Note 2. Amplitude Modulation

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Amplitude Modulation

Analog transmission

- Analog signals
  - Examples : Speech, music, images, and video signals
  - Modulated and transmitted directly
  - Converted into digital data & transmitted using digital-modulation techniques
  - Characterized by its bandwidth, dynamic range, and the nature of the signal
- Speech signals : Bandwidth of up to 4 kHz
- Audio and black-and-white video
  - Just one component, which measures air pressure or light intensity
- Music signal : Bandwidth of 20 kHz
- Color video
  - Four components : RGB color (3) components + intensity (1) component
  - Four video signals + audio signal (audio info.) in Color-TV broadcasting
  - Higher bandwidth, about 6 MHz

Amplitude Modulation

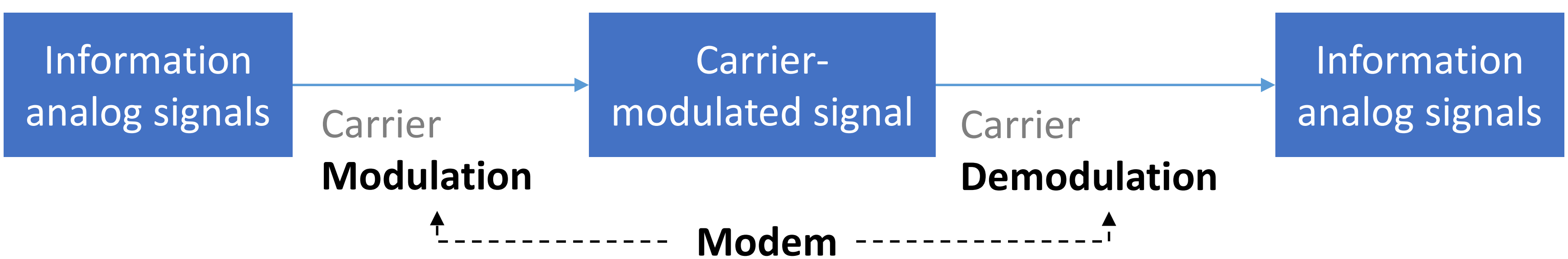
Transmission of analog signals by *carrier modulation*

- Sinusoidal carrier wave

$$c(t) = A_c \cos(2\pi f_c t + \theta)$$

  - Some characteristic of a carrier is changed according to variation in a message signal
- ***Amplitude* : *AM*** – The message signals change the amplitude
- ***Phase* : *PM*** – Phase of the carrier
- ***Frequency*** : ***FM*** – Frequency of the carrier

Described *Methods for demodulation* of the carrier-modulated signal to recover the analog information signal



Performance of these systems in the presence of noise : Later

Introduction to Modulation

$m(t)$

- Analog signal to be transmitted
- Baseband (lowpass) signal
- Bandwidth = W
  - $M(f) = 0$, for $|f| > W$
- Modulating signal

$$u(t) = m(t)c(t) = A_c m(t)\cos(2\pi f_c t)$$

- Passband signal
- Modulated signal
- Transmitted signal

$$c(t) = A_c \cos(2\pi f_c t + \phi_c)$$

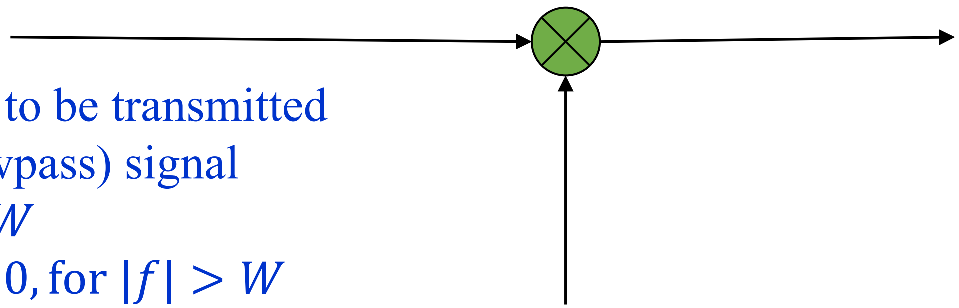
- Carrier signal
- Sinusoidal carrier wave
- A : Carrier amplitude
- f_c : Carrier frequency
- ϕ_c : Carrier phase = 0

- Transmit the message signal $m(t)$ through the communication channel
- Impressing the message signal on a carrier signal $c(t)$: Modulation
- The message signal $m(t)$ modulates the carrier signal $c(t)$

  ⇨ Amplitude, frequency, or phase of the signal : Functions of the message signal
- Modulation converts the message signal $m(t)$ from lowpass to bandpass, in the neighborhood of the carrier frequency f_c

Introduction to Modulation

$m(t)$



- Analog signal to be transmitted
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$u(t) = m(t)c(t)$

$= A_c m(t)\cos(2\pi f_c t)$

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$c(t) = A_c \cos(2\pi f_c t + \phi_c)$

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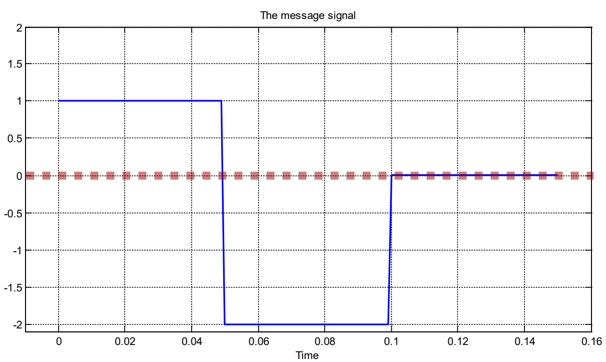
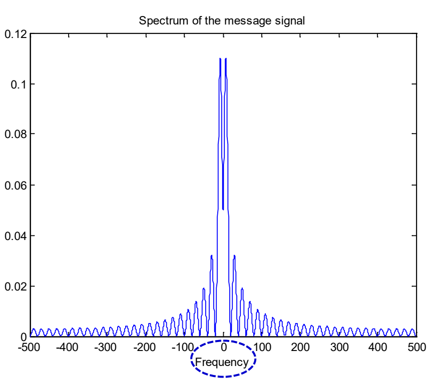
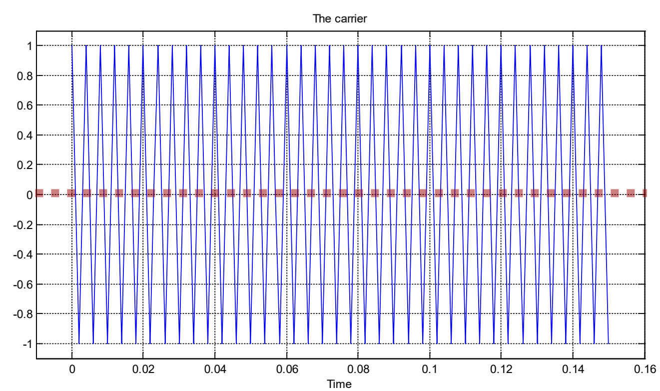
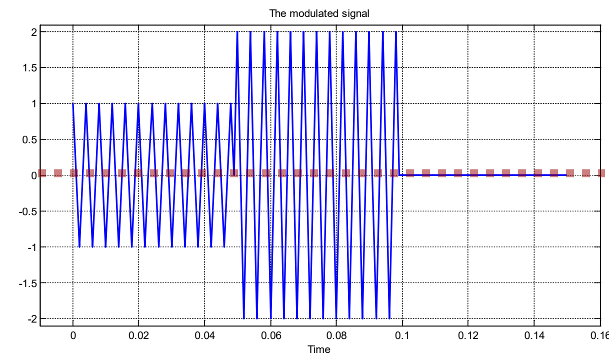
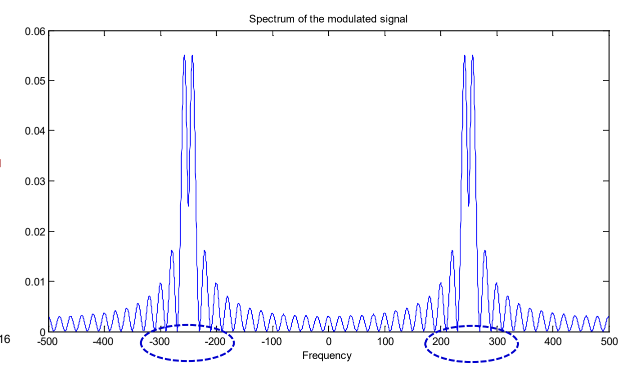




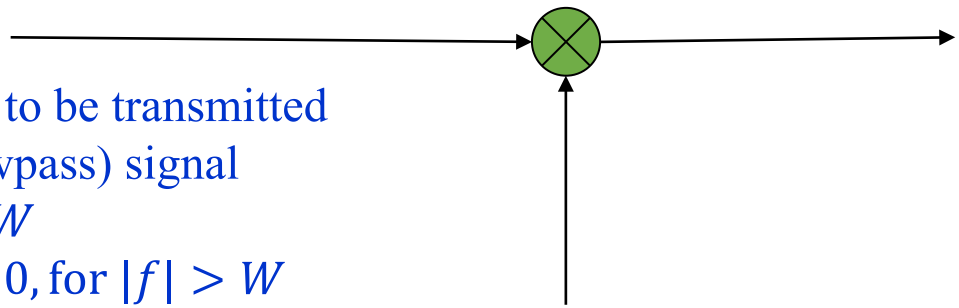


$c(t) = A_c \cos(2\pi 250t + 0)$





Introduction to Modulation



$m(t)$

- Analog signal to be transmitted
- Baseband (lowpass) signal
- Bandwidth = W

  : $M(f) = 0$, for $|f| > W$
- Modulating signal

$$u(t) = m(t)c(t)$$
$$= A_c m(t)\cos(2\pi f_c t)$$

- Passband signal
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$$c(t) = A_c \cos(2\pi f_c t + \phi_c)$$

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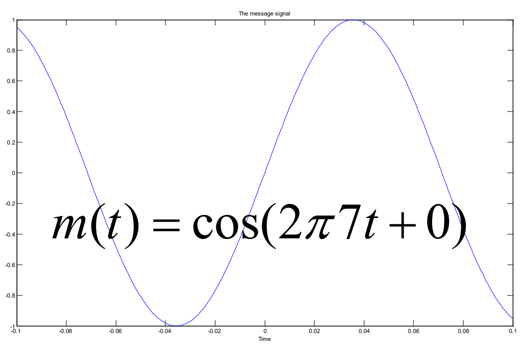
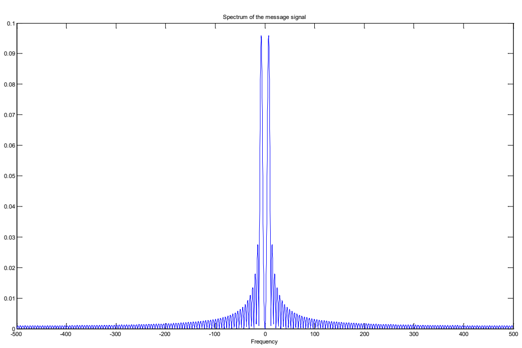
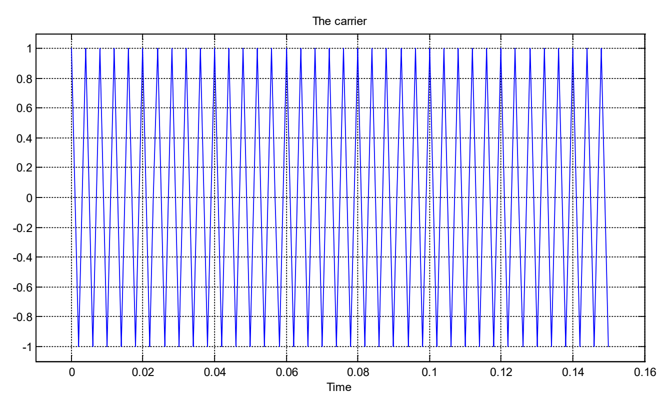
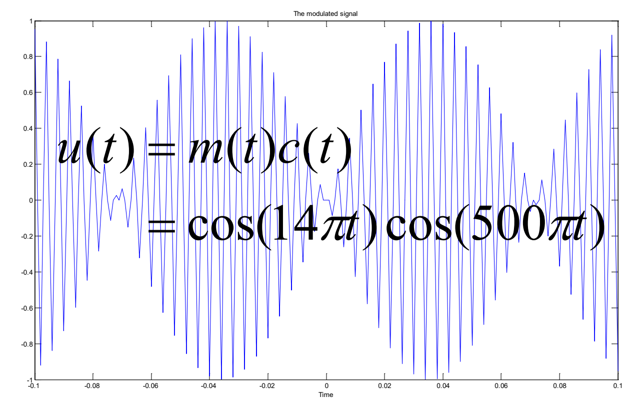


$$m(t) = \cos(2\pi 7t + 0)$$

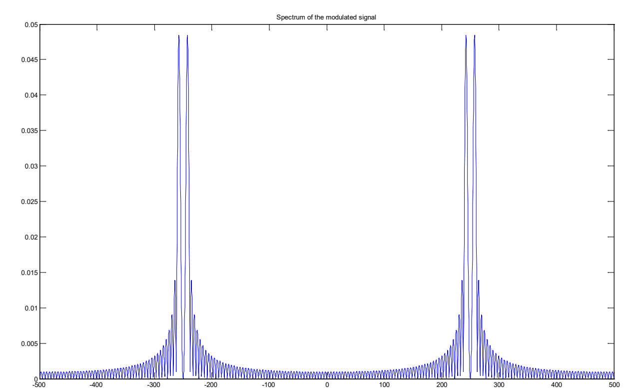




$$c(t) = \cos(2\pi 250t + 0)$$

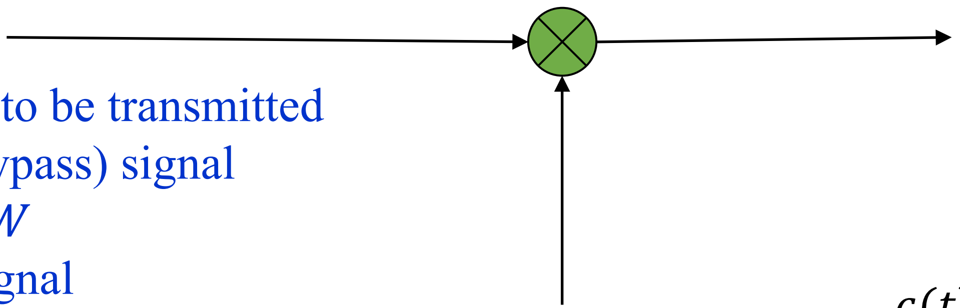


$$u(t) = m(t)c(t)$$
$$= \cos(14\pi t)\cos(500\pi t)$$



Introduction to Modulation

$m(t)$



$$u(t) = m(t)c(t)$$
$$= A_c m(t)\cos(2\pi f_c t)$$

- Analog signal to be transmitted
- Baseband (lowpass) signal
- Bandwidth = W
- Modulating signal

