Minimum Cost Multi-Commodity Problem

Yuanjun Yao, Keping Wang

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1 Problem Definition

Suppose we have a capacitated undirected graph (V, E, C, A), where c_e is the capacity (or bandwidth) of edge e, and a_e is the cost of unit capacity of edge e. There are pairs of nodes (s_i, t_i) that require exclusive bandwidth of d_i between them. A path is defined as a set of edges and the needed bandwidth of each edge, that connects the pair (s_i, t_i) with required bandwidth b_i. The goal is to find the paths that maximize the

As for the course project, we aim to do the following. First, find an approximation algorithm that solve the problem efficiently, when compared with solving it with a LP solver. Second, what's the complexity of the algorithm. Thirdly, analyze the approximation ratio of the algorithm.

2 Literature

[PST95] introduced a fast approximation algorithm for fractional packing problem, which could be used to solve the multi-commodity problem.

The definition of fractional packing problem is as follows: find $x¶$ such that $Ax < b$, where A is an $m \times N$ matrix, $b > 0$ and P is a convex set in R^n such that $Ax \geq 0$ for each $x \in P$.

For the multi-commodity problem, let P_l be the all possible flows for request pair r_l, then $P = p_1 p_2 \cdots p^r$. Thus, to find a point $x¶$, we can find a flow for each request pair and combine them.

Their approximation algorithm use the procedure improve-packing repeatedly to find a $\epsilon_0 - \mathit{approximate}$ solution. They start from an solution, which could be infeasible, and a relatively large ϵ, then call improve-packing procedure to find a solution x that is $6\epsilon - \mathit{approximate}$ solution or determine that the problem has no exact solution. Then they scales ϵ down, and repeat improve-packing procedure with the previous solution x, until they find a $\epsilon_0 - \mathit{approximate}$ solution or they determine that there is no such solution.

The approximation algorithms is polynomial. And we can use this algorithm to solve minimum cost multi-commodity flow problem. We can add another constraint $cx < \mathit{budget}$ to $Ax < b$. We use binary search to determine the value of buget, then we run the approximation algorithm to find an approximate solution.

3 Linear Programming of the Problem

We could write the LP of the problem in two different ways, one is formulate this as a minimum cost flow problem with multiple bandwidth constraints.

Capacity $u(e)$, unit-cost $c(e)$. Source-sink pairs (s_j, t_j) with non-negative demand d_j, $j = 1, \ldots, K$, specifying the K commodities.

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{e \in E} cost(e) f(e) \\
\text{Subject To} \quad & \sum_{w: wv \in E} f_j(wv) - \sum_{w: vw \in E} f_j(vw) = 0 \quad \text{for each } v \notin \{s_j, t_j\}, j = 1, \dots, K. \quad \text{conservation} \\
& sum_{w: vw \in E} f_j(vv) - \sum_{w: wv \in E} f_j(wv) = d_j \quad \text{for } v = s_j. \quad \text{requirements} \\
& \sum_j f_j(e) \le capacity(e) \quad \text{for each } e \quad \text{capacity} \\
& f_j(e) \ge 0 \quad \text{non-negativity}
\end{aligned}
$$

3.1 Minimum Cost Flow with Multiple Bandwidth Constraints

Capacity $u(e)$, unit-cost $c(e)$. Source-sink pairs (s_j, t_j) with non-negative demand d_j, $j = 1, \dots, K$, specifying the K commodities.

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{e \in E} cost(e) f(e) \\
\text{Subject To} \quad & \sum_{w: wv \in E} f_j(wv) - \sum_{w: vw \in E} f_j(vw) = 0 \quad \text{for each } v \notin \{s_j, t_j\}, j = 1, \dots, K. \quad \text{conservation} \\
& sum_{w: vw \in E} f_j(vv) - \sum_{w: wv \in E} f_j(wv) = d_j \quad \text{for } v = s_j. \quad \text{requirements} \\
& \sum_j f_j(e) \le capacity(e) \quad \text{for each } e \quad \text{capacity} \\
& f_j(e) \ge 0 \quad \text{non-negativity}
\end{aligned}
$$

3.2 Analysis with Total Capacity of Any Cuts that Separate Any Pairs

Motivated by the LP of Steiner tree and Steiner forest problem, we write the LP of the problem requiring the total capacity of the cut no less than the total requirements of pairs that are separated by the cut.

