 2) - (2\cdot 3) - (3\cdot 4) - \cdots - (n-2)(n-1) - (n-1)n \Big]$$

$$= \frac{1}{2}\left(n(n+1)(n-1) - \sum_{i=1}^{n-1} i(i+1) \right)$$

$$(4+7)+(5+8)+(6+9)$$

$$\sum_{i=1}^{n-1} i^2 + i$$

From sum of squares formula

$$\sum_{i=1}^{n-1} i^2 + \sum_{i=1}^{n-1} i$$

$$\frac{n(n+1)(2n+1)}{6} - n^2 + \frac{(n-1\cancel{-1+1})(n\cancel{+1})}{2}$$

$$= \frac{n(n+1)(n-1)}{2} - \left(\frac{n(n+1)(2n+1) - 12n^2}{12} + \frac{n(n-1)}{4} \right) = \frac{n^3}{3} + \frac{n^2}{2} - \frac{n}{3}$$

(thanks, Wolfram)

$$= O(n^3)$$

(as predicted)

2. $Pash(n) = \sum^{n} \sum^{i} \sum^{i+j} 1$

$$\text{Pesky}(n) = \sum_{i=1} \sum_{j=1} \sum_{k=j} 1$$

term is independent of j!

$$= \sum_{i=1}^{n} \sum_{j=1}^{i} (i + \cancel{j} - \cancel{j} + 1) = \sum_{i=1}^{n} \sum_{j=1}^{i} (i+1)$$

$$= \sum_{i=1}^{n} (i+1)(i - \cancel{1} + \cancel{1})$$

$$= \sum_{i=1}^{n} i^2 + i = \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= O(n^3)$$

3. $$\text{Pestiferous}(n) = \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=j}^{i+1} \sum_{l=1}^{i+j-k} 1$$

$$= \sum \sum \sum i + j - k \cancel{-1+1}$$

$$= \sum \sum \left[(i+j)(i + \cancel{1} - j + \cancel{1}) - \sum_{k=j}^{i+1} k \right]$$

$$= \sum \sum \left[i^2 - j^2 - \frac{(i+1-j+1)(i+1+j)}{2} \right]$$

$$= \sum \sum \left[i^2 - j^2 - \frac{(i-j+2)(i+j+1)}{2} \right] \quad i^2 \cancel{-ji} + 2i \cancel{+ji} - j^2 + 2j + i - j + 2$$

$$= \sum \sum \left[\frac{2(i^2 - j^2) - i^2 - j^2 + 3i + 3j + 2}{2} \right]$$

$$= \frac{1}{2} \sum \sum \left(i^2 - 3j^2 + 3i + 3j + 2 \right)$$

$$= \frac{1}{2} \sum \left[2(i - \cancel{1} + \cancel{1}) + \sum i^2 - 3 \sum j^2 + 3 \sum i + 3 \sum j \right]$$

j from 1 to i

$\dots i(i+1)(2i+1) \quad i(i+1)) \dots$

$$= \frac{1}{2}\sum\left(2i + i^3 + 3i^2 \qquad - 3\frac{i(i+1)(2i+1)}{6} + 3\frac{i(i+1)}{2}\right)$$

$$\downarrow$$

$$= \frac{1}{2}\sum\left(\frac{4i+2i^3+6i^2 - (i^2+i)(2i+1) + 3i^2+3i}{2}\right)$$

$$(i^2+i)(2i+1) \rightarrow 2i^3 + i^2 + 2i^2 + i$$

$$\downarrow$$

$$\frac{4i^3+12i^2+8i}{2}$$

$$\downarrow$$

$$2i^3+6i^2+4i$$

$$\downarrow$$

$$= \cancel{\frac{1}{2}}\sum_{i=1}^{n} \cancel{2}i(i^2+3i+2)$$

$$\downarrow$$

$$= \sum i^3 + 3\sum i^2 + 2\sum i$$

$$= \frac{n^2(n+1)^2}{4} + 3\frac{n(n+1)(2n+1)}{6} + \cancel{2}\frac{(n(n+1))}{\cancel{2}}$$

$$= \frac{n^4}{4} + \frac{3n^3}{2} + \frac{11n^2}{4} + \frac{3n}{2} = O(n^4)$$

4. Conundrum(n)

$$= \sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{k=i+j-1}^{n} 1$$

$$= \sum_{i=1}^{n}\sum_{j=i+1}^{n}(\underbrace{n-i-j+1}+1)$$

$$= \sum\sum(n-i-j+2)$$

move out

$$= \sum_{i=1}^{n}\left[n(n-i-\cancel{1}+\cancel{1}) + 2(n-i) - i(n-i) - \frac{(n+i+1)(n-i-\cancel{1}+\cancel{1})}{2}\right]$$

$$= \sum \left[n^2 - ni + 2n - 2i - ni + i^2 - \frac{n^2 + \cancel{ni} + n - \cancel{ni} - i^2 - i}{2} \right]$$

$$= \sum \frac{2n^2 - 4ni + 4n - 4i + 2i^2 - n^2 + n - i^2 - i}{2}$$

$$= \frac{1}{2}\sum_{i=1}^{n}\left(n^2 - 4ni + 5n - 5i + i^2\right)$$

$$= \frac{1}{2}\left[n^3 - 4n\frac{n(n+1)}{2} + 5n^2 - 5\frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{1}{2}\left(n^3 - 2n^3 - 2n^2 + 5n^2 - \frac{(5n^2+5n)3}{6} + \frac{(n^2+n)(2n+1)}{6}\right)$$

$$-n^3 \quad +3n^2$$

$$= \frac{1}{2} \cdot \frac{-6n^3 + 18n^2 + 15n^2 + 15n + 2n^3 + 2n^2 + n^2 + n}{6}$$

$$= \frac{1}{12} \cdot -4n^3 + 36n^2 + 16n$$

$$= \frac{-n^3 + 9n^2 + 4n}{3} = O(n^3)$$

notice no inner summation

5.

n=5	K	X
	1	1*2*4*8
	2	2*4*8
	3	3*6
	4	4*8
	5	8

← notice doubling!

7 asterisks

$$\underbrace{f(n)}_{\text{no. of points}} = \sum_{k=1}^{n} \left\lfloor \lg \frac{2^{\lceil \lg n \rceil}}{k} \right\rfloor$$

no. of pairs

let y = round n up to next 2^x, aka $2^{\lceil \lg n \rceil}$
then $\lfloor \lg y - \lg k \rfloor$

1 2 4 8

$k=1$

1 2 4 8

$\lfloor 3 \rfloor = 3$

$k=3$ 1 2 3 $\lfloor 1.x \rfloor$ =

$k=5$ 1 2 4 5 $\lfloor 0.x \rfloor = 0$

$k=2$ 2 4 8

$\lfloor 2 \rfloor = 2$

Big Oh : round up / simplify

$$\sum_{k=1}^{n} \left\lfloor \lg \left(\frac{2^{\lceil \lg n \rceil}}{k} \right) \right\rfloor$$

round n up s.t. we can drop ceil

drop floor cuz it rounds down; we don't want that

$$\approx \sum_{k=1}^{n} \left(\lg(n+1) - \lg k \right)$$

$$= \quad n \cdot \lg(n+1) - \left(\lg 1 + \lg 2 + \cdots + \lg n \right)$$

$$= n \cdot \lg(n+1) - \lg(n!) = \lg\left(\frac{(n+1)^n}{n!} \right)$$

$f(x) = O\left(\lg \frac{(n+1)}{n!}\right)$ lets simplify more...

