

# C5: Physics of Atmospheres and Oceans Lecture Notes

Authors

Date of Last Update

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# Introduction

These set of lecture notes for the C5: Atmospheres and Oceans options are based on the lectures delivered during the academic year 2024-2025. The lectures that year were delivered by **Raymond T. Pierrehumbert** (for **Thermodynamics** and **Radiative Transfer**), **Philip Stier** (for **Clouds**), **Neil Bowles** (for **Instrumentation**), **Myles Allen** (for **Climate Dynamics**), and **Tim Woollings** (for **Geophysical Fluid Dynamics**).

C5 is a massive course, and because it's very 'applications'-focused, it doesn't feel as unified as some other options are. This document is the result of an attempt to compile a self-consistent and thorough set of lecture notes on the course. As such, these notes will be long: I will at many points prioritise rigour and detail over brevity, and will link to where you can look if you want more information. However, I will try to emphasise (using coloured boxes!) key ideas to make these notes more skimmable. Furthermore, there is a limit to how thorough I can be: because C5 is very applied, it will build on the results of a great many sub-fields, many of which take a whole lecture series themselves to explain.

In reading these notes, you should be aware of two sources of biases: those of the lecturers and those of the author(s).

In the first case, it is not uncommon for lecturers gloss over material in lectures. For example, the 2024-2025 lectures did not feature scattering in **Radiative Transfer**, and so I will not cover scattering in these lecture notes. Furthermore, some of what is covered is on the very frontiers of modern research. As such, it's not surprising that the lecturers will have certain beliefs that have not (yet?) made it into the scientific consensus.

The second case is far more problematic. First, the author is just not passionate about Instrumentation. That chapter will be empty. Sorry. If you know someone (or you yourself) would be interested in filling that in, please let me know! Second, in some cases I present material differently to how the lecturers present them in attempt to present material in a way I personally find less confusing and more physically intuitive.

Third, and most importantly, I can't claim to be an expert in the topic, and so will get things wrong! Please get in touch if you spot any errors or believe any section is written confusingly. I would much rather you contact me about an 'error' that's actually correct than I miss an error!

**Part I**

**Thermodynamics**

# Introduction

This section of the course was lectured by [Raymond T. Pierrehumbert](#) covering basic Atmospheric Thermodynamics. While most concepts here will be applicable to oceanic physics, we will only be explicitly applying the concepts learnt in this section to atmospheric physics.

This section consists of three chapters:

1. [Basic Thermodynamic Concepts](#):

We recap some basic thermodynamic concepts you should be familiar with, like pressure, [ideal gases](#), and [heat capacity](#). We then explain importantly how we extend such concepts to deal with gases consisting of [multiple constituents](#), and introduce the important approximation of [Hydrostatic Balance](#).

2. [Dry Thermodynamics](#):

We focus primarily on the vertical temperature structure of the atmosphere. We explain some of this by deriving the [Dry Adiabats](#), which governs the temperature of a convecting parcel of air. Next, we consider convection, and derive a [criterion](#) of whether an atmosphere will be unstable to convection and define a [few terms](#) relating to convection.

3. [Moist Thermodynamics](#):

We extend the previous section to apply to atmospheres which have constituents which condense. We derive the [Moist Pseudo-Adiabats](#), the moist counterpart to the [Dry Adiabats](#). We do not discuss moist convection here, and instead discuss it in Part [III](#).



# Chapter 1

## Basic Thermodynamic Concepts

### 1.1 Definitions

#### 1.1.1 Basic Thermodynamic Definitions

We should review (or learn) the basic state variables of thermodynamics. For Atmospheric and Oceanic Physics, we will almost always work with *Intensive* variables: variables that are independent of the size of a system. For example, ‘number of particles’ is not an intensive variable, because the number of particles changes (doubles) if you double the size of the system. However, ‘number of particles *per unit volume*’ is an intensive variable. In this case, we’ll need to know:

Symbol	Name	Units	Meaning
$p$	Pressure	Pa (Pascal); bar (Bar); $\text{N m}^{-2}$ (Newtons per square metre);	The force exerted by the fluid in <i>all</i> directions at some location. $1 \text{ bar} \approx 10^5 \text{ Pa}$ .
$T$	Temperature	K (Kelvin)	A measure of the heat content of the system.
$\rho$	Density	$\text{kg m}^{-3}$ (Kilograms per metre cubed)	The mass per unit volume.
$n_a$	Number Density	$\text{m}^{-3}$ (Inverse metre cubed)	The number of molecules of some substance $a$ in one metre cubed.
$M_a$	Molar mass	$\text{kg mol}^{-1}$ (Kilograms per mole)	The mass of $N_A \sim 6.02 \times 10^{23}$ molecules (one mole) of $a$ .

Table 1.1: Basic Intensive Thermodynamic Variables

#### 1.1.2 Multiple Constituents

In general, the atmosphere is composed of multiple constituents. We (or the examiner) must make two ultimately arbitrary decisions when referring to a constituent’s abundance. First, we can either refer to a constituent’s abundance by its *mass*, or by its *number* (in *moles*). Second, we can either refer to it as a *concentration* or a *ratio*. These choices are ultimately arbitrary because one may freely convert between if one knows the molar mass of all constituents.

Suppose our atmosphere is composed of constituents  $\mathbb{S} = \{a, b, c, \dots\}$ . Each species  $i \in \mathbb{S}$  has a molar mass of  $M_i$  and a number density of  $n_i$ . Then we can refer to its abundance as follows. All quantities are dimensionless.

#### Key Idea 1.1: Ways of Referring to Constituent Quantities

We can refer to the amount of a constituent in an atmosphere as follows:

	Fraction/Concentration	Mixing Ratio
Mole	Mole Fraction or Molar Concentration $\frac{n_a}{\sum_{i \in \mathbb{S}} n_i}$	Molar Mixing Ratio or Volume Mixing Ratio $\frac{n_a}{\sum_{i \in \mathbb{S}, i \neq a} n_i}$
Mass	Mass Fraction or Mass Concentration $\frac{\rho_a}{\sum_{i \in \mathbb{S}} \rho_i} = \frac{M_a n_a}{\sum_{i \in \mathbb{S}} M_i n_i}$	Mass Mixing Ratio $\frac{\rho_a}{\sum_{i \in \mathbb{S}, i \neq a} \rho_i} = \frac{M_a n_a}{\sum_{i \in \mathbb{S}, i \neq a} M_i n_i}$

All formulations are equivalent, as one can freely convert between all four expressions using algebra or by using the following formula:

$$\rho_A = \frac{n_A M_A}{N_A} \quad (1.1)$$

where  $N_A = \text{Avogadro's Number} \sim 6.02 \times 10^{23} \text{ mol}^{-1}$  (it's on your formula sheet).<sup>1</sup>

We also introduce the concept of 'dilute': some constituent  $a$  is in the dilute limit if and only if:

$$n_a \ll \sum_{i \in \mathbb{S}, i \neq a} n_i \text{ and/or } \rho_a \ll \sum_{i \in \mathbb{S}, i \neq a} \rho_i$$

In the dilute limit, molar and/or mass fractions/concentrations are equivalent to mixing ratios. However, note that you're not always allowed to make this assumption (especially in exams!). There is also the ambiguity in the 'and/or', however oftentimes we are in both the molar and mass dilute limit since the molar masses  $M_i$  are all of similar size.

Oftentimes, one refers to dilute constituents in terms of *ppm* (parts per million) or *ppmv* (parts per million volume). *ppm* is the *Mass Fraction* or *Mass Concentration* multiplied by  $10^6$  while *ppmv* is the *Mole Fraction* or *Molar Concentration* multiplied by  $10^6$ .

## Atmospheric Composition: Planetary Examples

In *Mole-Fraction*:

- Earth's Atmosphere:
  - Nitrogen  $\text{N}_2$ : 0.78
  - Oxygen  $\text{O}_2$ : 0.21
  - Argon Ar: 0.0093
  - Carbon Dioxide  $\text{CO}_2$ : 0.000430 (430 *ppmv*)<sup>2</sup>
  - Water Vapour  $\text{H}_2\text{O}$ : A few percent.<sup>3</sup>
- Venus' Atmosphere:
  - Carbon Dioxide  $\text{CO}_2$ : 0.965
  - Nitrogen  $\text{N}_2$ : 0.035
  - Sulfur Dioxide  $\text{SO}_2$ : 150 *ppmv*
- Jupiter's (Outer) Atmosphere:
  - Hydrogen  $\text{H}_2$ : 0.86
  - Helium  $\text{He}_2$ : 0.136

## 1.2 Ideal Gases

### 1.2.1 Single Constituent Atmosphere

An **Ideal Gas** is a theoretical (imaginary) gas consisting of molecules which interact only via perfectly elastic collisions. No real gas is ideal, but many gases, including 99% of Earth's atmosphere (Nitrogen,

<sup>1</sup>To avoid a small confusion, note how only number density  $n_i$  is used in the definition for the *molar* concentrations/mixing ratios. This is because the number of moles per volume of a substance is directly proportional to the number density:  $n_i^{\text{mol}} = n_i N_A$ . As such the  $N_A$ 's simply cancel top and bottom in the fraction.

<sup>2</sup>At time of writing!

<sup>3</sup>This strongly depends on time and location due to dynamics discussed in [Moist Thermodynamics](#).

Oxygen, and Argon), behave approximately like ideal gases under atmospheric conditions like ours. We start from the version of ideal gas law you have probably seen before, now focusing on a gas made up of a single constituent  $a$ :

$$pV = N_a k_B T \quad (1.2)$$

where  $p$  = pressure,  $V$  = volume,  $N_a$  = number of molecules of  $a$ ,  $k_B$  = Boltzmann's constant, and  $T$  = temperature. We can divide by the volume, then multiply and divide by  $\frac{M_a}{N_A}$  to get:

$$\begin{aligned} p &= n_a k_B T && \text{; Divide 1.2 by } V \\ &= \left( \frac{n_a M_a}{N_A} \right) \left( \frac{k_B N_A}{M_a} \right) T && \text{; Multiply and divide by } \frac{M_a}{N_A} \\ &= \rho_a \left( \frac{k_B N_A}{M_a} \right) T && \text{; Use 1.1 to substitute for } \rho_a \\ &= \rho_a \left( \frac{R^*}{M_a} \right) T && \text{; } R^* \equiv k_B N_A = \text{gas constant} \\ &= \rho_a R_a T && \text{; } R_a \equiv \frac{R^*}{M_a} = \text{specific gas constant} \end{aligned}$$

The **Gas Constant**  $R^*$  is defined as:  $R^* \equiv k_B N_A \approx 8.314 \text{ J mol}^{-1} \text{ K}^{-1}$ . We further define the **Specific Gas Constant**  $R_a \equiv \frac{R^*}{M_a}$  which has units of  $\text{J kg}^{-1} \text{ K}^{-1}$ . **From now on I will refer to the specific gas constant  $R_a$  by just writing just  $R$ .** The final line gives us the version of the ideal gas law we will use most often as it features only *intensive* variables:

#### Key Idea I.2: Ideal Gas Law in Intensive Variables

The ideal gas law featuring only intensive variables of  $p$ ,  $\rho$ ,  $R$ , and  $T$ .

$$p = \rho R T \quad (1.3)$$

$R$  is the specific gas constant, defined as the universal gas constant  $R^*$  divided by the molar mass  $M_a$ .

$$R = \frac{R^*}{M_a} \quad (1.4)$$

### 1.2.2 Multiple Constituents in an Atmosphere and Dalton's Law

What if we now have multiple constituents in an atmosphere, all with different molar masses  $M_i$ ? We define the *partial pressure*  $p_a$  of some constituent  $a$  as the pressure the gas *would* have if you removed all other constituents and left  $a$  on its own. The total pressure is the sum of all the partial pressures. Dalton's Law (1.5) allows us to relate the partial pressure of an individual constituent  $p_a$  to the total pressure if all constituents are ideal gases:

$$\frac{p_a}{\sum_{i \in \mathbb{S}} p_i} = \frac{n_a}{\sum_{i \in \mathbb{S}} n_i} \quad (1.5)$$

So 1.5 says that the partial pressure (in an ideal gas) is set by the *mole fraction/molar concentration*.<sup>4</sup> It is not affected in any way by the *mass fraction* or *mass mixing ratio*. Now we can apply ideal gas

<sup>4</sup>In the dilute limit, it is equivalently set by the *molar mixing ratio*.

law individually to each constituent:

$$\begin{aligned} p_a &= n_a k_B T \\ p_b &= n_b k_B T \\ &\vdots \end{aligned}$$

Summing up then creatively multiplying by 1 we get:

$$\begin{aligned} p &= \left( \sum_{i \in \mathbb{S}} n_i \right) k_B T && \text{; Summing 1.2 for each constituent} \\ &= \left( \frac{\sum_{i \in \mathbb{S}} n_i M_i}{\sum_{i \in \mathbb{S}} n_i M_i} \right) \left( \frac{N_A}{N_A} \right) \left( \sum_{i \in \mathbb{S}} n_i \right) k_B T && \text{; Divide \& Multiply} \\ &= \left( \frac{\sum_{i \in \mathbb{S}} n_i M_i}{N_A} \right) \left( \frac{\sum_{i \in \mathbb{S}} n_i M_i}{\sum_{i \in \mathbb{S}} n_i} \right)^{-1} (N_A k_B) T && \text{; Regroup terms} \\ &= \left( \sum_{i \in \mathbb{S}} \frac{n_i M_i}{N_A} \right) (\bar{M}^{-1} R^*) T && \text{; Define the effective molar mass } \bar{M} \\ p &= \rho R T && \text{; Define the specific gas constant } R \end{aligned}$$

We should focus on the last three lines of algebra to remember two key facts:

1. The specific gas constant  $R$  for a multiple constituent gas is the gas constant  $R^*$  divided by the total effective molar mass  $\bar{M}$  (analogous to the single component case):

$$R = \frac{R^*}{\bar{M}} \quad (1.6)$$

2. The effective molar mass is the average of the individual molar masses  $M_i$  **weighted by particle number**  $n_i$ :

$$\bar{M} = \frac{\sum_{i \in \mathbb{S}} n_i M_i}{\sum_{i \in \mathbb{S}} n_i} \quad (1.7)$$

In the single component case,  $\bar{M} = M_a$ .

For the Earth (exercise),  $\bar{M} \approx 29 \times 10^{-3} \text{ kg mol}^{-1}$  so the **Specific Gas Constant**  $R \approx 0.287 \text{ J kg}^{-1} \text{ K}^{-1}$  (it will be on your formula sheet).

### 1.3 Heat Capacity and the Equipartition Theorem

The heat capacity of an object is a measure of how much heat (which has dimensions of energy) is required to change an object's temperature. The heat capacity itself often depends on the temperature. Let  $dQ$  = the heat transferred to an object and  $dT$  = the change in temperature of the object. Then the heat capacities of the object are:

$$c_v = \left( \frac{dQ}{dT} \right)_v \quad (1.8)$$

$$c_p = \left( \frac{dQ}{dT} \right)_p \quad (1.9)$$

where the subscripts of  $V$  and  $p$  mean that the volume and pressure, respectively, are held constant as heat is added/removed from the system. Notationally, when we write  $c_v$  or  $c_p$ , we usually mean the specific heat capacities: the heat capacities per mole or, more commonly, per mass. Similar to  $R$ , if not specified, assume that  $c_p, c_v$  are heat capacities *per unit mass*. (Remember, *intensive* variables!)

It turns out, and one can derive this via statistical mechanics, that at high enough temperatures systems obey the Equipartition Theorem:

$$c_v \approx \frac{f}{2}R \quad (1.10)$$

where  $f$  = the number of *fully excited* degrees of freedom an individual constituent has. So for a hot enough monatomic gas,  $f = 3$ , since monatomic gases have 3 translational degrees of freedom (forwards/backwards, up/down, right/left). For a hot enough linear diatomic gas (as seen in Fig. 1.1),  $f = 7$ , since diatomic gases have 3 translational degrees of freedom, 3 rotational degrees of freedom (spinning about the  $x, y, z$  axes), and 2 vibrational degrees of freedom<sup>5</sup>.

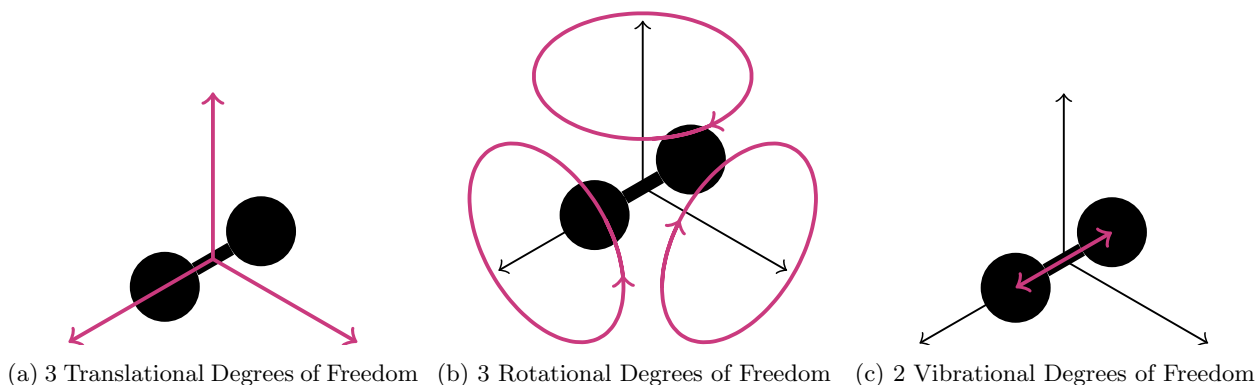


Figure 1.1: The 7 Degrees of Freedom for a Diatomic Molecules

### 1.3.1 "Excited" and "Frozen-out" Degrees of Freedom

But what does *fully excited* degrees of freedom mean? The constituents of gases obey the laws of quantum mechanics (as do all things, we think). One way in which quantum mechanics differs from classical mechanics is that oftentimes (but not always) the energy a system can have is discretised rather than continuous, i.e., a system can only have energies specifically equal to  $E$  or  $2E$  or  $1.8E$ . This means that the system can only receive or give energy in discrete amounts called 'quanta' – if you can't give them that exact amount of energy (or integer multiples of that), then you can't give it to them at all.

This leads to a uniquely quantum mechanical effect (that also explains the ultraviolet catastrophe): if the temperature is too low, the molecule might only get  $0.001E$  energy, and is highly unlikely to get energy even close to  $E$ . As such, the molecule will never get enough energy to excite that degree of freedom: we say that that degree of freedom is 'frozen' out. More rigourously, if  $k_B T \ll E$ , where  $E$  is the size of the energy quanta of some degree of freedom, then that degree of freedom is frozen out.

The size of this quanta of energy  $E$  depends on the system, but for the constituents in atmospheres, the quanta for vibrational degrees of freedom tend to be very large. For example, the energy of the

<sup>5</sup>One might expect there to be only 1 vibrational degree of freedom (i.e., they can only vibrate in and out... right?). The reason why it's 2 is slightly complicated, and requires some background knowledge in statistical mechanics. In the derivation for the equipartition theorem,  $f$  = the number of quadratic terms in the Hamiltonian. For translations,  $H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)$  so  $f = 3$ , and similarly for rotations. For vibrations, we approximate the Hamiltonian as a simple harmonic oscillator (which it will be if the vibrations are small enough), so there are *two* quadratic terms, arising from the kinetic and potential energies in the Hamiltonian (i.e., a spring):  $H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2$ .

Looking ahead, one can calculate the characteristic energy scale  $E$  of  $N_2$ 's vibrational mode by solving analytically for the eigenvalues of the simple harmonic oscillator and finding the difference. One then only needs to quote two empirical constants:  $m$  = the reduced mass of the  $N_2$ , and  $\omega$  (which encodes the strength of the bond between the N atoms).

first vibrational mode of Nitrogen is  $E \sim 4.7 \times 10^{-20}$  J. This seems small until you remember that this is *per molecule*. The temperature required for  $k_B T \sim E$  is  $T \sim E/k_B \sim 3400$  K.

Earth's mainly consists of a diatomic gas, so we might expect  $f = 7$ . However,  $f \approx 5$ , because the translational degrees of freedom are all fully excited, but only 2 rotational degrees of freedom are excited and none of the vibrational degrees of freedom are. Here I have drawn the excited degrees of freedom as thick magenta lines, and the frozen degrees of freedom as dashed thin magenta lines:

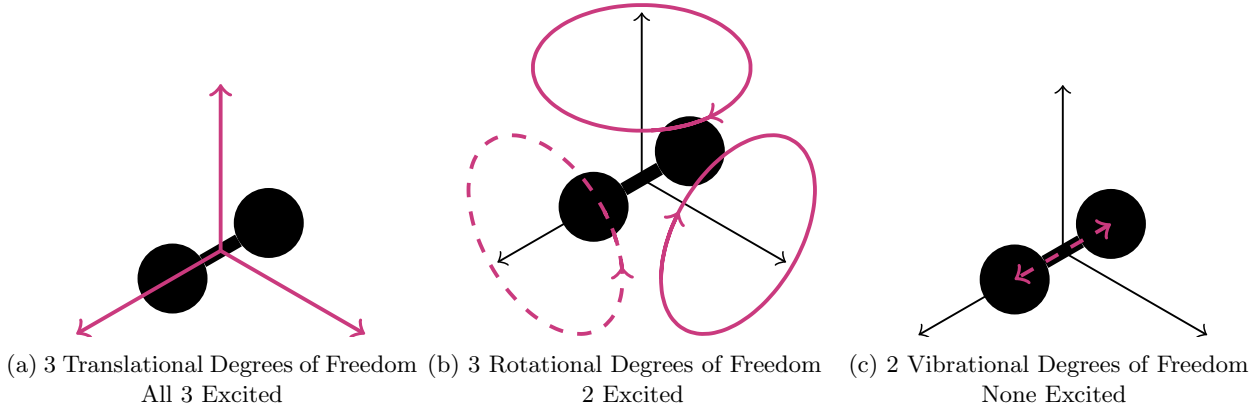


Figure 1.2: The 5 Excited Degrees of Freedom for a Diatomic Molecules in Earth-Like Conditions

## 1.4 Hydrostatic Balance

We now consider the force balance of a slab of atmosphere atmosphere:

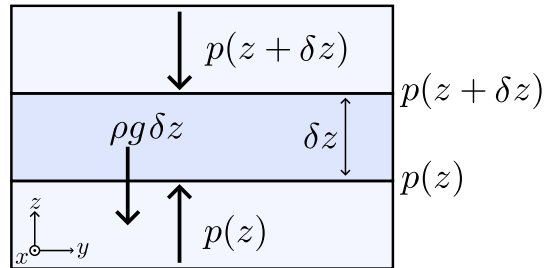


Figure 1.3: Forces (per unit area) on the middle slab of air.

The *hydrostatic approximation* assumes that these are the dominant forces on the slab, and that the acceleration of the slab of air is negligible (which is a pretty good assumption). Therefore:

$$0 = p(z + \delta z) - \rho g \delta z + p(z)$$

Rearranging and letting  $\delta z \rightarrow 0$  gives us **Hydrostatic Balance**:

### Key Idea I.3: Hydrostatic Balance

*Hydrostatic Balance governs the vertical pressure variation in a fluid if we assume that vertical acceleration (and advection) is small.*

$$\frac{dp}{dz} = -\rho g$$

(1.11)

*Pressure decreases with height in order to balance the force of gravity.*

There are two very useful facts that should be noted at this time.

First note that  $\rho, g > 0$ , so  $\frac{dp}{dz} > 0$  always. This means that  $p$  is monotonic in  $z$ , implying that there is always a one-to-one relation between  $p$  and  $z$ . This allows us to convert between **height coordinates** and **pressure coordinates**. For example, in *height coordinates*, we might let  $\vec{u} = \vec{u}(x, y, z)$ ,  $p = p(x, y, z)$ . In this case, we would have pressure  $p$  at some height  $z$  (and horizontal position) as a variable. In *pressure coordinates*, on the other hand, we let  $\vec{u} = \vec{u}(x, y, p)$ ,  $z = z(x, y, p)$ . In this case, we would have the height  $z$  of some given pressure surface  $p$  (and horizontal position) as a variable.

We will consistently be plotting plotting atmospheric properties (e.g., humidity) in pressure coordinates, rather than height coordinates, throughout this course. This is for three reasons. First, it is much easier for instruments to measure the pressure than the height. In fact, this is how airplanes measure ‘barometric’ height using a pressure altimeter [1]. Second, most problems we encounter will require only knowledge of a variable’s dependence on pressure. Third, pressure is a fundamental thermodynamic variable, and as such also gives us information about properties of the system, as well as the (rough) height.

Second note that if we rewrite Eqn. 1.11 in differential form, we get a very useful relation between the difference in pressure between two vertically separated points, and the mass per unit area within that layer.

$$\begin{aligned} dp &= -g(\rho dz) \\ &= -g dm \end{aligned} \quad (1.12)$$

where  $dm = \rho dz$  is the mass per unit area within that layer. Therefore (integrating), we get that pressure difference  $\Delta p$  between the bottom and top of a layer of atmosphere is directly proportional to the mass per unit area within that layer. Take a moment to appreciate this: we can measure the mass of a layer of atmosphere by only measuring the pressure difference between the top and the bottom! Integrating Equation 1.12 from the bottom of the atmosphere ( $p = p_s, m = 0$ ) to the top of the atmosphere ( $p = 0, m = m_{tot}$ ) (and assuming that variation in  $g$  is negligible) gives us:

$$m_{tot} = \frac{p_s}{g} \quad (1.13)$$

The surface pressure thus gives an immediate estimate of the amount of *stuff* in a planet’s atmosphere. Again considering Earth and Venus’ atmospheres:

- |  |  |
|--|--|
| • Earth:   | • Venus:   |
| – Surface Pressure:                                  | – Surface Pressure:                                  |
| $p_s \approx 1 \times 10^5 \text{ Pa}$               | $p_s \approx 93 \times 10^5 \text{ Pa}$              |
| – Gravitational Acceleration:                        | – Gravitational Acceleration:                        |
| $g \approx 9.81 \text{ m s}^{-2}$                    | $g \approx 8.87 \text{ m s}^{-2}$                    |
| – Atmospheric Mass:                                  | – Atmospheric Mass:                                  |
| $m_{tot} \approx 1.02 \times 10^4 \text{ kg m}^{-2}$ | $m_{tot} \approx 1.05 \times 10^6 \text{ kg m}^{-2}$ |

Let us now apply all this to find the mass of  $\text{CO}_2$  in the atmosphere:

$$\begin{aligned} m_{\text{CO}_2} &= \underbrace{\frac{p_s}{g}}_{\text{Mass of Atmosphere}} \underbrace{\frac{\overbrace{12 \times 10^{-3}}^{\text{molar mass of CO}_2}}{\underbrace{29 \times 10^{-3}}_{\text{M of air}}} \underbrace{430 \times 10^{-6}}_{\text{mole-fraction of CO}_2}}_{\text{mass-fraction of CO}_2} \\ &\approx 1.5 \text{ kg m}^{-2} \end{aligned}$$

So not much actually (but a lot radiatively)!

### 1.4.1 Example 1: Constant Density

In the ocean, for example, we can make the (sometimes) good approximation that  $\rho = \text{const.}$  We can then integrate 1.11 with the boundary condition that  $p(z = \eta) = p_s$  where  $p_s$  is, for example, the pressure at the surface of the ocean and  $\eta$  is the height of the surface. Note that this derivation is completely general, and that both  $p_s$  and  $\eta$  may in general be functions of  $x, y$ , and  $t$ , and  $\eta$  could be, for example, below the ocean surface at some interface.

$$\begin{aligned} \frac{dp}{dz} &= -\rho g \\ \int_{p_s}^p dp &= - \int_{\eta}^z \rho g dz \\ p(x, y, z, t) - p_s(x, y, t) &= \rho g(\eta(x, y, t) - z) \\ \therefore \boxed{p(x, y, z, t) &= \rho g(\eta(x, y, t) - z) + p_s(x, y, t)} \end{aligned} \quad (1.14)$$

We use this to derive the Shallow Water/Reduced Gravity System in Chapter 17.

### 1.4.2 Example 2: Ideal Gas

The atmosphere can be roughly regarded as an ideal gas, we can substitute in Equation 1.3 to find that Equation 1.11 transforms into:

$$\boxed{\frac{d \ln p}{dz} = -\frac{g}{RT}} \quad (1.15)$$

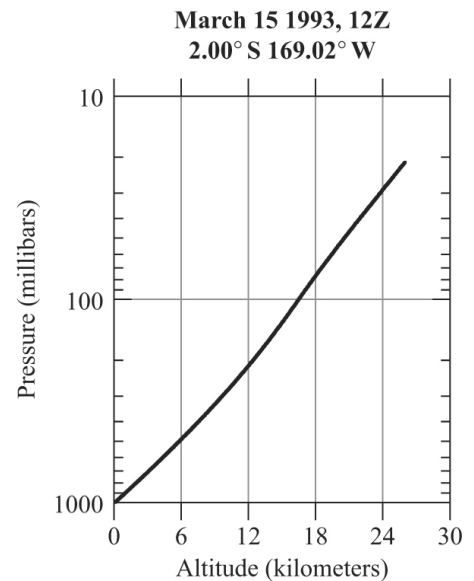
If we make the approximation that  $T(z) \sim \text{const}$  (which is not a bad approximation in Kelvin) we can integrate to find that  $p$  falls off exponentially.

$$p(z) = p(z = 0)e^{-z/H}$$

where  $H = \frac{RT}{g}$  is the scale height. As you can see below, this is actually pretty accurate!



(a) A Radiosonde! Also called a weather balloon/sounding balloon. It's released and floats upwards, continually measuring the pressure, temperature, and humidity of the air around it. Image from the Radiosonde Museum of North America [2].



(b) Radiosonde measurement of the Earth. Note the log scale on the pressure axis, so if  $p \sim e^{-z}$  we should expect a straight line, which we remarkably see. Notice how smooth the measurement is, despite the fact that it's a single radiosonde at a single (horizontal) point in space. Figure from Ray's book [3].



### 1.4.3 Example 3: Solid Rock Planet

If we want to include curvature (since planets are spheres), we can derive an analogous [Hydrostatic Relation](#) where:

$$\begin{aligned}\frac{dp}{dr} &= -\rho g \\ &= -\rho \frac{GM(r)}{r^2}\end{aligned}$$

We can pair this with a coupled differential equation for the mass  $M(r)$ :

$$\frac{dM}{dr} = 4\pi r^2 \rho$$

To solve these equations, as before, we need an expression for  $\rho(r)$  and boundary conditions. The former is typically given by the equation of state governing the material in question. For the Earth, for example, we might approximate  $\rho \approx \text{const}$  below the ground and  $\rho$  as ideal as in [Example 2](#). For the latter, we know that  $M(r=0) = 0$  and we can specify  $p(r=0) = p_0$  where  $p_0$  is unknown. We can then integrate the equations, then solve for  $p_0$  (e.g., if we know  $p(r=r_s) = p_s$  we can write  $p_0$  in terms of  $p_s$ ).

## Chapter 2

# Dry Thermodynamics

### 2.1 The Vertical Structure of Atmospheres

We begin with some empirical observations of planetary atmospheres:

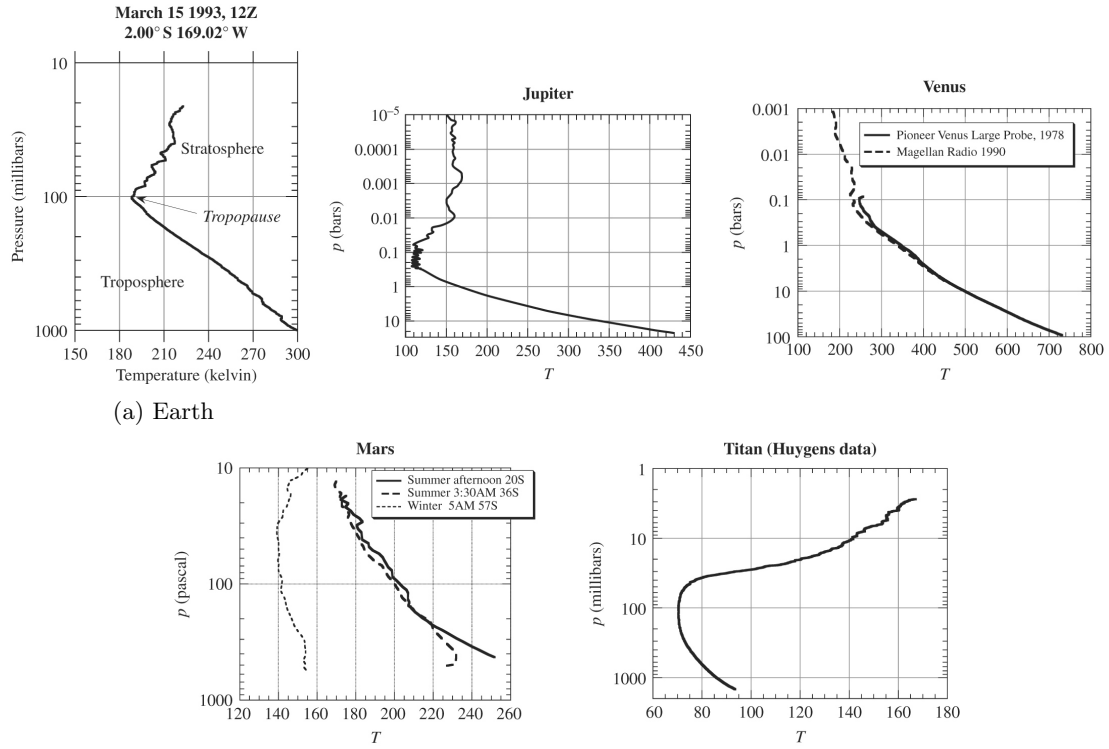


Figure 2.1: Vertical Temperature Profiles of Planets from Ray's book [3].

