Basic Concept

1. ODE (ordinary differential equation)

↳ 해를 구하는 방식

(equation) $F(x, y, \underline{y'}, y'', y^{(n')}) = 0$

달라진 형태 - 저런게 들어감

ex) $y' = \cos x \Leftrightarrow y' - \cos x = 0$, $y'' + 9y = e^{-2x}$, $y'y'' - \frac{3}{2}y'' = 0$

- 주로 x는 복잡하고 y는 덜 복잡하다
- $y^{(\,)}$ 개수가 차수

2. first-order ODE

↳ $F(x, y, y') = 0$: implicit form

$\Downarrow$

⇔ explicit form ⇒ 정리해서 $y' = f(x, y)$ 가 됨

ex) $x^3y' - 4y^2 = 0 \rightarrow y' = \frac{4y^2}{x^3}$ $(x \neq 0)$ 이게 푸는데요

$y = \frac{1}{2}x^2$

- 아무튼 해를 구해야 해요 → 해가 뭔데요
  - Solution : $y = h(x)$: function (differentiable on some open interval) $a < x < b$?

    → $F(x, h(x), h'(x)) = 0$ on $a < x < b$: 진짜 solution

    ex) $y' = \cos x$ 빼먹으면 안된다 $y = \sin x + \boxed{C}$ at $x \in \mathbb{R}$

    $y' = \cos x \rightarrow \cos x = \cos x$ on $\mathbb{R}'$

+) Constant

$y = h(x) \ni C$: C에 따라서 달라짐

↳ General Solution $\ni$ particular solution : C가 정해짐

3. Particular Solution

- Initial Value Problem (IVP) : Find General solution

  ↳ $y' = f(x, y)$, $y(x_0) = y_0$ (Initial condition) → particular S.

  $\Rightarrow$ $y = h(x)$: Solution $\begin{cases} h'(x) = f(x, h(x)) \text{ on } a<x<b \\ h(x_0) = y_0 \end{cases}$

  Ex) $y' = 3y$, $y = h(x) = ce^{3x}$ on $\mathbb{R}^1$

  $y' = 3ce^{3x} = h'(x) = 3ce^{3x}$ on $\mathbb{R}^1$

  If $y(0) = 5.7$, $y(0) = c \cdot 1 = 5.7$, so $c = 5.7$ → $y = 5.7e^{3x}$ on $\mathbb{R}^1$

4. Direction Field

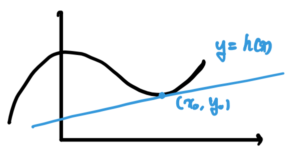
- $y' = \underbrace{f(x, y)}_{\text{function}}$: f. ODE

  $(x_0, y_0) \in \mathbb{R}^2$

  $y = h(x)$ $y_0 = h(x_0)$

  $\Rightarrow$ $y'(x_0) = h'(x_0) = \underbrace{f(x_0, y_0)}_{\text{real number}}$

  궁극) 그래프 → 기울기 → 해를 몰라도 알 수 있다



  $y = ax + b$ ⟶ $y - y_0 = f(x_0, y_0)(x - x_0)$

  $\Rightarrow a = y'(x_0)$ Linear approximation

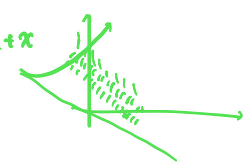
  기하적 측면인거로

  $y = h(x)$ 의

  $\Rightarrow$ 기울기를 표현하는 식이구나!

  Direction

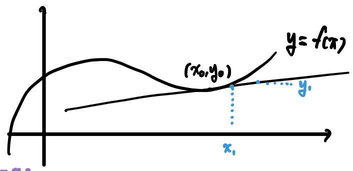
  ex) $y' = y + x$



  그리는 법) Sagemath

5. Euler's Method

- $y' = f(x, y)$, $y(x_0) = y_0$, $y = h(x)$



$\Rightarrow$ $x_1 = x_0 + c$. $y_1 = \underbrace{y_0 + c f(x_0, y_0)}$: 근사값 (오차 - $h(x_1) - y_1$)
$\left(y - y_0 = f(x_0, y_0)(x - x_0) \right)$

$x_2 = x_1 + c$, $y_2 = y_1 + c f(x_1, y_1)$

$\vdots$

$x_n = x_{n-1} + c$, $y_n = y_{n-1} + c f(x_{n-1}, y_{n-1})$ WHY? : 해를 모르는데 $y = f(x_n)$을 구하고 싶음

↳ Solution과 Direction Field가 유사함을 이용한 거군 ↳ (x_0, y_0), $f(x, y)$를 알 때 $h(x_n)$을 알고 싶다

6. Seperable ODE

↳ $g(y) y' = f(x)$

How? $y = h(x)$ then $\int dy = \int \frac{dh}{dx} dx$,

$\int g(y) \underline{y' \, dx} = \int f(x) \, dx + C$

$\int g(y) \cdot \underbrace{\frac{dh}{dx} \, dx}_{dy} = \int f(x) \, dx + C$ ($g(y)$, $f(x)$ 가 연속)

생각해보니까 $g(h(x))$ 형태만 다른거였음

Ex) $y' = 1 + y^2$ $\rightarrow$ $\frac{1}{1+y^2} y' = 1$ $\Rightarrow$ $\int \frac{1}{1+y^2} dy = \int dx$

$\Rightarrow$ $\tan^{-1} y = x + C$ $\Rightarrow$ $y = \tan(x + C)$ on $\mathbb{R}'$