- Passband signal
- Modulated signal
- Transmitted signal

$$c(t) = A_c \cos(2\pi f_c t + \phi_c)$$

- Carrier signal
- Sinusoidal carrier wave
- A : Carrier amplitude
- f_c : Carrier frequency
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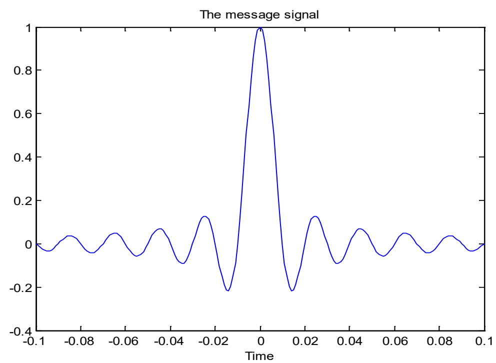
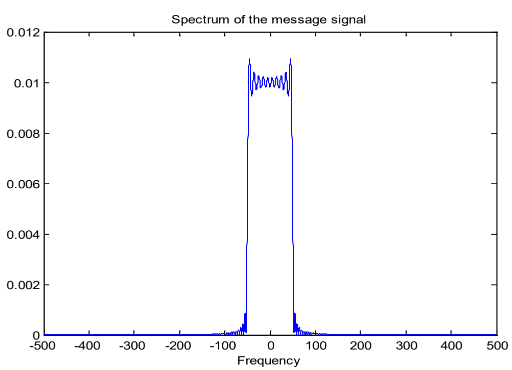
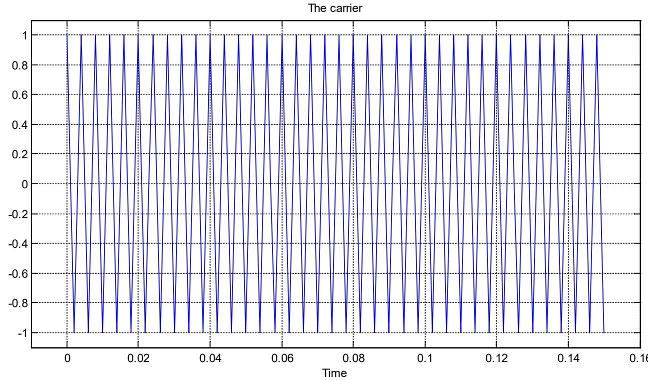
$$m(t) = \mathrm{sinc}(100t)$$

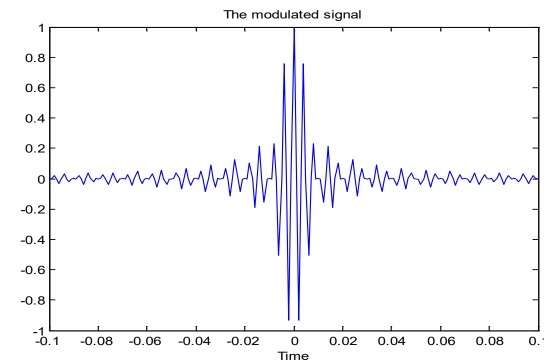


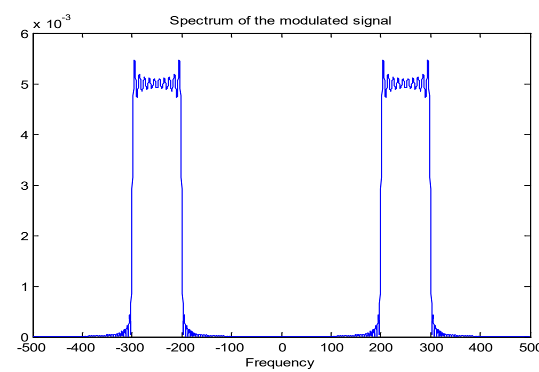


$$c(t) = \cos(2\pi 250t)$$



$$c(t) = m(t)c(t)$$
$$= \sin c\,(100t)\cos(500\pi t)$$

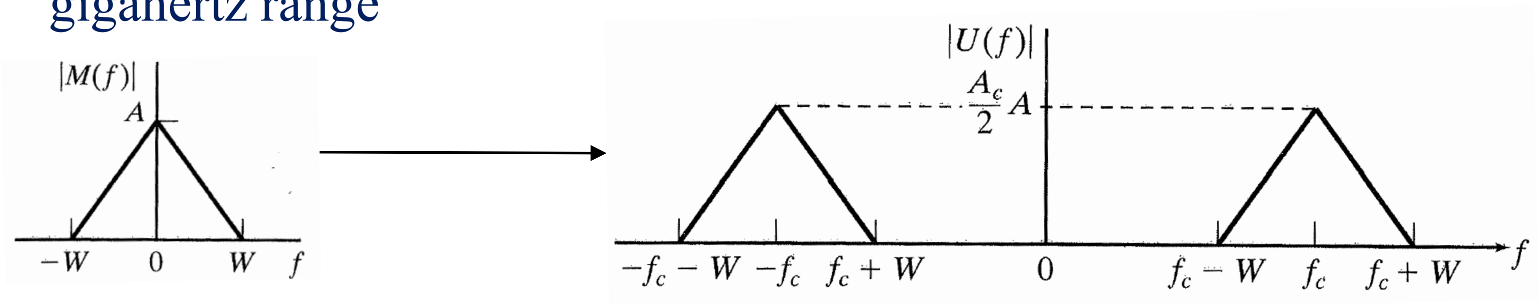




Objectives of Modulation

1. Match the passband characteristics of the channel

- Translate the frequency of the lowpass signal to the passband of the channel
  - ⇒ Spectrum of the transmitted bandpass signal : Match the passband characteristics of the channel
  - ⇒ Translate the speech signal from the low-freq. range (up to 4 kHz) to the gigahertz range



2. Simplify the structure of the transmitter

- Simplify the structure of the transmitter by employing higher frequencies
  - Tx of the signal : Low freq. – Huge, Higher freq. – Smaller antennas
  - Antenna size ∝ Wave length :
    - Baseband speech signal 4KHz, Passband signal 1GHz :

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{4\times10^3} = \frac{3}{4}\times10^5 m \quad \rightarrow \quad \lambda = \frac{c}{f} = \frac{3\times10^8}{10^9} = 0.3m = 30cm$$

Objectives of Modulation

1. Match the passband characteristics of the channel

2. Simplify the structure of the transmitter

3. Frequency-division multiplexing

- To accommodate for the simultaneous transmission of signals from several message sources, by means of frequency-division multiplexing.

4. Noise and interference immunity

- To expand the bandwidth of the transmitted signal in order to increase its noise and interference immunity in transmission over a noisy channel, as we will see in our discussion of angle-modulation

- **Objectives (1), (2), and (3) are met by all of the modulation methods described in this note – Amplitude, phase, frequency modulation**
- **Objective (4) is met by employing angle modulation to spread the signal $m(t)$ over a larger bandwidth – Phase/Frequency modulation**

Fourier Transform Pairs

Time Function	Fourier Transform
$\text{rect}\left(\frac{t}{T}\right)$	$T\,\text{sinc}\,(fT)$
$\text{sinc}\,(2Wt)$	$\frac{1}{2W}\text{rect}\left(\frac{f}{2W}\right)$
$\exp(-at)u(t), \quad a > 0$	$\frac{1}{a + j2\pi f}$
$\exp(-a\lvert t\rvert), \quad a > 0$	$\frac{2a}{a^2 + (2\pi f)^2}$
$\exp(-\pi t^2)$	$\exp(-\pi f^2)$
$\begin{cases} 1 - \frac{\lvert t\rvert}{T}, & \lvert t\rvert < T \\ 0, & \lvert t\rvert \geq T \end{cases}$	$T\,\text{sinc}^2(fT)$
$\delta(t)$	1
1	$\delta(f)$

Time Function	Fourier Transform
$\delta(t - t_0)$	$\exp(-j2\pi f t_0)$
$\exp(j2\pi f_c t)$	$\delta(f - f_c)$
$\cos(2\pi f_c t)$	$\frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$
$\sin(2\pi f_c t)$	$\frac{1}{2j}[\delta(f - f_c) - \delta(f + f_c)]$
$\text{sgn}(t)$	$\frac{1}{j\pi f}$
$\frac{1}{\pi t}$	$-j\,\text{sgn}(f)$
$u(t)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
$\sum_{i=-\infty}^{\infty} \delta(t - iT_0)$	$\frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_0}\right)$

Fourier Transform Properties

Property	*Mathematical Description*
1. Linearity	$ag_1(t) + bg_2(t) \rightleftharpoons aG_1(f) + bG_2(f)$ where a and b are constants
2. Dilation (time scaling)	$g(at) \rightleftharpoons \dfrac{1}{\lvert a \rvert} G\left(\dfrac{f}{a}\right)$ where a is a constant
3. Duality	If $g(t) \rightleftharpoons G(f)$, then $G(t) \rightleftharpoons g(-f)$
4. Time shifting	$g(t - t_0) \rightleftharpoons G(f) \exp(-j2\pi f t_0)$
5. Frequency shifting	$\exp(j2\pi f_c t) g(t) \rightleftharpoons G(f - f_c)$
6. Area under $g(t)$	$\int_{-\infty}^{\infty} g(t)\, dt = G(0)$
7. Area under $G(f)$	$g(0) = \int_{-\infty}^{\infty} G(f)\, df$
8. Differentiation in the time domain	$\dfrac{d}{dt} g(t) \rightleftharpoons j2\pi f G(f)$
9. Integration in the time domain	$\int_{-\infty}^{t} g(\tau)\, d\tau \rightleftharpoons \dfrac{1}{j2\pi f} G(f) + \dfrac{G(0)}{2} \delta(f)$
10. Conjugate functions	If $g(t) \rightleftharpoons G(f)$, then $g^*(t) \rightleftharpoons G^*(-f)$,
11. Multiplication in the time domain	$g_1(t) g_2(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda) G_2(f - \lambda)\, d\lambda$
12. Convolution in the time domain	$\int_{-\infty}^{\infty} g_1(\tau) g_2(t - \tau)\, d\tau \rightleftharpoons G_1(f) G_2(f)$
13. Correlation theorem	$\int_{-\infty}^{\infty} g_1(t) g_2^*(t - \tau)\, dt \rightleftharpoons G_1(f) G_2^*(f)$
14. Rayleigh's energy theorem	$\int_{-\infty}^{\infty} \lvert g(t) \rvert^2\, dt = \int_{-\infty}^{\infty} \lvert G(f) \rvert^2\, df$

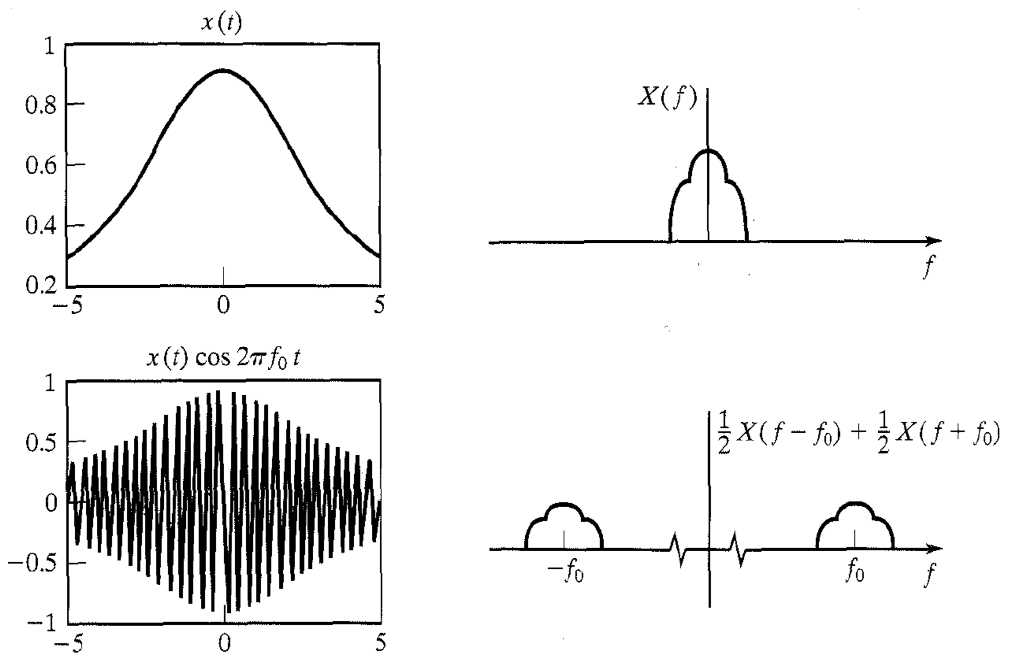
Example

- **Determine the Fourier transform of the signal $x(t)\cos(2\pi f_0 t)$**

- **Solution**

$$F[x(t)\cos(2\pi f_0 t)] = F\left[\frac{1}{2}x(t)e^{j2\pi f_0 t} + \frac{1}{2}x(t)e^{-j2\pi f_0 t}\right] = \frac{1}{2}X(f-f_0) + \frac{1}{2}X(f+f_0)$$

– This relation is the basis of the operation of amplitude modulation systems



Effect of modulation in both the time and frequency domain

AMPLITUDE MODULATION (AM)

Amplitude modulation

- Message signal $m(t)$: Impressed on the amplitude of the carrier signal

$$c(t) = A_c cos(2\pi f_c t + \phi_c)$$

  - Sinusoidal signal: Its amplitude = A function of the message signal $m(t)$

- Several different ways of amplitude modulating the carrier signal by $m(t)$
  - Each results in different spectral characteristics for the transmitted signal

    (a) Double sideband, suppressed-carrier AM (***DSB-SC AM***)

    (b) Conventional double-sideband ***AM*** (***DSB-LC AM***)

    (c) Single-sideband AM (***SSB AM***)

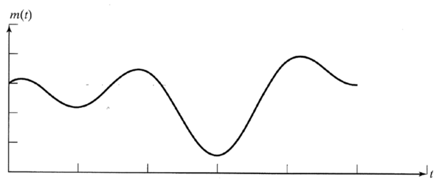
    (d) Vestigial-sideband AM (***VSB AM***)