$G(V, E, C, A)$, where C_e is the capacity of edge e, and a_e is the unit cost of edge e.

Prime LP:

Minimize $\sum_{e \in E} a_e x_e$, s.t.

$y_s : \forall (s, \bar{s}), \sum_{e \in (s, \bar{s}) x_e \ge Sum(s, \bar{s})}$, where $Sum(s, \bar{s}) is the sum of required bandwidth by pairs across t the cut.$

$x_e \ge 0$

$\beta_e : x_e \le c_e$

The dual of the problem is:

Maximize $\sum_{s, \bar{s}} y_s Sum(s, \bar{s}) - \sum_{e \in E} c_e \beta_e$ s.t.

$\forall e \in E, \sum_{s: e \in (s, \bar{s})} y_s - \beta_e \le a_e$

$y_s \ge 0$

$\beta_e \ge 0$

We can not directly propose an algorithm that is similar to the 2-approximation algorithm for Steiner forest. However, the interpretation of β_e in dual constraint inspired us with the following algorithm.

4 Our heuristic Solution

We developed an initial algorithm for this problem. The idea is, iteratively run minimum-cost flow algorithm on G with the remaining edge capacities for each unsatisfied pair and take the required capacity from the graph. Once we fail to find a feasible flow for a requirement, or the cost of the feasible flow is much higher than the infeasible flow using saturated links. We will locate the bottleneck edges, and increase the cost bottleneck edges dynamically and remove affected paths from the solution. The algorithms keep on running until finding a feasible solution.

Algorithm 1: HEURISTIC algorithm for minimum cost multi-commodity flow

Input: A graph $G = (V, E, C, A)$, and set of requirements $R = \{(s_i, t_i, b_i)\}$
Output: Set of paths that meet the requirements

$UnMetR \leftarrow R$
$P = \emptyset$
$G_{res} = G$
$G_\beta = G$
while $|P| < |R|$ **do**
  for $r_i \in R$ **do**
    $p_i = minimum\_cost\_flow(G_res, s_i, t_i, b_i)$
    if $p_i == \emptyset$ **then**
      $p'_i = minimum\_cost\_flow(G_{res}, s_i, t_i, b_i)$
      for $(e, c_e) \in p'_i$ **do**
        if $c_e > Available\_Capacity\_in\_G_{res}$ **then**
          Increase cost of e in G_{res} and G_β
          for $p \in P$ **do**
            if $e \in p$ **then**
              $P \leftarrow P - p$
              Return the capacity used by p to G_{res}
            end
          end
        end
      end
    else
      Reduce the capacity of edges in p_i from G_{res} $P \leftarrow P \cup p_i$
    end
  end
end
return P

There are 2 problems to be analyzed. First, will the algorithm terminates if there is a feasible solution, and what's the complexity of the algorithm, how is it when compared to solving the LP directly. Second, can we analyze the approximation ratio of this algorithm?

5 Improve-Packing Algorithm

We then implement the Improve-Packing Algorithm from [GK07], which is theoretically faster than a simple LP.

The general format of a family of problems is the following: given a set of m inequalities on n variables, and an oracle that produces the solution of an appropriate optimization problem over a convex set $P \in \mathcal{R}^n$, find a solution $x \in P$ that satisfies the inequalities, or detect that no such x exists. The basic idea is to start from an infeasible solution x, and use the optimization oracle to find a direction in which the violation of the inequalities can be decreased; this is done by calculating a vector y that is a dual solution corresponding to x. Then, x is carefully updated towards that direction, and the process is repeated until x becomes "approximately" feasible. For different kinds of problems, the optimization oracles and the notions of "approximation" are different.