$\lg(n+1) + \lg(n+1) + \dots \lg(n+1)$

$- \lg 1 - \lg 2 - \dots - \lg n$

$\lg \frac{n+1}{1} + \lg \frac{n+1}{2} + \lg \frac{(n+1)}{3} + \dots + \lg \frac{(n+1)}{n}$

round up each term in lgs to n+1

$\approx \lg (n+1)^n = n \lg(n+1)$

$\approx O(n \lg n)$

Ω? Find worst case... but all cases are similar.

Lets just show $\exists$ c st. $c \cdot n \lg n < f(x)$ $\forall x > x_0$

$f(5) = 7$

$5 \cdot \lg 5 = \lg 5^5 = 11.6$

let $c = \frac{7}{11.6} = 0.6$

$c \cdot 11.6 \leq 7$ for $x_0 = 5$

Try $f(16) = \sum_{k=1}^{16} \left\lfloor \lg \frac{2^{\lceil \lg 16 \rceil}}{k} \right\rfloor$

$= \sum^{16} \lfloor \lg \frac{16}{k} \rfloor$

$k=1$

$$= \lfloor \lg \tfrac{16}{1} \rfloor + \lfloor \lg \tfrac{16}{2} \rfloor + \cdots + \lfloor \lg \tfrac{16}{16} \rfloor$$

$$= 4 + 3 + \cdots + 0$$

round each term up

$$\approx 16 \cdot 4 = 64$$

$$\lg 16^{16} = 16 \lg 16$$
$$= 16 \cdot 4 = 64$$

since $0.6 \times 64 \leq 64$

$f(n) = \Omega(n \lg n)$

thus $f(n) = \Theta(n \lg n)$

6.

$n=5$ i	xpower	p_i
1	x	$a_1 x$
2	x^2	$a_2 x^2$
3	x^3	$a_3 x^3$
4	x^4	$a_4 x^4$
5	x^5	$a_5 x^5$

(a) worst case multiplication ops: $2n$

(same as best case)

best/worse case add ops: n

(b) "

(c) Can we do $O(\lg n)$?

$$n^5 = \overset{5 \text{ ops}}{(n \cdot n)(n \cdot n) n} = \overbrace{(n^2)^2 \cdot n}^{3 \text{ ops}}$$

$$n^8 = ((n^2)^2)^2 \quad 3 \text{ ops (vs. 8 w/ mult)}$$

I don't think we can do the same for this prob because we have n+1 distinct terms ($a_0, a_1, \dots a_n$)

7. Proof by induction!

Base: If $A = [1]$,
$m = A[1] = 1$, which is correct

Inductive: Assume $\max(A[1..n-1])$ is correct.
Then
for $A[1..n]$ array, on nth iteration of the loop,
- if $A[n]$ is bigger than anything in $A[1..n-1]$ it will be returned, which is correct.
- if $A[n]$ is equal or smaller, $\max(A[1..n-1])$ is returned, which is correct.

8. (a) $2^{n+1} = 2^n \cdot 2 = O(2^n)$

$\uparrow$ constant, so removable

(b) $2^{2n} = (2^n)^2 = 2^n \cdot \underbrace{2^n}_{\text{not constant}} \neq O(2^n)$

$= (2^2)^n = 4^n = O(4^n)$

9. (a) $f(n) = \log n^2$ $\quad g(n) = \log n + 5$

$= \underbrace{2}_{\text{drop}} \log n \quad \therefore f(n) = \Theta(g(n))$

(b) $f(n) = \sqrt{n}$ $\quad g(n) = \log n^2$

$= n^{\frac{1}{2}}$ dominates $\rightarrow$ $= 2 \log n$

$\therefore$ f can't be upper bounded by g

$\therefore f(n) = \Omega(g(n))$

(c) $f(n) = (\log n)^2$ $\quad g(n) = \log n$

$= \log^2 n$

Because $x^2 > x$ for $\forall x > 1$,

$\log_2 n$ dominates $\log n$ for $\log n > 1$
i.e. $n > 2$

$\therefore f(n) = \Omega(g(n))$

(d) $f(n) = n$; $g(n) = \log^2 n$

Assuming $x^{\frac{1}{2}} : \log$ relationship applies to $X : \log^2$, then X dominates $\log^2$

$f = \Omega(g)$

(e) $f(n) = n \log n + \underbrace{n}_{\text{drop}}$ $\qquad g(n) = \log n$

$$\lim_{n\to\infty}\left(\frac{n \lg \cancel{n}}{\cancel{\lg n}}\right) = \infty \quad \therefore f \text{ doms } g$$

$f = \Omega(g)$

(f) $f(n) = 10$ $\qquad g(n) = \log 10$
$= 1$

$$\lim_{n\to\infty} \frac{10}{1} = 10 \neq 0 \qquad f = \Theta(g)$$
$\neq \infty$

(g) $f(n) = 2^n$ $\qquad g(n) = \cancel{10} n^2$

exp doms quad $\therefore$ $f = \Omega(g)$

(h) $f(n) = 2^n$ $\qquad g(n) = 3^n$

$$\lim_{x\to\infty} \frac{2^n}{3^n} = \lim\left(\frac{2}{3}\right)^n \overset{?}{=} \infty \therefore g \text{ doms } f \therefore \quad f = O(g)$$

10. (a) $f(n) = \frac{(n^2 - n)}{2}$ $\qquad g(n) = 6n$

$$\lim_{x\to\infty} \frac{g}{f} = \lim_{x\to\infty} \frac{12\cancel{n}}{(n^2-n)}$$
$\cancel{n}(n-1)$

$$= \lim_{x\to\infty} \frac{12}{n-1} = 0$$

$\therefore$ f dominates g

$g = O(f)$

(b) $f = n + 2\sqrt{n}$ $\quad g = n^2$

$= n + 2n^{\frac{1}{2}}$

$$\lim_{n \to \infty} \frac{n^2}{n + 2n^{\frac{1}{2}}} = \lim_{n \to \infty} \frac{2n}{1 + \cancel{1}} = \infty$$

