Kattis

# Strings & Geometry

Atli FF

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School of Computer Science

Reykjavík University

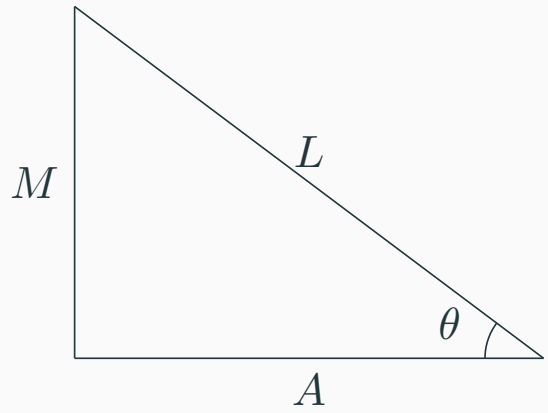
# Today we're going to cover

- Trigonometry
- Geometry
- Computational geometry
- String matching (naive, KMP)
- Tries
- Aho-Corasick

# Trigonometry

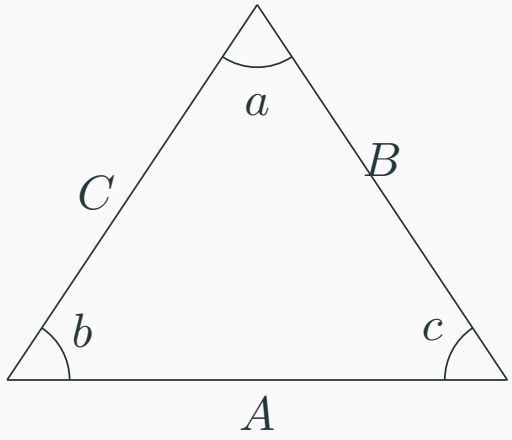
# Trigonometry

- Before we even dive into the geometry and how to do it on a computer, let's jog your memories.
- You should all hopefully be familiar with the trigonometric functions.
- We consider a triangle to be right-angled if it has a corner that's $90°$.
- For such triangles we have:
  - $\frac{A}{L} = \cos\theta.$
  - $\frac{M}{L} = \sin\theta.$
  - $\frac{M}{A} = \frac{M}{L}\frac{L}{A} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$
- We also have the pythagorean theorem $L^2 = A^2 + M^2$.



# More trig

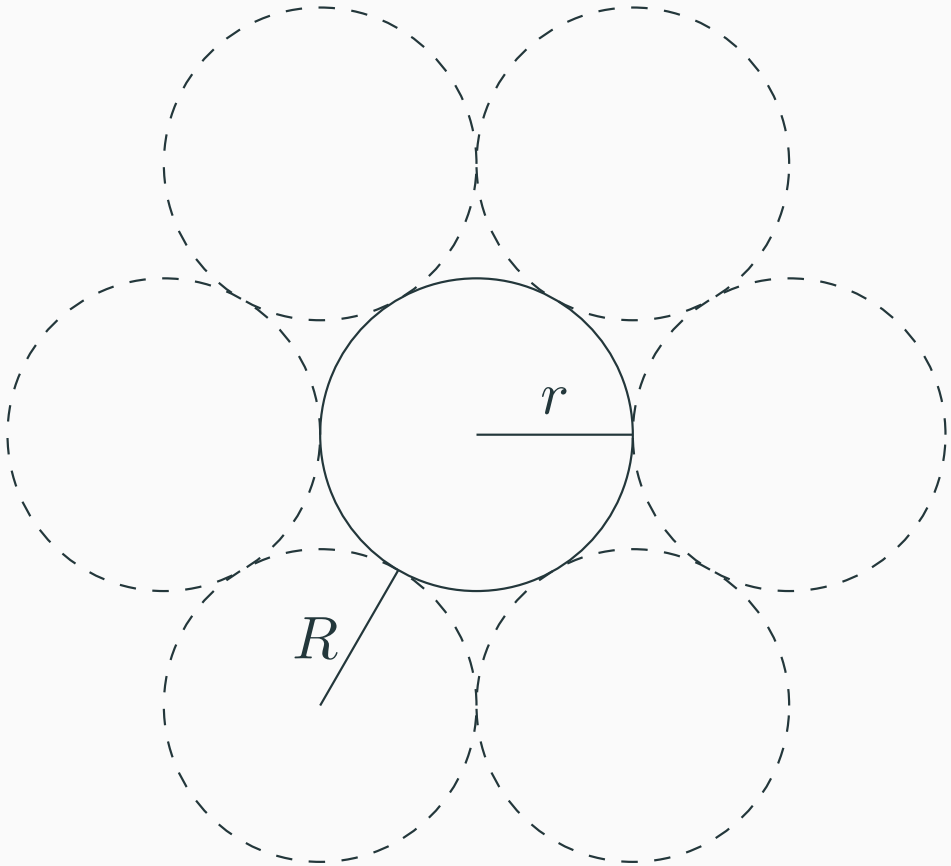
- More generally we have:
  - $\frac{\sin a}{A} = \frac{\sin b}{B} = \frac{\sin c}{C}$ (sine law).
  - $A^2 = B^2 + C^2 - 2BC\cos a$ (cosine law)
- **Exercise:** Prove the pythagorean theorem using the cosine law.



# Example: NN and the Optical Illusion

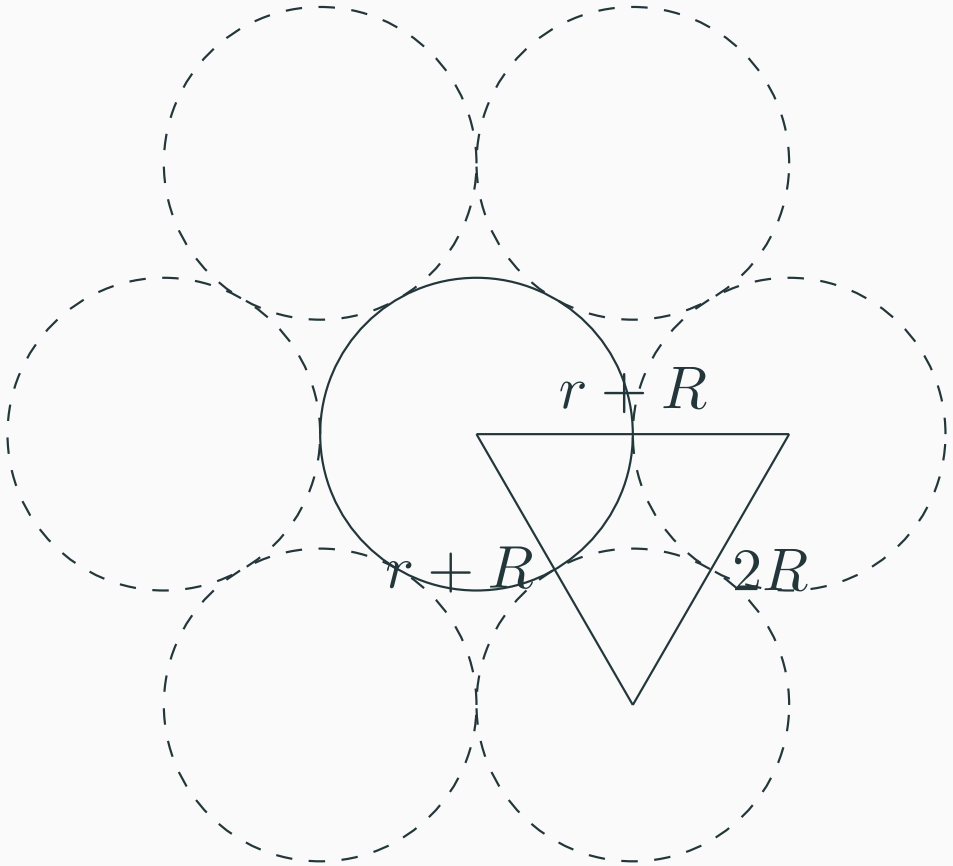
- You are given an integer $n$ and a real number $r$.
- You then draw a circle of radius $r$.
- You then want to draw $n$ circles of the same size tangent to the outside of this circle and such that they are tangent to their neighbours.
- What radius will the outer circles have?

# $N = 6$ image



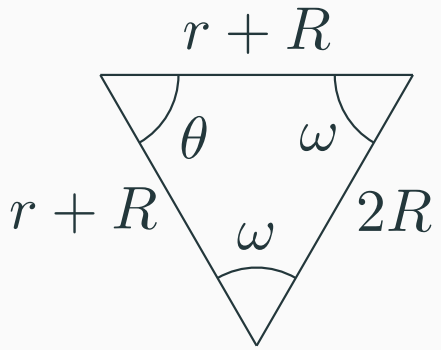
# Towards a solution

We see that the distance from the center of the circle in the middle to the center of an outer circle is $r + R$. We thus get an isosceles triangle.



# Closer and closer

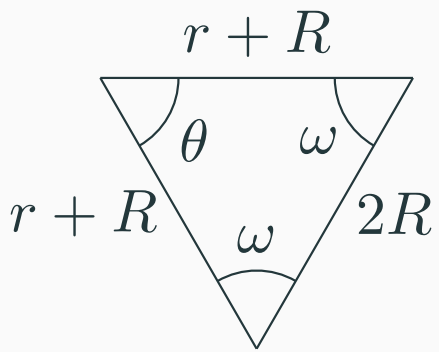
- Now we have $\theta = \dfrac{360^\circ}{n}$ and $\omega = \dfrac{180^\circ - \theta}{2}$.



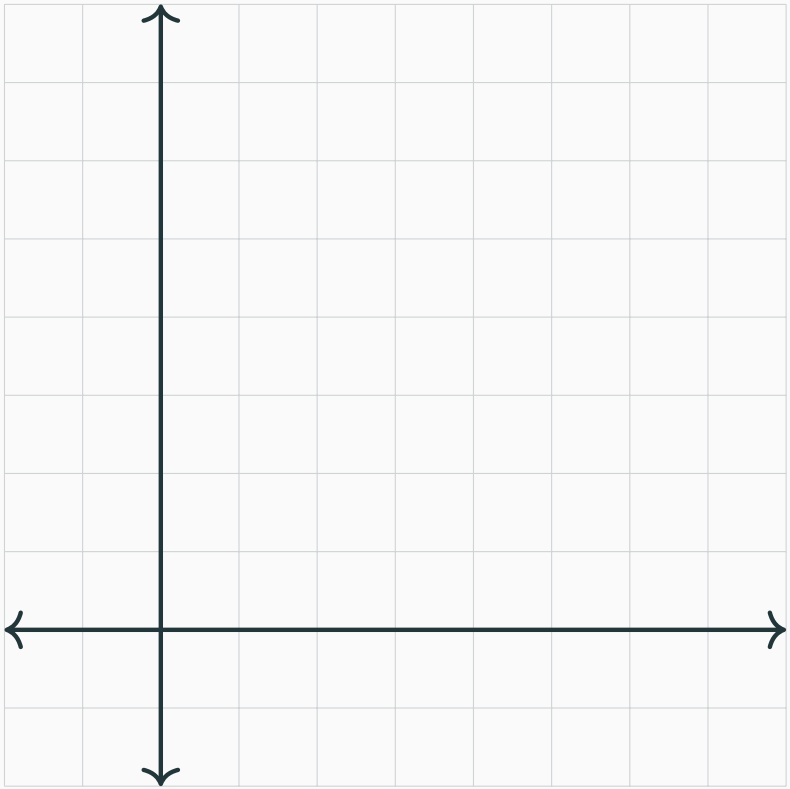
# Solution

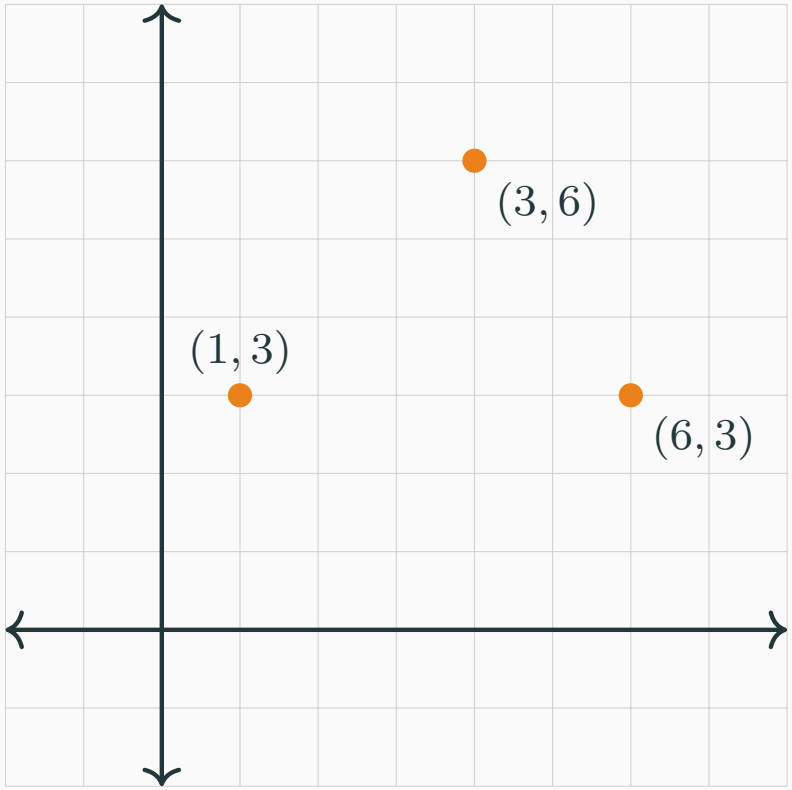
- Finally the law of sines gives us

$$\frac{2R}{\sin\theta} = \frac{r+R}{\sin\omega} \Rightarrow 2R\sin\omega = r\sin\theta + R\sin\theta$$
$$\Rightarrow 2R\sin\omega - R\sin\theta = r\sin\theta$$
$$\Rightarrow R = \frac{r\sin\theta}{2\sin\omega - \sin\theta}.$$

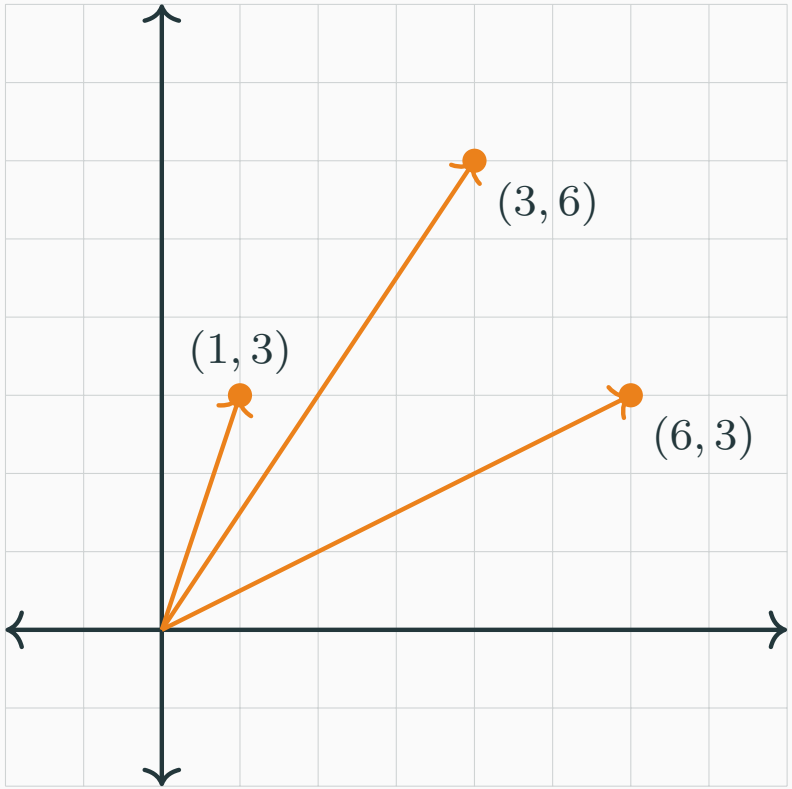


# Computer representation

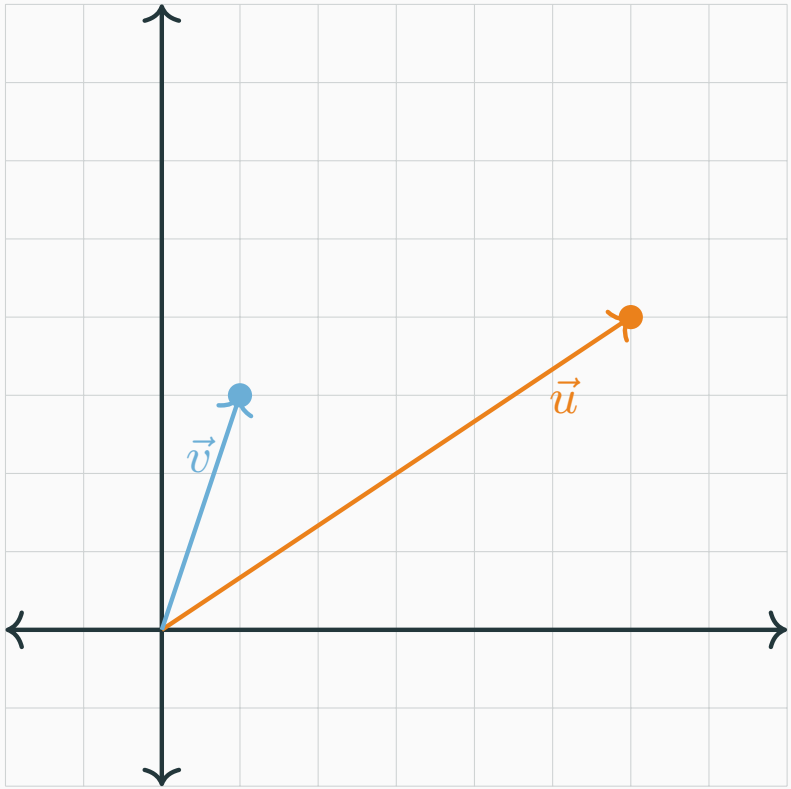


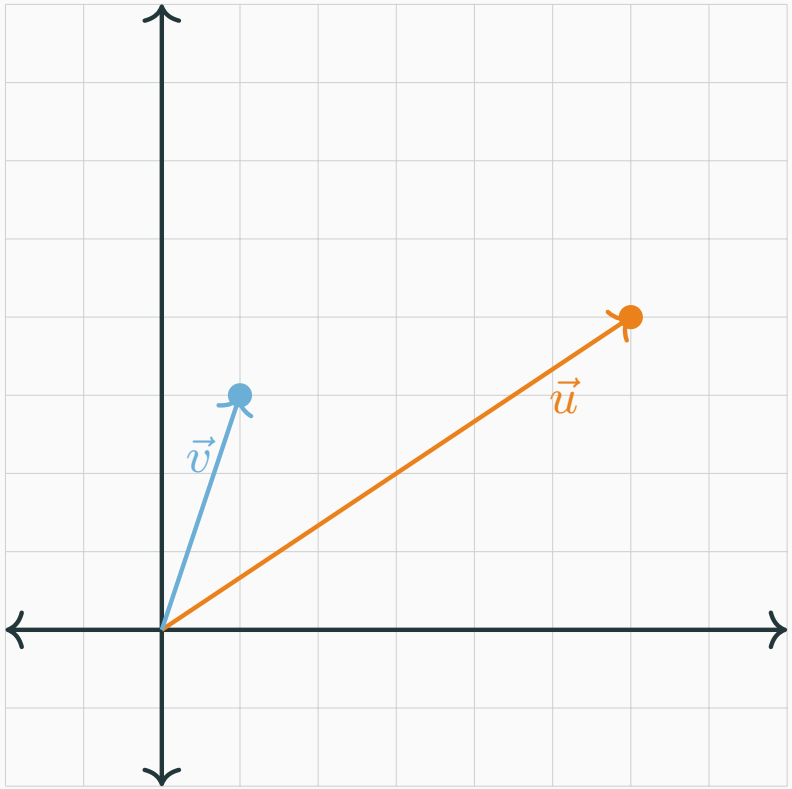



- Points are represented by a pair of numbers, $(x, y)$.



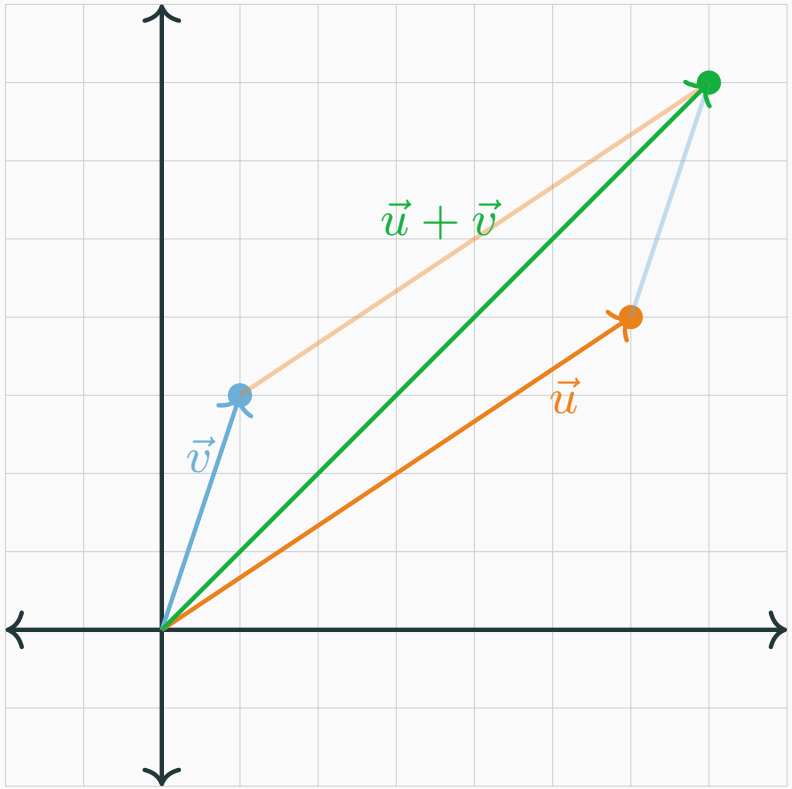

- Points are represented by a pair of numbers, $(x, y)$.
- Vectors are represented in the same way.
- Thinking of points as vectors allows us to do many things.


$\vec{v}$
$\vec{u}$




- Simplest operation, addition is defined as

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 + x_1 \\ y_0 + y_1 \end{pmatrix}$$

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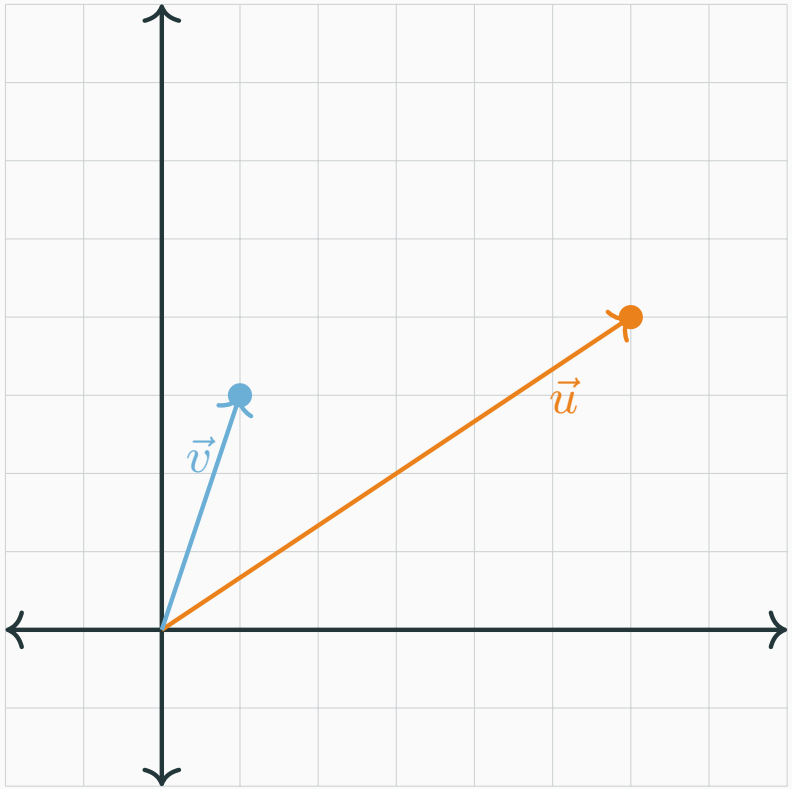


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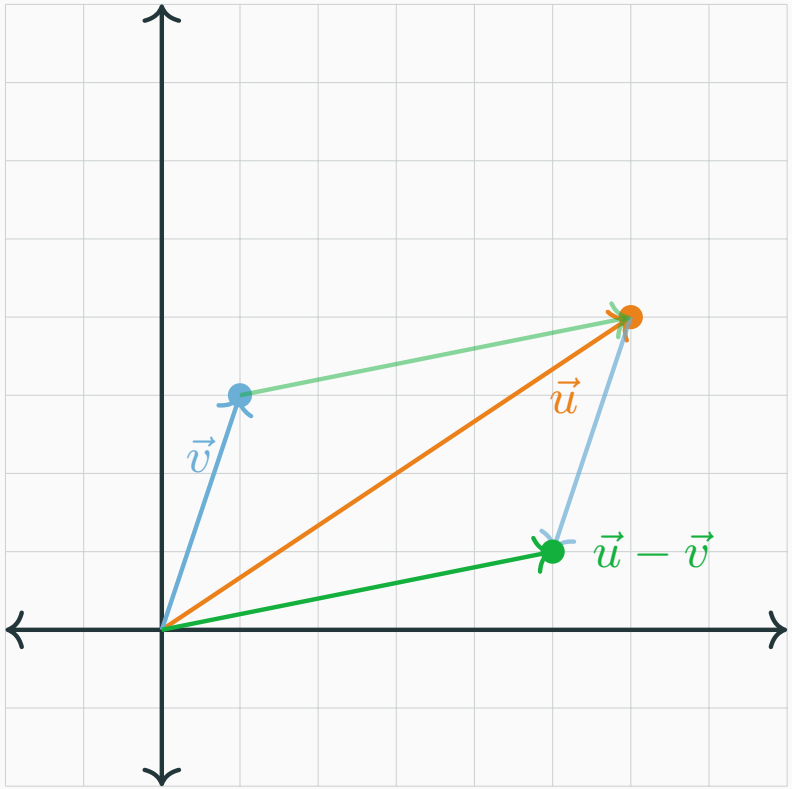
- Subtraction is defined in the same manner

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 - x_1 \\ y_0 - y_1 \end{pmatrix}$$

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```
struct point {
    double x, y;
    point(double _x, double _y) {
        x = _x, y = _y;
    }

    point operator+(const point &oth){
        return point(x + oth.x, y + oth.y);
    }

    point operator-(const point &oth){
        return point(x - oth.x, y - oth.y);
    }
};
```

. . . or we could use the `complex<double>` class.

```
using point = complex<double>;
```

. . . or we could use the `complex<double>` class.

```
using point = complex<double>;
```

The `complex` class in `C++` and `Java` has methods defined for

- Addition
- Subtraction
- Multiplication by a scalar
- Length
- Trigonometric functions
- And much more!

# Complex numbers

- We define $\mathbb{C} := \mathbb{R} \times \mathbb{R}$.
- Then we define addition on $\mathbb{C}$ such that for $(a, b), (c, d) \in \mathbb{C}$ we get
$$(a, b) + (c, d) = (a + c, b + d).$$
- We also define multiplication on $\mathbb{C}$ such that for $(a, b), (c, d) \in \mathbb{C}$ we get
$$(a, b) \cdot (c, d) = (ac - bd, ad + bc).$$
- We usually denote $(0, 1) \in \mathbb{C}$ by $i$ and $(x, y) \in \mathbb{C}$ by $x + yi$.
- Note that $(x, y) = (x, 0) + i \cdot (y, 0)$ here.
- We call these numbers in $\mathbb{C}$ *complex numbers*.

# Complex numbers ctd.

- If $z = x + yi \in \mathbb{C}$ then
  - We call $x$ the *real part* of $z$ and $y$ the *imaginary part* of $z$.
  - We define the *magnitude* of $z$ by $|z| = \sqrt{x^2 + y^2}$.
  - We call $x - yi$ the *conjugate* of $z$, denoted by $\overline{z}$.
  - We call the angle $(x, y)$ makes with the positive $x$-axis the *argument* of $z$ and denote it by $\mathrm{Arg}(z)$.

# Operations

- Let $w, z \in \mathbb{C}$.
- Then $w + z$ will be $z$ translated by $w$, as if we were adding vectors.
- If $|w| = 1$ then $z \cdot w$ will be $z$ rotated around $0$ by $\mathrm{Arg}(w)$ radians.
- If $|z| = r$ and $\mathrm{Arg}(z) = \theta$ we can write $z = re^{i\theta}$.
- If $z = r_1 e^{i\theta_1}$ and $w = r_2 e^{i\theta_2}$ then $z \cdot w = r_1 r_2 e^{i(\theta_1 + \theta_2)}$.

# Using complex in C++

- Usually we do `typedef complex<double> point`
- Then we can initialize a point with `point z(x, y)`
  - `real(z)` returns the $x$-coordinate
  - `imag(z)` returns the $y$-coordinate
  - `abs(z)` returns the magnitude $|z|$
  - `abs(z - w)` returns the distance from $z$ to $w$
  - `arg(z)` returns the argument of $z$
  - `conj(z)` returns the conjugate $\overline{z}$

# Example

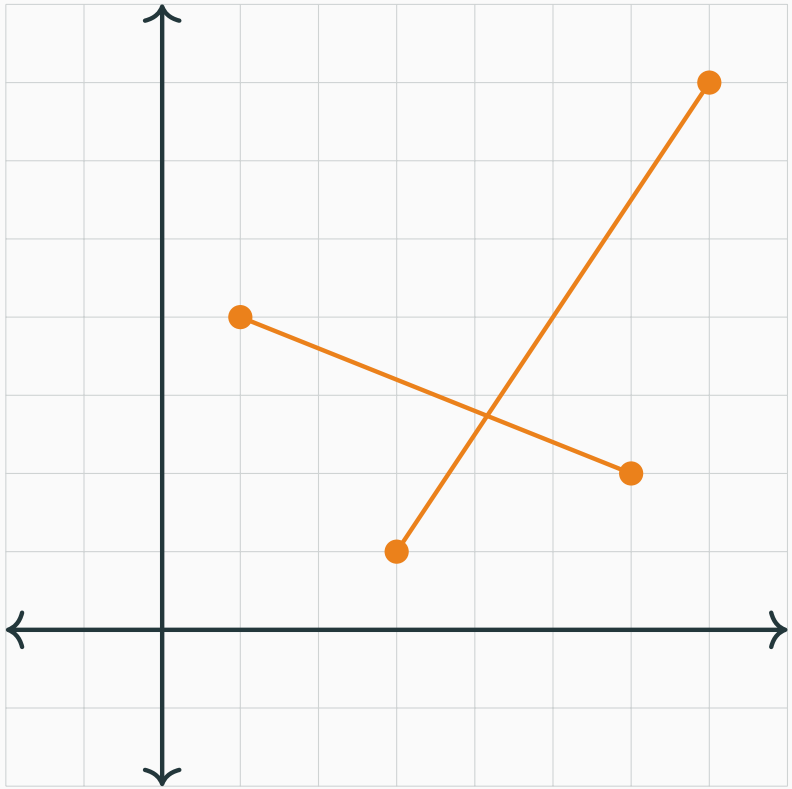
- Let us consider a problem.
- You start at $(0, 0)$ and get a sequence of commands.
- All the commands consist of a single letter and a number. The commands are:
  - ...`f x` you move forward $x$ meters..
  - ...`b x` you move backwards $x$ meters.
  - ...`r x` you rotate $x$ radians to the right.
  - ...`l x` you rotate $x$ radians to the left.
- How far from $(0, 0)$ do you end up after following the commands?

# Solution

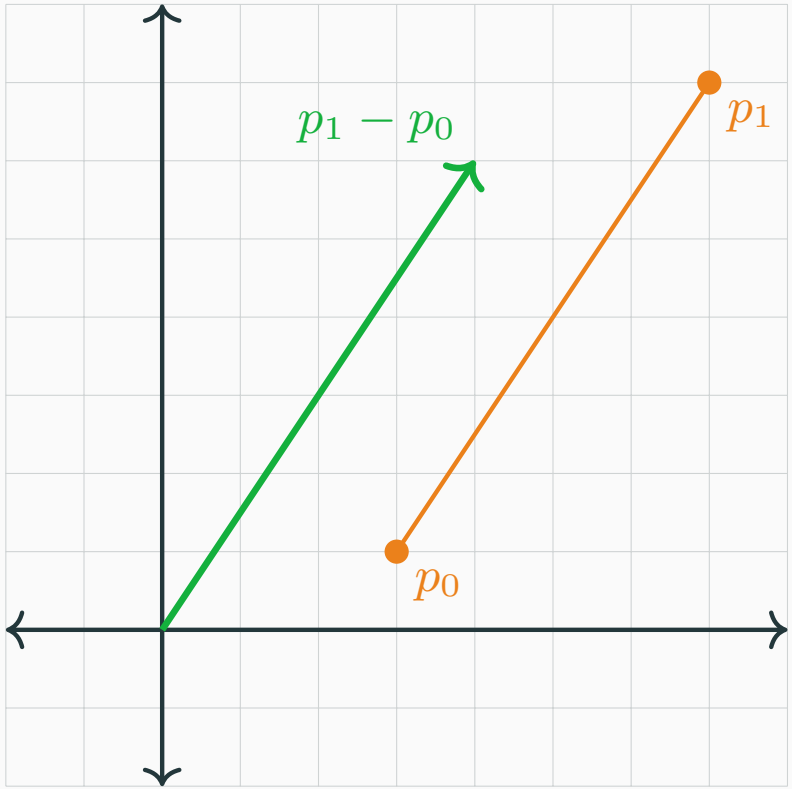
- If we are stood at $p \in \mathbb{C}$ and want to take a step of $r$ meters in the direction $\theta$ we simply add $re^{i\theta}$ to $p$.
- What direction we are facing at the start makes no difference since it gives the same distance at the end.

# Code

```cpp
#include <bits/stdc++.h>
using namespace std;
typedef complex<double> point;
int main() {
    int n; cin >> n;
    double x, r = 0.0;
    point p(0, 0);
    while (n--) {
        char c; cin >> c >> x;
        if (c == 'f')       p += x*exp(1i*r);
        else if (c == 'b') p -= x*exp(1i*r);
        else if (c == 'l') r += x;
        else if (c == 'r') r -= x;
        else assert(0);
        }
    cout << setprecision(15) << abs(p) << endl;
}
```



- Line segments are represented by a pair of points, $((x_0, y_0), (x_1, y_1))$.

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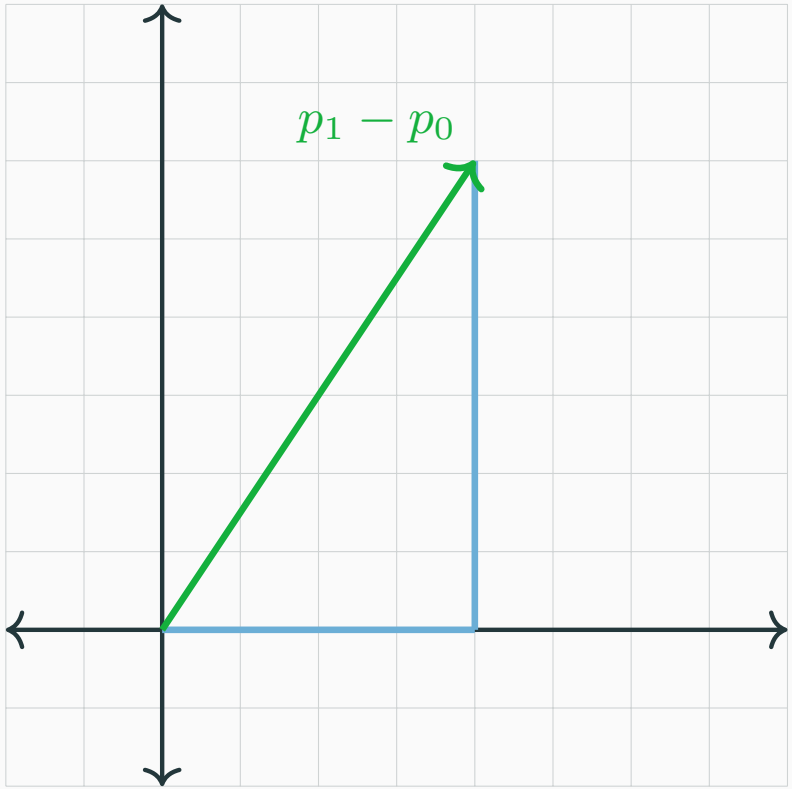


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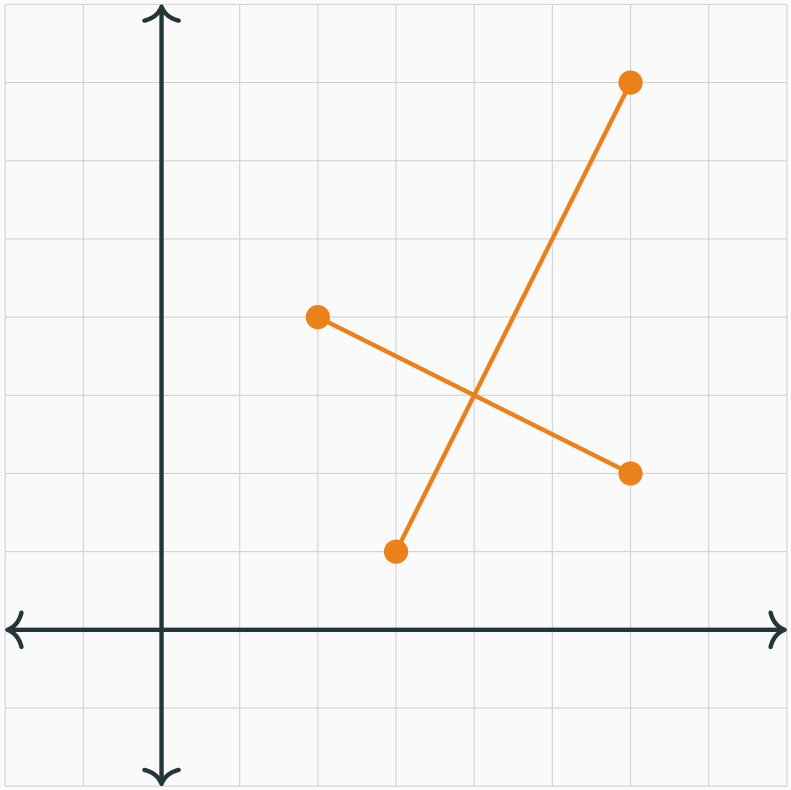
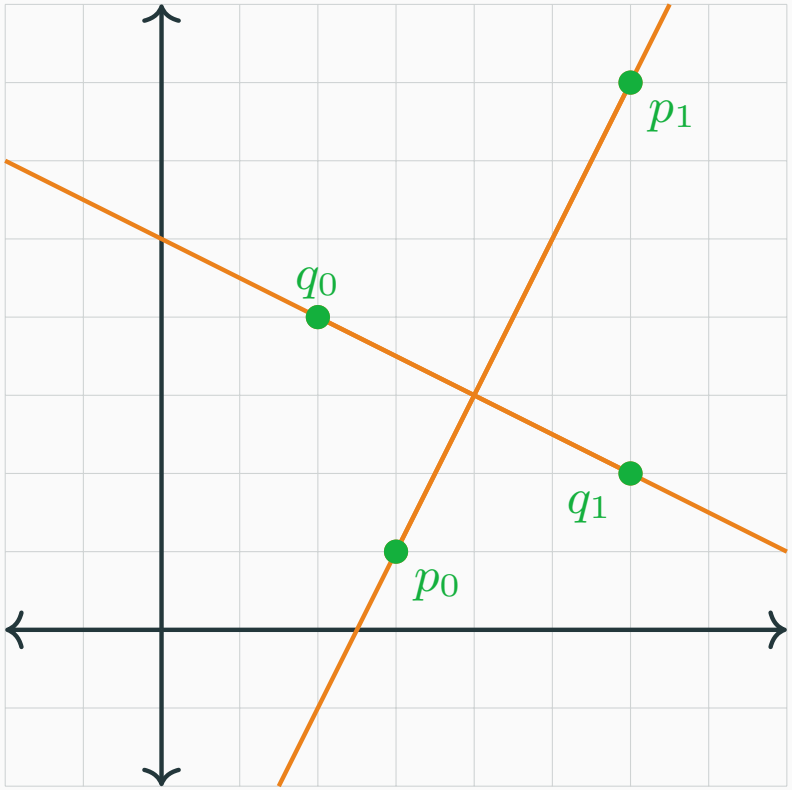
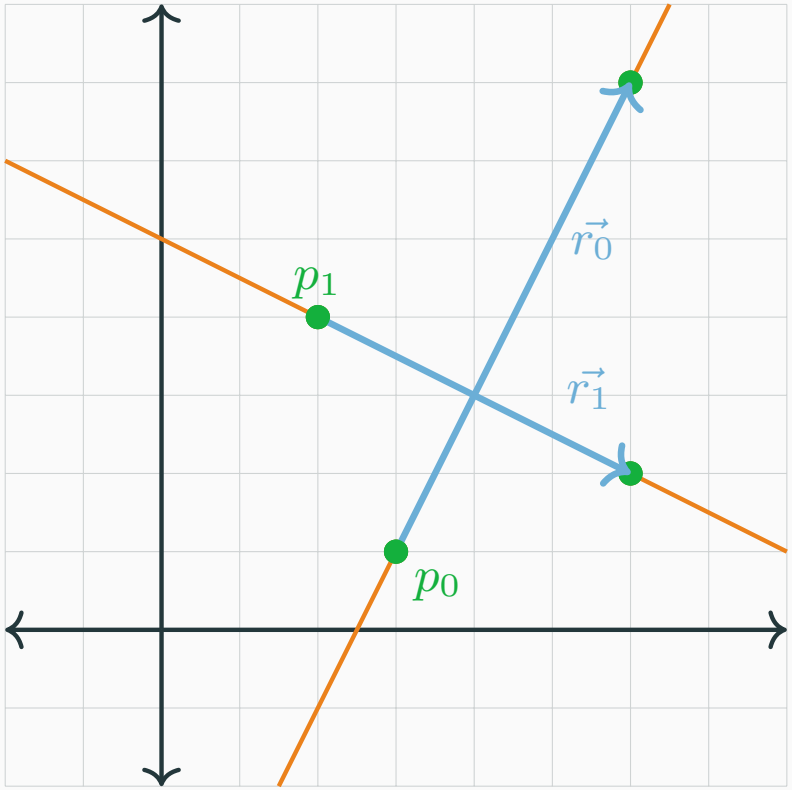


- Line representation same as line segments.

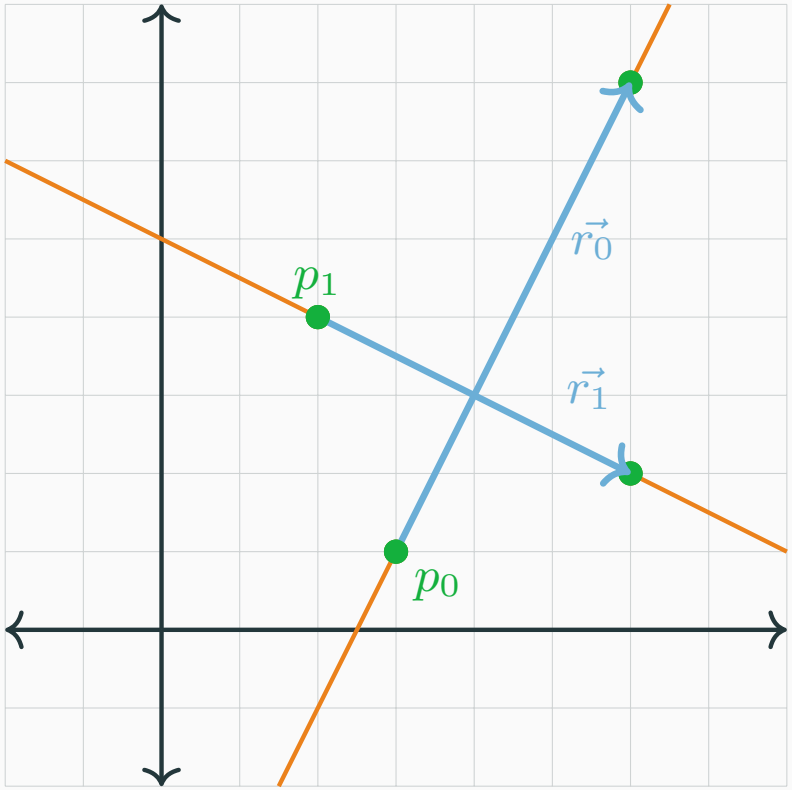



- Line representation same as line segments.
- Treat them as lines passing through the two points.




- Line representation same as line segments.
- Treat them as lines passing through the two points.
- Or as a point and a direction vector.

$$p + t \cdot \vec{r}$$

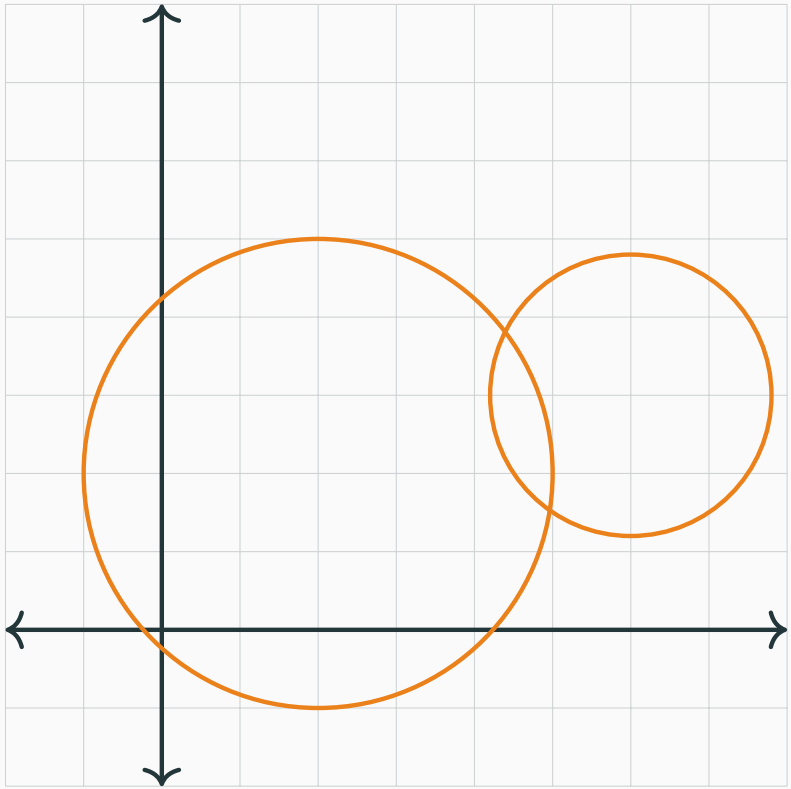



- Line representation same as line segments.
- Treat them as lines passing through the two points.
- Or as a point and a direction vector.

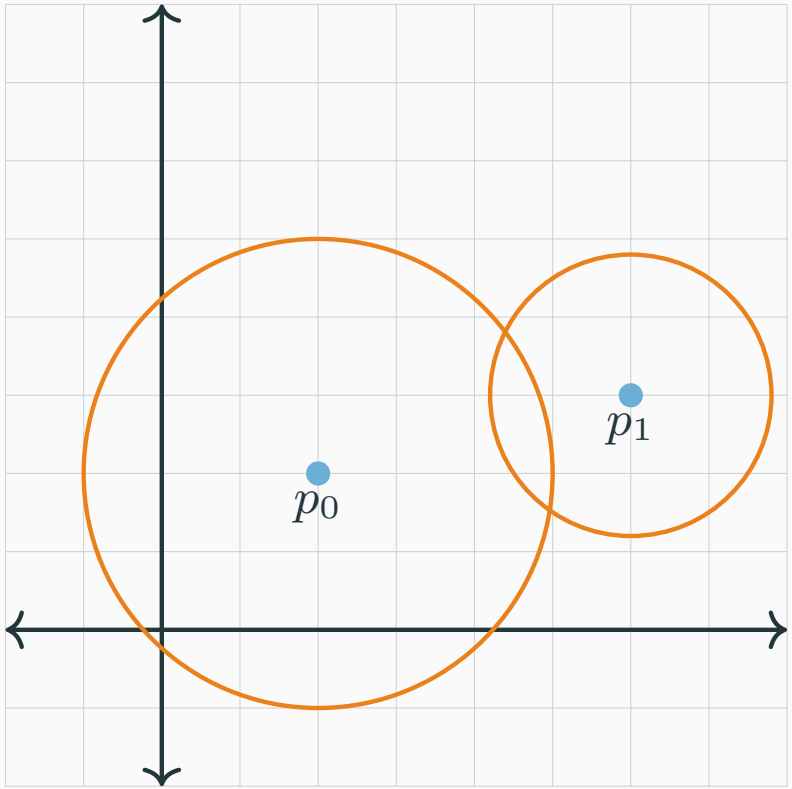
$$p + t \cdot \vec{r}$$

- Either way

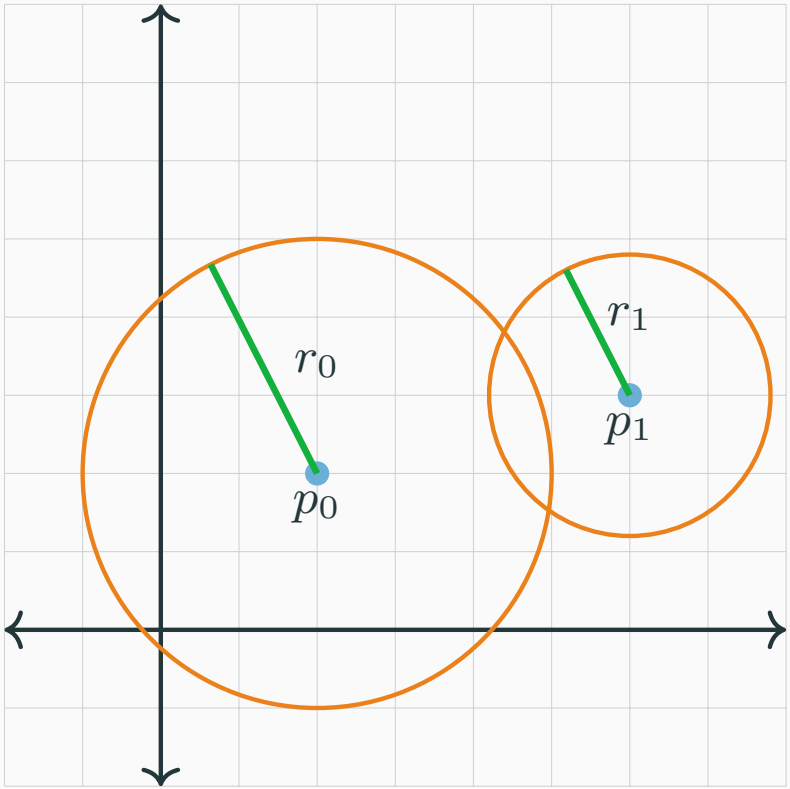
`pair<point,point>`



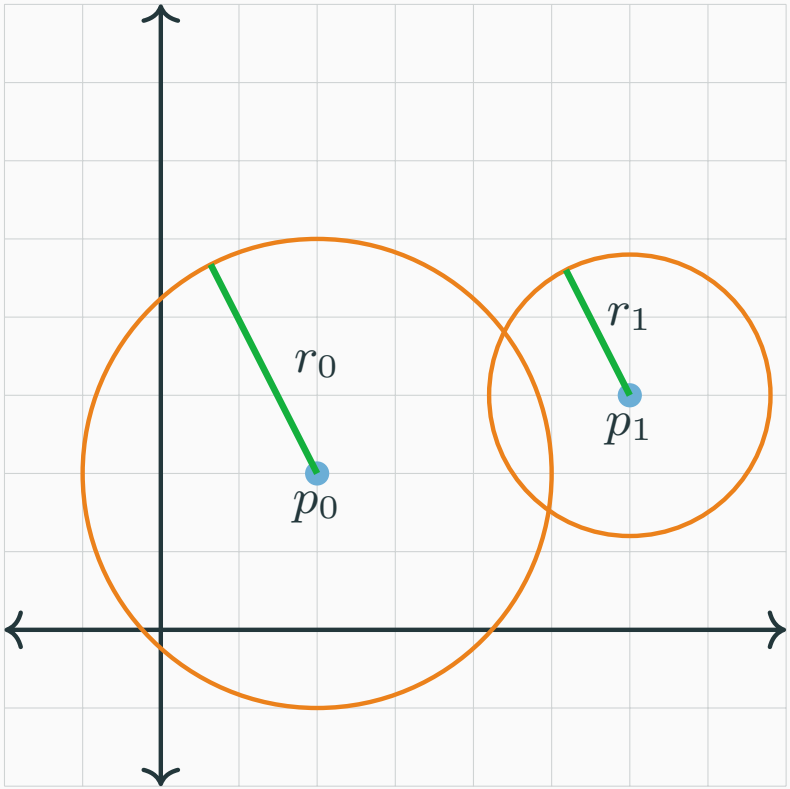
- Circles are very easy to represent.




- Circles are very easy to represent.
- Center point $p = (x, y)$.




- Circles are very easy to represent.
- Center point $p = (x, y)$.
- And the radius $r$.




- Circles are very easy to represent.
- Center point $p = (x, y)$.
- And the radius $r$.

`pair<point,double>`

# Computational geometry

Given two vectors

$$\vec{u} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

the dot product of $\vec{u}$ and $\vec{v}$ is defined as

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = x_0 \cdot x_1 + y_0 \cdot y_1$$

Given two vectors

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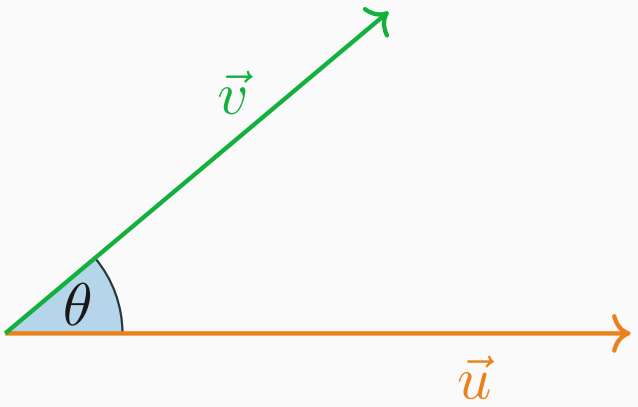
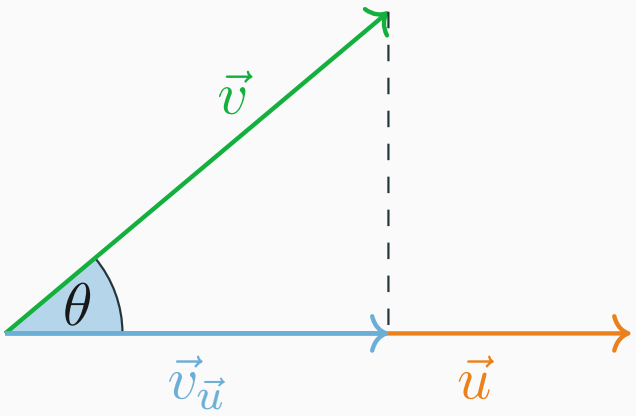
Which in geometric terms is

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$$




- Allows us to calculate the angle between $\vec{u}$ and $\vec{v}$.

$$\theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$$

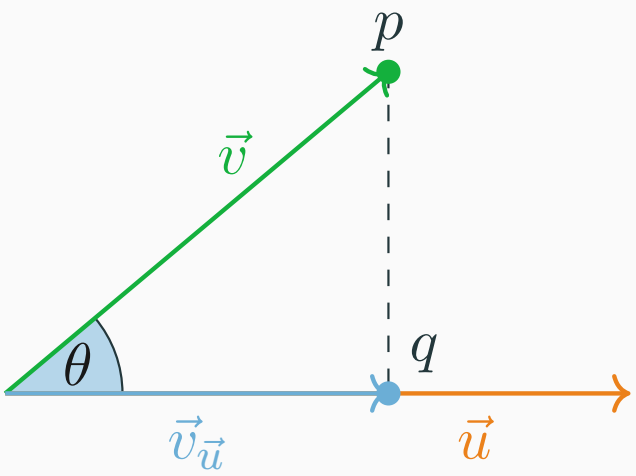



- Allows us to calculate the angle between $\vec{u}$ and $\vec{v}$.

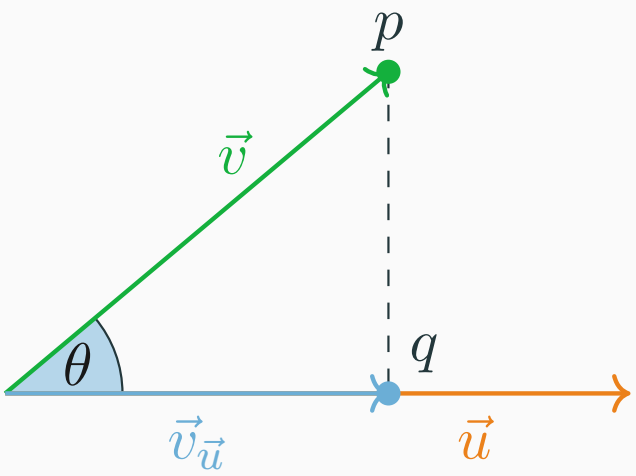
$$\theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$$

- And the projection of $\vec{v}$ onto $\vec{u}$.

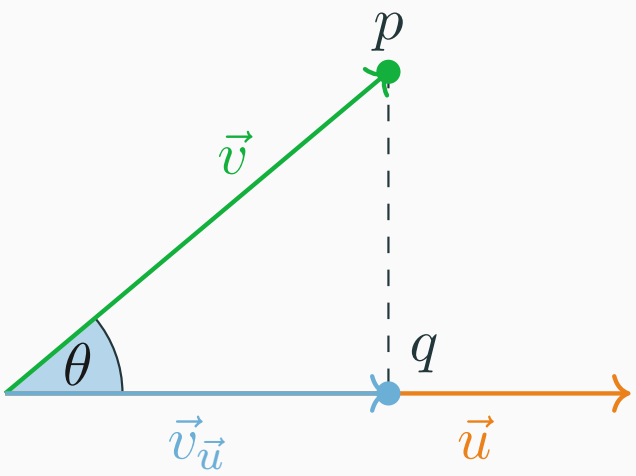
$$\vec{v}_{\vec{u}} = \left(\frac{\vec{u}\cdot\vec{v}}{|u|^2}\right)\vec{u}$$




- The closest point on $\vec{u}$ to $p$ is $q$.




- The closest point on $\vec{u}$ to $p$ is $q$.
- The distance from $p$ to $\vec{u}$ is the distance from $p$ to $q$.




- The closest point on $\vec{u}$ to $p$ is $q$.
- The distance from $p$ to $\vec{u}$ is the distance from $p$ to $q$.
- Unless $q$ is outside $\vec{u}$, then the closest point is either of the endpoints.

Rest of the code will use the `complex` class.

