C5: Physics of Atmospheres and Oceans Lecture Notes

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Introduction

These set of lecture notes for the C5: Atmospheres and Oceans Option are based on the lectures delivered during the academic year 2024 - 2025. The lectures that year were delivered by Raymond T. Pierrehumbert (for Thermodynamics and Radiative Transfer), Philip Stier (for Clouds), Neil Bowles (for Instrumentation), Myles Allen (for Climate Dynamics), and Tim Woollings (for Geophysical Fluid Dynamics).

C5 is a massive course, and because it's very 'applications'-focused, it doesn't feel as unified as some of the other C options. Because of this, it can feel difficult to distinguish between what you need to know and what you don't. These notes are the result of an attempt to rectify this by compiling a self-consistent and thorough set of notes on the course. As such, these notes will be long: we will at many points prioritise rigour and detail over brevity, and we'll link to where you can look if you want more information. However, we will try to emphasise (using coloured boxes!) key ideas to make these notes more skimmable. Furthermore, there is a limit to how thorough we can be. We've already mentioned that C5 is very applied, so we will build on and take for granted the results of a great many sub-fields (many of which take a whole lecture series themselves to explain).

In reading these notes, you should be aware of two sources of biases: the lecturers and the author(s).

In the first case, it is not uncommon for lecturers to gloss over certain material in lectures. For example, the 2024-2025 lectures did not feature scattering in Radiative Transfer, and so I will not cover scattering in these lecture notes. Furthermore, some of what is covered is on the very frontiers of modern research. As such, it's not surprising that the lecturers will have certain beliefs that have not (yet?) made it into the scientific consensus. We will try to flag when this happens, but there will be points where we represent some material very authoritatively, when in reality it is very disputed. Please let us know if you believe this happens at any point.

The second case is far more problematic. First, the author is just not passionate about Instrumentation. That chapter will be empty. Sorry. If you know someone (or you yourself) would be interested in filling that in, please let us know! Second, in some cases we present material differently to how the lecturers present them in attempt to present material in a way we personally find less confusing and more physically intuitive.

Third, and most importantly, I can't claim to be an expert in the topic, and so will get things wrong! Please get in touch if you spot any errors or believe any section is written confusingly. I would much rather you contact me about an 'error' that's actually correct than I miss an error!

Part I

Thermodynamics

Introduction

This section of the course was lectured by Raymond T. Pierrehumbert covering basic Atmospheric Thermodynamics. While most concepts here will be applicable to oceanic physics, we will only be explicitly applying the concepts learnt in this section to atmospheric physics.

This section consists of three chapters:

1. Basic Thermodynamic Concepts:

   We recap some basic thermodynamic concepts you should be familiar with, like pressure, ideal gases, and heat capacity. We then explain how to extend such concepts to deal with gases consisting of multiple constituents. Finally, we introduce the important approximation of Hydrostatic Balance.

2. Dry Thermodynamics:

   We focus primarily on the vertical temperature structure of the atmosphere. We predict and explain some observations by deriving the Dry Adiabat, which governs the temperature of a convecting parcel of air. Next, we consider convection, and derive a criterion of whether an atmosphere will be unstable to convection and define a few terms relating to convection.

3. Moist Thermodynamics:

   We extend the previous section to apply to atmospheres which have constituents which condense. We derive the Moist Pseudo-Adiabat, the moist counterpart to the Dry Adiabat. We do not discuss moist convection here, and instead discuss this in Part III.

Chapter 1

Basic Thermodynamic Concepts

1.1 Definitions

1.1.1 Basic Thermodynamic Definitions

We should review (or learn) the basic state variables of thermodynamics.

For Atmospheric and Oceanic Physics, we will almost always work with '*Intensive*' variables. These are variables that are independent of the size of a system. For example, 'number of particles' is not an intensive variable, because the number of particles changes (it doubles) if you double the size of the system. However, 'number of particles *per unit volume*' is an intensive variable, because this remains unchanged when you double the size of a system.

As a brief review, we'll need to know the following *intensive* variables:

Symbol	Name	Units	Meaning
p	Pressure	Pa (Pascal); bar (Bar); $N\,m^{-2}$ (Newtons per square metre);	The force exerted by the fluid in *all* directions at some location. $1\,\text{bar} \approx 10^5\,\text{Pa}$.
T	Temperature	K (Kelvin)	A measure of the heat content of the system.
ρ	Density	$kg\,m^{-3}$ (Kilograms per metre cubed)	The mass per unit volume.
n_a	Number Density	m^{-3} (Inverse metre cubed)	The number of molecules of some substance a in one metre cubed.
M_a	Molar mass	$kg\,mol^{-1}$ (Kilograms per mole)	The mass of $N_A \sim 6.02 \times 10^{23}$ molecules (one mole) of a.

Table 1.1: Basic Intensive Thermodynamic Variables

Note that many resources present the molar mass M_a in units of $g\,mol^{-1}$ (grams per mole). This is in order to produce more pretty numbers: for example, the molar mass of N_2 is $28.01 \times 10^{-3}\,kg\,mol^{-1}$ = $28.01\,g\,mol^{-1}$. Be wary of units, and always remember to convert everything to SI units when performing calculations. In this case you can convert the molar mass from units of $g\,mol^{-1}$ to units of $kg\,mol^{-1}$ by dividing by 1000:

$$\underbrace{M_a}_{M_a \text{ in SI units of } kg\,mol^{-1}} = \underbrace{M_a}_{M_a \text{ in non-SI units of } g\,mol^{-1}} \cdot \frac{1\,\text{kg}}{1000\,\text{g}}$$

1.1.2 Multiple Constituents

In general, a planet's atmosphere is composed of multiple constituents. Our goal now is to introduce some vocabulary so we can specify the abundance of a certain constituent in the atmosphere.

It turns out that we (or the examiner) must make two ultimately arbitrary decisions if we wish to refer to a certain constituent's abundance. First, we can either refer to a constituent's abundance

by its *mass*, or by its *number* (in *moles*). Second, we can either refer to a *concentration* or a *ratio*. These choices are ultimately arbitrary because one may freely convert between them with the molar mass using Equation 1.1.

Suppose our atmosphere is composed of constituents $\mathbb{S} = \{a, b, c, \ldots\}$. Each species $i \in \mathbb{S}$ has a molar mass of M_i and a number density of n_i. Then we can refer to its abundance as follows. All quantities are dimensionless.

Key Idea I.1: Definitions Regarding Constituent Abundance

We can refer to the abundance of a certain constituent a in an atmosphere/ocean as follows:

	Fraction/Concentration	Mixing Ratio
Mole	*Mole Fraction* or *Molar Concentration* $$x_a = \frac{n_a}{\sum\limits_{i \in \mathbb{S}} n_i}$$	*Molar Mixing Ratio* or *Volume Mixing Ratio* $$\frac{n_a}{\sum\limits_{i \in \mathbb{S}, i \neq a} n_i}$$
Mass	*Mass Fraction* or *Mass Concentration* $$q_a = \frac{\rho_a}{\sum\limits_{i \in \mathbb{S}} \rho_i} = \frac{M_a n_a}{\sum\limits_{i \in \mathbb{S}} M_i n_i}$$	*Mass Mixing Ratio* $$\frac{\rho_a}{\sum\limits_{i \in \mathbb{S}, i \neq a} \rho_i} = \frac{M_a n_a}{\sum\limits_{i \in \mathbb{S}, i \neq a} M_i n_i}$$

where n_i = the number density of i; M_i = the molar mass of i; and ρ_i = the density of i. We let x_a and q_a denote the mole fraction and mass fraction of a, respectively.

Which term you use is arbitrary, as you can freely convert between all four expressions using algebra or by using the following formula:

$$\rho_A = \frac{n_A M_A}{N_A} \tag{1.1}$$

where N_A = Avogadro's Number $\sim 6.02 \times 10^{23}\,\text{mol}^{-1}$ (it's on your formula sheet).[1]

We also introduce the concept of '**dilute**': some constituent a is in the **dilute** limit if and only if:

$$\boxed{n_a \ll \sum_{i \in \mathbb{S}, i \neq a} n_i \qquad \text{and/or} \qquad \rho_a \ll \sum_{i \in \mathbb{S}, i \neq a} \rho_i} \tag{1.2}$$

An important upshot of this is that, in the dilute limit, fractions/concentrations and mixing ratios are equivalent. In many problems, this simplifies the algebra massively, but you're not always allowed to assume that constituents are dilute (especially in exams!).

Note also that there is some ambiguity in the 'and/or' in Equation 1.2. For example, we might have a situation where $n_a \ll \sum n_i$ but **not** $\rho_a \ll \sum \rho_i$ (or vice versa). This occurs only if the molar masses M_a and M_i are not all of similar size (convince yourself that this is the case using Eqn. 1.1). This is almost never the case in scenarios we consider, so you can just treat the 'and/or' as just an 'and' in Equation 1.2.

Sometimes, we'll see people refer to dilute constituents in terms of *ppm* (parts per million) or *ppmv* (parts per million volume). *ppm* is defined as the *Mass Fraction* or *Mass Concentration* multiplied by 10^6 ($q_a \times 10^6$) while *ppmv* is defined the *Mole Fraction* or *Molar Concentration* multiplied by 10^6 ($x_a \times 10^6$).

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[1]To avoid a small confusion, note how only number densities n_i, not molar densities, are used in the definition for the *molar* concentrations/mixing ratios. Using molar densities or number densities give the same result. This is because the number of moles per volume of a substance is directly proportional to the number density: $n_i^{mol} = n_i N_A$. As such, the N_A's simply cancel top and bottom in the fraction.

Atmospheric Composition: Planetary Examples

In *Mole Fraction x_a*:

- Earth's Atmosphere:
  - Nitrogen N_2: 0.78
  - Oxygen O_2: 0.21
  - Argon Ar: 0.0093
  - Carbon Dioxide CO_2: 0.000430 (430 *ppmv*)[2]
  - Water Vapour H_2O: A few percent.[3]

- Venus' Atmosphere:
  - Carbon Dioxide CO_2: 0.965
  - Nitrogen N_2: 0.035
  - Sulfur Dioxide SO_2: 150 *ppmv*
- Jupiter's (Outer) Atmosphere:
  - Hydrogen H_2: 0.86
  - Helium He_2: 0.136

In Earth's ocean, the absolute salinity in units of $g\,kg^{-1}$ is defined as the mass fraction of salt. You can think of this as a *ppt* (parts per thousand). The ocean hovers around $30\,g\,kg^{-1}$.

1.2   Ideal Gases

1.2.1   Single Constituent Atmosphere

An **Ideal Gas** is a theoretical gas consisting of molecules which interact only via perfectly elastic collisions. No real gas is ideal, but many gases, including 99% of Earth's atmosphere (Nitrogen, Oxygen, and Argon), behave approximately like ideal gases under atmospheric conditions like ours. Ideal gases obey the ideal gas law, and we'll start from the version of the ideal gas law that you've probably seen before (and assume that the gas is made up of a single constituent a for simplicity):

$$\boxed{pV = N_a k_B T} \tag{1.3}$$

where p = pressure, V = volume, N_a = number of molecules of a, k_B = Boltzmann's constant, and T = temperature. We can divide by the volume, then multiply and divide by $\frac{M_a}{N_A}$ to get:

$$
\begin{aligned}
p &= n_a k_B T && \text{; Divide 1.3 by } V \\
&= \left(\frac{n_a M_a}{N_A}\right)\left(\frac{k_B N_A}{M_a}\right) T && \text{; Multiply and divide by } \frac{M_a}{N_A} \\
&= \rho_a \left(\frac{k_B N_A}{M_a}\right) T && \text{; Use 1.1 to substitute for } \rho_a \\
&= \rho_a \left(\frac{R^*}{M_a}\right) T && \text{; } R^* \equiv k_B N_A = \text{gas constant} \\
&= \rho_a R_a T && \text{; } R_a \equiv \frac{R^*}{M_a} = \text{specific gas constant}
\end{aligned}
$$

The **Gas Constant** R^* is defined as: $R^* \equiv k_B N_A \approx 8.314\,J\,mol^{-1}\,K^{-1}$. We further define the **Specific Gas Constant** $R_a \equiv \frac{R^*}{M_a}$ which has units of $J\,kg^{-1}\,K^{-1}$. **From now on I will refer to the specific gas constant R_a by just writing just R.** The final line gives us the version of the ideal gas law we will use most often as it features only *intensive* variables:

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[2]At time of writing!

[3]This strongly depends on time and location due to dynamics discussed in Moist Thermodynamics.

> **Key Idea I.2: Ideal Gas Law in Intensive Variables**
>
> The ideal gas law featuring only intensive variables of p, ρ, and T.
>
> $$p = \rho R T \tag{1.4}$$
>
> R is the specific gas constant, defined as the universal gas constant R^* divided by the molar mass M_a.
>
> $$R = \frac{R^*}{M_a} \tag{1.5}$$

1.2.2 Multiple Constituents in an Atmosphere and Dalton's Law

What if we now have multiple constituents in an atmosphere, all with different molar masses M_i? We first must define the *partial pressure p_a* of some constituent a as the pressure the gas *would* have if you removed all other constituents and left a on its own. Dalton's Law (1.6) allows us to relate the partial pressure of an individual constituent p_a to the total pressure p if all constituents are ideal gases:

$$\boxed{\frac{p_a}{p} = \frac{n_a}{\sum\limits_{i \in \mathbb{S}} n_i} = x_a} \tag{1.6}$$

So Equation 1.6 says that the partial pressure (of an ideal gas) is set by the *mole fraction/molar concentration x_a*[4]: it is not affected in any way by the *mass fraction q_a* or *mass mixing ratio*. We can further verify that, according to Dalton's Law, $p = \sum\limits_{i \in \mathbb{S}} p_i$ (since $\sum\limits_{i \in \mathbb{S}} x_i = 1$): the total pressure is the sum of all the partial pressures.

Now we apply the ideal gas law (Eqn. 1.3) individually to each constituent:

$$p_a = n_a k_B T$$
$$p_b = n_b k_B T$$
$$\vdots$$

We then sum up the equation above for each constituent and recall that $p = \sum\limits_{i \in \mathbb{S}} p_i$ (from 1.6) to get:

$$
\begin{aligned}
p &= \left(\sum_{i \in \mathbb{S}} n_i \right) k_B T \\[2mm]
&= \left(\frac{\sum\limits_{i \in \mathbb{S}} n_i M_i}{\sum\limits_{i \in \mathbb{S}} n_i M_i} \right) \left(\frac{N_A}{N_A} \right) \left(\sum_{i \in \mathbb{S}} n_i \right) k_B T && \text{; Creatively multiply by 1} \\[2mm]
&= \left(\frac{\sum\limits_{i \in \mathbb{S}} n_i M_i}{N_A} \right) \left(\frac{\sum\limits_{i \in \mathbb{S}} n_i M_i}{\sum\limits_{i \in \mathbb{S}} n_i} \right)^{-1} (N_A k_B) T && \text{; Regroup terms} \\[2mm]
&= \left(\sum_{i \in \mathbb{S}} \frac{n_i M_i}{N_A} \right) \left(\bar{M}^{-1} R^* \right) T && \text{; Define the \textbf{effective molar mass} } \bar{M} \\[2mm]
p &= \rho \, R \, T && \text{; Define the \textbf{specific gas constant} } R
\end{aligned}
$$

We should focus on the last three lines of algebra to remember two key facts:

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[4]In the dilute limit, it is equivalently set by the *molar mixing ratio*.

1. The specific gas constant R for a gas consisting of multiple constituents is the gas constant R^* divided by the **total effective molar mass** $\bar{M}$ (analogous to the single component case):

$$R = \frac{R^*}{\bar{M}} \tag{1.7}$$

2. The effective molar mass is the average of the individual molar masses M_i **weighted by particle number** n_i/**mole fraction** x_i:

$$\boxed{\bar{M} = \frac{\sum\limits_{i \in \mathbb{S}} n_i M_i}{\sum\limits_{i \in \mathbb{S}} n_i} = \sum_{i \in \mathbb{S}} x_i M_i} \tag{1.8}$$

In the single component case, $\bar{M} = M_a$.

For the Earth (excercise: consider only Nitrogen, Oxygen, and Argon), $\bar{M} \approx 29 \times 10^{-3} \, \text{kg mol}^{-1}$ so the **Specific Gas Constant** $R \approx 287 \, \text{J kg}^{-1} \, \text{K}^{-1}$ (it will be on your formula sheet).

1.3 Heat Capacity and the Equipartition Theorem

The heat capacity of an object is a measure of how much heat (which has dimensions of energy) is required to change an object's temperature. Let $đQ =$ the heat transferred to an object and $dT =$ the change in temperature of the object. Then the heat capacities of the object are:

$$c_v = \left(\frac{đQ}{dT}\right)_V \tag{1.9}$$

$$c_p = \left(\frac{đQ}{dT}\right)_p \tag{1.10}$$

where the subscripts of V and p mean that the volume and pressure, respectively, are held constant as heat is added/removed from the system. Notationally, when we write c_v or c_p, we usually mean the specific heat capacities: the heat capacities per mole or, more commonly, per mass. In exactly the same way as the specfic gas constant R, if not specified, assume that c_p and c_v denote the heat capacities *per unit mass* (remember, *intensive* variables!). For multiple constituents, we can calculate the specific heat capacities from molar heat capacities using the effective molar mass $\bar{M}$.

It's useful to remember that the heat capacity itself often depends on the temperature. However, sometimes we'll assume that it does not for analytical progress (and not realism).

It turns out, and one can derive this via statistical mechanics, that at high enough temperatures systems obey the Equipartition Theorem:

$$\boxed{c_v \approx \frac{f}{2} R} \tag{1.11}$$

where $f =$ the number of *fully excited* degrees of freedom an individual constituent has. So for a hot enough monatomic gas, $f = 3$, since monatomic gases have 3 translational degrees of freedom (forwards/backwards, up/down, right/left). For a hot enough linear diatomic gas (as seen in Fig. 1.1), $f = 7$, since diatomic gases have 3 translational degrees of freedom, 3 rotational degrees of freedom (spinning about the x, y, z axes), and 2 vibrational degrees of freedom[5].

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[5]One might expect there to be only 1 vibrational degree of freedom (i.e., they can only vibrate in and out...right?). The reason why it's 2 is slightly complicated, and requires some background knowledge in statistical mechanics. In the derivation for the equipartition theorem, $f =$ the number of quadratic terms in the Hamiltonian. For translations, $H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right)$ so $f = 3$, and similarly for rotations. For vibrations, we approximate the Hamiltonian as a simple harmonic oscillator (which it will be if the vibrations are small enough), so there are *two* quadratic terms, arising from the kinetic and potential energies in the Hamiltonian (i.e., a spring): $H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2$.

Looking ahead, one can calculate the characteristic energy scale E of N_2's vibrational mode by solving analytically for the eigenvalues of the simple harmonic oscillator and finding the difference. One then only needs to quote two empirical constants: $m =$ the reduced mass of the N_2, and ω (which encodes the strength of the bond between the N atoms).

(a) 3 Translational Degrees of Freedom (b) 3 Rotational Degrees of Freedom (c) 2 Vibrational Degrees of Freedom
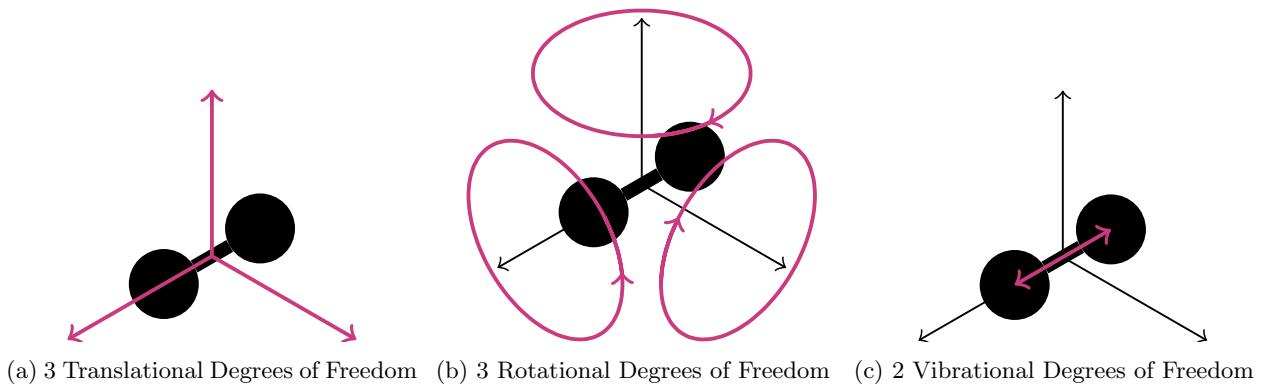
Figure 1.1: The 7 Degrees of Freedom for a Diatomic Molecules. Degrees of Freedom are shown in magenta arrows ■.

1.3.1   "Excited" and "Frozen-out" Degrees of Freedom

But what does it mean for a degree of freedom to be '*fully excited*' or for the temperature to be '*high enough*'?

To answer these questions, we must briefly discuss quantum mechanics, because the constituents of gases obey the laws of quantum mechanics (as do all things, we think). One way in which quantum mechanics differs from classical mechanics is that often (but not always) the energy that a system can have is **discretised** rather than **continuous**. For example, a system may only have energies specifically equal to 0, E, $1.8E$,... This means that the system can only receive or give energy in discrete amounts called 'quanta' (equal to the difference in energy between two discretely spaced energy levels) – if the molecule cannot receive these exact amounts of energy, then the molecule cannot receive energy at all.

This leads to a uniquely quantum mechanical effect[6]. Consider a molecule currently with energy equal to 0, with the lowest energy level above this with energy E. If the temperature is too low, the molecule might only get $0.001E$ energy on average, and is highly unlikely to get energy even close to E. As such, the molecule will never get enough energy to jump to the higher energy level (or any energy level higher than that). We say that there is not enough energy to 'excite' that degree of freedom and that that degree of freedom is 'frozen' out. More rigourously, if $k_B T \ll E$, where E is the size of the energy quanta of some degree of freedom, then that degree of freedom is frozen out. If $k_B T \gg E$, then that degree of freedom is '*fully excited*', and this is what it means for the temperature to be '*high enough*'.

The size of this quanta of energy E depends on the system, but for the constituents in atmospheres, the quanta for vibrational degrees of freedom tend to be very large. For example, the energy of the first vibrational mode of Nitrogen is $E \sim 4.7 \times 10^{-20}$ J. This seems small until you remember that this is *per molecule*. The temperature required for $k_B T \sim E$ is $T \sim E/k_B \sim 3400$ K.

Earth's atmosphere mainly consists of a diatomic gas, so we might expect $f = 7$. However, $f \approx 5$ because the translational degrees of freedom are all fully excited, but only 2 rotational degrees of freedom are excited and none of the vibrational degrees of freedom are (See Figure 1.2. I have drawn the excited degrees of freedom as thick magenta lines, and the frozen degrees of freedom as dashed thin magenta lines).

In reality, the heat capacity *per mole* (and *not* per mass) at constant volume is $c_v^* \approx 20.83$ J K^{-1} mol^{-1}. $R^* \approx 8.314$ J K^{-1} mol^{-1}, so our prediction would be $c_v^* \approx 5R^*/2 \approx 20.79$ J K^{-1} mol^{-1}, which is actually very close to the measured value!

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[6]This also explains the ultraviolet catastrophe!

(a) 3 Translational Degrees of Freedom All 3 Excited (b) 3 Rotational Degrees of Freedom 2 Excited (c) 2 Vibrational Degrees of Freedom None Excited

Figure 1.2: The 5 Excited Degrees of Freedom for a Diatomic Molecules in Earth-Like Conditions. Excited Degrees of Freedom are shown in filled magenta arrows ■, while frozen out Degrees of Freedom are shown in dashed magenta arrows ■.

Of course, the Equipartition Theorem says nothing regarding situations in which a degree of freedom is not fully excited nor frozen out. We are (perhaps only by coincidence) lucky that the situation is relatively simple on Earth; since the temperature lies within a range such that certain degrees of freedom are very excited and others are very frozen, we can very easily explain and predict that the atmospheric heat capacity is approximately $5R^*/2$ using the Equipartition Theorem. However, you should be aware that the situation is not so simple on many other planetary atmospheres.

1.4   Hydrostatic Balance

We now consider the forces on a slab of atmosphere:

Figure 1.3: Forces (per unit area) on the middle slab of air.

The *hydrostatic approximation* assumes that these are the dominant forces on the slab, and that the acceleration of the slab of air is negligible. These turn out to be good assumptions, but a full justification would be out of the scope of these notes.[7] Therefore:

$$0 = p(z + \delta z) - \rho g \delta z + p(z)$$

Rearranging and letting $\delta z \to 0$ gives us **Hydrostatic Balance**:

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[7]A full justification would have to start from *dynamics*. One would have to show that, if an arbitrary system were *not* in hydrostatic balance, then it would rapidly (relative to the timescales under consideration, which in this case might be days, months, or years) evolve to put itself approximately in hydrostatic balance.

> **Key Idea I.3: Hydrostatic Balance**
>
> Hydrostatic Balance governs the vertical pressure variation in a fluid if we assume that vertical acceleration (and advection) is small.
>
> $$\frac{dp}{dz} = -\rho g \qquad (1.12)$$
>
> Pressure decreases with altitude in order to balance the force of gravity.

There are two very useful facts that should be noted at this time.

First note that $\rho, g > 0$, so $\frac{dp}{dz} < 0$ always. This means that p is monotonic in z, implying that there is always a one-to-one relation between p and z. This allows us to convert between **height coordinates** and **pressure coordinates**. For example, in *height coordinates*, we might let $\vec{u} = \vec{u}(x, y, z)$, $p = p(x, y, z)$. In this case, we would have pressure p at some height z (and horizontal position) as a variable. In *pressure coordinates*, on the other hand, we let $\vec{u} = \vec{u}(x, y, p)$, $z = z(x, y, p)$. In this case, we would have the height z of some given pressure surface p (and horizontal position) as a variable.

We will consistently be plotting atmospheric properties (e.g., humidity) in pressure coordinates (as opposed to height coordinates) throughout this course. This is for three reasons. First, it is much easier for instruments to measure the pressure than the height. In fact, this is how airplanes measure 'barometric' height using a pressure altimeter [1]. Second, most problems we encounter will require only knowledge of a variable's dependence on pressure. Third, pressure is a fundamental thermodynamic variable, and as such also gives us information regarding the thermodynamic properties of the system, as well as the (rough) height. We convert to pressure coordinates explicitly in Section 17.4.

Second, note that if we rewrite Eqn. 1.12 in differential form, we get a very useful relation between the difference in pressure between two vertically separated points, and the mass per unit area within that layer.

$$dp = -g(\rho \, dz) \qquad (1.13)$$
$$= -g \, dm$$

where $dm = \rho \, dz$ is the mass per unit area within that layer. Therefore (integrating), we get that the pressure difference Δp between the bottom and top of a layer of atmosphere is directly proportional to the mass per unit area within that layer. Take a moment to appreciate this: we can measure the mass of a layer of atmosphere by only measuring the pressure difference between the top and the bottom! Integrating Equation 1.13 from the bottom of the atmosphere ($p = p_s, m = 0$) to the top of the atmosphere ($p = 0, m = m_{tot}$) (and assuming that variation in g is negligible) gives us:

$$m_{tot} = \frac{p_s}{g} \qquad (1.14)$$

The surface pressure on a rocky planet thus gives an immediate estimate of the amount of *stuff* in that planet's atmosphere. Again considering Earth and Venus' atmospheres:

- Earth:
  - Surface Pressure:
    $p_s \approx 1 \times 10^5 \, \text{Pa}$
  - Gravitational Acceleration:
    $g \approx 9.81 \, \text{m s}^{-2}$
  - Atmospheric Mass:
    $m_{tot} \approx 1.02 \times 10^4 \, \text{kg m}^{-2}$

- Venus:
  - Surface Pressure:
    $p_s \approx 93 \times 10^5 \, \text{Pa}$
  - Gravitational Acceleration:
    $g \approx 8.87 \, \text{m s}^{-2}$
  - Atmospheric Mass:
    $m_{tot} \approx 1.05 \times 10^6 \, \text{kg m}^{-2}$

Let us now apply all this to find the mass of CO_2 in the atmosphere:

$$m_{CO_2} = \underbrace{\frac{p_s}{g}}_{\text{Mass of Atmosphere}} \underbrace{\overbrace{\frac{12 \times 10^{-3}}{29 \times 10^{-3}}}^{\text{molar mass of } CO_2}}_{\hat{M} \text{ of air}} \underbrace{430 \times 10^{-6}}_{\text{mole fraction of } CO_2}$$

$$\underbrace{\phantom{\frac{12 \times 10^{-3}}{29 \times 10^{-3}} 430 \times 10^{-6}}}_{\text{mass fraction of } CO_2}$$

$$\approx 1.5 \, \text{kg} \, \text{m}^{-2}$$

So not much actually (but a lot radiatively, as you'll see in Part II)!

1.4.1 Example 1: Constant Density

In the ocean, for example, we can make the (sometimes) good approximation that $\rho = $ const. We can then integrate 1.12 with the boundary condition that $p(z = \eta) = p_s$ where p_s is, for example, the pressure at the surface of the ocean and η is the height of the surface. Note that this derivation is completely general, and that both p_s and η may in general be functions of x,y, and t, and η could be, for example, below the ocean surface at some interface.

$$\frac{dp}{dz} = -\rho g$$

$$\int_{p_s}^{p} dp = -\int_{\eta}^{z} \rho g \, dz$$

$$p(x, y, z, t) - p_s(x, y, t) = \rho g(\eta(x, y, t) - z)$$

$$\therefore \boxed{p(x, y, z, t) = \rho g(\eta(x, y, t) - z) + p_s(x, y, t)} \tag{1.15}$$

We use this to derive the Shallow Water/Reduced Gravity System in Chapter 18.

1.4.2 Example 2: Ideal Gas

The atmosphere can be roughly regarded as an ideal gas, so we can substitute in Equation 1.4 to find that Equation 1.12 transforms into:

$$\boxed{\frac{d \ln p}{dz} = -\frac{g}{RT}} \tag{1.16}$$

If we make the approximation that $T(z) \sim$ const (which is not a bad approximation in Kelvin) we can integrate to find that p falls off exponentially.

$$p(z) = p(z = 0)e^{-z/H}$$
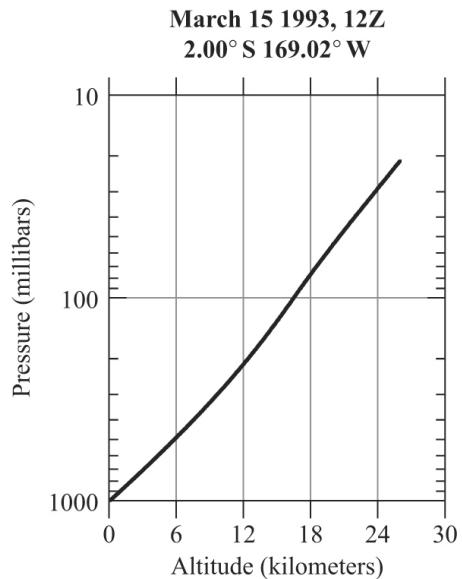
where $H = \frac{RT}{g}$ is the scale height. Plugging in Earth-like numbers[8] gives $H \approx 7.1 \, \text{km}$. This means that, roughly, the pressure falls off by a factor of e every 7.1 kilometers. Equivalently, the pressure falls off by a factor of 10 every $H \ln 10 \approx 16 \, \text{km}$.

Referring to the figure below, you can see that the pressure falls by a factor of 10 approximately every 17 km, which is pretty close!

---

[8]$R \approx 290 \, \text{J} \, \text{kg}^{-1} \, \text{K}^{-1}$, $T \approx 240 \, \text{K}$, $g \approx 9.8 \, \text{m} \, \text{s}^{-2}$. Only keeping two significant figures because we're interested in rough estimates here!

(a) A Radiosonde! Also called a weather balloon/sounding balloon. It's released and floats upwards, continually measuring the pressure, temperature, and humidity of the air around it. Image from the Radiosonde Museum of North America [2].

(b) Radiosonde measurement of the Earth. Note the log scale on the pressure axis, so if $p \sim e^{-z}$ we should expect a straight line, which we remarkably see. Notice how smooth the measurement is, despite the fact that it's a single radiosonde at a single (horizontal) point in space. Figure from Ray's book [3].

Recall that $R = R^*/\bar{M}$. This means that if we know the temperature, gravitational acceleration, and scale height of another planet's atmosphere, we can deduce the effective molar mass of the constituents of the atmosphere, and thus gain information on the planet's atmospheric composition!

1.4.3 Example 3: Solid Rock Planet

If we want to include curvature (since planets are more like spheres than infinite planes), we can derive an analogous Hydrostatic Relation where:

$$\frac{dp}{dr} = -\rho g$$
$$= -\rho \frac{GM(r)}{r^2}$$

We can couple this with a differential equation governing the mass below some radius r called $M(r)$:

$$\frac{dM}{dr} = 4\pi r^2 \rho$$

To solve these equations, as before, we need an expression for $\rho(r)$ and boundary conditions. $\rho(r)$ is typically given by the equation of state governing the material in question. For the Earth, for example, we might approximate $\rho \approx$ const below the ground ($r < r_s$) and ρ as ideal as in Example 2 above the ground ($r > r_s$). For boundary conditions, we know that $M(r = 0) = 0$ and we can let $p(r = 0) = p_0$ where p_0 is unknown. We can then integrate the equations, then solve for p_0 in terms of some other known reference pressure (e.g., if we know $p(r = r_s) = p_s$ we can write p_0 in terms of p_s).

Chapter 2

Dry Thermodynamics

2.1   The Vertical Structure of Atmospheres

We begin with some empirical observations of planetary atmospheres:

Figure 2.1: Vertical Temperature Profiles of Planets from Ray's book [3].

In all five plots, we see a sharp decrease of temperature with pressure in the lower portion of the atmosphere (which we call the **troposphere**), until a critical point (**tropoause**) in which the temperature decreases more slowly with height or even increases (**stratosphere**)[1].

In this chapter, we will attempt to explain some of this. We want to explain this because the vertical structure has profound consequences on many other phenomena of interest: the vertical structure affects, among other phenomena, where (or if) constituents condense (Moist Thermodynamics), the outgoing radiation to space (Radiative Transfer), and much of fluid dynamics (Geophysical Fluid Dynamics).

---

[1]I should note that I'm adhering to Ray's preferred terminology here. Ray uses **stratosphere** to refer to places in the atmosphere where radiative transfer is more dominant, which (we will see) causes a shallower decrease of temperature with height. This contrasts the more typical usage of the term **stratosphere**, which requires an increase of temperature with height. This only occurs if there is some heat source from above. In the case of the Earth, this heat source is due to the location of ozone, which is a very effective absorber of UV radiation. However, I prefer Ray's definition, due to reasons elaborated after I derive the Dry Adiabat and the Dry Stability Criterion.

As a spoiler:

- A temperature gradient is an indicator that there is an uneven heat source. The uneven heat source could be because:
  - Radiation from the sun is absorbed unevenly. For example, on Earth, the ground absorbs much more visible light from the sun than the atmosphere, and ozone in the stratosphere absorbs much more UV light than the rest of the atmosphere.
  - Energy is still leaking out from formation, as is the case with Jupiter.
- The atmosphere redistributes heat away from the aforementioned uneven heat source. Heat is transferred *down* the temperature gradient:
  - If temperature decreases with height, the heat is coming from below.
  - If temperature increases with height, the heat is coming from above.
- There are two dominant heat transport mechanisms:
  - Convection, which we will explain in this section. This is dominant in the **troposphere**.
  - Radiation, which will explain in Part II. This is dominant in the **stratsophere**.

2.2   The Dry Adiabat: Rising and Falling Parcels of Air

2.2.1   Derivation

To explain (some of) the aforementioned phenomenon, we start from the first law of thermodynamics governing a parcel of air:

$$dU = -p\,dV + T\,dS + \mu\,dN \tag{2.1}$$

where U = internal energy; p = pressure; V = volume; T = temperature; S = entropy; μ = chemical potential; and N = number of molecules. Roughly, the $-p\,dV$ term corresponds to work done by the parcel (i.e., energy transferred by the parcel pushing or being pushed by its surroundings), $T\,dS$ corresponds to the heat transfer between the parcel and its surroundings, and $\mu\,dN$ corresponds to energy gained or lost by exchanging particles with its surroundings.

We assume that the system is closed ($dN = 0$, so the parcel does not exchange air with its environment[2]) and divide by the mass of the air parcel to find:

$$du = -p\,d\left(\frac{1}{\rho}\right) + T\,ds$$

where u and s refer to the internal energy and entropy *per unit mass* (remember, we want intensive variables only!). We now approximate the air parcel as an ideal gas. This means that:

1. The gas obeys the Ideal Gas Law (1.4)
2. The internal energy is a function of temperature only: $u = c_v T$, where c_v = the specific heat capacity at constant volume.
3. The specific heat capacities are related in the following way: $c_v + R = c_p$ (also per unit mass)

$$
\begin{aligned}
c_v\,dT &= -p\,d\left(\frac{1}{\rho}\right) + T\,ds && ; du = c_v dT \text{ and assuming } \frac{\partial c_v}{\partial T} \approx 0 \\
&= -p\,d\left(\frac{RT}{p}\right) + T\,ds && ; \text{Ideal gas using Eqn. 1.4} \\
&= -pR\left(\frac{dT}{p} - \frac{T\,dp}{p^2}\right) + T\,ds && ; \text{Assume } \bar{M} \text{ is constant.} \\
&= -R\,dT + \frac{RT\,dp}{p} + T\,ds
\end{aligned}
$$

---

[2]This amounts to the assumption that entrainment does not occur. This is usually false, but it's not a bad assumption for what we aim to explain.

Dividing by T and rearranging, we find that:

$$ds = \frac{R_a + c_v}{T} dT + R\frac{dp}{p}$$

$$= c_p \, d\ln T - R \, d\ln p \qquad\qquad ; \, d(\ln x) = \frac{dx}{x} \text{ and } R + c_v = c_p$$

$$= c_p \, d\left(\ln\left(T(p)^{\frac{-R}{c_p}}\right)\right) \qquad\qquad ; \, \ln(x) - a\ln(y) = \ln\left(x(y)^{-a}\right)$$

If we make the crucial assumption that the parcel exchanges a negligible amount of heat with its surroundings, then we can approximate $ds = 0$[3]. In reality, there is some level of heat transfer ($ds \neq 0$), but these processes are typically much slower (timescale $\sim$ days/weeks) than the changes in pressure caused by upwards/downwards parcel motion (timescale $\sim$ hours). We can thus conclude that:

$$0 = c_p \, d\left(\ln\left(T(p)^{\frac{-R}{c_p}}\right)\right)$$

and so we derive:

Key Idea I.4: The Dry Adiabat

The dry adiabat governs the temperature of an air parcel that is adiabatically lifted or dropped (due to, e.g., convection or dynamics). The temperature increases as pressure increases. This is because, as the parcel moves to a location where the ambient atmospheric pressure is higher, the atmosphere does work on the air parcel to compress it and increase its energy and therefore increase its temperature.

$$\boxed{\frac{d\ln T}{d\ln p} = \frac{R}{c_p}} \qquad (2.2) \qquad\qquad \boxed{T(p) = T_0 \left(\frac{p}{p_0}\right)^{\frac{R}{c_p}}} \qquad (2.3)$$

If we crucially assume that **convection** is the **dominant vertical energy transport mechanism** in the atmosphere then the **atmosphere follows a dry adiabat**. This is somewhat accurate for Earth's troposphere.

2.2.2 Physical Interpretation

Equation 2.3 is the **Dry Adiabat**. '**Dry**' because we have no condensation, and '**Adiabat**' because we assume $dS = 0$ (and the word 'adiabatic' refers to processes where $dS = 0$). Constants (p_0, T_0) refer to the initial temperature and pressure of the air parcel.

Physically 2.3 corresponds to a parcel starting at some temperature and pressure (T_0, p_0) and being displaced upwards/downwards. When it is displaced upwards/downwards, it undergoes negligible diabatic heating and entrainment, so it's temperature does not change that way. However, when the ambient pressure changes, it expands/contracts, and the air parcel does positive/negative work on its surroundings. This *decreases/increases* the internal energy of the parcel, thus decreasing/increasing the temperature. That's why T decreases as p decreases in 2.3.

However, we should be wary of the limits of our conclusions here. Recall that we started this section considering an air parcel, not the entire atmosphere! We must make a final, crucial assumption that rising and falling air parcels (convection) are the dominant mechanisms for heat transport within some domain in the atmosphere (e.g., from the ground to a certain height – for example the tropopause). *This* implies that the temperature profile of the atmosphere within this domain follows the **Dry**

---

[3]Why can we only approximate $ds = 0$ and not set $ds = 0$? It is because the relation $dQ = T\,ds$ (where dQ is the heat transfer) holds only if the process is reversible, but we have made no such assumptions here. If the process is irreversible, as it probably is, then $dQ \leq T\,ds$ and so it is possible for $dQ = 0$ and $ds \neq 0$.
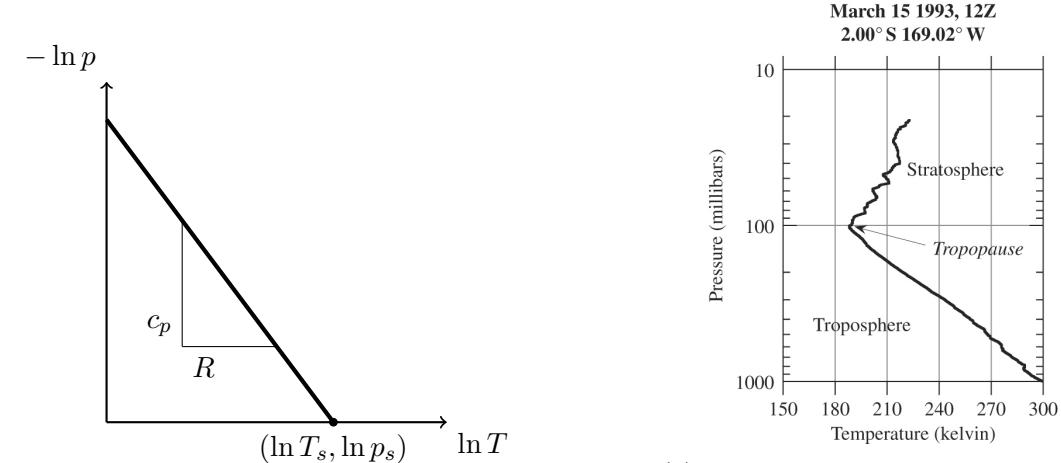
Adiabat. If this domain extends to the ground, we may take the initial conditions initial conditions (p_0, T_0) to be the temperature and pressure of the surface (p_s, T_s).

This assumption is not bad in the Earth's troposphere, partly because here the atmosphere is heated from below: the atmosphere is transparent to the radiation from the sun, which penetrates through the atmosphere and heats the ground. The ground then heats the air above it, which becomes buoyant and convects. Similarly, Jupiter's internal energy leakage maintains a convective troposphere. However, not all planetary atmospheres and not even all of Earth's atmosphere obey this assumption.

Finally, now, we can see now why many atmospheres have temperatures decreasing as pressure decreases or height increases. The first fact is readily noted from Eqn. 2.3. We can show the second fact by combining the Dry Adiabat 2.3, Hydrostatic Balance, and Ideal Gas to obtain an equation linking temperature with height:

$$
\begin{aligned}
\frac{R}{c_p} &= \frac{d\ln T}{d\ln p} \\
&= \frac{dT}{dz}\frac{p}{T}\frac{dz}{dp} && \text{; chain rule} \\
&= \frac{dT}{dz}\frac{p}{T}(-\rho g)^{-1} && \text{; 1.12} \\
&= \frac{dT}{dz}\frac{p}{T}\frac{RT}{p}(-g)^{-1} && \text{; 1.4} \\
\therefore\ \boxed{\frac{dT}{dz} = -\frac{g}{c_p} = -\Gamma_d} && \text{; } \Gamma_d = \text{ Dry Adiabatic Lapse Rate} \approx 10\,\text{K}\,\text{km}^{-1} && (2.4)
\end{aligned}
$$

(a) The Dry Adiabat, as predicted by Equation 2.3.

(b) Radiosonde measurement of Earth's Temperature Profile.

Figure 2.2: The Dry Adiabat: Theory and Observation

Note that R and c_p have the same units/dimensions, so $\frac{R}{c_p}$ is dimensionless. We can use the Equipartition Theorem to estimate how quickly T falls off with decreasing p. We know that $c_v \approx \frac{f}{2}R \approx \frac{5}{2}R$ for the Earth, and we know that $c_p \approx c_v + R$ for an ideal gas. Therefore, $c_p \approx \frac{7}{2}R$, so for the Earth:

$$
\frac{R}{c_p} \approx \frac{R}{\frac{7}{2}R} = \frac{2}{7} \approx \frac{d\ln T}{d\ln p} \approx 0.286
$$

Of course, the atmosphere does not actually follow a **Dry Adiabat**, as clearly shown by Figure 2.2b. If we look at only the troposphere (i.e., the lower portion of Figure 2.2b where $p > 100\,\text{mbar}$), we can estimate from the figure that $d\ln T/d\ln p \approx \frac{\ln(300/180)}{\ln(1000/100)} \approx 0.222$. This is pretty close to the predicted 0.286, but decidedly much lower! Of course outside of the troposphere our prediction is completely off.

This is because some of our assumptions were not as accurate as we would like. In reality, $dN \neq 0$, and entrainment occurs, and $dS \neq 0$, and diabatic heating occurs. Some of this diabatic heating is due to condensation, which we will learn how to deal with in the next chapter on Moist Thermodynamics. We'll find that this will act to shallow the temperature profile (i.e., decrease $d \ln T / d \ln p$). Some of this is radiative heating, which we will discuss in the next part on Radiative Transfer.

However, (dry) convection as a mechanism always perturbs an existing atmospheric profile towards an **Adiabat** and a temperature slope of 2/7, and it is other processes (radiation, fluid dynamics, moisture) that pull the atmosphere away from an adiabat. In some cases, like in the **stratosphere**, radiation pulls the atmospheric profile to be super-stable, in which case convection is prohibited due to the stability of the atmosphere. In other cases, these other processes pull the atmospheric profile to be unstable, in which case convection takes over to push the atmosphere back to the **Adiabat**. What we have just touched upon is Radiative-Convective Equilibrium, which we will discuss more in Part III.

2.3   Potential Temperature

In basic thermodynamics, we often introduce the concept of temperature operationally in the following way: suppose we have two objects A and B, with temperatures T_A and T_B, respectively, and we place them in contact with each other. 'Temperature' encodes whether heat will flow, and which way the heat will flow. If $T_A = T_B$, no heat will flow. If $T_B > T_A$, heat will flow from B to A, and vice versa if $T_B < T_A$.

However, this will not work in atmospheric thermodynamics. Suppose we have two parcels of air called A and B at pressures p_A and p_B[4] with temperatures T_A and T_B, respectively. Suppose we displace parcel A in order to place it in contact with parcel B (with no entrainment ($dN = 0$) or diabatic heating ($dS = 0$)).

Which way will heat flow?

We cannot naively use temperature as we did before, because parcel A's temperature will change following the Dry Adiabat as we move it to a different pressure. In other words, temperature is not a variable which is Materially Conserved in Atmospheric/Oceanic Thermodynamics. A variable is Materially Conserved if and only if it remains constant following a air/water parcel. This is a concept which will become very important in Part VI.

We now want to define a quantity which serves the same role as temperature did before (i.e., a variable which encodes which way heat will flow) but which is *also*, unlike temperature, materially conserved. Let us consider a more general scenario, and move each parcel to some third pressure p_{ref}[5].

We define the **Potential Temperature** θ as the temperature a parcel *would* have if it were perfectly adiabatically moved to some reference pressure level p_{ref}:

$$\theta(T, p) = T \left(\frac{p}{p_{ref}} \right)^{-\frac{R}{c_p}} \tag{2.5}$$

Recall that a few pages ago in our derivation of the Dry Adiabat we found that:

$$ds = c_p \, d \left(\ln \left(T(p)^{\frac{-R}{c_p}} \right) \right) \tag{2.6}$$

Since the $\ln T(p)^{\frac{-R}{c_p}}$ term is within the differential, we can freely add and subtract constants within

---

[4]We are getting used to using pressure as a coordinate rather than height, remember!

[5]Set $p_{ref} = p_B$ to recover our first example.

the differential, which corresponds to multiplying within the logarithm.[6] We get then that:

$$ds = c_p \, d\ln T \left(\frac{p}{p_{ref}}\right)^{\frac{-R}{c_p}} \tag{2.7}$$

$$= c_p \, d\ln\theta \tag{2.8}$$

As you can see, if $ds = 0$, as we assumed during convection, then $d\theta = 0$. Therefore, as we lift or drop a parcel, θ is materially conserved.[7]

Recalling that $TdS = dQ = $ heat, we find that the potential temperature is a measure of the extent to which diabatic/heat transfer processes have heated or cooled our parcel of air. It can be a proxy for how wrong we were to assume that $dS = 0$ in our derivation of Dry Adiabat.

2.4 Convection

2.4.1 Buoyancy

The atmosphere (and ocean) is a fluid, so even in the absence of most forces like friction, gravity is not the only force acting on a fluid parcel. There are also pressure gradient forces, which act to push fluid parcels away from areas of high pressure to areas of low pressure.

We already accounted for the effects of pressure in deriving Hydrostatic Balance and the Dry Adiabat. In the former case, the pressure gradient force acts to push fluid parcels upwards away from lower altitude regions of high pressure to higher altitude regions of low pressure in order to counteract the force of gravity on the ambient air. In the latter case, we only consider large-scale pressure gradients in the ambient atmosphere and we ignored pressure gradients across a fluid parcel.

Now we wish to take into account pressure gradients across a fluid parcel, and consider a fluid parcel which does *not* have the same density as the air around it. Suppose we have a fluid parcel of mass m occupying some volume V. For simplicity assume the fluid parcel is rectangular, with a horizontal area of A and a height of h such that $V = Ah$.

Let us consider the forces per unit mass on the fluid parcel in the vertical direction. There are three forces of interest: the gravitational force $F_g = -g$, the pressure force on the *bottom* pushing the parcel *upwards* $F_{bot} = p(z)A/m$, and the pressure force on the *top* pushing the parcel *downwards* $F_{top} = p(z+h)A/m$. $p(z)$ is the pressure at height z. The net force is then:

$$F_z = -g - \frac{A}{m}(p(z+h) - p(z))$$

We now make two assumptions. First, we assume that h is small, therefore, $p(z+h) - p(z) \approx \frac{dp}{dz}h$. Second, we assume that the pressure in the fluid is set by Hydrostatic Balance, therefore $p(z+h) - p(z) \approx \frac{dp}{dz}h = -\rho_a g h$, where ρ_a is the density of the ambient air. We define $\rho_p = m/V$ as the density of the fluid parcel, and find that the total force on fluid parcel is:

$$F_b = -g + \frac{Ah}{m}\rho_a g$$

$$= g\left(-1 + \frac{\rho_a}{\rho_p}\right)$$

$$\therefore \quad \boxed{F_b = g\left(\frac{\rho_a - \rho_p}{\rho_p}\right)} \tag{2.9}$$

---

[6]Another way of looking at this is noting that $d\ln\left(\frac{1}{p_{ref}}\right)^{\frac{-R}{c_p}} = 0$, and so we add $d\ln\left(\frac{1}{p_{ref}}\right)^{\frac{-R}{c_p}} = 0$ to both sides to go from 2.6 to 2.7.

[7]In the Ocean, even without radiative heating processes, θ is not conserved. This is because there is a third thermodynamic state variable, salinity, which will change the temperature of sea water without adding any heat. Oceanographers have come up with another kind of materially conserved temperature in the ocean: *Conservative Temperature*.

where F_b = the buoyancy force. Physically, the buoyant force is intuitive: if the air parcel is lighter than the ambient air ($\rho_p < \rho_a$) it will convect and rise, and vice versa. Furthermore, we can write F_b in terms of the temperature T (using 1.4) or potential temperature θ (using 2.5) in very neat forms, since many terms cancel top and bottom:

$$F_b = g\left(\frac{\rho_a - \rho_p}{\rho_p}\right)$$

$$\boxed{F_b = g\left(\frac{T_p - T_a}{T_a}\right)} \tag{2.10}$$

$$\boxed{F_b = g\left(\frac{\theta_p - \theta_a}{\theta_a}\right)} \tag{2.11}$$

So a parcel will convect and rise if it is warmer or has a higher potential temperature than its ambient surroundings.

Finally, another way of thinking of the buoyant force is as a kind of '*reduced gravity*'. The fluid acts like its on a planet of gravitational acceleration $g' = g\left(\frac{\rho_a - \rho_p}{\rho_p}\right)$ not g. This will be how we think of things in Chapter 18.

2.4.2 Convective Instability Criterion

Now we wish to find whether our atmosphere is convectively stable, i.e., whether the atmosphere will spontaneously undergo convection. If pressure and density did not appreciably change by height, then this would be easy: if $T(z) > T(z+c)$ where $c > 0$ (and assuming that $\frac{\partial \rho}{\partial T} < 0$), then we will have denser air overlying lighter air, and the parcels will convect.[8]

As before, the situation is a bit more complicated in an atmosphere when density and pressure vary significantly with height. Suppose again that we have an atmospheric temperature profile $T = T(p)$ (not necessarily following the dry adiabat). Suppose we lift a parcel at some pressure p to some (smaller) pressure $p - \delta p$.

That parcel will be convectively unstable and continue rising if it is lighter (and therefore hotter) than the ambient surrounding air. We assume that the parcel is initially in equilibrium with its surroundings ($T_{parcel}(p) = T_{ambient}(p)$). We then Taylor Expand:

$$T_{parcel}(p - \delta p) > T_{ambient}(p - \delta p)$$

$$T_{parcel}(p) - \frac{dT_{parcel}}{dp}\delta p > T_{ambient}(p) - \frac{dT_{ambient}}{dp}\delta p$$

$$\frac{d\ln T_{parcel}}{d\ln p} < \frac{d\ln T_{ambient}}{d\ln p}$$

Substituting in the Dry Adiabat for $\frac{d\ln T_{parcel}}{d\ln p}$ we find the dry stability criterion:

Key Idea I.5: The Dry Stability Criterion

Suppose a dry atmosphere has a temperature profile $T(p)$. It will be:

Unstable to convection if:	Neutrally stable to convection if:	and Stable to convection if:
$\boxed{\dfrac{d\ln T}{d\ln p} > \dfrac{R}{c_p}}$ (2.12)	$\boxed{\dfrac{d\ln T}{d\ln p} = \dfrac{R}{c_p}}$ (2.13)	$\boxed{\dfrac{d\ln T}{d\ln p} < \dfrac{R}{c_p}}$ (2.14)

---

[8]This is the case if we're considering two layers of different temperature and composition that are in contact. This is because, at the point of contact, pressure changes only infinitessimaly, so the Dry Adiabat is inapplicable (since $dp \approx 0$). To determine whether these two layers are convectively stable, you must compare the densities directly, which depend on both the temperature of each layer and the molecular composition.

Note that 2.12 and 2.14 are not automatically ruled out due to 2.3. That is because 2.3 governs a perfectly adiabatic rising/falling parcel of air. While this is the dominant contribution to the atmospheric temperature profile in the troposphere, it is not the only one (even within the troposphere). Other processes (water vapour, radiation, dynamics) can push the atmospheric temperature profile away from the dry adiabat and make it stable or unstable. However, typically, unstable atmospheres are removed very quickly through spontaneous convection.

Physically, the idea is as follows: an atmosphere is stable if the temperature decreases slow enough with height (2.14). If it decreases too quickly (2.12), a lifted parcel of air will have cooled *less* than its surroundings, and thus be warmer and lighter than its surroundings, and continue to rise due to buoyancy (2.10).

If 2.14 obtains then an atmosphere will be *super-stable* and will inhibit convection. A quick sanity check to remember which way the inequality is in 2.14 is the following: an isothermal[9] atmosphere is always super-stable. An isothermal atmosphere will have $\frac{d\ln T}{d\ln p} = 0$, so $\left(\frac{d\ln T}{d\ln p} < \frac{R}{c_p}\right)$ corresponds to stability. If one has a *super-stable* atmosphere then that indicates that there has been some other dominant energy transport mechanism that has kept the atmosphere in this state. This is characteristic of a **stratosphere**.[10]
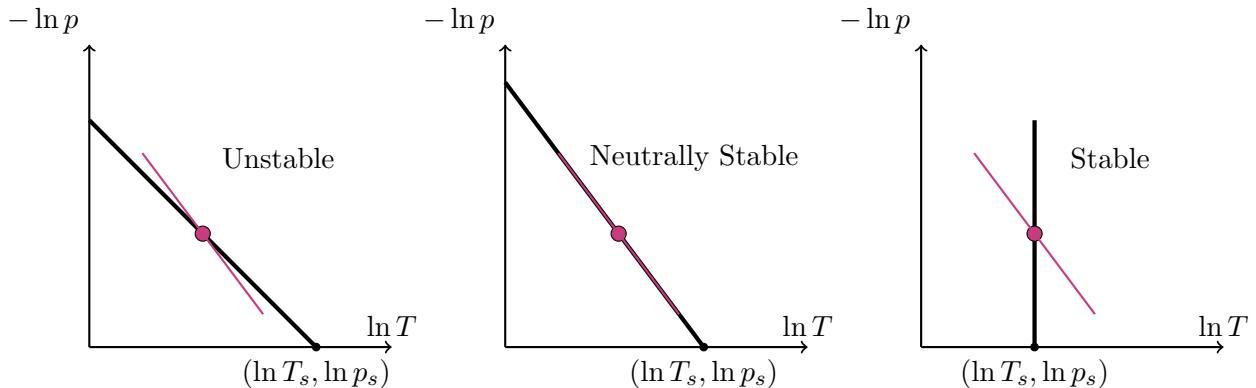
Figure 2.3: Dry Stability: From left to right: unstable, neutrally stable, and super stable atmospheric temperature profiles. Thick black is the atmospheric profile, thin magenta is the dry adiabatic slope.