In all five plots, we see observe a sharp decrease of temperature with pressure in the lower portion of the atmosphere (which we call the **troposphere**), until a critical point (**tropopause**) in which the temperature decreases more slowly with height or even increases (**stratosphere**)<sup>1</sup>.

In this chapter, we will attempt to explain this. We want to explain this because the vertical structure has profound consequences on many other phenomena of interest: the vertical structure affects, among other phenomena, where (or if) constituents condense ([Moist Thermodynamics](#)), the outgoing radiation to space ([Radiative Transfer](#)), and much of fluid dynamics ([Geophysical Fluid Dynamics](#)).

---

<sup>1</sup>I should note that I'm adhering to Ray's preferred terminology here. Ray uses **stratosphere** to refer to places in the atmosphere where radiative transfer is more dominant, which (we will see) causes a shallower decrease of temperature with height. This contrasts the more typical usage of the term **stratosphere**, which requires an increase of temperature with height. This only occurs if there is some heat source from above. In the case of the Earth, this heat source is due to the location of ozone, which is a very effective absorber of UV radiation. However, I prefer Ray's definition, due to reasons elaborated after I derive the [Dry Adiabats](#) and the [Dry Stability Criterion](#).

As a spoiler:

- A temperature gradient is an indicator that there is an uneven heat source. The uneven heat source could be because:
  - Radiation from the sun is absorbed unevenly. For example, on Earth, the ground absorbs much more visible light from the sun than the atmosphere, and ozone in the stratosphere absorbs much more UV light than the rest of the atmosphere.
  - Energy is still leaking out from formation, as is the case with Jupiter.
- The atmosphere redistributes heat away from the aforementioned uneven heat source. Heat is transferred *down* the temperature gradient:
  - If temperature decreases with height, the heat is coming from below.
  - If temperature increases with height, the heat is coming from above.
- There are two dominant heat transport mechanisms:
  - Convection, which we will explain in this section. This is dominant in the **troposphere**.
  - Radiation, which will explain in Part II. This is dominant in the **stratosphere**.

## 2.2 The Dry Adiabatic: Rising and Falling Parcels of Air

### 2.2.1 Derivation

To explain (some of) the aforementioned phenomenon, we start from the first law of thermodynamics governing a parcel of air:

$$dU = -p dV + T dS + \mu dN \quad (2.1)$$

where  $U$  = internal energy;  $p$  = pressure;  $V$  = volume;  $T$  = temperature;  $S$  = entropy;  $\mu$  = chemical potential; and  $N$  = number of molecules. Roughly, the  $-p dV$  term corresponds to work done by the parcel (i.e., energy transferred by the parcel pushing or being pushed by its surroundings),  $T dS$  corresponds to the heat transfer between the parcel and its surroundings, and  $\mu dN$  corresponds to energy gained or lost by exchanging particles with its surroundings.

We assume that the system is closed ( $dN = 0$ , so the parcel does not exchange air with its environment<sup>2</sup>) and divide by the mass of the air parcel to find:

$$du = -p d\left(\frac{1}{\rho}\right) + T ds$$

where  $u$  and  $s$  refer to the internal energy and entropy *per unit mass* (remember, we want intensive variables only!). We now approximate the air parcel as an ideal gas. This means that:

1. The gas obeys the Ideal Gas Law (1.3)
2. The internal energy is a function of temperature only:  $u = c_v T$ , where  $c_v$  = the specific heat capacity at constant volume.
3. The specific heat capacities are related in the following way:  $c_v + R = c_p$  (also per unit mass)

$$\begin{aligned} c_v dT &= -p d\left(\frac{1}{\rho}\right) + T ds && ; du = c_v dT \text{ and assuming } \frac{\partial c_v}{\partial T} \approx 0 \\ &= -p d\left(\frac{RT}{p}\right) + T ds && ; \text{Ideal gas using Eqn. 1.3} \\ &= -pR \left( \frac{dT}{p} - \frac{T dp}{p^2} \right) + T ds && ; \text{Assume } \bar{M} \text{ is constant.} \\ &= -R dT + \frac{RT dp}{p} + T ds \end{aligned}$$

---

<sup>2</sup>This amounts to the assumption that entrainment does not occur. This is usually false, but it's not a bad assumption for what we aim to explain.

Dividing by  $T$  and rearranging, we find that:

$$\begin{aligned}
 ds &= \frac{R_a + c_v}{T} dT + R \frac{dp}{p} \\
 &= c_p d \ln T - R d \ln p & ; d(\ln x) = \frac{dx}{x} \text{ and } R + c_v = c_p \\
 &= c_p d \left( \ln \left( T(p)^{\frac{-R}{c_p}} \right) \right) & ; \ln(x) - a \ln(y) = \ln(x(y)^{-a})
 \end{aligned}$$

If we make the crucial assumption that the parcel exchanges a negligible amount of heat with its surroundings, then we can approximate  $ds = 0$ <sup>3</sup>. In reality, there is some level of heat transfer ( $ds \neq 0$ ), but these processes are typically much slower (timescale  $\sim$  days/weeks) than the changes in pressure caused by upwards/downwards parcel motion (timescale  $\sim$  hours). We can thus conclude that:

$$0 = c_p d \left( \ln \left( T(p)^{\frac{-R}{c_p}} \right) \right)$$

and so we derive:

#### Key Idea I.4: The Dry Adabat

The dry adiabat governs the temperature of an air parcel that is adiabatically lifted or dropped (due to, e.g., convection or dynamics). The temperature increases as pressure increases. This is because, as the parcel moves to a location where the ambient atmospheric pressure is higher, the atmosphere does work on the air parcel to compress it and increase its energy and therefore temperature.

$$\frac{d \ln T}{d \ln p} = \frac{R}{c_p} \quad (2.2)$$

$$T(p) = T_0 \left( \frac{p}{p_0} \right)^{\frac{R}{c_p}} \quad (2.3)$$

If we crucially assume that **convection** is the **dominant vertical energy transport mechanism** in the atmosphere then the **atmosphere follows a dry adiabat**. This is somewhat accurate for Earth's troposphere.

### 2.2.2 Physical Interpretation

Equation 2.3 is the **Dry Adabat**. ‘**Dry**’ because we have no condensation, and ‘**Adiabat**’ because we assume  $dS = 0$  (and the word ‘adiabatic’ refers to processes where  $dS = 0$ ). Constants  $(p_0, T_0)$  refer to the initial temperature and pressure of the air parcel.

Physically 2.3 corresponds to a parcel starting at some temperature and pressure  $(T_0, p_0)$  and being displaced upwards/downwards. When it is displaced upwards/downwards, it undergoes negligible diabatic heating and entrainment, so its temperature does not change that way. However, when the ambient pressure changes, it expands/contracts, and the air parcel does positive/negative work on its surroundings. This *decreases/increases* the internal energy of the parcel, thus decreasing/increasing the temperature. That’s why  $T$  decreases as  $p$  decreases in 2.3.

<sup>3</sup>Why can we only approximate  $ds = 0$  and not set  $ds = 0$ ? It is because the relation  $dQ = T ds$  (where  $dQ$  is the heat transfer) holds only if the process is reversible, but we have made no such assumptions here. If the process is irreversible, as it probably is, then  $dQ \leq T ds$  and so it is possible for  $dQ = 0$  and  $ds \neq 0$ .

However, we should be wary of the limits of our conclusions here. Recall that we started this section considering an air parcel, not the entire atmosphere! We must make a final, crucial assumption that rising and falling air parcels (convection) is the dominant mechanism for heat transport within some domain in the atmosphere (e.g., from the ground to a certain height – for example the tropopause). *This* implies that the temperature profile of the atmosphere within this domain follows the **Dry Adiab**. If this domain extends to the ground, we may take the initial conditions  $(p_0, T_0)$  to be the temperature and pressure of the surface  $(p_s, T_s)$ .

However, not all planetary atmospheres and not even all of Earth's atmosphere obey this assumption. This assumption is not bad in the Earth's troposphere partly because here the atmosphere is heated from below: the atmosphere is transparent to the radiation from the sun, which penetrates through the atmosphere and heats the ground. The ground then heats the air above it, which becomes buoyant and convects. Similarly, Jupiter's internal energy leakage maintains a convective troposphere.

Finally, now, we can see now why many atmospheres have temperatures decreasing as pressure decreases or height increases. The first fact is readily noted from Eqn. 2.3. We can show the second fact by combining the Dry Adiab 2.3, Hydrostatic Balance, and Ideal Gas Ideal Gas to obtain an equation linking temperature with height:

$$\begin{aligned}
 \frac{R}{c_p} &= \frac{d \ln T}{d \ln p} \\
 &= \frac{dT}{dz} \frac{p}{T} \frac{dz}{dp} && \text{; chain rule} \\
 &= \frac{dT}{dz} \frac{p}{T} (-\rho g)^{-1} && \text{; 1.11} \\
 &= \frac{dT}{dz} \frac{p}{T} \frac{RT}{p} (-g)^{-1} && \text{; 1.3} \\
 \therefore \boxed{\frac{dT}{dz} = -\frac{g}{c_p} = -\Gamma_d} &&& \text{; } \Gamma_d = \text{Dry Adiabatic Lapse Rate} \approx 10 \text{ K km}^{-1}
 \end{aligned}$$

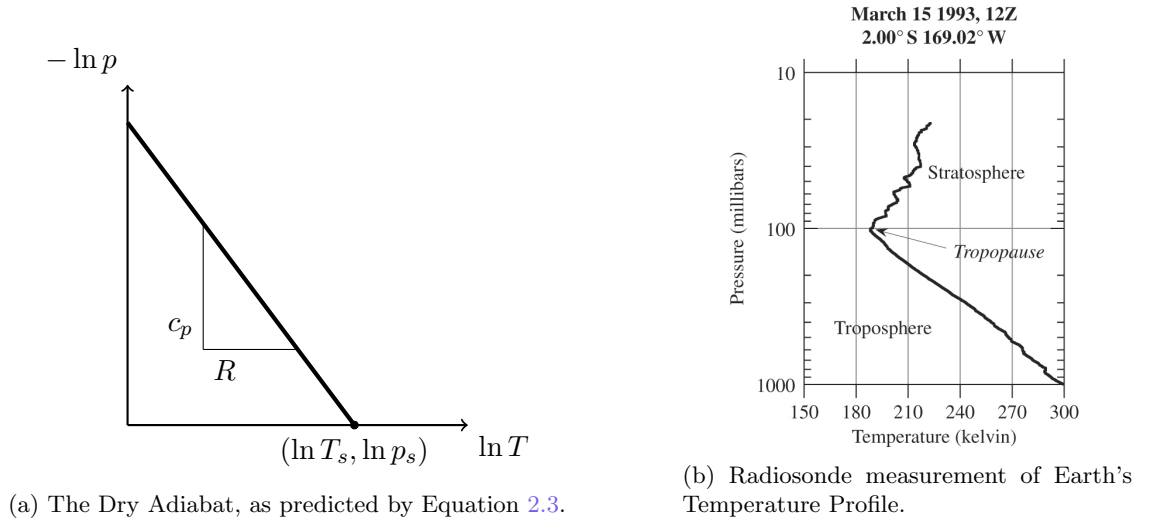


Figure 2.2: The Dry Adiab: Theory v. Observation

Note that  $R$  and  $c_p$  have the same units/dimensions, so  $\frac{R}{c_p}$  is dimensionless. We can use the Equipartition Theorem to estimate how quickly  $T$  falls off with decreasing  $p$ . We know that  $c_v \approx \frac{f}{2} R \approx \frac{5}{2} R$  for the Earth, and we know that  $c_p \approx c_v + R$  for an ideal gas. Therefore,  $c_p \approx \frac{7}{2} R$ , so for the Earth:

$$\frac{R}{c_p} \approx \frac{R}{\frac{7}{2} R} = \frac{2}{7} \approx \frac{d \ln T}{d \ln p}$$

Of course, the atmosphere does not actually follow a **Dry Adiat**, as clearly shown by Figure 2.2b. It is somewhat accurate in the troposphere, but not so much elsewhere. That is because some of our assumptions were not as accurate as we would like. In reality,  $dN \neq 0$ , and entrainment occurs, and  $dS \neq 0$ , and diabatic heating occurs. Some of this diabatic heating is due to condensation, which we will learn how to deal with in the next chapter on **Moist Thermodynamics**. Some of this is radiative heating, which we will discuss in the next part on **Radiative Transfer**.

However, convection as a mechanism always perturbs an existing atmospheric profile to an **Adiat**, and it is other processes (radiation, fluid dynamics) that pull the atmosphere away from an adiabat. In some cases, like in the **stratosphere**, radiation pulls the atmospheric profile to be superstable, in which case convection is prohibited due to the stability of the atmosphere. In other cases, these other processes pull the atmospheric profile to be unstable, in which case convection takes over to push the atmosphere back to the **Adiat**. What we have just touched upon is **Radiative-Convective Equilibrium**, which we will discuss more in Part III.

## 2.3 Potential Temperature

In basic thermodynamics, we often introduce the concept of temperature operationally in the following way: suppose we have two objects  $A$  and  $B$ , with temperatures  $T_A$  and  $T_B$ , respectively, and we place them in contact with each other. Temperature encodes whether heat will flow, and in which way. If  $T_A = T_B$ , no heat will flow. If  $T_B > T_A$ , heat will flow from  $B$  to  $A$ , and vice versa.

However, this will not work in atmospheric thermodynamics. Suppose we have two parcels of air called  $A$  and  $B$  at pressures  $p_A$  and  $p_B$ <sup>4</sup> with temperatures  $T_A$  and  $T_B$ , respectively. Suppose we displace parcel  $A$  in order to place it in contact with parcel  $B$  (with no entrainment ( $dN = 0$ ) or diabatic heating ( $dS = 0$ )).

Which way will heat flow?

We cannot naively use temperature as we did before, because parcel  $A$ 's temperature will change following the **Dry Adiat** as we move it to a different pressure. In other words, temperature is not a variable which is **Materially Conserved** in Atmospheric/Oceanic Thermodynamics. A variable is **Materially Conserved** if and only if it remains constant following a air/water parcel. This is a concept which will become very important in Part VI.

We now want to define a quantity which serves the same role as temperature did before (i.e., a variable which encodes which way heat will flow) but which is *also*, unlike temperature, **materially conserved**. Let us consider a more general scenario, and move each parcel to some third pressure  $p_{ref}$ <sup>5</sup>.

We define the **Potential Temperature**  $\theta$  as the temperature a parcel *would* have if it were perfectly adiabatically moved to some reference pressure level  $p_{ref}$ :

$$\theta(T, p) = T \left( \frac{p}{p_{ref}} \right)^{-\frac{R}{c_p}} \quad (2.4)$$

Recall that a few pages ago in our derivation of the **Dry Adiat** we found that:

$$ds = c_p d \left( \ln \left( T(p)^{\frac{R}{c_p}} \right) \right) \quad (2.5)$$

Since the  $\ln T(p)^{\frac{R}{c_p}}$  term is within the differential, we can freely add and subtract constants within

<sup>4</sup>We are getting used to using pressure as a coordinate rather than height, remember!

<sup>5</sup>Set  $p_{ref} = p_B$  to recover our first example.

the differential, which corresponds to multiplying within the logarithm. We get then that:

$$ds = c_p d \ln T \left( \frac{p}{p_{ref}} \right)^{\frac{-R}{c_p}} \quad (2.6)$$

$$= c_p d \ln \theta \quad (2.7)$$

As you can see, if  $ds = 0$ , as we assumed during convection, then  $d\theta = 0$ . Therefore, as we lift or drop a parcel,  $\theta$  is [materially conserved](#).<sup>6</sup>

Recalling that  $TdS = dQ = \text{heat}$ , we find that the potential temperature is a measure of the extent to which diabatic/heat transfer processes have heated or cooled our parcel of air. It's essentially a measure of how wrong we were to assume that  $dS = 0$  in our derivation of [Dry Adiabats](#).

## 2.4 Convection

### 2.4.1 Buoyancy

The atmosphere (and ocean) is a fluid, so even in the absence of most forces like friction, gravity is not the only force acting on a fluid parcel. There are also pressure gradient forces, which act to push fluid parcels away from areas of high pressure to areas of low pressure.

We already accounted for the effects of pressure in deriving [Hydrostatic Balance](#) and the [Dry Adiabats](#). In the former case, the pressure gradient force acts to push fluid parcels upwards away from lower altitude regions of high pressure to higher altitude regions of low pressure in order to counteract the force of gravity on the ambient air. In the latter case, we only consider large-scale pressure gradients in the ambient atmosphere and we ignored pressure gradients across a fluid parcel.

Now we wish to take into account pressure gradients across a fluid parcel, and consider a fluid parcel which does *not* have the same density as the air around it. Suppose we have a fluid parcel of density  $m$  occupying some volume  $V$ . For simplicity assume the fluid parcel is rectangular, with a horizontal area of  $A$  and a height of  $h$  such that  $V = Ah$ .

Let us consider the forces per unit mass on the fluid parcel in the vertical direction. There are three forces of interest: the gravitational force  $F_g = -g$ , the pressure force on the *bottom* pushing the parcel *upwards*  $F_{bot} = p(z)A/m$ , and the pressure force on the *top* pushing the parcel *downwards*  $F_{top} = p(z+h)A/m$ .  $p(z)$  is the pressure at height  $z$ . The net force is then:

$$F_z = -g - \frac{A}{m}(p(z+h) - p(z))$$

We now make two assumptions. First, we assume that  $h$  is small, therefore,  $p(z+h) - p(z) \approx \frac{dp}{dz}h$ . Second, we assume that the pressure in the fluid is set by [Hydrostatic Balance](#), therefore  $p(z+h) - p(z) \approx \frac{dp}{dz}h = -\rho_a g h$ , where  $\rho_a$  is the density of the ambient air. We define  $\rho_p = m/V$  as the density of the fluid parcel, and find that the total force on fluid parcel is:

$$\begin{aligned} F_b &= -g + \frac{Ah}{m}\rho_a g \\ &= g \left( -1 + \frac{\rho_a}{\rho_p} \right) \\ \therefore F_b &= g \left( \frac{\rho_a - \rho_p}{\rho_p} \right) \end{aligned} \quad (2.8)$$

where  $F_b$  = the buoyancy force. Physically, the buoyant force is intuitive: if the air parcel is lighter than the ambient air ( $\rho_p < \rho_a$ ) it will convect and rise, and vice versa. Furthermore, we can write  $F_b$

<sup>6</sup>In the Ocean, even without radiative heating processes,  $\theta$  is not conserved. This is because there is a third thermodynamic state variable, salinity, which will change the temperature of sea water without adding any heat. Oceanographers have come up with another kind of materially conserved temperature in the ocean: *Conservative Temperature*.

in terms of the temperature  $T$  (using 1.3) or potential temperature  $\theta$  (using 2.4) in very neat forms, since many terms cancel top and bottom:

$$F_b = g \left( \frac{\rho_a - \rho_p}{\rho_p} \right)$$

$$F_b = g \left( \frac{T_p - T_a}{T_a} \right) \quad (2.9)$$

$$F_b = g \left( \frac{\theta_p - \theta_a}{\theta_a} \right) \quad (2.10)$$

So a parcel will convect if it is warmer or has a higher potential temperature than its ambient surroundings.

Finally, another way of thinking of the buoyant force is as a kind of ‘*reduced gravity*’. The fluid acts like its on a planet of gravitational acceleration  $g' = g \left( \frac{\rho_a - \rho_p}{\rho_p} \right)$  not  $g$ , and  $g' < g$  always since  $\rho_a > 0$ . This will be how we think of things in Chapter 17.

## 2.4.2 Convective Instability Criterion

Now suppose we wish to find whether our atmosphere is convectively stable, i.e., whether the atmosphere will spontaneously undergo convection. If pressure and density did not appreciably change by height, then this would be easy: if  $T(z) > T(z+c)$  where  $c > 0$  (and assuming that  $\frac{\partial \rho}{\partial T} < 0$ ), then we will have denser air overlying lighter air, and the parcels will convect.<sup>7</sup>

As before, the situation is a bit more complicated in an atmosphere when density and pressure vary significantly with height. Suppose again that we have an atmospheric temperature profile  $T = T(p)$  (not necessarily following the dry adiabat). Suppose we lift a parcel at some pressure  $p$  to some (smaller) pressure  $p - \delta p$ .

That parcel will be convectively unstable and continue rising if it is lighter (and therefore hotter) than the ambient surrounding air. We assume that the parcel is initially in equilibrium with its surroundings ( $T_{\text{parcel}}(p) = T_{\text{ambient}}(p)$ ). We then Taylor Expand:

$$T_{\text{parcel}}(p - \delta p) > T_{\text{ambient}}(p - \delta p)$$

$$T_{\text{parcel}}(p) - \frac{dT_{\text{parcel}}}{dp} \delta p > T_{\text{ambient}}(p) - \frac{dT_{\text{ambient}}}{dp} \delta p$$

$$\frac{d \ln T_{\text{parcel}}}{d \ln p} < \frac{d \ln T_{\text{ambient}}}{d \ln p}$$

Substituting in the [Dry Adiat](#) for  $\frac{d \ln T_{\text{parcel}}}{d \ln p}$  we find the dry stability criterion:

### Key Idea I.5: The Dry Stability Criterion

Suppose a dry atmosphere has a temperature profile  $T(p)$ . It will be:

Unstable to convection if:

$$\frac{d \ln T}{d \ln p} > \frac{R}{c_p}$$

(2.11)

Neutrally stable to convection if:

$$\frac{d \ln T}{d \ln p} = \frac{R}{c_p}$$

(2.12)

and Stable to convection if:

$$\frac{d \ln T}{d \ln p} < \frac{R}{c_p}$$

(2.13)

<sup>7</sup>This is the case if we’re considering two layers of different temperature and composition that are in contact. This is because, at the point of contact, pressure changes only infinitesimally, so the [Dry Adiat](#) is inapplicable (since  $dp \approx 0$ ). To determine whether these two layers are convectively stable, you must compare the densities directly, which depend on both the temperature of each layer and the molecular composition.



Note that 2.11 and 2.13 are not automatically ruled out due to 2.3. That is because 2.3 governs a perfectly adiabatic rising/falling parcel of air. While this is the dominant contribution to the atmospheric temperature profile in the troposphere, it is not the only one (even within the troposphere). Other processes (water vapour, radiation, dynamics) can push the atmospheric temperature profile away from the dry adiabat and make it stable or unstable. However, typically, unstable atmospheres are removed very quickly through spontaneous convection.

Physically, the idea is as follows: an atmosphere is stable if the temperature decreases slow enough with height (2.13). If it decreases too quickly (2.11), a lifted parcel of air will have cooled *less* than its surroundings, and thus be warmer and lighter than its surroundings, and continue to rise due to buoyancy (2.9).

If 2.13 obtains then an atmosphere will be *super-stable* and will inhibit convection. A quick sanity check to remember which way the inequality is in 2.11 is the following: an isothermal<sup>8</sup> atmosphere is always super-stable. An isothermal atmosphere will have  $\frac{d \ln T}{d \ln p} = 0$ , so  $\left(\frac{d \ln T}{d \ln p} < \frac{R}{c_p}\right)$  corresponds to instability. If one has a *super-stable* atmosphere then that indicates that there has been some other dominant energy transport mechanism that has kept the atmosphere in this state. This is characteristic of a **stratosphere**.<sup>9</sup>

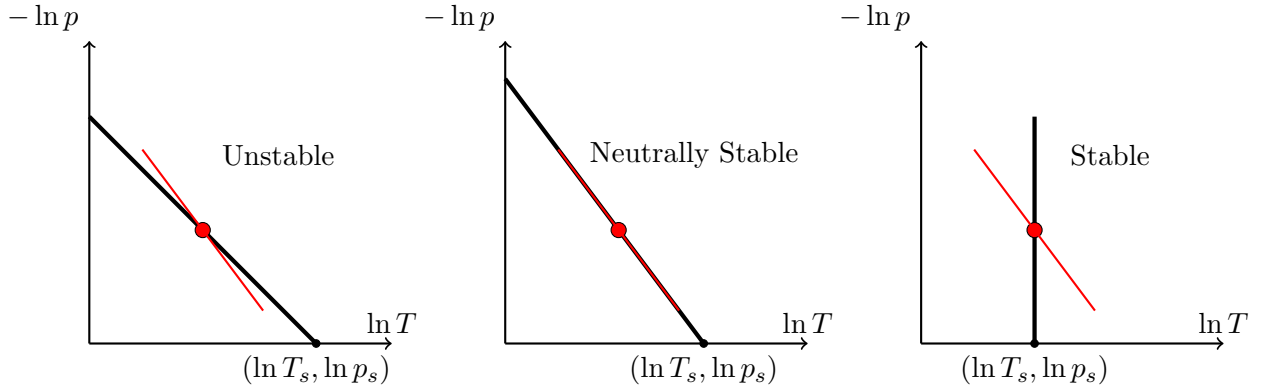


Figure 2.3: Dry Stability: From left to right: unstable, neutrally stable, and super stable atmospheric temperature profiles. Thick black is the atmospheric profile, thin red is the dry adiabatic slope.

### 2.4.3 Convective Stability Criterion with Potential Temperature

We can formulate an equivalent criterion for potential temperature. Recall that if two parcels of air have potential temperatures  $\theta_A$  and  $\theta_B$  such that  $\theta_A > \theta_B$ , then parcel  $A$  will have a higher temperature than parcel  $B$  if they're at the same pressure.

We can very straightforwardly derive an equivalent criterion for convective instability, this time in terms of potential temperature  $\theta$ , by rearranging 2.4 for  $T(\theta)$  and substituting that into 2.11.

$$\begin{aligned}
 1 &< \frac{c_p}{R} \frac{d \ln T(\theta)}{d \ln p} && ; 2.11 \\
 &= \frac{c_p}{R} \frac{d \ln \left( \theta \left( \frac{p}{p_{ref}} \right)^{\frac{R}{c_p}} \right)}{d \ln p} && ; \text{substitute 2.4} \\
 &= \frac{c_p}{R} \left( \frac{d \ln \theta}{d \ln p} + \frac{d \ln \left( p^{\frac{R}{c_p}} \right)}{d \ln p} + \frac{d \ln \left( p_{ref}^{\frac{-R}{c_p}} \right)}{d \ln p} \right) && ; \text{split logarithm} \\
 &= \frac{c_p}{R} \frac{p}{\theta} \frac{d \theta}{d p} + \frac{c_p}{R} \frac{R}{c_p} && ; \ln p^{R/c_p} = R/c_p \ln p
 \end{aligned}$$

<sup>8</sup>'Iso' meaning same and 'thermal' meaning temperature, so 'Isothermal' means constant temperature.

<sup>9</sup>Recall the footnote on [this](#) page. This is why Ray prefers that definition of **stratosphere**: a stratosphere should be characterised by what energy transport mechanism is dominant, not whether it happens to have shortwave absorbers.

Therefore, since  $\theta > 0$  and  $p > 0$  always, we get that our atmosphere is unstable if and only if:

$$\boxed{\frac{d\theta}{dp} > 0} \quad (2.14)$$

Using the hydrostatic relation and the fact that  $g > 0, \rho > 0$ , we get our criterion for instability in height coordinates:

$$\boxed{\frac{d\theta}{dz} < 0} \quad (2.15)$$

Therefore, if potential temperature decreases with height, then our atmosphere is unstable. This makes physical sense. Recall that potential temperature takes the place of temperature in atmospheric thermodynamics, in the sense that it is potential temperature that stays the same when you lift/drop an air parcel. If you lift an air parcel, its potential temperature stays the same. However, if the ambient potential temperature decreases with height, then it will be colder than the air parcel, and the air parcel will keep rising.

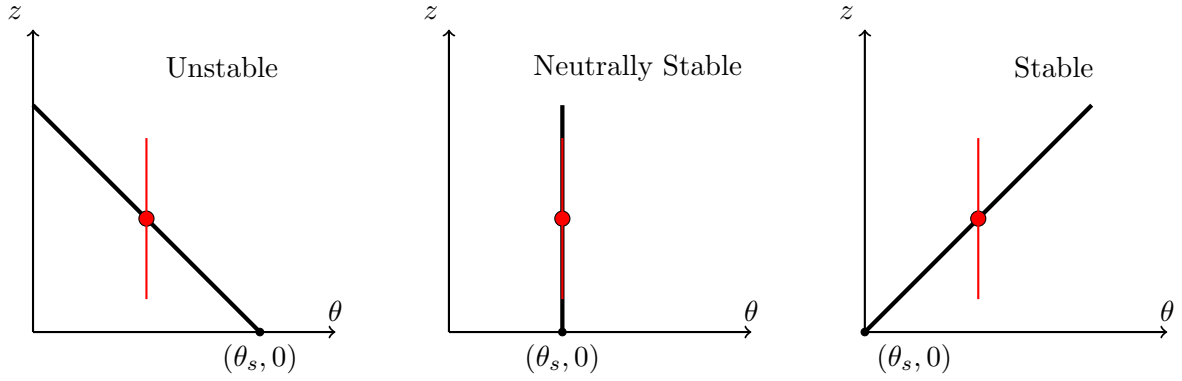


Figure 2.4: Dry Stability: From left to right: unstable, neutrally stable, and super stable atmospheric temperature profiles. Thick black is the atmospheric profile, thin red is the dry adiabatic slope.

#### 2.4.4 CAPE, LFC, LNB

We have so far characterised *whether* convection will occur in our dry atmosphere, but now we wish to know what will happen *during* convection. We start again by considering a parcel of air at an initial pressure  $p_0$  with a temperature of  $T_p(p_0)$ . Let  $T_a(p)$  be the temperature profile of the ambient atmosphere, and assume that initially the parcel is in equilibrium with its surroundings (i.e., assume  $T_p(p_0) = T_a(p_0)$ ).

We define the **Level of Free Convection (LFC)**  $p_{LFC}$  as the *first* location where  $T_p(p) = T_a(p)$ . We define the **Level of Neutral Buoyancy (LNB)** as the *second* location where  $T_p(p) = T_a(p)$ .

We define the **Convective Available Potential Energy (CAPE)** as the total available potential energy (per unit mass) for a parcel of air from the Level of Free Convection to the Level of Neutral Buoyancy. In other words, *CAPE* is equal to the total kinetic energy per unit mass a parcel would gain if it was allowed to convect from the Level of Free Convection to the Level of Neutral Buoyancy:

$$CAPE = \int_{z_{LFC}}^{z_{LNB}} F_b dz \quad (2.16)$$

We can substitute for  $F_b$  using 2.9 and convert to pressure coordinates using 1.15 to find that:

$$\begin{aligned}
 CAPE &= \int_{z_{LFC}}^{z_{LNB}} \frac{T_p - T_a}{T_a} g dz \\
 &= \int_{p_{LFC}}^{p_{LNB}} \frac{T_p - T_a}{T_a} (-R T_a d \ln p) \\
 \therefore CAPE &= R \int_{p_{LNB}}^{p_{LFC}} (T_p - T_a) d \ln p
 \end{aligned}$$

We know  $T_p(p)$ , since  $T_p(p)$  follows the **Dry Adiab** with  $(p_0, T_0) = (p_{LFC}, T_{LFC})$ , so we can straightforwardly calculate  $CAPE$  if we are given the atmospheric temperature profile  $T_a(p)$ .

We further define **Convective IN**hibition (**CIN**) as the energy a fluid parcel would need to overcome to reach the **LFC**:

$$CIN = R \int_{p_{LFC}}^{p_0} (T_p - T_a) d \ln p$$

We expect  $CIN < 0$  and  $CAPE > 0$ . Above **LNB**, we expect there to be more ‘ $CIN$ ’, which will prevent the parcel from rising further. However, at this point, the parcel will have already gained kinetic energy from  $CAPE$ . We call where the parcel ‘stops’ the **Level of Maximum Ascent (LMA)**, which occurs at  $p_{LMA}$  where the ‘ $CIN$ ’ above **LNB** equals the  $CAPE$ :

$$R \int_{p_{LMA}}^{p_{LNB}} (T_p - T_a) d \ln p = R \int_{p_{LNB}}^{p_{LFC}} (T_p - T_a) d \ln p$$

### Key Idea I.6: Convection Definitions

Let the ambient air temperature be  $T_a(p)$  (black line ■). Consider an air parcel at an initial pressure  $p_0$  with a temperature of  $T_p(p_0) = T_a(p_0)$  that is then lifted. Upon lifting, the parcel’s temperature  $T_p(p)$  follows the **Dry Adiab** (dashed magenta line ■). Then:

**Level of Free Convection (LFC):**  $p = p_{LFC}$  if  $T_p(p) = T_a(p)$ ,  $dT_p/dp < dT_a/dp$ , and this is the first pressure  $p < p_0$  where this occurs.

**Level of Neutral Buoyancy (LNB):**  $p = p_{LNB}$  if  $T_p(p) = T_a(p)$ ,  $dT_p/dp > dT_a/dp$ , and this is the first pressure  $p < p_0$  where this occurs.

