Kattis

More Dynamic Programming

Arnar Bjarni Arnarson & Atli FF

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School of Computer Science

Reykjavík University

What is dynamic programming?

- A problem solving paradigm
- Similar in some respects to both divide and conquer and backtracking
- Divide and conquer recap:
  - Split the problem into *independent* subproblems
  - Solve each subproblem recursively
  - Combine the solutions to subproblems into a solution for the given problem
- Dynamic programming:
  - Split the problem into *overlapping* subproblems
  - Solve each subproblem recursively
  - Combine the solutions to subproblems into a solution for the given problem

Dynamic programming formulation

- Formulate the problem in terms of smaller versions of the problem (recursively)
- Turn this formulation into a recursive function
- Memoize the function (remember results that have been computed)

Dynamic programming formulation

```
map<problem, value> memory;

value dp(problem P) {
    if (is_base_case(P)) {
        return base_case_value(P);
    }

    if (memory.find(P) != memory.end()) {
        return memory[P];
    }

    value result = some value;
    for (problem Q in subproblems(P)) {
        result = combine(result, dp(Q));
    }

    memory[P] = result;
    return result;
}
```

DP over bitmasks

- Remember the bitmask representation of subsets?
- Each subset of n elements is mapped to an integer in the range $0, \ldots, 2^n - 1$
- This makes it easy to do dynamic programming over subsets

Traveling salesman problem

- We have a graph of n vertices, and a cost $c_{i,j}$ between each pair of vertices i, j. We want to find a cycle through all vertices in the graph so that the sum of the edge costs in the cycle is minimal.
- This problem is NP-Hard, so there is no known deterministic polynomial time algorithm that solves it
- Simple to do in $O(n!)$ by going through all permutations of the vertices, but that's too slow if $n > 11$
- Can we go higher if we use dynamic programming?

Traveling salesman problem

- Without loss of generality, assume we start and end the cycle at vertex 0
- Let $\mathrm{tsp}(i, S)$ represent the cheapest way to go through all vertices in the graph and back to vertex 0, if we're currently at vertex i and we've already visited the vertices in the set S
- Base case: $\mathrm{tsp}(i, \text{all vertices}) = c_{i,0}$
- Otherwise $\mathrm{tsp}(i, S) = \min_{j \notin S} \{ c_{i,j} + \mathrm{tsp}(j, S \cup \{j\}) \}$

Traveling salesman problem

```
const int N = 20;
const int INF = 100000000;
int c[N][N];
int mem[N][1<<N];
memset(mem, -1, sizeof(mem));
int tsp(int i, int S) {
    if (S == ((1 << N) - 1)) {
        return c[i][0];
    }
    if (mem[i][S] != -1) {
        return mem[i][S];
    }
    int res = INF;
    for (int j = 0; j < N; j++) {
        if (S & (1 << j))
            continue;
        res = min(res, c[i][j] + tsp(j, S | (1 << j)));
    }

    mem[i][S] = res;
    return res;
}
```