```
#define P(p) const point &p
#define L(p0, p1) P(p0), P(p1)
double dot(P(a), P(b)) {
    return real(a) * real(b) + imag(a) * imag(b);
}
double angle(P(a), P(b), P(c)) {
    return acos(dot(b - a, c - b) / abs(b - a) / abs(c - b));
}
point closest_point(L(a, b), P(c), bool segment = false) {
    if (segment) {
        if (dot(b - a, c - b) > 0) return b;
        if (dot(a - b, c - a) > 0) return a;
    }
    double t = dot(c - a, b - a) / norm(b - a);
    return a + t * (b - a);
}
```

Given two vectors

$$\vec{u} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

the cross product of $\vec{u}$ and $\vec{v}$ is defined as

$$\left| \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \times \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \right| = x_0 \cdot y_1 - y_0 \cdot x_1$$

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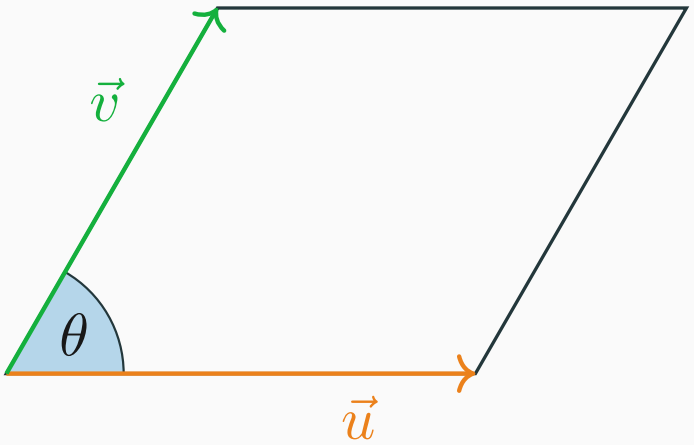
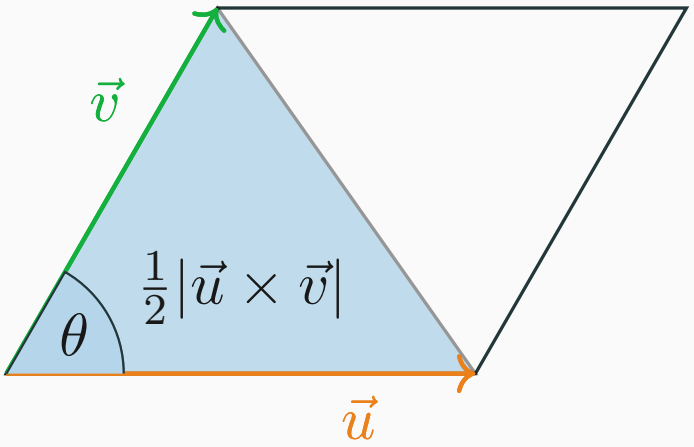
Which in geometric terms is

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$$




- Allows us to calculate the area of the triangle formed by $\vec{u}$ and $\vec{v}$.

$$\frac{|\vec{u} \times \vec{v}|}{2}$$

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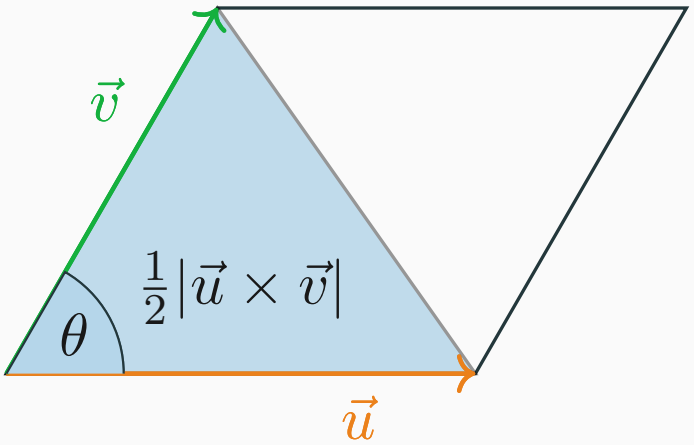
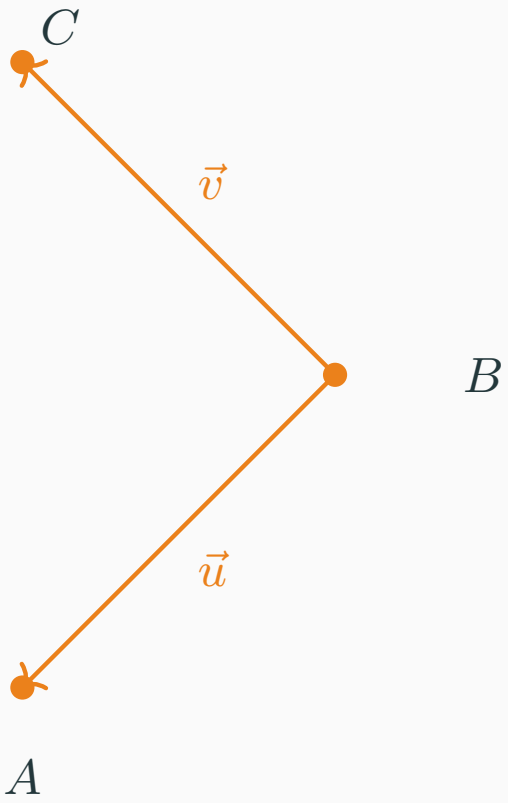



- Allows us to calculate the area of the triangle formed by $\vec{u}$ and $\vec{v}$.

$$\frac{|\vec{u} \times \vec{v}|}{2}$$

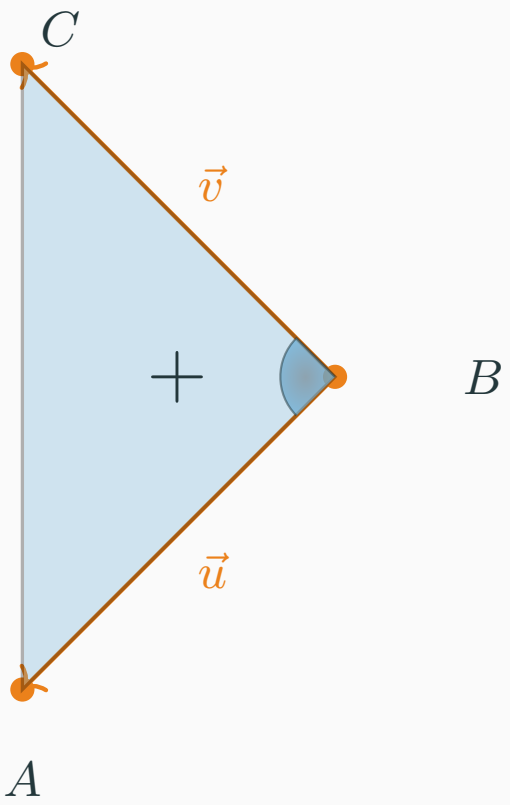
- And can tell us if the angle between $\vec{u}$ and $\vec{v}$ is positive or negative.

$$|\vec{u} \times \vec{v}| < 0 \quad \text{iff} \quad \theta < \pi$$
$$|\vec{u} \times \vec{v}| = 0 \quad \text{iff} \quad \theta = \pi$$
$$|\vec{u} \times \vec{v}| > 0 \quad \text{iff} \quad \theta > \pi$$




- Given three points $A$, $B$ and $C$, we want to know if they form a counter-clockwise angle in that order.

$$A \to B \to C$$



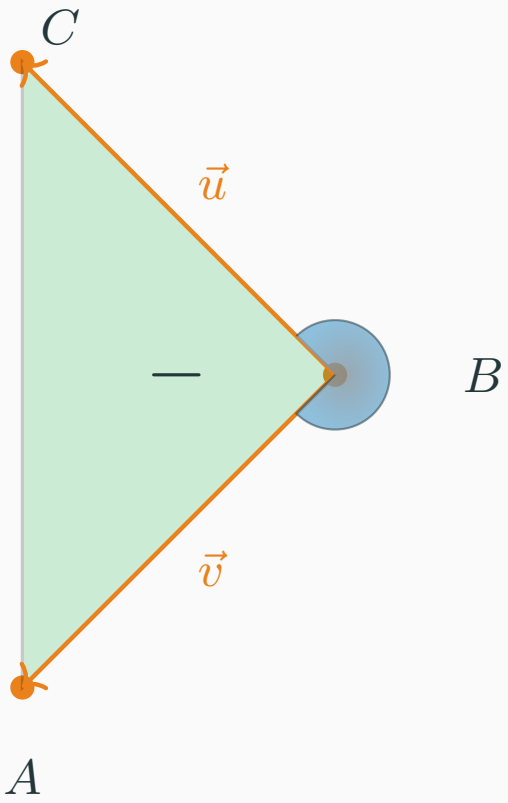

- Given three points $A$, $B$ and $C$, we want to know if they form a counter-clockwise angle in that order.

$$A \to B \to C$$

- We can examine the cross product of and the area of the triangle formed by

$$\vec{u} = B - C \quad \vec{v} = B - A$$

$$\vec{u} \times \vec{v} > 0$$




- The points in the reverse order do not form a counter clockwise angle.

$$C \to B \to A$$

- In the reverse order the vectors swap places

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$$C \to B \to A$$

- In the reverse order the vectors swap places

$$\vec{u} = B - A \quad \vec{v} = B - C$$

$$\vec{u} \times \vec{v} < 0$$

- If the points $A$, $B$ and $C$ are on the same line, then the area will be $0$.

```
double cross(P(a), P(b)) {
    return real(a)*imag(b) - imag(a)*real(b);
}
double ccw(P(a), P(b), P(c)) {
    return cross(b - a, c - b);
}
bool collinear(P(a), P(b), P(c)) {
    return abs(ccw(a, b, c)) < EPS;
}
```

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$$B = x_0 - x_1$$
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- If we have two lines given by such equations, we simply need to solve for the two unknowns, $x$ and $y$.

For two lines

$$A_0x + B_0y = C_0$$
$$A_1x + B_1y = C_1$$

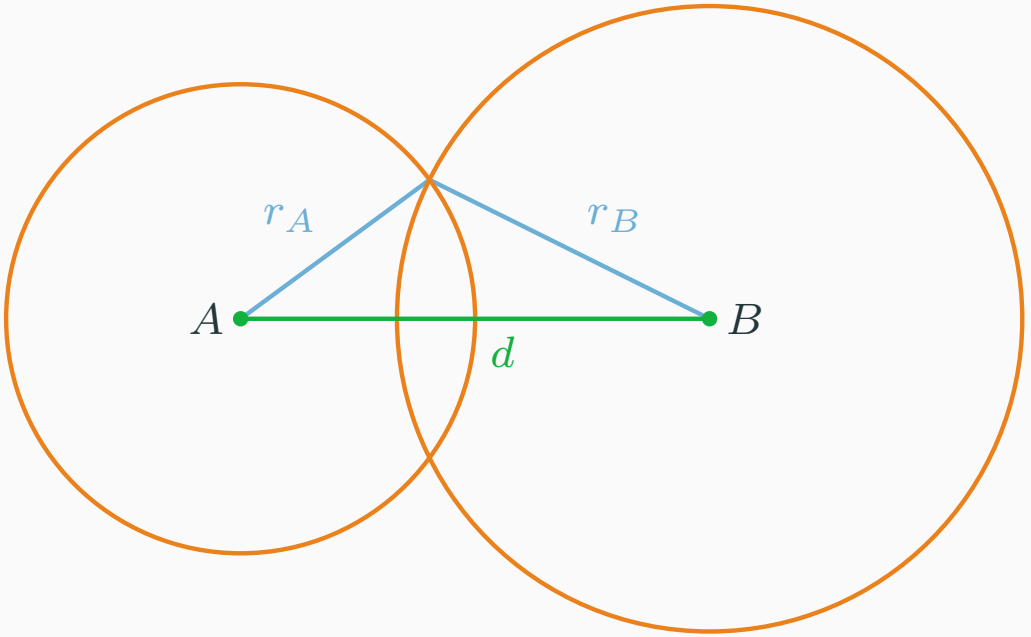
The intersection point is

$$x = \frac{(B_1 \cdot C_0 - B_0 \cdot C_1)}{D}$$
$$y = \frac{(A_0 \cdot C_1 - A_1 \cdot C_0)}{D}$$

Where

$$D = A_0 \cdot B_1 - A_1 \cdot B_0$$

Quite similar problem is to find the intersections of two circles.



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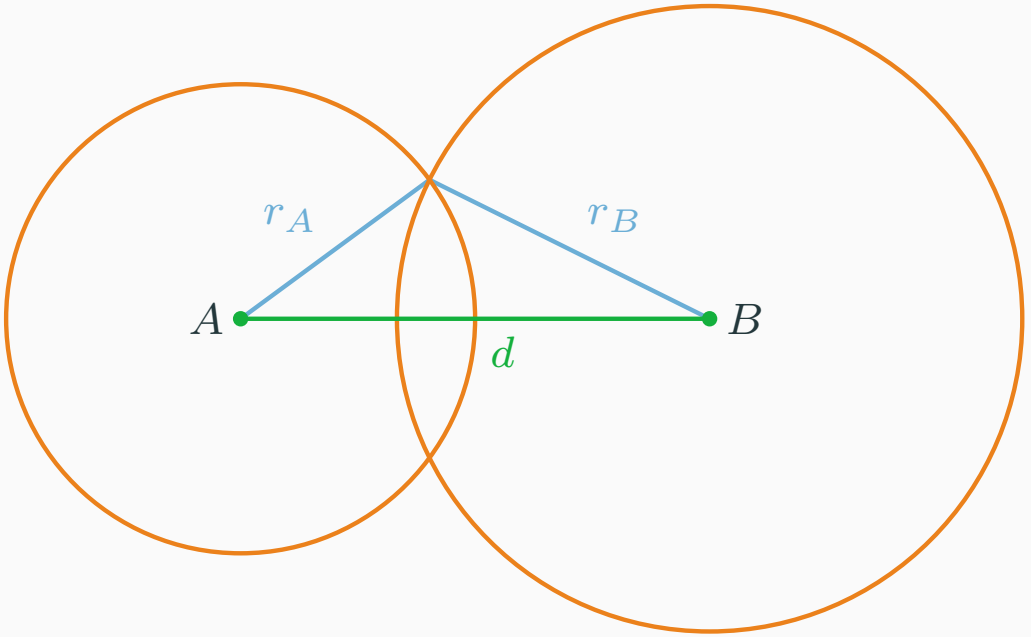


- If $d > r_0 + r_1$ the circles do not intersect.

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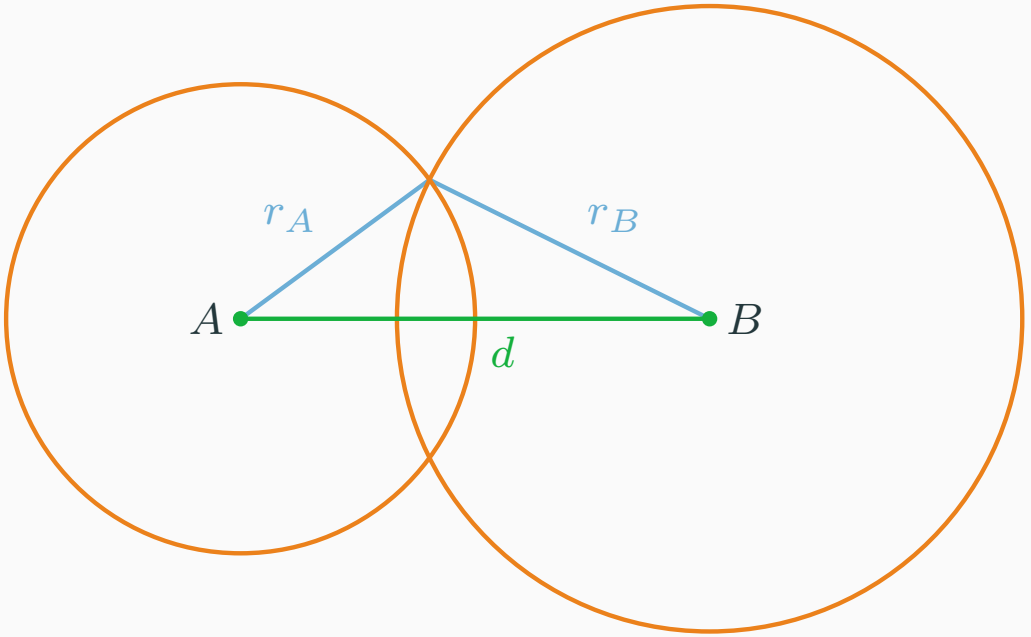


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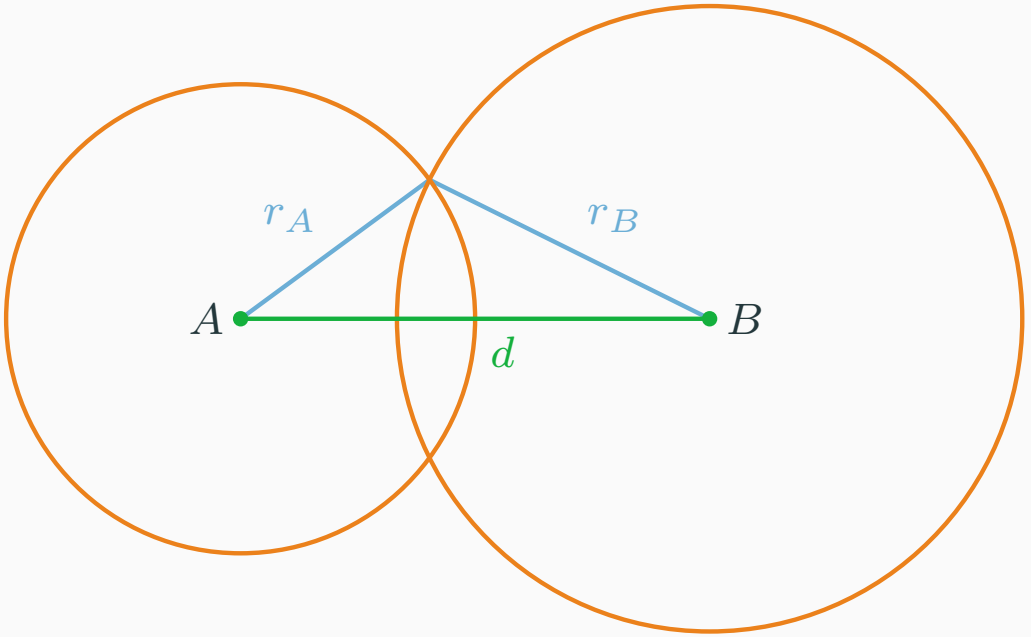


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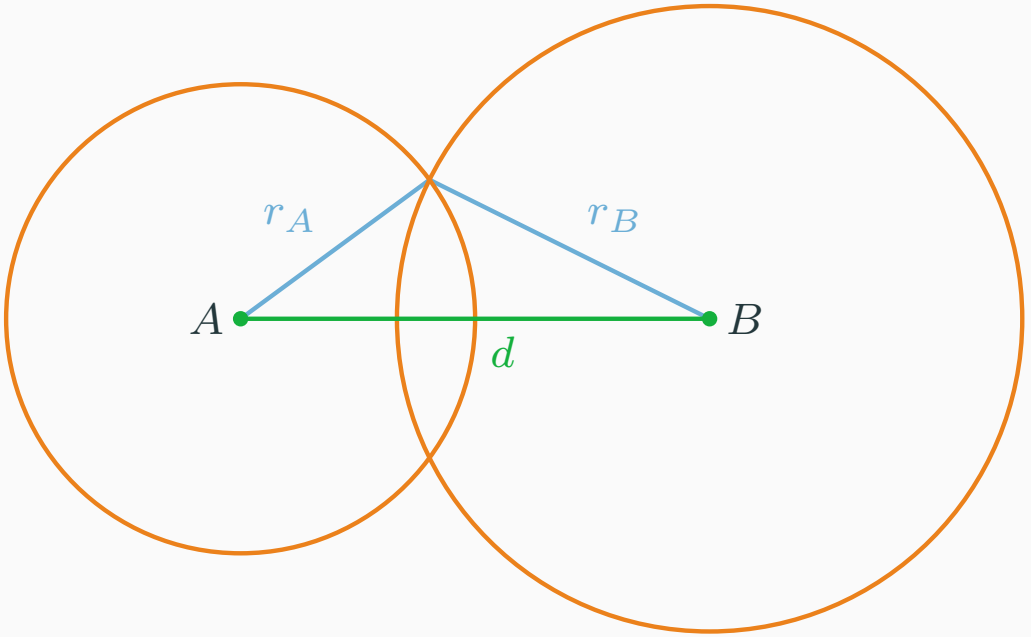


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- Let's look at the last case.

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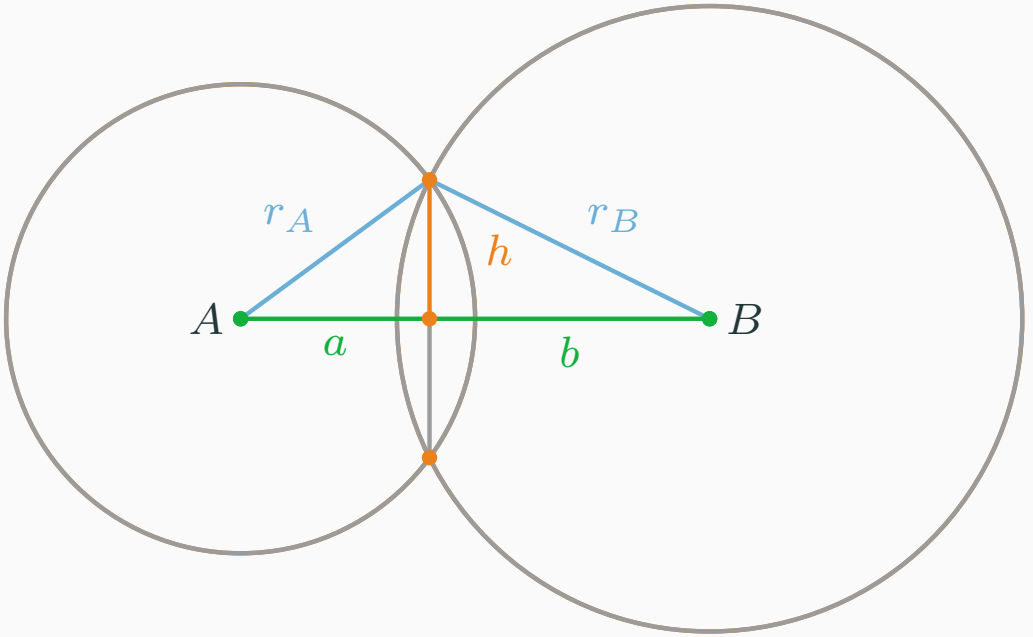


- We can solve for the vectors $a$ and $h$ from the equations

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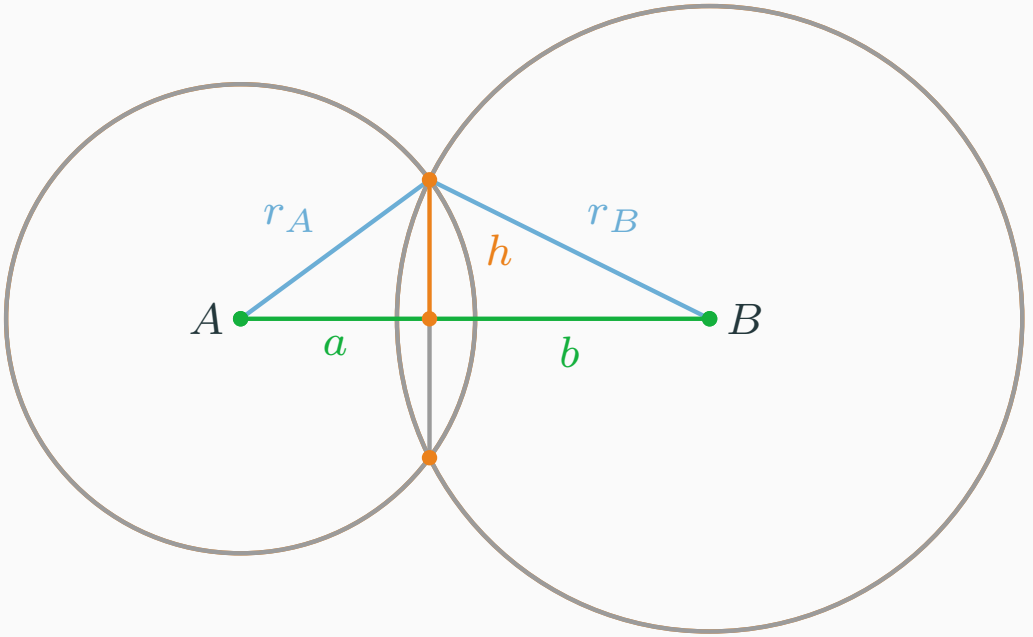


- We can solve for the vectors $a$ and $h$ from the equations

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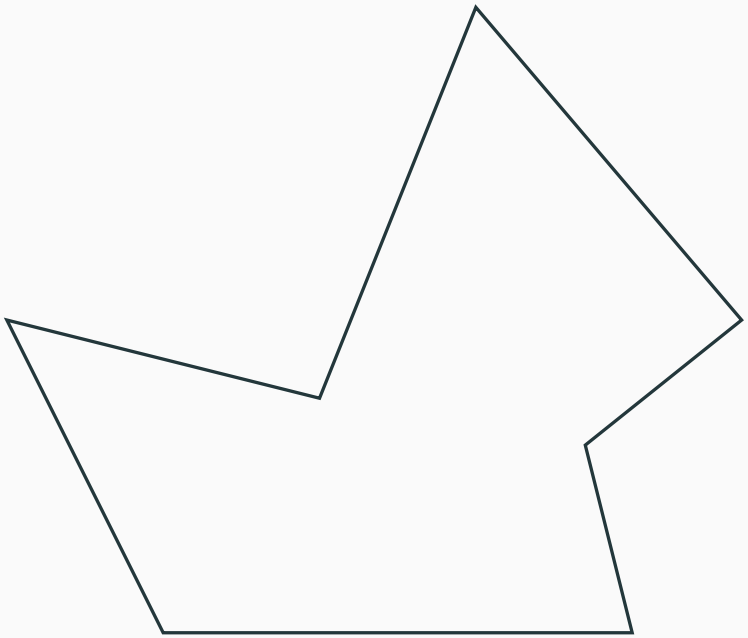
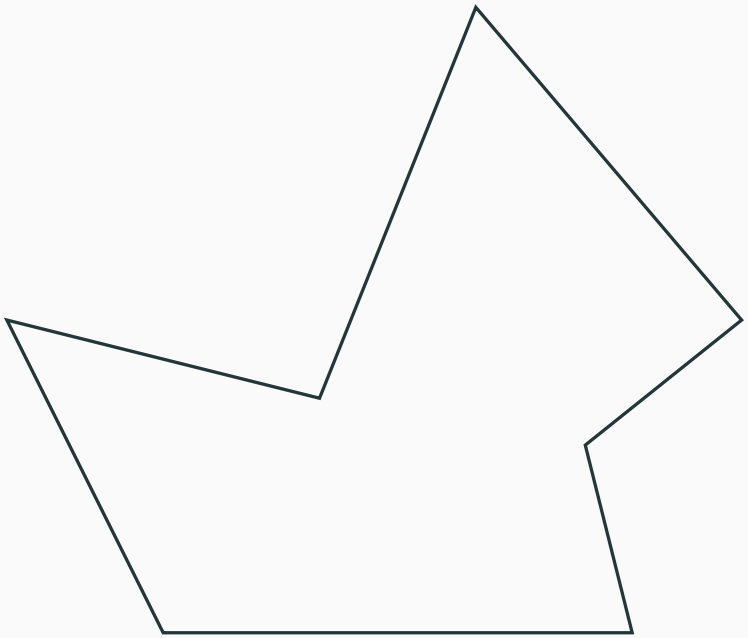
- We get

$$a = \frac{r_A^2 - r_B^2 + d^2)}{2 \cdot d}$$

$$h^2 = r_A^2 - a^2$$



- Polygons are represented by a list of points in the order representing the edges.



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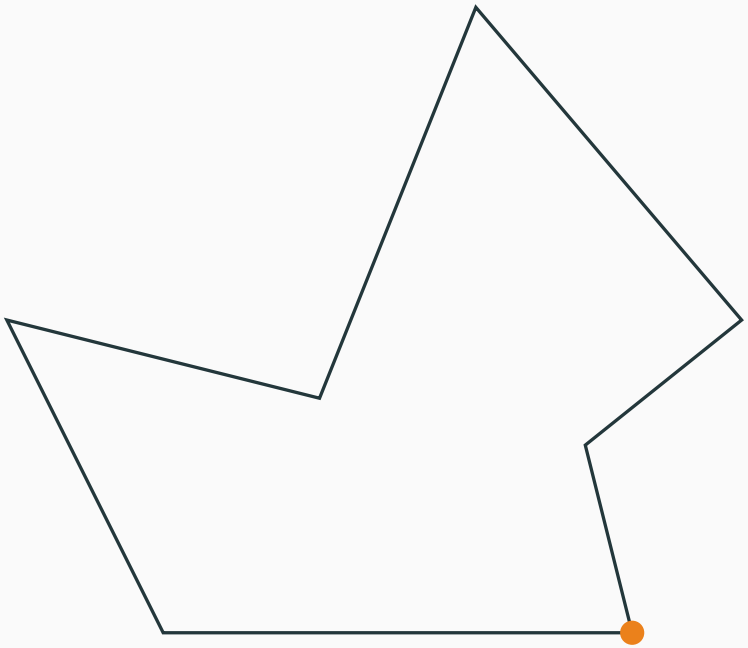

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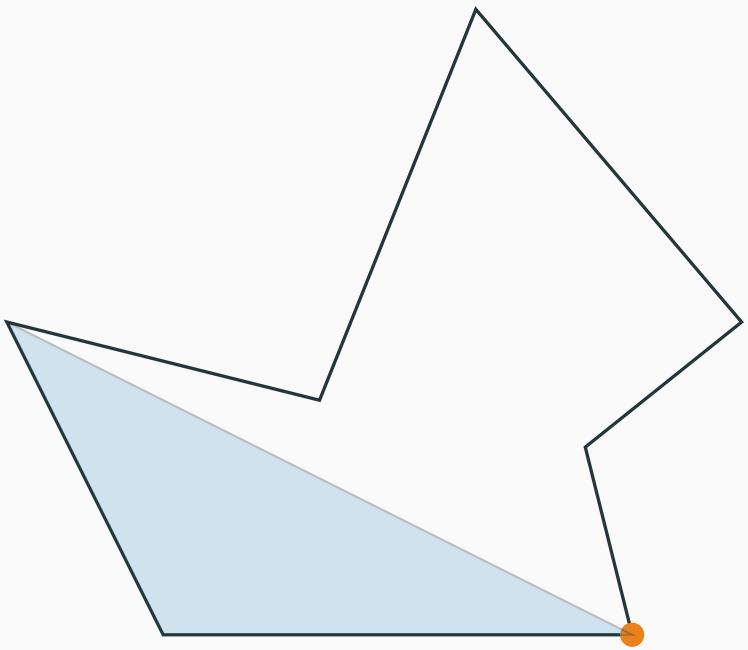

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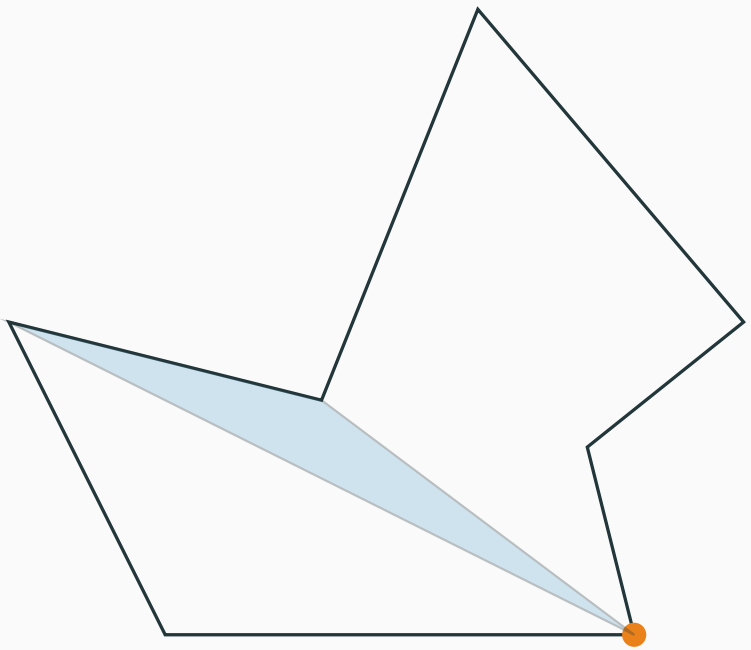

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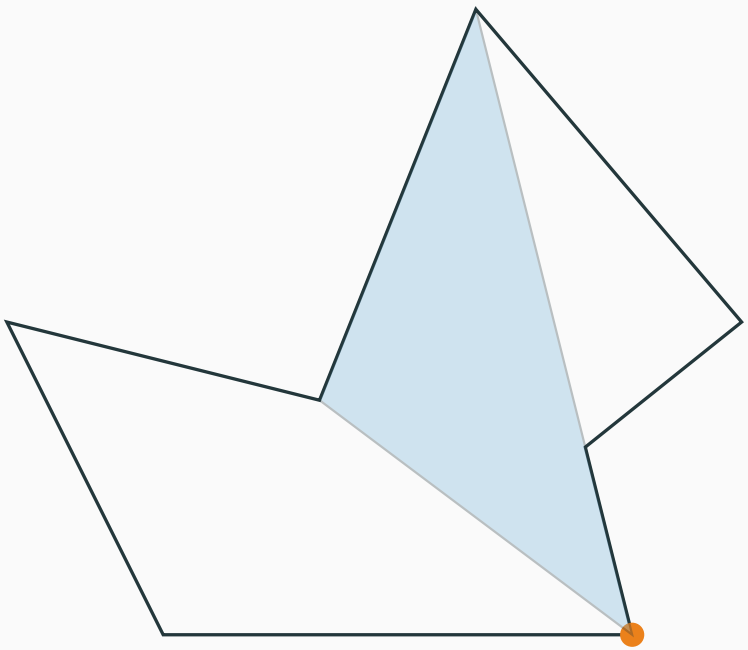

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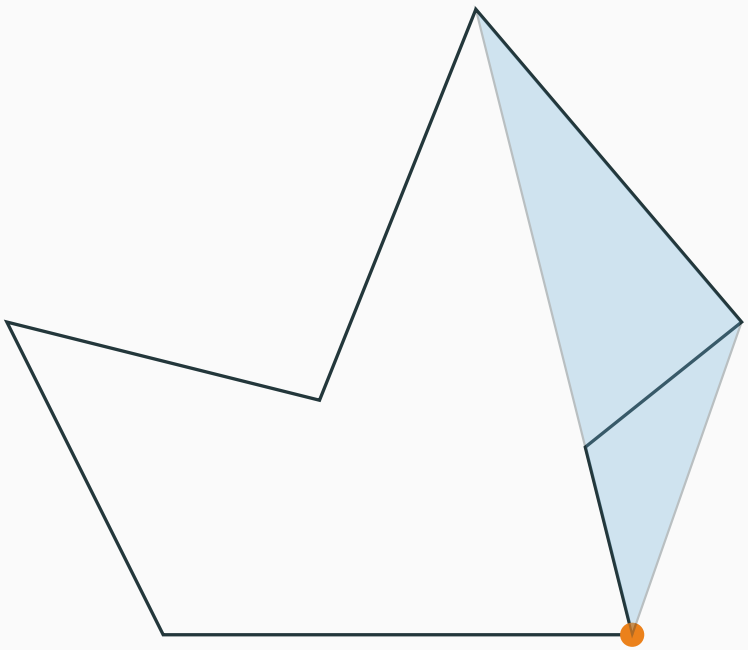

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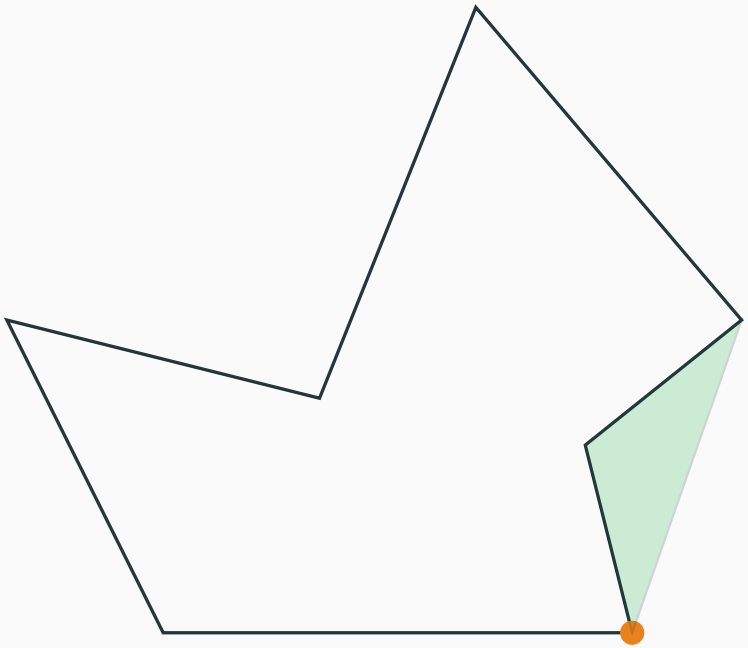

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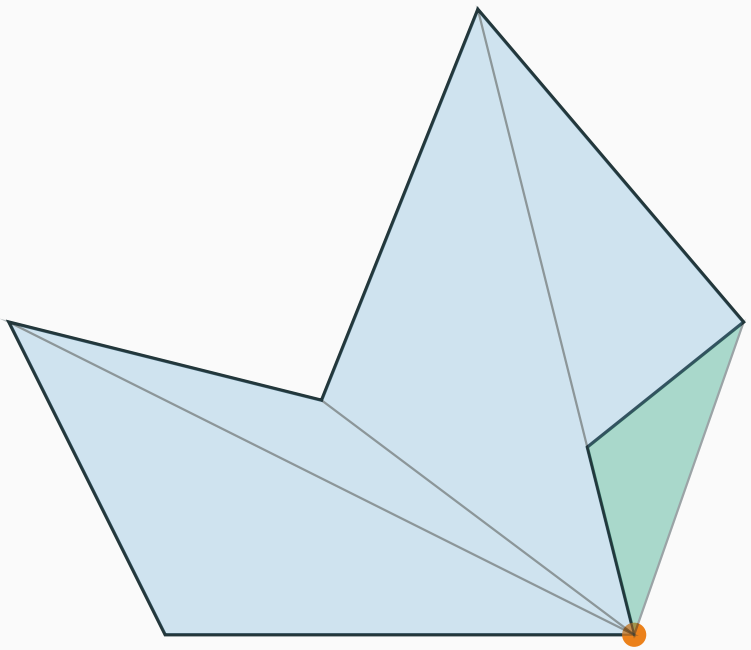

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```
double polygon_area_signed(const vector<point> &p) {
    double area = 0;
    int cnt = size(p);
    for (int i = 1; i + 1 < cnt; i++){
        area += cross(p[i] - p[0], p[i + 1] - p[0])/2;
    }
    return area;
}
double polygon_area(vector<point> &p) {
    return abs(polygon_area_signed(p));
}
```

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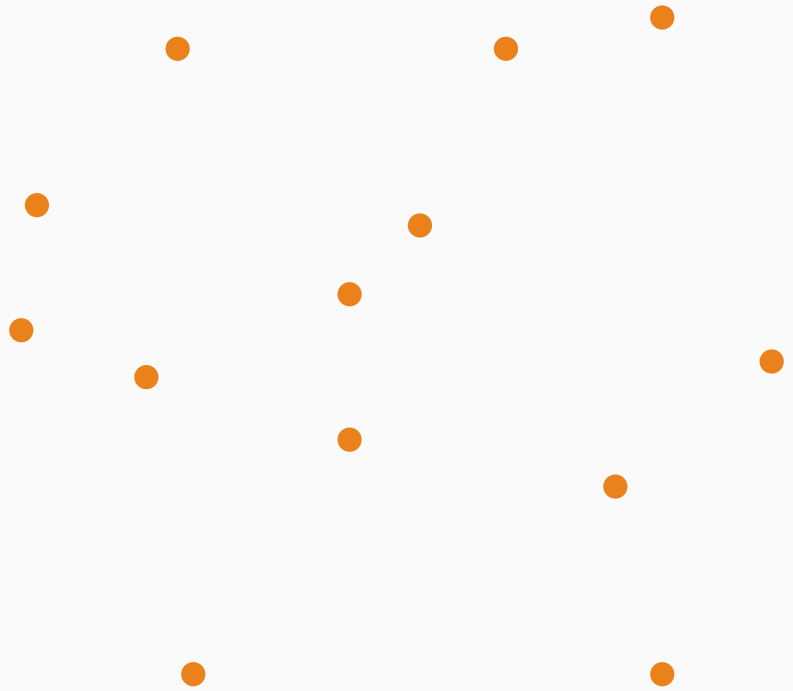

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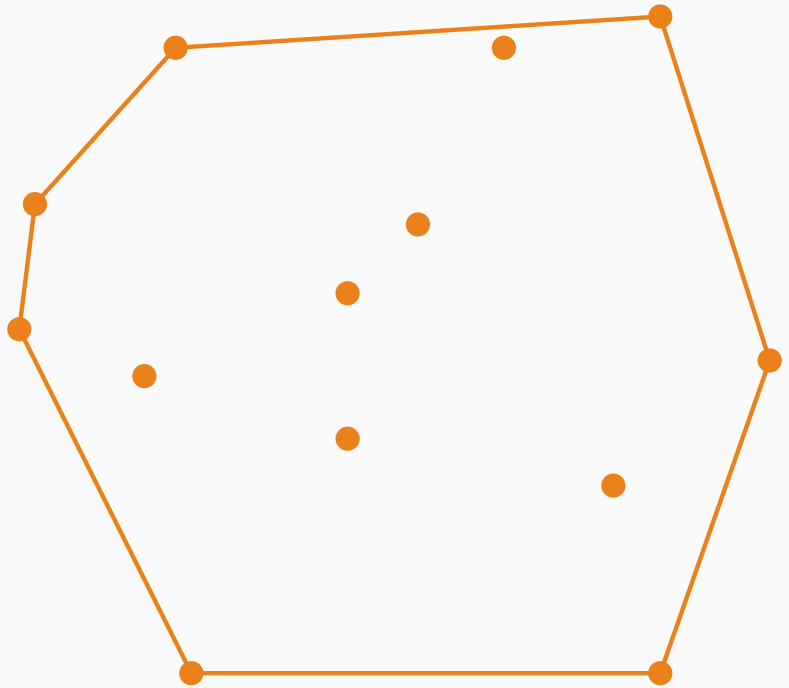

Graham scan:

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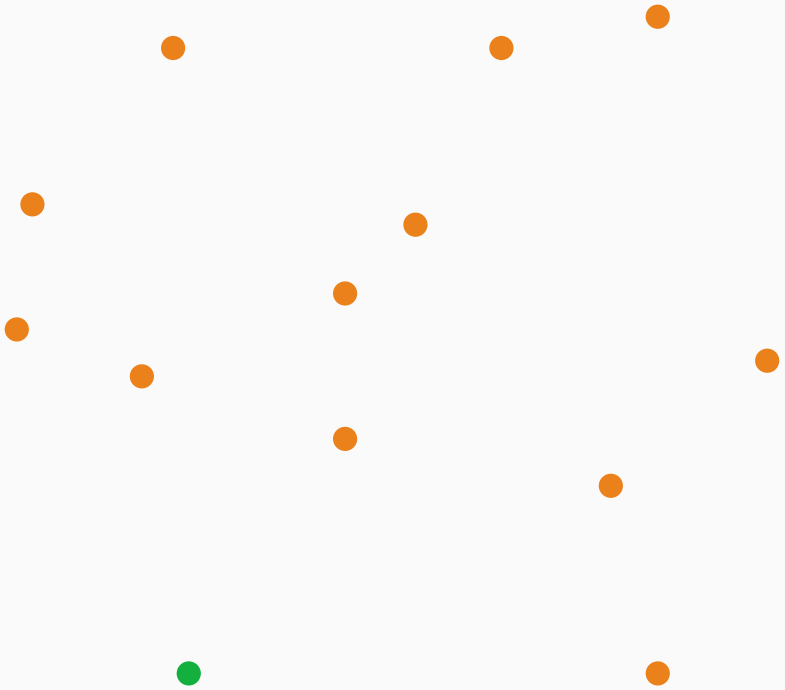
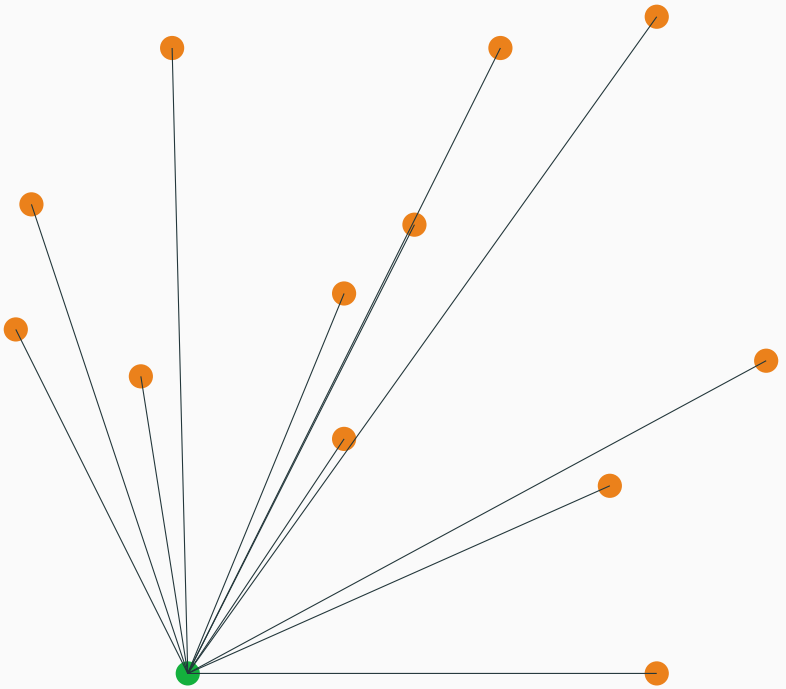
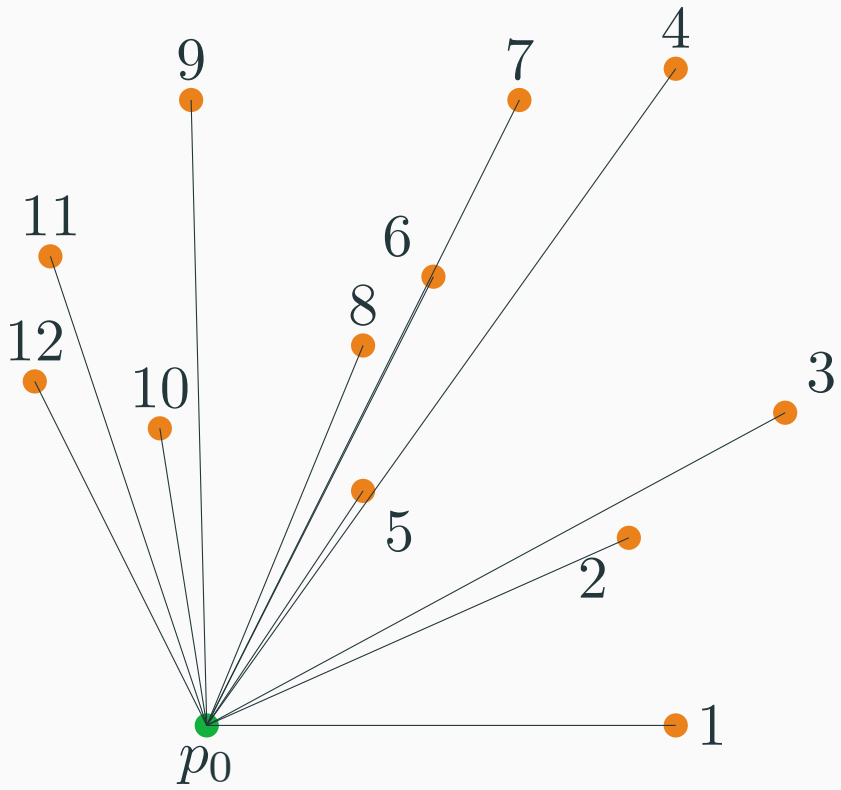
Time complexity $O(N \log N)$.