**Level of Maximum Ascent (LMA):**  $p = p_{LMA}$  where  $p_{LMA}$  is determined by Equation 2.19 (I.e., where the blue area ■ equals the orange area ■).

$$CAPE = R \int_{p_{LNB}}^{p_{LFC}} (T_p - T_a) d \ln p$$

(2.17)

$$CIN = R \int_{p_{LFC}}^{p_0} (T_p - T_a) d \ln p$$

(2.18)

$$R \int_{p_{LMA}}^{p_{LNB}} (T_p - T_a) d \ln p = CAPE$$

(2.19)

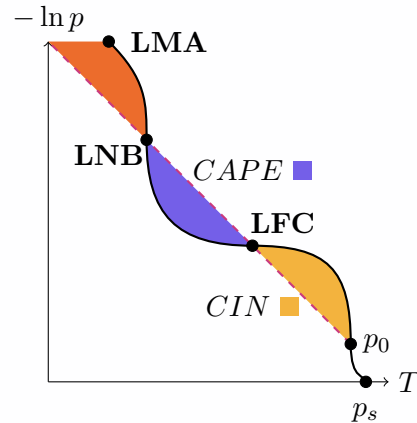


Figure 2.5: Atmospheric Profile. The parcel is lifted from  $p_0$ .  $CIN$  is proportional to the area shaded in **Yellow** ■ and  $CAPE$  is proportional to the area shaded in **Blue** ■.

### 2.4.5 Conservation of Enthalpy during Convection

We define the (specific) **Enthalpy**  $h$  as follows:

$$h = u + p \frac{1}{\rho} \quad (2.20)$$

For an ideal gas, we can substitute for  $p$  using 1.3 and  $u = c_v T$  to write  $h$  in terms of  $T$  only:

$$\begin{aligned} h &= \underbrace{c_v T}_u + \underbrace{\rho R T \frac{1}{\rho}}_p \\ &= \underbrace{(c_v + R)}_{c_p} T \\ &= c_p T \end{aligned}$$

Therefore we can interpret the enthalpy  $h$  as an expression of the heat content of a system at constant pressure, like for example in the atmosphere.

We introduce the enthalpy because we can show that, under suitable assumptions, the total enthalpy  $H$  of an atmospheric column is conserved during convection:

$$H = \iiint \rho h dV = \text{const}$$

We make three assumptions:

1. The initial and final states are both in Hydrostatic Balance 1.11
2. The kinetic energy of the initial and final state is negligible compared to the potential and internal energy.
3. No mass or heat leaves domain.

The last assumption requires some elaboration. First, we *don't* assume that no energy leaves the domain: we allow the column to do work on the air above or below itself as it expands/contracts. Second, we *don't* assume that there is no heating within the column. In fact, we expect there to be some heating within the column in general: If a column is convectively unstable, then there will be **CAPE**, and we expect this to be converted into an appreciable amount of kinetic energy as the air parcels within the column convect. Once convection acts to remove the **CAPE**, we expect friction to slow the parcels down and convert the kinetic energy into heat.

We first consider the total energy in an atmospheric column per horizontal area between altitudes  $z_0$  and  $z_1$ , which is the sum of the internal energy  $u$  and the potential energy  $gz$  (and we have assumed kinetic energy  $\frac{1}{2}v^2$  is negligible):

$$\begin{aligned} E &= \int_{z_0}^{z_1} \rho (u + gz) dz \\ &= \frac{1}{g} \int_{p_1}^{p_0} (u + gz) dp \end{aligned}$$

We then integrate the potential energy term  $gz$  by parts:

$$\begin{aligned} g \int_{p_1}^{p_0} z dp &= g [zp]_{p_0}^{p_1} - g \int_{z_1}^{z_0} p dz \\ &= g(z_0 p_0 - z_1 p_1) + \int_{p_1}^{p_0} \frac{p}{\rho} dp \end{aligned}$$

Therefore the total energy per unit area is:

$$E = \frac{1}{g} \int_{p_1}^{p_0} \left( u + \frac{p}{\rho} \right) dp + (z_0 p_0 - z_1 p_1)$$

We make a final, fourth assumption here that the pressure at the bottom  $p_0$  and top  $p_1$  of the column is unchanged between the final and initial states. This is ensured if we assume the entire column (inside and outside the column under consideration) is in hydrostatic balance and the mass of the air above, within, and below the column under consideration is unchanged (recall 1.12). The change in energy  $\Delta E$  between the final and initial states is then:

$$\Delta E = \frac{1}{g} \Delta \int_{p_1}^{p_0} h dp + p_0 \Delta z_0 - p_1 \Delta z_1 \quad (2.21)$$

We now wish to equate this with the change in energy of the atmospheric column governed by 2.1. Since we have assumed that there is no heat or mass transfer to or from the system, we can set  $dN = 0$  and approximate  $dS = 0$ . Therefore, the change in energy (per unit area), to a good approximation, is set by:

$$\begin{aligned} dE &= -p dV/A \\ &= p_0 dz_0 - p_1 dz_1 \\ \therefore \Delta E &= p_0 \Delta z_0 - p_1 \Delta z_1 \end{aligned} \quad (2.22)$$

Therefore, setting 2.21 equal to 2.22 we can cancel the  $p\Delta z$  to find that:

$$\Delta H = \Delta \iiint \rho h dz dx dy = 0 \quad (2.23)$$

The enthalpy of a column is conserved under convection. If we assume that the atmosphere is an ideal gas (so that  $h = c_p T$ ) and that  $c_p$  is approximately independent of temperature (so that we can pull  $c_p$  out of the integral), then we can convert to pressure coordinates to find that:

$$H = \int \int \frac{c_p}{g} \int T dp dx dy$$

Therefore we find that the enthalpy is proportional to  $\int T dp$ , and so:

$$\Delta \int T dp = 0 \quad (2.24)$$

during convection.

## Chapter 3

# Moist Thermodynamics

### 3.1 Phase Transitions

In the previous chapter, we considered only atmospheres made up completely of constituents that do not undergo any phase transitions. However, Earth's atmosphere (and many others) clearly do not obey this assumption. Earth's atmosphere has a non-negligible amount of condensible substances: water vapour (look outside and you might see a cloud!), which clearly condense, evaporate, and freeze in Earth-like conditions.

So what is a phase transition? We will only be dealing with so-called **First-Order Phase Transitions**, which occur when there is a *discontinuity* in a thermodynamic state variable and an absorption or release of a large amount of energy, called **latent heat**. Phase transitions occur at certain temperatures and pressures<sup>1</sup>. We can represent this with a **Phase Diagramme**, as shown below:

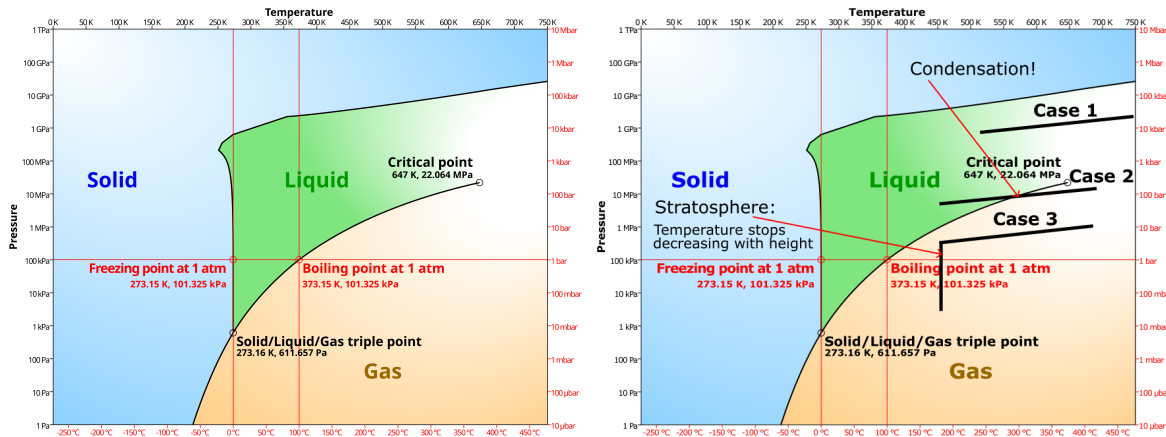


Figure 3.1: Phase Diagramme for Water from Wikipedia [4].

Focus on the left-hand plot for now. The coloured regions indicate what phase water will be in at a certain the partial pressure (of the water) and temperature: gas (vapour), liquid (steam), and solid (ice). Crucially, this does not depend on the total pressure of the atmosphere or surrounding gases, but only the partial pressure of the condensible substance (in this case, the partial pressure of water). The solid black lines indicate **Phase Boundaries**. The **Triple Point** is a point where all three states can coexist. The **Critical Point** is the point past which there is no phase transition between gas and liquid.

The pressure at the **Phase Boundaries** is the **Saturation Vapour Pressure**  $p_{sat}(T)$ . If the partial pressure  $p < p_{sat}$ , then more vapour (gas form) can be added without condensing. If the  $p \geq p_{sat}$ , then the gas will condense until the partial pressure reaches the saturation vapour pressure. However, it's important to remember that condensation is not instantaneous, which is partly why we can have atmospheric conditions where  $p > p_{sat}$ . This will be important when we consider **Clouds**.

<sup>1</sup>Whether a phase transition will occur might also depend on other thermodynamic state variables. For example, salinity is a thermodynamic variable which affects when sea water evaporates.

In fact all gases can condense if the temperature is low enough/partial pressure is high enough. CO<sub>2</sub> can condense on Mars, and N<sub>2</sub> condenses on Neptune. Why doesn't CO<sub>2</sub> and N<sub>2</sub> condense here on Earth? It all depends on the vertical structure of the atmosphere: the precise profile of  $T = T(p)$ . Recall that, as  $p$  decreases, so too does  $T$  (if our atmosphere follows the [Dry Adiabats](#)). There are three cases, all labeled in the right-hand plot of Figure 3.1.

1. Case 1: The atmosphere has such a high pressure/temperature that it avoids a phase transition altogether by meandering around the critical point.
2. Case 2: The atmospheric profile cuts right through a phase boundary, and the condensible condenses (clouds form!).
3. Case 3: The stratosphere starts before the atmosphere hits the phase boundary, in which case temperature no longer decreases with height, so it never crosses the phase boundary.

For O<sub>2</sub> and N<sub>2</sub> in Earth like conditions, we are in case 3: it never gets cold enough for O<sub>2</sub> and N<sub>2</sub> to condense, which is why we have no O<sub>2</sub> and N<sub>2</sub> clouds. However, we are in Case 2 with water (H<sub>2</sub>O), and so we do get water clouds!

When a gas condenses from phase  $A$  to phase  $B$ , it typically releases a large amount of **Latent Heat**. Again, we would like to work in intensive variables, so we often work with the **Specific Latent Heat**  $L_{A \rightarrow B}$  (units of J kg<sup>-1</sup>) which is the amount of energy released per kilogram of substance when undergoing a phase transition from  $A$  to  $B$  ( $L_{A \rightarrow B} > 0$  if heat is released). Like heat capacity, this often has a temperature dependence, but we will in most cases treat this as constant.

### 3.2 The Clausius-Clapeyron Relation

We can derive the shape of the phase boundary with the **Clausius Clapeyron Relation**, which relates the saturation vapour pressure to the temperature using the latent heat and the densities. We will not derive it here, but it would be useful for you to memorise the following relation:

$$\frac{dp_{sat}}{dT} = \frac{L_{A \rightarrow B}}{T \left( \frac{1}{\rho_A} - \frac{1}{\rho_B} \right)} \quad (3.1)$$

If we assume that  $\rho_g \ll \rho_l$  (i.e., the gaseous form is much less dense than the liquid form), and that the gas is ideal, we find that:

$$\begin{aligned} \frac{dp_{sat}}{dT} &\approx \frac{L_{g \rightarrow l}}{T \frac{1}{\rho_g}} \\ &= \frac{L_{g \rightarrow l} p}{R_c T^2} \\ \therefore \frac{d \ln p_{sat}}{d \left( \frac{1}{T} \right)} &\approx - \frac{L_{g \rightarrow l}}{R_c} \end{aligned} \quad (3.2)$$

Note that I have written  $R_c$  to make clear that  $R_c$  is the specific gas constant for the condensible (e.g., water) *not* for the atmosphere. We can now derive an analytic expression for the saturation vapour pressure if we further assume that the latent heat is constant.

$$p_{sat} \approx p_0 e^{\frac{L}{R_c} \left( \frac{1}{T_0} - \frac{1}{T} \right)} \quad (3.3)$$

where I now just write  $L$  for the latent heat of condensation and  $(T_0, p_0)$  are any point on the saturation vapour pressure curve. This could be, for example, the **triple point** or the **critical point**.

As we can see then, the saturation vapour pressure depends approximately *exponentially* on temperature, so saturation vapour pressure decreases exponentially as temperature decreases. Conversely, if saturation vapour pressure decreases,  $T$  changes only logarithmically (i.e., very slowly).

### 3.3 The Moist Pseudo-Adiabat: Lifting Moist Parcels of Air

Our goal now is to derive an analogous [Dry Adiab](#) relation for moist atmospheres consisting of some condensible substance  $c$ . We call this the **Moist Pseudo-Adiabat**. We call this a ‘*Pseudo*’-Adiabat because we assume, for mathematical convenience, that the condensible substance  $c$  is instantaneously removed from the air parcel. This can occur if, for example,  $c$  instantaneously rains out (but rain is not the only way in which this may occur!). When the condensible is removed, this constitutes a diabatic process ( $dN \neq 0$ ), so strictly speaking we will not derive an adiabat in this section.

In reality, rising moist air parcels that condense will not instantaneously eject its condensed  $c$ , and so will not follow the pseudo-adiabat. However, the pseudo-adiabat is a good approximation if the amount of condensate left in the air parcel is small, which is at least the case on Earth.

Recall that we are using the same logic as we did before in deriving the [Dry Adiab](#). As such, we are still assuming that the dominant mechanism of heat transport within the atmosphere is vertical convection, only this time the air parcels convecting are moist (have condensible substances in them).

We will now consider two limiting cases. In both cases, we make the crucial assumption that the air parcels remain saturated with the condensible. In other words, we assume that the partial pressure of the condensible is equal to the saturation vapour pressure  $p_{sat}$ .

#### 3.3.1 Limit I: Single Component Condensible Atmosphere

We consider the first limiting case, with an atmosphere made up wholly of one condensible substance  $c$ . Since there is only one constituent, the total pressure is equal to the partial pressure of the  $c$ . There are two possibilities now.

The first possibility obtains if the surface at the ground (and therefore partial pressure at the ground) is too high ( $p_g > p_{sat}(T_g)$ ). Then there will be an ocean of  $c$  at the surface. The pressure within the ocean will obey [Hydrostatic Relation](#), and the surface of the ocean is determined by where the pressure and temperature intersect the phase boundary ( $p = p_s = p_{sat}(T_s)$ ).

The temperature above the surface of the ocean is set by convecting air parcels. These air parcels originate from the surface of the ocean, and are initially saturated. However, recall that we have assumed that the air parcel remains saturated.<sup>2</sup> As such, the atmospheric profile must lie on the phase boundary, so the temperature is fixed to the phase boundary and is as follows (found by rearranging [3.3](#) for  $T$  in terms of  $p_{sat}$ ):

$$\text{If } p_g > p_{sat}(T_g): T(p) = \frac{T_s}{1 - \frac{RT_s}{L} \ln \frac{p}{p_s}} \quad (3.4)$$

$$p_s = p_{sat}(T_s) \quad (3.5)$$

The surface pressure  $p_s$ , then, is set by the surface temperature  $T_s$ . Recalling Equation [1.13](#), the mass of the atmosphere is set by how hot it is. Physically, this corresponds to how much of  $C$  you can evaporate from the ocean!

Second, if the surface at the ground is too low ( $p_g = p_s < p_{sat}(T_s)$ ), then there will be no surface  $c$  ocean, and the atmosphere will follow the [Dry Adiab](#) until it reaches the phase boundary at the **lifted condensation level**, after which it will follow Equation [3.4](#):

$$\text{If } p_s < p_{sat}(T_s): T(p) = \begin{cases} T_s \left( \frac{p}{p_s} \right)^{\frac{R}{c_p}} & \text{for } p < p_{LCL} \\ \frac{T_s}{1 - \frac{RT_s}{L} \ln \frac{p}{p_{LCL}}}, & \text{for } p > p_{LCL} \end{cases} \quad (3.6)$$

$$\text{where } p_{LCL} = p_{sat} \left( T_s \left( \frac{p_{LCL}}{p_s} \right)^{\frac{R}{c_p}} \right) \quad (3.7)$$

In other words, the temperature follows the Dry Adiab until the **lifted condensation level**. We can solve for the pressure at the **lifted condensation level** by finding where the Dry Adiab intersects the phase boundary (the latter of which is calculated using [Clausius Clapeyron](#)).

<sup>2</sup>I'm not sure how justified this assumption is. I plan to ask Ray about this at some point, but if this footnote is still here when you're reading it I have not (yet).



### 3.3.2 Limit II: Dilute Condensible

#### Derivation

We now consider the second limiting case, where the atmosphere is mainly made up of a non-condensable substance  $a$ , and a dilute condensable substance  $c$ . We follow a similar derivation to the [Dry Adiabatic](#), and start from the first law of thermodynamics, but this time include the extra latent heat term  $-L dq$ :

$$du = -p d\left(\frac{1}{\rho}\right) + T ds - L dq \quad (3.8)$$

where  $L$  = the latent heat of condensation and  $q = \frac{\rho_c}{\rho_a}$  the mass mixing ratio of  $c$  in its gaseous form (and *not* the  $c$  in its liquid form, which we assume (recall) are removed from the parcel of air). For water,  $q$  = the mass mixing ratio of the water vapour and the dry air, *not* the water droplets and the dry air. In the dilute limit,  $q \ll 1$ , and  $q$  is equal to the mass fraction.

Take care to note the assumptions we're making here, and what the letters we've written actually mean.  $u$ ,  $\rho$ ,  $T$ , and  $s$  refer to properties of  $a$  in the parcel of air, while  $p$  refers to the total pressure of both the  $a$  and  $c$ . Here we are implicitly making three assumptions. First, we assume that we are in the dilute limit  $q \ll 1$ , therefore  $\rho = \rho_a + \rho_c \approx \rho_a$ . Second, we assume the heat capacities of  $a$  and  $c$  are of similar size, and that therefore the specific internal energy  $u$  is primarily made up of the internal energy of  $a$ .

As a sanity check, let us consider the minus sign on the  $L dq$  term. This makes physical sense: we expect the internal energy of the  $a$  to *increase* if condensation occurs, since latent heat is released when  $c$  condenses, and heat is added to  $a$  in the air parcel. If condensation occurs, then the mass of the liquid  $c$  (e.g., water droplets) must increase, and therefore (by mass conservation) the mass of the gaseous  $c$  (e.g., water vapour) must decrease. Therefore, the mass mixing ratio of  $c$  must decrease, so  $dq < 0$ , therefore  $-L dq > 0$ , as expected.

Let us first consider the  $L dq$  term:

$$\begin{aligned} L dq &= L d\left(\frac{\rho_c}{\rho_a}\right) \\ &= L d\left(\frac{M_c e}{M_a(p - e)}\right) \\ &\approx L d\left(\frac{M_c e}{M_a p}\right) \\ L dq &\approx L \epsilon d\left(\frac{e}{p}\right) \end{aligned}$$

where  $\rho_c$  = the density of the condensable  $c$ ;  $\rho_a$  = the density of the air  $a$ ;  $M_c$  = the molar mass of the condensable;  $M_a$  = the molar mass of the air;  $p = p_a + p_c$  = the total pressure;  $e = p_c$  = the partial pressure of the condensable;  $\epsilon = \frac{M_c}{M_a}$  = the ratio of molar masses  $c$  and  $a$ .

We can now apply product rule and assume that the condensable is always saturated. This allows us

to use the simplified version of Clausius Clapyeron (3.2) to simplify things.

$$\begin{aligned}
L dq &\approx L \epsilon d \left( \frac{e}{p} \right) \\
&= L \epsilon \left( \frac{de}{p} - \frac{e}{p^2} dp \right) && ; \text{Chain Rule} \\
&= L \epsilon \frac{e}{p} \left( \frac{de}{e} - \frac{dp}{p} \right) \\
&= Lq d \ln p_{sat} - Lq d \ln p && ; q \approx L \epsilon \frac{e}{p} \text{ and assume } e \approx p_{sat} \\
&= -q \frac{L^2}{R_c} d \left( \frac{1}{T} \right) - Lq d \ln p && ; \text{Clausius Clapeyron 3.2} \\
&= q \frac{L^2}{R_c T^2} dT - Lq d \ln p \\
&= q \frac{L^2}{R_c T} d \ln T - Lq d \ln p
\end{aligned}$$

We now substitute this into 3.8 and, for simplicity, let  $ds = 0$  from the outset.<sup>3</sup> Following identical algebra in the derivation of the [Dry Adiabatic](#), and carefully distinguishing between  $R_a$  and  $R_c$  (the specific gas constants of the non-condensable and dilute condensable), we find that:

$$\begin{aligned}
c_v dT &= -R_a dT + \frac{R_a T dp}{p} - L dq \\
\therefore c_p dT &= \frac{R_a T dp}{p} - q \frac{L^2}{R_c T} d \ln T + Lq d \ln p \\
\therefore c_p d \ln T - \frac{qL^2}{R_c T^2} d \ln T &= R_a d \ln p + \frac{Lq}{T} d \ln p
\end{aligned}$$

Solving for  $\frac{d \ln T}{d \ln p}$  and rearranging, we the **Moist Pseudo-Adiabatic** in the dilute limit:

#### Key Idea 1.7: Dilute Moist Pseudo-Adiabatic

The adiabat for an atmosphere consisting of a dilute condensable is as follows:

$$\frac{d \ln T}{d \ln p} = \frac{R}{c_p^a} \frac{1 + \frac{L q_{sat}}{R_a T}}{1 + \frac{L q_{sat}}{c_p^a T} \frac{L}{R_c T}} \quad (3.9)$$

This is in general a shallower slope than the [Dry Adiabatic](#) due to the latent heat of condensation keeping the parcel warm.

where I have written  $q = q_{sat}$  and  $c_p = c_p^a$  to remind us that, first, we have made the crucial assumption that the air parcel remains saturated, and second, that  $c_p$  is the heat capacity of the non-condensable  $a$ .

### Physical Interpretation

Equation 3.9 is, in general, not analytically solvable. However, we have written it in the form that provides easy interpretation. The first term is the slope of the adiabat if there were no condensation

<sup>3</sup>We could allow  $ds \neq 0$  and derive what's called an **Equivalent Moist Potential Temperature**  $\theta_E$  (analogous to [Potential Temperature](#)  $\theta$ ), which is similarly [materially conserved](#) for moist saturated parcels of air. We don't for brevity, but if you wish to read up more on it, here is a resource: [whoops I forgot to insert the link, please let me know if this is still here!](#)

(the [Dry Adiab](#)), while the second complicated looking fraction are the changes in the slope due to condensation. Setting  $q = 0$  recovers the Dry Adiab.

Note that although we have made the dilute assumption that  $q \ll 1$ , the corrections on the right hand side are generally of order 1. This is because the corrections go as  $\frac{Lq_{sat}}{R_a T}$  and  $\frac{Lq_{sat}}{c_p T} \frac{L}{R_c T}$ , and it is generally the case that, for water vapour on Earth at least,  $\frac{L}{R_a T} \gg 1$ .

It is almost always the case (although I cannot show algebraically) that the moist adiabat is shallower than the dry adiabat, i.e., that temperature decreases less with a decrease in pressure (and increase in height):

$$\frac{d \ln T_{moist}}{d \ln p} < \frac{d \ln T_{dry}}{d \ln p}$$

Physically, this corresponds to the effect of the condensible. As a parcel rises, the pressure decreases, so it expands and cools (following the dry adiabat). However, since the temperature decreases, so too does the saturation vapour pressure (3.1). So if the parcel is saturated, as we have assumed,  $c$  must condense and release latent heat into the surrounding air parcel, warming it a bit and offsetting the cooling from expansion.

Finally, we must note that we have made the assumption that the parcel remains saturated. In general, this will be the case for rising saturated parcels of air, which will only condense so much as to remain approximately saturated. However, if a parcel is *sinking*, then clearly it is not always realistic to assume that it will automatically acquire some more water vapour to remain saturated. As such, if a parcel of air is sinking, it will most likely follow the [Dry Adiab](#).

### What Happens if $q < q_{sat}$ ?

Suppose we start an air parcel off sub-saturated (i.e.,  $q < q_{sat}$ , initially). What temperature profile will it follow as it is lifted?

Initially, as it is lifted, its mass mixing ratio  $q$  will remain constant, and its temperature will decrease following the [Dry Adiab](#). This will continue until the temperature has cooled so much (following the Dry Adiab) causing the saturation vapour pressure to decrease sufficiently such that  $q = q_{sat}$ . After this point, it will follow the [Dilute Moist Pseudo-Adiab](#).

### 3.3.3 Moist Convection

Ideas from Section 2.4 carry over to here, but are changed subtly. However, to avoid repetition, please refer to Chapter 8.

**Part II**

**Radiative Transfer**

# Introduction

This section of the course was lectured by [Raymond T. Pierrehumbert](#) covering basic atmospheric Radiative Transfer. In the academic year 2024-2025, Ray did not cover scattering, so I will not cover scattering in these notes.

This section consists of four chapters:

1. [Basic Radiation Concepts](#):

We introduce the basic vocabulary needed to describe radiative transfer. We first characterise the radiation field, including how to encode the [direction](#) and [magnitude](#) of radiation. Next, we describe how objects emit and absorb radiation.

2. [Molecular Spectroscopy](#):

We detail how we characterise [spectral lines](#) in terms of certain parameters. We then discuss how those parameters are determined physically, and introduce some explicit functional forms such as the [Lorentz Line Shape](#).

3. [Schwarzschild Equation](#):

We formulate an equation for radiative transfer throughout the atmosphere and solve it. We

4. [Radiative Equilibrium](#):

We make the grey gas approximation in order to analyse how an atmosphere would behave in pure radiative equilibrium: i.e., if radiation were the dominant energy transport mechanism.

## Chapter 4

# Basic Radiation Concepts

### 4.1 Direction of Radiation: Rays and Solid Angle

In general, radiation will not only be a function of position  $\vec{r}$  but will also be a function of direction<sup>1</sup>. We can encode the direction in which radiation is propagating with a **ray**, a vector  $\vec{\omega}$  of (dimensionless) length 1 on a unit sphere.  $\vec{\omega}$  encodes information regarding the direction of propagation, but not regarding the magnitude (e.g., energy) of the radiation. In Cartesian coordinates, this is:

$$\vec{\omega} = \begin{pmatrix} \sin(\theta) \cos(\phi) \\ \sin(\theta) \sin(\phi) \\ \cos(\theta) \end{pmatrix} \quad (4.1)$$

We also define the **Solid Angle**, which is a measure of a set of rays, with units of **Steradians** (sr). In other words, **solid angle** encodes from *how many* directions some radiation is coming from. For example, if radiation is propagating towards you from every direction, the **solid angle** of that will be larger than if radiation were only propagating towards you from a single direction.

We can make this more intuitive by drawing an analogy between the solid angle in 3D (measured in **steradians** (sr)) and the angle in 2D angle (measured in radians (rad)). In 2D, radians measure an angle by relating the circumference of a circle to the arc length subtended by that angle (see Fig. 4.1a). For example,  $1\pi$  rad corresponds to an angle that subtends half a unit circle (180 degrees) and thus an arc length of  $1\pi$ . Analogously, steradians measure a solid angle by relating the surface area of a sphere to the patch subtended by a solid angle (see Fig. 4.1b). For example, a set of rays pointing in all directions has a solid angle of  $4\pi$  sr, while a set of rays with some positive  $x$  component (i.e.,  $\sin(\theta) \cos(\phi) > 0$ ) has a solid angle of  $2\pi$  sr.

Often we will have to integrate over solid angles/directions (as radiation will be a continuous function of  $\vec{\omega}$ ) in the same way that we might integrate over angles. We therefore introduce the differential solid angle:

#### Key Idea II.1: Solid Angle

*The solid angle  $\Omega$  is a dimensionless area encoding how many directions we are considering, and is measured in steradians sr. We define the differential solid angle  $d\Omega$ :*

$$d\Omega = d\theta \sin \theta d\phi \quad (4.2)$$

$$\Omega = \iint_{\text{directions considered}} d\Omega \quad (4.3)$$

<sup>1</sup>This is why your eyes will hurt if you look directly at the sun but they won't if you look at these lecture notes.

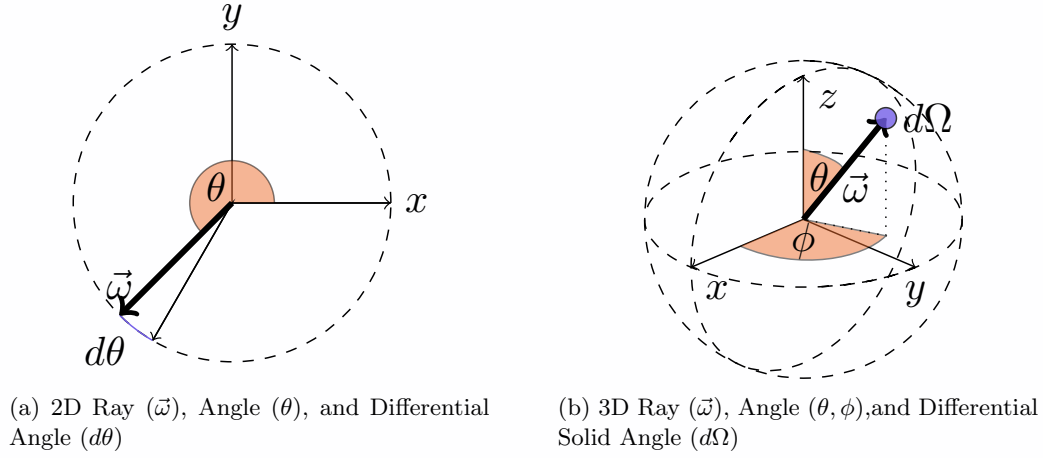


Figure 4.1: Rays, Angles, and Differential Solid Angle

One can integrate this differential over the entire sphere ( $\theta \in [0, \pi], \phi \in [0, 2\pi]$ ) to readily verify that the solid angle over an entire sphere is  $4\pi$  sr, and over half a sphere ( $\theta \in [0, \pi/2], \phi \in [0, 2\pi]$ ) is  $2\pi$  sr. More specifically, consider integrating over a cone of half-angle  $\theta_0$ . Then the solid angle is:

$$\begin{aligned}
 \Omega &= \iint_{\text{half cone}} d\Omega \\
 &= \int_0^{2\pi} \int_0^{\theta_0} d\theta \sin \theta d\phi \\
 \Omega &= 2\pi(1 - \cos \theta_0)
 \end{aligned} \tag{4.4}$$

Intuitively, the differential solid angle is the dimensionless area element in spherical coordinates. Think of the small rectangle with side lengths  $r d\theta$  and  $r \sin \theta d\phi$ , but you divide by  $r^2$  to make it dimensionless. This is analogous to the case in 2D, where the differential angle is a dimensionless line element in radial coordinates. Think, analogously, of a small line of side length  $r d\theta$ , but again you divide by  $r$  to make it dimensionless.

We also define the **projected** differential solid angle. This accounts for the fact that if we have some surface which is absorbing radiation, radiation parallel to the surface will not be absorbed.

$$d\Omega_{\perp} = \cos \zeta d\Omega$$

(4.5)

where  $\zeta$  is the **zenith angle**, the angle between the ray ( $\vec{\omega}$ ) and the ray normal to the surface in question. Sometimes we orient our axes such that  $\theta = \zeta$  because it simplifies the system, but note that we do not always do this nor does this always simplify the situation.

## 4.2 Magnitude of Radiation: (Spectral) Radiance and (Spectral) Irradiance

Now that we have learnt how to characterise the *direction* in which radiation is propagating, we wish to characterise the *magnitude* of the radiation.

Let us consider some surface *perpendicular* to some radiation propagating through it. The **Radiance**  $I$  is the energy flux of the radiation flowing through that surface per unit *time* per unit *area* per unit *solid angle*.  $I$  has units of  $\text{J s}^{-1} \text{m}^{-2} \text{sr}^{-1}$ . Now consider an *arbitrarily* oriented surface, which is oriented with at some **zenith angle**  $\zeta$  to some radiation. Then the energy flux  $\phi$  (in units of  $\text{J s}^{-1}$ ) is calculated by integrating the **radiance** over solid angles and the area of the patch as follows:

$$\phi = \iint I d\Omega_{\perp} dA \tag{4.6}$$

The **Irradiance**  $E$  is the energy flux flowing perpendicular through that surface per unit time and unit area.  $E$  has units of  $\text{J s}^{-1} \text{m}^{-2}$ . The two are related as follows:

$$E = \int_{\Omega_{\pm}} I d\Omega_{\perp}$$

$$\phi = \int E dA$$

Often times radiation is only propagating from one hemisphere (i.e., from one direction) so we integrate over the upwards or downwards propagating hemisphere, which we denote as  $\Omega_+$  and  $\Omega_-$ , respectively. I also remind you that  $E$  cannot be a function of  $\vec{\omega}$  but  $I$  generally is.