Double-Sideband Suppressed-Carrier AM

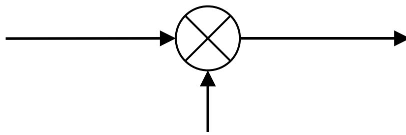
■ *Double-sideband, suppressed-carrier (DSB-SC) AM* signal

- Multiplying the message signal $m(t)$ with the carrier signal $c(t) = A_c \cos(2\pi f_c t)$

  - lowpass signal
  - Bnadwidth = W
  - Modulating signal



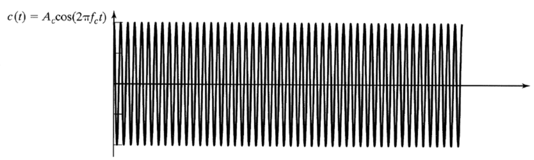
$m(t)$



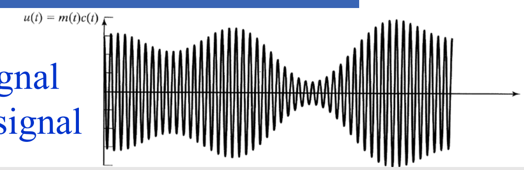
$$u(t) = m(t)c(t) = A_c m(t)\cos(2\pi f_c t)$$

$c(t) = A_c \cos(2\pi f_c t)$

- Sinusoidal carrier wave



- Passband signal
- Modulated signal



- Not contain a carrier component
- All the transmitted power contained in the modulating signal $m(t)$: Spectrum of $U(f)$
- $m(t)$: Not have any DC component

⇨ No impulse in $U(f)$ at $f = f_c$

⇨ $u(t)$: Called a ***suppressed-carrier*** signal

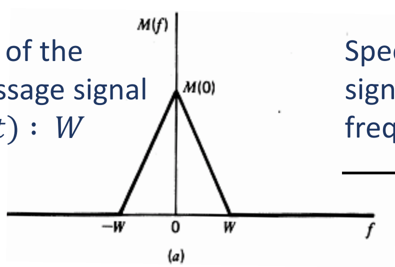
⇨ $u(t)$: ***DSB-SC AM*** signal

- Spectrum of the modulated signal : FT of $u(t)$

$$u(t) = A_c m(t)\cos(2\pi f_c t)$$

$$\longrightarrow \quad U(f) = \frac{A_c}{2}[M(f - f_c) + M(f + f_c)]$$

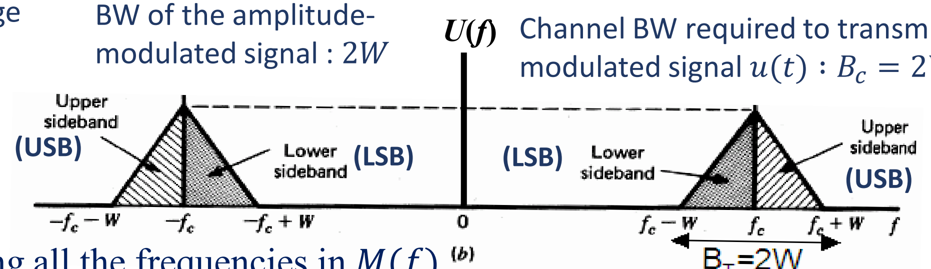
BW of the message signal $m(t)$: W



Spectrum of the message signal $m(t)$: Shifted in freq. by an amount f_c

BW of the amplitude-modulated signal : $2W$

Channel BW required to transmit the modulated signal $u(t)$: $B_c = 2W$



- USB (or LSB) of $U(f)$ containing all the frequencies in $M(f)$

Example 3.2.1

- **Suppose that the modulating signal *m*(*t*) is a sinusoidal of the form**

$$m(t) = a\cos(2\pi f_m t) \quad f_m << f_c$$

- **Determine the DSB-SC AM signal and its upper and lower sidebands.**

- ***Solution***

$F[\cos(2\pi f_c t] = \frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$

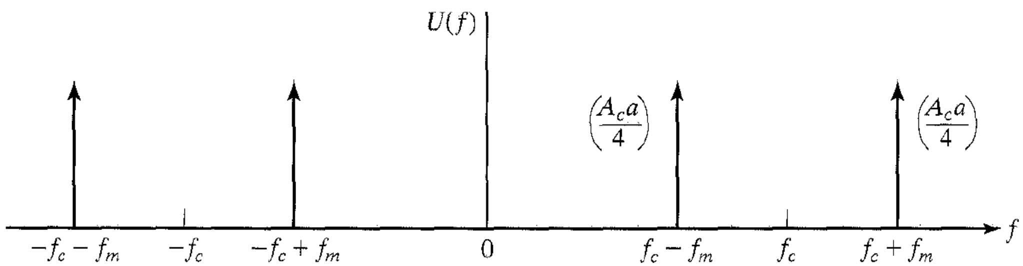
  - The DSB-SC AM is expressed in the time domain as

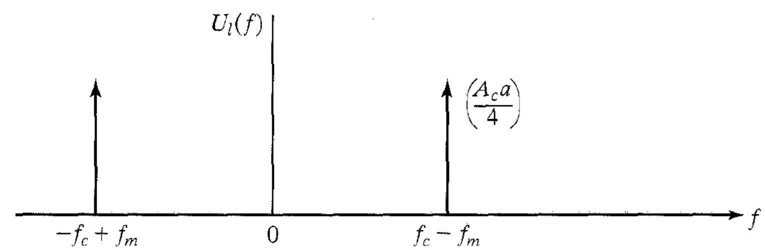
$$u(t) = m(t)c(t) = A_c a\cos(2\pi f_m t)\cos(2\pi f_c t)$$

$$= \frac{A_c a}{2}\cos[2\pi(f_c - f_m)t] + \frac{A_c a}{2}\cos[2\pi(f_c + f_m)t]$$

  - Taking the FT, the modulated signal in the frequency domain will have the following form:

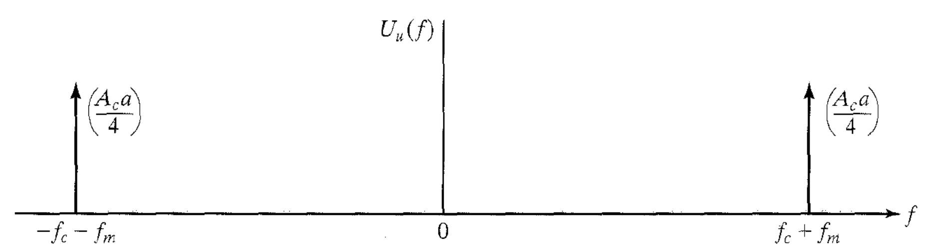
$$U(f) = \frac{A_c a}{4}[\delta(f - f_c + f_m) + \delta(f + f_c - f_m)] + \frac{A_c a}{4}[\delta(f - f_c - f_m) + \delta(f + f_c + f_m)]$$





  - LSB of $u(t)$: $u_l(t) = \frac{A_c a}{2}\cos[2\pi(f_c - f_m)t]$

  - USB of $u(t)$: $u_u(t) = \frac{A_c a}{2}\cos[2\pi(f_c + f_m)t]$



Power Content of DSB-SC Signals

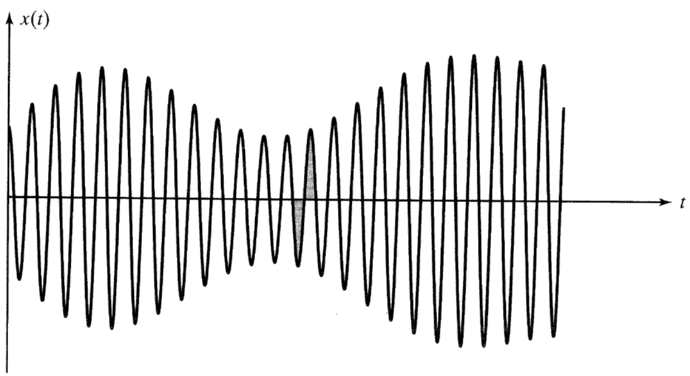
- **Power content of the DSB-SC signal :** $u(t) = A_c m(t)\cos(2\pi f_c t)$

$$P_u = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} u^2(t)dt = \lim_{T\to\infty}\int_{-T/2}^{T/2} A_c^2 m^2(t)\cos^2(2\pi f_c t)dt$$

$$= \frac{A_c^2}{2}\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} m^2(t)[1+\cos(4\pi f_c t)]dt$$

$$= \frac{A_c^2}{2}\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} m^2(t)dt + \frac{A_c^2}{2}\lim_{T\to\infty}\underbrace{\frac{1}{T}\int_{-T/2}^{T/2} m^2(t)\cos(4\pi f_c t)dt}_{0} = \frac{A_c^2}{2}P_m$$

- $m^2(t)$: Slowly varying signal, $\cos(4\pi f_c t)$: High frequency sinusoid
- Multiplying result : High-frequency sinusoid with a slowly varying envelope
- Envelope : Slowly varying
  - ⇨ Positive and Negative halves of each cycle
  - ⇨ Almost the same amplitude
  - ⇨ When they are integrated, they cancel each other
  - ⇨ Overall integral of $m^2(t)\cos(4\pi f_c t)$: Almost zero
  - ⇨ Divide the result of the integral by T (= Very large)
  - ⇨ Second term in Equation : Zero



Plot of $m^2(t)\cos(4\pi f_c t)$

- Power in the message signal $m(t)$: $P_m = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} m^2(t)dt$

Example 3.2.3

- Determine the power in the modulated signal and the power in each of the sidebands in Example 3.2.1.

- **Solution**

$$m(t) = a\cos(2\pi f_m t) \quad f_m << f_c \quad \Rightarrow \quad P_m = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |m(t)|^2 dt$$

$$= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} a^2\cos^2(2\pi f_m t)dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\frac{a^2}{2}[1+\cos(4\pi f_m t)]dt$$

$$= \lim_{T\to\infty}\left[\frac{a^2 T}{2T} + \left[\frac{a^2}{8\pi f_m t}\cos(4\pi f_0 t)\right]_{-T/2}^{T/2}\right] = \frac{a^2}{2}$$

$$u(t) = \frac{A_c a}{2}\cos[2\pi(f_c - f_m)t] + \frac{A_c a}{2}\cos[2\pi(f_c + f_m)t] \Rightarrow P_u = \frac{A_c^2}{2}P_m = \frac{A_c^2 a^2}{4}$$

- Due to the symmetry of the sidebands, the power in the upper and lower sidebands, P_{USB} and P_{LSB}, are equal and given by

$$u_l(t) = \frac{A_c a}{2}\cos[2\pi(f_c - f_m)t]$$

$$u_u(t) = \frac{A_c a}{2}\cos[2\pi(f_c + f_m)t]$$

$$\longrightarrow P_{ls} = P_{us} = \frac{A_c^2 a^2}{8}$$

Demodulation of DSB-SC AM Signals

- Transmit the DSB-SC AM signal $u(t)$ through an ideal channel (No channel distortion & no noise)
- Received signal = Modulated signal : $r(t) = u(t) = m(t)c(t) = A_c m(t)\cos(2\pi f_c t)$

How to demodulate the received signal

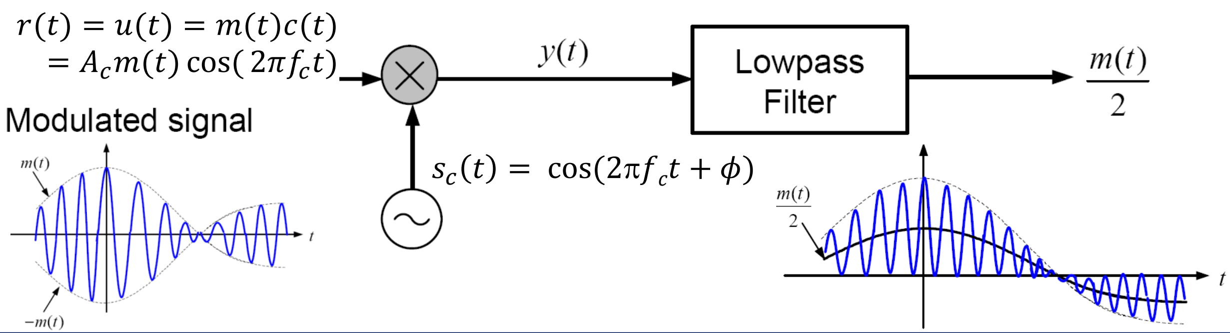
1. Multiplying $r(t)$ by a locally generated $\cos(2\pi f_c \pi f_c t + \phi)$: $r(t)\cos(2\pi f_c t + \phi) = A_c m(t)\cos(2\pi f_c t)\cos(2\pi f_c t + \phi) = \frac{1}{2}A_c m(t)\cos(\phi) + \frac{1}{2}A_c m(t)\cos(4\pi f_c t + \phi)$
   - Frequency content of the message signal $m(t)$: Limited to W Hz, $W \ll f_c$
2. Pass the product signal through an ***ideal lowpass filter*** with the BW W
   - Eliminate the signal components centered at frequency $2f_c$
   - Pass the signal components centered at frequency $f = 0$ without distortion

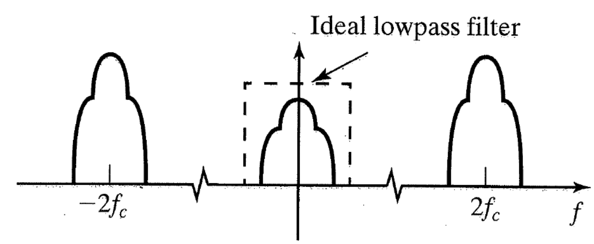
⇨ Output of the ideal lowpass filter : $y_l(t) = \frac{1}{2}A_c m(t)\cos(\phi)$ → Reduction of the power by a factor of $\cos^2\phi$

⇨ Scale the desired signal by the phase ϕ

⇨ $\phi \neq 0$: Amplitude reduced by the factor $\cos(\phi)$ → $\phi = 45°$: Amplitude by $2^{1/2}$ & power reduced by a factor of two → $\phi = 90°$: Desired signal component vanishes