6 Evaluation

In this section, we will show the evaluation of 3 kinds solutions to the minimum cost multi-commodity problem, the fast approximation problem, our heuristic algorithm, and LP solution of the problem.

6.1 Graph Generation

We randomly generate graphs and requirements. We specify the number of vertices and the number of requirement pairs. For each requirement pair, we randomly add a path of random length with average length of 6 to the graph,

6.2 Performance Comparison

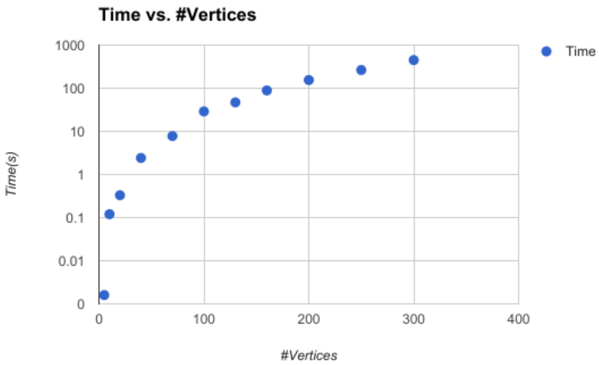
we generate different graphs and requirements and solve the problems using three methods, and compare their performance, in forms of running time and the cost of the solution.

Problem Size			Heuristic		Fast Approx		LP	
#Vertices	#Edges	#Requests	Time(s)	Cost	Time(s)	Cost	Time(s)	Cost
5	8	3	0.0016	3651	4.79	3561	0.00	2637
10	34	8	0.12	6253	66	5849	0.00	4744
20	97	16	0.33	19396			0.01	12103
40	284	39	2.42	36773			0.15	22911
70	483	62	7.79	74228			0.46	38311
100	865	100	29	104671			2.32	45633
130	1100	127	47	158250			4.75	70895
160	1392	58	89	175745			9.78	78136
200	1681	198	155	248484			15.62	97311
250	2295	247	265	278546				
300	2571	298	450	393k				

As shown in the table, the approximation algorithm based on improve-packing subroutine is slow. The reason is that, it uses minimum-cost flow as subroutine (There is alternative way of using shortest-path algorithm as subroutine. We haven't figured the decomposition techniques to decrease the ρ). And we need to run improve-packing algorithm for many times.

The heuristic algorithm also use minimum-cost flow as subroutine. It is relatively fast, but the cost of the solution is larger than the exact solution of LP, since we added penalty factor to congested edges and this may lead to some pair of nodes using longer path.

We draw the graph of running time vs. number of vertices. Note that the running time is in log scale. This curve shows that as far as in our experiments, the running time of the heuristic algorithm is not exponential.



Discussions

For multi-path multi-commodity flow problem, solving the linear programming with a solver is fast enough for graphs with hundreds of vertices and edges. If only one path is allowed, it would request more integer constraints in linear programming and the performance is questionable. Our heuristic algorithm uses min-cost flow as subroutine to find a feasible solution for a pair of request, we could directly use shortest path instead to meet the single-path requirement. However, this won't work for the fast approximation algorithm for multi-commodity flow problem, since in this algorithm, the solution x is a linear combination of multiple solutions.

In our implementation heuristic algorithm, the amount penalty factor increment for a congested edge is a fixed small number. This makes the updates slow, but the cost smaller. We may be able to dynamically adjust the amount for better timing and cost balance.

References

[GK07] Naveen Garg and Jochen Koenemann. Faster and simpler algorithms for multicommodity flow and other fractional packing problems. *SIAM Journal on Computing*, 37(2):630–652, 2007.

[PST95] Serge A Plotkin, David B Shmoys, and Éva Tardos. Fast approximation algorithms for fractional packing and covering problems. *Mathematics of Operations Research*, 20(2):257–301, 1995.