

Convex Hull Optimization

Arnar Bjarni Arnarson

Árangursrík forritun og lausn verkefna

School of Computer Science

Reykjavík University

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Kalila and Dimna in the Logging Industry

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- We want to minimize the total charge cost to cut all trees.

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- Since $b_n = 0$ we must only cut the largest tree, at that point all cuts are free.
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- It is also quite clear that once we start cutting a tree, we should finish cutting it before starting to cut others.

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- A line!

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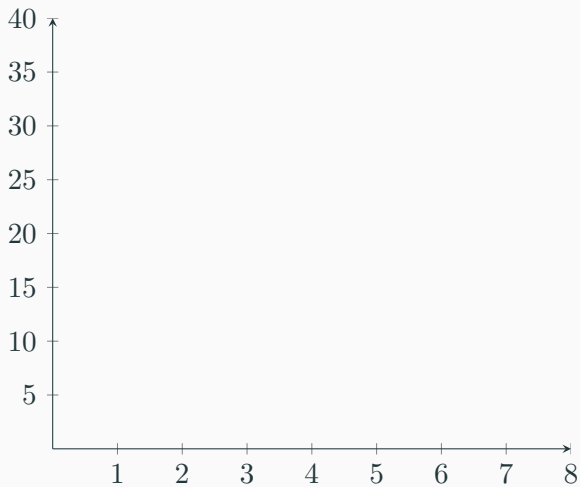
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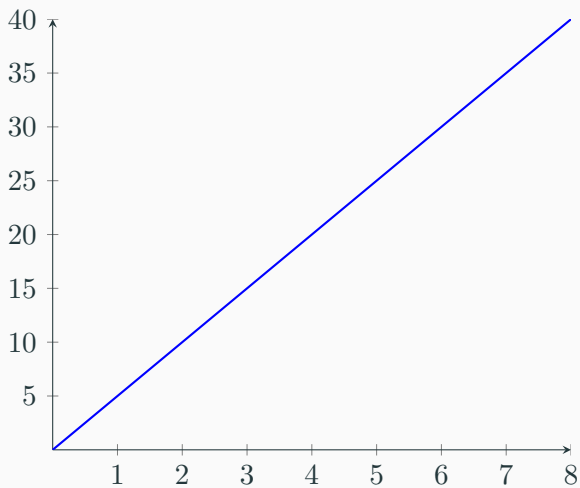
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- We need both operations to be sub-linear in time complexity.