⇨ Need for a phase-coherent or synchronous demodulator





Frequency-domain representation of the DSB-SC AM demodulation

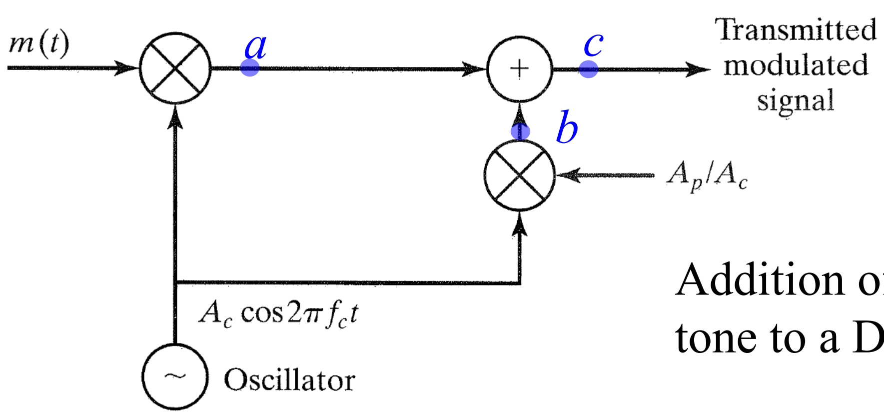
Demodulation of DSB-SC AM Signals

- **Two ways of generating a sinusoid that is phase-locked to the phase of the received carrier at the receiver**

  1. ***Phase lock loop (PLL)***
     - Generate a phase-locked sinusoidal carrier from the received signal $r(t)$ without the need of a pilot signal (angle modulation).
  2. ***Pilot tone***

     $$u(t) = A_c m(t)\cos(2\pi f_c t) \quad \Rightarrow \quad u(t) = A_c m(t)\cos(2\pi f_c t) + A_p \cos(2\pi f_c t)$$

     - Add a carrier component "***a pilot tone***." into the transmitted signal in Fig
     - Amplitude A_p & power $A_p^2/2$: Selected to be significantly smaller than those of the modulated signal $u(t)$
     - Transmitted signal : Double-sideband, No longer a suppressed carrier signal $\Rightarrow$ Similar to DSB-LC AM



Addition of a pilot tone to a DSB-AM signal

Envelope Detector

■ Envelope Detector : Circuit diagram : Figure

- Easily demodulate conventional DSB-AM
- Diode and an RC circuit = Basically a simple LPF

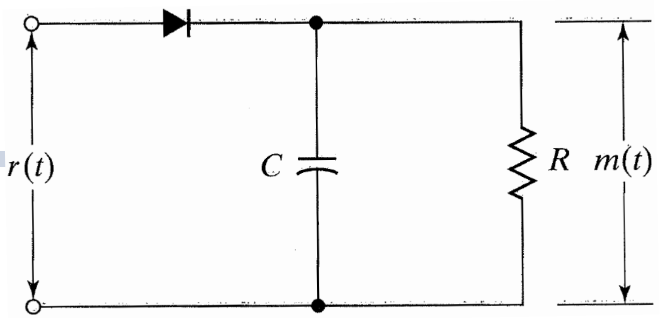
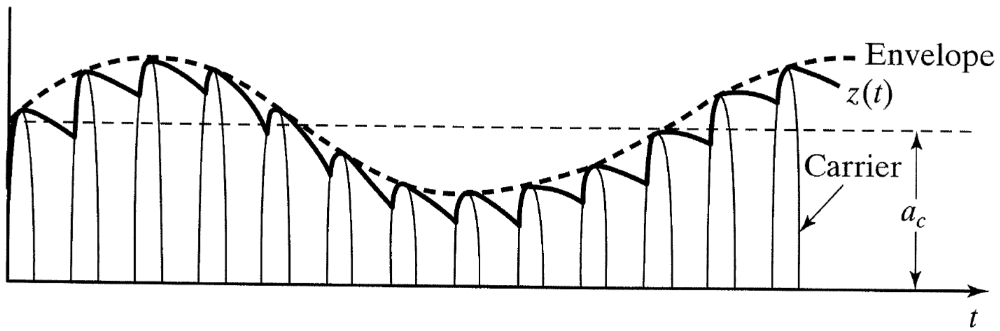


Figure 3.27 An envelope detector



- Selection of the time constant RC
  - RC = Too large : Too slow discharge of the capacitor $\Rightarrow$ Not follow the envelope $\Rightarrow$ Too small BW of the LPF
  - RC = Too small : Very rapidly fall $\Rightarrow$ Not follow the envelope $\Rightarrow$ Too large BW of the LPF
  - For good performance of the envelope detector

$$\frac{1}{f_c} \ll RC \ll \frac{1}{W} \quad \Leftrightarrow \quad W \ll \frac{1}{RC} \ll f_c$$

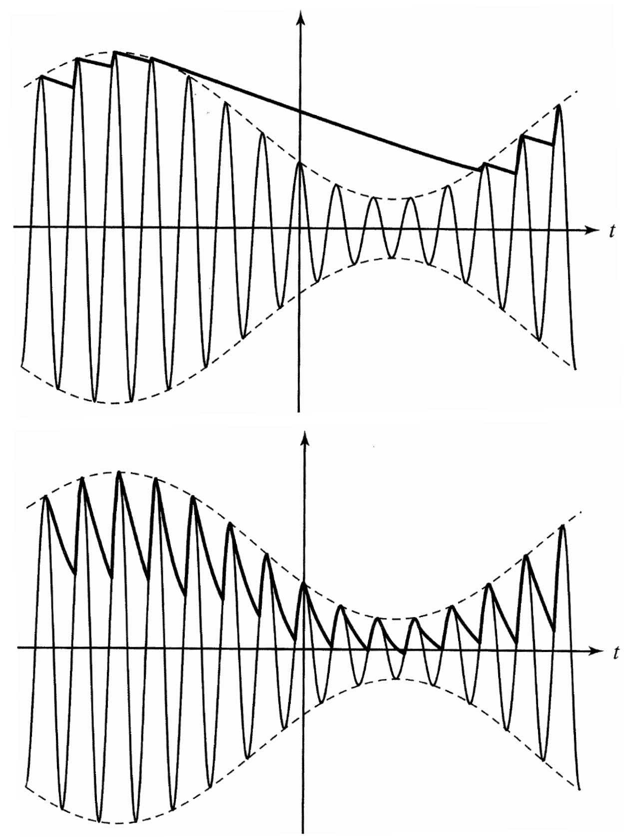
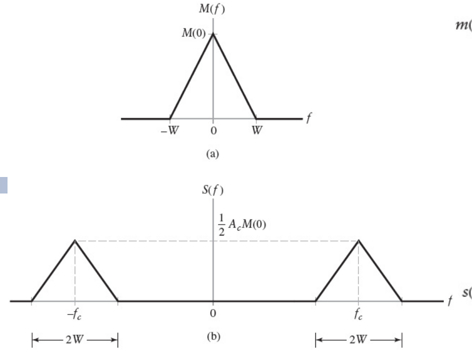
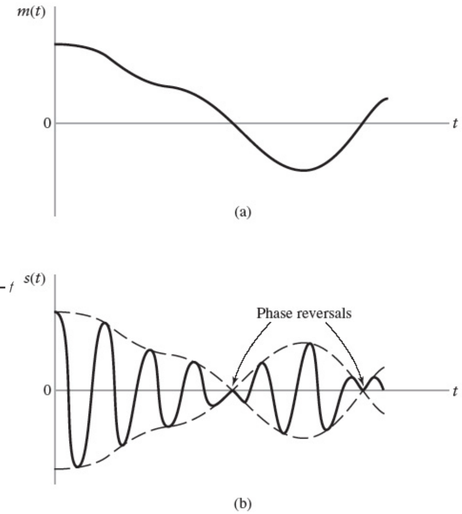


Figure 3.28 Effect of (a) large and (b) small RC values on the performance of the envelope detector

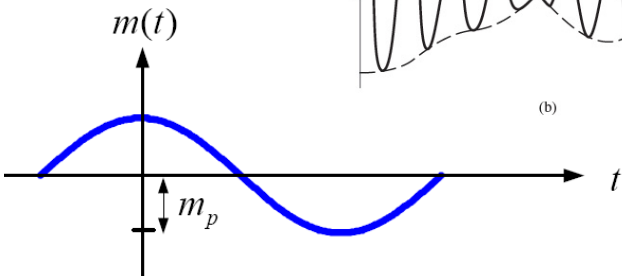
Conventional AM





■ Phase reversal

- Message signal $m(t)$: Zero crossing
  - ⇒ Different envelope of a DSB-SC AM
  - ⇒ Not a simple demodulation using envelope detection



■ Conventional AM signal

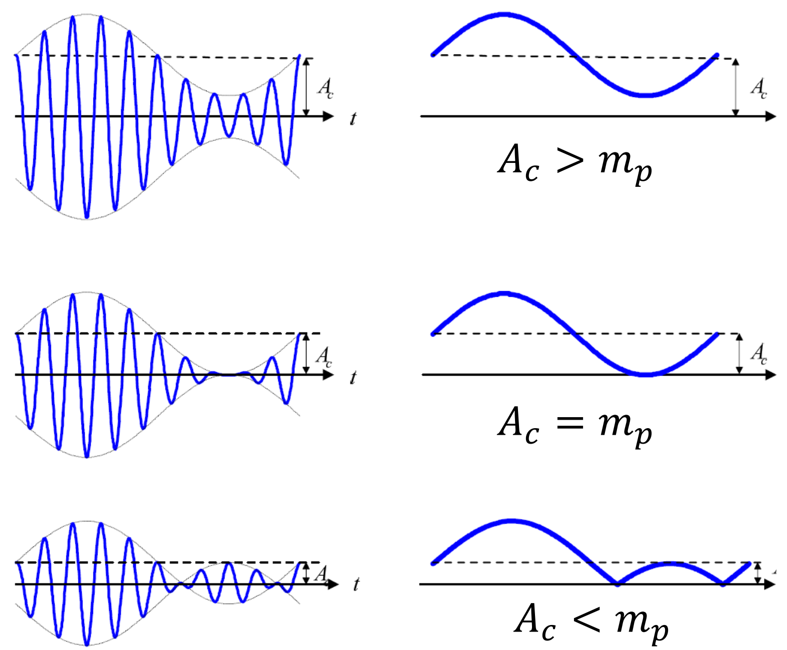
- Satisfy the condition that $|m(t)| \leq 1$
- Large carrier component + DSB-SC AM signal

$$u(t) = A_c[1 + m(t)]\cos(2\pi f_c t)$$
$$= \underbrace{A_c m(t)\cos(2\pi f_c t)}_{\text{DSB AM signal}} + \underbrace{A_c\cos(2\pi f_c t)}_{\text{Carrier component}}$$

DSB AM signal

Carrier component : Extra carrier
⇨ Very simple demodulator structure
⇨ Commercial AM broadcasting

1. $|m(t)| \leq 1$
   - Amplitude $A_c[1 + m(t)]$: Always positive
   - Desired condition & Easy to demodulate
2. $m(t) < -1$ for some t
   - ***Overmodulated***
   - Complex demodulation

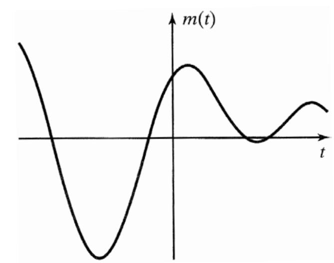


Conventional Amplitude Modulation

- **Conventional AM signal** $u(t) = A_c[1 + m(t)]\cos(2\pi f_c t)$
  - $m(t)$: Scale magnitude : Always less than unity

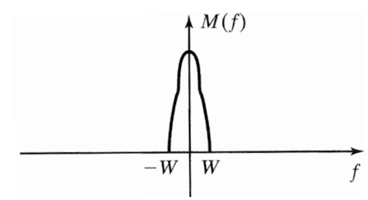
    $\Rightarrow$ Normalize $m(t)$: Minimum value -1

$$m(t) = a m_n(t) \quad \Leftarrow \quad m_n(t) = \frac{m(t)}{\max|m(t)|}$$



  - ***Modulation index*** : Scale factor a – Generally a constant less than 1
    - $|m_{n(t)}| \le 1$ and $0 < a < 1 \quad \Rightarrow \quad 1 + m(t) = 1 + a m_n(t) > 0$

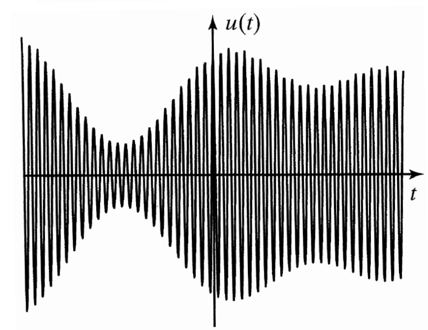
      $\Rightarrow$ Modulated signal : Never overmodulated



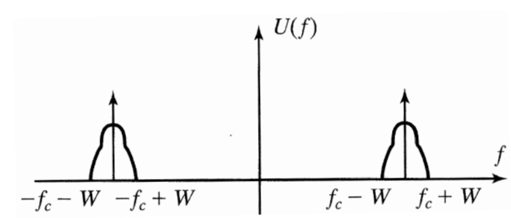
  - **Spectrum of the Conventional AM Signal** $[m(t) \Leftrightarrow M(f)]$

$$u(t) = A_c[1 + a m_n(t)]\cos(2\pi f_c t)$$
$$= A_c a m_n(t)\cos(2\pi f_c t) + A_c\cos(2\pi f_c t)$$

$$U(f) = F[u(t)] = F[A_c a m_n(t)\cos(2\pi f_c t)] + F[A_c\cos(2\pi f_c t)]$$
$$= \frac{A_c a}{2}[M_n(f - f_c) + M_n(f + f_c)] + \frac{A_c}{2}[\delta(f - f_c) + \delta(f + f_c)]$$



- BW = Twice the BW of the message signal
- DC component : Impulse in $U(f)$ at $f = fc$
- Carrier component the modulated signal $u(t)$



Example

- Suppose that the modulating signal $m(t)$ is a sinusoid of the form

$$m(t) = a\cos(2\pi f_m t) \quad f_m << f_c$$

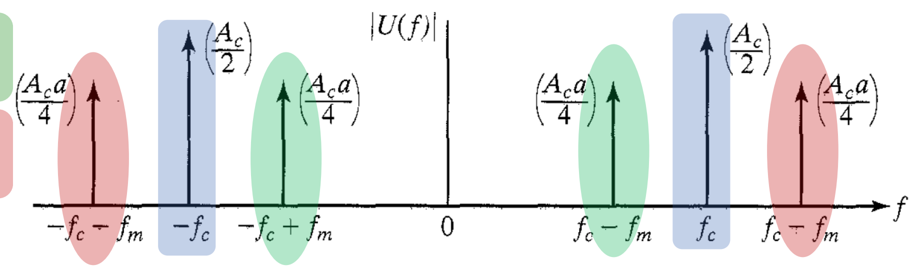
- Determine the DSB-AM signal, its upper & lower sidebands, & its spectrum, assuming a modulation index of a.