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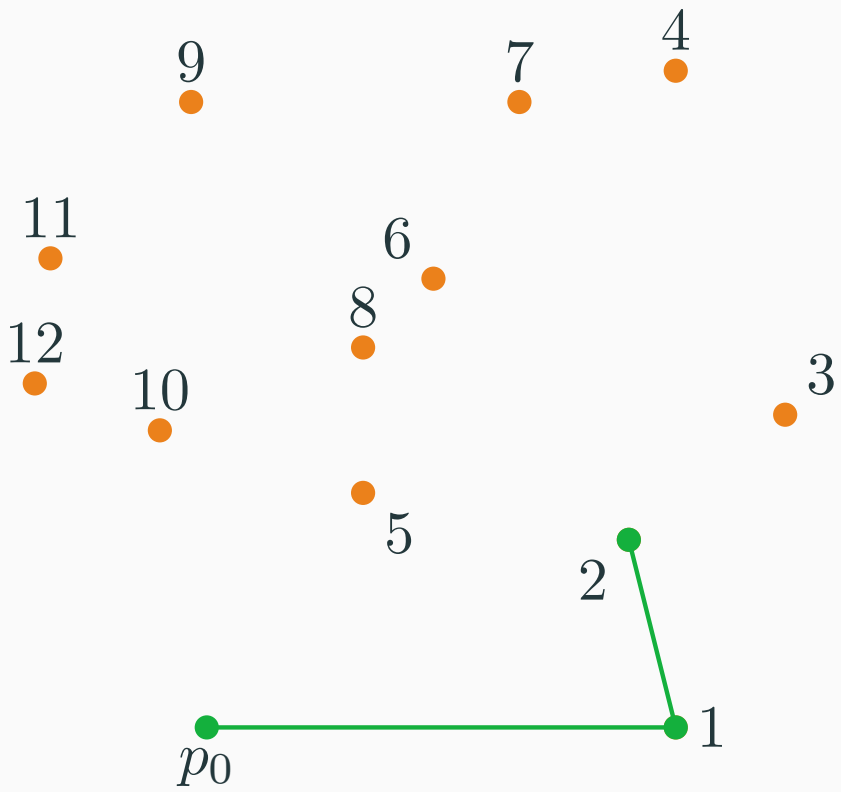
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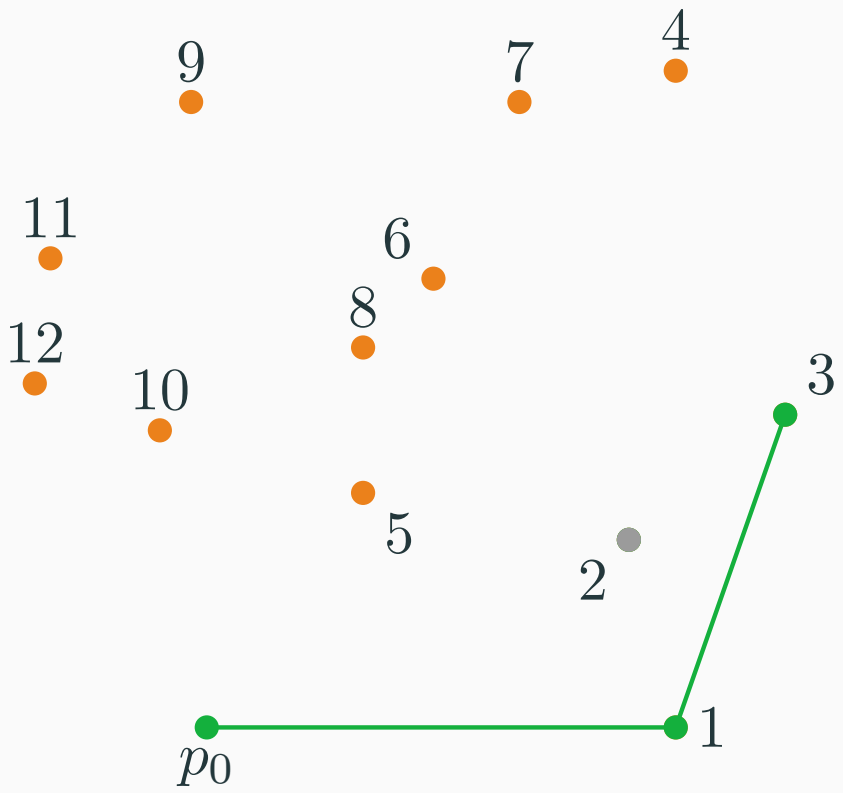
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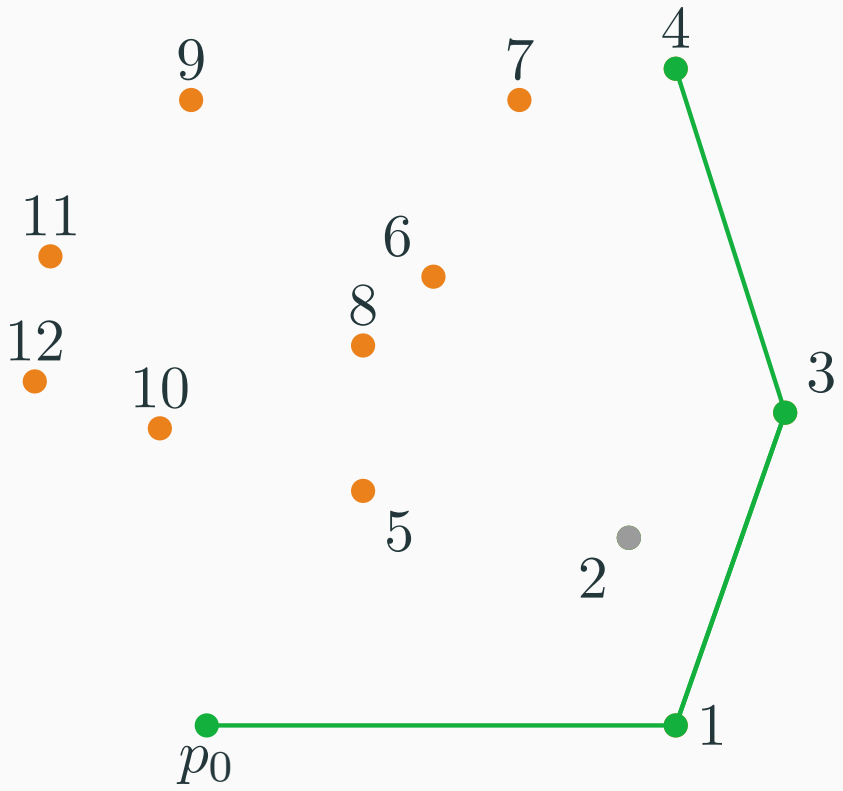
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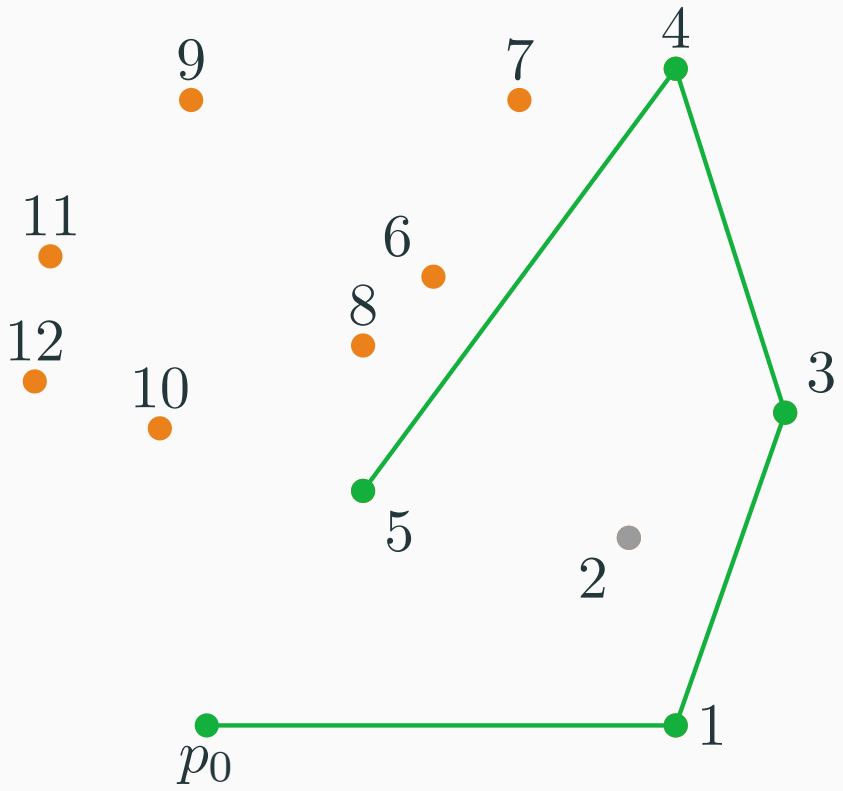
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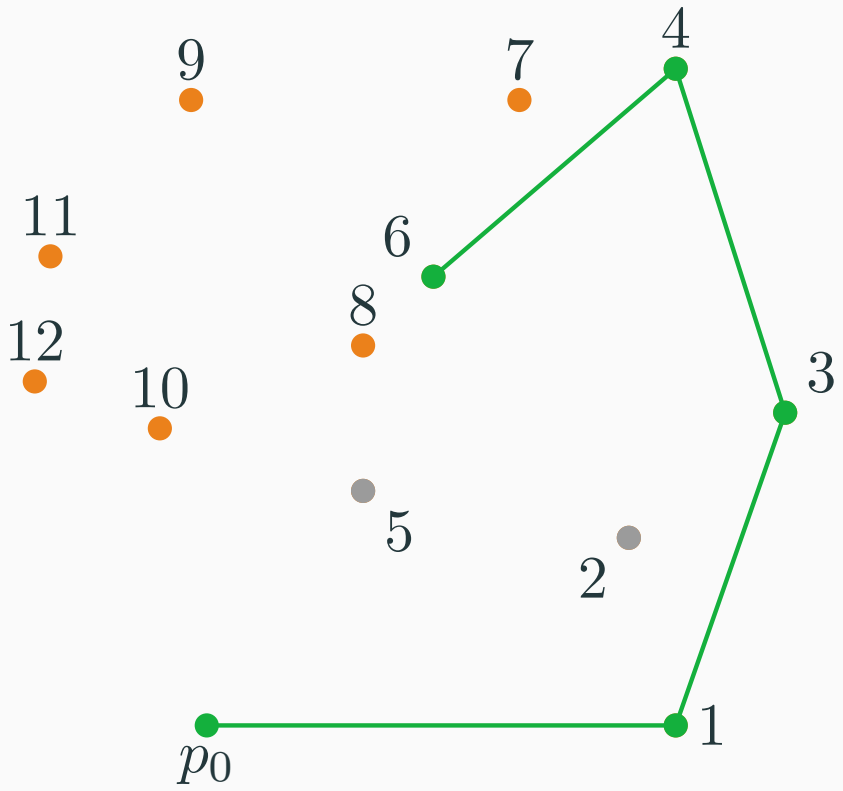
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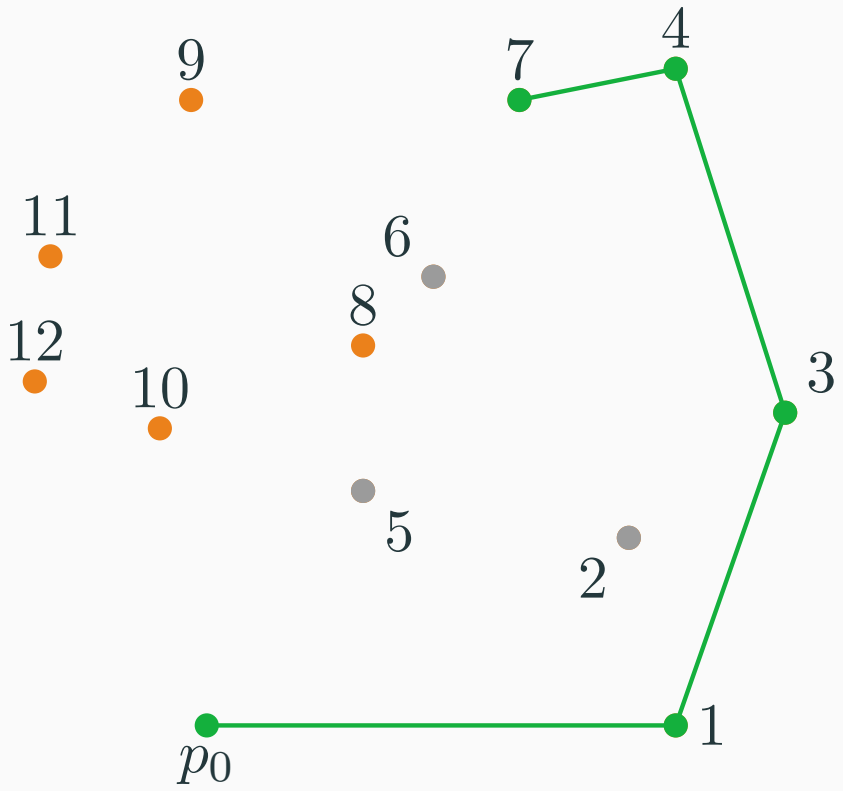
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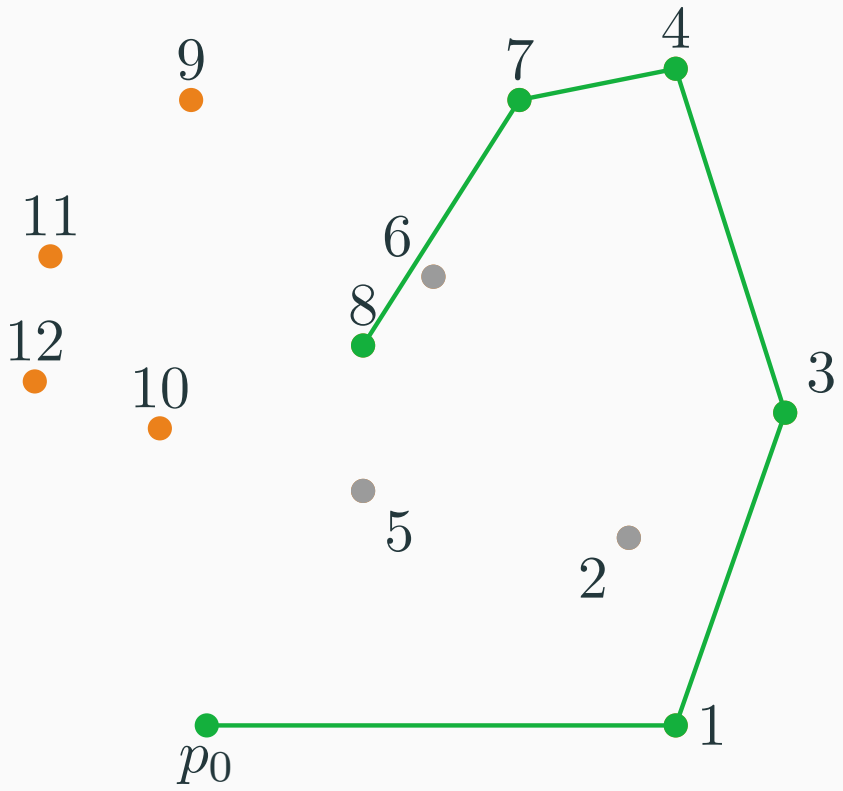
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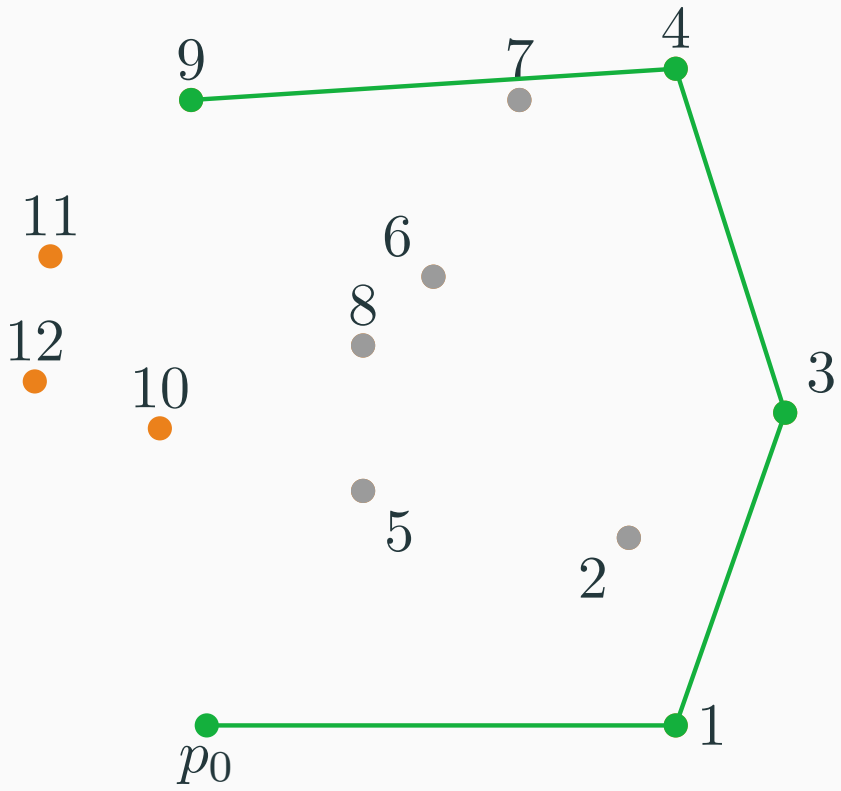
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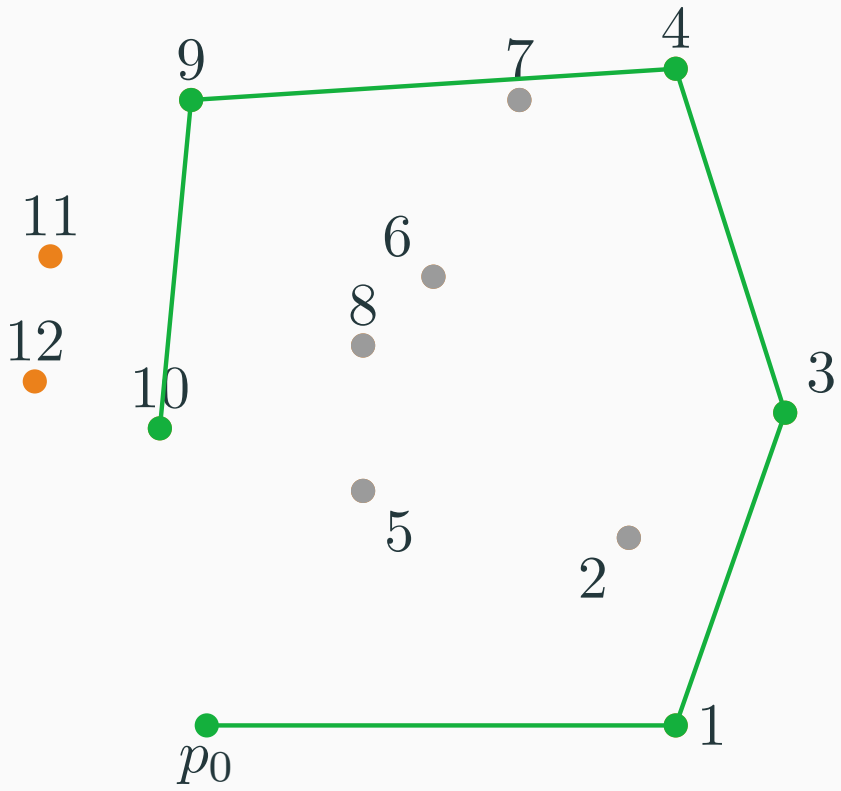
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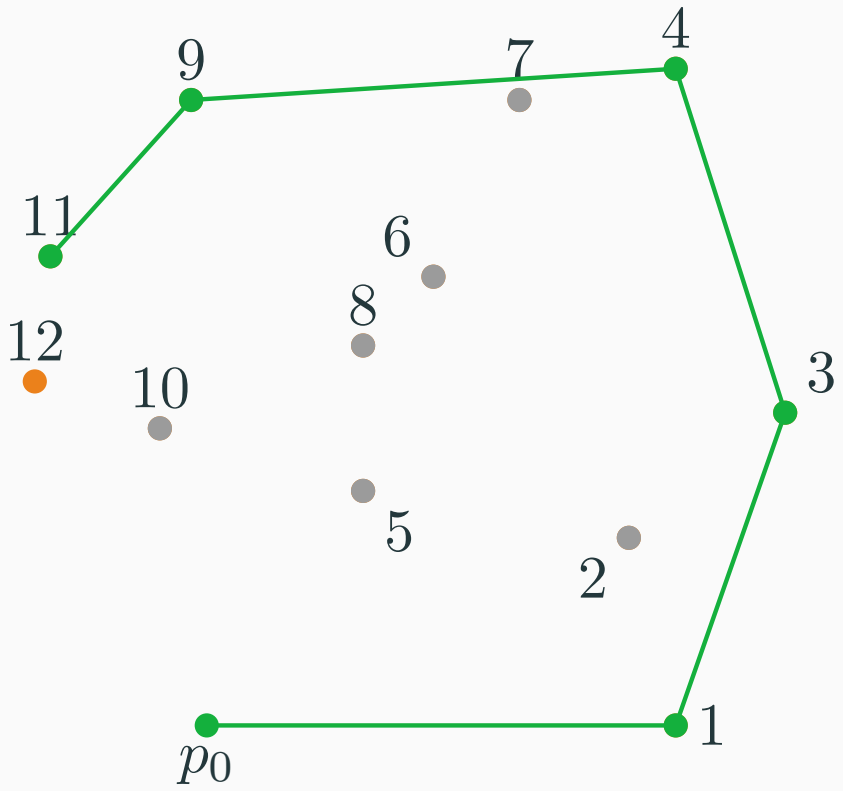
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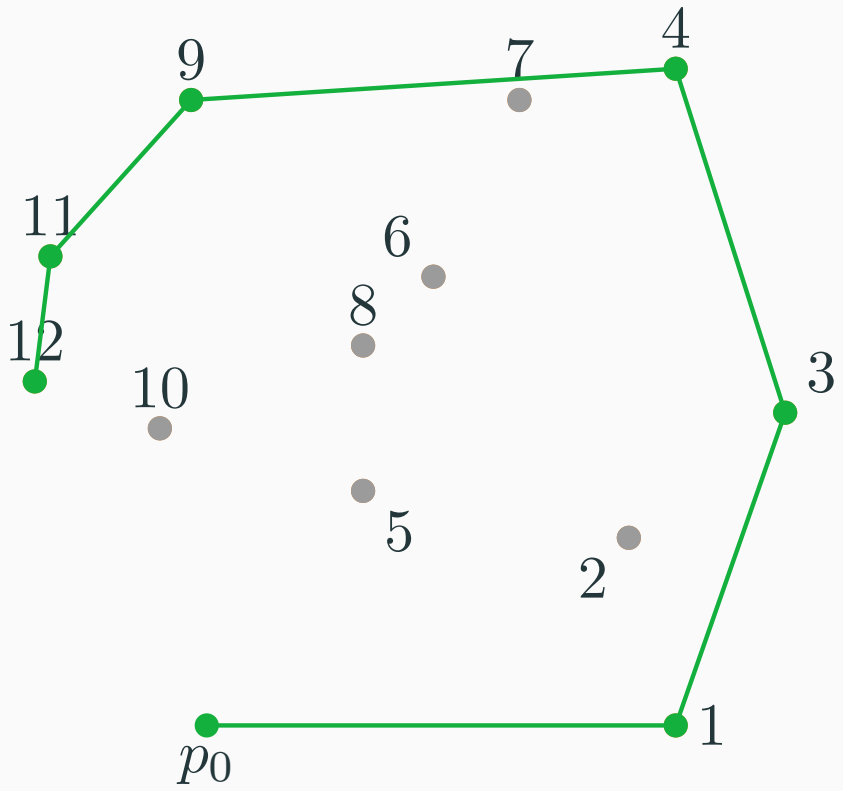
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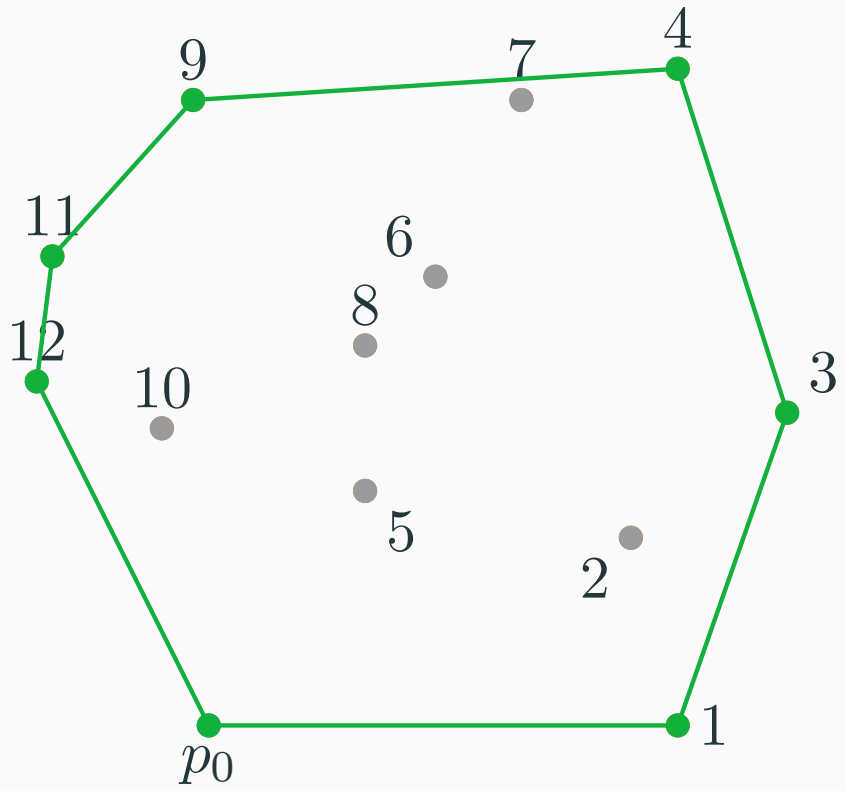
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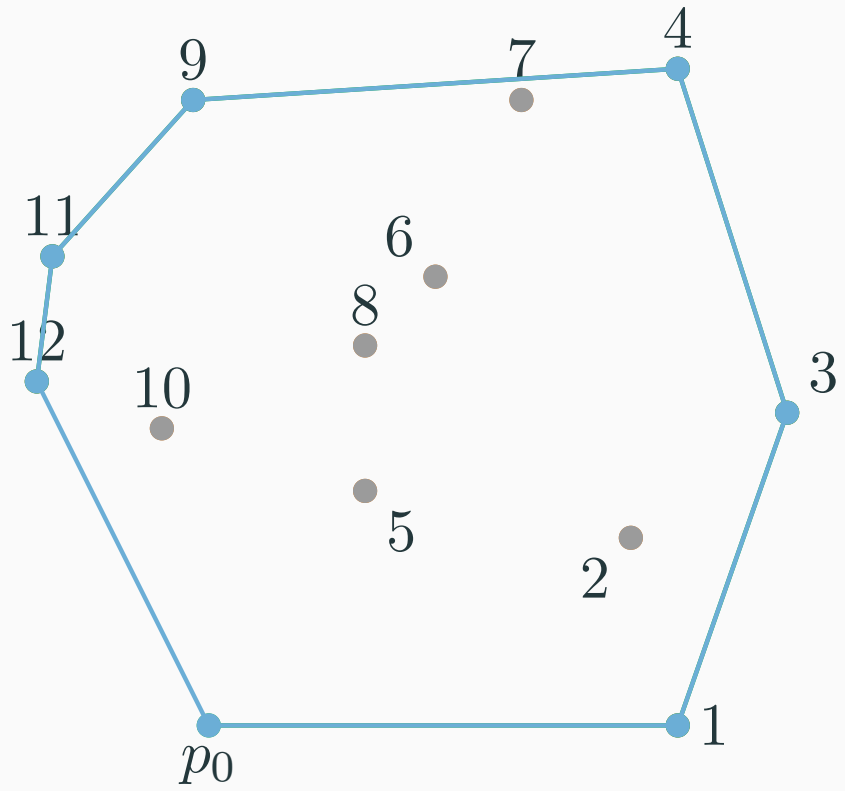
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2
1
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```
point hull[MAXN];
bool cmp(const point &a, const point &b) {
  return abs(real(a) - real(b)) > EPS ?
    real(a) < real(b) : imag(a) < imag(b); }
int convex_hull(vector<point> p) {
    int n = size(p), l = 0;
    sort(p.begin(), p.end(), cmp);
    for (int i = 0; i < n; i++) {
        if (i > 0 && p[i] == p[i - 1])
            continue;
        while (l >= 2 && ccw(hull[l - 2], hull[l - 1], p[i]) >= 0)
            l--;
        hull[l++] = p[i]; }
    int r = l;
    for (int i = n - 2; i >= 0; i--) {
        if (p[i] == p[i + 1])
            continue;
        while (r - l >= 1 && ccw(hull[r - 2], hull[r - 1], p[i]) >= 0)
            r--;
        hull[r++] = p[i]; }
    return l == 1 ? 1 : r - 1; }
```

Many other algorithms exist

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- Quick hull, similar idea to quicksort.

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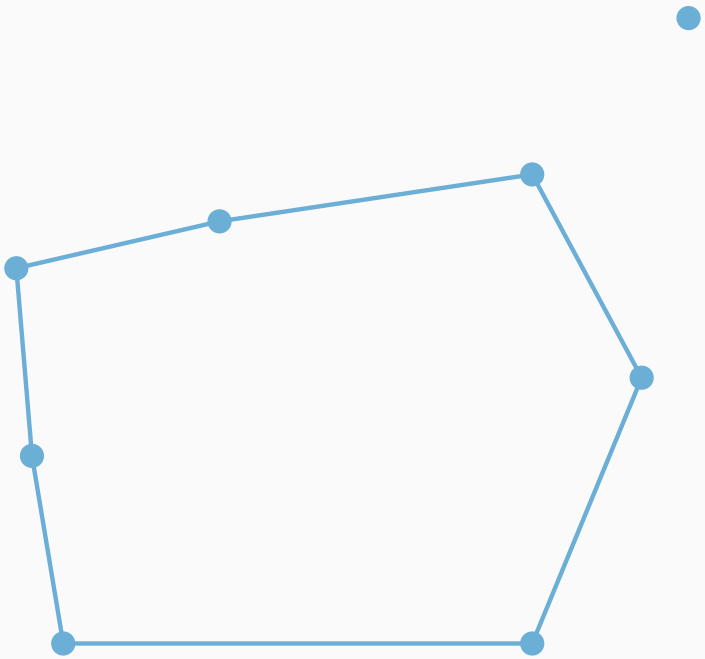
Some can be extended to three dimensions, or higher.

Simple algorithm to check if a point is in a convex polygon.



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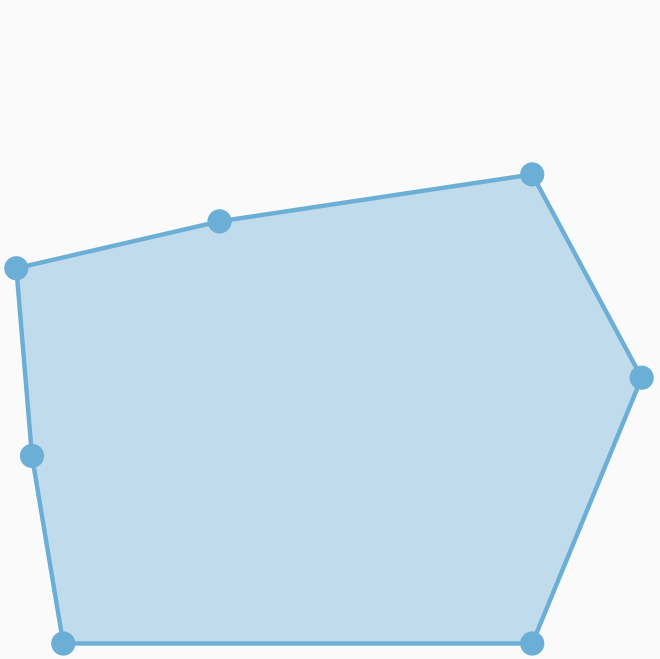

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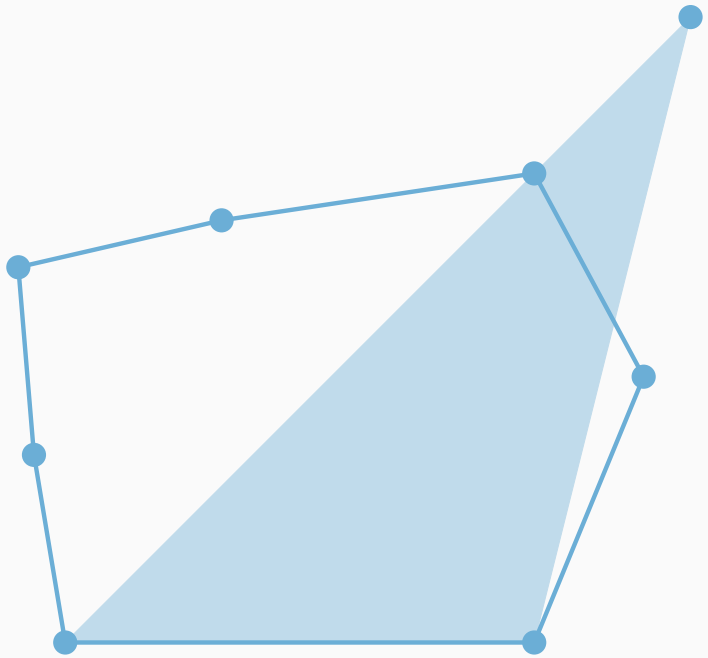

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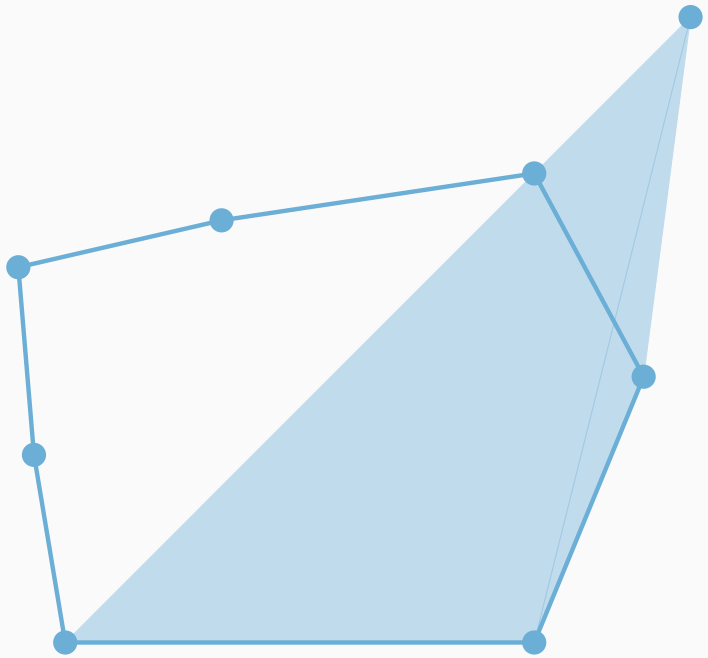

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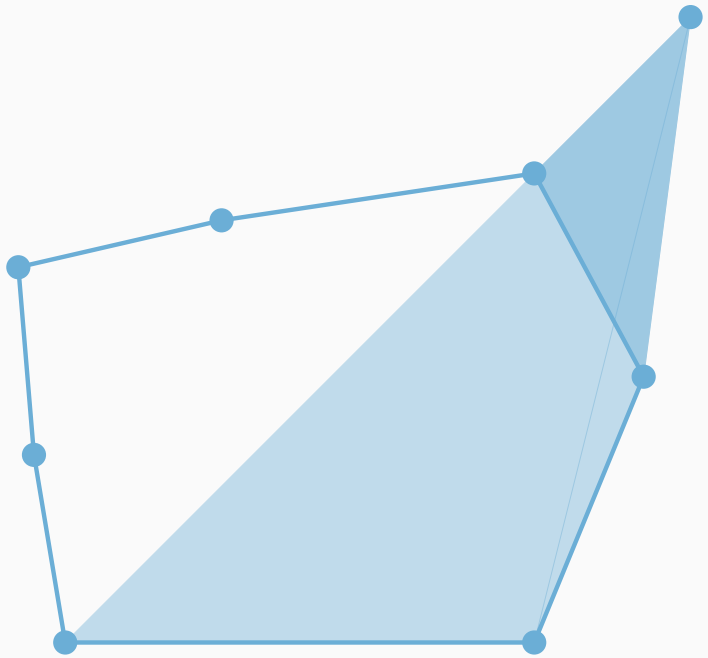

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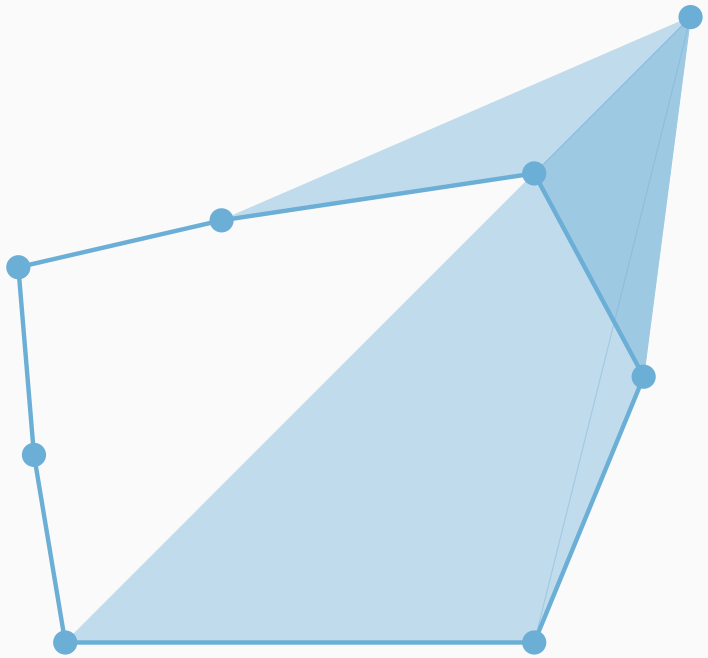

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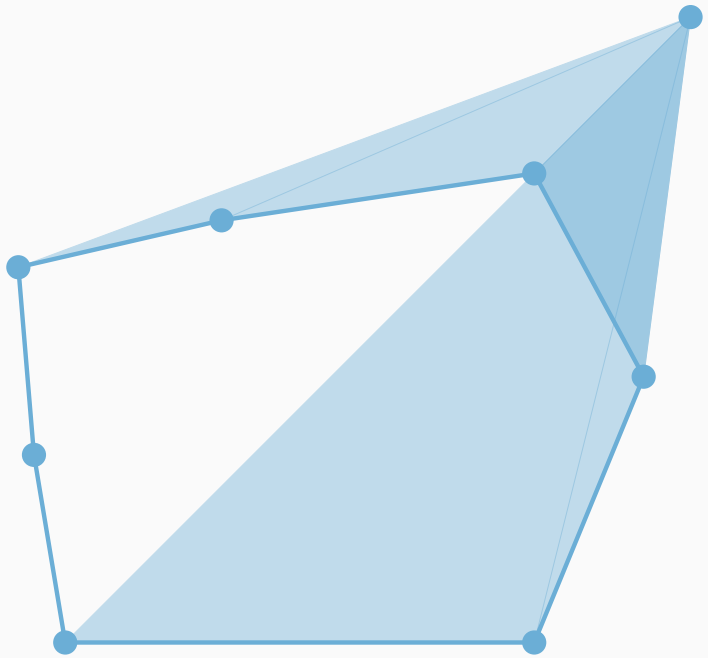

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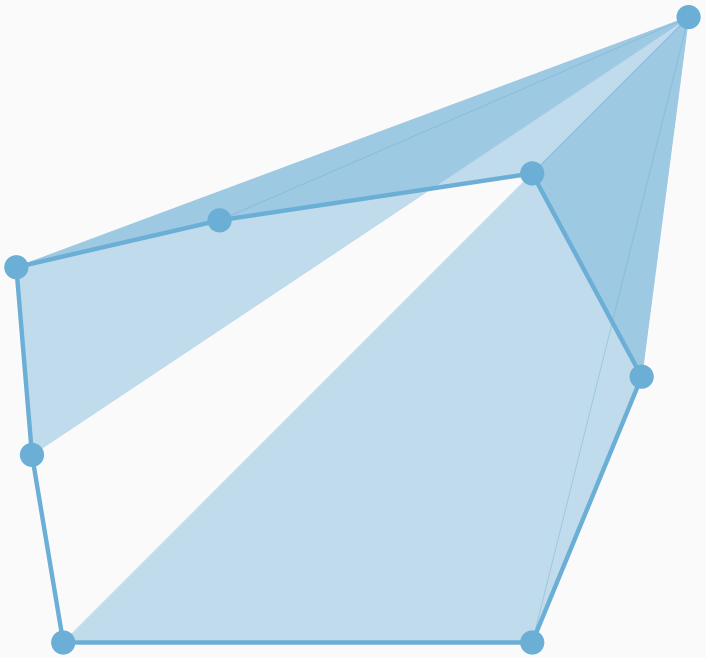

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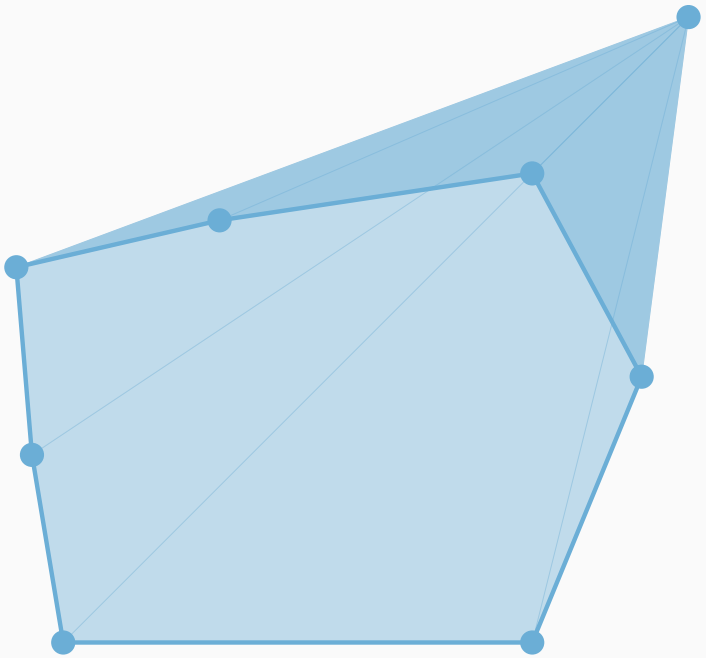

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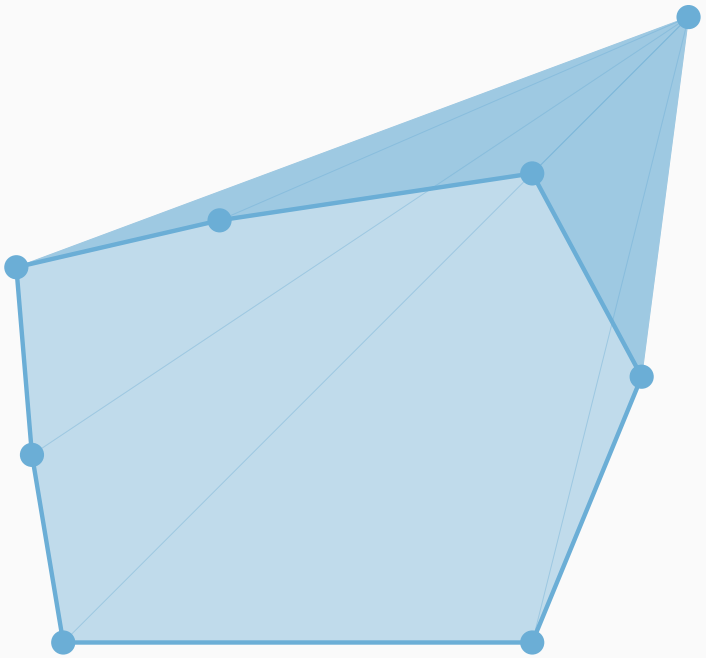

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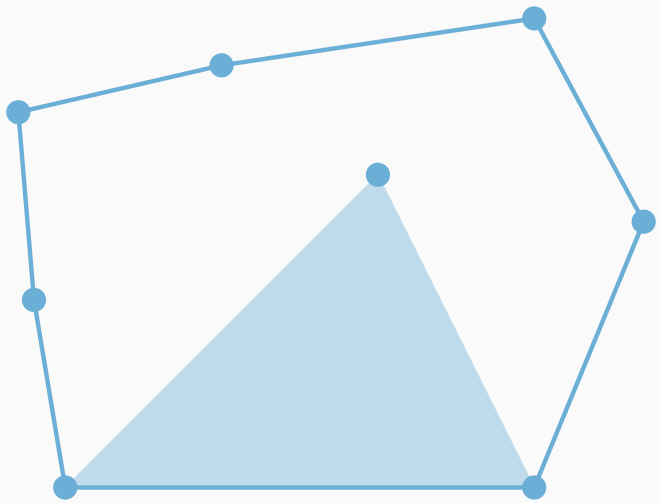

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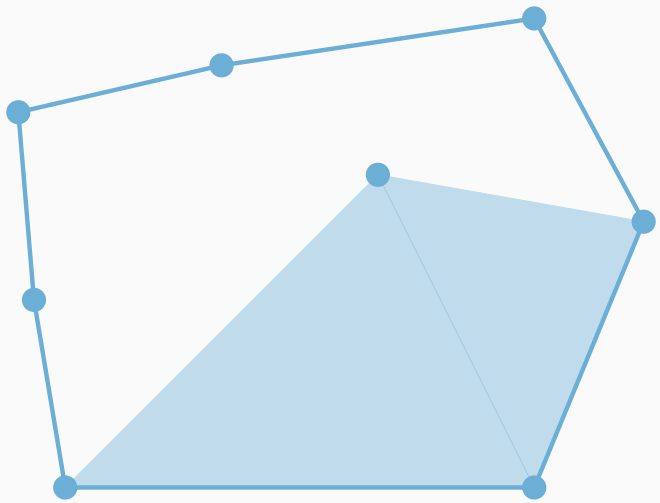

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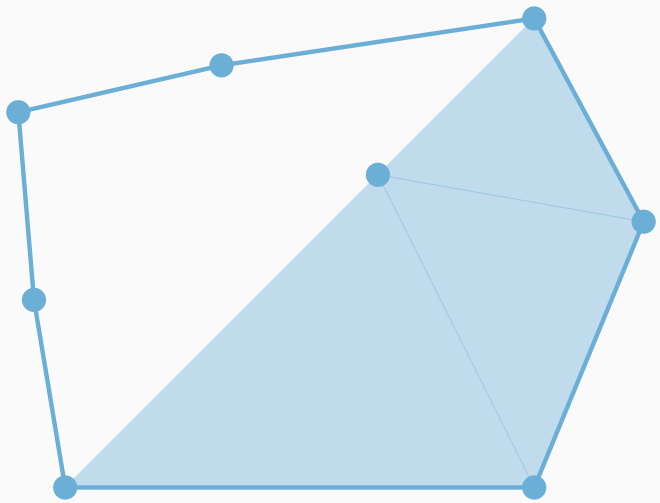

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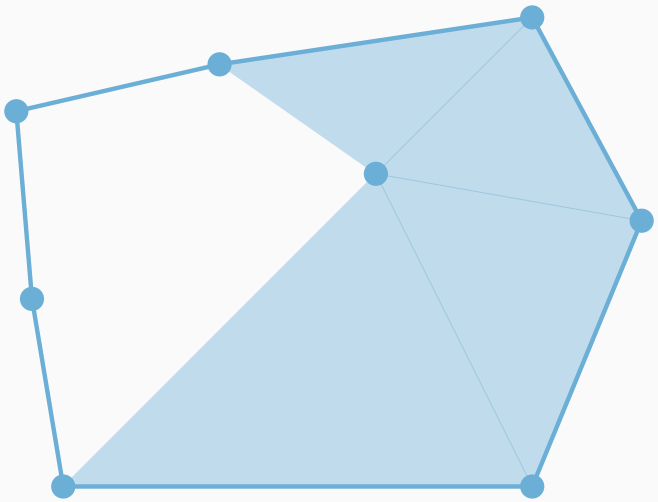

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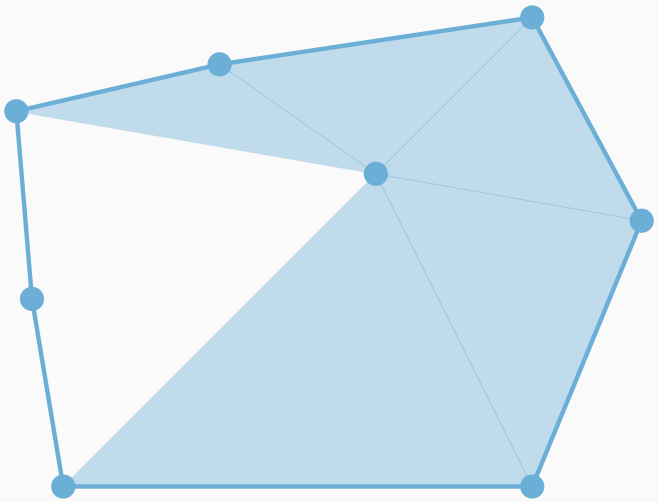

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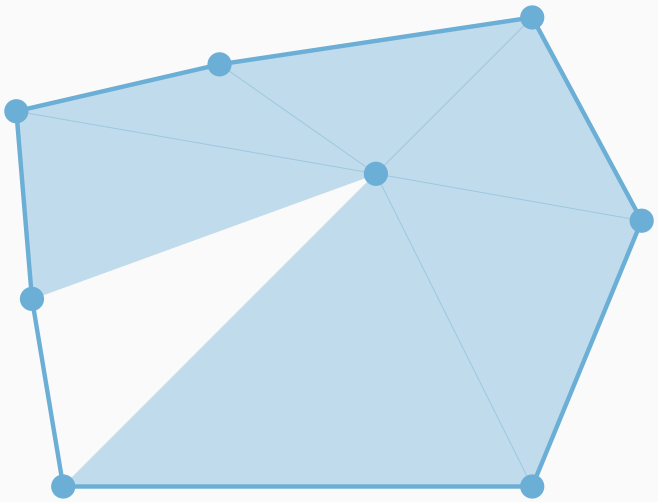

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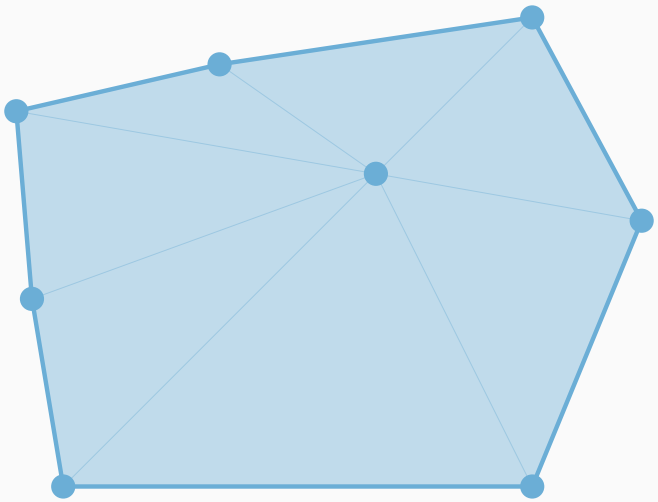

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How about non convex polygon?

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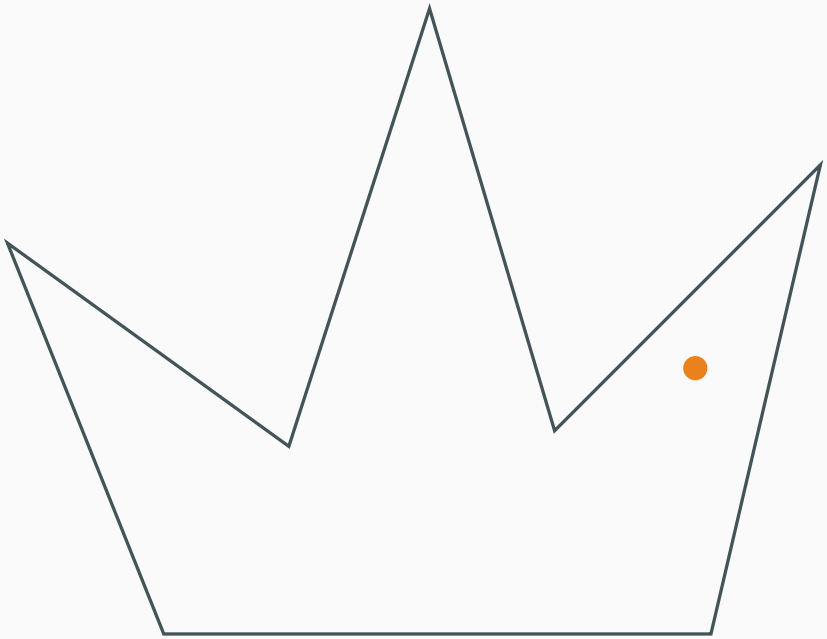
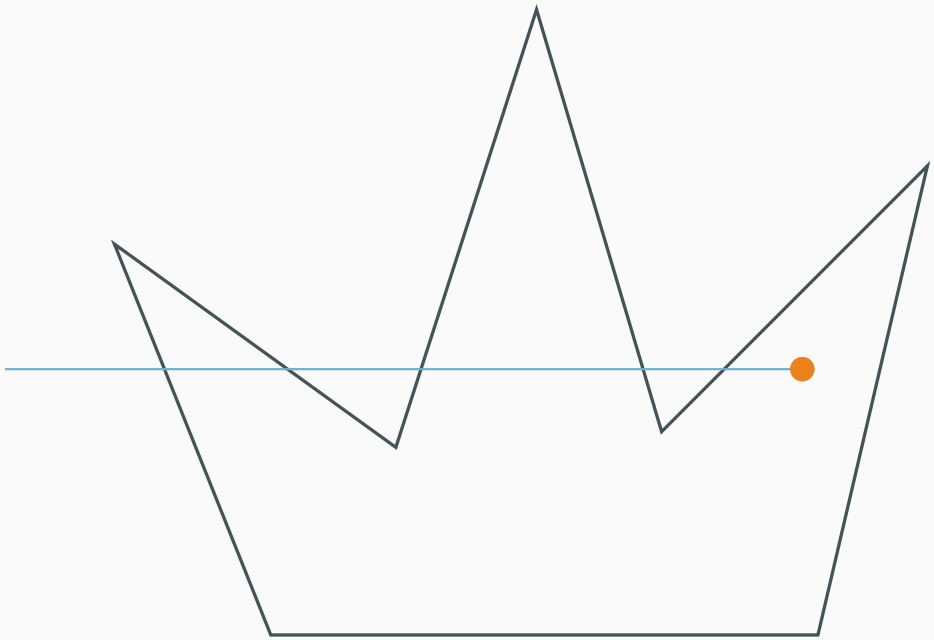
- We examine a ray passing through the polygon to the point.

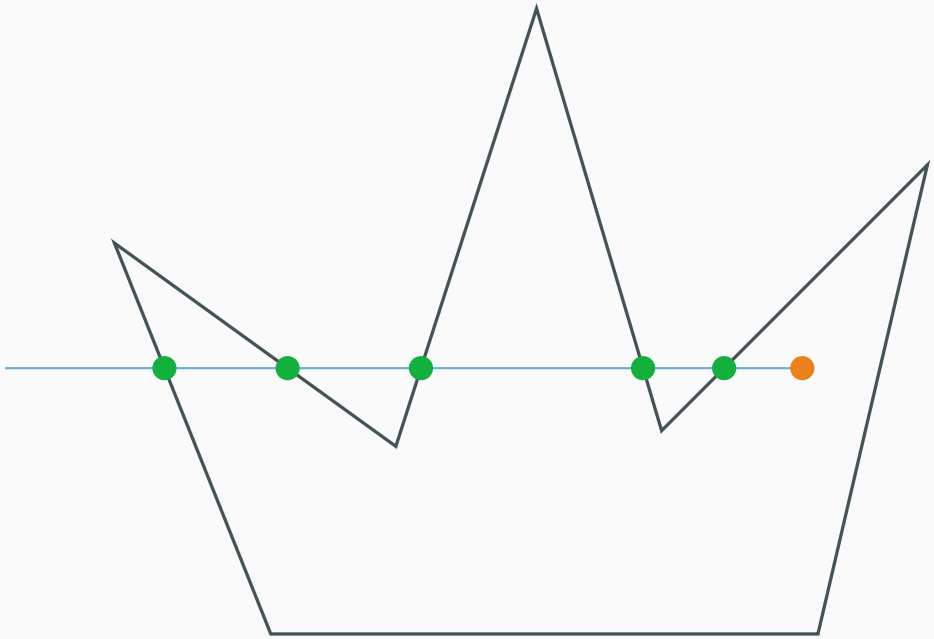
How about non convex polygon?

- The *even-odd rule* algorithm.
- We examine a ray passing through the polygon to the point.
- If the ray crosses the boundary of the polygon, then it alternately goes from outside to inside, and outside to inside.

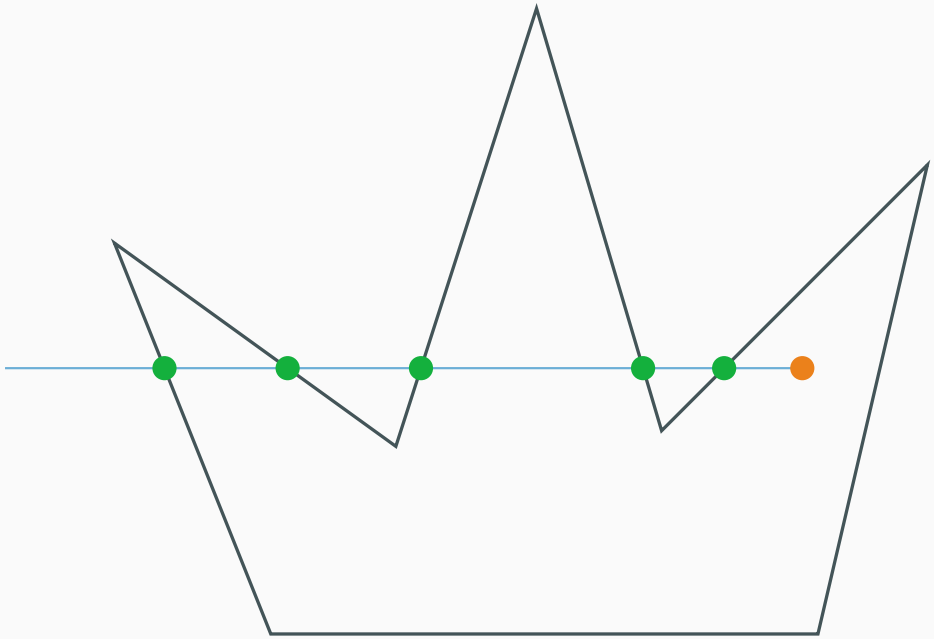




- Ray from the outside of the polygon to the point.



- Ray from the outside of the polygon to the point.
- Count the number of intersection points.



- Ray from the outside of the polygon to the point.
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- If odd, then the point is inside the polygon.
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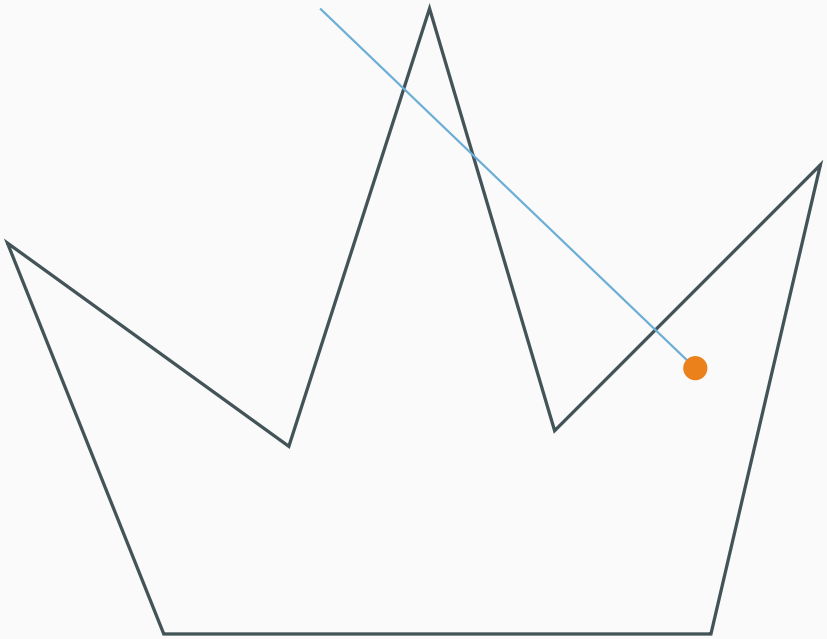

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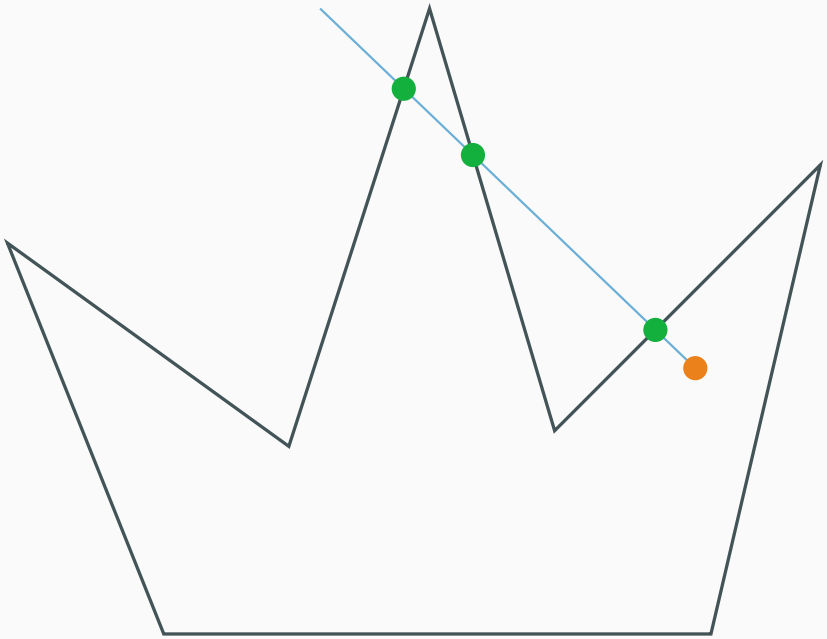

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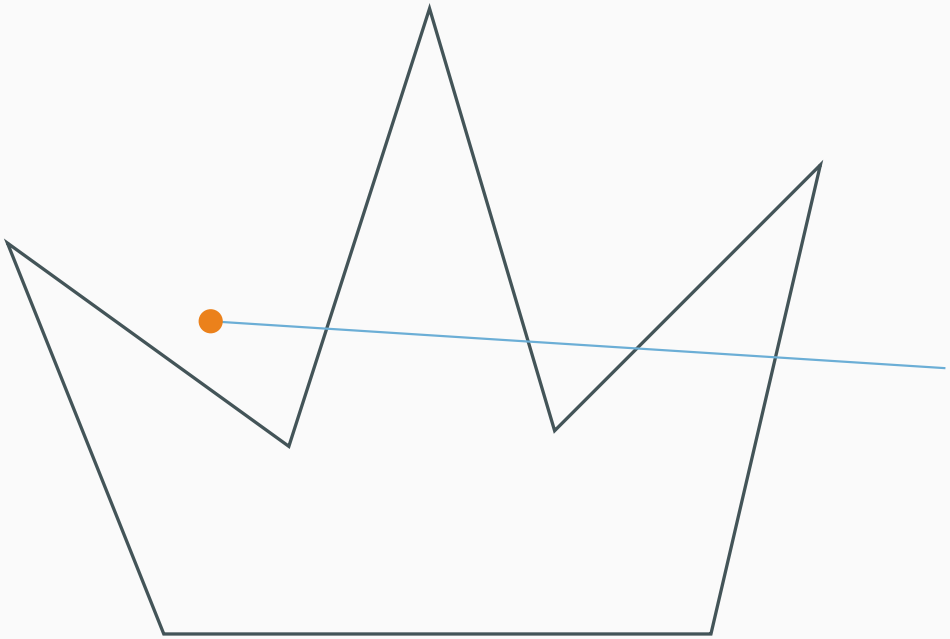

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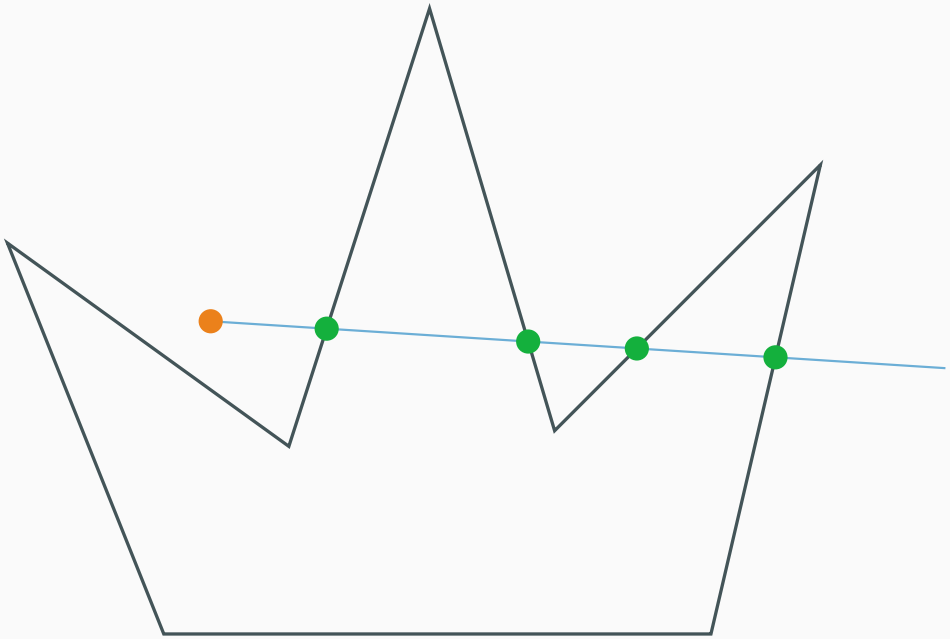

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# An algorithm

- Computational geometry has a lot of impressive and technical algorithms.
- The most famous one is probably Delaunay triangulation.
- But that one is a bit too hard for this course, so we will instead look at the classical closest point algorithm.
- We are given $n$ points in the plan, find the pair of points that are closest to one another.
- We can clearly solve this in $\mathcal{O}(n^2)$ time, but can we do better?

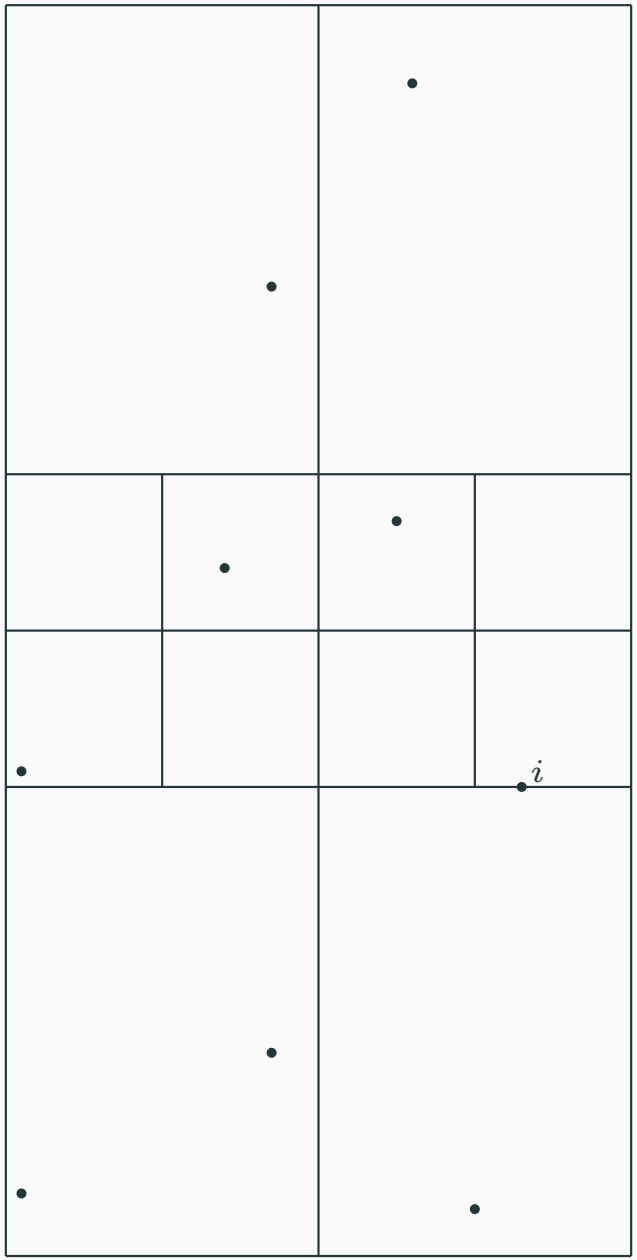
# Divide and conquer

- We sort the points by $x$-coordinate and split the list in half.
- Let $x_0$ be such that it's between the coordinates of the left and right halves.
- Start by solving each half recursively.
- We now have to find if there's some pair with one point in each half that does better.
- We can't simply try all pairs, that's too slow. Suppose the smallest distance we found recursively was $d$.
- Then we can ignore all points with $x$-coordinte outside $[x_0 - d, x_0 + d]$.
- Sort the points inside of this interval by their $y$-coordinate.
- The big trick is now that we only need to consider a few neighbours for each point.

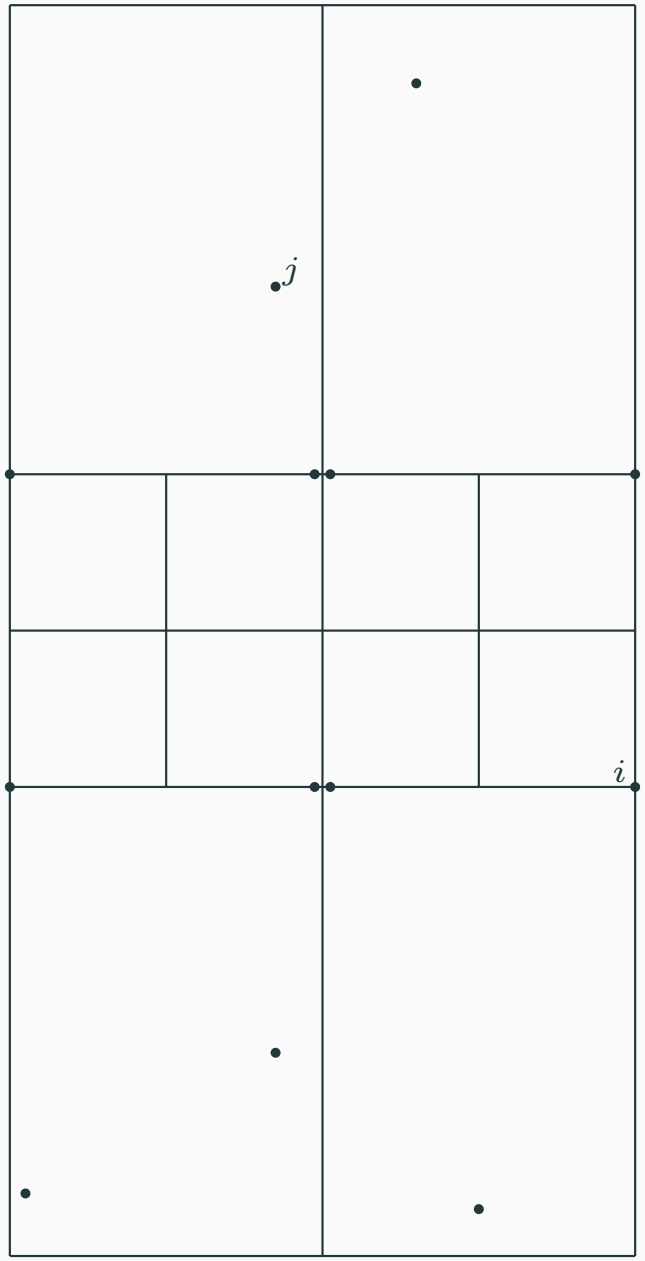
# Neighbours

- Divide the area above $x_i$ into 8 squares, each with side length $d/2$.
- If he distance between all points in each half is at least $d$, then we can have at most each point per square.
- All points outside these squares are at a distance of at least $d$ from $x_i$, so we can ignore them.
- Thus we only need to look at the distance from $x_i$ to $x_j$ when $j - i \leq 7$.

# Diagram



# Diagram



# Complexity

- Each recursive call is $\mathcal{O}(n \log(n))$.
- Thus by the master theorem the total complexity is $\mathcal{O}(n \log^2(n))$.
- If we sort the $y$ values as we go using mergesort, we can actually do each call in $\mathcal{O}(n)$.
- This way the complexity is actually $\mathcal{O}(n \log(n))$.

# Strings

# String problems

- Strings frequently appear in our kind of problems
  - I/O
  - Parsing
  - Identifiers/names
  - Data
- But sometimes strings play the key role
  - We want to find properties of some given strings
  - Is the string a palindrome?
- Here we're going to talk about things related to the latter type of problems
- These problems can be hard, because the length of the strings are often huge

# String Matching

# String matching

- Given a string $S$ of length $n$,
- and a string $T$ of length $m$,
- find all occurrences of $T$ in $S$
- Note:
  - Occurrences may overlap
  - Assume strings contain characters from some alphabet $\Sigma$

# String matching

Example:

- $S = \mathrm{cabcababacaba}$
- $T = \mathrm{aba}$

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- $S = \text{cabcababacaba}$
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- Three occurrences:
  - $\text{cabc}\color{red}{\text{aba}}\text{bacaba}$

# String matching

Example:

- $S = \text{cabcababacaba}$
- $T = \text{aba}$
- Three occurrences:
  - $\text{cabc}\textcolor{red}{\text{aba}}\text{bacaba}$
  - $\text{cabcab}\textcolor{red}{\text{aba}}\text{caba}$

# String matching

Example:

- $S = \text{cabcababacaba}$
- $T = \text{aba}$
- Three occurrences:
  - $\text{cabc}\textcolor{red}{\text{aba}}\text{bacaba}$
  - $\text{cabcab}\textcolor{red}{\text{aba}}\text{caba}$
  - $\text{cabcababac}\textcolor{red}{\text{aba}}$

# Naive string matching algorithm

- For each substring of length $m$ in $S$,
- check if that substring is equal to $T$.

# Naive string matching algorithm

- $S$: bacbababaabcbab
- $T$: ababaca

# Naive string matching algorithm

- $S$: `bacbababaabcbab`
- $T$: `  ababaca`

# Naive string matching algorithm

- $S$: `bacbababaabcbab`
- $T$: `  ababaca`

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- $S$: `bacbababaabcbab`
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- $S$: `bacbababaabcbab`
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# Naive string matching algorithm

- $S$: `bacbababaabcbab`
- $T$: `ababaca`

# Naive string matching algorithm