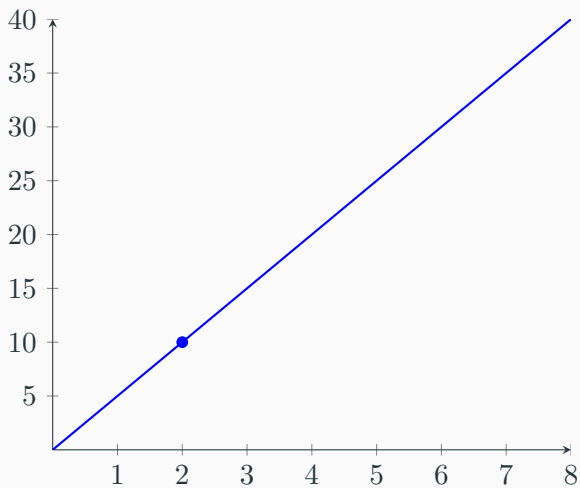
Sample 1 - Illustrated



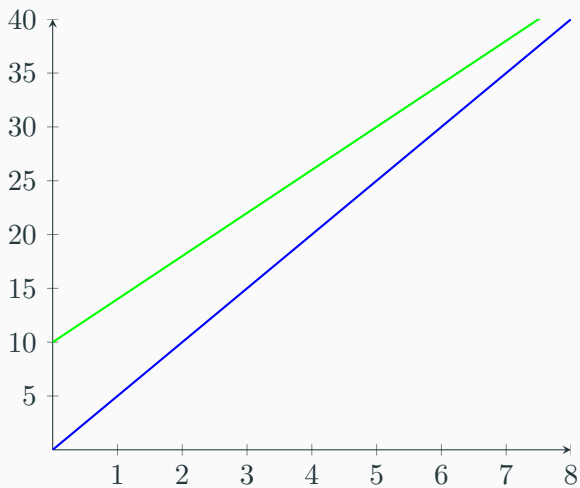
Sample 1 - Illustrated



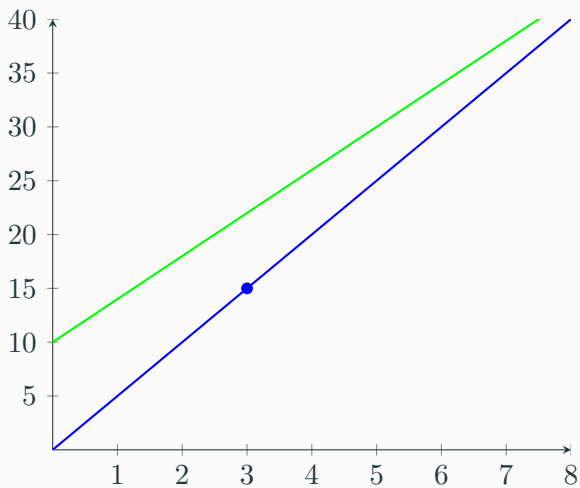
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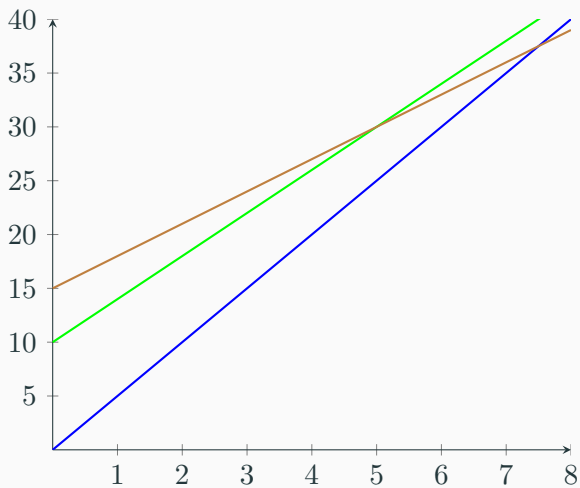
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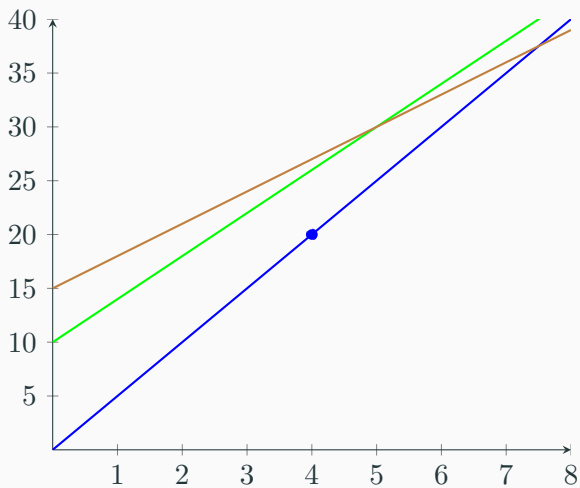
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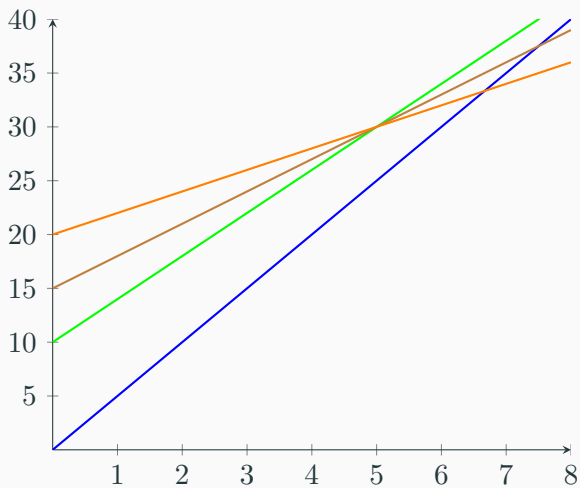