2.4.3 Convective Stability Criterion with Potential Temperature

We can formulate an equivalent criterion for potential temperature. Recall that if two parcels of air have potential temperatures θ_A and θ_B such that $\theta_A > \theta_B$, then parcel A will have a higher temperature than parcel B if they're at the same pressure.

We can very straightfoward derive an equivalent criterion for convective instability, this time in terms of potential tempreature θ, by rearranging 2.5 for $T(\theta)$ and substituting that into 2.12.

---

[9]'Iso' meaning same and 'thermal' meaning temperature, so 'Isothermal' means constant temperature.

[10]Recall the footnote on this page. This is why Ray prefers that definition of **stratosphere**: a stratsophere should be characterised by what energy transport mechanism is dominant, not whether it happens to have shortwave absorbers.

$$1 < \frac{c_p}{R} \frac{d\ln T(\theta)}{d\ln p} \qquad\qquad ; 2.12$$

$$= \frac{c_p}{R} \frac{d\ln\left(\theta\left(\frac{p}{p_{ref}}\right)^{\frac{R}{c_p}}\right)}{d\ln p} \qquad\qquad ; \text{substitute } 2.5$$

$$= \frac{c_p}{R}\left(\frac{d\ln\theta}{d\ln p} + \frac{d\ln\left(p^{\frac{R}{c_p}}\right)}{d\ln p} + \frac{d\ln\left(p_{ref}^{\frac{-R}{c_p}}\right)}{d\ln p}\right) \qquad\qquad ; \text{split logarithm}$$

$$= \frac{c_p}{R}\frac{p}{\theta}\frac{d\theta}{dp} + \frac{c_p}{R}\frac{R}{c_p} \qquad\qquad ; \ln p^{R/c_p} = R/c_p \ln p$$

Therefore, since $\theta > 0$ and $p > 0$ always, we get that our atmosphere is unstable if and only if:

$$\boxed{\frac{d\theta}{dp} > 0} \qquad\qquad (2.15)$$

Using the hydrostatic relation and the fact that $g > 0, \rho > 0$, we get our criterion for instability in height coordinates:

$$\boxed{\frac{d\theta}{dz} < 0} \qquad\qquad (2.16)$$

Therefore, if potential temperature decreases with height, then our atmosphere is unstable. This makes physical sense. Recall that potential temperature takes the place of temperature in atmospheric thermodynamics in the sense that it is *potential temperature* that stays the same when you lift/drop and air parcel. If you lift an air parcel, its potential temperature stays the same. However, if the ambient potential temperature decreases with height, then the ambient air will be colder than the air parcel, and the air parcel will keep rising.

Figure 2.4: Dry Stability: From left to right: unstable, neutrally stable, and super stable atmospheric temperature profiles. Thick black is the atmospheric profile, thin magenta is the dry adiabatic slope.

2.4.4 CAPE, LFC, LNB

We have so far characterised *whether* convection will occur in our dry atmosphere, but now we wish to know what will happen *during* convection. We start again by considering a parcel of air at an initial pressure p_0 with a temperature of $T_p(p_0)$. Let $T_a(p)$ be the temperature profile of the ambient atmosphere, and assume that initially the parcel is in equilibrium with its surroundings (i.e., assume $T_p(p_0) = T_a(p_0)$).

We define the **L**evel of **F**ree **C**onvection (**LFC**) p_{LFC} as the *closest* location to p_0 where $T_p(p) = T_a(p)$, $dT_p/dp < dT_a/dp$ (the parcel is convectively unstable), and $p < p_0$ (it's higher).

We define the **L**evel of **N**eutral **B**uoyancy (**LNB**) p_{LNB} as the location where $T_p(p) = T_a(p)$, $dT_p/dp > dT_a/dp$ (the parcel is convectively stable), and $p < p_{LFC}$ (it's higher than the **LFC**).

We define the **C**onvective **A**vailable **P**otential **E**nergy (**CAPE**) as the total available potential energy (per unit mass) for a parcel of air from the Level of Free Convection to the Level of Neutral Buoyancy. In other words, $CAPE$ is equal to the total kinetic energy per unit mass a parcel would gain if it was allowed to convect from the Level of Free Convection to the Level of Neutral Buoyancy:

$$CAPE = \int_{z_{LFC}}^{z_{LNB}} F_b \, dz \tag{2.17}$$

We can substitute for F_b using 2.10 and convert to pressure coordinates using 1.16 to find that:

$$CAPE = \int_{z_{LFC}}^{z_{LNB}} \frac{T_p - T_a}{T_a} g \, dz$$
$$= \int_{p_{LFC}}^{p_{LNB}} \frac{T_p - T_a}{\cancel{T_a}} (-R\cancel{T_a} \, d\ln p)$$
$$\therefore \boxed{CAPE = R \int_{p_{LNB}}^{p_{LFC}} (T_p - T_a) \, d\ln p}$$

We know $T_p(p)$, since $T_p(p)$ follows the Dry Adiabat with $(p_0, T_0) = (p_{LFC}, T_{LFC})$, so we can straightfowardly calculate $CAPE$ if we are given the atmospheric temperature profile $T_a(p)$.

We further define **C**onvective **IN**hibition (**CIN**) as the energy a fluid parcel would need to overcome to reach the **LFC**:

$$\boxed{CIN = R \int_{p_{LFC}}^{p_0} (T_p - T_a) \, d\ln p}$$

We expect $CIN < 0$ and $CAPE > 0$. Above **LNB**, we expect there to be more 'CIN', which will prevent the parcel from rising further. However, at this point, the parcel will have already gained kinetic energy from $CAPE$. We call where the parcel 'stops' the **L**evel of **M**aximum **A**scent (**LMA**), which occurs at p_{LMA} where the 'CIN' above **LNB** equals the $CAPE$:

$$\boxed{R \int_{p_{LMA}}^{p_{LNB}} (T_p - T_a) \, d\ln p = R \int_{p_{LNB}}^{p_{LFC}} (T_p - T_a) \, d\ln p}$$

Key Idea I.6: Convection Definitions

Let the ambient air temperature be $T_a(p)$ (black line ■). Consider an air parcel at an initial pressure p_0 with a temperature of $T_p(p_0) = T_a(p_0)$ that is then lifted. Upon lifting, the parcel's temperature $T_p(p)$ follows the Dry Adiabat (dashed magenta line ■). Then:

Level of Free Convection (LFC): $p = p_{LFC}$ if $T_p(p) = T_a(p)$, $dT_p/dp < dT_a/dp$, and this is the first pressure $p < p_0$ where this occurs.

Level of Neutral Buoyancy (LNB): $p = p_{LNB}$ if $T_p(p) = T_a(p)$, $dT_p/dp > dT_a/dp$, and this is the first pressure $p < p_{LFC}$ where this occurs.

Level of Maximum Ascent (LMA): $p = p_{LMA}$ where p_{LMA} is determined by Equation 2.20 (I.e., where the blue area ■ equals the orange area ■).

$$CAPE = R \int_{p_{LNB}}^{p_{LFC}} (T_p - T_a) \, d\ln p$$

(2.18)

$$CIN = R \int_{p_{LFC}}^{p_0} (T_p - T_a) \, d\ln p$$

(2.19)

$$R \int_{p_{LMA}}^{p_{LNB}} (T_p - T_a) \, d\ln p = CAPE$$

(2.20)
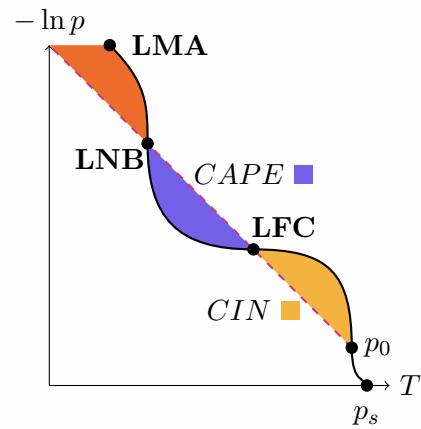
Figure 2.5: Atmospheric Profile. The parcel is lifted from p_0. CIN is proportional to the area shaded in Yellow ■ and $CAPE$ is proportional to the area shaded in Blue ■.

2.4.5 Conservation of Enthalpy during Convection

We define the (specific) **Enthalpy** h as follows:

$$h = u + p\frac{1}{\rho}$$

(2.21)

For an ideal gas, we can substitute for p using 1.4 and $u = c_v T$ to write h in terms of T only:

$$h = \underbrace{c_v T}_{u} + \underbrace{\rho R T}_{p} \frac{1}{\rho}$$
$$= \underbrace{(c_v + R)}_{c_p} T$$
$$= c_p T$$

Therefore we can interpret the enthalpy h as an expression of the heat content of a system at constant pressure, like for example in the atmosphere.

We introduce the enthalpy because we can show that, under suitable assumptions, the total enthalpy H of an atmospheric column is conserved during convection:

$$H = \iiint \rho h \, dV = const$$

We make three assumptions:

1. The initial and final states are both in Hydrostatic Balance 1.12
2. The kinetic energy of the initial and final state is negligible compared to the potential and internal energy.
3. No mass or heat leaves domain.

The last assumption requires some elaboration. First, we *don't* assume that no energy leaves the domain: we allow the column to do work on the air above or below itself as it expands/contracts. Second, we *don't* assume that there is no heating within the column. In fact, we expect there to be some heating within the column in general: If a column is convectively unstable, then there will be **CAPE**, and we expect this to be converted into an appreciable amount of kinetic energy as the air parcels within the column convect. Once convection acts to remove the **CAPE**, we expect friction to slow the parcels down and convert the kinetic energy into heat.

We first consider the total energy in an atmospheric column per horizontal area between altitudes z_0 and z_1, which is the sum of the internal energy u and the potential energy gz (and we have assumed kinetic energy $\frac{1}{2}v^2$ is negligible):

$$E = \int_{z_0}^{z_1} \rho \left(u + gz \right) dz$$
$$= \frac{1}{g} \int_{p_1}^{p_0} \left(u + gz \right) dp$$

We then integrate the potential energy term gz by parts:

$$g \int_{p_1}^{p_0} z \, dp = g \left[zp \right]_{p_0}^{p_1} - g \int_{z_1}^{z_0} p \, dz$$
$$= g(z_0 p_0 - z_1 p_1) + \int_{p_1}^{p_0} \frac{p}{\rho} dp$$

Therefore the total energy per unit area is:

$$E = \frac{1}{g} \int_{p_1}^{p_0} \left(u + \frac{p}{\rho} \right) dp + (z_0 p_0 - z_1 p_1)$$

We make a final, fourth assumption here that the pressure at the bottom p_0 and top p_1 of the column is unchanged between the final and initial states. This is ensured if we assume the entire column (inside and outside the column under consideration) is in hydrostatic balance and the mass of the air above, within, and below the column under consideration is unchanged (recall 1.13). The change in energy ΔE between the final and initial states is then:

$$\Delta E = \frac{1}{g} \Delta \int_{p_1}^{p_0} h \, dp + p_0 \Delta z_0 - p_1 \Delta z_1 \tag{2.22}$$

We now wish to equate this with the change in energy of the atmospheric column governed by 2.1. Since we have assumed that there is no heat or mass transfer to or from the system, we can set $dN = 0$ and approximate $dS = 0$. Therefore, the change in energy (per unit area), to a good approximation, is set by:

$$dE = -p \, dV/A$$
$$= p_0 dz_0 - p_1 dz_1$$
$$\therefore \Delta E = p_0 \Delta z_0 - p_1 \Delta z_1 \tag{2.23}$$

Therefore, setting 2.22 equal to 2.23 we can cancel the $p\Delta z$ to find that:

$$\Delta H = \Delta \iiint \rho h \, dz \, dx \, dy = 0 \tag{2.24}$$

The enthalpy of a column is conserved under convection. If we assume that the atmosphere is an ideal gas (so that $h = c_p T$) and that c_p is approximately independent of temperature (so that we can pull c_p out of the integral), then we can convert to pressure coordinates to find that:

$$H = \int \int \frac{c_p}{g} \int T \, dp \, dx \, dy$$

Therefore we find that the enthalpy is proportional to $\int T \, dp$, and so:

$$\boxed{\Delta \int T dp = 0} \tag{2.25}$$

during convection.

Chapter 3

Moist Thermodynamics

3.1 Phase Transitions

In the previous chapter, we considered only atmospheres made up completely of constituents that do not undergo any phase transitions. However, Earth's atmosphere (and many others) clearly do not obey this assumption. Earth's atmosphere has a non-negligible amount of condensible substances: water vapour (look outside and you might see a cloud!), which clearly condense, evaporate, and freeze in Earth-like conditions.

So what is a phase transition? We will only be dealing with what are called **First-Order Phase Transitions**, which occur when there is a *discontinuity* in a thermodynamic state variable and an absorption or release of a large amount of energy, called **latent heat**. Phase transitions occur at certain temperatures and pressures[1]. We can represent this with a **Phase Diagramme**, as shown below:
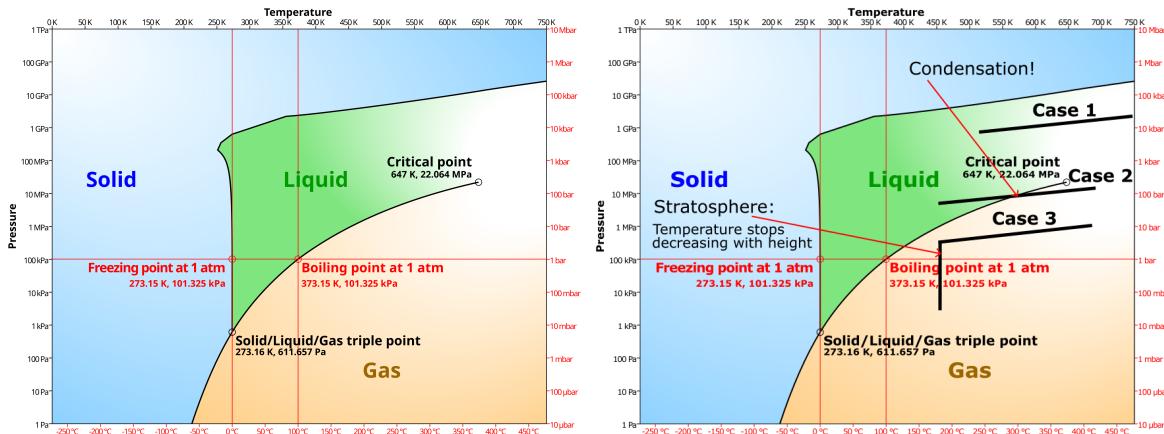
Figure 3.1: Phase Diagramme for Water from Wikipedia [4].

Focus on the left-hand plot for now. The coloured regions indicate what phase water will be in at a certain the partial pressure (of the water) and temperature: gas (vapour), liquid (steam), and solid (ice). Crucially, this does not depend on the total pressure of the atmosphere or surrounding gases, but only the partial pressure of the condensible substance (in this case, the partial pressure of water). The solid black lines indicate **Phase Boundaries**. The **Triple Point** is the point where all three states can coexist. The **Critical Point** is the point past which there is no phase transition between gas and liquid.

The pressure at the **Phase Boundaries** is the **Saturation Vapour Pressure** $p_{sat}(T)$. If the partial pressure $p < p_{sat}$, then more vapour (gas form) can be added without condensing. If the $p \geq p_{sat}$, then the gas will condense until the partial pressure reaches the saturation vapour pressure.

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[1]Whether a phase transition will occur might also depend on other thermodynamic state variables. For example, salinity is a thermodynamic variable which affects when sea water evaporates/freezes.

However, it's important to remember that condensation is not instantaneous, which is partly why we can have atmospheric conditions where $p > p_{sat}$. This will be important when we consider Clouds.

In fact all gases can condense if the temperature is low enough/partial pressure is high enough. CO_2 condenses on Mars, and N_2 condenses on Neptune. Why doesn't CO_2 and N_2 condense here on Earth? It all depends on the vertical structure of the atmosphere: the precise profile of $T = T(p)$. Recall that, as p decreases, so too does T (if our atmosphere follows the Dry Adiabat). There are three cases, all labeled in the right-hand plot of Figure 3.1.

1. Case 1: The atmosphere has such a high pressure/temperature that it avoids a phase transition altogether by meandering around the critical point.
2. Case 2: The atmospheric profile cuts right through a phase boundary, and the condensible condenses (clouds form!).
3. Case 3: The stratosphere starts before the atmosphere hits the phase boundary, in which case temperature no longer decreases with height, so it never crosses the phase boundary.

For O_2 and N_2 in Earth like conditions, we are in case 3: it never gets cold enough for O_2 and N_2 to condense, which is why we have no O_2 and N_2 clouds. However, we are in Case 2 with water (H_2O), and so we do get water clouds!

When a gase condenses from phase A to phase B, it typically releases a large amount of **Latent Heat**. Again, we would like to work in intensive variables, so we often work with the **Specific Latent Heat** $L_{A \to B}$ (units of $J\,kg^{-1}$) which is the amount of energy released per kilogram of substance when underoing a phase transition from A to B ($L_{A \to B} > 0$ if heat is released). Like heat capacity, this often has a temperature dependence, but we will in most cases treat this as constant.

3.2 The Clausius-Clapeyron Relation

We can derive the shape of the phase boundary with the **Clausius Clapeyron Relation**, which relates the saturation vapour pressure to the temperature using the latent heat and the densities. We will not derive it here, but it would be useful for you to memorise the following relation:

$$\frac{dp_{sat}}{dT} = \frac{L_{A \to B}}{T\left(\frac{1}{\rho_A} - \frac{1}{\rho_B}\right)} \tag{3.1}$$

If we assume that $\rho_g \ll \rho_l$ (i.e., the gaseous form is much less dense than the liquid form), and that the gas is ideal, we find that:

$$\frac{dp_{sat}}{dT} \approx \frac{L_{g \to l}}{T\frac{1}{\rho_g}}$$

$$= \frac{L_{g \to l}p}{R_c T^2}$$

$$\therefore \quad \boxed{\frac{d\ln p_{sat}}{d\left(\frac{1}{T}\right)} \approx -\frac{L_{g \to l}}{R_c}} \tag{3.2}$$

Note that I have written R_c to make clear that R_c is the specific gas constant for the condensible (e.g., water) *not* for the atmosphere. We can now derive an analytic expression for the saturation vapour pressure if we further assume that the latent heat is constant.

$$\boxed{p_{sat} \approx p_0\, e^{\frac{L}{R_c}\left(\frac{1}{T_0} - \frac{1}{T}\right)}} \tag{3.3}$$

where I now just write L for the latent heat of condensation and (T_0, p_0) are any point on the saturation vapour pressure curve. This could be, for example, the **triple point** or the **critical point**.

As we can see then, the saturation vapour pressure depends approximately *exponentially* on temperature, so saturation vapour pressure decreases exponentially as temperature decreases. Conversely, if saturation vapour pressure decreases, T changes only logarithmically (i.e., very slowly).

3.3   The Moist Pseudo-Adiabat: Lifting Moist Parcels of Air

Our goal now is to derive an analogous Dry Adiabat relation for moist atmospheres consisting of some condensible substance c. We call this the **Moist Pseudo-Adiabat**. We call this a '*Pseudo*'-Adiabat because we assume, for mathematical convenience, that the condensible substance c is instantaneously removed from the air parcel. This occurs if, for example, c instantaneously rains out (but rain is not the only way in which this may occur!). When the condensible is removed, this constitutes a diabatic process ($dN \neq 0$), so strictly speaking we will not derive an adiabat in this section.

In reality, rising moist air parcels that condense will not instantaneously eject its condensed c, and so will not follow the pseudo-adiabat. However, the pseudo-adiabat is a good approximation if the amount of condensate left in the air parcel is small, which is at least the case on Earth.

Recall that we are using the same logic as we did before in deriving the Dry Adiabat. As such, we are still assuming that the dominant mechanism of heat transport within the atmosphere is vertical convection, only this time the air parcels convecting are moist (have condensible substances in them).

We will now consider two limiting cases. In both cases, we make the crucial assumption that the air parcels remain saturated with the condensible. In other words, we assume that the partial pressure of the condensible is equal to the saturation vapour pressure p_{sat}.

3.3.1   Limit I: Single Component Condensible Atmosphere

We consider the first limiting case, with an atmosphere made up wholly of one condensible substance c. Since there is only one constituent, the total pressure is equal to the partial pressure of the c. There are two possibilities now.

The first possibility obtains if the temperature at the ground $T_g(p_g)$ is too low such that $p_g > p_{sat}(T_g(p_g))$, i.e., the pressure at the ground is higher than the saturation vapour pressure at the ground. Then there will be an ocean of c at the surface. The pressure within the ocean will obey Hydrostatic Relation, and the surface of the ocean is determined by where the pressure and temperature intersect the phase boundary ($p = p_s = p_{sat}(T_s(p_s))$).

The temperature above the surface of the ocean is set by convecting air parcels. These air parcels originate from the surface of the ocean, and are initially saturated. However, recall that we have assumed that the air parcel remains saturated.[2] As such, the atmospheric profile must lie on the phase boundary, so the temperature is fixed to the phase boundary and is as follows (found by rearranging 3.3 for T in terms of p_{sat}):

$$\text{If } p_g > p_{sat(T_g)}\colon\ T(p) = \frac{T_s}{1 - \frac{RT_s}{L}\ln\frac{p}{p_s}} \tag{3.4}$$

$$p_s = p_{sat}(T_s) \tag{3.5}$$

The surface pressure p_s, then, is set by the surface temperature T_s. Recalling Equation 1.14, the mass of the atmosphere is set by how hot it is. Physically, this corresponds to how much c you can evaporate from the ocean!

The second possibility obtains if the temperature at the ground $T_g(p_g)$ is too high such that $p_g < p_{sat}(T_g(p_g))$, i.e., the pressure at the ground is lower than the saturation vapour pressure at the ground. Then there will be no surface c ocean, and the atmosphere will follow the Dry Adiabat until it reaches the phase boundary at the **lifted condensation level**, after which it will follow Equation 3.4:

$$\text{If } p_s < p_{sat(T_s)}\colon\ T(p) = \begin{cases} T_s\left(\dfrac{p}{p_s}\right)^{\frac{R}{c_p}} & \text{for } p < p_{LCL} \\[2ex] \dfrac{T_s}{1 - \frac{RT_s}{L}\ln\frac{p}{p_{LCL}}}, & \text{for } p > p_{LCL} \end{cases} \tag{3.6}$$

$$\text{where } p_{LCL} = p_{sat}\left(T_s\left(\frac{p_{LCL}}{p_s}\right)^{\frac{R}{c_p}}\right) \tag{3.7}$$

---

[2]I'm not sure how justified this assumption is. I plan to ask Ray about this at some point, but if this footnote is still here when you're reading it I have not (yet).

In other words, the temperature follows the Dry Adiabat until the **lifted condensation level**. We can solve for the pressure at the **lifted condensation level** by finding where the Dry Adiabat intersects the phase boundary (the latter of which is calculated using Clausius Clapeyron).

3.3.2 Limit II: Dilute Condensible

Derivation

We now consider the second limiting case, where the atmosphere is mainly made up of a non-condensible substance a, and a dilute condensible substance c. We follow a similar derivaiton to the Dry Adiabat, and start from the first law of thermodynamics, but this time include the extra latent heat term $-L\,dq$:

$$du = -p\,d\left(\frac{1}{\rho}\right) + T\,ds - L\,dq \tag{3.8}$$

where $L =$ the latent heat of condensation and $q = \rho_c/\rho_a =$ the mass mixing ratio of c in its gaseous form (and *not* c in its liquid form, which we assume (recall) is removed immediately from the parcel of air). For water, $q =$ the mass mixing ratio of the water vapour and the dry air, *not* the water droplets and the dry air. In the dilute limit, $q \ll 1$, and q is equal to the mass fraction.

Take care to note the assumptions we're making here, and what the letters we've written actually mean. u, ρ, T, and s refer to properties of a in the parcel of air, while p refers to the total pressure of both the a and c. Here we are implicitly making three assumptions. First, we assume that we are in the dilute limit $q \ll 1$, therefore $\rho = \rho_a + \rho_c \approx \rho_a$. Second, we assume the heat capacities of a and c are of similar size, and that therefore the specific internal energy u is primarily made up of the internal energy of a.

As a sanity check, let us consider the minus sign on the $L\,dq$ term. This makes physical sense: we expect the internal energy of the a to *increase* if condensation occurs, since latent heat is released when c condenses, and heat is added to a in the air parcel. If condensation occurs, then the mass of the liquid c (e.g., water droplets) must increase, and therefore (by mass conservation) the mass of the gaseous c (e.g., water vapour) must decrease. Therefore, the mass mixing ratio of c must decrease, so $dq < 0$, therefore $-L\,dq > 0$, as expected.

Let us first consider the $L\,dq$ term:

$$
\begin{aligned}
L\,dq &= L\,d\left(\frac{\rho_c}{\rho_a}\right) && \text{; Definition of } q \\
&= L\,d\left(\frac{M_c e}{M_a(p-e)}\right) && \text{; Ideal Gas (1.4) and assume } T_c = T_a \\
&\approx L\,d\left(\frac{M_c}{M_a}\frac{e}{p}\right) && \text{; Dilute so } e \ll p \\
L\,dq &\approx L\,\epsilon\,d\left(\frac{e}{p}\right) && \text{; Define} \epsilon = \frac{M_c}{M_a}
\end{aligned}
$$

where $\rho_c =$ the density of the condensible c; $\rho_a =$ the density of the air a; $M_c =$ the molar mass of the condensible; $M_a =$ the molar mass of the air; $p = p_a + p_c =$ the total pressure; $e = p_c =$ the partial pressure of the condensible; and $\epsilon = M_c/M_a =$ the ratio of molar masses c and a.

We can now apply product rule and assume that the condensible is always saturated. This allows

us to use the simplified version of Clausius Clapeyron (3.2):

$$L\,dq \approx L\,\epsilon\,d\left(\frac{e}{p}\right)$$

$$= L\,\epsilon\left(\frac{de}{p} - \frac{e}{p^2}\,dp\right) \qquad\qquad ; \text{Chain Rule}$$

$$= L\,\epsilon\frac{e}{p}\left(\frac{de}{e} - \frac{dp}{p}\right)$$

$$= Lq\,d\ln p_{sat} - Lq\,d\ln p \qquad\qquad ; q \approx L\,\epsilon\frac{e}{p} \text{ and assume } e \approx p_{sat}$$

$$= -q\frac{L^2}{R_c}\,d\left(\frac{1}{T}\right) - Lq\,d\ln p \qquad\qquad ; \text{Clausius Clapeyron (3.2)}$$

$$= q\frac{L^2}{R_c T^2}\,dT - Lq\,d\ln p$$

$$= q\frac{L^2}{R_c T}\,d\ln T - Lq\,d\ln p$$

We now subsitute this into 3.8 and, for simplicity, let $ds = 0$ from the outset.[3] Following identical algebra in the derivation of the Dry Adiabat, and carefully distinguishing between R_a and R_c (the specific gas constants of the non-condensible and dilute condensible), we find that:

$$c_v\,dT = -R_a\,dT + \frac{R_a T\,dp}{p} - L\,dq$$

$$\therefore c_p\,dT = \frac{R_a T\,dp}{p} - q\frac{L^2}{R_c T}\,d\ln T + Lq\,d\ln p$$

$$\therefore c_p\,d\ln T - \frac{qL^2}{R_c T^2}\,d\ln T = R_a\,d\ln p + \frac{Lq}{T}\,d\ln p$$

Solving for $\frac{d\ln T}{d\ln p}$ and rearranging, we find the **Moist Pseudo-Adiabat** in the dilute limit:

Key Idea I.7: Dilute Moist Pseudo-Adiabat

The adiabat for an atmosphere consisting of a dilute condensible is as follows:

$$\frac{d\ln T}{d\ln p} = \frac{R}{c_p^a}\frac{1 + \frac{L\,q_{sat}}{R_a T}}{1 + \frac{L\,q_{sat}}{c_p^a T}\frac{L}{R_c T}} \qquad (3.9)$$

This is in general a *shallower* slope than the Dry Adiabat due to the latent heat of condensation keeping the parcel warm.

I have written $q = q_{sat}$ and $c_p = c_p^a$ to remind us that, first, we have made the crucial assumption that the air parcel remains saturated, and second, that c_p is the heat capacity of the non-condensible a.

Physical Interpretation

Equation 3.9 is, in general, not analytically solvable. However, we have written it in the form that provides easy interpretation. The first term on the right-hand-side (R/c_p^a) is the slope of the adiabat if there were no condensation (the Dry Adiabat), while the second complicated looking fraction on

---

[3]We could allow $ds \neq 0$ and derive what's called an **Equivalent Moist Potential Temperature** θ_E (analogous to Potential Temperature θ), which is similarly materially conserved for moist saturated parcels of air. We don't for brevity, but if you wish to read up more on it, here is a resource: whoops I forgot to insert the link, please let me know if this is still here!

the right-hand-side are the changes in the slope due to condensation. Setting $q_{sat} = 0$ recovers the Dry Adiabat.

Note that although we have made the dilute assumption that $q_{sat} \ll 1$, the corrections on the right hand side are generally of order 1. This is because the corrections go as $\frac{L q_{sat}}{R_a T}$ and $\frac{L q_{sat}}{c_p T} \frac{L}{R_c T}$, and it is gernally the case that, for water vapour on Earth at least, $\frac{L}{R_a T} \gg 1$.

It is almost always the case (although I cannot show algebraically) that the moist adiabat is shallower than the dry adiabat, i.e., that temperature decreases less with a decrease in pressure (and increase in height):

$$\frac{d \ln T_{moist}}{d \ln p} < \frac{d \ln T_{dry}}{dp}$$

Physically, this corresponds to to the effect of the condensible. As a parcel rises, the pressure decreases, so it expands and cools (following the dry adiabat). However, since the temperature decreases, so too does the saturation vapour pressure (3.1). So if the parcel is saturated, as we have assumed, c must condense and release latent heat into the surrounding air parcel, warming the parcel a bit and offsetting the cooling from expansion.

Finally, we must note that we have made the assumption that the parcel remains saturated. In general, this will be the case for rising saturated parcels of air, which will only condense so much as to remain approximatly saturated. However, if a parcel is *sinking*, then clearly it is not always realistic to assume that it will automatically acquire some more water vapour to remain saturated. As such, if a parcel of air is sinking, it will most likely follow the Dry Adiabat.

What Happens if $q < q_{sat}$?

Suppose we start an air parcel off sub-saturated (i.e., $q < q_{sat}$, initially). What temperature profile will it follow as it is lifted?

Initially, as it is lifted, its mass mixing ratio q will remain constant, and its temperature will decrease following the Dry Adiabat. This will continue until the temperature has cooled so much (following the Dry Adiabat) causing the saturation vapour pressure to decrease sufficiently such that $q = q_{sat}$. After this point, it will follow the Dilute Moist Pseudo-Adiabat.

3.3.3 Moist Convection

Ideas from Section 2.4 carry over to here, but are changed subtly. To avoid repetition the bulk of this discussion will be in Chapter 8.

However, for completion, it's useful to note that, similar to dry convection, moist convection as a mechanism always perturbs an existing atmospheric profile towards a **Moist Pseudo-Adiabat**. It is other processes (now just radiation and fluid dynamics) that pull the atmosphere away from this pseudo-adiabat.

Part II

Radiative Transfer

Introduction

This section of the course was lectured by Raymond T. Pierrehumbert covering basic atmospheric Radiative Transfer. In the academic year 2024-2025, Ray did not cover scattering, so I will not cover scattering in these notes.

This section consists of four chapters:

1. Basic Radiation Concepts:

   We introduce the basic vocabulary needed to describe radiative transfer. We first characterise the radiation field, including how to encode the direction and magnitude of radiation. Next, we introduce the terminology to describe how objects absorb (especially in the case of something like the atmosphere) and emit radiation.

2. Molecular Spectroscopy:

   We characterise spectral lines in terms of certain parameters and discuss what physically sets those parameters, and, in some cases, introduce some explicit functional forms such as the Lorentz Line Shape. Finally, we discuss continuum absorption.

3. The Schwarzschild Equation:

   This is pretty beefy section.

   We derive the Schwarzschild Equation, which governs the spectral radiance throughout the atmosphere. We approximate the angular distribution using the Two-Stream Approximation and thus derive the Angle-Averaged Upwards and Downwards Schwarzschild Equations, which governs the spectral irradiance.

   We solve these equations explicitly to find an expression for the OLR, the Outgoing Longwave Radiation and intrepret the solution physically by defining the Radiating Level.

   Next, we aim to find the OLR in terms of irradiances, and so integrate the OLR over a spectral line. We find analytic forms in the weak and strong-line limits.

   We next define a few terms to do with Radiative Forcing, including the ECS (**Equilibrium Climate Sensitivity**) and DRF (**Differential Radiative Forcing**). We calculate these explicitly in the weak and strong-line limits.

   Finally, to build some more intuition, we calculate the OLR explicitly for a grey gas (a gas whose optical properties exhibit no frequency dependence).

4. Radiative Equilibrium:

   We consider a grey gase in order to analyse how an atmosphere would behave in pure radiative equilibrium: i.e., if radiation were the dominant energy transport mechanism.

   We solve the system to find the temperature profile of the grey gas in radiative equilbrium, and intepret the solution.

   Finally, we consider some cases where the planet has an internal leakage energy source (from, for example, the formation of the planet) or an atmosphere with shortwave absorbers.

Chapter 4

Basic Radiation Concepts

4.1 Direction of Radiation: Rays and Solid Angle

In general, radiation will not only be a function of position $\vec{r}$ but will also be a function of direction[1]. We can encode the direction that some radiation is propagates with a **ray**, a vector $\vec{\omega}$ of (dimensionless) length 1 on a unit sphere (see Figure 4.1b). $\vec{\omega}$ encodes information regarding the direction of propagation, but not regarding the magnitude (e.g., energy) of the radiation. In Cartesian coordinates, this is:

$$\vec{\omega} = \begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix} \tag{4.1}$$

We also wish to define a term which can measure *how many* directions some radiation is coming from. More intuitively, we want the term to encode how much of your field of vision is taken up by some radiation; for example, if radiation is propagating towards you from every direction, we want this term to be larger than if radiation were only propagating towards you from a single direction.

We thus define the **Solid Angle** Ω, which is a measure of a set of rays, with units of **Steradians** (sr). The **Solid Angle** Ω is defined as the area on a unit sphere taken up by some radiation.

It is easier to understand the **solid angle** (measured in **steradians** (sr)) in 3D if we draw an analogy to the more familiar **angle** (measured in **radians** (rad)) in 2D.

In 2D, radians measure an angle by relating the circumference of a circle to the arc length subtended by that angle (see Fig. 4.1a). For example, 1π rad corresponds to an angle that subtends half a unit circle (180 degrees) and thus an arc length of 1π. Another example is 2π rad, which corresponds to an angle that subtends a unit circle (360 degrees) and thus an arc length of 2π.

Analogously, steradians measure a solid angle by relating the surface area of a sphere to the patch subtended by a solid angle (see Fig. 4.1b). For example, 2π sr corresponds to an solid angle that subtends half a unit sphere and thus an area of 2π. This might correspond with a set of rays with some positive x component (i.e., $\sin(\theta)\cos(\phi) > 0$). Another example is 4π sr, which corresponds to an angle that subtends a unit sphere and thus an area of 4π. This example corresponds to a set of rays pointing in all directions.[2]

Often we will have to integrate over solid angles/directions (as radiation will be a continuous function of $\vec{\omega}$) in the same way that we might integrate over angles. We therefore introduce the differential solid angle:

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[1]This is why your eyes will hurt if you look directly at the sun but they won't if you look at these lecture notes.

[2]If we wish to take the analogy further, we can consider radiation propogating in 2D space. In 2D, a ray takes the form of $\vec{\omega} = (\cos(\theta), \sin(\theta))$. If we focus on a single point, we can encode from how many directions radiation is coming from by considering the field of view.

In 2D space, a field of view is 1D (in the same way that in 3D space, a field of view is 2D). As such, we can encode this by the **angle**, which is defined as the length on a unit circle taken up by some radiation.

Key Idea II.1: Solid Angle

The solid angle Ω is a dimensionless area encoding how many directions we are considering, and is measured in steradians sr. We define the differential solid angle $d\Omega$ and the solid angle Ω:

$$d\Omega = d\theta \sin\theta \, d\phi \tag{4.2}$$

$$\Omega = \iint_{\text{directions considered}} d\Omega = \frac{A}{r^2} \tag{4.3}$$

Intuitively, the solid angle is the area A/r^2 taken up on the surface of a unit sphere by the directions considered. It encodes the amount of field of view we are considering. The analogy to a 2D angle θ is shown below:

(a) 2D Ray ($\vec{\omega}$), Angle (θ), and Differential Angle ($d\theta$)

(b) 3D Ray ($\vec{\omega}$), Angle (θ, ϕ),and Differential Solid Angle ($d\Omega$)

Figure 4.1: Rays, Angles, and Differential Solid Angle

One can integrate this differential over the entire sphere ($\theta \in [0, \pi], \phi \in [0, 2\pi]$) to readily verify that the solid angle over an entire sphere is 4π sr, and over half a sphere ($\theta \in [0, \pi/2], \phi \in [0, 2\pi]$) is 2π sr. More specifically, consider integrating over a cone of half-angle θ_0. Then the solid angle is:

$$\Omega = \iint_{\text{half cone}} d\Omega$$
$$= \int_0^{2\pi} \int_0^{\theta_0} d\theta \sin\theta \, d\phi$$
$$\Omega = 2\pi(1 - \cos\theta_0) \tag{4.4}$$

Intuitively, the differential solid angle is the dimensionless area element in spherical coordinates. Think of the small rectangle with side lengths $r \, d\theta$ and $r \sin\theta \, d\phi$, but you divide by r^2 to make it dimensionless. This is analogous to the case in 2D, where the differential angle is a dimensionless line element in radial coordinates. Think, analogously, of a small line of side length $r \, d\theta$, but again you divide by r to make it dimensionless.

We also define the **projected** differential solid angle $d\Omega_\perp$. This accounts for the fact that if we have some surface which is absorbing radiation, radiation parallel to the surface will not be absorbed.

$$d\Omega_\perp = \cos\zeta \, d\Omega \tag{4.5}$$

where ζ is the **zenith angle**, the angle between the ray ($\vec{\omega}$) and the ray normal to the surface in question. Sometimes we orient our axes such that $\theta = \zeta$ because it simplifies the system, but note that we do not always do this nor does this always simplify the situation.

4.2 Magnitude of Radiation: (Spectral) Radiance and (Spectral) Irradiance

Now that we have learnt how to characterise the *direction* in which radiation if propogating, we wish to characterise the *magnitude* of the radiation.

Let us consider some surface *perpendicular* to some radiation propogating through it. The **Radiance** I is the energy flux of the radiation flowing through that surface per unit *time* per unit *area* per unit *solid angle*. I has units of $\mathrm{J\,s^{-1}\,m^{-2}\,sr^{-1}}$. Now consider an *arbitrarily* oriented surface, which is oriented with at some **zenith angle** ζ to some radiation. Then the energy flux per unit time ϕ (in units of $\mathrm{J\,s^{-1}}$) is calculated by integrating the **radiance** over solid angles and the area of the patch as follows:

$$\phi = \iint I \, d\Omega_\perp \, dA \tag{4.6}$$

The **Irradiance** E is the energy flux flowing perpendicular through that surface per unit time and unit area. E has units of $\mathrm{J\,s^{-1}\,m^{-2}}$. The two are related as follows:

$$E = \int_{\Omega_\pm} I \, d\Omega_\perp$$
$$\phi = \int E \, dA$$

Usually we'll consider radiation propagating only from one hemisphere (i.e., from one direction) so we'll integrate over the upwards or downwards propagating hemisphere, which we denote as Ω_+ and Ω_-, respectively. I also remind you that E cannot be a function of $\vec{\omega}$ but I generally is.

So far our discussion has been completely general, but now we focus on light/electromagnetic radiation, which is a wave and thus has a frequency/wavelength. The **Spectral Radiance** L and **Spectral Irradiance** F is the radiance and irradiance (respectively) at a given frequency per unit frequency. L and F have units of $\mathrm{J\,s^{-1}\,m^{-2}\,sr^{-1}\,Hz^{-1}}$ and $\mathrm{J\,s^{-1}\,m^{-2}\,Hz^{-1}}$, respectively. Equivalently, $L(\nu)\,d\nu$ is the radiance carried by waves with some frequency $\nu' \in [\nu, \nu + d\nu]$ and $F(\nu)\,d\nu$ is the irradiance carried by waves with some frequency $\nu' \in [\nu, \nu + d\nu]$.

Due to the laws of electromagnetism, the total energy of a superposition of electromagnetic waves is given by the sum of the energy of the waves at each frequency. As such:

$$I = \int L \, d\nu \tag{4.7}$$

$$E = \int F \, d\nu \tag{4.8}$$

$$\phi = \iint L \, d\nu \, d\Omega_\perp \, dA \tag{4.9}$$

Since the spectral radiance at any point is typically a function of both ν and $\vec{\omega}$, we must integrate over both ν and $\vec{\omega}$ to obtain the radiative energy fluxes per unit area.

Practically, this integration cannot be solved analytically and so is done numerically. In these notes, we will make a few approximations in order to make analytical progress/extract physical intuition. We will approximate the distribution over $\vec{\omega}$ (the *Two-Stream Approximation* in 6.2), integrate over frequency bands (6.5), and make the (bad) grey gas approximation ($\frac{\partial L}{\partial \nu} = 0$) in Section 7.1.

As a summary:

Key Idea II.2: Characterising the Magnitude of Radiation

Consider some radiation propagating through a surface at some **zenith angle** ζ (i.e., the radiation is at angle ζ to the vector normal to the surface). Let the power flux in $\mathrm{J\,s^{-1}}$ through the surface be ϕ and the area A. Then we define:

Radiance I: The energy flux per unit time, area, and solid angle parallel to the direction the radiation is propogating in.

$$I = \frac{\partial^2 \phi}{\partial A \, \partial \left(\Omega \cos \zeta\right)}$$
$$= \frac{\text{Power flux \textbf{parallel} to \textbf{radiation}}}{\text{per (Area)(Solid Angle)}}$$

Irradiance E: The energy flux per unit time and area through the surface in question.

$$F = \frac{\partial \phi}{\partial A}$$
$$= \frac{\text{Power flux \textbf{through} the \textbf{surface}}}{\text{per (Area)}}$$

Spectral Radiance L: The energy flux per unit time, area, solid angle, and frequency parallel to the direction the radiation is propogating in.
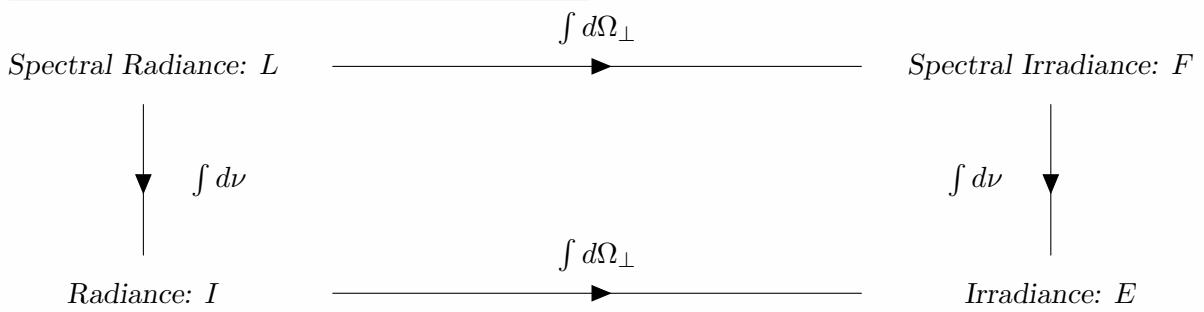
$$L = \frac{\partial^3 \phi}{\partial A \, \partial \left(\Omega \cos \zeta\right) \, \partial \nu}$$
$$= \frac{\text{Power flux \textbf{parallel} to \textbf{radiation}}}{\text{per (Area)(Solid Angle)(Frequency)}}$$

Spectral Irradiance F: The energy flux per unit time, area, solid angle, and frequency through the surface in question.

$$F = \frac{\partial^2 \phi}{\partial A \, \partial \nu}$$
$$= \frac{\text{Power flux \textbf{through} the \textbf{surface}}}{\text{per (Area)(Frequency)}}$$

Spectral Radiance: L $\xrightarrow{\ \int d\Omega_\perp\ }$ Spectral Irradiance: F

$\downarrow \int d\nu$ $\qquad\qquad$ $\int d\nu \downarrow$

Radiance: I $\xrightarrow{\ \int d\Omega_\perp\ }$ Irradiance: E

4.2.1 Frequency, Wavelength, and Wavenumber

So far, we have specified how quickly a wave oscillates by refering to its frequency ν, but we could have referred to its wavelength λ or its wavenumber $\tilde{\nu}$. All of these give you the same information, and you can freely convert between them if you know the speed of the wave c. For light, this is $c \approx 3 \times 10^8 \, \mathrm{m\,s^{-1}}$. You can find the information summarised in the table below.

Symbol	Name	Units	Meaning
ν	Frequency	$\mathrm{s^{-1}}$; Hz	The number of oscillations per second.
λ	Wavelength	m	The peak-to-peak length of the wave. $\nu\lambda = c$
$\tilde{\nu}$	Wavenumber	$\mathrm{m^{-1}}$; $\mathrm{cm^{-1}}$	The wavelengths per unit metre. $\tilde{\nu} = \lambda^{-1}$. In spectroscopy, we often refer to wavenumber in terms of wavelengths per *centimetre* ($\mathrm{cm^{-1}}$) for convenient numbers.

Finally, we should note two nuances. First, I introduced **spectral radiance/irradiance** as the radiance/irradiance per unit frequency, but many people refer to the **spectral radiance/irradiance** as the radiance/irradiance per unit wavelength or per unit wavenumber. The choice of description is ultimately arbitrary, but you should be aware of this potential confusion.

The second nuance is more important. It is the $L(\nu)\,d\nu$ term which is physically meaningful, not the $L(\nu)$ term by itself. This is because, as already mentioned, ν is only one of the ways we could have chosen represent the property of wavelength/frequency. We could equally use the wavelength λ. To illustrate the point, let's convert the spectral radiance $L(\nu)\,d\nu$ carried by the radiation with frequency $\nu' \in [\nu, \nu + d\nu]$ to something in terms of the wavelength λ:

$$\nu\lambda = c \therefore$$
$$L(\nu)\,d\nu = L\left(\frac{c}{\lambda}\right) d\left(\frac{c}{\lambda}\right)$$
$$= \frac{c}{\lambda^2} L\left(\frac{c}{\lambda}\right)\,d\lambda$$
$$= \hat{L}(\lambda)\,d\lambda$$

where we have defined $\hat{L}(\lambda) \equiv \frac{c}{\lambda^2} L\left(\frac{c}{\lambda}\right)$. Notice how the functional form fundamentally changes when we undergo an ultimately arbitrary coordinate transformation from ν to λ: the '*frequency* spectral radiance' $L(\nu)$ and '*wavelength* spectral radiance' $L(\lambda)$ are not the identical! But nothing physically *out there* should depend on how we choose to write it down, so we conclude that $L(\nu)$ by itself is an artefact arising from our choice to represent it in terms of ν.

4.3   Absorption: Optical Thickness and Transmission Function

Now that we can characterise radiation in terms of direction and magnitude, we move to lay the foundations of characterising how matter interacts with radiation. We first consider absorption.

Suppose some spectral radiance is traveling through a slab of radiation-absorbing *stuff* (e.g., air). We define the **absorptivity** $\alpha(\nu, T, p)$ of the slab as the (dimensionless) fraction of spectral radiance absorbed by the material:

$$L_{fin}(\nu) = L_{in}(\nu)(1 - \alpha(\nu, T, p)) \text{ or}$$
$$\Delta L(\nu) = -\alpha(\nu, T, p)L(\nu)$$

We now focus only on a slab of radiation-absorbing stuff (e.g., a slab of atmosphere) that is only infinitessimaly thick. We define the **mass path** $d\mu$ as the mass per area (perpendicular to the propogation of radiation) and define the infinitesimal **Optical Thickness** $d\tau_\nu$ of the slab as the absorptivity α of that slab. In other words, suppose some spectral radiance $L(\nu)$ is traveling through the radiation-absorbing **stuff**. Then the infinitesimal change of spectral radiation by the slab $dL(\nu)$ is given by:

$$dL = -\alpha(\nu, T, p)L(\nu)$$
$$= d\tau_\nu(T, p)\,L(\nu)$$

where I have written $d\tau_\nu$ with a subscript ν to remind us that $d\tau_\nu$ is a strong function of ν.

Finally, we define the absorption cross section $\kappa(\nu, T, p)$ as the effective area of spectral radiance absorbed per unit mass of absorber encountered. κ has units of $\text{m}^2\,\text{kg}^{-1}$, and will generally depend on frequency, temperature, and pressure. Therefore:

$$\boxed{d\tau_\nu = \kappa(\nu, T, p)\,d\mu} \tag{4.10}$$

We can integrate this between two points in the atmosphere p_1 and p_2 to find the final spectral

radiance L_2 in terms of the initial spectral radiance L_1:

$$\frac{dL}{L} = -d\tau_\nu$$

$$\therefore \int_{p_1}^{p_2} \frac{dL}{L} = -\left| \int_{p_1}^{p_2} d\tau_\nu \right|$$

$$\ln\left(\frac{L_2}{L_1}\right) = -\left| \int_{p_1}^{p_2} d\tau_\nu \right|$$

$$\therefore L_2 = L_1 \exp\left(-\left| \int_{p_1}^{p_2} d\tau_\nu \right|\right)$$

Notice the absolute value in the second line. If we did not have this, we could simply reverse the limits of integration (e.g., integrate from p_2 to p_1) and obtain that the spectral radiance *increases* as it passes through. However, this is clearly unphysical: the spectral radiance absorbed should be the same regardless of whether the radiation is going upwards through the absorber or downwards through the absorber. Finally, if we note that:

$$\tau_\nu(p_1) - \tau_\nu(p_2) = \int_{p_1}^{p_2} d\tau_\nu = \int_{p_1}^{p_2} \kappa(\nu, T, p) \, d\mu$$

then we can define the transmission function $\mathcal{T}_\nu$ which encodes the absorption of the spectral irradiance between those two points:

$$\mathcal{T}_\nu(p_1, p_2) = e^{-|\tau_\nu(p_1) - \tau_\nu(p_2)|} \tag{4.11}$$
$$L_2 = L_1 \, \mathcal{T}_\nu(p_1, p_2)$$

Note that it is only the transmission function which is physically and dynamically significant here, as *that* is what determines absorption. We could work directly with the transmission function, but for convenience, we'd much rather work in terms of optical depth.

Meanwhile, the optical depth itself is not directly physically meaningful. This is because the transmission function only cares about absolute value differences in optical depth.[3] As such, we must make two arbitrary choices when calculating our optical depth (analogous to choosing, e.g., the origin of our coordinate system in Newtonian mechanics):

1. It's ordering: we can let τ_ν *increase* with height (decrease with pressure) or *decrease* with height (increase with pressure). This is because the transmission function only cares about the absolute values.
2. The reference value: we can pick $\tau_\nu = \tau_{\nu,ref}$ at any reference level. This is because the transmission function only cares about differences in optical depth.

I will adopt Ray's convention here, although I should note he switches it up for convenience (as he is well within his rights to!) in section 7.3. As such I will choose:

1. τ_ν to increase with height, i.e., τ_ν to decrease with pressure.
2. τ_ν to be equal to 0 at the ground.

---

[3]The situation is analogous to Classical Electromagnetism or Newtonian Mechanics. In Electromagnetism and Mechanics, the only objects of physical consequence are the *forces*, which depend only on the electric/magnetic fields $(\vec{E}, \vec{B})$[4]and accelerations $(\ddot{\vec{r}})$ respectively. However, for convenience, we'd rather work with the electromagnetic scalar/vector potentials $(\phi, \vec{A})$ or positions $(\vec{r})$, so we have some freedom to choose some properties of $(\phi, \vec{A})$ or $(\vec{r})$. In the former case, we must fix the gauge (e.g., pick the Lorenz or Coulomb gauge) and in the latter case, we must pick the origin and veloctiy of our coordinate system (e.g., pick the origin as the centre of the Earth). In fancy speak, we say that electromagnetism has a gauge symmetry and Newtonian mechanics has galillean symmetry. We must do a similar thing here for optical depth.

[4]Actually the situation in electromagnetism is a bit more complicated than I've made it. In fact, only the motion of charged objects is directly observable. Since $\vec{B}$ interacts with matter through equations involve its cross product (the Biot-Savart Law and $\vec{F} = q\vec{v} \times \vec{B}$), we've used a convention of the right-handed cross product $(\vec{x} \times \vec{y} = \vec{z})$. We could have equally used the left handed cross product and had equivalent empirical predictions. I'm also not going to go into bivectors and pseudovectors here, because this footnote is already too long.

We now specify that the spectral radiance is travelling through a slab of air with some density ρ and with some absorber with mass-fraction q. We can see that $d\mu = q\,\rho\,dl$ where dl is the length of atmosphere the radiation travels through. We further specify that the spectral radiance is traveling at some angle θ relative to the vertical so that $dl = +dz/\cos\theta$ where $z = $ the height[5]. Finally, we relate dz to dp through Hydrostatic Balance (1.12). Then:

Key Idea II.3: Optical Depth Coordinates and the Transmission Function

We define the optical depth coordinate τ_ν, which encodes how much absorbing *stuff* is below a pressure level p:

$$d\tau_\nu = -\kappa(\nu, T(p), p)\, q(p) \frac{dp}{g\cos\theta} \quad (4.12) \qquad \tau_\nu(p) = \int_{p_s}^{p} -\kappa(\nu, T(p'), p')\, q(p') \frac{dp'}{g\cos\theta} \quad (4.14)$$

$$\tau_\nu = 0 \text{ at } p = p_s \quad (4.13) \qquad \tau_\nu(p=0) = \tau_{\nu,\infty} \quad (4.15)$$

The transmission function $\mathcal{T}_\nu$ between two points p_1 and p_2 encodes how much the spectral radiance is attenuated between the two points and is:

$$\mathcal{T}_\nu(p_1, p_2) = e^{-|\tau_\nu(p_1) - \tau_\nu(p_2)|} \quad (4.16) \qquad L(\nu, p_2) = L(\nu, p_1)\,\mathcal{T}_\nu(p_1, p_2) \quad (4.17)$$

A taylor expansion of Equation 4.17 shows that the fraction of spectral radiance absorbed in an infinitesimmaly thin layer is the optical thickness of that layer $|d\tau_\nu|$.

Notice that I write $T = T(p)$ here. Until 7.1, we assume that the vertical temperature profile of the atmosphere is given by some known relation (perhaps the Dry Adiabat if we are in the troposphere).

Similar to the hydrostatic relation allowing us to use both pressure and height coordinates interchangably, Equation 4.12 and the fact that $\kappa, q, g > 0$ and $\theta = const$ ensures that $d\tau_\nu \sim -dp$[6] for all p which in turn ensures that there is a one-to-one relation betweeen p and τ_ν, allowing us to use both pressure and optical depth coordinates interchangably.

4.3.1 Example: Grey Atmosphere and Well Mixed Absorber

Let us make three assumptions. First, assume that the atmosphere is *grey*. This amounts to the assumption that the atmosphere interacts with light in the same way regardless of frequency, so $\frac{\partial \kappa}{\partial \nu} = 0$. Second, assume that κ has no T or p dependence. Third, assume that the absorber is *well-mixed*. This means that the concentration q is independent of height $\frac{\partial q}{\partial p} = 0$. We can now integrate the optical depth to find:

$$\begin{aligned}
\tau_\nu(p) &= \int_{p_s}^{p} -\kappa\, q(p) \frac{dp'}{g\cos\theta} \\
&= -\kappa\, q \frac{1}{g\cos\theta} \int_{p_s}^{p} dp' \\
&= \frac{\kappa\, q}{g\cos\theta}(p_s - p)
\end{aligned}$$

---

[5] I could have equally chosen that $dl = -dz/\cos\theta$ here.
[6] Or that $d\tau_\nu \sim dp$ if $\theta > \frac{\pi}{2}$!

We define $\tau_\infty = \int_{p_s}^0 d\tau_\nu$ as the optical depth of the *entire* atmosphere (at that angle θ). Therefore:

$$\tau_\infty = \tau(0)$$
$$= \frac{\kappa\,q}{g\,\cos\theta}(p_s - 0)$$
$$= \frac{\kappa\,q\,p_s}{g\,\cos\theta}$$
$$\therefore \tau_\nu(p) = \tau_\infty\left(1 - \frac{p}{p_s}\right)$$

However, typically, κ is a strong function of ν. We will discuss how exactly κ depends on ν in the next chapter.

4.4 Emission: BlackBody Radiation

We now turn to emission.

We first define a **blackbody** as a theoretical object which absorbs all incident electromagnetic radiation. To a good approximation, many stellar bodies are blackbodies, including the Sun. One can show that such a body must emit spectral radiance at all times given by $L_{BB}(\nu, T)$, where T is the temperature of the blackbody:

$$\boxed{L_{BB}\,d\nu = B(\nu, T)\,d\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}\,d\nu} \tag{4.18}$$

You do not have to memorise this, but you should be aware of a few properties.

First, at all fixed frequencies ν, $B(\nu, T)\,d\nu$ is monotonically increasing with T. As T increases, so too does B and every frequency, so if a blackbody warms up it releases more energy (See Figure 4.2a).

