Matrix

: a rectangular array.

ex) $\begin{bmatrix} 1, & 2 \\ 3 & 4 \end{bmatrix}$ $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ ...

↳ The numbers are called **entries**.

row : $\longrightarrow$, 행, 횡

column : $\downarrow$, 열, 종

Vector : a single row or column.

ex) $[1, 2\ 3\ 4]$: row vector

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$: column vector.

∘ $m \times n$ matrix A : m rows and n columns matrix

$$A = [\underbrace{a}_{}{}_{\underset{\text{row}}{i}\,\underset{\text{column}}{j}}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{m1} & & & a_{mn} \end{bmatrix}$$

∘ **square matrix** : $m = n$.

main diagonal : $a_{11}, a_{22}, a_{nn} \cdots$

ex) $\begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix}$

∘ **equality of matrix**

→ A, B : matrix ① A, B : $m \times n$ matrix.

$A = [a_{ij}]$, $B = [b_{ij}]$ $a_{ij} = b_{ij}$ for $\forall i, j$ $(1 \le i \le m, 1 \le j \le n)$.

∘ **Addition of matrix**

★ A, B : $m \times n$ matrix , $A + B = [a_{ij} + b_{ij}]$.

∘ **scalar multiplication**

A : $m \times n$ matrix , a : scalar , $aA = [aa_{ij}]$

- Rules of matrix

(a) $A+B = B+A$

(b) $A+(B+C) = (A+B)+C$

(c) $A+0 = A$ → 0 : $m \times n$ matrix, zero matrix

(d) $A+(-A) = 0$

(1) $c(A+B) = cA+cB$

(2) $(c+k)A = cA+kA$

(3) $c(kA) = k(cA)$

(4) $1 \cdot A = A$

Matrix Multiplication

: $A = [a_{ij}]$: $m \times n$ matrix

$B = [b_{jk}]$: $n \times r$ matrix

condition size ↙ ↘ result size

$$\Rightarrow C = [c_{ik}], \quad c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk} = a_{i1}b_{1k} + a_{i2}b_{2k} \cdots$$

☆ c_{ik} — A's row, B's column.

ex) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+4+9 \\ 4+10+18 \end{bmatrix} = \begin{bmatrix} 14 \\ 32 \end{bmatrix}$

- ☆ $AB \neq BA$. Rule

(a) $(kA)B = k(AB) = A(kB)$

(b) $A(BC) = (AB)C$

(c) $(A+B)C = AB+BC$

(d) $C(A+B) = CA+CB$

] 순서 유의

- $C = AB$, C's 1th column → Ab_1. C's nth column → Ab_n.

C's 1th row → a_1B C's nth row → a_nB ...

- Transposition

A : $m \times n$ matrix $\xrightarrow{\text{Trans.}}$ $A^T = n \times m$ matrix, $[a_{ji}]$

↳ Rule

(a) $(A^T)^T = A$

(b) $(A+B)^T = A^T + B^T$

(c) $(cA)^T = cA^T$

(d) $(AB)^T = B^TA^T$

pf

$$A = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \quad B = [b_1 \cdots b_r] \quad AB = \begin{bmatrix} a_1b_1 & \cdots & a_1b_r \\ a_mb_1 & \cdots & a_mb_r \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} a_1b_1 & \cdots & a_mb_1 \\ \vdots & & \vdots \\ a_1b_r & & a_mb_r \end{bmatrix} = \begin{bmatrix} b_1a_1 & \cdots & b_1a_m \\ b_ra_1 & \cdots & b_ra_m \end{bmatrix} = B^TA^T$$

Gaussian

- equation → matrix form.

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

: equation

A : coefficient mat. $A\vec{x} = \vec{b}$

$$\Rightarrow \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

↳ m×n matrix.