So far our discussion has been completely general, but now we focus on light/electromagnetic radiation, which is a wave and thus has a frequency/wavelength. The **Spectral Radiance**  $L$  and **Spectral Irradiance**  $F$  is the radiance and irradiance (respectively) at a given frequency per unit frequency.  $L$  and  $F$  have units of  $\text{J s}^{-1} \text{m}^{-2} \text{sr}^{-1} \text{Hz}^{-1}$  and  $\text{J s}^{-1} \text{m}^{-2} \text{Hz}^{-1}$ , respectively. Equivalently,  $L(\nu) d\nu$  is the radiance carried by waves with some frequency  $\nu' \in [\nu, \nu + d\nu]$  and  $F(\nu) d\nu$  is the irradiance carried by waves with some frequency  $\nu' \in [\nu, \nu + d\nu]$ . Due to the laws of electromagnetism, the total energy of a superposition of electromagnetic waves is given by the sum of the energy of the waves at each frequency. As such:

$$I = \int L d\nu \quad (4.7)$$

$$E = \int F d\nu \quad (4.8)$$

$$\phi = \iint L d\nu d\Omega_{\perp} dA \quad (4.9)$$

Since the spectral radiance at any point is typically a function of both  $\nu$  and  $\vec{\omega}$ , we must integrate over both  $\nu$  and  $\vec{\omega}$  to obtain the radiative energy fluxes per unit area. Practically, this integration cannot be solved analytically and so is done numerically. In these notes, we will make a few approximations in order to make analytical progress/extract physical intuition. We will approximate the distribution over  $\vec{\omega}$  (the *Two-Stream Approximation* in 6.2), integrate over frequency bands (6.4), and make the (bad) grey gas approximation ( $\frac{\partial L}{\partial \nu} = 0$ ) in Section 7.1.

As a summary:

### Key Idea II.2: Characterising the Magnitude of Radiation

Consider some radiation propagating through a surface at some **zenith angle**  $\zeta$  (i.e., the radiation is at angle  $\zeta$  to the vector normal to the surface). Let the power flux in  $\text{J s}^{-1}$  through the surface be  $\phi$  and the area  $A$ . Then we define:



**Radiance  $I$ :** The energy flux per unit time, area, and solid angle parallel to the direction the radiation is propagating in.

$$I = \frac{\partial^2 \phi}{\partial A \partial (\Omega \cos \zeta)}$$

$$= \frac{\text{Power flux **parallel to radiation**}}{\text{per (Area)(Solid Angle)}}$$

**Irradiance  $E$ :** The energy flux per unit time and area through the surface in question.

$$F = \frac{\partial \phi}{\partial A}$$

$$= \frac{\text{Power flux **through the surface**}}{\text{per (Area)}}$$

**Spectral Radiance  $L$ :** The energy flux per unit time, area, solid angle, and frequency parallel to the direction the radiation is propagating in.

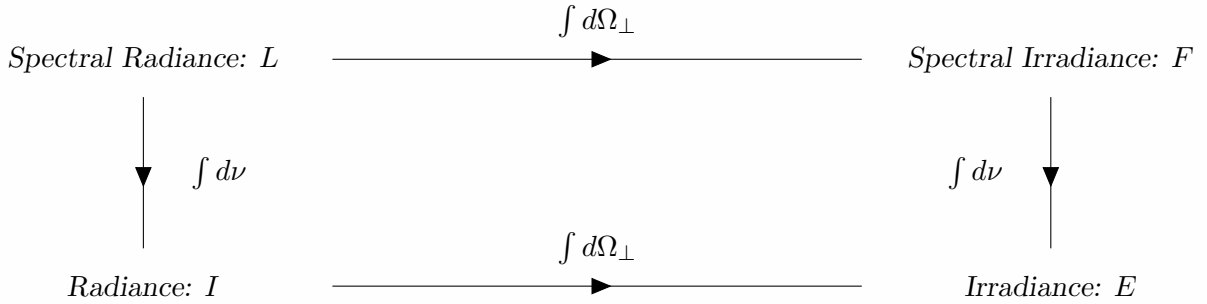
$$L = \frac{\partial^3 \phi}{\partial A \partial (\Omega \cos \zeta) \partial \nu}$$

$$= \frac{\text{Power flux **parallel to radiation**}}{\text{per (Area)(Solid Angle)(Frequency)}}$$

**Spectral Irradiance  $F$ :** The energy flux per unit time, area, solid angle, and frequency through the surface in question.

$$F = \frac{\partial^2 \phi}{\partial A \partial \nu}$$

$$= \frac{\text{Power flux **through the surface**}}{\text{per (Area)(Frequency)}}$$



#### 4.2.1 Frequency, Wavelength, and Wavenumber

So far, we have specified how quickly a wave oscillates by referring to its frequency  $\nu$ , but we could have referred to its wavelength  $\lambda$  or its wavenumber  $\tilde{\nu}$ . All of these give you the same information, and you can freely convert between them if you know the speed of the wave  $c$ . For light, this is  $c \approx 3 \times 10^8 \text{ m s}^{-1}$ . You can find the information summarised in the table below.

Symbol	Name	Units	Meaning
$\nu$	Frequency	$\text{s}^{-1}$ ; Hz	The number of oscillations per second.
$\lambda$	Wavelength	m	The peak-to-peak length of the wave. $\nu\lambda = c$
$\tilde{\nu}$	Wavenumber	$\text{m}^{-1}$ ; $\text{cm}^{-1}$	The wavelengths per unit metre. $\tilde{\nu} = \lambda^{-1}$ . In spectroscopy, we often refer to wavenumber in terms of wavelengths per <i>centimetre</i> ( $\text{cm}^{-1}$ ) for convenient numbers.

Finally, we should note two nuances. First, I introduced **spectral radiance/irradiance** as the radiance/irradiance per unit frequency, but many people refer to the **spectral radiance/irradiance** as the radiance/irradiance per unit wavelength or per unit wavenumber. The choice of description is ultimately arbitrary, but you should be aware of this potential confusion.

The second nuance is more important. It is the  $L(\nu) d\nu$  term which is physically meaningful, not the  $L(\nu)$  term by itself. This is because, as already mentioned,  $\nu$  is only one of the ways we could have chosen represent the property of wavelength/frequency. We could equally use the wavelength  $\lambda$ . To illustrate the point, let's convert the spectral radiance  $L(\nu) d\nu$  carried by the radiation with frequency

$\nu' \in [\nu, \nu + d\nu]$  to something in terms of the wavelength  $\lambda$ :

$$\begin{aligned}\nu\lambda &= c \therefore \\ L(\nu) d\nu &= L\left(\frac{c}{\lambda}\right) d\left(\frac{c}{\lambda}\right) \\ &= \frac{c}{\lambda^2} L\left(\frac{c}{\lambda}\right) d\lambda \\ &= \hat{L}(\lambda) d\lambda\end{aligned}$$

where we have defined  $\hat{L}(\lambda) \equiv \frac{c}{\lambda^2} L\left(\frac{c}{\lambda}\right)$ . Notice how the functional form fundamentally changes when we undergo an ultimately arbitrary coordinate transformation from  $\nu$  to  $\lambda$ : the ‘frequency spectral radiance’  $L(\nu)$  and ‘wavelength spectral radiance’  $L(\lambda)$  are not the identical! But nothing physically *out there* should depend on how we choose to write it down, so we conclude that  $L(\nu)$  by itself is an artefact arising from our choice to represent it in terms of  $\nu$ .

### 4.3 Emission: Black Body Radiation

Now that can characterise radiation in terms of direction and magnitude, we now aim to lay the foundations of characterising how matter interacts with radiation. We first consider emission here.

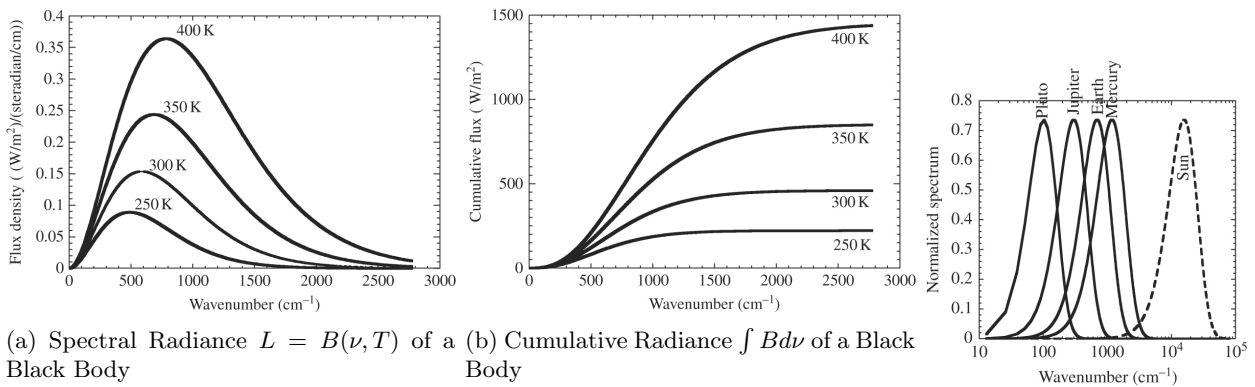
A **black body** is a theoretical object which absorbs all incident electromagnetic radiation. To a good approximation, many stellar bodies are black bodies, including the Sun. One can show that such a body must emit spectral radiance at all times given by  $L_{BB}$ :

$$L_{BB} d\nu = B(\nu, T) d\nu = \frac{2h\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu}{k_B T}\right) - 1} d\nu \quad (4.10)$$

You do not have to memorise this, but you should be aware of a few properties.

First, at all fixed frequencies  $\nu$ ,  $B(\nu, T) d\nu$  is monotonically increasing with  $T$ . As  $T$  increases, so too does  $B$  and every frequency, so if a blackbody warms up it releases more energy (See Figure 4.2a).

Second, while blackbodies emits radiation at all frequencies, its ‘characteristic frequency’ increases as temperature increases.<sup>2</sup> Since oftentimes the star will always be hotter than its planet, this results in what’s called a *spectral gap*: the planet emits radiation at a much longer wavelength than its star, which is what enables the greenhouse effect (See Figure 4.2b).



<sup>2</sup>We define the ‘characteristic frequency’ as follows. First, define the cumulative blackbody radiance as  $I_{BB}(\nu) = \int_0^\nu B(\nu', T) d\nu'$ .  $I_{BB}$  monotonically increases as  $\nu$  increases (since  $B(\nu') > 0$ ), so  $I_{BB}$  reaches its maximum value at  $\nu = \infty$ . We define the characteristic frequency  $\nu_{char}$  as the median emission frequency: the frequency where  $I_{BB}$  is half its maximum value. More rigourously,  $\nu_{BB}$  satisfies the following relation:  $\int_0^{\nu_{char}} B(\nu', T) d\nu' = \frac{1}{2} \int_0^\infty B(\nu', T) d\nu'$ .<sup>3</sup>

<sup>3</sup>Why all the hassle? Why not find where  $B(\nu)$  is maximum and define that as the characteristic frequency? We do not do this because this definition is not physically meaningful. Recall the discussion we had just last section: it is  $B(\nu) d\nu$  that is meaningful, not  $B(\nu)$ . (Exercise: differentiate  $B(\nu)$  to find where  $B$  is maximum with respect to  $\nu$ . Then, transform into wavelength coordinates, making sure to transform  $d\nu$  as well, then differentiate with respect to wavelength. You should find that the maximum wavelength is not the maximum frequency).

Third and finally,  $B$  is not a function of  $\vec{\omega}$ :  $B$  is isotropic! This final fact allows us to calculate the spectral irradiance of a blackbody (where we have integrated over  $d\Omega_{\perp}$  not  $d\Omega$ , since only the radiation perpendicular to the surface of the blackbody is emitted):

$$\begin{aligned} F(\nu, T) &= \int_{\Omega_{\perp}} B(\nu, T) d\Omega_{\perp} \\ &= \left( \int_{\Omega_{\perp}} d\Omega_{\perp} \right) B(\nu, T) \\ &= \pi B(\nu, T) \end{aligned}$$

where we have set the zenith angle  $\zeta$  equal to  $\theta$  and integrated the projected differential solid angle:

$$\begin{aligned} \int_{\Omega_{\perp}} d\Omega_{\perp} &= \int_{\Omega_{\perp}} \cos \zeta d\Omega \\ &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos \zeta d\theta \sin \theta d\phi \\ &= 2\pi \int_0^{\frac{\pi}{2}} \sin \theta \cos \theta d\theta \\ &= 2\pi \left[ \frac{1}{2} \sin^2 \theta \right]_0^{\frac{\pi}{2}} \\ &= \pi \end{aligned}$$

To find the (non-spectral) irradiance of a black body, we integrate over all frequencies to recover the Stefan-Boltzmann Law governing the power output per unit area  $P$  of a black body:

$$P = \sigma T^4 = \int_0^{\infty} \pi B(\nu, T) d\nu \quad (4.11)$$

#### 4.3.1 Emissivity and Kirchoff's Law

Many bodies are not like blackbodies at all, even approximately. We thus define the **emissivity**  $\epsilon \in [0, 1]$  of an object as follows:

$$\epsilon(\nu, T, p) = \frac{L(\nu, T, p)}{B(\nu, T)} \quad (4.12)$$

where  $T$  = the temperature and  $p$  = the pressure of the object in question. Physically, the emissivity encodes how much the object is like a blackbody at each frequency, temperature, and pressure. For a perfect blackbody,  $\epsilon = 1$  for all frequencies, temperatures, and pressures. One can show that, in local thermodynamic equilibrium, the emissivity of an object is exactly equal to its absorptivity  $\alpha$ , where  $\alpha$  is (dimensionless) the fraction of spectral radiance absorbed by the material.

$$\epsilon(\nu, T, p) = \alpha(\nu, T, p) \quad (4.13)$$

We will use this to derive the [Schwarzschild Equation](#), which governs radiative transfer in the atmosphere. However, we should note that the derivation assumes local thermodynamic equilibrium. This is a good approximation in most parts of the atmosphere, but fails near the top of the atmosphere.

#### 4.3.2 Calculating the Stellar Constant

To a good approximation, stellar bodies are black bodies ( $\epsilon = 1$ ). Using this, we can estimate the incident (not absorbed) radiative energy flux on the Earth (or any planet) from the Sun (or any star)

by approximating that star as black body. We thus define the **Stellar Constant**  $L_\star$  as the incident stellar irradiance on a *flat* surface *facing* the sun.

There are two important nuances here. First, note that the **Stellar Constant** is *not* necessarily equal to the incident stellar irradiance on the surface of the planet because the planet is *curved*. The stellar constant is only equal to the incident stellar irradiance on the surface of the planet at the equator at 12:00 if the planet has no axial tilt. Second, note that this is also different from the *absorbed* stellar power – to obtain that we have to account for albedo (some incident stellar irradiance is reflected) and integrate over area.

Let us now calculate  $L_\star$ . We do this by integrating the spectral radiance of the star (which is approximately  $B(\nu, T_\star)$ , where  $T_\star$  is the temperature of the star) over all frequencies (to obtain the radiance) and over the solid angle  $\Omega_\star$  subtended by the star at the planet (to obtain the irradiance):

$$L_\star = \int_{\Omega_\star} \int_0^\infty B_\star(T, \nu) d\nu d\Omega_\perp$$

Calculating this explicitly using 4.11, we get:

$$L_\star = \int_{\Omega_{Sun}} \frac{1}{\pi} \sigma T^4 d\Omega_\perp$$

We now make the assumption that the star is very far away from the planet: that  $d \gg R_\star, R$ , where  $d$  = the distance between the star, and the planet and  $R_\star, R$  = the radius of the star and planet, respectively. This means we can make two approximations.

Our first approximation is that  $\cos \zeta = 1$  or  $d\Omega \approx d\Omega_\perp$ . This if the radiation reaches the planet, they all must be approximately parallel to each other and perpendicular to the surface considered. Otherwise, even if the radiation were only slightly tilted, it would miss the planet. So:

$$\begin{aligned} L_\star &= \int_{\Omega_\star} \frac{1}{\pi} \sigma T^4 d\Omega_\perp \\ &\approx \int_{\Omega_\star} \frac{1}{\pi} \sigma T^4 d\Omega \\ &= \Omega_\star \frac{1}{\pi} \sigma T_\star^4 \end{aligned}$$

where we have assumed in the last line that the radiance is independent of solid angle. Our goal now is to calculate  $\Omega_\star$ , the solid angle subtended by the star at the planet, which we can calculate by using 4.4.

Our second approximation is that  $\theta_0 \ll 1$ , and so  $\cos \theta_0 \approx 1 - \theta_0^2/2$  and  $\theta_0 \approx R_\star/d$ . As such:

$$\begin{aligned} \Omega_\star &= 2\pi(1 - \cos(\theta_0)) \\ &\approx 2\pi \left( 1 - \left( 1 - \frac{1}{2}\theta_0^2 \right) \right) \\ &\approx \pi \frac{1}{2} \frac{R_\star^2}{d^2} \end{aligned}$$

Another (in my opinion more intuitive) way to think of this calculation is to consider a sphere centred on the planet with a radius  $d$  so that the star lies on the surface of this imaginary sphere. The total solid angle of the entire sphere is  $4\pi$ , but the star is only occupying a small area on the surface of that sphere. The solid angle is thus approximately the total solid angle  $4\pi$  multiplied by the fraction of area the star covers. This fraction is  $\pi R_\star^2 / 4\pi d^2$ . This is only approximate, because if it weren't the case that  $R_\star \ll d$  then the area covered by the sun would not be  $\pi R_\star^2$ . Anyways, plugging it in gives the same results:

$$\begin{aligned} \Omega_\star &= 4\pi \frac{\pi R_\star^2}{4\pi d^2} \\ &= \pi \frac{R_\star^2}{d^2} \end{aligned}$$

Therefore the stellar constant is;

$$L_{\star} = \frac{R_{\star}^2}{d^2} \sigma T_{\star}^4 \quad (4.14)$$

However, as already noted, the power actually absorbed by the planet is different. We must factor in albedo of the planet, as some radiation is reflected, and we must also factor in the curvature of the planet (which we assume is spherical), as the radiation is not perpendicular to the surface of the planet.

To do this, we must integrate over area and accounting for reflection and curvature. Suppose some fraction  $a \in [0, 1]$  of the instellation is reflected. Therefore:

$$\begin{aligned} P_{in} &= \int (1 - a) L_{\star} \cos \zeta \, dA \\ &= (1 - a) L_{\star} \int \cos \zeta \, dA \end{aligned}$$

where the zenith angle term  $\cos \zeta$  represent the fact that the sun is only hitting the Earth from one side, and only the component perpendicular to the ground is absorbed. As such,  $\int \cos \zeta \, dA$  is the *projected* area of the Earth due to the  $\cos \zeta$  term, and therefore is  $\pi R^2$ , not  $4\pi R^2$  (exercise). Therefore:

$$P_{in} = (1 - a) \pi R^2 \underbrace{\frac{R_{\star}^2}{d^2} \sigma T_{\star}^4}_{L_{\star}} \quad (4.15)$$

If we define the total **Outgoing Longwave Radiation**  $OLR$  as the total irradiance leaving the planet, then we get a radiation budget for the planet:

$$\begin{aligned} G &= (1 - a) \pi R^2 L_{\star} - 4\pi R^2 OLR \\ G &\sim (1 - a) \frac{L_{\star}}{4} - OLR \end{aligned} \quad (4.16)$$

Almost all of **Radiative Transfer** will be dedicated to calculating this  $OLR$ , and we'll have to introduce a lot of machinery to calculate it.

If we wish to make progress now, though, we can also (crudely) approximate the planet as having an emissivity independent of  $T$  or  $\nu$  and its radiation as isotropic to estimate the outgoing power:

$$\begin{aligned} P_{out} &= \int \int_{\Omega_+} \int_0^{\infty} \epsilon B(\nu, T) \, d\nu \, d\Omega_{\perp} \, dA \\ &= \epsilon \left( \int dA \right) \int_{\Omega_+} d\Omega_{\perp} \int_0^{\infty} B(\nu, T) \, d\nu \\ &= (4\pi R^2) \epsilon \sigma T^4 \\ \therefore OLR &= \epsilon \sigma T^4 \end{aligned}$$

In equilibrium,  $G = 0$ , which means we can solve for the equilibrium temperature of the Earth:

$$\begin{aligned} 0 &= (1 - a) \frac{L_{\star}}{4} - OLR \\ &= (1 - a) \frac{L_{\star}}{4} - \epsilon \sigma T^4 \\ \therefore T &= \left( (1 - a) \frac{L_{\star}}{4\epsilon\sigma} \right)^{\frac{1}{4}} \\ &= T_{\star} \left( \frac{1 - a}{4\epsilon} \frac{R_{\star}^2}{d^2} \right)^{\frac{1}{4}} \end{aligned}$$

## 4.4 Absorption: Optical Thickness and Transmission Function

Suppose some spectral radiance is traveling through a slab of radiation-absorbing *stuff* (e.g., air) with an infinitesimal mass-path of  $d\mu$ , where  $d\mu$  is the mass per area (perpendicular to the propagation of radiation). We define the infinitesimal **Optical Thickness**  $d\tau_\nu$  of the slab as the absorptivity  $\alpha$  of the slab. In other words, suppose some spectral radiance  $L$  is traveling through the radiation-absorbing **stuff**. Then the infinitesimal change of spectral radiation by the slab  $dL$  is given by:

$$\begin{aligned} dL &= -\alpha(\nu, T)L \\ &= d\tau_\nu L \end{aligned}$$

where I have written  $\tau_\nu$  with a subscript  $\nu$  to remind us that  $\tau_\nu$  is a strong function of  $\nu$ . We further define the absorption cross section  $\kappa(\nu, T, p)$  as the effective area of spectral radiance absorbed per unit mass of absorber encountered.  $\kappa$  has units of  $\text{m}^2 \text{kg}^{-1}$ , and will generally depend on frequency, temperature, and pressure. Therefore:

$$\boxed{d\tau_\nu = \kappa(\nu, T, p) d\mu} \quad (4.17)$$

We can integrate this between two points in the atmosphere  $p_1$  and  $p_2$  to find the final spectral radiance  $L_2$  in terms of the initial spectral radiance  $L_1$ :

$$\begin{aligned} \frac{dL}{L} &= -d\tau_\nu \\ \therefore \int_{p_1}^{p_2} \frac{dL}{L} &= - \left| \int_{p_1}^{p_2} d\tau_\nu \right| \\ \ln \left( \frac{L_2}{L_1} \right) &= - \left| \int_{p_1}^{p_2} d\tau_\nu \right| \\ \therefore L_2 &= L_1 \exp \left( - \left| \int_{p_1}^{p_2} d\tau_\nu \right| \right) \end{aligned}$$

Notice the absolute value in the second line. If we did not have this, we could simply reverse the limits of integration (e.g., integrate from  $p_2$  to  $p_1$ ) and obtain that the spectral radiance *increases* as it passes through. However, this is clearly unphysical: the spectral radiance absorbed should be the same regardless of whether the radiation is going upwards through the absorber or downwards through the absorber. Finally, if we note that:

$$\tau_\nu(p_1) - \tau_\nu(p_2) = \int_{p_1}^{p_2} d\tau_\nu = \int_{p_1}^{p_2} \kappa(\nu, T, p) d\mu$$

then we can define the transmission function  $\mathcal{T}_\nu$  which encodes the absorption of the spectral irradiance between those two points:

$$\begin{aligned} \mathcal{T}_\nu(p_1, p_2) &= e^{-|\tau_\nu(p_1) - \tau_\nu(p_2)|} \\ L_2 &= L_1 \mathcal{T}_\nu(p_1, p_2) \end{aligned} \quad (4.18)$$

Note that it is only the transmission function which is physically and dynamically significant here, as *that* is what determines absorption. We could work directly with the transmission function, but for convenience, we'd much rather work in terms of optical depth.

However, the optical depth itself is not directly physically meaningful. This is because the transmission function only cares about absolute value differences in optical depth.<sup>4</sup> As such, we must

<sup>4</sup>The situation is analogous to Classical Electromagnetism or Newtonian Mechanics. In Electromagnetism and Mechanics, the only objects of physical consequence are the *forces*, which depend only on the electric/magnetic fields ( $\vec{E}, \vec{B}$ )<sup>5</sup> and accelerations ( $\vec{r}$ ) respectively. However, for convenience, we'd rather work with the electromagnetic scalar/vector potentials ( $\phi, \vec{A}$ ) or positions ( $\vec{r}$ ), so we have some freedom to choose some properties of ( $\phi, \vec{A}$ ) or ( $\vec{r}$ ). In the former case, we must fix the gauge (e.g., pick the Lorenz or Coulomb gauge) and in the latter case, we must pick the origin and velocity of our coordinate system (e.g., pick the origin as the centre of the Earth). In fancy speak, we say that electromagnetism has a gauge symmetry and Newtonian mechanics has galilean symmetry. We must do a similar thing here for optical depth.

<sup>5</sup>Actually the situation in electromagnetism is a bit more complicated than I've made it. In fact, only the motion of

make two choices when calculating our optical depth (analogous to choosing, e.g., the origin of our coordinate system in Newtonian mechanics):

1. It's ordering: we can let  $\tau_\nu$  *increase* with height (decrease with pressure) or *decrease* with height (increase with pressure). This is because the transmission function only cares about the absolute values.
2. The reference value: we can pick  $\tau_\nu = \tau_{\nu,ref}$  at any reference level. This is because the transmission function only cares about differences in optical depth.

I will adopt Ray's convention here, although I should note he switches it up for convenience (as he is well within his rights to!) in section 7.2.4. As such I will choose:

1.  $\tau_\nu$  to increase with height, i.e.,  $\tau_\nu$  to decrease with pressure.
2.  $\tau_\nu$  to be equal to 0 at the ground.

We now specify that the spectral radiance is travelling through a slab of air with some density  $\rho$  and with some absorber with mass-fraction  $q$ . We can see that  $d\mu = q\rho dl$  where  $dl$  is the length of atmosphere the radiation travels through. We further specify that the spectral radiance is traveling at some angle  $\theta$  relative to the vertical so that  $dl = +dz/\cos\theta$  where  $z$  = the height<sup>6</sup>. Finally, we relate  $dz$  to  $dp$  through Hydrostatic Balance (1.11). Then:

### Key Idea II.3: Optical Depth Coordinates and the Transmission Function

We define the optical depth coordinate  $\tau_\nu$ , which encodes how much absorbing *stuff* is below a pressure level  $p$ :

$$d\tau_\nu = -\kappa(\nu, T(p), p) q(p) \frac{dp}{g \cos\theta}$$

(4.19)

$$\tau_\nu(p) = \int_{p_s}^p -\kappa(\nu, T(p'), p') q(p') \frac{dp'}{g \cos\theta}$$

(4.21)

$$\tau_\nu = 0 \text{ at } p = p_s$$

(4.20)

$$\tau_{\nu,\infty} = \tau_\nu(0)$$

(4.22)

The transmission function  $\mathcal{T}_\nu$  between two points  $p_1$  and  $p_2$  encodes how much the spectral radiance is attenuated between the two points and is:

$$\mathcal{T}_\nu(p_1, p_2) = e^{-|\tau_\nu(p_1) - \tau_\nu(p_2)|}$$

(4.23)

$$L(\nu, p_2) = L(\nu, p_1) \mathcal{T}_\nu(p_1, p_2)$$

(4.24)

Notice that I write  $T = T(p)$  here. Until 7.1, we assume that the vertical temperature profile of the atmosphere is given by some known relation (perhaps the [Dry Adiabats](#) if we are in the troposphere).

#### 4.4.1 Example: Grey Atmosphere and Well Mixed Absorber

Let us make three assumptions. First, assume that the atmosphere is *grey*. This amounts to the assumption that the atmosphere interacts with light in the same way regardless of frequency, so  $\frac{\partial \kappa}{\partial \nu} = 0$ . Second, assume that  $\kappa$  has no  $T$  or  $p$  dependence. Third, assume that the absorber is *well-mixed*. This means that the concentration  $q$  is independent of height  $\frac{\partial q}{\partial p} = 0$ . We can now integrate

---

charged objects is directly observable. Since  $\vec{B}$  interacts with matter through equations involve its cross product (the Biot-Savart Law and  $\vec{F} = q\vec{v} \times \vec{B}$ ), we've used a convention of the right-handed cross product ( $\vec{x} \times \vec{y} = \vec{z}$ ). We could have equally used the left handed cross product and had equivalent empirical predictions. I'm also not going to go into bivectors and pseudovectors here, because this footnote is already too long.

<sup>6</sup>I could have equally chosen that  $dl = -dz/\cos\theta$  here.

the optical depth to find:

$$\begin{aligned}
\tau_\nu(p) &= \int_{p_s}^p -\kappa q(p) \frac{dp'}{g \cos \theta} \\
&= -\kappa q \frac{1}{g \cos \theta} \int_{p_s}^p dp' \\
&= \frac{\kappa q}{g \cos \theta} (p_s - p)
\end{aligned}$$

We define  $\tau_\infty = \int_{p_s}^0 d\tau_\nu$  as the optical depth of the *entire* atmosphere (at that angle  $\theta$ ). Therefore:

$$\begin{aligned}
\tau_\infty &= \tau(0) \\
&= \frac{\kappa q}{g \cos \theta} (p_s - 0) \\
&= \frac{\kappa q p_s}{g \cos \theta} \\
\therefore \tau_\nu(p) &= \tau_\infty \left( 1 - \frac{p}{p_s} \right)
\end{aligned}$$

However, typically,  $\kappa$  is a strong function of  $\nu$ . We will discuss how exactly  $\kappa$  depends on  $\nu$  in the next chapter.



# Chapter 5

## Molecular Spectroscopy

### 5.1 Spectral Lines

Here we cover how the absorption cross section  $\kappa$  depends on  $\nu$ . In almost all cases, save [Continuum Absorption](#),  $\kappa$  is made up of many ‘spectral lines’ (see Figure 5.1b and ignore the white lines).

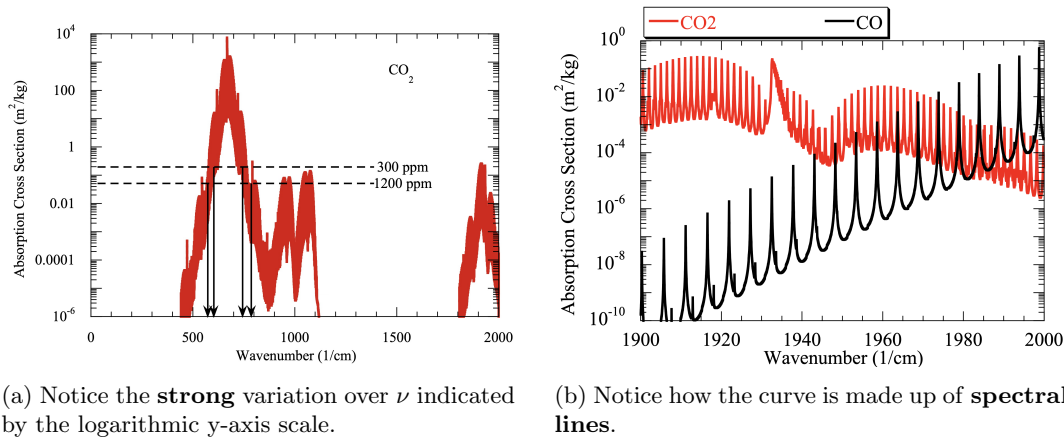


Figure 5.1: CO<sub>2</sub>'s Absorption Cross Section at 100 mbar and 260 K. Figures from Ray's lecture slides.

We will discuss how to characterise the **line shape** of each individual spectral line, which we will refer to as  $\kappa_{\nu_c}$ :

#### Key Idea II.4: Line Shape

An individual spectral line is characterised as follows. All parameters have units of  $\text{s}^{-1}$ .

$$\kappa_{\nu_c}(\nu) = \frac{S}{\delta} F\left(\frac{\nu - \nu_c}{\delta}\right) \quad (5.1)$$

**Line Centre**  $\nu_c$ : encodes the centre of the spectral line.

**Line Strength**  $S$ : encodes how strong of an absorber the spectral line is. Defined as follows:

$$S = \int_{-\infty}^{\infty} \kappa(\nu) d\nu \quad (5.2)$$

Generally a function of  $T$

**Line Width**  $\delta$ : encodes how wide the spectral line is. Generally a function of  $p$  and  $T$ .

**Line shape**  $F(x)$ : encodes the shape of the line (e.g., how does the line decay as  $\nu$  goes away from  $\nu_c$ ). Normalised such that:

$$\int_{-\infty}^{\infty} F(x) dx = 1$$

Why do gases absorb/emit electromagnetic radiation in the first place? Gases absorb/emit electromagnetic radiation because its constituents, the molecules, have different states corresponding to different energy levels. If a molecule receives just enough energy from a photon, it can transition from a lower energy level to a higher energy level, and thus absorb the photon. If a molecule is already in a higher energy level, it can spontaneously emit a photon to transition to a lower energy level.

There are three types of transitions that a molecule can undergo:

1. **Electron Transitions:** An electron on the molecule hops to a higher energy state. Typically visible and UV frequency photons cause/are emitted from these transitions.
2. **Vibrational Transitions:** The molecule begins vibrating/vibrates faster. Typically IR (Infrared) frequency photons cause/are emitted from these transitions.
3. **Rotational Transitions:** The molecule begins rotating/rotates faster. Typically microwave frequency photons cause/are emitted from these transitions.

We will now discuss what determines each parameter of  $\nu_c$ ,  $S$ ,  $\delta$  and  $F(x)$ .

### 5.1.1 Line Centre $\nu_c$

In order to discuss what determines the line centre  $\nu_c$ , we must discuss why spectral lines exist at all, rather than  $\kappa$  having a very smooth dependence on  $\nu$ .

When a molecule absorbs a photon of frequency  $\nu$ , it must receive energy  $E = h\nu$ . However, as discussed in Section 1.3, the energy levels of molecules are quantised, so molecules can only absorb a photon with energy equal to the energy level spacings  $\Delta E$ . As such,  $\nu$  is constrained such that  $\nu = \Delta E/h$ , and this is what sets the line centre.  $\Delta E$  itself is set by the quantum mechanical properties of the molecule, which is found through a variety of empirical and theoretical approaches.

