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1:

a): $E = \sum_{i=1}^{n} \left\| \begin{bmatrix} x_i' \\ y_i' \end{bmatrix} - \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} - \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \right\|^2$

$$= \sum_{i=1}^{n} \left\| \begin{matrix} x_i' - m_1 x_i - m_2 y_i - t_1 \\ y_i' - m_3 x_i - m_4 y_i - t_2 \end{matrix} \right\|^2$$

$$\frac{dE}{dm_1} = \sum_{i=1}^{n} -2x_i (x_i' - m_1 x_i - m_2 y_i - t_1)$$

$$\frac{dE}{dm_2} = \sum_{i=1}^{n} -2y_i (x_i' - m_1 x_i - m_2 y_i - t_1)$$

$$\frac{dE}{dm_3} = \sum_{i=1}^{n} -2x_i (y_i' - m_3 x_i - m_4 y_i - t_2)$$

$$\frac{dE}{dm_4} = \sum_{i=1}^{n} -2y_i (y_i' - m_3 x_i - m_4 y_i - t_2)$$

$$\frac{dE}{dt_1} = \sum_{i=1}^{n} -2 (x_i' - m_1 x_i - m_2 y_i - t_1)$$

$$\frac{dE}{dt_2} = \sum_{i=1}^{n} -2 (y_i' - m_3 x_i - m_4 y_i - t_2)$$

b): setting gradient to 0:

$$\sum_{i=1}^{n} -x_i y_i' + m_3 x_i^2 + m_4 x_i y_i + x_i t_2 = 0$$

$$\vdots$$

$$\sum_{i=1}^{n} -y_i' + m_3 x_i + m_4 y_i + t_2 = 0$$

$$\begin{bmatrix} \sum_i^n x_i^2 & \sum_i^n x_i y_i & 0 & 0 & \sum_i^n x_i & 0 \\ \sum_i^n x_i y_i & \sum_i^n y_i^2 & 0 & 0 & \sum_i^n y_i & 0 \\ 0 & 0 & \sum_i^n x_i^2 & \sum_i^n x_i y_i & 0 & \sum_i^n x_i \\ 0 & 0 & \sum_i^n x_i y_i & \sum_i^n y_i^2 & 0 & \sum_i^n y_i \\ \sum_i^n x_i & \sum_i^n y_i & 0 & 0 & n & 0 \\ 0 & 0 & \sum_i^n x_i & \sum_i^n y_i & 0 & n \end{bmatrix} \cdot \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} \sum_i^n x_i x_i' \\ \sum_i^n y_i x_i' \\ \sum_i^n x_i y_i' \\ \sum_i^n y_i y_i' \\ \sum_i^n x_i' \\ \sum_i^n y_i' \end{bmatrix}$$

c) $h = S^{-1} U$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 & 3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \\ 5 \\ 8 \end{bmatrix}$$

$$h = [2 \quad 0 \quad 0 \quad 2 \quad 1 \quad 2]^T$$

$$\therefore E = \sum_{i=1}^{n} \left\| \begin{bmatrix} x_i' \\ y_i' \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\|^2$$

2:

a) $v = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$,

$$v' = \begin{pmatrix} x_2' - x_1' \\ y_2' - y_1' \end{pmatrix} = s \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$\cos\theta = \frac{v'v}{\|v\|\|v'\|} = \frac{(x_2'-x_1')(x_2-x_1) + (y_2'-y_1')(y_2-y_1)}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \cdot \sqrt{(x_2'-x_1')^2+(y_2'-y_1')^2}}$$

$$\theta = \arccos\left( \frac{(x_2'-x_1')(x_2-x_1) + (y_2'-y_1')(y_2-y_1)}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \cdot \sqrt{(x_2'-x_1')^2+(y_2'-y_1')^2}} \right)$$

b) $$s = \frac{\|v'\|}{\|v\|} = \frac{\sqrt{(x_2'-x_1')^2+(y_2'-y_1')^2}}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}$$

c) $\begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} - s \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$t_x = x' - s\cos\theta\, x + s\sin\theta\, y$

$t_y = y' - s\sin\theta\, x - s\cos\theta\, y$

d): Because points correspondences are: $\{(\frac{1}{2}, 0) \to (0,0)\}$, $\{(0, \frac{1}{2}) \to (-1,-1)\}$

$\therefore\ v = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \qquad v' = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

With the equations in a), b), c), we can compute:

$\theta = \arccos 0 \Rightarrow \theta = \frac{\pi}{2}$

$s = \frac{\sqrt{2}}{\sqrt{\frac{1}{4}}} = 2$

$t_x = 0 - 2 \times 0 \times \frac{1}{2} + 2 \times 1 \times 0 = 0$

$t_y = 0 - 2 \times 1 \times \frac{1}{2} + 2 \times 0 \times 0 = -1$

$\therefore \begin{bmatrix} x' \\ y' \end{bmatrix} = 2 \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$