$\therefore$ g dominates f

$f = O(g)$

(c) $f = n \lg n$ $\quad g = n\sqrt{n}/2$

$$= \frac{n \cdot n^{\frac{1}{2}}}{2} = \frac{n^{\frac{3}{2}}}{2}$$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{2\cancel{n} \lg n}{\cancel{n}\sqrt{n}} = \lim_{n \to \infty} \frac{2 \lg n}{n^{\frac{1}{2}}}$$

$$= \lim_{n \to \infty} \frac{\frac{2}{x}}{\frac{1}{2}} = \lim_{n \to \infty} \frac{4}{x}$$

$$= 0$$

$\therefore$ g dominates f

$f = O(g)$

(d) $f = n$ $\quad g = \log^2 n$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{n}{\log^2 n} = \lim_{n \to \infty} \frac{1}{\frac{\lg n}{n} + \frac{\lg n}{n}} = \lim_{n \to \infty} \frac{1}{2 \frac{\lg n}{n}}$$

$$= \lim_{n \to \infty} \frac{n}{2 \lg n} = \lim_{n \to \infty} \frac{1}{\frac{2}{n}} = \lim_{n \to \infty} \frac{n}{2}$$

$$= \infty$$

f dominates g

$g = O(f)$

$g = \omega(1)$

(e) $f(n) = 2(\log n)^2$ $\qquad g(n) = \log n + 1$

$$\lim \frac{f}{g} = \lim \frac{2(\log n)^2}{\log n + 1} = \lim \frac{2(2\log n \cdot \cancel{\frac{1}{n}})}{\cancel{\frac{1}{n}}} = \lim_{n\to\infty} 4\lg n$$

$$= \infty$$

f dominates g

$\therefore g = O(f)$

(f) $f = 4n\log n + n$ $\qquad g = \frac{n^2 - n}{2} = \frac{n(n-1)}{2}$

$= n(4\log n + 1)$

$$\lim_{n\to\infty} \frac{f}{g} = \lim_{n\to\infty} \frac{2\cancel{n}(4\log n + 1)}{\cancel{n}(n-1)}$$

$$= \lim \frac{8\log n + 2}{n-1}$$

$$= \lim \frac{\frac{8}{n}}{1} = \lim_{n\to\infty} \frac{8}{n}$$

$$= 0$$

$\therefore$ g dominates f

$f = O(g)$

---

11. (a) $f = 3n^2$ $\quad g = n^2$ $\quad f(n) = \Theta(g(n))$

(b) O

(c) Θ. $\frac{1}{n}$ diminishes as $n \to \infty$ so not a factor

(d) $2^{k\log n} = 2^{\log n^k} = n^k \therefore \Theta$

(e) $2^{2n} = (2^2)^n = 4^n$

$\therefore O$

12. $n^3 - 3n^2 - n + 1 = \Theta(n^3)$

$$cn^3 \geq n^3 - 3n^2 - n + 1$$

$$c \geq 1 - \frac{3}{n} - \frac{1}{n^2} + \frac{1}{n^3}$$

$$\left. d \leq 1 - \frac{3}{n} - \frac{1}{n^2} + \frac{1}{n^3} \right\} \text{ solve for this}$$

There must $\exists$ x st $d \leq x \leq c$

$n \approx 3.38$

So when $c \geq 3.38$, $d \leq 3.38$ $x =$

$cn^3 \geq n^3 - 3n^2 - n + 1$ for $n \geq n_0$

& $dn^3 \leq n^3 - 3n^2 - n + 1$ for $n \geq n_0$

13. $n^2 = O(2^n)$

$$\lim_{n \to \infty} \frac{n^2}{2^n} = \lim_{n \to \infty} \frac{2n}{2^n \ln 2} = \lim_{n \to \infty} \frac{2}{2^n \ln^2 2}$$

$$= 0$$

$\therefore$ 2^n dominates n^2

14. $\Theta(n^2) = \Theta(n^2 + 1)$

any function upper bounded by $n^2 + 1$ can also be upper bounded by n^2 with a slightly bigger constant.

for lower bound n^2's constant can even be the same as n^2+1 and still have $cn^2 \leq cn^2 + 1$.

15.

	double	+1
n^2	$(2n)^2 = 4n^2$	$(n+1)^2 = n^2+2n+1$
n^3	$(2n)^3 = 8n^3$	$(n+1)^3$
$100n^2$	$100(2n)^2 = 400n^2$	$100(n+1)^2 = 100n^2+200n+100$
$n \lg n$	$2n\lg 2n = 2n\lg n + 2n\lg 2 = 2n\lg n + 2n$	$(n+1)\lg(n+1)$
2^n	$2^{2n} = 4^n$	$2^{n+1} = 2 \cdot 2^n$

16. 10^{10} ops/sec $\times \frac{60\,sec}{1\,m} \cdot \frac{60\,m}{1\,hr} = 3600 \times 10^{10}$ ops/hr

$f(n) = n^2 \Rightarrow$ for input size $n \rightarrow n^2$ ops complete

(a) $n^2 = 3.6 \times 10^{13} = 36 \times 10^{12}$

$n = 6 \times 10^6$ max input size to get 36×10^{12} ops $(6{,}000{,}000)$

(b) $n^3 = 36 \times 10^{12}$

$n \approx 33{,}019$

(c) $100n^2 = 36 \times 10^{12}$

$n^2 = 36 \times 10^{10}$

$n = 6 \times 10^5 = 600{,}000$

(d) $n \lg n = 36 \times 10^{12} \longrightarrow n \approx 9 \times 10^{11}$ from Wolfram

let $y = W(36 \times 10^{12}) \therefore n = e^{W(\log(2^{36 \times 10^{12}}))}$ Lambert-W

(e) $2^n = 36 \times 10^{12}$

$\lg(36 \times 10^{12}) = n$

$n \approx 45$

(f) $2^{2^n} = 36 \times 10^{12}$

$\lg(36 \times 10^{12}) = 2^n$

$2^n \approx 45$

$\lg 45 = n$

$n \approx 5$

17.

(a) $f(n) \leq c \cdot g(n)$ for $\forall\, n > 1$

$n^2 + n + 1 \leq c \cdot 2n^3$

$c \geq \frac{1}{2} \cdot \left(\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3}\right)$ ← as n increases, the right side decreases, meaning c can be lowered.

$n = 1$

$c \geq \frac{1}{2}(1 + 1 + 1)$

$c \geq \frac{3}{2}$

(b)

$cn^2 \geq n\sqrt{n} + n^2$

$c \geq \dfrac{n^{\frac{3}{2}} + n^2}{n^2}$

$c \geq n^{-\frac{1}{2}} + 1$

$c \geq \frac{1}{\sqrt{n}} + 1$ ← c can be lowered

$c \geq \ldots$ can be ...