- **Solution**
  - From Equation, $u(t) = A_c[1 + m(t)]\cos(2\pi f_c t)$
  - DSB-AM signal

$$u(t) = A_c[1 + a\cos(2\pi f_m t)]\cos(2\pi f_c t) = A_c\cos(2\pi f_c t) + A_c a\cos(2\pi f_m t)\cos(2\pi f_c t)$$
$$= A_c\cos(2\pi f_c t) + \frac{A_c a}{2}\cos[2\pi(f_c - f_m)t] + \frac{A_c a}{2}\cos[2\pi(f_c + f_m)t]$$

  - LSB : $u_l(t) = \frac{A_c a}{2}\cos[2\pi(f_c - f_m)t]$, USB component : $u_u(t) = \frac{A_c a}{2}\cos[2\pi(f_c + f_m)t]$
  - Spectrum of the DSB-AM signal $u(t)$:

$$U(f) = \frac{A_c}{2}[\delta(f - f_c) + \delta(f + f_c)]$$
$$+ \frac{A_c a}{4}[\delta(f - f_c + f_m) + \delta(f + f_c - f_m)]$$
$$+ \frac{A_c a}{4}[\delta(f - f_c - f_m) + \delta(f + f_c + f_m)]$$



Power for the Conventional AM Signal

- **Conventional AM (DSB-LC) : Similar to a DSB-SC $m(t)$ substituted with $1 + m_n(t)$**
- **DSB-SC : $u(t) = A_c m(t)\cos(2\pi f_c t)$**

Power in the message signal

$\Rightarrow$ Power in the modulated signal $P_u = \frac{A_c^2}{2} P_m$

Assumed that the average of $m_n(t)$ is zero
$\Rightarrow$ Valid assumption for many signals, including audio signals.

- **Conventional AM : $u(t) = A_c[1 + am_n(t)]\cos(2\pi f_c t)$**

$$P_m = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}[1 + am_n(t)]^2 dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}[1 + 2\overbrace{am_n(t)}^{0} + a^2 m_n^2(t)]dt$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}[1 + a^2 m_n^2(t)]dt = 1 + a^2 P_{m_n}$$

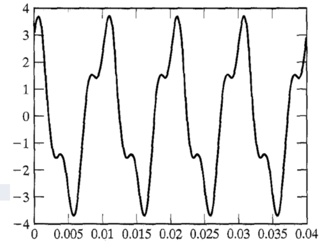
$$\Rightarrow P_m = 1 + a^2 P_{m_n} \qquad \Rightarrow \qquad P_u = \underline{\frac{A_c^2}{2}} + \underline{\frac{A_c^2}{2} a^2 P_{m_n}}$$

- Applies to the existence of the carrier
- Does not carry any information

- Information-carrying component
- Usually much smaller than the 1st component
- $a < 1, |m_n(t)| < 1$ & for signals with a large dynamic range, $P_{m_n} \ll 1$

- Conventional AM systems are far less power efficient than the DSB-SC systems
- Advantage of conventional AM : Easily demodulated

Example – DSB-LC AM



Message signal in Example

- $m(t) = 3\cos(200\pi t) + \sin(600\pi t)$, $c(t) = cos(2 \times 10^5 t)$, $a = 0.85$
- Determine the power in the carrier component & in the sideband components

- **Solution**
  - Determine $m_n(t)$: Normalized message signal

    → Determine max $|m(t)|$: Extrema of $m(t)$: Derivative = zero

    $$m'(t) = -600\pi \sin(200\pi t) + 600\pi \cos(600\pi t) = 0$$

    $$\Rightarrow \cos(600\pi t) = \sin(200\pi t) = \cos\left(\frac{\pi}{2} - 200\pi t\right) \quad \rightarrow 600\pi t = \frac{\pi}{2} - 200\pi t$$

    $\Rightarrow$ One solution : $800\pi t = \frac{\pi}{2}, or\ t = \frac{1}{1600} \Rightarrow m(t) = 3\cos(200\pi t) + \sin(600\pi t)$

    $$\Rightarrow m\left(\frac{1}{1600}\right) = 3\cos\left(200\pi \frac{1}{1600}\right) + \sin\left(600\pi \frac{1}{1600}\right) = 3\cos\left(\frac{\pi}{8}\right) + \sin\left(\frac{3\pi}{8}\right) = 3.6955$$

    $\Rightarrow$ Maximum value of the signal $m(t)$

    $$\Rightarrow m_n(t) = \frac{3\cos(200\pi t) + \sin(600\pi t)}{3.6955} = 0.8118\cos(200\pi t) + 0.2706\sin(600\pi t)$$

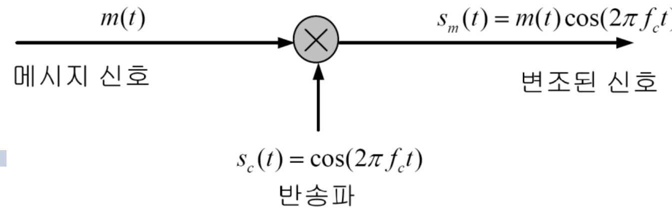
  - Power in the sum of two sinusoids with different frequencies = Sum of powers in them

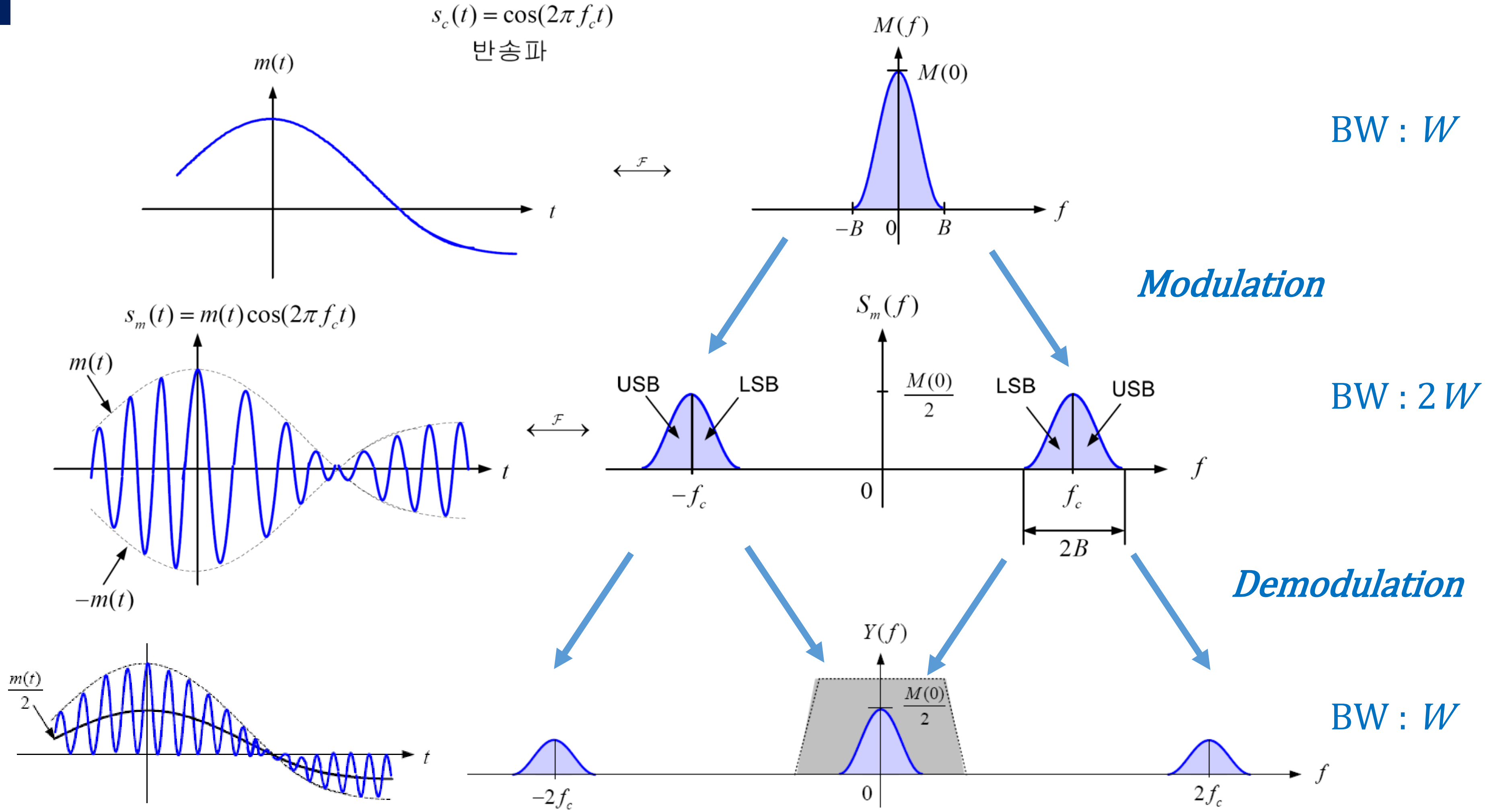
    $$u(t) = A_c[1 + am_n(t)]\cos(2\pi f_c t) \rightarrow P_u = \frac{A_c^2}{2} + \frac{A_c^2}{2} a^2 P_{m_n} \rightarrow P_{m_n} = \frac{1}{2}[0.8188^2 + 0.2706^2]$$

    $$= 0.3661$$

    $$\text{Power} \Rightarrow \ \text{Carrier} : \frac{A_c^2}{2} = \frac{1^2}{2} 0.5, \quad \text{Sideband} : \frac{A_c^2}{2} a^2 P_{m_n} = \frac{1}{2} \times 0.85^2 \times 0.3661 = 0.1323$$

DSB AM signals





Single-Sideband AM

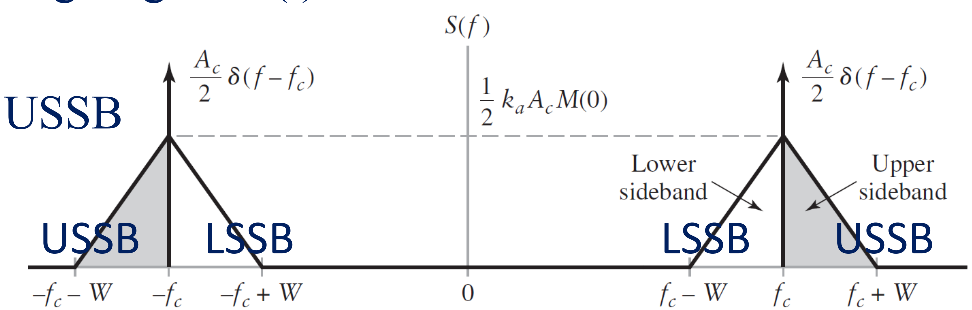
- Channel BW of a DSB-SC AM signal : $B_c = 2W$ Hz for transmission
  - W : BW of the message signal ⇒ Two sidebands $2W$: Redundant

Single-sideband (SSB) AM signal

- Reconstruct the message signal $m(t)$ at the receiver
- Transmission of either SSB

  $$u(t) = A_c m(t)\cos(2\pi f_c t) \mp A_c \hat{m}(t)\sin(2\pi f_c t)$$

  - Tx signal BW = BW baseband message signal $m(t)$
  - $\hat{m}(t)$: Hilbert transform of $m(t)$
    - Plus (+) = LSSB, Minus (–) = USSB



Hilbert transform (HT)

- Not involve a change of domain :
  - Fourier, Laplace, & z-transforms : Equivalent signal
- Hilbert transform : Not a transform, not involve a domain change
  - Not equivalent to the original signal : Completely different signal

    ⇒ Another signal in the same domain

    ⇒ Time domain signal with the same argument t

Hilbert Transform

- $\hat{x}(t)$ **: Hilbert transform of a signal $x(t)$**
  - Lag the frequency components of $x(t)$ by 90° ⇒ Exactly the same frequency components present in $x(t)$ with the same amplitude–except a 90° phase delay
  - Carrying it out twice ⇒180° phase shift ⇒ Sign reversal of the original signal
  - Hilbert transform of $x(t) = A\cos(2\pi f_0 t + \theta)$

$$A\cos(2\pi f_0 t + \theta) \Rightarrow A\cos(2\pi f_0 t + \theta - 90°) = A\sin(2\pi f_0 t + \theta)$$

- **Delay of $\pi/2$ at all frequencies**
  - Positive freqs. $e^{j2\pi f_0 t} \Rightarrow e^{j2\pi f_0 t - \frac{\pi}{2}} = -je^{j2\pi f_0 t}$: Spectrum of the signal multiplied by $-j$
  - Negative freqs. $e^{-j2\pi f_0 t} \Rightarrow e^{-j(2\pi f_0 t - \frac{\pi}{2})} = je^{-j2\pi f_0 t}$ ⇒ Spectrum of the signal multiplied by $+j$

  ⇒Spectrum (Fourier transform) of the signal : Multiplied by $-j\,\mathrm{sgn}(f)$

- **Hilbert Transform and Its Properties**
  - **Evenness and Oddness** : The Hilbert transform of an even signal is odd, and the Hilbert transform of an odd signal is even
  - **Sign Reversal** : Applying the Hilbert-transform operation to a signal twice causes a sign reversal of the signal
  - **Energy** : The energy content of a signal is equal to the energy content of its Hilbert transform
  - **Orthogonality** : The signal $x(t)$ and its Hilbert transform are orthogonal

Example

- **Determine the Hilbert transform of the signal $x(t) = 2sinc(2t)$**
- **Solution**
  - We use the frequency-domain approach to solve this problem.
  - Using the scaling property of the Fourier transform,

$$x(t) \leftrightarrow X(f)$$
$$\Rightarrow x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{f}{a}\right)$$

$$F[x(t)] = 2 \times \frac{1}{2}\Pi\left(\frac{f}{2}\right) = \Pi\left(\frac{f}{2}\right) = \Pi\left(f + \frac{1}{2}\right) + \Pi\left(f - \frac{1}{2}\right)$$

1st term : All the negative frequencies   2nd term : All the positive frequencies

  - Frequency-domain representation of the Hilbert transform of $x(t)$:

$\Rightarrow$ Use the relation $F[\hat{x}(t)] = -j\,\mathrm{sgn}(f)\,F[x(t)]$

$$F[\hat{x}(t)] = j\Pi\left(f + \frac{1}{2}\right) - j\Pi\left(f - \frac{1}{2}\right)$$

$$x(t) \leftrightarrow X(f)$$
$$\Rightarrow x(t - t_0) \leftrightarrow X(f)e^{-j2\pi f t_0}$$
$$\Rightarrow x(t)e^{j2\pi f_0 t} \leftrightarrow X(f - f_0)$$

