Kattis

Weighted Graphs

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Today we're going to cover

- Minimum spanning tree
- Shortest paths
  - Single source
  - Negative weights
  - All pairs

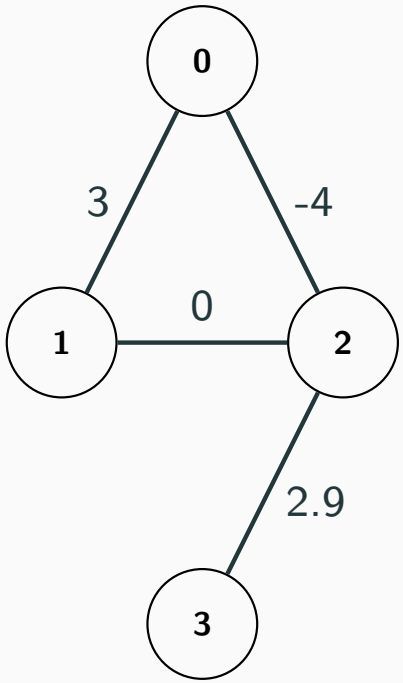
Weighted graphs

- Now the edges in our graphs may have weights, which could represent
  - the distance of the road represented by the edge
  - the cost of going over the edge
  - some capacity of the edge
- We can use a modified adjacency list to represent weighted graphs

Weighted graphs

```
struct edge {
    // Depending on context,
    // having both endpoints
    // can be overkill
    int u, v;
    int weight;
    // Sometimes weight has to
    // be a double, or something else

    edge(int _u, int _v, int _w) {
        u = _u;
        v = _v;
        weight = _w;
    }
};
```



Weighted graphs

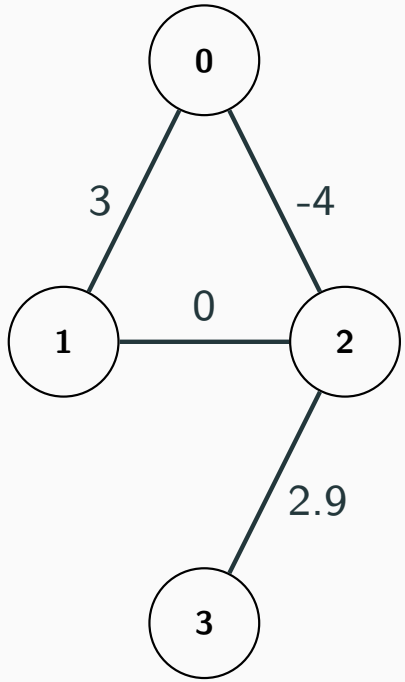
```
vector<edge> adj[4];

adj[0].push_back(edge(0, 1, 3));
adj[0].push_back(edge(0, 2, -4));

adj[1].push_back(edge(1, 0, 3));
adj[1].push_back(edge(1, 2, 0));

adj[2].push_back(edge(2, 0, -4));
adj[2].push_back(edge(2, 1, 0));
adj[2].push_back(edge(2, 3, 2.9));

adj[3].push_back(edge(3, 2, 2.9));
```



Minimum spanning trees

Minimum spanning tree

- We have an undirected weighted graph
- The vertices along with a subset of the edges in the graph is called a spanning tree if
  - it forms a tree (i.e. does not contain a cycle) and
  - the tree spans all vertices (all vertices can reach all other vertices)
- The weight of a spanning tree is the sum of the weights of the edges in the subset
- We want to find a minimum spanning tree

Minimum spanning tree

- This can model many things, the cheapest way to connect a set of objects is one of them
- For some graph problems we can remove everything but a spanning tree to simplify calculations
- Note however that it does not have to be unique, but the minimum weight is unique

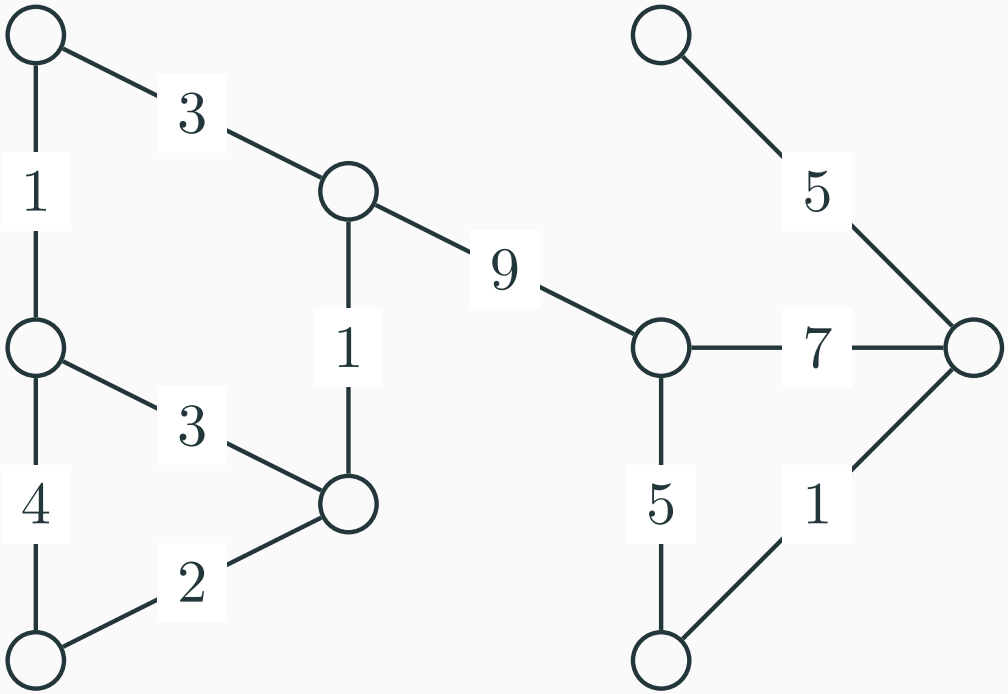
Minimum spanning tree

- Several greedy algorithms work

- Go through the edges in the graph in increasing order of weight
- Greedily pick an edge if it doesn't form a cycle (Union-Find can be used to keep track of when we would get a cycle)
- When we've gone through all edges, we have a minimum spanning tree

- This is Kruskal's algorithm
- Time complexity is $O(E \log E)$

Example



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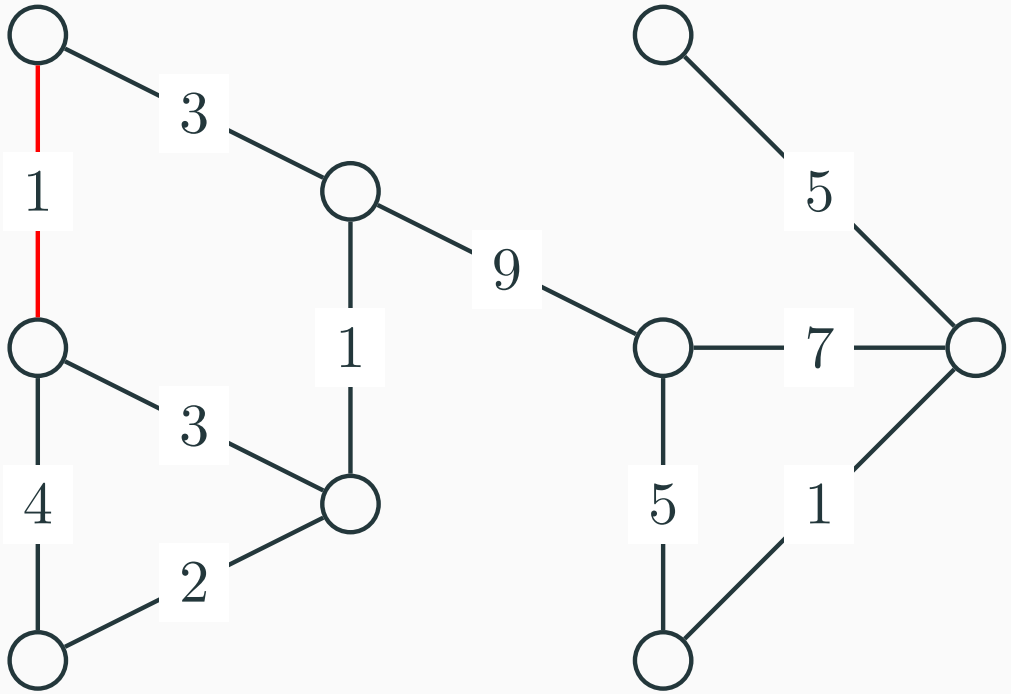