```
int string_match(const string &s, const string &t) {
    int n = s.size(),
        m = t.size();

    for (int i = 0; i + m - 1 < n; i++) {
        bool found = true;
        for (int j = 0; j < m; j++) {
            if (s[i + j] != t[j]) {
                found = false;
                break;
            }
        }
        if (found) {
            return i;
        }
    }

    return -1;
}
```

# Naive string matching algorithm

- Double for-loop
  - outer loop is $O(n)$ iterations
  - inner loop is $O(m)$ iterations worst case
- Time complexity is $O(nm)$ worst case

# Naive string matching algorithm

- Double for-loop
  - outer loop is $O(n)$ iterations
  - inner loop is $O(m)$ iterations worst case
- Time complexity is $O(nm)$ worst case
- Can we do better?

# Knuth–Morris–Pratt algorithm

- The KMP algorithm avoids useless comparisons:
  - $S$: `bacbababaabcbab`
  - $T$: `ababaca`

# Knuth–Morris–Pratt algorithm

- The KMP algorithm avoids useless comparisons:
  - $S$: `bacbababaabcbab`
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# Knuth–Morris–Pratt algorithm

- The KMP algorithm avoids useless comparisons:
  - $S$: `bacbababaabcbab`
  - $T$: `ababaca`
- The number of shifts depend on which characters are currently matched

# Knuth–Morris–Pratt algorithm

- How are the number of shifts determined?
- Let $\pi[q] = \max\{k : k < q \text{ and } T[1 \ldots k] \text{ is a suffix of } T[1 \ldots q]\}$

# Knuth–Morris–Pratt algorithm

- How are the number of shifts determined?
- Let $\pi[q] = \max\{k : k < q \text{ and } T[1 \ldots k] \text{ is a suffix of } T[1 \ldots q]\}$
- Example:

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $T[i]$ | a | b | a | b | a | c | a |
| $\pi[i]$ | 0 | 0 | 1 | 2 | 3 | 0 | 1 |

# Knuth–Morris–Pratt algorithm

- How are the number of shifts determined?
- Let $\pi[q] = \max\{k : k < q \text{ and } T[1 \ldots k] \text{ is a suffix of } T[1 \ldots q]\}$
- Example:

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $T[i]$ | a | b | a | b | a | c | a |
| $\pi[i]$ | 0 | 0 | 1 | 2 | 3 | 0 | 1 |

- If, at position $i$, $q$ characters match (i.e. $T[1 \ldots q] = S[i \ldots i + q - 1]$), then
  - if $q = 0$, shift pattern $1$ position right
  - otherwise, shift pattern $q - \pi[q]$ positions right

# Knuth–Morris–Pratt algorithm

- Example:
  - $S$: `bacbababaabcbab`
  - $T$: `ababaca`

# Knuth–Morris–Pratt algorithm

- Example:
  - $S$: `bacbababaabcbab`
  - $T$: `ababaca`
  - $5$ characters match, so $q = 5$

# Knuth–Morris–Pratt algorithm

- Example:
  - $S$: `bacbababaabcbab`
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  - $\pi[q] = \pi[5] = 3$

# Knuth–Morris–Pratt algorithm

- Example:
  - $S$: `bacbababaabcbab`
  - $T$: `ababaca`
  - $5$ characters match, so $q = 5$
  - $\pi[q] = \pi[5] = 3$
  - Then shift $q - \pi[q] = 5 - 3 = 2$ positions

# Knuth–Morris–Pratt algorithm

- Example:
  - $S$: `bacbababaabcbab`
  - $T$: `ababaca`
  - $5$ characters match, so $q = 5$
  - $\pi[q] = \pi[5] = 3$
  - Then shift $q - \pi[q] = 5 - 3 = 2$ positions
  - $S$: `bacbababaabcbab`
  - $T$: `ababaca`

# Knuth–Morris–Pratt algorithm

- Given $\pi$, matching only takes $O(n)$ time
- $\pi$ can be computed in $O(m)$ time
- Total time complexity of KMP therefore $O(n + m)$ worst case

# Knuth–Morris–Pratt algorithm

```
vi kmppi(string &p) {
  int m = p.size(), i = 0, j = -1;
  vi b(m + 1, -1);
  while(i < m) {
    while(j >= 0 && p[i] != p[j]) j = b[j];
    b[++i] = ++j;
  }
  return b;
}