Second, while blackbodies emit radiation at all frequencies, its 'characteristic frequency' increases as temperature increases.[7] Since the star will almost always be hotter than its planet, this results in what's called a *spectral gap*: the planet tends to emit radiation at a much longer wavelength than its star, which is what enables the greenhouse effect (See Figure 4.2c). For this reason, we often call the radiation emitted by the planet '**longwave**' radiation and the radiation emitted by the star '**shortwave**' radiation.

---

[7]We define the 'characteristic frequency' as follows. First, define the cumulative blackbody radiance as $I_{BB}(\nu) = \int_0^\nu B(\nu', T)\,d\nu'$. I_{BB} monotonically increases as ν increases (since $B(\nu') > 0$), so I_{BB} reaches its maximum value at $\nu = \infty$. We define the characteristic frequency $\nu_{char}(T)$ as the median emission frequency: the frequency where I_{BB} is half its maximum value.

More rigourously, ν_{BB} satisfies the following relation: $\int_0^{\nu_{char}(T)} B(\nu', T)\,d\nu' = \frac{1}{2}\int_0^\infty B(\nu', T)\,d\nu'$.[8]It is the case that $\frac{d\nu_{char}}{dT} > 0$ for all T.

[8]Why all the hassle? Why not find where $B(\nu, T)$ is maximum and define that as the characteristic frequency? We do not do this becuase this definition is not physically meaningful. Recall the discussion we had just last section: it is $B(\nu)\,d\nu$ that is meaningful, not $B(\nu)$. (Excercise: differentiate $B(\nu)$ to find where B is maximum with respect to ν. Then, transform into wavelength coordinates, making sure to transform $d\nu$ as well, then differentiate with respect to wavelength. You should find that the maximum wavelength is not the maximum frequency).

(a) Spectral Radiance $L = B(\nu, T)$ of a BlackBody.

(b) Cumulative Radiance $\int B d\nu$ of a BlackBody.

(c) Normalised spectral radiance of bodies in the solar system.

Figure 4.2: Properties of the **blackbody** spectral emission $B(\nu, T)$.

Third and finally, B is not a function if $\vec{\omega}$: B is isotropic! This final fact allows us to calculate the spectral irradiance of a blackbody by integrating over the projected differential solid angle $d\Omega_\perp$. We integrate over $d\Omega_\perp$ and not $d\Omega$ because only the radiation perpendicular to the surface of the blackbody is emitted:

$$F(\nu, T) = \int_{\Omega_-} B(\nu, T) \, d\Omega_\perp$$
$$= \left(\int_{\Omega_-} d\Omega_\perp \right) B(\nu, T)$$
$$= \pi B(\nu, T)$$

The final line is found by setting the zenith angle ζ equal to θ and integrating the projected differential solid angle explicitly:

$$\int_{\Omega_-} d\Omega_\perp = \int_{\Omega_-} \cos \zeta \, d\Omega$$
$$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos \zeta \, d\theta \, \sin \theta \, d\phi$$
$$= 2\pi \int_0^{\frac{\pi}{2}} \sin \theta \cos \theta \, d\theta$$
$$= 2\pi \left[\frac{1}{2} \sin \theta^2 \right]_0^{\frac{\pi}{2}}$$
$$= \pi$$

To find the (non-spectral) irradiance of a blackbody, we integrate over all frequencies to recover the Stefan-Boltzmann Law governing the power output per unit area P of a blackbody:

$$\boxed{P = \sigma T^4 = \int_0^\infty \pi B(\nu, T) \, d\nu} \tag{4.19}$$

4.4.1 Emissivity and Kirchoff's Law

Many bodies are not like blackbodies at all, even approximately. We thus define the **emissivity** $\epsilon \in [0, 1]$ of an object as follows:

$$\boxed{\epsilon(\nu, T, p) = \frac{L(\nu, T, p)}{B(\nu, T)}} \tag{4.20}$$

where $T =$ the temperature; $p =$ the pressure; and L is the spectral radiance emission of the object in question. Physically, the emissivity encodes how much the object is like a blackbody at each frequency, temperature, and pressure. For a perfect blackbody, $\epsilon = 1$ for all frequencies, temperatures, and pressures. One can show that, in local thermodynamic equilibrium, the emissivity of an object is exactly equal to its absorptivity α:

$$\boxed{\epsilon(\nu, T, p) = \alpha(\nu, T, p)} \tag{4.21}$$

We will use this to derive the Schwarzschild Equation, which governs radiative transfer in the atmosphere. However, we should note that the derivation assumes local thermodynamic equilibrium. This is a good approximation in most parts of the atmosphere, but fails near the top of the atmosphere.

As a summary:

Key Idea II.4: Blackbody Radiation, Emissivity, and Kirchoff's Law

A blackbody emits radiation at all times with a well known spectral radiance function $B(\nu, T)$ (given by Equation 4.19). This spectral radiance:
1. Monotonically increases at every freuqency with temperature.
2. Increases its characteristic frequency with temperature.
3. Is isotropic in direction.

The (dimensionless) **emissivity** ϵ of an object encodes the spectral radiance L an object emits such that:

$$L(\nu, T, p) = \epsilon(\nu, T, p)B(\nu, T) \tag{4.22}$$

In local thermodynamic equilibrium, the **emissivity** is equal to the **absorptivity** α according to Kirchoff's Law:

$$\boxed{\epsilon(\nu, T, p) = \alpha(\nu, T, p)} \tag{4.23}$$

4.4.2 Example: Calulating the Stellar Constant

We've covered a lot of what are probably unfamiliar concepts to you at this point, so it would be useful to apply such concepts in the explicit calculation of something. This will hopefully build some intuition.

To a good approximation, stellar bodies are black bodies ($\epsilon = 1$). We can use this to estimate the incident (not absorbed) radiative energy flux on the Earth (or any planet) from the Sun (or any star) by approximating that star as blackbody.

Our goal now is to calculate the **Stellar Constant** $L_\star$, defined as the incident stellar irradiance on a *flat* surface *facing* the sun.

There are two important nuances here. First, note that the **Stellar Constant** is *not* necessarily equal to the incident stellar irradiance on the surface of the planet because the planet is *curved*. The stellar constant is only equal to the incident stellar irradiance on the surface of the planet at the equator at 12:00 if the planet has no axial tilt. Second, note that this is also different from the *absorbed* stellar power – to obtain that we have to do three things: first, account for the aforementioned nuance (i.e., that only the radiation perpendicular to the curved surface is absorbed); second, account for the albedo of the planet (some incident stellar irradiance is reflected); and third, integrate over the area of the planet.

Let us now calculate $L_\star$. Our strategy is to integrate the spectral radiance of the star (which is approximately $B(\nu, T_\star)$, where $T_\star$ is the temperature of the star) over all frequencies (to obtain the radiance) and over the solid angle $\Omega_\star$ subtended by the star at the planet (to obtain the irradiance):

$$L_\star = \int_{\Omega_\star} \int_0^\infty B(T_\star, \nu) \, d\nu \, d\Omega_\perp$$

Calculating this explicitly using 4.19, we get:

$$L_\star = \int_{\Omega_{Sun}} \frac{1}{\pi} \sigma T_\star^4 \, d\Omega_\perp$$

We now make the assumption that the star is very far away from the planet: that $d \gg R_\star, R$, where $d =$ the distance between the star and the planet, and $R_\star, R =$ the radius of the star and planet, respectively. This allows us to make two approximations.

Our first approximation is that $\cos\zeta \approx 1$ or that $d\Omega \approx d\Omega_\perp$. This is because the radiation reaching the planet must all be approximately parallel to each other and perpendicular to the surface considered. If this weren't the case, and the radiation was only slightly tilted, the radiation would completely miss the planet. So:

$$L_\star = \int_{\Omega_\star} \frac{1}{\pi} \sigma T_\star^4 \, d\Omega_\perp$$
$$\approx \int_{\Omega_\star} \frac{1}{\pi} \sigma T_\star^4 \, d\Omega$$
$$= \Omega_\star \frac{1}{\pi} \sigma T_\star^4$$

where we have assumed in the last line that the radiance is independent of solid angle. Our goal now is to calculate $\Omega_\star$, the solid angle subtended by the star at the planet, which we can calculate by using 4.4.

Our second approximation is that $\theta_0 \ll 1$, and so $\cos\theta_0 \approx 1 - \theta_0^2/2$ and $\theta_0 \approx R_\star/d$. As such:

$$\Omega_\star = 2\pi(1 - \cos(\theta_0))$$
$$\approx 2\pi \left(1 - \left(1 - \frac{1}{2}\theta_0^2 \right) \right)$$
$$\approx \pi \frac{1}{2} \frac{R_\star^2}{d^2}$$

Another (in my opinion more intuitive) way to think of this calculation is to consider a sphere centred on the planet of radius d so that the star lies on the surface of this imaginary sphere. The total solid angle of the entire sphere is 4π, but the star is only occupying a small area on the surface of that sphere. The solid angle is equal to the total solid angle 4π multiplied by the fraction of field of view the star covers.

We approximate the fraction of the field of view covered by the star as equal to the fraction of the *surface area* of the imaginary sphere the star covers. This fraction is itself approximately equal to $\pi R_\star^2/4\pi d^2$. This is only approximate, because if it weren't the case that $R_\star \ll d$, then the area covered by the sun would not be $\pi R_\star^2$ (and furthermore, this wouldn't be the fraction of the field of view covered by the sun).

Anyways, it is the case that $R_\star, R \gg d$, so plugging it in gives the same results:

$$\Omega_\star = 4\pi \frac{\pi R_\star^2}{4\pi d^2}$$
$$= \pi \frac{R_\star^2}{d^2}$$

Therefore the stellar constant is;

$$L_\star = \frac{R_\star^2}{d^2} \sigma T_\star^4 \tag{4.24}$$

Let's calculate this for our sun! Plugging in the radius of the Sun $R_\star \approx 696.3 \times 10^3$ km, the distance between the Earth and the Sun $d \approx 149.6 \times 10^6$ km, and the Sun's surface temperature $T_\star \approx 5772$ K,

we get that $L_\star = \approx 1364\,\mathrm{W\,m^{-2}}$. The literature value is approximately $1361\,\mathrm{W\,m^{-2}}$, so we have gotten remarkably close!

However, as already noted, the power actually absorbed by the planet is different. Again, we must factor in three things. We must factor in the curvature of the planet (which we assume is spherical), as the radiation is not perpendicular to the surface of the planet, and it is only radiation perpendicular to the surface. We must factor in albedo of the planet, as some radiation is reflected. And finally, we must integrate over area to find the power (rather than the power per unit area).

We now integrate over the area of the planet and account for for reflection and curvature. Suppose some fraction $a \in [0,1]$ of the instellation is reflected. Therefore:

$$P_{in} = \int (1-a) L_\star \cos\zeta\, dA$$

$$= (1-a) L_\star \int \cos\zeta\, dA$$

where the zenith angle term $\cos\zeta$ represent the fact that the sun is only hitting the Earth from one side, and only the component perpendicular to the ground is absorbed. As such, $\int \cos\zeta\, dA$ is the *projected* area of the Earth due to the $\cos\zeta$ term, and therefore is πR^2, not $4\pi R^2$ (exercise). Therefore:

$$P_{in} = (1-a)\pi R^2 \underbrace{\frac{R_\star^2}{d^2}\sigma T_\star^4}_{L_\star} \tag{4.25}$$

If we define the total **Outgoing Longwave Radiation** OLR as the total irradiance leaving the planet, then we get a radiation budget for the planet:

$$G = (1-a)\pi R^2 L_\star - 4\pi R^2 OLR$$

$$\boxed{G \sim (1-a)\frac{L_\star}{4} - OLR} \tag{4.26}$$

Almost all of Radiative Transfer will be dedicated to calculating this OLR, and we'll have to introduce a lot of machinery to calculate it.

If we wish to make progress now, though, we simply define the effective emissivity ϵ_{eff} such that:

$$\epsilon_{eff} \int \int_{\Omega_+} \int_0^\infty B(\nu, T)\, d\nu\, d\Omega_\perp\, dA \equiv \int \int_{\Omega_+} \int_0^\infty \epsilon(\nu, T) B(\nu, T)\, d\nu\, d\Omega_\perp\, dA$$

While we are perfectly within our rights to theoretically define such an effective emissivity, this does not help us at all in practically calculating OLR. Regardless, it will be useful for some intuition. Now that we've defined this, we can estimate the outgoing power:

$$P_{out} = \int \int_{\Omega_+} \int_0^\infty \epsilon(\nu, T) B(\nu, T)\, d\nu\, d\Omega_\perp\, dA$$

$$= \epsilon_{eff} \left(\int dA \right) \int_{\Omega_+} d\Omega_\perp \int_0^\infty B(\nu, T)\, d\nu$$

$$= (4\pi R^2)\, \epsilon_{eff} \sigma T^4$$

$$\therefore OLR = \epsilon_{eff} \sigma T^4$$

In equilibrium, $G = 0$, which means we can solve for the equilibrium temperature of the Earth:

$$0 = (1-a)\frac{L_\star}{4} - OLR$$

$$= (1-a)\frac{L_\star}{4} - \epsilon_{eff}\sigma T^4$$

$$\therefore T = \left((1-a)\frac{L_\star}{4\epsilon_{eff}\sigma} \right)^{\frac{1}{4}}$$

$$= T_\star \left(\frac{1-a}{4\epsilon_{eff}} \frac{R_\star^2}{d^2} \right)^{\frac{1}{4}}$$

Clearly then, the temperature of the planet should be proportional to the temperature of the star, and the constant of proportionality $\left(\frac{1-a}{4\epsilon_{eff}}\frac{R_\star^2}{d^2}\right)^{\frac{1}{4}}$ increases as the albedo decreases or as the emissivity decreases. The former might be due to reduced ice cover, and the latter might be due to an increase in greenhouse gas concentration, for example.

Chapter 5

Molecular Spectroscopy

5.1   Spectral Lines

Here we cover how the absorption cross section κ depends on ν. In almost all cases, save Continuum Absorption, κ is made up of many 'spectral lines' (see Figure 5.1b and ignore the white lines).

(a) Notice the **strong** variation over ν indicated by the logarithmic y-axis scale.

(b) Notice how the curve is made up of **spectral lines**.

Figure 5.1: CO_2's Absorption Cross Section at $100\,\mathrm{mbar}$ and $260\,\mathrm{K}$. Figures from Ray's lecture slides.

We will discuss how to characterise the **line shape** of each individual spectral line, which we will refer to as κ_{ν_c}:

Key Idea II.5: Line Shape

An individual spectral line is characterised as follows. All parameters have units of s^{-1}.

$$\kappa_{\nu_c}(\nu) = \frac{S}{\delta} F\left(\frac{\nu - \nu_c}{\delta}\right)$$

(5.1)

Line Centre ν_c: encodes the centre of the spectral line.

Line Strength S: encodes how strong of an absorber the spectral line is. Defined as follows:

$$S = \int_{-\infty}^{\infty} \kappa(\nu) \, d\nu$$

(5.2)

Generally a function of T

Line Width δ: encodes how wide the spectral line is. Generally a function of p and T.

Line shape $F(x)$: encodes the shape of the line (e.g., how does the line decay as ν goes away from ν_c). Normalised such that:

$$\int_{-\infty}^{\infty} F(x) \, dx = 1$$

Why do gases absorb/emit electromagnetic radiation in the first place? Gases absorb/emit electromagnetic radiation because its consitutents, the molecules, have different states corresponding to different energy levels. If a molecule recieves just enough energy from a photon, it can transition from a lower energy level to a higher energy level, and thus absorb the photon. If a molecule is already in a higher energy level, it can spontaneously emit a photon to transition to a lower energy level.

There are three types of transitions that a molecule can undergo:

1. **Electron Transitions**: An electron on the molecule hops to a higher energy state. Typically visible and UV frequency photons cause/are emitted from these transitions.
2. **Vibrational Transitions**: The molecule begins vibrating/vibrates faster. Typically IR (Infrared) frequency photons cause/are emitted from these transitions.
3. **Rotational Transitions**: The molecule begins rotating/rotates faster. Typically microwave frequency photons cause/are emitted from these transitions.

We will now discuss what determines each parameter of ν_c, S, δ and $F(x)$.

5.1.1 Line Centre ν_c

In order to discuss what determines the line centre ν_c, we must discuss why spectral lines exist at all, rather than κ having a very smooth dependence on ν.

When a molecule absorbs a photon of frequency ν, it must recieve energy $E = h\nu$. However, as discussed in Section 1.3, the energy levels of molecules are quantised, so molecules can only absorb a photon with energy equal to the energy level spacings ΔE. As such, ν is constrained such that $\nu = \Delta E/h$, and this is what sets the line centre. ΔE itself is set by the quantum mechanical properties of the molecule, which is found through a variety of empirical and theoretical approaches.

A question you might (rightfully) have now is the following: if energy levels are discritised such that the molecule can only absorb energy of E_1, E_2, E_3, $\ldots$, then it can only absorb photons with a frequency *exactly* equal to $\nu_1 = h/E_1$, $\nu_2 = h/E_2$, $\nu_3 = h/E_3$, $\ldots$. Therefore, we should expect κ to be a collection of infinitely thin lines, and nothing like what we see in Figure 5.1, which is a collection of thin bumps! Furthermore, since only infinitely thin lines are absorbed, then nothing is actually absorbed at all, so $\kappa = 0$ for all materials! What went wrong?

This question will be answered when we discuss Broadening Mechanisms.

5.1.2 Line Strength S

The line strength S is determined by two factors.

The first factor is how strongly the molecule couples with the electromagnetic field. Typically, **electron** transitions already couple strongly to the electromagnetic field, as electrons are already charged, and moving charges couple strongly to electromagnetic radiation.

For **vibrational** and **rotational** transitions, the distortions of the molecule caused by the vibrations or rotations must change the molecules electric **dipole-moment**. A full explanation of what 'dipole-moment' means isn't possible here, but the gist of it is this: a molecule has a dipole moment if it has an uneven distribution of electric charge. For example, if it is positively charged on one end and negatively charged on the other.

H_2O has an intrinsic dipole moment, since the electrons in the bonds are more closely bound to the oxygen, and so the oxygen is a little bit negatively charged and the hydrogens are a little bit positively charged. Vibrations and rotations alter this, and this makes H_2O a very strong infrared absorber.

CO_2 has no intrinsic dipole moment, but it has two vibrational modes (see Figure 5.2) which alter its dipole moment. This is typical of linear triatomic molecules. As such, it is a strong infrared absorber as well.
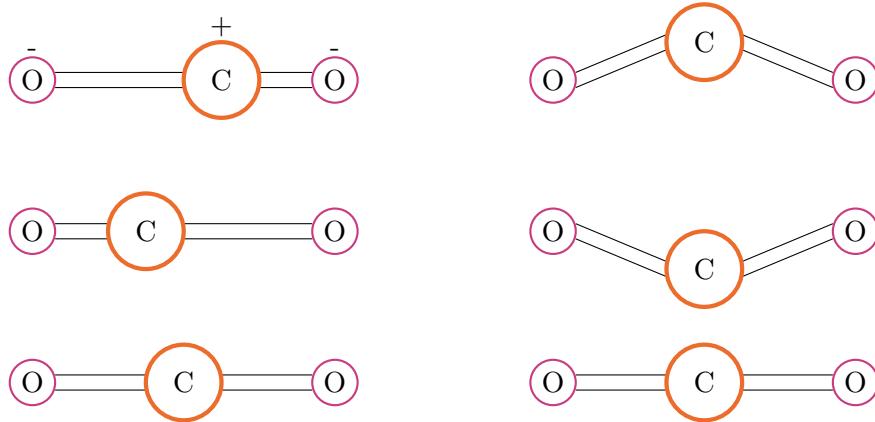
Figure 5.2: The two dipole moments of CO_2 that alter its dipole moment, thus making it a strong absorber/emitter in the infrared. Arrows indicate the movement of the central carbon molecule.

Meanwhile, N_2 and O_2 are both diatomic molecules made up of identical atoms, and so don't couple strongly to the electromagnetic field, since none of the rotational or vibrational modes change the molecules' dipole moment. This is why N_2 and O_2 are both optically inactive.

The second factor depends on the average occupation of the energy states. The intuition is as follows. Suppose a spectral line has a line centre at ν_c, and thus relies on a transition from energy levels from E_i to E_j (such that $E_j - E_i = h\nu_c$). A gas that has, say, half of its molecules in energy state E_i will have half of its molecules able to absorb a molecule of frequency ν_c, but a gas that has a higher proportion (say, three-quarters) will have a higher proportion of its molecules (three-quarters) able to absorb a molecule of frequency ν_c. Therefore the more occupied the energy states are, the higher the line strength.

The occupation of energy states depends on the temperature of a system, and one can derive this using statistical mechanics. We won't derive it here, but the result for the functional form of S for IR (Infra-red) radiation is as follows:

$$S(T) = S(T_0)\frac{Q(T_0)}{Q(T)} \left(\frac{1 - \exp\left(\frac{-h\nu_c}{k_B T}\right)}{1 - \exp\left(\frac{-h\nu_c}{k_B T_0}\right)} \exp\left(-\frac{h\nu_l}{k_B}\left(\frac{1}{T} - \frac{1}{T_0}\right)\right) \right)$$

where $h\nu_l$ is the energy of the lower energy state and $Q(T)$ is some function of T depending on the molecule. You do not need to memorise or understand much of this equation, but it's important to note that the main dependence comes from the exponential terms. S **strongly** increases with temperature (it has an exponential dependence): as temperature increases, line-strength also increases.

5.2 Broadening Mechanisms: Line Shape F and Width δ

If we stop our discussion here, then we would have a collection of infinitely thin lines,and so $F(x) = \delta(x)$ for all lines, where δ is the dirac delta function. However, lines are clearly **broadened** (see Figure 5.1b) and are not delta functions. There are three physical mechanisms which explain why lines are broadened in our context, and why molecules may absorb/emit photons with frequency ν close to ν_c but not equal to ν_c.

5.2.1 Uncertainty Broadening

One can show, but we won't show it here, that the following relation holds:

$$Var(\hat{E})\,\tau = h/2$$

where $Var(\hat{E})$ is the variance of the energy of the system, τ is the expected lifetime of the system, and h is Planck's constant. To be perfectly honest, I do not fully understand how this equation arises, and what relation this bears to the Heisenberg Uncertainty Relation other than a superficial resemblance.[1]

Let's now interpret the consequences of the equation on the molecules we're considering. The lifetime of a system τ encodes how stable the system is. Recall that a molecule is in an 'excited' state before emitting a photon or after absorbing a photon. This state is, generally, meta-stable, and thus has a finite lifetime ($\tau \neq \infty$) after which it has a high probability of emitting a photon and transitioning to a lower energy state.

The variance of the energy $Var(\hat{E})$ represents the fact that, because of this, these molecules aren't actually in a state of exact/definite energy (called an energy eigenstate). This means that when a system transitions from one energy state to another energy state, the energy difference can actually be approximately equal to ΔE, with some spread determined by the lifetime of the system τ.

One can determine the line width and shape from these principles more rigourously, but it turns out that in normal atmospheric circumstances this line width is negligible, so we do not discuss this in C5. It will suffice for you simply to know that, regardless of whether the upcoming two broadening mechanisms come into effect, there is some natural Uncertainty Broadening mechanism which ensures that *all* lines have finite width.

5.2.2 Doppler Broadening

Our second broadening mechanism is doppler broadening, and this occurs because the molecules in the gas might be moving away/towards us when they absorb or emit a photon. As such, the photon will be red or blue shifted.

Since the molecules are moving at non-relativistic speeds, the (doppler-shifted) frequency ν we see is related to the actual linecentre ν_c by:

$$\nu = \nu_c + (1 + v/c)$$

where c = the speed of light and v = the velocity the particle is moving in towards/away us.

The actual line-width is thus fixed by the velocity distribution of the molecules. This is predicted by the Maxwell-Boltzmann Distribution using Kinetic Theory, and so we get that:

$$\boxed{F(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}} \tag{5.3}$$

$$\boxed{\delta(T) = \frac{\nu_c}{c} \left(\frac{2k_B T}{m} \right)^{\frac{1}{2}}} \tag{5.4}$$

---

[1]The Heisenberg Uncertainty Relation relates the variance in momentum and variance in position of any quantum system. Despite what I see as a superficial resemblance, I will be using the name 'Uncertainty Broadening' because that's what it's called in C5 and I don't want to confuse you. If I could choose though, I'd probably use the sometimes-used 'Lifetime Broadening' or 'Natural Broadening'.

So the line-width is scales as $\sqrt{T}$ – the line gets thicker as the gas gets hotter. Conversely, if the temperature falls to 0 this equation predicts that the line width would fall to 0, and so $\kappa(\nu = \nu_c) \to \infty$. However, this never happens, as Uncertainty Broadening maintains a non-zero line width.

This is mechanism of broadening is much more substantial than Uncertainty Broadening, but it is still very thin for two reasons. First, as seen from 5.3, the line shape is exponential in x. As such, F falls off extremley rapidly as ν moves away from ν_c. Second, as seen from 5.4, generally $\delta \ll \nu_c$ since $\left(\frac{2k_B T}{m}\right)^{\frac{1}{2}} \ll c^2$. However, it is dominant in the stratosphere.

5.2.3 Collisional/Pressure Broadening

In the troposphere, this is by far the most important type of broadening. This arises because the kinetic energy of a molecule is not actually quantised. As such, if a molecule is colliding with another molecule while a photon hits it, it can borrow/give some energy to the other molecule to absorb a photon of slightly different frequency than ν_c.

The broadening thus scales with the frequency of collisions, which depends on the pressure and temperature. This gives us the Lorentz Profile:

Key Idea II.6: Lorentz Line Shape

The shape F and width δ of a spectral line is, in planetary climate contexts, mainly set by the Lorentz Line shape:

$$F(x) = \frac{1}{\pi} \frac{1}{x^2 + 1} \qquad (5.5)$$

$$\delta \approx \delta_0 \frac{p}{p_0} \left(\frac{T_0}{T}\right)^n \qquad (5.6)$$

where δ_0, p_0, T_0 are all reference widths, pressures, and temperatures and n is a dimensionless constant. All constants are, in practice, found empirically. Oftentimes, $n \approx 1/2$.

As we can see then, pressure broadening results in pressure *increasing* the line width and temperature *decreasing* the line width. If, for example, the pressure were to fall to 0 this equation predicts that the line width would fall to 0, and so $\kappa(\nu = \nu_c) \to \infty$. However, again, this never happens, as Uncertainty Broadening (and Doppler Broadening if $T \neq 0$) would maintain a non-zero line width.

You should memorise this for the exam Key Idea 5.2.3 for the exam.

5.3 Continuum Absorption

[Under Construction]

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[2]Physically, $\left(\frac{2k_B T}{m}\right)^{\frac{1}{2}}$ is proportional to the mean speed of the molecules.

Chapter 6

The Schwarzschild Equation

6.1   Derivation of the Schwarzschild Equation

Consider some radiation traveling through a slab of atmosphere at some zenith angle θ with some spectral radiance L with some frequency ν. We wish to derive how $L(\nu)$ varies as it propogates through the atmosphere. Suppose the slab of atmosphere has vertical infintesimal optical thickness of $d\tau_\nu^*$. Since the path of the radiation is slanted, the spectral radiance travels through an infinitesimal optical depth of $d\tau_\nu = d\tau_\nu^*/\cos\theta$. The slab absorbs some spectral radiance, but it also emits some spectral radiance:

The spectral radiance going into the slab is $L(\tau_\nu^*)$. The spectral radiance going out of the slab is $L(\tau_\nu^* + d\tau_\nu^*)$, which we wish to express in terms of $L(\tau_\nu^*)$. We know that some proportion α of $L(\tau_\nu^*)$ is absorbed by the radiation and that this proportion is equal to the absorptivity $\alpha(\nu, T) = d\tau_\nu$. However, we also know that the slab of atmosphere also emits radiation equal to $\epsilon(\nu, T)\, B(\tau_\nu^*)$, where ϵ = the emissivity. Therefore:

$$L(\tau_\nu^* + d\tau_\nu^*) = L(\tau_\nu^*)\,\underbrace{(1-\alpha)}_{\text{absorption}} + \underbrace{\epsilon B(\tau_\nu^*)}_{\text{emission}}$$

$$= L(\tau_\nu^*) - L(\tau_\nu^*)\,d\tau_\nu + \epsilon B(\tau_\nu^*)$$

We now assume that the slab is in local thermodynamic equilibrium, and so $\epsilon = \alpha = d\tau_\nu$ and rearrange to get:

$$\frac{L(\tau_\nu^* + d\tau_\nu^*) - L(\tau_\nu^*)}{d\tau_\nu} = -L(\tau_\nu^*) + B(\tau_\nu^*)$$

Inconveniently, our coordinate $d\tau_\nu$ depends on θ, but we wish to have a coordinate be the same at every horizontal position. We thus define $d\tau_\nu^* = d\tau_\nu \cos\theta$ and derive the **Schwarszchild Equation**:

$$\cos(\theta)\frac{dL}{d\tau_\nu^*}(\tau_\nu^*) = -L(\tau_\nu^*) + B(\nu, T(\tau_\nu^*)) \tag{6.1}$$

6.2 Angle Averaging: The Two-Stream Approximation

Note that generally, L is a function of direction ($\frac{\partial L}{\partial \vec{\omega}} \neq 0$). We proceed by making what's called the *Two-Stream Approximation*, but we need not make this approximation if we were coding up a full radiative transfer model. We only assume this here to make analytical progress in order to gain some physical intuition.

We define the net upwards and downward fluxes of irradiance as follows:

$$F_+ = \int_{\Omega_+} L(\vec{\omega}) \, d\Omega_\perp \tag{6.2}$$

$$F_- = - \int_{\Omega_-} L(\vec{\omega}) \, d\Omega_\perp \tag{6.3}$$

Note that Ray refers to these as I_+ and I_- in his notes. However, we shall refer to them here by the letter F, because these are (spectral) **irradiances**, **not** (spectral) **radiances**, as we have integrated over solid angles here. We then integrate Equation 6.1 in the upwards and downwards directions, and make the assumption that B is isotropic.

$$\int_{\Omega_+} \cos(\theta) \frac{dL}{d\tau_\nu^*} \, d\Omega = \int_{\Omega_+} -L + B \, d\Omega$$

$$\int_{\Omega_+} \frac{dL}{d\tau_\nu^*} \, d\Omega_\perp = - \int_{\Omega_+} L \, d\Omega + \int_{\Omega_+} B \, d\Omega$$

$$\frac{dF_+}{d\tau_\nu^*} = - \int_{\Omega_+} L \, d\Omega + 2\pi B$$

So far, everything is roughly very accurate. We now make the *Two-Stream Approximation* in order to relate F_+ to $\int_{\Omega_+} L \, d\Omega$. We make (counter-intuitively) the assumption that $L(\vec{\omega})$ is roughly isotropic enough to be pulled outside of the integral over solid angle. This allows us to find that:

$$\int_{\Omega_+} L \, d\Omega \approx L \int_{\Omega_+} d\Omega$$

$$= L \, 2\pi$$

$$= 2\pi \, L \frac{\int_{\Omega_+} d\Omega_\perp}{\int_{\Omega_+} d\Omega_\perp}$$

$$\approx 2\pi \frac{\int_{\Omega_+} L \, d\Omega_\perp}{\int_{\Omega_+} d\Omega_\perp}$$

$$= 2\pi \frac{F_+}{\pi}$$

$$\int_{\Omega_+} L \, d\Omega \approx 2F_+$$

We can go through a similar reasoning with F_-, which allows us to derive the angle-averaged upwards and downwards Schwarszchild Equations:

$$\frac{1}{2} \frac{dF_+}{d\tau_\nu^*} = -F_+ + \pi B \tag{6.4}$$

$$-\frac{1}{2} \frac{dF_-}{d\tau_\nu^*} = -F_- + \pi B \tag{6.5}$$

Note the extra '−' sign on Equation 6.5. Comparing 6.4 with 6.1, we can conclude that this approximation is equivalent to setting $\cos(\theta) = \frac{1}{2}$ on average! With this in mind, we define the effective propagation angle $\tilde{\theta}$ as $\tilde{\theta} = \cos^{-1}(1/2)$. We now abuse notation and define $\tau_\nu \equiv \tau_\nu^* / \cos(\tilde{\theta}) = 2\tau_\nu^*$ to get rid of that pesky $\frac{1}{2}$ factor to obtain the:

> **Key Idea II.7: Angle-Averaged Schwarzschild Equations**
>
> The following equations govern the net upward (F_+) and downwards (F_-) spectral irradiance, derived from the Schwarszchild Equation using the *Two-Stream Approximation*.
>
> $$\frac{dF_+}{d\tau_\nu}(\tau_\nu) = -F_+(\tau_\nu) + \pi B(\tau_\nu) \qquad (6.6) \qquad -\frac{dF_-}{d\tau_\nu}(\tau_\nu) = -F_-(\tau_\nu) + \pi B(\tau_\nu) \qquad (6.7)$$
>
> $B(\tau_\nu)$ is the blackbody radiation at some optical depth τ_ν. To solve these equations we require some functional form of $B(\tau_\nu)$.
>
> We can relate optical depth coordinates τ_ν to pressure coordinates p using 4.12:
>
> $$d\tau_\nu = -\kappa(\nu, T(p), p)\, q(p) \frac{dp}{g^{\frac{1}{2}}} \qquad (6.8)$$
>
> $$= -\kappa(\nu, T(p), p)\, q(p) \frac{dp}{g \cos \tilde{\theta}} \qquad (6.9)$$
>
> where $\tilde{\theta}$ = the effective propagation angle.

6.3 General Solutions and Radiating Level

We can integrate 6.6 and 6.7 using an integrating factor and a single boundary condition to find:

$$F_+(\nu, \tau_\nu) = F_+(\nu, 0)e^{-\tau_\nu} + \int_0^{\tau_\nu} \pi B(\nu, T(\tau_\nu'))e^{-(\tau_\nu - \tau_\nu')}\, d\tau_\nu' \qquad (6.10)$$

$$F_-(\nu, \tau_\nu) = F_-(\nu, \tau_{\nu,\infty})e^{-(\tau_{\nu,\infty} - \tau_\nu)} + \int_{\tau_\nu}^{\tau_{\nu,\infty}} \pi B(\nu, T(\tau_\nu'))e^{-(\tau_\nu' - \tau_\nu)}\, d\tau_\nu' \qquad (6.11)$$

where we have used the boundary conditions:

$$\text{at the ground} \quad \boxed{\tau_\nu = 0\ ,\ F_+(\nu, \tau_\nu) = F_+(\nu, 0)}$$

$$\text{at the top fo the atmosphere} \quad \boxed{\tau_\nu = \tau_{\nu_\infty}\ ,\ F_-(\nu, \tau_\nu) = F_-(\nu, \tau_\infty)}$$

There are three subtleties we should note. First, note the sign flips between 6.10 and 6.11, arising due to the sign flip on the derivative in 6.6 and 6.7. Second, note that the limits of integration are swapped between 6.10 and 6.11, arising due to the difference in boundary conditions.

The final subtlety is the most important: in general $\tau_\nu = \tau_\nu(\nu)$. Suppose we wish to the total radiative upwards flux of energy at some height/pressure level. We cannot simply find $F_+(\tau = \tau(p))$ then integrate over all frequencies, as in general $\tau_{\nu_1}(p) \neq \tau_{\nu_2}(p)$.

Physically, the solution is intuitive: the total upwards radiance at some optical depth τ is simply the sum of the irradiance at each level τ', each attenuated by the transmission function between τ and τ'. To be more explicit, we sum:

1. $F_+(\nu, 0)e^{-\tau_\nu}$: The irradiance ($F_+(\nu, 0)$) from the bottom ($\tau_\nu = 0$) of the atmosphere, attenuated by the optical depth $e^{-(\tau_\nu - 0)}$.[1]
2. $\pi B(\nu, T(\tau_\nu'))\, d\tau_\nu'\, e^{-(\tau_\nu - \tau_\nu')}$: The irradiance ($B(\nu, T(\tau_\nu'))\, d\tau_\nu'$) from each layer ($\tau_\nu = \tau_\nu'$), also attenuated by the optical depth $e^{-(\tau_\nu - \tau_\nu')}$

---

[1]Generally, this is the upwards irradiance emitted by the ground on a rocky planet, but one could also choose $\tau_\nu = 0$ in the middle of the atmosphere (as you must on a gaseous planet), in which case $F_+(\nu, 0)$ will be the total upwards irradiance at that level (originating from all the atmosphere (and ground) underneath it).

We can also rewrite 6.4 and 6.5 in terms of the transmission function $\mathcal{T}_\nu(p_1, p_2)$ if we find the differential of the transmission function with respect to p_2 and assume that $\tau_\nu(p_1) \geq \tau_\nu(p_2)$ (i.e., if $p_1 \leq p_2$):

$$d\left(\mathcal{T}_\nu(p_1, p_2)\right) = d\left(e^{-|\tau_\nu(p_1) - \tau_\nu(p_2)|}\right)$$

$$= e^{-\tau_\nu(p_1)} d\left(e^{\tau_\nu(p_2)}\right)$$

$$= e^{-\tau_\nu(p_1)} e^{\tau_\nu(p_2)} d\tau_\nu(p_2)$$

$$= e^{-|\tau_\nu(p_1) - \tau_\nu(p_2)|} d\tau_\nu(p_2)$$

$$\therefore \ d\mathcal{T}_\nu(p_1, p_2) = \mathcal{T}_\nu(p_1, p_2)\, d\tau_\nu(p_2)$$

If, conversely, $\tau_\nu(p_1) \leq \tau_\nu(p_2)$ (i.e., if $p_1 \geq p_2$), then $d\mathcal{T}_\nu(p_1, p_2) = -\mathcal{T}_\nu(p_1, p_2)\, d\tau_\nu(p_2)$.

Key Idea II.8: Solution to the Angle-Averaged Schwarszchild Equations

The total upwards $F_+(\nu, p)$ and downwards $F_-(\nu, p)$ spectral irradiance at frequency ν and pressure level p is given by the following:

$$F_+(\nu, p) = F_+(\nu, 0)\mathcal{T}_\nu(p, p_s) + \int_{p'=p_s}^{p} \pi B(\nu, T(p'))\, d\mathcal{T}_\nu(p, p') \tag{6.12}$$

$$F_-(\nu, p) = F_-(\nu, 0)\mathcal{T}_\nu(0, p) - \int_{p'=p}^{0} \pi B(\nu, T(p'))\, d\mathcal{T}_\nu(p, p') \tag{6.13}$$

where $d\mathcal{T}_\nu(p, p')$ is the differential transmission function with respect to p'. $d\mathcal{T}_\nu(p, p') = \mathcal{T}_\nu(p, p')\, d\tau_\nu(p')$ if $p \leq p'$ (as in 6.12) or $d\mathcal{T}_\nu(p, p') = -\mathcal{T}_\nu(p, p')\, d\tau_\nu(p')$ if $p \geq p'$ (as in 6.13). The **Outgoing Longwave Radiation** OLR (in spectral irradiances) is then defined as $F_+(\nu, p = 0) = F_+(\nu, \tau = \tau_{\nu, \infty})$ and given by:

$$OLR = F_+(\nu, 0)\mathcal{T}_\nu(0, p_s) + \int_{p_s}^{0} \pi B(\nu, T(p'))\, d\mathcal{T}_\nu(0, p') \tag{6.14}$$

Note that even though the integral in 6.12 is taken from p to $p < p_s$, the integral is positive because $d\mathcal{T}_\nu(p, p') < 0$, because $d\mathcal{T}_\nu(p, p') \sim d\tau(p') \sim -dp'$.

6.4 The OLR and the Radiating Level

The **Outgoing Longwave Radiation** OLR (in spectral irradiances) is then defined as the total spectral irradiance escaping to space, and given by $F_+(\nu, p = 0) = F_+(\nu, \tau = \tau_{\nu, \infty})$:

$$OLR = F_+(\nu, 0)e^{-\tau_{\nu, \infty}} + \int_0^{\tau_{\nu, \infty}} \pi B(\nu, T(\tau'_\nu))e^{-(\tau_{\nu, \infty} - \tau'_\nu)}\, d\tau'_\nu \tag{6.15}$$

The physical interpretation is identical to F_+ at any optical depth: the total upwards spectral irradiance escaping is the spectral irradiance from each level attenuated by the amount of absorption in between that level and space.

Since the attenuation is exponential, this gives us a privileged scale of optical depth set by when $(\tau_{\nu, \infty} - \tau'_\nu) \sim 1$. This allows us to define:

> **Key Idea II.9: The Radiating Level p_{rad}**
>
> We define the **radiating level** at $\tau_\nu = \tau_{\nu,rad}$ and $p = p_{rad}$ which satisfy the following relations:
>
> $$\tau_{\nu,\infty} - \tau_{\nu,rad} = 1 \qquad (6.16)$$
>
> $$\tau_\nu(p_{rad}) = \tau_{\nu,rad} \qquad (6.17)$$
>
> Physically, the interpretation is as follows: any radiation coming from an optical depth $\tau_\nu < \tau_{\nu,rad}$ ($p > p_{rad}$) will be attenuated **much** more than radiation coming from an optical depth $\tau_\nu > \tau_{\nu,rad}$ ($p < p_{rad}$) (see Figure 6.1).
>
> As such, the OLR radiation actually escaping into space will disproportionately look like the radiation emitted by 'stuff' at or above p_{rad}. In other words, if you were to look down at the atmosphere, you would not 'see' anything from an optical depth coming from $p > p_{rad}$.
>
>
>
> Figure 6.1: Schematic showing where radiation escapes looking at the vertical temperature profile $T(p)$ of the Earth. Squiggly magenta arrows ■ indicate radiation emitted from an optical depth of $\tau_\nu > \tau_{\nu,rad}$ which escapes to space. Squiggly orange arrows ■ indicate radiation emitted from an optical depth of $\tau_\nu < \tau_{\nu,rad}$, which is quickly absorbed by the atmosphere above and cannot escape to space.
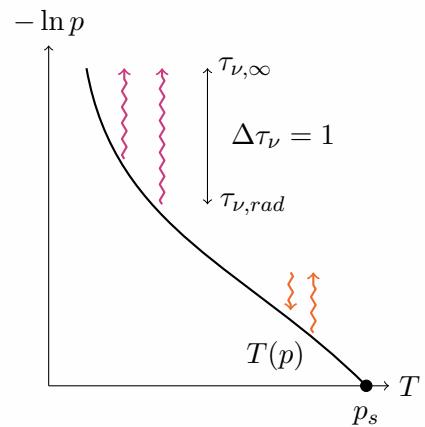
Most of the time, the OLR looks disproportionately like the stuff remitted *at* p_{rad} (rather than simply at or above p_{rad}). This is for the following reason, which I emphasise may fail to obtain. Temperature typically decreases with height for most of the mass in an atmosphere. Recall that $B(\nu, T)$ is monotonic in T. Therefore, OLR will look like emission high enough to escape ($p \leq p_{rad}$) but low enough such that emission is large.

Note that I've only defined the radiating level for an atmosphere that is optically thick, i.e., in an atmosphere where $\tau_{\nu,\infty} > 1$ obtains. This is the case on the Eart where ν is in the IR light region, for example.

In an atmosphere that is optically thin ($\tau_{\nu,\infty} < 1$) this definition breaks down, and pretty much all radiation from throughout the atmosphere can escape into space. What the OLR at this frequency looks like, then, is mainly set by the precise emission profile of the atmosphere.

This is further complicated on rocky planets where there is a distinct surface. If we consider the Earth for the case where ν is in the visible light region, for example, we will find that (ignoring clouds for now) the atmosphere is optically thin but the ground is optically thick (you can see through air but you cannot see through the ground). As such, the OLR at this frequency is essentially set by the emission at the ground: $p_{rad} = p_s$, since $\tau_\nu \to -\infty$ at the ground due to the discontinuity of switching between something optically thin (the atmosphere) and something optically thick (the ground).

We can also express everything in terms of the transmission function. Defining the radiating level in terms of the transmission function is straightforward: p_{rad} satisfies the following relation[2]:

$$\mathcal{T}_\nu(0, p_{rad}) = e^{-1} \qquad (6.18)$$

6.5 Frequency Bands

We're not finished yet! Recall that we have expressed OLR in terms of *spectral* irradiances, but we require the (non-spectral) irradiances if we wish to compute the total outgoing radiative energy flux. What we may do now is integrate the OLR over the centre of a single absorption band centred about

---

[2]And, of course, $p_{rad} > 0$.

ν_c. Typically, both $F_+(\nu, 0)$ and $B(\nu, T)$ are roughly constant as a function of ν compared to τ_ν, so we can treat $F_+(\nu, 0) \approx F_+(\nu_c, 0)$ and $B(\nu) \approx B(\nu_c)$ for $\nu \in [\nu_c - \frac{\Delta}{2}, \nu_c + \frac{\Delta}{2}]$.

Integrating 6.14 over frequency, we find the following:

$$OLR_{\nu_c} = F_+(\nu_c, 0)\bar{\mathcal{T}}_{\nu_c}(0, p_s) + \int_{p_s}^{0} \pi B(\nu_c, T(p')) \, d\bar{\mathcal{T}}_{\nu_c}(p', p_s) \qquad (6.19)$$

where we have assumed that $\mathcal{T}_\nu \approx 0$ at $\nu = \nu_c \pm \frac{\Delta}{2}$ (the line is thin) and defined the **Band Integrated Transmission Function** $\bar{\mathcal{T}}_{\nu_c}$ as:

$$\bar{\mathcal{T}}_{\nu_c}(p_1, p_2) = \int_{-\infty}^{\infty} \mathcal{T}_\nu \, d\nu \qquad (6.20)$$

$$\approx \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} \mathcal{T}_\nu \, d\nu \qquad (6.21)$$

$$= \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} e^{-|\tau_\nu(p_1) - \tau_\nu(p_2)|} \, d\nu \qquad (6.22)$$

Note that OLR_{ν_c} has dimenions of an irradiance (power per area), since $\bar{\mathcal{T}}_{\nu_c}$ has dimensions of frequency and F_+ and πB have dimensions of spectral irradiance (power per area per frequency). We now abuse notation and write the total OLR as the total power leaving the Earth per unit area. We've already assumed that each line is thin, which implies that the spectral lines don't overlap, so we can find OLR by summing up OLR_{ν_c} for each spectral line centred about each ν_c:

$$OLR = \sum_{\nu_c} OLR_{\nu_c}$$

Now that our OLR is in terms of known quantities and $\bar{\mathcal{T}}_{\nu_c}$, our next goal is to calculate $\bar{\mathcal{T}}_{\nu_c}$. Typically, this can be done numerically, but to make analytical progress we will look at three cases: the no-line limit, weak line limit, and strong line limit.

6.5.1 The No-Line Limit

I should make you aware that this is my own personal addition and not talked about in the lecture notes. This limit is a bit non-sensical[3], but I think it builds some intuition.

The no-line limit is if $\kappa = 0$ at that frequency band, and so $|\tau_\nu(p_1) - \tau_\nu(p_2)| = 0$. As such, the transmission function $\mathcal{T}_\nu(p_1, p_2) = e^{-|\tau_\nu(p_1) - \tau_\nu(p_2)|} = 1$ and so:

$$\bar{\mathcal{T}}_{\nu_c}(p_1, p_2) = \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} 1 \, d\nu$$

$$= \Delta$$

We can therefore substitute Δ for $\bar{\mathcal{T}}_{\nu_c}$ everywhere. Physically, this means that *nothing* is absorbed out of this absorption band: everything in the band from $\nu = \nu - \frac{\Delta}{2}$ to $\nu = \nu + \frac{\Delta}{2}$ is let through. Substituting explicitly that $\bar{\mathcal{T}}_{\nu_c}(p_1, p_2) = \Delta$ and $d\bar{\mathcal{T}}_{\nu_c}(p_1, p_2) = 0$ for all p_1 and p_2, our solution is:

$$OLR_{\nu_c} = F_+(\nu_c, 0)\Delta$$

In other words, the OLR_{ν_c} at this frequency ν_c is simply set by the upward radiative flux at frequency ν_c from the ground, Physically, the solution looks like this because of two reasons.

---

[3]I think it's non-sensical because optical depth coordinates make little physical sense if $\kappa = 0$. Recall that $d\tau \sim \kappa$, so if $\kappa = 0$ then $\tau(p) = 0$ for all p: there is no longer a one-to-one mapping between p and τ. Furthermore, if $\kappa = 0$, then this is hardly a spectral line! That being said, our solution is still mathematically and physically valid.

1. $\alpha = 0$: Since $\kappa = 0$, $\alpha = 0$, which means the atmosphere is completely transparent at this frequency. As such, all the spectral radiance from the ground is let through, and the OLR_{ν_c} is completely ignorant of the concentration.

2. $\epsilon = 0$: Due to Kirchoff's law, $\epsilon = 0$ as well, which means the atmosphere emits no radiation at this frequency either. As such, none of the OLR_{ν_c} comes from the atmosphere.

Because of these two reasons, the OLR_{ν_c} is completely ignorant of any atmospheric properties[4].

We'll find that in the weak and strong line limits (and indeed in all cases where $\kappa \neq 0$) that things change dramatically. First, $\bar{\mathcal{T}}_{\nu_c} < \Delta$, and so the effective 'width' will be smaller than Δ, representing the chunk of radiation from the ground that is absorbed by the fact that $\kappa \neq 0$. Second, the integral term will not be negative, reflecting the fact that the atmosphere contributes to the OLR. Importantly, this will depend on the vertical temperature profile of the atmosphere, as the temperature sets the Blackbody radiation.

6.5.2   The Weak Line Limit

The weak line limit obtains if $|\tau_\nu(p_1) - \tau_\nu(p_2)| \ll 1$ for all frequencies ν. If $\tau(p)$ is well-behaved, this will be arbitrarily satisfied if p_1 and p_2 are sufficiently close. We can then Taylor Expand $e^{|\tau_\nu(p_1) - \tau_\nu(p_2)|}$ and keep only first order terms:

$$e^{-|\tau_\nu(p_1) - \tau_\nu(p_2)|} \approx 1 - |\tau_\nu(p_1) - \tau_\nu(p_2)|$$

We can now integrate over frequencies to find that:

$$\bar{\mathcal{T}}_{\nu_c}(p_1, p_2) \approx \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} e^{-|\tau_\nu(p_1) - \tau_\nu(p_2)|} \, d\nu$$

$$\approx \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} 1 - |\tau_\nu(p_1) - \tau_\nu(p_2)| \, d\nu$$

$$= \Delta - \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} \left| \int_{p_2}^{p_1} -\kappa(\nu, T(p), p) \, q(p) \frac{dp}{g \cos \tilde{\theta}} \right| d\nu$$

$$= \Delta - \frac{1}{g \cos \tilde{\theta}} \left| \int_{p_2}^{p_1} q(p) \left(\underbrace{\int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} \kappa(\nu, T(p), p) \, d\nu}_{S(T(p))} \right) dp \right|$$

$$= \Delta - S(T_0) \left(\frac{1}{g \cos \tilde{\theta}} \left| \int_{p_1}^{p_2} \frac{S(T(p))}{S(T_0)} q(p) \, dp \right| \right)$$

The keen reader will note that this result **differs** from the result obtains in Ray's lecture slides. I am confident, however, that this expression is correct, as this is the expression that shows up in Ray's book [3] and the algebra is more rigourous.

We now define[5] the effective strength $S = S(T_0)$ such that:

$$\int_{p_1}^{p_2} S(T(p)) q(p) \, dp \equiv S(T_0) \int_{p_1}^{p_2} q(p) \, dp \tag{6.23}$$

This means that the band-integrated transmission function in the weak line limit is:

$$\boxed{\bar{\mathcal{T}}_{\nu_c}(p_1, p_2) \approx \Delta - S\mu} \tag{6.24}$$

where μ is the mass path between those pressures such that $\mu_p = \frac{1}{g \cos \tilde{\theta}} \left| \int_{p_2}^{p_1} q(p) \, dp \right|$. I have also abused notation and written just S for the effective strength.

---

[4]Other than, of course, the property that the atmosphere is completely transparent at this frequency!

[5]Again, we are perfectly well within our rights to theoretically define this 'effective strength', but it does not help us much in actually practically calculating the band-integrated transmission function.

Intuitively, $\Delta - S\mu$ is the effective width of this line, and the $S\mu$ term represents the 'chunk' of outwards irradiance taken out by this line (the chunk absorbed). As you can see, the absorption is **linear** in mass-path, not exponential as we might expect from the exponential attenuation of spectral radiance in (Equation 4.11). This is possible becuase the attenuation of **spectral** irradiance is not the same as the attenuation of **non-spectral** radiance: we found the former by integrating the latter over the spectral line, which has its own dependence on ν.

Let us now calculate the OLR_{ν_c} in the weak line limit. We again need to find the differential $d\bar{\mathcal{T}}_{\nu_c}(0, p')$ with respect to p':

$$
\begin{aligned}
d\bar{\mathcal{T}}_{\nu_c}(p', p_s) &= d\left(\Delta - S\frac{1}{g\cos\tilde{\theta}}\left|\int_0^{p'} q(p)\,dp\right|\right) \\
&= -\frac{S}{g\cos\tilde{\theta}}\,d\left(\int_0^{p'} q(p)\,dp\right) \\
&= -\frac{S}{g\cos\tilde{\theta}}q(p')\,dp'
\end{aligned}
$$

where we have used integral properties to go from the second line to the third line. We substitute this into 6.19 to find that:

$$
OLR_{\nu_c} = F_+(\nu_c, 0)(\Delta - S\mu) + \int_0^{p_s} \pi B(\nu_c, T(p'))\frac{Sq(p')}{g\cos\tilde{\theta}}\,dp' \tag{6.25}
$$

We interpret the terms physically as follows:

$$
OLR_{\nu_c} = \underbrace{F_+(\nu_c, 0)(\Delta \overbrace{-S\mu}^{\text{Width taken out}})}_{\text{Emission from Ground}} + \underbrace{\int_0^{p_s} \pi B(\nu_c, T(p'))\frac{Sq(p')}{g\cos\tilde{\theta}}\,dp'}_{\text{Emission from Atmosphere}}
$$

So if $\kappa = 0$, we see that OLR_{ν_c} *decreases* due to the atmosphere absorbing some emission from the ground, but it also *increases* due to the atmosphere itself emitting some radiation. Whether this ultimately increases or decreases OLR_{ν_c} will depend crucially on both $T(p)$, the vertical temperature profile of the atmosphere, and $q(p)$, the distribution of the absorber in the atmosphere.

In fact, in general, we *cannot* say whether introducing a greenhouse gas will increase or decrease OLR_{ν_c} without any knowledge of the vertical temperature profile of the atmosphere.

6.5.3   The Strong Line Limit

For the strong line limit, we require that $|\tau_\nu(p_1) - \tau_\nu(p_2)| \gg 1$ at the *centre* of the line (i.e. at $\nu = \nu_c$). This is weaker than the opposite of the weak-line-limit assumption, because any line far enough from the centre is weak. We also assume a Lorentz line shape and width from Key Idea 5.2.3 so our treatment here will not be completely general.

If $|\tau_\nu(p_1) - \tau_\nu(p_2)| \gg 1$ at the centre of the line, then we can approximate it as effectively infinite. We thus approximate the Lorentz line-shape as:

$$
\begin{aligned}
F(x) &= \frac{1}{\pi}\frac{1}{x^2 + 1} \\
&\approx \frac{1}{\pi}\frac{1}{x^2}
\end{aligned}
$$

We then integrate the transmission function over frequencies again:

$$
\begin{aligned}
\bar{\mathcal{T}}_{\nu_c}(p_1, p_2) &\approx \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} \exp(-|\tau_\nu(p_1) - \tau_\nu(p_2)|)\, d\nu \\
&= \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} \exp\left(-\left|\int_{p_2}^{p_1} -\kappa(\nu, T(p), p) q(p) \frac{dp}{g \cos\tilde{\theta}}\right|\right) d\nu \\
&\approx \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} \exp\left(-\left|\int_{p_2}^{p_1} \frac{1}{\pi} \frac{S(T)}{\delta(T,p)} \left(\frac{\nu - \nu_c}{\delta(T,p)}\right)^{-2} q(p) \frac{dp}{g \cos\tilde{\theta}}\right|\right) d\nu \\
&= \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} \exp\left(-\left|\int_{p_2}^{p_1} \frac{1}{\pi} S(T) \delta(T,p) \frac{1}{(\nu - \nu_c)^2} q(p) \frac{dp}{g \cos\tilde{\theta}}\right|\right) d\nu \\
&= \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} \exp\left(-\left|\int_{p_2}^{p_1} \frac{1}{\pi} S(T) \frac{p}{p_0} \left(\frac{T_0}{T}\right)^n \frac{1}{(\nu - \nu_c)^2} q(p) \frac{dp}{g \cos\tilde{\theta}}\right|\right) d\nu \\
&= \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} \exp\left(-\frac{\delta_0}{\pi g \cos\tilde{\theta}} \frac{1}{(\nu - \nu_c)^2} \left|\int_{p_2}^{p_1} S(T(p)) \frac{p}{p_0} \left(\frac{T_0}{T(p)}\right)^n q(p)\, dp\right|\right) d\nu
\end{aligned}
$$

Again, to get rid of the pesky T dpendence within the integral with respect to p, we define[6] the effective strength $S = S(T_0)$ such that:

$$
\int_{p_1}^{p_2} S(T(p)) \frac{p}{p_0} \left(\frac{T_0}{T(p)}\right)^n q(p)\, dp \equiv S(T_0) \int_{p_1}^{p_2} \frac{p}{p_0} q(p)\, dp
$$

Therefore:

$$
\bar{\mathcal{T}}_{\nu_c}(p_1, p_2) = \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} \exp\left(-\frac{1}{(\nu - \nu_c)^2} \frac{S \delta_0}{\pi g \cos\tilde{\theta}} \left|\int_{p_2}^{p_1} \frac{p}{p_0} q(p)\, dp\right|\right) d\nu
$$

To make the integral easier, we note that the integral is symmetric about $\nu = \nu_c$ and define x such that:

$$
\frac{1}{x^2} \equiv \frac{1}{(\nu - \nu_c)^2} \frac{S \delta_0}{g \cos\tilde{\theta}} \left|\int_{p_2}^{p_1} \frac{p}{p_0} q(p)\, dp\right|
$$

Therefore:

$$
\bar{\mathcal{T}}_{\nu_c}(p_1, p_2) = 2 \sqrt{\frac{S \delta_0}{g \cos\tilde{\theta}} \left|\int_{p_2}^{p_1} \frac{p}{p_0} q(p)\, dp\right|} \int_0^X \exp\left(-\frac{1}{\pi x^2}\right) dx
$$

where

$$
X = \frac{\Delta}{2} \left(\frac{S \delta_0}{g \cos\tilde{\theta}} \left|\int_{p_2}^{p_1} \frac{p}{p_0} q(p)\, dp\right|\right)^{-1/2}
$$

One can use the result that the following integral holds if $X \gg 1$.

$$
\int_0^X \exp\left(-\frac{1}{\pi x^2}\right) \approx X - 1
$$

Note that this assumption is satisfied if we assume that the lines are thin. If the line is thin, then the integrand decays to 0 at the linewidth, which implies that $X \gg 1$ as required. This finally gives us that the tranmission function is as follows in the strong-line limit:

$$
\boxed{\bar{\mathcal{T}}_{\nu_c}(p_1, p_2) = \Delta - 2\sqrt{S \delta_0 \mu_p}} \tag{6.26}
$$

---

[6]Read the footnote on the previous page.

where μ_p is the pressure weighted mass path between those two pressures:

$$\mu_p = \frac{1}{g\cos\tilde{\theta}} \left| \int_{p_2}^{p_1} \frac{p}{p_0} q(p)\, dp \right|$$

Again, $\Delta - 2\sqrt{S\delta_0\mu_p}$ is the effective width of this line, and the $2\sqrt{S\delta_0\mu_p}$ term represents the 'chunk' of outwards irradiance taken out by this line (the chunk absorbed). As you can see, the absorption scales as the **square-root** of the (pressure-weighted) mass-path, not exponential. Again, this is possible becuase the attenuation of **spectral** irradiance is not the same as the attenuation of **non-spectral** radiance. A new point is that this is *slower* than a weak line.