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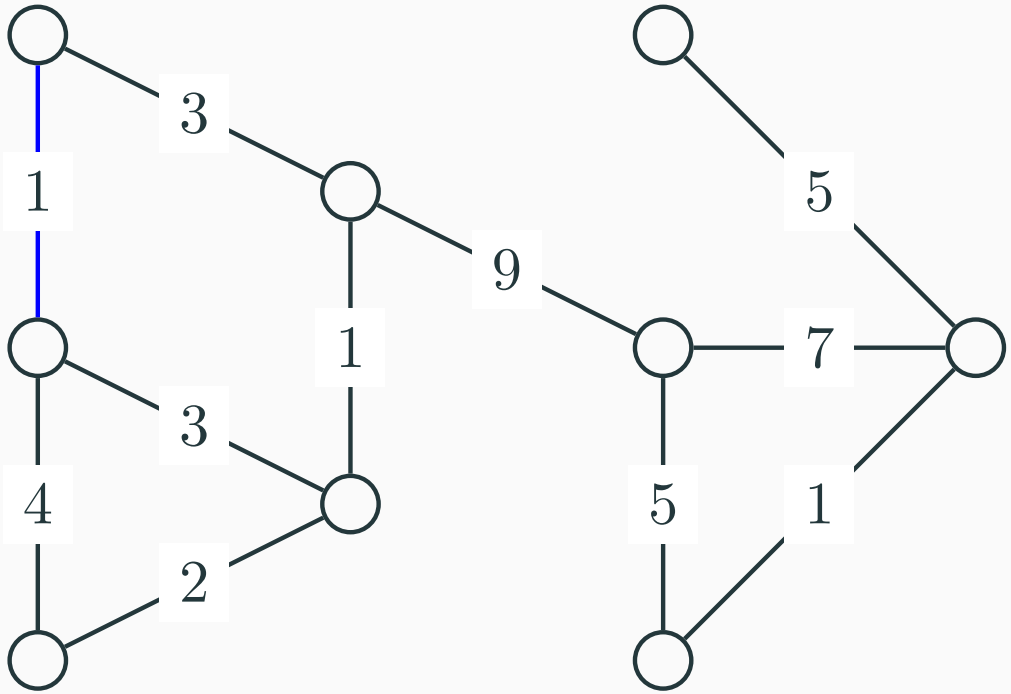

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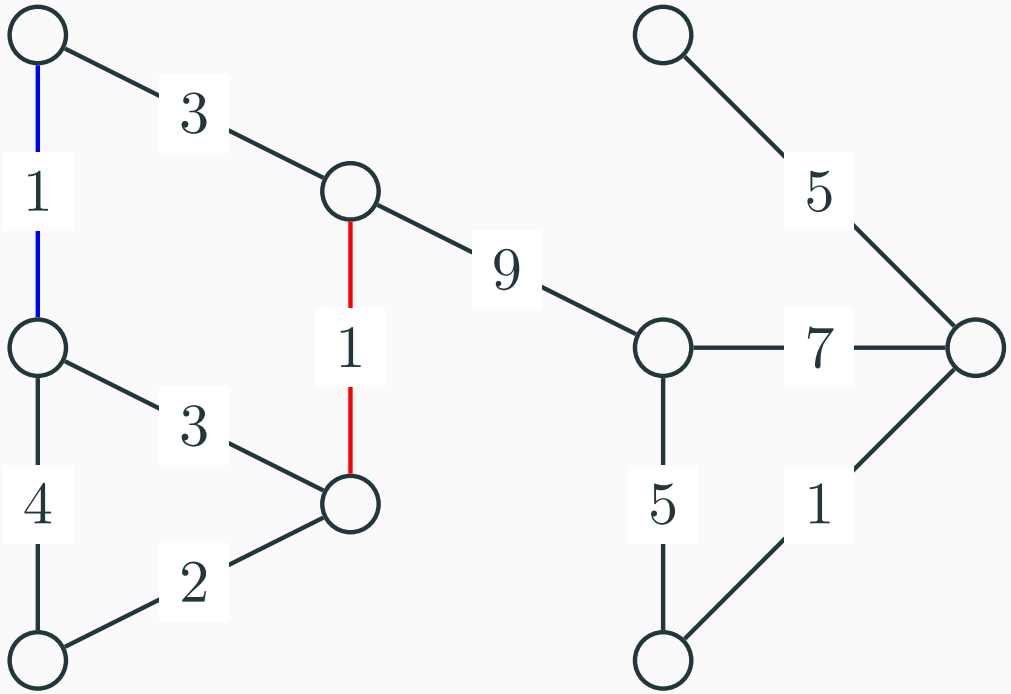

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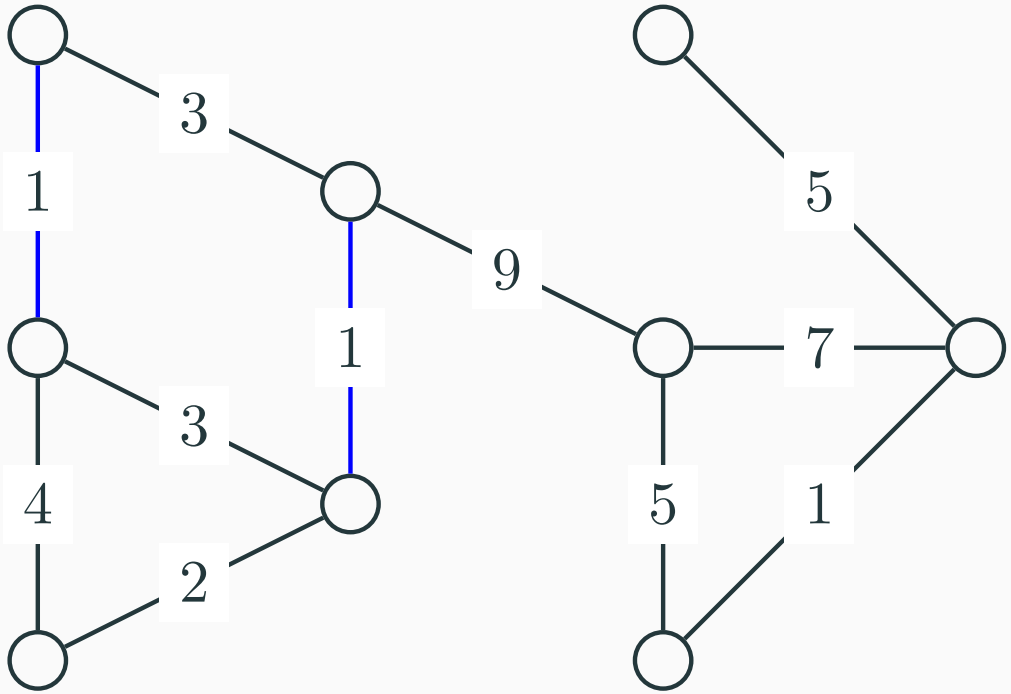

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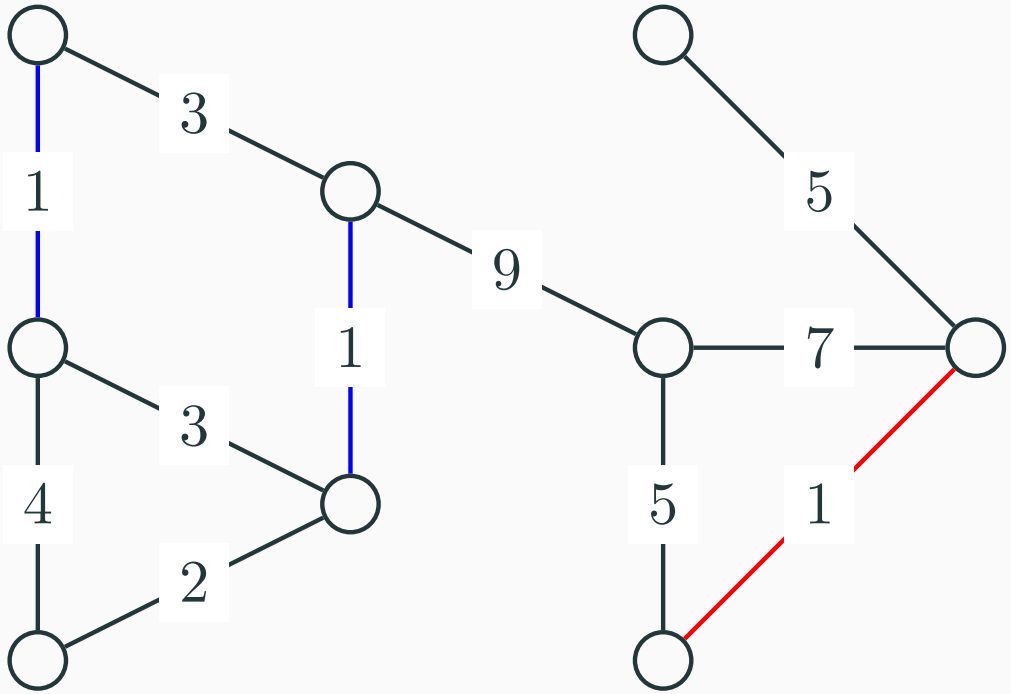