A question you might (rightfully) have now is the following: if energy levels are discretised such that the molecule can only absorb energy of  $E_1, E_2, E_3, \dots$ , then it can only absorb photons with a frequency *exactly* equal to  $\nu_1 = h/E_1, \nu_2 = h/E_2, \nu_3 = h/E_3, \dots$ . Therefore, we should expect  $\kappa$  to be a collection of infinitely thin lines, and nothing like what we see in Figure 5.1, which is a collection of thin bumps! Furthermore, since only infinitely thin lines are absorbed, then nothing is actually absorbed at all, so  $\kappa = 0$  for all materials! What went wrong?

This question will be answered when we discuss [Broadening Mechanisms](#).

### 5.1.2 Line Strength $S$

The line strength  $S$  is determined by two factors.

The first factor is how strongly the molecule couples with the electromagnetic field. Typically, **electron** transitions already couple strongly to the electromagnetic field, as electrons are already charged, and moving charges couple strongly to electromagnetic radiation.

For **vibrational** and **rotational** transitions, the distortions of the molecule caused by the vibrations or rotations must change the molecule's electric **dipole-moment**. A full explanation of what 'dipole-moment' means isn't possible here, but the gist of it is this: a molecule has a dipole moment if it has an uneven distribution of electric charge. For example, if it is positively charged on one end and negatively charged on the other.

H<sub>2</sub>O has an intrinsic dipole moment, since the electrons in the bonds are more closely bound to the oxygen, and so the oxygen is a little bit negatively charged and the hydrogens are a little bit positively charged. Vibrations and rotations alter this, and this makes H<sub>2</sub>O a very strong infrared absorber.

CO<sub>2</sub> has no intrinsic dipole moment, but it has two vibrational modes (INSERT FIGURE HERE) which alter its dipole moment. This is typical of linear triatomic molecules. As such, it is a strong infrared absorber as well.

Meanwhile, N<sub>2</sub> and O<sub>2</sub> are both diatomic molecules made up of identical atoms, and so don't couple strongly to the electromagnetic field, since none of the rotational or vibrational modes change the molecules' dipole moment. This is why N<sub>2</sub> and O<sub>2</sub> are both optically inactive.

The second factor depends on the average occupation of the energy states. The intuition is as follows. Suppose a spectral line has a line centre at  $\nu_c$ , and thus relies on a transition from energy levels from  $E_i$  to  $E_j$  (such that  $E_j - E_i = h\nu_c$ ). A gas that has, say, half of its molecules in energy state

$E_j$  will have half of its molecules able to absorb a molecule of frequency  $\nu_c$ , but a gas that has a higher proportion (say, three-quarters) will have a higher proportion of its molecules (say, three-quarters) able to absorb a molecule of frequency  $\nu_c$ .

The occupation of energy states depends on the temperature of a system, and one can derive this using statistical mechanics. We won't derive it here, but the result for the functional form of  $S$  for IR (Infra-red) radiation is as follows:

$$S(T) = S(T_0) \frac{Q(T_0)}{Q(T)} \left( \frac{1 - \exp\left(\frac{-h\nu_c}{k_B T}\right)}{1 - \exp\left(\frac{-h\nu_c}{k_B T_0}\right)} \exp\left(-\frac{h\nu_l}{k_B} \left(\frac{1}{T} - \frac{1}{T_0}\right)\right) \right)$$

where  $h\nu_l$  is the energy of the lower energy state and  $Q(T)$  is some function of  $T$  depending on the molecule. You do not need to memorise or understand much of this equation, but it's important to note that the main dependence comes from the exponential terms.  $S$  **strongly** increases with temperature (it has an exponential dependence): as temperature increases, line-strength also increases.

## 5.2 Broadening Mechanisms: Line Shape $F$ and Width $\delta$

If we stop our discussion here, then we would have a collection of infinitely thin lines, and so  $F(x) = \delta(x)$  for all lines, where  $\delta$  is the dirac delta function. However, lines are clearly **broadened** (see Figure 5.1b) and are not delta functions. There are three physical mechanisms which explain why lines are broadened in our context, and why molecules may absorb/emit photons with frequency  $\nu$  close to  $\nu_c$  but not equal to  $\nu_c$ .

### 5.2.1 Uncertainty Broadening

One can show, but we won't show it here, that the following relation holds:

$$Var(\hat{E})\tau = h/2$$

where  $Var(\hat{E})$  is the variance of the energy of the system,  $\tau$  is the expected lifetime of the system, and  $h$  is Planck's constant. To be perfectly honest, I do not fully understand how this equation arises, and what relation this bears to the Heisenberg Uncertainty Relation other than a superficial resemblance.<sup>1</sup>

Let's now interpret the consequences of the equation on the molecules we're considering. The lifetime of a system  $\tau$  encodes how stable the system is. Recall that a molecule is in an 'excited' state before emitting a photon or after absorbing a photon. This state is, generally, meta-stable, and thus has a finite lifetime ( $\tau \neq \infty$ ) after which it has a high probability of emitting a photon and transitioning to a lower energy state.

The variance of the energy  $Var(\hat{E})$  represents the fact that, because of this, these molecules aren't actually in a state of exact/definite energy (called an energy eigenstate). This means that when a system transitions from one energy state to another energy state, the energy difference can actually be approximately equal to  $\Delta E$ , with some spread determined by the lifetime of the system  $\tau$ .

One can determine the line width and shape from these principles more rigourously, but it turns out that in normal atmospheric circumstances this line width is negligible, so we do not discuss this in C5. It will suffice for you simply to know that, regardless of whether the upcoming two broadening mechanisms come into effect, there is some natural Uncertainty Broadening mechanism which ensures that *all* lines have finite width.

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<sup>1</sup>The Heisenberg Uncertainty Relation relates the variance in momentum and variance in position of any quantum system. Despite what I see as a superficial resemblance, I will be using the name 'Uncertainty Broadening' because that's what it's called in C5 and I don't want to confuse you. If I could choose though, I'd probably use the sometimes-used 'Lifetime Broadening' or 'Natural Broadening'.

### 5.2.2 Doppler Broadening

Our second broadening mechanism is doppler broadening, and this occurs because the molecules in the gas might be moving away/towards us when they absorb or emit a photon. As such, the photon will be red or blue shifted.

Since the molecules are moving at non-relativistic speeds, the (doppler-shifted) frequency  $\nu$  we see is related to the actual linecentre  $\nu_c$  by:

$$\nu = \nu_c + (1 + v/c)$$

where  $c$  = the speed of light and  $v$  = the velocity the particle is moving in towards/away us.

The actual line-width is thus fixed by the velocity distribution of the molecules. This is predicted by the Maxwell-Boltzmann Distribution using Kinetic Theory, and so we get that:

$$F(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} \quad (5.3)$$

$$\delta(T) = \frac{\nu_c}{c} \left( \frac{2k_B T}{m} \right)^{\frac{1}{2}} \quad (5.4)$$

So the line-width scales as  $\sqrt{T}$  – the line gets thicker as the gas gets hotter. Conversely, if the temperature falls to 0 this equation predicts that the line width would fall to 0, and so  $\kappa(\nu = \nu_c) \rightarrow \infty$ . However, this never happens, as Uncertainty Broadening maintains a non-zero line width.

This mechanism of broadening is much more substantial than Uncertainty Broadening, but it is still very thin for two reasons. First, as seen from 5.3, the line shape is exponential in  $x$ . As such,  $F$  falls off extremely rapidly as  $\nu$  moves away from  $\nu_c$ . Second, as seen from 5.4, generally  $\delta \ll \nu_c$  since  $\left( \frac{2k_B T}{m} \right)^{\frac{1}{2}} \ll c^2$ . However, it is dominant in the stratosphere.

### 5.2.3 Collisional/Pressure Broadening

In the troposphere, this is by far the most important type of broadening. This arises because the kinetic energy of a molecule is not actually quantised. As such, if a molecule is colliding with another molecule while a photon hits it, it can borrow/give some energy to the other molecule to absorb a photon of slightly different frequency than  $\nu_c$ .

The broadening thus scales with the frequency of collisions, which depends on the pressure and temperature. This gives us the Lorentz Profile:

#### Key Idea II.5: Lorentz Line Shape

The shape  $F$  and width  $\delta$  of a spectral line is, in planetary climate contexts, mainly set by the Lorentz Line shape:

$$F(x) = \frac{1}{\pi} \frac{1}{x^2 + 1} \quad (5.5)$$

$$\delta \approx \delta_0 \frac{p}{p_0} \left( \frac{T_0}{T} \right)^n \quad (5.6)$$

where  $\delta_0$ ,  $p_0$ ,  $T_0$  are all reference widths, pressures, and temperatures and  $n$  is a dimensionless constant. All constants are, in practice, found empirically. Oftentimes,  $n \approx 1/2$ .

As we can see then, pressure broadening results in pressure *increasing* the line width and temperature *decreasing* the line width. If, for example, the pressure were to fall to 0 this equation predicts

<sup>2</sup>Physically,  $\left( \frac{2k_B T}{m} \right)^{\frac{1}{2}}$  is proportional to the mean speed of the molecules.

that the line width would fall to 0, and so  $\kappa(\nu = \nu_c) \rightarrow \infty$ . However, again, this never happens, as Uncertainty Broadening (and Doppler Broadening if  $T \neq 0$ ) would maintain a non-zero line width.

You should memorise this for the exam Key Idea [5.2.3](#) for the exam.

## 5.3 Continuum Absorption

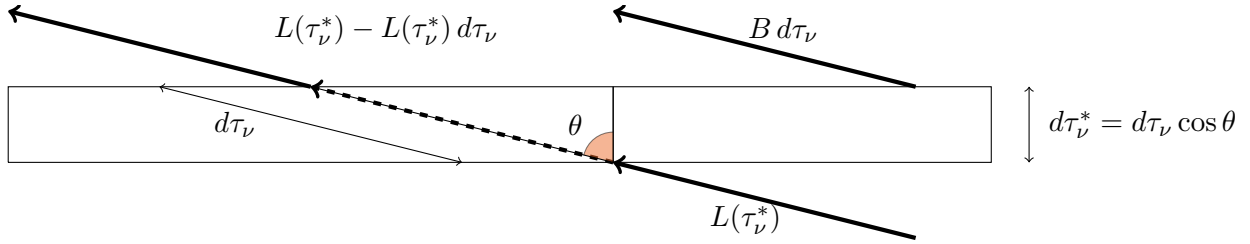
[Under Construction]

## Chapter 6

# The Schwarzschild Equation

### 6.1 The Equations of Motion

Consider some radiation traveling through a slab of atmosphere at some zenith angle  $\theta$  with some spectral radiance  $L$  with some frequency  $\nu$ . We wish to derive how  $L(\nu)$  varies as it propagates through the atmosphere. Suppose the slab of atmosphere has vertical infinitesimal optical thickness of  $d\tau_\nu^*$ . Since the path of the radiation is slanted, the spectral radiance travels through an infinitesimal optical depth of  $d\tau_\nu = d\tau_\nu^* / \cos \theta$ . The slab absorbs some spectral radiance, but it also emits some spectral radiance:



The spectral radiance going into the slab is  $L(\tau_\nu^*)$ . The spectral radiance going out of the slab is  $L(\tau_\nu^* + d\tau_\nu^*)$ , which we wish to express in terms of  $L(\tau_\nu^*)$ . We know that some proportion  $\alpha$  of  $L(\tau_\nu^*)$  is absorbed by the radiation and that this proportion is equal to the absorptivity  $\alpha(\nu, T) = d\tau_\nu$ . However, we also know that the slab of atmosphere also emits radiation equal to  $\epsilon(\nu, T) B(\tau_\nu^*)$ , where  $\epsilon$  = the emissivity. Therefore:

$$\begin{aligned} L(\tau_\nu^* + d\tau_\nu^*) &= L(\tau_\nu^*) \underbrace{(1 - \alpha)}_{\text{absorption}} + \underbrace{\epsilon B(\tau_\nu^*)}_{\text{emission}} \\ &= L(\tau_\nu^*) - L(\tau_\nu^*) d\tau_\nu + \epsilon B(\tau_\nu^*) \end{aligned}$$

We now assume that the slab is in local thermodynamic equilibrium, and so  $\epsilon = \alpha = d\tau_\nu$  and rearrange to get:

$$\frac{L(\tau_\nu^* + d\tau_\nu^*) - L(\tau_\nu^*)}{d\tau_\nu} = -L(\tau_\nu^*) + B(\tau_\nu^*)$$

Inconveniently, our coordinate  $d\tau_\nu$  depends on  $\theta$ , but we wish to have a coordinate be the same at every horizontal position. We thus define  $d\tau_\nu^* = d\tau_\nu \cos \theta$  and derive the **Schwarzschild Equation**:

$$\cos(\theta) \frac{dL}{d\tau_\nu^*}(\tau_\nu^*) = -L(\tau_\nu^*) + B(\nu, T(\tau_\nu^*))$$

(6.1)

## 6.2 Angle Averaging: The Two-Stream Approximation

Note that generally,  $L$  is a function of direction ( $\frac{\partial L}{\partial \vec{\omega}} \neq 0$ ). We proceed by making what's called the *Two-Stream Approximation*, but we need not make this approximation if we were coding up a full radiative transfer model. We only assume this here to make analytical progress in order to gain some physical intuition.

We define the net upwards and downward fluxes of irradiance as follows:

$$F_+ = \int_{\Omega_+} L(\vec{\omega}) d\Omega_{\perp} \quad (6.2)$$

$$F_- = - \int_{\Omega_-} L(\vec{\omega}) d\Omega_{\perp} \quad (6.3)$$

Note that Ray refers to these as  $I_+$  and  $I_-$  in his notes. However, we shall refer to them here by the letter  $F$ , because these are (spectral) **irradiances**, **not** (spectral) **radiances**, as we have integrated over solid angles here. We then integrate Equation 6.1 in the upwards and downwards directions, and make the assumption that  $B$  is isotropic.

$$\begin{aligned} \int_{\Omega_+} \cos(\theta) \frac{dL}{d\tau_{\nu}^*} d\Omega &= \int_{\Omega_+} -L + B d\Omega \\ \int_{\Omega_+} \frac{dL}{d\tau_{\nu}^*} d\Omega_{\perp} &= - \int_{\Omega_+} L d\Omega + \int_{\Omega_+} B d\Omega \\ \frac{dF_+}{d\tau_{\nu}^*} &= - \int_{\Omega_+} L d\Omega + 2\pi B \end{aligned}$$

So far, everything is roughly very accurate. We now make the *Two-Stream Approximation* in order to relate  $F_+$  to  $\int_{\Omega_+} L d\Omega$ . We make (counter-intuitively) the assumption that  $L(\vec{\omega})$  is roughly isotropic enough to be pulled outside of the integral over solid angle. This allows us to find that:

$$\begin{aligned} \int_{\Omega_+} L d\Omega &\approx L \int_{\Omega_+} d\Omega \\ &= L 2\pi \\ &= 2\pi L \frac{\int_{\Omega_+} d\Omega_{\perp}}{\int_{\Omega_+} d\Omega_{\perp}} \\ &\approx 2\pi \frac{\int_{\Omega_+} L d\Omega_{\perp}}{\int_{\Omega_+} d\Omega_{\perp}} \\ &= 2\pi \frac{F_+}{\pi} \\ \int_{\Omega_+} L d\Omega &\approx 2F_+ \end{aligned}$$

We can go through a similar reasoning with  $F_-$ , which allows us to derive the angle-averaged upwards and downwards Schwarzschild Equations:

$$\frac{1}{2} \frac{dF_+}{d\tau_{\nu}^*} = -F_+ + \pi B \quad (6.4)$$

$$-\frac{1}{2} \frac{dF_-}{d\tau_{\nu}^*} = -F_- + \pi B \quad (6.5)$$

Note the extra ‘-’ sign on Equation 6.5. Comparing 6.4 with 6.1, we can conclude that this approximation is equivalent to setting  $\cos(\theta) = \frac{1}{2}$  on average! With this in mind, we define the effective propagation angle  $\tilde{\theta}$  as  $\tilde{\theta} = \cos^{-1}(1/2)$ . We now abuse notation and define  $\tau_{\nu} \equiv \tau_{\nu}^* / \cos(\tilde{\theta}) = 2\tau_{\nu}^*$  to get rid of that pesky  $\frac{1}{2}$  factor to obtain the:

## Key Idea II.6: Angle-Averaged Schwarzschild Equations

The following equations govern the net upward ( $F_+$ ) and downwards ( $F_-$ ) spectral irradiance, derived from the [Schwarzschild Equation](#) using the *Two-Stream Approximation*.

$$\frac{dF_+}{d\tau_\nu}(\tau_\nu) = -F_+(\tau_\nu) + \pi B$$

(6.6)

$$-\frac{dF_-}{d\tau_\nu}(\tau_\nu) = -F_-(\tau_\nu) + \pi B$$

(6.7)

We can relate optical depth coordinates  $\tau_\nu$  to pressure coordinates  $p$  using [4.19](#):

$$d\tau_\nu = -\kappa(\nu, T(p), p) q(p) \frac{dp}{g \frac{1}{2}} \quad (6.8)$$

$$= -\kappa(\nu, T(p), p) q(p) \frac{dp}{g \cos \tilde{\theta}} \quad (6.9)$$

where  $\tilde{\theta}$  = the effective propagation angle.

## 6.3 General Solutions and Radiating Level

We can integrate [6.6](#) and [6.7](#) using an integrating factor and a single boundary condition to find:

$$F_+(\nu, \tau_\nu) = F_+(\nu, 0)e^{-\tau_\nu} + \int_0^{\tau_\nu} \pi B(\nu, T(\tau'_\nu))e^{-(\tau_\nu - \tau'_\nu)} d\tau'_\nu \quad (6.10)$$

$$F_-(\nu, \tau_\nu) = F_-(\nu, \tau_{\nu, \infty})e^{-(\tau_{\nu, \infty} - \tau_\nu)} + \int_{\tau_\nu}^{\tau_{\nu, \infty}} \pi B(\nu, T(\tau'_\nu))e^{-(\tau'_\nu - \tau_\nu)} d\tau'_\nu \quad (6.11)$$

where we have used the boundary conditions:

$$\text{at the ground } \boxed{\tau_\nu = 0, F_+(\nu, \tau_\nu) = F_+(\nu, 0)}$$

$$\text{at the top of the atmosphere } \boxed{\tau_\nu = \tau_{\nu, \infty}, F_-(\nu, \tau_\nu) = F_-(\nu, \tau_\infty)}$$

There are three subtleties we should note. First, note the sign flips between [6.10](#) and [6.11](#), arising due to the sign flip on the derivative in [6.6](#) and [6.7](#). Second, note that the limits of integration are swapped between [6.10](#) and [6.11](#), arising due to the difference in boundary conditions.

The final subtlety is the most important: in general  $\tau_\nu = \tau_\nu(\nu)$ . Suppose we wish to find the total radiative upwards flux of energy at some height/pressure level. We cannot simply find  $F_+(\tau = \tau(p))$  then integrate over all frequencies, as in general  $\tau_{\nu_1}(p) \neq \tau_{\nu_2}(p)$ .

Physically, the solution is intuitive: the total upwards radiance at some optical depth  $\tau$  is simply the sum of the irradiance at each level  $\tau'$ , each attenuated by the transmission function between  $\tau$  and  $\tau'$ . To be more explicit, we sum:

1.  $F_+(\nu, 0)e^{-\tau_\nu}$ : The irradiance ( $F_+(\nu, 0)$ ) from the bottom ( $\tau_\nu = 0$ ) of the atmosphere, attenuated by the optical depth  $e^{-(\tau_\nu - 0)}$ .<sup>1</sup>
2.  $\pi B(\nu, T(\tau'_\nu)) d\tau'_\nu e^{-(\tau_\nu - \tau'_\nu)}$ : The irradiance ( $B(\nu, T(\tau'_\nu)) d\tau'_\nu$ ) from each layer ( $\tau_\nu = \tau'_\nu$ ), also attenuated by the optical depth  $e^{-(\tau_\nu - \tau'_\nu)}$

<sup>1</sup>Generally, this is the upwards irradiance emitted by the ground on a rocky planet, but one could also choose  $\tau_\nu = 0$  in the middle of the atmosphere (as you must on a gaseous planet), in which case  $F_+(\nu, 0)$  will be the total upwards irradiance at that level (originating from all the atmosphere (and ground) underneath it).



The **Outgoing Longwave Radiation**  $OLR$  (in spectral irradiances) is then defined as  $F_+(\nu, p = 0) = F_+(\nu, \tau = \tau_{\nu, \infty})$  and given by:

$$OLR = F_+(\nu, 0)e^{-\tau_{\nu, \infty}} + \int_0^{\tau_{\nu, \infty}} \pi B(\nu, T(\tau'_\nu))e^{-(\tau_{\nu, \infty} - \tau'_\nu)} d\tau'_\nu \quad (6.12)$$

The physical interpretation is identical. The total upwards spectral irradiance escaping is the spectral irradiance from each level attenuated by the amount of absorption in between those levels.

Since the attenuation is exponential, this gives us an optical depth ‘lengthscale’ set by when  $(\tau_{\nu, \infty} - \tau'_\nu) \sim 1$ . This allows us to define:

#### Key Idea II.7: The Radiating Level $p_{rad}$

We define the **radiating level** at  $\tau_\nu = \tau_{\nu, rad}$  and  $p = p_{rad}$  which satisfy the following relations:

$$\tau_{\nu, \infty} - \tau_{\nu, rad} = 1$$

(6.13) *Figure here!*

$$\tau_\nu(p_{rad}) = \tau_{\nu, rad}$$

(6.14)

Physically, the interpretation is as follows: any radiation coming from an optical depth  $\tau_\nu < \tau_{\nu, rad}$  ( $p > p_{rad}$ ) will be attenuated **much** more than radiation coming from an optical depth  $\tau_\nu > \tau_{\nu, rad}$  ( $p < p_{rad}$ ).

As such, the  $OLR$  radiation actually escaping into space will disproportionately look like the radiation emitted by ‘stuff’ at  $p_{rad}$ . In other words, if you were to look down at the atmosphere, you would not ‘see’ anything from an optical depth coming from  $p > p_{rad}$ .

Note that this definition only really works if  $\tau_{\nu, \infty} > 1$ . If  $\nu$  is in the visible light region, for example,  $\tau_{\nu, \infty} < 0$ , which is why you can see all the way to the ground (since  $p_{rad} = p_s$ , since  $\tau_\nu \rightarrow -\infty$  at the ground due to the discontinuity of switching between something optically thin (the atmosphere) and something optically thick (the ground)).

We can also express everything in terms of the transmission function. Defining the radiating level in terms of the transmission function is straightforward:  $p_{rad}$  satisfies the following relation<sup>2</sup>:

$$\mathcal{T}_\nu(0, p_{rad}) = e^{-1} \quad (6.15)$$

We can also rewrite 6.4 and 6.5 in terms of the transmission function  $\mathcal{T}_\nu(p_1, p_2)$  if we find the differential of the transmission function with respect to  $p_2$  and assume that  $\tau_\nu(p_1) \geq \tau_\nu(p_2)$  (i.e., if  $p_1 \leq p_2$ ):

$$\begin{aligned} d(\mathcal{T}_\nu(p_1, p_2)) &= d\left(e^{-|\tau_\nu(p_1) - \tau_\nu(p_2)|}\right) \\ &= e^{-\tau_\nu(p_1)} d\left(e^{\tau_\nu(p_2)}\right) \\ &= e^{-\tau_\nu(p_1)} e^{\tau_\nu(p_2)} d\tau_\nu(p_2) \\ &= e^{-|\tau_\nu(p_1) - \tau_\nu(p_2)|} d\tau_\nu(p_2) \\ \therefore d\mathcal{T}_\nu(p_1, p_2) &= \mathcal{T}_\nu(p_1, p_2) d\tau_\nu(p_2) \end{aligned}$$

If, conversely,  $\tau_\nu(p_1) \leq \tau_\nu(p_2)$  (i.e., if  $p_1 \geq p_2$ ), then  $d\mathcal{T}_\nu(p_1, p_2) = -\mathcal{T}_\nu(p_1, p_2) d\tau_\nu(p_2)$ .

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<sup>2</sup>And, of course,  $p_{rad} > 0$ .

## Key Idea II.8: Solution to the Angle-Averaged Schwarzschild Equations

The total upwards  $F_+(\nu, p)$  and downwards  $F_-(\nu, p)$  spectral irradiance at frequency  $\nu$  and pressure level  $p$  is given by the following:

$$F_+(\nu, p) = F_+(\nu, 0)\mathcal{T}_\nu(p, p_s) + \int_{p'=p_s}^p \pi B(\nu, T(p')) d\mathcal{T}_\nu(p, p') \quad (6.16)$$

$$F_-(\nu, p) = F_-(\nu, 0)\mathcal{T}_\nu(0, p) - \int_{p'=p}^0 \pi B(\nu, T(p')) d\mathcal{T}_\nu(p, p') \quad (6.17)$$

where  $d\mathcal{T}_\nu(p, p')$  is the differential transmission function with respect to  $p'$ .  $d\mathcal{T}_\nu(p, p') = \mathcal{T}_\nu(p, p') d\tau_\nu(p')$  if  $p \leq p'$  (as in 6.16) or  $d\mathcal{T}_\nu(p, p') = -\mathcal{T}_\nu(p, p') d\tau_\nu(p')$  if  $p \geq p'$  (as in 6.17).

The **Outgoing Longwave Radiation OLR** (in spectral irradiances) is then defined as  $F_+(\nu, p=0) = F_+(\nu, \tau = \tau_{\nu, \infty})$  and given by:

$$OLR = F_+(\nu, 0)\mathcal{T}_\nu(0, p_s) + \int_{p_s}^0 \pi B(\nu, T(p')) d\mathcal{T}_\nu(0, p') \quad (6.18)$$

Note that even though the integral in 6.16 is taken from  $p$  to  $p < p_s$ , the integral is positive because  $d\mathcal{T}_\nu(p, p') < 0$ , because  $d\mathcal{T}_\nu(p, p') \sim d\tau(p') \sim -dp'$ .

## 6.4 Frequency Bands

We're not finished yet! Recall that we have expressed *OLR* in terms of *spectral* irradiances, but we require the (non-spectral) irradiances if we wish to compute the total outgoing radiative energy flux. What we may do now is integrate the *OLR* over the centre of a single absorption band centred about  $\nu_c$ . Typically, both  $F_+(\nu, 0)$  and  $B(\nu, T)$  are roughly constant as a function of  $\nu$  compared to  $\tau_\nu$ , so we can treat  $F_+(\nu, 0) \approx F_+(\nu_c, 0)$  and  $B(\nu) \approx B(\nu_c)$  for  $\nu \in [\nu_c - \frac{\Delta}{2}, \nu_c + \frac{\Delta}{2}]$ .

Integrating 6.18 over frequency, we find the following:

$$OLR_{\nu_c} = F_+(\nu_c, 0)\bar{\mathcal{T}}_{\nu_c}(0, p_s) + \int_{p_s}^0 \pi B(\nu_c, T(p')) d\bar{\mathcal{T}}_{\nu_c}(p', p_s) \quad (6.19)$$

where we have assumed that  $\mathcal{T}_\nu \approx 0$  at  $\nu = \nu_c \pm \frac{\Delta}{2}$  (the line is thin) and defined the **Band Integrated Transmission Function**  $\bar{\mathcal{T}}_{\nu_c}$  as:

$$\bar{\mathcal{T}}_{\nu_c}(p_1, p_2) = \int_{-\infty}^{\infty} \mathcal{T}_\nu d\nu \quad (6.20)$$

$$\approx \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} \mathcal{T}_\nu d\nu \quad (6.21)$$

$$= \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} e^{-|\tau_\nu(p_1) - \tau_\nu(p_2)|} d\nu \quad (6.22)$$

Note that  $OLR_{\nu_c}$  has dimensions of an irradiance (power per area), since  $\bar{\mathcal{T}}_{\nu_c}$  has dimensions of frequency and  $F_+$  and  $\pi B$  have dimensions of spectral irradiance (power per area per frequency). We now abuse notation and write the total *OLR* as the total power leaving the Earth per unit area. We've already

assumed that each line is thin, which implies that the spectral lines don't overlap, so we can find  $OLR$  by summing up  $OLR_{\nu_c}$  for each spectral line centred about each  $\nu_c$ :

$$OLR = \sum_{\nu_c} OLR_{\nu_c}$$

Now that our  $OLR$  is in terms of known quantities and  $\bar{\mathcal{T}}_{\nu_c}$ , our next goal is to calculate  $\bar{\mathcal{T}}_{\nu_c}$ . Typically, this can be done numerically, but to make analytical progress we will look at three cases: the no-line limit, weak line limit, and strong line limit.

### 6.4.1 The No-Line Limit

I should make you aware that this is my own personal addition and not talked about in the lecture notes. This limit is a bit non-sensical<sup>3</sup>, but I think it builds some intuition.

The no-line limit is if  $\kappa = 0$  at that frequency band, and so  $|\tau_{\nu}(p_1) - \tau_{\nu}(p_2)| = 0$ . As such, the transmission function  $\mathcal{T}_{\nu}(p_1, p_2) = e^{-|\tau_{\nu}(p_1) - \tau_{\nu}(p_2)|} = 1$  and so:

$$\begin{aligned} \bar{\mathcal{T}}_{\nu_c}(p_1, p_2) &= \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} 1 d\nu \\ &= \Delta \end{aligned}$$

We can therefore substitute  $\Delta$  for  $\bar{\mathcal{T}}_{\nu_c}$  everywhere. Physically, this means that *nothing* is absorbed out of this absorption band: everything in the band from  $\nu = \nu - \frac{\Delta}{2}$  to  $\nu = \nu + \frac{\Delta}{2}$  is let through. Substituting explicitly that  $\bar{\mathcal{T}}_{\nu_c}(p_1, p_2) = \Delta$  and  $d\bar{\mathcal{T}}_{\nu_c}(p_1, p_2) = 0$  for all  $p_1$  and  $p_2$ , our solution is:

$$OLR_{\nu_c} = F_+(\nu_c, 0)\Delta$$

In other words, the  $OLR_{\nu_c}$  at this frequency  $\nu_c$  is simply set by the upward radiative flux at frequency  $\nu_c$  from the ground. Physically, the solution looks like this because of two reasons.

1.  $\alpha = 0$ : Since  $\kappa = 0$ ,  $\alpha = 0$ , which means the atmosphere is completely transparent at this frequency. As such, all the spectral radiance from the ground is let through, and the  $OLR_{\nu_c}$  is completely ignorant of the concentration.
2.  $\epsilon = 0$ : Due to Kirchoff's law,  $\epsilon = 0$  as well, which means the atmosphere emits no radiation at this frequency either. As such, none of the  $OLR_{\nu_c}$  comes from the atmosphere.

Because of these two reasons, the  $OLR_{\nu_c}$  is completely ignorant of any atmospheric properties.

We'll find that in the weak and strong line limits (and indeed in all cases where  $\kappa \neq 0$ ) that things change dramatically. First,  $\bar{\mathcal{T}}_{\nu_c} < \Delta$ , and so the effective 'width' will be smaller than  $\Delta$ , representing the chunk of radiation from the ground that is absorbed by the fact that  $\kappa \neq 0$ . Second, the integral term will not be negative, reflecting the fact that the atmosphere contributes to the  $OLR$ . Importantly, this will depend on the vertical temperature profile of the atmosphere, as the temperature sets the Blackbody radiation.

### 6.4.2 The Weak Line Limit

The weak line limit obtains if  $|\tau_{\nu}(p_1) - \tau_{\nu}(p_2)| \ll 1$  for all frequencies  $\nu$ . If  $\tau(p)$  is well-behaved, this will be arbitrarily satisfied if  $p_1$  and  $p_2$  are sufficiently close. We can then Taylor Expand  $e^{|\tau_{\nu}(p_1) - \tau_{\nu}(p_2)|}$  and keep only first order terms:

$$e^{-|\tau_{\nu}(p_1) - \tau_{\nu}(p_2)|} \approx 1 - |\tau_{\nu}(p_1) - \tau_{\nu}(p_2)|$$

---

<sup>3</sup>I think it's non-sensical because optical depth coordinates make no sense if  $\kappa = 0$ . Recall that  $d\tau \sim \kappa$ , so if  $\kappa = 0$  then  $\tau(p) = 0$  for all  $p$ . Furthermore, if  $\kappa = 0$ , then this is hardly a spectral line! That being said, our solution is still valid.

We can now integrate over frequencies to find that:

$$\begin{aligned}
\bar{\mathcal{T}}_{\nu_c}(p_1, p_2) &\approx \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} e^{-|\tau_{\nu}(p_1) - \tau_{\nu}(p_2)|} d\nu \\
&\approx \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} 1 - |\tau_{\nu}(p_1) - \tau_{\nu}(p_2)| d\nu \\
&= \Delta - \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} \left| \int_{p_2}^{p_1} -\kappa(\nu, T(p), p) q(p) \frac{dp}{g \cos \bar{\theta}} \right| d\nu \\
&= \Delta - \frac{1}{g \cos \bar{\theta}} \left| \int_{p_2}^{p_1} q(p) \underbrace{\left( \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} \kappa(\nu, T(p), p) d\nu \right)}_{S(T(p))} dp \right| \\
&= \Delta - S(T_0) \left( \frac{1}{g \cos \bar{\theta}} \left| \int_{p_1}^{p_2} \frac{S(T(p))}{S(T_0)} q(p) dp \right| \right)
\end{aligned}$$

The keen reader will note that this result **differs** from the result obtains in Ray's lecture slides. I am confident, however, that this expression is correct, as this is the expression that shows up in Ray's book [3] and the algebra is more rigorous.