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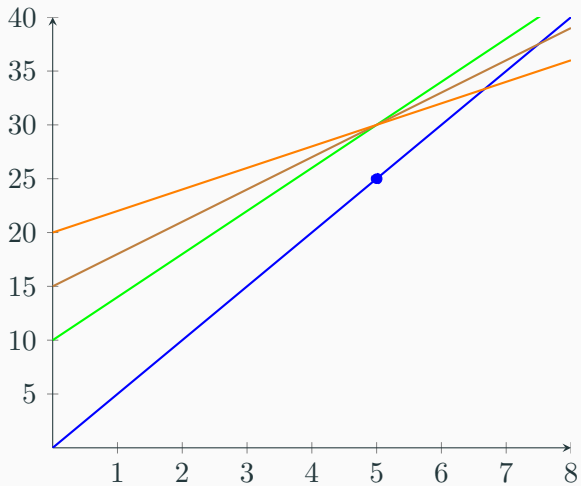
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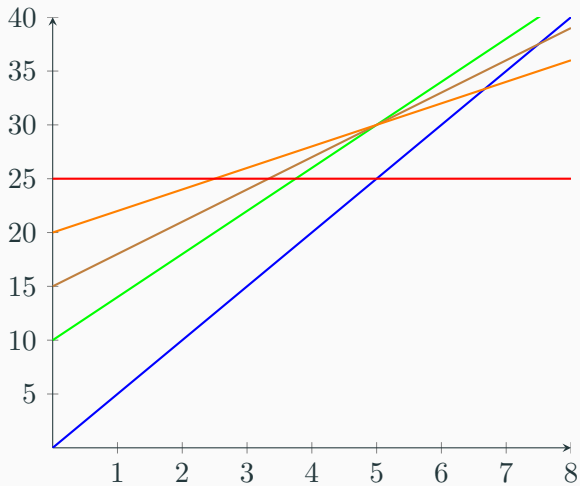
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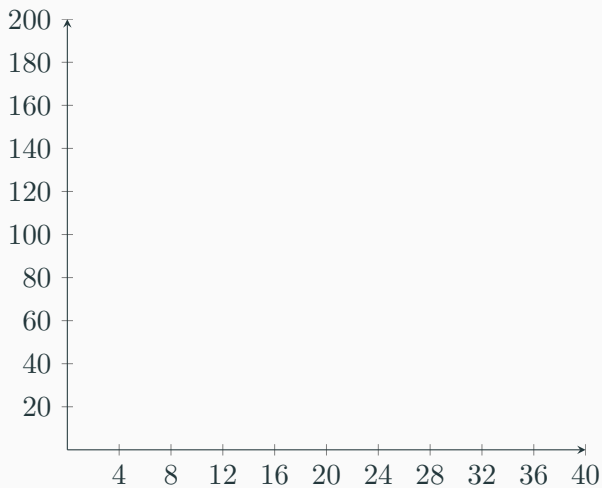
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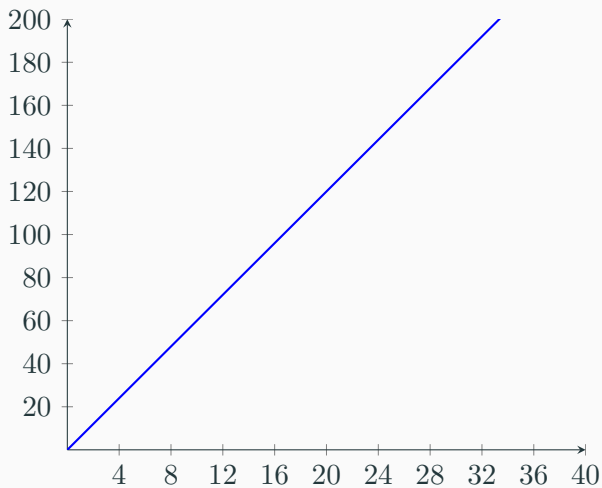
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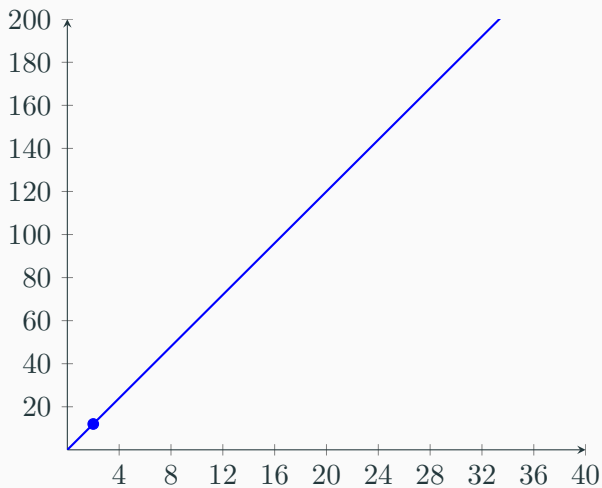
Sample 2 - Illustrated