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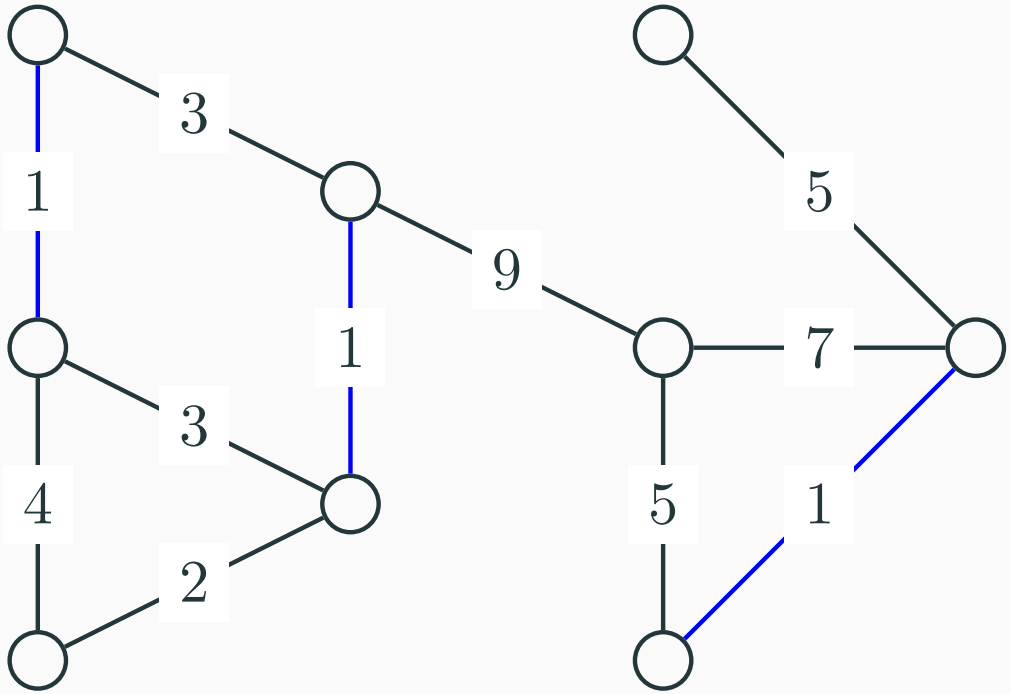

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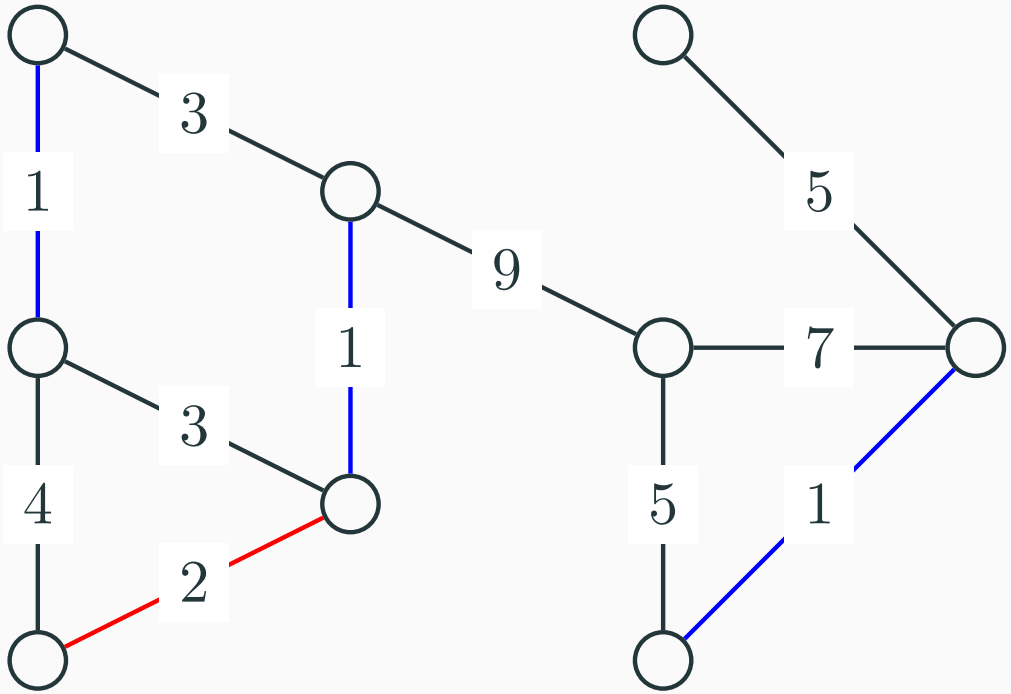

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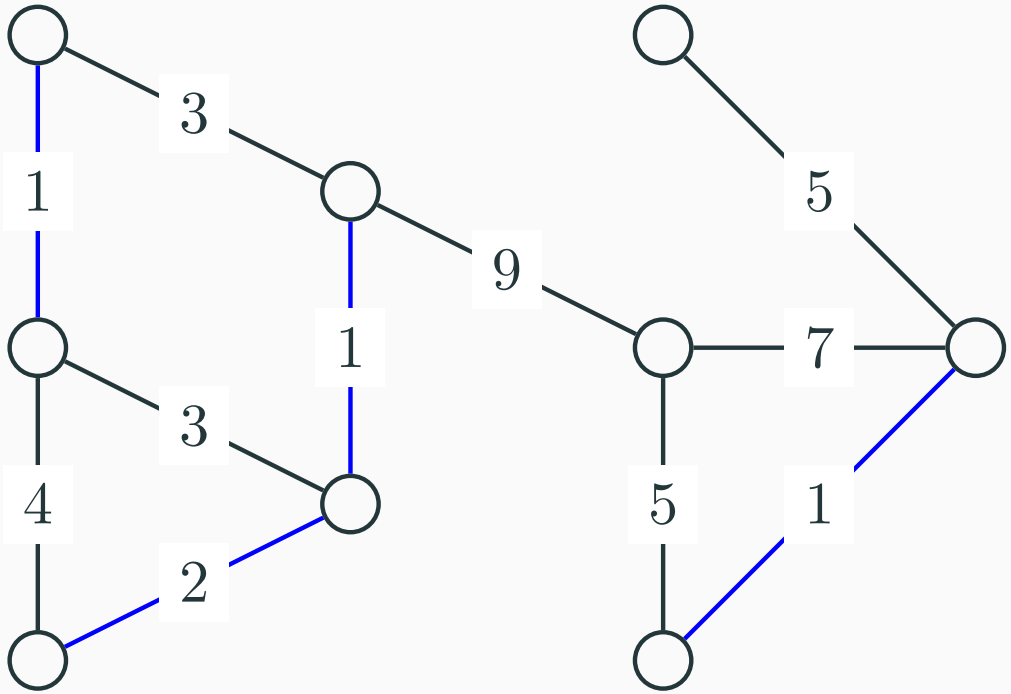

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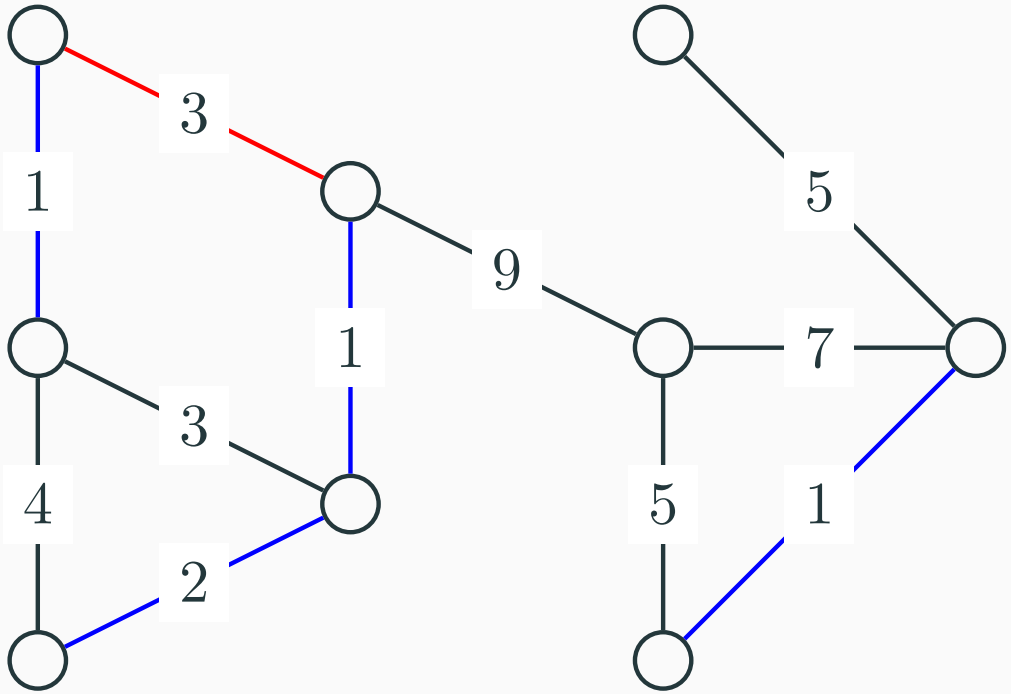