Traveling salesman problem

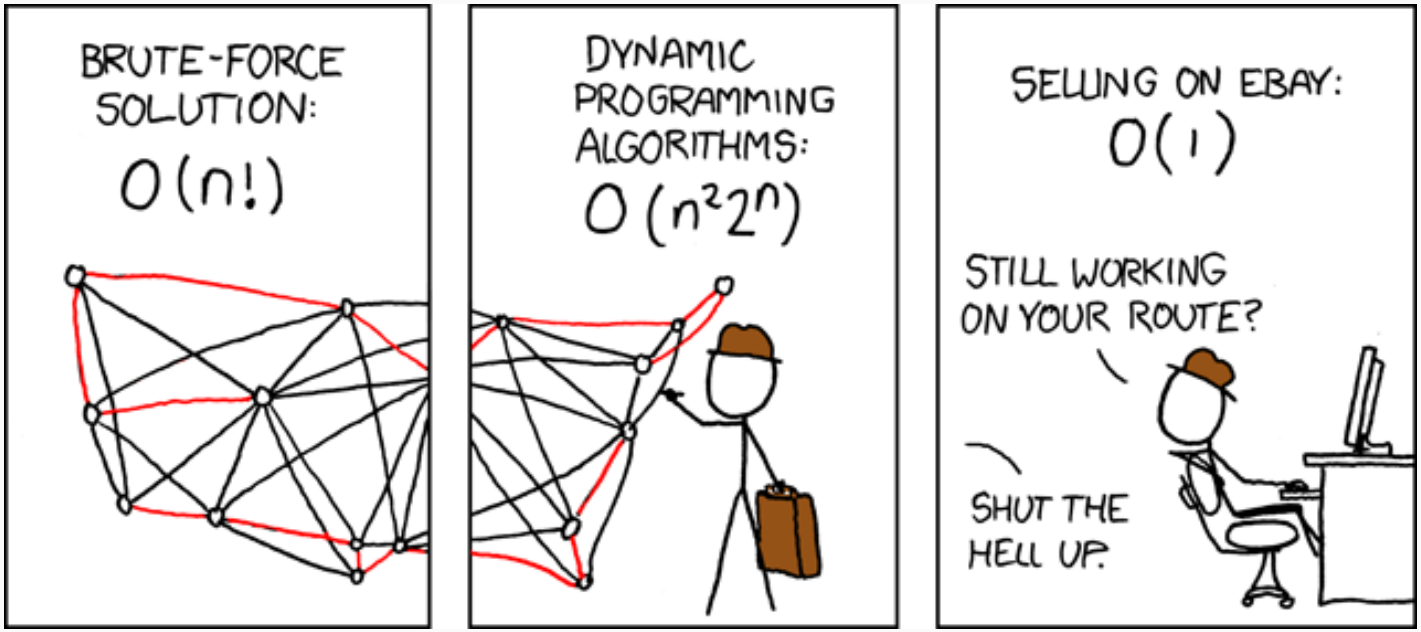
- Then the optimal solution can be found as follows:

```
printf("%d\n", tsp(0, 1<<0));
```

Traveling salesman problem

- Time complexity?
- There are $n \times 2^n$ possible inputs
- Each input is computed in $O(n)$ assuming recursive calls are $O(1)$
- Total time complexity is $O(n^2 2^n)$
- Now n can go up to about 20

Traveling salesman problem



Subset sum problem

- Another common dynamic programming task is known as the subset sum problem.
- Given n positive integers $a_1, \ldots, a_n$ find if there is a subset with sum c. Variants also include finding the sum closest to c, greatest sum not exceeding c and so on.
- The naïve solution here would involve checking every subset, which if done efficiently (for example with gray codes) takes $O(2^n)$, which is quite slow.

Subset sum problem

- Let $f(i, s)$ be a boolean function answering whether there exists a subset of $a_1, \ldots, a_i$ with sum s.
- Then

$$f(i,s) = \begin{cases} \texttt{true} \text{ if } i = s = 0 \\ \texttt{false} \text{ if } i = 0, s \neq 0 \\ \texttt{false} \text{ if } s < 0 \\ f(i-1,s) \ \texttt{ or } \ f(i-1,s-a_i) \text{ otherwise} \end{cases}$$

- The different variants can then be read from the values of f. Each state takes $O(1)$ to compute, and there are $n(a_1 + \cdots + a_n)$ states. Denoting the sum by Σ this makes our time complexity $O(n\Sigma)$, which isn't great, but is often better than $O(2^n)$.

Subset sum problem

```
const int N = 20;
const int SIGMA = 10000;
int a[N];
int dp[N][SIGMA];
// use int so -1 is unmemoized
// 0 and 1 are the bools as usual
bool subsetsum(int i, int s) {
    if(i < 0) return s == 0;
    if(s < 0) return false;
    if(dp[i][s] != -1) return dp[i][s];
    return dp[i][s] = subsetsum(i - 1, s) || subsetsum(i - 1, s - a[i]);
}
```

Subset sum problem - variant

- Say we want to find the most even way to split the numbers into two groups, that is to say in a way that minimizes the difference of the sums of the two groups.
- Furthermore we want to actually output these numbers rather than just the difference in sum.
- We can use the subset sum solution to do this, simply adding a table that keeps track of what choices we made at what point.