$n=1 \downarrow$ to ≈ 1 as $n \to \infty$

$c \geq 2$

(c) $\frac{cn^2}{2} \geq n^2-n+1$

$c \geq \frac{(n^2-n+1)2}{n^2} = \frac{2n^2-2n+2}{n^2}$ ← c must grow as $n \to \infty$

$n=1 \downarrow$

$c \geq \frac{2\cancel{-2}+2}{1} = 2$

18. $f_1(n)=O(g_1(n))$, $f_2(n)=O(g_2(n))$ → $f_1(n)+f_2(n) = O(g_1(n)+g_2(n))$?

$= O(g_1(n)) + O(g_2(n))$

$\leq \underbrace{c_1 g_1(n_1) + c_2 \cdot g_2(n_2)}$ for some $n_1 > n_x$, $n_2 > n_y$

is less than

$\leq \overbrace{c_3 g_1(n) + c_3 g_2(n)}$ for some $n > \max(n_x, n_y)$ and $c_3 = \max(c_1, c_2)$

$\leq c_3(g_1(n)+g_2(n))$

$\downarrow$

$O(g_1(n)+g_2(n))$

19.

$= \Omega(g_1(n)) + \Omega(g_2(n))$

$\geq c_1 g_1(n_1) + c_2 \cdot g_2(n_2)$ for some $n_1 > n_x,\ n_2 > n_y$

is greater than

$\geq c_3 g_1(n) + c_3 g_2(n)$ for some $n > \max(n_x, n_y)$ and $c_3 = \min(c_1, c_2)$

$\geq c_3 (g_1(n) + g_2(n))$

$\downarrow$

$\Omega(g_1(n)+g_2(n))$

20. $f_1(n) \cdot f_2(n) = O(g_1(n) \cdot g_2(n))$?

$= O(g_1(n)) \cdot O(g_2(n))$

$\leq c_1 g_1(n_1) \cdot c_2 g_2(n_2)$ for $n_1 > n_x,\ n_2 > n_y$

$\leq c_3\ g_1(n) g_2(n)$ for $n > \max(n_x, n_y)$ and $c_3 = c_1 c_2$

$= O(g_1(n) g_2(n))$

21. $a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n^1 + a_0 = O(n^k)$

Proof by induction.

base case:

$$k=0 \rightarrow a_0 = O(1) = O(n^0)$$

Assuming $\underbrace{a_{k-1} n^{k-1} + \dots}$ $= O(n^{k-1})$

$$a_k n^k + \underbrace{a_{k-1} n^{k-1} + \dots a_0}$$

$$\leq a_k n^k + c n^{k-1} \quad \text{for } n > n_0$$

$$\leq a_k n^k + \frac{c n^k}{n}$$

$$\leq \left(a_k + \frac{c}{n}\right) n^k$$

as n increases, $\frac{c}{n} \to 0$,

so $\lim_{n \to \infty} a_k + \frac{c}{n} = a_k$. a constant

$$\leq d n^k \quad \therefore a_k n^k + \dots + a_0 = O(n^k)$$

22. $(n+a)^b = \Theta(n^b)$ for $b > 0$

Base: $b=1 \rightarrow c_1 n \leq n+a \leq c_2 n$ for $n > n_0$

Ind: Assume $(n+a)^{b-1} = \Theta(n^{b-1})$

$$(n+a)^b = (n+a)\,\Theta(n^{b-1})$$

$$(n+a)\,c_1 n^{b-1} \le (n+a)^b \le (n+a)\,c_2 n^{b-1}$$

$$\le \left(\frac{n+a}{n} c_2\right) n^b$$

$$\le \frac{\cancel{n}\left(1+\frac{ac_2}{n}\right)}{\cancel{n}} n^b$$

$$\le \left(1+\frac{ac_2}{n}\right) n^b$$

$$\downarrow$$

$$\lim_{n\to\infty}\left(1+\frac{ac_2}{n}\right) = 1, \text{ a constant}$$

$$\therefore \quad (n+a)^b = \Theta(n^b)$$

23.

$\lg \lg n$

$\ln n$

$\lg n$

$\sqrt{n}$

$\lg^2 n$

n

$n \lg n$

$n^{1+\varepsilon}$

Same $\left[\begin{array}{l} n^2 + \lg n \\ n^2 \end{array}\right.$

n^3

$n - n^3 + 7n^5$

Same $\left[\begin{array}{l} 2^n \\ 2^{n-1} \end{array}\right.$

e^n

$n!$

24.

$2^{\frac{\sqrt{n}}{2}}$

$\sqrt{2^{\sqrt{n}}} = \left(2^{(n^{\frac{1}{2}})}\right)^{\frac{1}{2}} \ll 2^{n^{\frac{1}{2}}} \ll 2^n$ ← placement due to experimentation. Limit test didn't simplify...

n^{π}

$\sqrt{2^{\sqrt{n}}} \gg \sqrt{\sqrt{n}^2} = \sqrt{n}$

$5 \lg^2 n$

$n^{\dots} \ll n^{\dots}$

$\lim_{n \to \infty} \frac{2^{\frac{\sqrt{n}}{2}}}{n^2}$

$2n$

2

$f'(g(x)) \cdot g'(x)$ Chain rule

$= 2^{\frac{\sqrt{n}}{2}} (\ln 2) \cdot \frac{1}{4\sqrt{n}}$

$= \underbrace{\frac{1}{4} n^{-\frac{1}{2}}} \underbrace{2^{\frac{\sqrt{n}}{2}}}$ Product rule

$(fg)' = f'g + g'f$

$-\frac{1}{8} n^{-\frac{3}{2}} 2^{\frac{\sqrt{n}}{2}} + 2^{\frac{\sqrt{n}}{2}} \ln 2 \cdot \frac{1}{4} n^{-\frac{1}{2}}$

doesn't make it simpler...

$2^{\log^4 n} \ll 2^n$

π^n

Worst case

$$\binom{n}{5} = \frac{n!}{n-5!\, 5!} = \frac{n!}{\left(\frac{n}{2}\right)! \cdot 2} = O(n!)$$

$$\binom{n}{n-4} = \frac{n!}{(\cancel{n-n}+4)!\,(n-4)!} = O(n!)$$

$$n^4 \binom{n}{n-2}$$

25.

$\sum_{i=1}^{n} i^i = 1 + 2^2 + 3^3 + 4^4 + \dots + n^n$

n^n

$n!$

$\binom{n}{n-\frac{n}{2}} \left\{ \overline{2\left(\frac{n}{2}\right)!} \right.$

$\lim_{n\to\infty} \frac{\frac{\cancel{n!}}{2\left(\frac{n}{2}\right)!}}{\cancel{n!}} = \lim \frac{1}{2\left(\frac{n}{2}\right)!} = 0 \therefore \binom{n}{n-\frac{n}{2}} = o(n!)$

little o

$2^{\log^4 n} \ll 2^n$

$(\log n)^{\log n} \ll (\log n)^n$

$n^{\log^2 n} \ll n^n$

$2^{2\log n} = 4^{\log n} \ll 4^n$

$n^{\log^2 n} \gg$

Order is $\sum_{i=1}^{n} i^i \gg n^n \gg n! \gg \binom{n}{n-4} \gg 4^{\log n} \gg 2^{\log^4 n}$

$\gg (\log n)^{\log n}$

via experimentation

26.