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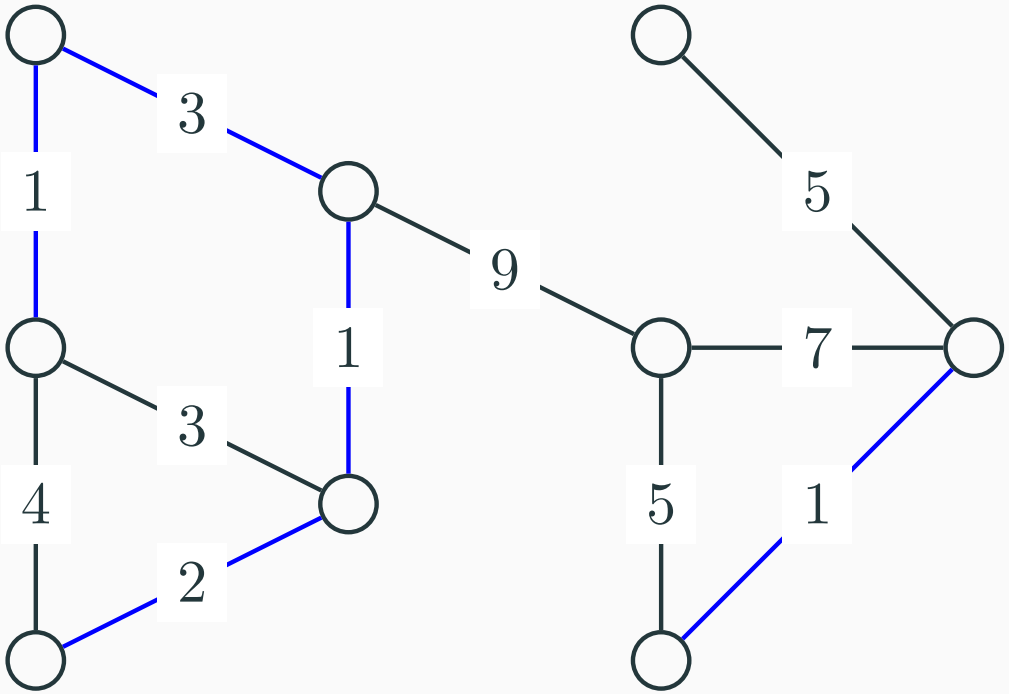

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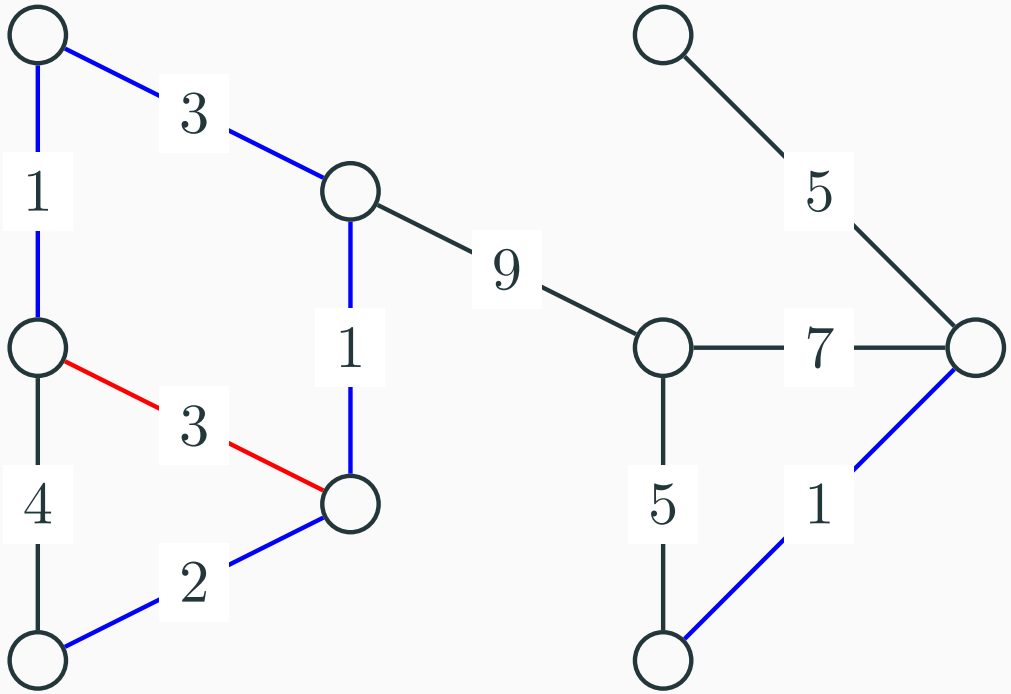

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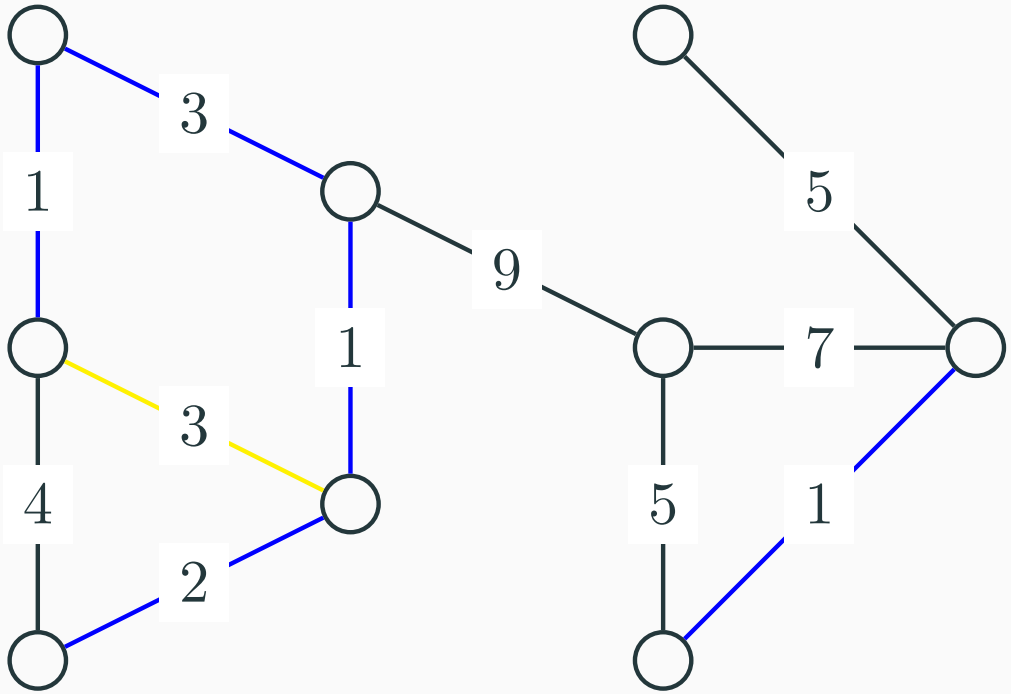

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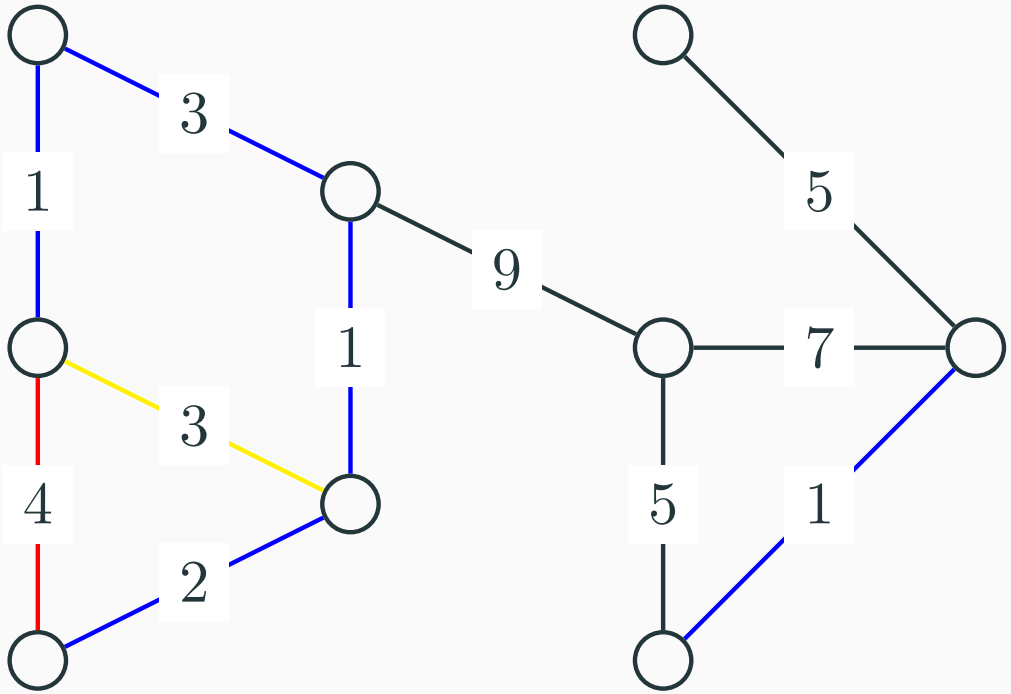