vi kmp(string &s, string &p) {
  int n = s.size(), m = p.size(), i = 0, j = 0;
  vi b = kmppi(p), a = vi();
  while(i < n) {
    while(j >= 0 && s[i] != p[j]) j = b[j];
    ++i; ++j;
    if(j == m) {
      a.push_back(i - j);
      j = b[j];
    }
  }
  return a; }
```

# Tries

# Sets of strings

- We often have sets (or maps) of strings
- Insertions and lookups usually guarantee $O(\log n)$ comparisons

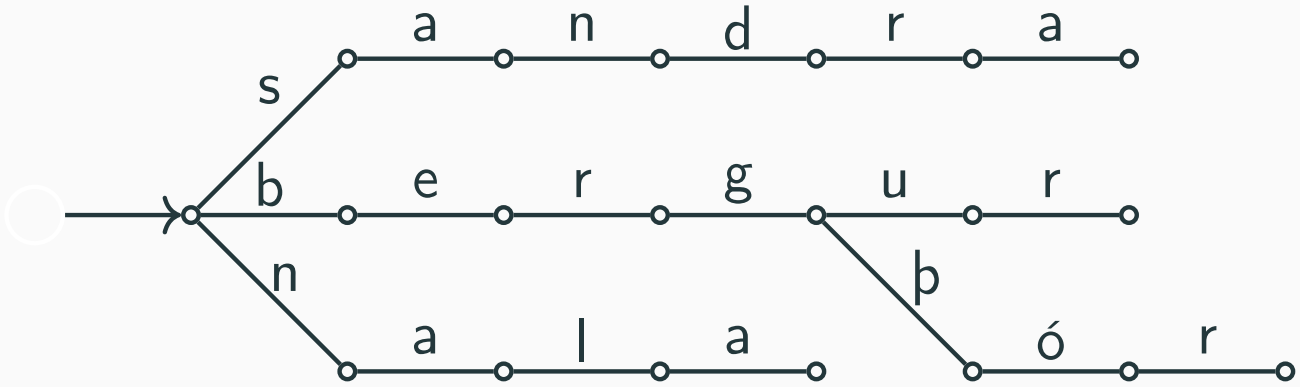
- But string comparisons are actually pretty expensive...

- There are other data structures, like tries, which do this in a more clever way

# Tries

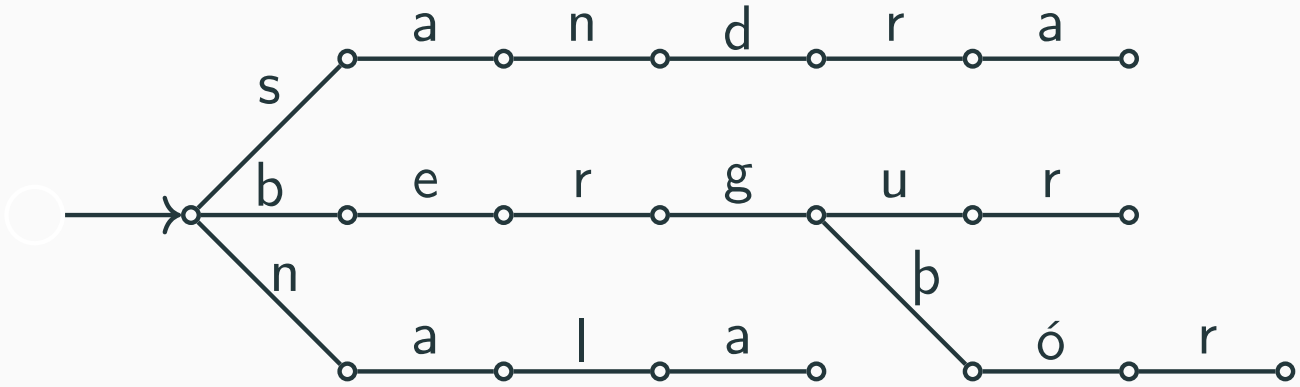
- Tries contain strings not at every node, but as paths in a tree.
- Each node only has a character and we say the trie contains the string if you can get it by walking along nodes starting at the root.
- The nodes can also carry additional data, quite a lot in fact, as we will see later.

# Example



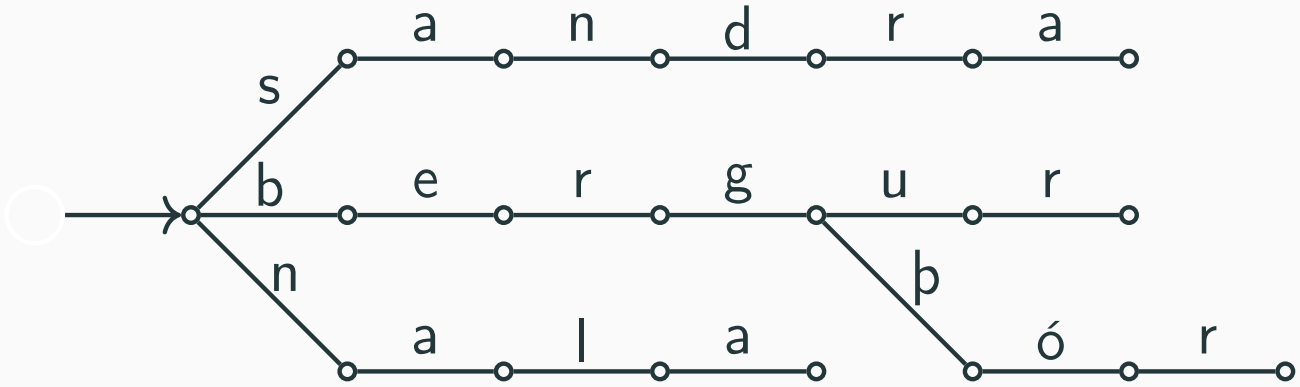

- Examples of strings in this trie include:

# Example



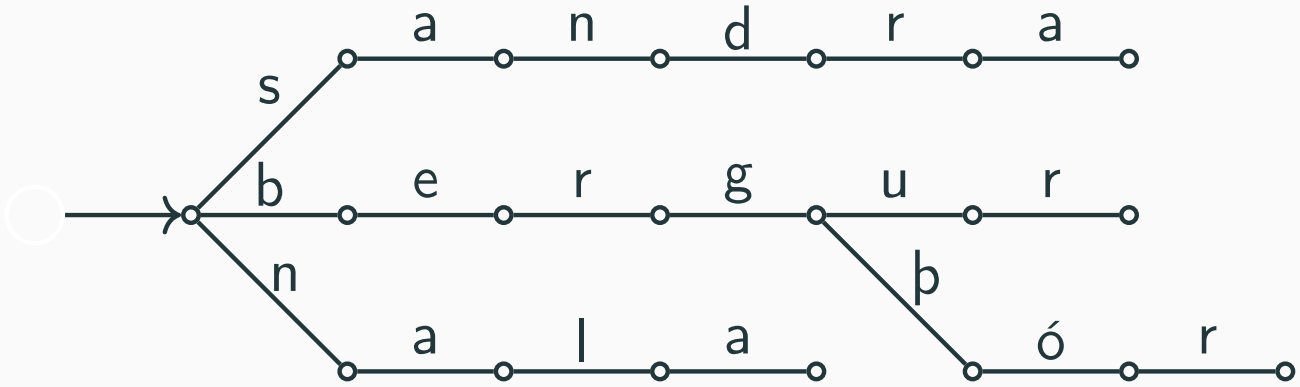

- Examples of strings in this trie include:

# Example



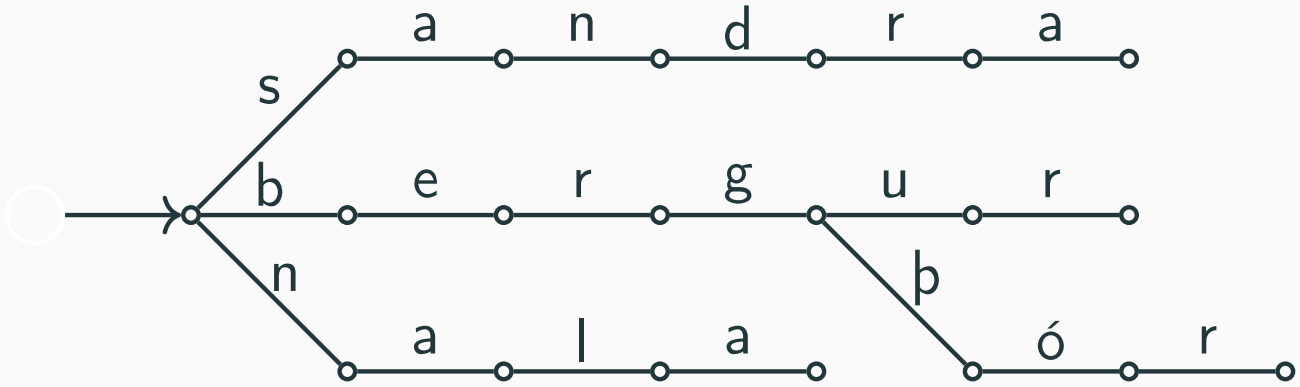

- Examples of strings in this trie include:
  - „sandra”,

# Example



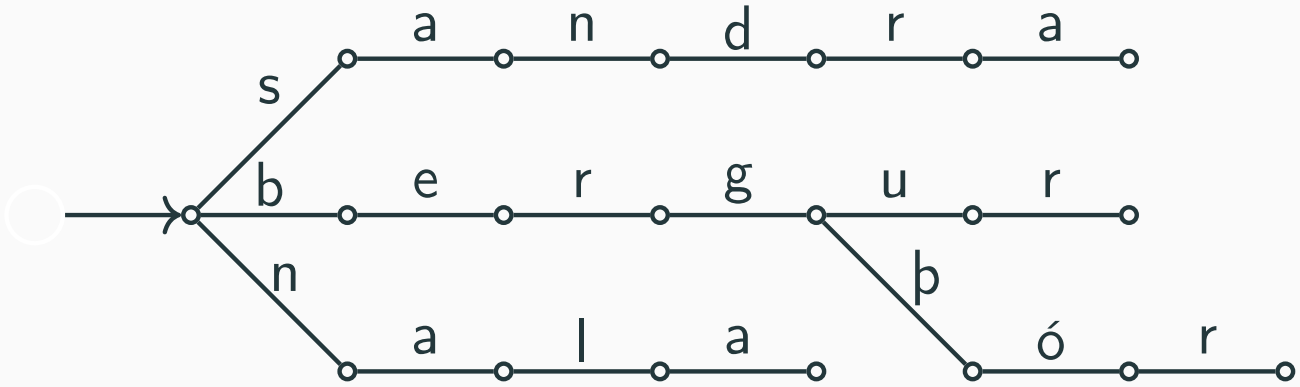

- Examples of strings in this trie include:
  - „sandra”,
  - „nala”,

# Example



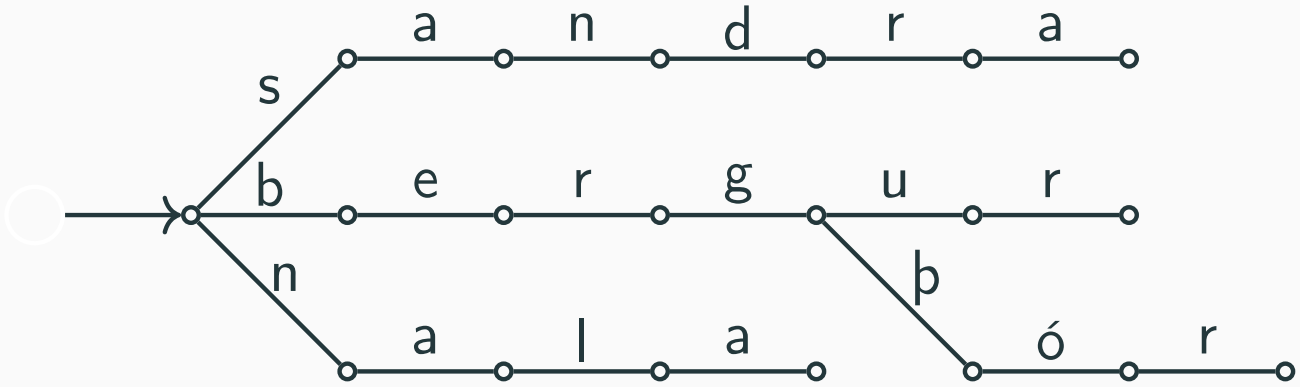

- Examples of strings in this trie include:
  - „sandra”,
  - „nala”,
  - „bergur”,

# Example



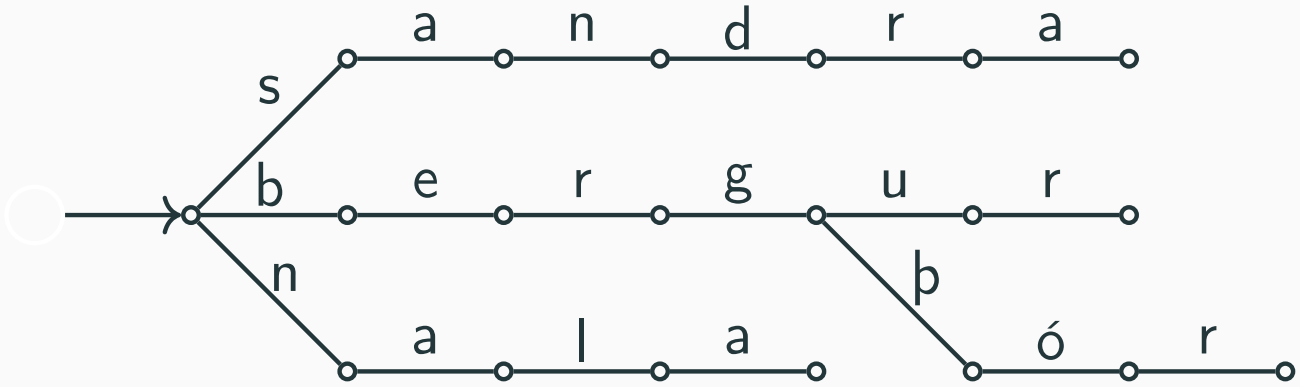

- Examples of strings in this trie include:
  - „sandra”,
  - „nala”,
  - „bergur”,
  - „bergþór”,

# Example




- Examples of strings in this trie include:
  - „sandra”,
  - „nala”,
  - „bergur”,
  - „bergþór”,
  - „san” and

# Example

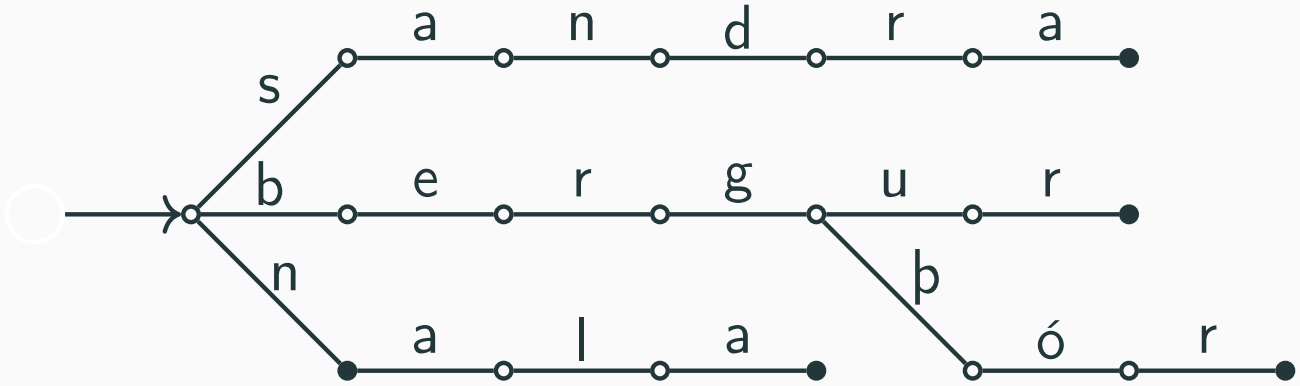



- Examples of strings in this trie include:
  - „sandra”,
  - „nala”,
  - „bergur”,
  - „bergþór”,
  - „san” and
  - „” (empty string)

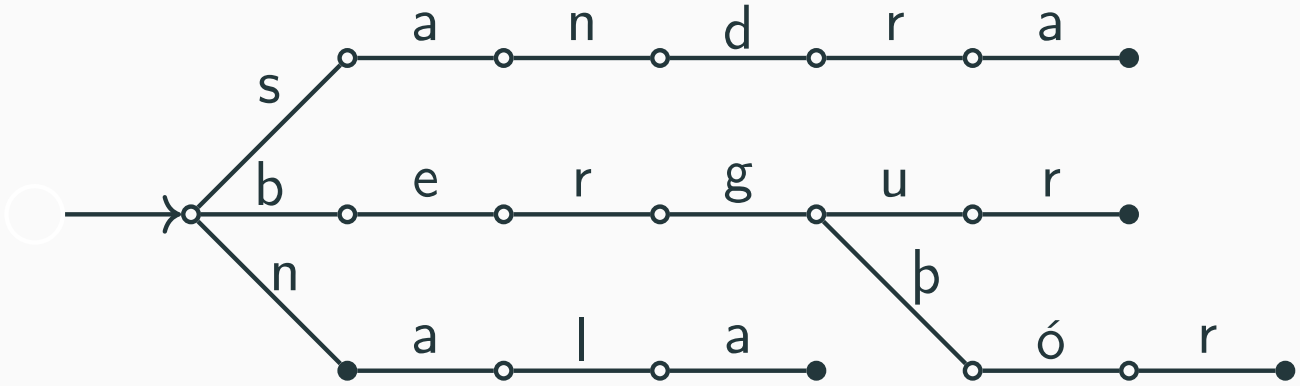
# End nodes

- It is common to mark some nodes as end nodes.
- This is an example of extra data to put into nodes.
- Then we can consider a string $s$ to be in the tree if you can walk through the tree to get the string **and** end at an end node.

# Example

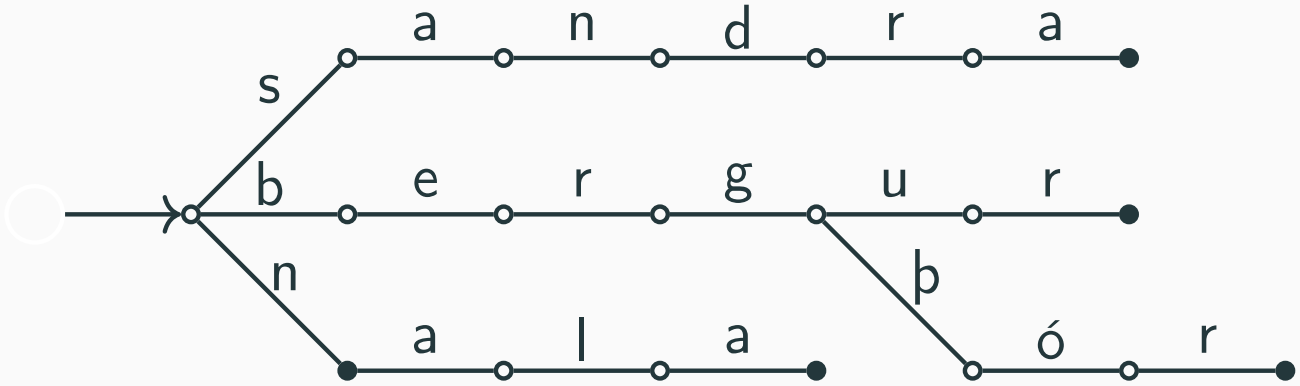


# Example



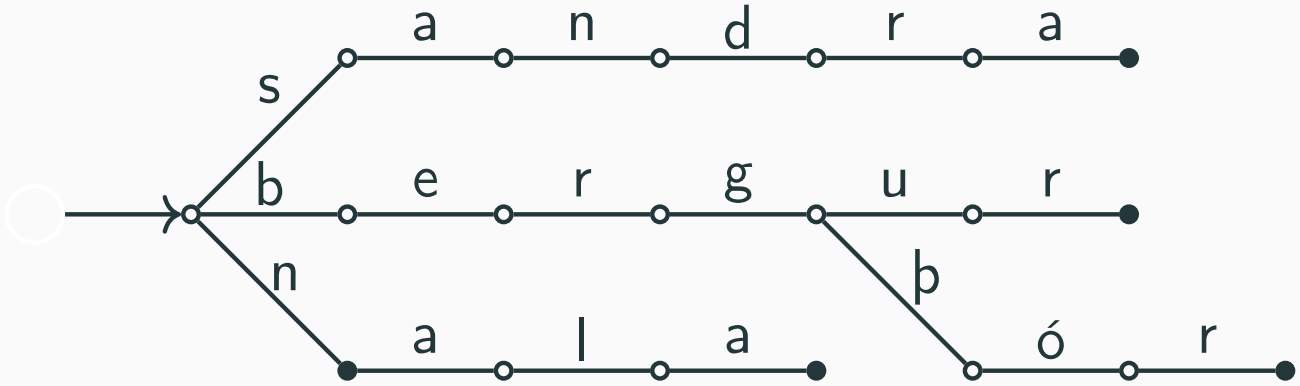

- The strings in the trie are:

# Example



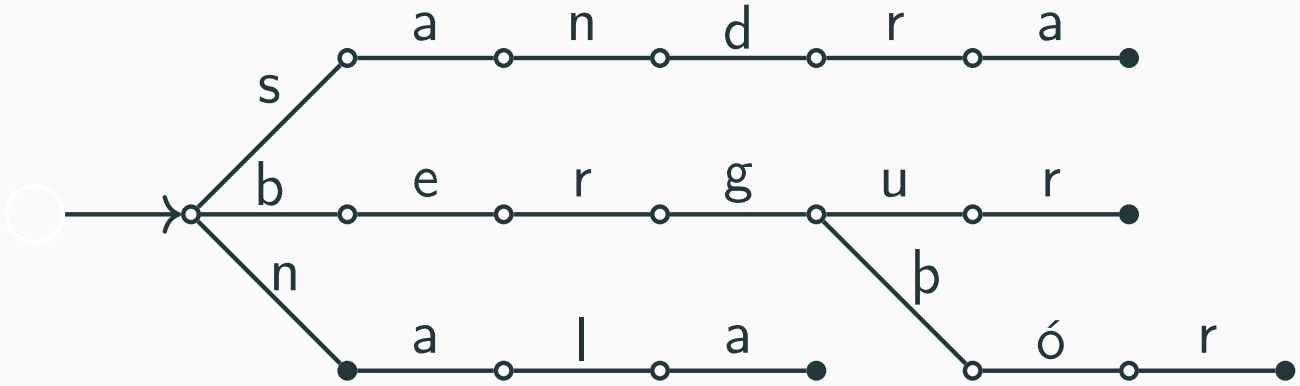

- The strings in the trie are:
  - „sandra”,

# Example



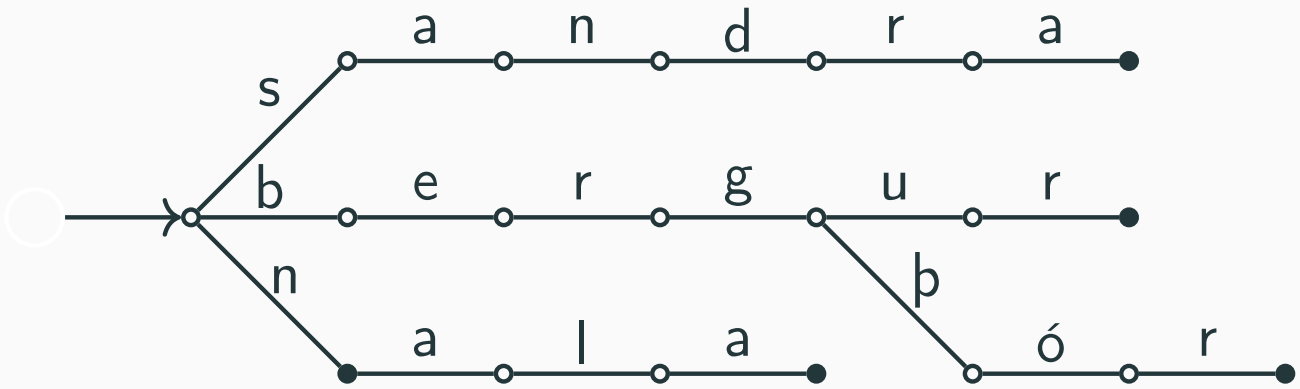

- The strings in the trie are:
  - „sandra”,
  - „nala”,

# Example



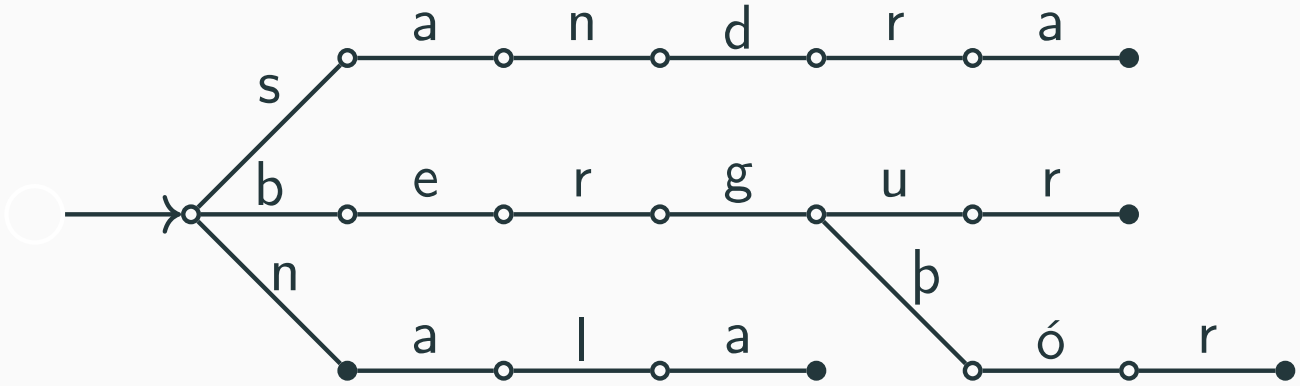

- The strings in the trie are:
