• Linear independence of vectors

$\mathbb{R}^n \ni (x_1 \dots x_n)$, $a_{(1)} \in \mathbb{R}^n$ $\quad a_{(1)} = (a_{11}, a_{12}\ a_{13} \cdots a_{1n}) \Rightarrow$ vector

$\vdots$

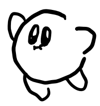
$a_{(m)} \in \mathbb{R}^n$

◦ linear combination

$c_1 a_{(1)} + \cdots + c_m a_{(m)}$, $c_i \in \mathbb{R}^1$ $\quad c_1 a_{(1)} = (c_1 a_{11}\ c_1 a_{12} \cdots\ c_1 a_{1n})$

$a_{(1)} + a_{(2)} = (a_{11} + a_{21}\quad a_{12} + a_{22} \cdots\ a_{1n} + a_{2n})$

$\rightarrow x_1 a_{(1)} + \cdots + x_m a_{(m)} = \vec{0}$ $\quad (0 \in \mathbb{R}^n,\ 0 = (0, 0, \cdots 0))$



(쓰기 귀찮아 아 아) → 선형 연립 방정식

◦ linearly independent

$x_1 a_{(1)} + \cdots + x_m a_{(m)} = \vec{0}$ if $x_{(m)} = \vec{0}$ is an unique solution,

then $a_{(1)}, a_{(2)} \dots a_{(m)}$ are linearly independent

ex) $a_{(1)} \dots a_{(m)} \in \mathbb{R}^n$ are linearly dependent

$x_1 a_{(1)} + \cdots + x_m a_{(m)} = \vec{0}$, $\quad (c_1 \dots c_m) \neq \vec{0}$ (solution)

$c_1 a_{(1)} + \cdots + c_m a_{(m)} = \vec{0}$ $\quad$ assume that $c_1 \neq 0$

$\rightarrow a_{(1)} = -\frac{1}{c_1}(c_2 a_{(2)} + \cdots + c_m a_{(m)})$

근데 그럼 결국 연방 해의 계수로 판단하는 거 아님?

+) span

independent 한 vectors 를 가지고

$\mathbb{R}^2$의 모든 vector를 표현할 수 있다

목적 : 선형 연립방정식에서 필요없는 방정식 제거

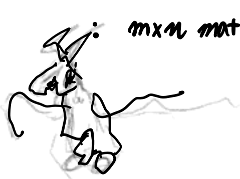
· Rank

A : m×n matrix

rank(A) = r (r>0) : 해당 matrix의 방정식을 풀기 위한 최소 식 개수

real def : the maximum number of linearly independent row vectors of A

• row-equivalent matrix



: m×n matrix $A, \hat{A}$ $\quad A \xrightarrow[\text{② multiply by constant.}]{\text{① change row}} \hat{A}$

- row equivalent : same rank (∴ row

- row - echelon form

  ① 0이 아닌 것을 찾는다 $\begin{bmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \end{bmatrix}$

  $\Rightarrow \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \boxed{S} \\ \boxed{0} & \end{bmatrix}$

  $\Rightarrow (1, 0, 0)$

- $\text{rank}(A) = \text{rank}(A^T)$

  $A : \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \boxed{S} \\ \boxed{0} & \end{bmatrix} \rightarrow A^T = \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \boxed{0} \\ \boxed{S} & \end{bmatrix}$

  찾는 방법은 같다

  0 (다 곱해서 뺀다)

Vector space

$\mathbb{R}^n$ (dimension) : vector space , $(x_1 \dots x_n) \in \mathbb{R}^n \rightarrow c(x_1 \dots x_n) \in \mathbb{R}^n$

$\dim \mathbb{R}^n = n$: 무슨 그리...라구요 or 하나 생략하여

> $e_1 \dots e_n \in \mathbb{R}^n$ are linearly inde. $\alpha \in \mathbb{R}^n$, $(= c_i \alpha_n)$ $c_i \in \mathbb{R}^n$

Homogeneous linear system

$$\Rightarrow \begin{matrix} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \vdots \qquad \ddots \qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{matrix} \Rightarrow \begin{bmatrix} a_{11}x_1 + \cdots + a_{m1}x_1 \\ \vdots \\ \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \underline{0}$$

homogeneous

$$\Leftrightarrow \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix} x_1 + \cdots + \cdots = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- if rank A < n then there is a nontrivial solution

- **Null Space** $= N$ (vector space $\mathbb{R}^n$)

A : $m \times n$ matrix $\quad A\vec{x} = 0, \quad x_1, x_2$ solutions

$Ax_1 = 0, \quad Ax_2 = 0$

N : the set of all solutions of $A\vec{x} = 0$

$\dim N$ = nullity.

A : $n \times m$ matrix $\qquad$ rank A + nullity $= n$.

Non-homogeneous system

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$$

(1) $A\vec{x} = \vec{b}$ consistent (has a solution)

$\tilde{A}$: augmented matrix of A $\rightarrow$ $\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ & \ddots & & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$

rank$(A) \le$ rank$(\tilde{A})$ $\quad$ ($\underbrace{\text{rank}(\tilde{A}) = \text{rank}(A)}_{\text{해가 있다}}$ or rank$(A) + 1 =$ rank$(\tilde{A})$)

rank$(A) = r \le n$

$$c_1 \underbrace{\begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + c_r \begin{bmatrix} a_{1r} \\ \vdots \\ a_{mr} \end{bmatrix}}_{\text{linearly indep.}} + c_{r+1} \begin{bmatrix} b_1 \\ \\ b_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \qquad -\frac{(c_1 + c_r + \cdots)}{c_{r+1}} : \text{solution}$$

(2) #(solution)

① $A = n \times m$ matrix, rank $A = n$.

Then the system has an unique solution

rank(A) = rank$(\tilde{A}) = n$

$a_{(1)} \cdots a_{(n)}$: columns of A

$\sum_{i=1}^{n} c\, a_{(i)} = \sum_{i=1} \bar{c}\, a_{(i)} \qquad \bar{c} = c$

rank(A) = rank$(\tilde{A}) = r < n$.

c : solution of $A\vec{x} = \vec{b}$

$Ac = b$, $A\vec{x}$'s Null space N

$\bar{B} = \{c + \alpha \mid \alpha \in N\}$, $\bar{c} = c + \alpha$, $A\bar{c} = Ac + \cancel{A\alpha} = b$.

$\dim B = n - r$