Wronskian Eq. (반복이다)

- **Thm 1.**

y_1, y_2 are the solutions of a homo. linear. 2nd-order ODE on I.

If $W[y_1, y_2] = y_1 y_2' - y_1' y_2 \neq 0$ for $\forall x_0 \in I$

$\rightarrow$ $y = c_1 y_1 + c_2 y_2$ is the general solution (y_1 and y_2 are linear independent)

$\Rightarrow$ y^*: particular solution

$y^* = c_1^* y_1 + c_2^* y_2$

pf

$x_0 \in I$, $\left.\begin{aligned} y^*(x_0) &= c_1 y_1(x_0) + c_2 y_2(x_0) \\ y^{*\prime}(x_0) &= c_1 y_1'(x_0) + c_2 y_2'(x_0) \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= (y(x_0) y_2'(x_0) - y'(x_0) y_2(x_0)) / W[y_1, y_2](x_0) \\ c &= (y(x_0) y_1'(x_0) - y'(x_0) y_1(x_0)) / W[y_1, y_2](x_0) \end{aligned}$

$y^*(x_0) = k_0$, $y^{*\prime}(x_0) = k_1$ $\rightarrow$ $y = c_1 y_1 + c_2 y_2$, $y(x_0) = k_0$, $y'(x_0) = k_1$

$\Rightarrow y^* = y$ on I

- **Thm 2.**

If there is $\exists x_0 \in I$ such that $W[y_1, y_2](x_0) = 0$

$\Rightarrow$ $y \neq c_1 y_1 + c_2 y_2$, y_1, y_2 are linearly dependent

pf

$W[y_1, y_2](x_0) = y_1(x_0) y_2'(x_0) - y_1'(x_0) y_2(x_0) = 0$

$\Rightarrow$ $y_1(x_0) = c y_2(x_0)$, $y = y_1 - c y_2$

$y^* \equiv 0$ / $y \leftarrow$ general unique. $\rightarrow$ $y_1 = c y_2$: linearly dependent

- Fact : $W[y_1, y_2]$ is either identically zero or never zero

proc Lemma W is Wronskian, and a solution of $w' + p(x)w = 0$.

$W = y_1 y_2' - y_1' y_2$, $W' = y_1' y_2' + y_1 y_2'' - y_1'' y_2 - y_1' y_2'$

$= y_1 y_2'' - y_1'' y_2$

$y_1 y_2'' - y_1'' y_2 + p(y_1 y_2' - y_1' y_2) = 0$.

$= y_1 (y_2'' + p y_2') - y_2 (y_1'' + p y_1')$

$= y_1 \cdot (-q y_2) - y_2 (-q y_1) = 0$ ⌟

$\frac{1}{w} w' = -p$

$\ln|w| = -\int p\, ds + \ln|w(x_0)|$ ⌟ Lemma

$W = \underbrace{w(x_0) e^{-\int p\, ds}}_{\text{not zero}}$

$w = 0 \iff w(x_0) = 0$ $\quad$ $w \neq 0 \iff w(x_0) \neq 0$

Show that $y(x) = x^2$ can't be a solution of ODE

if p, q are conti at $x=0$

assume that $y = x^2$ is a solution, problem → x^2 is not a solution!!

$\left.\begin{matrix} x^2 \\ g(x) \end{matrix}\right]$: linearly independent. → $y = c_1 x^2 + c_2 g(x)$.

$W[x^2, g(x)] = g'(x)x^2 - 2x g(x) = x(g'(x)x - 2g(x))$, $W = 0$ at $x = 0$.

Non-homogeneous ODE

- $y'' + p(x)y' + q(x)y = r(x)$ — ① $r(x) \neq 0$

  $y'' + p(x)y' + q(x)y = 0$ — ②

- General solution

  : ②'s general sol. + ①'s particular sol. ⇒ $y = y_h + y_p$

  ↳ y_h ↳ y_p $(y = c_1y_1 + c_2y_2 + y_p)$

Thm.1

$\left\{\begin{matrix} y_h + y_p : \text{sol. of ① on } I \\ y_1, y_2 : \text{sol. of ① on } I \end{matrix}\right.$ ⇒ $y_1 - y_2$: sol. of ② on I.

pf

note.) $L[y] = y'' + p(x)y' + q(x)y$

$L[y_h + y_p] = r(x)$ on I → $L[y_h] + L[y_p] = 0 + r(x)$

$L[y_1 - y_2] = L[y_1] - L[y_2] = r(x) - r(x) = 0$ ⇒

Thm 2.