Subset sum problem

```
#include <bits/stdc++.h>
using namespace std;
vector<int> a;
vector<vector<int>> dp, mv;
bool subsetsum(int i, int s) {
        if(i < 0) return s == 0;
        if(s < 0) return false;
    if(dp[i][s] != -1) return dp[i][s];
    dp[i][s] = 0;
    if(subsetsum(i - 1, s)) {
        dp[i][s] = 1;
        mv[i][s] = 1;
    } else if(subsetsum(i - 1, s - a[i])) {
        dp[i][s] = 1;
        mv[i][s] = 0;
    }
        return dp[i][s];
}
```

Subset sum problem

```
int main() {
        int n, sm = 0; cin >> n;
        a = vector<int>(n);
        for(int i = 0; i < n; ++i)
                cin >> a[i], sm += a[i];
        dp = mv = vector<vector<int>>(n, vector<int>(sm, -1));
        int bst = sm / 2;
        while(!subsetsum(n - 1, bst)) bst--;
        vector<bool> group1(n, false);
        for(int i = n - 1; i >= 0; --i) {
                if(!mv[i][bst]) {
                        group1[i] = true;
                        bst -= a[i];
                }
        }
        cout << "Difference: " << abs(bst - (sm - bst)) << '\n';
        cout << "Group 1: ";
        for(int i = 0; i < n; ++i) if(group1[i]) cout << a[i] << ' ';
        cout << "\nGroup 2: ";
        for(int i = 0; i < n; ++i) if(!group1[i]) cout << a[i] << ' ';
        cout << '\n';
}
```

Multidimensional DP comment

- The order in which you put the dimensions in a multidimensional dp can affect the running time.
- This is due to cache locality, so if you are fetching sequentially from one dimension and not the other, this can make one order faster.
- Usually it doesn't matter for determining AC/TLE, but in a few cases it might.

Knapsack problem

- The subset sum problem time complexity is exponential, as the sums of the numbers is exponential in the size of the actual input.
- Among similar "hard" problems (in terms of time complexity) is the knapsack problem.
- We have n items, each with some value and some weight. We also have a knapsack with a maximum weight capacity and want to maximize the value with respect to this condition.

Knapsack problem

- We can once more use dynamic programming to solve this.
- We let $f(i, j)$ be the maximum value we can get from the first i items if our maximum weight is j.
- Let $v_1, \ldots, v_n$ be the values and $w_1, \ldots, w_n$ the weights. Then

$$f(i,j) = \begin{cases} -\infty \text{ if } j < 0 \\ 0 \text{ otherwise if } i = 0 \\ \max(f(i-1,j), f(i-1,j-w_i) + v_i) \text{ otherwise} \end{cases}$$

- The time complexity is then $O(nS)$ where S is the sum of the weights.
- We'll leave translating this into code as an exercise.

Egg dropping problem

- The previous problems are all very well known and classic in computer science. Let us also take a slightly less common example called the egg dropping problem.
- We have a building with k floors and we wish to figure out from which floor we have to drop an egg so it breaks. I.e. for some x dropping it from the x-th floor the egg will break, but dropping it from the $(x-1)$-st floor the egg won't break it.
- If we have n eggs, how few trials can we get away with in the worst case?

Egg dropping problem

- Say we drop the egg from a floor y.
- If the egg breaks, we only need to check floors $< y$ and have one less egg. This is essentially the same problem again but with $y - 1$ floors and $n - 1$ eggs.
- If the egg doesn't break we only need to check floors $> y$, so the problem is again the same with $k - y$ floors and n eggs.
- Since we are looking at the worst case, our result is the worse of these two.
- Since we can choose any y, we take the best result among all y.

Egg dropping problem

- Let $f(n, k)$ be the minimum number of trials for n eggs and k floors.
- We note that if we have one egg, we must always go through the floors in order since we can't afford to break an egg.
- All together this gives us

$$f(i,j) = \begin{cases} 1 \text{ if } k = 1 \\ 0 \text{ if } k = 0 \\ k \text{ if } i = 1 \\ \min_{1 \leq x \leq k} 1 + \max(f(i-1, x-1), f(i, j-x)) \end{cases}$$

- We see that this takes $O(k)$ per state and we have $O(nk)$ states, so the time complexity is $O(nk^2)$.

Egg dropping problem

- As a side note, this can also be solved in $O(nk)$ with a different dynamic programming approach.
- Try considering calculating the maximum number of floors you can check with n eggs and k trials using dynamic programming.
- Using some more clever ideas this can even be brought down to $O(n \log(k))$, but we won't need this here.