  - Taking the inverse Fourier transform

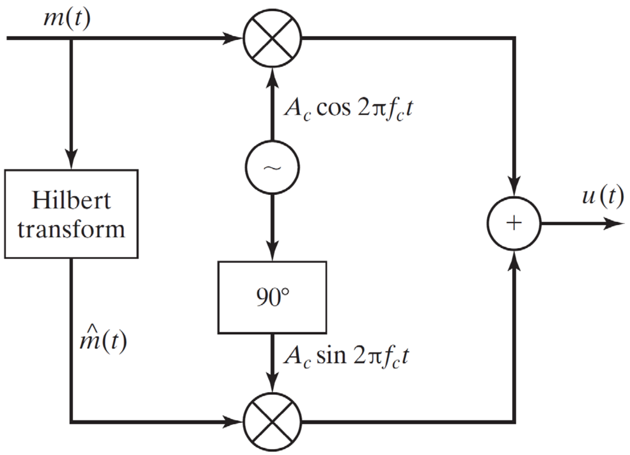
$$\hat{x}(t) = je^{-j\pi t}\mathrm{sinc}(t) - je^{j\pi t}\mathrm{sinc}(t) = -j(e^{j\pi t} - e^{-j\pi t})\mathrm{sinc}(t)$$
$$= -j \times 2j\sin(\pi t)\mathrm{sinc}(t) = 2\sin(\pi t)\mathrm{sinc}(t)$$

Single-Sideband AM

- Time-domain representation of a SSB-AM signal

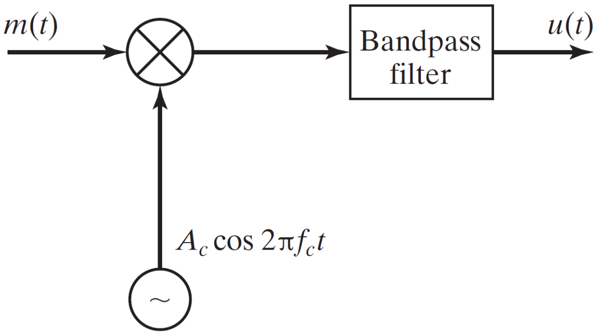
$$u_{SSB}(t) = A_c m(t)\cos(2\pi f_c t) \mp A_c \hat{m}(t)\sin(2\pi f_c t)$$

  - Minus sign : USSB-AM signal
  - Plus sign : LSSB-AM signal



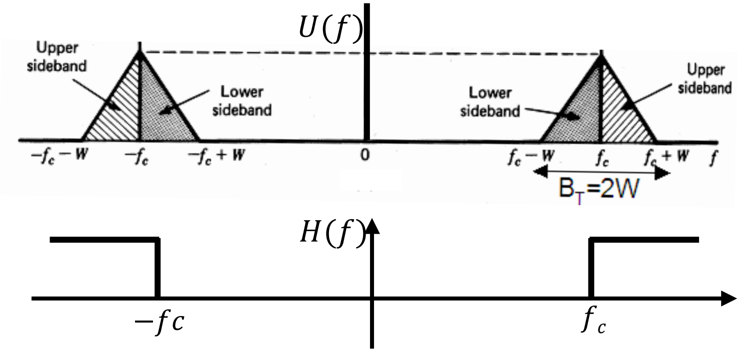
Generation of a lower single-sideband AM signal

- Generation of the SSB-AM signal $u(t)$
  - Using the system configuration in Figure Hilbert-transform filter



Generation of a single-sideband AM signal by filtering one of the sidebands of a DSB-SC AM signal

- Another method, illustrated in Figure
  1. Generates a DSB-SC AM signal
  2. Employs a filter that selects either the upper sideband or the lower sideband of the double-sideband AM signal



Example

- Suppose that the modulating signal is a sinusoid of the form

$$m(t) = \cos(2\pi f_m t) \quad f_m \ll f_c$$

- Determine the two possible SSB-AM signals

- **Solution**

$\cos\alpha\cos\beta \mp \sin\alpha\sin\beta = \cos(\alpha \pm \beta)$

  - Hilbert transform of $m(t)$: $\hat{m}(t) = \sin(2\pi f_m t)$

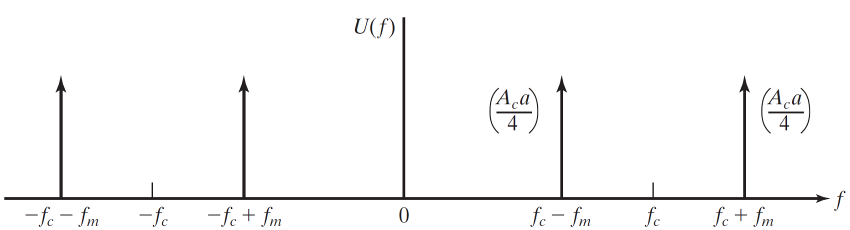
$$\Rightarrow u_{SSB}(t) = A_c m(t)\cos(2\pi f_c t) \mp A_c \hat{m}(t)\sin(2\pi f_c t)$$
$$= A_c\cos(2\pi f_m t)\cos(2\pi f_c t) \mp A_c\sin(2\pi f_m t)\sin(2\pi f_c t)$$

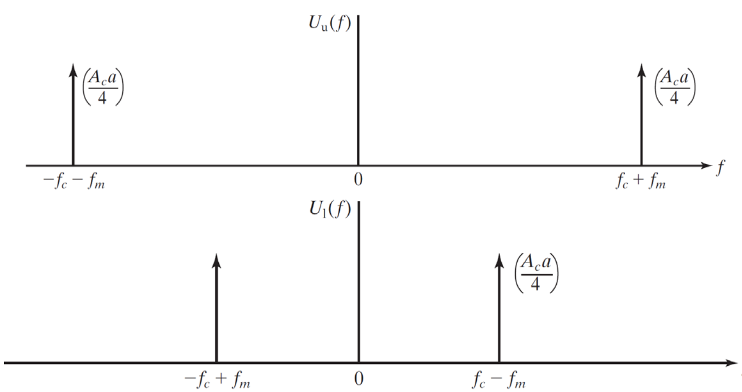
  - If we take the upper (–) sign, we obtain the upper-sideband signal

$$u_u(t) = u_{USB}(t) = A_c\cos(2\pi f_m t)\cos(2\pi f_c t) - A_c\sin(2\pi f_m t)\sin(2\pi f_c t) = A_c\cos 2\pi(f_c + f_m)t$$

  - If we take the lower (+) sign, we obtain the lower-sideband signal

$$u_l(t) = u_{LSB}(t) = A_c\cos(2\pi f_m t)\cos(2\pi f_c t) + A_c\sin(2\pi f_m t)\sin(2\pi f_c t) = A_c\cos 2\pi(f_c - f_m)t$$





Demodulation of SSB-AM Signals

Demodulation of SSB-AM Signals

- To recover the message signal $m(t)$ in the received SSB-AM signal
- Require a phase-coherent or synchronous demodulator, as for DSB-SC AM signals
- USSB signal : $u_{USSB}(t) = A_c m(t)\cos(2\pi f_c t) - A_c \hat{m}(t)\sin(2\pi f_c t)$

$$
\begin{aligned}
r(t)\cos(2\pi f_c t + \phi) &= u(t)\cos(2\pi f_c t + \phi) \\
&= A_c m(t)\cos(2\pi f_c t)\cos(2\pi f_c t + \phi) - A_c \hat{m}(t)\sin(2\pi f_c t)\cos(2\pi f_c t + \phi) \\
&= \frac{1}{2}A_c m(t)[\cos(\phi) + \cos(4\pi f_c t + \phi)] - \frac{1}{2}A_c \hat{m}(t)[\sin(-\phi) + \sin(4\pi f_c t + \phi)] \\
&= \frac{1}{2}A_c m(t)\cos(\phi) + \frac{1}{2}A_c \hat{m}(t)\sin(\phi) + \text{double frequency terms}
\end{aligned}
$$

LPF

$$
y_l(t) = \frac{1}{2}A_c m(t)\cos(\phi) + \frac{1}{2}A_c \hat{m}(t)\sin(\phi)
$$

- Phase offset
  - Reduces the amplitude of the desired signal $m(t)$ by $\cos\phi$
  - Results in an undesirable sideband signal due to the presence of $\hat{m}(t)$ in $y_l(t)$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$

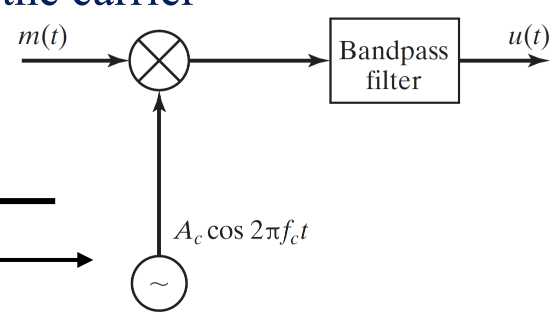
$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)]$$

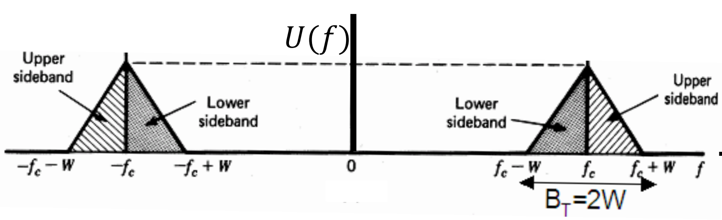
- Not present in the demodulation of a DSB-SC signal
- A factor that contributes to the distortion of the demodulated SSB signal

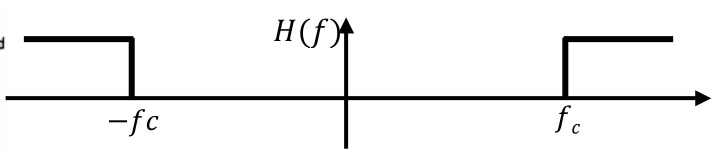
Demodulation of SSB-AM Signals

- **Transmission of a *pilot tone* at the carrier frequency**
  - Phase-coherent reference signal for synchronous demodulation at the Rx
    ⇒ Eliminate the undesirable sideband-signal component
    ⇒ Allocate a portion of the transmitted power to the Tx of the carrier

- **Spectral efficiency of SSB AM**
  ⇒ Very attractive for use in voice communications over telephone channels (wirelines and cables)
  - Filter method
    - Selects one of the two signal sidebands for transmission
    - Difficult to implement when the message signal $m(t)$ has a large power concentrated in the vicinity of $f = 0$
    - Sideband filter : Extremely sharp cutoff in the vicinity of the carrier
    - Very difficult to implement in practice







Vestigial-Sideband AM

■ *Vestigial-sideband* (*VSB*) *AM*

- Relax the stringent-frequency response requirements on the sideband filter in an SSB-AM system
- Allowing vestige – Portion of the unwanted sideband
- Simplify the design of the sideband filter at the cost of a modest increase in the channel bandwidth required to transmit the signal
- Appropriate for signals that have a strong low-frequency component, such as video signals
- That is why this type of modulation is used in standard TV broadcasting

Vestigial-Sideband AM

Vestigial-sideband (VSB) AM

- Relax the stringent-freq. response requirements on the SB filter in an SSB-AM system
- Allowing vestige – Portion of the unwanted sideband (SB)
- Simplify the design of the SB filter at the cost of a modest increase in the channel BW required to transmit the signal
- Appropriate for signals that have a strong low-freq. component, such as video signals
- That is why this type of modulation is used in standard TV broadcasting

Generation a VSB-AM signal

- Generate a DSB-SC AM signal ⇒ Pass it through a SB filter with the freq. response $H(f)$ in Fig 1 ⇒ VSB signal in the time domain

  $$u(t) = [A_c m(t) \cos 2\pi f_c t] * h(t)$$

  $h(t)$: Impulse response of the VSB filter
- Frequency domain expression

$$U(f) = \frac{A_c}{2}[M_n(f - f_c) + M_n(f + f_c)]H(f)$$

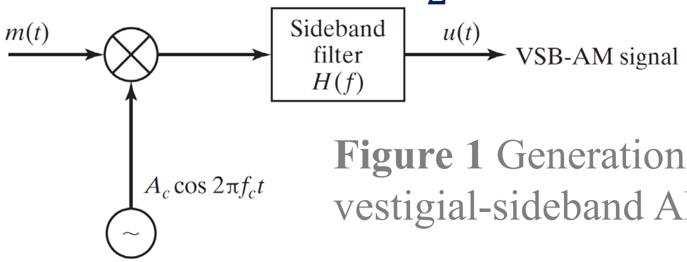


Figure 1 Generation of vestigial-sideband AM signal

Demodulation of a VSB-AM

- Multiply $u(t)$ by the carrier component $\cos 2\pi f_c t$

$\Rightarrow v(t) = u(t) \cos 2\pi f_c t$

$\Rightarrow V(f) = \frac{1}{2}[U(f - f_c) + U(f + f_c)]$

- Filtering : Ideal LPF

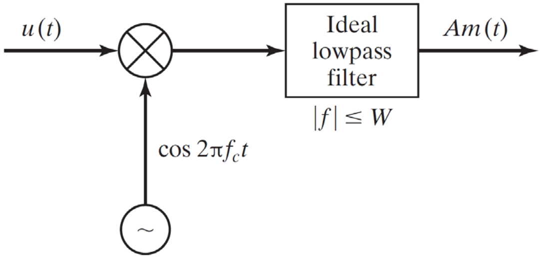


Figure 2 Demodulation of VSB signal

Vestigial-Sideband AM

- **Vestigial-sideband (VSB) AM**
  - Substitute $U(f)$ into $V(f)$: $U(f) = \frac{A_c}{2}[M_n(f-f_c) + M_n(f+f_c)]H(f)$

$$V(f) = \frac{1}{2}[U(f-f_c) + U(f+f_c)]$$

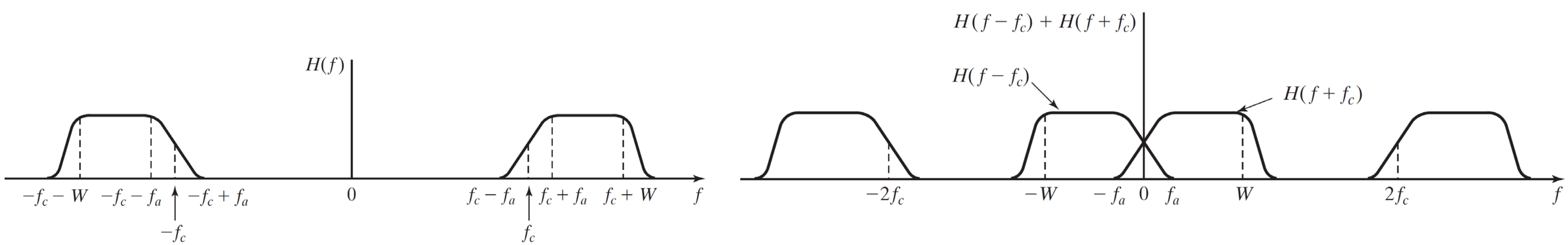
$$V(f) = \frac{A_c}{4}[M(f-2f_c) + M(f)]H(f-f_c) + \frac{A_c}{4}[M(f) + M(f+2f_c)]H(f+f_c)$$