$n!$

2^n

$\left(\frac{3}{2}\right)^n$

$n - n^3 + 7n^5$
n^3

$\left[\begin{matrix} n^2 + \log n \\ n^2 \end{matrix}\right.$

$n \log n$

n < $\frac{n}{\log n}$ via experiment

$n^{\frac{1}{3}} + \log n$

$n^{\frac{1}{2}}$

$\log^2 n$

$\ln n$

$\log n$

$\log \log n$

6

$\left(\frac{1}{3}\right)^n$ via experiment

27. (a) if $f(n) = o(g(n))$ g doms f and is in a higher class, so $f \neq \Omega(g)$ $\therefore f \neq \Theta(g)$

So $f = n$, $g = n^2$ works

(b) None. See reason why here

(c) Θ needs Ω & O, so if no O, no Θ $\therefore$ None

(d) $g = n$, $f = n^2$

28

a	T	e	T
b	F	f	F
c	F	g	F
d	F		

29. (a) $f = \Omega(g)$ (c) $f = \Omega(g)$

(b) $f = O(g)$

30 (a) Yes. Some inputs may require a nested loop, necessitating n^2 ops, while

Some might not, keeping it $O(n)$.

(b) Yes. $f = O(n^2)$ just means f can be upperbounded by the n^2 class; it doesn't say any inputs have to hit that upper bound

(c) Yes. For same reason as (a)

(d) No. If all inputs are upperbounded by the n-class, then there is no input lowerbounded by n^2-class, so $\neq \Theta(n^2)$

(e) In both parity cases, n^2 dominates, so yes.

31. (a) No. Can show empirically, but the limit test doesn't work here.
Instead:
$3^n \leq c2^n$?

$$c \geq \frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n \leftarrow \text{not a constant; } \underline{c \text{ must}} \text{ grow as } n \text{ increases}$$

(b) $\log 3^n = n \log 3 = \Theta(n)$
$O(\log 2^n) = O(n)$, so yes

(c) $3^n = \Omega(2^n)$?
$3^n \geq c2^n$
$c \leq \left(\frac{3}{2}\right)^n \leftarrow$ as n grows, c can remain the same

Discovery: If $c < g(n)$, where $g(n) > n$ for $\forall n \in \mathbb{R}$, then c exists.
OTOH, if $c > g(n)$, c can't exist (as a constant)

(d) $\Omega(\log 2^n) = \Omega(n)$, so yes

32. (a) $f(n) = \sum^n \frac{1}{i} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-1} + \frac{1}{n}$

$\underset{\text{input size}}{\uparrow}\ \underbrace{i=1}_{\text{ops}} \qquad = \sum_{i=1}^{n} i^{-1}$

Suspicion: $\ln x$

From online:

$$\frac{1}{k} \geq \frac{1}{x} \geq \frac{1}{k+1} \quad \text{for } x \in [k, k+1]$$

$$\sum_{k=1}^{n} \frac{1}{k} \geq \int_{1}^{n+1} \frac{dx}{x} \geq \sum_{k=2}^{n+1} \frac{1}{k} \qquad \approx \ln|x| + C \quad \text{(from integration rules, ignoring intervals)}$$

$$\therefore \Theta(\ln n)$$

(b) $\sum_{i=1}^{n} \lceil i^{-1} \rceil = \left\lceil \frac{1}{1} \right\rceil + \left\lceil \frac{1}{2} \right\rceil + \dots \left\lceil \frac{1}{n} \right\rceil$

$= n \cdot 1$

$= \Theta(n)$

(c) $\sum_{i=1}^{n} \log i = \log 1 + \log 2 + \dots + \log(n-1) + \log n \quad = O(n \log n)$

$\approx \int_{1}^{n+1} \log x \, dx \approx x \log x - \frac{x}{\ln 10} = \Theta(x \log x)$

(d) $f(n) = \log(n!) = \log(1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot n)$

$= \log 1 + \log 2 + \dots + \log n = \Theta(n \log n)$

33. $f_1 = n^2 \lg n$

$f_2 = n \lg^2 n = n(\lg n)(\lg n)$

$f_3 = \sum_{i=0}^{n} 2^i = \underset{1}{2^0} + \underset{2}{2^1} + \underset{4}{2^2} + \underset{8}{2^3} + \dots 2^n$

$= \frac{2^{n+1} - 1}{2 - 1} = 2^{n+1} - 1$

$f_4 = \lg(2^{n+1} - 1) \approx \lg(2^n) = n$

$2^{\lg n} = n$

$\lg(2^n) = n$

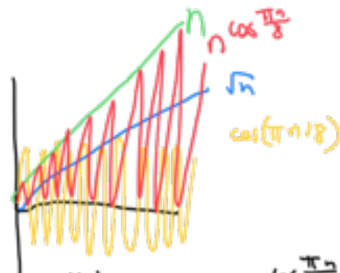
$n \ll n \lg^2 n \ll n^2 \lg n \ll 2^n$

34. All are true; 3^{n-1}, 3^n, 3^{n+1} are all in the same class and

can be converted among each other.

35. (a) $1000 \cdot 2^n + 4^n = \Theta(4^n)$ (c) $\log(n^{20}) + \log^{10} n = \Theta(\log^{10} n)$
(b) $n + n\log_2 n + \sqrt{n} = \Theta(n\log n)$ (d) $99^n + n^{100} = \Theta(n^{100})$

36 $A = O(B)?$ (a) O, o
$o(B)?$
$\Omega(B)?$ (b) O, o
$\omega(B)?$
$\Theta(B)?$ (c) $n^{\cos(\pi n/8)}$ can be between n^{-1} (tiny) and n^{1}



Because of the oscillation, $c\, n^{\cos\frac{\pi n}{8}} \not\leq \sqrt{n}$ $\therefore$ can't be O, Ω, or Θ
" $\not\geq \sqrt{n}$

Dominance relation also can't yield 0 or ∞ since it's periodic, so no o, ω either

(d) O, o

(e) $(\lg n)^n = n\lg n \ll n^{\lg n}$ $\because ab \leq a^b$ for $a>1, b>1$
$\therefore \Omega, \omega$

(f) $\lg n! = \sum_{i=1}^{n} \log i = n\log n - \frac{n}{\ln 6}$
$\therefore O, \Omega, \Theta$

37
1	$i=0$	$1 = 3^0$
1 1 1	$i=1$	$3 = 3^1$
1 2 3 2 1	$i=2$	$9 = 3^2$
1 3 6 7 6 3 1	$i=3$	$27 = 3^3$
1 4 10 16 19 16 10 4 1	$i=4$	$81 = 3^4$

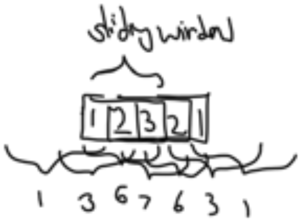
$\text{sum}_i = 3^{i-1}$

Proof: $s_0 = 3^0 = 1$ (base case) which corresponds to first row

Proof. $sum_1 = 3$ = (base case), [illegible]

Now assume $Sum_{i-1} = 3^{i-2}$

Then $Sum_i = 3 \cdot Sum_{i-1}$ because we have a 3-width sliding window such that each value is added into the next row 3 times.