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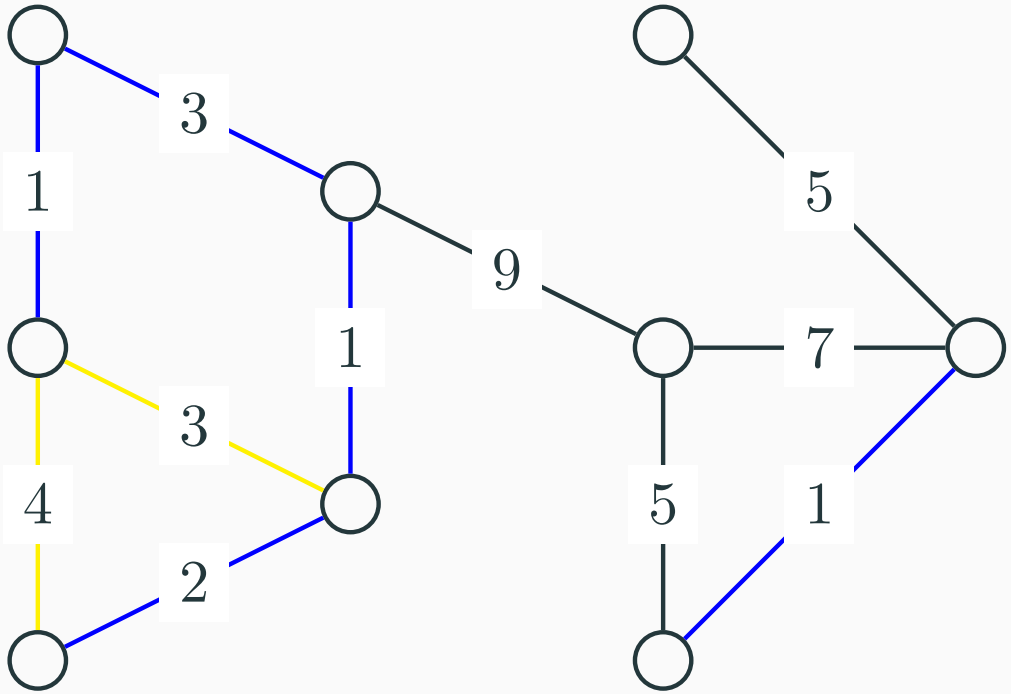

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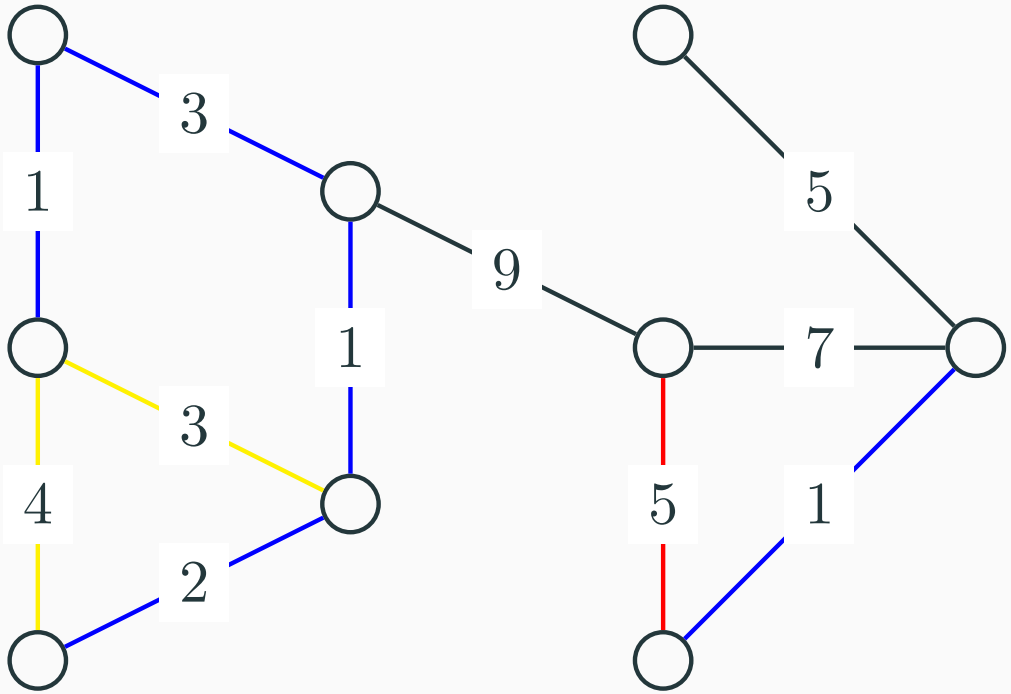

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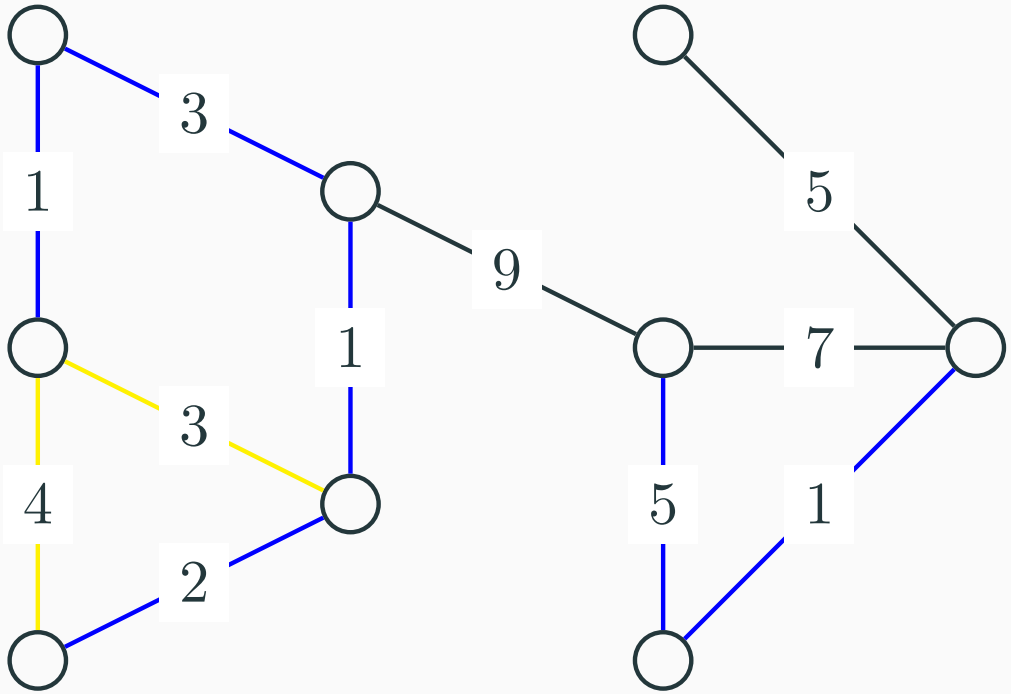