  - LPF
    - Rejects the double-frequency terms
    - Passes only the components in the frequency range $|f| \le W$
    - Signal spectrum at the output of the ideal lowpass filter

$$V_l(f) = \frac{A_c}{4}M(f)[H(f-f_c) + H(f+f_c)]$$

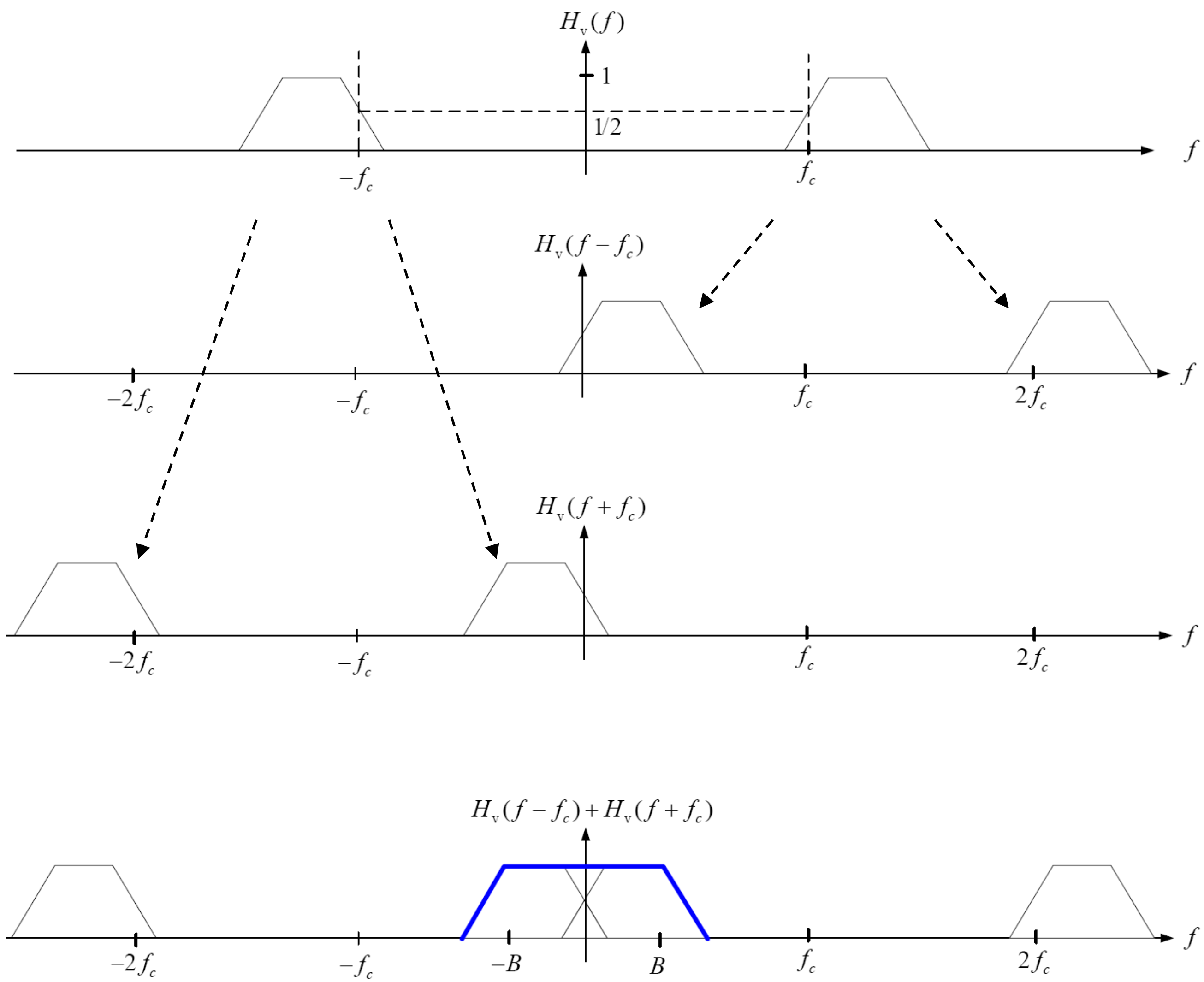
    - Message signal at the output of the LPF : Must be undistorted
  - Condition for the VSB-filter characteristic

$$H(f-f_c) + H(f+f_c) = \textbf{constant} \quad |f| \le W$$



VSB-filter characteristics

VSB Filter


$H_v(f)$
1
1/2
$-f_c$
f_c
f
$H_v(f-f_c)$
$-2f_c$
$-f_c$
f_c
$2f_c$
f
$H_v(f+f_c)$
$-2f_c$
$-f_c$
f_c
$2f_c$
f
$H_v(f-f_c)+H_v(f+f_c)$
$-2f_c$
$-f_c$
$-B$
B
f_c
$2f_c$
f

Example

- Message signal : $m(t) = 10 + 4\cos(2\pi t) + 8\cos(4\pi t) + 10\cos(20\pi t)$
- Specify both the frequency-response characteristic of a VSB filter that passes the upper sideband and the first frequency component of the lower sideband
- **Solution : Spectrum of the DSB-SC AM signal** $u(t) = m(t)\cos 2\pi f_c t$

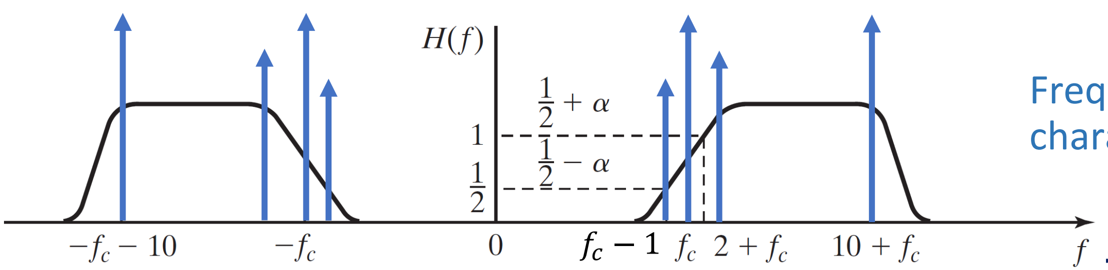
$$U(f) = 5[\delta(f - f_c) + \delta(f + f_c)] + 2[\delta(f - f_c - 1) + \delta(f + f_c + 1)]$$
$$+4[\delta(f - f_c - 2) + \delta(f + f_c + 2)] + 5[\delta(f - f_c - 10) + \delta(f + f_c + 10)]$$

- VSB filter : Designed to have
  - Unity gain in the range $2 \leq |f - f_c| \leq 10$
  - Gain of 1/2 at $f = f_c$
  - Gain of $1/2 + \alpha$ at $f = f_c + 1$
  - Gain of $1/2 - \alpha$ at $f = f_c - 1$

  α is some conveniently selected parameter satisfying the condition $0 < \alpha < 1/2$

- VSB filter : Designed to have

$$H(f) = \begin{cases} 1 & 2 \leq |f - f_c| \leq 10 \\ \frac{1}{2} & f = f_c \\ \frac{1}{2} + \alpha & f = f_c + 1 \\ \frac{1}{2} - \alpha & f = f_c - 1 \end{cases}$$



Frequency-response characteristic of the VSB filter

Signal Multiplexing

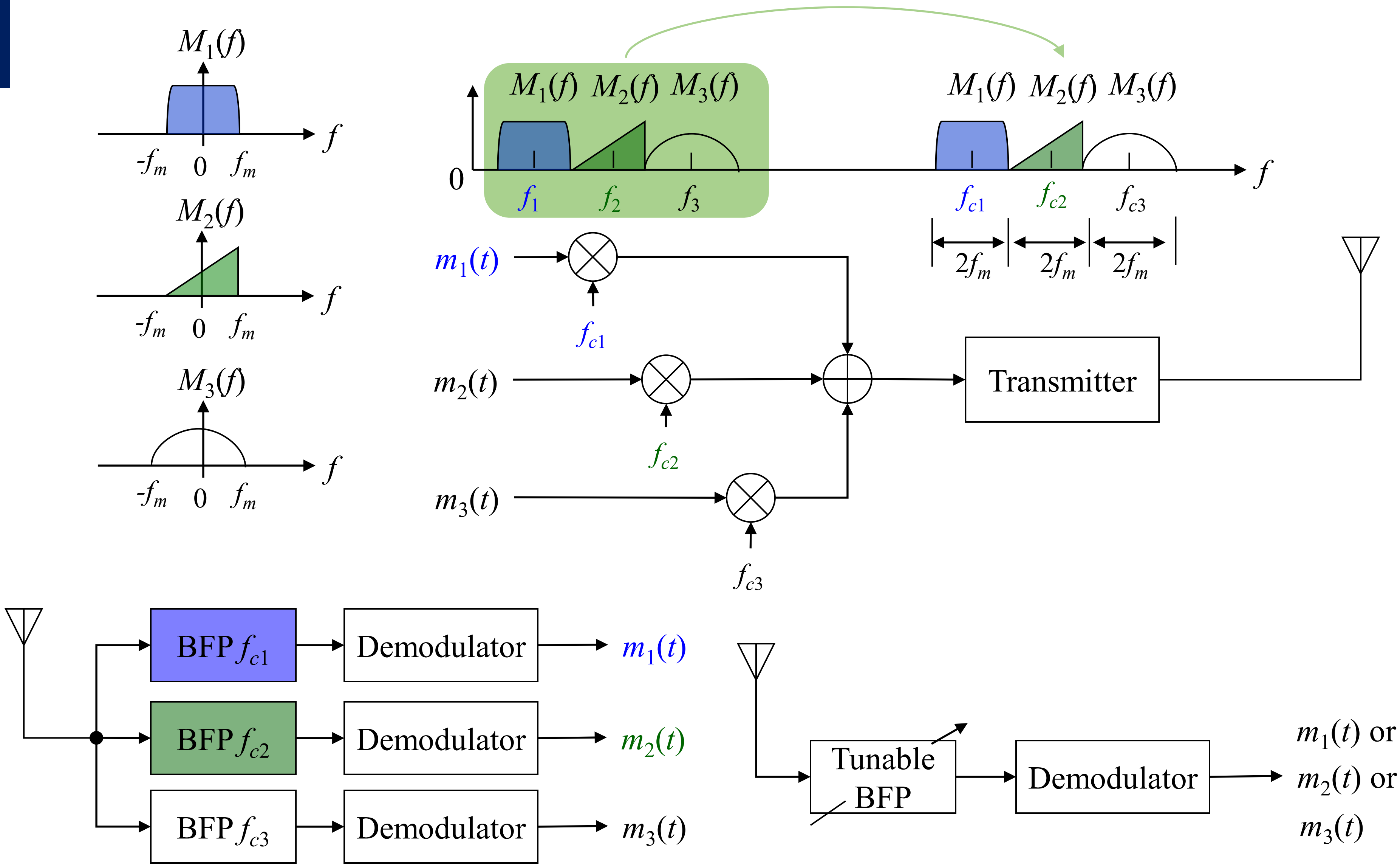
Multiplexing

- Combining separate message signals into a composite signal for transmission over a common channel

Frequency-Division multiplexing

- Modulation
  - Translate the message signal by an amount equal to the carrier frequency f_c
- Simultaneous transmission of two or more message signals
  - Object
    - Transmit simultaneously two or more message signals over the communication channel
  - How
    - Each message signal modulate a carrier of a different frequency
    - Minimum separation between two adjacent carriers : $2W$ for DSB AM and W for SSB AM ($W =$ BW of each of the message signals)
  - The various message signals occupy separate frequency bands of the channel and do not interfere with one another during transmission

Signal Multiplexing

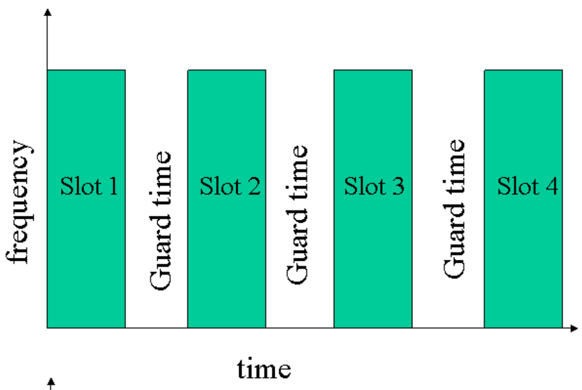


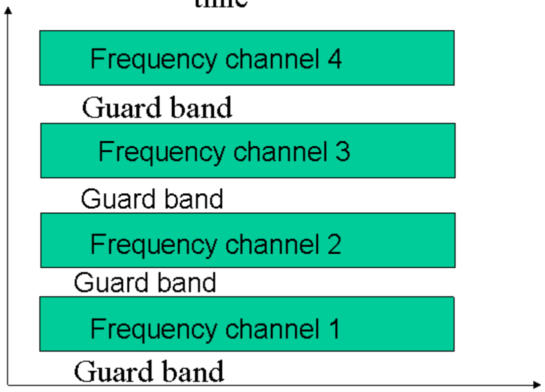
Signal Multiplexing

Multiplexing

- Combine a number of independent signals into a composite signal
- The signals must be kept apart : Not interfere with each other
  - ⇨ Separate the signals in frequency (or time, code)
  - ⇨ Separate the signals at the receiver

- Two commonly used methods for signal multiplexing:
  1) Time-division multiplexing : Usually used to transmit digital information
  2) Frequency-division multiplexing : Used with either analog or digital signal transmission





AM-Radio Broadcasting

- AM-radio broadcasting
  - Frequency band of commercial AM-radio broadcasting : 535-1605kHz
  - Transmission of voice and music
  - Carrier-frequency range : 540-1600 kHz with 10 kHz spacing
- Radio stations employ conventional AM for signal transmission
  - Limit the baseband message signal $m(t)$ to a BW of approximately 5 kHz
  - Major objective : Reduce the cost of implementing the receiver from an economic standpoint