We continue by calculating OLR_{ν_c} in the strong line limit. We again need to find the differential $d\bar{\mathcal{T}}_{\nu_c}(0, p')$ with respect to p':

$$d\bar{\mathcal{T}}_{\nu_c}(p', p_s) = d\left(\Delta - 2\sqrt{S\delta_0 \frac{1}{g\cos\tilde{\theta}} \left| \int_0^{p'} \frac{p}{p_0} q(p)\, dp \right|} \right)$$

$$= -2\sqrt{\frac{S\delta_0}{g\cos\tilde{\theta}}}\, d\left(\sqrt{\int_0^{p'} \frac{p}{p_0} q(p)\, dp} \right)$$

$$= -\sqrt{\frac{S\delta_0}{\mu_p g\cos\tilde{\theta}}} \frac{p'}{p_0} q(p')\, dp'$$

We substitute this into 6.19 to find that:

$$OLR_{\nu_c} = F_+(\nu_c, 0)(\Delta - 2\sqrt{S\delta_0\mu_p}) + \int_0^{p_s} \pi B(\nu_c, T(p')) \sqrt{\frac{S\delta_0}{\mu_p g\cos\tilde{\theta}}} \frac{p'}{p_0} q(p')\, dp' \qquad (6.27)$$

Again, in the strong line limit, the OLR_{ν_c} *decreases* due to the atmosphere absorbing some emission from the ground and *increases* due to the atmosphere itself emitting some radiation, and whether this ultimately increases or decreases OLR_{ν_c} will depend on $T(p)$.

6.5.4 Non-Isolated Lines

I note here that, sadly, none of what we learnt will be applicable to calculating the radiative forcing of CO_2 or H_2O. That is becuase both CO_2 and H_2O have strongly **overlapping** lines, and so we cannot simply integrate over a single spectral line and sum up each contribution, as that would result in double counting.

We also won't cover how to deal with this here. Dissapointing I know, but it is what it is. . .

6.6 Radiative Forcing

Suppose we wish to know how varying some parameter λ (for example, CO_2 concentrations) varies the temperature of the Earth. Consider again the radiation budget of the Earth (4.26), now per unit area, which is a function of λ and the mean surface temperature T_s:

$$G = \frac{1}{4}(1-a)L_\star - OLR(T_s, \lambda) \qquad (6.28)$$

We know that in equilibrium, $G = 0$. In this section, we will only be considering equilibrium solutions, and will *not* consider transient response. This us allows to use some tricks. Suppose that initially, $\lambda_{in} = \lambda$, and we perturb λ such that $\lambda_{fin} = \lambda + \Delta\lambda$. Before:

$$G(T_s, \lambda) = 0$$

and after:

$$G(T_s, \lambda + \Delta\lambda) = 0$$

so

$$\frac{\Delta G}{\Delta\lambda} = 0$$

But λ_{in} and λ_{fin} are completely arbitrary, so we can take the limit as $\Delta\lambda \to 0$ and find that the derivative of G with respect to λ is 0:

$$\frac{dG}{d\lambda} = 0$$

Using chain rule, we find that:

$$\frac{dG}{d\lambda} = \frac{\partial G}{\partial\lambda} + \frac{\partial G}{\partial T_s}\frac{\partial T_s}{\partial\lambda} = 0$$

This allows us to solve for $\frac{\partial T_s}{\partial\lambda}$, which is what, at least to a first order, we actually care about. There are some terms that pop up that we define as the **Differential Radiative Forcing** (**DRF**) and the **Equilibrium Climate Sensitivity Parameter** (**ECS**):

$$\frac{\partial T_s}{\partial\lambda} = \frac{\overbrace{-\partial G/\partial\lambda}^{DRF}}{\underbrace{\partial G/\partial T_s}_{ECS}} \tag{6.29}$$

Note the negative sign on the **DRF**.

Physically, **DRF** is the extra power per unit area staying in (or leaving if $DRF < 0$) the Earth per unit change in λ. This is closely connected to but distinct from the *Radiative Forcing*, which is simply the the extra power or the extra power per unit area staying in (or leaving if $DRF < 0$) the Earth. Now generally, this will encode how this parameter *immediately* changes the radiative forcing. For example, if λ = aerosol concentrations, this will immediately affect clouds and how they affect. Or, if λ encodes some ice dynamics resulting in less ice-cover, this might affect the albedo.

The **ECS** encodes how sensitive the climate (or more specifically, T_s) is to a change in radiative forcing. If there is some radiative forcing $G \neq 0$, the temperature will change, and this will affect other parts of the climate system to bring G back to 0. Intuitively, you can think of this parameter as encoding all the *feedbacks*. For example, when the temperature increases, some ice will melt, which will decrease the albedo and increase G, thus causing T_s to increase even more! The most straightforward feedback is the **Planck Feedback**, which essentially encodes the fact that a hotter planet will radiate more blackbody radiation.

As a summary:

> ## Key Idea II.10: Radiative Forcing: DRF and ECS
>
> Let $G = G(\lambda, T_s)$ be the radiation energy budget for the Earth per unit area, where λ is some parameter (e.g., the concentration of CO_2) and T_s is the mean surface temperature. The following relations hold (explained elsewhere):
>
> $$G = \frac{1}{4}(1-a)L_\star - OLR(T_s, \lambda) \qquad\qquad G = \frac{dG}{d\lambda} = 0$$
>
> We define the **Differential Radiative Forcing (DRF)** and the **Equilibrium Climate Sensitivity Parameter (ECS)** as follows:
>
> $$DRF = -\frac{\partial G}{\partial \lambda} \qquad (6.30) \qquad\qquad ECS = \frac{\partial G}{\partial T_s} \qquad (6.31)$$
>
> **DRF** encodes the extra power per unit area staying in the Earth caused by a perturbation λ. **ECS** encodes how much T_s would have to change to affect G and encodes all the feedbacks.

Now we'll calculate the **DRF** and let $\lambda = q$, the concentration of an absorber (a greenhouse gas) in the weak and strong line limits. We won't calculate the **ECS**, as it doesn't bring much insight at this point.

We will make four assumptions here just for simplicity and to derive some physical intuition. We assume first that a do not depend on λ or T_s for simplicity, which allows us to substitute OLR for every G in 6.29. Second, we consider just one spectral line such that $OLR = OLR_{\nu_c}$, also for simplicity. Third, we assume that the absorber is well-mixed, so that $q(p) = const$. Fourth, we assume that $F_+(\nu_c, 0)$, the spectral irradiance at the ground, is equal to $\pi B(\nu_c, T_s)$.

6.6.1 The Weak Line Limit

Using our expression for the weak line OLR from 6.25 we now calculate the **DRF** and let $\lambda = q$, the concentration of our weak line absorber. I also remind you that the temperature profile of the atmosphere is a function of T_s.

$$
\begin{aligned}
DRF &= -\frac{\partial}{\partial q} OLR_{\nu_c} \\
&= -\frac{\partial}{\partial q}\left(-F_+(\nu_c, 0)S\frac{qp_s}{g\cos\tilde\theta} + \frac{Sq}{g\cos\tilde\theta}\int_0^{p_s} \pi B(\nu_c, T(T_s, p'))\, dp' \right) \\
&= \frac{S\pi}{g\cos\tilde\theta}\left(B(\nu_c, T_s)p_s - \int_0^{p_s} B(\nu_c, T(T_s, p'))\, dp' \right)
\end{aligned}
\qquad (6.32)
$$

There are two key takeaways. First, note that OLR_{ν_c} is proportional to q in the weak line limit, so the DRF is constant (with respect to q). The second takeaway is more important, and more general, note that the DRF could be positive or negative depending on $T(T_s, p')$. Recall that $B(\nu_c, T)$ is monotonically increasing with T.

If the temperature decreases with height (so that $T(T_s, p') < T_s$), then $B(\nu_c, T(T_s, p')) < B(\nu_c, T_s)$ such that the first term is larger than the second term. Therefore, $DRF > 0$, and an increase in q will result in a *positive* radiative forcing.

If the temperature increases with height (so that $T(T_s, p') > T_s$), then $B(\nu_c, T(T_s, p')) > B(\nu_c, T_s)$ such that the first term is smaller than the second term. Therefore, $DRF < 0$, and an increase in q will result in a *negative* radiative forcing.

The first case is generally the case for most of Earth's atmosphere, but the second case actually obtains in the stratosphere (recall Figure 2.2b). As such, if we increase the concentration there, we actually increase the OLR_{ν_c} escaping the Earth. This is actually what we see on Earth currently, but the increase from the OLR_{ν_c} is actually very small, because the stratosphere makes up such a small proportion of the atmosphere's mass (again, look at Figure 2.2b and note the logarithmic pressure axis, and recall that dp is proportional to the mass in a layer).

6.6.2   The Strong Line Limit

We now use our expression for the strong line OLR from 6.25 to calculate the **DRF**. We again assume that q is constant.

$$
DRF = -\frac{\partial}{\partial q} OLR_{\nu_c}
$$

$$
= -\frac{\partial}{\partial q} \left(-F_+(\nu_c, 0)\sqrt{S\delta_0 \frac{qp_s^2}{p_0 g \cos\tilde{\theta}}} + \int_0^{p_s} \pi B(\nu_c, T(p'))\sqrt{\frac{S\delta_0}{\mu_p g \cos\tilde{\theta}}} \frac{p'}{p_0} q(p')\, dp' \right)
$$

$$
= \pi\sqrt{\frac{S\delta_0}{2p_0 g \cos\tilde{\theta}}} \frac{1}{\sqrt{q}} \left(B(\nu_c, T_s)p_s - \int_0^{p_s} B(\nu_c, T(p'))\, dp' \right) \tag{6.33}
$$

There are again, two key takeaways. First, that this time OLR_{ν_c} is proportional to $\sqrt{q}$ in the strong line limit, so the $DRF \sim 1/\sqrt{q}$, there are diminishing returns! This makes sense, as the centre of the line is already completely absorbing all radiation.

The second takeaway is exactly the same second takeaway in the previous section: the DRF could be positive or negative depending on $T(T_s, p')$. This is clear since the right hand term in brackets is exactly the same in both 6.33 and 6.32.

Key Idea II.11: Radiative Forcing Key Takeaways

The two key takeaways for isolated line radiative forcing are:

Takeaway 1:
How efficient radiative forcing is per molecule of absorber added depends on how much absorber is in the atmosphere already.
- If the concentration of absorbers is low (weak line limit), the $OLR \sim q$.
- If the concentration of absorbers is high (strong line limit), the $OLR \sim \sqrt{q}$.

Takeaway 2:
Whether $DRF > 0$ or $DRF < 0$ depends on the vertical temperature profile of the atmosphere $T(p)$.
- If $\frac{dT}{dp} > 0$, then $DRF > 0$. More q results in less OLR escaping.
- If $\frac{dT}{dp} < 0$, then $DRF < 0$. More q results in more OLR escaping.

6.7   Grey Gas OLR

To build more intuition and to consider, in detail, how $T(p)$ affects OLR, we make the (highly unrealistic) **Grey Gas Approximation**. The **Grey Gas Approximation** assumes that κ is not a function of ν, so that τ is not a function of ν. This allows us to integrate 6.6 and 6.7 over all frequencies. Recalling 4.19, we derive the frequency integrated upwards and downwards Schwarzschild equations:

$$
\frac{dE_+}{d\tau} = -E_+ + \sigma T^4 \tag{6.34}
$$

$$
-\frac{dE_-}{d\tau} = -E_- + \sigma T^4 \tag{6.35}
$$

where now the Schwarzschild equations are in terms of **irradiances** (E_+ and E_-). We can integrate this (as before) to find the general solutions for a grey gas:

$$
E_+(\tau) = \sigma T_s^4 e^{-\tau} + \int_0^{\tau} \sigma T(\tau')^4 e^{-(\tau-\tau')}\, d\tau' \tag{6.36}
$$

$$
E_-(\tau) = E_-(\tau_\infty)e^{-(\tau_\infty-\tau)} + \int_\tau^{\tau_\infty} \sigma T(\tau')^4 e^{-(\tau'-\tau)}\, d\tau' \tag{6.37}
$$

where we have assumed that $E_+(0) = \sigma T_s^4$. The total OLR (in watts per square metre) is therefore:

$$OLR = \sigma T_s^4 e^{-\tau_\infty} + \int_0^{\tau_\infty} \sigma T(\tau')^4 e^{-(\tau_\infty - \tau')} \, d\tau' \tag{6.38}$$

Again, we need $T(\tau(p))$ to calculate the OLR, which in turn requires us relating $T(p)$ and $p(\tau)$. Let us assume first that $T(p)$ follows the dry adiabat so that $T(p) = T_s \left(\frac{p}{p_s} \right)^{R/c_p}$, and second that κ has no T, p dependence and that q is well mixed (as in Section 4.3.1) so that $\tau(p) = \tau_\infty \left(1 - \frac{p}{p_s} \right)$. Therefore, $T(\tau)$ is given by:

$$T(\tau) = T_s \left(1 - \frac{\tau}{\tau_\infty} \right)^{\frac{R}{c_p}}$$

Substituting this into Equation 6.38, then making a change of coordinates within the integral:

$$OLR = \sigma T_s^4 e^{-\tau_\infty} + \int_0^{\tau_\infty} \sigma T_s^4 \left(1 - \frac{\tau'}{\tau_\infty} \right)^{\frac{4R}{c_p}} e^{-(\tau_\infty - \tau')} \, d\tau'$$

$$= \sigma T_s^4 \left(e^{-\tau_\infty} + \int_0^{\tau_\infty} \left(1 - \frac{\tau'}{\tau_\infty} \right)^{\frac{4R}{c_p}} e^{-(\tau_\infty - \tau')} \, d\tau' \right)$$

$$= \sigma T_s^4 \underbrace{\left(e^{-\tau_\infty} + \left(\frac{1}{\tau_\infty} \right)^{\frac{4R}{c_p}} \int_0^{\tau_\infty} x^{\frac{4R}{c_p}} e^{-x} \, dx \right)}_{\epsilon}$$

where $x = \tau_\infty - \tau'$.[7] Recalling the discussion in 4.4.2, where we defined the OLR such that $OLR = \epsilon \sigma T^4$, we have defined everything in the brackets as our effective emissivity ϵ of the planet.

Calculating the ECS and DRF explicitly (and recalling that $\tau_\infty = \kappa q p_s / g$):

$$ECS = \frac{\partial OLR}{\partial T_s}$$

$$= \epsilon \, 3\sigma \, T_s^3$$

$$DRF = -\frac{\partial OLR}{\partial q} = -\sigma T_s^4 \frac{\partial \epsilon}{\partial q} = -\sigma T_s^4 \frac{\partial \epsilon}{\partial \tau_\infty} \frac{\partial \tau_\infty}{\partial q}$$

$$= -\sigma T_s^4 \left(-e^{-\tau_\infty} - \frac{4R}{c_p} \left(\tau_\infty \right)^{-\frac{4R}{c_p} - 1} \int_0^{\tau_\infty} x^{\frac{4R}{c_p}} e^{-x} \, dx + e^{-\tau_\infty} \right) \frac{\kappa p_s}{g}$$

$$= \sigma T_s^4 \frac{4R}{c_p} \frac{\kappa p_s}{g} \left(\frac{1}{\tau_\infty} \right)^{\frac{4R}{c_p} + 1} \int_0^{\tau_\infty} x^{\frac{4R}{c_p}} e^{-x} \, dx$$

We thus find that $DRF > 0$ in an atmosphere with a vertical temperature profile set by the dry adiabat: more greenhouse gas concentration results in a decrease in the outgoing longwave radiation. Recalling our previous discussion regarding radiative forcing, it's not a surprise that $DRF > 0$, since the dry adiabat is one of the many mechanisms that results in an atmosphere where temperature decreases with height.

---

[7]The keen reader would have noticed that we could have completely avoided this change of coordinates if we simply defined our optical depth coordinate differently such that $\tau = 0$ where $p = 0$ (rather than where $p = p_s$) than and instead τ decreased with height (rather than increased with height)!

However, recalling our discussion in Section 4.11, we're fine, since these were all conventions anyways.

Chapter 7

Radiative Equilibrium

7.1 The Radiative Equilibrium Assumption

Recall the frequency and angle integrated downwards and upwards Schwarszchild Equations for a grey gas:

$$\frac{dE_+}{d\tau} = -E_+ + \sigma T^4$$
$$-\frac{dE_-}{d\tau} = -E_- + \sigma T^4$$

where E_+, E_+ are now irradiances (rather than spectral irradiances).

As already noted, if we're not given some functional form of $T(\tau)$, then we cannot solve this system, because we have three variables (E_+, E_+, T) but only two equations. Recall that the general solutions to the non-frequency averaged upwards and downwards Schwarszschild Equations (6.11, 6.10) featured $B(\tau')$ within the integral, which depends on the atmospheric temperature profile as a function of optical depth. In the previous chapters, we just took this as something given. For example, we might assume that T follows the dry adiabat.

We do not make these assumptions this chapter, and instead aim to solve for T. To solve for T, we must impose a constraint. This constraint will be that of **radiative equilibrium**: there is no net warming or heating from radiation at each optical depth. We now consider conservation of energy on some infinitesimal slab of atmosphere with optical thickness $\delta\tau$ and area A by considering the upwards and downwards irradiances entering and leaving the slab.

$$P_{in} = (E_+(\tau) + E_+(\tau + \delta\tau))A$$
$$P_{out} = (E_+(\tau + \delta\tau) + E_+(\tau))A$$

Radiative equilibrium, therefore:

$$P_{in} - P_{out} = 0$$
$$= (E_+(\tau) + E_-(\tau + \delta\tau) - E_+(\tau + \delta\tau) - E_-(\tau))A$$
$$= A\,\delta\tau\left(-\frac{E_+(\tau + \delta\tau) - E_+(\tau)}{\delta\tau} + \frac{E_-(\tau + \delta\tau) - E_-(\tau)}{\delta\tau}\right)$$

We can freely divide by $A \neq 0$ and take $\delta\tau \to 0$ to find our third and final equation that expresses radiative equilibrium:

$$\frac{d}{d\tau}(E_+ - E_-) = 0 \tag{7.1}$$

We can now solve the coupled ODEs 6.34, 6.35, and 7.1 to give us expressions for E_+, E_-, and T in terms of τ. Finally, we can find τ as a function of p (or z) in order to give us the temperature profile of an atmosphere in **radiative equilibrium**.

There are two final nuances to note. First, we are effectively making the *opposite* assumption to the one we made in deriving the adiabat (2.3): we are now assuming that radiation is the dominant energy transfer mechanism, rather than convection. It's worth noting that neither limit is entirely correct, whether radiation or convection wins out as the dominant mechanism depends on the situation. Having said this, which one turns out to be the dominant mechanism is not a coincidence, but is, of course, a consequence of the dynamics governing the system.

The second nuance is related to why we make the radiative equilibrium approximation now, *after* we have made the grey gas approximation, and not before. While non-grey gases in radiative equilibrium can and do occur, it does not result in a simple equation as in Equation 7.1. This is because *it need not be the case that the gas is in radiative equilibrium at every wavenumber.*

In other words, let $F_\pm(\nu)$ denote the total upwards/downwards spectral irradiance such that $E_\pm = \int F_\pm(\nu)\, d\nu$, where $E_\pm$ is the total upwards/downwards irradiance. For a non-grey gas in radiative equilibrium, it *is* the case that:

$$\frac{d}{d\tau}(E_+ - E_-) = 0$$

However, it is almost never the case that for some ν:

$$\frac{d}{d\tau}(F_+(\nu) - F_-(\nu)) = 0 \tag{7.2}$$

Radiative equilibrium only requires that a layer of atmosphere recieves as much energy from radiation as it radiates away, it does *not* require that this much happen at each frequency (i.e., that 7.2 obtains)! 7.2 is a *horrible* assumption, but it is the assumption we would need if we wish to solve the Upwards and Downwards *non-frequency integrated* Schwarszchild Equations (6.6, 6.7) governing the spectral irradiances.

7.2 General Solution

7.2.1 Derivation

The general solution is quickly revealed by adding and subtracting equations 6.34 and 6.35. Adding the equations give:

$$\frac{d}{d\tau}(E_+ - E_-) = -E_+ - E_- + 2\sigma T^4$$

Because the gas is in radiative equilibrium (Equation 7.1), the left hand side is 0. So we find that:

$$E_+ + E_- = 2\sigma T^4 \tag{7.3}$$

Subtracting Equations 6.34 and 6.35:

$$\frac{d}{d\tau}(E_+ + E_-) = -E_+ + E_-$$

The derivative of the right hand side wrt τ is 0 (again due to 7.1) so is constant. We can find the value of the right-hand-side by applying the boundary conditions. We apply the boundary condition at the top of the atmosphere:

$$E_+(\tau_\infty) = OLR \; ; \; E_-(\tau_\infty) = 0 \; \therefore$$
$$E_+ - E_- = OLR \tag{7.4}$$

Therefore (substituting our expression for $E_+ - E_-$ from Equation 7.4 into Equation 7.3):

$$\frac{d}{d\tau}(E_+ + E_-) = -OLR$$

Integrating (and using the boundary conditions again) gives:

$$E_+ + E_- = OLR(1 + \tau_\infty - \tau) \tag{7.5}$$

We are now in a position to find expressions for all three variables. We can find T by substituting 7.3 into 7.5. We can find E_+ and E_+ by adding or subtracting (respectively) 7.4 and 7.5. The solutions are:

Key Idea II.12: Radiative Equilibrium Solutions

A grey gas in pure radiative equilibrium will have a temperature profile $(T(\tau))$, upwards irradiance $(E_+(\tau))$, and downwards irradiance $(E_-(\tau))$ that obey the following relations:

$$T^4 = \frac{OLR}{2\sigma}(1 + \tau_\infty - \tau) \tag{7.6}$$

$$E_+ = \frac{OLR}{2}(2 + \tau_\infty - \tau) \tag{7.7}$$

$$E_- = \frac{OLR}{2}(\tau_\infty - \tau) \tag{7.8}$$

We can rewrite these equations using global energy conservation. We further assume that the entire atmosphere is in radiative equilibrium, so it must be receiving as much energy as it is emitting. Therefore the following holds:

$$OLR = S_\star \tag{7.9}$$

where $S_\star = (1 - \alpha) \cos \zeta \, L_\star$ is the absorbed stellar flux per unit area at the surface of the planet.

$$T^4 = \frac{S_\star}{2\sigma}(1 + \tau_\infty - \tau) \tag{7.10}$$

$$E_+ = \frac{S_\star}{2}(2 + \tau_\infty - \tau) \tag{7.11}$$

$$E_- = \frac{S_\star}{2}(\tau_\infty - \tau) \tag{7.12}$$

7.2.2 Interpretation

Having analytic expressions for a grey gas in radiative equilibrium, we can now extract some physical intuition.

The upwards irradiance E_+ and the OLR

Let's first consider the upwards irradiance E_+. At the *top* of the atmosphere $E_+ = OLR$ (obviously). We can equate this to an effective emitting temperature of the planet T_{eff} such that $\sigma T_{eff}^4 = OLR = E_+(\tau_\infty)$. At the *bottom* of the atmosphere, $\tau = 0$, so $E_+(0) = \frac{OLR}{2}(2 + \tau_\infty) = \frac{\sigma T_{eff}^4}{2}(2 + \tau_\infty)$. We can match this with our boundary condition, that all the upwards radiation at $\tau = 0$ is from the ground, and so $E_+(0) = \sigma T_g^4$ where T_g =the temperature of the ground (note that I am being careful not to say the temperature of the air at the ground!). We can relate T_g and T_{eff}: $E_+(0) = \sigma T_g^4 = \frac{\sigma T_{eff}^4}{2}(2 + \tau_\infty)$. Therefore:

$$T_{eff} = T_g \left(\frac{1}{1 + \frac{\tau_\infty}{2}}\right)^{\frac{1}{4}} , \text{ or, } OLR = \sigma T_g^4 \frac{1}{1 + \frac{\tau_\infty}{2}}$$

OLR and therefore T_{eff} are effectively fixed by $L_\star$, the properties of the star (and albedo), so it's clear that as τ_∞ gets arbitrarily large, so too must T_g, whereas if $\tau \to 0$, $T_g \to T_{eff}$, intuitively.

The temperature profile T

Next let's consider the temperature profile. As we can clearly see, as τ increases, T decreases, so radiative equilibrium also results in a temperature profile that decreases with height. Let's consider two limiting cases: air high up in the atmosphere, and air low in the atmosphere. Air right near the ground is located at $\tau = 0$, so $\sigma T(0)^4 = \frac{OLR}{2}(1 + \tau_\infty)$. But we just derived that $\sigma T_g^4 = \frac{OLR}{2}(2 + \tau_\infty)$. So in fact, $T_g > T(0)$: there is a temperature discontinuity at the ground, and the ground is warmer than the air immediately above it!

Is it stable to convection? Recall that an atmosphere is stable if it satisfies 2.12. We know it will be unstable at the ground, because at the ground the temperature gradient is infinite due to the discontinuity. Therefore, the air immediately above the ground will be warmed by the ground, and will convect. More generally though, we can directly differentiate 7.6 to see if it satisfies 2.12 to find where (if any) the temperature profile is unstable:

$$\frac{d}{dp}T^4 = -\frac{OLR}{2\sigma}\frac{d}{dp}(1 + \tau_\infty - \tau)$$

$$4T^3\frac{dT}{dp} = -\frac{OLR}{2\sigma}\frac{d\tau}{dp}$$

$$\therefore \frac{d\ln T}{d\ln p} = -\frac{p}{4T^4}\frac{OLR}{2\sigma}\frac{d\tau}{dp}$$

$$= -\frac{p}{4}\frac{1}{OLR(1 + \tau_\infty - \tau)/2\sigma}\frac{OLR}{2\sigma}\left(-\kappa(T(p), p)\frac{q(p)}{g}\right)$$

$$= -\frac{\kappa(T(\tau(p)), p)\,q(p)\,p}{4\,g\,(1 + \tau_\infty - \tau(p))}$$

We therefore, given some expression for the functional dependence of κ, an expression for the vertical temperature structure of the grey gas atmosphere in radiative equilibrium.

For now, to make progress, we just assume that $\kappa = const$. This allows us to use the relation derived before for $\tau(p)$:

$$\tau(p) = \tau_\infty\left(1 - \frac{p}{p_s}\right)$$

$$\tau_\infty = \tau(p_s) = \frac{\kappa\,q\,p_s}{g}$$

Therefore, our expression becomes:

$$\frac{d\ln T}{d\ln p} = \frac{1}{4}\left(\frac{\tau_\infty p}{p_s + \tau_\infty p}\right)$$

Clearly then, the temperature gradient of the atmosphere has no dependence whatsoever on the OLR. The only property the OLR affects in a radiative equilibrium atmosphere is how hot the atmosphere is, but it does not affect how quickly temperature falls off with height.

Let's look at some limits. At the top of the atmosphere, $p \to 0$, and as $p \to 0$, $\frac{d\ln T}{d\ln p} \to 0$ too. So near the top of the atmosphere, referring back to our Dry Stability Criterion, an atmosphere in radiative equilibrium is almost always super-stable. This explains why stratospheres, where the vertical temperature structure is set by radiative equilibrium, only really occurs up high in the atmosphere and results in temperature profiles that are stable to convection.

At the bottom of the atmosphere, where $p \to p_s$, $\frac{d\ln T}{d\ln p} \to \frac{1}{4}\left(\frac{\tau_\infty}{1 + \tau_\infty}\right)$. The temperature profile at the ground then depends strongly on the optical thickness of the entire atmosphere τ_∞. If the atmosphere is optically thin ($\tau_\infty \ll 1$), then $\frac{d\ln T}{d\ln p}$ can also become arbitrarily small close to the surface (i.e., as $\tau_\infty \to 0, \frac{d\ln T}{d\ln p} \to 0$). If the atmosphere is optically thick ($\tau_\infty \gg 1$), though, $\frac{d\ln T}{d\ln p} \to \frac{1}{4}$.

Now, again, whether this ends up as actually unstable or not depends on the Dry Stability Criterion, which depends on the heat capacity c_p of the atmosphere, which, recalling our discussion on the Equipartition Theorem, depends on the number of degrees of freedom of the molecules that compose

that atmosphere. For the Earth, where $c_p \approx 7/2R$, an optically thick grey gas atmosphere would be *just about* stable at the surface.

More generally, an optically thick atmosphere at the surface will be unstable if $\frac{4R}{c_p} < 1$.

Note though that our model is extremely limited. First, we have assumed that the atmosphere is grey, and *further* assumed that κ has no dependence on T or p. Recalling from Section 5.1.2 that S and therefore κ has a strong exponential dependence on T (as T increases, κ strongly increases), we expect κ to be much larger near the surface of the planet (where T is larger) than near the top of the atmopshere (where T is smaller).

It is not difficult to verify that, if this is the case, $\frac{d\ln T}{d\ln p}$ would be larger at $p \to p_s$ in an optically thick atmosphere, and therefore would be more prone to going unstable. (Exercise: let $\kappa = ap$, then integrate explicitly to find $d\ln T/d\ln p$. You should find that, now, $d\ln T/d\ln p \to 1/2$ instead of $1/4$).

Second, we have not considered the diurnal cycle: we have assumed that the [UNDER CON-STRUCTION]

7.2.3   Internal Leakage

In many cases, though, an atmosphere in radiative equilibrium is not only receiving energy from its star. It is also receiving energy from the planet: after formation, a planet often releases energy leftover from the formation. The Earth itself releases energy left over from formation (see here: [5]).

In such a case, we should be clear what we mean by 'Radiative Equilibrium'. 'Radiative Equilibirum' in this context applies to the *atmosphere*, not to the planet. We are assuming that the atmosphere receives as much energy as it releases, *not* that the planet receives as much energy as it releases. Because of this, again the system is sort of in a quasi-steady state: the atmosphere is in a 'steady state', but the planet is not: it is cooling and receiving less energy than it is emitting. However, because the atmosphere relaxes to equilibrium much faster than the planet, we can treat the atmosphere as simply in time-dependent equilibrium.

Suppose that the planet's surface leaks energy at a rate of F_{int}. Then the energy budget of the atmosphere must sum to 0, so the following criterion must hold (analogous to Equation 7.9):

$$OLR = S_\star + F_{int}$$

Equations 7.10-7.12 are modified accordingly, but note that really the only affect of internal leakage is that, like OLR, F_{int} only increases the temperature of the entire atmosphere monotonically – it does not affect the vertical temperature structure of the atmosphere.

7.3   Shortwave absorbers

We now consider an atmosphere where there are shortwave absorbers/emitters in the atmosphere, i.e., we consider an atmosphere that can absorb shortwave from the star. So far we have only dealt with a gas that is a longwave absorber/emitter: it can absorb the radiation emitted by the surface and other components of the atmosphere but not the radiation emitted by the star.

What I just said should be slightly confusing. After all, we explicitly assumed earlier that that the atmosphere was grey ($\kappa = const$). Therefore, if the gas absorbs/emits longwave radiation it must also absorb/emit shortwave radiation.

This is technically correct, so we should be explicit as to what we are modifying in the model, and why the physical interpretation of this modification corresponds to adding shortwave absorbers/emitters to the atmosphere. The modification we make is as follows. We revise the Radiative Equilibrium criterion (Equation 7.1) to be below:

$$\frac{d}{d\tau}\left(E_+ - E_- - E_\star\right) = 0 \tag{7.13}$$

where $E_\star(\tau) =$ is the monotonically increasing ($E_\star$ is largest at the top of the atmosphere) downwards irradiance of the star. In this section we simply take $E_\star$ as a given function (e.g., we let $E_\star \sim e^\tau$) – we do not solve for it.

What we are representing in this model, now, is that there is some downwards irradiance from the star at the top of the atmosphere that is (and here's the crucial part) **absorbed by shortwave absorbers in the atmosphere**. This implies that $E_\star$ decreases as it propagates downwards through the atmosphere, and this is how we are indirectly representing the effect of shortwave absorbers.[1]

Let us now solve this system as before, this time with the $E_\star$.

7.3.1 Derivation

Before we derive the vertical profile, we first redefine our coordinate system for convenience (which we are well within our rights to - see Section 4.3). We now set $\tau = 0$ at $p = 0$ rather than $p = p_s$, but allow τ to increase with height as before. This means that $\tau = -\tau_\infty$ at $p = p_s$. We don't *need* to do this, so the following derivation would be exactly the same (save swapping out a few boundary conditions).

We derive the solutions analogous to how we derived the solutions without shortwave absorbers. As before, we add and subtract Equations 6.34 and 6.35. Adding the equations gives (as before):

$$\frac{d}{d\tau}(E_+ - E_-) = -E_+ - E_- + 2\sigma T^4$$

This time we must subtract $\frac{dE_\star}{d\tau}$ to both sides to fulfil our revised Radiative Equilibrium criterion (Equation 7.13) to obtain the following:

$$\frac{d}{d\tau}(E_+ - E_- - E_\star) = -E_+ - E_- + 2\sigma T^4 - \frac{dE_\star}{d\tau}$$

Now we use Equation 7.13 to find that the left hand side is 0:

$$E_+ + E_- = 2\sigma T^4 - \frac{dE_\star}{d\tau} \tag{7.14}$$

which is the shortwave version of Equation 7.3

Again, analogous to the derivation before, we subtract Equations 6.34 and 6.35:

$$\frac{d}{d\tau}(E_+ + E_-) = -E_+ + E_-$$

To massage this into the version we would like, we add $E_\star$ to both sides. Therefore:

$$\frac{d}{d\tau}\left(E_+ + E_- + \int_0^\tau E_\star(\tau')\, d\tau'\right) = -E_+ + E_- + E_\star$$

The derivative of the right hand side wrt τ is 0 (again due to Equation 7.13) so is constant. We can find the value of the right-hand-side by applying the boundary conditions. We apply the boundary condition at the top of the atmosphere:

$$E_+(0) = OLR \; ; \; E_-(0) = 0 \; ; \; E_\star(0) = E_{\star,top} \; \therefore$$
$$E_+ - E_- - E_\star = OLR - E_{\star,top} \tag{7.15}$$

Therefore (substituting our expression for $E_+ - E_-$ from Equation 7.15 into Equation 7.14):

$$\frac{d}{d\tau}\left(E_+ + E_- + \int_0^\tau E_\star(\tau')\, d\tau'\right) = -OLR + E_{\star,top}$$

---

[1]Of course, another difference is that now the boundary condition at the top of the atmosphere is different. Whereas before there was no downwards irradiance ($E_-(\tau = \tau_\infty) = 0$), now there is some non-zero downwards irradiance ($E_-(\tau_\infty) + E_\star(\tau_\infty) = E_\star(\tau_\infty) \neq 0$). However, I am of the opinion that this is not the crucial difference. The crucial difference is that this downwards irradiance of the star is *monotonic in* τ. This ensures the effect of shortwave absorbers high in the atmosphere: there are shortwave absorbers which block the shortwave irradiance propagating downwards (by absorbing them), and re-emit some of the energy as longwave radiation (which is encoded in the E_- term).

Integrating (again using the boundary conditions gives):

$$E_+ + E_- + \int_0^\tau E_\star(\tau')\, d\tau' = (OLR - E_{\star,top})(1 - \tau) + E_{\star,top}$$

$$\therefore E_+ + E_- = (OLR - E_{\star,top})(1 - \tau) + E_{\star,top} + \int_\tau^0 E_\star(\tau')\, d\tau' \tag{7.16}$$

Again, we are in a position to find expressions for all three variables. We can find T by substituting 7.14 into 7.16. We can find E_+ and E_+ by adding or subtracting (respectively) 7.15 and 7.16. The solutions are:

Key Idea II.13: Radiative Equilibrium Solutions (with Shortwave Absorber)

A grey gas in pure radiative equilibrium with a shortwave absorber will have a temperature profile $(T(\tau))$, upwards irradiance $(E_+(\tau))$, and downwards irradiance $(E_-(\tau))$ that obey the following relations:

$$2\sigma T^4 = (OLR - E_{\star,top})(1 - \tau) + E_{\star,top} + \int_\tau^0 E_\star(\tau')\, d\tau' + \frac{dE_\star}{d\tau} \tag{7.17}$$

We can rewrite this equation as follows (exercise):

$$2\sigma T^4 = OLR(1 - \tau) + \frac{dE_\star}{d\tau} - \int_\tau^0 (E_{\star,top} - E_\star(\tau'))\, d\tau' \tag{7.18}$$

$$[\text{UNDER CONSTRUCTION}] \tag{7.19} \qquad [\text{UNDER CONSTRUCTION}] \tag{7.20}$$

where $E_\star(\tau)$ is the downwards shortwave irradiance which monotonically increases with τ. In this problem, we simply treat this as given.

7.3.2 Interpretation

First, we should check that we recover the no-short-wave-absorbers solution if we set $E_\star = E_{\star,top}$ for all τ. Looking at Equation 7.18, we can see that our temperature profile reduces to $2\sigma T^4 = OLR(1 - \tau)$, which is identical to the temperature profile from before (up to, of course, our arbitrary redefinition of τ).

Now, lets examine how the presence of shortwave absorbers modifies our temperature profile (Equation 7.18). The first $\frac{dE_\star}{d\tau}$ term represents shortwave warming from absorption. If $\frac{dE_\star}{d\tau} > 0$, then $E_\star$ increases with optical depth τ, and therefore decreases as the shortwave irradiance propagates *downwards* through the atmosphere. This indicates that there is *strong* shortwave absorption at this optical depth, which results in local warming at this point.

The second $-\int_\tau^0 (E_{\star,top} - E_\star(\tau'))\, d\tau'$ term represents

$$2\sigma T^4 = \underbrace{OLR(1 - \tau)}_{\text{original solution}} + \overbrace{\frac{dE_\star}{d\tau}}^{\text{shortwave warming}} - \underbrace{\int_\tau^0 (E_{\star,top} - E_\star(\tau'))\, d\tau'}_{\text{????}}$$

To make these discussions more concrete, we now let $E_\star(\tau)$ be given by the following:

$$E_\star = E_{\star,top} e^{a\tau}$$

[UNDER CONSTRUCTION]

Part III

Clouds

Introduction

This section of the course was lectured by Philip Stier covering Cloud Physics. THIS SECTION IS STILL UNDER CONSTRUCTION.

This section consists of three chapters:

1. Dynamical Systems:

    Meow

2. Predictability:

    Meow

3. Estimation:

    Meow

Chapter 8

Convection and Thermodynamics

8.1 Definitions of Humidity

8.1.1 hi

8.1.2 Clausius-Clapeyron Relation

8.2 Convection and Tephigrams

8.3 Radiative-Convective Equilibrium

Chapter 9

Warm Cloud Microphysics

9.1 Growth in Thermodynamic Equilibrium

We first consider the very initial formation of a cloud droplet, formed when water vapour (gas) condenses into a liquid water droplet. If a water droplet is to begin forming, the process must be thermodynamically favourable; i.e., the process is set by what thermodynamic equilibrium is. This is an assumption we are making: it is a good assumption for now, but it will break down when other factors limit growth (e.g., kinetic non-equilibrium processes). One reason it is a good assumption now is because the processes that take place which push the system towards thermodynamic equilibrium occur on very small time-scales, and so the system is what is called *quais-steady* (a concept we will be reintroduced to in Part V): it evolves so quickly that it is effectively always in equilibrium.

9.1.1 Homogenous Nucleation: The Kelvin Equation

We first consider the pure water vapour with no aerosols. In the atmosphere, these tiny cloud droplets are coupled to an effectively infinite heat bath and are held at fixed pressure (from atmospheric temperature). We thus assume that the pressure and temperature are held fixed.

If the pressure and temperature are feld fixed, it can be shown that any spontaneous thermodynamic process must, in equilibrium, evolve in order to decrease the Gibbs Free Energy G. The Gibbs Free Energy G is defined as:

$$G = U - TS + pV \tag{9.1}$$

Actually, the situation is a bit more complicated than simply decreasing the Gibbs Free Energy. It is more accurate to require the system to result in a *local* decrease in Gibbs Free Energy. To explain what *local* decrease means, consider a system with a Gibbs Free Energy $G(\lambda)$ that depends on some continuous parameter λ. Suppose that the system is initially in a state of $\lambda = \lambda_0$. Then λ will evolve such that $\delta\lambda \sim -\frac{\partial G}{\partial \lambda}(\lambda_0)$. The idea is that the system does not 'know' where the global minimum of the Gibbs Free Energy is, so it will only evolve locally to decrease the Gibbs Free Energy.

We now consider the formation of a cloud droplet, and let our parameter $\lambda = r$ where r is the radius of our cloud droplet. The Gibbs Free Energy of the system is the sum of the Gibbs Free Energy of $N_{drop}(r)$ molecules of condensed water vapour in the droplet and $N - N_{drop}$ molecules of *un*condensed water vapour, where $N =$ the total number of water molecules. We keep N fixed to represent conservation of water molecules.

$$G(r) = \underbrace{N_{drop}(r)\, g_v + 4\pi r^2\sigma}_{G \text{ of droplet}} + \underbrace{(N - N_{drop}(r))g_l}_{G \text{ of vapour}}$$

where g_v, g_l are Gibbs Free Energy per molecule of the vapour and liquid phases of water; $\sigma =$ the surface tension; and $r =$ the radius of the water droplet. The Gibbs Free Energy of the droplet is the sum of the Gibbs Free Energy of the liquid water molecules in the droplet ($N_{drop}\, g_v$) and the Gibbs Free Energy of the liquid-gas interface (represented by the $4\pi r^2\sigma$ surface tension term).

We write $N_{drop}(r)$ in terms of r by assuming that the droplet is spherical, therefore $N_{drop} = \frac{4}{3}\pi r r^3/v_l$, where $v_l =$ the volume of one liquid molecule of water. Therefore:

$$G = \frac{4\pi}{3v_l}r^3(g_l - g_v) + 4\pi\sigma r^2 + Ng_v \tag{9.2}$$

Our goal now is to calculate $g_l - g_v$. We can take the differential of G in 9.1 then divide by the number of water molecules to find that:

$$\begin{aligned} dG &= dU - T\,dS - dT\,S + p\,dV + dp\,V \\ &= \underbrace{T\,dS - p\,dV}_{dU} - T\,dS - dT\,S + p\,dV + dp\,V \\ dG &= -S\,dT + V\,dp \\ \therefore dg &= -\frac{1}{N}S\,dT + \frac{V}{N}\,dp \end{aligned}$$

We make two assumptions here. First, we assume that the temperature is constant (recall the effective infinite heat bath of the atmosphere!), and therefore neglect the $S\,dT$ term. Second, we assume that $v_g = \frac{V_g}{N} \gg v_l = \frac{V_l}{N}$ (i.e., that the volume per gaseous water molecules is much bigger than the volume per liquid water molecule). Therefore, we ignore the change in Gibbs Free Energy in the liquid and so $dg = (v_l - v_g)dp \approx -v_g\,dp$.

$$g_l - g_v = \int_{\text{no droplet}}^{\text{droplet}}$$

assuming that the 1.3, $\frac{k_BT}{p_g} \gg \frac{k_BT}{p_l}$, and so the

$$g_l - g_v =$$

Therefore:

$$e_S(r) = e_S(\infty)\exp\left(\frac{2\sigma v_l}{k_BTr}\right) \tag{9.3}$$

For ease of notation, we let $A = \frac{2\sigma v_l}{k_BT} = \frac{2\sigma}{R_v\rho T}$ (recalling the definition of v_l and R_v in 1.5). Therefore:

$$e_S(r) = e_S(\infty)\exp\left(\frac{A}{r}\right)$$

9.1.2 Homogenous Nucleation: The Raoult Equation

$$e_S^{sol}(\infty) = e_S^0(\infty)\frac{n_w}{n_{sol} + n_w} \tag{9.4}$$

In other words, the vapour pressure is set by the mole-fraction of water. Generally, $n_{sol} \ll n_w$, so:

$$\begin{aligned} \frac{n_w}{n_{sol} + n_w} &= \frac{1}{n_{sol}/n_w + 1} \\ &= \left(1 + \frac{n_{sol}}{n_w}\right)^{-2} \\ &\approx 1 - \frac{n_{sol}}{n_w} \end{aligned}$$

9.1.3 The Köhler Equation

We can combine 9.3 and 9.4 into one equation by considering a cloud droplet which has a curved interface (taking into account 9.3) and has a dissolved solute (taking into account 9.4).

The saturation vapour pressure is therefore:

$$e_S^{sol}(r) = e_S^{sol}(\infty) \exp\left(\frac{2\sigma v_l}{k_B T r}\right)$$

$$= e_S^0(0) \frac{n_w}{n_{sol} + n_w} \exp\left(\frac{2\sigma v_l}{k_B T r}\right)$$

$$e_S^{sol}(r) = e_S^0(0)\left(1 - \frac{B}{r^3}\right) e^{\frac{A}{r}} \tag{9.5}$$

9.2 Growth by Condensation

We now assume that the droplet has grown to such as size that the rate of droplet formation now becomes kinetically limited: that it is limited primarily by the rate at which water molecules and heat may diffuse away/towards the droplet.

9.2.1 Diffusion of Water Molecules

We begin from the diffusion equation governing the diffusion of water vapour *outside* the droplet:

$$\frac{\partial n}{\partial t} = \vec{\nabla} \cdot \vec{\Phi} \tag{9.6}$$

$$\vec{\Phi} = D\vec{\nabla} n \tag{9.7}$$

where $n =$ the number concentration of water molecules; $\vec{\Phi} =$ the flux of water molecules; and $D =$ the diffusivity of water in air. We assume, again, that the droplet is quasi-steady, and so that $\frac{\partial n}{\partial t} = 0$. Furthermore, we assume that the water vapour outside the droplet is spherically symmetric, and so $n = n(r)$ only. We can then solve $\nabla^2 n = 0$. In spherical coordinates, the solution is:

$$n(r) = C_1 - \frac{C_2}{r}$$

To find the constants, we apply the boundary conditions that far away from the droplet $(r \to \infty)$ the number density is the ambient vapour density (n_∞) and near the droplet $(r = R$, where $R =$ the radius of the droplet), the number density is the vapour density of the surface. This gives:

$$n(r) = n_\infty - \frac{R}{r}(n_\infty - n_r)$$

We now apply mass conservation to the water molecules *within* the droplet:

$$\frac{d}{dt}\int_V (t)\rho\, dV = \int_S (t)m_{\text{H}_2\text{O}}\vec{\Phi} \cdot d\vec{S} \tag{9.8}$$

where $m_{\text{H}_2\text{O}} =$ the mass of one water molecule. The left-hand-side is the rate of change of the mass of the water droplet and the right-hand-side is the flux of water molecules into/out of the boundaries. $V(t)$ and $S(t)$ indicate the volume of the droplet and surface of the droplet over which we are integrating, and these time-vary due to the fact that the droplet is changing size.

We write the left-hand-side as just dM/dt where $M =$ the total mass of the droplet, and substitute in 9.7 and 9.8 for the right-hand-side to obtain:

$$\boxed{\frac{dM}{dt} = 4\pi R D(\rho_\infty - \rho_R)} \tag{9.9}$$

9.2.2 Diffusion of Heat

9.3 Growth by Collision/Collection/Coallescence

9.3.1 Collision Processes

Chapter 10

Cold Cloud Microphysics

Chapter 11

Cloud Morphology, Radiation, and Climate

Part IV

Instrumentation

There's nothing here!

Part V

Climate Dynamics

Introduction

This section of the course was lectured by Myles Allen, covering a miscellaneous set of topics. His stated goal (at time of writing) was to introduce you to certain more general methods using climate/weather as an example.

This section consists of three chapters:

1. Dynamical Systems:

   We consider the Earth as a simple, low-dimensional, linear, deterministic dynamical system. We solve the system using normal modes and a further method (not discussed by Myles) using the quasi-steady approximation.

   In many cases we find this to reveal multiple timescales of response: a shorter timescale response (where the slower component stays constant) and a longer timescale response (where the faster component is always in time-varying equilibrium).

2. Predictability:

   We discuss the growth of errors and perturbations in a linearised N dimensional dynamical system, and how to minimise such errors in, for example, a weather forecast.

   We briefly discuss the growth of non-linear errors.

3. Estimation:

   We introduce the maximum likelihood estimator, the estimator that minimises the squared difference between the prediction and the data (this is just finding a line of best fit).

   We discuss errors.

Chapter 12

Dynamical Systems

12.1 The Equations of Motion

We first consider the climate as a simple dynamical system, the simplest of which is a linear dynamical deterministic system.

In reality, the climate is highly non-linear and stochastic. As such, we should consider this model as representing only small perturbations about some reference state. If one considers small enough perturbations, any (well-behaved) state will be linear (think: Taylor expansions). The state will, in general, be given by:

$$\dot{\vec{x}} = \hat{J}\vec{x} + \vec{f} \tag{12.1}$$

where $\vec{x} \in \mathbb{R}^{n}$[1] is the state-vector of the system, $\hat{J} \in \mathbb{R}^n \times \mathbb{R}^n$ is the linear(ised) Jacobian, and $\vec{f} \in \mathbb{R}^n$ is the external forcing.

We apply this first to an energy-budget equation for the climate, characterised by the state-vector (T_s, T_d) representing perturbations of the mean atmospheric/surface-ocean temperature and deep ocean temperature, respectively, governed by the equations:

$$\boxed{C_s\dot{T}_s = F_{ext}(t) - \lambda T_s - \gamma(T_s - T_d) - \lambda'(T_s - T_d)} \tag{12.2}$$

$$\boxed{C_d\dot{T}_d = -\gamma(T_d - T_s)} \tag{12.3}$$

where C_s, C_d are the effective heat capacities of the atmosphere/surface oceans and deep oceans per unit area, respectively. The left-hand side represents the change in the energy (per unit area) and the right-hand side represents various energy fluxes (per unit area). The various terms are explained below:

Term	Interpretation
$F_{ext}(t)$	The perturbation in external forcing due to albedo, aerosol, greenhouse gas, solar, etc. forcing.
λT_s	The additional energy radiated to space (per unit area) due to warming. If $\lambda T_s > 0$ then *more* energy is being radiated into space, which typically results in cooling. λ is the *climate sensitivity parameter*.
$-\gamma(T_s - T_d)$	The energy flux from the atmosphere/surface oceans *into* the deep oceans. This is typically small, due to the fact that the oceans are very stably stratified. As such, mixing/convection is severely inhibited, and surface waters can only penetrate into the deep oceans at a select few places (cf. Sec.20.3).
$-\lambda'(T_s - T_d)$	The additional energy radiated into space due to the system being out of equilibrium. We have reason to believe (empirically, due to climate models) that this occurs. We encode this crudely by letting $(T_s - T_d)$ represent the degree to which the climate is out of equilibrium.

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[1]This is just fancy-speak for an n dimensional vector.

Note that if we add Equations 12.2 and 12.3, we do not get 0 on the right-hand side: the climate is not an isolated system. There is a net energy flux from F_{ext}, $-\lambda T_s$, and $-\lambda'(T_s - T_d)$.

12.2 Solving the System by Diagonalising the Matrix

Before you read this section, I should note that, while this method is completely general to linear systems, it is very time-consuming to use this method in the exam. Furthermore, I think more physical intuition comes from another method of solving the system which I explain in Section 12.3. If you're low on time, and the question asks you to sketch it, it might be easier to think in terms of the method in Section 12.3.

This method relies on some prior knowledge of linear algebra, specifically diagonalisation, eigenvectors, and eigenvalues. I'll review them quickly here, but if you're familiar with the content you can skip to Section 12.2.2.

12.2.1 Deriving the Method (Review of ODEs and Linear Algebra)

Reframing the ODE Problem into a Linear Algebra Problem

First we write Equations 12.2 and 12.3 in matrix form:

$$\underbrace{\begin{pmatrix} \dot{T}_s \\ \dot{T}_d \end{pmatrix}}_{\dot{\vec{T}}} = \underbrace{\begin{pmatrix} -\frac{\lambda+\gamma+\lambda'}{C_s} & \frac{\gamma+\lambda'}{C_s} \\ \frac{\gamma}{C_d} & -\frac{\gamma}{C_d} \end{pmatrix}}_{\hat{J}} \underbrace{\begin{pmatrix} T_s \\ T_d \end{pmatrix}}_{\vec{T}} + \underbrace{\begin{pmatrix} F_{ext}(t)/C_s \\ 0 \end{pmatrix}}_{\vec{f}} \tag{12.4}$$

We now attempt to **diagonalise** $\hat{J}$. We find **eigenvectors** $\vec{v}_i =$ that satisfy the following relation:

$$\hat{J}\vec{v}_i = \lambda_i \vec{v}_i \tag{12.5}$$

where $\lambda_i \in \mathbb{R}$ is an **eigenvalue**. I'll confine our attention to 2D systems here for brevity, but everything I say is readily extendable to N dimensions. In 2D, we write $\vec{v}_i = ([\vec{v}_i]_1, [\vec{v}_i]_2)^T$.[2]

We now slap our eigenvectors $\vec{v}_i$ together into a matrix $\hat{E}$ defined as follows:x

$$\hat{E} = \begin{pmatrix} \uparrow & \uparrow \\ \vec{v}_1 & \vec{v}_2 \\ \downarrow & \downarrow \end{pmatrix}$$

We then define $\hat{\Sigma} = \hat{E}^{-1}\hat{J}\hat{E}$ and show then the following:

$$\hat{E}^{-1}\hat{J}\hat{E} = \hat{E}^{-1}\hat{J} \begin{pmatrix} \uparrow & \uparrow \\ \vec{v}_1 & \vec{v}_2 \\ \downarrow & \downarrow \end{pmatrix}$$

$$= \hat{E}^{-1} \begin{pmatrix} \uparrow & \uparrow \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 \\ \downarrow & \downarrow \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \hat{E}^{-1}\hat{E}$$

$$\hat{\Sigma} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Therefore, multiplying on the left by $\hat{E}$ and right by $\hat{E}^{-1}$:

$$\hat{J} = \hat{E} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \hat{E}^{-1}$$

---

[2]Note further that each eigenvector is non-unique – it is specified only up to a constant of proportionality. I.e., if $\vec{v}_i$ is an eigenvector, then $\alpha\vec{v}_i$ is also an eigenvector for all $\alpha \in \mathbb{R}$.

We can then multiple Equation 12.4 on the left by $\hat{E}^{-1}$ and define $\vec{S} \equiv \hat{E}^{-1}\vec{T}$ and $\vec{g} \equiv \hat{E}^{-1}\vec{f}$:

$$\hat{E}^{-1}\dot{\vec{T}} = \hat{E}^{-1}\hat{J}\vec{T} + \hat{E}^{-1}\vec{f}$$
$$= \cancel{\hat{E}^{-1}}\cancel{\hat{E}}\hat{\Sigma}\hat{E}^{-1}\vec{T} + \hat{E}^{-1}\vec{f}$$
$$\dot{\vec{S}} = \hat{\Sigma}\vec{S} + \vec{g}$$

Recall that $\hat{\Sigma}$ is completely diagonal, so our coupled equations are now uncoupled in the new coordinate system $\vec{S}$! We write the ith component of the vectors with a subscript i and thus write that:

$$\dot{S}_i = \lambda_i S_i + g_i$$

Therefore, using an integrating factor, our solution is:

$$S_i(t) = S_i(0)e^{\lambda_i t} + e^{\lambda_i t}\int_0^t g_i(t')e^{-\lambda_i t'}\,dt' \tag{12.6}$$

Physically, $S_i(t)$ is the ith normal mode, a mode that is decoupled completely from the other modes: this mode will evolve completely ignorant to the behaviour of the other modes. The eigenvectors λ_i, which have dimensions of inverse-time, represent the timescales of the ith normal mode $\tau_i \sim 1/\lambda_i$.