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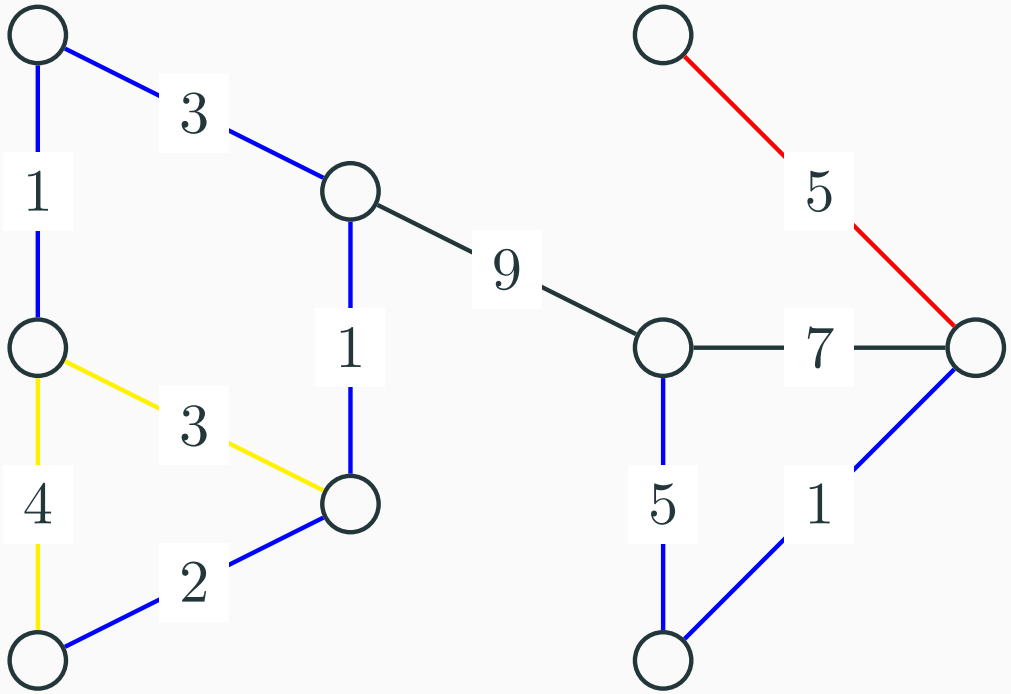

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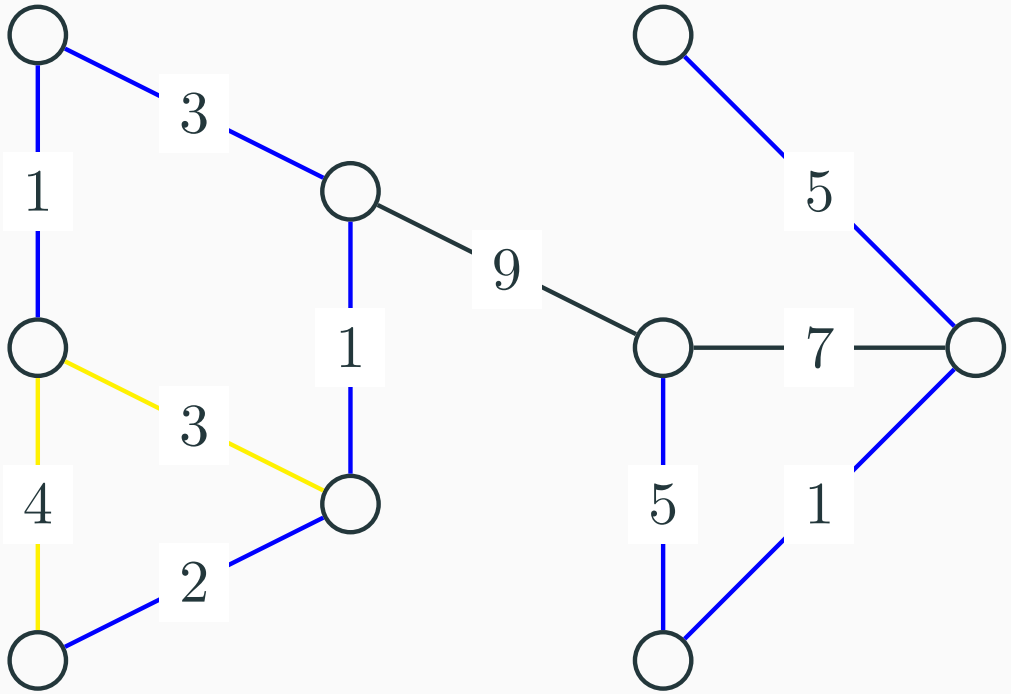

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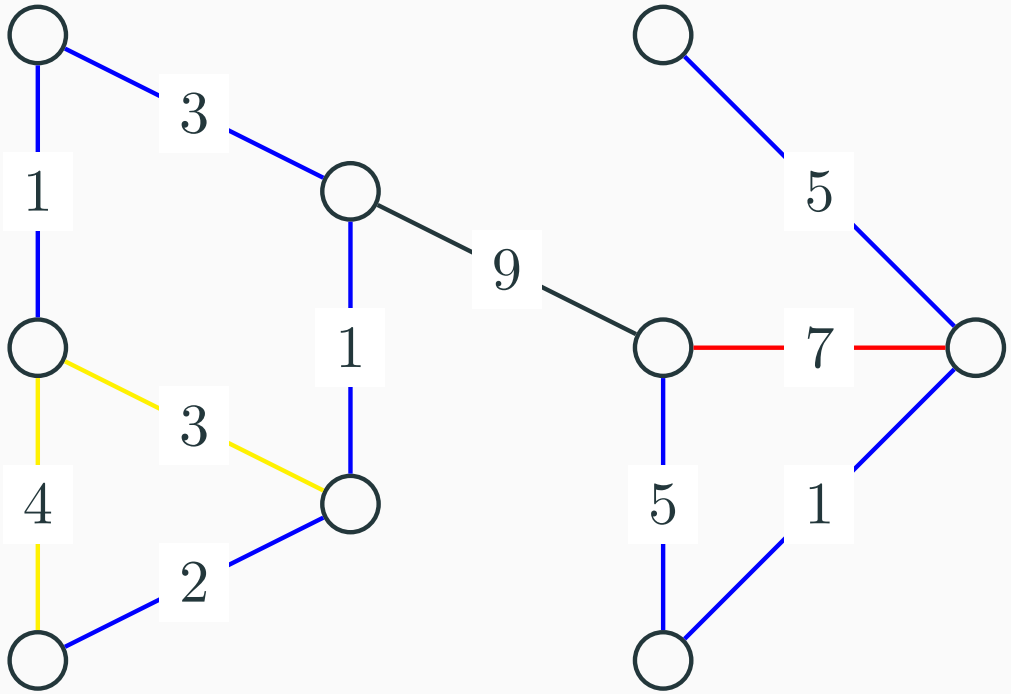

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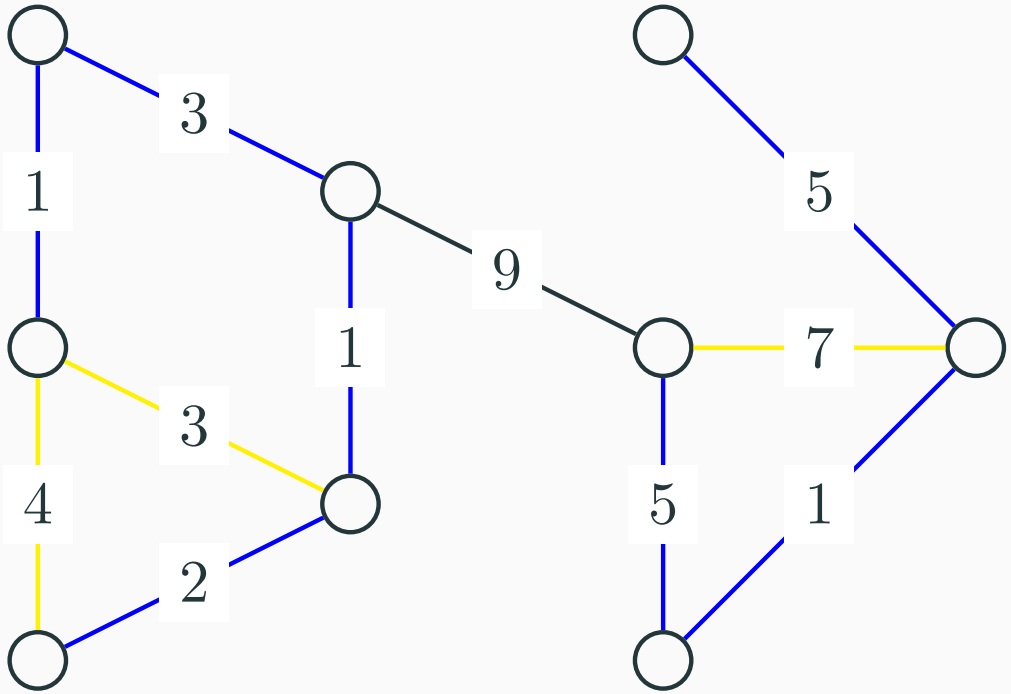