$\therefore \; = 3 \cdot 3^{i-2}$

$sum_i = 3^{i-1}$

38. X-mas days: n Presents: $\sum_{i=1}^{n} i$ $\rightarrow \sum_{i=1}^{n} \sum_{j=1}^{i} j$

$P_1 \quad P_2 \quad P_3$

$1 + (2+1) + (3+2+1)$

$$= \sum_{i=1}^{n} \frac{i(i+1)}{2}$$

$$= \frac{1}{2} \sum_{i=1}^{n} (i^2 + i) = \frac{1}{2} \sum_{i=1}^{n} i^2 + \frac{1}{2} \sum_{i=1}^{n} i$$

$$= \frac{1}{2} \cdot \frac{(n)(n+1)(2n+1)}{6} + \frac{n(n+1)}{4}$$

$$= \frac{(n^2+n)(2n+1)}{12} + \frac{3n^2+3n}{12}$$

$$= \frac{2n^3 + 2n^2 + n^2 + n + 3n^2 + 3n}{12}$$

$$= \frac{2n^3 + 6n^2 + 4n}{12}$$

$$= \frac{n^3 + 3n^2 + 2n}{6}$$

$$= \frac{n(n^2 + 3n + 2)}{6}$$

$$Presents_n = \frac{n(n+2)(n+1)}{6}$$

Testing:

$Present_1 = \frac{1(3)(2)}{6} = 1$ ✓

$Present_2 = \frac{2(4)(3)}{6} = 4$ ✓

$Present_3 = \frac{3 \cdot 5 \cdot 4}{6} = 10$ ✓

39. $n=3$ $([1,4])$ $n=4$ $([1,5])$

$[1,3,2]$ missing 4 $\quad [4,2,3,5]$ missing 1

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2} \leftarrow$$ Value if no numbers missing

$\therefore$ solution is to sum the array, then diff with $\sum_{i=1}^{n+1} i$ to obtain the missing value.

$$\text{missing} = \sum_{i=1}^{n+1} i - \text{Sum(array)}$$

Test : $n=3 \rightarrow \sum_{i=1}^{n+1} i = 1+2+3+4$
$= 10$
$= 10 - \underbrace{\text{sum}[1,3,2]}_{6}$
$= 4$ ✓

$n=4 \rightarrow 15 - \underbrace{\text{sum}[4,2,3,5]}_{14}$
$= 1$ ✓

40 $T(n) = \sum_{i=1}^{n} \sum_{j=i}^{2i} 1$

(a)

(b) $\sum_{i=1}^{n} (2i - i + 1) = \sum i + 1 = \frac{\overbrace{n(n+1)}^{n^2+n}}{2} + \frac{n \cdot 2}{2} = \frac{n^2+3n}{2}$

Test : $T(1) = \frac{1+3}{2} = 2$ ✓ $T(3) = \frac{9+9}{2} = 9$ ✓
$T(2) = \frac{4+6}{2} = 5$ ✓

41 $T(n) = \sum_{i=1}^{n/2} \sum_{j=i}^{n-i} \sum_{k=1}^{j} 1$

a.

b. $\sum \sum_{j=i}^{n-i} j$

$= \sum \frac{(n-i-i+1)n}{2}$

$= \sum_{i=1}^{n/2} \frac{n^2 - 2in + n}{2}$

$= \frac{1}{2}\left[\sum n^2 - 2n\sum i + \sum n\right]$

$= \frac{1}{2}\left[n^2\left(\frac{n}{2}\right) + n\left(\frac{n}{2}\right) - \cancel{2}n\left(\frac{\left(\frac{n}{2}+1\right)\left(\frac{n}{2}\right)}{\cancel{2}}\right)\right]$

$= \frac{1}{2}\left[\frac{n^3}{2} + \frac{n^2}{2} - \underbrace{\left(\frac{n^2}{2}+n\right)\left(\frac{n}{2}\right)}\right]$

Test:

$T(1) = \frac{(1)^3}{8} = \frac{1}{8}$ ✗

$T(2) = \frac{2^3}{8} = 1$ ✓

$T(3) = \frac{3^3}{8} = \frac{27}{8} = 3\frac{3}{8}$ ✗

$T(4) = \frac{4^3}{8} = \frac{64}{8} = 8$ ✓

Conclusion: Need to change expression to

$$\left\lfloor \sum_{i=1}^{n/2} \sum_{j=i}^{n-i} \sum_{k=1}^{j} 1 \right\rfloor = \left\lfloor \frac{n^3}{8} \right\rfloor$$

I wonder if you can express floor function in algebraic terms...

$$= \frac{1}{2}\left[\frac{\cancel{2}n^3}{4} + \cancel{\frac{n^2}{2}} - \cancel{\frac{n^3}{4}} - \cancel{\frac{n^2}{2}}\right]$$

$$= \frac{1}{2}\cdot\frac{n^3}{4} = \frac{n^3}{8}$$

42. single digit, base 10 (eg. 5×4)

$\Rightarrow$ $5+5+5+5 = 3$ ops

1 operation ea.

$5\times 1 = 0$ op

$5\times 9 = 8$ ops

$9\times 9 = 8$ ops

$11\times 11 = (10 \text{ ops})\times 2 = 20$ ops

$100\times 100 = 99 \text{ ops} \times 3 = 297$ ops

Observation: with base 10, max 9 (~ 10) additions can be done from multiplying 1 digits

with 2 digits, max 99 (~ 100) additions, each addition 2 ops

with n digits, max $\sim 10^n$ additions, each addition n ops

with base 2, max 1 (~ 2) additions on 1 digit

$1\cdot 1 = 1+1 = 10 = 1$ op

$\underbrace{1}_{3}\underbrace{1\cdot 11}_{3} = 11+11+11 = 2 \text{ ops} \times 2 = 4$

$11111\cdot \underbrace{11111}_{2^5-1=31} =$

with 2 digits, max 3 (~ 4) additions, each addition 2 ops

with n digits, max $\sim 2^n$ additions, each addition n ops

-2 because:
- -1 from $x-1$ additions for x terms
- -1 from no addition if all digits are 0

$$\text{Pattern: } \overset{\text{Max}}{\text{Ops}}(n,b) = \overbrace{(b^n-2)}^{\text{additions}}n$$

$$= nb^n - 2n$$

$$= \Theta(nb^n)$$

43
$$\begin{array}{r} \dot{1}27 \\ \times\ 211 \\ \hline 127 \\ 1270 \\ +\ 25400 \\ \hline 26{,}797 \end{array}$$