Complex states

- Most problems so far have had a state representable with a fixed number of integer values.
- This has allowed us to use static arrays for memoization.
- What if the state is not easily representable by integers or parameters are dynamic in count?
- For example: a larger range of integers, a string, a vector, or a custom type

Complex states

```
map<vector<int32_t>, int32_t> mem;
int32_t dp(const vector<int32_t>& state) {
    vector<int32_t> next_state(state);
    int32_t result{ 0 };
    for (auto& x : next_state) {
        if (x > 0) {
            int32_t old = x;
            x /= 2;
            result += dp(next_state);
            x = old;
        }
    }
    return result;
}
```

Complex states

```
map<tuple<type1_t, type2_t, ...>, result_t> mem;
// alternatively use unordered_map, need to ensure state is hashable
result_t dp(type1_t param1, type2_t param2, ...) {
    const auto state{ tie(param1, param2, ...) };
    if (is_base_case(state)) {
        return 0; // or 1 or whatever it should be
    }
    if (const auto it{ mem.find(state) }; it != mem.end()) {
        return it->second;
    }
    result_t result{};
    // compute answer recursively
    return mem[state] = result;
}
```

Advanced Dynamic Programming

- There are some methods to apply to optimize dynamic programming solutions.
- Here memory optimizations can be just as important as time optimizations.
- Binary search can often be used to find the other end of a transition.
- Instead of storing the memory in an array, using another data structure may fit your dynamic programming solution better.
- A few more involved tricks exist, for example the Knuth optimization.

Simple memory optimization

- DP structure is a D-dimensional table. For example, $D = 3$.
- Often $dp[i][j][k]$ depends only on $dp[i-1][a][b]$ for some a and b, meaning there are clear *levels*.

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- DP structure is a D-dimensional table. For example, $D = 3$.
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- If our approach is top-down, this does not help us at all.
- But if our approach is bottom-up, it is building a new level on top of the most recently computed level each time.

Simple memory optimization

- DP structure is a D-dimensional table. For example, $D = 3$.
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- If our approach is top-down, this does not help us at all.
- But if our approach is bottom-up, it is building a new level on top of the most recently computed level each time.
- Only need to store two levels at a time: the most recently computed and the one we're currently computing.
- Same idea works if dependencies have max depth m, then have $m+1$ levels and index the level modulo $m+1$.

Kaffiskömmtun

- Want to maximize amount of coffee given
- $N \leq 10^6$ professors, each batch fills $C \leq 500$ cups, new batch takes T seconds to make.
- Given $1 \leq t_1 \leq t_2 \leq \cdots \leq 10^9$, the times at which each person arrives.
- Each person can take coffee, if batch is not empty, and then choose to make a new batch or leave.
- How to formulate a solution?

Kaffiskömmtun - first attempt

- Let $dp(i, j)$ return max coffee cups people $i, \ldots, N-1$ can get if j cups remain in the batch when person i arrives.
- Then $dp(0, 0)$ is the final answer.
- Base case is $dp(n, j) = 0$.
- Recursive case is

$$dp(i,j) = \max \begin{pmatrix} \mathrm{dp}(k, C) & \text{if } j = 0 \\ 1 + \mathrm{dp}(i+1, j-1) & \text{if } j \neq 0 \\ 1 + \mathrm{dp}(k, C) & \text{if } j \neq 0 \end{pmatrix}$$

  where k is the smallest index such that $t_i + T \leq t_k$.

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  where k is the smallest index such that $t_i + T \leq t_k$.
- There are two issues with this solution.
- Time complexity is $\mathcal{O}(N^2 C)$.
- Memory used is $\mathcal{O}(NC)$ or $4 \cdot 10^6 \cdot 500 = 2 \cdot 10^9$ bytes.

Kaffiskömmtun - second attempt

- Lets reframe $dp(i)$ to mean how many people of $i, \ldots N-1$ can get coffee if person i arrives to meet an empty batch.

```
vector<int> dp(n+1, 0);
int res = 0;
for (int i = n-1; i >= 0; i--) {
    int start = i+1;
    while (start < n && times[start] < times[i]+t) start++;
    int end = min(start+c, n);
    for (int j = start; j < end; j++) {
        dp[i] = max(dp[i], j - start + dp[j] + 1);
    }
    res = max(res, dp[i]);
}
```

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}
```

- Note that times are in increasing order, so we can binary search the first that works.
- Memory becomes $\mathcal{O}(N)$ and time $\mathcal{O}(N \log N + NC)$.