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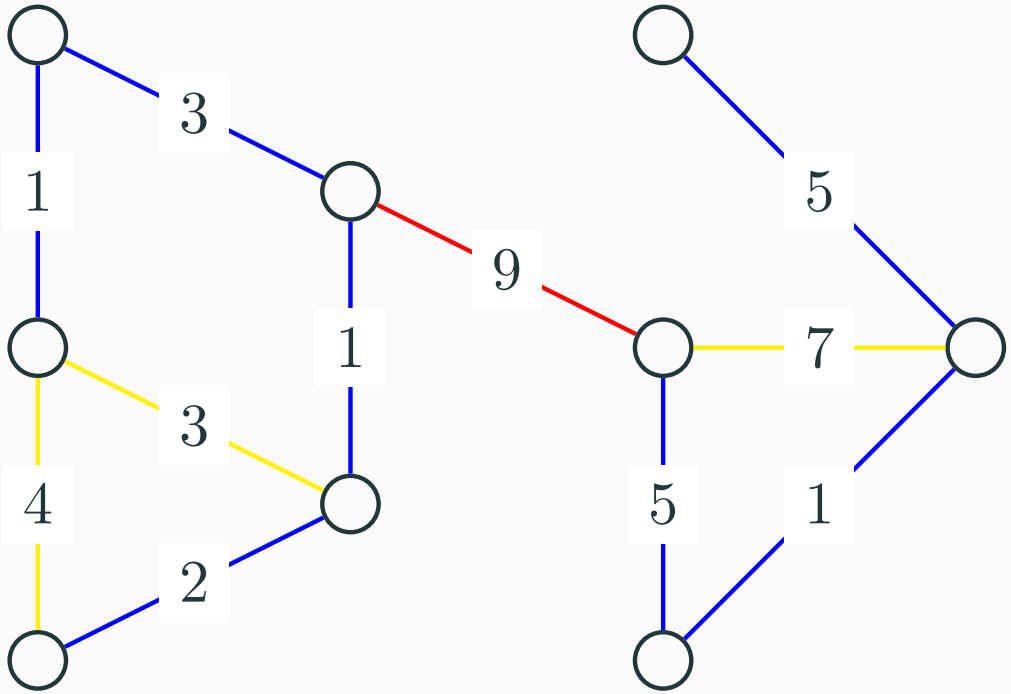

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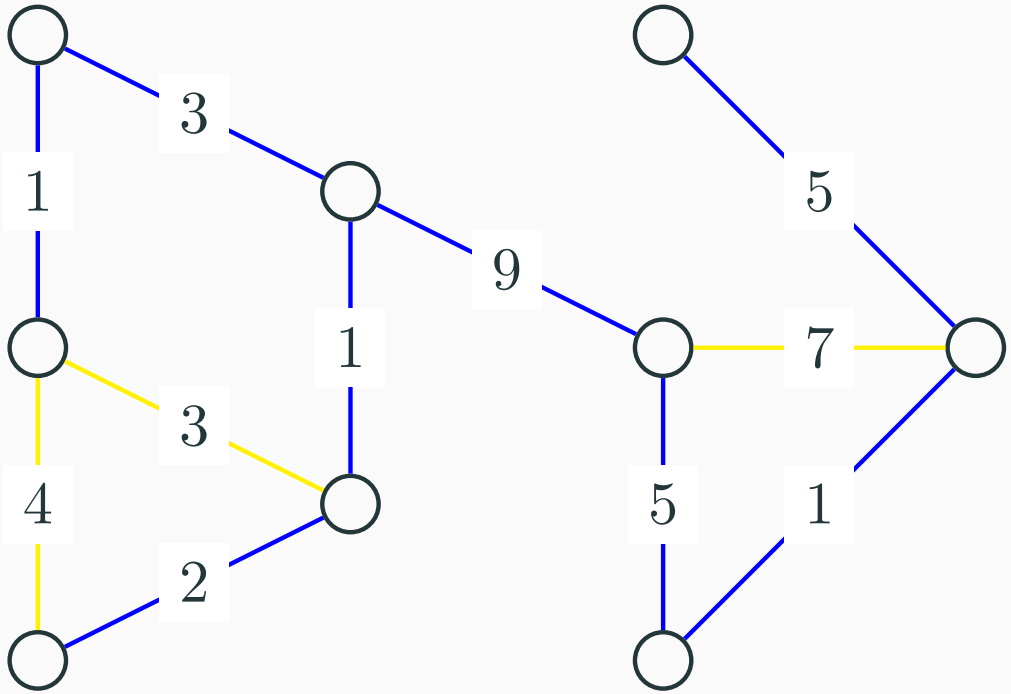

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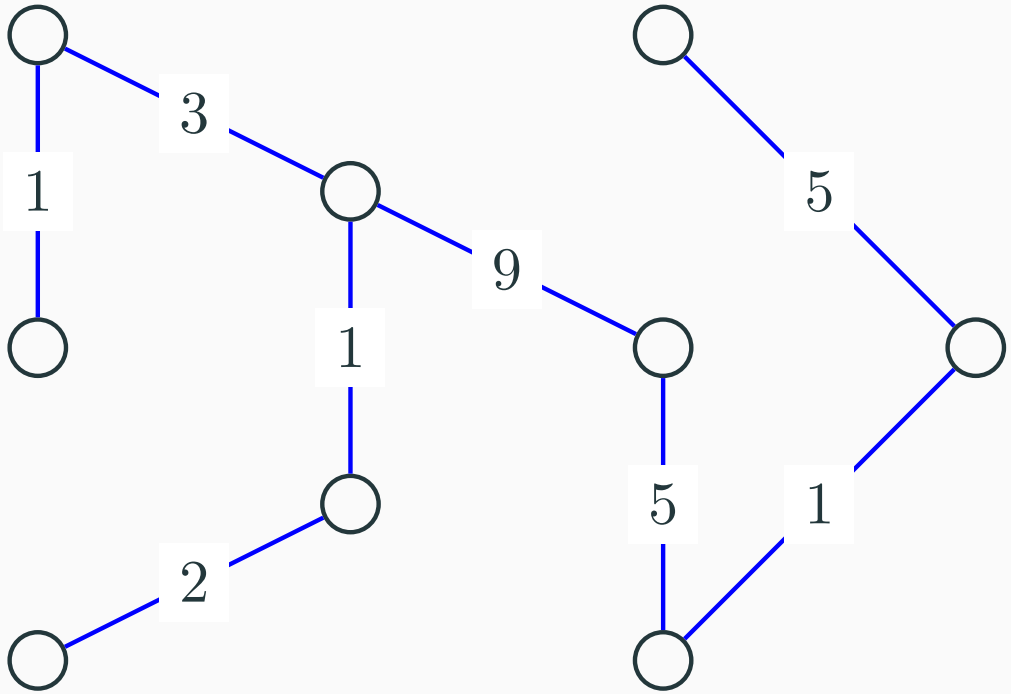


Minimum spanning tree

```
bool edge_cmp(const edge &a, const edge &b) {
    return a.weight < b.weight;
}

vector<edge> mst(int n, vector<edge> edges) {
    union_find uf(n);
    sort(edges.begin(), edges.end(), edge_cmp);

    vector<edge> res;
    for (int i = 0; i < edges.size(); i++) {
        int u = edges[i].u,
            v = edges[i].v;

        if (uf.find(u) != uf.find(v)) {
            uf.unite(u, v);
            res.push_back(edges[i]);
        }
    }

    return res;
}
```

Minimum spanning tree

- Another one that works is Prim's algorithm
- We start with vertex 0, then grow the tree one vertex at a time
- Each time we take the cheapest edge that connects our tree to something not in the tree
- This can also be done in $\mathcal{O}(E \log(E))$
- For our purposes Kruskal will suffice

Single source shortest path

Shortest paths

- We have a weighted graph (undirected or directed)
- Given two vertices u, v, what is the shortest path from u to v?
- If all weights are the same (or 0/1), this can be solved with breadth-first search
- Of course, this is usually not the case...