2 adds, 3 mults

2 adds, 3 mults

3 mults

4

3 digits $\Rightarrow$ 13 ops

$3+(3+1)$ digits long

n digits $\Rightarrow$ n^2 mults,
$(n-1)^2$ adds

$$\underset{(n)}{\text{MaxOps}} = n^2 + (n-1)^2$$
$$= n^2 + n^2 - 2n + 1$$
$$= 2n^2 - 2n + 1$$
$$= \Theta(n^2)$$

44. a. $$a^{\log_a xy} = xy$$
$$= a^{\log_a x} \cdot a^{\log_a y}$$
$$= a^{\log_a x + \log_a y}$$

From online

b. $$\log_a x^y = \log_a (x^{y-1} \cdot x)$$
$$= \log_a x^{y-1} + \log_a x$$
$$= \log_a x^{y-2} + \log_a x + \log_a x$$
$$= \cancel{\log_a x^{y-y}} + \underbrace{\log_a x + \cdots + \log_a x}_{y \text{ terms}}$$
$$= y(\log_a x)$$

c. $$\log_a x = y$$
$$\Downarrow$$
$$a^y = x$$
$$\log_b a^y = \log_b x$$

From online again

Trick: solve for y, using exp:log inverse rule and also taking the $\log_b$ of both sides (that's possible?!)

$\log_b a = \log_b x$ $\quad y = \frac{\log_b x}{}$

$\log_a x$ $\log_b a$

$\log_a x \cdot \log_b a = \log_b x$

d. $x^{\log_b y} = z$ Finding z...

$\Downarrow$

$\log_y(x^{\log_b y}) = \log_y z$

$(\log_b y)(\log_y x) = \log_y z$

$\log_b x = \log_y z$

$z = y^{\log_b x}$

$\boxed{\log_a(b) = c \Downarrow a^{(c)} = b}$

$1 = \frac{3}{a} \leftarrow \frac{a}{1} = 3$

$1 = \frac{\log 3}{\log a} \leftarrow \log a = \log 3$ makes sense

$e^a = e^3$ also makes sense

$e^y = e^{mx+b}$ now too?

$1 = \frac{48/12}{4} \leftarrow 4 = \frac{48}{12}$

$1 = \frac{\log(\frac{48}{12})}{\log 4} \leftarrow \log 4 = \log(\frac{48}{12})$

I guess exp or log of both sides is always fine, but not of both numerator and denominator.

45.

$\lceil \lg(n+1) \rceil$

$\lg 1 = 0$

$\lg 2 = 1$

$\lg 4 = 2$

$\lg 8 = 3$

$\lceil \lg 3.5 \rceil = \lceil 1.8 \rceil$

$= \lfloor 1.3 \rfloor + 1$

$= 2$

$\lceil \lg 5 \rceil = \lceil 2.3 \rceil = 3$

$\lfloor \lg 4 \rfloor + 1 = \lfloor 2 \rfloor + 1$

Base case: $n=1$

$\lceil \lg 2 \rceil = \lceil 1 \rceil = 1$

$= \lfloor \lg 1 \rfloor + 1 = 0 + 1$

Inductive Case:

Assume $\lceil \lg(n) \rceil = \lfloor \lg(n-1) \rfloor + 1$

Facts:

$\lg(n+1) > \lg(n)$

$\lg(n+1) < \lg(n) + 1$ $\therefore$ input increase of +1 means smaller than +1 output increase

$\lceil x \rceil = \lceil x - 1 \rceil + 1$

$\lg(n) < \lg(n+1) < \lg(n) + 1$

$\lfloor \lg n \rfloor + 1 = \lfloor \lg(n) + 1 \rfloor$

$$\lceil \lg(n+1) \rceil = \lceil \lg(n+1) - 1 \rceil + 1$$

$$\lg n + 1 > \lceil \lg n \rceil$$

Biggest increase is if $= \lceil \lg 1.00..01 \rceil = \lceil 0.00..01 \rceil = 1$,
which is still smaller than
$\lg 1.00..01 + 1 = 1.00..01$
Smallest increase is if $\lg n$ is an integer,
so $\lceil \lg n \rceil = \lg n$

Question rewritten:
$\lceil \lg n \rceil - \lfloor \lg(n-1) \rfloor = 1$ prove

$$\lg(n) - \lg(n-1) = \begin{cases} 1 & \text{iff } n = 2 \\ \text{between } (0, 1) & \text{otherwise} \end{cases}$$

If $n \in \mathbb{Z}$, then 4 scenarios

eg $\lg 6 = 2.58$
A $\lg 5 = 2.3$
B $\lg 4 = 2$
C $\lg 3 = 1.58$

	$\lg n \in \mathbb{Z}$	$\lg(n) \notin \mathbb{Z}$
$\lg(n-1) \in \mathbb{Z}$	$n=2$	B
$\lg(n-1) \notin \mathbb{Z}$	C	A

Case B: $\lceil \lg n \rceil - \lfloor \lg(n-1) \rfloor = \lceil \lg n \rceil - \lg(n-1)$
$= \lg(n-1) + 1 - \cancel{\lg(n-1)}$ $\because$ we know $\lg(n-1)$ is next smallest integer
$= 1$

Case C: $\lceil \lg n \rceil - \lfloor \lg(n-1) \rfloor = \lg n - \lfloor \lg(n-1) \rfloor$
$= \cancel{\lg n} - (\lg(n) - 1) = 1$ $\because$ we know $\lg n$ is next largest integer.

Case A: $\lceil \lg n \rceil - \lfloor \lg(n-1) \rfloor = 1$ $\because$ both terms b/w same 2 integers.

If $n \notin \mathbb{Z}$, this rule doesn't hold, it seems.
Ex. $\lceil \lg 4.1 \rceil \approx \lceil 2.01 \rceil = 3$
$\lfloor \lg 3.1 \rfloor + 1 = \lfloor 1.99 \rfloor + 1 = 2$

46.

$n = 1 \Rightarrow 1$ (1 bit)
$= 2 \Rightarrow 10$ (2)
$= 3 \Rightarrow 11$ (2)
$= 4 \Rightarrow 100$ (3)

Obs: x bits holds numbers from 0 to $2^x - 1$

Doesn't seem like induction is useful here...

Base case: see this

Inductive case: Assume $n-1$ is represented by $\lfloor \lg(n-1) \rfloor + 1$ bits (ie. $\text{bits}_{n-1} = \lfloor \lg(n-1) \rfloor + 1$)

Maybe only in summation/ recursive sequences!