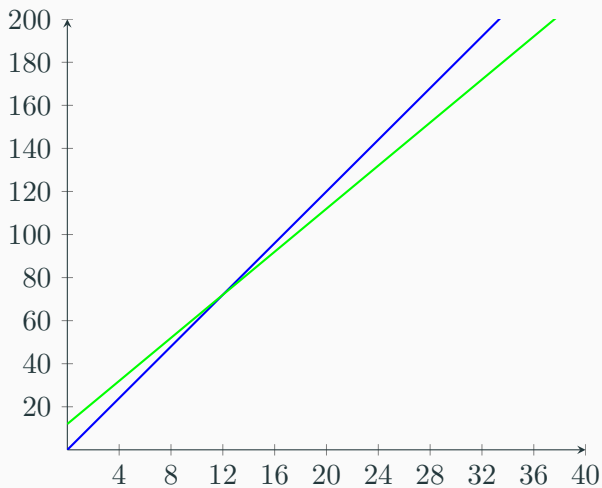
Sample 2 - Illustrated



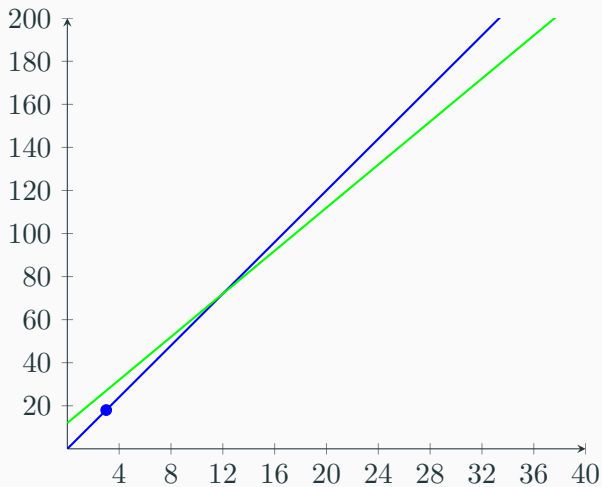
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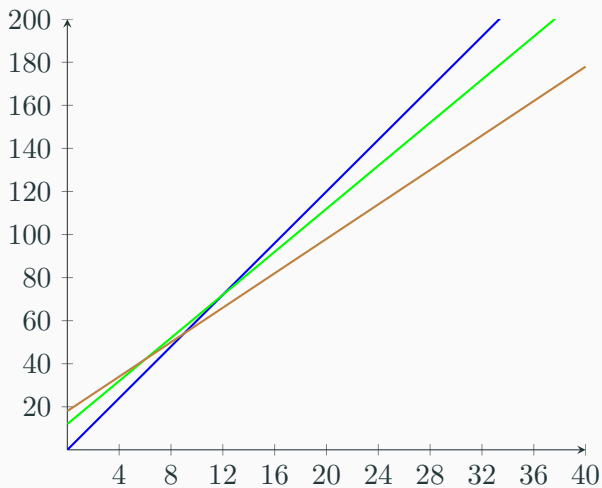
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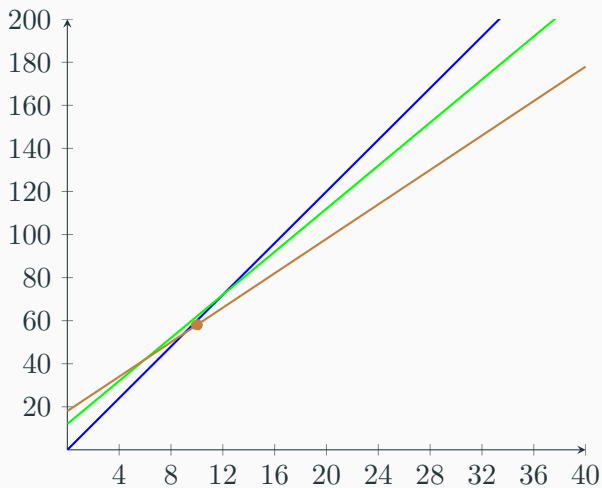
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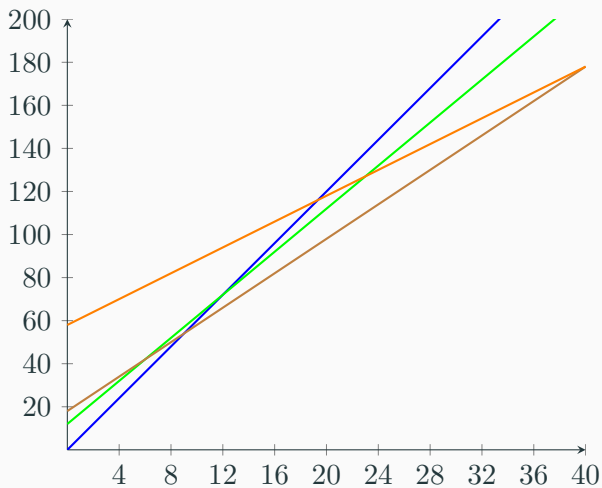
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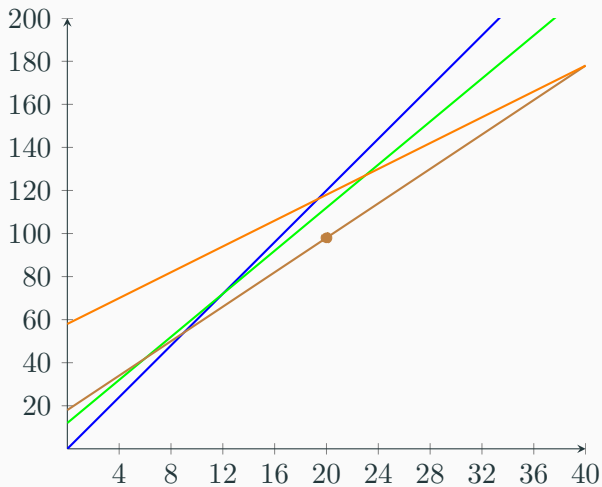
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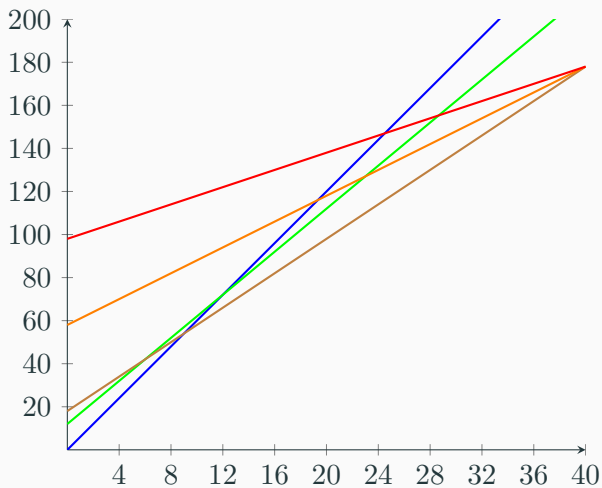