Our final step is to relate S_i back to T_i, which is easy if we note that $\vec{S} \equiv \hat{E}^{-1}\vec{T}$, therefore $\vec{T} = \hat{E}\vec{S}$. Therefore, $T_i = [v_i]_1 S_1 + [v_i]_2 S_2$ so our general solution is:

$$T_j(t) = \sum_i \left([v_j]_i \left(S_i(0)e^{\lambda_i t} + e^{\lambda_i t}\int_0^t g_i(t')e^{-\lambda_i t'}\,dt'\right)\right) \tag{12.7}$$

where $g_i(t) = \sum_j [\hat{E}^{-1}]_{ij}[\vec{f}]_j$ and $S_i(0) = \sum_j [\hat{E}^{-1}]_{ij}T_j(0)$. You'll have to calculate this explicitly for the problem at hand.

So we can solve our ODE if we find the **eigenvectors** and **eigenvalues** of $\hat{J}$! We've transformed our problem into a linear algebra problem! So how do we solve this now?

Solving the Linear Algebra Problem

We rewrite Equation 12.5 as follows:

$$\hat{J}\vec{v}_i = \lambda_i \vec{v}_i$$
$$= \lambda_i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\vec{v}_i$$
$$\therefore \vec{0} = \left(\vec{J} - \lambda_i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\vec{v}_i$$

where $\vec{0} = (0,0)^T$.

Now clearly, an easy solution is if we set $\vec{v}_i = \vec{0}$, but this is only one solution, and we wish to find the non-trivial solutions. One can show in linear algebra that a non-zero matrix times a non-zero vector can be equal to $\vec{0}$ only if the determinant of that matrix is zero. We thus solve for:

$$\det\left(\vec{J} - \lambda_i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

to obtain our eigenvalues λ_i. This should give us a quadratic in λ_i in 2-dimensions, but if we have N dimensions we should get a N-dimensional polynmial in λ. Once we have done that we can go through each λ_i one at a time and substitute the λ_i into Equation 12.5 to find the corresponding $\vec{v}_i$. And we're done!

12.2.2 Application to Our Dynamical System

Application without Simplifications

Let us refer again to our solution written in Equation 12.8. We can make a few simplifications when applying this to our model. First, $\vec{f} = (F_{ext}/C_s, 0)^T$, so we can set $[\vec{f}]_2 = 0$.

Application with Simplifications

Let us now apply this method to our dynamical system. We first write the Jacobian as follows to simplify the algebra:

$$\hat{J} = \begin{pmatrix} -\frac{\lambda+\gamma+\lambda'}{C_s} & \frac{\gamma+\lambda'}{C_s} \\[2mm] \frac{\gamma}{C_d} & -\frac{\gamma}{C_d} \end{pmatrix} = \begin{pmatrix} -a & b \\[2mm] c & -c \end{pmatrix}$$

Typically, $C_s \ll C_d$, representing the massive heat capacity of the deep oceans compared to the atmosphere and land. If we assume further that λ, γ, and λ' are of similar size, we can conclude that $|a|, |b| \gg |c|$. This will be important later for an approximation that we will make.

Let us now continue with our method. We want to find the eigenvalues of $\hat{J}$. Let us refer to the eigenvalues as r_i instead of λ_i to avoid confusion. We want to solve:

$$\det \begin{pmatrix} -a - r_i & b \\[2mm] c & -c - r_i \end{pmatrix} = 0$$

This is a straightforward quadratic in r_i, which has the solution:

$$r_i = \frac{1}{2}\left(-a - c \pm \sqrt{a^2 - 2ac + 4bc + c^2}\right)$$

Because $|a| \gg |c|$, we keep only first order in $\frac{c}{a}$, and write:

$$r_i = \frac{1}{2}\left(-a - c \pm \sqrt{a^2 - 2ac + 4bc + c^2}\right)$$
$$= \frac{1}{2}\left(-a - c \pm a\sqrt{1 - \frac{2c}{a} + \frac{4bc}{a^2} + \left(\frac{c}{a}\right)^2}\right)$$
$$\approx \frac{1}{2}\left(-a - c \pm a\left(1 - \frac{c}{a} + \frac{2bc}{a^2}\right)\right)$$
$$r_2 \approx \underbrace{c}_{\mathcal{O}(c)}\overbrace{\left(-1 + \frac{b}{a}\right)}^{\mathcal{O}(1)} \; , \; r_1 \approx -\underbrace{a}_{\mathcal{O}(a)} - \overbrace{\frac{c}{a}}^{\mathcal{O}\left(\frac{c}{a}\right)\ll 1}\underbrace{\frac{b}{}}_{\mathcal{O}(a)}$$

Recall again that $|c| \ll |a|, |b|$. The eigenvalue on the left r_2 is of order c, so it is small, but note both terms are of order c. As such, we cannot neglect any terms in r_2.[3] The eigenvector on the right r_1 has the second $-bc/a$ term much smaller than the first $-a$ term, so we can neglect the second term. Our eigenvectors r_i are therefore:

$$r_1 \approx -a \; , \; r_2 \approx c\left(-1 + \frac{b}{a}\right)$$

---

[3]Well, nothing is stopping us from neglecting both terms, but this would be a worse approximation. Exercise: approximate $r_2 \approx 0$ and show that this approximation results in $T_d = 0$ for all time. This is the result of the assumption we've implicitly made by approximating $r_2 = 0$. In effect we've assumed that C_s is ridiculously small compared to C_d, therefore any flux from T_s will result in no change from T_d.

Plugging in values for a, b, c, we get:

$$r_1 \approx -\frac{\lambda + \gamma + \lambda'}{C_s} \quad , \quad r_2 \approx -\frac{\lambda \gamma}{(\lambda + \gamma + \lambda') C_s}$$

Having found the (approximate) eigenvalues, our goal now is to find the (approximate) eigenvectors. We can estimate this directly by substituting in the eigenvalues directly into Equation 12.5 and solving for $\vec{v}_i$. Following some algebra (and again being consistent that $|c| \ll |a|, |b|$), we find that the corresponding eigenvectors are:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \vec{v}_2 = \begin{pmatrix} 1 \\ a/b \end{pmatrix} = \begin{pmatrix} 1 \\ (\lambda + \gamma + \lambda') / (\gamma + \lambda') \end{pmatrix}$$

Key Idea V.1: Dynamical Systems Recipe

Suppose we have a coupled set of ODEs governing the time-evolution of $x(t)$ and $y(t)$ given by:

$$A\dot{x} = Cx + Dy + G(t)$$
$$B\dot{y} = Ex + Fy + H(t)$$

Solve this system in the following way:
1. Write your ODE in matrix form:

$$\dot{\vec{x}} = \hat{J}\vec{x} + \vec{f} \quad \text{where} \quad \hat{J} = \begin{pmatrix} C/A & D/A \\ E/B & F/B \end{pmatrix} \text{ and } \vec{f}(t) = \begin{pmatrix} G(t)/A \\ H(t)/B \end{pmatrix}$$

2. Find the eigenvectors and eigenvalues of $\hat{J}$.
3. See if you can exploit any separation of scales to neglect terms. For example, if $A \ll B$, then we can keep only first order in E/B and F/B.
4. Plug the approximate eigenvalues and eigenvectors into 12.6.

Application without Simplifications

Of course, if you wish to solve the system exactly without making the approximations associated with the $Cs \ll C_d$ condition, then this is completely possible as well, albeit with slightly uglier algebra.

You will eventually find that the first normal mode with the eigenvector $\vec{v}_1 = (1, 0)^T$ is actually more like $\vec{v}_1 = (1, \epsilon)^T$, where ϵ is pretty small – the deep oceans do warm on the short timescale, although very slowly compared to the atmosphere!

$$T_j = \sum_i \left([v_j]_i \left(S_i(0) e^{\lambda_i t} + e^{\lambda_i t} \int_0^t g_i(t') e^{-\lambda_i t'} \, dt' \right) \right) \tag{12.8}$$

where $g_i(t) = [\hat{E}^{-1}]_{i1}$.

Note though that, either way, it's not as simple as just plugging the eigenvalues and eigenvectors into 12.6. If there is any forcing (i.e., if $\vec{f} \neq 0$) we would also need to find $g_i(t)$, which means we need to invert the matrix of eigenvalues $\hat{E}$ to find $\hat{E}^{-1}$. This is fine on a computer, but in an exam this is incredibly time consuming (but it is possible!), so I'd recommend reading up on this next unofficial method.

12.3 Solving the System by Separating Timescales

Here we present an alternative method, and have Ian Hewitt from the Maths department to thank for this way of thinking.

This method is both less general than the former method, as it requires the assumption that $C_s \ll C_d$, and more general than the former method, as it does *not* require the ODEs to be linear. We present this method because we believe first that this method provides more physical intuition than the previous method, and second that this method is far more time-efficient than the former method, and that most equations presented in this course will be simple enough for this method to save a lot of time, especially in the exam if you have to sketch the general behaviour.

However I should emphasise that this is an '**unofficial**' method. I cannot guarantee that the marker will understand what you are doing if you do this in the exam.

As already mentioned, we must exploit the separation of timescales here. Here, T_s evolves much faster than T_d, which allows us to make two approximations.[4]

The System on Short Timescales

The first approximation we make is valid on *short* time-scales. The terms on the right-hand side of equation 12.2 and 12.3 are of both around the same size, therefore $C_s \dot{T}_s$ and $C_d \dot{T}_d$ are around the same size. However, since $C_d \gg C_s$, we require that $\dot{T}_d \ll \dot{T}_s$. Are first approximation thus involves letting $\dot{T}_d \approx 0$, and we integrate this to find that $T_d(t) = 0$ for all times. Physically, this represents the fact that T_d does not respond on very short timescales due to the massive heat capacity C_d. Equations 12.2 and 12.3 thus reduce to:

$$C_s \dot{T}_s = F_{ext}(t) - (\lambda + \gamma + \lambda')T_s \tag{12.9}$$

$$C_d \dot{T}_d = 0 \tag{12.10}$$

on short timescales. We can integrate Equation 12.9 using an integrating factor, with the general solution in Section 12.4. The timescale is set by comparing the T_s term to the $\dot{T}_s$ term: $\tau_s \sim \frac{C_s}{\lambda + \lambda' + \gamma} = 1/r_1$. Thus on an $\mathcal{O}(\tau_s)$ timescale, T_s equilibrates and T_d remains constant (at 0). We have derived our first eigenvalue $(1/\tau_s)$ and eigenvector $((1,0)^T)$ here without diagonalising anything!

The System on Long Timescales

The second approximation we make is valid on *long* time-scales. Here we consider when T_d is of the same size as T_s. For this to be the case, the terms on the right-hand side of equation 12.3 must be massive (because C_d) is massive, and so the terms on the right-hand side of equation 12.2 must be massive, and so $C_s \dot{T}_s$ must be massive as well. However, this is impossible, since $C_s \dot{T}_s \ll C_d \dot{T}_d$, therefore, the only way equation 12.2 can be satisfied is if the terms on the right-hand side balance each other. Therefore, we assume that $C_s \dot{T}_s = 0$ on these timescales.

Another way of thinking about this is as follows. We anticipate that under these conditions (where the terms on the right-hand side of Equation 12.2 are massive), $\dot{T}_s$ will initially be very large, and so T_s will evolve rapidly into equilibrium, thus ensuring that $C_s \dot{T}_s = 0$ most of the time. On this timescale, we say that T_s is **quasi-steady**: T_s evolves rapidly into equilibrium with T_d, and so we set $\dot{T}_s = 0$ in Equation 12.2. Equations 12.2 and 12.3 thus reduce to:

$$0 = F_{ext}(t) - \lambda T_s - \gamma(T_s - T_d) - \lambda'(T_s - T_d) \tag{12.11}$$

$$C_d \dot{T}_d = -\gamma(T_d - T_s) \tag{12.12}$$

on long timescales. We now solve for $T_s(t)$ in terms of $T_d(t)$ and substitute $T_s(t)$ into Equation 12.3. Solving Equation 12.11 gives the following quasi-equilibrium expression for $T_s(t)$:

$$T_s(t) = \frac{1}{\lambda + \gamma + \lambda'} \left(F_{ext}(t) + (\gamma + \lambda')T_d(t) \right)$$

---

[4]A full rigorous treatment requires non-dimensionalising the equations, but we believe this is ultimately not needed for physical intuition in this case.

Comparing the ratios of T_s to T_d given by this expression gives us the same eigenvector $\left(\left(1, \frac{\lambda+\gamma+\lambda'}{\gamma+\lambda'}\right)^T \right)$ we derived from before. Substituting this into 12.12 gives the following ODE governing T_d:

$$C_d \dot{T}_d = \frac{\gamma}{\lambda+\gamma+\lambda'} F_{ext}(t) - \frac{\gamma\lambda}{\lambda+\gamma+\lambda'} T_d \tag{12.13}$$

on long timescales. The timescale is again set again by comparing the T_d term to the $\dot{T}_d$ term: $\tau_d \sim \frac{C_d(\lambda+\gamma+\lambda')}{\gamma\lambda} = 1/r_2$. Again, we have derived our second eigenvalue $(1/\tau_d)$ and eigenvector here without diagonalising anything!

We can now say, in retrospect, that the short timescale approximation (Eqns. 12.9 and 12.10) is valid on timescales much shorter than τ_d, that the long timescale approximation (Eqns. 12.11 and 12.12) is valid on timescales much longer than τ_s, and that we require $\tau_d \gg \tau_s$. This is satisfied if and only if $\frac{C_d(\lambda+\gamma+\lambda')}{\gamma\lambda} \gg \frac{C_s}{\lambda+\lambda'+\gamma}$, which is indeed satisfied if $\gamma, \lambda, \lambda'$ are of similar size (which they are) and $C_d \gg C_s$, as originally assumed.

Note the nuance in these two approximations, where we confusingly seem to set the time derivatives to 0 in both cases. In the first case, we set $\dot{T}_d = 0$ because the timescales are too *short*, and T_d essentially does not react on this timescale. In the second case, we set $\dot{T}_s = 0$ because the timescales are too *long*. The timescales are so long that we assume that $\dot{T}_s$ rapidly equilibrates such that at each time, it is always in equilibrium, and thus $\dot{T}_s = 0$

Key Idea V.2: Timescale Approximations

Suppose we have a coupled set of ODEs governing the time-evolution of $x(t)$ and $y(t)$ given by:

$$A\dot{x} = f(x, y)$$
$$B\dot{y} = g(x, y)$$

such that $B \gg A$ and f and g are of similar size. We further assume that $x(t = 0) = y(t = 0) = 0$. Then we can make two approximations which indicate the different responses:

The **Transient Response**: On **short** timescales, τ_s, we assume that y has no time to respond so that:	The **Quasi-Steady Response**: On **long** timescales, τ_d, we assume that x responds rapidly into equilibrium so that:
$$\dot{y} = 0$$	$$f(x, y) = 0$$
Therefore the transient response is governed by:	Therefore the transient response is governed by:
$$\boxed{A\dot{x} = f(x, 0)} \tag{12.14}$$ $$\boxed{y = 0} \tag{12.15}$$	$$\boxed{0 = f(x, y)} \tag{12.16}$$ $$\boxed{B\dot{y} = g(x(y), y)} \tag{12.17}$$
We find the timescale τ_s by comparing the $\dot{x}$ term to the x term in Equation 12.14.	We find the timescale τ_d by comparing the $\dot{y}$ term to the y terms in Equation 12.17.

12.4 Example Solutions

We can use either of the methods above to solve the ODEs. Let us now specify that $\vec{f}(t) = (F_{ext}(t), 0)^T$ (i.e., that only the atmosphere/surface oceans are forced directly) and assume that $T_s(0) = T_d(0) = 0$. In the solution, we get two different responses for an arbitrary $F_{ext}(t)$:

On short timescales, i.e., the normal mode with the larger eigenvalue and thus shorter timescale (we

define $\tau_s \equiv \frac{C_s}{\lambda + \lambda' + \gamma}$):

$$T_s(t) \approx \frac{1}{C_s} \int_0^t F_{ext}(\hat{t}) e^{-\frac{t+\hat{t}}{\tau_s}} \, d\hat{t} \tag{12.18}$$

$$T_d(t) \approx 0 \tag{12.19}$$

On long timescales, i.e., the normal mode with the smaller eigenvalue and thus longer timescale (we define $\tau_d \equiv \frac{C_d(\lambda + \lambda' + \gamma)}{\gamma \lambda}$):

$$T_s(t) \approx \frac{1}{\lambda + \gamma + \lambda'} \left(F_{ext}(t) + (\gamma + \lambda') T_d(t) \right) \tag{12.20}$$

$$T_d(t) \approx \frac{\gamma}{C_d(\lambda + \gamma + \lambda')} \int_0^t F_{ext}(\hat{t}) e^{-\frac{t+\hat{t}}{\tau_d}} \, d\hat{t} \tag{12.21}$$

Let us now solve this system explicitly for some given $F_{ext}(t)$.

Constant F_{ext}

Suppose $F_{ext}(t)$ is a step-function, representing, for example, an instantaneous emission of CO_2 which remains in the atmosphere indefinitely:

$$F_{ext} = \begin{cases} 0 & \text{if } t < 0 \\ F & \text{if } t > 0 \end{cases}$$

Now clearly, for $t < 0$, $T_s(t) = T_d(t) = 0$. For $t > 0$, we integrate explicitly to find the solutions in each case. On a short timescale:

$$\begin{aligned}
T_s(t) &\approx \frac{1}{C_s} e^{-\frac{t}{\tau_s}} \int_0^t F e^{\frac{\hat{t}}{\tau_s}} \, d\hat{t} \\
&= \frac{1}{C_s} e^{-\frac{t}{\tau_s}} \left(F \tau_s \left(e^{\frac{t}{\tau_s}} - 1 \right) \right) \\
&= \frac{F \tau_s}{C_s} \left(1 - e^{-\frac{t}{\tau_s}} \right) \\
&= \frac{F}{\lambda + \lambda' + \gamma} \left(1 - e^{-\frac{t}{\tau_s}} \right) \\
T_d(t) &\approx 0
\end{aligned}$$

We find then that on an $\mathcal{O}(\tau_s)$ timescale, T_s equilibrates with a timescale of τ_s to its new equilibrium value $\frac{F}{\lambda + \lambda' + \gamma}$ (which I emphasise is completely ignorant of the timescale τ_s or the heat capacity C_s!), and T_d remains constant. Note how, despite the fact that $T_d = 0 = const$, the deep oceans are not 'invisible'. The impact of the deep oceans is encoded in the timescale τ_s and the parameter γ and λ'. The atmosphere/surface-ocean thus equilibrates to a level where the heat flux in (from the increase in $F_{ext}(t)$) equals the heat flux out (from more heat flux into space, and, crucially, more heat flux into the deep oceans).
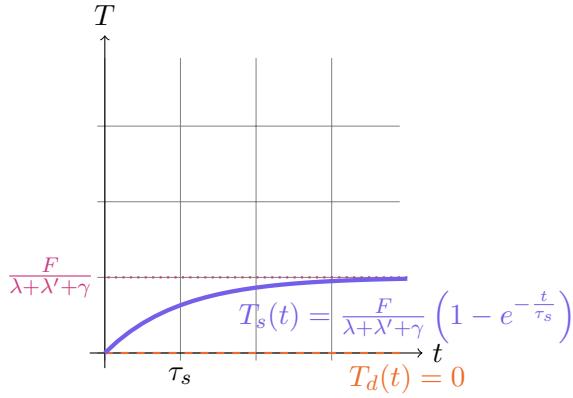
Figure 12.1: The short timescale response of the system subject to constant forcing. The response of T_s and T_d are drawn as a filled blue line ■ and dashed orange line ■, respectively. The dotted magenta line ■ indicates what T_s tends towards on a short timescale.

On long timescales:

$$T_s(t) \approx \frac{1}{\lambda + \gamma + \lambda'} \left(F + (\gamma + \lambda')T_d(t)\right)$$

$$T_d(t) \approx \frac{\gamma}{C_d(\lambda + \gamma + \lambda')} e^{-\frac{t}{\tau_d}} \int_0^t F e^{\frac{\hat{t}}{\tau_d}} d\hat{t}$$

$$= \frac{\gamma}{C_d(\lambda + \gamma + \lambda')} e^{-\frac{t}{\tau_d}} \left(F\tau_d \left(e^{\frac{t}{\tau_d}} - 1\right)\right)$$

$$= \frac{\gamma F \tau_d}{C_d(\lambda + \gamma + \lambda')} \left(1 - e^{-\frac{t}{\tau_d}}\right)$$

$$= \frac{F}{\lambda} \left(1 - e^{-\frac{t}{\tau_d}}\right)$$

Notice how the response of T_d on this long timescale is pretty much exactly analogous to the response of T_s on a short timescale. We find how T_s evolves by substituting our expression for T_d into T_s:

$$T_s(t) = \frac{F}{\lambda} \left(1 - \frac{\gamma + \lambda'}{\lambda + \gamma + \lambda'} e^{-\frac{t}{\tau_d}}\right)$$

We find then that on an $\mathcal{O}(\tau_d)$ timescale, T_d equilibrates with a timescale of τ_d to its new equilibrium value $\frac{F}{\lambda}$. Note that this equilibrium value is completely ignorant to λ' and γ, which represent out-of-equilibrium effects – as expected. This is why we call λ the **Equilibrium Climate Sensitivity Parameter** – it is what the climate tends towards in equilibrium. Meanwhile, T_s also equilibrates towards this $\frac{F}{\lambda}$ value as well, but note that it already has a 'headstart' at time $t = 0$ to get to this equilibrium value. It gets to this point via the short timescale response, where T_d effective doesn't respond fast enough.

$$T_s(t) = \frac{F}{\lambda}\left(1 - \frac{\gamma+\lambda'}{\lambda+\gamma+\lambda'}e^{-\frac{t}{\tau_s}}\right)$$

$$T_d(t) = \frac{F}{\lambda}\left(1 - e^{-\frac{t}{\tau_d}}\right)$$
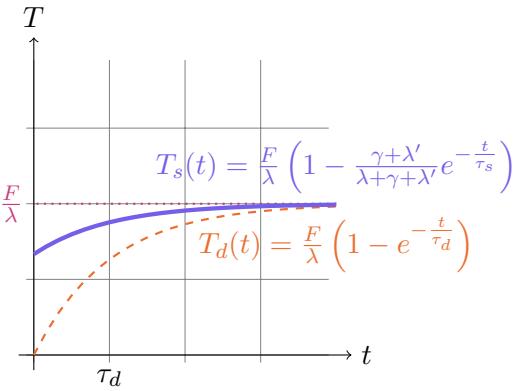
Figure 12.2: The long timescale response of the system subject to constant forcing. The response of T_s and T_d are drawn as a filled blue line ■ and dashed orange line ■, respectively. The dotted magenta line ■ indicates what both T_s and T_d tend toward on a long timescale.

Linearly Increasing F_{ext}

Suppose $F_{ext}(t)$ is a function that is initially 0, but then after time $t = 0$ begins linearly increasing. This might represent, for example, emissions of CO_2 which begin at time $t = 0$:

$$F_{ext} = \begin{cases} 0 & \text{if } t < 0 \\ Gt & \text{if } t > 0 \end{cases}$$

Again, it's clear that, for $t < 0$, $T_s(t) = T_d(t) = 0$. For $t > 0$, we again integrate explicitly to find the solutions in each case. On a short timescale:

$$\begin{aligned}
T_s(t) &\approx \frac{1}{C_s}e^{-\frac{t}{\tau_s}}\int_0^t G\hat{t}\, e^{\frac{\hat{t}}{\tau_s}}\, d\hat{t} \\
&= \frac{1}{C_s}e^{-\frac{t}{\tau_s}}G\left(\left[\hat{t}\tau_s e^{\frac{\hat{t}}{\tau_s}}\right]_0^t - \left[\tau_s^2 e^{\frac{\hat{t}}{\tau_s}}\right]_0^t\right) \\
&= \frac{G\tau_s}{C_s}\left(t - \tau_s + \tau_s e^{-\frac{t}{\tau_s}}\right) \\
&= \frac{G}{\lambda + \lambda' + \gamma}\left(t - \tau_s + \tau_s e^{-\frac{t}{\tau_s}}\right) \\
T_d(t) &\approx 0
\end{aligned}$$

Interpretation is similar to before, with a few twists. On an $\mathcal{O}(\tau_s)$ timescale, T_s equilibrates with a timescale of τ_s to its new equilibrium value, but this time its new equilibrium value is one where it increases linearly. Note again that this equilibrium value is completely ignorant of the timescale τ_s or the heat capacity C_s. As before, T_d remains constant, but its effect is still encoded in the timescale τ_s and the parameters γ and λ'.
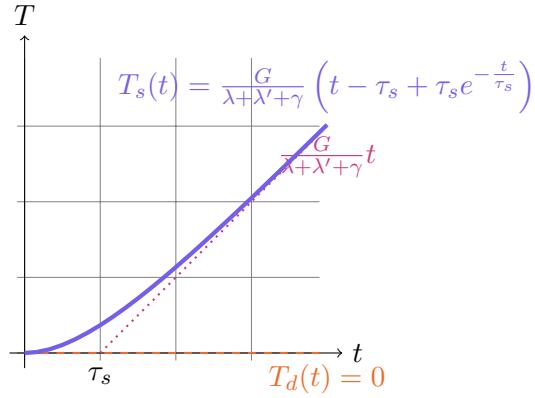
Figure 12.3: The short timescale response of the system subject to linearly increasing forcing. The response of T_s and T_d are drawn as a filled blue line ■ and dashed orange line ■, respectively. The dotted magenta line ■ indicates what T_s tends towards on a short timescale.

On long timescales:

$$T_s(t) \approx \frac{1}{\lambda + \gamma + \lambda'}\left(Gt + (\gamma + \lambda')T_d(t)\right)$$

$$T_d(t) \approx \frac{\gamma}{C_d(\lambda + \gamma + \lambda')}e^{-\frac{t}{\tau_d}}\int_0^t G\hat{t}e^{\frac{\hat{t}}{\tau_d}}\,d\hat{t}$$

$$= \frac{G\gamma}{C_d(\lambda + \gamma + \lambda')}e^{-\frac{t}{\tau_d}}\left(\left[\hat{t}\tau_d e^{\frac{\hat{t}}{\tau_d}}\right]_0^t - \left[\tau_d^2 e^{\frac{\hat{t}}{\tau_d}}\right]_0^t\right)$$

$$= \frac{G\gamma\tau_d}{C_d(\lambda + \gamma + \lambda')}\left(t - \tau_d + \tau_d e^{-\frac{t}{\tau_d}}\right)$$

$$= \frac{G}{\lambda}\left(t - \tau_d + \tau_d e^{-\frac{t}{\tau_d}}\right)$$

T_s is again found by substituting our expression for T_d into T_s:

$$T_s(t) = \frac{G}{\lambda}\left(t - \tau_d + \tau_d\frac{\gamma + \lambda'}{\lambda + \gamma + \lambda'}e^{-\frac{t}{\tau_d}}\right)$$

Analogous to before, on an $\mathcal{O}(\tau_d)$ timescale, T_d equilibrates with a timescale of τ_d to its new 'equilibrium' value $\frac{Gt}{\lambda}$. Note that, again, the 'equilibrium' trend is completely ignorant to λ' and γ, which represent out-of-equilibrium effects. T_d, T_s also equilibrates towards this equilibrium trend as well, and, as before, it already has a 'headstart' at time $t = 0$ to get to this equilibrium value. It gets to this point via the short timescale response, where T_d effective doesn't respond fast enough.
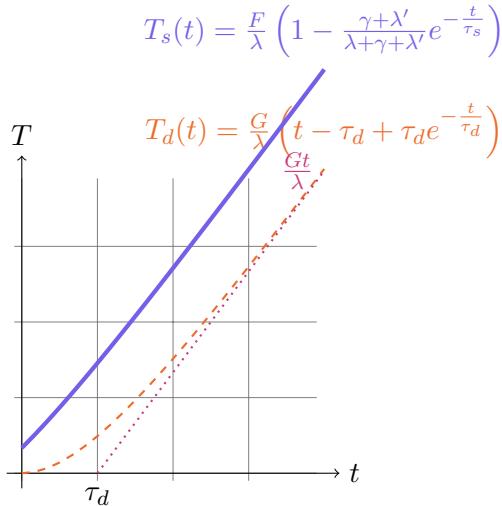
Figure 12.4: The long timescale response of the system subject to constant forcing. The response of T_s and T_d are drawn as a filled blue line ■ and dashed orange line ■, respectively. The dotted magenta line ■ indicates what both T_s and T_d tend toward on a long timescale.

[EXAMPLE SOLUTIONS UNDER CONSTRUCTION]

12.4.1 Rewriting the Solutions in Terms of Emission

So far we have only written the solution in terms of $F_{ext}(t)$, the external radiative forcing. However, we do not directly control radiative forcing – we only control CO_2 emissions! To represent this, let us first

[UNDER CONSTRUCTION]

12.5 Analogous Systems

We can apply this to similar systems. Consider, for example, a carbon budget equation, given by:

$$R_a \dot{C}_a = E(t) - r(C_a - C_b) \tag{12.22}$$

$$R_b \dot{C}_b = r(C_a - C_b) - s(C_b - C_d) \tag{12.23}$$

$$R_d \dot{C}_d = s(C_b - C_d) \tag{12.24}$$

where C_a, C_b, C_d are the effective CO_2 anomalies in the atmosphere, biosphere/surface-ocean, and deep ocean, respectively. The left hand side is the change in CO_2 concentrations, and the right-hand side are carbon fluxes. It's typically the case that $R_d \gg R_b \gg R_a$ due to similar reasons as before, as well as a bit of ocean chemistry. We can use the exact same method as before to define three timescales:

1. A short timescale ($t_a \sim \mathcal{O}\left(\frac{R_a}{r}\right)$): We set $C_b \approx C_d \approx 0$, as C_b, C_d respond very slowly. C_a evolves according to $R_a \dot{C}_a = E(t) - rC_a$.

2. A medium timescale ($t_b \sim \mathcal{O}\left(\frac{R_b}{s}\right)$): We set $C_d \approx 0$ since C_d responds very slowly, and we set $\dot{C}_a \approx 0$ as C_a evolves rapidly into equilibrium with C_b. We then get that $C_a(t)$ evolves quasi-steadily ($C_a(t) \approx C_b(t) + \frac{E(t)}{r}$) and C_b evolves according to $R_b \dot{C}_b = E(t) - sC_b$.

3. A long timescale ($t_d \gg \mathcal{O}\left(\frac{R_b}{s}\right)$): We set $\dot{C}_a \approx \dot{C}_b \approx 0$ since C_a, C_b evolve rapidly into equilibrium with C_d. We then get that $C_a(t)$ and $C_b(t)$ evolve quasi-steadily ($C_a(t) \approx C_b(t) + \frac{E(t)}{r}$ and $C_b(t) \approx C_d(t) + \frac{E(t)}{s}$), and C_d evolves according to $R_d \dot{C}_d = E(t)$. Note that it is difficult to define a timescale here as from our second method – we cannot simply compare C_d to $\dot{C}_d$.

   If we defer to diagonalising the matrix, we find that the eigenvalue is 0, and so the timescale is actually $t_d \sim \infty$! This reflects the fact that our carbon budget system is closed, and once you

pump in carbon (represented by the $E(t)$ term), it has nowhere to go other than to stay in the system. As such, if $E(t) > 0$ for all t, the system will never relax to a steady state, and Ca, C_b, and C_d will all grow without bound.

If you wish to solve what happens in this system, I would recommend doing the returning quasi-steady method, and solving the single ODE governing C_d (assuming that C_a and C_b are quasisteady).

This system, again, may be solved by finding eigenvalues and eigenvectors.

Chapter 13

Predictability

13.1 The Linear Error Propagator

We continue considering a linear dynamical system here, for example, a weather system. Consider a state characterised by some state-vector $\vec{x} \in \mathbb{R}^n$ (perhaps encoding the pressure, temperature, humidity, and velocity at each point in space[1]), whose time-evolution (or perhaps, linearised time-evolution) is given by:

$$\dot{\vec{x}} = \hat{J}\vec{x} \tag{13.1}$$

where $\hat{J} \in \mathbb{R}^n \times \mathbb{R}^n$ is the Jacobian given by:

$$\hat{J} = \begin{pmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} & \cdots \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix} \tag{13.2}$$

Given that $\vec{x}$ has n-dimensions, we require n initial conditions given by $\vec{x}(t = t_0)$ to integrate this ODE. This will typically achieved via data assimilation, but all that needs to be known at this point is that we cannot know the initial conditions to a sufficient precision. That's just life ¯_(ツ)_/¯

The system is also **chaotic**: small initial errors in initial conditions will not remain small final errors, once we've integrated the system. Our aim now is to investigate how errors in the initial conditions propagate and evolve. Suppose that the actual state of the system is given by $\vec{y}(t)$ and our prediction is given by $\vec{x}(t)$. We assume $\vec{y}$ time-evolves by Equation 13.1.[2] We define some initial error or perturbation[3] $\vec{\delta x}(t)$ such that, at all times, the following holds:

$$\vec{y}(t) = \vec{x}(t) + \vec{\delta x}(t) \tag{13.3}$$

At time $t = t_0$ then, our initial error is $\vec{\delta x}(t = t_0)$. We wish to find how this error evolves as we integrate the system forwards.

We know that

$$\vec{y}(t_0 + \delta t) = \vec{y}(t_0) + \int_{t_0}^{t_0 + \delta t} \hat{J}\vec{y}(t')dt'$$

$$= \vec{x}(t_0) + \vec{\delta x}(t_0) + \int_{t_0}^{t_0 + \delta t} \hat{J}(\vec{x}(t') + \vec{\delta x}(t'))dt'$$

$$= \vec{x}(t_0) + \int_{t_0}^{t_0 + \delta t} \hat{J}\vec{x}(t')dt' + \vec{\delta x}(t_0) + \int_{t_0}^{t_0 + \delta t} \hat{J}\vec{\delta x}(t')dt'$$

$$= \vec{x}(t_0 + \delta t) + \vec{\delta x}(t_0 + \delta t)$$

---

[1]In reality, of course, such a state-vector would be infinite-dimensional, as the temperature (for example) is a function of position $T(\vec{r}, t)$, which is continuous. However, all numerical models must discretise space (i.e., divide space into 'boxes'), reducing $\vec{x}$ to a very large but finite dimensional vector.

[2]This amounts to the assumption that our model is perfect, i.e., there is no model error. This is, of course, false, so this method is only applicable to dealing with errors arising from errors in initial conditions.

[3]We use error and perturbation interchangeably in this chapter

where we have used the definition of $\vec{\delta x}(t)$ and the fact that $\hat{J}$ is a linear(ised) operator. Therefore,

$$\vec{\delta x}(t_0 + \delta t) = \vec{\delta x}(t_0) + \int_{t_0}^{t_0 + \delta t} \hat{J} \vec{\delta x}(t') dt'$$
$$\approx \vec{\delta x}(t_0) + \hat{J} \delta t \, \vec{\delta x}(t_0)$$
$$= (\mathbb{I} + \hat{J} \delta t) \, \vec{\delta x}(t_0)$$

Where we have used the fact that δt is small. We define the linear error propagator $\hat{A}$ as:

$$\hat{A} = (\mathbb{I} + \hat{J} \delta t) \tag{13.4}$$

$$\vec{\delta x}(t_0 + \delta t) \approx \hat{A} \, \vec{\delta x}(t_0) \tag{13.5}$$

13.2 Local Perturbation Growth

13.2.1 Criterion for Maximum Perturbation Growth

Suppose we want to find which initial errors will grow the most. This may because we want to run an ensemble weather forecast: instead of integrating from a single initial condition, we will integrate our weather model from multiple initial conditions compatible with the observations. Generally, the final states will be different from each other as they depend on the initial conditions, and the final spread will indicate how predictable the current state is or whether there are any low probability high impact weather events. However, we want to prudently choose which initial conditions we integrate from, given our limited computing resources, which is why we want to find which initial errors will grow the most.

Finding which errors will grow the most amounts to trying to trying to maximise $||\vec{\delta x}(t_0 + \delta t)||$ with respect to $\vec{\delta x}(t_0)$. This can't be the whole story though, since $\vec{\delta x}(t_0 + \delta t)$ is a linear function of $\vec{\delta x}(t_0)$ (Equation 13.5), so we can maximise $\vec{\delta x}(t_0 + \delta t)$ simply by letting $\vec{\delta x}(t_0) \to \infty$. More precisely, then, we want to find the *direction* of $\vec{\delta x}(t_0)$ which maximises $||\vec{\delta x}(t_0 + \delta t)||$. We can do this by introducing a constraint on the magnitude of the initial error (i.e., enforce $||\vec{\delta x}(t_0)|| - C = 0, C \in \mathbb{R}$)[4] and maximising $||\vec{\delta x}(t_0 + \delta t)||$ using Lagrange multipliers. So we introduce the Lagrange multiplier λ and maximise our Lagrangian $\mathscr{L}$.

$$\mathscr{L} = \underbrace{||\vec{\delta x}(t_0 + \delta t)||^2}_{\text{function to be maximised}} - \lambda \underbrace{(||\vec{\delta x}(t_0)||^2 - C^2)}_{\text{constraint}} \tag{13.6}$$

Substituting in Equation 13.5 and differentiating with respect to $\vec{\delta x}(t_0)$, we find that:

$$\frac{\partial \mathscr{L}}{\partial \vec{\delta x}(t_0)} = 2(\hat{A}^T \hat{A} \vec{\delta x}(t_0) - \lambda \vec{\delta x}(t_0))$$
$$= 0$$

if and only if:

$$\hat{A}^T \hat{A} \, \vec{\delta x}(t_0) = \lambda \, \vec{\delta x}(t_0) \tag{13.7}$$

This is a stationary point only if the initial error $\vec{\delta x}(t_0)$ is an eigenvector of $\hat{A}^T \hat{A}$. However, not all stationary points are maximisers (e.g., it could be a minimiser or a saddle point). It turns out that the final error is maximised only if the initial error is the *largest* eigenvector of $\hat{A}^T \hat{A}$[5]. Note that $\hat{A}^T \hat{A}$

---

[4]Myles sets $C = 1$, but C can actually just be any number.

[5]This is deduced by considering the Hessian: the matrix of second partial derivatives. It can be shown, analogous to the 1D case, that the stationary point is a maximiser only if the Hessian is negative semidefinite (i.e., each eigenvalue ≤ 0). In this case, the hessian is the matrix $\hat{A}^T \hat{A} - \lambda \mathbb{I}$, which is clearly only negative semidefinite if $\lambda \geq$ the largest eigenvalue of $\hat{A}^T \hat{A}$. However, λ itself is an eigenvalue (so it cannot be larger than the largest eigenvalue), so it must be equal to the largest eigenvalue!

is symmetric, so (by a theorem in linear algebra) eigenvectors will be orthogonal and eigenvalues will be real.

13.2.2 Generic Perturbation Growth

We now have a criterion for which error(s) will grow the most. Now we turn to a more general analysis of how these perturbations, including the perturbation that grows the most, evolve. To do this, we aim to sort these initial perturbations by the direction they point. We do this by simply finding the eigenvalues and eigenvectors of $\hat{A}^T\hat{A}$. You'll find that one (or more) of these eigenvectors will maximise the error following Equation 13.7. In an exam, you might be given the explicit form of $\hat{A}^T\hat{A}$, and you will have to find the eigenvalues and eigenvectors which will maximise (or minimise or anywhere in between) the final error (following the method outlined here).

However, we'll do this in a slightly roundabout way. I promise this will be worth it. First, we introduce a concept called **Singular Value Decomposition** (SVD). An SVD is a decomposition of any matrix as follows:

$$\hat{A} = \hat{P}\,\hat{Q}\,\hat{R}^T$$

where $\hat{P}$, $\hat{R} \in \mathbb{R}^n \times \mathbb{R}^n$ are orthonormal matrices and $\hat{Q} \in \mathbb{R}^n \times \mathbb{R}^n$ is a diagonal matrix. A matrix $\hat{P}$ is orthonormal if and only if it satisfies the following relation: $\hat{P}\hat{P}^T = \mathbb{I}$. All you need to know is that an SVD exists for any matrix, so it certainly exists for $\hat{A}$. Calculating $\hat{A}^T\hat{A}$ gives us:

$$\hat{A}^T\hat{A} = \hat{R}\,\hat{Q}^T\,\hat{P}^T\,\hat{P}\,\hat{Q}\,\hat{R}^T$$
$$= \hat{R}\,\hat{Q}^2\,\hat{R}^T$$
$$\therefore \hat{R}^T\left(\hat{A}^T\hat{A}\right)\hat{R} = \hat{Q}^2$$

But $\hat{Q}$ is already diagonal! So actually, if we find the SVD of the matrix $\hat{A}$ – more specifically if we find $\hat{Q}$ and $\hat{R}$, we find the eigenvalues and eigenvectors for $\hat{A}^T\hat{A}$.

'Growth' of Perturbations: Local Lyapunov Exponents

Suppose now that $\vec{\delta x}_i(t_0)$ is the ith[6] eigenvector of $\hat{A}^T\hat{A}$ (i.e., it is the ith column of $\hat{R}$) with the eigenvalue q_i^2, where q_i is the ith entry on the diagonal in $\hat{Q}$ (i.e., $\hat{Q} = diag(\ldots, q_{i-1}, q_i, q_{i+1}, \ldots)$). After some finite time, the eigenvector will have evolved to

'Rotation' of Perturbations

However, it is important to note that, in general, eigenvectors of $\hat{A}^T\hat{A}$ are not also eigenvectors of $\hat{A}$. Therefore, these initial perturbations will not only stretch (at a rate set by the Local Lyapunov Exponent), but they will also rotate. So far, we have only characterised how quickly errors will grow (again, via the Local Lyapunov Exponent), but we have not yet characterised how the errors will propagate into other errors.

To do this, we must refer to the SVD of $\hat{A}$. We can write it as follows:

$$\hat{P}^T\,\hat{A}\,\hat{R} = \hat{Q}$$
$$\therefore \hat{A}\,\hat{R} = \hat{Q}\hat{P}$$

where we have used the fact that $\hat{R}$ and $\hat{P}$ are both orthonormal and that $\hat{Q}$ is diagonal.

Recall that the ith column of $\hat{R}$ is parallel to our initial perturbation. This is mapped to the ith column of $\hat{P}$ via the following rule (the following rule is simply matrix multiplication)

$$[\hat{P}]_{ji} = \sum_k [\hat{A}]_{jk}[\hat{P}]_{ki}$$
$$= \sum_k [\hat{A}]_{jk}[\vec{\delta x}_i(t_0)]_k$$

---

[6]i is just an arbitrarily label here. There is no objective 'order' to the eigenvalues/eigenvectors. I'm saying this just for clarity.

The linear error propagator therefore maps the initial perturbation $\vec{\delta x_i}(t_0)$ to a column of $\hat{P}$. We can then examine the difference between the columns of $\hat{P}$ and columns of $\hat{Q}$ to find the way in which initial perturbations rotate. If the columns are identical (i.e., $\hat{P} = \hat{Q}$), then the initial perturbations do not rotate and only grow (or shrink).

Let us now consider $\hat{A}^T \hat{A}$ and $\hat{A} \hat{A}^T$:

$$\hat{A}^T \hat{A} = (\mathbb{I} + \hat{J}\,\delta t)^T (\mathbb{I} + \hat{J}\,\delta t)$$
$$= \mathbb{I} + (\hat{J} + \hat{J}^T)\delta t + \mathcal{O}\left(\delta t^2\right)$$
$$\hat{A}\,\hat{A}^T = (\mathbb{I} + \hat{J}\,\delta t)(\mathbb{I} + \hat{J}\,\delta t)^T$$
$$= \mathbb{I} + (\hat{J} + \hat{J}^T)\delta t + \mathcal{O}\left(\delta t^2\right)$$

So actually, if δt is small, then to first order in δt $\hat{A}^T \hat{A} = \hat{A}\,\hat{A}^T$. We call matrices that satisfy this property 'normal' matrices, and they have the unique property that, under an SVD, $\hat{P} = \hat{Q}$.

In other words, then, to first order in δt, errors do not rotate, and simply grow or shrink. The rotation of errors is a second order effect in δt, and so we must consider perturbation growth over a finite time to investigate this.

13.3   Finite-Time Perturbation Growth

[UNDER CONSTRUCTION]

Chapter 14

Estimation

14.1 The Pseudo-Inverse $\hat{K}$

Suppose we take n measurements, which we encode in some matrix $\vec{y} \in \mathbb{R}^m$, with some random error $\vec{\epsilon} \in \mathbb{R}^m$, in order to estimate some values $\vec{x} \in \mathbb{R}^n$ where $n < m$. The measurements $\vec{y}$ are some (known) function of $\vec{x}$ like so: $\vec{y} = f(\vec{x}) + \vec{\epsilon}$ where $f : \mathbb{R}^n \to \mathbb{R}^m$.

In some cases, f is linear, and we can represent $f(\vec{x}) = \hat{H}\vec{x}$ where $\hat{H} \in \mathbb{R}^m \times \mathbb{R}^n$ (i.e., $\hat{H}$ is an $m \times n$ matrix). We know $\vec{y}$, but we wish to estimate $\vec{x}$. Since $m < n$, the system is overdetermined: we cannot simply invert $\hat{H}$ and solve for $\vec{x}$, which is exacerbated by the error $\vec{\epsilon}$.

Instead, we we apply linear regression, and minimise the squared difference between the actual (error-prone) measurements $\vec{y}$ and the predicted measurements $\hat{\vec{y}} = \hat{H}\vec{x}$, where $\hat{\vec{x}} \in \mathbb{R}^m$ is our estimate of $\vec{x}$. (We denote our prediction/estimates by hats and the real values without hats).

One can differentiate $||\vec{y} - \hat{\vec{y}}||^2 = ||\vec{y} - \hat{H}\hat{\vec{x}}||^2$ with respect to $\hat{\vec{x}}$ (since everything is smooth) and churn through some algebra to obtain the least-squares estimator.

$$\hat{\vec{x}} = \hat{K}\vec{y} \tag{14.1}$$

$$= (\hat{H}^T\hat{H})^{-1}\hat{H}^T\vec{y} \tag{14.2}$$

where the pseudo-inverse $\hat{K}$ of $\hat{H}$ is:

$$\boxed{\hat{K} = (\hat{H}^T\hat{H})^{-1}\hat{H}^T} \tag{14.3}$$

We call $\hat{K}$ the pseudo-inverse of $\hat{H}$ because $\hat{K}\hat{H} = \mathbb{I}$ (but note that generally $\hat{H}\hat{K} \neq \mathbb{I}$). In practice in exams, you will probably have to do this quickly on your calculator. The strategy is as follows:

> **Key Idea V.3: Estimator Recipe**
>
> Suppose you need to estimate some quantities x_i based on some data y_j. Find the **least-squares estimator** in the following way:
>   1. Rearrange the system into a linear system as follows:
>
> $$\vec{y} = \hat{H}\vec{x}$$
>
>   2. Enter the matrix $\hat{H}$ into your calculator and calculate the **pseudo-inverse** $\hat{K} = (\hat{H}^T\hat{H})^{-1}\hat{H}^T$.
>   3. Yeet the $\hat{H}$ onto the observations $\vec{y}$ (which you should be given), i.e., estimate $\vec{x}$ as $\hat{\vec{x}} = \hat{K}\vec{y}_0$.
>   4. Watch the examiners award you The Gibbs Prize for Performance in the MPhys examination (£500).
>   5. Donate those £500 (and maybe the prize as well) to the author(s) of these lecture notes.

14.2   Errors

We assume, for simplicity, that there are negligible errors on the parameters within the operator $\hat{H}$. We assume only that there are some errors $\vec{\epsilon}$ on the observations $\vec{y}_0$. We assume that the errors are given by some well defined $n \times n$ covariance matrix $\hat{S}$:

$$\langle \vec{\epsilon}\vec{\epsilon}^T \rangle = \hat{S}$$

$$[\hat{S}]_{ij} = [\vec{\epsilon}]_i [\vec{\epsilon}]_j$$

We also assume that the average, indicated by brakets $\langle \rangle$, of the noise is 0:

$$\langle \vec{\epsilon} \rangle = 0$$

This allows us to show that, on average:

$$\langle \vec{\hat{x}} \rangle = \langle \hat{K}\,\vec{y} \rangle$$
$$= \langle \hat{K}\,(\hat{H}\,\vec{x} + \vec{\epsilon}) \rangle$$
$$= \langle \vec{x} \rangle$$

$$Var\left(\vec{\hat{x}} \right) = \langle (\vec{x} - \vec{\hat{x}})(\vec{x} - \vec{\hat{x}})^T \rangle$$
$$= \langle (\vec{x} - \hat{K}\,\vec{y})(\vec{x} - \hat{K}\,\vec{y})^T \rangle$$
$$= \langle (\vec{x} - \hat{K}\,(\hat{H}\,\vec{x} + \vec{\epsilon}))(\vec{x} - (\hat{H}\,\vec{x} + \vec{\epsilon}))^T \rangle$$
$$= \langle \hat{K}\,\vec{\epsilon}\vec{\epsilon}^T \hat{K}^T \rangle$$
$$= \hat{K}\hat{S}\hat{K}^T$$

In other words, the mean of our estimation is the real value, and the covariance of our estimation is the covariance of
[UNDER CONSTRUCTION]

14.3   Interpretation

[UNDER CONSTRUCTION]

14.4   Examples

[UNDER CONSTRUCTION]

Part VI

Geophysical Fluid Dynamics

Introduction

This section of the course was lectured by Tim Woollings, covering Basic and Advanced Geophysical Fluid Dynamics (GFD). Basic GFD is covered in HT, while Advanced GFD is covered in Trinity term.

This section consists of six chapters. Basic GFD covers the first four chapters, and Advanced GFD covers the last two:

1. Preliminaries:

   This chapter covers some preliminaries worth knowing prior to . This includes a reminder on Vector Calculus Definitions and Interpretations, Fluid Mechanics, and brief discussions on differential heating and the effects of stratification and rotation.

2. The Equations of Motion:

   To be perfectly honest, you could skip almost all of this section except for the final coloured box should you wish. Here we introduce the primitive equations of motion governing the fluid mechanics of the atmospheres and oceans. We main aim of this chapter is to derive a final simplified set of equations.

3. The Fundamental Diagnostic Relations:

   We introduce some fundamental diagnostic (time-independent) relations, including hydrostatic balance and geostrophic balance, then discuss the implications on (anti-) cyclones and Thermal Wind Balance. We use hydrostatic balance to transform explicitly into pressure coordinates, and express the prior relations in terms of pressure coordinates.

   Even though such relations are diagnostic (i.e., they feature no time-derivatives), these relations might still hold even if the system is changing: in this case, the relations may hold in a quasi-steady manner (a concept introduced in Part V).

4. The Shallow Water System:

   We finally analyse a simple time-dependent system. In doing so, we sacrifice the vertical dimension (i.e., dispense of the vertical coordinate z) and work solely in a 2D system. We use this sytem to introduce energy in GFD, including KE (Kinetic Energy) and APE (Available Potential energy), as well as Potential Vorticity.

   Finally, we show that the (linearised) Shallow Water Systems allows certain wave solutions, including Poincaré, Rossby, and Kelvin Waves, which redistribute energy in the Atmosphere and Oceans.

5. 3D Systems:

   We reintroduce the vertical coordinate z and analyse 3D time-dependent responses in GFD. After a briefly disussing 3D Gravity Waves (the analogue to the Shallow Water Poincaré waves), we derive and interpret Quasi-Geostrophic Theory.

   We then analyse Rossby Waves through this framework, then give a very cursory introduction to Instability and Geostrophic Turbulence.

6. Ocean Circulation:

   The final section in GFD focuses exclusively on the Oceans, although some concepts may be applicable to the atmosphere.

   We first introduce Ekman Transport, and analyse analyse its implications, which include up/downwelling of fluid and Sverdrup Balance, the implication of both Ekman up/downwelling and conservation of potential vorticity.

   Finally, we briefly discuss the Meridional Overturning Circulation, and introduce two simple models which aim to capture the dynamics: the Stommel and Gnanadesikan Box Models.

Chapter 15

Preliminaries

15.1 A Quick Reminder on Vector Calculus

Before we jump into **Geophysical Fluid Dynamics** (**GFD**) proper, we start with a quick reminder of some vector calculus concepts. I'll mainly focus on the interpretation of these concepts, as this will be important for the interpretation of physical results later on.

You should be familiar with all the information in the box below:

Key Idea VI.1: Vector Calculus Definitions

Let $\vec{r} = (x, y, z)^T$ denote the position vector and t denote the time.

We write the **partial derivative** with respect to coordinate x as follows (the subscripts indicate which variables are held constant while differentiating; in this case, coordinates y, z, t are held constant while differentiating with respect to x):

$$\partial_x = \frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x} \right)_{y, z, t} \qquad (15.1)$$

Often we will omit the large brackets and leave implicit which coordinates are being held constant. Analogous for y, z, t.

We define the **nabla** operator $\vec{\nabla}$ as follows (in Cartesian coordinates):

$$\vec{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \qquad (15.2)$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)^T$$

Let $p(\vec{r}, t)$ be an arbitrary scalar field and $\vec{u}(\vec{r}, t) = (u_x, u_y, u_z)^T$ be an arbitrary vector field. We define the **Gradient**, **Curl**, and **Divergence** as follows:

$$\text{Grad } p \equiv \vec{\nabla} p = \begin{pmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{pmatrix} \qquad (15.3)$$

$$\text{Curl } \vec{u} \equiv \vec{\nabla} \times \vec{u} = \begin{pmatrix} \frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \\ \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \\ \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \end{pmatrix} \qquad (15.4)$$

$$\text{Div } \vec{u} \equiv \vec{\nabla} \cdot \vec{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \qquad (15.5)$$

In this box we mentioned a **scalar field** and a **vector field**. A field is just anything that has a value at each point in space and time. A **scalar field** is a field that takes a scalar value at each point.

For example, the pressure field $p = p(\vec{r}, t)$ is a scalar field, which denotes the pressure at each point in space and time. Meanwhile, a **vector field** is a field that takes a vector value at each point. The velocity field $\vec{u} = \vec{u}(\vec{r}, t)$ is a vector field, which denotes the velocity at each point in space and time. While this isn't strictly correct, you can just think of a vector field as three scalar fields stacked on top of each other (one for each component).

Now we will interpret the **Gradient**, the **Curl**, and the **Divergence**.

15.1.1 The Gradient

We can only take the gradient of a **scalar field**, and the gradient is always a **vector field**.

The gradient encodes the local change properties of the **scalar field** with respect to space. More specifically, at each point in space and time, the value of the gradient is a vector with the following properties: the direction (in 3D space) of the vector is in the direction of steepest ascent, and the magnitude of the vector denotes how much the scalar field ascends at that point.

You should be familiar with the one-dimensional Taylor expansions. Consider some function $f : \mathbb{R} \to \mathbb{R}^1$. The following is the Taylor expansion of f, which holds for small δx:

$$f(x + \delta x) = f(x) + \frac{df}{dx} \delta x$$

An analogous version holds in three dimensions as well:

$$p(\vec{r} + \delta \vec{r}, t) = p(\vec{r}, t) + \left(\vec{\nabla} p(\vec{r}, t) \right) \cdot \delta \vec{r}$$
$$= p(\vec{r}, t) + (\text{Grad } p(\vec{r}, t)) \cdot \delta \vec{r}$$

Consider the case where $\delta \vec{r}$ is perpendicular to $\vec{\nabla} p$ (i.e., $\vec{\nabla} p \cdot \delta \vec{r} = 0$). In this direction, there is no local change in p, therefore $p(\vec{r} + \delta \vec{r}, t) = p(\vec{r}, t)$ for all $\delta \vec{r} \perp \vec{\nabla} p$. This implies that contours of constant p are **always** perpendicular to the gradient of p.

15.1.2 The Curl

We can only take the curl of a **vector field**, and the curl is always a **vector field**.

Broadly, the curl measures the local rotation of the **vector field**. More specifically, at each point in space and time, the value of the curl is a vector with the following properties: the direction (in 3D space) of the vector is in the direction perpendicular to the plain of rotation (following the right-hand rule)[2], and the magnitude of the vector denotes how much local rotation there is at this point.

Note crucially that this only measures the **local** rotation, not the global rotation. The example often given to explain this distinction is a paddleboard: consider a very small paddleboard suspended in a fluid with a velocity field $\vec{u}(\vec{r})$. The paddleboard is pushed around by the flow. The curl of $\vec{u}$ encodes whether the paddleboard itself will turn, not whether it will follow a circular path.

(a) Paddleboard rotates (orange arrow ■) ⟷ Curl is non-zero, i.e., $\vec{\nabla} \times \vec{u} \neq 0$

(b) Despite following a circular path, the paddleboard does *not* rotate ⟷ Curl is zero, i.e., $\vec{\nabla} \times \vec{u} = 0$
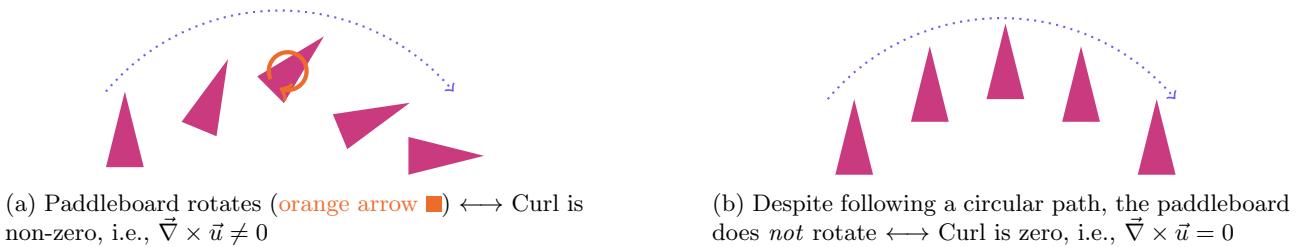
Figure 15.1: Paddleboard (magenta triangle ■) demonstration of curl. The Dotted Blue Arrow ■ indicates the direction of time evolution (the paddleboard is being advected by the velocity field $\vec{u}(\vec{r})$).