+) 해는 항상 하나다

Ex) $y' = (x+1) e^{-x} y^2 \xrightarrow{(y \neq 0)} \frac{1}{y^2} y' = \frac{x+1}{e^x}$ $\rightarrow$ $-\frac{1}{y} = -(x+2) e^{-x} + C$

$\rightarrow$ $y = \frac{1}{(x+2) e^{-x} + C}$

7. Reduction to Seperable Form

↳ $y' = f(\frac{y}{x})$ (f is differentiable) $x \neq 0$

⇒ $\frac{y}{x} = u$, $y = ux$, $y' = u'x + u = f(u)$ → $u'x = f(u) - u$ → $\frac{u'}{f(u)-u} = \frac{1}{x}$

⇒ $\int \frac{1}{f(u)-u}\, du = \ln|x| + C$

Ex) $2xyy' = y^2 - x^2$

$\frac{2y}{x} y' = (\frac{y}{x})^2 - 1$ ($x \neq 0$) $\frac{y}{x} = u$, $2uy' = u^2 - 1$ $y' = \frac{(u - \frac{1}{u})}{2} = f(u)$

$\int -\frac{2u}{1+u^2}\, du = \ln|x| + C$ → $1 + u^2 = |x|^{-1} C_3$. → $x^2 + y^2 = cx$

8. Exact ODE

↳ $u(x,y)$: function with continuous partial derivatives

$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$: differential form / total differential

If $u(x,y) = c$, then $du = 0$ → $y' = -\frac{\partial u}{\partial x} / \frac{\partial u}{\partial y}$ ⟶

solution

Ex) $u(x,y) = x + x^2y^3 = c$

$du = (1 + 2xy^3)\, dx + 3x^2y^2\, dy = 0$

$\frac{dy}{dx} = -\frac{1 + 2xy^3}{3x^2y^2}$ ⇒ $y' = -\frac{1 + 2xy^3}{3x^2y^2}$

↑ 모양새만 나누기

$f : \mathbb{R}^2 \to \mathbb{R}^1$ $f \circ \phi : \mathbb{R}^1 \to \mathbb{R}^1$

$\phi : \mathbb{R}^1 \to \mathbb{R}^2$

$\frac{d}{dt}(f \circ \phi) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{dh}{dt} \end{bmatrix} = 0$

$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dh}{dt} = 0$

$= f(t, h(t)) = 0$

→ implicit solution

→ $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \underbrace{\frac{dy}{dx}}_{y'} = 0$ 2개 연이별 곱이

↑ 이거의 역순

- $M(x,y) + N(x,y)\, y' = 0$ (diff, conti.)

→ $M(x,y)\, dx + N(x,y)\, dy = 0$ (c.f.) $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$)

$M \downarrow \frac{\partial u}{\partial x}$ $N \downarrow \frac{\partial u}{\partial y}$ ⇒ Find $u(x,y)$

→ $u(x,y)$ 가 존재하는 ODE = Exact

- 판별 방법

$$M = \frac{\partial u}{\partial x},\ N = \frac{\partial u}{\partial y} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ → exact

- 찾는 방법

$$M = \frac{\partial u}{\partial x} \rightarrow u = \int M\,dx + k(y)$$

$$N = \frac{\partial u}{\partial y} \rightarrow \frac{\partial}{\partial y}\left(\int M\,dx + k(y)\right) = N$$

↳ 찾아졌다 → ~~$du = 0$ 임을 이용해~~ $u(x,y)$ 자체가 해다

Ex) $\cos(x+y) + (3y^2 + 2y + \cos(x+y))\,y' = 0$

$$\frac{\partial}{\partial y}(\cos(x+y)) = -\sin(x+y)$$

$$\frac{\partial}{\partial x}(\ '' \ ) = -\sin(x+y). \quad \Rightarrow \text{exact ODE,}$$

$$\sin(x+y) + k(y) + c'$$

$$c^2 + y^3 + y^2 + \sin(x+y). \quad \rightarrow u(x,y) = \sin(x+y) + y^3 + y^2 + c^*$$

general solution

- $u(x,y) = c$ 가 주어졌을 때

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0,$$

$$\rightarrow y' = -\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y}$$ → first-order ODE

→ $u(x,y) = c$: implicit solution

Ex) $\cos y \sinh x + 1 - \sin y \cosh x\ y' = 0.$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -\sin x \cosh x = 0.$$

$$u(x,y) = +\cos y \cosh x + x = c$$

Ex) $-y + xy' = 0.$

not Exact, $y' = \frac{y}{x}$ $(x \neq 0)$, $u = \frac{y}{x}$. $y' = u = f(u)$

$y = ux$. $y' = u + u'x = u$. $u'x = 0$. $y = 0$!

($\frac{1}{x^2}$ integration factor) exact ← $-\frac{y}{x^2} + \frac{y'}{x} = 0$. $u = \frac{y}{x} + c$

- $P(x,y) + Q(x,y)\,y' = 0$ is not exact

→ $FP(x,y) + FQ(x,y)\,y' = 0$ is exact.

→ $F_y P + F P_y = F_x Q + F Q_x$ ($F(x)$ 이다)

→ $F P_y = F' Q + F Q_x$ → $F' Q = F(P_y - Q_x)$ → $\frac{F'}{F} = \frac{P_y - Q_x}{Q} = R(x)$

⇒ $F(x) = e^{\int R(x)\,dx}$

① $F(x)$, $F(y)$ 찾아내기

② Exact 풀기