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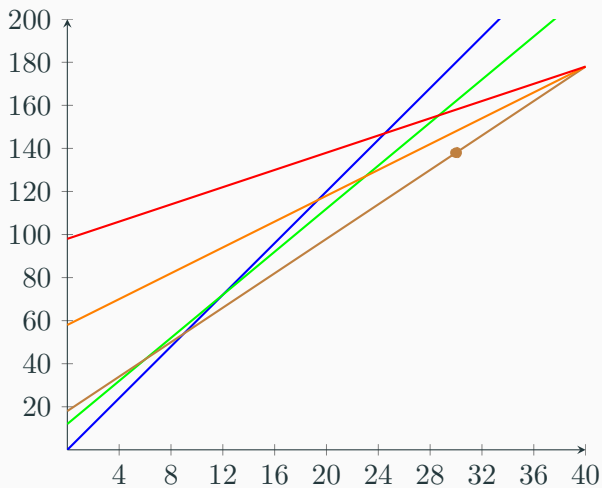
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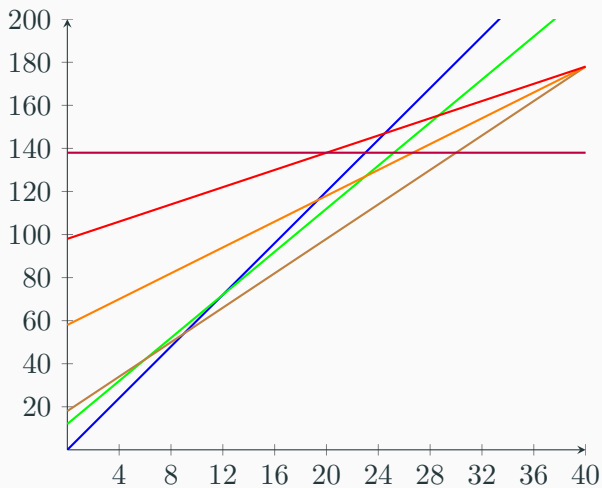
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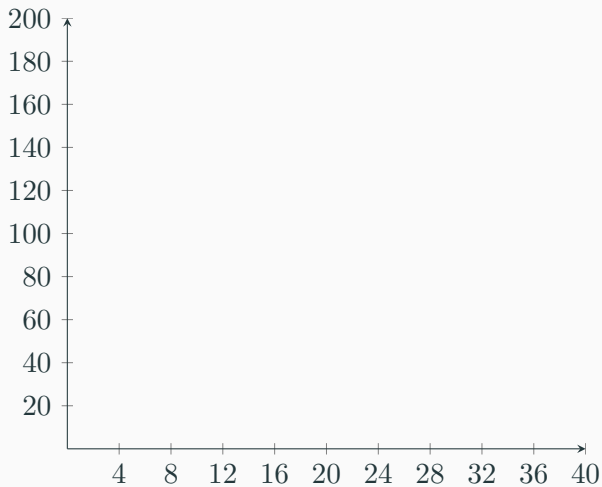
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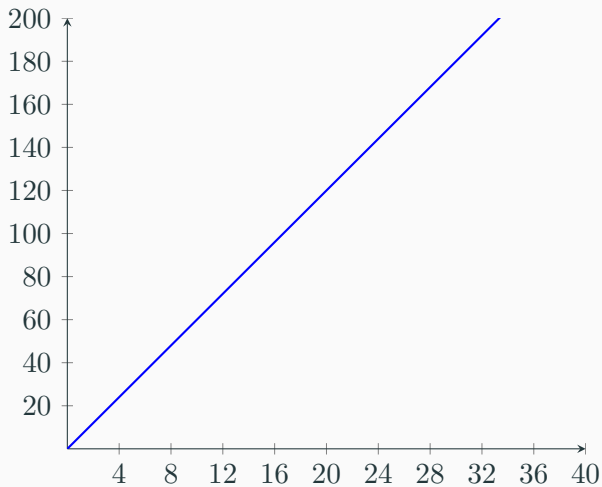
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- We can therefore iteratively pop redundant lines from the back before adding a line.