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[1]This is fancy speak for the function $f(x)$ takes real numbers as an input and outputs real numbers.

[2]This is just a convention. The right hand rule means that if you form a thumbs up with your right hand, your thumb is parallel with the curl if and only if the rotation is in the direction from your knuckles to the end of your fingers (following your fingers).

15.1.3   The Divergence

We can only take the divergence of a **vector field**, and the divergence is always a **scalar field**.

Broadly, the divergence measures the local 'spreading out' properties of a **vector field**. More specifically, at each point in space and time, the value of the divergence is a scalar that is larger the more

I find the divergence theorem explains divergence the easiest:

$$\iiint_v \vec{\nabla} \cdot \vec{u} \, dV \equiv \iint_s \vec{u} \cdot d\vec{S} \tag{15.6}$$

where v and s denote that we integrate over the volume and surface, respectively, of some closed 3D blob (closed means there are no holes on the blob). Suppose that $\vec{u}$ is some flux per unit area. If, for example, $\vec{u}$ is the velocity, then $\vec{u}$ would be the volume flux per unit area.

The divergence theorem says that the *total flux* out of some blob (equal to the integral of $\vec{u}$ over the surface of the blob, i.e., the right-hand side of 15.6) is equal to the *total divergence* of the blob (equal to the integral of $\vec{\nabla} \cdot \vec{u}$ over the volume of the blob, i.e., the left-hand side of 15.6).

The divergence can therefore be thought of as the outwards flux of $\vec{u}$ at some point, i.e., as encoding how many vectors point towards or away from a point. If the divergence is positive/negative, more/less arrows point outward than inward at that point (respectively). If the divergence is 0, then exactly the same 'amount' of arrows point inwards as outwards.

As a summary:

Key Idea VI.2: Vector Calculus Interpretation

$\vec{\nabla}p$	$\vec{\nabla} \times \vec{u}$	$\vec{\nabla} \cdot \vec{u}$
The **Gradient** $\vec{\nabla}p$ is a vector field which encodes how the scalar field p varies with space. $\vec{\nabla}p$ always points towards the direction of steepest ascent in p, and is perpendicular to contours of constant p.	The **Curl** $\vec{\nabla} \times \vec{u}$ is a vector field which encodes the local rotation properties of the vector field $\vec{u}$. The curl does *not* encode the large-scale rotation properties of $\vec{u}$.	The **Divergence** $\vec{\nabla} \cdot \vec{u}$ is a scalar field which encodes the local 'spreading out' properties of the vector field $\vec{u}$.

15.2   A Quick Reminder of Fluid Mechanics

To aid understanding, we quickly introduce some concepts of fluid mechanics. Tim should have a more comprehensive review on canvas should you need it.

First we introduce the concept of a fluid element. This is an infinitesimal volume of fluid, whose properties may change as it is advected around with the flow. For example, in deriving the Dry Adiabat, we were considering a fluid element of air – except we called it an 'air parcel' then.

Second, we introduce the Material Derivative:

$$\frac{D}{Dt} = \underbrace{\frac{\partial}{\partial t}}_{\text{Time Derivative at a Fixed Point}} + \underbrace{\vec{u} \cdot \vec{\nabla}}_{\text{Advection}} \tag{15.7}$$

The Material Derivative of a quantity is the time-derivative of a quantity *following a fluid element*. This is useful because it is much easier to consider what effects might change a quantity of a fluid element. For example, the forces per unit mass $\vec{F}$ affect a fluid parcel as follows: $\frac{D\vec{u}}{Dt} = \vec{F}$ (this is just Newton's second law).

However, in GFD, we will mostly be interested in how a quantity changes at a **fixed point in space** (indicated by the $\frac{\partial}{\partial t}$ term), *not following a fluid element.* If we rearrange the Equation 15.7 for $\left(\frac{\partial}{\partial t}\right)$, we find that the quantity at some point is modified by both whatever affects the fluid element $\left(\frac{D}{Dt}\right)$, for example the force $\vec{F}$, and the properties of fluid elements which are advected into that location $-\vec{u} \cdot \vec{\nabla}$:

$$\underbrace{\frac{\partial A}{\partial t}}_{\text{Local change in } A} = \underbrace{\frac{DA}{Dt}}_{\text{Change in } A \text{ following a fluid element}} - \underbrace{\vec{u} \cdot \vec{\nabla} A}_{\text{Advection of fluid elements with different } A} \tag{15.8}$$

Hopefully you can see why this physically makes sense. The local change in A is the is comprised of the the change in A following a fluid element, i.e., comprised of what happens exactly at that point, **and** advection, i.e., the effect of fluid elements that low into that point. Advection is the negative dot product of $\vec{u}$ with the gradient of A. Hopefully this makes sense too: the gradient points in the direction of increasing A. If A is increasing in the same direction as $\vec{u}$, then $\vec{u}$ is 'blowing' areas of lower A into areas of higher A, so this should be a negative contribution to the local change in A.

Third, we introduce the idea of Material Conservation. You might have heard about what it means for a quantity to be **conserved**. For example, if you've solved a mechanics problem, you might recall that the total energy was conserved, meaning that $\frac{dE}{dt} = 0$ where $E = $ the total energy.

Material Conservation is a closely related idea, where certain quantities are conserved following a fluid parcel. Mathematically, a quantity A is materially conserved if and only if it obeys the following relation:

$$\frac{DA}{Dt} = 0 \tag{15.9}$$

If Equation 15.9 obtains for some quantity A, this is equivalent to demanding that, if one follows a fluid parcel, the quantity A does not change. However, as a whole, A, need not be globally conserved. If one considers Equation 15.8 and sets $\frac{DA}{Dt} = 0$, then you can see that the only local change in A is from fluid parcels with different A being advected into that location.

Exercise: refer forwards to Equation 16.3. Note even if we assume that $Q = 0$, we will find that $\frac{DT}{Dt} \neq 0$. In other words, temperature is not materially conserved (so long as $\beta \neq 0$). This is why we introduced the idea of Potential Temperature, as it is the case that $\frac{D\theta}{Dt} = 0$ (excercise: take 16.3, solve for β using the ideal gas law, set $Q = 0$, and show that $\frac{D\theta}{Dt} = 0$ using the product rule).

Finally, we define the vorticity $\vec{\omega}$ as the curl of the velocity:

$$\vec{\omega} = \vec{\nabla} \times \vec{u} \tag{15.10}$$

If you refer back to the box on Vector Calculus Interpretation, this should remind you that the curl of the velocity is the **local** rotation of the velocity field, not the **global** rotation. We will usually be considering the vertical component of the vorticity $[\vec{\omega}]_z$, which we will often write just as ξ.

15.3   Why There Is Motion in the First Place

It's important to consider why there is motion in Atmospheres and Ocean in the first place.

Regarding the first point, if you are familiar with the Navier Stokes Equations governing fluid flow, then you will know that $\vec{u} = \vec{\nabla}p = 0$ is a solution, and is a solution a system will tend towards if there is friction and forcing[3], and planets certainly have friction. We therefore need a non-uniform energy source. In this case, it is the Sun, and it drives motion because it provides energy to the Earth unevenly. On small scales, this may be because of variations in reflectivity (albedo), but on large scales this is mostly due to the spherical geometry of the Earth: more solar radiation is absorbed at the equator than at the poles (think: **zenith angle** from Radiative Transfer, which goes as $\cos\zeta$).

This differential heating causes horizontal temperature gradients, which in turn cause horizontal density and pressure gradients, which drive motion. Another way to think of it is as follows: in

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[3]This forcing can be encoded in the boundary conditions, for example.

equilibrium, the Earth absorbs more solar radiation in the equator than in the poles (see Figure 15.2). As such, there must be an energy transport mechanism from the equator to the poles, and this is what drives fluid motion.[4]
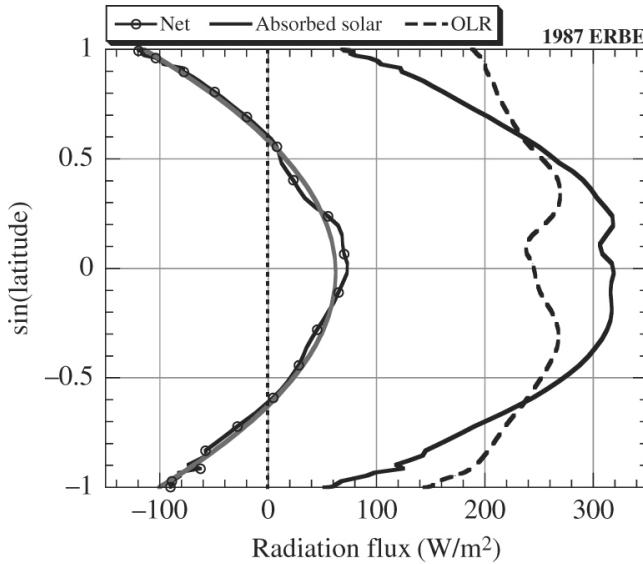

Figure 15.2: Figure from Ray's book showing differential heating. More heat is

15.4 What is Special About 'Geophysical' Fluid Mechanics?

It is also important to consider what sets **Geophysical Fluid Dynamics** apart from most other Fluid Dynamics. There are two important properties that **Geophysical Fluid Dynamics** aim to investigate: **stratification** and **rotation**. The effects of these will pop up again and again throughout GFD.

15.4.1 Stratification

Stratification is important because the fluids in the atmospheres and oceans are affected by gravity and vary in density.

You have already seen the effect of stratification when we derived the Hydrostatic Balance. We found that pressure decreases with height

$$\frac{dp}{dz} = -\rho g$$

$$\frac{d}{dz}(\rho R T) = -\rho g$$

$$R T \frac{d\rho}{dz} +$$

REFER TO THE BUOYANCY FREQUENCY SECTION IN THE CLOUDS SECTION [UNDER CONSTRUCTION]

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[4]I'm personally somewhat ambivalent about this explanation, but I'm not very confident so be sceptical of what I'm about to say. Here's why I'm uncomfortable about this explanation: I cannot see why it must be the case that the variation in OLR cannot be identical to the variation in absorbed solar radiation. To put it another way, I cannot see why the dotted curve cannot be identical to the solid curve in Figure 15.2. It feels like putting the cart before the horse: surely the fluid dynamics also itself affects the OLR into space.

I prefer the following explanation. Assume that differential heating did not obtain, and that the poles and equator were in perfect equilibrium, such that the absorbed solar at each point exaclty equalled the OLR. Because the absorbed solar is more at the equator than the poles, that means the OLR must be more at the equator than the poles. Loosely following the properties of emission (recall this information on emission) suggests that the poles must be colder than the poles.

This implies that there must be horizontal temperature gradients, and where there are temperature gradients there are density gradients. These horizontal density gradients generate instability, because a system with the minimum amount of gravitational potential energy will only have *vertical* density gradients: hence motion.

15.4.2 Rotation

Rotation is important because, generally, planets rotate, and the equations of motion ensure that fluid parcels must obey conservation of angular momentum. The idea is as follows: if a fluid parcel is at rest relative to the surface of the Earth, its total angular momentum will actually differ depending on where it is. If it is at the poles, it will have no angular momentum, whereas if it is at the equator, it will have a large amount of angular momentum.

In the previous section, we discussed how differential heating implies that there must be some motion. This motion is required to close the energy budget. Becuase there both is net radiative heating at the equator and net radiative cooling at the poles, and the equator and poles are not heating and cooling over time, there must be some mechanism transfering energy from the equator to the poles. This is achieved by the atmospheres and oceans.

Let us then consider a extremely simple model of this: **Hadley's Model** of convection as a zonally (zonally means east-west) symmetric equator-to-pole convection cell:
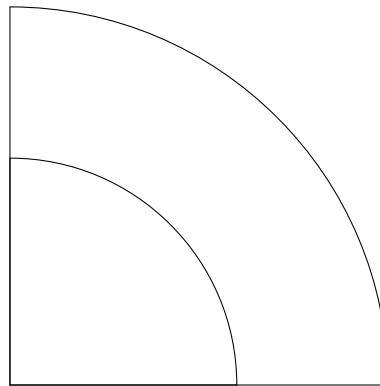

Figure 15.3: Convection Cell: [FIGURE UNDER CONSTRUCTION]

Let us not consider a fluid parcel in this convection cell. We assume that this fluid parcel starts out at the equator at rest (relative to the surface of the Earth) and moves north. We also assume that there are no zonal forces (i.e., no east-west forces) acting on the fluid parcel. This is a highly unrealistic assumption. We further assume that the radius of the Earth a is much larger than the height of the fluid parcel above the Earth.

We know that the total angular momentum of the parcel is approximately equal to the following:

$$L(\phi, u) = \underbrace{a \cos \phi}_{\vec{r}} \underbrace{(\Omega a \cos \phi + u)}_{\vec{u}}$$

where a = the radius of the Earth, ϕ = the latitude, Ω = the angular velocity of the Earth, and u = the zonal velocity of the fluid parcel relative to the surface of the Earth. The angular momentum is equal to $\vec{r} \times \vec{u}$, the radius $\vec{r}$ times the velocity $\vec{u}$. The radius is equal to $a \cos \phi$ (as it is the radius from the axis of rotation), and the velocity is equal to $\Omega a \cos \phi + u$, the sum of the velocity from the rotation of the Earth and the velocity relative to the Earth.[5]

Since there are no zonal forces on the air parcel, this implies that the angular momentum is conserved. Setting $L(\phi, u) = const$ imposes the constraint $u = u(\phi)$. We use our assumption that, at $\phi = 0$ (the equator), the $u = 0$ (fluid parcel is at rest). Therefore, some simple algebra gives us the following relation:

$$\boxed{u = a\Omega \left(\frac{1}{\cos \phi} - \cos \phi \right)}$$

---

[5]This is, strictly speaking, incorrect. The angular momentum is a vector (actually a pseudo-vector), so it's strange to say that the angular momentum is equal to a scalar value. To be more precise, $L(\phi, u)$ is the component of the angular momentum in the direction aligned with the angular velocity of the Earth. Equivalently, this is the component of the angular momentum parallel to the vector pointing directly upwards from the ground at the north pole.

Essentially, what this says is that, as the air parcel moves north (as ϕ increases), the zonal velocity u also increases! Conservation of angular velocity imposes that the air parcel is deflected as it moves northwards, so fluid parcel motion cannot always be as straightforward on a rotating planet as on a non-rotating planet.

Of course, this model is highly unrealistic. Firstly, it predicts that $u \to \infty$ at $\cos\phi \to \pi/2$, which is clearly unphysical. Secondly, from observations, we pretty much never observe $u \geq 150\,\mathrm{m\,s^{-1}}$, so the model fails much earlier than $u \to \infty$. In reality, our assumption that there were no zonal forces was a very inaccurate assumption: as horizontal velocities become arbitrarily large, these induce large shears (a shear is when a layer of stuff (e.g., air) is sliding over another layer of stuff (e.g., air)), which in turn induce large forces either from viscous or turbulent eddy stresses.[6]

However, the general moral of the story does apply: there are conservation laws that constrain motion more on rotating planets than non-rotating planets. We find that this will actually be the Material Conservation of Potential Vorticity, but this is jumping ahead a bit.

Looking back, this should be seen as as cursory, highly unrealistic taste of the effect that stratification and rotation might have on dynamics. It won't be much of an exaggeration ot say that we'll spend most of GFD discussing the effects of rotation.

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[6]For a cursory look at the effect of vertical stresses, refer to this section on Ekman Transport!

Chapter 16

The Equations of Motion

16.1 The Primitive Equations of Motion

For simplicity we ignore state variables like salinity, humidity, etc., and restrict our attention to a system fully characterised by 6-state variables: the velocity $\vec{u} = (u, v, w)$, pressure p, density ρ, and temperature T. These are governed by six equations: 3 momentum budget equations (16.1), one mass budget equation (16.2), one energy budget equation (16.3), and an equation of state (16.4). Each term is labelled below:

$$\underbrace{\frac{D\vec{u}}{Dt}}_{\text{Acceleration and Advection}} = \overbrace{-\frac{1}{\rho}\vec{\nabla}p}^{\text{Pressure Gradients}} \underbrace{-g\vec{k}}_{\text{Gravity}} + \overbrace{\nu\vec{\nabla}^2\vec{u}}^{\text{Viscous Dissipation}} \underbrace{+\frac{\nu}{3}\vec{\nabla}(\vec{\nabla}\cdot\vec{u})}_{} + \overbrace{\vec{F}}^{\text{Other Forces}} \tag{16.1}$$

$$\underbrace{\frac{D\rho}{Dt}}_{\text{Change in Mass of Fluid Parcel}} + \overbrace{\rho\vec{\nabla}\cdot\vec{u}}^{\text{Convergence/Divergence of Mass}} = 0 \tag{16.2}$$

$$\underbrace{c_p\frac{DT}{Dt}}_{\text{Energy of Air Parcel}} - \overbrace{\frac{\beta T}{\rho}\frac{Dp}{Dt}}^{\text{Heating by Compression}} = \underbrace{Q}_{\text{Diabatic Heating (e.g., by radiation)}} \tag{16.3}$$

$$\rho = \underbrace{\rho(T, p, \dots)}_{\text{Equation of State}} \tag{16.4}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}$ is the material derivative; g = gravitational acceleration; $\vec{k}$ = unit vector towards the centre of the Earth; ν = kinematic viscocity; $\vec{F}$ = other forces per unit mass (e.g., friction); c_p = heat capacity per unit mass; $\beta = -\frac{1}{\rho}\frac{\partial\rho}{\partial T}$ thermal expansion coefficient; and Q = heating per unit mass.

However, the equations currently are not fit for purpose. First, these equations are written for a coordinate system that is *inertial* (non-accelerating) and *cartesian*. However, the surface of the Earth is curved and the Earth is rotating[1], and we wish to describe what is going on *here* with us. Furthermore, it would be quite demanding (computationally and conceptually) to adopt an inertial non-curved coordinate system, as then you'd have to, for example, keep track of the fact that the mountains keep moving.

Second, these equations are currently too complicated to be analytically tractable. We will make various approximations later to simplify the equations of motion, however we should note that many of the terms we neglect cannot be neglected in a weather forecast or climate model.

[1]or so NASA and the lizard overlords would have you think!

16.2 Simplifications

16.2.1 Rotating Coordinate Systems

As already mentioned, the momentum budget equation applies for an *inertial coordinate system*, i.e., a coordinate system which is not accelerating. However we wish to describe the dynamics in a rotating coordinate system which rotates with the Earth (or planet). Our goal now is to find a relation between the time derivative of some arbitrary vector $\vec{A}$ in an inertial coordinate system $\left(\frac{d\vec{A}}{dt}\right)_I$ and the time derivative in a rotating coordinate system $\left(\frac{d\vec{A}}{dt}\right)$.

We consider two coordinate systems: an *inertial* coordinate system and a *rotating* coordinate system. Both systems share an origin, but the *inertial* coordinate system rotates with a constant angular velocity $\vec{\Omega}$ where $|\vec{\Omega}|$ is the angular speed (in $\mathrm{rad\,s^{-1}}$) and $\vec{\Omega}$ points along the axis of rotation.

Now consider some arbitrary vector $\vec{A}(t)$ in or rotating coordinate system. The time derivative in the inertial coordinate system $\left(\frac{d\vec{A}}{dt}\right)_I$ is related to the time derivative in the rotating coordinate system $\left(\frac{d\vec{A}}{dt}\right)_R$ as follows (see Figure 16.1):

(a) Change in $\vec{A}$ in the Rotating Coordinate System

(b) Change in $\vec{A}$ in the Inertial Coordinate System
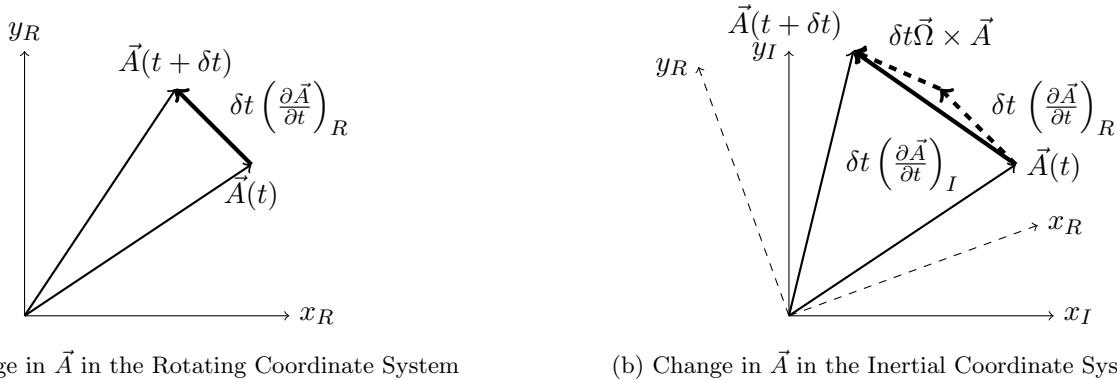
Figure 16.1: The change in $\vec{A}$ in a rotating (16.1a) and inertial (16.1b) coordinate system. As seen in 16.1b, the change in the inertial coordinate system $\delta t \left(\frac{\partial \vec{A}}{\partial t}\right)_I$ (thick arrow) is equal to sum of the change in the rotating coordinate system $\left(\delta t \left(\frac{\partial \vec{A}}{\partial t}\right)_R\right)$ *and* the rotation of the rotating coordinate system $\left(\delta t\, \vec{\Omega} \times \vec{A}\right)$ (thick dashed arrows).

$$\left(\frac{d\vec{A}}{dt}\right)_I = \left(\frac{d\vec{A}}{dt}\right)_R + \vec{\Omega} \times \vec{A} \tag{16.5}$$

If we let $\vec{A} = \vec{r}$, where $\vec{r}$ is the position of our fluid parcel, we can apply Equation 16.5 twice to

obtain an expression for the acceleration of a fluid parcel in a rotating coordinate system.

$$\vec{u}_I = \left(\frac{d\vec{r}}{dt}\right)_I$$
$$= \left(\frac{d\vec{r}}{dt}\right)_R + \vec{\Omega} \times \vec{r}$$
$$\vec{u}_I = \vec{u}_R + \vec{\Omega} \times \vec{r}$$
$$\therefore \vec{a}_I = \left(\frac{d\vec{u}_I}{dt}\right)_I$$
$$= \left(\frac{d\vec{u}_I}{dt}\right)_R + \vec{\Omega} \times \vec{u}_I$$
$$= \left(\frac{d\vec{u}_R}{dt}\right)_R + \vec{\Omega} \times \left(\frac{d\vec{r}}{dt}\right)_R + \vec{\Omega} \times \left(\vec{u}_R + \vec{\Omega} \times \vec{r}\right)$$
$$= \vec{a}_R + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \vec{\Omega} \times \vec{r}$$

Therefore the acceleration in the rotating frame of reference $\vec{a}$ is equal to the acceleration in a inertial coordinate system summed with two fictitious[*] forces: the **centrifugal force** and the **coriolis force**.

$$\vec{a}_R = \vec{a}_I + \underbrace{2\vec{\Omega} \times \vec{u}_R}_{\text{Coriolis}} + \underbrace{\vec{\Omega} \times \vec{\Omega} \times \vec{r}}_{\text{Centrifugal}} \qquad (16.6)$$

The **centrifugal force** is the force that pushes objects away from the axis of rotation. This is the force that makes your arms fling out if you spin. The **coriolis force** is the force that diverts moving objects which move away or towards the axis of rotation.

We have thus transformed Equation 16.1 into a rotating coordinate system:

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} + \frac{1}{\rho}\vec{\nabla}p + g\vec{k} = -\vec{\Omega} \times \vec{\Omega} \times \vec{r} + \nu\vec{\nabla}^2\vec{u} + \frac{\nu}{3}\vec{\nabla}(\vec{\nabla} \cdot \vec{u}) + \vec{F} \qquad (16.7)$$

16.2.2   Local Cartesian Coordinates

We have just learnt how to deal with coordinate systems that vary with time. We will now skim over (since the details aren't that interesting) how to deal with coordinate systems that vary with space.

We wish to use a coordinate system that follows the surface of the Earth, with the basis vectors $\hat{x}$, $\hat{y}$, $\hat{z}$, pointing in the east, north, and upwards (away from the centre of the Earth) directions. In this coordinate system, we define the position and velocity coordinates (x, y, z) and (u, v, w) as indicating distance and velocity in the east, north, and upwards directions, respectively. We further define the latitude $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ ($\phi = 0$ indicates the equator, $\phi = \pm\frac{\pi}{2}$ is the north/south pole) and the radial distance r where $r = a + z$ where $a =$ the radius of the Earth (see Figure 16.2).
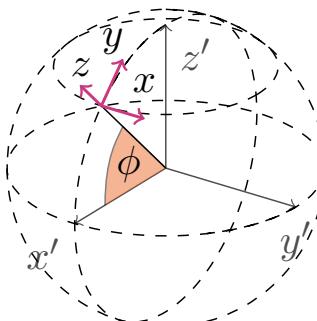

Figure 16.2: Local Carteisan Coordinates. (x', y', z') indicate our global cartesian coordinate system, while (x, y, z) indicate our local cartesian coordinate system in magenta arrows.

Churning through the algebra, we get:

$$\frac{Du}{Dt} - \left(2\Omega + \frac{u}{r\cos\phi}\right)(v\sin\phi - w\cos\phi) + \frac{1}{\rho}\frac{\partial p}{\partial x} = F_x$$

$$\frac{Dv}{Dt} + \frac{wv}{r} + \left(2\Omega + \frac{u}{r\cos\phi}\right)u\sin\phi + \frac{1}{\rho}\frac{\partial p}{\partial y} = F_y$$

$$\frac{Dw}{Dt} - \frac{u^2 + v^2}{r} - 2\Omega u\cos\phi + \frac{1}{\rho}\frac{\partial p}{\partial z} + g = F_z$$

where we have lumped together all the terms on the right-hand-side of 16.7 into the F_x, F_y, F_z. Many of the extra terms $\left(\text{e.g., the } \frac{wv}{r} \text{ term}\right)$, are due to the fact that we have adopted a local cartesian system, in which the coordinate system changes as we move over the Earth.

16.2.3 Incompressibility

Our final assumption is regarding the compressibility of fluids. If a fluid has constant density ($\rho = \rho_0 = const$), then by Equation 16.2 $\vec{\nabla} \cdot \vec{u} = 0$ applies. We say that this fluid is **incompressible**: every bit of fluid that enters some volume must also leave.

We will end up assuming that $\vec{\nabla} \cdot \vec{u} = 0$. Note that this should seem, to you, a **highly** dubious assumption. First, we know that sound waves are possible (after all we can hear, both on land and in water), but sound waves are impossible in an incompressible fluid. Second, and more importantly, we know that the atmosphere is approximately an ideal gas, which can have its density changed if warmed or cooled. For the ocean, we know that temperature, salinity, and pressure all change the density of seawater.

However, we are justified to use this equation if we assume that density variations $\delta p = \delta p(x, y, z, t)$ are much smaller than the background density ρ_0.

$$\therefore \boxed{\vec{\nabla} \cdot \vec{u} = 0} \tag{16.8}$$

However, I should emphasise that this does *not* imply that $\frac{D\rho}{Dt} = \frac{D\delta\rho}{Dt} = 0$ (but it also doesn't prohibit it).

16.2.4 Scale Analysis

Finally, we make a few phenomenological simplifications. We do this here for simplicity, but our results are not general or particularily robust: these terms are essential in climate/weather models. We do this here by performing a **scale analysis**: we estimate the size of quantities by their typical values (found empirically), then ignore terms that are much smaller than the other terms. We never neglect the pressure gradient term.

Scale	Symbol	Terms Approximated	Typical Magnitude
Horizontal Scale	L	$\frac{\partial}{\partial x} \sim \frac{1}{L}$, $\frac{\partial}{\partial y} \sim \frac{1}{L}$	$10^6\,\mathrm{m}$
Vertical Scale	H	$\frac{\partial}{\partial z} \sim \frac{1}{H}$	$10^4\,\mathrm{m}$
Horizontal Velocity	U	$u \sim U$, $v \sim U$	$10\,\mathrm{m\,s^{-1}}$
Vertical Velocity	W	$w \sim W$	$10^{-2}\,\mathrm{m\,s^{-1}}$
Time Scale	T	$\frac{\partial}{\partial t} \sim \frac{1}{T}$	$10^5\,\mathrm{s}$
Density	ρ	$\rho \sim \rho$	$1\,\mathrm{kg\,s^{-1}}$
Earth's Radius	a	$r \sim a$	$6.4 \times 10^6\,\mathrm{m}$
Rotation Rate	Ω	$\Omega \sim \Omega$	$10^{-4}\,\mathrm{s^{-1}}$
Acceleration of Gravity	g	$g \sim g$	$10\,\mathrm{m\,s^{-2}}$

I do not write out all the details here, as they would easily take multiple pages. If you are curious about the details, refer to the mini-scale analysis I do in deriving 17.1. A more rigorous method would be non-dimensionalisation, but this is overkill for our purposes. Again, if you are curious about the details, refer to the mini-non-dimensionalisation I do in deriving the Rossby Number.

Suffice it to say that we eliminate many terms through this scale analysis. This includes any terms that feature the viscosity ν and many terms arising from our local cartesian coordinate system. The centrifugal force is *not* eliminated universally by our scale analysis, but we neglect this term for a different reason.[2]

After eliminating the small terms, we arrive at our final simplified set of equations.

16.3   The Simplified Equations of Motion

> **Key Idea VI.3: The Simplified Equations of Motion for GFD**
>
> We define the horizontal gradient operator as $\vec{\nabla}_h = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}, 0 \right)^T$, the horizontal velocity as $\vec{u}_h = (u, v, 0)^T$, the horizontal forces as $\vec{F}_h$, and the **coriolis parameter** as $f = 2\Omega \sin \phi$ where $\phi = $ the latitude.
>
> The simplified equations of motion we will be using are as follows:
>
> **Vertical Momentum Balance:**
>
> $$\frac{Dw}{Dt} + \frac{1}{\rho} \frac{\partial p}{\partial z} + g = F_z \qquad (16.9)$$
>
> **Incompressibility:**
>
> $$\vec{\nabla} \cdot \vec{u} = 0 \qquad (16.10)$$
>
> **Horizontal Momentum Balance:**
>
> $$\underbrace{\frac{D\vec{u}_h}{Dt}}_{\text{acceleration and advection}} + \overbrace{f\vec{k} \times \vec{u}_h}^{\text{coriolis}} + \underbrace{\frac{1}{\rho}\vec{\nabla}_h p}_{\text{horizontal pressure gradients}} = \overbrace{\vec{F}_h}^{\text{other forces}} \qquad (16.11)$$

Physically, the **coriolis parameter** $f = 2\Omega \sin \phi$ is the planetary vorticity: it is the vorticity that a fluid parcel at rest (relative to the surface of the Earth) has in the z-direction. Note that we've ignored the Energy Equation (16.3). This equation is important, and it is used to derive an approximate mass conservation equation later in section , but the details are slightly convoluted and we don't really discuss it in C5.

We will mostly be dealing with the horizontal momentum equation 16.11, so it is worth detailing the interpretation of each terms. The first term, $\frac{D\vec{u}_h}{Dt}$ is the material derivative of $\vec{u}_h$. It represents the acceleration and advection of $\vec{u}_h$. The $f\vec{k} \times \vec{u}_h$ term is the acceleration due to coriolis. The $\frac{1}{\rho}\vec{\nabla}_h p$ is

---

[2]The reason why we neglect the centrifugal force is for two reasons.

First, the centrifugal force is a conservative force, i.e., we can write it as the gradient of a scalar field: i.e., there exists a ϕ such that $-\vec{\Omega} \times \vec{\Omega} \times \vec{r} = -\vec{\nabla}\phi$. Therefore, we can simply define an 'effective gravitational potential' $\Phi = \phi + \psi$, where ψ is the gravitational potential energy field.

The second reason is a bit more subtle.

First, consider the shape of the Earth (or any planet). Any non-rotating sufficiently-large planet (which has had (sufficient) time to come into equilibrium) is a sphere. The reason why is because it is *this* configuration which minimises the gravitational potential energy, and it minimises the gravitational potential energy because the surface of the planet aligns with the surfaces/contours of constant gravitational potential energy ψ. In other words, the planet is spherical because surfaces of constant ψ are also spherical.

For a rotating planet, the surfaces of constant potential energy are now deformed (the total potential $\Phi = \phi + \psi =$ the sum of the gravitational and the centrifugal potential energies). Now, the contours of constant Φ are oblate spheroids, and so the planet's shape tends towards this.

Recalling (from the Vector Calculus Intepretation Box) that the gradient of a scalar field Φ is perpendicular to the contours of constant Φ, *and* recalling that the force experienced by an object in this field is this gradient $\vec{F} = -\vec{\nabla}\Phi$, it turns out that the force experienced by an object will always be perpendicular to the surface of the planet. Therefore, the combined gravitational and centrifugal forces still points 'downwards' at each point on the surface, similar to a non-rotating spherical planet.

This is the second reason why we neglect the centrifugal force. Because the combined centrifugal and gravitational force always points downwards, it is phenomenologically very similar to a situation with only the gravitational force. As such, we choose to neglect the centrifugal force for simplicity.

the force due to the horizontal pressure gradient. Finally, the $\vec{F}_h$ term is external forces, e.g., friction, wind-forcing, etc.

Chapter 17

The Fundamental Diagnostic Relations

17.1 Hydrostatic Balance (Approximate Vertical Momentum Balance)

We now make further approximations to extract physical intuition. Consider 16.9, and again perform a scale analysis and assume that $F_z \approx 0$.

$$\frac{Dw}{Dt} + \qquad\qquad \frac{1}{\rho}\frac{\partial p}{\partial z} + g = 0$$

$$\underbrace{\frac{\partial w}{\partial t}}_{\frac{W}{T}} + \qquad \underbrace{u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y}}_{U\frac{W}{L}} + \underbrace{w\frac{\partial w}{\partial z}}_{W\frac{W}{H}} + \qquad \frac{1}{\rho}\frac{\partial p}{\partial z} + g = 0$$

$$\sim \frac{W}{T} + \qquad\qquad \sim U\frac{W}{L} + \sim \frac{W^2}{H} + \qquad \frac{1}{\rho}\frac{\partial p}{\partial z} + g = 0$$

$$\therefore \frac{10^{-2}\,\mathrm{m\,s^{-1}}}{10^5\,\mathrm{s}} + \qquad 10\,\mathrm{m\,s^{-1}}\frac{10^{-2}\,\mathrm{m\,s^{-1}}}{10^6\,\mathrm{m}} + \frac{\left(10^{-2}\,\mathrm{m\,s^{-1}}\right)^2}{10^4\,\mathrm{m}} + \qquad \frac{1}{\rho}\frac{\partial p}{\partial z} + 10\,\mathrm{m\,s^{-2}} = 0$$

$$10^{-7}\,\mathrm{m\,s^{-2}} + \qquad\qquad 10^{-7}\,\mathrm{m\,s^{-2}} + 10^{-8}\,\mathrm{m\,s^{-2}} + \qquad \frac{1}{\rho}\frac{\partial p}{\partial z} + 10\,\mathrm{m\,s^{-2}} = 0$$

$$\cancel{10^{-7}\,\mathrm{m\,s^{-2}}} + \qquad\qquad \cancel{10^{-7}\,\mathrm{m\,s^{-2}}} + \cancel{10^{-8}\,\mathrm{m\,s^{-2}}} + \qquad \frac{1}{\rho}\frac{\partial p}{\partial z} + 10\,\mathrm{m\,s^{-2}} = 0$$

As seen then, many of the acceleration terms (the terms in the $\frac{Dw}{Dt}$) are 8 orders of magnitutde smaller than gravity! As such, the only term that can balance the gravity term is the pressure gradient. We thus neglect all other terms and derive again:

Key Idea VI.4: Hydrostatic Balance

Hydrostatic Balance governs the vertical pressure variation in a fluid if we assume that vertical acceleration (and advection) is small.

$$\frac{\partial p}{\partial z} = -\rho g \qquad (17.1)$$

Pressure decreases with height in order to balance the force of gravity.

Physically, hydrostatic balance is a force/momentum balance in the vertical direction, where the force of gravity is balanced by the vertical pressure gradient. Notice now the difference between the Hydrostatic Balance I have written here (17.1) and the Hydrostatic Balance I have written in the Thermodynamics Part (1.12): I have written it here with a partial derivative. Strictly, it should have been a partial derivative before too, but I write it here to note that we are now considering horizontal variations as well as vertical variations.

For an ideal gas, one can substitute for ρ using 1.4:

$$\boxed{\frac{\partial \ln p}{\partial z} = -\frac{g}{RT}}$$

(17.2)

17.2 Geostrophic Balance (Approximate Horizontal Momentum Balance)

17.2.1 Rossby Number and Non-Dimensionalisation

Let us now consider the horizontal momentum equation 16.11 and let $\vec{F}_h \approx 0$ for now. We nondimensionalise the equations by defining dimensionless hatted variables and choosing characteristic scales such that the dimensionless hatted variables are of order 1:

$$t = [t]\hat{t} \; ; \; \vec{r} = L\hat{\vec{r}} \; ; \; \vec{u}_h = [u_h]\hat{\vec{u}}_h \; ; \; p = [p]\hat{p}$$

So, for example, $[t] =$ the timescale with dimensions of time and $L =$ the lengthscale with dimesions of length, while $\hat{t}$ and $\hat{\vec{r}}$ is the dimensionless time/position variables. We *must* pick these scales $[t]$, L, $[u_h]$, $[p]$ *such that* the hatted variables $(\hat{t}, \hat{\vec{r}}, \hat{\vec{u}}_h, \hat{p})$ are of order one, so we're constrained by the system under consideration.[1] We now substitute these variables into 16.11:

$$\frac{[u_h]}{[t]}\frac{\partial \hat{\vec{u}}_h}{\partial t} + \frac{[u_h]^2}{L}\left(\hat{\vec{u}}_h \cdot \hat{\vec{\nabla}}\right)\hat{\vec{u}}_h + [u_h]f\vec{k} \times \hat{\vec{u}}_h + [p]\frac{1}{\rho}\vec{\nabla}_h\hat{p} = 0$$

where $\hat{\vec{\nabla}} = (\partial/\partial\hat{x}, \partial/\partial\hat{y})^T$ is the non-dimesional nabla operator. Dividing by $f[u_h]$, we get:

$$\boxed{\frac{1}{f[t]}}\frac{\partial \hat{\vec{u}}_h}{\partial t} + \boxed{\frac{[u_h]}{fL}}\left(\hat{\vec{u}}_h \cdot \hat{\vec{\nabla}}\right)\hat{\vec{u}}_h + \vec{k} \times \hat{\vec{u}}_h + \boxed{\frac{[p]}{\rho f[u_h]}}\vec{\nabla}_h\hat{p} = 0$$

$$\underbrace{\phantom{\frac{1}{f[t]}}}_{\equiv St} \quad \underbrace{\phantom{\frac{[u_h]}{fL}}}_{\equiv Ro} \quad \underbrace{\phantom{\frac{[p]}{\rho f[u_h]}}}_{\equiv P}$$

We thus see three non-dimesional coefficients appear in front of each of the terms in each equation: St, Ro, and P.[2] These non-dimesional coefficients compare the relative size and therefore importance each term in the Horizontal Momentum equation. You might expect there to be four non-dimesional coefficients because there were four scales we could choose, but there are only three because this fourth non-dimensional coefficient is set by the Horizontal Momentum equation itself: all the terms on the left hand side *must* sum to 0.

Now, written just in terms of the non-dimensional coefficients, we write the (isotropic) non-dimensional horizontal momentum equation:

$$\underbrace{St\frac{\partial \hat{\vec{u}}_h}{\partial t}}_{\mathcal{O}(St)} + \underbrace{Ro\left(\hat{\vec{u}}_h \cdot \hat{\vec{\nabla}}\right)\hat{\vec{u}}_h}_{\mathcal{O}(Ro)} + \underbrace{\vec{k} \times \hat{\vec{u}}_h}_{\mathcal{O}(1)} + \underbrace{P\vec{\nabla}_h\hat{p}}_{\mathcal{O}(P)} = 0$$

(17.3)

Note that because we have nondimensionalised such that each variable is of order one, the $\vec{k} \times \hat{\vec{u}}_h$ is always of order one, while the other terms are of order St, Ro, and P, respectively.

The physical interpretation of each coefficient is as follows: St represents the ratio of the intrinsic timescale of the system $[t]$ (set by dynamics of the particular problem under consideration) and the timescale of the rotation of the planet $1/f$. This intrinsic timescale could be set, for example, by the stratification (for example the Brunt Vasalla frequency you encountered in Clouds PUT REFERENCE HERE $[t] = 1/N$). Other times, if there is no intrinsic timescale, the timescale is often set by the advection $[t] = L/[u_h]$. If this timescale is much shorter than the rotation timescale (i.e., if $[t] \ll 1/f$)

---

[1]In fact we've already made an implicit assumption that the system is roughly isotropic, which means that we can choose $[v] = [u] = [u_h]$ and $[x] = [y] = L$.

[2]While St and Ro are standard nomenclature for these non-dimesional quantities, P is not.

then $St \gg 1$, so the left-hand most term is much larger than the rotation $\vec{k} \times \vec{u}_h$ term. Physically, this means that the local rate of change 'doesn't see' the fact that the planet is rotating. This is because everything happens so quickly that the planet hasn't rotated much to affect events. This is what happens on human scales: if I throw a ball at you then this takes much less time than $1/f \sim 1$ day, so we don't notice the Earth rotating.

For simplicity, we simply assume at this point that the following obtains: that the timescale is set by advection or something slower than advection such that $[t] \geq L/[u_h]$ (i.e., we are never interested in stuff happening on very time-scales shorter than the advective timescale). As such, we can approximate $St \leq Ro$ so that we get:

$$Ro \left(\frac{\partial \vec{u}_h}{\partial t} + \left(\vec{u}_h \cdot \vec{\nabla} \right) \vec{u}_h \right) + \vec{k} \times \vec{u}_h + P \vec{\nabla}_h \hat{p} = 0$$

We can now explain the **Rossby Number** Ro very easily. Ro compares the effect of advection/acceleration to rotation: if $Ro \ll 1$ (the **Rossby Number** is small), then rotation is important and dominates dynamics. If $Ro \gg 1$ (the **Rossby Number** is large), then the flow evolves in such a way where coriolis is negligible.

Key Idea VI.5: Rossby Number

We define the dimensionless **Rossby Number** Ro, which encodes the relative importance of (planetary) rotation and advection/acceleration terms, as follows:

$$Ro = \frac{U}{fL} \tag{17.4}$$

where U = the horizontal velocity scale, f = the coriolis parameter, and L = the horizontal length scale.
If $Ro \ll 1$, then the rotation of the planet is important, and the flow should be approximately **Geostrophic**. If $Ro \gg 1$, then acceleration/advection is important.

Note that I have not specified P. This is because we expect the pressure scale P to be set by the dynamics governed by the Horizontal Momentum Equation. If $Ro \ll 1$, then we expect P to be of order 1 in order to balance the coriolis $\vec{k} \times \vec{u}_h$ term. If $Ro \gg 1$, then we expect P to be of order Ro in order to balance the advection/acceleration $D\vec{u}_h/Dt$ term.

We can also write the Rossby Number in terms of a ratio between the planetary vorticity and relative vorticity. The planetary vorticity $\xi_{planetary}$ is the angular velocity of a fluid parcel due to the rotation of the Earth, and scales as f. The relative vorticity is the vorticity of our fluid parcel in our coordinate system which is rotating with the Earth, and scales as $\xi_{relative} = [\vec{\omega}]_z = [\vec{\nabla} \times \vec{u}_h]_z \sim U/L$. Therefore:

$$Ro = \frac{\xi_{relative}}{\xi_{planetary}} \tag{17.5}$$

$$\sim \frac{U/L}{f} \tag{17.6}$$

The actual size of Ro depends on the kind of system we are considering: i.e., what are the length-scales and velocity-scales of the system. But it also, crucially, depends on the coriolis parameter $f = 2\Omega \sin \phi$. $f = 0$ at or near the equator, and so it is unlikely that Ro will ever be small near the equator. However, let us be clear. This *does not* imply that rotation in general is unimportant near the equator. However, this does *suggest* that the flow is probably not geostrophic near the equator.

Much of what we will discuss here in GFD will require Ro to be small and for the system to be geostrophic, or approximately geostrophic. This means that what we discuss is really only guaranteed to be accurate in the mid-/high-latitudes (coincidentally where we are!). It is worth bearing this in

mind. However, note that Ro being small is a *sufficient* condition (provided that our other assumptions as sound) – it is not a *necessary* condition. Ro being large does not imply that our theories must fail, or that the flow cannot be geostrophic or approximately so.

17.2.2 Geostrophic Balance

If $Ro \ll 1$ and all other forces are negligible, then rotation dominates dynamics. We can therefore neglect external forces (set $\vec{F}_h \approx 0$) and acceleration and advection (set $D\vec{u}_h/Dt \approx 0$) in Equation 17.3 (the horizontal momentum equation). Because the equation must equal 0, we are forced to conclude that P is of order one, so Equation 17.3 and therefore Equation 16.11 must be a balance between the horizontal pressure gradient force and the coriolis force.[3]

Key Idea VI.6: Geostrophic Balance

A flow is geostrophic if the dominant terms in the horizontal momentum balance equation (Equation 16.11) are the coriolis force and the pressure gradient force. As such, these terms must balance as follows:

$$f\vec{k} \times \vec{u}_h = -\frac{1}{\rho}\vec{\nabla}_h p \tag{17.7}$$

This results in a velocity field determined by the pressure gradient as follows (Exercise: Show that Equation 17.8 satisfies Equation 17.7 for any arbitrary pressure field p if we assume $\vec{\nabla}_h f = 0$):

$$\vec{u}_h = \frac{1}{\rho f}\vec{k} \times \vec{\nabla}_h p \tag{17.8}$$

Written out, each component of $\vec{u}_h = (u, v)^T$ is as follows:

$$u = -\frac{1}{\rho f}\frac{\partial p}{\partial y} \qquad\qquad v = \frac{1}{\rho f}\frac{\partial p}{\partial x}$$

Recall that the horizontal gradient of pressure $\vec{\nabla}_h p$ points in the direction of increasing p. Equation 17.7 shows that the coriolis force, which pushes a fluid at right angles to its velocity, balances the pressure gradient force, which pushes a fluid from areas of high pressure to areas of low pressure.

We can use Equation 1.4 to show that, for an ideal gas, geostrophic balance may be written as follows:

$$\vec{u}_h = \frac{RT}{f}\vec{k} \times \vec{\nabla}_h \ln p \tag{17.9}$$

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[3]You might be suspicious from what I've said. After all, Ro depends on velocity-scale U and length-scale L, which we expect should be set by the dynamics and governed by the underlying equations of motion!

In other words, *if* $Ro \ll 1$, then we expect the system to be in geostrophic balance, but why is $Ro \ll 1$ in the first place, and why would our atmosphere/ocean system tend towards a state such that $Ro \ll 1$ holds? If we set $Ro \ll 1$ *a priori*, doesn't this mean we're putting in the answer by hand? Conversely, if we set $Ro \ll 1$ *a posteriori* (based on observations), then this seems like putting the cart before the horse: again, $Ro \ll 1$ *because* of the underlying equations of motion, not the other way around!

The answer lies in the phenomenon of **geostrophic adjustment**, which you can read about in Vallis' textbook on GFD [6]. The gist is this: anomalies and instabilities on a rotating planet cannot simply propagate away (e.g., via gravity flattening the surface of a fluid, for example) from the source to make the fluid more homogenous. It cannot because the fluid is on a rotating planet, and so must preserve some form of angular momentum.

As such, after anomalies begin propagating away, there will come a point (where the anomaly reaches the Rossby Deformation Scale) where it is deflected by the coriolis force, and the system evolves *towards* geostrophic balance.

More details, again, can be found in Vallis' book.
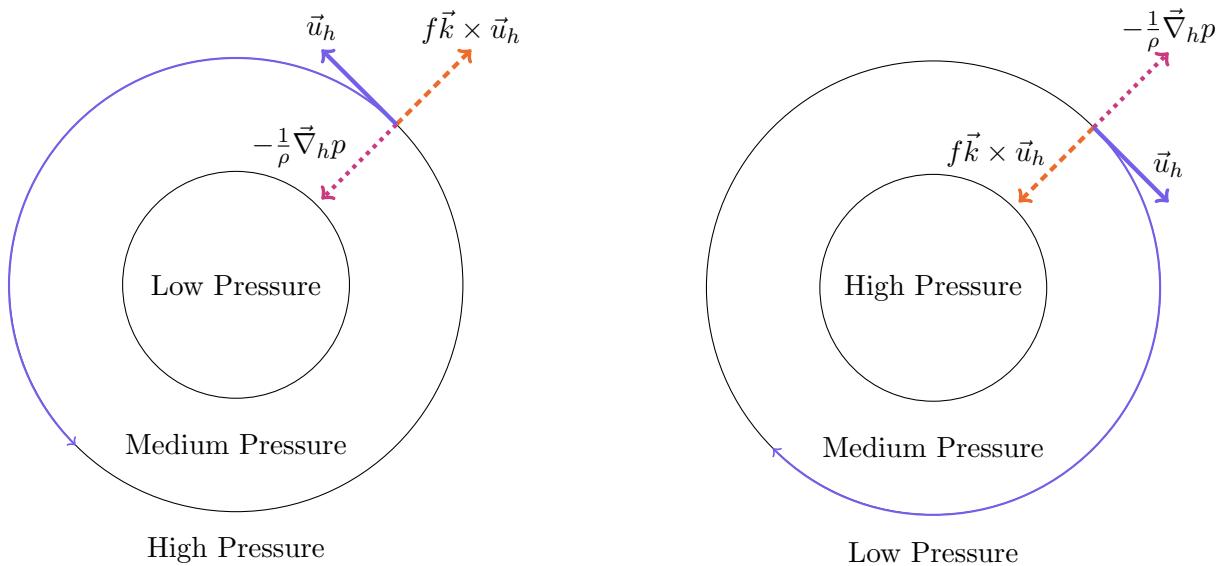
17.2.3 Cyclones and Anti-Cyclones

The horizontal velocity field is thus set by the horizontal pressure field in Equation 17.8.

Recall from our Vector Calculus Interpretation Box that the contours of constant p are always perpendicular to the gradient $\vec{\nabla}_h p$ of the pressure field p, and point from areas of low pressure to areas of high pressure. Recall further that a cross product of two vectors $\vec{a} \times \vec{b}$ is always perpendicular to both $\vec{a}$ and $\vec{b}$.[4]

Therefore, if the flow is geostrophic, then $\vec{u}_h$ flows parallel **isobars** (an isobar is defined as a contour of constant p). It is not difficult to verify this by showing that $\vec{u}_h \cdot \vec{\nabla}_h p = 0$.

If you consider blobs of higher or lower pressure (which I will draw with circles for simplicity), you will see that the horizontal velocity flows either clockwise or anticlockwise around the areas of high or low pressure. Whether it is clockwise or anticlockwise will depend if we are in the northern hemisphere (so that $f = 2\Omega \sin\phi > 0$ obtains) or in the southern hemisphere (so that $f = 2\Omega \sin\phi < 0$ obtains).

(a) Cyclone: a **Low** Pressure System. The fluid flows anti-clockwise around a cyclone in the northern hemisphere.

(b) Anti-Cyclone: a **High** Pressure System. The fluid flows clockwise around an anti-cyclone in the northern hemisphere.

Figure 17.1: Cyclones and Anticyclones in the northern hemisphere ($f > 0$). Isobars are drawn as solid black lines ▪., the pressure gradient force $\left(-\frac{1}{\rho}\vec{\nabla}_h p\right)$ as thick dotted magenta arrows ▪, the coriolis force $\left(f\vec{k} \times \vec{u}_h\right)$ as thick dashed orange arrows ▪, and the velocity ($\vec{u}_h$) as thick filled dark blue arrows ▪.

Now, of course, cyclones and anti-cyclones as I have described them here are theoretical approximations to *actual* cyclones and anti-cyclones. One way in which actual cyclones/anti-cyclones differ is that they are subject to other forces, like friction with the ground or centrifugal forces. Another way in which actual cyclones/anti-cyclones differ is that they are three-dimensional, and we have not considered the vertical velocity in cyclones or anti-cyclones.

However, we do not yet have the vocabulary needed to fully describe such systems. We will have to postpone discussion regarding this for now.

Suffice it to say that cyclones and anti-cyclones are weather systems that propogate across the mid-/high-latitudes. They can be generated if there is energy to cause an area of high or low pressure,

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[4]The way I like to remember how this works is the right-hand rule. Make a cartesian coordinate system with your right hand: first make a thumbs up with your right hand, then point your index finger away from you, then point your middle fingle to your left.

Now consider $\vec{a} \times \vec{b} = \vec{c}$. If you align $\vec{a}$ with your thumb, then align $\vec{b}$ with your index finger, then $\vec{c}$ will be pointing in the direction of your middle finger!

Further note: (I think?) this is why we call certain cartesian coordinate systems a right handed coordinate system, because $\hat{x} \times \hat{y} = \hat{z}$, where $\hat{x}, \hat{y}, \hat{z}$ are the unit vectors pointing in the x, y, and z directions.

and they dissipate due to the effects of surface friction and potential vorticity (a concept we will cover later). Because of how cyclones and anti-cyclones interact with the ground and friction, cyclones result in areas of rising air, while anti-cyclones result in areas of sinking air. Thinking back to convection, this brings unsettled, cloudy, and rainy conditions in the former case.

17.2.4   Gradient-Wind Balance
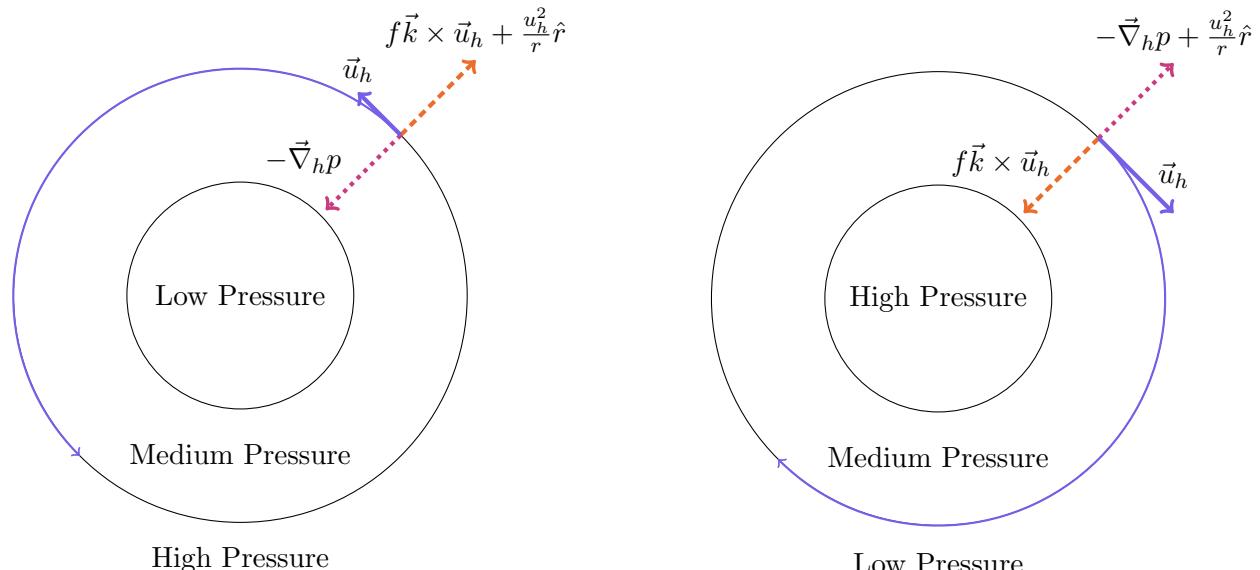
We will now make a brief detour to analyse how the centrifugal force affects cyclones and anti-cyclones.

I should be clear *which* centrifugal force we are considering in this subsection. In this case, we are considering the the centrifugal force on the air parcel arising from the fact that the air parcel is moving in a circle *around the centre of the cyclone or anti-cyclone*, and is therefore accelerating inwards towards the cyclone. This gives rise to a centrifugal force that pushes the air parcel away from the centre of the cyclone (in the frame of reference rotating with the cyclone/anti-cyclone).

We are *not* considering the centrifugal force on the air parcel arising from the fact that air parcel is moving in a circle *around the centre of the Earth* (due to the fact that the Earth is spinning), and is therefore accelerating inwards towards the centre of the Earth. This gives rise to a centrifugal force that pushes the air parcel away from the centre of the Earth (in the frame of reference rotating with the Earth but not the cyclone/anti-cyclone).

Now that we've gotten this potential confusion out the way, consider an air parcel moving in circular motion around a centre of high or low pressure. We adopt a local polar coordinate system, with the coordinates (r, θ) where $x = r\cos\theta$ and $y = r\sin\theta$. We assume there are no forces in the θ-direction and only consider forces in the r-direction.

In this direction, there are three forces on the air parcel. We already considered the first two: the pressure gradient force $\left(-\frac{1}{\rho}\vec{\nabla}_h p = -\frac{1}{\rho}\frac{\partial p}{\partial r}\hat{r}\right)$ and the coriolis force $\left(f\vec{k} \times \vec{u}_h = fu_h\hat{r}\right)$ where $u_h = |\vec{u}_h|$ and $\hat{r}$ is the unit vector pointing in the r-direction. Since the air parcel is moving in circular motion, we know that it is subject to a centrifugal force away from the vortex with magnitude $\frac{u_h^2}{r}\hat{r}$.[5]. For an cyclone and an anti-cyclone in the northern hemisphere, it will look as follows:

(a) Cyclone: a **Low** Pressure System. The fluid flows anti-clockwise around a cyclone in the northern hemisphere.