We now define (what seems to me as a dodgy definition!) the effective strength  $S = S(T_0)$  such that:

$$\int_{p_1}^{p_2} S(T(p)) q(p) dp \equiv S(T_0) \int_{p_1}^{p_2} q(p) dp \quad (6.23)$$

This means that the band-integrated transmission function in the weak line limit is:

$$\bar{\mathcal{T}}_{\nu_c}(p_1, p_2) \approx \Delta - S\mu \quad (6.24)$$

where  $\mu$  is the mass path between those pressures such that  $\mu_p = \frac{1}{g \cos \bar{\theta}} \left| \int_{p_2}^{p_1} q(p) dp \right|$ . I have also abused notation and written just  $S$  for the ffective strength.

Intuitively,  $\Delta - S\mu$  is the effective width of this line, and the  $S\mu$  term represents the 'chunk' of outwards irradiance taken out by this line (the chunk absorbed). As you can see, the absorption is **linear** in mass-path, not exponential as we might expect from the exponential attenuation of spectral radiance in (Equation 4.18). This is possible because the attenuation of **spectral** irradiance is not the same as the attenuation of **non-spectral** radiance: we found the former by integrating the latter over the spectral line, which has its own dependence on  $\nu$ .

Let us now calculate the  $OLR_{\nu_c}$  in the weak line limit. We again need to find the differential  $d\bar{\mathcal{T}}_{\nu_c}(0, p')$  with respect to  $p'$ :

$$\begin{aligned}
d\bar{\mathcal{T}}_{\nu_c}(p', p_s) &= d \left( \Delta - S \frac{1}{g \cos \bar{\theta}} \left| \int_0^{p'} q(p) dp \right| \right) \\
&= -\frac{S}{g \cos \bar{\theta}} d \left( \int_0^{p'} q(p) dp \right) \\
&= -\frac{S}{g \cos \bar{\theta}} q(p') dp'
\end{aligned}$$

where we have used integral properties to go from the second line to the third line. We substitute this into 6.19 to find that:

$$OLR_{\nu_c} = F_+(\nu_c, 0)(\Delta - S\mu) + \int_0^{p_s} \pi B(\nu_c, T(p')) \frac{Sq(p')}{g \cos \bar{\theta}} dp' \quad (6.25)$$

We interpret the terms physically as follows:

$$OLR_{\nu_c} = \underbrace{F_+(\nu_c, 0)(\Delta \overbrace{-S\mu}^{\text{Width taken out}})}_{\text{Emission from Ground}} + \underbrace{\int_0^{p_s} \pi B(\nu_c, T(p')) \frac{Sq(p')}{g \cos \tilde{\theta}} dp'}_{\text{Emission from Atmosphere}}$$

So if  $\kappa = 0$ , we see that  $OLR_{\nu_c}$  *decreases* due to the atmosphere absorbing some emission from the ground, but it also *increases* due to the atmosphere itself emitting some radiation. Whether this ultimately increases or decreases  $OLR_{\nu_c}$  will depend crucially on  $T(p)$ , the vertical temperature profile of the atmosphere.

### 6.4.3 The Strong Line Limit

For the strong line limit, we require that  $|\tau_\nu(p_1) - \tau_\nu(p_2)| \gg 1$  at the *centre* of the line (i.e. at  $\nu = \nu_c$ ). This is weaker than the opposite of the weak-line-limit assumption, because any line far enough from the centre is weak. We also assume a Lorentz line shape and width from Key Idea 5.2.3 so our treatment here will not be completely general.

If  $|\tau_\nu(p_1) - \tau_\nu(p_2)| \gg 1$  at the centre of the line, then we can approximate it as effectively infinite. We thus approximate the Lorentz line-shape as:

$$F(x) = \frac{1}{\pi} \frac{1}{x^2 + 1} \approx \frac{1}{\pi} \frac{1}{x^2}$$

We then integrate the transmission function over frequencies again:

$$\begin{aligned} \bar{\tau}_{\nu_c}(p_1, p_2) &\approx \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} \exp(-|\tau_\nu(p_1) - \tau_\nu(p_2)|) d\nu \\ &= \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} \exp\left(-\left|\int_{p_2}^{p_1} -\kappa(\nu, T(p), p) q(p) \frac{dp}{g \cos \tilde{\theta}}\right|\right) d\nu \\ &\approx \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} \exp\left(-\left|\int_{p_2}^{p_1} \frac{1}{\pi} \frac{S(T)}{\delta(T, p)} \left(\frac{\nu - \nu_c}{\delta(T, p)}\right)^{-2} q(p) \frac{dp}{g \cos \tilde{\theta}}\right|\right) d\nu \\ &= \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} \exp\left(-\left|\int_{p_2}^{p_1} \frac{1}{\pi} S(T) \delta(T, p) \frac{1}{(\nu - \nu_c)^2} q(p) \frac{dp}{g \cos \tilde{\theta}}\right|\right) d\nu \\ &= \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} \exp\left(-\left|\int_{p_2}^{p_1} \frac{1}{\pi} S(T) \frac{p}{p_0} \left(\frac{T_0}{T}\right)^n \frac{1}{(\nu - \nu_c)^2} q(p) \frac{dp}{g \cos \tilde{\theta}}\right|\right) d\nu \\ &= \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} \exp\left(-\frac{\delta_0}{\pi g \cos \tilde{\theta}} \frac{1}{(\nu - \nu_c)^2} \left|\int_{p_2}^{p_1} S(T(p)) \frac{p}{p_0} \left(\frac{T_0}{T(p)}\right)^n q(p) dp\right|\right) d\nu \end{aligned}$$

Again, to get rid of the pesky  $T$  dependence within the integral with respect to  $p$ , we define the effective strength  $S = S(T_0)$  such that:

$$\int_{p_1}^{p_2} S(T(p)) \frac{p}{p_0} \left(\frac{T_0}{T(p)}\right)^n q(p) dp \equiv S(T_0) \int_{p_1}^{p_2} \frac{p}{p_0} q(p) dp$$

Therefore:

$$\bar{\tau}_{\nu_c}(p_1, p_2) = \int_{\nu_c - \frac{\Delta}{2}}^{\nu_c + \frac{\Delta}{2}} \exp\left(-\frac{1}{(\nu - \nu_c)^2} \frac{S\delta_0}{\pi g \cos \tilde{\theta}} \left|\int_{p_2}^{p_1} \frac{p}{p_0} q(p) dp\right|\right) d\nu$$

To make the integral easier, we note that the integral is symmetric about  $\nu = \nu_c$  and define  $x$  such that:

$$\frac{1}{x^2} \equiv \frac{1}{(\nu - \nu_c)^2} \frac{S\delta_0}{\pi g \cos \tilde{\theta}} \left|\int_{p_2}^{p_1} \frac{p}{p_0} q(p) dp\right|$$

Therefore:

$$\bar{\mathcal{T}}_{\nu_c}(p_1, p_2) = 2\sqrt{\frac{S\delta_0}{g \cos \bar{\theta}} \left| \int_{p_2}^{p_1} \frac{p}{p_0} q(p) dp \right|} \int_0^X \exp\left(-\frac{1}{\pi x^2}\right) dx$$

where

$$X = \frac{\Delta}{2} \left( \frac{S\delta_0}{g \cos \bar{\theta}} \left| \int_{p_2}^{p_1} \frac{p}{p_0} q(p) dp \right| \right)^{-1/2}$$

One can use the result that the following integral holds if  $X \gg 1$ .

$$\int_0^X \exp\left(-\frac{1}{\pi x^2}\right) \approx X - 1$$

Note that this assumption is satisfied if we assume that the lines are thin, and that the integrand decays to 0 at the linewidth, which implies that  $X \gg 1$ . This finally gives us that the transmission function is as follows in the strong-line limit:

$$\bar{\mathcal{T}}_{\nu_c}(p_1, p_2) = \Delta - 2\sqrt{S\delta_0\mu_p} \quad (6.26)$$

where  $\mu_p$  is the pressure weighted mass path between those two pressures:

$$\mu_p = \frac{1}{g \cos \bar{\theta}} \left| \int_{p_2}^{p_1} \frac{p}{p_0} q(p) dp \right|$$

Again,  $\Delta - 2\sqrt{S\delta_0\mu_p}$  is the effective width of this line, and the  $2\sqrt{S\delta_0\mu_p}$  term represents the ‘chunk’ of outwards irradiance taken out by this line (the chunk absorbed). As you can see, the absorption scales as the **square-root** of the (pressure-weighted) mass-path, not exponential. Again, this is possible because the attenuation of **spectral** irradiance is not the same as the attenuation of **non-spectral** radiance. A new point is that this is *slower* than a weak line.

We continue by calculating  $OLR_{\nu_c}$  in the strong line limit. We again need to find the differential  $d\bar{\mathcal{T}}_{\nu_c}(0, p')$  with respect to  $p'$ :

$$\begin{aligned} d\bar{\mathcal{T}}_{\nu_c}(p', p_s) &= d \left( \Delta - 2\sqrt{\frac{S\delta_0}{g \cos \bar{\theta}} \left| \int_0^{p'} \frac{p}{p_0} q(p) dp \right|} \right) \\ &= -2\sqrt{\frac{S\delta_0}{g \cos \bar{\theta}}} d \left( \sqrt{\int_0^{p'} \frac{p}{p_0} q(p) dp} \right) \\ &= -\sqrt{\frac{S\delta_0}{\mu_p g \cos \bar{\theta}}} \frac{p'}{p_0} q(p') dp' \end{aligned}$$

We substitute this into 6.19 to find that:

$$OLR_{\nu_c} = F_+(\nu_c, 0)(\Delta - 2\sqrt{S\delta_0\mu_p}) + \int_0^{p_s} \pi B(\nu_c, T(p')) \sqrt{\frac{S\delta_0}{\mu_p g \cos \bar{\theta}}} \frac{p'}{p_0} q(p') dp' \quad (6.27)$$

Again, in the strong line limit, the  $OLR_{\nu_c}$  *decreases* due to the atmosphere absorbing some emission from the ground and *increases* due to the atmosphere itself emitting some radiation, and whether this ultimately increases or decreases  $OLR_{\nu_c}$  will depend on  $T(p)$ .

#### 6.4.4 Non-Isolated Lines

I note here that, sadly, none of what we learnt will be applicable to calculating the radiative forcing of CO<sub>2</sub> or H<sub>2</sub>O. That is because both CO<sub>2</sub> and H<sub>2</sub>O have strongly **overlapping** lines, and so we cannot simply integrate over a single spectral line and sum up each contribution, as that would result in double counting.

We also won't cover how to deal with this here. Dissapointing I know, but it is what it is...

### 6.5 Radiative Forcing

Suppose we wish to know how varying some parameter  $\lambda$  (for example, CO<sub>2</sub> concentrations) varies. Consider the radiation budget of the Earth:

$$G = \frac{1}{4}(1 - \alpha)S_0 - OLR(T_s, \lambda) \quad (6.28)$$

We know that in equilibrium,  $G = 0$ . Now consider the differential of  $G$  with respect to some parameter:

$$dG =$$

#### 6.5.1 The Weak Line Limit

#### 6.5.2 The Strong Line Limit

### 6.6 Grey Gases

We consider a simple (highly unrealistic) analytical case where  $\tau$  is not a function of  $\nu$ . This allows us to integrate 6.6 and 6.7 over all frequencies. Recalling 4.11, we derive the frequency integrated upwards and downwards Schwarzschild equations:

$$\frac{dE_+}{d\tau} = -E_+ + \sigma T^4 \quad (6.29)$$

$$-\frac{dE_-}{d\tau} = -E_- + \sigma T^4 \quad (6.30)$$

where now the Schwarzschild equations are in terms of **irradiance** ( $E_+$  and  $E_-$ ). We can integrate this (as before) to find the general solutions for a grey gas:

$$E_+(\tau) = E_+(0)e^{-\tau} + \int_0^\tau \sigma T(\tau')^4 e^{-(\tau-\tau')} d\tau' \quad (6.31)$$

$$E_-(\tau) = E_-(\tau_\infty)e^{-(\tau_\infty-\tau)} + \int_\tau^{\tau_\infty} \sigma T(\tau')^4 e^{-(\tau'-\tau)} d\tau' \quad (6.32)$$

We can now see that the total

## Chapter 7

# Radiative Equilibrium

### 7.1 Grey Gases, but in Radiative Equilibrium

Recall the

$$\begin{aligned}\frac{dE_+}{d\tau} &= -E_+ + \sigma T^4 \\ -\frac{dE_-}{d\tau} &= -E_- + \sigma T^4\end{aligned}$$

where  $E_+$ ,  $E_-$  are now irradiances (rather than spectral irradiances).

You'll note that we cannot actually solve this system, because we have three variables ( $E_+$ ,  $E_-$ ,  $T$ ) but only two equations. However, this should be the case: recall the general solutions to the non-frequency averaged upwards and downwards Schwarzschild Equations (6.11, 6.10) featured  $B(\tau')$  within the integral, which depends on the atmospheric temperature profile and composition as a function of optical depth.

We now make the crucial assumption of radiative equilibrium: there is no net warming or heating from radiation at each optical depth. We now consider conservation of energy on some infinitesimal slab of atmosphere with optical thickness  $\delta\tau$  and area  $A$  by considering the upwards and downwards irradiances entering and leaving the slab.

$$\begin{aligned}P_{in} &= (E_+(\tau) + E_+(\tau + \delta\tau))A \\ P_{out} &= (E_-(\tau + \delta\tau) + E_-(\tau))A\end{aligned}$$

Radiative equilibrium, therefore:

$$\begin{aligned}P_{in} - P_{out} &= 0 \\ &= (E_+(\tau) + E_-(\tau + \delta\tau) - E_-(\tau + \delta\tau) - E_+(\tau))A \\ &= A\delta\tau \left( -\frac{E_+(\tau + \delta\tau) - E_+(\tau)}{\delta\tau} + \frac{E_-(\tau + \delta\tau) - E_-(\tau)}{\delta\tau} \right)\end{aligned}$$

We can freely divide by  $A \neq 0$  and take  $\delta\tau \rightarrow 0$  to find our third and final equation for radiative equilibrium:

$$\frac{d}{d\tau}(E_+ - E_-) = 0 \tag{7.1}$$

We can now solve the coupled ODEs 6.29, 6.30, and 7.1 to give us expressions for  $E_+$ ,  $E_-$ , and  $T$  in terms of  $\tau$ . Finally, we can find  $\tau$  as a function of  $p$  (or  $z$ ) in order to give us the temperature profile of an atmosphere in **radiative equilibrium**.

There are two final nuances to note. First, we are effectively making the *opposite* assumption to the one we made in deriving the adiabat (2.3): we are now assuming that radiation is the dominant



energy transfer mechanism, rather than convection. It's worth noting that neither limit is entirely correct, whether radiation or convection wins out as the dominant mechanism depends on the situation. Having said this, which one turns out to be the dominant mechanism is not a coincidence, but is, of course, a consequence of the dynamics governing the system.

The second nuance is why we are making this approximation now, *after* we have made the grey gas approximation and not before. While non-grey gases in radiative equilibrium can occur, it does not result in a simple equation as in Equation 7.1. This is because *it need not be the case that the gas is in radiative equilibrium at every wavenumber*.

In other words, let  $E^\nu$  denote the spectral irradiance and  $E = \int E^\nu d\nu$  denote the irradiance. For a non-grey gas in radiative equilibrium, it *is* the case that:

$$\frac{d}{d\tau}(E_+ - E_-) = 0$$

However, it is pretty much never the case that:

$$\frac{d}{d\tau}(E_+^\nu - E_-^\nu) = 0 \quad (7.2)$$

7.2 is a *horrible* assumption, but it is the assumption we would need if we wish to solve the Upwards and Downwards *non-frequency integrated* Schwarzschild Equations (6.6, 6.7) governing the spectral irradiances.

## 7.2 General Solution

### 7.2.1 Derivation

The general solution is quickly revealed by adding and subtracting equations 6.29 and 6.30. Adding the equations give:

$$\frac{d}{d\tau}(E_+ - E_-) = -E_+ - E_- + 2\sigma T^4$$

Because the gas is in radiative equilibrium (Equation 7.1), the left hand side is 0. So we find that:

$$E_+ + E_- = 2\sigma T^4 \quad (7.3)$$

Subtracting Equations 6.29 and 6.30:

$$\frac{d}{d\tau}(E_+ + E_-) = -E_+ + E_-$$

The derivative of the right hand side wrt  $\tau$  is 0 (again due to 7.1) so is constant. We can find the value of the right-hand-side by applying the boundary conditions. We apply the boundary condition at the top of the atmosphere:

$$\begin{aligned} E_+(\tau_\infty) &= OLR ; E_-(\tau_\infty) = 0 \quad \therefore \\ E_+ - E_- &= OLR \end{aligned} \quad (7.4)$$

Therefore:

$$\frac{d}{d\tau}(E_+ + E_-) = -OLR$$

Integrating again using the boundary conditions gives:

$$E_+ + E_- = OLR(1 + \tau_\infty - \tau) \quad (7.5)$$

We are now in a position to find expressions for all three variables. We can find  $T$  by substituting 7.3 into 7.5. We can find  $E_+$  and  $E_-$  by adding or subtracting (respectively) 7.4 and 7.5. The solutions are:

## Key Idea II.9: Radiative Equilibrium Solutions

$$T^4 = \frac{OLR}{2\sigma}(1 + \tau_\infty - \tau) \quad (7.6)$$

$$E_+ = \frac{OLR}{2}(2 + \tau_\infty - \tau) \quad (7.7)$$

$$E_- = \frac{OLR}{2}(\tau_\infty - \tau) \quad (7.8)$$

We can rewrite these equations using global energy conservation:  $OLR = L_\star$  where  $L_\star$  is the absorbed stellar flux per unit area at the surface of the planet.

$$T^4 = \frac{L_\star}{2\sigma}(1 + \tau_\infty - \tau) \quad (7.9)$$

$$E_+ = \frac{L_\star}{2}(2 + \tau_\infty - \tau) \quad (7.10)$$

$$E_- = \frac{L_\star}{2}(\tau_\infty - \tau) \quad (7.11)$$

### 7.2.2 Interpretation

Having analytic expressions for a grey gas in radiative equilibrium, we can now extract some physical intuition.

Let's first consider the upwards irradiance  $E_+$ . At the *top* of the atmosphere  $E_+ = OLR$  (obviously). We can equate this to an effective emitting temperature of the planet  $T_{eff}$  such that  $\sigma T_{eff}^4 = OLR = E_+(\tau_\infty)$ . At the *bottom* of the atmosphere,  $\tau = 0$ , so  $E_+(0) = \frac{OLR}{2}(2 + \tau_\infty) = \frac{\sigma T_{eff}^4}{2}(2 + \tau_\infty)$ . We can match this with our boundary condition, that all the upwards radiation at  $\tau = 0$  is from the ground, and so  $E_+(0) = \sigma T_g^4$  where  $T_g$  = the temperature of the ground (note that I am being careful not to say the temperature of the air at the ground!). We can relate  $T_g$  and  $T_{eff}$ :  $E_+(0) = \sigma T_g^4 = \frac{\sigma T_{eff}^4}{2}(2 + \tau_\infty)$ . Therefore:

$$T_{eff} = T_g \left( \frac{1}{1 + \frac{\tau_\infty}{2}} \right)^{\frac{1}{4}}, \text{ or, } OLR = \sigma T_g^4 \frac{1}{1 + \frac{\tau_\infty}{2}}$$

$OLR$  and therefore  $T_{eff}$  are effectively fixed by  $L_\star$ , the properties of the star (and albedo), so it's clear that as  $\tau_\infty$  gets arbitrarily large, so too must  $T_g$ , whereas if  $\tau \rightarrow 0$ ,  $T_g \rightarrow T_{eff}$ , intuitively.

Next let's consider the temperature profile. As we can clearly see, as  $\tau$  increases,  $T$  decreases, so radiative equilibrium also results in a temperature profile that decreases with height. Let's consider two limiting cases: air high up in the atmosphere, and air low in the atmosphere. Air right near the ground is located at  $\tau = 0$ , so  $\sigma T(0)^4 = \frac{OLR}{2}(1 + \tau_\infty)$ . But we just derived that  $\sigma T_g^4 = \frac{OLR}{2}(2 + \tau_\infty)$ . So in fact,  $T_g > T(0)$ : there is a temperature discontinuity at the ground, and the ground is warmer than the air immediately above it!

Is it stable to convection? Recall that an atmosphere is stable if it satisfies 2.11. We know it will be unstable at the ground, because at the ground the temperature gradient will be very large: the air immediately above the ground will be warmed by the ground, and will convect. More generally

though, we can directly differentiate 7.6 to see if it satisfies 2.11:

$$\begin{aligned}
\frac{d}{dp} T^4 &= -\frac{OLR}{2\sigma} \frac{d}{dp} (1 + \tau_\infty - \tau) \\
4T^3 \frac{dT}{dp} &= -\frac{OLR}{2\sigma} \frac{d\tau}{dp} \\
\therefore \frac{d \ln T}{d \ln p} &= -\frac{p}{4T^4} \frac{OLR}{2\sigma} \frac{d\tau}{dp} \\
&= -\frac{p}{4} \frac{1}{OLR(1 + \tau_\infty - \tau)/2\sigma} \frac{OLR}{2\sigma} \left( -\kappa(T(p), p) \frac{q(p)}{g} \right)
\end{aligned}$$

Therefore, using :

$$\frac{d \ln T}{d \ln p} = \frac{p \kappa(T(p), p) q(p)}{4} \tag{7.12}$$

Note that we have not considered the diurnal cycle: we have assumed that the

### 7.2.3 Internal Leakage

### 7.2.4 Shortwave absorbers

## Part III

# Clouds

# Introduction

This section of the course was lectured by **Philip Stier** covering Geophysical Fluid Dynamics.

This section consists of three chapters:

1. **Dynamical Systems:**

Meow

2. **Predictability:**

Meow

3. **Estimation:**

Meow

## Chapter 8

# Convection and Thermodynamics

### 8.1 Definitions of Humidity

#### 8.1.1 $h_i$

#### 8.1.2 Clausius-Clapeyron Relation

### 8.2 Convection and Tephigrams

### 8.3 Radiative-Convective Equilibrium

# Chapter 9

## Warm Cloud Microphysics

### 9.1 Growth in Thermodynamic Equilibrium

We first consider the very initial formation of a cloud droplet, formed when water vapour (gas) condenses into a liquid water droplet. If a water droplet is to begin forming, the process must be thermodynamically favourable; i.e., the process is set by what thermodynamic equilibrium is. This is an assumption we are making: it is a good assumption for now, but it will break down when other factors limit growth (e.g., kinetic non-equilibrium processes). One reason it is a good assumption now is because the processes that take place which push the system towards thermodynamic equilibrium occur on very small time-scales, and so the system is what is called *quasi-steady* (a concept we will be reintroduced to in Part V): it evolves so quickly that it is effectively always in equilibrium.

#### 9.1.1 Homogenous Nucleation: The Kelvin Equation

We first consider the pure water vapour with no aerosols. In the atmosphere, these tiny cloud droplets are coupled to an effectively infinite heat bath and are held at fixed pressure (from atmospheric temperature). We thus assume that the pressure and temperature are held fixed.

If the pressure and temperature are held fixed, it can be shown that any spontaneous thermodynamic process must, in equilibrium, evolve in order to decrease the Gibbs Free Energy  $G$ . The Gibbs Free Energy  $G$  is defined as:

$$G = U - TS + pV \quad (9.1)$$

Actually, the situation is a bit more complicated than simply decreasing the Gibbs Free Energy. It is more accurate to require the system to result in a *local* decrease in Gibbs Free Energy. To explain what *local* decrease means, consider a system with a Gibbs Free Energy  $G(\lambda)$  that depends on some continuous parameter  $\lambda$ . Suppose that the system is initially in a state of  $\lambda = \lambda_0$ . Then  $\lambda$  will evolve such that  $\delta\lambda \sim -\frac{\partial G}{\partial \lambda}(\lambda_0)$ . The idea is that the system does not 'know' where the global minimum of the Gibbs Free Energy is, so it will only evolve locally to decrease the Gibbs Free Energy.

We now consider the formation of a cloud droplet, and let our parameter  $\lambda = r$  where  $r$  is the radius of our cloud droplet. The Gibbs Free Energy of the system is the sum of the Gibbs Free Energy of  $N_{drop}(r)$  molecules of condensed water vapour in the droplet and  $N - N_{drop}$  molecules of *uncondensed* water vapour, where  $N$  = the total number of water molecules. We keep  $N$  fixed to represent conservation of water molecules.

$$G(r) = \underbrace{N_{drop}(r) g_v + 4\pi r^2 \sigma}_{G \text{ of droplet}} + \underbrace{(N - N_{drop}(r)) g_l}_{G \text{ of vapour}}$$

where  $g_v$ ,  $g_l$  are Gibbs Free Energy per molecule of the vapour and liquid phases of water;  $\sigma$  = the surface tension; and  $r$  = the radius of the water droplet. The Gibbs Free Energy of the droplet is the sum of the Gibbs Free Energy of the liquid water molecules in the droplet ( $N_{drop} g_v$ ) and the Gibbs Free Energy of the liquid-gas interface (represented by the  $4\pi r^2 \sigma$  surface tension term).

We write  $N_{drop}(r)$  in terms of  $r$  by assuming that the droplet is spherical, therefore  $N_{drop} = \frac{4}{3}\pi r^3/v_l$ , where  $v_l$  = the volume of one liquid molecule of water. Therefore:

$$G = \frac{4\pi}{3v_l}r^3(g_l - g_v) + 4\pi\sigma r^2 + N g_v \quad (9.2)$$

Our goal now is to calculate  $g_l - g_v$ . We can take the differential of  $G$  in 9.1 then divide by the number of water molecules to find that:

$$\begin{aligned} dG &= dU - T dS - dT S + p dV + dp V \\ &= \underbrace{T dS - p dV}_{dU} - T dS - dT S + p dV + dp V \\ dG &= -S dT + V dp \\ \therefore dg &= -\frac{1}{N}S dT + \frac{V}{N} dp \end{aligned}$$

We make two assumptions here. First, we assume that the temperature is constant (recall the effective infinite heat bath of the atmosphere!), and therefore neglect the  $S dT$  term. Second, we assume that  $v_g = \frac{V_g}{N} \gg v_l = \frac{V_l}{N}$  (i.e., that the volume per gaseous water molecules is much bigger than the volume per liquid water molecule). Therefore, we ignore the change in Gibbs Free Energy in the liquid and so  $dg = (v_l - v_g)dp \approx -v_g dp$ .

$$g_l - g_v = \int_{\text{no droplet}}^{\text{droplet}}$$

assuming that the 1.2,  $\frac{k_B T}{p_g} \gg \frac{k_B T}{p_l}$ , and so the

$$g_l - g_v =$$

Therefore:

$$e_S(r) = e_S(\infty) \exp\left(\frac{2\sigma v_l}{k_B T r}\right) \quad (9.3)$$

For ease of notation, we let  $A = \frac{2\sigma v_l}{k_B T} = \frac{2\sigma}{R_v \rho T}$  (recalling the definition of  $v_l$  and  $R_v$  in 1.4). Therefore:

$$e_S(r) = e_S(\infty) \exp\left(\frac{A}{r}\right)$$

### 9.1.2 Homogenous Nucleation: The Raoult Equation

$$e_S^{sol}(\infty) = e_S^0(\infty) \frac{n_w}{n_{sol} + n_w} \quad (9.4)$$

In other words, the vapour pressure is set by the mole-fraction of water. Generally,  $n_{sol} \ll n_w$ , so:

$$\begin{aligned} \frac{n_w}{n_{sol} + n_w} &= \frac{1}{n_{sol}/n_w + 1} \\ &= \left(1 + \frac{n_{sol}}{n_w}\right)^{-1} \\ &\approx 1 - \frac{n_{sol}}{n_w} \end{aligned}$$



### 9.1.3 The Köhler Equation

We can combine 9.3 and 9.4 into one equation by considering a cloud droplet which has a curved interface (taking into account 9.3) and has a dissolved solute (taking into account 9.4).

The saturation vapour pressure is therefore:

$$\begin{aligned} e_S^{sol}(r) &= e_S^{sol}(\infty) \exp\left(\frac{2\sigma v_l}{k_B T r}\right) \\ &= e_S^0(0) \frac{n_w}{n_{sol} + n_w} \exp\left(\frac{2\sigma v_l}{k_B T r}\right) \\ e_S^{sol}(r) &= e_S^0(0) \left(1 - \frac{B}{r^3}\right) e^{\frac{A}{r}} \end{aligned} \quad (9.5)$$

## 9.2 Growth by Condensation

We now assume that the droplet has grown to such a size that the rate of droplet formation now becomes kinetically limited: that it is limited primarily by the rate at which water molecules and heat may diffuse away/towards the droplet.

### 9.2.1 Diffusion of Water Molecules

We begin from the diffusion equation governing the diffusion of water vapour *outside* the droplet:

$$\frac{\partial n}{\partial t} = \vec{\nabla} \cdot \vec{\Phi} \quad (9.6)$$

$$\vec{\Phi} = D \vec{\nabla} n \quad (9.7)$$

where  $n$  = the number concentration of water molecules;  $\vec{\Phi}$  = the flux of water molecules; and  $D$  = the diffusivity of water in air. We assume, again, that the droplet is quasi-steady, and so that  $\frac{\partial n}{\partial t} = 0$ . Furthermore, we assume that the water vapour outside the droplet is spherically symmetric, and so  $n = n(r)$  only. We can then solve  $\nabla^2 n = 0$ . In spherical coordinates, the solution is:

$$n(r) = C_1 - \frac{C_2}{r}$$

To find the constants, we apply the boundary conditions that far away from the droplet ( $r \rightarrow \infty$ ) the number density is the ambient vapour density ( $n_\infty$ ) and near the droplet ( $r = R$ , where  $R$  = the radius of the droplet), the number density is the vapour density of the surface. This gives:

$$n(r) = n_\infty - \frac{R}{r}(n_\infty - n_r)$$

We now apply mass conservation to the water molecules *within* the droplet:

$$\frac{d}{dt} \int_V (t) \rho dV = \int_S (t) m_{\text{H}_2\text{O}} \vec{\Phi} \cdot d\vec{S} \quad (9.8)$$

where  $m_{\text{H}_2\text{O}}$  = the mass of one water molecule. The left-hand-side is the rate of change of the mass of the water droplet and the right-hand-side is the flux of water molecules into/out of the boundaries.  $V(t)$  and  $S(t)$  indicate the volume of the droplet and surface of the droplet over which we are integrating, and these time-vary due to the fact that the droplet is changing size.

We write the left-hand-side as just  $dM/dt$  where  $M$  = the total mass of the droplet, and substitute in 9.7 and 9.8 for the right-hand-side to obtain:

$$\boxed{\frac{dM}{dt} = 4\pi R D (\rho_\infty - \rho_R)} \quad (9.9)$$

### **9.2.2 Diffusion of Heat**

## **9.3 Growth by Collision/Collection/Coalescence**

### **9.3.1 Collision Processes**

## Chapter 10

# Cold Cloud Microphysics

## Chapter 11

# Cloud Morphology, Radiation, and Climate

**Part IV**

**Instrumentation**

**Part V**

**Climate Dynamics**

## Introduction

This section of the course was lectured by [Myles Allen](#) covering a sort of miscellaneous set of topics.

This section consists of three chapters:

1. [Dynamical Systems](#):

Meow

2. [Predictability](#):

Meow

3. [Estimation](#):

Meow

# Chapter 12

## Dynamical Systems

### 12.1 The Equations of Motion

We first consider the climate as a simple dynamical system, the simplest of which is a linear dynamical deterministic system.

In reality, the climate is highly non-linear and stochastic. As such, we should consider this model as representing only small perturbations about some reference state. If one considers small enough perturbations, any (well-behaved) state will be linear (think: Taylor expansions). The state will, in general, be given by:

$$\dot{\vec{x}} = \hat{J}\vec{x} + \vec{f} \quad (12.1)$$

where  $\vec{x} \in \mathbb{R}^n$  is the state-vector of the system,  $\hat{J} \in \mathbb{R}^n \times \mathbb{R}^n$  is the linear(ised) Jacobian, and  $\vec{f} \in \mathbb{R}^n$  is the external forcing.

We apply this first to an energy-budget equation for the climate, characterised by the state-vector  $(T_s, T_d)$  representing perturbations of the mean atmospheric/surface-ocean temperature and deep ocean temperature, respectively, governed by the equations:

$$C_s \dot{T}_s = F_{ext}(t) - \lambda T_s - \gamma(T_s - T_d) - \lambda'(T_s - T_d) \quad (12.2)$$

$$C_d \dot{T}_d = -\gamma(T_d - T_s) \quad (12.3)$$

where  $C_s$ ,  $C_d$  are the effective heat capacities of the atmosphere/surface oceans and deep oceans per unit area, respectively. The left-hand side represents the change in the energy (per unit area) and the right-hand side represents various energy fluxes (per unit area). The various terms are explained below:

Term	Interpretation
$F_{ext}(t)$	The perturbation in external forcing due to albedo, aerosol, greenhouse gas, solar, etc. forcing.
$\lambda T_s$	The additional energy radiated to space (per unit area) due to warming. If $\lambda T_s > 0$ then <i>more</i> energy is being radiated into space, which typically results in cooling. $\lambda$ is the <i>climate sensitivity parameter</i> .
$-\gamma(T_s - T_d)$	The energy flux from the atmosphere/surface oceans <i>into</i> the deep oceans. This is typically small, due to the fact that the oceans are very stably stratified. As such, mixing/convection is severely inhibited, and surface waters can only penetrate into the deep oceans at a select few areas (cf. Sec.19.3).
$-\lambda'(T_s - T_d)$	The additional energy radiated into space due to the system being out of equilibrium. We have reason to believe (empirically, due to climate models) that this occurs. We encode this crudely by letting $(T_s - T_d)$ represent the degree to which the climate is out of equilibrium.