  - „sandra”,
  - „nala”,
  - „bergur”,

# Example




- The strings in the trie are:
  - „sandra”,
  - „nala”,
  - „bergur”,
  - „bergþór” and

# Example




- The strings in the trie are:
  - „sandra”,
  - „nala”,
  - „bergur”,
  - „bergþór” and
  - „n”

# Adding strings

- What if we want to add a string to a trie?
- We walk through it as usual, but simply add nodes when we find ourself at a dead end with letters left to walk through.
- This increases the size of the tree by at most the size of the string.

# Example

# Example

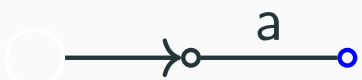
„api”

# Example

„api”

# Example

„pi”



# Example

# Example

# Example

„api“

# Example

„apar”

# Example

„apar”

# Example

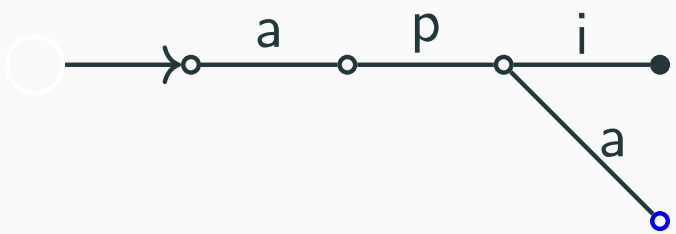
„par”

# Example

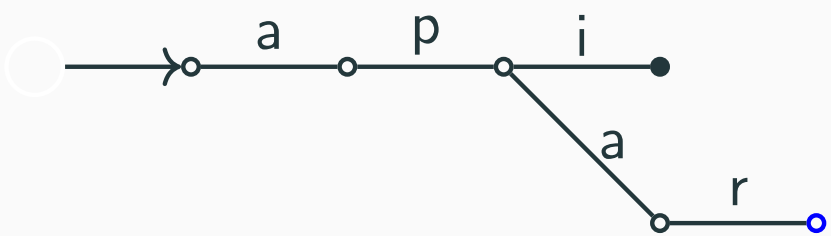
„ar“

# Example

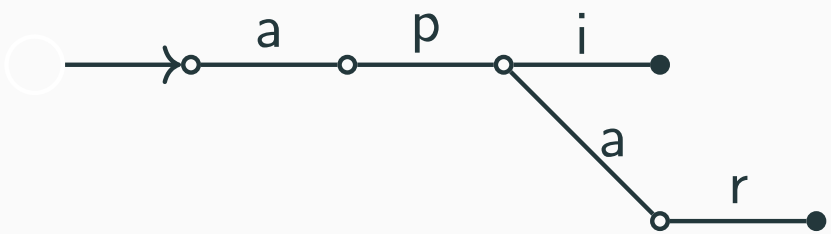
„r”



# Example

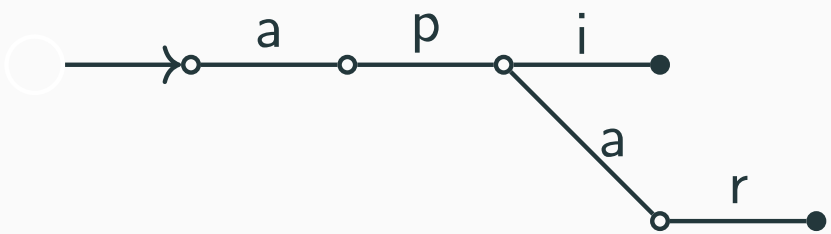


# Example



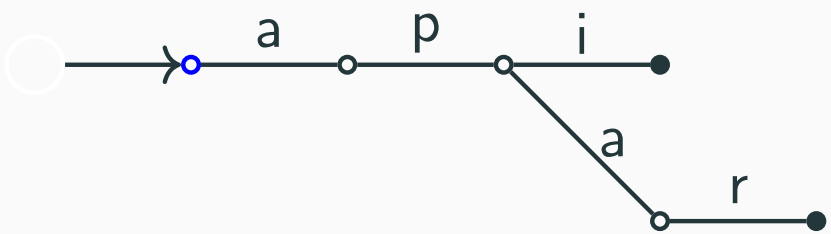
# Example

„apaköttur”



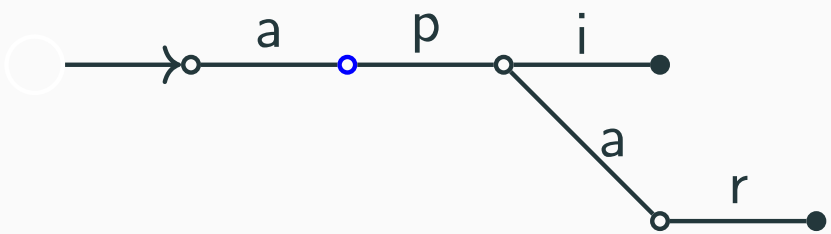
# Example

„apaköttur”



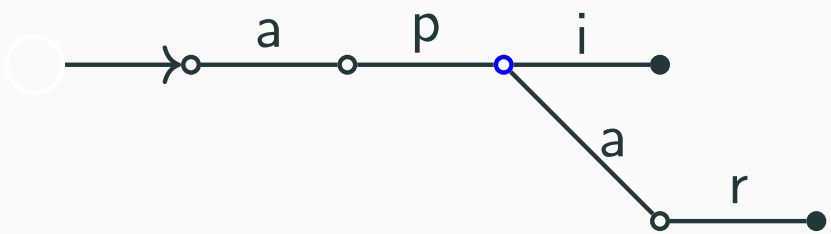
# Example

„paköttur”



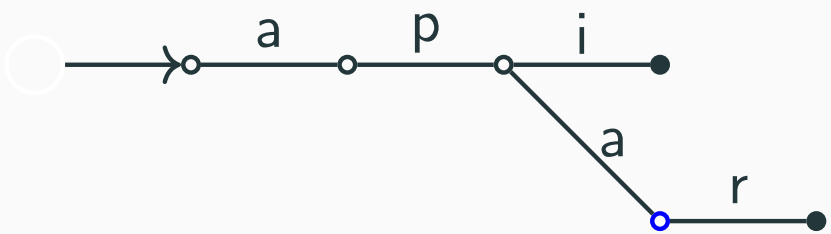
# Example

„aköttur”



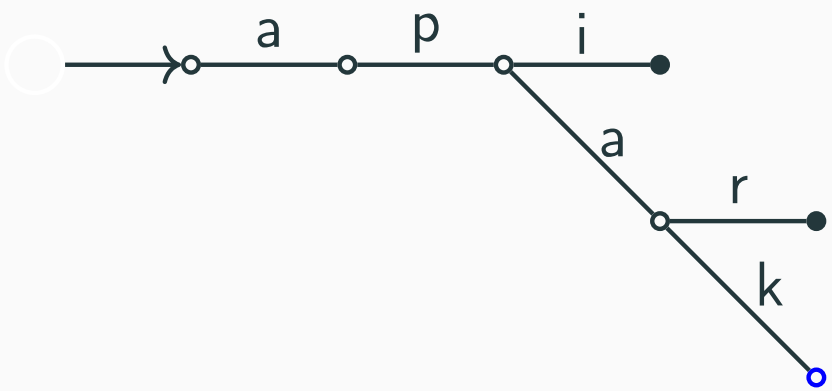
# Example

„köttur”



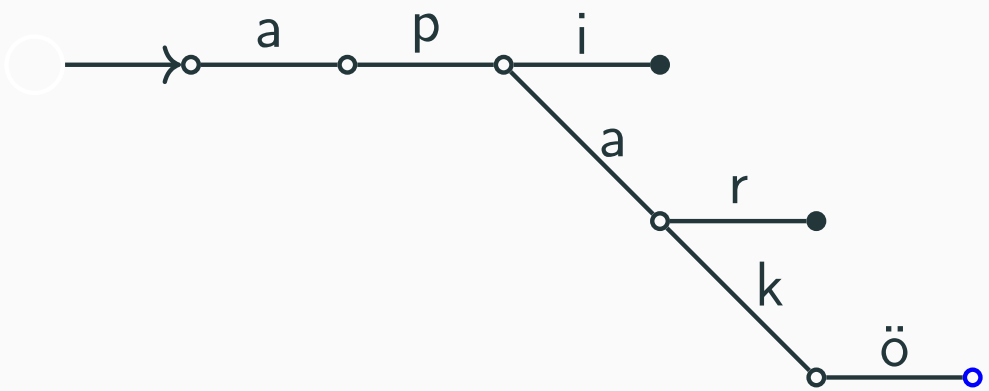
# Example

„öttur“



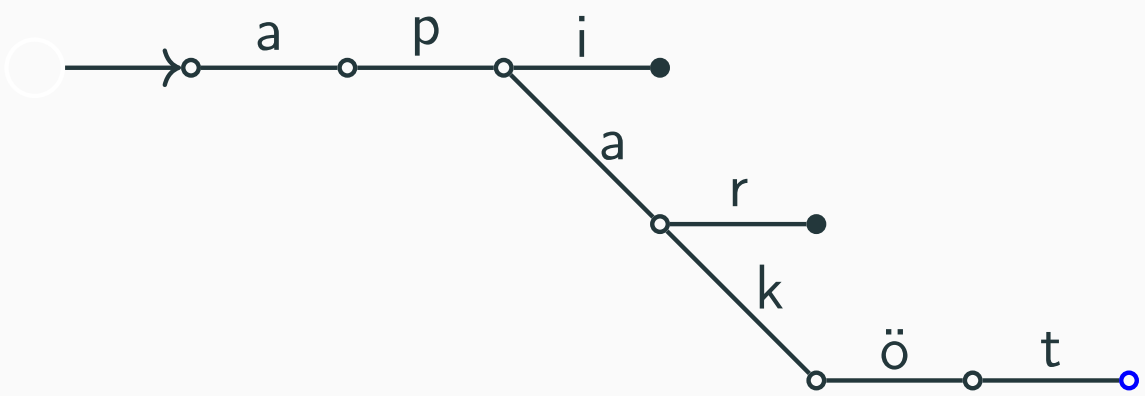
# Example

„ttur”



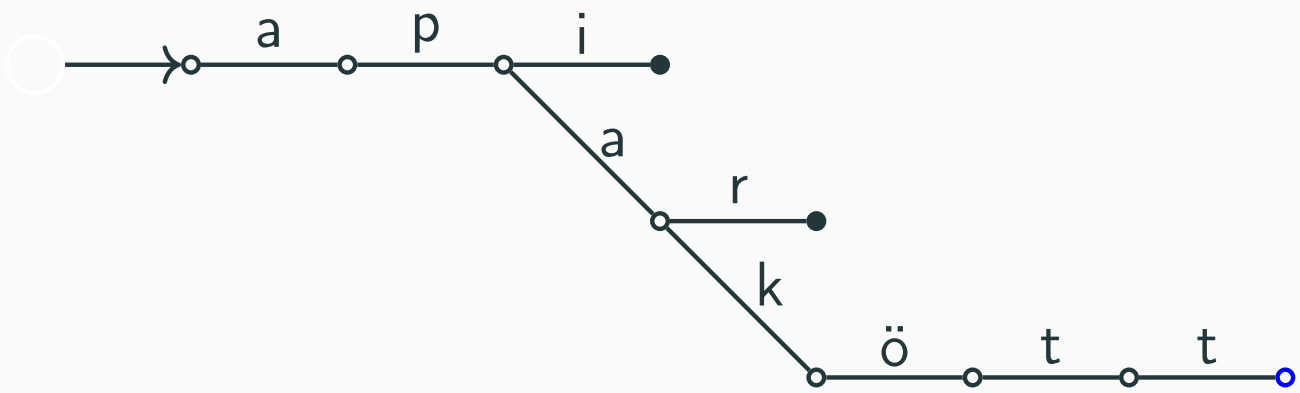
# Example

„tur”



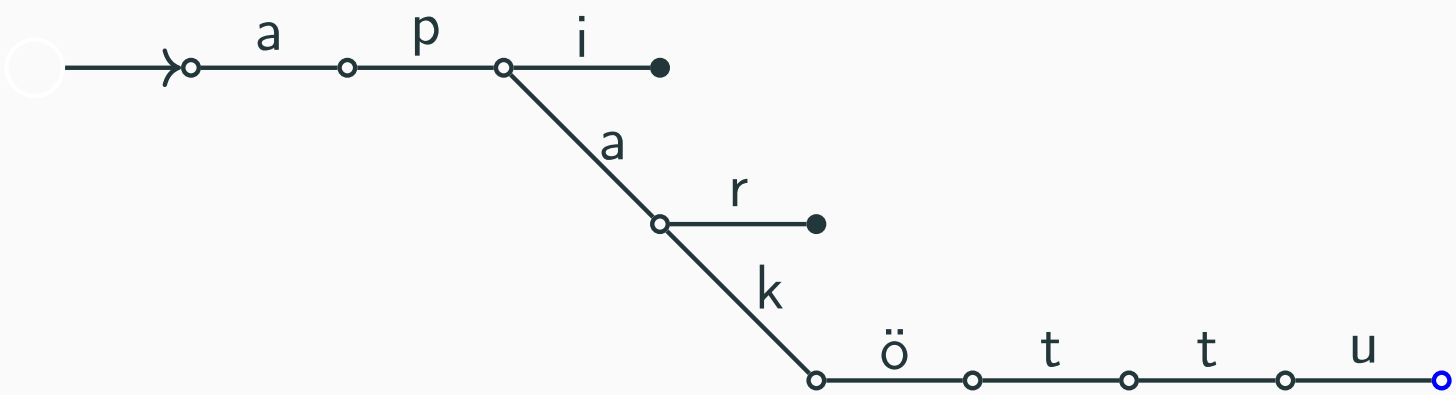
# Example

„ur“

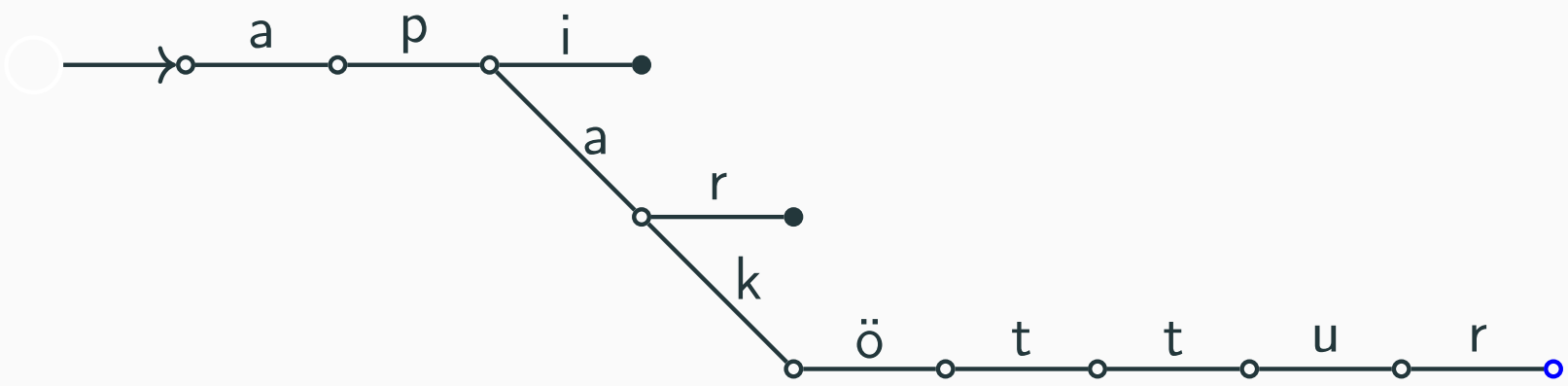


# Example

„r”

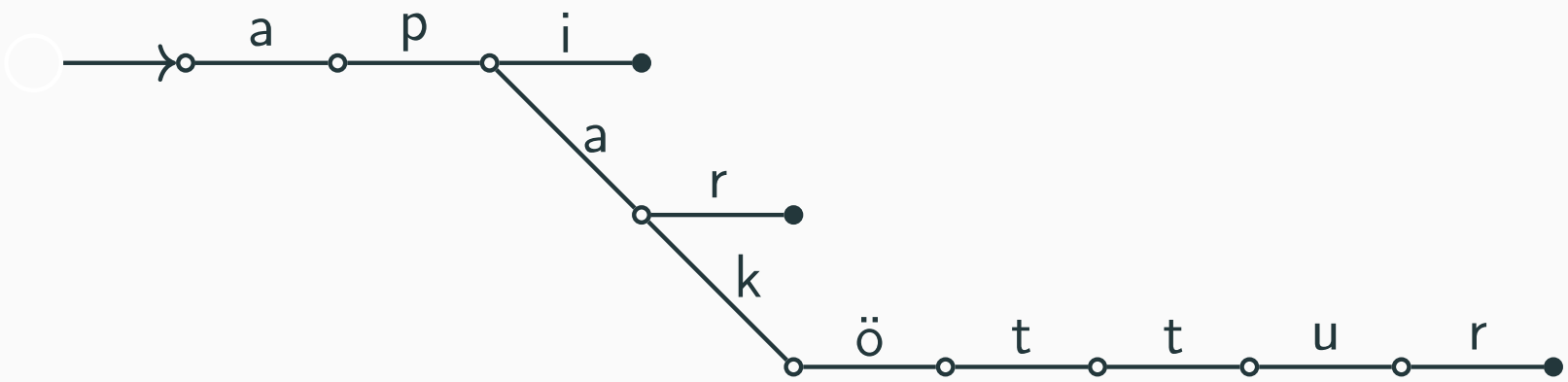


# Example



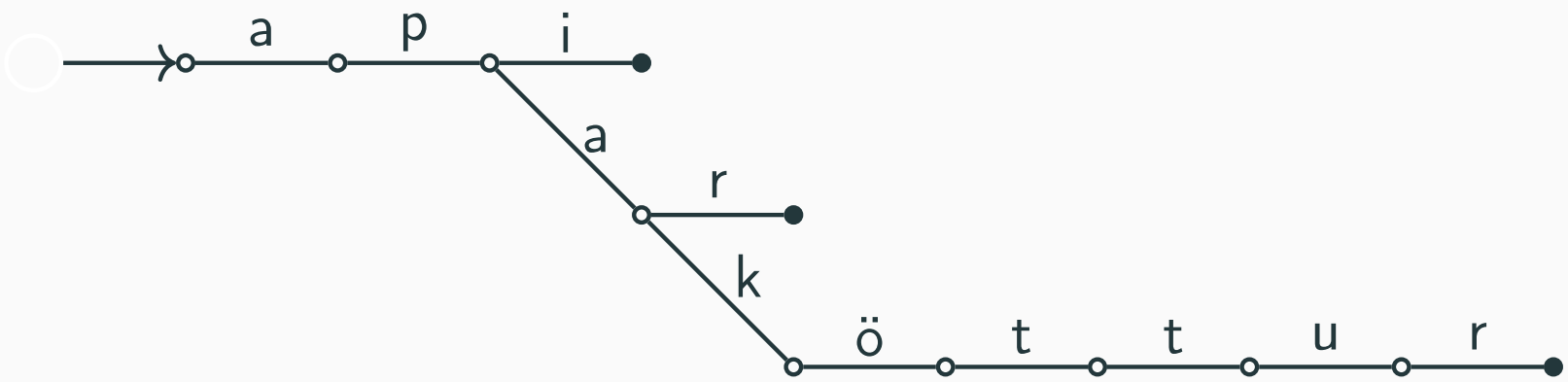
# Example

„api“



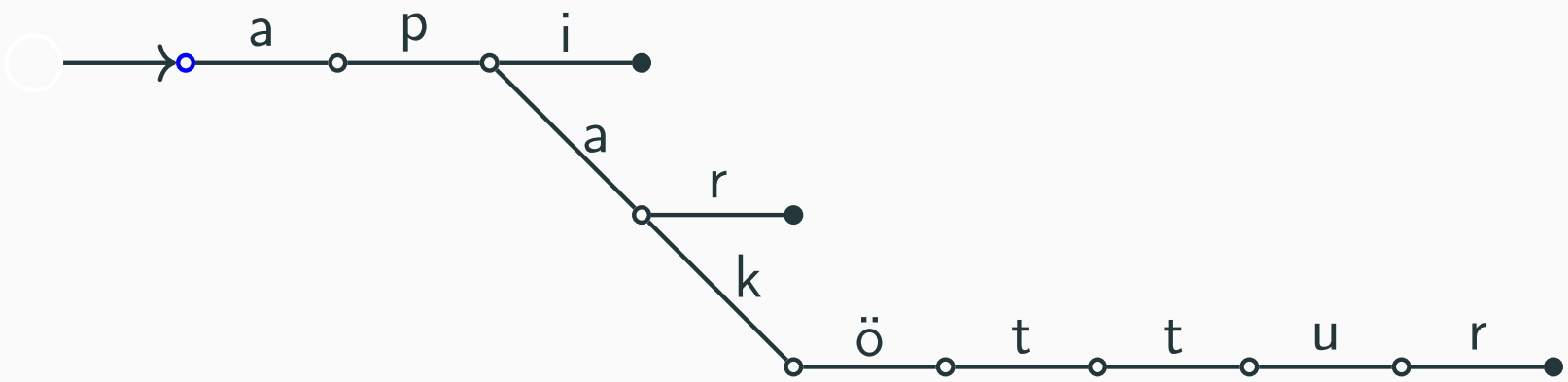
# Example

„altari”



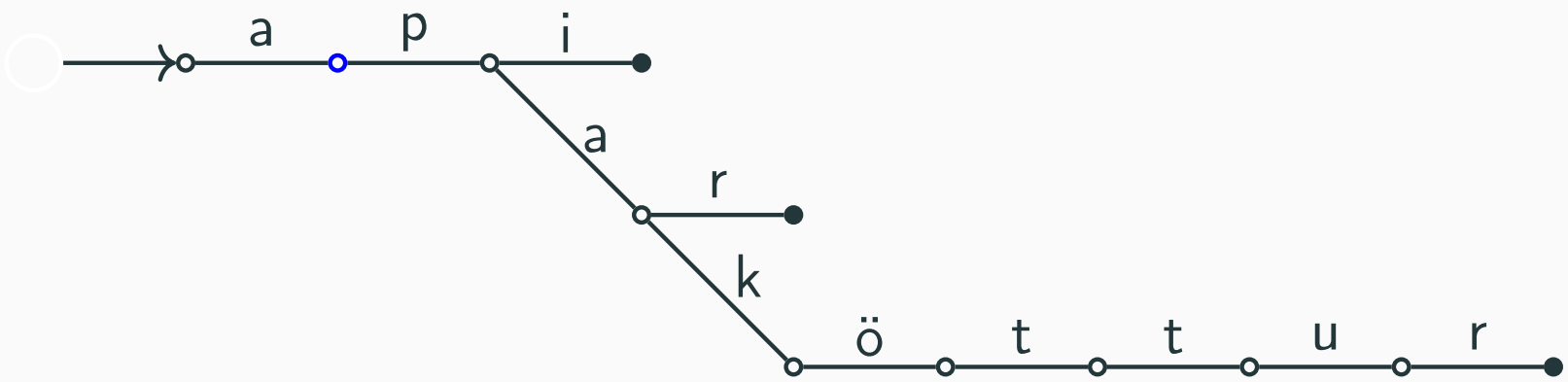
# Example

„altari”



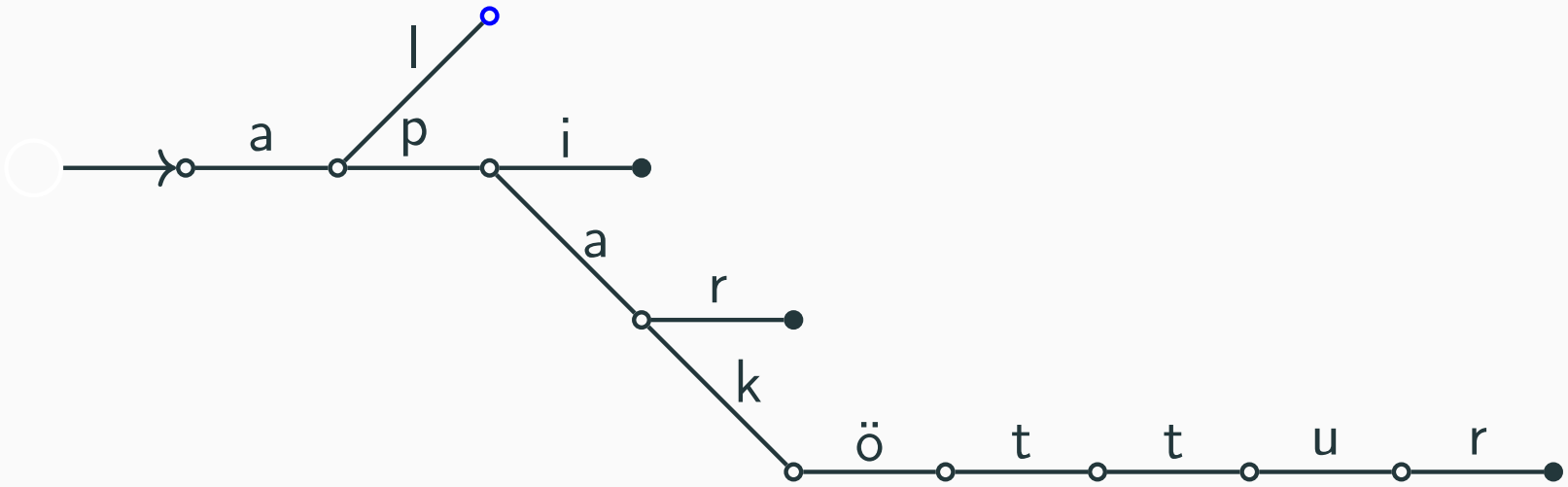
# Example

„ltari”



# Example

„tari”



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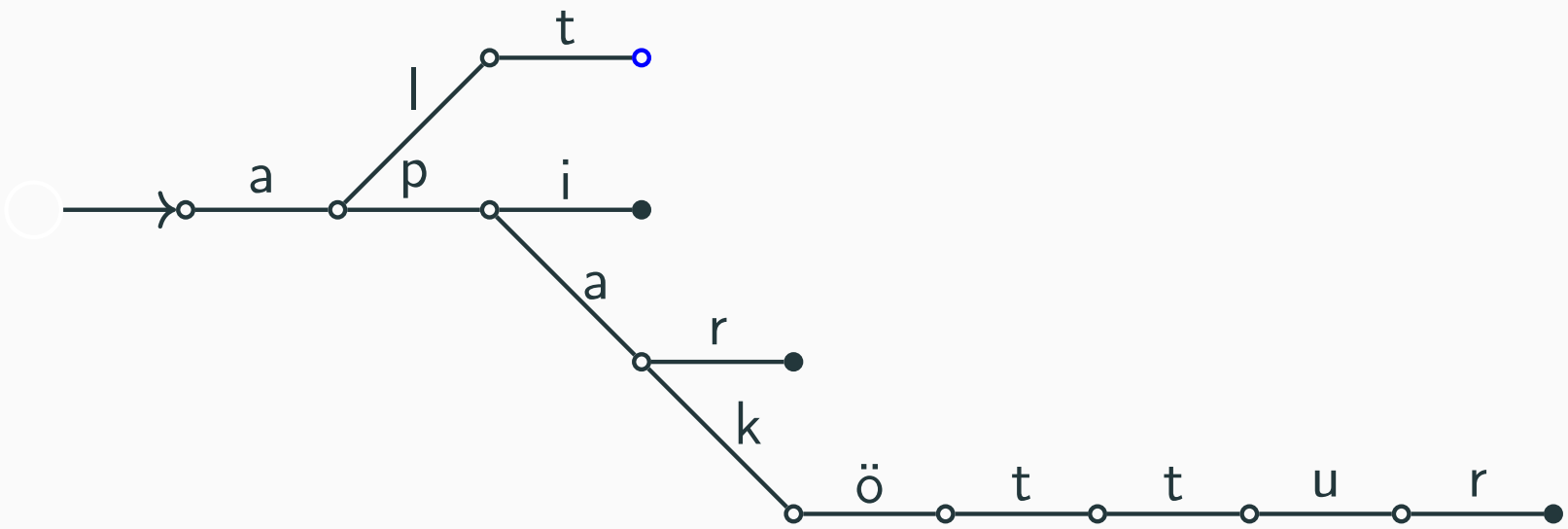

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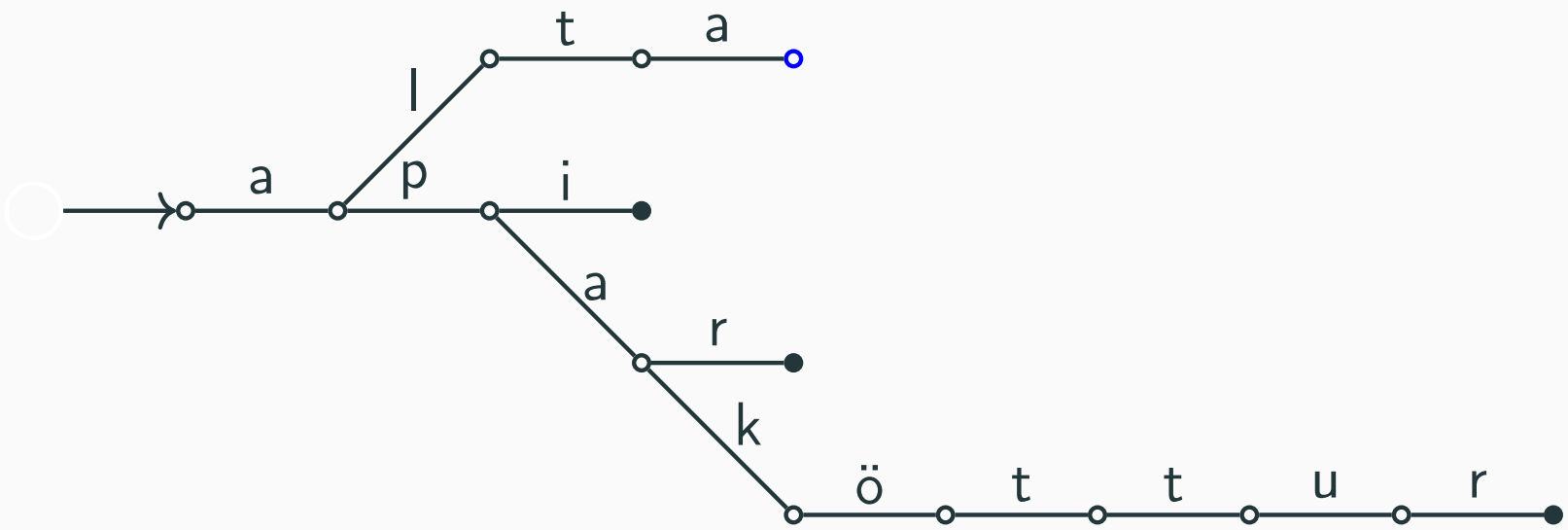

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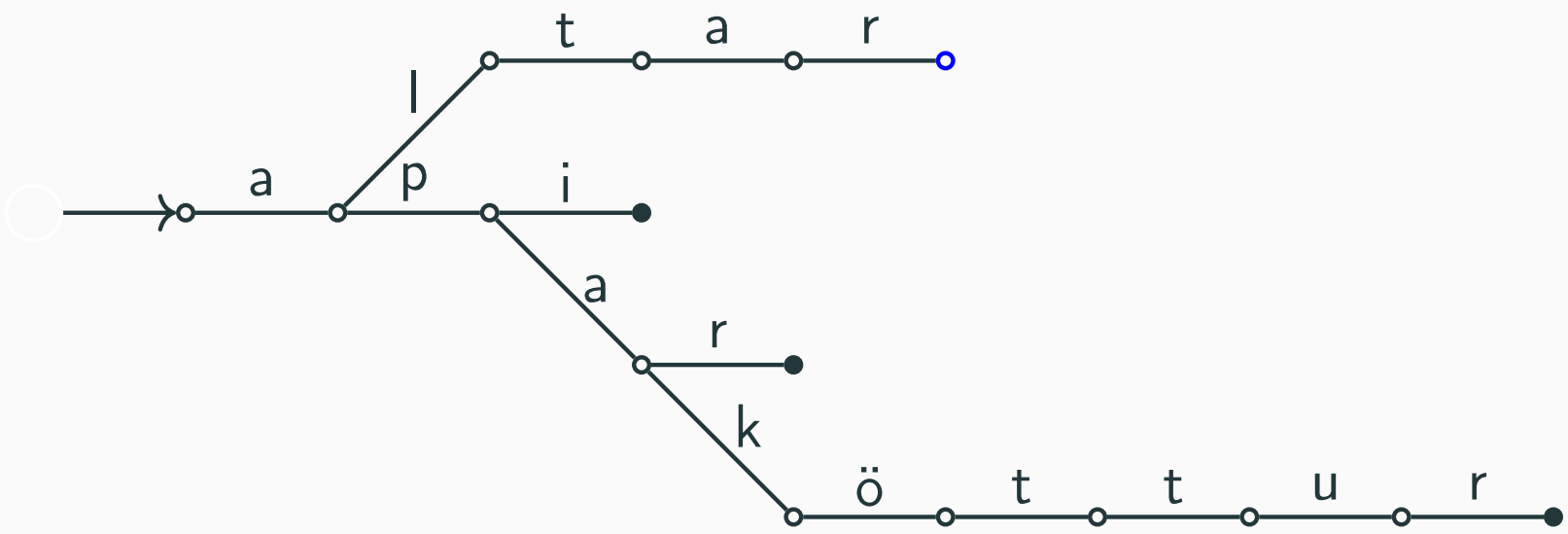

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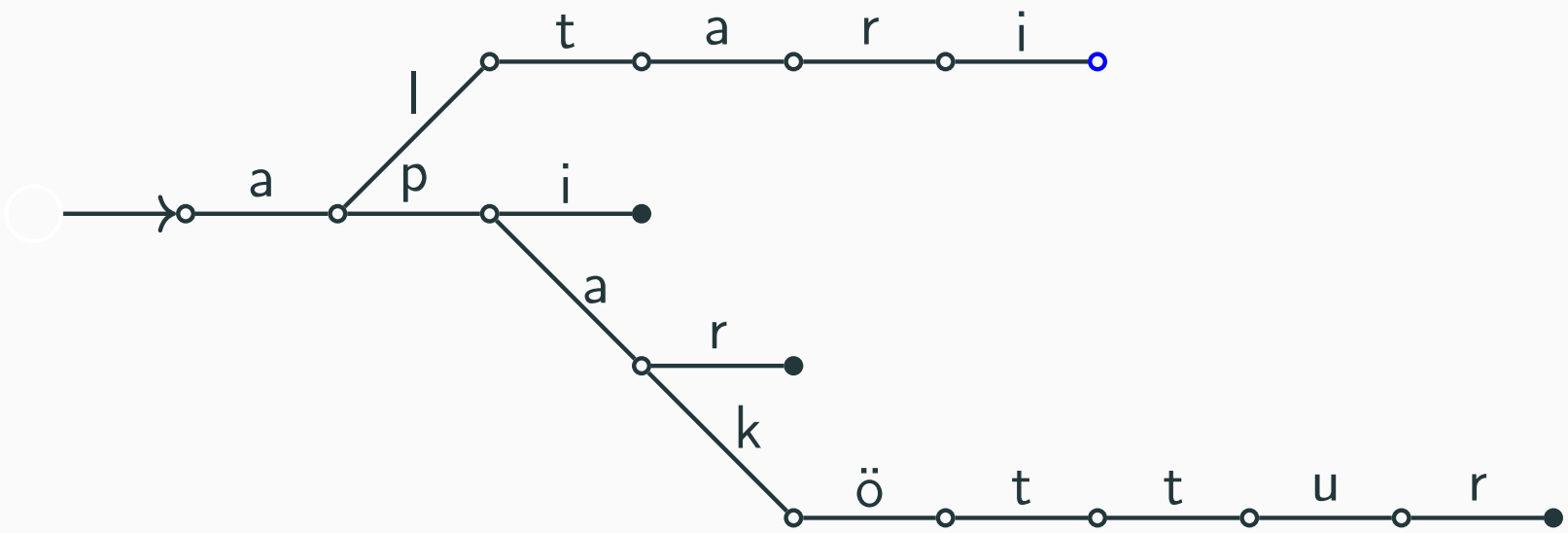

# Example

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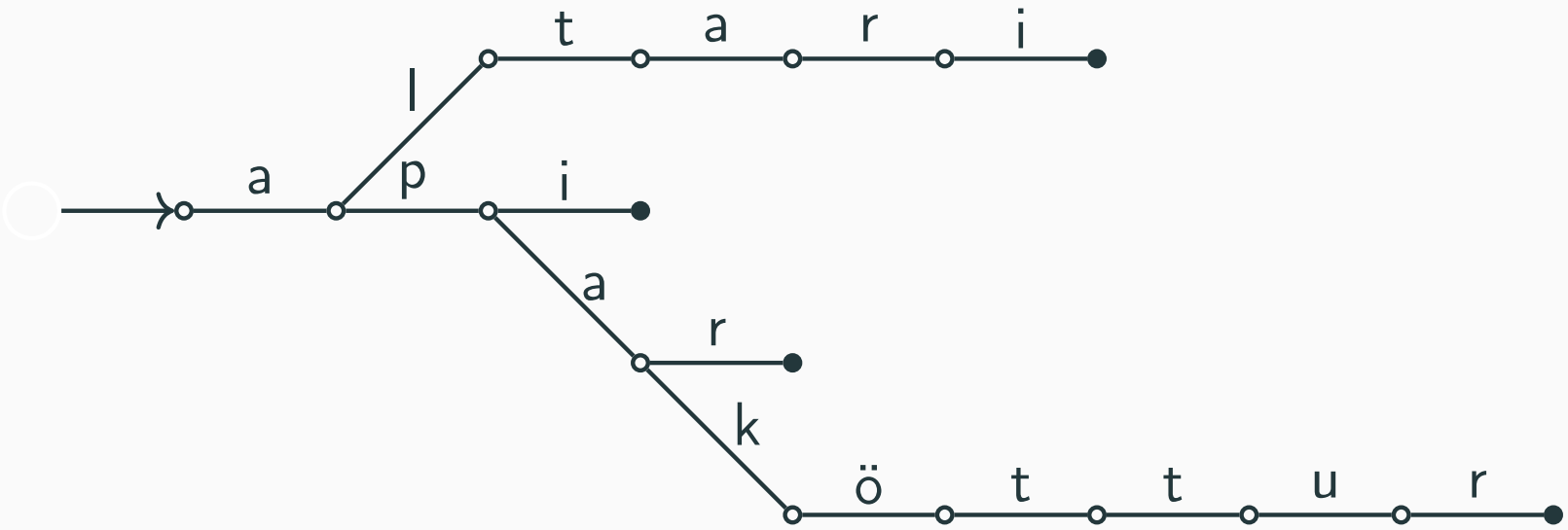

# Example

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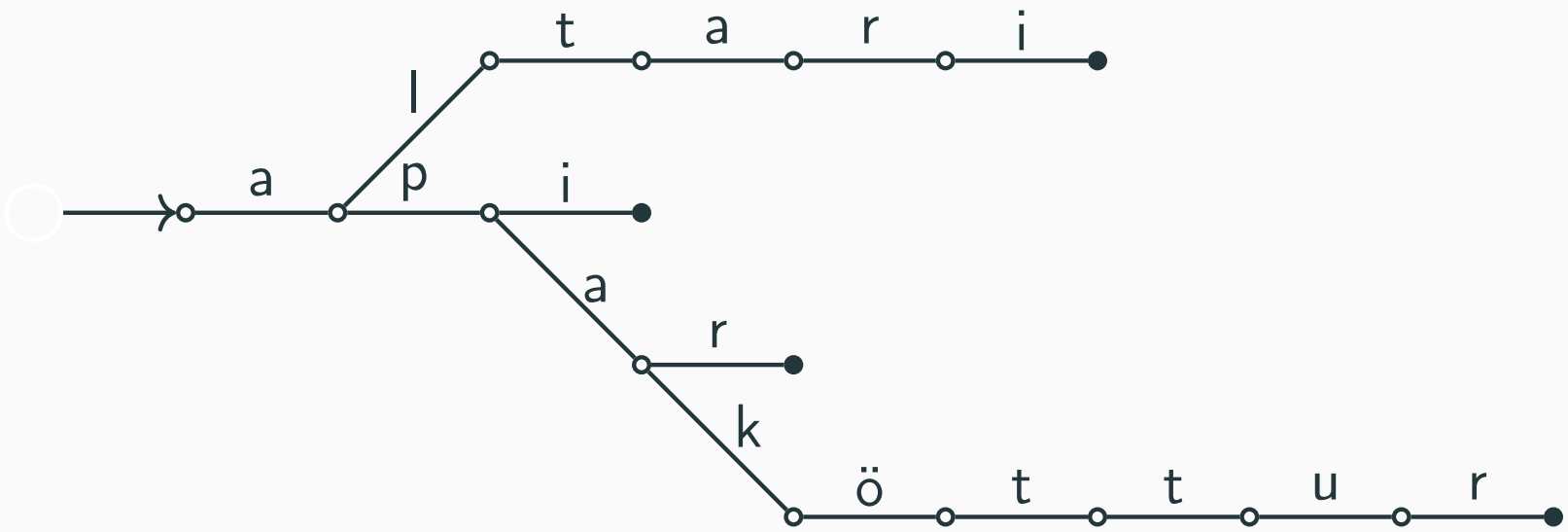

# Example

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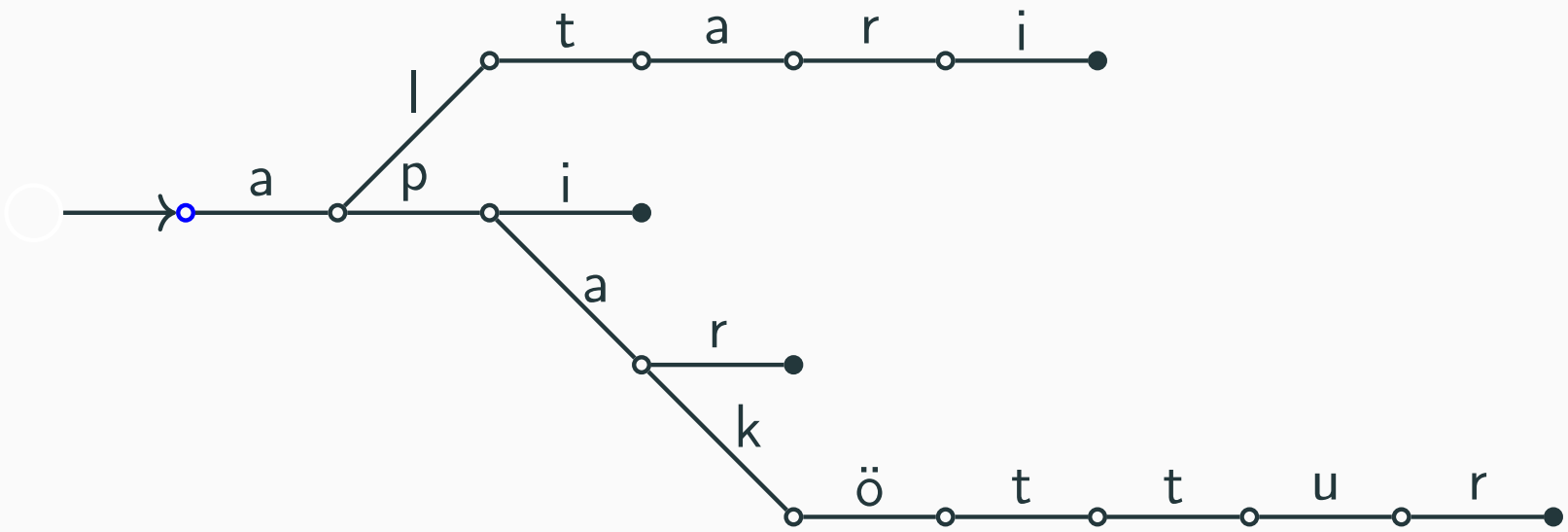

# Example

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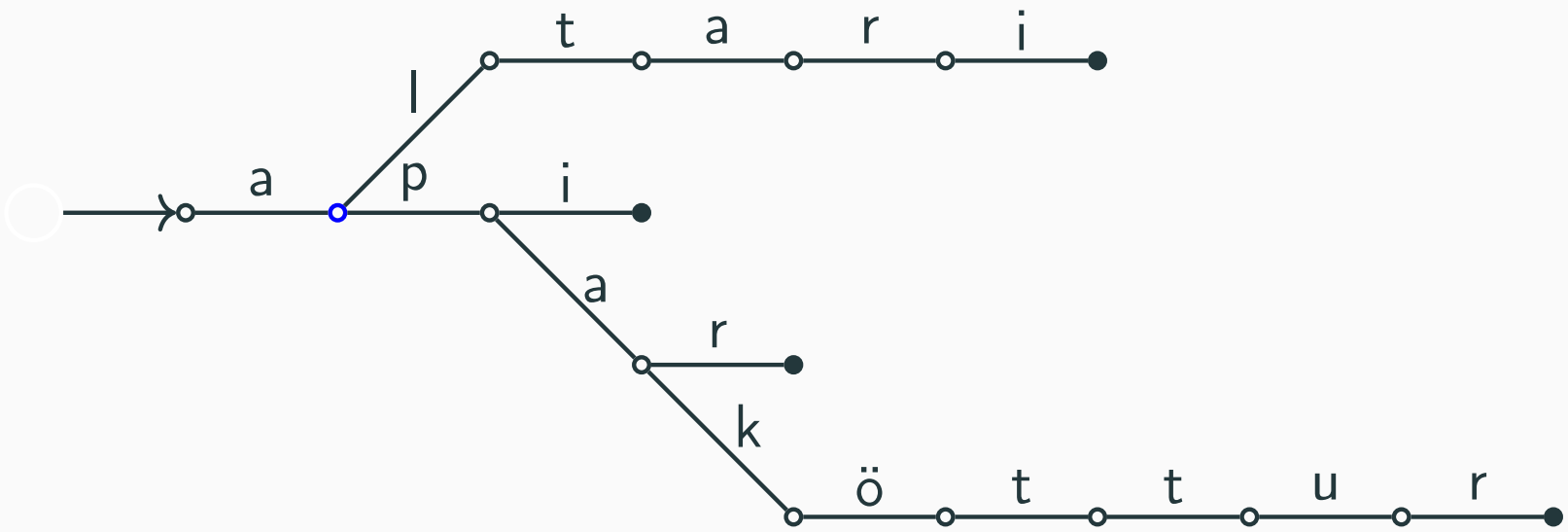
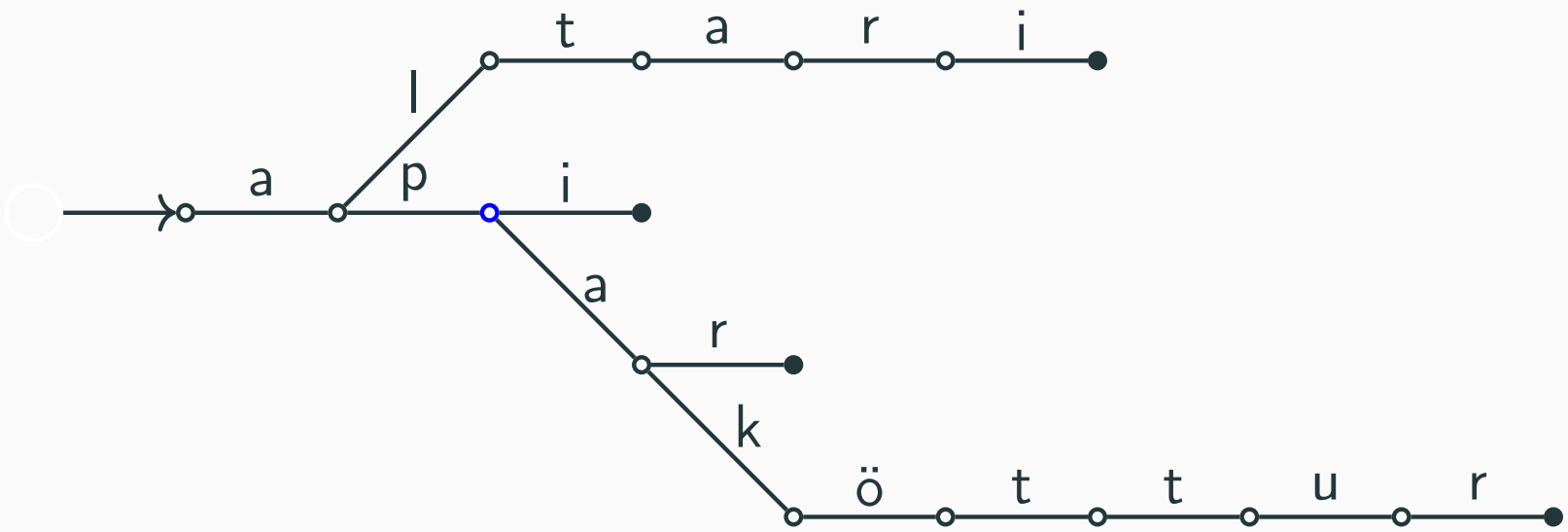