General solution of non-homo ODE includes all solutions

⇒ $y = y_h + y_p = c_1y_1 + c_2y_2 + y_p$ → $y^* = c_1^*y_1 + c_2^*y_2 + y_p$ on I (particular solution)

pf

y^* : sol. of ① on I, $\hat{y} = y^* - (ay_1 + by_2 + y_p)$

$\hat{y}$: sol. of ② on I,

$x_0 \in I$, $\hat{y}(x_0) = k_0$, $\hat{y} = \hat{c}_1y_1 + \hat{c}_2y_2$ on I → $y^* = (\hat{c}_1 + a)y_1 + (\hat{c}_2 + b)y_2 + y_p$

$\hat{y}'(x_0) = k_1$

How to find y_p

- $y'' + ay' + by = r(x)$ (a, b : constants)

Ex) $y'' + y = 0.001x^2$

① $y_p = \alpha x^2$ $\quad y'' = 2\alpha$ → 안됨

② $y = \alpha_2 x^2 + \alpha_1 x + \alpha_0$ → $2\alpha_2 + \alpha_2 x^2 + \alpha_1 x + \alpha_0$.

$\alpha_2 = 0.001 \quad \alpha_1 = 0 \quad \alpha_0 = -0.002$.

↳ 운이 좋다

Ex) $y'' + 3y' + 2.25y = -10e^{-1.5x}$

$y_h = (C_1 + C_2 x)e^{-1.5x}$

$y_p = \underline{x}e^{1.5x}$ (x) $\quad y_p = x^2 e^{-1.5x}$

우차성 곱하기

- ke^{rx} : $ce^{rx}, xe^{rx}, x^2e^{rx} \cdots$

  kx^n : $a_n x^n \cdots + a_0$

  $\left.\begin{array}{l} k\cos wx \\ k\sin wx \end{array}\right\}$ → $a\cos wx + b\sin bx$

$ke^{rx}\cos wx$ - $e^{rx}(a\cos wx + b\sin wx)$

$ke^{rx} + kx^n = ce^{rx} + \sum_{n=0}^{n} a_{pm}$.

+) 망함 : x 곱하기

ex) $L[y] = y'' + ay' + by$, $L[y_{p_i}] = r_i$ $\quad L[y_{pm}] = r_m$, $\quad L\left[\sum_{i=1}^{m} y_{p_i}\right] = \sum_{i=1}^{m} r_m$

ex) $y'' + 2y' + 0.75y = 2\cos x - 0.25\sin x + 0.09x$.

$y_p = a\cos x - b\sin x + \alpha x + \beta$

$y_p' = -a\sin x - b\cos x + \alpha$

Variation of Parameters

- $y'' + py' + qy = r(x)$, $y_p = ?$, $y_h = c_1 y_1 + c_2 y_2$

  idea

  assume that $y_p = \underline{u(x)} y_1 + \underset{\text{find}}{\underline{v(x)}} y_2$

  $y_p' = (\underbrace{u'y_1 + v'y_2}_{=0\,?}) + uy_1' + vy_2'$ ☆ $u'y_1 + v'y_2 = 0$

  =0 ? (u' and v' is variable) 안되면 망했다

  $y_p'' = u'y_1' + v'y_2' + uy_1'' + vy_2''$

  $$\left.\begin{aligned} &\rightarrow u(\underbrace{y_1'' + py_1' + qy_1}_{=0}) + v(\underbrace{y_2'' + py_2 + qy_2}_{=0}) + u'y_1' + v'y_2' = r(x) \\ &u'y_1 + v'y_2 = 0 \end{aligned}\right] \rightarrow \text{find } u(x), v(x)$$

  $$\rightarrow v' = \frac{ry_1}{\underbrace{y_1y_2' - y_1'y_2}_{W[y_1, y_2]}} , \quad u' = \frac{ry_2}{y_1'y_2 - y_1y_2'}$$

ex) $y'' + y = \tan x$ $\qquad y_h = c_1 \cos x + c_2 \sin x$

$W[y_1, y_2] = \cos x \cdot \cos x + \sin x \cdot \sin x = 1$

$u' = \tan x \sin x \rightarrow u = \sin x - \ln|\sec x + \tan x|$

$v = \tan x \cos x \rightarrow v = -\cos x$

ex) $y'' + 4y' + 3y = 65 \cos 2x$

$-1, -3,$ $\quad y_p = a \cos 2x + b \sin 2x$ $\qquad$ a,b 연립 / $y_h = c_1 e^{-3x} + c_2 e^{-x} \ldots$

$y_p' = 2a \cos 2x - 2b \sin 2x$ $\qquad a = -1 , b = 8$, $y_p = -\cos 2x + 8 \sin 2x$.

$y_p'' = -4a \cos 2x - 4a \sin 2x$ $\qquad W = -e^{-4x} + 3e^{-4x} = 2e^{-4x}$

$\underline{ry_2}$ $\quad \int \frac{65 \cos 2x \cdot e^{-x}}{2e^{-4x}} \ldots$

이께까지 쓰고 되긴한데

$[uv] - \int uv'$ 쓰면 그냥 풀려요...