Note that if we add Equations 12.2 and 12.3, we do not get 0 on the right hand side: the climate is not an isolated system. There is a net energy flux from  $F_{ext}$ ,  $-\lambda T_s$ , and  $-\lambda'(T_s - T_d)$ .



## 12.2 Solving the System by Diagonalising the Matrix

Before you read this section, while this method is completely general to linear systems, I should note that this is quite time-consuming to do in the exam. Furthermore, I think more physical intuition comes from another method of solving the system which I explain in Section 12.3. If you're low on time, and the question asks you to sketch it, it might be easier to think in terms of the method in Section 12.3.

This method relies on some prior knowledge of linear algebra, specifically diagonalisation, eigenvectors, and eigenvalues. I'll review them quickly here, but if you're familiar with the content you can skip to Section 12.2.2.

### 12.2.1 Deriving the Method (Review of ODEs and Linear Algebra)

#### Reframing the ODE Problem into a Linear Algebra Problem

First we write Equations 12.2 and 12.3 in matrix form:

$$\underbrace{\begin{pmatrix} \dot{T}_s \\ \dot{T}_d \end{pmatrix}}_{\dot{\vec{T}}} = \underbrace{\begin{pmatrix} -\frac{\lambda+\gamma+\lambda'}{C_s} & \frac{\gamma+\lambda'}{C_s} \\ \frac{\gamma}{C_d} & -\frac{\gamma}{C_d} \end{pmatrix}}_{\hat{J}} \underbrace{\begin{pmatrix} T_s \\ T_d \end{pmatrix}}_{\vec{T}} + \underbrace{\begin{pmatrix} F_{ext}(t)/C_s \\ 0 \end{pmatrix}}_{\vec{f}} \quad (12.4)$$

We now attempt to **diagonalise**  $\hat{J}$ . We find **eigenvectors**  $\vec{v}_i$  that satisfy the following relation:

$$\hat{J}\vec{v}_i = \lambda_i\vec{v}_i \quad (12.5)$$

where  $\lambda_i \in \mathbb{R}$  is an **eigenvalue**. I'll confine our attention to 2D systems here for brevity, but everything I say is readily extendable to  $N$  dimensions. In 2D, we write  $\vec{v}_i = ([\vec{v}_i]_1, [\vec{v}_i]_2)^T$ .

We now slap our eigenvectors  $\vec{v}_i$  together into a matrix  $\hat{E}$  defined as follows:

$$\hat{E} = \begin{pmatrix} \uparrow & \uparrow \\ \vec{v}_1 & \vec{v}_2 \\ \downarrow & \downarrow \end{pmatrix}$$

We then define  $\hat{\Sigma} = \hat{E}^{-1}\hat{J}\hat{E}$  and show then the following:

$$\begin{aligned} \hat{E}^{-1}\hat{J}\hat{E} &= \hat{E}^{-1}\hat{J} \begin{pmatrix} \uparrow & \uparrow \\ \vec{v}_1 & \vec{v}_2 \\ \downarrow & \downarrow \end{pmatrix} \\ &= \hat{E}^{-1} \begin{pmatrix} \uparrow & \uparrow \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 \\ \downarrow & \downarrow \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \hat{E}^{-1}\hat{E} \\ \hat{\Sigma} &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \end{aligned}$$

Therefore, multiplying on the left by  $\hat{E}$  and right by  $\hat{E}^{-1}$ :

$$\hat{J} = \hat{E} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \hat{E}^{-1}$$

We can then multiple Equation 12.4 on the left by  $\hat{E}^{-1}$  and define  $\vec{S} \equiv \hat{E}^{-1}\vec{T}$  and  $\vec{g} \equiv \hat{E}^{-1}\vec{f}$ :

$$\begin{aligned} \hat{E}^{-1}\dot{\vec{T}} &= \hat{E}^{-1}\hat{J}\vec{T} + \hat{E}^{-1}\vec{f} \\ &= \hat{\Sigma}\hat{E}^{-1}\vec{T} + \hat{E}^{-1}\vec{f} \\ \dot{\vec{S}} &= \hat{\Sigma}\vec{S} + \vec{g} \end{aligned}$$

Recall that  $\hat{\Sigma}$  is completely diagonal, so our coupled equations are now uncoupled in the new coordinate system  $\vec{S}$ ! We write the  $i$ th component of the vectors with a subscript  $i$  and thus write that:

$$\dot{S}_i = \lambda_i S_i + g_i$$

Therefore, using an integrating factor, our solution is:

$$S_i(t) = S_i(0)e^{\lambda_i t} + e^{\lambda_i t} \int_0^t g_i(t')e^{-\lambda_i t'} dt'$$

Our final step is to relate  $S_i$  back to  $T_i$ , which is easy if we note that  $\vec{S} \equiv \hat{E}^{-1}\vec{T}$ , therefore  $\vec{T} = \hat{E}\vec{S}$ . Therefore,  $T_i = [v_i]_1 S_1 + [v_i]_2 S_2$  so our general solution is:

$$T_j = \sum_i \left( [v_j]_i \left( S_i(0)e^{\lambda_i t} + e^{\lambda_i t} \int_0^t g_i(t')e^{-\lambda_i t'} dt' \right) \right) \quad (12.6)$$

So we can solve our ODE if we find the **eigenvectors** and **eigenvalues** of  $\hat{J}$ ! We've transformed our problem into a linear algebra problem! So how do we solve this now?

### Solving the Linear Algebra Problem

We rewrite Equation 12.5 as follows:

$$\begin{aligned} \hat{J}\vec{v}_i &= \lambda_i \vec{v}_i \\ &= \lambda_i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{v}_i \\ \therefore \vec{0} &= \left( \vec{J} - \lambda_i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \vec{v}_i \end{aligned}$$

where  $\vec{0} = (0, 0)^T$ .

Now clearly, an easy solution is if we set  $\vec{v}_i = \vec{0}$  and  $\lambda_i = 0$ , but this is only one solution, and we wish to find the non-trivial solutions. One can show in linear algebra that a non-zero matrix times a non-zero vector can be equal to  $\vec{0}$  only if the determinant of the matrix is zero. We thus solve for:

$$\det \left( \vec{J} - \lambda_i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = 0$$

to obtain our eigenvalues  $\lambda_i$ . Once we have done that we can go through each  $\lambda_i$  one at a time and substitute the  $\lambda_i$  into Equation 12.5 to find the corresponding  $\vec{v}_i$ . And we're done!

### 12.2.2 Application to Our Dynamical System

Before we apply this method to our dynamical system, we make an important simplification. Typically,  $C_s \ll C_d$ , representing the massive heat capacity of the deep oceans compared to the atmosphere and land. If we write  $\hat{J}$  as:

$$\hat{J} = \begin{pmatrix} -\frac{\lambda + \gamma + \lambda'}{C_s} & \frac{\gamma + \lambda'}{C_s} \\ \frac{\gamma}{C_d} & -\frac{\gamma}{C_d} \end{pmatrix} = \begin{pmatrix} -a & b \\ c & -c \end{pmatrix}$$

and assume further that  $\lambda$ ,  $\gamma$ , and  $\lambda'$  are of similar size, we can conclude that  $|a|, |b| \gg |c|$ .

Let us now continue with our method. We want to find the eigenvalues of  $\hat{J}$ . Let us refer to the eigenvalues as  $r_i$  to avoid confusion. We want to solve:

$$\det \begin{pmatrix} -a - r_i & b \\ c & -c - r_i \end{pmatrix} = 0$$

This is a straightforward quadratic in  $r_i$ , which has the solution:

$$r_i = \frac{1}{2} \left( -a - c \pm \sqrt{(a - c)^2 - 4(ac - bc)} \right)$$

We now keep only first order in  $\frac{c}{a}$ , and write:

$$\begin{aligned} r_i &= \frac{1}{2} \left( -a - c \pm \sqrt{a^2 - 2ac - 4c(a - b) + c^2} \right) \\ &= \frac{1}{2} \left( -a - c \mp a \sqrt{1 - \frac{2c}{a} - \frac{4c}{a} \frac{a - b}{a} + \left(\frac{c}{a}\right)^2} \right) \\ &\approx \frac{1}{2} \left( -a - c \mp a \left( 1 - \frac{c}{a} - \frac{2c}{a} \frac{a - b}{a} \right) \right) \\ &= \end{aligned}$$

Our eigenvectors  $r_i$  are therefore:

$$r_i =$$

$$\det \begin{pmatrix} -\frac{\lambda + \gamma + \lambda'}{C_s} - r_i & \frac{\gamma + \lambda'}{C_s} \\ \frac{\gamma}{C_d} & -\frac{\gamma}{C_d} - r_i \end{pmatrix} = 0$$

### Key Idea V.1: Dynamical Systems Recipe

Suppose we have a coupled set of ODEs governing the time-evolution of  $x(t)$  and  $y(t)$  given by:

$$\begin{aligned} A\dot{x} &= Cx + Dy + G(t) \\ B\dot{y} &= Ex + Fy + H(t) \end{aligned}$$

Solve this system in the following way:

1. Write your ODE in matrix form:

$$\dot{\vec{x}} = \hat{J}\vec{x} + \vec{f} \quad \text{where} \quad \hat{J} = \begin{pmatrix} C/A & D/A \\ E/B & F/B \end{pmatrix}$$

2. Find the eigenvectors and eigenvalues of  $\hat{J}$ .
3. See if you can exploit any separation of scales to neglect terms. For example, if  $A \ll B$ , then we can keep only first order in  $E/B$  and  $F/B$ .
4. Plug the approximate eigenvalues and eigenvectors into [12.6](#).

## 12.3 Solving the System by Separating Timescales

Here we present an alternative method. This method requires the assumption that  $C_s \ll C_d$ , but is more general than the former system in that it does *not* require the ODE to be linear. We present this method because we believe this way provides more physical intuition. We have [Ian Hewitt](#) from the Maths department to thank for this way of thinking. Furthermore, most equations presented in this course will be simple enough for this method to save a lot of time, especially in the exam if you have to sketch the general behaviour.

As already mentioned, we must exploit the separation of timescales here. Here,  $T_s$  evolves much faster than  $T_d$ , which allows us to make two approximations.<sup>1</sup>

<sup>1</sup>A full rigorous treatment requires non-dimensionalising the equations, but we believe this is ultimately not needed for physical intuition in this case.

The first approximation we make is valid on *short* time-scales. The terms on the right hand side of equation 12.3 are of order one, and thus  $C_d \dot{T}_d$  is also of order one. However, since  $C_d$  is massive,  $\dot{T}_d$  is very small, and so we let  $\dot{T}_d \approx 0$  and integrate to find that  $T_d(t) = 0$  for all times. Physically, this represents the fact that  $T_d$  does not respond on very short timescales due to the massive heat capacity  $C_d$ . Equations 12.2 and 12.3 thus reduce to:

$$C_s \dot{T}_s = F_{ext}(t) - (\lambda + \gamma + \lambda') T_s \quad (12.7)$$

$$C_d \dot{T}_d = 0 \quad (12.8)$$

on short timescales. We can integrate Equation 12.7 using an integrating factor, with the general solution in Section 12.4. The timescale is set by comparing the  $T_s$  term to the  $\dot{T}_s$  term:  $\tau_s \sim \frac{C_s}{\lambda + \lambda' + \gamma}$ . Thus on an  $O(\tau_s)$  timescale,  $T_s$  equilibrates and  $T_d$  remains constant (at 0). We have derived our first eigenvector and eigenvalue here, but without diagonalising anything!

The second approximation we make is valid on *long* time-scales. On this timescale,  $T_s$  is **quasi-steady**:  $T_s$  evolves rapidly into equilibrium with  $T_d$ , and so we set  $\dot{T}_s = 0$  in Equation 12.2. Equations 12.2 and 12.3 thus reduce to:

$$0 = F_{ext}(t) - \lambda T_s - \gamma(T_s - T_d) - \lambda'(T_s - T_d) \quad (12.9)$$

$$C_d \dot{T}_d = -\gamma(T_d - T_s) \quad (12.10)$$

on long timescales. We now solve for  $T_s(t)$  in terms of  $T_d(t)$  and substitute  $T_s(t)$  into Equation 12.3. Solving Equation 12.9 gives  $T_s(t) = \frac{1}{\lambda + \gamma + \lambda'} (F_{ext}(t) + (\gamma + \lambda') T_d(t))$ . Substituting this into 12.10 gives the following ODE governing  $T_d$ :

$$C_d \dot{T}_d = \frac{\gamma}{\lambda + \gamma + \lambda'} F_{ext}(t) - \frac{\gamma \lambda}{\lambda + \gamma + \lambda'} T_d \quad (12.11)$$

on long timescales. The timescale is set again by comparing the  $T_d$  term to the  $\dot{T}_d$  term:  $\tau_d \sim \frac{C_d(\lambda + \gamma + \lambda')}{\gamma \lambda}$ . Again, we have derived our second eigenvector and eigenvalue here without diagonalising anything!

We can now say, in retrospect, that the short timescale approximation (Eqns. 12.7 and 12.8) is valid on timescales much shorter than  $\tau_d$ , the long timescale approximation (Eqns. 12.9 and 12.10) is valid on timescales much longer than  $\tau_s$ , and that we require  $\tau_d \gg \tau_s$ . This is satisfied if and only if  $\frac{C_d(\lambda + \gamma + \lambda')}{\gamma \lambda} \gg \frac{C_s}{\lambda + \lambda' + \gamma}$ , which is indeed satisfied if  $\gamma, \lambda, \lambda'$  are of similar size (which they are) and  $C_d \gg C_s$ , as originally assumed.

Note the nuance in these two approximations, where we confusingly seem to set the time derivatives to 0 in both cases. In the first case, we set  $\dot{T}_d = 0$  because the timescales are too *short*, and  $T_d$  essentially does not react on this timescale. In the second case, we set  $\dot{T}_s = 0$  because the timescales are too *long*. The timescales are so long that we assume that  $\dot{T}_s$  rapidly equilibrates such that at each time, it is always in equilibrium, and thus  $\dot{T}_s = 0$ .

### Key Idea V.2: Timescale Approximations

Suppose we have a coupled set of ODEs governing the time-evolution of  $x(t)$  and  $y(t)$  given by:

$$A\dot{x} = f(x, y)$$

$$B\dot{y} = g(x, y)$$

such that  $B \gg A$  and  $f$  and  $g$  are of similar size. We further assume that  $x(t = 0) = y(t = 0) = 0$ . Then we can make two approximations which indicate the different responses:

**The Transient Response:** On *short* timescales,  $\tau_s$ , we assume that  $y$  has no time to respond so that:

$$\dot{y} = 0$$

Therefore the transient response is governed by:

$$A\dot{x} = f(x, 0) \quad (12.12)$$

$$y = 0 \quad (12.13)$$

We find the timescale  $\tau_s$  by comparing the  $\dot{x}$  term to the  $x$  term in Equation 12.12.

**The Quasi-Steady Response:** On *long* timescales,  $\tau_d$ , we assume that  $x$  responds rapidly into equilibrium so that:

$$f(x, y) = 0$$

Therefore the transient response is governed by:

$$0 = f(x, y) \quad (12.14)$$

$$B\dot{y} = g(x(y), y) \quad (12.15)$$

We find the timescale  $\tau_d$  by comparing the  $\dot{y}$  term to the  $y$  terms in Equation 12.15.

## 12.4 Solutions

We thus get two different responses for an arbitrary external forcing (assuming  $T_s(0) = T_d(0) = 0$ ):

On short timescales (we define  $\tau_s \equiv \frac{C_s}{\lambda + \lambda' + \gamma}$ ):

$$T_s(t) \approx \frac{1}{C_s} \int_0^t F_{ext}(\hat{t}) e^{-\frac{t+\hat{t}}{\tau_s}} d\hat{t} \quad (12.16)$$

$$T_d(t) \approx 0 \quad (12.17)$$

On long timescales (we define  $\tau_d \equiv \frac{C_d(\lambda + \lambda' + \gamma)}{\gamma\lambda}$ ):

$$T_s(t) \approx \frac{1}{\lambda + \gamma + \lambda'} (F_{ext}(t) + (\gamma + \lambda')T_d(t)) \quad (12.18)$$

$$T_d(t) \approx \frac{\gamma}{C_d(\lambda + \gamma + \lambda')} \int_0^t F_{ext}(\hat{t}) e^{-\frac{t+\hat{t}}{\tau_d}} d\hat{t} \quad (12.19)$$

Suppose  $F_{ext}(t)$  is a step-function, representing, for example, an instantaneous emission of  $\text{CO}_2$  which remains in the atmosphere indefinitely.

The solutions are as follows:

$$T_s(t) \approx \frac{1}{C_s} \int_0^t F_{ext}(\hat{t}) e^{-\frac{t+\hat{t}}{\tau_s}} d\hat{t} \quad (12.20)$$

$$T_d(t) \approx 0 \quad (12.21)$$

On long timescales, we define  $\tau_d \equiv \frac{C_d(\lambda + \lambda' + \gamma)}{\gamma\lambda}$ :

$$T_s(t) \approx \frac{1}{\lambda + \gamma + \lambda'} (F_{ext}(t) + (\gamma + \lambda')T_d(t)) \quad (12.22)$$

$$T_d(t) \approx \frac{\gamma}{C_d(\lambda + \gamma + \lambda')} \int_0^t F_{ext}(\hat{t}) e^{-\frac{t+\hat{t}}{\tau_d}} d\hat{t} \quad (12.23)$$

## 12.5 Analogous Systems

We can apply this to similar systems. Consider, for example, a carbon budget equation, given by:

$$R_a \dot{C}_a = E(t) - r(C_a - C_b) \quad (12.24)$$

$$R_b \dot{C}_b = r(C_a - C_b) - s(C_b - C_d) \quad (12.25)$$

$$R_d \dot{C}_d = s(C_b - C_d) \quad (12.26)$$

where  $C_a, C_b, C_d$  are the effective CO<sub>2</sub> anomalies in the atmosphere, biosphere/surface-ocean, and deep ocean, respectively. The left hand side is the change in CO<sub>2</sub> concentrations, and the right hand side are carbon fluxes. It's typically the case that  $R_d \gg R_b \gg R_a$  due to similar reasons as before, as well as a bit of ocean chemistry. We can use the exact same method as before to define three timescales:

1. A short timescale ( $t_a \sim \mathcal{O}\left(\frac{R_a}{r}\right)$ ): We set  $C_b \approx C_d \approx 0$ , as  $C_b, C_d$  respond very slowly.  $C_a$  evolves according to  $R_a \dot{C}_a = E(t) - rC_a$ .
2. A medium timescale ( $t_b \sim \mathcal{O}\left(\frac{R_b}{s}\right)$ ): We set  $C_d \approx 0$  since  $C_d$  responds very slowly, and we set  $\dot{C}_a \approx 0$  as  $C_a$  evolves rapidly into equilibrium with  $C_b$ . We then get that  $C_a(t)$  evolves quasi-steadily ( $C_a(t) \approx C_b(t) + \frac{E(t)}{r}$ ) and  $C_b$  evolves according to  $R_b \dot{C}_b = E(t) - sC_b$ .
3. A long timescale ( $t_d \gg \mathcal{O}\left(\frac{R_b}{s}\right)$ ): We set  $\dot{C}_a \approx \dot{C}_b \approx 0$  since  $C_a, C_b$  evolve rapidly into equilibrium with  $C_d$ . We then get that  $C_a(t)$  and  $C_b(t)$  evolve quasi-steadily ( $C_a(t) \approx C_b(t) + \frac{E(t)}{r}$  and  $C_b(t) \approx C_d(t) + \frac{E(t)}{s}$ ), and  $C_d$  evolves according to  $R_d \dot{C}_d = E(t)$ . Note that it is difficult to define a timescale here as from our second method – we cannot simply compare  $C_d$  to  $\dot{C}_d$ . If we defer to diagonalising the matrix, we find that the eigenvalue is 0, and so the timescale is actually  $t_d \sim \infty$ ! This reflects the fact that our carbon budget system is closed, and once you pump in carbon (represented by the  $E(t)$  term), it has nowhere to go other than to stay in the system. As such, if  $E(t) > 0$  for all  $t$ , the system will never relax to a steady state, and  $C_a, C_b$ , and  $C_d$  will all grow without bound.

This system, again, may be solved by finding eigenvalues and eigenvectors.

# Chapter 13

## Predictability

### 13.1 The Linear Error Propagator

We continue considering a linear dynamical system here, for example, a weather system. Consider a state characterised by some state-vector  $\vec{x} \in \mathbb{R}^n$  (perhaps encoding the pressure, temperature, humidity, and velocity at each point in space<sup>1</sup>), whose time-evolution (or perhaps, linearised time-evolution) is given by:

$$\dot{\vec{x}} = \hat{J}\vec{x} \quad (13.1)$$

where  $\hat{J} \in \mathbb{R}^n \times \mathbb{R}^n$  is the Jacobian given by:

$$\hat{J} = \begin{pmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} & \dots \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} & \dots \\ \vdots & \vdots & \vdots \end{pmatrix} \quad (13.2)$$

Given that  $\vec{x}$  has  $n$ -dimensions, we require  $n$  initial conditions given by  $\vec{x}(t = t_0)$  to integrate this ODE. This will typically be achieved via data assimilation, but all that needs to be known at this point is that we cannot know the initial conditions to a sufficient precision, because the system is chaotic.

Therefore, we aim to find how errors in the initial conditions propagate and evolve. Suppose that the actual state of the system is given by  $\vec{y}(t)$  and our prediction is given by  $\vec{x}(t)$ . We assume  $\vec{y}$  time-evolves by Equation 13.1.<sup>2</sup> We define some initial error or perturbation<sup>3</sup>  $\vec{\delta x}(t)$  such that, at all times, the following holds:

$$\vec{y}(t) = \vec{x}(t) + \vec{\delta x}(t) \quad (13.3)$$

At time  $t = t_0$  then, our initial error is  $\vec{\delta x}(t = t_0)$ . We wish to find how this error evolves as we integrate the system forwards.

We know that

$$\begin{aligned} \vec{y}(t_0 + \delta t) &= \vec{y}(t_0) + \int_{t_0}^{t_0 + \delta t} \hat{J}\vec{y}(t') dt' \\ &= \vec{x}(t_0) + \vec{\delta x}(t_0) + \int_{t_0}^{t_0 + \delta t} \hat{J}(\vec{x}(t') + \vec{\delta x}(t')) dt' \\ &= \vec{x}(t_0) + \int_{t_0}^{t_0 + \delta t} \hat{J}\vec{x}(t') dt' + \vec{\delta x}(t_0) + \int_{t_0}^{t_0 + \delta t} \hat{J}\vec{\delta x}(t') dt' \\ &= \vec{x}(t_0 + \delta t) + \vec{\delta x}(t_0 + \delta t) \end{aligned}$$

<sup>1</sup>In reality, of course, such a state-vector would be infinite-dimensional, as the temperature (for example) is a function of position  $T(\vec{r}, t)$ , which is continuous. However, all numerical models must discretise space (i.e., divide space into 'boxes'), reducing  $\vec{x}$  to a very large but finite dimensional vector.

<sup>2</sup>This amounts to the assumption that our model is perfect, i.e., there is no model error. This is, of course, false, so this method is only applicable to dealing with errors arising from errors in initial conditions.

<sup>3</sup>We use error and perturbation interchangeably in this chapter

where we have used the definition of  $\vec{\delta x}(t)$  and the fact that  $\hat{J}$  is a linear(ised) operator. Therefore,

$$\begin{aligned}\vec{\delta x}(t_0 + \delta t) &= \vec{\delta x}(t_0) + \int_{t_0}^{t_0 + \delta t} \hat{J} \vec{\delta x}(t') dt' \\ &\approx \vec{\delta x}(t_0) + \hat{J} \delta t \vec{\delta x}(t_0) \\ &= (\mathbb{I} + \hat{J} \delta t) \vec{\delta x}(t_0)\end{aligned}$$

Where we have used the fact that  $\delta t$  is small. We define the linear error propagator  $\hat{A}$  as:

$$\boxed{\hat{A} = (\mathbb{I} + \hat{J} \delta t)} \quad (13.4)$$

$$\boxed{\vec{\delta x}(t_0 + \delta t) \approx \hat{A} \vec{\delta x}(t_0)} \quad (13.5)$$

## 13.2 Local Perturbation Growth

Suppose we want to find which initial errors will grow the most.<sup>4</sup> This amounts to trying to maximise  $\|\vec{\delta x}(t_0 + \delta t)\|$  with respect to  $\vec{\delta x}(t_0)$ . This can't be the whole story though, since  $\vec{\delta x}(t_0 + \delta t)$  is a linear function of  $\vec{\delta x}(t_0)$  (Equation 13.5), so we can maximise  $\vec{\delta x}(t_0 + \delta t)$  simply by letting  $\vec{\delta x}(t_0) \rightarrow \infty$ . More precisely, then, we want to find the *direction* of  $\vec{\delta x}(t_0)$  which maximises  $\|\vec{\delta x}(t_0 + \delta t)\|$ . We can do this by introducing a constraint on the magnitude of the initial error (i.e., enforce  $\|\vec{\delta x}(t_0)\| - C = 0, C \in \mathbb{R}$ )<sup>5</sup> and maximising  $\|\vec{\delta x}(t_0 + \delta t)\|$  using Lagrange multipliers. So we introduce the Lagrange multiplier  $\lambda$  and maximise our Lagrangian  $\mathcal{L}$ .

$$\mathcal{L} = \|\vec{\delta x}(t_0 + \delta t)\|^2 - \lambda(\|\vec{\delta x}(t_0)\|^2 - C^2) \quad (13.6)$$

Substituting in Equation 13.5 and differentiating with respect to  $\vec{\delta x}(t_0)$ , we find that:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \vec{\delta x}(t_0)} &= 2(\hat{A}^T \hat{A} \vec{\delta x}(t_0) - \lambda \vec{\delta x}(t_0)) \\ &= 0\end{aligned}$$

if and only if:

$$\hat{A}^T \hat{A} \vec{\delta x}(t_0) = \lambda \vec{\delta x}(t_0)$$

So the final error is maximised when the initial error is the largest eigenvector of  $\hat{A}^T \hat{A}$ <sup>6</sup>. Note that  $\hat{A}^T \hat{A}$  is symmetric, so eigenvectors will be orthogonal and eigenvalues will be real.

---

<sup>4</sup>This may be because we want to run an ensemble weather forecast: instead of integrating from a single initial condition, we will integrate our weather model from multiple initial conditions compatible with the observations. Generally, the final states will be different from each other, and their clustering will indicate how predictable the current state is, or whether there are any low probability high impact weather events. However, we want to prudently choose which initial conditions we integrate from, given our limited computing resources, which is why we want to find which initial errors will grow the most.

<sup>5</sup>Myles sets  $C = 1$ , but  $C$  can actually just be any number.

<sup>6</sup>This is deduced by considering the Hessian - the matrix of second partial derivatives. It can be shown, analogous to the 1D case, that the stationary point is a maximiser only if the Hessian is negative semidefinite (i.e., each eigenvalue  $\leq 0$ ). In this case, the Hessian is the matrix  $\hat{A}^T \hat{A} - \lambda \mathbb{I}$ , which is clearly only negative semidefinite if  $\lambda \geq$  the largest eigenvalue of  $\hat{A}^T \hat{A}$ .



# Chapter 14

## Estimation

### 14.1 The Pseudo-Inverse $\hat{K}$

Suppose we take  $n$  measurements, which we encode in some matrix  $\vec{y} \in \mathbb{R}^m$ , with some random error  $\vec{\epsilon} \in \mathbb{R}^m$ , in order to estimate some values  $\vec{x} \in \mathbb{R}^n$  where  $n < m$ . The measurements  $\vec{y}$  are some (known) function of  $\vec{x}$  like so:  $\vec{y} = f(\vec{x}) + \vec{\epsilon}$  where  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ .

In some cases,  $f$  is linear, and we can represent  $f(\vec{x}) = \hat{H}\vec{x}$  where  $\hat{H} \in \mathbb{R}^m \times \mathbb{R}^n$  (i.e.,  $\hat{H}$  is an  $m \times n$  matrix). We know  $\vec{y}$ , but we wish to estimate  $\vec{x}$ . Since  $m < n$ , the system is overdetermined: we cannot simply invert  $\hat{H}$  and solve for  $\vec{x}$ , which is exacerbated by the error  $\vec{\epsilon}$ .

Instead, we we apply linear regression, and minimise the squared difference between the actual (error-prone) measurements  $\vec{y}$  and the predicted measurements  $\hat{\vec{y}} = \hat{H}\hat{\vec{x}}$ , where  $\hat{\vec{x}} \in \mathbb{R}^n$  is our estimate of  $\vec{x}$ . (We denote our prediction/estimates by hats and the real values without hats).

One can differentiate  $\|\vec{y} - \hat{\vec{y}}\|^2 = \|\vec{y} - \hat{H}\hat{\vec{x}}\|^2$  with respect to  $\hat{\vec{x}}$  (since everything is smooth) and churn through some algebra to obtain the least-squares estimator.

$$\hat{\vec{x}} = \hat{K}\vec{y} \quad (14.1)$$

$$= (\hat{H}^T \hat{H})^{-1} \hat{H}^T \vec{y} \quad (14.2)$$

where the pseudo-inverse  $\hat{K}$  of  $\hat{H}$  is:

$$\hat{K} = (\hat{H}^T \hat{H})^{-1} \hat{H}^T \quad (14.3)$$

We call  $\hat{K}$  the pseudo-inverse of  $\hat{H}$  because  $\hat{K}\hat{H} = \mathbb{I}$  (but note that generally  $\hat{H}\hat{K} \neq \mathbb{I}$ ). In practice in exams, you will probably have to do this quickly on your calculator. The strategy is as follows:

#### Key Idea V.3: Estimator Recipe

Suppose you need to estimate some quantities  $x_i$  based on some data  $y_j$ . Find the **least-squares estimator** in the following way:

1. Rearrange the system into a linear system as follows:

$$\vec{y} = \hat{H}\vec{x}$$

2. Enter the matrix  $\hat{H}$  into your calculator and calculate the **pseudo-inverse**  $\hat{K} = (\hat{H}^T \hat{H})^{-1} \hat{H}^T$ .
3. Yeet the  $\hat{H}$  onto the observations  $\vec{y}$  (which you should be given), i.e., estimate  $\vec{x}$  as  $\hat{\vec{x}} = \hat{K}\vec{y}_0$ .
4. Watch the examiners award you The Gibbs Prize for Performance in the MPhys examination (£500).
5. Donate those £500 (and maybe the prize as well) to the author(s) of these lecture notes.

## 14.2 Errors

We assume, for simplicity, that there are negligible errors on the parameters within the operator  $\hat{H}$ . We assume only that there are some errors  $\vec{\epsilon}$  on the observations  $\vec{y}_0$ . We assume that the errors are given by some well defined  $n \times n$  covariance matrix  $\hat{S}$ :

$$\begin{aligned}\langle \vec{\epsilon} \vec{\epsilon}^T \rangle &= \hat{S} \\ [\hat{S}]_{ij} &= [\vec{\epsilon}]_i [\vec{\epsilon}]_j\end{aligned}$$

We also assume that the average of the noise is 0:

$$\langle \vec{\epsilon} \rangle = 0$$

This allows us to show that, on average:

$$\begin{aligned}\langle \vec{\hat{x}} \rangle &= \langle \hat{K} \vec{y} \rangle \\ &= \langle \hat{K} (\hat{H} \vec{x} + \vec{\epsilon}) \rangle \\ &= \langle \vec{x} \rangle\end{aligned}$$

$$\begin{aligned}Var(\vec{\hat{x}}) &= \langle (\vec{x} - \vec{\hat{x}})(\vec{x} - \vec{\hat{x}})^T \rangle \\ &= \langle (\vec{x} - \hat{K} \vec{y})(\vec{x} - \hat{K} \vec{y})^T \rangle \\ &= \langle (\vec{x} - \hat{K} (\hat{H} \vec{x} + \vec{\epsilon}))(\vec{x} - (\hat{H} \vec{x} + \vec{\epsilon}))^T \rangle \\ &= \langle \hat{K} \vec{\epsilon} \vec{\epsilon}^T \hat{K}^T \rangle \\ &= \hat{K} \hat{S} \hat{K}^T\end{aligned}$$

In other words, the mean of our estimation is the real value, and the covariance of our estimation is the covariance of

## 14.3 Interpretation

## 14.4 Examples

## Part VI

# Geophysical Fluid Dynamics

# Introduction

This section of the course was lectured by **Tim Woollings** covering Geophysical Fluid Dynamics.

This section consists of three chapters:

1. **Dynamical Systems:**

Meow

2. **Predictability:**

Meow

3. **Estimation:**

Meow

## Chapter 15

# The Equations of Motion

### 15.1 Why There Is Motion in the First Place

It's important to consider why there is motion in the Atmospheres and the Oceans in the first place, and why we call the study **Geophysical Fluid Dynamics**, rather than simply Fluid Dynamics.

Regarding the first point, if you are familiar with the [Navier Stokes Equations](#) governing fluid flow, then you will know that  $\vec{u} = \vec{\nabla}p = 0$  is a solution, and is a solution a system will tend towards if there is friction, and planets certainly have plenty of friction. We therefore need a non-uniform energy source. In this case, it is the sun, and it drives motion because it provides energy to the Earth unevenly. On small scales, this may be because of variations in reflectivity (albedo), but on large scales this is also due to the spherical geometry of the Earth: more solar radiation is absorbed at the equator than at the poles (think: **zenith angle** from [Radiative Transfer](#), which goes as  $\cos \zeta$ ).