(b) Anti-Cyclone: a **High** Pressure System. The fluid flows clockwise around an anti-cyclone in the northern hemisphere.

Figure 17.2: Gradient Wind Balance with cyclones and anticyclones in the northern hemisphere ($f > 0$). Isobars are drawn as solid black lines ■., the pressure gradient force $\left(-\frac{1}{\rho}\vec{\nabla}_h p\right)$ as thick dotted magenta arrows ■, the coriolis force $\left(f\vec{k} \times \vec{u}_h\right)$ as thick dashed orange arrows ■, and the velocity ($\vec{u}_h$) as thick filled dark blue arrows ■.

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[5]Equivalently, we can say that the parcel must be accelerating inwards with an acceleration of $-\frac{u_h^2}{r}\hat{r}$

Notice how, for a cyclone, the centrifugal force points in the same direction as the coriolis force. For an anti-cyclone, the centrifugal force points in the opposite driection as the coriolis force (i.e., in the direction of the pressure gradient force).

Therefore, if the pressure gradient force is of a similar size for both a cyclone and anti-cyclone, we should expect the coriolis force to be *smaller* in a cyclone than in an anticyclone. Since the corilis force scales as the wind speed u_h, we expect a cyclone to produce *weaker* winds than an anti-cyclone for the same pressure gradients. Turning this on its head, if a cyclone and an anti-cyclone have similar wind-speeds, we expect the cyclone to have *stronger* pressure gradients.

We can understand this algebraically as well. The force balance in the r-direction is as follows:

$$\boxed{\frac{u_h^2}{r} + fu_h - \frac{1}{\rho}\frac{\partial p}{\partial r} = 0}$$

(17.10)

which is a quadratic equation for the velocity u_h, where $u_h > 0$ if u_h is spiraling in the anti-clockwise direction (in the northern hemisphere).

For a cyclone, $u_h > 0$ For an anti-cyclone, $u_h < 0$

$$\frac{u_h^2}{r} + fu_h = \frac{1}{\rho}\frac{\partial p}{\partial r} \qquad\qquad \frac{u_h^2}{r} - \frac{1}{\rho}\frac{\partial p}{\partial r} = -fu_h$$

If we solve Equation 17.10 directly, we find that u_h is given by the following:

$$u_h = \frac{r}{2}\left(-f \pm \sqrt{f^2 + \frac{4}{\rho r}\frac{\partial p}{\partial r}}\right)$$

So it turns out anticyclones (where $\frac{\partial p}{\partial r} < 0$) really can't form in this simple model if $\left|\frac{1}{\rho}\frac{\partial p}{\partial r}\right| > \frac{rf^2}{4}$, i.e., if the pressure gradient is too large.

17.3   Thermal Wind Balance

Now that we have covered the fundamental diagnostic relations of Hydrostatic Balance and Geostrophic Balance, we now aim to put these two pieces of information together to find a relation between the horizontal velocity $\vec{u}_h$ and the temperature in an ideal gas atmosphere.[6]

We begin with Equations 17.2 and 17.9 expressing respectively hydrostatic and geostrophic balance for an ideal gas. We first take the partial derivative with respect to z of Equation 17.9:

$$\frac{\partial \vec{u}_h}{\partial z} = \frac{\partial}{\partial z}\left(\frac{R}{f}\vec{k}\times\left(T\vec{\nabla}_h\ln p\right)\right) \qquad ; \frac{\partial}{\partial z} \text{ of Eqn. 17.9}$$

$$= \frac{R}{f}\vec{k}\times\left(\frac{\partial T}{\partial z}\vec{\nabla}_h\ln p + T\vec{\nabla}_h\frac{\partial \ln p}{\partial z}\right) \qquad ; \text{Assume } \frac{\partial \bar{M}}{\partial z} = 0 \text{ and that partial derivatives commute}$$

$$= \frac{R}{f}\vec{k}\times\left(\frac{\partial T}{\partial z}\vec{\nabla}_h\ln p - T\vec{\nabla}_h\frac{g}{RT}\right) \qquad ; \text{Substitute Eqn. 17.2 for } \frac{\partial \ln p}{\partial z}$$

$$= \frac{R}{f}\vec{k}\times\left(\frac{\partial T}{\partial z}\vec{\nabla}_h\ln p + \frac{g}{RT}\vec{\nabla}_hT\right) \qquad ; \text{Apply chain rule}$$

$$= \frac{R}{f}\vec{k}\times\left(\frac{\partial T}{\partial z}\frac{f}{RT}\vec{k}\times\vec{u}_h + \frac{g}{RT}\vec{\nabla}_hT\right) \qquad ; \text{Substitute Eqn. 17.7 for } \vec{\nabla}_h\ln p$$

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[6]I should note that when oceanographers refer to the '**thermal wind relation**', they do not require the fluid to follow the ideal gas equation of state. This should be expected, as seawater has a different, horribly complicated equation of state. As such, the '**thermal wind relation**' is often written as a relation between the horizontal velocity $\vec{u}_h$ and the density ρ.

I should also note a further subtlety. Observationally, pressure is very difficult to measure. [UNDER CONSTRUC-TION]

Now we estimate each term in the final brackets in the final equation:

$$\frac{\partial \vec{u}_h}{\partial z} = \frac{1}{fT}\vec{k} \times \left(\underbrace{\overbrace{\frac{\partial T}{\partial z}}^{\sim 1 \times 10^{-2}\,\mathrm{K\,m^{-1}}} \overbrace{f}^{\sim 1 \times 10^{-5}\,\mathrm{s^{-1}}} \vec{k} \times \overbrace{\vec{u}_h}^{\sim 10\,\mathrm{m\,s^{-1}}}}_{\sim 1 \times 10^{-6}\,\mathrm{K\,s^{-2}}} + \underbrace{\overbrace{g}^{\sim 10\,\mathrm{m\,s^{-2}}} \overbrace{\vec{\nabla}_h T}^{\sim 1 \times 10^{-5}\,\mathrm{K\,m^{-1}}}}_{\sim 1 \times 10^{-4}\,\mathrm{K\,s^{-2}}} \right)$$

So we see that even though horizontal temperature gradients $\vec{\nabla}_h T$ are typically 1000 times weaker than vertical temperature gradients $\frac{\partial T}{\partial z}$ (recall that temperature temperature gradients roughly follow the Lapse Rate in Equation 2.4), it is the term that features the horizontal temperature gradient that ends up begin much larger than the term that features the vertical temperature gradient. We would not be able to make this approximation if the planet rotated much faster (f were larger) or if typical horizontal gradients were much smaller ($\vec{\nabla}_h T$ were smaller)[7].

As such, we neglect[8] the term on the left in the equation, and derive the **Thermal Wind Equations**:

$$\boxed{\frac{\partial \vec{u}_h}{\partial z} \approx \frac{g}{fT}\vec{k} \times \vec{\nabla}_h T} \tag{17.11}$$

Written out, each component of $\vec{u}_h = (u, v)^T$ is as follows:

$$\boxed{\frac{\partial u}{\partial z} \approx -\frac{g}{fT}\frac{\partial T}{\partial y}} \tag{17.12}$$

$$\boxed{\frac{\partial v}{\partial z} \approx \frac{g}{fT}\frac{\partial T}{\partial x}} \tag{17.13}$$

The physical interpretation is as follows: if the system is approximately both Geostrophic and Hydrostatic, and obeys ideal gas law, then horizontal temperature gradients imply vertical velocity gradients (and vice versa).

Let us now consider the vertical component of the vorticity. Recall that the voriticity is defined as follows: $\vec{\omega} = \vec{\nabla} \times \vec{u}$. Therefore the vertical component is given by: $[\vec{\omega}]_z = \xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$. If we differentiate Equations 17.12 and 17.13 with respect to y and x (respectively), then subtract them we can find an equation governing ξ:

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) \approx \frac{g}{fT}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$\therefore \boxed{\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{\partial \xi}{\partial z} \approx \frac{g}{fT}\vec{\nabla}_h^2 T} \tag{17.14}$$

Now in the Vector Calculus Interpretation Box we interpreted the meaning of the gradient of a scalar field, but we are yet to interpret the **divergence** of the **gradient** of a scalar field. In this case, we specifically considering $\vec{\nabla}_h^2 T$.

This encodes the *curvature* of the temperature scalar field $T(\vec{r}, t)$. You will find that, if the temperature is constant or if the temperature increases at a constant rate in some direction, that this will be 0. However, if the temperature increases then decreases, or increases more and more, then this will be non-zero.

Therefore, Equation 17.14 essentially says that the vertical gradient of the vorticity ξ is set by the curvature of the temperature field T.

This might make more sense with a concrete example. Suppose that we have a blob of cold air, i.e., we have a temperature field $T(\vec{r})$ that is minimum at the centre of the blob and warmer around the blob. If we consider $\vec{\nabla}_h T(\vec{r})$ (which I remind you is a vector field which points from low T to high

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[7]Although, while f is a parameter set by the planet and not the fluid, we expect $\vec{\nabla}_h T$ to be set by the dynamics governing the fluid, so whether/where this obtains is a bit more compliated.

[8]We do not actually have to neglect this term when we derive these equations in pressure coordinates!

T), we would expect the vector field to look like arrows pointing away from the cold centre. Therefore the divergence of this vector field $\vec{\nabla}_h^2 T$ will be **positive** at this cold blob.

If you wan't to think of this in terms of curvature, since the temperature decreases then increases again when you pass through this cold blob, this implies that the temperature field $T(\vec{r})$ must be curved.

Using Equation 17.14 $\vec{\nabla}_h^2 T > 0$ therefore implies that $\frac{\partial \xi}{\partial z} > 0$ (in the northern hemisphere), therefore the vertical vorticity increases with height in this cold blob! Vice versa for a hot blob in the northern hemisphere: in a hot blob will have vorticity decreasing with height.

17.4 Pressure Coordinates

17.4.1 Partial Derivative Identities

To convert to pressure coordinates we must first introduce some identities from vector calculus involving partial derivatives. We will skip the proof, so you'll just have to take the following identities on trust:

$$\left(\frac{\partial a}{\partial b}\right)_{c,d} = \left(\frac{\partial b}{\partial a}\right)_{c,d}^{-1}$$

$$\left(\frac{\partial a}{\partial b}\right)_c \left(\frac{\partial b}{\partial c}\right)_a \left(\frac{\partial c}{\partial a}\right)_b = -1$$

where we have explicitly written which variables are held constant in the subscripts. We apply this to p and z:

$$\left(\frac{\partial p}{\partial z}\right)_{x,y} = \left(\frac{\partial z}{\partial p}\right)_{x,y}^{-1} \tag{17.15}$$

$$\left(\frac{\partial p}{\partial x}\right)_z \left(\frac{\partial x}{\partial z}\right)_p \left(\frac{\partial z}{\partial p}\right)_x = -1$$

$$\left(\frac{\partial p}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_p \left(\frac{\partial z}{\partial p}\right)_y = -1$$

where the last two equations become:

$$\left(\vec{\nabla}_h p\right)_z = - \left(\vec{\nabla}_h z\right)_p \left(\frac{\partial p}{\partial z}\right)_{x,y} \tag{17.16}$$

We must be careful with which variables we are actually holding constant when transforming from height coordinates to pressure coordinates. Before this section, we have implicitly held z constant when calculating horizontal derivatives like $\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial x}$. In practice, you should memorise Equations 17.15 and 17.16 if you wish to reproduce this in an exam.

We will now apply this to four equations: hydrostatic balance (17.1), geostrophic balance (17.8), thermal wind balance (17.11), and mass conservation (16.2) to derive their equivalent form in pressure coordinates.

17.4.2 Hydrostatic Balance

Starting first from the hydrostatic relation (Equation 17.1), we derive the equivalent relation expressing hydrostatic balance but in pressure coordinates using Equation 17.15:

$$-\rho g = \left(\frac{\partial p}{\partial z}\right)_{x,y}$$

$$= \left(\left(\frac{\partial z}{\partial p}\right)_{x,y}\right)^{-1}$$

$$\therefore \quad \boxed{\left(\frac{\partial \Phi}{\partial p}\right)_{x,y} = -\frac{1}{\rho}} \tag{17.17}$$

where we have defined the **geopotential height** Φ as follows:

$$\boxed{\Phi = g\,z} \tag{17.18}$$

There is nothing special about **geopotential height**. You can literally just think of **geopotential height** Φ as the height variable z multiplied by g.

If we assume that the atmosphere consists of an ideal gas, then the following relation is the analogue for Equation 1.16:

$$\boxed{\frac{\partial \Phi}{\partial \ln p} = -RT} \tag{17.19}$$

17.4.3   Geostrophic Balance

We now turn to geostrophic balance. Again, starting from Equation 17.8 but this time applying Equaiton 17.16:

$$\vec{u}_h = \frac{1}{\rho f}\vec{k}\times\left(\vec{\nabla}p\right)_z$$

$$= \frac{1}{\rho f}\vec{k}\times\left(-\left(\vec{\nabla}_h z\right)_p\left(\frac{\partial p}{\partial z}\right)_{x,y}\right)$$

$$= \frac{1}{\rho f}\vec{k}\times\left(-\left(\vec{\nabla}_h z\right)_p(-\rho g)\right)$$

$$\therefore \quad \boxed{\vec{u}_h = \frac{1}{f}\vec{k}\times\left(\vec{\nabla}_h \Phi\right)_p} \tag{17.20}$$

where we have used hydrostatic balance to substitute for $\frac{\partial p}{\partial z}$. Each component of $\vec{u}_h$ is as follows:

$$\boxed{u = -\frac{1}{f}\left(\frac{\partial \Phi}{\partial y}\right)_p} \qquad\qquad \boxed{v = \frac{1}{f}\left(\frac{\partial \Phi}{\partial x}\right)_p}$$

17.4.4   Thermal Wind Balance

Thermal Wind Balance is particularly elegant in pressure coordinates, as we do not need to make the awkward scale analysis approximation we did in deriving Equation 17.11 in height coordinates. We still need to assume that the system is in hydrostatic and geostrophic balance, and that the system is an ideal gas. To derive thermal wind balance in pressure coordinates, we simply take the derivative of Equation 17.20 with respect to $\ln p$ (and make use of Equation 17.19):

$$\left(\frac{\partial}{\partial \ln p}\right)_{x,y}(\vec{u}_h) = \left(\frac{\partial}{\partial \ln p}\right)_{x,y}\left(\frac{1}{f}\vec{k}\times\left(\vec{\nabla}_h \Phi\right)_p\right)$$

$$= \frac{1}{f}\vec{k}\times\left(\vec{\nabla}_h\left(\frac{\partial \Phi}{\partial \ln p}\right)_{x,y}\right)_p$$

$$= \frac{1}{f}\vec{k}\times\left(\vec{\nabla}_h(-RT)\right)_p$$

$$\therefore \quad \boxed{\left(\frac{\partial \vec{u}_h}{\ln p}\right)_{x,y} = -\frac{R}{f}\vec{k} \times \left(\vec{\nabla}_h T\right)_p} \tag{17.21}$$

Again, each component of $\vec{u}_h$ is as follows:

$$\boxed{\left(\frac{\partial u}{\partial \ln p}\right)_{x,y} = \frac{R}{f}\left(\frac{\partial T}{\partial y}\right)_p}$$

$$\boxed{\left(\frac{\partial v}{\partial \ln p}\right)_{x,y} = -\frac{R}{f}\left(\frac{\partial T}{\partial x}\right)_p}$$

17.4.5 Mass Conservation

[DERIVATION UNDER CONSTRUCTION]

A final implication of converting to pressure coordinates is that we now have a new 'vertical velocity'. The old vertical velocity was defined as follows:

$$w_z = \frac{Dz}{Dt}$$

However, in pressure coordinates, we now define 'vertical velocity' as follows:

$$w_p = \frac{Dp}{Dt}$$

Again, assuming that the system is hydrostatic allows us to approximate w_p the following way[9]:

$$w_p = \left(\underbrace{\frac{\partial}{\partial t} + u\left(\frac{\partial}{\partial x}\right)_z + v\left(\frac{\partial}{\partial y}\right)_z}_{\text{small} \, \therefore \, \text{neglect}} + \underbrace{w_z\frac{\partial}{\partial z}}_{\text{big}}\right)p$$

$$\approx w_z\left(\frac{\partial p}{\partial z}\right)_{x,y}$$

$$= -w_z\rho g$$

Recall Equation 16.2 which states the following (this applies for both incompressible and compressible fluids):

$$\frac{D\rho}{Dt} + \rho\vec{\nabla} \cdot \vec{u} = 0$$

Writing this in pressure coordinates gives us:

$$0 = \left(\frac{\partial}{\partial t} + \vec{u}_h \cdot \left(\vec{\nabla}_h\right)_z + w_z\frac{\partial}{\partial z}\right)\rho + \rho\left(\left(\vec{\nabla}_h\right)_z \cdot \vec{u}_h + \frac{\partial}{\partial z}w_z\right)$$

$$= \left(\frac{\partial}{\partial t} + \vec{u}_h \cdot \left(\vec{\nabla}_h\right)_z - \frac{w_p}{\rho g}\frac{\partial p}{\partial z}\frac{\partial}{\partial p}\right)\rho + \rho\left(\left(\vec{\nabla}_h\right)_z \cdot \vec{u}_h - \frac{\partial p}{\partial z}\frac{\partial}{\partial p}\frac{w_p}{\rho g}\right)$$

$$= \left(\frac{\partial}{\partial t} + \vec{u}_h \cdot \left(\vec{\nabla}_h\right)_z + w_p\frac{\partial}{\partial p}\right)\rho + \rho\left(\left(\vec{\nabla}_h\right)_z \cdot \vec{u}_h + \rho g\frac{\partial}{\partial p}\frac{w_p}{\rho g}\right)$$

Recall Equation 16.2 which states the following (for an incompressible fluid):

$$\frac{D\rho}{Dt} = \left(\frac{\partial}{\partial t} + u\left(\frac{\partial}{\partial x}\right)_z + v\left(\frac{\partial}{\partial y}\right)_z + w_z\frac{\partial}{\partial z}\right)\rho = 0$$

---

[9]Recall that the hydrostatic relation assumes that vertical variations in p are dominant, therefore we can approximate the material derivative of p as just the vertical advection component.

17.4.6 Some Pressure Coordinate Intuition

It's useful to just go over a single example to gain some intuition regarding pressure coordinates and this 'geopotential height' object, and how the height z could enter dynamically into the equations of motion.

Consider a system, like the Earth for example, that has an area of high pressure at

[UNDER CONSTRUCTION]

Chapter 18

The Shallow Water System: 2D System

18.1 The Equations of Motion

In the previous chapter, we considered diagnostic relations: Hydrostatic Balance and Geostrophic Balance. These are very useful and important relations, but notice that there are no time derivatives in 17.1 and 17.8. If there are no time-derivatives, the equations cannot reproduce non-steady state solutions and time-evolution. However, clearly, the atmosphere and ocean evolve and are not in a steady state, and we wish to understand this.

Our first tool to consider time-dependent changes is the shallow water system. We consider a system governed the following equations and looks like this:

Key Idea VI.7: The Shallow Water System

The Shallow Water System is a simple two-dimensional time-varying system governing the evolution of a thin layer of incompressible fluid subject only to the pressure gradient and coriolis force.

The three state variables are the horizontal velocity $\vec{u}_h(x, y, t) = (u, v)^T$ and the layer thickness $h(x, y, t)$. Equivalently, we can express the state variables as the horizontal velocity $\vec{u}_h(x, y, t) = (u, v)^T$ and the layer height $\eta(x, y, t)$.

$$\frac{D_h \vec{u}_h}{Dt} + f\vec{k} \times \vec{u}_h + g\vec{\nabla}_h \eta = 0 \qquad (18.1)$$

$$\frac{\partial h}{\partial t} + \vec{\nabla}_h \cdot (h\vec{u}_h) = 0 \qquad (18.2)$$

$$h = \eta - b \qquad (18.3)$$

where $\frac{D_h}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} =$ the horizontal material derivative, $\eta(x, y, t) =$ the height of the surface interface, and $b(x, y) =$ the bottom interface (which is given).

$$\vec{u}_h = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\rho = \rho_0 = const$$

$z = \eta(x, y, t)$

$h(x, y, t)$

$z = b(x, y)$
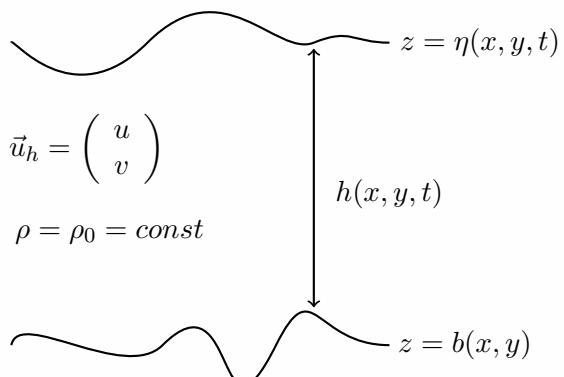
Figure 18.1: The Shallow Water System

We now aim to derive these equations from three primitive (approximate) equations: the horizontal momentum equation (16.11), the vertical momentum equation (16.9), and mass conservation (16.2). First, we neglect all vertical and horizontal forces F_z and $\vec{F}_h$ (e.g. neglect friction) other than gravity and pressure gradient forces.

Second, we assume that the density is constant $\rho = \rho_0$ for all time and at all points. This

immediately implies (from Equation 16.2) that the system must be incompressible, so we know that Equation 16.10 holds.

Third, we assume that the system is **shallow**: $H/L \ll 1$ where H is the vertical lengthscale and L is the horizontal lengthscale. If we non-dimensionalise 16.10, we derive that the vertical velocity scale W is much smaller than the horizontal velocity scale U: $W \ll U$:

$$\vec{\nabla} \cdot \vec{u} = 0$$

$$\underbrace{\vec{\nabla}_h \cdot \vec{u}_h}_{U/L} + \underbrace{\frac{\partial w}{\partial z}}_{W/H} = 0$$

$$\therefore \frac{U}{L} \sim \frac{W}{H} \quad \therefore \frac{W}{U} \sim \frac{H}{L}$$

$$\therefore \boxed{W \ll U}$$

We take this as justification for our fourth assumption: that vertical acceleration and advection is small compared to gravity and pressure gradient forces. As such, the system is in approximately in Hydrostatic Balance (we have simplified Equation 16.9 here). Importantly, though, we do *not* assume that the vertical acceleration/advection is 0 everywhere. We simply assume that they are much smaller than vertical pressure gradients and gravity.

This allows us to integrate the hydrostatic relation from some reference height η (the height of the surface) and pressure p_s (the pressure at the surface) to find that the following holds (refer to Equation 1.15 to see the derivation).

$$p(x, y, z, t) = \rho g(\eta(x, y, t) - z) + p_s(x, y, t)$$

Fifth, we assume that the fluid above the layer of fluid under consideration has a much lower viscocity than the current layer or that it is much less dense than the other positions. This allows us to assume that $\vec{\nabla}_h p_s \ll \vec{\nabla}_h \eta$, and that therefore $\vec{\nabla}_h p \approx \rho_0 g \vec{\nabla}_h \eta$. As such, Equation 16.11 becomes the following:

$$\frac{D\vec{u}_h}{Dt} + f\vec{k} \times \vec{u}_h = g\vec{\nabla}_h \eta$$

Notice that everything on the right hand side has no dependence on z. However, it must hold for each all z within the layer, therefore, we can conclude that $\vec{u}_h$ cannot depend on z either.[1] Therefore, we can neglect the vertical advection term in the material derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\cancel{\frac{\partial}{\partial z}}$$

This allows us to derive Equation 18.1, which is an approximation of the horizontal momentum equation (16.11):

$$\boxed{\frac{D_h\vec{u}_h}{Dt} + f\vec{k} \times \vec{u}_h = g\vec{\nabla}_h \eta}$$

where $\frac{D_h}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$ is the horizontal material derivative.

The interpretation is as follows: this is just horizontal momentum balance, where the horizontal acceleration is set by the coriolis force and the pressure gradient force, and no other forces. The pressure gradient force only depends on the height of the surface of the fluid, and the height affects the pressure through the hydrostatic relation.

---

[1] If it did, then the $f\vec{k} \times \vec{u}_h$ term would depend on z and so left hand side would depend on z. Depending on z implies that its value would change depending on z, but the right hand side cannot change its value depending on z, and so the equation would not hold for some z.

Let us now turn to Mass Conservation Equation 16.2. We already derived from the $\rho = \rho_0 = const$ assumption that the system must be incompressible, so the following holds:

$$\vec{\nabla} \cdot \vec{u} = \vec{\nabla}_h \cdot \vec{u}_h + \frac{\partial w}{\partial z} = 0$$

Let us integrate this equation with respect to z from $z = b(x, y)$ to $z = \eta(x, y, t)$ and recall first that $\vec{u}_h$ is not a function of z and second that we assumed that w was much smaller than u, v, not that $w = 0$. This allows us to derive the following:

$$\int_b^\eta 0 \, dz = \int_b^\eta \vec{\nabla}_h \cdot \vec{u}_h + \frac{\partial w}{\partial z} \, dz$$
$$0 = h(x, y, t)\vec{\nabla}_h \cdot \vec{u}_h + [w]_b^\eta$$
$$= h \vec{\nabla}_h \cdot \vec{u}_h + w(z = \eta) - w(z = b)$$

$$\therefore \boxed{0 = h \vec{\nabla}_h \cdot \vec{u}_h + w(z = \eta) - w(z = b)}$$

Now it turns out (and this is justified by some fluid mechanical boundary conditions) that $w(z = \eta) = \frac{D_h \eta}{Dt}$ and $w(z = b) = \frac{D_h b}{Dt}$. The interpretation is as follows: the vertical velocity at an interface must be exactly equal to the change in interface height/depth following a fluid parcel, because if this weren't the case, we would get a fluid parcel phasing through the ground (as is the case at the bottom where $z = b$) or the fluid parcel is exactly what sets the change in the interface depth (as is the case at the top where $z = \eta$).

As such:

$$w(z = \eta) - w(z = b) = \frac{D_h \eta}{Dt} - \frac{D_h b}{Dt}$$
$$= \frac{D_h h}{Dt}$$
$$= \frac{\partial h}{\partial t} + \vec{u}_h \cdot \vec{\nabla}_h h$$

Therefore, we get that:

$$0 = h \vec{\nabla}_h \cdot \vec{u}_h + w(z = \eta) - w(z = b)$$
$$= h \vec{\nabla}_h \cdot \vec{u}_h + \frac{\partial h}{\partial t} + \vec{u}_h \cdot \vec{\nabla}_h h$$
$$0 = \frac{\partial h}{\partial t} + \vec{\nabla}_h \cdot (h\vec{u}_h)$$

We have derived Equation 18.2. The interpretation for this equation is as follows: the height of a fluid can change, but it is constrained by the velocity of the fluid becauce the fluid is incompressible. If the height increases at some point, there must be more fluid flowing into that point than out of it, but since the fluid cannot compress, the height of the layer must increase.

We have derived all the equations of motion for the Shallow Water System, but the crux of the issue is whether or not this is a faithful representation of time-dependent geophysical fluid dynamics. Sadly, the details are quite complicated, and I'm not sure if I'll even get to them. If I do, they are at the end of this chapter in Section 18.9.

For now, just take this as granted, and let us start analysing this system.

18.2 Energetics

18.2.1 Kinetic Energy and Available Potential Energy

Let us first consider the energy in the shallow water system, which the sum of the kinetic energy and the (gravitational) potential energy. Again, we would prefer to work in intensive variables, so we find these per unit area.

The kinetic energy per unit area KE is as follows:

$$KE = \frac{1}{2}\underbrace{\rho_0\,V}_{m}\,u_h^2/A$$

$$= \frac{1}{2}\rho_0\,A\,h/A\,u_h^2$$

$$\therefore \boxed{KE = \frac{1}{2}\rho_0\,h\,u_h^2}$$

where $u_h = |\vec{u}_h|$. The idea is simple: it's just the kinetic energy divided by the unit area.

The potential energy per unit area is a little bit more complicated. To find this, we need to consider all the fluid parcels from $z = b$ to $z = \eta$ and sum the potential energy of all the fluid parcels.

Let us first consider a fluid parcel at height z with some height dz and width A (therefore, it has a volume equal to $A\,dz$ and mass equal to $\rho_0\,A\,dz$). The potential energy of *this* parcel (dPE) is as follows:

$$dPE = \underbrace{\rho_0\,a\,dz}_{m}\,g\,z$$

We have to sum this over all the fluid parcels from $z = b$ to $z = \eta$ so our total potential energy per unit area PE is as follows:

$$PE = \int_{z=b}^{z=\eta} dPE/A$$

$$= \int_{z=b}^{z=\eta} \underbrace{\rho_0\,A\,dz}_{m}\,g\,z/A$$

$$= \rho_0\,g\left[\frac{1}{2}z^2\right]_b^{\eta}$$

$$= \frac{1}{2}\rho_0\,g\left(\eta^2 - b^2\right)$$

Now the final fiddly thing we have to do is to say that we don't actually care about the total potential energy – we care about the **available** potential energy. This is the potential energy available to be extracted by the system (by, e.g., conversion to kinetic energy). To find this, we must subtract the minimum potential energy from the total potential energy. The minimum potential energy state occurs if the surface is flat, i.e., if $\eta = const$. We can simply define our coordinate system such that $z = 0$ at this flat surface so that $\eta = 0$ and that $b < 0$ in this minimum PE state. Therefore, the minimum PE density is:

$$PE_{min} = -\frac{1}{2}\rho_0\,g\,b^2$$

Subtracting this off from our total PE per unit area gives us the total available potential energy per unit area:

$$\boxed{PE = \frac{1}{2}\rho_0\,g\,\eta^2}$$

So the kinetic energy scales as $h\,u_h^2$ while the available potential energy scales as $g\,\eta^2$.

18.2.2 Conservation of Energy

We will now derive conservation of energy in a shallow water system. We first take the dot product of Equation 18.1 and $\rho_0\, h\, \vec{u}_h$. This gives us:

$$0 = \rho_0\, h\, \vec{u}_h \cdot \frac{\partial \vec{u}_h}{\partial t} + \rho_0\, h\, \vec{u}_h \cdot \left(\vec{u}_h \cdot \vec{\nabla}_h\right) \vec{u_h} + f\rho_0\, h\, \underbrace{\vec{u}_h \cdot \vec{k} \times \vec{u}_h}_{=0} + g\rho_0\, h\, \vec{u}_h \cdot \vec{\nabla}_h \eta$$

$$= h\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0 u_h^2\right) + h\vec{u}_h \cdot \vec{\nabla}\left(\frac{1}{2}\rho_0 u_h^2\right) + h\vec{u}_h \cdot \vec{\nabla}\left(\rho_0 g\eta\right)$$

where we have used chain rule to bring the u_h^2's inside the derivatives. We next multiply Equation 18.2 with $\frac{1}{2}\rho_0 u_h^2$ to get:

$$\frac{1}{2}\rho_0 u_h^2 \frac{\partial h}{\partial t} + \frac{1}{2}\rho_0 u_h^2 \vec{\nabla}_h \cdot (h\vec{u}_h) = 0$$

Adding these two equations together, and recalling product rule for derivatives, gives an equation governing the kinetic energy:

$$\underbrace{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0 h u_h^2\right)}_{\text{Change in } KE} + \underbrace{\vec{\nabla}.\left(\vec{u}_h \frac{1}{2}\rho_0 h u_h^2\right)}_{\text{Convergence/Divergence of } KE} + \underbrace{h\vec{u}_h \cdot \vec{\nabla}(\rho_0 g\eta)}_{\text{Conversion of } KE \text{ to } APE} = 0$$

Let us take some time to interpret this equation. The first term is straightforward: it is just the local change in KE per unit time. The second term is the convergence or divergence of KE.

indicated by the second term, or from APE being converted into KE. The second term is a flux because is the divergence of the kinetic

We can form an equation governing potential energy directly by multiplying Equation 18.2 with $\rho_0 g\eta$:

$$\underbrace{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0 g\eta^2\right)}_{\text{Change in } PE} + \underbrace{\rho_0 g\eta\vec{\nabla} \cdot (h\vec{u}_h)}_{\text{Conversion of } APE \text{ to } KE}$$

Adding these equations together, we get an energy budget equation:

$$\frac{\partial}{\partial t}\left(\underbrace{\frac{1}{2}\rho_0 h u_h^2 + \frac{1}{2}\rho_0 g\eta^2}_{E}\right) + \vec{\nabla} \cdot \left(\underbrace{h\rho_0\vec{u}_h\left(\frac{1}{2}u_h^2 + g\eta\right)}_{\vec{S}}\right) = 0 \tag{18.4}$$

$$\frac{\partial E}{\partial t} + \vec{\nabla} \cdot \vec{S} = 0 \tag{18.5}$$

where E is the total energy per unit area and $\vec{S}$ is the energy flux vector.

This is not just an energy conservation equation, it is a *local* energy conservation equation. A local energy conservation equation imposes the constraint that change in the total energy at any point E must be exactly equal to the energy flux $\vec{S}$ entering or leaving that area.

If we integrate equation 18.5 around some closed area A with some boundary L, we can apply the divergence theorem to find the following holds:

$$\frac{\partial}{\partial t}\iint_A E\, dx\, dy = -\int_L \vec{S} \cdot \vec{n}\, dl$$

Physically, E and $\vec{S}$ are straightforward to interpret. E, is the total energy, which is the sum of the kinetic and available potential energies. $\vec{S}$ is the energy flux vector.

18.3 Potential Vorticity

We define the **potential vorticity** (**PV**) Q in this context as follows:

$$Q = \frac{f + \xi}{h}$$

Q is the absolute vorticity in the z-direction $[\vec{\omega}_a]_z$, which is the sum of the planetary vorticity in the z-direction f (the coriolis parameter!) and the relative vorticity in the z direction $\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$, divided by the thickness of the layer h.

In a way, we can think of **PV** as encoding the two important properties of GFD: rotation and stratification. The rotation is encoded by the $f + \xi$ term, which is the total (planetary+relative) vorticity in the z-direction. The stratification is encoded by the layer thickness h. If h is small, then this implies that the layer of constant density is very thin, which implies that density in the broader fluid increases rapidly with height. Conversely, if h is large, then this implies that the layer of constant density is very thick, which implies that density in the broader fluid increases very slowly with height.

18.3.1 Material Conservation of Potential Vorticity

Derivation

We first take the z component of the curl Equation 18.1 (and recalling that $f \neq const$):

$$\frac{D_h}{Dt}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \left(\frac{\partial(fv)}{\partial y} + \frac{\partial(fu)}{\partial x}\right) = 0$$

$$\therefore \frac{D_h\xi}{Dt} + \vec{\nabla}_h \cdot (f\vec{u}_h) = \frac{D_h\xi}{Dt} + f\vec{\nabla}_h \cdot \vec{u}_h + \vec{u}_h \cdot \vec{\nabla}_h f = 0$$

Next, we rearrange Equation 18.2 as follows:

$$\frac{D_h h}{Dt} + h\vec{\nabla}_h \cdot \vec{u}_h = 0$$

We now cancel out the $\vec{\nabla}_h \cdot \vec{u}_h$ in both equations to obtain:

$$\frac{D_h\xi}{Dt} - \frac{f}{h}\frac{D_h h}{Dt} + \vec{u_h} \cdot \vec{\nabla}f = 0$$

Recalling that $\frac{\partial f}{\partial t} = 0$, therefore that $\vec{u_h} \cdot \vec{\nabla}f = \frac{D_h f}{Dt}$, we rewrite the equation just in terms of material derivatives then divide the whole equation by h:

$$\frac{1}{h}\frac{D_h\xi}{Dt} + \frac{1}{h}\frac{D_h f}{Dt} - \frac{f}{h^2}\frac{D_h h}{Dt} = 0$$

Using product rule, this implies the following:

Key Idea VI.8: Material Conservation of Potential Vorticity

For a shallow water system, the **Potential Vorticity** (**PV**) Q is materially conserved:

$$\frac{D_h Q}{Dt} = 0 \qquad (18.6)$$

$$Q = \frac{f + \xi}{h} \qquad (18.7)$$

where $\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ the relative vorticity. Equivalently, following a fluid parcel, the potential vorticity remains constant.

Interpretation

We can interpret material conservation of potential vorticity by considering an infinitesimally thin column of fluid.

This column of fluid lies between the bottom ($z = b$) and top ($z = \eta$) of the layer, and is infintesimally thin. We impose that the column is infinitesimally thin to ensure that u, v, and h do not vary appreciably over the entire column. However, we *must* consider the entire column of fluid because our assumption (in the derivation) that u and v do not vary with z.

As this column of fluid is advected around, it must conserve its angular momentum in the z-direction. Suppose the column is initially rotating with some (total) angular velocity. If the column is stretched (i.e., if h increases), it become thinner (because the total volume of the column remains constant), and so it must therefore rotate faster to conserve angular momentum.
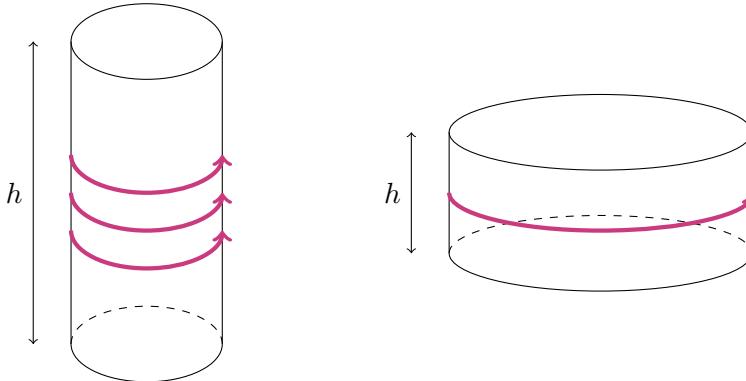

Figure 18.2: Intuition behind the Conservation of Potential Vorticity. When the fluid column gets stretched vertically (i.e., h increases), the fluid must rotate (magenta arrows ■) faster, and vice versa.

18.4 A Brief Primer on Waves

In the previous section, we derived geostrophic balance and hydrostatic balance.

However, the Earth is never in perfect balance, and perturbations and anomalies will occur, either because we have built a wind farm, or the sun is radiating differently, or because freshwater is being injected into certain areas in the oceans.

In this course, we analyse how systems respond to these anomalies by looking only for wave-like solutions: we plug in a solution that is proportional to $\exp(i(\vec{k} \cdot \vec{x} - \omega t))$, where $\vec{k} = (k_x, k_y, k_z)^T \in \mathbb{R}^3$ is the angular wavevector (the vector analogue of the wavenumber – refer to Section 4.2.1 if you've forgotton what a wavenumber is) and ω is the angular frequency.

The justification for plugging in this somewhat convoluted function is as follows. Practically, we want a function that is simple enough such that the solution is easy to interpret. However, we also want a function complicated enough such that it can encode certain spatial and time dependencies. This is why we pick the exponential function, which is simple, which depends on a linear sum of space and time, which. More fundamentally, any function[2] can be approximated by an infinite sum of periodic complex function like $\exp(i(\vec{k} \cdot \vec{x} - \omega t))$ – we assume that these can be accurately approximated by a large enough sum.

In practice, though, you can just keep this in the back of your mind, and you'll mostly just be plugging in $\exp(i(\vec{k} \cdot \vec{x} - \omega t))$ into the equations of motion. After plugging these in, we will find that $\vec{k}$ and ω are constrained by these equations, and solving for this constraint gives us a **Dispersion Relation**:

$$\omega = \omega(\vec{k})$$

In other words, the frequency with which the perturbation oscillates is set by the spatial scale (or vice versa). This will give us valuable information regarding how these waves propogate. Furthermore,

---

[2]I think, please correct my maths if it's bad.

we expect the anomalies that generate/excite these waves to have a similar timescale/lengthscale based on general principles of how waves resonate. However, again, this is just a very cursory approximation. We know (e.g., from turbulence for example – think Kolmogorov's $k^{-\frac{5}{3}}$ kinetic energy spectrum[3]) that energy can propogate from one lengthscale to another.

18.4.1   The wave-vector and the angular frequency

Physically, the wave-vector encodes the direction the wavefronts face. To see this, consider some field p which is proportional to $\exp(i(\vec{k} \cdot \vec{x} - \omega t))$, i.e., let $p = p_0 \exp(i(\vec{k} \cdot \vec{x} - \omega t))$. Let us draw contours of constant p at some time t. To draw these contours, we solve for the values of $\vec{x}$ that solve the following equation:

$$p(\vec{x}, t) = const$$

where $const$ is just any value (picking a different $const$ will give you a different contour).

Since $p \sim \exp(i(\vec{k} \cdot \vec{x} - \omega t))$, this implies that p is constant only for the values where $\vec{k} \cdot \vec{x}$ is constant. These are picked out by the spatial points $\vec{x}$ that can be written as follows:

$$\vec{x} = \vec{x}_0 + \delta\vec{x}$$
$$\text{where } \delta\vec{x} \perp \vec{k} \text{ and } \vec{x}_0 = another\ const$$

Therefore the contours of constant p, or the wavefronts, are perpendicular to $\vec{k}$:
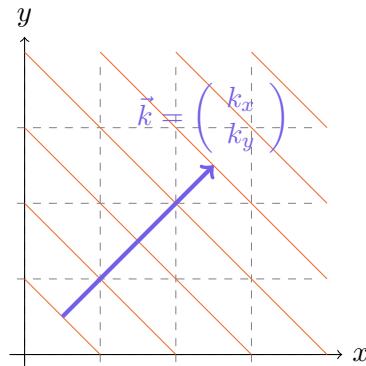

Figure 18.3: A wave vector $\vec{k}$ in the (x, y) plane drawn with a thick blue arrow ▉. Wavefronts lie perpendicular to the wavevector, drawn with thin solid lines ▉. Note, crucially, that it need not be the case that the energy or the wavefronts propagate in the direction of $\vec{k}$.

However, $\vec{k}$ only really tells you about the wavelengths in each dimension and the orientation of the wavefronts. It **does not** tell you which direction the energy or the wavefronts propagate in. These propagate at the **phase** and **group** velocities, which we will talk about now.

18.4.2   The Phase and Group Velocities

The peaks and troughs in the wave (i.e., the wavefronts) will propogate with a **phase velocity** denoted by $\vec{c}_p$, where the ith component $[\vec{c}_p]_i$ is found as follows:

$$[\vec{c}_p]_i = \frac{\omega}{[\vec{k}]_i} \tag{18.8}$$

---

[3]As always, the atmosphere/oceans are far more complicated than this due at least in part to the presence of rotation. See Vallis [6] Ch. 12.

Meanwhile, the wave-packet and the energy will propogate with a **group velocity** denoted by $\vec{c}_g$, where the ith component $[\vec{c}_g]_i$ is found as follows:

$$[\vec{c}_g]_i = \frac{\partial \omega}{\partial [\vec{k}]_i} \qquad (18.9)$$

For example, in the x direction, the **group** and **phase velocities** are found as follows:

$$[\vec{c}_p]_x = \frac{\omega}{k_x}$$

$$[\vec{c}_g]_x = \frac{\partial \omega}{\partial k_x}$$

This is a slightly confusing distinction, as you've most likely only encountered non-dispersive waves in the past, and non-dispersive waves are waves such that the phase velocity is equal to the group velocity ($\vec{c}_g = \vec{c}_p$). Since I can't insert any animations into these pdf notes, I'd highly recommend clicking on this website on dispersive waves. I'd especially pay attention to the little animations.

Now that you've thoroughly read that resource, we can interpret what this means in our context. Suppose that the pressure perturbation is proportional to $\exp(i(\vec{k} \cdot \vec{x} - \omega t))$ where $\omega = \omega(\vec{k})$ such that $\vec{c}_p = \vec{c}_p(\vec{k})$ and $\vec{c}_g = \vec{c}_g(\vec{k})$. Peaks and troughs propagate with a velocity set by the **phase velocity** $\vec{c}_p = \vec{c}_p(\vec{k})$, so areas of low or high pressure (i.e., cyclones and anti-cyclones in mid-/high-latitudes) propogate at the velocity $\vec{c}_p$. Meanwhile, the amplitude of this low or high pressure perturbation propagates with a velocity set by the **group velocity** $\vec{c}_g = \vec{c}_g(\vec{k})$. You saw that, in general, this means that the energy propagates with the **group velocity**, *not* the **phase velocity**.

18.5 Inertia-Gravity/Poincaré Waves

18.5.1 Derivation

We first linearise the Shallow Water Equations about a state of rest $\vec{U} = 0$ and layer thickness H. For simplicity we assume a flat bed $b = -H = const$. This is the first time we linearise in this course, so I will go through this a bit more carefully.

First, we write our variables as a sum of the unvarying quantity and the small perturbation:

$$\vec{u}_h = \vec{U} + \vec{u}'$$
$$h = H + h'$$
$$\eta = N + \eta'$$

where $H \gg h'$ and $N \gg \eta'$.[4]

Second, we substitute these equations into the shallow water equations, then eliminate terms that are not zeroth or first order in the primed quantites. For example, for Equation 18.3:

$$\frac{\partial}{\partial t}\left(H + h'\right) + \vec{\nabla}_h \cdot \left((H + h')(\vec{U} + \vec{u}')\right) = 0$$

$$\frac{\partial}{\partial t}\left(H + h'\right) + \vec{\nabla}_h \cdot \left(H\vec{U} + H\vec{u}' + h'\vec{U} + \cancel{h'\vec{u}'}\right) = 0$$

Third, we use the fact that H, N, and $\vec{U}$ are constant (and that $\vec{U} = 0$).

$$\cancel{\frac{\partial H}{\partial t}} + \frac{\partial h'}{\partial t} + \cancel{\vec{\nabla}_h \cdot \left(H\vec{U}\right)} + \vec{\nabla}_h \cdot \left(H\vec{u}'\right) + \vec{\nabla}_h \cdot \left(h'\cancel{\vec{U}}\right) = 0$$

$$\frac{\partial h'}{\partial t} + H\vec{\nabla}_h \cdot \vec{u}' = 0$$

---

[4]You might notice that we haven't required $|\vec{U}| \gg \vec{u}'$. This is because, strictly speaking, $\vec{U} = 0$, so there is no way for $\vec{U}$ to be much larger than $\vec{u}'$. We need some other criterion on what we require for $\vec{u}'$ to be 'small'.

Doing the same for equation 18.1 allows us to derive the **Linearised Shallow Water Equations**.

Key Idea VI.9: The Linearised Shallow Water Equations

Linearising the Shallow Water System with a flat bottom ($b = -H$) about a state of rest where $u = v = \eta = 0$ and $h = H$ by keeping only terms first order in small primed variables gives rise to the following equations:

$$\frac{\partial \eta'}{\partial t} + H\vec{\nabla}_h \cdot \vec{u}' = 0 \tag{18.10}$$

$$\frac{\partial \vec{u}'}{\partial t} + f\vec{k} \times \vec{u}' + g\vec{\nabla}_h \eta' = 0 \tag{18.11}$$

Note that it is also possible to linearise the shallow water system about other background states. E.g., we could have lineraised about a state of $v = \eta = 0$, $h = H$, and $u = U$. This would analyse perturbations about a mean zonal flow.

While we use these equations to analyse the time-dependent response of anomalies, note that actual phenomena are not always sufficiently represented by a linearised system – sometimes the anomalies are not small relative to the background state!

I will now derive it slightly different to how Tim does it in his lecture series. In his lecture series, he does some cross differentiating and adding to eliminate $\vec{u}'$ for an equation only in η'. While this is more general than what I'm about to do, it is also (in my humble opinion) a bit less straightforward and hard to remember than what i'm about to do. Plus, both methods generate the same answer anyways.

We now let the following obtain for η' and $\vec{u}'$:[5]

$$\eta' = \eta_0 \exp(ikx + ily - i\omega t)$$

$$\vec{u}' = \vec{u}_0' \exp(ikx + ily - i\omega t) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \exp(ikx + ily - i\omega t)$$

Substituting this into the linearised shallow water equations gives us the following three equations:

$$(-i\omega\eta_0 + H(iku_0 + ilv_0))\exp(ikx + ily - i\omega t) = 0$$
$$(-i\omega u_0 - fv_0 + gik\eta_0)\exp(ikx + ily - i\omega t) = 0$$
$$(-i\omega v_0 + fu_0 + gil\eta_0)\exp(ikx + ily - i\omega t) = 0$$

Because these equations must apply for all x, y, t, we must impose that the coefficient in front of the exponential must be 0. We therefore get the following equations:

$$-i\omega\,\eta_0 + iHk\,u_0 + iHl\,v_0 = 0$$
$$igk\,\eta_0 - i\omega\,u_0 - f\,v_0+ = 0$$
$$igl\,\eta_0 + f\,u_0 - i\omega\,v_0 = 0$$

$$\therefore \begin{pmatrix} -\omega & Hk & Hl \\ igk & -i\omega & -f \\ igl & f & -i\omega \end{pmatrix} \begin{pmatrix} \eta_0 \\ u_0 \\ v_0 \end{pmatrix} = \vec{0} \tag{18.12}$$

Our goal is to find the dispersion relation governing $\omega = \omega(k, l)$, but we want this dispersion relation to be arbitrary regardless of η_0, u_0, and v_0 (and we expect these to be set by the nature of

---

[5]You might be suspicious about what we've done here. After all, how can the height of a fluid be a complex number? You're right in that, strictly speaking, we should require that $\eta' \in \mathbb{R}$. We can remedy this by simply stipulating that $\eta' = \mathrm{Re}\,(\eta')$ (to abuse notation).

the initial perturbation that generates the waves). We also require that these η_0, u_0, $v_0 \neq 0$ so that we have non-trivial solutions (the trivial solution, is, of course, $\eta' = 0$, $\vec{u}'_h = \vec{0}$).

We can go about this two ways. First, we can just eliminate η_0, u_0, v_0 via some algebra and this is fairly straightforward. Second, we write this in matrix form, as I have above, and recall(?) that if a matrix times a non-zero vector is equal to 0, then the determinant of the matrix must be zero. Using this second time-saving method we get:

$$
\begin{aligned}
0 = \det &\begin{pmatrix} -\omega & Hk & Hl \\ igk & -i\omega & -f \\ igl & f & -i\omega \end{pmatrix} \\
= -\omega &\begin{vmatrix} -i\omega & -f \\ f & -i\omega \end{vmatrix} - Hk \begin{vmatrix} igk & -f \\ igl & -i\omega \end{vmatrix} + Hl \begin{vmatrix} igk & -i\omega \\ igl & f \end{vmatrix} \\
= -\omega &\left(-\omega^2 + f^2 \right) - Hk \left(gk\omega + \cancel{igfl} \right) + Hl \left(\cancel{igfk} - gl\omega \right) \\
0 = \omega &\left(\omega^2 - f^2 - gH \left(k^2 + l^2 \right) \right)
\end{aligned}
$$

We therefore obtain the following dispersion relation for **Poincaré/Inertia-Gravity Waves**:

$$
\boxed{\omega = 0} \quad \text{or} \quad \boxed{\omega^2 = f^2 + gH \left(k^2 + l^2 \right)} \tag{18.13}
$$

The first $\omega = 0$ solution implies that the wave is in a steady state, and does not propogate (since $\vec{c}_p = \vec{c}_g = 0$). Plugging this into Equation 18.11 (exercise!) shows that $\vec{u}'_h$ is geostrophic! Furthermore, the phase and group velocities are also 0, thus implying (as expected) that geostrophic anomalies do not propagate.

The second $\omega^2 = f^2 + gH \left(k^2 + l^2 \right)$ solution is more interesting. We now make an important simplification for ease of analysis. We call this the f-**plane approximation**.

Key Idea VI.10: The f-Plane Approximation

The f-**plane approximation** is the approximation where the coriolis parameter f is treated as constant:

$$
\boxed{f \approx 2\Omega \sin \phi_0 = const}
$$

where ϕ_0 is the approximate latitude.
Recall that, in reality, the coriolis parameter is actually $f = 2\Omega \sin \phi$, where $\phi = $ the latitude. Recalling that y is the north-ward pointing coordinate, we expect $\phi = \phi(y)$ so that $f = f(y)$. The **f-plane approximation** ignores this, and approximates $\phi = const$.

Having made this simplification, we can make two further simplifications. First, we let $l = 0$ without loss of generality[6]. Second, we only consider the $\omega > 0$ solution. There is a difference between the $\omega > 0$ and $\omega < 0$ solution, but this isn't too relevant to what we wish to discuss (Exercise: solve for u_0, v_0 in terms of η_0 and show that the sign of ω affects this).

18.5.2 Interpretation

With our simplifications, our dispersion relation looks like this:

---

[6]We can make this simplification without loss of generality because fo the following reason: the shallow water system with an $f-$plane approximation is isotropic in the x, y, plane, so, for any wave-vector $\vec{k} = (k, l)^T$ we can just rotate the coordinate system to align with the k vector and the dynamics remain unchanged. As such, we can set $l = 0$, because for any wave with a non-zero l, we can just reorient our coordinate system such that our x-axis aligns with the wavevector $\vec{k}$ so that $l = 0$.

> **Key Idea VI.11: Inertial-Gravity/Poincaré Dispersion Relation**
>
> The dispersion relation for Poincaré/Inertia-Gravity Waves in a Shallow Water System is as follows:
>
> $$\omega(k) = (f^2 + \underbrace{gH}_{gH \equiv c^2} k^2)^{\frac{1}{2}}$$
>
> where we have defined $c \equiv \sqrt{gH}$ = the speed[7] at which gravitational waves propagate. If we plot $\omega(k)$, it looks like this:
>
>
>
> Figure 18.4: The dispersion relation for Poincaré Waves, drawn in filled blue ■. The long- and short-wave limits, respectively, are drawn in dotted magenta ■ and dashed orange ■.
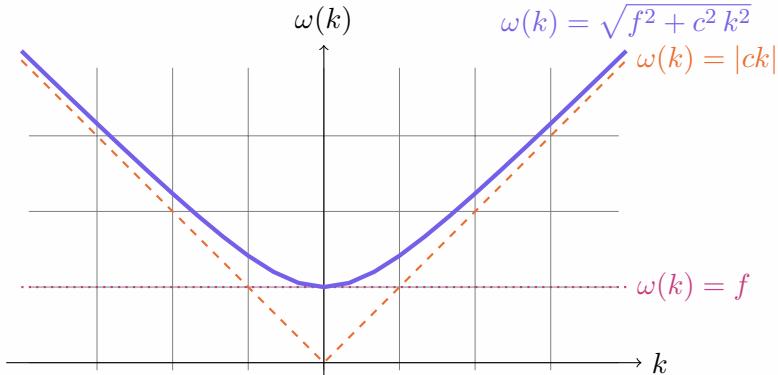
The Phase and Group Velocities

Let us now calculate the phase and group velocities c_p and c_g, which, I remind you, is the speed at which peaks/troughs and wave-packets/energy (respectively) propagate.

$$c_p = \frac{\omega}{k} = \frac{\sqrt{f^2 + c^2 k^2}}{k}$$
$$c_g = \frac{\partial \omega}{\partial k} = \frac{c^2 k}{\sqrt{f^2 + c^2 k^2}} = \frac{c^2}{c_p^2}$$

Doing some algebraic malipulation allows us to derive the following relation:

$$\frac{c_g}{c_p} = \frac{\omega^2 - f^2}{\omega^2}$$

which implies that it is always the case that $c_p > c_g$. Therefore, peaks and troughs travel faster than a wavepacket or energy does.

 Let us now interpret directly the mechanisms that control the propagation of this wave (i.e., how Poincaré waves may propagate of anomalies). To do this, let us look at what $\omega(k)$ in the limit of very large and very small k. Of course, when we say k is large or small, we must specify large or small *relative to what*. To this end, we define the **Rossby Deformation Radius** L_d as follows:

---

[7]The phase or the group velocity? Wait and see...

Key Idea VI.12: The Rossby Deformation Radius for a Shallow Water System

We define the **Rossby Deformation Radius** L_d for a Shallow Water System as follows:

$$L_d = \frac{\sqrt{gH}}{f} \qquad (18.14)$$

The Rossby deformation radius is a lengthscale which demarcates roughly when phenomena start 'feeling' the effects of planetary rotation and the coriolis force.
Phenomena occuring on lengthscales much shorter than L_d do not 'feel' the effects of rotation.
Phenomena occuring on lengthscales much larger than L_d 'feel' the effects of rotation strongly.

To convince you of this fact, consider the following argument. Suppose that our horizontal velocity scale U is set by the speed at which gravitational waves propagate, i.e., $U \sim \sqrt{gH}$. We know that the Rossby Number indicates whether rotation is important relative to inertia: if $Ro \ll 1$, then rotation is important and vice versa.

$Ro = 1$ exactly when the lengthscale $L = L_d$. Therefore if $L \gg L_d$, then $Ro \ll 1$, therefore rotation is important, adn vice versa.

We now consider the large and small k limits. Recall that k is the wavenumber, and so the wavelength of the wave $\lambda \sim 1/k$. Therefore the lengthscale of the wave $L \sim \lambda \sim 1/k$. Therefore if k is very large, then the lengthscale L is very short, and vice versa. It is for this reason that we call the following limits the **short-wave** and **long-wave** limits.

The Short-Wave Limit and the Gravity Wave Mechanism

In the **short-wave limit**, the wavelength of the wave is **very small**, so we impose that $L \ll L_d$ and therefore that $k \gg 1/L_d$. In this limit, $c^2 k^2 \gg f^2$, and so in the shortwave limit:

$$\boxed{\omega(k) \approx ck = \sqrt{gH}k}$$

In the **short-wave** limit then, the wave is what we call a **gravity wave**, and feels no effects of rotation! We can also think of this in terms of the Rossby Number. If $U \sim \sqrt{gH}$ and $L \ll L_d$, this implies that $Ro \gg 1$. Since there is no other intrinsic timescale here, we can conclude that the the the $f\vec{k} \times \vec{u}_h$ is much smaller than both terms in Equation 18.11.