Show

$bits_n = \lfloor \lg n \rfloor + 1$ for $n \geq 1$

$\lfloor \lg n \rfloor$ :: Jumps from 1 to n in multiples of 2, rounded down st. $\ln 3 = 2 \cdot x \rightarrow 2$

Since x bits holds 2^x possible values, we can conclude x bits holds numbers 0 to $2^x - 1$

So to hold number n st $n = 2^x - 1$, solve for X to find bits needed: $2^x = n + 1$

$$x = \lg(n+1)$$

Since we know $n, x \in \mathbb{Z}$, we can modify the equation thus:

$x = \lceil \lg(n+1) \rceil$ :: a noninteger value for x implies we need more bits than $\lfloor x \rfloor$, thus $\lceil x \rceil$

(As per problem 45,) one $n \in \mathbb{Z}$

$$x = \lfloor \lg n \rfloor + 1 \quad \blacksquare$$

Observation

whats the relationship b/w floor & ceil?

if $x \in \mathbb{Z}$,

$\lceil x \rceil = x = \lfloor x \rfloor$

if $x \notin \mathbb{Z}$,

$\lceil x \rceil = \lfloor x \rfloor + 1$

Obs: proofs are hard. They need you to understand clearly the chain of inferences that you'll need.

But it'll help me become a better logician.

using ln instead of lg because derivative is simpler, and ln is equiv to lg in Big Oh terms

47. $\lim_{n\to\infty} \dfrac{\cancel{n} \ln \sqrt{n}}{\cancel{n} \ln n} = \lim_{n\to\infty} \dfrac{\frac{1}{\sqrt{n}}\left(\frac{1}{2} n^{-\frac{1}{2}}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \dfrac{n}{\sqrt{n}} \cdot \dfrac{1}{2\sqrt{n}} = \lim_{n\to\infty} \dfrac{\cancel{n}}{2\cancel{n}} = \dfrac{1}{2} \Rightarrow$ no dominance; same class.

$\therefore O(n \lg \sqrt{n}) = O(n \lg n)$, so a function can be $O(n \lg \sqrt{n})$ while still maintaining $\Omega(n \lg n)$

48. 5 element $\rightarrow$ 2 element st. a,b,c,d,e each have $\frac{2}{5} = 40\%$ chance of being x, y.

$\{a\ b\ c\ d\ e\}$ $\{x, y\}$

n element $\rightarrow$ 2 element set : How to select?

$\{a \ldots n\}$ $\{x, y\}$

Proposal: for e in set:
if randint(1,5) $\geq$ 4, put in subset

then mark as added

Stop after subset has 2 elements

Plan: Loop over set of n numbers to find length.
then for each element e in subset, set its value to
set[randint(0, n)] (then mark as used)

number

maybe this step doesn't ensure a uniform distribution over S?

Alt: Use array of random numbers of size n, pick indices from array to get elements for subset until full. But generating that array (rng) could be done with same method as above and be considered uniform, right? If so, then original plan is still valid.

49

A B C D

nodes=4, data=4

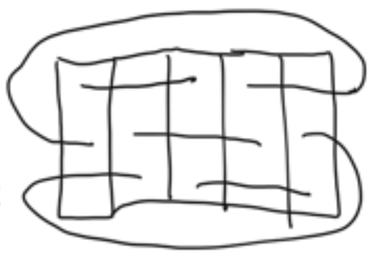
ABC
ABD
ACD
BCD

EV=1

$\binom{4}{3}=4$

$\binom{5}{3}=\frac{5\cdot 4}{2}=10$

$\frac{5 \text{ failure configs}}{10} \Rightarrow EV=.5$



← n=5, d=5

Pattern: interleave replication across nodes. Seems to scale to any n, d quantity.

$\frac{6}{\binom{6}{3}}=\frac{3}{10}=.3$

n=6, d=6

expected loss when 1 random node fails:
0, since every entry is replicated twice on other nodes. on the failed node

Obs:
Number of failure combinations is equal to number of data entries.

Expected loss when 3 random nodes fail:
$= 1\cdot P$(those 3 nodes hold same data entry) + $0 \cdot P$(nodes don't hold any same entries)
$= 1\cdot P$(those 3 nodes hold same entry)

1, since that one entry will be lost

0 since no loss if all of them hold different entries

Since P(those 3 nodes hold same entry)

$= \frac{1000}{\binom{1000}{3}} = \frac{1}{166,167}$, then that is the EV.

times trip is assigned

50. $EV = \sum_{i=1}^{n} i \cdot p_i$ prob. that

$[5, 1, 3, 6]$ — $i=2$ — $[5, 6, 1, 3]$ — $i=3$

$$= 1p_1 + 2p_2 + 3p_3 + \dots + np_n$$

when min is 1st $= \frac{(n-1)!}{n!}$

prob

What is the pattern here?

prob of reverse order: that's when tmp is assigned each iteration

$$\left(\frac{1}{n}\right)\left(\frac{1}{n-1}\right)\left(\frac{1}{n-2}\right)\cdots 1$$

$$= \frac{1}{\prod_{i=0}^{n}(n-i)}$$

$$= \frac{1}{n!}$$

ex.

123, 132, 213, 231, 312, 321

$$1p_1 + 2p_2 + 3p_3$$

$$= 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{6}$$

$$= \frac{1}{3} + 1 + \frac{1}{2}$$

$$= \frac{2}{6} + \frac{6}{6} + \frac{3}{6} = \frac{11}{6} = 1\frac{5}{6} = 1.8\overline{3}$$

ex when $n=4$

$$1\left(\frac{6}{24}\right) + 2\left(\frac{11}{24}\right) + 3\left(\frac{6}{24}\right) + 4\left(\frac{1}{24}\right)$$

$$= 2.08\overline{3}$$

Online: Try another angle.

let $X_{i \in [1,N]} = 1$ if A_i ever set to max

$\therefore$ min_set_ct $= X_i + \dots + X_n$

$$E(\text{min\_set\_ct}) = E(X_i + \dots + X_n) = E(X_i) + \dots + E(X_n)$$

$$= 1 \cdot P_{1 \text{ ever set as min}} + 1 \cdot P_{2 \text{ ever set as min}} + \dots + 1 \cdot P_{n \text{ ever set as min}}$$

Pattern: A_i ever set as min if its min of its left side (incl. self). Assuming uniform dist, that happens with prob $\frac{1}{\text{leftside(incl. self)}}$

$$= 1 \cdot \frac{1}{1} + 1 \cdot \frac{1}{2} + \dots + 1 \cdot \frac{1}{n} = \underbrace{\sum_{i=1}^{n} \frac{1}{i}}_{\text{harmonic}} \approx \ln n + \gamma$$

51. Take 1 coin from bag 1, 2 from 2, etc. Diff actual weight with expected weight if all were the same weight. The number of ...

weight if all were the same weight. The number of grams off corresponds to the bag number that has the false coins.