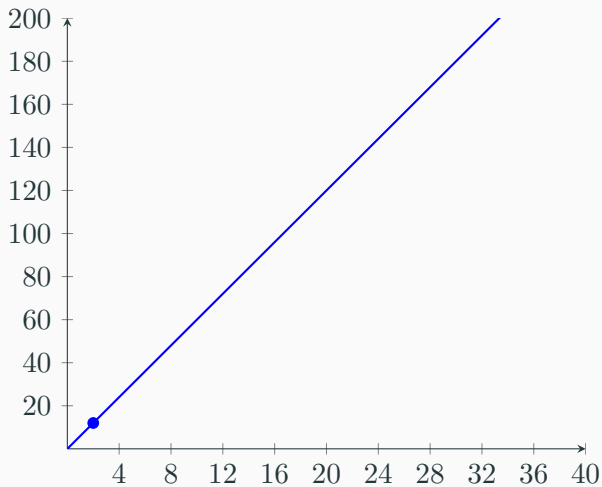
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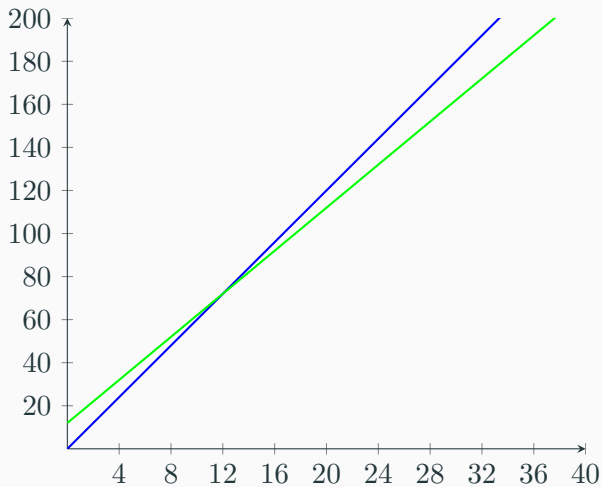
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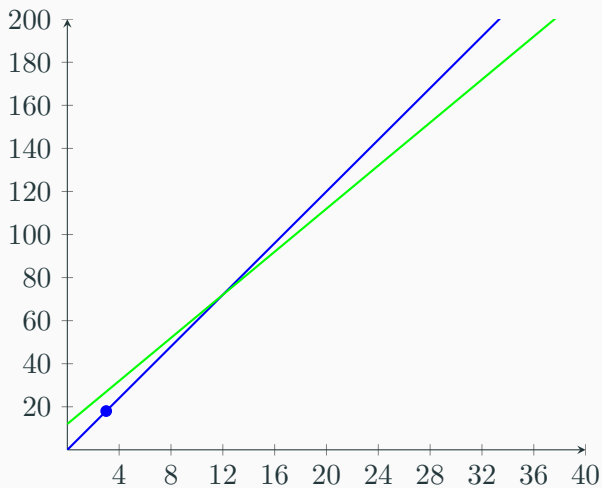
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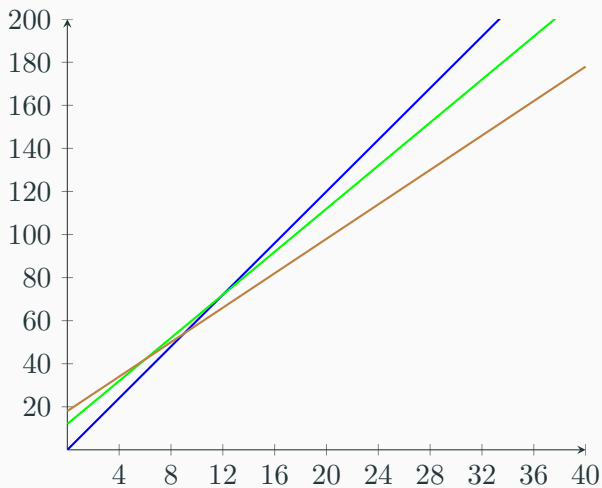
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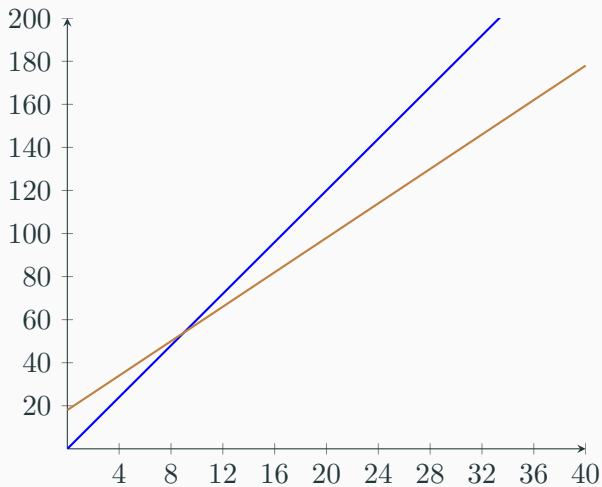
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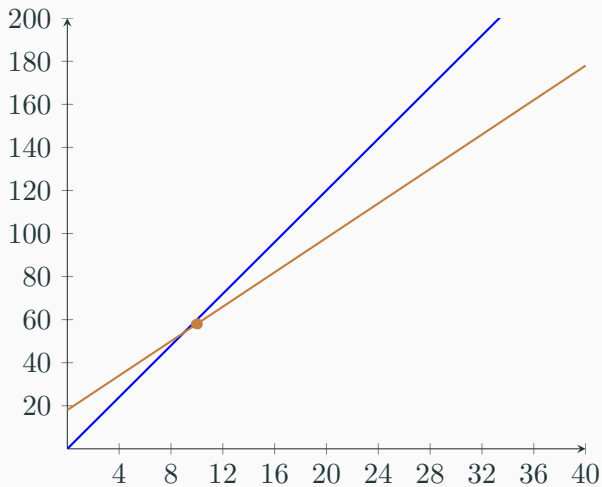
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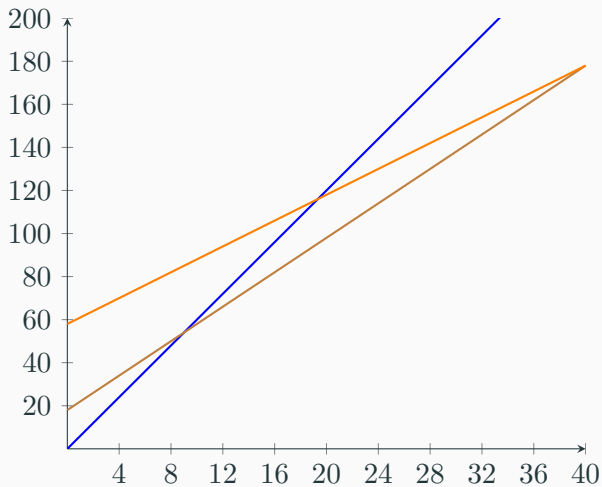
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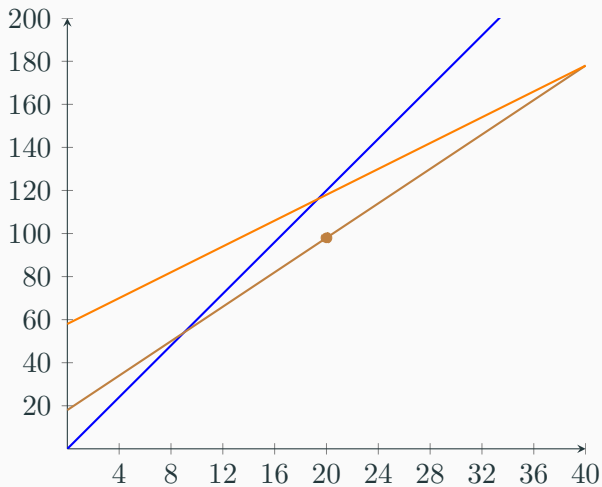