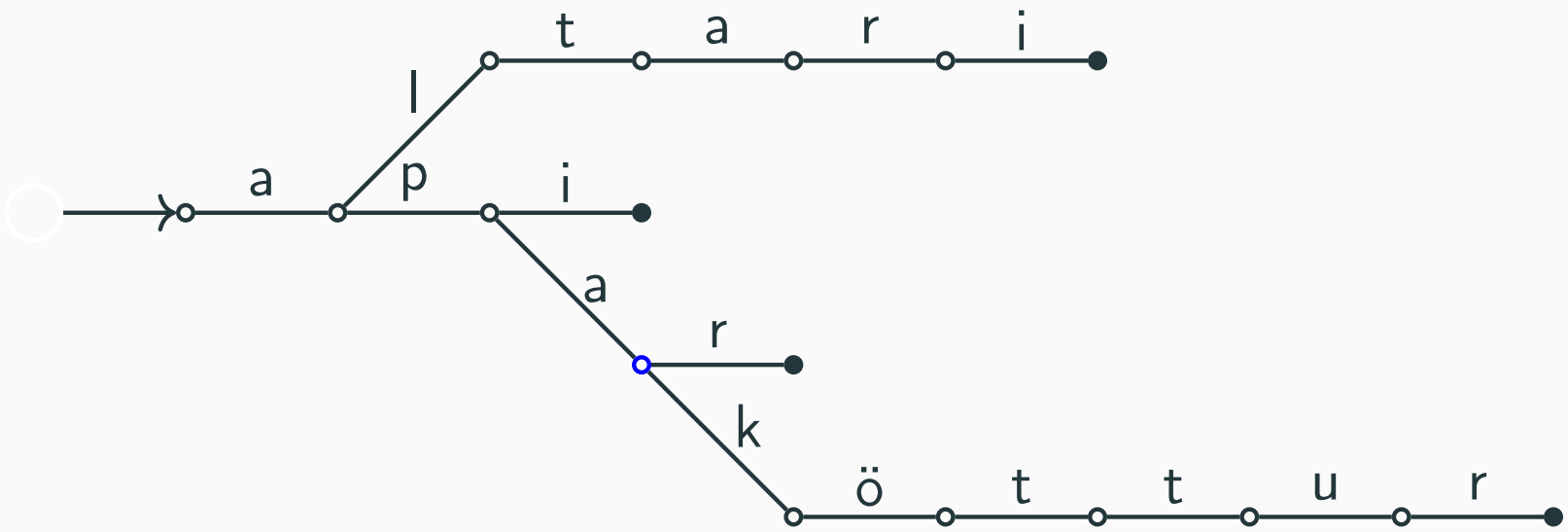
# Example

„aspil”



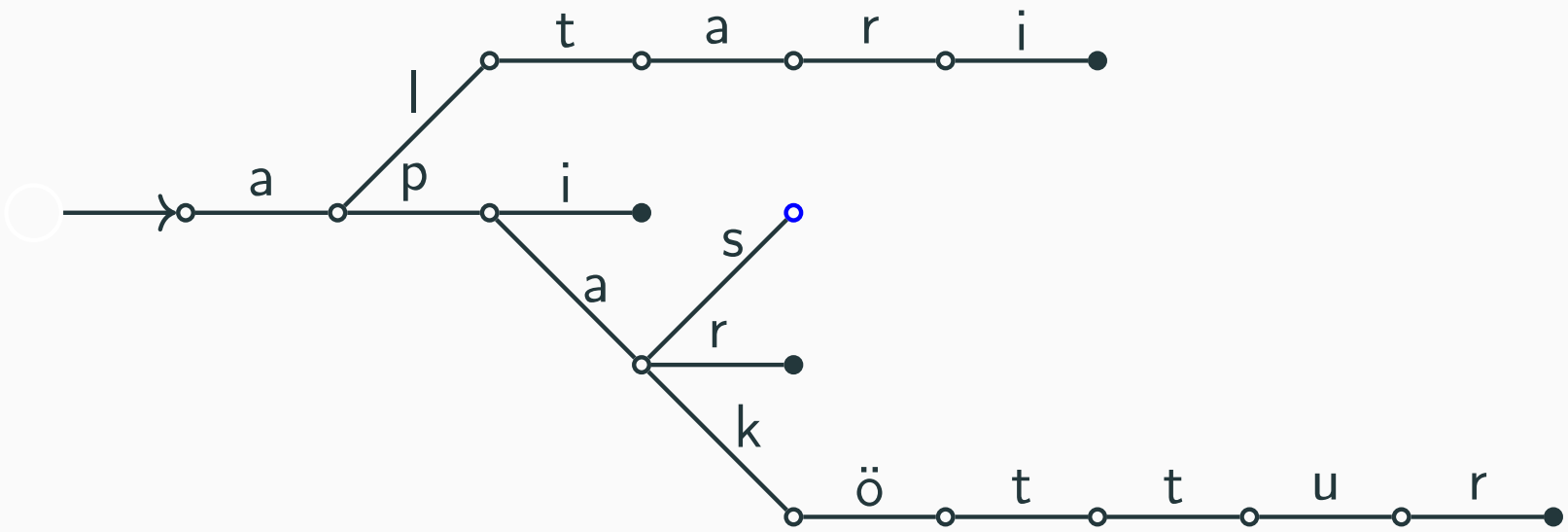
# Example

„spil”



# Example

„pil”



# Example

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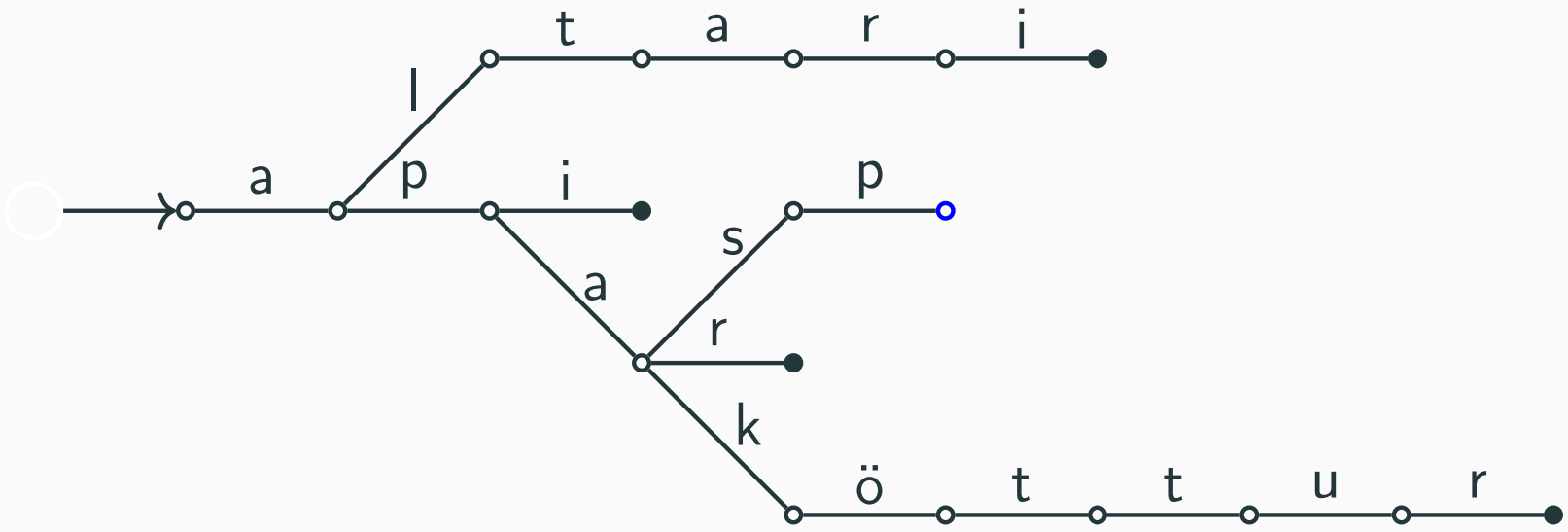

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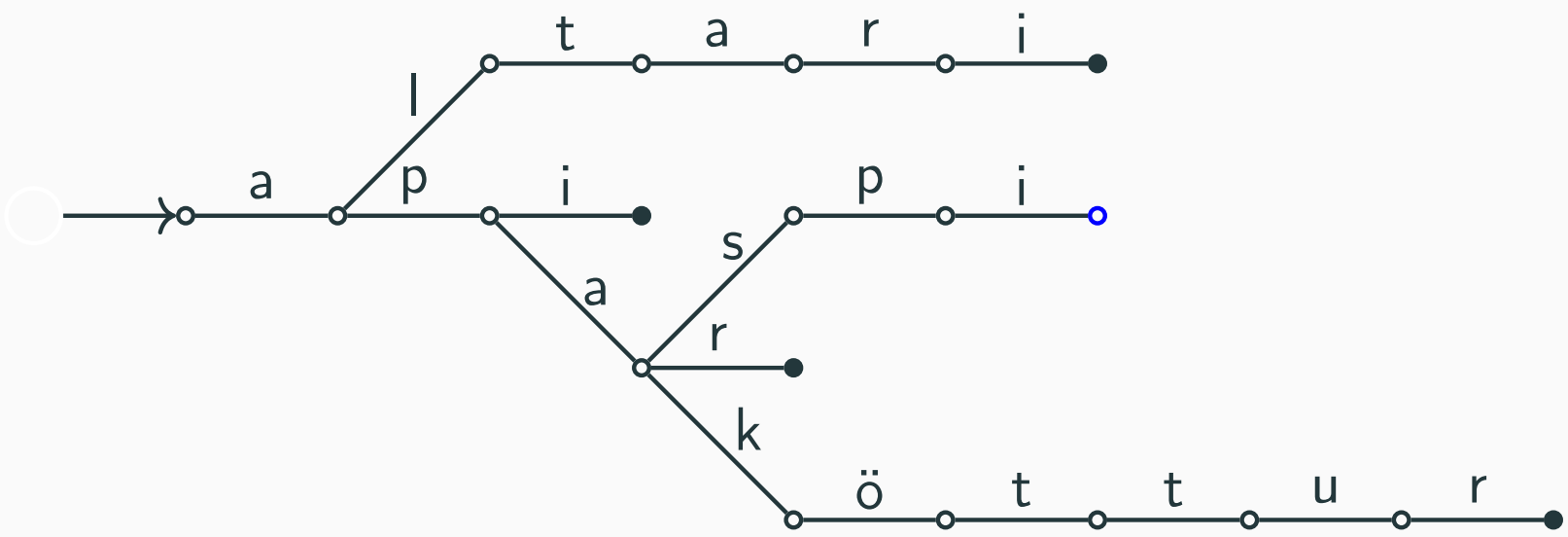
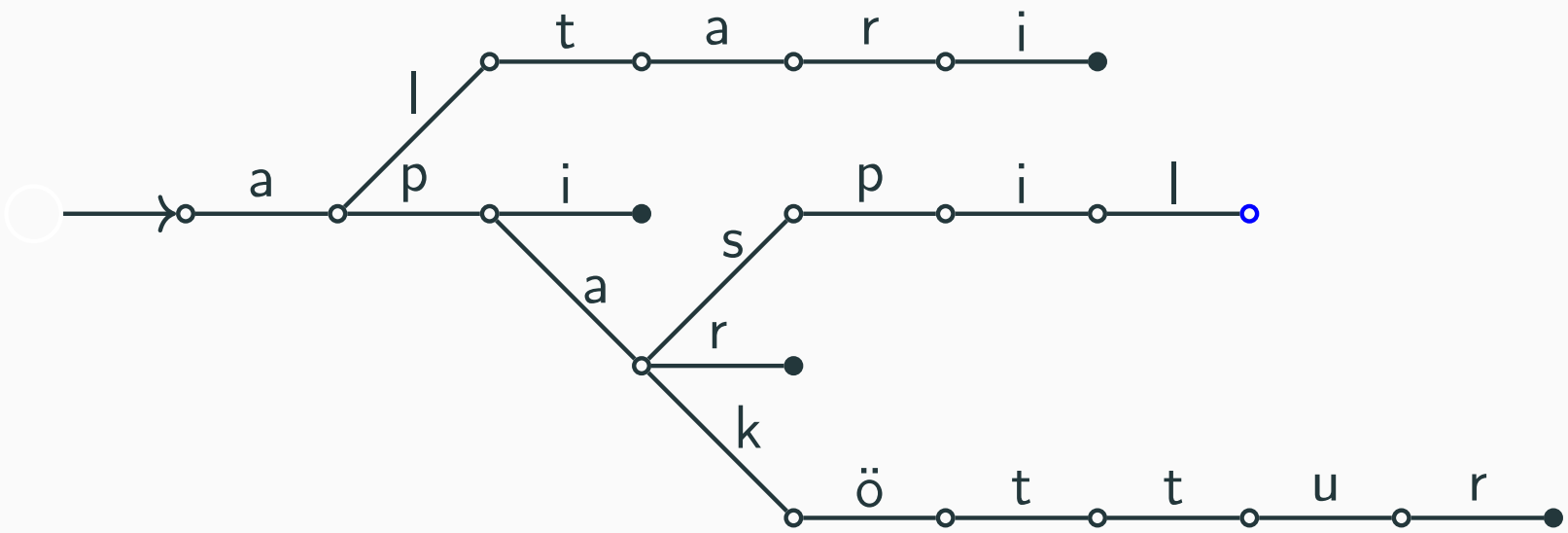
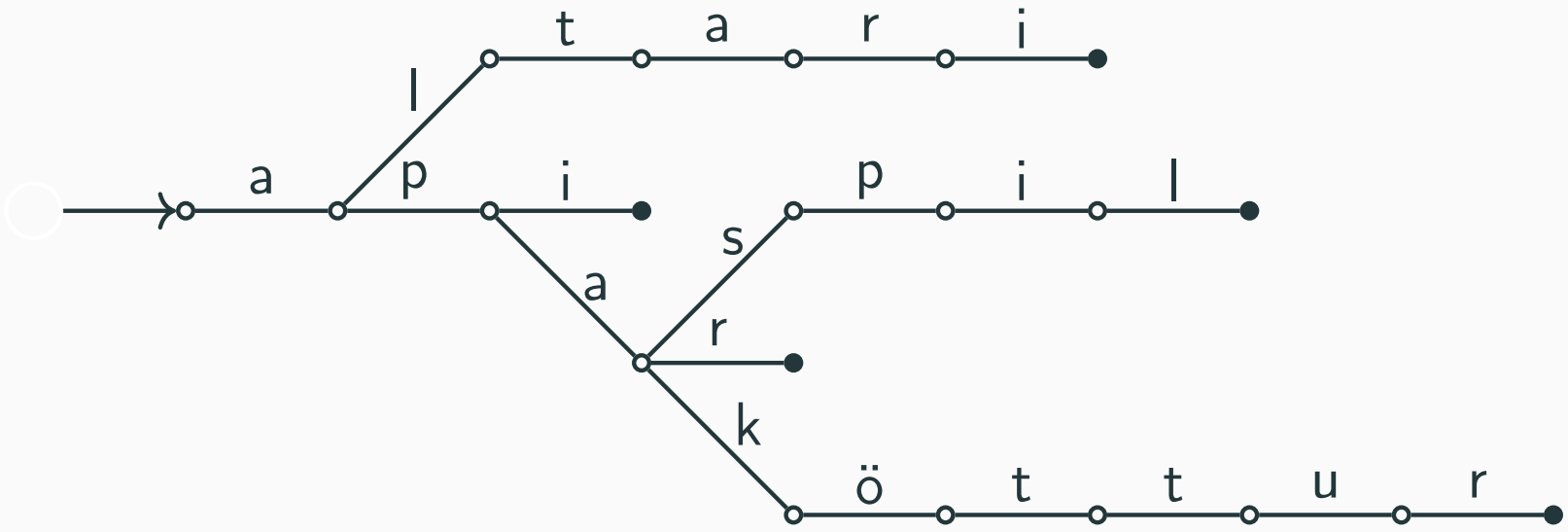


# Example



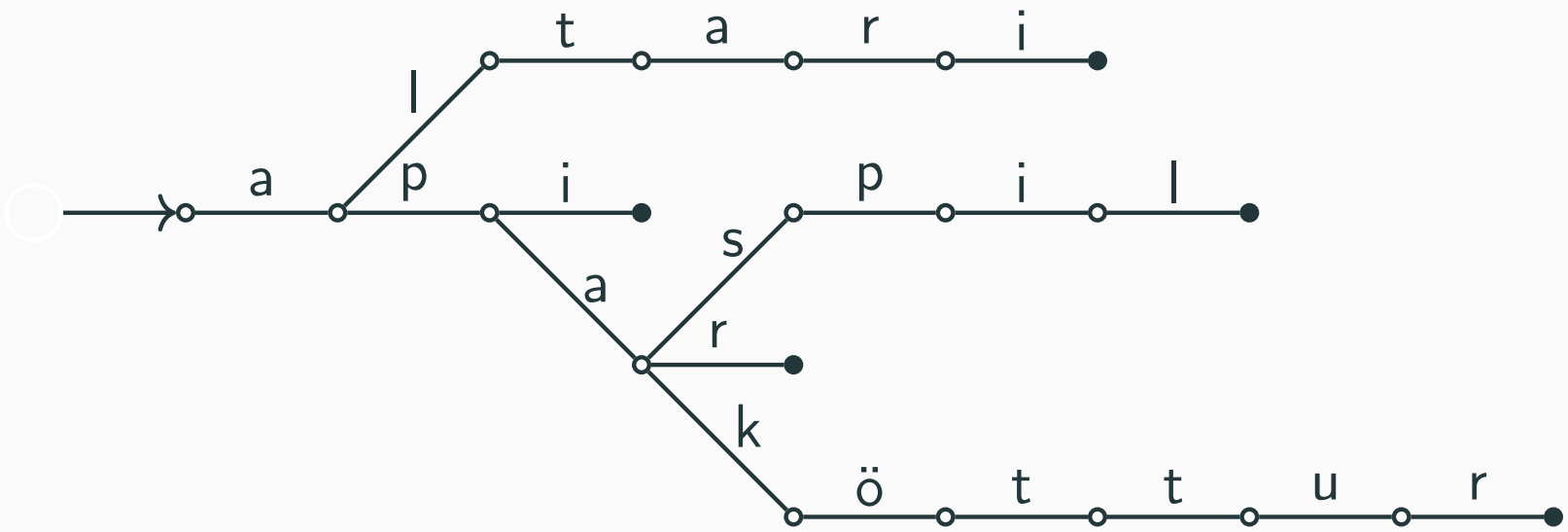
# Example

„api“



# Example

„altaristafla”



# Example

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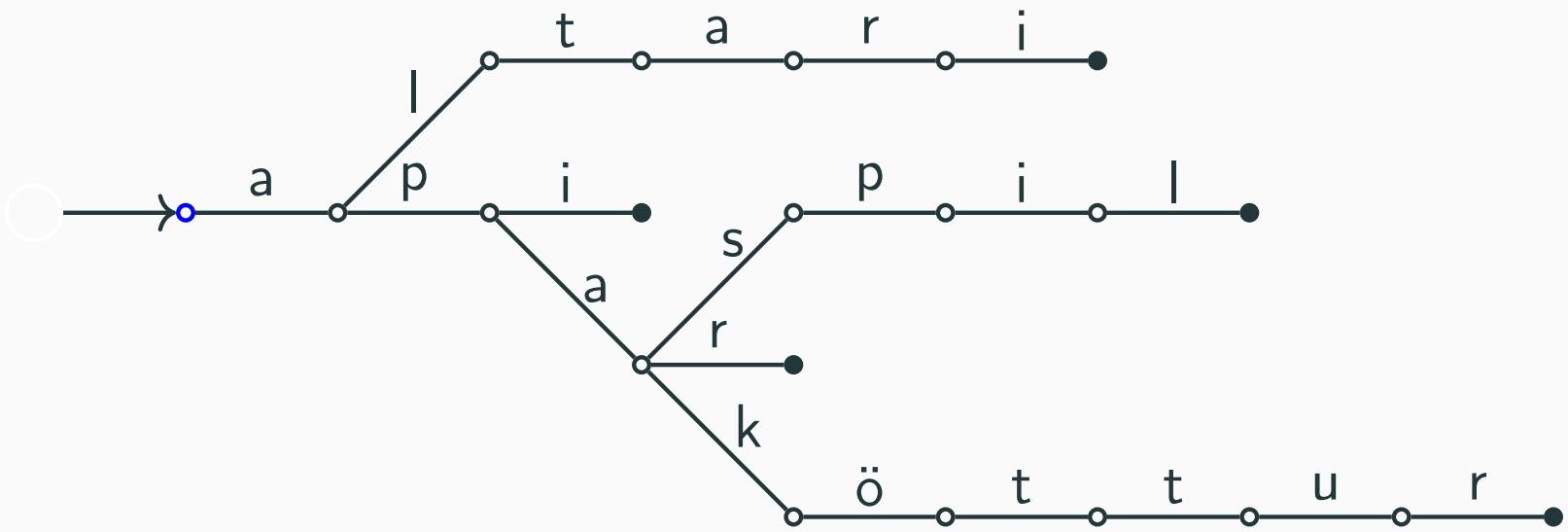
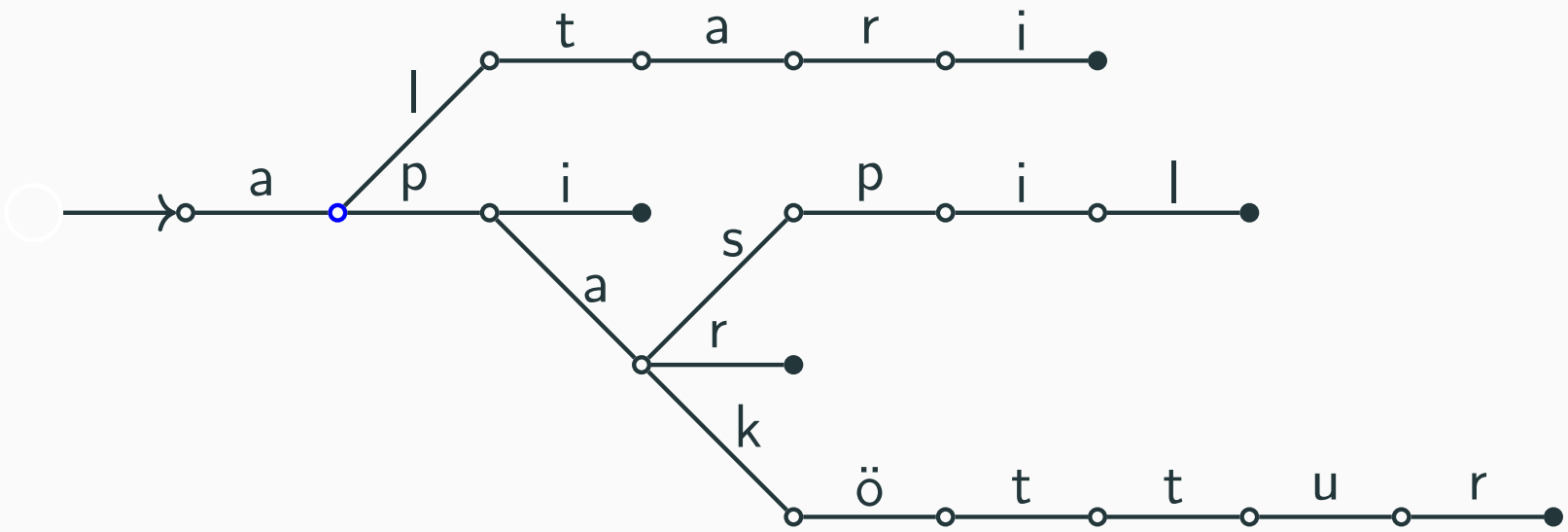


# Example

„ltaristafla”



# Example

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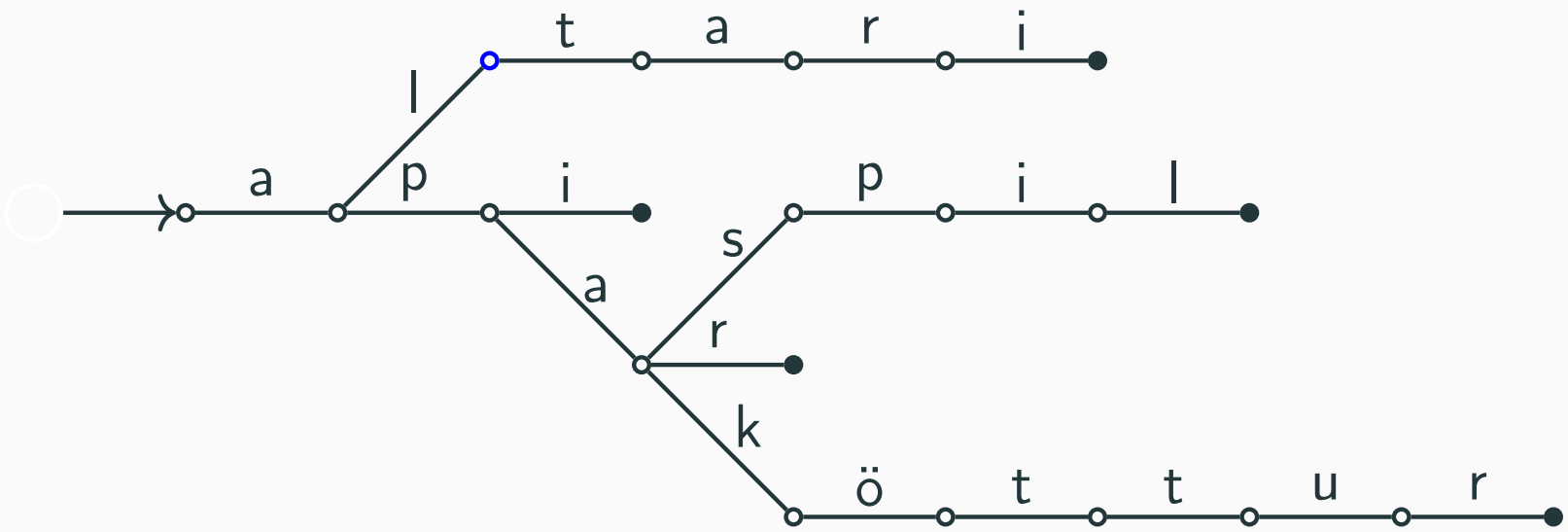
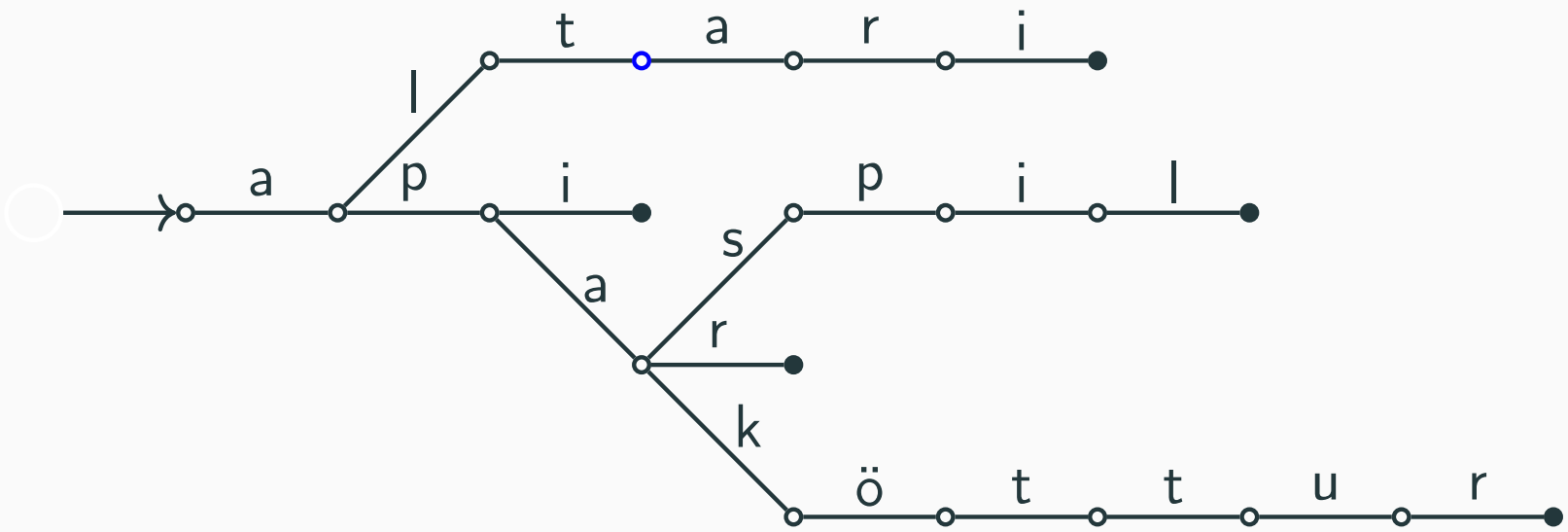


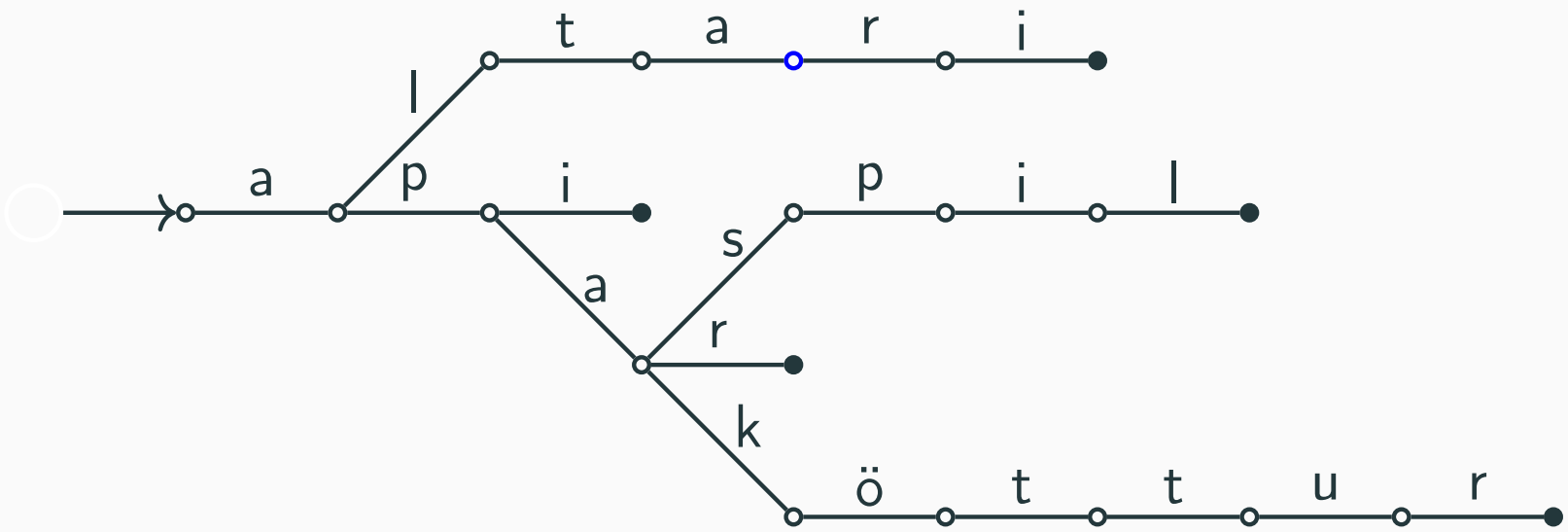
# Example

„aristafla”



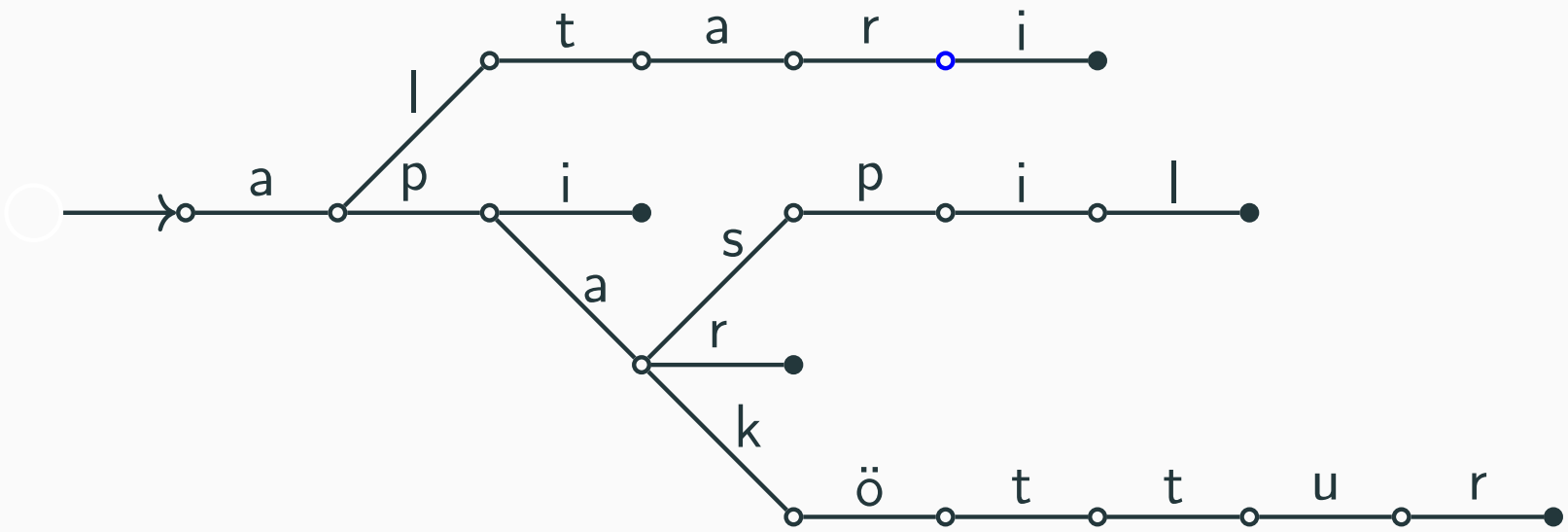
# Example

„ristafla”



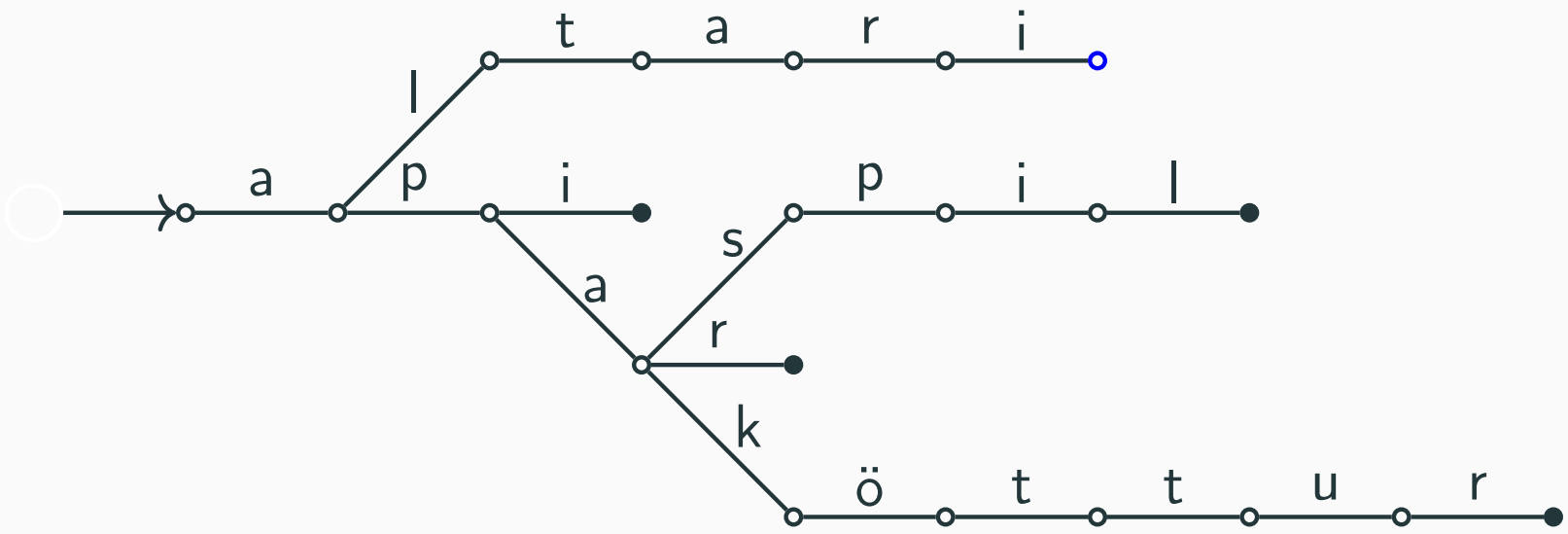
# Example

„istafla”



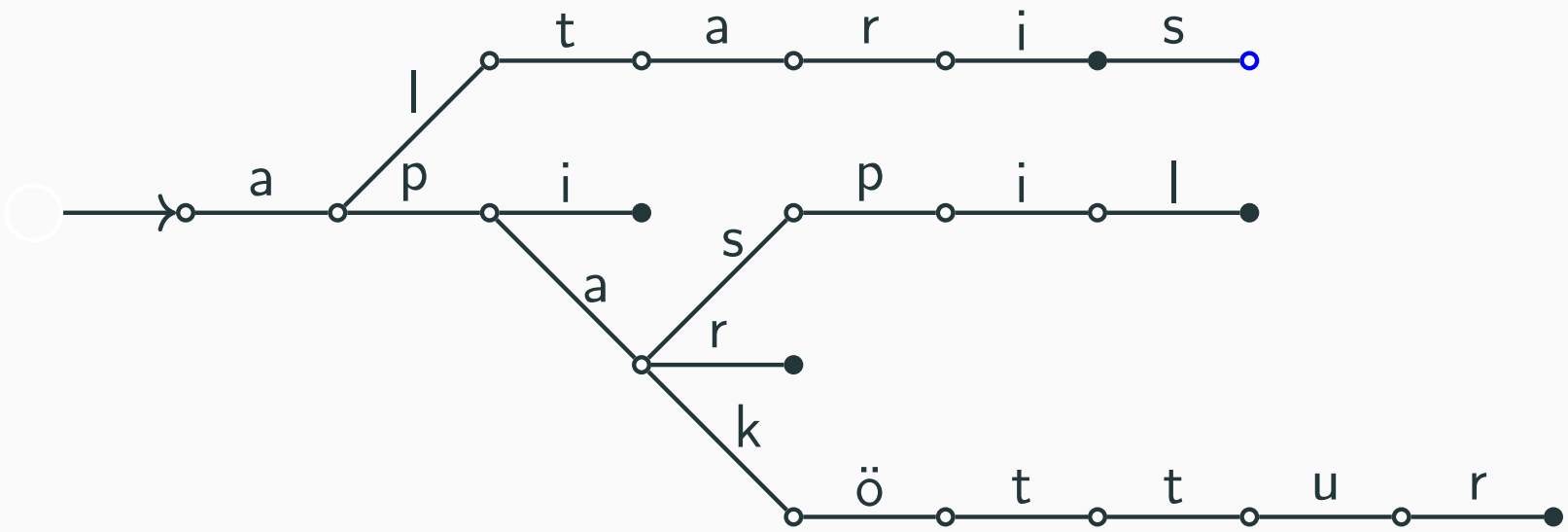
# Example

„stafla”



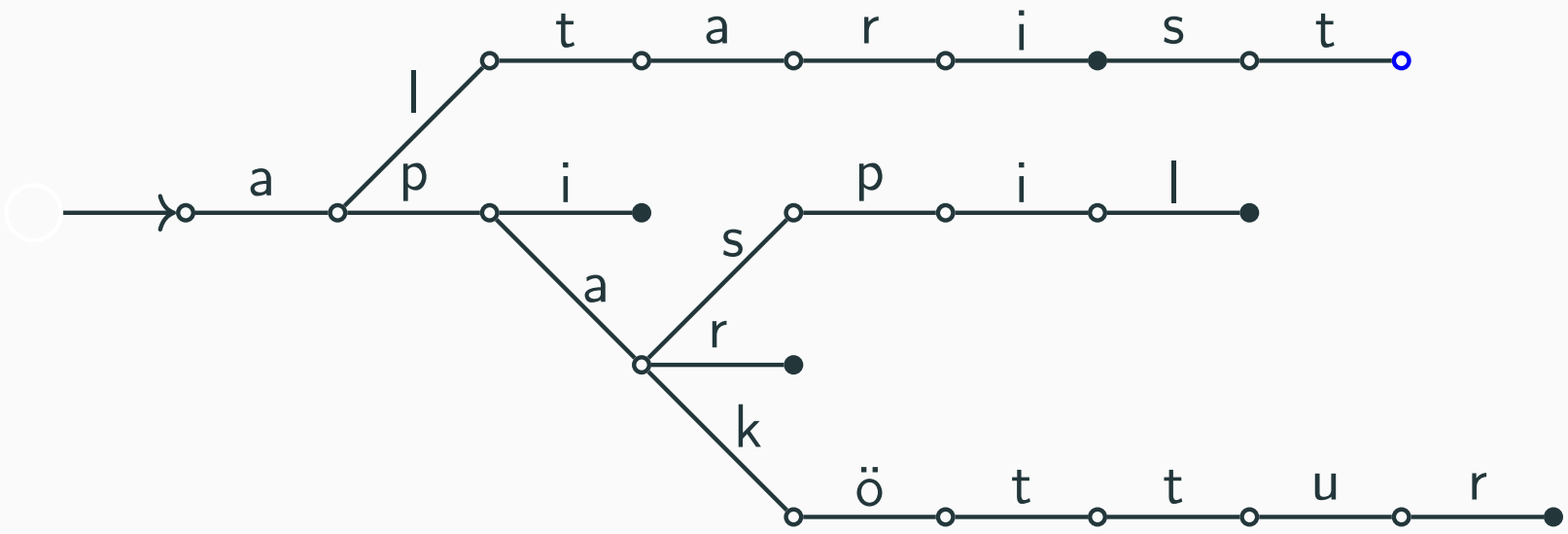
# Example

„tafla”



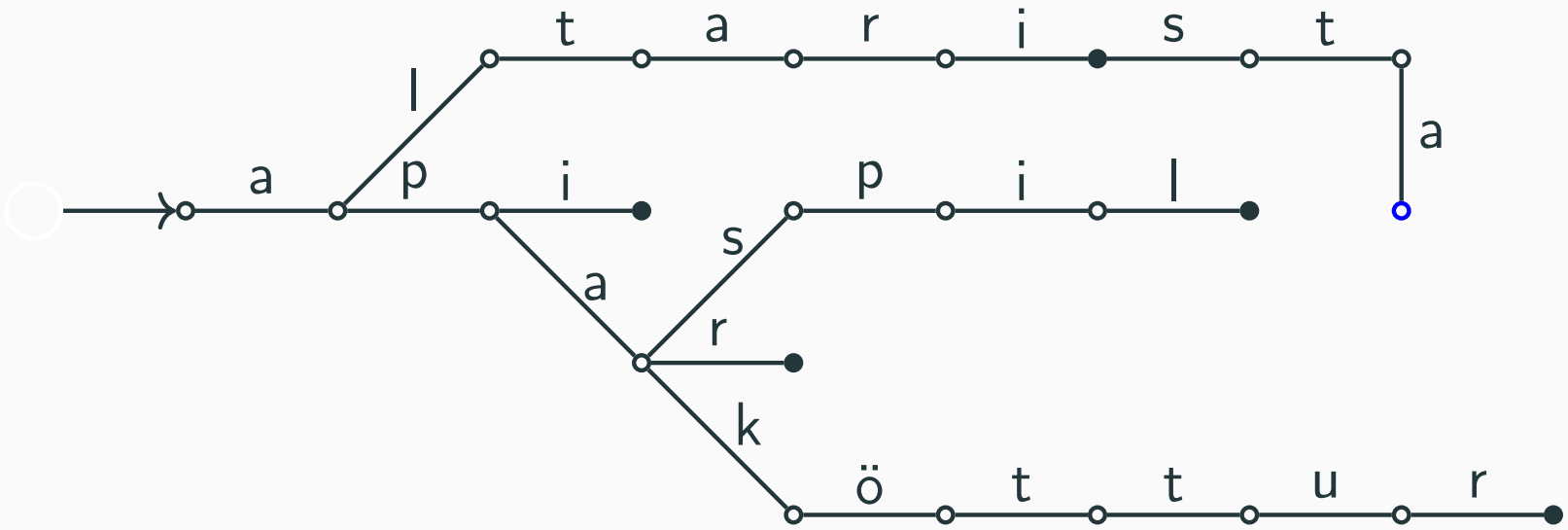
# Example

„afla”



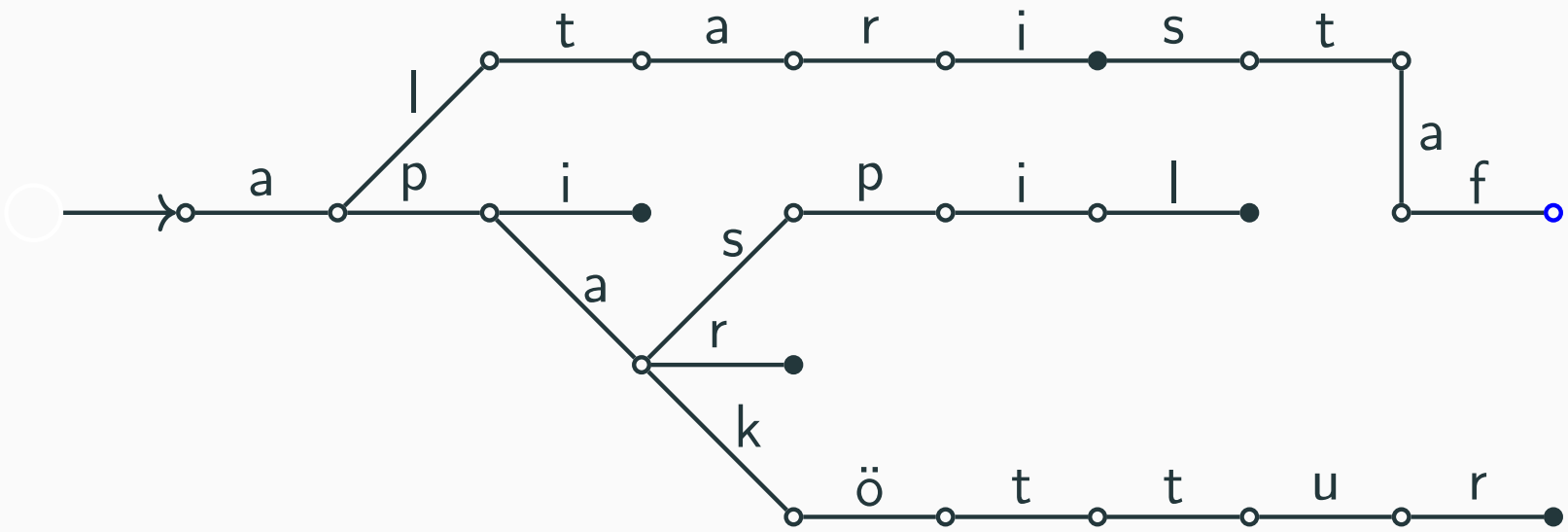
# Example

„fla”



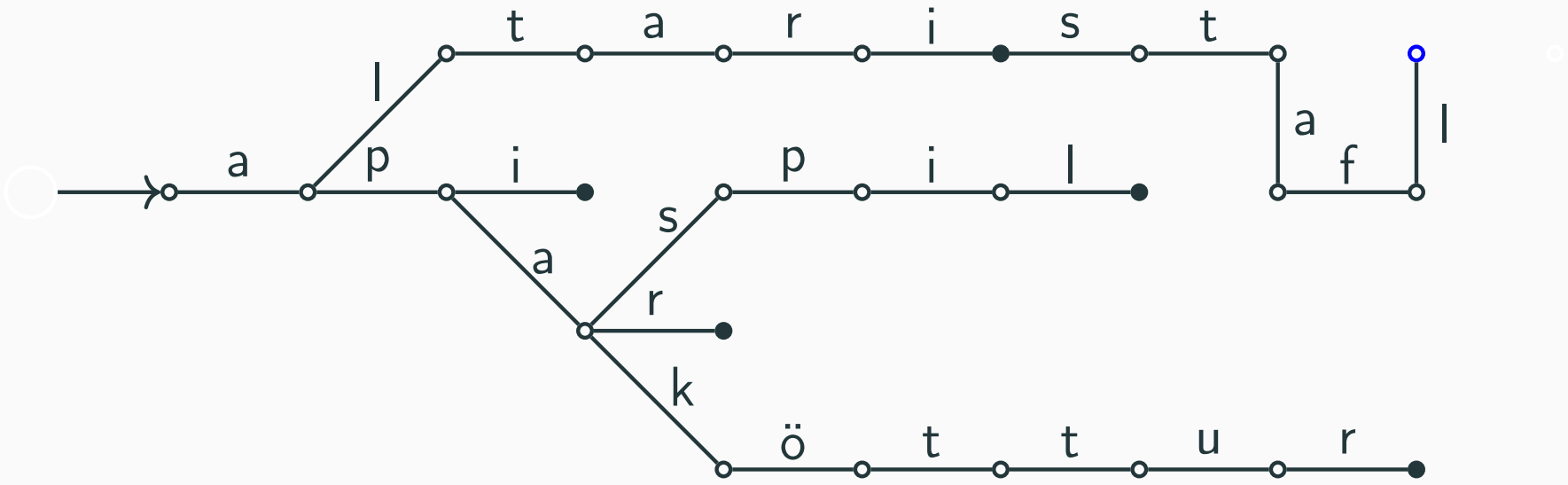
# Example

„la”

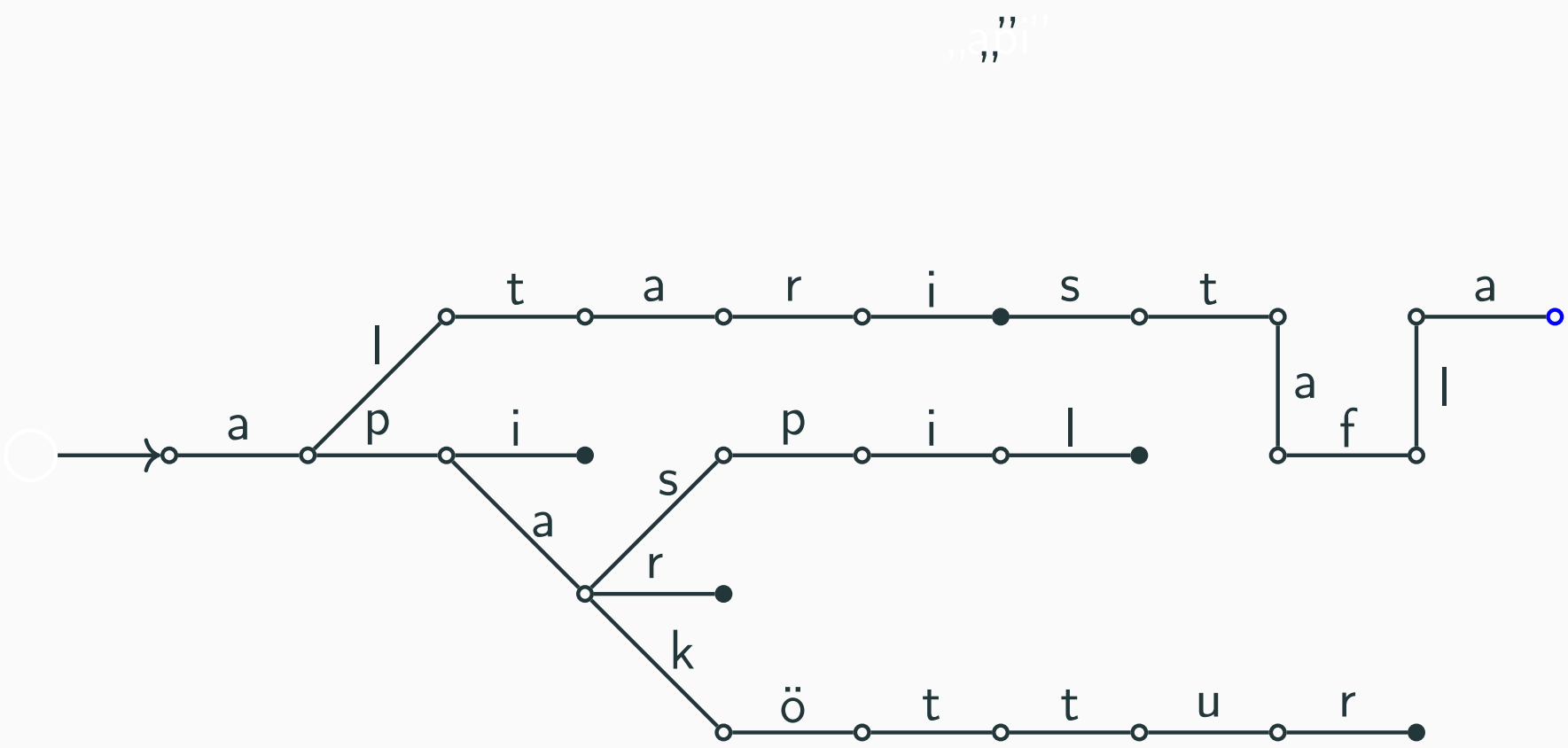


# Example

„a“

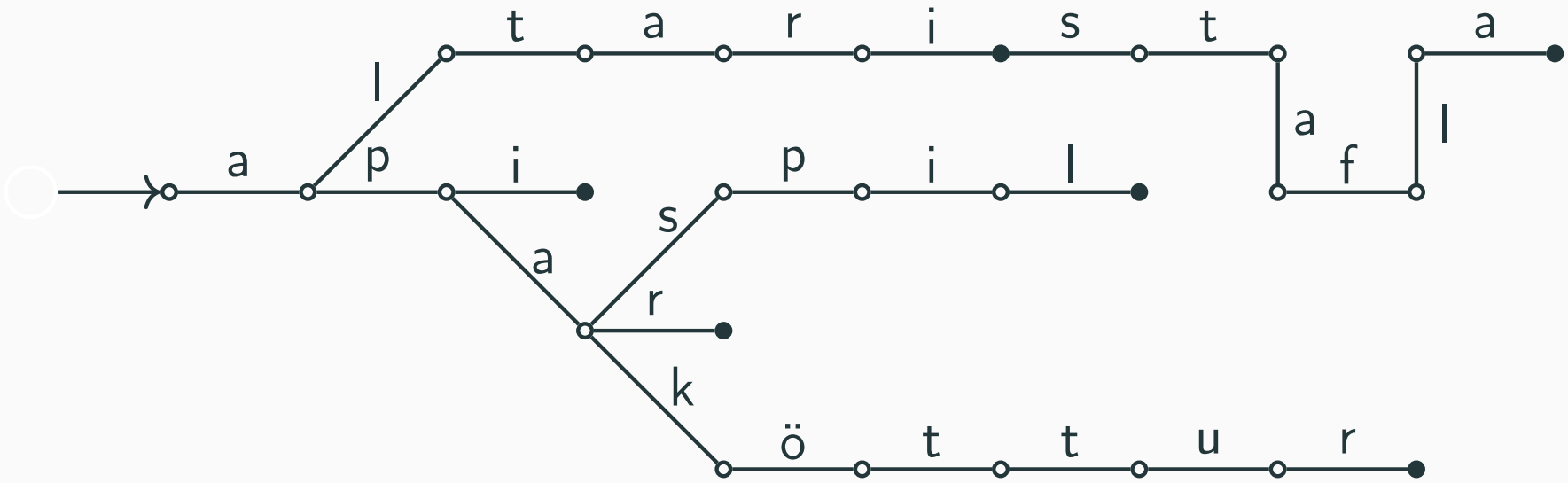


# Example



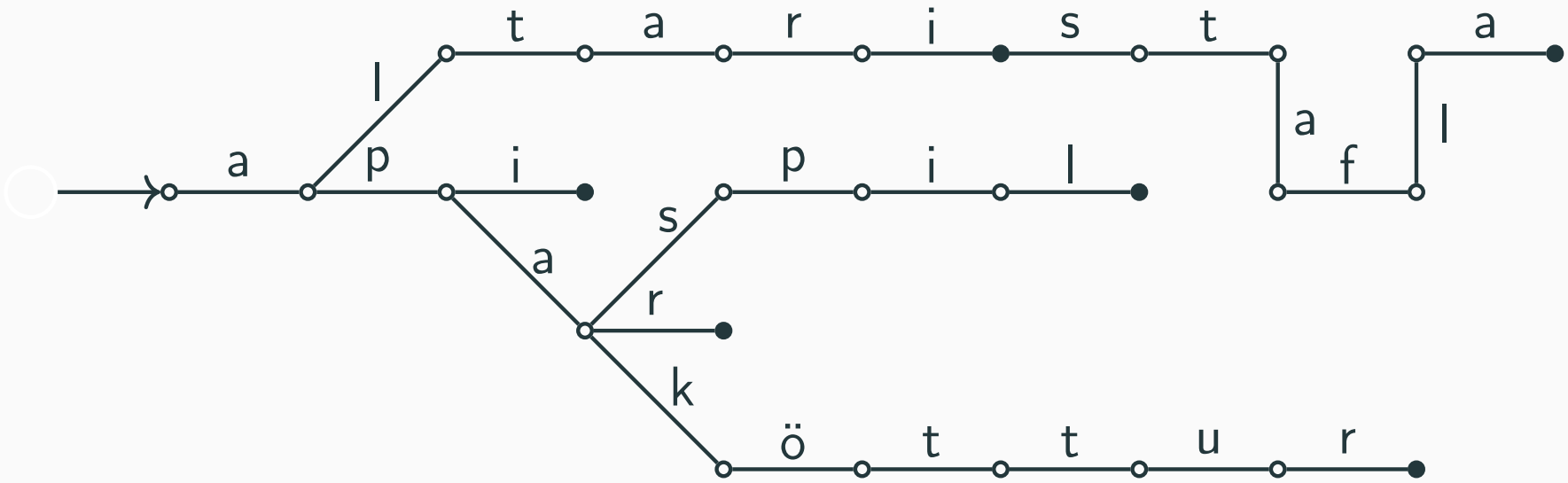
# Example

„api“



# Example

„altarisganga”



# Example

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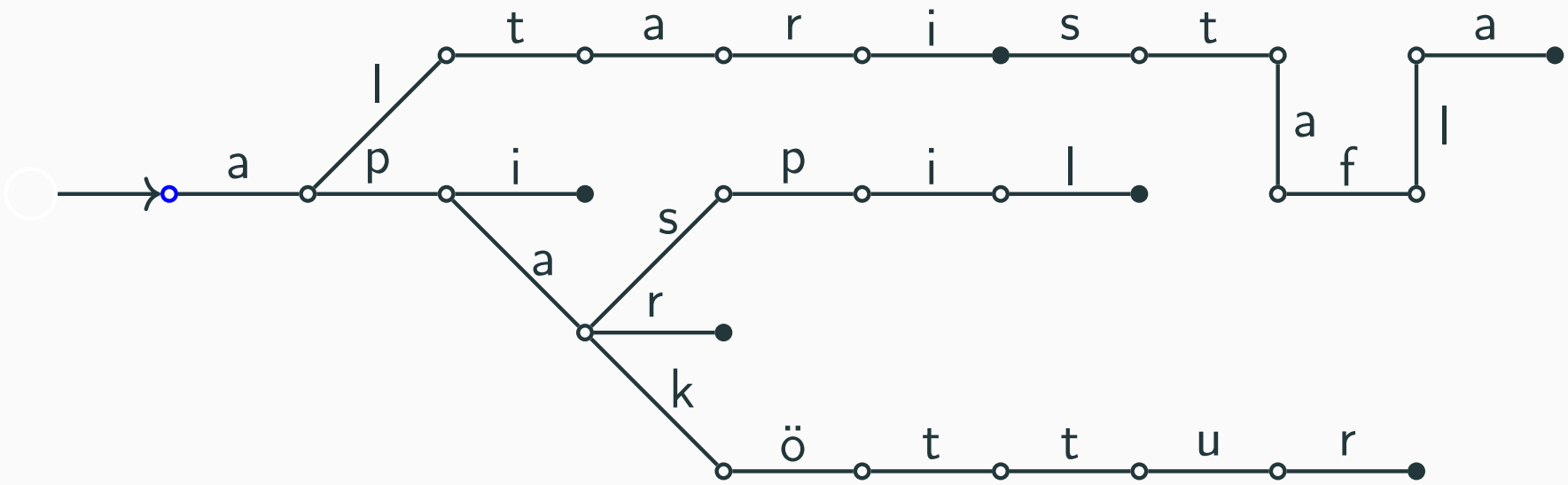

# Example

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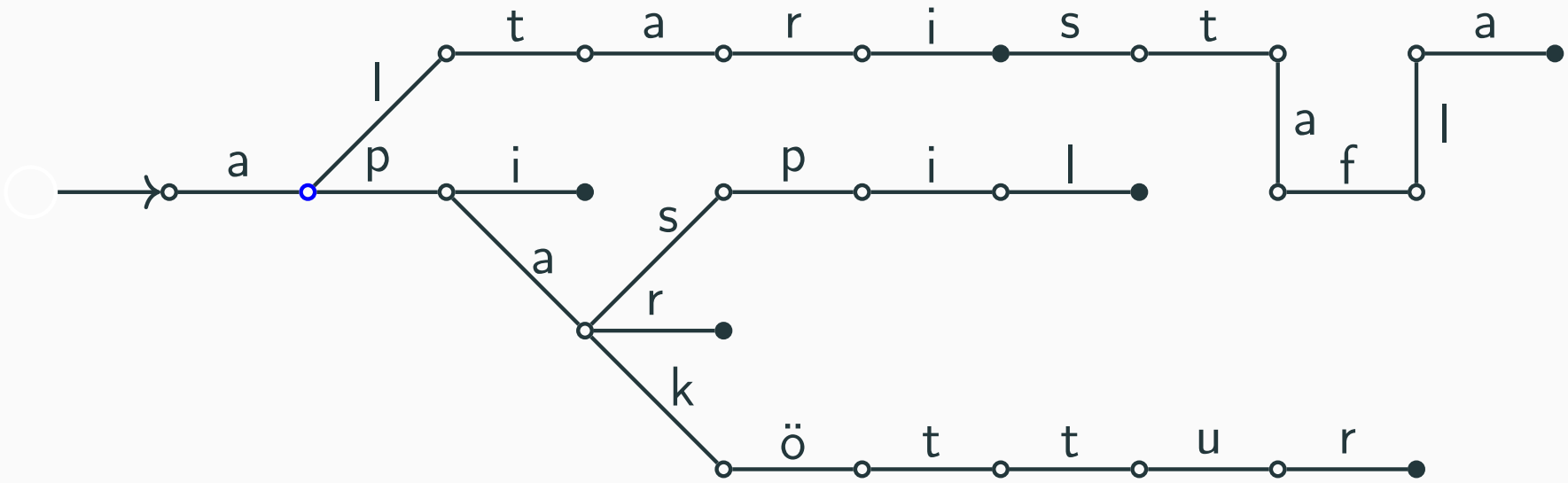

# Example

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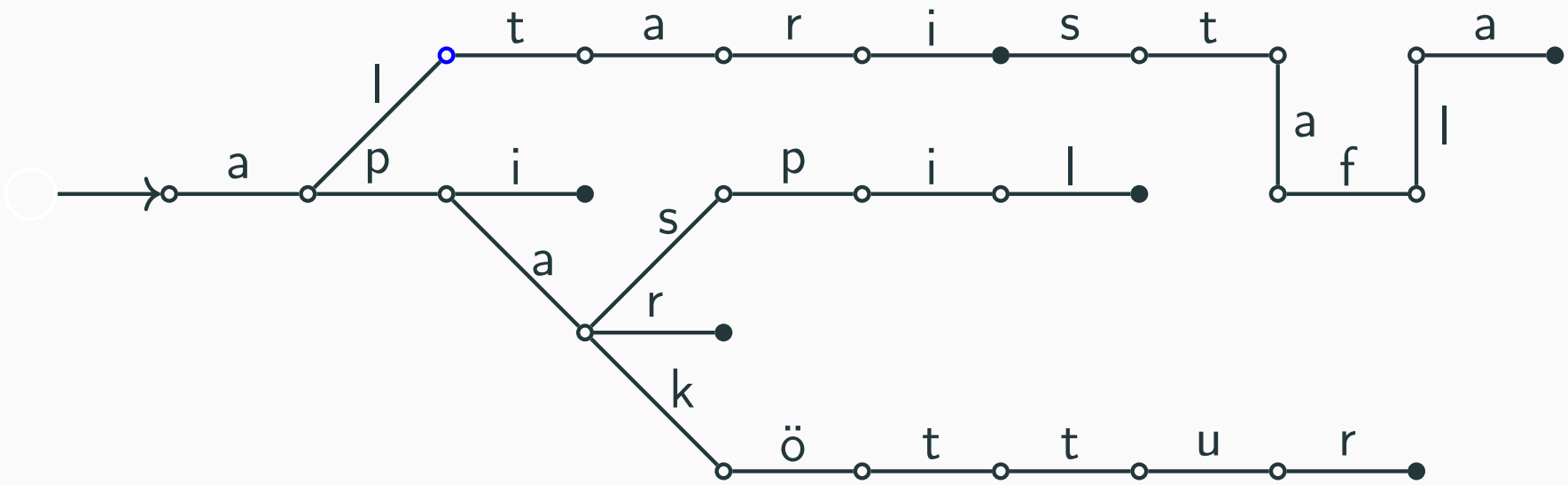

# Example

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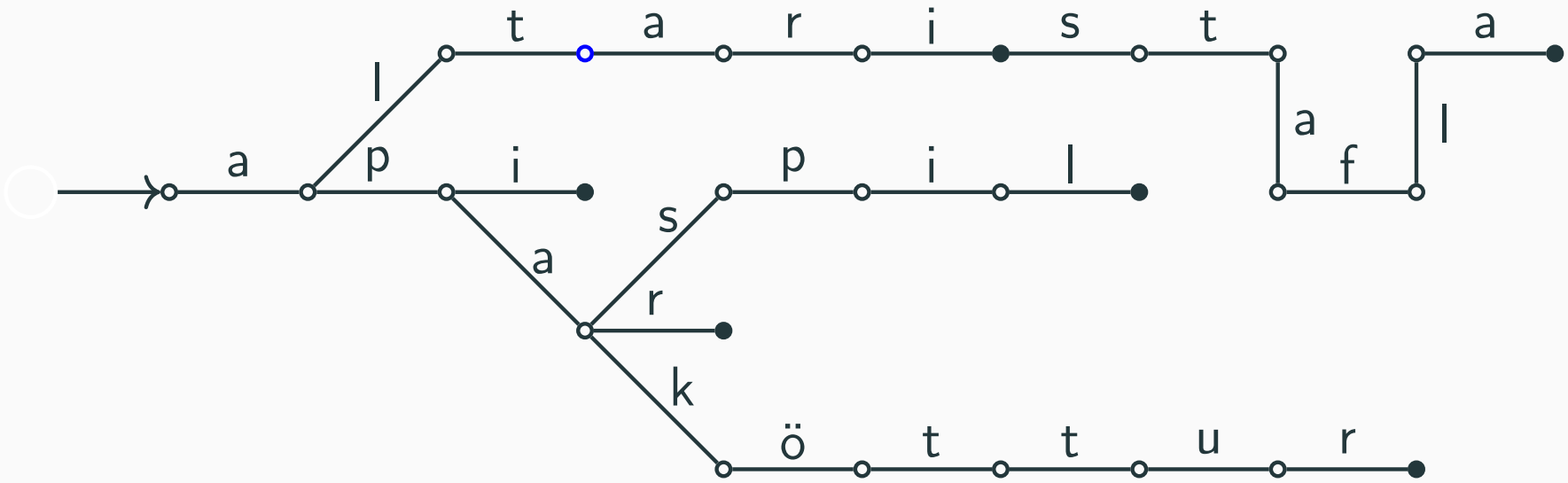

# Example

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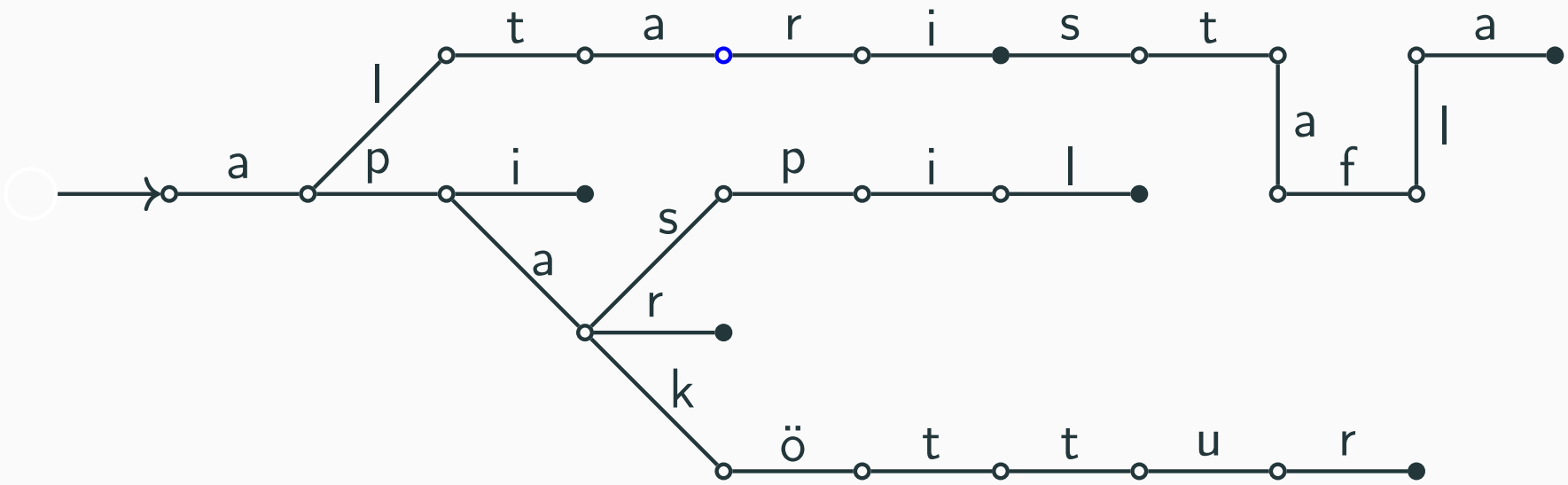

# Example

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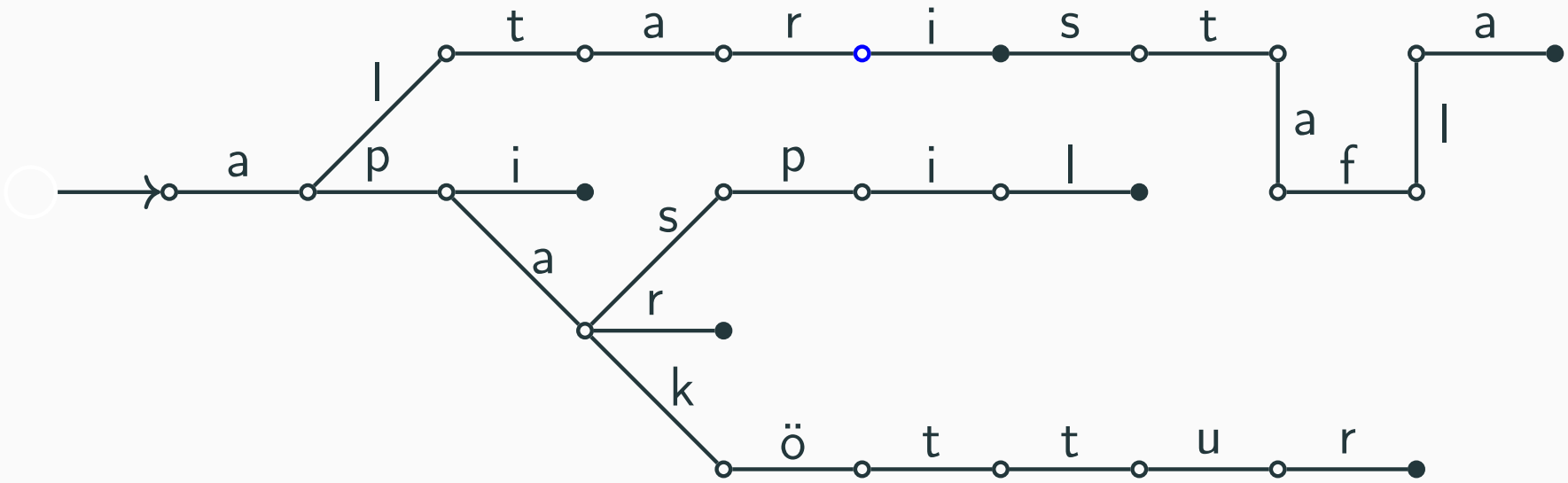
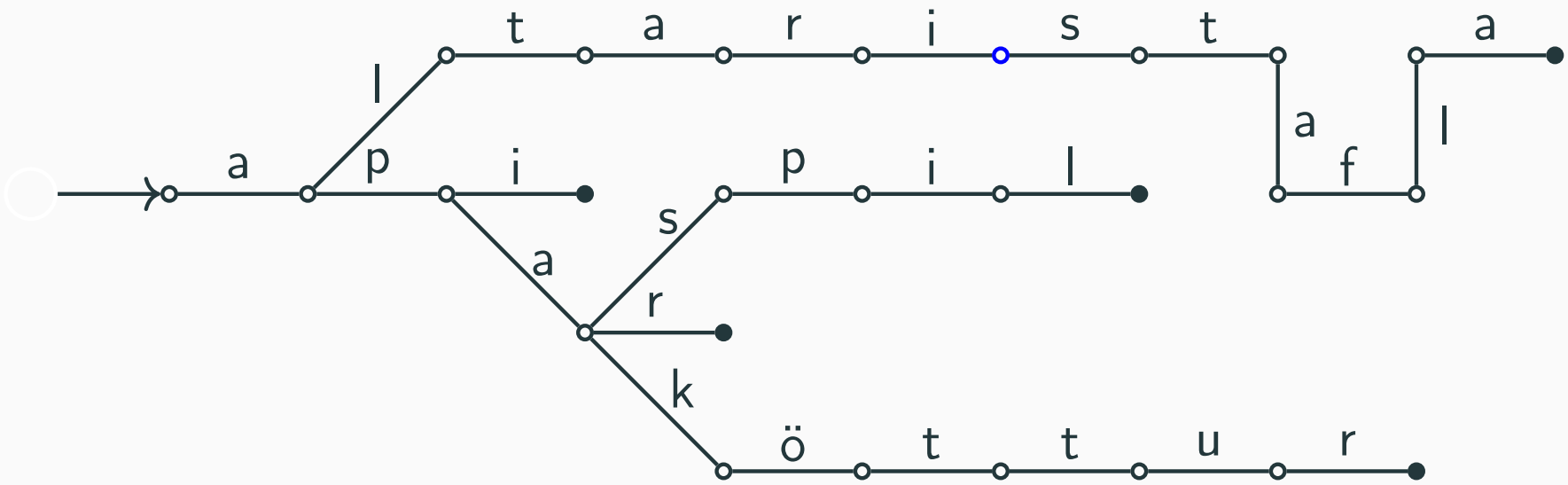


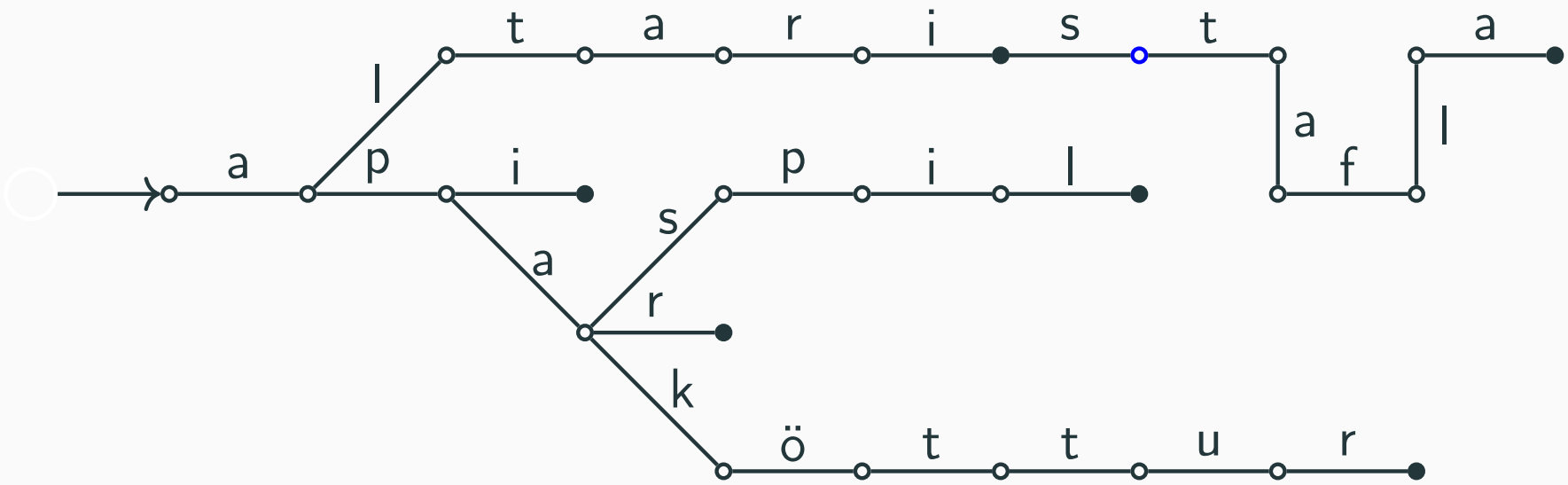
# Example

„sganga”

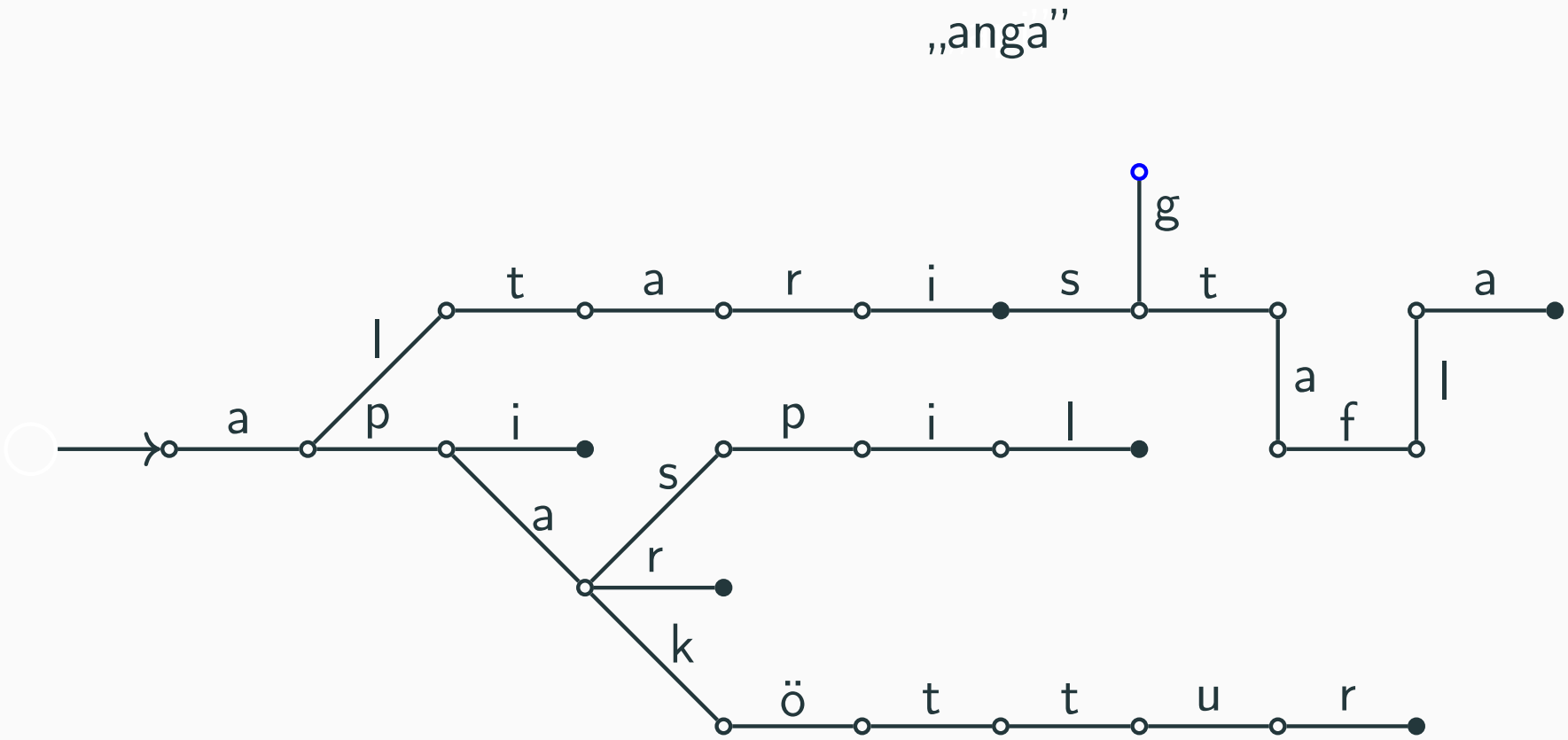


# Example

„ganga”



# Example



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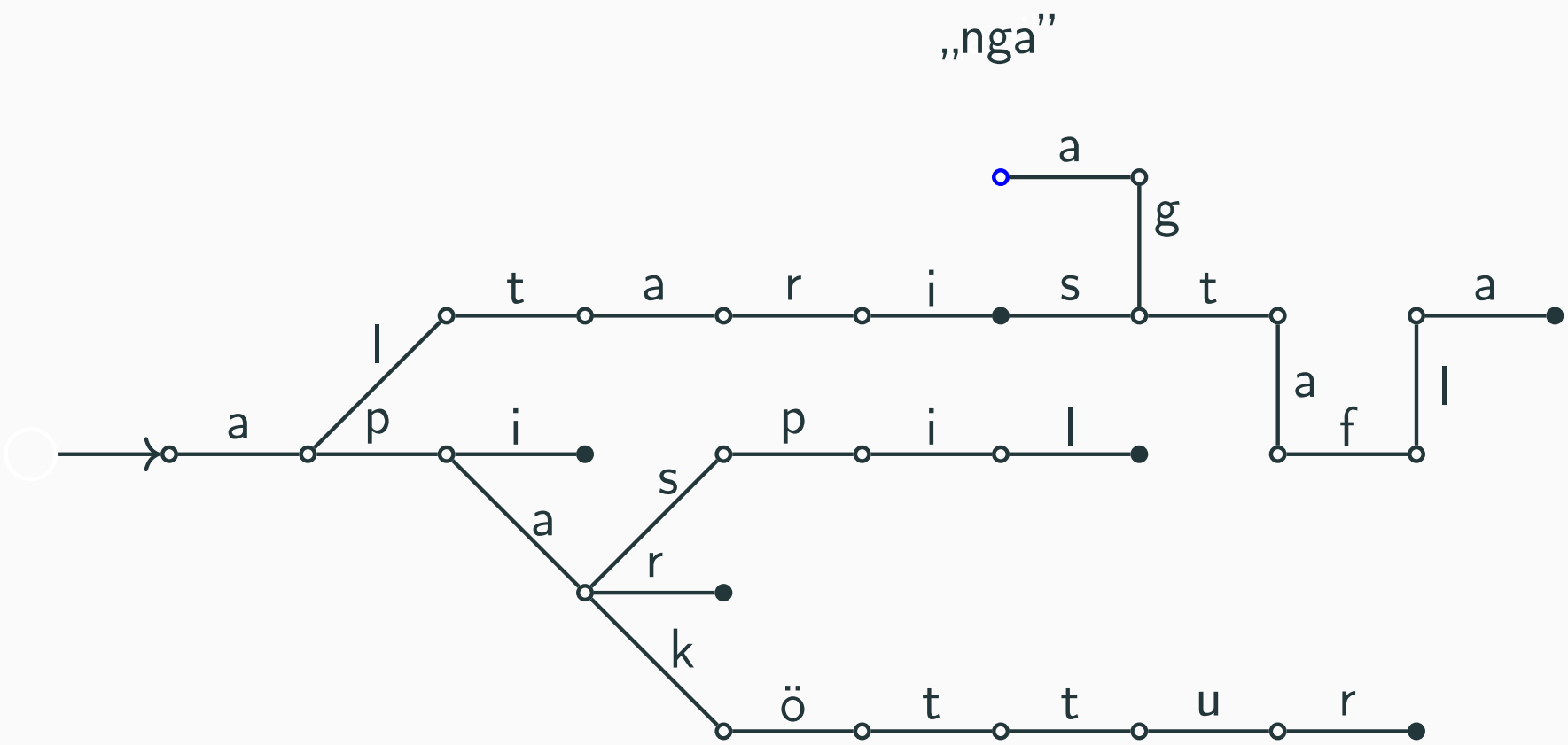
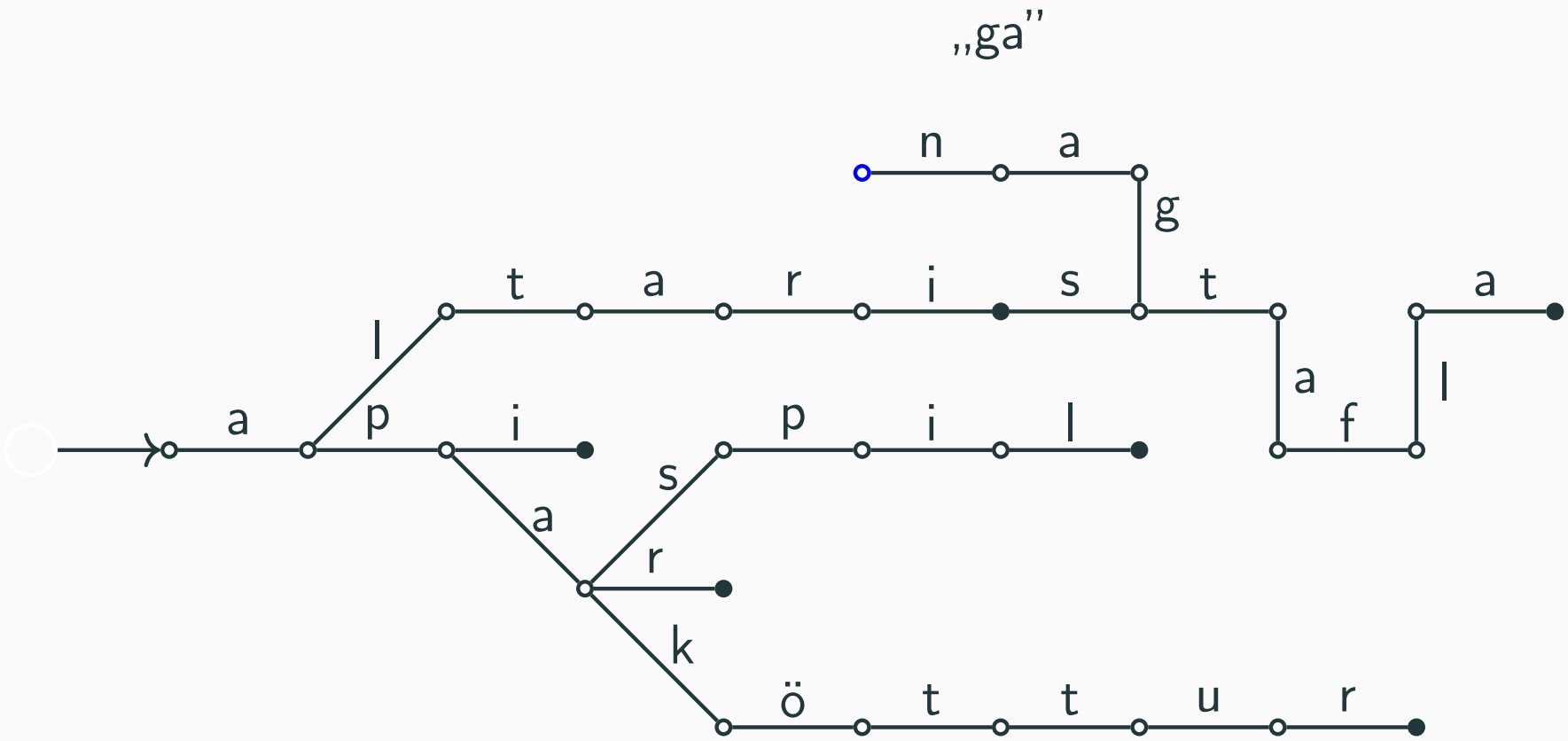
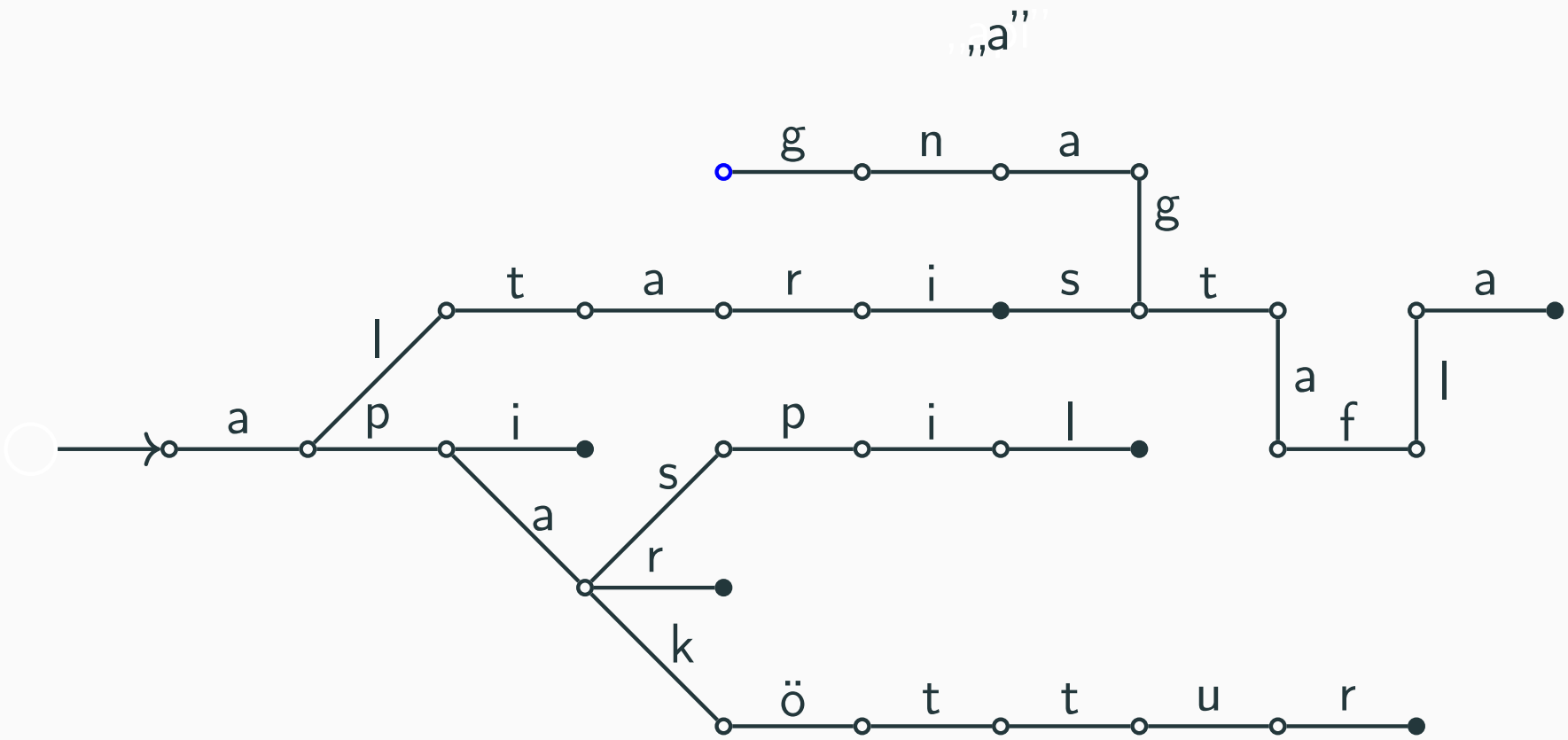


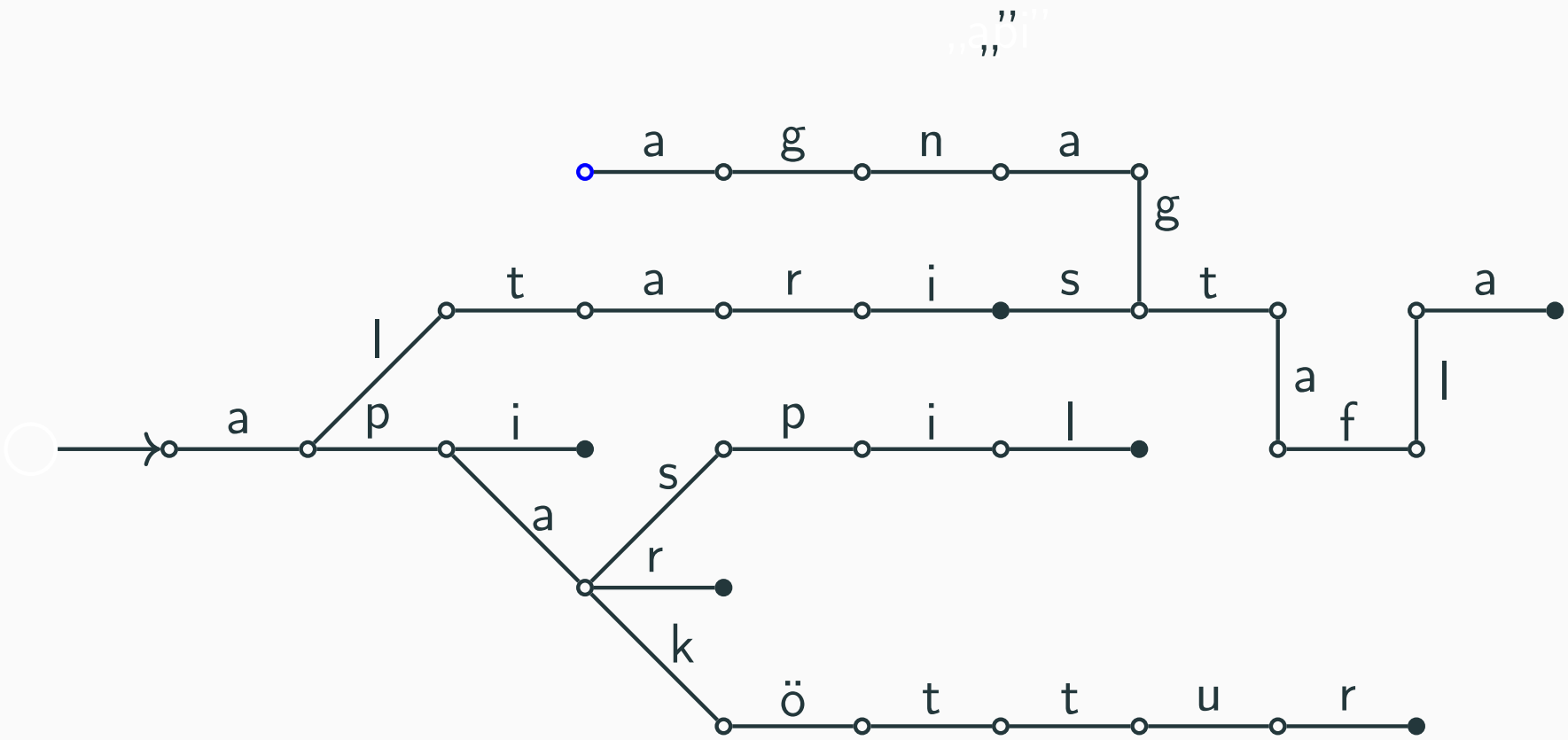
# Example



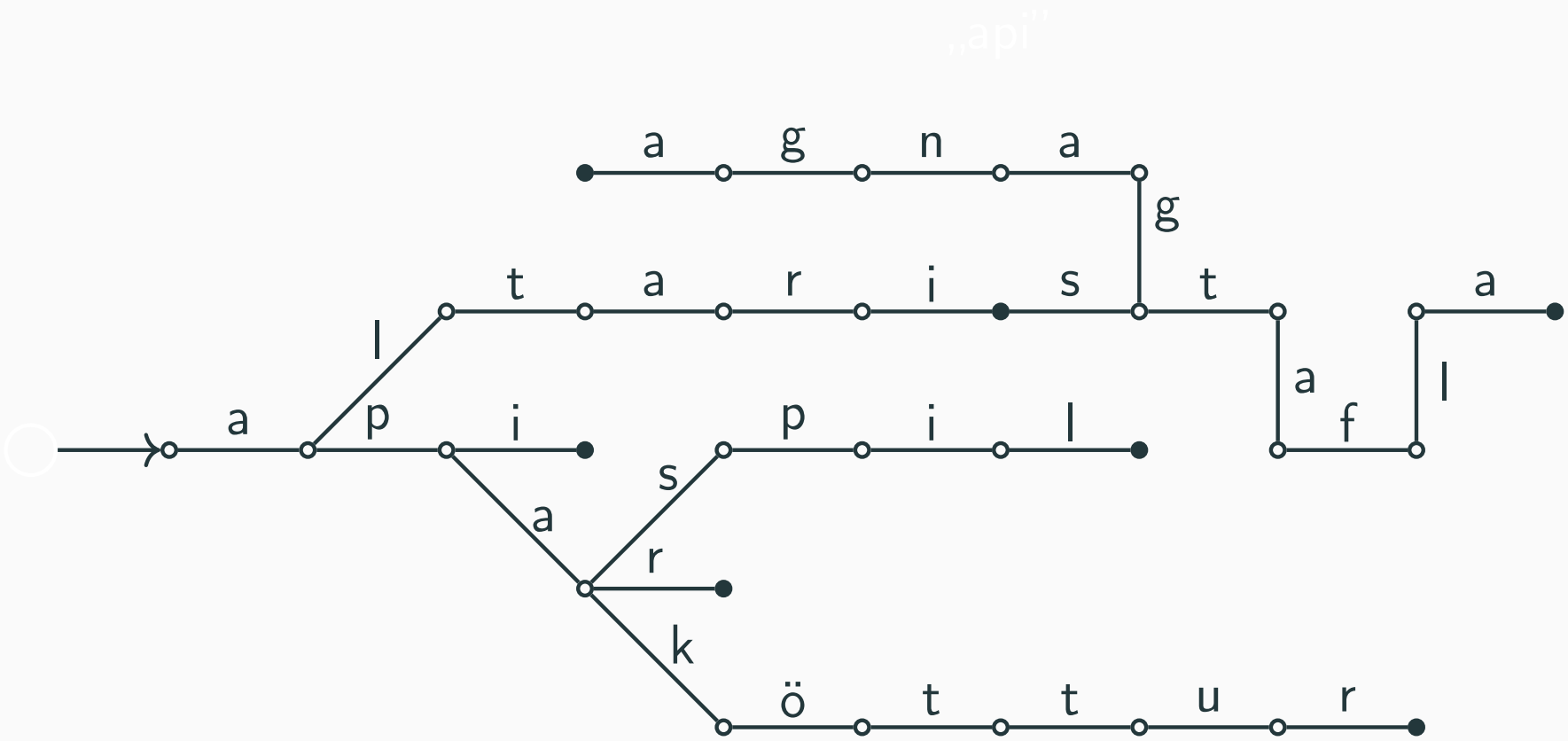
# Example



# Example



# Example



# Tries