This differential heating causes horizontal temperature gradients, which in turn cause horizontal density and pressure gradients, which drive motion. Another way to think of it is as follows: in equilibrium, the Earth absorbs more solar radiation in the equator than in the poles (see Figure 15.1). As such, there must be an energy transport mechanism from the equator to the poles, and this is what drives fluid motion.<sup>1</sup>

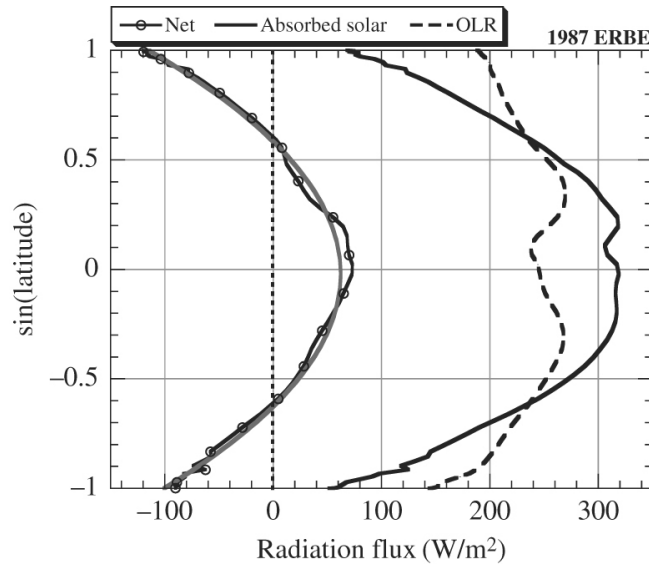


Figure 15.1: Figure from Ray's book showing differential heating. More heat is

<sup>1</sup>I'm personally somewhat ambivalent about this explanation, but I'm not very confident so be sceptical of what I'm about to say. Here's why I'm uncomfortable about this explanation: I cannot see why it must be the case that the variation in OLR cannot be identical to the variation in absorbed solar radiation. To put it another way, I cannot see why the dotted curve cannot be identical to the solid curve in Figure 15.1. The absorbed energy per unit area scales as  $\cos \phi$ , where  $\phi$  is the latitude. It feels like putting the cart before the horse: surely the fluid dynamics also itself affects the OLR into space.

## 15.2 What is Special About ‘Geophysical’ Fluid Mechanics?

Regarding the second point, there are two important properties that separate **Geophysical Fluid Dynamics** into its own field: that of **stratification** and **rotation**. The effects of these will pop up again and again throughout this part.

**Stratification** is

## 15.3 A Quick Primer in Fluid Mechanics

To aide understanding, we quickly introduce some concepts of fluid mechanics. Tim should have a more comprehensive primer on the canvas should you need it.

First we introduce the concept of a fluid element. This is an infinitesimal volume of fluid, whose properties may change as it is advected around with the flow. For example, in deriving the [Dry Adiabats](#), we were considering an air parcel – a fluid element of air.

Second, we introduce the [Material Derivative](#):

$$\frac{D}{Dt} = \underbrace{\frac{\partial}{\partial t}}_{\text{Time Derivative at a Fixed Point}} + \underbrace{\vec{u} \cdot \vec{\nabla}}_{\text{Advection}} \quad (15.1)$$

The [Material Derivative](#) of a quantity is the time-derivative of a quantity *following a fluid element*. In GFD, we will at most times be interested in how a quantity changes at a fixed point in space, not following a fluid element. The [Material Derivative](#) takes into account that a quantity at some point can be modified by both local change ( $\frac{\partial}{\partial t}$ ), for example by heating, and fluid parcels with different properties being advected into that location  $\vec{u} \cdot \vec{\nabla}$ .

Third, we introduce the idea of [Material Conservation](#). You might have heard about what it means for a quantity to be **conserved**. For example, if you’ve solved a mechanics problem, you might recall that the total energy was conserved, meaning that  $\frac{dE}{dt} = 0$  where  $E$  = the total energy.

This does occur in fluid mechanics, but but we will also have situations where certain quantities are [Materially Conserved](#). Mathematically, a quantity  $A$  is materially conserved if and only if it obeys the following relation:

$$\frac{DA}{Dt} = 0 \quad (15.2)$$

In other words, if one follows a fluid parcel, that quantity does not change. However, as a whole,  $A$ , need not be globally conserved.

Now recall Equation 15.5. Note even if we assume that  $Q = 0$ , we will find that  $\frac{DT}{Dt} \neq 0$ . In other words, temperature is not materially conserved (so long as  $\beta \neq 0$ ). This is why we introduced the idea of [Potential Temperature](#), as it is the case that  $\frac{D\theta}{Dt} = 0$  (exccercise: take 15.5, solve for  $\beta$  using the ideal gas law, set  $Q = 0$ , and show that  $\frac{D\theta}{Dt} = 0$ ).

## 15.4 The Primitive Equations of Motion

For simplicity we ignore state variables like salinity, humidity, etc., and restrict our attention to a system fully characterised by 6-state variables: the velocity  $\vec{u} = (u, v, w)$ , pressure  $p$ , density  $\rho$ , and temperature  $T$ . These are governed by six equations: 3 momentum budget equations (15.3), one mass budget equation (15.4), one energy budget equation (15.5), and an equation of state (15.6). Each term is labelled below:

$$\begin{array}{c}
\text{Acceleration and Advection} \\
\frac{D\vec{u}}{Dt}
\end{array}
=
\begin{array}{c}
\text{Pressure Gradients} \\
-\frac{1}{\rho}\vec{\nabla}p
\end{array}
+
\begin{array}{c}
\text{Gravity} \\
-g\vec{k}
\end{array}
+
\begin{array}{c}
\text{Viscous Dissipation} \\
\nu\vec{\nabla}^2\vec{u}
\end{array}
+
\begin{array}{c}
\text{Convergence/Divergence of Mass} \\
+\frac{\nu}{3}\vec{\nabla}(\vec{\nabla}\cdot\vec{u})
\end{array}
+
\begin{array}{c}
\text{Other Forces} \\
\vec{F}
\end{array}
\quad (15.3)$$

$$\begin{array}{c}
\text{Change in Mass of Fluid Parcel} \\
\frac{D\rho}{Dt}
\end{array}
+
\begin{array}{c}
\text{Convergence/Divergence of Mass} \\
+\rho\vec{\nabla}\cdot\vec{u}
\end{array}
= 0 \quad (15.4)$$

$$c_p \frac{DT}{Dt} - \frac{\beta T}{\rho} \frac{Dp}{Dt} = Q \quad (15.5)$$

$$\rho = \rho(T, p, \dots) \quad (15.6)$$

where  $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla}$  is the material derivative;  $g$  = gravitaional acceleration;  $\vec{k}$  = unit vector towards the centre of the Earth;  $\nu$  = kinematic viscosity;  $\vec{F}$  = other forces per unit mass (e.g., friction);  $c_p$  = heat capacity per unit mass;  $\beta = -\frac{1}{\rho} \frac{\partial \rho}{\partial T}$  thermal expansion coefficient; and  $Q$  = heating per unit mass.

However, the equations currently are not fit for purpose. First, these equations are written for a coordinate system that is *inertial* (non-accelerating) and *cartesian*. However, the surface of the Earth is curved and the Earth is rotating<sup>2</sup>, and we wish to describe what is going on *here* with us. Furthermore, it would be quite demanding (computationally and conceptually) to adopt an inertial non-curved coordinate system, as then you'd have to, for example, keep track of the fact that the mountains keep moving.

Second, these equations are currently too complicated to be analytically tractable. We will make various approximations later to simplify the equations of motion, however we should note that many of the terms we neglect cannot be neglected in an, e.g., weather forecast or climate model.

## 15.5 Simplifications

### 15.5.1 Rotating Coordinate Systems

As already mentioned, the momentum budget equation applies for an *inertial coordinate system*, i.e., a coordinate system which is not accelerating. However we wish to describe the dynamics in a rotating coordinate system, which rotates with the Earth (or planet). It is generally the case that the time derivative of some arbitrary vector  $\vec{A}$  in the inertial coordinate system  $\left(\frac{d\vec{A}}{dt}\right)_I$  will be *different* from the time derivative in the rotating coordinate system  $\left(\frac{d\vec{A}}{dt}\right)_R$ .

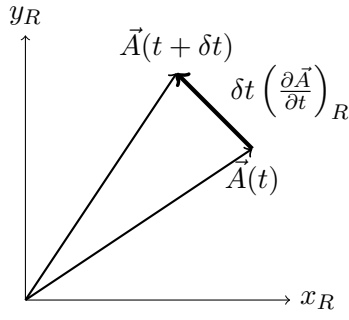
We consider two coordinate systems: an *inertial* coordinate system, and a *rotating* coordinate system. Both systems share an origin, but the *inertial* coordinate system rotates with a constant angular velocity  $\vec{\Omega}$  where  $|\vec{\Omega}|$  is the angular speed and  $\vec{\Omega}$  points along the axis of rotation.

Now consider some arbitrary vector  $\vec{A}(t)$  in or rotating coordinate system. It is the case that  $\left(\frac{d\vec{A}}{dt}\right)_I$  and  $\left(\frac{d\vec{A}}{dt}\right)_R$  are related  $\left(\frac{d\vec{A}}{dt}\right)_I$  as follows (see Figure 15.2):

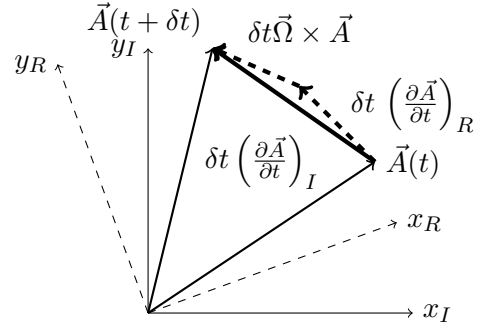
$$\left(\frac{d\vec{A}}{dt}\right)_I = \left(\frac{d\vec{A}}{dt}\right)_R + \vec{\Omega} \times \vec{A} \quad (15.7)$$

If we let  $\vec{A} = \vec{r}$ , where  $\vec{r}$  is the position of our fluid parcel, we can apply Equation 15.7 twice to obtain an expression for the acceleration of a fluid parcel in a rotating coordinate system. We let subscripts  $I$  and  $R$  refer to inertial and rotating coordinate systems and  $\vec{r}$ ,  $\vec{u}$ , and  $\vec{a}$  refer to positions, velocities, and accelerations.

<sup>2</sup>or so NASA and the lizard overlords would have you think!



(a) Change in  $\vec{A}$  in the Rotating Coordinate System



(b) Change in  $\vec{A}$  in the Inertial Coordinate System

Figure 15.2: The change in  $\vec{A}$  in a rotating (15.2a) and inertial (15.2b) coordinate system. As seen in 15.2b, the change in the inertial coordinate system  $\delta t \left( \frac{\partial \vec{A}}{\partial t} \right)_I$  (thick arrow) is equal to sum of the change in the rotating coordinate system  $\left( \delta t \left( \frac{\partial \vec{A}}{\partial t} \right)_R \right)$  and the rotation of the rotating coordinate system  $\left( \delta t \vec{\Omega} \times \vec{A} \right)$  (thick dashed arrows).

$$\begin{aligned}
 \vec{u}_I &= \left( \frac{d\vec{r}}{dt} \right)_I \\
 &= \left( \frac{d\vec{r}}{dt} \right)_R + \vec{\Omega} \times \vec{r} \\
 \vec{u}_I &= \vec{u}_R + \vec{\Omega} \times \vec{r} \\
 \therefore \vec{a}_I &= \left( \frac{d\vec{u}_I}{dt} \right)_I \\
 &= \left( \frac{d\vec{u}_I}{dt} \right)_R + \vec{\Omega} \times \vec{u}_I \\
 &= \left( \frac{d\vec{u}_R}{dt} \right)_R + \vec{\Omega} \times \left( \frac{d\vec{r}}{dt} \right)_R + \vec{\Omega} \times (\vec{u}_R + \vec{\Omega} \times \vec{r}) \\
 &= \vec{a}_R + 2\vec{\Omega} \times \vec{u}_R + \vec{\Omega} \times \vec{\Omega} \times \vec{r}
 \end{aligned}$$

Therefore the acceleration in the rotating frame of reference  $\vec{a}$  is equal to the acceleration in a inertial coordinate system summed with two fictitious\* forces: the **centrifugal force** and the **coriolis force**.

$$\vec{a}_R = \vec{a}_I + \underbrace{2\vec{\Omega} \times \vec{u}_R}_{\text{Coriolis}} + \underbrace{\vec{\Omega} \times \vec{\Omega} \times \vec{r}}_{\text{Centrifugal}} \quad (15.8)$$

The **centrifugal force** is the force that pushes objects away from the axis of rotation. This is the force that makes your arms fling out if you spin. The **coriolis force** is the force that diverts moving objects which move away or towards the axis of rotation.

We have thus transformed Equation 15.3 into a rotating coordinate system:

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} + \frac{1}{\rho} \vec{\nabla} p + g\vec{k} = -\vec{\Omega} \times \vec{\Omega} \times \vec{r} + \nu \vec{\nabla}^2 \vec{u} + \frac{\nu}{3} \vec{\nabla} (\vec{\nabla} \cdot \vec{u}) + \vec{F} \quad (15.9)$$

### 15.5.2 Local Cartesian Coordinates

We have just learnt how to deal with coordinate systems that vary with time. We will now skim over (since the details aren't that interesting) how to deal with coordinate systems that vary with space.



We wish to use a coordinate system that follows the surface of the Earth, with the basis vectors  $\hat{x}$ ,  $\hat{y}$ ,  $\hat{z}$ , pointing in the east, north, and upwards (away from the centre of the Earth) directions. In this coordinate system, we define the position and velocity coordinates  $(x, y, z)$  and  $(u, v, w)$  as indicating distance and velocity in the east, north, and upwards directions, respectively. We further define the latitude  $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$  ( $\phi = 0$  indicates the equator,  $\phi = \pm\frac{\pi}{2}$  is the north/south pole), the radial distance  $r$  where  $r = a + z$  where  $a$  = the radius of the Earth.

Churning through the algebra, we get:

$$\begin{aligned}\frac{Du}{Dt} - \left(2\Omega + \frac{u}{r \cos \phi}\right) (v \sin \phi - w \cos \phi) + \frac{1}{\rho} \frac{\partial p}{\partial x} &= F_x \\ \frac{Dv}{Dt} + \frac{wv}{r} + \left(2\Omega + \frac{u}{r \cos \phi}\right) u \sin \phi + \frac{1}{\rho} \frac{\partial p}{\partial y} &= F_y \\ \frac{Dw}{Dt} - \frac{u^2 + v^2}{r} - 2\Omega u \cos \phi + \frac{1}{\rho} \frac{\partial p}{\partial z} + g &= F_z\end{aligned}$$

where we have lumped together all the terms on the right-hand-side of 15.9 into the  $F_x, F_y, F_z$ . Many of the extra terms (e.g., the  $\frac{wv}{r}$  term), are due to the fact that we have adopted a local cartesian system, in which the coordinate system changes as we move over the Earth.

### 15.5.3 Incompressibility

This is in preparation for the next simplification we make: we assume that the fluid we are dealing with is **incompressible**<sup>3</sup>. This implies that density is **Materially Conserved** following a fluid parcel. Therefore, for density  $\rho$ :

$$\begin{aligned}\frac{D\rho}{Dt} &= 0 \\ \therefore \boxed{\vec{\nabla} \cdot \vec{u} = 0}\end{aligned}\tag{15.10}$$

where we have obtained 15.10 by substituting Mass Conservation (15.4).

### 15.5.4 Scale Analysis

Finally, we make a few phenomenological simplifications. We do this here for simplicity, but our results are not general or particularly robust: these terms are essential in climate/weather models. We do this here by performing a **scale analysis**: we estimate the size of quantities by their typical values (found empirically), then ignore terms that are much smaller than the other terms. We never neglect the pressure gradient term.

Scale	Symbol	Terms Approximated	Typical Magnitude
Horizontal Scale	$L$	$\frac{\partial}{\partial x} \sim \frac{1}{L}, \frac{\partial}{\partial y} \sim \frac{1}{L}$	$10^6 \text{ m}$
Vertical Scale	$H$	$\frac{\partial}{\partial z} \sim \frac{1}{H}$	$10^4 \text{ m}$
Horizontal Velocity	$U$	$u \sim U, v \sim U$	$10 \text{ m s}^{-1}$
Vertical Velocity	$W$	$w \sim W$	$10^{-2} \text{ m s}^{-1}$
Time Scale	$T$	$\frac{\partial}{\partial t} \sim \frac{1}{T}$	$10^5 \text{ s}$
Density	$\rho$	$\rho \sim \rho$	$1 \text{ kg s}^{-1}$
Earth's Radius	$a$	$r \sim a$	$6.4 \times 10^6 \text{ m}$
Rotation Rate	$\Omega$	$\Omega \sim \Omega$	$10^{-4} \text{ s}^{-1}$
Acceleration of Gravity	$g$	$g \sim g$	$10 \text{ m s}^{-2}$

<sup>3</sup>This should seem, to you, a highly dubious assumption. First, we know that sound waves are possible (after all we can hear, both on land and in water), but sound waves are impossible in an incompressible fluid. Second, we know that the atmosphere is approximately an ideal gas, which can have its density changed if warmed or cooled.

I do not write out all the details here, as they would easily take multiple pages. If you are curious about the details, refer to the mini-scale analysis I do in deriving 16.1. A more rigorous method would be non-dimensionalisation, but this is overkill for our purposes. Again, if you are curious about the details, refer to the mini-non-dimensionalisation I do in deriving the Rossby Number. After eliminating small terms, we get our final simplified set of equations.

### 15.5.5 The Simplified Equations of Motion

#### Key Idea VI.1: The Equations of Motion for GFD

We define the horizontal gradient operator taken at constant  $z$  as  $\vec{\nabla}_h = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right)^T$ , the horizontal velocity as  $\vec{u}_h = (u, v, 0)^T$ , the horizontal forces as  $\vec{F}_h$ , and the **coriolis parameter** as  $f = 2\Omega \sin \phi$  where  $\phi$  = the latitude.

**Horizontal Momentum Balance:**

$$\frac{D\vec{u}_h}{Dt} + f\vec{k} \times \vec{u}_h + \frac{1}{\rho}\vec{\nabla}_h p = \vec{F}_h$$

(15.11)

**Mass Conservation:**

$$\frac{D\rho}{Dt} = 0$$

(15.13)

**Vertical Momentum Balance:**

$$\frac{Dw}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial z} + g = F_z$$

(15.12)

**Incompressibility:**

$$\vec{\nabla} \cdot \vec{u} = 0$$

(15.14)

We will mostly be dealing with the horizontal momentum equation 15.11, so it is worth detailing the meanings of each terms. The first term,  $\frac{D\vec{u}_h}{Dt}$  is the material derivative of  $\vec{u}_h$ . It represents the acceleration of  $\vec{u}_h$  and the advection of  $\vec{u}_h$ . The  $f\vec{k} \times \vec{u}_h$  term is the acceleration due to coriolis. The  $\frac{1}{\rho}\vec{\nabla}_h p$  is the force due to the horizontal pressure gradient. Finally, the  $\vec{F}_h$  term is external forces, e.g., friction, wind-forcing, etc.

$$\underbrace{\frac{D\vec{u}_h}{Dt}}_{\text{acceleration and advection}} + \underbrace{f\vec{k} \times \vec{u}_h}_{\text{coriolis}} + \underbrace{\frac{1}{\rho}\vec{\nabla}_h p}_{\text{horizontal pressure gradients}} = \underbrace{\vec{F}_h}_{\text{external forces}}$$

## Chapter 16

# The Fundamental Diagnostic Relations

### 16.1 Hydrostatic Balance (Approximate Vertical Momentum Balance)

We now make further approximations to extract physical intuition. Consider 15.12, and again perform a scale analysis and assume that  $F_z \approx 0$ .

$$\begin{array}{lll}
 \underbrace{\frac{\partial w}{\partial t}}_{\frac{W}{T}} + & \underbrace{u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z}}_{U \frac{W}{L} + W \frac{W}{H}} + & \frac{Dw}{Dt} + \\
 \sim \frac{W}{T} + & \sim U \frac{W}{L} + \sim \frac{W^2}{H} + & \frac{1}{\rho} \frac{\partial p}{\partial z} + g = 0 \\
 \therefore \frac{10^{-2} \text{ m s}^{-1}}{10^5 \text{ s}} + & 10 \text{ m s}^{-1} \frac{10^{-2} \text{ m s}^{-1}}{10^6 \text{ m}} + \frac{(10^{-2} \text{ m s}^{-1})^2}{10^4 \text{ m}} + & \frac{1}{\rho} \frac{\partial p}{\partial z} + 10 \text{ m s}^{-2} = 0 \\
 10^{-7} \text{ m s}^{-2} + & 10^{-7} \text{ m s}^{-2} + 10^{-8} \text{ m s}^{-2} + & \frac{1}{\rho} \frac{\partial p}{\partial z} + 10 \text{ m s}^{-2} = 0 \\
 \cancel{10^{-7} \text{ m s}^{-2}} + & \cancel{10^{-7} \text{ m s}^{-2}} + \cancel{10^{-8} \text{ m s}^{-2}} + & \frac{1}{\rho} \frac{\partial p}{\partial z} + 10 \text{ m s}^{-2} = 0
 \end{array}$$

As seen then, many of the acceleration terms (the terms in the  $\frac{Dw}{Dt}$ ) are 8 orders of magnitude smaller than gravity! As such, the only term that can balance the gravity term is the pressure gradient. We thus neglect all other terms and derive again:

#### Key Idea VI.2: Hydrostatic Balance

*Hydrostatic Balance governs the vertical pressure variation in a fluid if we assume that vertical acceleration (and advection) is small.*

$$\frac{\partial p}{\partial z} = -\rho g$$

(16.1)

*Pressure decreases with height in order to balance the force of gravity.*

Physically, hydrostatic balance is a force/momentum balance in the vertical direction, where the force of gravity is balanced by the vertical pressure gradient. Notice now the difference between the Hydrostatic Balance I have written here (16.1) and the Hydrostatic Balance I have written in the Thermodynamics Section (1.11): I have written it here with a partial derivative. Strictly, it should

have been a partial derivative before too, but I write it here to note that we are now considering horizontal variations as well as vertical variations.

For an ideal gas, one can substitute for  $\rho$  using 1.3:

$$\boxed{\frac{\partial \ln p}{\partial z} = -\frac{g}{RT}} \quad (16.2)$$

## 16.2 Geostrophic Balance (Approximate Horizontal Momentum Balance)

### 16.2.1 Rossby Number and Non-Dimensionalisation

Let us now consider the horizontal momentum equation 15.11 and let  $\vec{F}_h \approx 0$  for now. We nondimensionalise the equations by defining dimensionless hatted variables and choosing characteristic scales such that the dimensionless hatted variables are of order 1:

$$t = [t]\hat{t} ; \vec{u}_h = [u_h]\vec{\hat{u}}_h ; p = [p]\hat{p}$$

So, for example,  $[t]$  = the timescale with dimensions of time, while  $\hat{t}$  is the dimensionless time. Note that we pick these scales  $[t]$ ,  $[u_h]$ ,  $[p]$ , but we have to be self-consistent: we can only make three choices.<sup>1</sup> We now substitute these variables into 15.11:

$$\frac{[u_h]}{[t]} \frac{D\vec{\hat{u}}_h}{D\hat{t}} + [u_h]f\vec{k} \times \vec{\hat{u}}_h + [p]\frac{1}{\rho}\vec{\nabla}_h\hat{p} = 0$$

Dividing by  $f[u_h]$ , we get:

$$\frac{1}{f[t]} \frac{D\vec{\hat{u}}_h}{D\hat{t}} + \vec{k} \times \vec{\hat{u}}_h + \frac{[p]}{\rho f[u_h]}\vec{\nabla}_h\hat{p} = 0$$

We now make a choice: we choose an advective timescale (making a choice for  $[t]$ )

$$[t] = [u_h]L ; L = [t]$$

We therefore derive the **Rossby Number**, which compares the effect of advection/acceleration to rotation. If the **Rossby Number** is small, then rotation is important and dominates dynamics. If the **Rossby Number** is large, then the flow evolves in such a way where coriolis is negligible.

$$Ro = \frac{U}{fL} \quad (16.3)$$

We can also write the Rossby Number in terms of a ratio between the planetary vorticity and relative vorticity. The planetary vorticity  $\omega_{planetary}$  is the angular velocity of a fluid parcel due to the rotation of the Earth, and scales as  $f$ . The relative vorticity is the vorticity of our fluid parcel in our coordinate system which is rotating with the Earth, and scales as  $\omega_{relative} \sim [\vec{\omega}]_z = [\vec{\nabla} \times \vec{u}_h]_z \sim U/L$ . Therefore:

$$Ro = \frac{\omega_{relative}}{\omega_{planetary}} \quad (16.4)$$

$$\sim \frac{U/L}{f} \quad (16.5)$$

---

<sup>1</sup>In fact we've already made an implicit choice that the  $[v] = [u] = [u_h]$ .

### Key Idea VI.3: Rossby Number

We define the dimensionless **Rossby Number**  $Ro$ , which encodes the relative importance of (planetary) rotation and advection/acceleration terms, as follows:

$$Ro = \frac{U}{fL} \quad (16.6)$$

where  $U$  = the horizontal velocity scale,  $f$  = the coriolis parameter, and  $L$  = the horizontal length scale.

If  $Ro \ll 1$ , then the rotation of the planet is important, and the flow should be approximately **Geostrophic**. If  $Ro \gg 1$ , then acceleration/advection is important.

#### 16.2.2 Geostrophic Balance

If  $Ro \ll 1$ , then rotation dominates dynamics, and we can neglect the acceleration/advection term. The horizontal momentum equation thus consists of a balance between the horizontal pressure gradients and the coriolis force.

$$f\vec{k} \times \vec{u}_h = -\frac{1}{\rho}\vec{\nabla}_h p$$

This results in a velocity field determined by the pressure gradient as follows:

$$\vec{u}_h = \frac{1}{\rho f}\vec{k} \times \vec{\nabla}_h p \quad (16.7)$$

For an ideal gas:

$$\vec{u}_h = \frac{RT}{f}\vec{k} \times \vec{\nabla}_h \ln p \quad (16.8)$$

#### 16.2.3 Cyclones and Anti-Cyclones

#### 16.2.4 Thermal Wind Balance

$$\frac{\partial \vec{u}_h}{\partial z} \approx \frac{1}{fT}\vec{k} \times \vec{\nabla}_h T \quad (16.9)$$

### 16.3 Gradient-Wind Balance

### 16.4 Pressure Coordinates

$$\left(\frac{\partial a}{\partial b}\right)_{c,d} = \left(\frac{\partial b}{\partial a}\right)_{c,d}^{-1}$$

$$\left(\frac{\partial a}{\partial b}\right)_c \left(\frac{\partial b}{\partial c}\right)_a \left(\frac{\partial c}{\partial a}\right)_b = -1$$

We apply this to  $p$  and  $z$ :

$$\left(\frac{\partial p}{\partial z}\right)_{x,y} = \left(\frac{\partial z}{\partial p}\right)_{x,y}^{-1} \quad (16.10)$$

$$\left(\frac{\partial p}{\partial x}\right)_z \left(\frac{\partial x}{\partial z}\right)_p \left(\frac{\partial z}{\partial p}\right)_x = -1$$

$$\left(\frac{\partial p}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_p \left(\frac{\partial z}{\partial p}\right)_y = -1$$

where the last two equations become:

$$\left(\vec{\nabla} p\right)_z = - \left(\vec{\nabla} z\right)_p \left(\frac{\partial p}{\partial z}\right)_{x,y} \quad (16.11)$$

We apply this to hydrostatic balance (16.1), geostrophic balance (16.7), and thermal wind balance (16.9) to derive their equivalent form in pressure coordinates.

$$\begin{aligned} -\frac{g}{RT} &= \left(\frac{\partial \ln p}{\partial z}\right)_{x,y} \\ -\frac{g}{RT} &= \left(\frac{\partial z}{\partial \ln p}\right)_{x,y}^{-1} \\ \therefore \\ \boxed{\frac{\partial \Phi}{\partial \ln p} = -RT} \end{aligned} \quad (16.12)$$

$$\boxed{\Phi = g z} \quad (16.13)$$

$$\left(\vec{\nabla} p\right)_z = \quad (16.14)$$

## Chapter 17

# The Shallow Water System: 2D System

### 17.1 The Equations of Motion

In the previous chapter, we considered diagnostic relations: hydrostatic balance and geostrophic balance. These are very important, and very useful, but notice that there are no time derivatives in 16.1 and 16.7. If there are no time-derivatives, the equations cannot reproduce non-steady state solutions and time-evolution. However, clearly, the atmosphere and ocean evolve and are not in a steady state, and we wish to understand this.

Our first tool to consider time-dependent changes is the shallow water system. We consider a system as follows: FIGURE HERE

We derive the equations that govern this system from the Equations 15.11 and 15.12. First, we neglect all vertical and horizontal forces  $F_z$  and  $\vec{F}_h$  (e.g. neglect friction).

Second, we assume that the system is **shallow**:  $H/L \ll 1$  where  $H$  is the vertical lengthscale and  $L$  is the horizontal lengthscale. If we non-dimensionalise 15.14, we derive that the vertical velocity scale  $W$  is much smaller than the horizontal velocity scale  $U$ :  $W \ll U$ :

$$\begin{aligned}\vec{\nabla} \cdot \vec{u} &= 0 \\ \underbrace{\vec{\nabla}_h \cdot \vec{u}_h}_{U/L} + \underbrace{\frac{\partial w}{\partial z}}_{W/H} &= 0 \\ \therefore \frac{U}{L} \sim \frac{W}{H} \quad \therefore \frac{W}{U} \sim \frac{H}{L} \\ \therefore \boxed{W \ll U}\end{aligned}$$

As such, we neglect the vertical advection term in the material derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + \cancel{w \frac{\partial}{\partial z}}$$

Third, we assume that the system is approximately in **Hydrostatic Balance**. However, we do not assume that the vertical velocities are 0 everywhere. We simply assume that they are much smaller than vertical pressure gradients and gravity.

Fourth, we

We neglect friction and the centripetal acceleration. The equations of motion, then, are

We derive the equations of motions

## Key Idea VI.4: The Shallow Water Equations

$$\frac{D\vec{u}}{Dt} + f\vec{k} \times \vec{u} + g\vec{\nabla}\eta = 0 \quad (17.1)$$

$$\frac{\partial h}{\partial t} + \vec{\nabla} \cdot (h\vec{u}) = 0 \quad (17.2)$$

$$h = \eta - b \quad (17.3)$$

## 17.2 Energetics

## 17.3 Potential Vorticity

## 17.4 Waves

### 17.4.1 A Quick Primer on Waves

In the previous section, we derived geostrophic balance and hydrostatic balance.

However, the Earth is never in perfect balance, and small perturbations will occur, either because we have built a wind farm, or the sun is radiating differently, or because freshwater is being injected into certain areas in the oceans.

We analyse how systems respond to these perturbations by looking for wavelike solutions: by plugging in

### 17.4.2 Poincaré Waves

We first linearise the Shallow Water Equations about a state of rest  $\vec{U} = 0$  and layer thickness  $H$ . For simplicity we assume a flat bed  $b = -H = \text{const.}$  This is the first time we linearise in this course, so I will go through this a bit more carefully.

First, we write our variables as a sum of the unvarying quantity and the small perturbation:

$$\begin{aligned} \vec{u} &= \vec{U} + \vec{u}' \\ h &= H + h' \\ \eta &= N + \eta' \end{aligned}$$

where  $|\vec{U}| \gg |\vec{u}'|$ ,  $H \gg h'$ , and  $N \gg \eta'$ .

Second, we substitute these equations into the shallow water equations, then eliminate terms that are not zeroth or first order in the primed quantities. For example, for Equation 17.3:

$$\begin{aligned} \frac{\partial}{\partial t} (H + h') + \vec{\nabla} \cdot ((H + h')(\vec{U} + \vec{u}')) &= 0 \\ \frac{\partial}{\partial t} (H + h') + \vec{\nabla} \cdot (H\vec{U} + H\vec{u}' + h'\vec{U} + \cancel{h'\vec{u}'}) &= 0 \end{aligned}$$



**17.4.3 Kelvin Waves**

**17.4.4 Rossby Waves**

## Chapter 18

# 3D Systems

### 18.1 Gravity Waves

#### 18.1.1 The Boussinesq Approximation

### 18.2 Quasi-Geostrophic Theory

### 18.3 Quasi-Geostrophic Rossby Waves

### 18.4 Instabilities and Geostrophic Turbulence

# Chapter 19

## Ocean Circulation

As already touched upon, the oceans, while similar to the atmosphere in the sense that it is a fluid affected by gravity (and so stratification) and rotation (and so coriolis), differs dynamically due to a few facts.

1. Equation of state:  $g'$  is much smaller, so  $L_d$  is much smaller
2. Rossby number
3. External forcing
4. Bathymetry

We'll be exploring some effects of external forcing [19.1](#) and the small rossby number [19.2](#).

### 19.1 Ekman Transport

Assume  $Ro \ll 1$  so we can

$$f\vec{k} \times \vec{u} = \frac{1}{\rho} \tag{19.1}$$

### 19.2 Sverdrup Balance

### 19.3 Stommel Box Model

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