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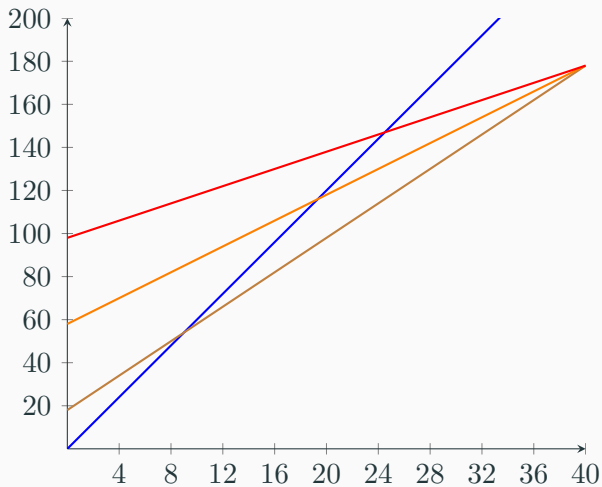
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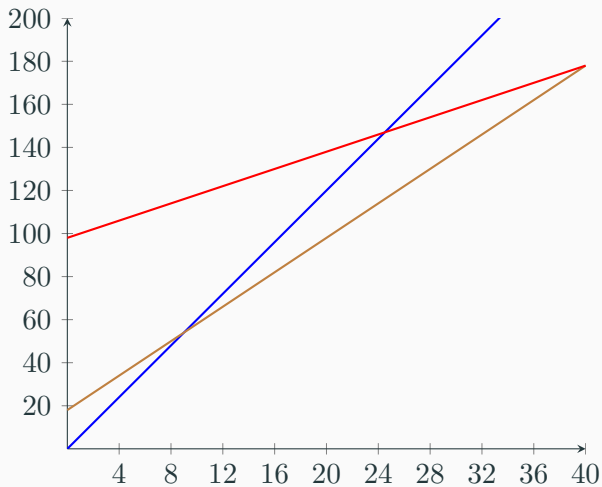
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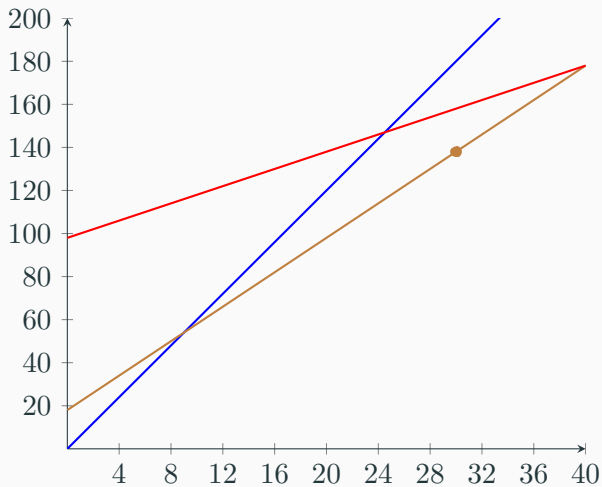
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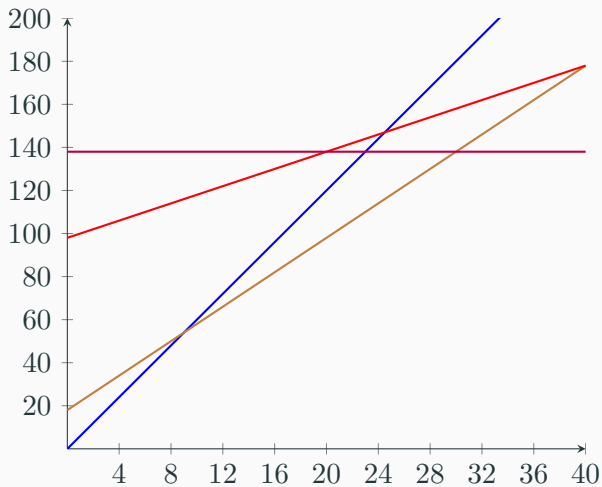
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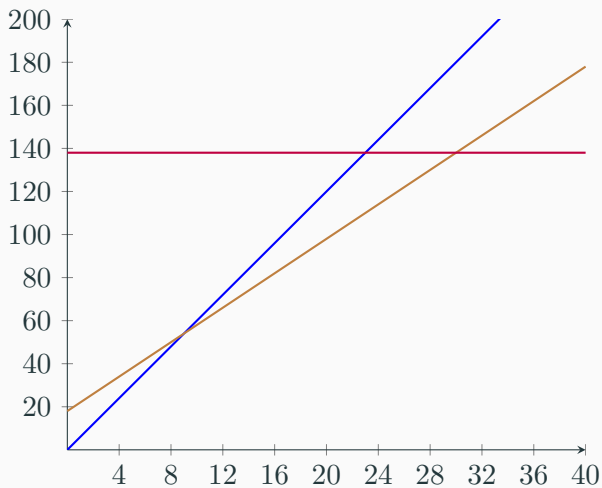
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- Construction takes $\mathcal{O}(n)$ time
- Each query takes $\mathcal{O}(\log n)$ time.
- We have improved the time complexity from $\mathcal{O}(n^2)$ to $\mathcal{O}(n \log n)$.