```
struct node {
    node* children[26];
    bool is_end;

    node() {
        memset(children, 0, sizeof(children));
        is_end = false;
    }
};
```

# Tries

```
void insert(node* nd, char *s) {
    if (*s) {
        if (!nd->children[*s - 'a'])
            nd->children[*s - 'a'] = new node();

        insert(nd->children[*s - 'a'], s + 1);
    } else {
        nd->is_end = true;
    }
}
```

# Tries

```
bool contains(node* nd, char *s) {
    if (*s) {
        if (!nd->children[*s - 'a'])
            return false;

        return contains(nd->children[*s - 'a'], s + 1);
    } else {
        return nd->is_end;
    }
}
```

# Tries

```
node *trie = new node();

insert(trie, "banani");

if (contains(trie, "banani")) {
    // ...
}
```

# Tries

- Time complexity?

- Let $k$ be the length of the string we're inserting/looking for
- Lookup is $\mathcal{O}(k)$ and insertion is both $\mathcal{O}(k|\Sigma|)$
- The insertion takes this time because we might have to make $k$ nodes, each needing $|\Sigma|$ pointers initialized
- This can be improved by using a map/dict for children instead, but that does make lookup slower, tradeoffs as usual

# String multimatching

# Aho-Corasick

- Let us now have some string $s$ and a list of $n$ strings $p$, where we denote the $j$-th string by $p_j$.
- Let $|s|$ be the length of $s$ and $|p| = |p_1| + \cdots + |p_n|$.
- We want to find all substrings of $s$ that are in the list $p$.
- We could run KMP $n$ times, once for each $p_j$, for a time complexity of $\mathcal{O}(n \cdot |s| + |p|)$.
- The Aho-Corasick algorithm improves on this.

# The algorithm

- We start by putting all strings in $p$ into a trie $T$.
- We then want to turn $T$ into a finite state automata.
- The nodes of the trie will be our states but the transitions from each state will correspond to a letter from $\Sigma$.

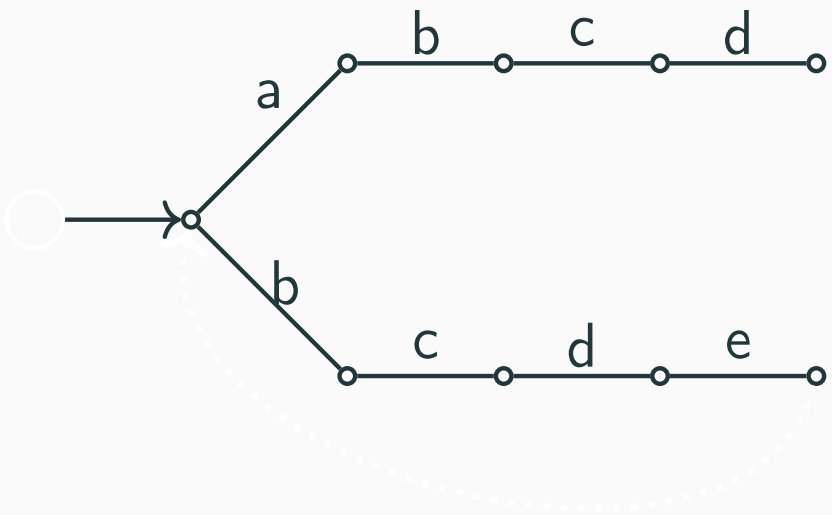
# The automata

- Suppose we are in node $v$ in $T$ and want to transition according to the letter $c$ in $\Sigma$.
- If there is an node corresponding to adding a $c$ after $v$ we can travel there.
- If not we need to travel back to some node $w$ so the string corresponding to $w$ is a suffix of the one corresponding to $v$.
- We want to drop the least amount of information, so we want $w$ to be as long as possible.
- We call these transitions *suffix links*. Note that they are essentially independent of $c$.
- We let the suffix link of the root point back to itself for simplicity's sake.

# Suffix links

- How do we find the suffix links?
- We will cheat and borrow a method from the future, dynamic programming.
- Let $f(w, c)$ denote the transition from node $w$ with the letter $c$ and let $g(w)$ be the suffix link of $w$.
- Also let $p$ be the parent of $p$ and $f(p, a) = v$. Then $g(v) = f(g(p), a)$.
- Thus we have a recursive formula we can use.

# Example



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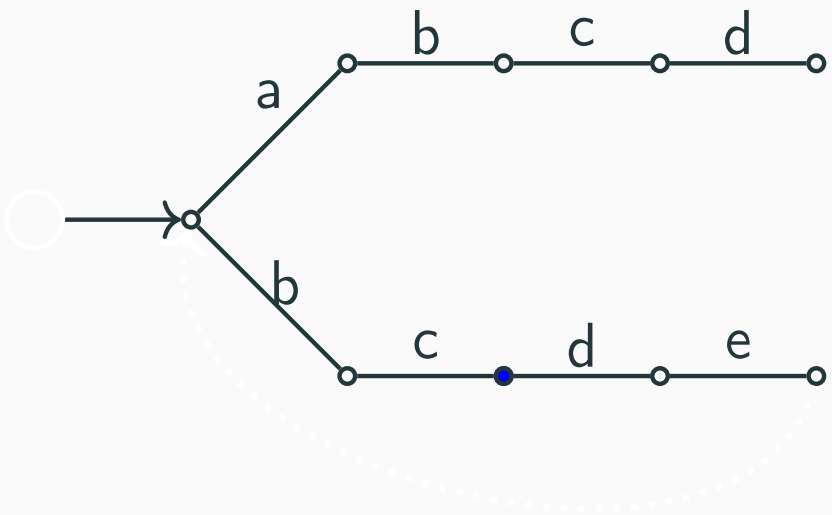
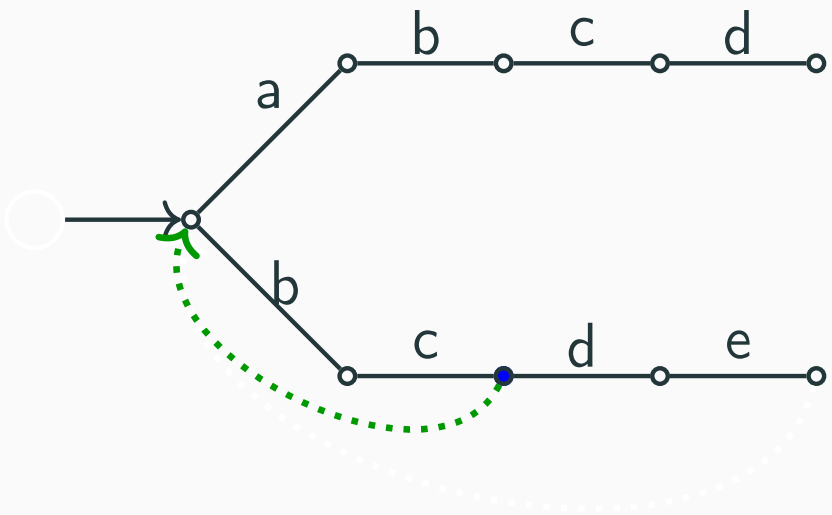
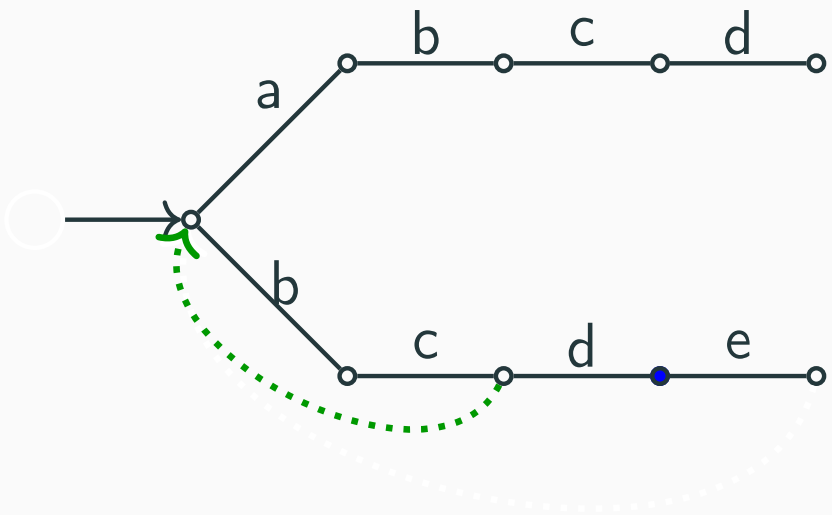


# Example



# Example



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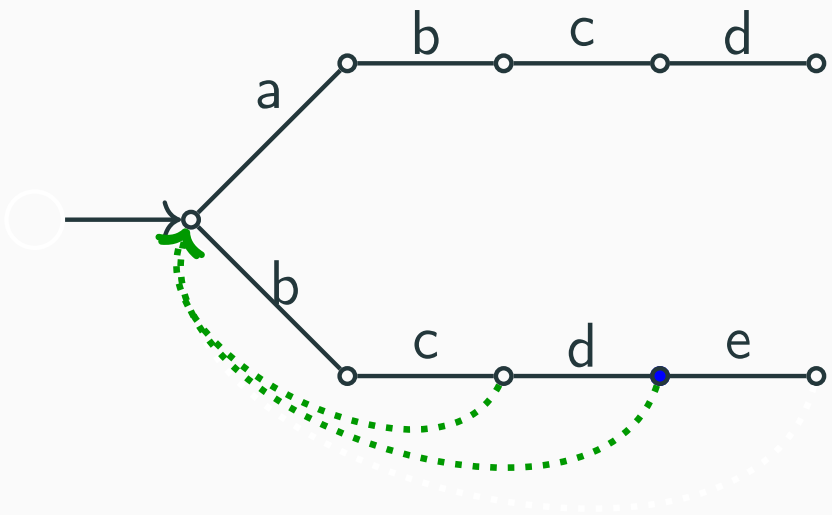

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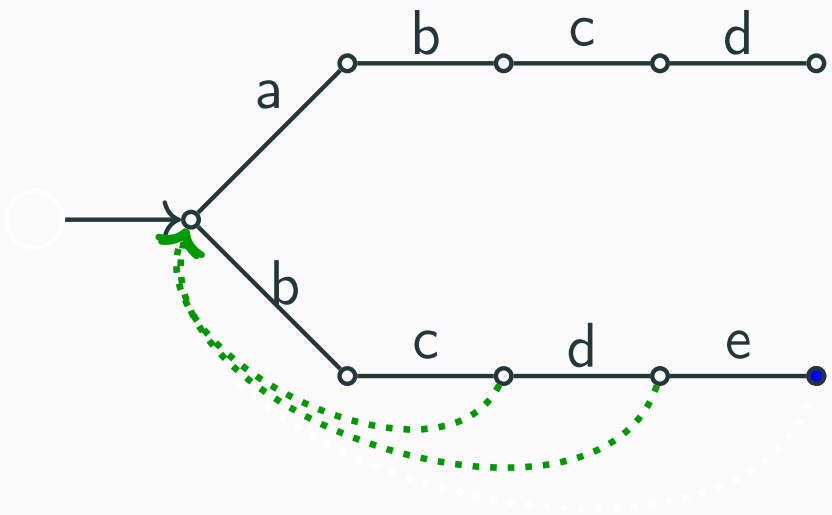

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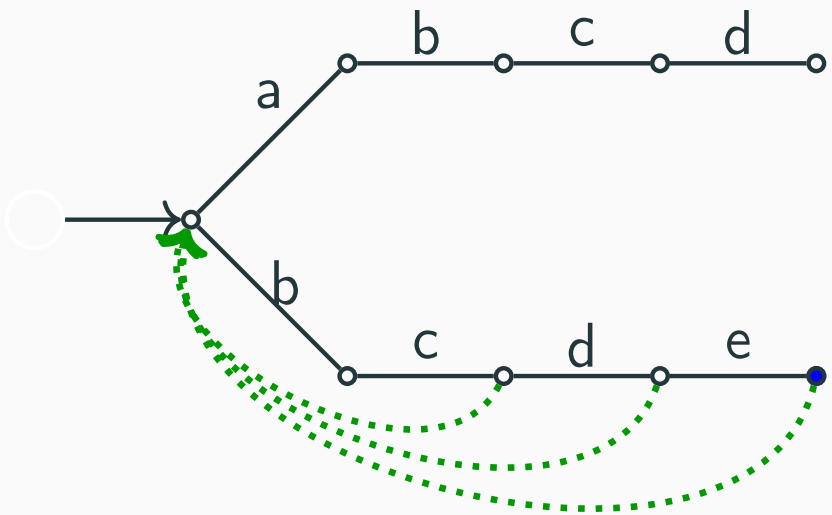

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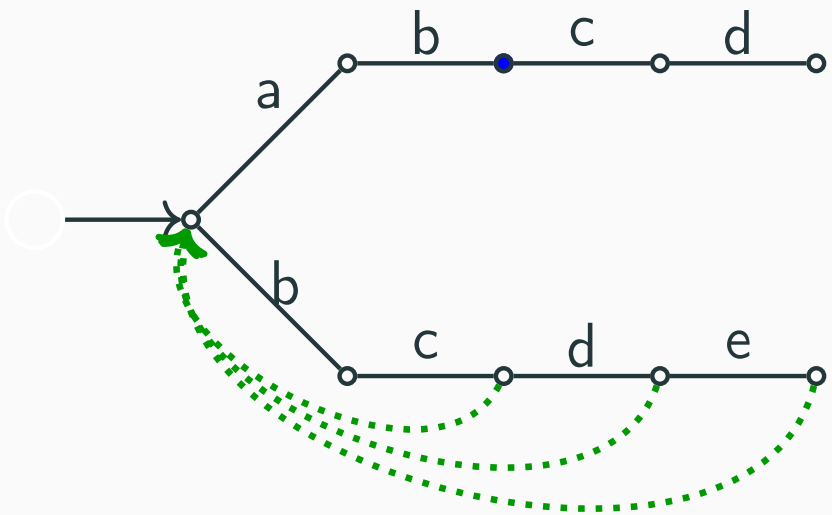

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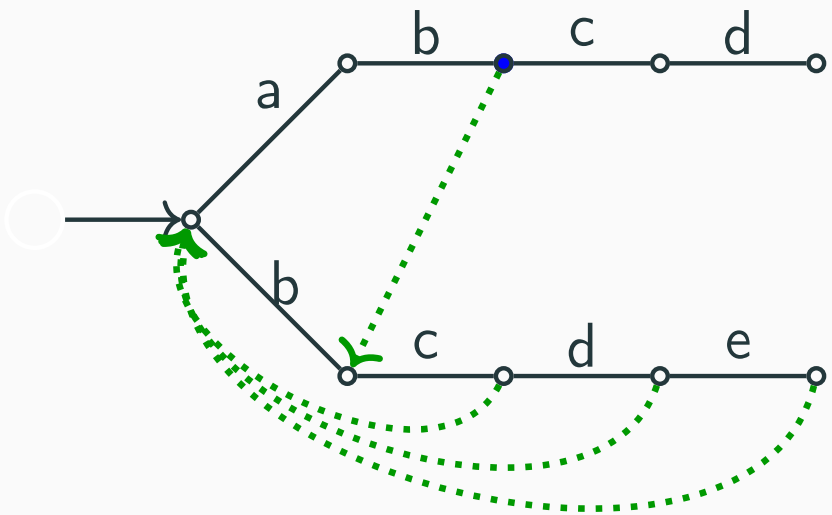

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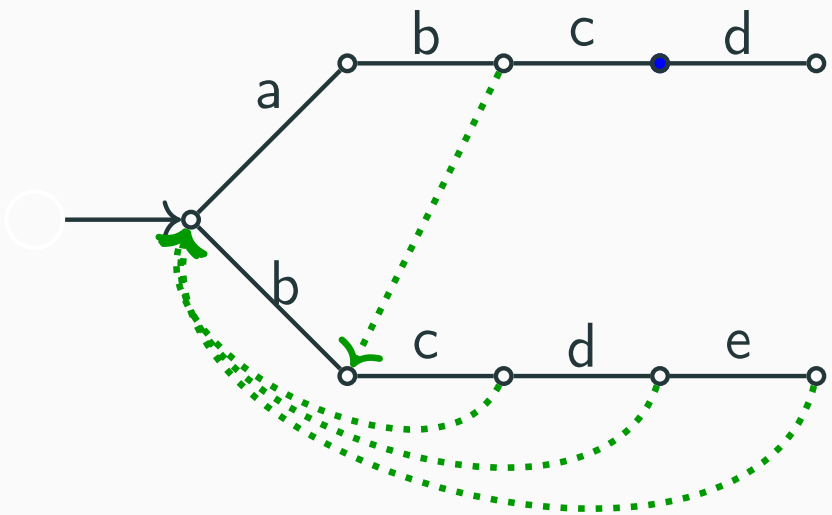

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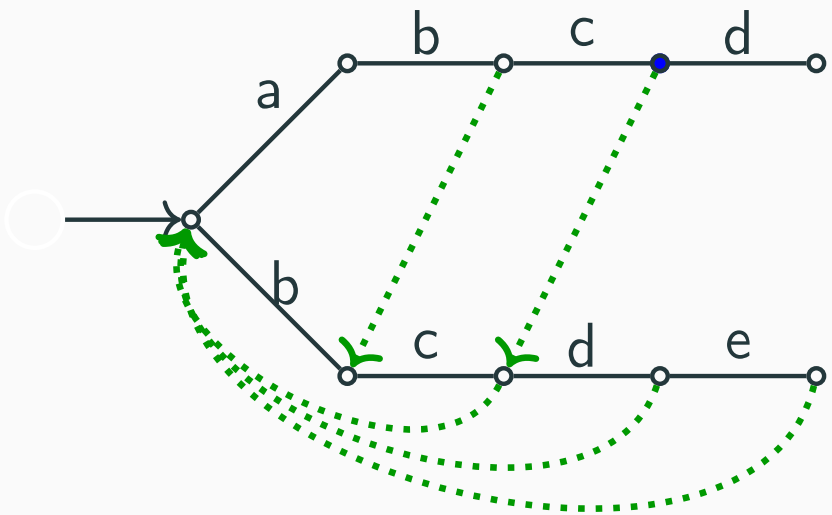

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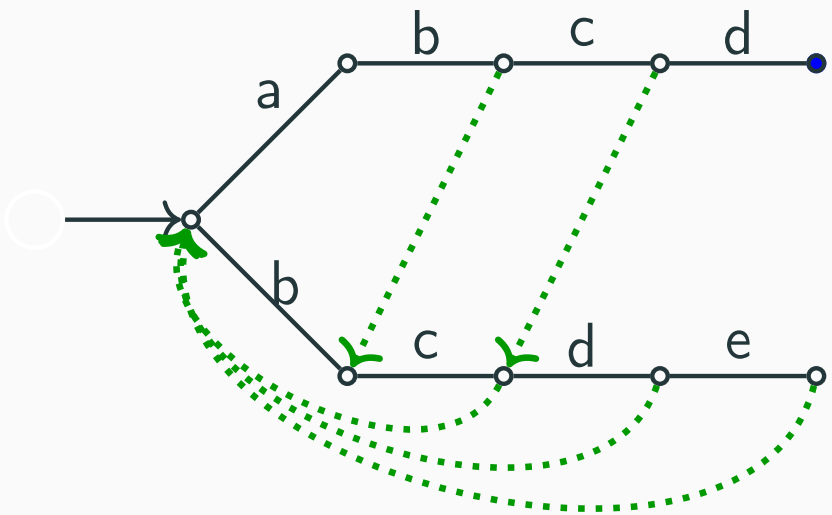

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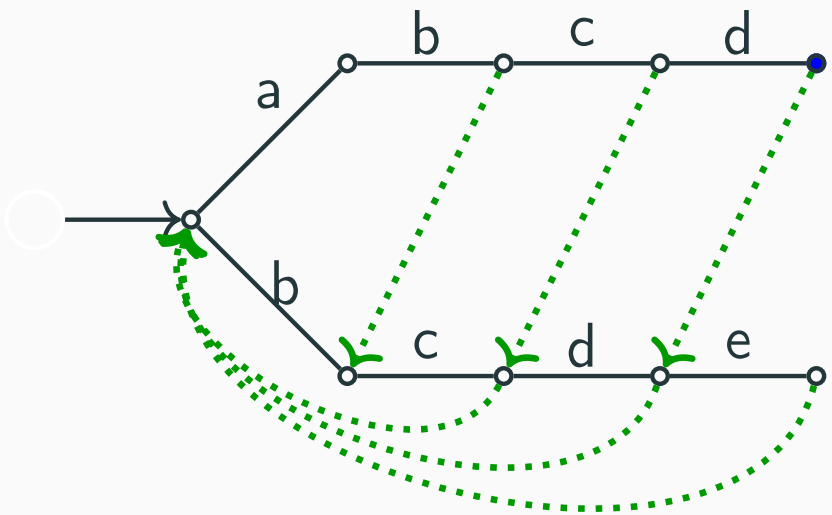
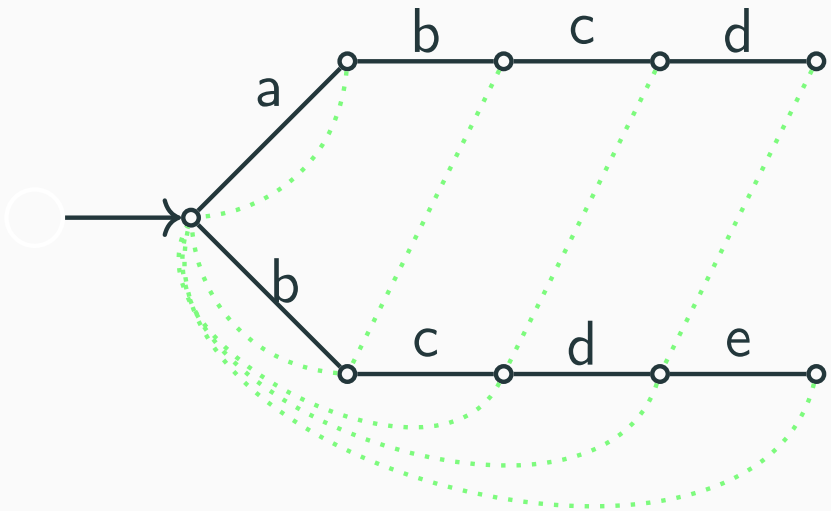


# End nodes

- We also have to mark end nodes in $T$.
- We then walk through $s$ and move around the state machine according to the letters encountered.

# Example

„abcdcdeaaabcdeabcxab”



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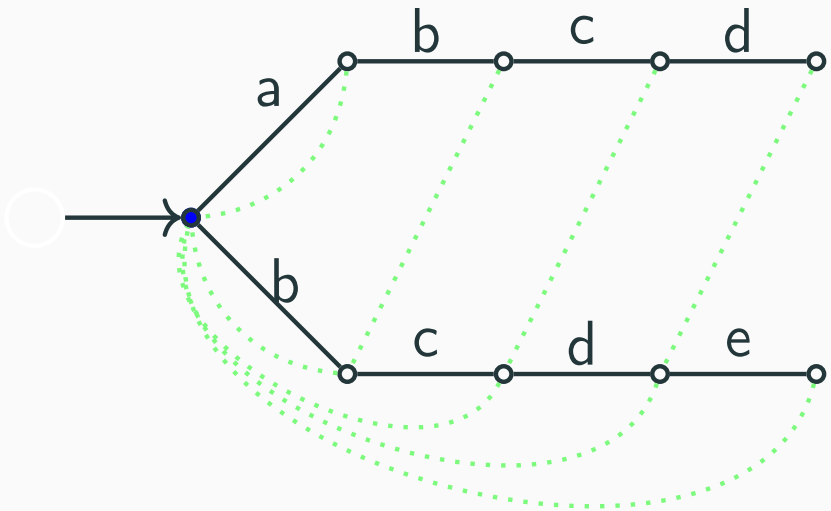

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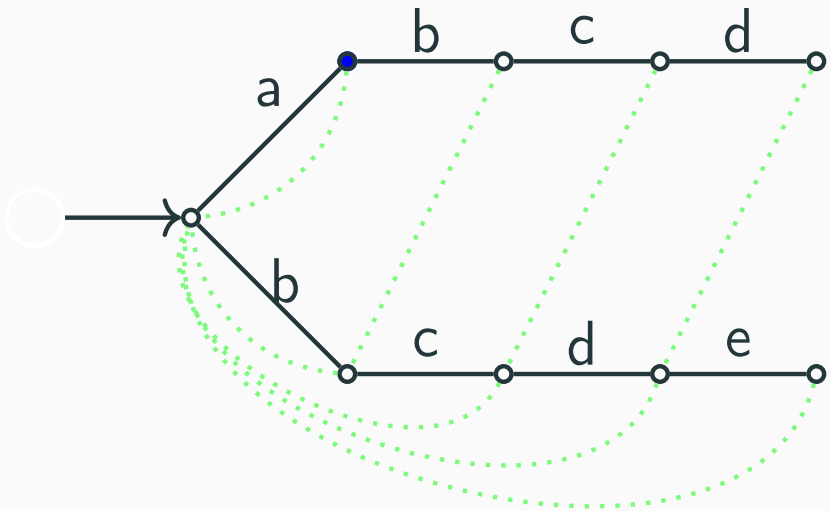
„bcdcdeaaabcdeabcxab”



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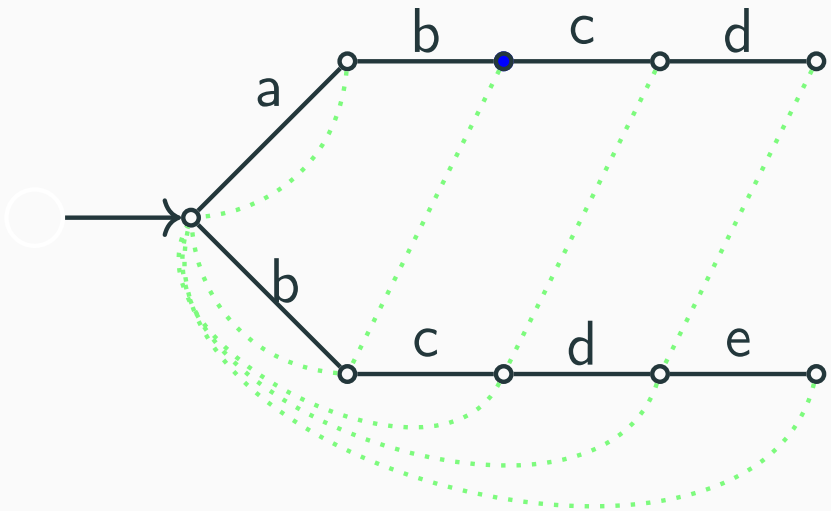

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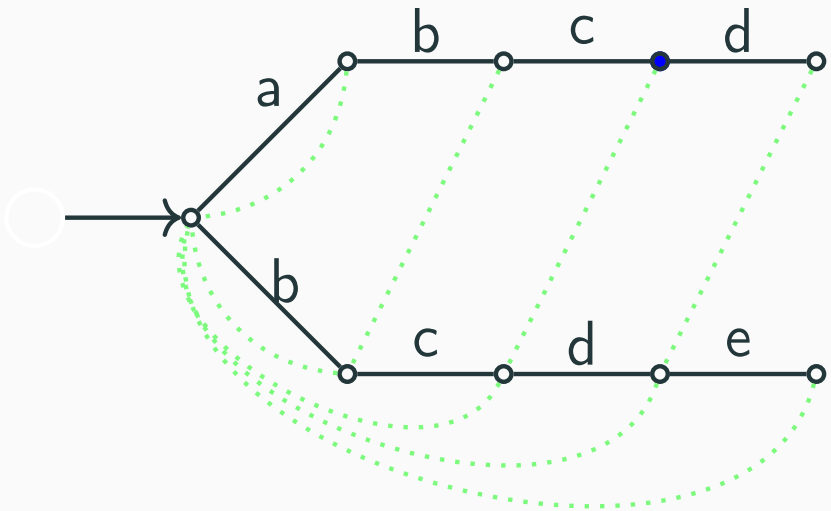

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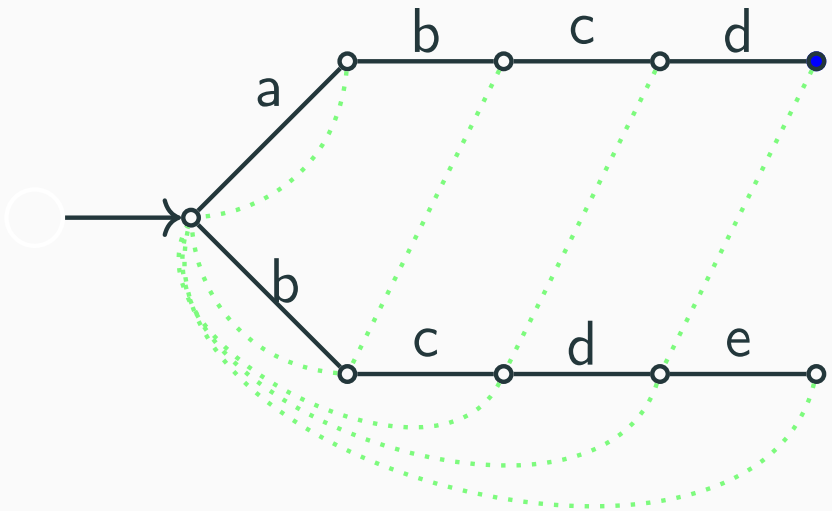

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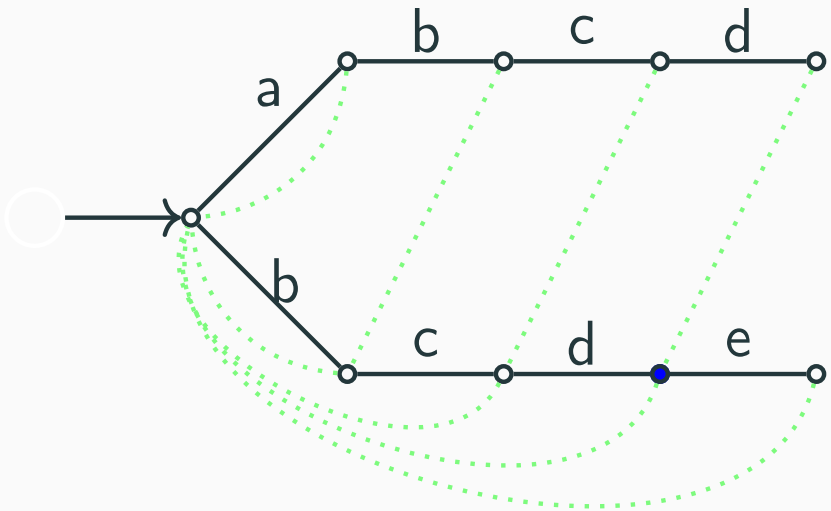

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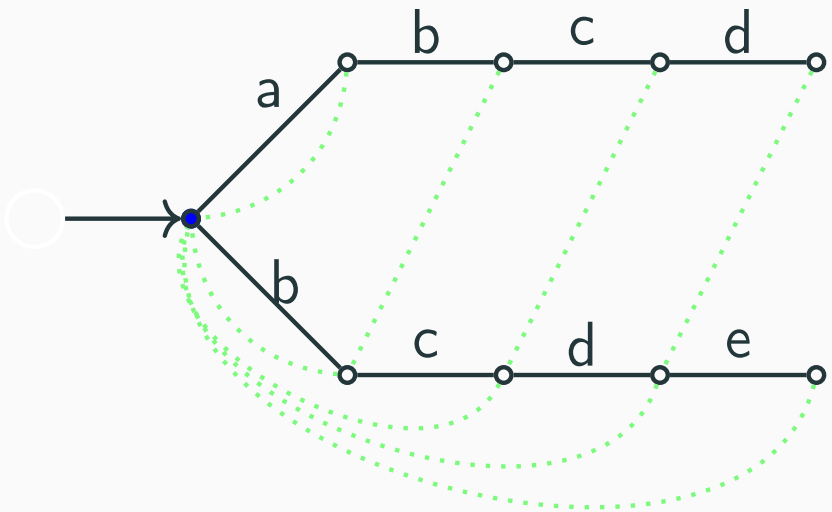
„cdeaaabcdeabcxab”



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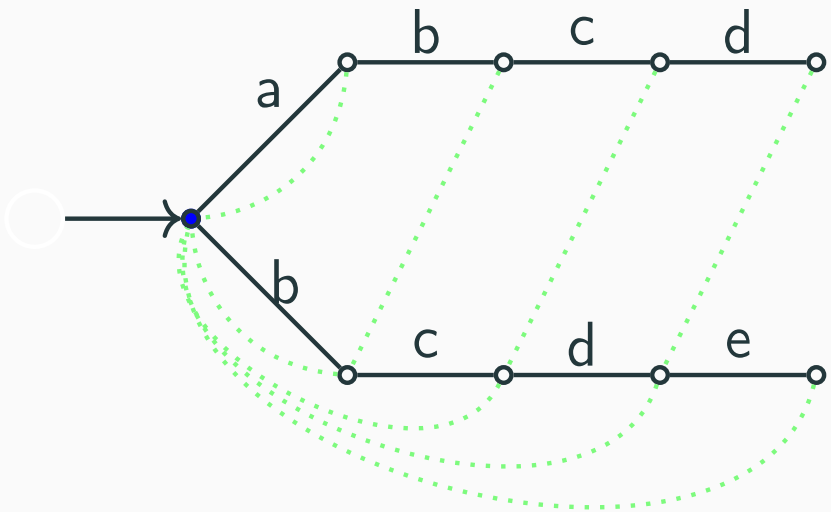

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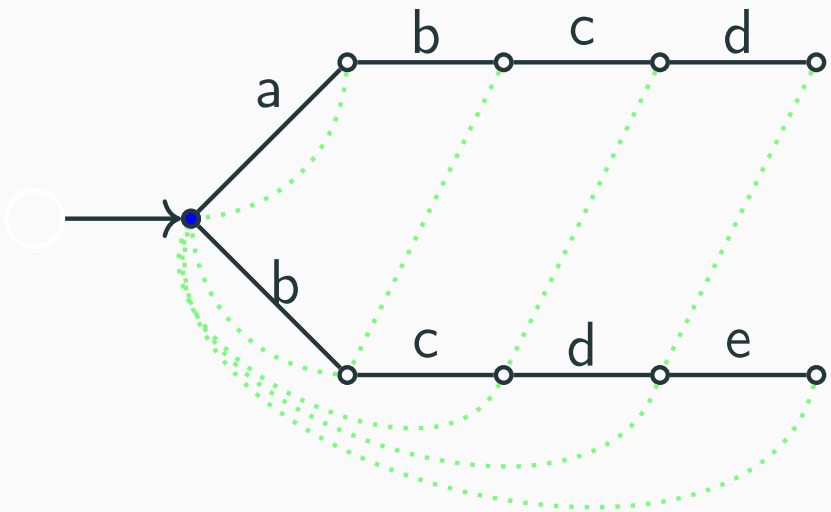

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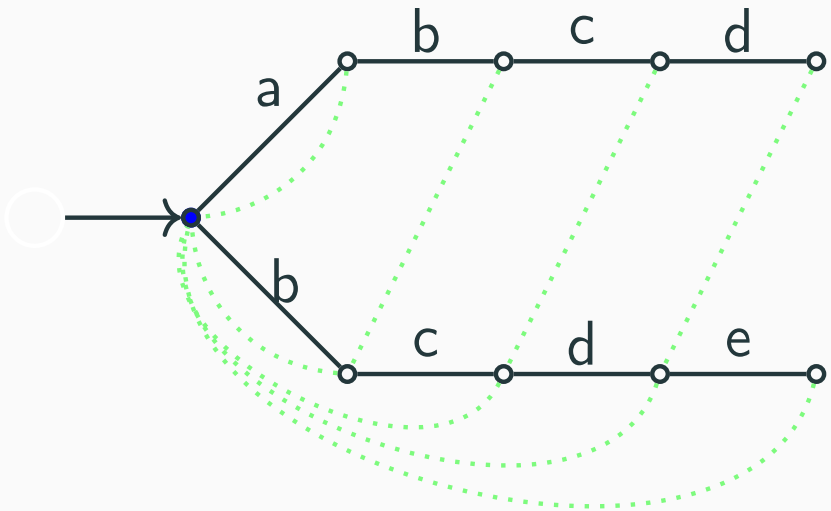

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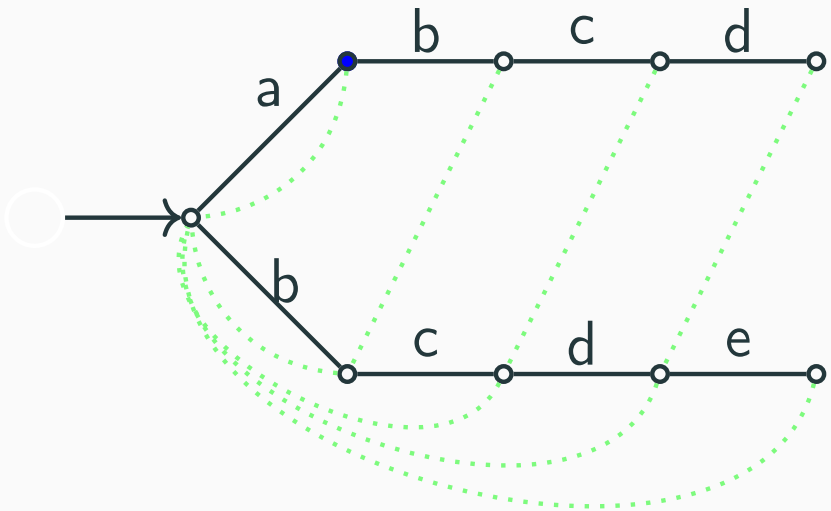

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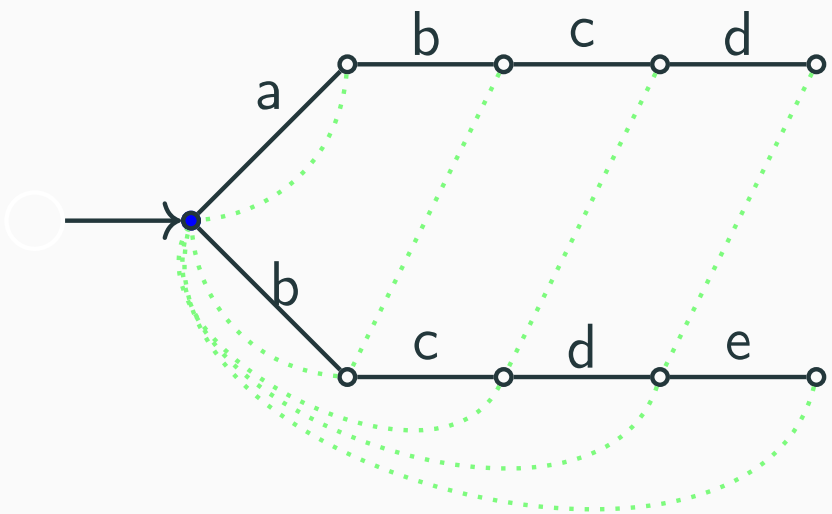

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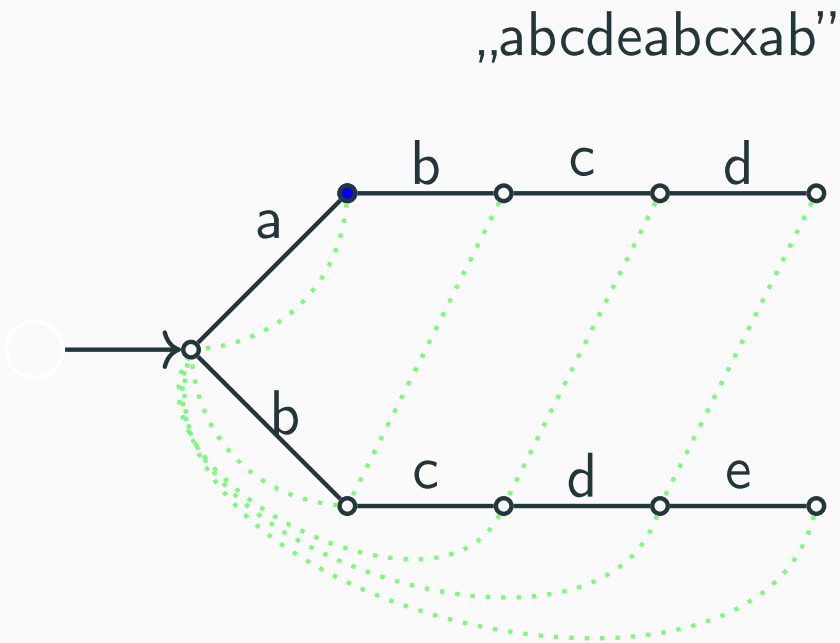

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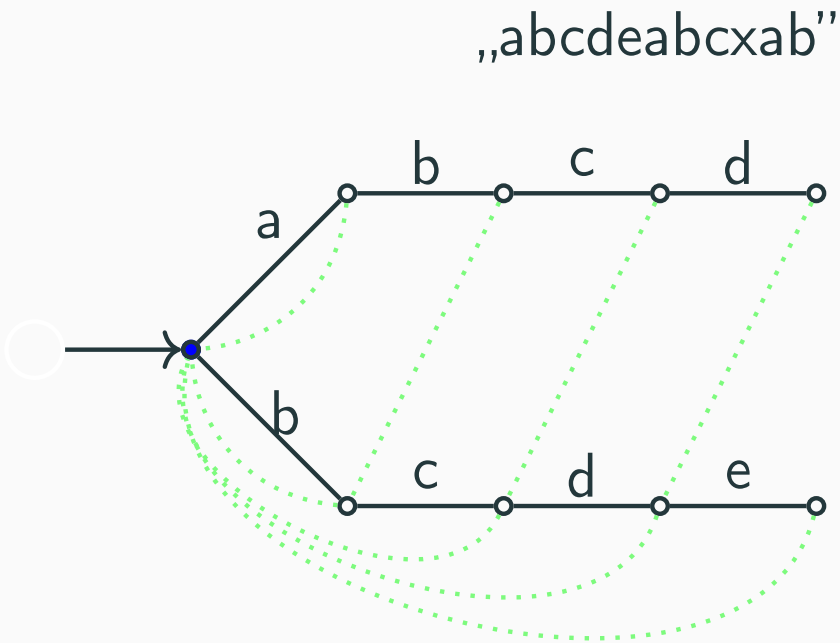

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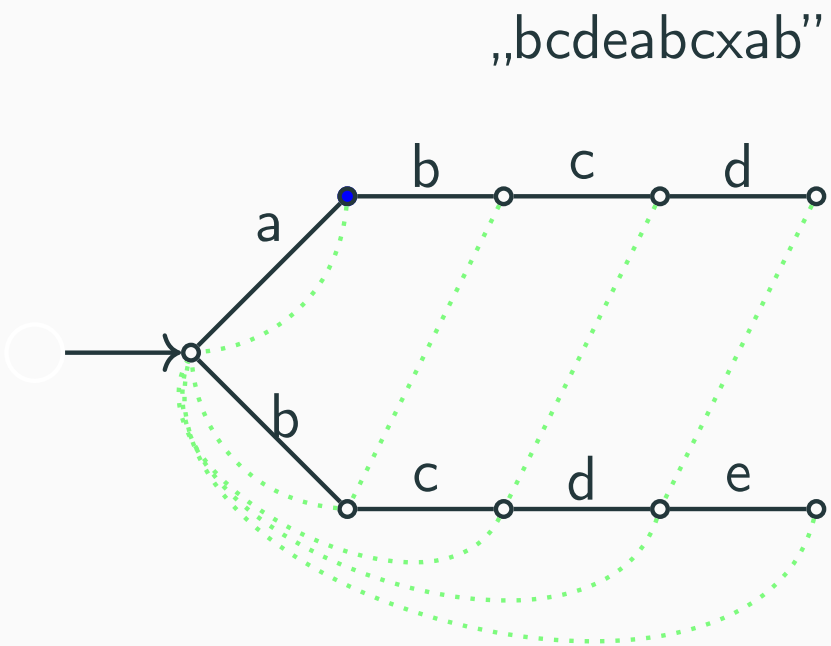

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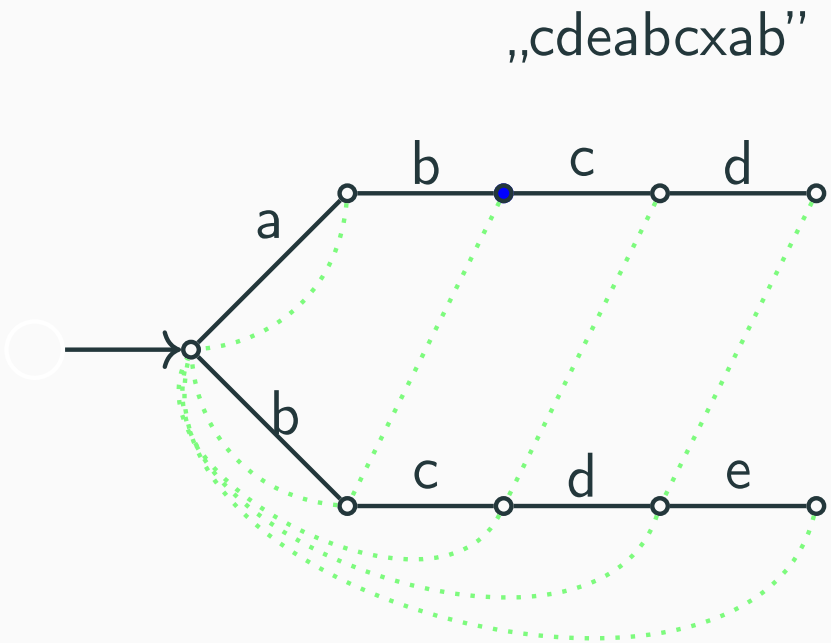

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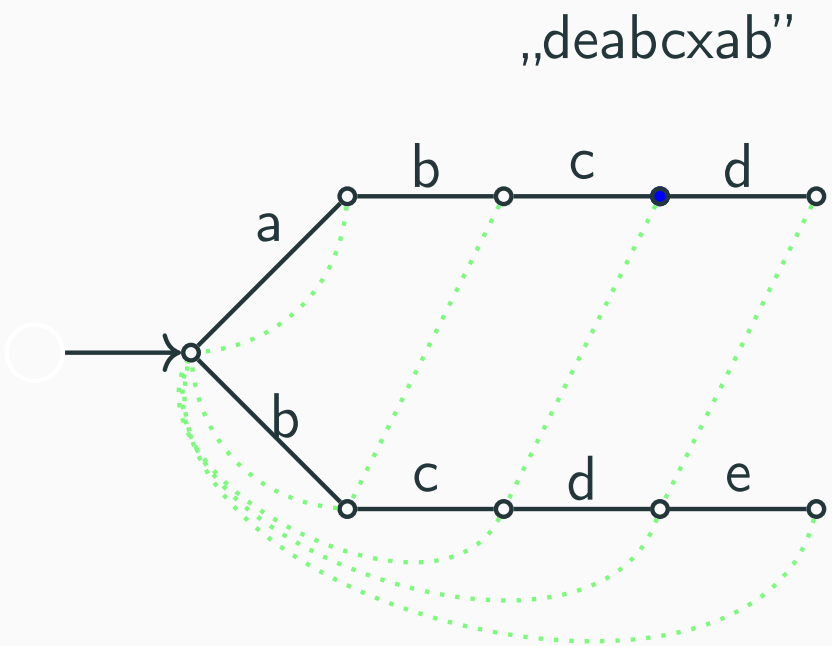

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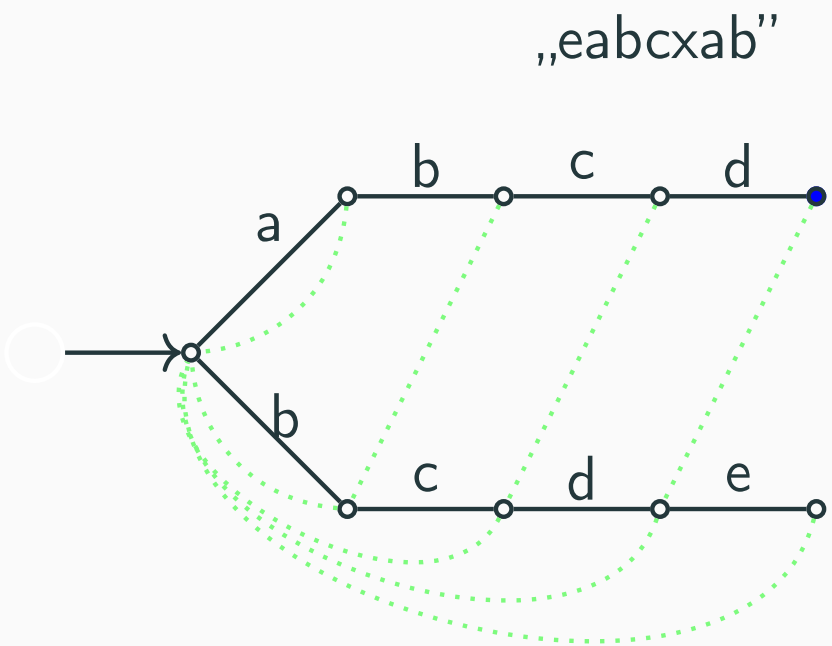

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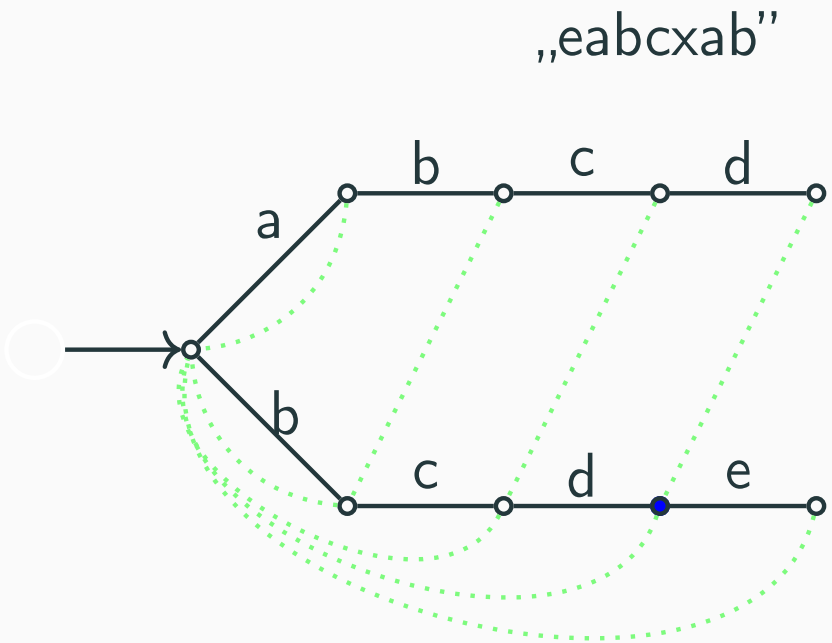

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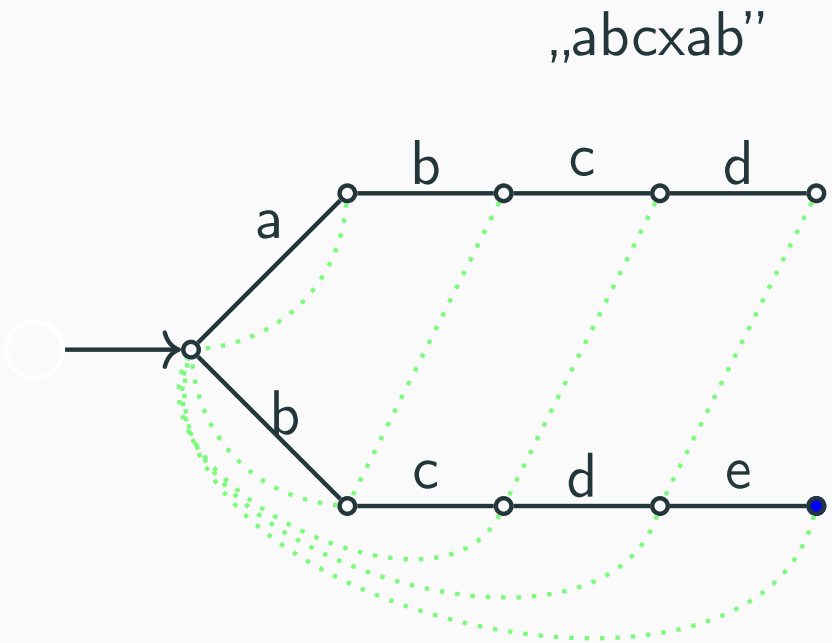

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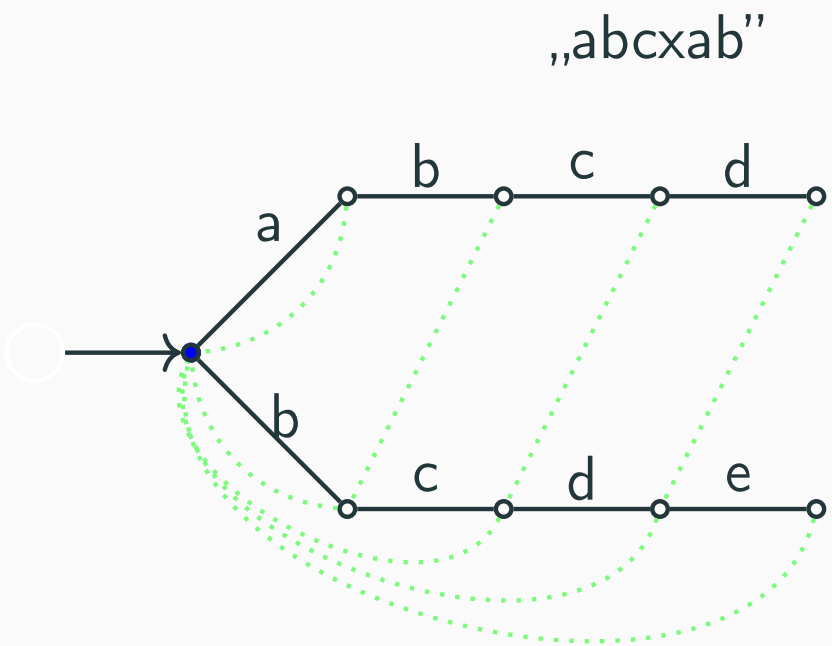

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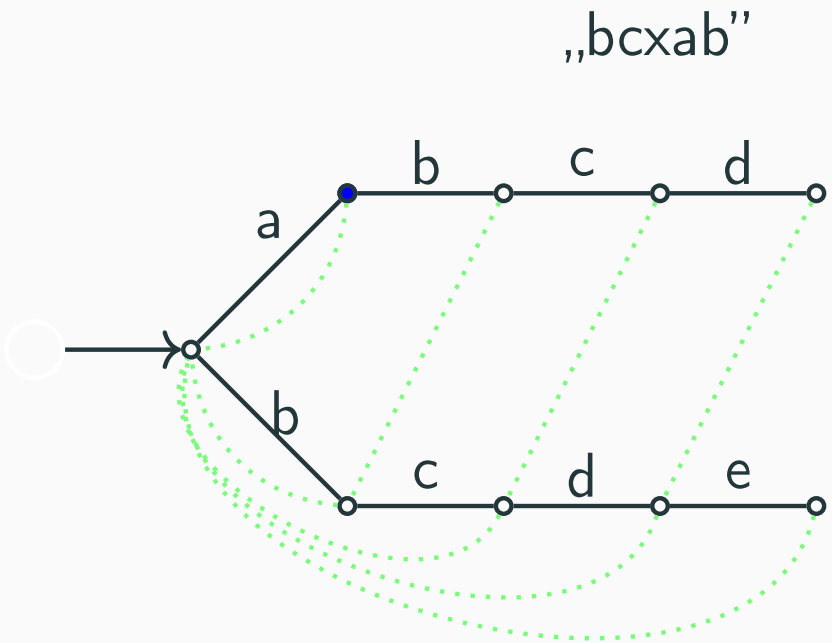

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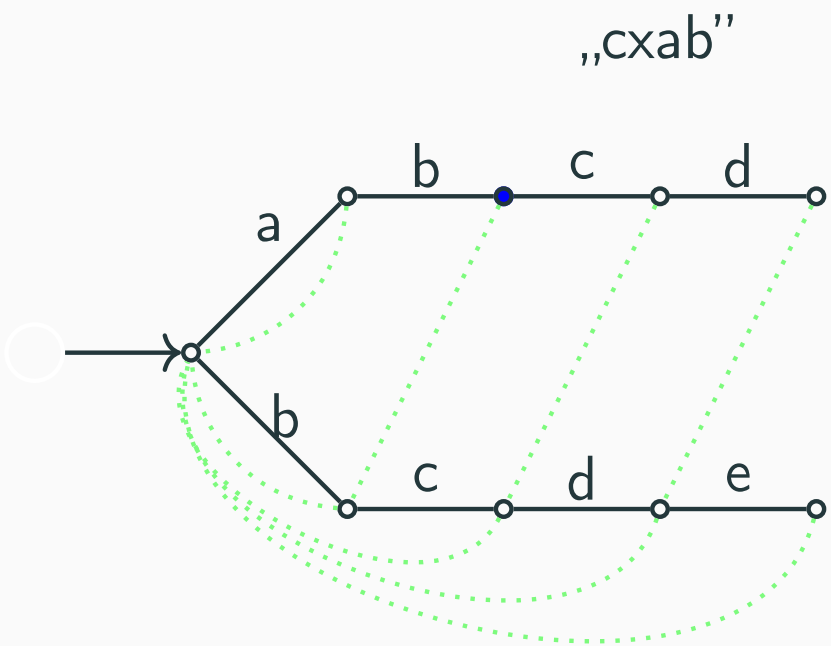

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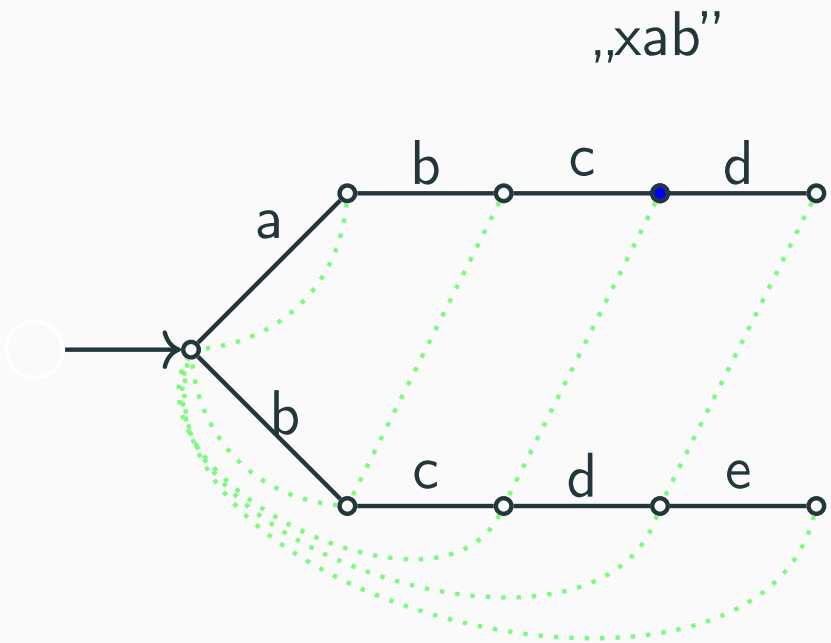

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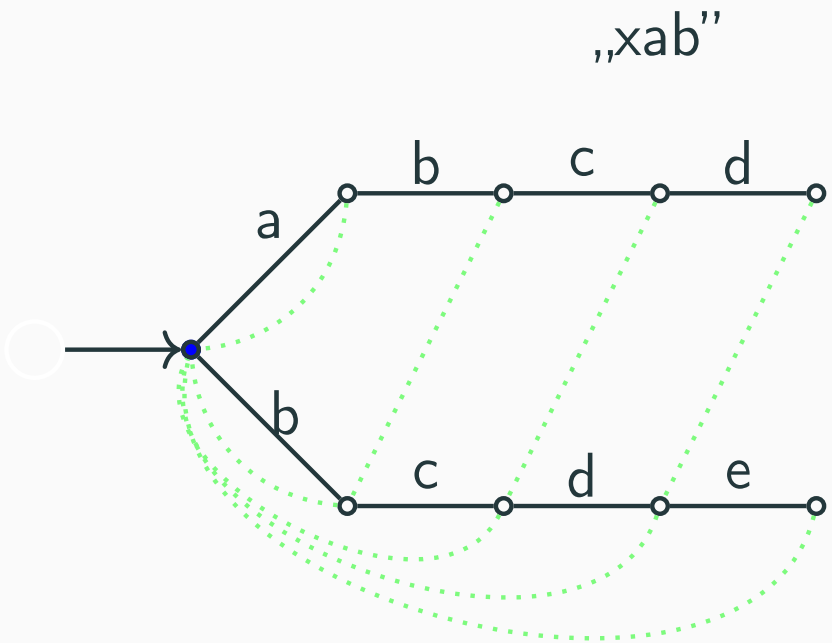

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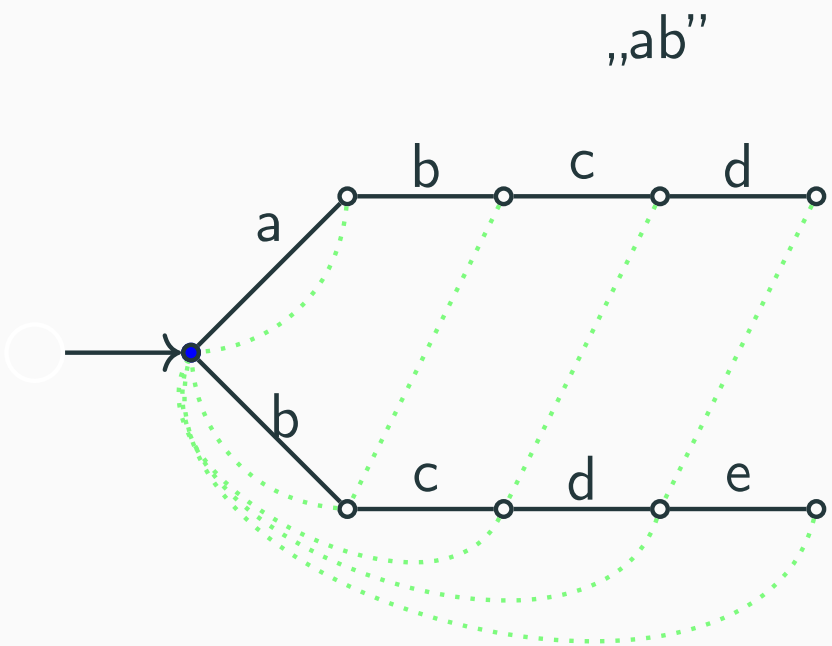

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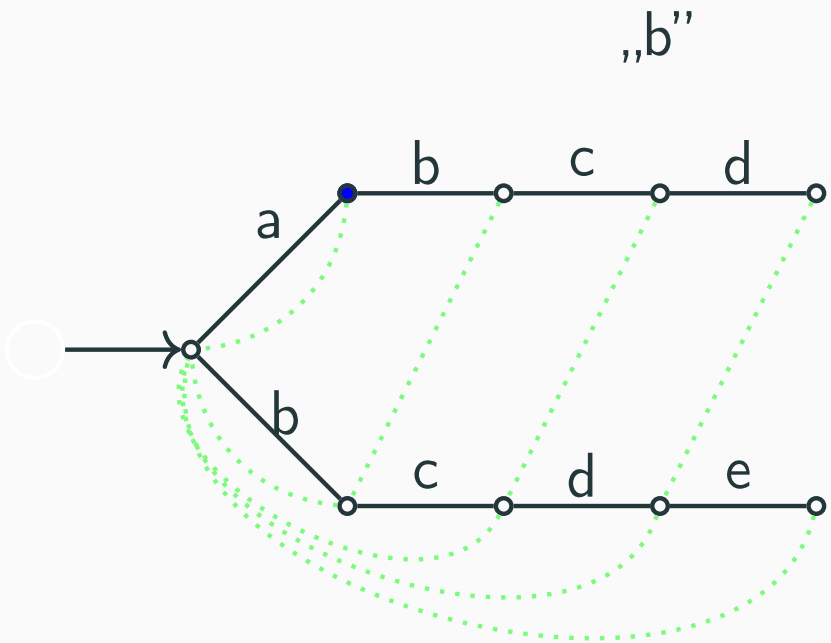

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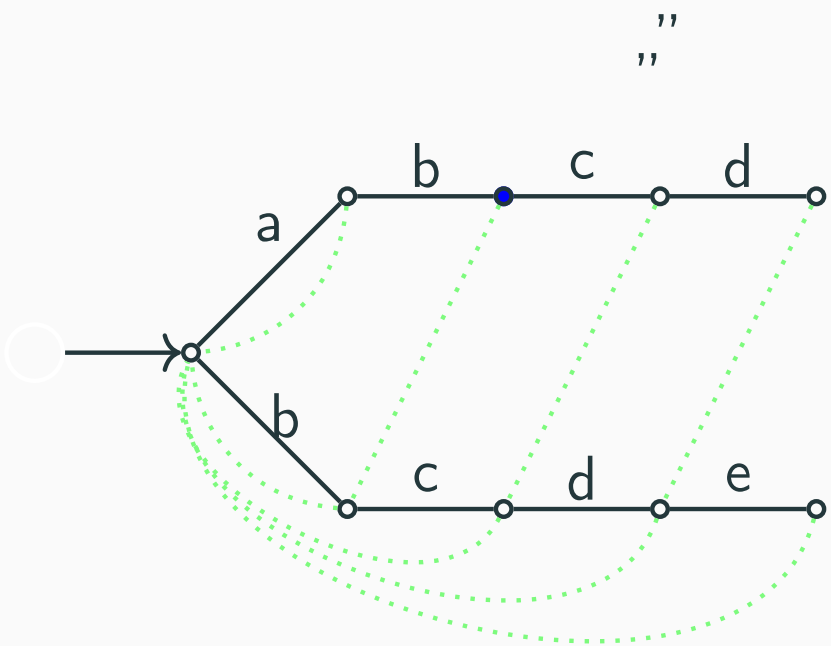

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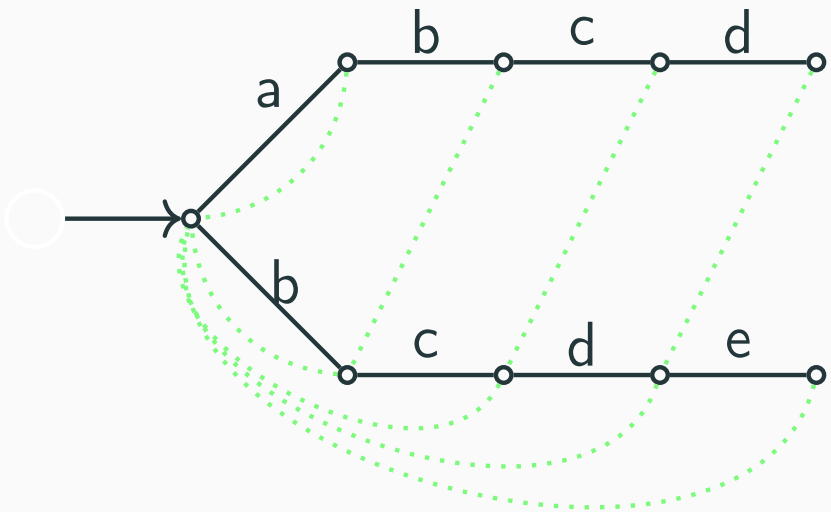

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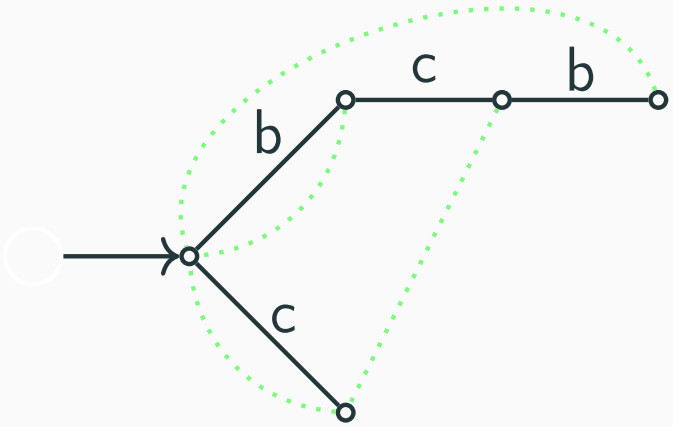

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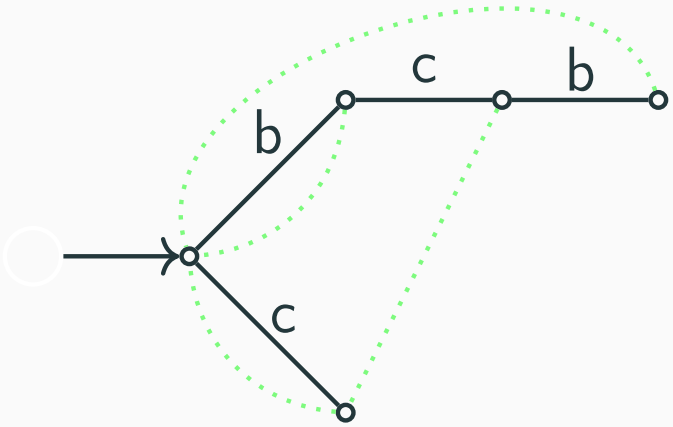

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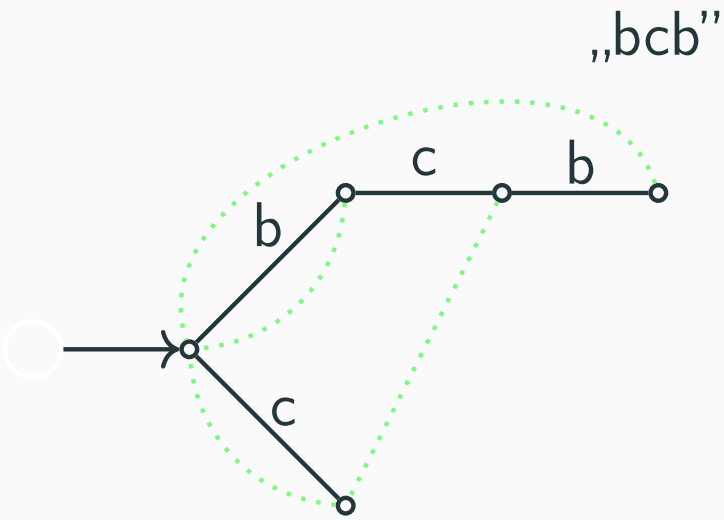

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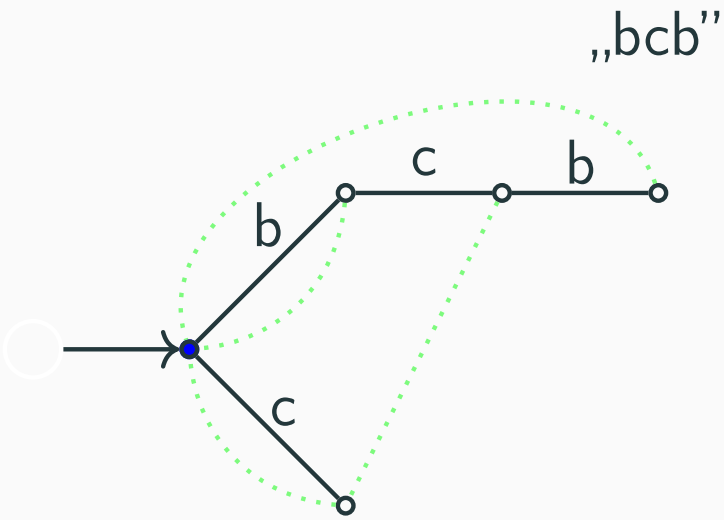

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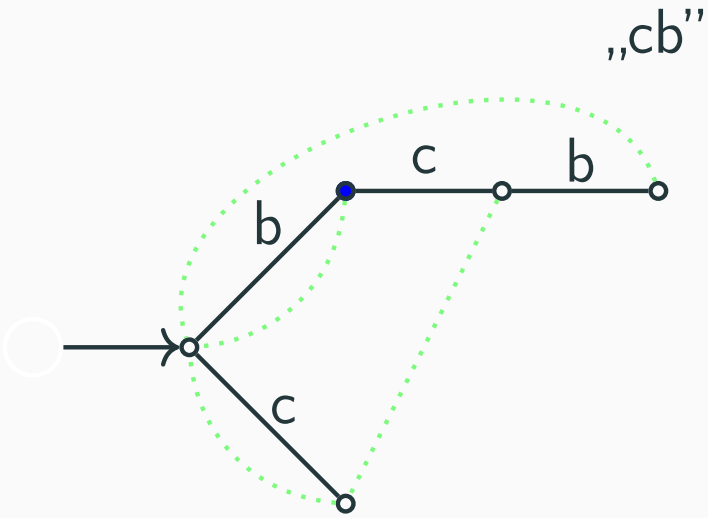

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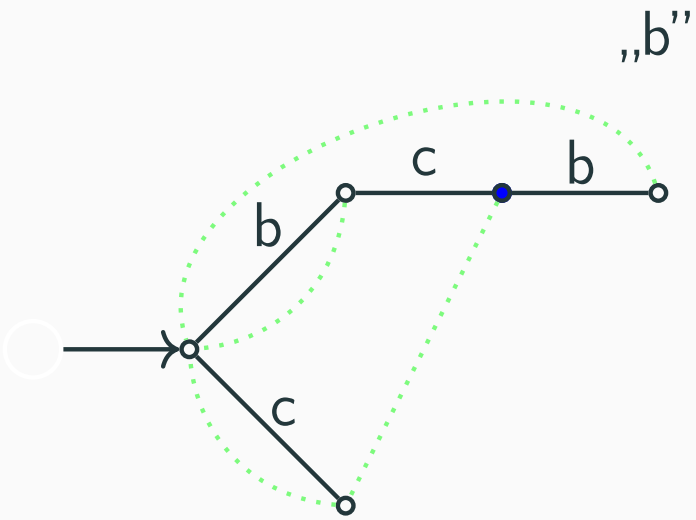

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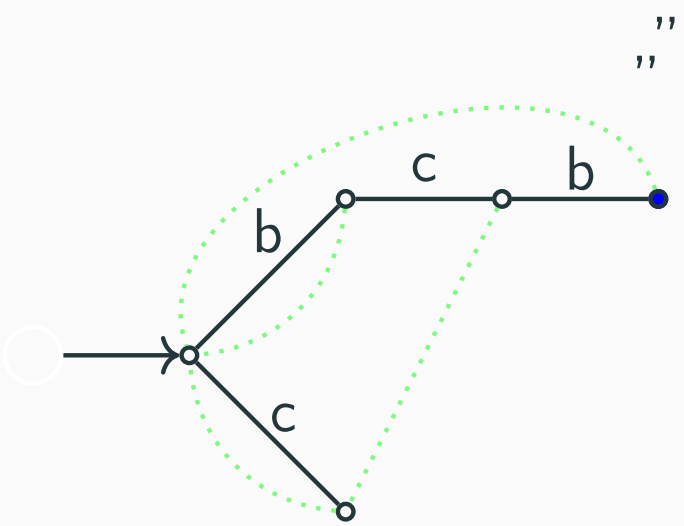

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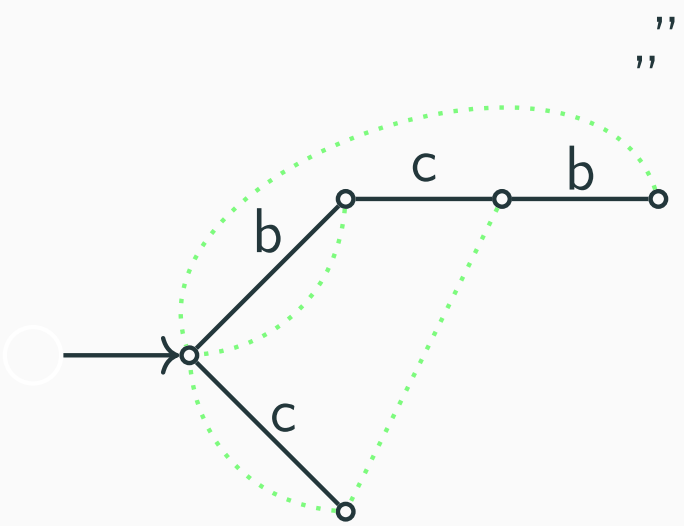

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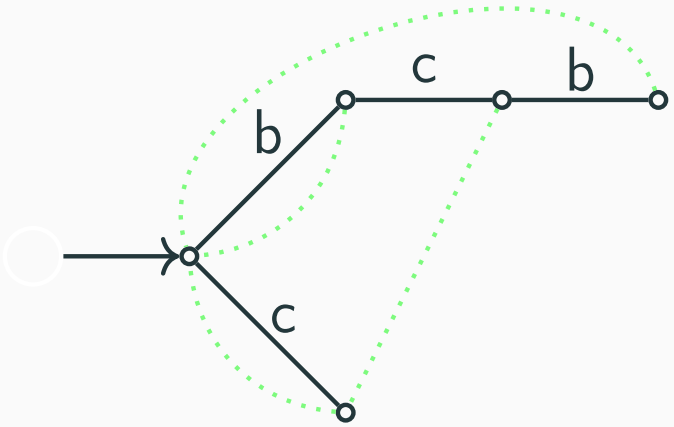


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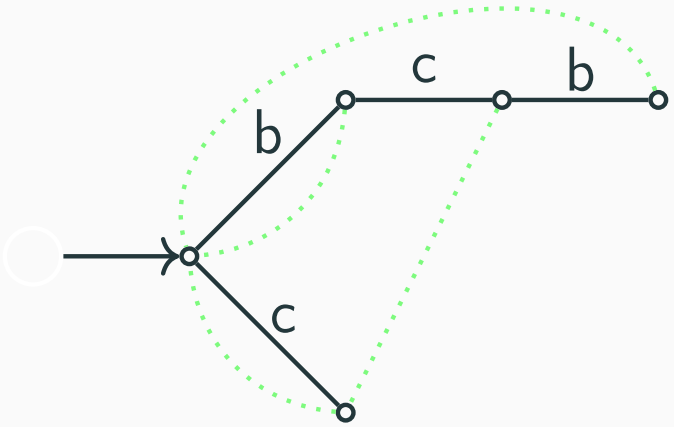


- To keep the complexity in check we again use dynamic programming.
- We add the concatenated links into the tree, calling them *exit links*.

# Speed

- Let us assume the strings in $p$ appear $k$ times in $s$.
- Then the time complexity is $\mathcal{O}(|s| + |\Sigma| \cdot |p| + k)$
- If we only want the number of matches, the implementation can be modified accordingly and then the complexity is $\mathcal{O}(|s| + |\Sigma| \cdot |p|)$.
- Note that for a bounded alphabet, this second complexity is linear.

# Implementation explanation

- The implementation contains three helper functions.
- The first is `trie_step(...)` which is used to move around the state machine.
- The second is `trie_suffix(...)` which is used to find suffix links.
- The third is `trie_exit(...)` which is used to find exit links.
- All these functions are recursive and memoized.

# Aho nodes

```
constexpr int ALPHABET{ 128 };
// Helper function to get index of letter
constexpr int val(char c) { return c; }
struct listnode {
    // n is index of next node, v is value of this node
    int v, n;
    listnode(int _v, int _n) : v(_v), n(_n) { }
};
struct trienode {
    // l is the index of the pattern that ends here or -1 if none
    // e is the exit link index, d is the suffix link index
    // p is the parent index
    // c is the character of the incoming edge
    // t is the transition table of the trie node
    int t[ALPHABET], l, e, p, c, d;
    trienode(int _p, int _c) :
        l(-1), e(-1), p(_p), c(_c), d(-1) {
        memset(t, -1, sizeof(t));
    }
};
```

# Aho trie

```
struct trie {
    // r is the index of the root
    int r;
    vector<trienode> m;
    vector<listnode> w;

    trie() {
        m = vector<trienode>();
        w = vector<listnode>();
        r = trie_node(-1, -1);
    }

    int list_node(int v, int n) {
        w.push_back(listnode(v, n));
        return w.size() - 1;
    }
    int trie_node(int p, int c) {
        m.push_back(trienode(p, c));
        return m.size() - 1;
    }

    void trie_insert(string &s, int x) {
        int h, i = 0;
        for(h = r; i < s.size(); h = m[h].t[val(s[i])], i++)
            if(m[h].t[val(s[i])] == -1)
                m[h].t[val(s[i])] = trie_node(h, val(s[i]));
        m[h].l = list_node(x, m[h].l);
    }

    int trie_suffix(int h) {
        if(m[h].d != -1) return m[h].d;
        if(h == r || m[h].p == r) return m[h].d = r;
        return m[h].d =
            trie_step(trie_suffix(m[h].p), m[h].c);
    }

    int trie_step(int h, int c) {
        if(m[h].t[c] != -1) return m[h].t[c];
        return m[h].t[c] = h == r ? r :
            trie_step(trie_suffix(h), c);
    }

    int trie_exit(int h) {
        if(m[h].e != -1) return m[h].e;
        if(h == 0 || m[h].l != -1) return m[h].e = h;
        return m[h].e = trie_exit(trie_suffix(h));
    }
};
```

# Aho implementation

```
int aho_corasick(string &s, vector<string> &p) {
    trie t; int h, i, j, k, w, m = p.size(), l[m];
    for(i = 0; i < m; i++) l[i] = p[i].size();
    for(i = 0; i < m; i++) t.trie_insert(p[i], i);
    s.push_back('\0');
    for(i = 0, j = 0, h = t.r; j < s.size(); j++) {
        k = t.trie_exit(h);
        while(t.m[k].l != -1) {
            for(w = t.m[k].l; w != -1; w = t.w[w].n) {
                cout << p[t.w[w].v] << " found at index " <<
                    j - l[t.w[w].v] << '\n';
            }
            k = t.trie_exit(t.trie_suffix(k));
        }
        h = t.trie_step(h, val(s[j]));
    }
    return i;
}
```