↳ matrix form

↳ if $\vec{b} = \vec{0}$, homogeneous system.

$$\Rightarrow \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$: augmented matrix (m×(n+1) matrix)

- Gauss Elimination & Back Substitution.

ex)
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 18 \\ 2 & 1 & 0 & 8 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & 1 & 0 & 8 \\ 0 & 2 & 5 & 18 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & -2 & 8 \\ 0 & 2 & 5 & 18 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & -2 & 8 \\ 0 & 0 & 19 & 38 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad z=2,\ y=4,\ x=2$$

① Interchange of two rows/equations.

② Addition of a constant multiple of one row to another row

③ Multiplication of row by a nonzero constant c.

- The Three possible Cases
  1) a unique solution ✕
  2) infinitely many solution ✓ ┘ consistant system
  3) no solution ✓ ┘ inconsistance system

  ⇒ Every homogeneous linear system is consistant. ($\vec{x} = 0$, trivial solution)

  → 2), free column : 값을 결정해주는 $\in \mathbb{R}$.

  pivot column : 값이 결정되는

- Row Echelon Form

  if... ① in each non-zero row, the first non-zero entry is in a column to the left of any leading entries below it. = upper-triangular form.

↓

- Reduced Row Echelon Form

  if... the leading entry in each non-zero row is 1.

  each column containing leading 1 has zeros everywher else.

- rank (A) = #(nonzero rows of A)

$[A|\vec{b}] \rightarrow [R|\vec{f}] = \left[\begin{array}{c|c} 0 & \vec{f}_1 \\ \hline & \vec{f}_2 \end{array}\right]$ rank (A) = r.

$A = [a_{ij}]_{m \times n}$.

#(equations) = m.

#(unknowns) = n.

i) $r < m \rightarrow \vec{f}_2 \neq \vec{0}$: no solutions, inconsistant

ii) $r = m \rightarrow$ a unique sol.

iii) o.w → many sol. $\vec{f}_2 = \vec{0}$.

Linear Independence, Rank of a matrix.

Let $\vec{a_1}, \vec{a_2}, \vec{a_3} \dots \vec{a_m} \in \mathbb{R}^n$ be vectors.

- Linear combination

  → $c_1\vec{a_1} + c_2\vec{a_2} + \cdots + c_m\vec{a_m}$

- Linearly independent

  if $c_1\vec{a_1} + \cdots + c_m\vec{a_m} = \vec{0} \iff c_1 = \cdots = c_m = 0$,

  $\vec{a_1}, \vec{a_2} \cdots \vec{a_m}$ are linearly independent

- Linearly dependent

  if $\exists\, c_1 \dots c_m$ s.t $c_i \neq 0$ for some i and $c_1\vec{a_1} + \cdots c_m\vec{a_m} = \vec{0}$.

  ↳ 최소 하나의 벡터가 다른 벡터로 표현이 가능함

  pf $c_1\vec{a_1} + \cdots + c_m\vec{a_m} = \vec{0}$ for some nonzero c_i,

  $c_1 \neq 0$. $\times \frac{1}{c_1}$, $\vec{a_1} + \frac{c_2}{c_1}\vec{a_2} + \cdots + \frac{c_m}{c_1}\vec{a_m} = \vec{0}$, → $-\vec{a} = \frac{1}{c_1}(c_2\vec{a_2} + \cdots + c_m\vec{a_m})$

  ↳ zero vector가 하나라도 있으면 linearly dependent

- Rank of A

  ↳ max #(linearly independent vector/row)

<u>Thm.</u>

Let $A = \begin{bmatrix} \vec{a_1} \\ \vdots \\ \vec{a_m} \end{bmatrix}_{m \times n}$, $B = [\vec{a_1} \cdots \vec{a_m}]$ (A^T) Then $\vec{a_1} \cdots \vec{a_m}$ are linearly independent iff $\mathrm{rank}(A) = \mathrm{rank}(B) = m$

if $n < m$, then vectors are linearly dependent

↳ n차원에서 n개 초과의 벡터는 필요가 없음