Such a wave is **non-dispersive**, meaning that $c_g = c_p$. Calculating c_g and c_p in this limit shows the following to obtain:

$$c_p = c = \sqrt{gH}$$
$$c_g = c = \sqrt{gH}$$

So gravity waves propagate at the speed c, and do not disperse: the wave-packet stays together!

Recall that energy propagates at the group velocity c_g. Therefore, small length-scale anomalies can propagate across the atmosphere/ocean at the speed c_g.

Now we can explain the mechanism for wave propagation in the **short-wave** limit. Suppose some phenomena generates a η' anomaly (for example, a hump in η'). Therefore, at this point, $\vec{\nabla}_h \eta' \neq 0$. Recalling that the gradient of η' is a vector field pointing from low η' to high η', $\vec{\nabla}_h \eta'$ will look like a vector field with vectors all pointing towards the centre of the hump.

Referring to Equation 18.11 and neglecting the coriolis term implies that $\frac{\partial \vec{u}'}{\partial t} \neq 0$, and must be equal and opposite to $\vec{\nabla}_h \eta'$. Therefore, $\frac{\partial \vec{u}'}{\partial t}$ looks like a vector field pointing outwards, so the fluid accelerates away from this hump.

The fluid accelerating away from this hump implies further that there is no a divergence of fluid at this hump and a convergence of fluid around the hump. Therefore $\vec{\nabla}_h \cdot \vec{u}' > 0$ at the hump and $\vec{\nabla}_h \cdot \vec{u}' < 0$ around the hump. Therefore, from Equation 18.10, at the hump $\frac{\partial \eta'}{\partial \eta'} < 0$ and around the hump $\frac{\partial \eta'}{\partial \eta'} > 0$ so the hump spreads outwards and this is how the wave propagates.

Physically, what is occuring is that η' anomalies generate in pressure anomalies (from the hydrostatic relation), and these pressure anomalies generate $\vec{u}'$ anomalies, which turn generate η' anomalies, which propagates the wave. This is why we call these **gravity waves**, because the restoring force is gravity (as gravity is what generates the pressure anomalies via the hydrostatic relation).

The Long-Wave Limit and the Inertia/Coriolis Wave Mechanism

Let us now look at the **long-wave limit**. In the **long-wave limit**, the wavelength of the wave is **very large**, so we impose that $L \gg L_d$ and therefore that $k \ll 1/L_d$. In this limit, $c^2 k^2 \ll f^2$, and so in the long-wave limit:

$$\boxed{\omega(k) \approx f}$$

In the **long-wave** limit, then, the wave is just an inertial oscillation: the wave feels only the effects of the coriolis force, and feels no effects of gravity. Again, we can also think of this in terms of the Rossby Number.

Let us now calculate c_g and c_p in this limit:

$$c_p = \frac{f}{k}$$
$$c_g = 0$$

Again, recalling that energy propagates at the group velocity, we can conclude that large length-scale anomalies are 'trapped': large length-scale anomalies *cannot* propagate away from the source.

To explain this, let us consider the mechanism for how this wave sustains itself. In this limit, rotation is dominant. Suppose, again, that there is some surface height η' anomaly. This, again, would imply that $\vec{\nabla}_h \eta' \neq 0$, which by Equation 18.11 would implies that $f\vec{k} \times \vec{u}' \neq 0$.

Since $c_g = 0$, **long-wave** Poincaré waves cannot transport energy, so in general this will not be the mechanism for propagation of long length-scale anomalies. We must turn to other mechanisms to explain this.

Explicit Solutions: Fluid Parcel Trajectories

We now wish to find explicit solutions for $\eta = \eta'$ and $\vec{u}_h = \vec{u}'$. To do this, we must first find u_0 and v_0 in terms of η_0 (we cannot solve for all three constants, as this will depend on properties of the initial perturbation, and we wish to keep analysis here general). Refering back to Equation 18.12, we find (after some algebra) that the following obtains:

$$u_0 = \frac{\omega}{kH}\eta_0$$
$$v_0 = \frac{f}{kH}e^{-\frac{i\pi}{2}}\eta_0$$

Next, we take only the real part of our solutions for η' and $\vec{u}'$ (since heights and velocities must be real) to find the following expressions:

$$\eta' = \eta_0 \cos(ikx - i\omega t)$$
$$u' = \frac{\omega}{kH}\eta_0 \cos(ikx - i\omega t)$$
$$v' = \frac{f}{kH}\eta_0 \sin(ikx - i\omega t)$$

Let's examine some behaviour. Both u' and v' scale inversely with kH, so as the vertical wavelength increases (i.e., k decreases) or the layer thickness decreases so too does u' and v' for a given surface height anomaly η_0.

u' scales with ω, while v' scales with f. This asymmetry is because we are only considering $\vec{k} = (k, 0)^T$ (i.e., $l = 0$), so the velocity *parallel* to the direction of $\vec{k}$ scales as ω while the velocity

perpendicular to $\vec{k}$ scales as f. Now $\omega \geq f$ for all k (refer back to the dispersion relation), so $u' > v'$ always. In the shortwave limit ($k \gg 1/L_d$) especially, where the wave is essentially just a **gravity wave** which feels little effect of rotation, $u' \gg v'$, and so the velocity is almost completely in the direction of propagation.

Finally, a quick rearrangement of u' and v' shows the following:

$$\frac{u'^2}{\omega^2} + \frac{v'^2}{f^2} = \left(\frac{\eta_0}{kH}\right)^2$$

The u' and v' therefore trace out an ellipse!

Energetics

We now wish to investigate the energy in these waves. To do this, we calculate $\bar{KE}$ and $\bar{APE}$, the kinetic and available potential energies (respectively) *averaged* spatially over a wavelength and temporally over a period:

$$\bar{KE} = \frac{1}{T}\frac{1}{L_x}\frac{1}{L_y}\int_0^T \int_0^{L_x} \int_0^{L_y} \underbrace{\left(\frac{1}{2}\rho_0 h u_h^2\right)}_{KE} dx\,dy\,dt$$

$$\bar{PE} = \frac{1}{T}\frac{1}{L_x}\frac{1}{L_y}\int_0^T \int_0^{L_x} \int_0^{L_y} \underbrace{\left(\frac{1}{2}\rho_0 g \eta^2\right)}_{PE} dx\,dy\,dt$$

where $T = \frac{2\pi}{\omega}$, $L_x = \frac{2\pi}{k}$, and $L_y = \frac{2\pi}{k}$.

For simplicity, again, we assume that $l = 0$, such that the meridional wavelength is infinite, i.e., there is no dependence on y. As such, the integral with respect to y simply cancels out with the $1/L_y$. Furthermore, since η' and $\vec{u}'_h$ are small, we linearise such that the h in the $\bar{PE}$ expression becomes a constant H:

$$\bar{KE} = \frac{1}{T}\frac{1}{L_x}\frac{1}{2}\rho_0 H \int_0^T \int_0^{L_x} u_h^2\,dx\,dy\,dt$$

$$\bar{PE} = \frac{1}{T}\frac{1}{L_x}\frac{1}{2}\rho_0 g \int_0^T \int_0^{L_x} \eta^2\,dx\,dy\,dt$$

[UNDER CONSTRUCTION]

18.6   Rossby Waves

18.6.1   The β-Plane Approximation

Before we discuss Rossby Waves, we must first introduce the β-plane approximation. The β-plane approximation is our step-up from the f-plain approximation of the coriolis f parameter. Now, instead of crudely approximating $f = const$, we allow β to vary linearly in y.

The idea is as follows. We first Taylor Expand the coriolis parameter f about some latitude ϕ_0:

$$f(\phi) = 2\Omega \sin \phi$$
$$\approx \underbrace{2\Omega \sin \phi_0}_{f(\phi_0)} + \underbrace{2\Omega \cos \phi_0}_{f'(\phi_0)}(\phi - \phi_0)$$

Next, we approximate the north-south coordinate $y \approx a\phi$ (this amounts to the assumption that the y is small enough such that the Earth can be treated as roughly flat at that point).

> **Key Idea VI.13: The β-Plane Approximation**
>
> The β-**plane approximation** is the approximation where the coriolis parameter f is treated as linear in y:
>
> $$f \approx \underbrace{2\Omega\sin\phi_0}_{\equiv f_0} + \underbrace{\frac{2\Omega\cos\phi_0}{a}}_{\beta}(y - y_0)$$
>
> where $\beta = \frac{\partial f}{\partial y}$.
> We require $y - y_0 \ll a$ for this approxmation to be accurate. This approximation can also be thought of as a Taylor Expansion of f about $y = y_0$.

18.6.2 Derivation

Our derivation of Rossby Waves is a little bit different from just plugging in an exponential solution. First, for simplicity, we assume that $b = const$ for simplicity. This implies that $\vec{\nabla}_h h' = \vec{\nabla}_h \eta'$ for simplicity.

Second, we rearange Equation 18.11 for the $\vec{u}'_h$ in the coriolis term to find the following:

$$u' = -\frac{g}{f}\frac{\partial h'}{\partial y} - \underbrace{\frac{1}{f}\frac{\partial v'}{\partial t}}_{\text{small}} \qquad (18.15) \qquad\qquad v' = \frac{g}{f}\frac{\partial h'}{\partial x} + \underbrace{\frac{1}{f}\frac{\partial u'}{\partial t}}_{\text{small}} \qquad (18.16)$$

Third, we assume that $Ro \ll 1$, therefore, we expect the flow to be approximately in geostrophic balance. We therefore approximate the second term in both of the above equations as small (as I've written).

Fourth, we rewrite those small terms by resubstituting in v' and u' and taking the time derivative. We do this by More taking our expression for v' in Equation 18.16, then substitute v' into the small right-hand term in Equation 18.15:

$$u' = -\frac{g}{f}\frac{\partial h'}{\partial y} - \frac{1}{f}\frac{\partial}{\partial t}v'$$

$$= -\frac{g}{f}\frac{\partial h'}{\partial y} - \underbrace{\frac{1}{f}\frac{\partial}{\partial t}\left(\frac{g}{f}\frac{\partial h'}{\partial x} + \overbrace{\frac{1}{f}\frac{\partial u'}{\partial t}}^{\text{small}}\right)}_{\text{small}}$$

$$= -\frac{g}{f}\frac{\partial h'}{\partial y} - \underbrace{\frac{g}{f^2}\frac{\partial^2 h'}{\partial t\partial x}}_{\text{small}} - \underbrace{\frac{1}{f^2}\frac{\partial^2 u'}{\partial t^2}}_{\text{doubly small}}$$

Fifth, we neglect the doubly small term (the third term) by approximating it as 0.

Sixth, we use the β-plane approximation with the f in the first term and the f-plane approximation with the f^2 in the second term. The justification for not using the β-plane approximation on both terms is as follows: the β-plane approximation is only valid if $y - y_0 \ll a$, therefore the $\beta(y - y_0)$ term is much smaller than the f_0 term. We therefore neglect this term for the same reason why we neglected the $\frac{1}{f^2}\frac{\partial^2 u}{\partial t^2}$ term: it is doubly small.

Repeating this for v' allows us to derive the following relations for u' and v':

$$u' = -\frac{g}{f}\frac{\partial h'}{\partial y} - \frac{g}{f_0^2}\frac{\partial^2 h'}{\partial t\partial x} \qquad\qquad v' = \frac{g}{f}\frac{\partial h'}{\partial x} - \frac{g}{f_0^2}\frac{\partial^2 h'}{\partial t\partial y}$$

Now we plug in our expressions for u' and v' into Equation 18.10 to derive the following equation

governing h:

$$\begin{aligned}
0 &= \frac{\partial h'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0 \\
&= \frac{\partial h'}{\partial t} + H\left(\underbrace{-\frac{g}{f}\frac{\partial^2 h'}{\partial y \partial x} - \frac{g}{f_0^2}\frac{\partial^3 h'}{\partial t \partial x^2}}_{\frac{\partial u'}{\partial x}} + \underbrace{\frac{g}{f}\frac{\partial^2 h'}{\partial x \partial y} - \frac{g}{f_0^2}\frac{\partial^3 h'}{\partial t \partial y^2} - \frac{g}{f^2}\beta\frac{\partial h'}{\partial y}}_{\frac{\partial v'}{\partial x}}\right) \\
&= L_d^2\left(\frac{1}{L_d^2}\frac{\partial}{\partial t} + \frac{\partial}{\partial t}\vec{\nabla}_h^2 + \beta\frac{\partial}{\partial x}\right)h'
\end{aligned}$$

where $L_d =$ the Rossby Deformation Scale.

Again, we seek wave-like solutions, so we substitute in $h' = h_0 \exp\left(ikx + ily - i\omega t\right)$ to derive the **dispersion relation** governing the propagation of Rossby Waves:

Key Idea VI.14: Rossby Wave Dispersion Relation

$$\omega = \frac{-\beta k}{k^2 + l^2 + \frac{1}{L_d^2}} \qquad (18.17)$$
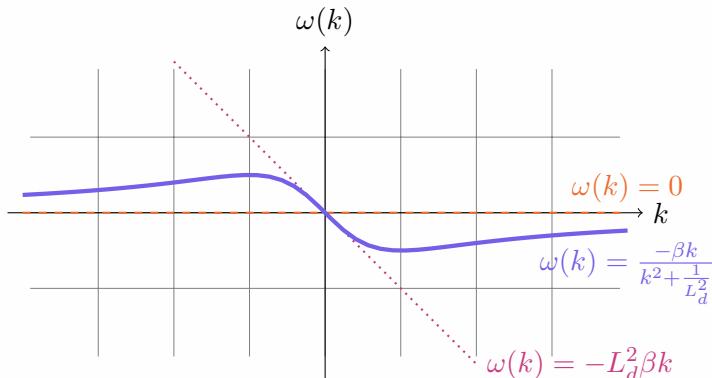
Figure 18.5: The dispersion relation for Rossby Waves, drawn in filled blue ■. The long- and short-wave limits, respectively, are drawn in dotted magenta ■ and dashed orange ■. For simplicity we have set $l = 0$.

18.6.3   Interpretation

Note that we can no longer set $l = 0$ without loss of generality, as the system is no longer isotropic. There is a difference between the $y-$direction and the $x-$direction: f increases in the $y-$direction, but f does not depend on the $x-$direction.

The Phase and Group Velocities

Again, let us calculate the phase and group velocites in the x-direction.

$$[\vec{c}_p]_x = -\frac{\beta}{k^2 + l^2 + \frac{1}{L_d^2}}$$

$$[\vec{c}_g]_x = \beta\left(\frac{k^2 - l^2 - \frac{1}{L_d^2}}{k^2 + l^2 + \frac{1}{L_d^2}}\right)$$

Notice that the $x-$component (i.e., the zonal/east-west component) of the phase velocity $\vec{c}_p$ is always negative (since $\beta > 0$ always). This implies that the phase velocity is always westwards, and so peaks and troughs *always* travel westward. We will explain this later when we explain the mechanism for Rossby Wave propagation.

Let us now turn to the $x-$component of the group velcoity $\vec{c}_g$. The group velocity can be positive or negative depending on both the zonal and meridional wavelengths and their size relative to the Rossby Deformation Radius L_d. If k is large relative to both l and $1/L_d$ (i.e., the zonal wavelength is small relative to both the meridional wavelength and the Rossby Deformation Radius) then energy can propagate eastward. Conversely, if either l or $1/L_d$ is large relative to k, then energy also propagates westward.

Rossby Wave Propagation Mechanism

Recall that Potential Vorticity is Materially Conserved in the shallow water system. As such, any fluid parcel that is displaced by this anomaly/wave must have its potential vorticity remain constant.

Suppose that some phenomena generates an anomaly in $\vec{u}'_h$ such that a fluid parcel is pushed northwards (i.e., in the positive y-direction). If the fluid parcel is advected northwards, then the planetary vorticity f increases. However, $(f + \xi)/h$ must be constant, and assuming that the change in h is small, we require ξ to decrease, which generates negative vorticity. The fluid parcels around this parcel are therefore forced accordingly: the parcels to the west are forced northward, and the parcels to the east are forced southward.

[FIGURE UNDER CONSTRUCTION]

Note that this explanation only works if the Ro is not extremely small. However, in the ocean, Ro is extremely small, and this implies that the relative vorticity is always much smaller than the planetary vorticity. Here is an alternative explanation for the ocean (courtesy of David Marshall). In the ocean, since $Ro \ll 1$, the perturbations must be geostrophic. Therefore, u is always geostrophic and the following obtains:

$$u = -\frac{g}{f(y)}\frac{\partial \eta}{\partial y}$$

Suppose we have a bump in the ocean (i.e., a positive height anomaly). Since there is a height anomaly, there is also a pressure anomaly, the pressure is larger in the bump. Since the flow must be geostrophic, this implies that the fluid must be anti-cylonic and must be flowing clockwise in the northern hemisphere (recall our discussion on cyclones and anti-cyclones in Section 17.2.3).

However, recall that $u \sim 1/f$ because it is geostrophic, therefore at the north edge of the height anomaly, the f is larger, therefore u is smaller, and at the south edge of the height anomaly f is smaller, therefore u is larger. This implies that there is more fluid flowing westward than there is fluid flowing eastward, therefore there is a convergence of fluid westward and a divergence of fluid eastwards. This means that the hump travels westwards, thus explaining the westward propagation of Rossby Waves in the ocean.

Westward Phase Propagation
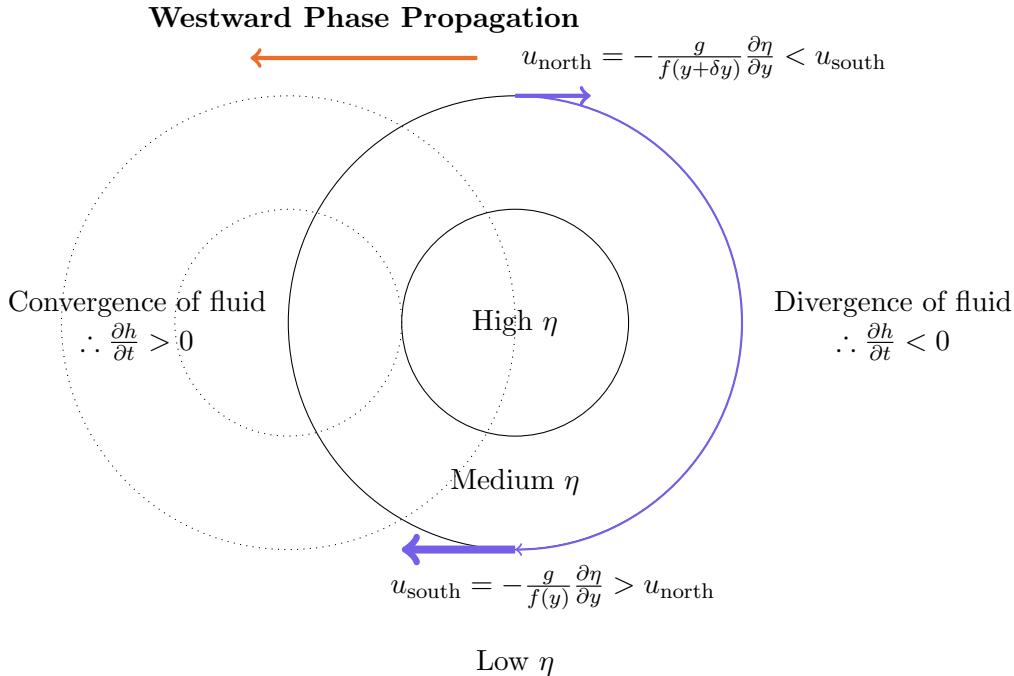
Figure 18.6: Alternative Explanation of the Westward Phase Propagation of Rossby Waves. Since $\beta = \frac{df}{dy} > 0$ and $u \sim 1/f$, $u_{\text{north}} < u_{\text{south}}$, therefore there is a convergence of fluid on the east and a divergence of fluid on the west. Solid black lines indicate η at time $t = t$, while dotted black lines indicate η at time $t = t + \delta t$.

18.7 Kelvin Waves

[KELVIN WAVES UNDER CONSTRUCTION]

For Rossby Waves, we assumed that the anomalies were to leading order geostrophic, which recall is a blance between the the corilis force and the pressure gradient force. There are at least two cases in which this assumption cannot hold.

The first case is along a boundary or a coastline.

The second case is along the equator, where $f \approx 0$, therefore $Ro \gg 1$.

18.7.1 Coastal Kelvin Waves

18.7.2 Equatorial Kelvin Waves

18.8 The Effects of Stratification: the Reduced Gravity System

As mentioned in Section 15.4, the two important properties of GFD are rotation and stratification. We already consider the effects of rotation in the Shallow Water System by introducing the coriolis force (the $f\vec{k} \times \vec{u}_h$ term) into Equation 18.1. However, we have not yet considered stratification.

Now we consider a twist on the shallow water system. Instead of considering one single layer of fluid, consider two layers of fluid. We will be modelling the motion of the fluid in the top layer with density $\rho = \rho_0$. We assume that this top layer is bounded from above by a fluid of negligible density, such that it is almost always flat. We assume further that this top layer is bounded from below by a fluid of density $\rho_1 = \rho_0 + \Delta\rho_0$ with no horizontal fluid motion (See Figure 18.7).

Analogous to how we derived the shallow water equations, we now derive the reduced gravity equations. We make the identical assumptions as before, but our treatment of the horizontal pressure gradients $\vec{\nabla}_h p$ must differ due to the different configuration.

As before, we assume that the system is approximately in hydrostatic balance. The pressure between $z = 0$ and $z = -h$ must be given by the following:

$$p(x, y, z, t) = -\rho_0 g z + p_s(x, y, t)$$

Taking the horizontal gradient of this gives us the following:

$$\vec{\nabla}_h p = \vec{\nabla}_h p_s$$

Note importantly that, as opposed to our derivation of the shallow water equations, we can no longer neglect the $\vec{\nabla}_h p_s$ term, as this term is the leading order contribution to $\vec{\nabla}_h p$ – if we neglected this term, $\vec{\nabla}_h p = 0$!

We now consider the pressure below $z = -h$, again found hydrostatically:

$$p(x, y, z, t) = -\rho_1 g(h(x, y, t) + z) + \rho_0 g h(x, y, t) + p_s(x, y, t)$$

However, we have assumed that the horizontal velocity in this layer is 0, which implies that horizontal pressure gradients within this layer must vanish, i.e., $\vec{\nabla}_h p(x, y, z < -h, t) = 0$. This condition gives un an expression for $\vec{\nabla}_h p_s$, which in turn gives us an expression for $\vec{\nabla}_h p$:

$$0 = -\rho_1 g \vec{\nabla}_h h + \rho_0 g \vec{\nabla}_h h + \vec{\nabla}_h p_s$$
$$\therefore -\vec{\nabla}_h p_s = -(\rho_1 - \rho_0) \vec{\nabla}_h h$$

The mass conservation equation is derived identically to before. We thus derive the **Reduced Gravity Equations**:

$$\frac{D_h \vec{u}_h}{Dt} + f\vec{k} \times \vec{u}_h + g' \vec{\nabla}_h h = 0 \qquad (18.18)$$

$$\frac{\partial h}{\partial t} + \vec{\nabla}_h \cdot (h \vec{u}_h) = 0 \qquad (18.19)$$
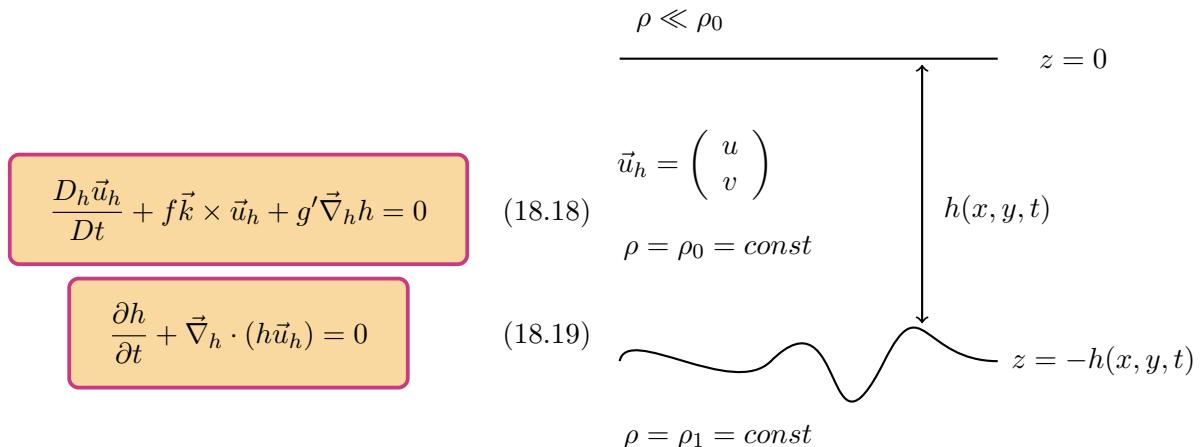
Figure 18.7: The Reduced Gravity System

Where g' is the reduced gravity and is defined as follows:

> **Key Idea VI.15: The Reduced Gravity**
>
> We define the reduced gravity g' as:
>
> $$g' = \frac{\Delta \rho_0}{\rho} g \qquad (18.20)$$
>
> where $g =$ is the acceleration due to the gravity; ρ is the density of the layer; and $\Delta \rho = \rho_1 - \rho_0$, where ρ_0 is the density of the layer under consideration and ρ_1 is the density of the underlying layer.
> The reduced gravity equations are exactly the same as the Shallow Water Equations but with all instances of g replaced with a g'.
> It is generally the case that $g' \ll g$.

Importantly, the reduced gravity system affects the Rossby Deformation Radius, now given by the following expression:

$$L_d = \frac{\sqrt{g'H}}{f}$$

Therefore, since generally $g' \ll g$, and $L_d \sim \sqrt{g'}$, then L_d is much smaller than we anticipated before.

18.9   Justifying the Shallow Water System

I strongly *strongly* emphasise that this section is off-syllabus. However, I thought I'd write it here to make the section more complete. I follow the Vallis' derivation in his book [6], where it can be found in chapter 3.

In these entire set of notes, we formulate and analyse models of the atmosphere and oceans. Of course, part of the accuracy relies on the accuracy of our analysis, but crucially part of the accuracy also relies on the accuracy of the models. The shallow water system would not be as interesting of a system (to us at least) if we did not have some evidence that the model was faithful and captured (some) dynamics of the atmosphere and oceans.

Chapter 19

Time-Dependent 3D Systems

In the previous chapter, we considered a time-dependent model of the atmosphere and oceans and analysed time-dependent responses. The key limitation of that system was that that system was 2-dimensional: we only considered a shallow layer at a time. We accounted for this somewhat when we introduced the Reduced Gravity System, which represented the effects of stratification insofar as g was replaced by g'.

Now we aim to represent time-dependent 3D responses of the atmospheres and oceans. We start with gravity waves, although I would prefer to call these 3D Poincaré waves, as I will argue that these are the 3D analogue to the 2D Inertia Gravity/Poincaré Waves we considered last chapter. After this, we move onto what will take up the bulk of the chapter: Quasi-Geostrophic Theory

19.1 Gravity Waves (3D Poincaré Waves)

19.1.1 The Boussinesq Approximation and Linearisation

Before linearising the system, we first invoke the **Boussinesq Approximation**. The **Boussinesq Approximation** is an approximation on the density field ρ. We first split the density field into three components that obey the following relations:

$$\rho(\vec{r}, t) = \rho_0 + \hat{\rho}(z) + \rho'(\vec{r}, t)$$
$$\bar{\rho}(z) = \rho_0 + \hat{\rho}(z)$$
$$\rho_0, \bar{\rho} \gg \hat{\rho}(z), \rho'(\vec{r}, t)$$

If we multiply Equations 16.11 (Horizontal Momentum) and 16.9 (Vertical Momentum) by ρ and expand every instance of ρ as above, we obtain the following equations:

$$(\rho_0 + \hat{\rho} + \rho'(\vec{r}, t))\frac{D\vec{u}_h}{Dt} + (\rho_0 + \hat{\rho} + \rho'(\vec{r}, t))f\vec{k} \times \vec{u}_h + \vec{\nabla}_h p = 0$$

$$(\rho_0 + \hat{\rho} + \rho'(\vec{r}, t))\frac{Dw}{Dt} + (\rho_0 + \hat{\rho} + \rho'(\vec{r}, t))g + \frac{\partial p}{\partial z} = 0$$

Recalling that $\rho_0 \gg \hat{\rho}, \rho'$, but also recalling from our scale analysis in Section 16.2.4 that g is large, we neglect *all* variations in ρ in these equations other than when ρ is multiplied by g. This gives us the Approximate Momentum Equations under the Boussinesq Approximation:

$$\boxed{\rho_0 \frac{D\vec{u}_h}{Dt} + \rho_0 f\vec{k} \times \vec{u}_h + \vec{\nabla}_h p = 0} \tag{19.1}$$

$$\boxed{\frac{Dw}{Dt} + \frac{\rho(\vec{r}, t)}{\rho_0}g + \frac{\partial p}{\partial z} = 0} \tag{19.2}$$

Take note that we can define a spatially/temporally varying analogue of the reduced gravity (in the shallow water system) $g' = \frac{\rho(\vec{r},t)}{\rho_0}g$, which, by stipulation, is much smaller than g. Furthermore, note

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that we did *not* need to assume that $\rho' \ll \hat{\rho}$ in deriving Equations 19.1 and 19.2. However, we will now assume that $\rho' \ll \hat{\rho}$ when we linearise the equations.

We now, in analogy to how we derived the dispersion relation before, linearise the equations of motion. We linearise about a state of rest $\vec{u} = 0$[1] and a continuous (given) stratification $\rho = \bar{\rho}(z)$ and pressure field $p = \bar{p}(z)$.

$$\vec{u}_h = \vec{0} + \vec{u}_h' \qquad\qquad\qquad p = \bar{p}(z) + p'$$
$$w = 0 + w' \qquad\qquad\qquad \rho = \bar{\rho}(z) + \rho'$$

We further assume that the system is approximately hydrostatic, therefore, the following applies:

$$\frac{d\bar{p}}{dz} = -\bar{\rho}g$$

Which equations are we linearising? We linearise Equations 19.1 (Horizontal Momentum), 19.2 (Vertical Momentum), and 16.3 (Energy Equation). We will also use Equation 16.10 (Incompressibility), but this equation does not need to be linearised. This will give us, in total, 5 equations.

Skipping over the details of linearisation, we derive the following linearised equations (Equation 19.1 becomes 19.3, 19.2 becomes 19.4, 16.10 is 19.5, and 16.3 becomes 19.6[2]):

$$\rho_0 \frac{\partial \vec{u}_h'}{\partial t} + \rho_0 f \vec{k} \times \vec{u}_h' + \vec{\nabla}_h p' = 0 \qquad (19.3) \qquad\qquad \vec{\nabla} \cdot \vec{u} = 0 \qquad (19.5)$$

$$\rho_0 \frac{\partial w'}{\partial t} + \rho' g + \frac{\partial p'}{\partial z} = 0 \qquad (19.4) \qquad\qquad \frac{\partial \rho'}{\partial t} + w \frac{\partial \bar{\rho}}{\partial z} = 0 \qquad (19.6)$$

We thus have five linear equations governing five variables: u, v, w, ρ, and p (I'm being lazy and have dropped the primes (the ′'s) for brevity). Note that $\bar{\rho}$ is a given background density field: you do not have to solve for this.

19.1.2 Aside: Buoyancy Oscillations

Let us first consider a state such that: $u = v = p = \frac{\partial w}{\partial z} = 0$, in which case all equations other than Equations 19.4 and 19.5 are trivially satisfied. If we differentiate Equation 19.4 with respect to t, we can substitute for $\frac{\partial \rho}{\partial t}$ using equation 19.6 to obtain the following equation:

$$\frac{\partial^2 w}{\partial t^2} + N^2 w = 0 \qquad\qquad\qquad (19.7)$$

where $N = \sqrt{-\frac{g}{\rho_0} \frac{\partial \bar{\rho}}{\partial z}}$ = the Brunt-Väisälä frequency. If you recall the discussion in section INSERT REFERENCE TO CLOUDS HERE in clouds, we have rederived the buoyancy oscillations as well as the buoyancy frequency!

This solutions describes columns of air (since $\frac{\partial w}{\partial z} = 0$) oscillating upwards and downwards with a frequency given by N. The restoring force is not simply the gravitational force, but the buoyancy force, which is weaker than gravity due to the buoyancy of the air in the background state.

19.1.3 Derivation

We now derive the dispersion relation for internal gravity waves. Tim derives the dispersion relation slightly different to how I personally prefer to derive it. As before, Tim's method is more general, but my method (in my humble opinion) is a bit easier to remember and churn out in an exam. I won't go

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[1]Note now that $\vec{U}$ is the 3-dimensional velocity, not just the horizontal velocity.

[2]The details for this are a bit complicated. I'd honestly recommend ignoring this, but if you're interested I'd refer to Vallis' textbook [6]. You'll need to first refer to page 41 where he rewrites the energy equation (he calls it the 'thermodynamic equation') in another form. Then skip to page 72 where he simplifies this equation using the Boussinesq approximation. Then finally he derives that $\frac{D\rho}{Dt} = 0$ if there is no heating.

through the details because they are straightforward. I encourage the reader to derive the equations themselves. The method is analogous to our derivation of the dispersion relation for Poincaré Waves: we set each variable proportional to an exponential function as follows:

$$\vec{u}_h = \vec{u}_0 \exp\left(ikx + ily + imz - i\omega t\right) \qquad p = p_0 \exp\left(ikx + ily + imz - i\omega t\right)$$
$$w = w_0 \exp\left(ikx + ily + imz - i\omega t\right) \qquad \rho = \rho_c \exp\left(ikx + ily + imz - i\omega t\right)$$

Next, we eliminate four out of the five coefficients (e.g., we eliminate the coefficients $\vec{u}_0$, w_0, and p_0, leaving everything written in terms of ρ_c, k, m, and ω). Next, we claim that $\rho_c \neq 0$ since we wish to find non-trivial solutions, and thus derive a relation between ω to k and m. It should be the following:

$$\omega^2 = \frac{N^2\left(k^2 + l^2\right) + f^2 m^2}{k^2 + l^2 + m^2} \tag{19.8}$$

19.1.4 Interpretation

We again assume the f-plane approximation so that we can take the meridional wavelength l to be infinite (i.e., set $l = 0$) without loss of generality. This gives us the following dispersion relation:

$$\omega^2 = \frac{N^2 k^2 + f^2 m^2}{k^2 + m^2} \tag{19.9}$$

First take a look at a few limits. If $N = 0$ (i.e., the density does not decrease or increase with height) or $k = 0$ (i.e., the meridional wavelength is infinite), then the following equation reduces to inertial oscillations. This is for two different reasons. If $N = \frac{\partial \rho}{\partial z} = 0$, there is no effect of gravity, because the buoyancy exactly cancels out the effect of gravity. Convsersely, if $k = 0$, the wavefronts are (recall Figure 18.3) parallel to the x-direction, hence heach

Conversely, if $f = 0$ or $m = 0$, we fe

To interpret this dispersion relation, let us first take the hydrostatic limit. Note that this is a *further* assumption we must make. While we have already assumed that the background state is hydrostatic, we have not assumed that the perturbations are hydrostatic yet.

If we assume that the *perturbations* are hydrostatic as well, which implies that the $\frac{\partial w'}{\partial t} = 0$ in Equation 19.4. If one follows through the derivation (I have not yet done this), this supposedly makes the k^2 term in the demoninator vanish, so that the dispersion relation is as follows:

$$\omega^2 = \left(\frac{N}{m}\right)^2 k^2 + f^2$$

We can now make a direct comparison between this dispersion relation and the analogous dispersion relation for Shallow Water Inertia-Gravity Waves. We can now identify the $\frac{N}{m}$ term as analogous to the $c = \sqrt{gH}$ term. We see now then that the speed of gravity waves is set by the background stratification (which is encoded, now, by the buoyancy frequency N but was encoded by the layer thickness H in the Shallow Water System) and now the vertical wavenumber m.

This is to be expected: now that we have moved to a fully three-dimensional system, we shoudl expect that the vertical dimension now plays some role in how waves propagate.

19.2 Quasi-Geostrophic Theory

19.2.1 Derivation

Assumptions

We first assume that $Ro < 1$, i.e., Ro is small but not extremely small. Second, we assume density variations are small so we can use the Boussinesq approximation. Third, we assume that the back-

ground pressure and density fields are hydrostatic. Fourth, we use the β-plane approximation. (Fifth) We further assume that $\frac{\beta y}{f_0} \sim \mathcal{O}(Ro)$.

Our strategy is to now decompose each field into a geostrophic and ageostrophic component:

$$\vec{u} = \vec{u}_g + \vec{u}_{ag}$$
$$p = \bar{p}(z)p_g + p_{ag}$$
$$\rho = \bar{\rho}(z)\rho_g + \rho_{ag}$$

such that

$$\frac{\partial \bar{p}}{\partial z} = -\bar{\rho}g$$

We further assume that the ratio of the magnitude of the ageostrophic comoponents to the geostrophic components are of order Ro. For example:

$$\frac{|\vec{u}_{ag}|}{|\vec{u}_g|} \sim O(Ro)$$

We aim to derive a simplified set of equations from Equations 16.11, 16.9, 16.10, and

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Key Idea VI.16: The Quasi-Geostrophic Equations

The Quasi-Geostrophic Equations, which govern a Boussinesq, hydrostatic fluid on the β-plane, evolves according to the Quasi-Geostrophic Potential Vorticity Equation if $Ro^2 \ll Ro \ll 1$.

$$\frac{D_g q}{Dt} = 0 \tag{19.10}$$

where $\frac{D_g}{Dt}$ = the geostrophic material derivative, q = the quasi-geostrophic potential vorticity, ψ = the streamfunction, $\vec{u}_g$ = the horizontal geostrophic velocity, p_g = the geostrophic pressure field, ρ_g = the geostrophic density field, and w_{ag} = the ageostrophic vertical velocity. These obey the following relations:

$$\frac{D_g}{Dt} = \frac{\partial}{\partial t} + \vec{u}_g \cdot \vec{\nabla} \tag{19.11}$$

$$q = f_0 + \beta y + \vec{\nabla}_h^2 \psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right) \tag{19.14}$$

$$\vec{u}_g = \vec{k} \times \vec{\nabla}_h \psi \tag{19.12}$$

$$p_g = \rho_0 f \psi \tag{19.15}$$

$$w_{ag} = -\frac{D_g}{Dt}\left(\frac{f_0}{N^2}\frac{\partial \psi}{\partial z}\right) \tag{19.13}$$

$$\rho_g = \frac{\rho_0 f_0}{g}\frac{\partial \psi}{\partial z} \tag{19.16}$$

19.2.2 Interpretation

Before interpreting all the equations, it's first important to note the *practical* achievement of these equations. We have drastically reduced the amount of variables we need to solve for by writing all the fields of interest (u_g, v_g, w_{ag}, p_g, and ρ_g) in terms of a single scalar field which is the streamfunction $\psi(\vec{r}, t)$. Furthermore, this streamfunction has a fully three-dimensional dependence, constituting our simple but fully 3D model of the atmosphere/ocean.

Interpretation of Equation 19.10 is straightforward and analogous to our interpretation of Conservation of Potential Vorticity for the Shallow Water System. I should note, though, that we should think of this equation as a linearised version, as we use the geostrophic material derivative, rather than the material derivative. Recall that we assume that the flow is to leading order geostrophic, so assuming that the advection is accurately characterised by just the geostrophic velocities is not a bad assumption.

Interpretation of 19.14 is also relatively straightforward. q is our quasi-geostrophic potential vorticity, an approximation of the potential vorticity. The terms are as follows:

$$q = \underbrace{f_0 + \beta y}_{f=\text{Planetary Vorticity}} + \overbrace{\vec{\nabla}_h^2 \psi}^{\xi=\text{Relative Vorticity}} + \underbrace{\frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right)}_{\text{Vortex Stretching/Squishing}}$$

The first two terms are identified straightforwardly. $f_0 + \beta y$ is manifestly a first approximation of the planetary vorticty, and $\vec{\nabla}_h^2 \psi$ is equal to the relative vorticity if one considers Equation 19.12.

The third term is a bit less straightforward. First consider the form of Conservation of Potential Vorticity for the Shallow Water System and expand Equation 18.6 as follows (after multiplying by h):

$$0 = h\frac{D}{Dt}\left(\frac{f+\xi}{h}\right)$$
$$= \frac{D}{Dt}(f+\xi) + \boxed{-\frac{f+\xi}{h}\frac{D}{Dt}(h)}$$

Writing in it this form separates the two components of potential vorticity. The first term expresses the change in the vorticity of a fluid parcel, while the second boxed term expresses the change in the thickness of the column. In, other words, the second term captures the vortex stretching/squishing effect of conservation of potential vorticity.

If we expand Equation 19.10 analogously we get the following:

$$0 = \frac{D_g}{Dt}\left(f + \xi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right)\right)$$
$$= \frac{D_g}{Dt}(f+\xi) + \boxed{f_0\frac{D_g}{Dt}\left(\frac{\partial}{\partial z}\left(\frac{f_0}{N^2}\frac{\partial \psi}{\partial z}\right)\right)}$$

We identify the boxed terms as analogous to each other: they both correspond to vortex stretching/squishing. Making this analogy allows us to interpret each component the third term more readily:

$$(f+\xi)\frac{1}{h}\frac{D}{Dt}(h) \longleftrightarrow f_0\frac{1}{\partial z}\left(-\frac{D_g}{Dt}\left(\frac{f_0}{N^2}\frac{\partial \psi}{\partial z}\right)\right)$$
$$f+\xi \longleftrightarrow f_0$$
$$\frac{1}{h} \longleftrightarrow \frac{\partial}{\partial z}$$
$$\frac{D}{Dt}(h) \longleftrightarrow -\frac{D_g}{Dt}\left(\frac{f_0}{N^2}\frac{\partial \psi}{\partial z}\right) = w_{ag}$$

Much of this can be argued from dimensional grounds, and, of course, they must match up dimensionally. The f_0 corresponds to the $f+\xi$ term, and is a good first order approximation because we assumed that Ro was small, and therefore that $\frac{\xi}{f}$ was small. The $\frac{1}{h}$ corresponds to the $\frac{\partial}{\partial z}$ [UNDER CONSTRUCTION IDK WHY LOL]

Finally, the The $\frac{D}{Dt}(h)$ corresponds to the w_{ag}. Recall that the interface height in the shallow water system is [ALSO UNDER CONSTRUCTION]

Equations 19.15 and 19.16 have a straightforward interpretation as well. The former is simply the constraint imposed by geostrophy, and the latter is the constraint imposed by hydrostasy.

19.3 Quasi-Geostrophic Rossby Waves

19.3.1 Derivation

As before with the Shallow Water System we linearise the Quasi-Geostrophic Potential Vorticity Equation about a background state. However, this time we assume that the background state is

a constant zonal flow, i.e., we linearise about the state $\vec{u}_g = (0, U, 0)$ with a constant background stratificaiton ($N = const$). In other words, we linearise about a streamfunction Ψ such that:

$$\psi = \Psi + \psi'$$
$$\psi' \ll \Psi$$
$$\Psi = -Uy$$

Specifying

We also use the β-plane approximation.
linearisation...

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\vec{\nabla}_h^2 \psi' + \frac{f_0^2}{N^2}\frac{\partial \psi'}{\partial z}\right) + \beta\frac{\partial \psi'}{\partial x} = 0$$

Key Idea VI.17: Quasi-Geostrophic Rossby Waves

$$\omega = Uk - \frac{\beta k}{k^2 + l^2 + \frac{f_0^2 m^2}{N^2}} \tag{19.17}$$

19.3.2   Interpretation

19.4   Instabilities and Turbulence

Chapter 20

Ocean Circulation

Throughout GFD, we have formulated models that apply to both the oceans and the atmosphere, but mostly with a focus on the atmosphere. Geostrophic and hydrostatic balance are accurate diagnostic relations in the ocean, our analysis using the shallow water system of waves is applicable to the oceans, aas is our discussion on instability and turbulence.

Now we pay explicit attention to just the oceans, but, as before, concepts here may be applicable to the atmosphere. As already touched upon, the oceans, while similar to the atmosphere in the sense that it is a fluid on a planet and so is affected by gravity (and so stratification) and rotation (and so coriolis), differs from the atmosphere dynamically in a few key ways.

First, the oceans are externally forced differently to the atmosphere. Recall our discussion in Section 15.3 that differential heating from the sun drives motion in the atmosphere: the atmosphere and oceans must transport energy poleward to close the energy budget. While the oceans accomplish this as well, the oceans are also, crucially, forced externally via atmospheric winds.

Second, unlike the atmosphere, the ocean has walls. This leads to effects in which we must consider. We considered this while discussing coastal 9.3 waves, for example, but we will consider this further when we incorporate effects like friction.

Third, the equation of state and typical sea-water conditions means that variations in density within the ocean are very small, and seawater is much denser than the atmosphere. This implies that, if we were to model the ocean using a reduced gravity model, g' would be much smaller, and therefore L_d is also shorter. If we were to model the ocean using a continuous stratification, N would be smaller, which, again, implies that L_d is shorter in the ocean.

Finally, Ro is much smaller in the ocean than in the atmosphere. f is comparable in both the atmospheres and oceans. Meanwhile, both L and U are smaller in the ocean. We can understand this somewhat as follows, but I emphasise that what I am about to say is by no means a *sufficient* explanation[1]. We expect L to be set by L_d, which recall depends on the stratification. Since N is smaller in the ocean, L is smaller in the ocean. Meanwhile, we expect U to be set by the energy and momentum budgets. While the atmosphere is directly forced by differential heating and latent heat releases, we expect (?) this energy source to be larger than the wind-stress the atmosphere exerts on the oceans, so we expect U to be larger in the atmosphere. I have only explained why both L and U are smaller in the ocean than in the atmospheres, but note crucially that I have not explained why U is smaller by a larger amount than L, and *this* is what is required to explain why Ro is smaller in the ocean than in the atmosphere.

We'll be exploring some effects of external forcing 20.1 and the small rossby number 20.2.

20.1   Ekman Transport (Shallow Ocean Circulation)

We begin by considering the effect of external forcing due to friction. For the ocean, this will be due to the atmosphere blowing over the ocean, which exerts a shear stress on the surface of the ocean, forcing the ocean to move in the same direction as the air. A **stress** is a force per unit area across a

---

[1] And to be honest I don't know a dynamically sound argument why Ro is smaller in the oceans. I think, however, an argument might go like this. First, note the energy recieved by the oceans and the atmospheres in $\mathrm{W\,m^{-2}}$

boundary. In this case, this is the boundary between the atmosphere adn the ocean. A **shear** stress is

Actually, phenomena we discuss here is applicable to the atmosphere as well: the ground exerts a shear stress on the atmosphere in the opposite direction that the atmosphere is flowing in!

Ekman transport is the transport due to the circulation of

Assume $Ro \ll 1$ so we can neglect the acceleration/advection term in Equation 16.11 and that $\rho = \rho_0 = const$:

$$f\vec{k} \times \vec{u}_h + \frac{1}{\rho_0}\vec{\nabla}_h p = \vec{F}_b$$

We now no longer neglect other forces, and reintroduce the horizontal stress force. We assume that the horizontal stresses vary the most vertically, which means that we can write F_b to a good approximation as follows:

$$\vec{F}_b = \frac{1}{\rho_0}\frac{\partial\vec{\tau}}{\partial z}$$

$$\vec{\tau} = \underbrace{\rho_0\nu\frac{\partial\vec{u}_h}{\partial z}}_{\text{viscous stress}} - \underbrace{\rho_0\overline{\vec{u}'_h w'}}_{\text{turbulent eddy stress}}$$

where $\vec{\tau}$ is the horizontal stress vector[2]. The second term is a bit more complicated, but just think of it as the effect of turbulent momentum fluxes which produce an effective force on the fluid (and read this[3] footnote if you want more). Ekman transport is therefore a three-way balance between coriolis, pressure gradient forces, and horizontal stress from friction.

We define the **Ekman Layer** as lying between $z = 0$, where the surface of the ocean is, and $z = -H$, where $z = -H$ when $\vec{\tau} = \vec{0}$. This specifies where $z = -H$, but note that we do not necessarily need to solve for H.

20.1.1 Ekman Spiral

20.1.2 Ekman Flux

We had to make the highly dubious approximation of the turbulent eddy stress term by a single (constant) turbulent eddy viscosity, we do not have to make the same assumption here. We again decompose the velocity $\vec{u}_h$ into a geostrophic component and an ekman (ageostrophic) component:

$$\vec{u}_h = \vec{u}_g + \vec{u}_{Ek}$$

where, again, $\vec{u}_g$ is defined as in Equation 17.8.

This implies that $\vec{u}_{Ek}$ is given by the following expression:

$$\vec{u}_{Ek} = -\frac{1}{\rho_0 f}\vec{k} \times \frac{\partial\vec{\tau}}{\partial z}$$

---

[2]If you are familiar with fluid mechanics, you will notice that, in reality, the stress is actually a tensor. [FOOTNOTE UNDER CONSTRUCTION]

[3]The explanation of the $\rho_0\overline{\vec{u}'_h w'}$ is as follows. First we perform a Reynolds Decomposition by splitting the velocity into a (time, space, or other suitably defined) average (indicated by a bar) and a fluctuation indicated by a prime:

$$\vec{u}_h = \overline{\vec{u}_h} + \vec{u}'_h$$
$$w = \overline{w} + w'$$

where the fluctuations average out to 0 (e.g., $\overline{w'} = 0$).

While the fluctuations by themselves might average out to 0, there might be correlations that don't average out to 0. The turbulent momentum flux is the average correlation between the horizontal velocity and the vertical velocity: if there are correlations, then this will result in momentum being advected downwards or upwards.

For example, if $\vec{u}'_h$ is more likely to be pointing eastward when $w' < 0$, then this will result in eastward momentum preferentially being advected downwards by vertical velocity fluctuations, which causes an effective momentum flux.

We now integrate this expression from $z = -H$ to $z = 0$, and note that we can remain completely ignorant of the precise expression for $\vec{\tau}$ and make use simply of our boundary conditions. We define $\vec{U}_{Ek}$ as follows:

$$\vec{U}_{Ek} \equiv \int_{z=-H}^{z=0} \vec{u}_{Ek} \, dz$$

$$= -\frac{1}{\rho_0 f} \vec{k} \times \int_{z=-H}^{z=0} \frac{\partial \vec{\tau}}{\partial z} \, dz$$

$$= -\frac{1}{\rho_0 f} \vec{k} \times \int_{\vec{\tau}=\vec{0}}^{\vec{\tau}=\vec{\tau}_s} \, d\vec{\tau}$$

$$\therefore \quad \boxed{\vec{U}_{Ek} = -\frac{1}{\rho_0 f} \vec{k} \times \vec{\tau}_s} \tag{20.1}$$

We therefore have an expression for the Ekman Flux $\vec{U}_{Ek}$: the column integrated fluid flux from wind-forcing. Note a key nuance that this flux is not a volume flux: the dimensions are not a volume per second, but an area per second. To find the volume flux, we would also need to integrate this expression horizontally perpendicular to $\vec{U}_{Ek}$. (E.g., if $\vec{U}_{Ek}$ is pointing northward, we would need to integrate this zonally, i.e., from west to east to find the volume flux.)

Physically, we have a very similar situation to Geostrophic Balance, for $\vec{U}_{Ek}$, we have a total balance between the horizontal stress $\vec{\tau}$ and the coriolis force. To make this more clear, we can write an expression for $\vec{U}_{Ek}$ another way:

$$f \vec{k} \times \vec{U}_{Ek} = \frac{1}{\rho_0} \vec{\tau}_s$$

which looks very analogous to Equation 17.7, which expresses geostrophic balance.

Note that, in the northern hemisphere, $\vec{U}_{Ek}$ points to the right of $\vec{\tau}_s$, and vice versal in the southern hemisphere.

We have so far considered the horizontal Ekman flux, but we now wish to consider the vertical Ekman flux. We again decompose the vertical velocity w into a geostrophic component[4] and an Ekman (ageostrophic) component.

We wish to find the Ekman vertical velocity at $z = -H$ which we write as w_{Ek}. To find w_{Ek}, we apply Equation 16.10, and integrate this equation again from $z = -H$ to $z = 0$, where we assume that $w = 0$ at $z = 0$[5].

We define w_{Ek} as follows:

$$w_{Ek} \equiv w(z = -H)$$

$$= w(z = -H) - w(z = 0)$$

$$= \int_{z=-H}^{z=0} \frac{\partial w}{\partial z} \, dz$$

$$= \int_{z=-H}^{z=0} \vec{\nabla} \cdot \vec{u} - \vec{\nabla}_h \cdot \vec{u}_{Ek} \, dz$$

$$= \vec{\nabla}_h \cdot \int_{z=-H}^{z=0} \vec{u}_{Ek} \, dz$$

$$= \vec{\nabla}_h \cdot \vec{U}_{Ek}$$

---

[4]Recall that $w_g = 0$ only if ρ and f do not vary, and while $\rho = \rho_0 = const$, we do not wish to make the f-plane approximation here!

[5]It suffices just to assume that $w(z = 0) \ll w(z = -H)$, which makes sense when we recall that $w = 0$ is set by the fluid interface, and the air-sea interface moves much slower than the the interface within th ocean (recall our derivation of the Reduced Gravity System)

$$\therefore \quad w_{Ek} = \vec{k} \cdot \vec{\nabla} \times \left(\frac{1}{\rho_0 f} \vec{\tau}_s \right) = \left[\vec{\nabla} \times \left(\frac{1}{\rho_0 f} \vec{\tau}_s \right) \right]_z \tag{20.2}$$

In other words, the ekman upwelling is equal to the (z-component of the) vorticity of the $\frac{1}{\rho_0 f} \vec{\tau}_s$. This could come from f, or it could come from the surface wind-stress field $\vec{\tau}_s$.

20.2 Sverdrup Balance (Deep Ocean Circulation)

20.2.1 Conservation of Potential Vorticity

We now turn our focus to below the Ekman layer, i.e., to $z < -H$. We crucially assume that the flow is geostrophic below the Ekman layer (owing to the small Ro in the ocean). However, we do *not* make the f-plane approximation, and instead allow the coriolis parameter to vary according to the β-plane approimation.

This implies (exercise) that geostrophic flow is no longer incompressible in the horizontal direction, and so to maintain incompressibility overall, there must be upwelling or downwelling. This, upwelling or downwelling will end up being set by the ekman upwelling or downwelling, which is why surface wind-stress can have an effect deep in the ocean. We also assume that the density $\rho \approx \rho_0$ is constant.

We first consider Equation 16.10, and allow the horizontal velocities u and v to be geostrophic (i.e., set by Equation 17.7):

$$
\begin{aligned}
0 &= \vec{\nabla} \cdot \vec{u} \\
&= \vec{\nabla}_h \cdot \vec{u}_h + \frac{\partial w}{\partial z} \\
&= \frac{1}{\rho_0} \vec{\nabla}_h \cdot \left(\frac{1}{f(y)} \vec{k} \times \vec{\nabla}_h p \right) + \frac{\partial w}{\partial z} \\
&= \frac{1}{\rho_0} \vec{\nabla}_h \left(\frac{1}{f} \right) \cdot \left(\vec{k} \times \vec{\nabla}_h p \right) + \frac{1}{\rho_0 f} \vec{\nabla}_h \cdot \vec{k} \times \vec{\nabla}_h p + \frac{\partial w}{\partial z} \\
0 &= -\frac{\beta}{f} v + \frac{\partial w}{\partial z}
\end{aligned}
$$

Integrating this now from $z = -D$ (the depth of the ocean) to $z = -H$ and assuming the $w(z = -D) = 0$, we find an equation for the depth-integrated meridional velocity: [UNDER CONSTRUCTION]

20.2.2 Western Boundary Currents: The Effects of Friction

20.3 Box Models of the Meridional Overturning Circulation

In this section we aim to construct a simple box model of the **Atlantic Meridional Overturning Circulation** (AMOC). The AMOC is a complex overturning circulation, consisting of warm waters which flow from the Southern Ocean northward, before turning cold and salty, sinking in the North Atlantic, and flowing southward to complete the loop.

This is a somewhat oversimplified version of the **MOC**, but I sadly don't have the time nor energy to get into it at this point.

Crucially, though, the water must sink in the North Atlantic, and this requires the water to be denser or at least as dense as the water below it. We must therefore consider the **equation of state** that governs the density of sea water.

The equation of state is horribly complicated, and there is (to the best of my knowledge) no theoretical way to derive it. Instead, we use an empirically found Gibbs function of seawater G to calculate the equation of state. Recalling that $dG = -S\,dT + \rho\,dp + \ldots$, you can see that $\rho = \partial G / \partial p$.

This, still, is too complicated for our purposes, so we simply linearise the equation of state and assume that water is incompressible (water is actually compressible).

$$\rho = \rho_0(1 - \alpha(T - T_0) + \beta(S - S_0)) \tag{20.3}$$

Definitions and Symbols

References

[1] Wikipedia. *Earth's internal heat budget.* URL: https://en.wikipedia.org/wiki/Earth's_internal_heat_budget. (accessed: 09.10.2025).

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[3] R T Pierrehumbert. *Principles of Planetary Climate.* Cambridge University Press, 2010. ISBN: 9780521865562.

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[5] Wikipedia. *Pressure altimeter.* URL: https://en.wikipedia.org/wiki/Pressure_altimeter. (accessed: 20.07.2025).

[6] G K Vallis. *Atmospheric and oceanic fluid dynamics : fundamentals and large-scale circulation.* Cambridge University Press, 2017, p. 847. ISBN: 9781107588417.