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- Use a multiset instead of a vector/array to keep track of lines and quickly find insertion point. A bit more hassle but powerful.
- Need to consider removing neighbouring lines with higher and lower slopes.

Simple Implementation

```
struct line {
    ll m, b;
    ll get(ll x) { return m*x + b; };
    ll intersect(line other) { return (other.b - b) / (m - other.m); }
};

struct convex_hull_trick {
    vector<line> lines;
    void add(line l) {
        auto sz = lines.size();
        while (
            sz >= 2 &&
            lines[sz-2].intersect(lines[sz-1]) >= lines[sz-2].intersect(l)
        ) {
            lines.pop_back();
            sz--;
        }
        lines.push_back(l);
    }
    // to be continued...
```

Simple Implementation - Continued

```
// ...continued
ll query(ll x) {
    int lo = 0, hi = static_cast<int>(lines.size()) - 2;
    int ind = hi+1;
    while (lo <= hi) {
        int mid = (lo+hi)/2;
        if (lines[mid].intersect(lines[mid+1]) >= x) {
            ind = mid;
            hi = mid-1;
        }
        else {
            lo = mid+1;
        }
    }
    return lines[ind].get(x);
}
};
```

Try on these problems!

- Kalila and Dimna in the Logging Industry
- Covered Walkway
- Commando
- Avoiding Airports