Receiver – Demodulating the incoming modulated signal

- Carrier frequency tuning : Select the desired signal : Desired radio or TV station
- Filtering : Separate the desired signal from other modulated signals that may be picked up along the way
- Amplification : Compensate for the loss of signal power incurred in the course of transmission

Summary

- **Introduction to Modulation**

- **Amplitude Modulation**
  - Double-Sideband Suppressed-Carrier AM : DSB-SC AM
  - Conventional Amplitude Modulation : DSB-LC AM
  - Single-Sideband AM : SSB AM & Vestigial Sideband AM – VSB AM

- **Implementation of AM Demodulators**
  - Demodulator : Envelope Detector = LPF

- **Signal Multiplexing**
  - Frequency-Division Multiplexing
  - Quadrature-Carrier Multiplexing

Fourier Series and Its Properties

- **Which signals can be expanded in terms of complex exponentials?**
  - To answer this question, we will give the conditions for a periodic signal to be expandable in terms of complex exponentials

- **Dirichlet conditions**
  - Conditions for $x(t)$ to be expanded i.t.o complex exponential signals
  - $x(t)$: A periodic signal with period T_0

1. $x(t)$: Absolutely integrable over its period, i.e., $\int_0^{T_0} |x(t)| dt < \infty \rightarrow \int_{t_1}^{t_1+T_1} |x(t)|^2 dt < \infty$
2. The signal $x(t)$ has a finite number of maxima and minima in the ***expansion interval***
3. The signal $x(t)$ has a finite number of discontinuities in the ***expansion interval***

- **Fourier series**

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi \frac{n}{T_0} t} = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi n f_0 t},$$

$$x_n = \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} x(t) e^{-j2\pi \frac{n}{T_0} t} dt = \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} x(t) e^{-j2\pi n f_0 t} dt$$

for some arbitrary α

Signals by Fourier Series

- Complex exponential Fourier series representation of the signal $x(t)$ over the expansion interval $t_1 < t < t_1 + T_1$
  - Basis signals ($f_1 = 1/T_1$)

$$\varphi_n(t) = e^{j2\pi(nf_1)t} \quad n = 0,\ \pm1,\ \pm2 \cdots, \qquad \int_{t_1}^{t_2} \varphi_n(t)\varphi_m(t)\,dt = \begin{cases} \lambda_n & n = m \\ 0 & n \neq m \end{cases}$$

  - Resulting complex-exponential Fourier expansion

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi n f_1 t}$$

  - A truncated version of the series containing $2N + 1$

$$\hat{x}_N(t) = \sum_{n=-N}^{N} x_n e^{j2\pi n f_1 t}$$

  - Computation of the coefficients x_n : $x_n = \frac{1}{\lambda_n}\int_{t_1}^{t_1+T_1} x(t)e^{-j2\pi n f_1 t}dt$
  - Parameter λ_n : $\lambda_n = \int_{t_1}^{t_1+T_1} e^{j2\pi n f_1 t}e^{-j2\pi n f_1 t}dt = \int_{t_1}^{t_1+T_1} dt = T_1$

Signals by Fourier Series

$$\lambda_{nm} = \int_{t_1}^{t_1+T_1} e^{j2\pi n f_1 t} e^{-j2\pi m f_1 t} dt$$

$$= \int_{t_1}^{t_1+T_1} e^{j2\pi(n-m)f_1 t} dt = \frac{1}{j2\pi(n-m)f_1} \left[e^{j2\pi(n-m)f_1 t}\right]_{t_1}^{t_1+T_1}$$

$$= \frac{1}{j2\pi(n-m)f_1} \left[e^{j2\pi(n-m)f_1(t_1+T_1)} - e^{j2\pi(n-m)f_1 t_1}\right]$$

$$= \frac{1}{j2\pi(n-m)f_1} \left[e^{j2\pi(n-m)f_1 t_1} e^{j2\pi(n-m)f_1 T_1} - e^{j2\pi(n-m)f_1 t_1}\right]$$

$$= \frac{1}{j2\pi(n-m)f_1} e^{j2\pi(n-m)f_1 t_1} \left[e^{j2\pi(n-m)f_1 T_1} - 1\right]$$

$$f_1 = \frac{1}{T_1}$$

$$= \frac{1}{j2\pi(n-m)f_1} e^{j2\pi(n-m)f_1 t_1} \left[e^{j2\pi(n-m)} - 1\right]$$

$$= 0$$

Fourier Transform

- Spectrum of an aperiodic signal ⇨ Fourier transform
- Fourier transform from the Fourier series
  1. Complex exponential Fourier series representation of an aperiodic signal over the interval $-T/2 < t < T/2$
  2. Interval increase until the entire time axis is encompassed
- Conditions of the Fourier transform existence
  - Fourier series of $x(t)$ exists if, for any T, $\int_{-T/2}^{T/2} |x(t)|^2 dt < \infty$
  - In the limit as $T \to \infty$, $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$
    - Fourier transform exists if the aperiodic signal is an energy signal
  - Dirichlet condition
    1. $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$
    2. The signal $x(t)$ has a finite number of maxima and minima in ***any finite interval***
    3. The signal $x(t)$ has a finite number of discontinuities in ***any finite interval***

Fourier Transform

■ Fourier Transform (FT)

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

- Spectrum of $x(t)$, Fourier transform of $x(t)$

▪ Amplitude $X(f)$: Double-sided amplitude-density spectrum or amplitude spectrum

▪ Angle of $X(f)$: Double-sided phase spectrum

▪ The spectrum $X(f)$ completely characterizes the energy signal $x(t)$ when $x(t)$ satisfies the Dirichlet conditions

■ Inverse Fourier Transform (IFT)

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

- IFT produces an energy signal satisfying the Dirichlet conditions from the signal's Fourier transform or spectrum
- The signal can be uniquely recovered from its Fourier transform ⇨ The Fourier transform of an energy signal satisfying the Dirichlet conditions is unique

Fourier Transform Pair

■ Fourier transform pair

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \equiv F[x(t)]$$

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df \equiv F^{-1}[X(f)]$$

$$x(t) \leftrightarrow X(f)$$

- $\omega = 2\pi f$

$$X_\omega(\omega) \equiv X\left(\frac{\omega}{2\pi}\right) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \int_{-\infty}^{\infty} X\left(\frac{\omega}{2\pi}\right)e^{j\omega t}d\left(\frac{\omega}{2\pi}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_\omega(\omega)e^{j\omega t}d\omega$$

Fourier Transform & Spectrum Properties

- **Time-limited signal**
  - A ***time-limited signal*** is one that is nonzero only for a finite length time interval
- **Band-limited signal**
  - A ***band-limited signal*** is one that has a nonzero spectrum only for a finite length frequency interval

- Property I
  - A signal that is time-limited cannot be band-limited and a signal that is band-limited cannot be time-limited

$$\hat{x}_N(t) = \sum_{n=-N}^{N} x_n e^{j2\pi n f_1 t}$$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \equiv F[x(t)]$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \equiv F^{-1}[X(f)]$$

Fourier Transform Theorems

- Theorem 1 : ***Linearity***

$$x(t) \leftrightarrow X(f) \qquad y(t) \leftrightarrow Y(f) \qquad \Rightarrow ax(t) + by(t) \leftrightarrow aX(f) + bY(f)$$

- Theorem 2 : ***Scale Change***

$$x(t) \leftrightarrow X(f) \qquad \Rightarrow \qquad x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

- Theorem 3 : ***Time Reversal***

$$x(t) \leftrightarrow X(f) \qquad \Rightarrow \qquad x(-t) \leftrightarrow X(-f)$$

- Theorem 4 : ***Complex Conjugation***

$$x(t) \leftrightarrow X(f) \qquad \Rightarrow \qquad x*(t) \leftrightarrow X*(-f)$$

- Theorem 5 : ***Duality***

$$x(t) \leftrightarrow X(f) \qquad \Rightarrow \qquad X(t) \leftrightarrow x(-f)$$

- Theorem 6 : ***Time Shift***

$$x(t) \leftrightarrow X(f) \qquad \Rightarrow \qquad x(t-t_0) \leftrightarrow X(f)e^{-j2\pi f t_0}$$

$$F[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0)e^{-j2\pi f t}dt = \int_{-\infty}^{\infty} x(\tau)e^{-j2\pi f(\tau+t_0)}d\tau$$

$$= \left[\int_{-\infty}^{\infty} x(\tau)e^{-j2\pi f\tau}d\tau\right]e^{-j2\pi f t_0} = X(f)e^{-j2\pi f t_0}$$

$$t - t_0 = \tau \rightarrow \begin{cases} t = \tau + t_0 \\ dt = d\tau \end{cases}$$

$$\rightarrow \begin{cases} t \rightarrow \infty, & t \rightarrow -\infty \\ \tau \rightarrow \infty, & \tau \rightarrow -\infty \end{cases}$$

Basic Properties of the Fourier Transform

■ Theorem 7 : ***Frequency Translation***

$$x(t) \leftrightarrow X(f) \qquad \Rightarrow \qquad x(t)e^{j2\pi f_0 t} \leftrightarrow X(f - f_0)$$

■ Theorem 8 : ***Modulation***

$$\begin{aligned} F\left[x(t)e^{j2\pi f_0 t}\right] &= \int_{-\infty}^{+\infty} x(t)e^{j2\pi f_0 t}e^{-j2\pi f t}dt \\ &= \int_{-\infty}^{+\infty} x(t)e^{-j2\pi(f-f_0)t}dt \\ &= X(f - f_0) \end{aligned}$$

- Dual of the time-shift theorem.
- A shift in the time domain results in a multiplication by a complex exponential in the frequency domain
- A multiplication in the time domain by a complex exponential results in a shift in the frequency domain
- A shift in the frequency domain is usually called modulation

Fourier Transform Theorems

■ Theorem 9 : ***Convolution***

$$x(t) \leftrightarrow X(f) \quad \text{and} \quad y(t) \leftrightarrow Y(f)$$
$$\Downarrow$$
$$x(t) * y(t) \leftrightarrow X(f)Y(f)$$

$$z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\lambda)y(t-\lambda)d\lambda$$

$$\begin{aligned}
Z(f) &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\lambda)y(t-\lambda)d\lambda\right] e^{-j2\pi ft} dt \\
&= \int_{-\infty}^{\infty} x(\lambda) \left[\int_{-\infty}^{\infty} y(t-\lambda)\, e^{-j2\pi ft} dt\right] d\lambda \\
&= \int_{-\infty}^{\infty} x(\lambda) \left[\int_{-\infty}^{\infty} y(\tau)\, e^{-j2\pi f(\tau+\lambda)} d\tau\right] d\lambda \\
&= \left[\int_{-\infty}^{\infty} x(\lambda) e^{-j2\pi f\lambda} d\lambda\right]\left[\int_{-\infty}^{\infty} y(\tau)\, e^{-j2\pi f\tau} d\tau\right] \\
&= X(f)Y(f) = F[x(t) * y(t)]
\end{aligned}$$

$$t-\lambda = \tau \rightarrow \begin{cases} t = \tau + \lambda \\ dt = d\tau \end{cases} \rightarrow \begin{cases} t \rightarrow \infty, & t \rightarrow -\infty \\ \tau \rightarrow \infty, & \tau \rightarrow -\infty \end{cases}$$

■ Theorem 10 : ***Multiplication***

$$x(t) \leftrightarrow X(f) \quad \text{and} \quad y(t) \leftrightarrow Y(f)$$
$$\Downarrow$$
$$x(t)y(t) \leftrightarrow X(f) * Y(f)$$

FTs and Spectra of Selected Signals

- Signal 1 : ***Impulse***

$$A\delta(t) \leftrightarrow A \qquad \Leftarrow F[A\delta(t)] = \int_{-\infty}^{\infty} A\delta(t)e^{-j2\pi ft}dt = A$$

$$A\delta(t-t_0) \leftrightarrow Ae^{-j2\pi ft_0} \quad \Leftarrow F[A\delta(t-t_0)] = \int_{-\infty}^{\infty} A\delta(t-t_0)e^{-j2\pi ft}dt = Ae^{-j2\pi ft_0}$$

$$x(t) \leftrightarrow X(f) \qquad \Rightarrow x(t-t_0) \leftrightarrow X(f)e^{-j2\pi ft_0}$$

- Signal 2 : ***Cosine***

$$\cos(2\pi f_a t) \leftrightarrow \frac{1}{2}\delta(f-f_a) + \frac{1}{2}\delta(f+f_a)$$

$$\sin(2\pi f_a t) \leftrightarrow \frac{1}{2j}\delta(f-f_a) - \frac{1}{2}\delta(f+f_a)$$

$$\cos(2\pi f_a t) = \frac{1}{2}e^{j2\pi f_a t} + \frac{1}{2}e^{-j2\pi f_a t}$$

$$\Rightarrow F[\cos(2\pi f_a t)] = \frac{1}{2}F\left[e^{j2\pi f_a t}\right] + \frac{1}{2}F\left[e^{-j2\pi f_a t}\right] = \frac{1}{2}\delta(f-f_a) + \frac{1}{2}\delta(f+f_a)$$

$$\sin(2\pi f_a t) = \frac{1}{2j}\left[e^{j2\pi f_a t} - e^{-j2\pi f_a t}\right]$$

$$\Rightarrow F[\sin(2\pi f_a t)] = \frac{1}{2j}\left[F\left[e^{j2\pi f_a t}\right] - F\left[e^{-j2\pi f_a t}\right]\right] = \frac{1}{2j}[\delta(f-f_a) - \delta(f+f_a)]$$

FTs and Spectra of Selected Signals

- Signal 3 : ***Rectangular***

$$g\left(\frac{t}{T}\right) = A\mathrm{rect}(t), \qquad \mathrm{rect}(t) = \begin{cases} 1, & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & Otherwise \end{cases}$$

$$\begin{aligned} G(f) &= F[g(t)] \\ &= \int_{-T/2}^{T/2} A\exp(-j2\pi f t)\,dt \\ &= \frac{A}{-j2\pi f}[\exp(-j2\pi f t)]_{-T/2}^{T/2} \\ &= \frac{A}{-j2\pi f}[\exp(-j\pi f T) - \exp(j\pi f T)] \\ &= \frac{A}{\pi f}\sin(\pi f T) = \frac{AT}{\pi f T}\sin(\pi f T) \\ &= AT\frac{\sin(\pi f T)}{\pi f T} \\ &= AT\mathrm{sinc}(fT), \mathrm{sinc}(\lambda) = \frac{\sin(\pi\lambda)}{\pi\lambda} \end{aligned}$$