Shortest paths

- There are many known algorithms to find shortest paths
- Like breadth-first search, these algorithms usually find the shortest paths from a given start vertex to all other vertices
- We will look at Dijkstra's algorithm, the Bellman-Ford algorithm and the Floyd-Warshall algorithm

Shortest paths

- We will start with the simplest case, where no weights are negative and we want the distance from a particular vertex to the other ones
- Why no negative weights? Well if there is a cycle with negative weight on the way to our destination there is no shortest path! You can always go a "shorter" way by cycling around the negative cycle an extra time
- In principle Dijkstra is not so dissimilar from BFS

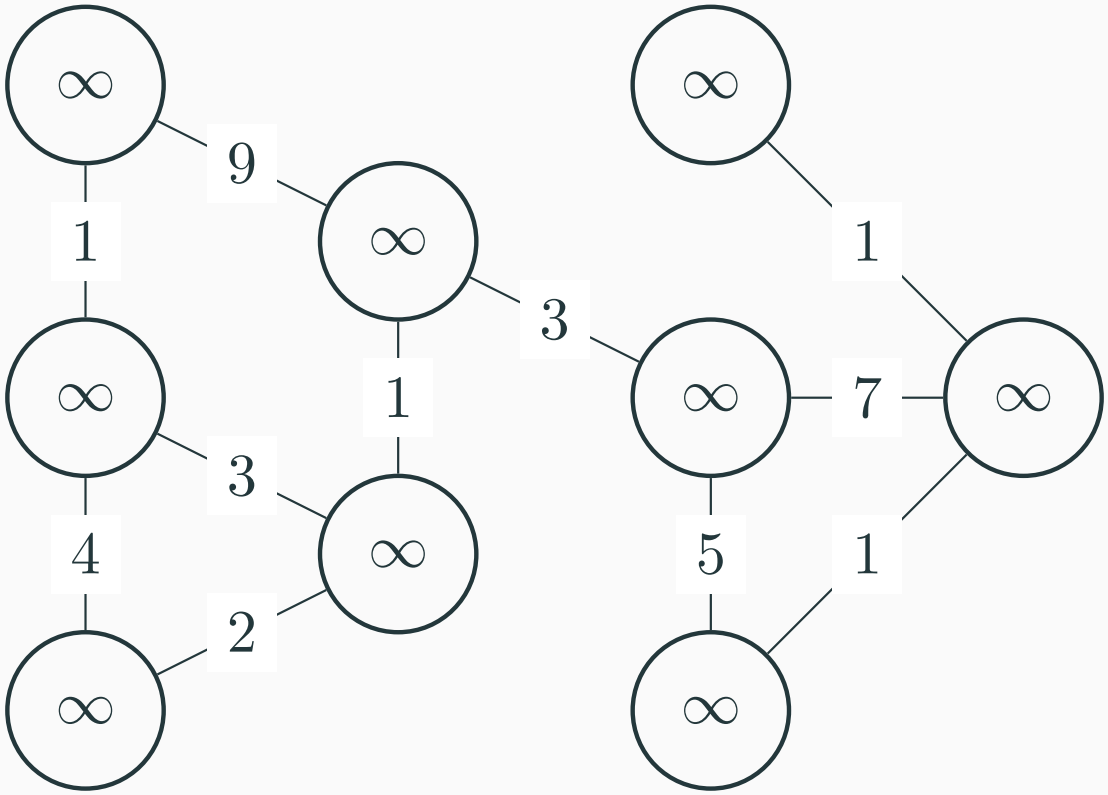
Shortest paths

- In BFS the next item in our queue was always the closest one to the source vertex we had not processed yet
- In Dijkstra's algorithm we have to do some extra work to ensure this is the case
- We use a min-heap, so that the next vertex we pick is always the closest unprocessed one

Shortest paths

- We initialize the source vertex to distance 0, others to ∞
- We then at any point take the closest vertex v that we haven't processed
- We then check all edges adjacent to v and update the distance to the neighbours
- We then mark v as done and don't check it again
- Note that if all weights are one this will just do the same things as BFS

Example



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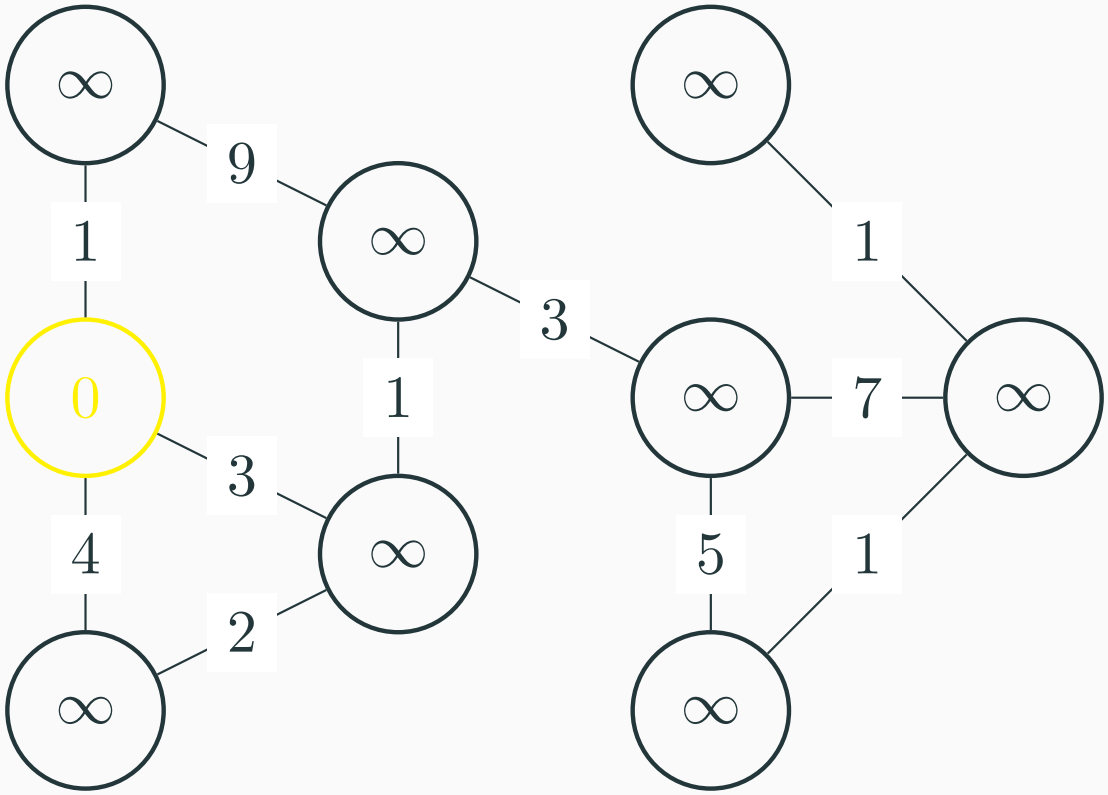

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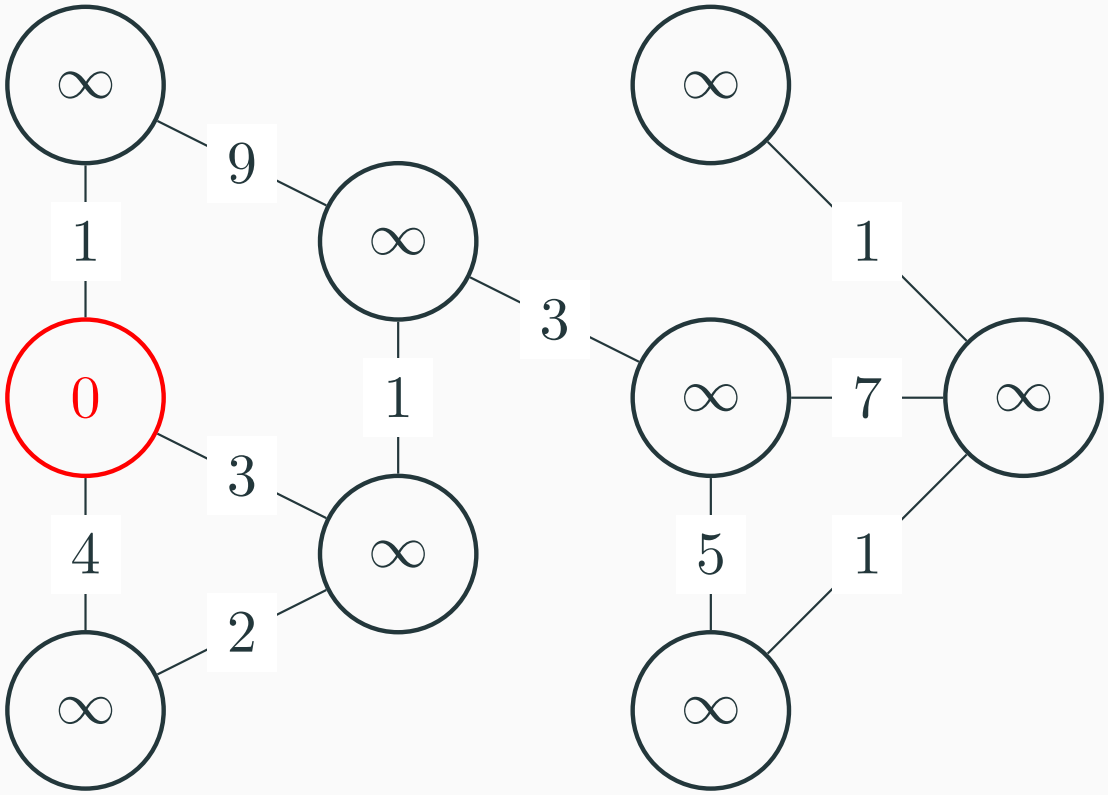

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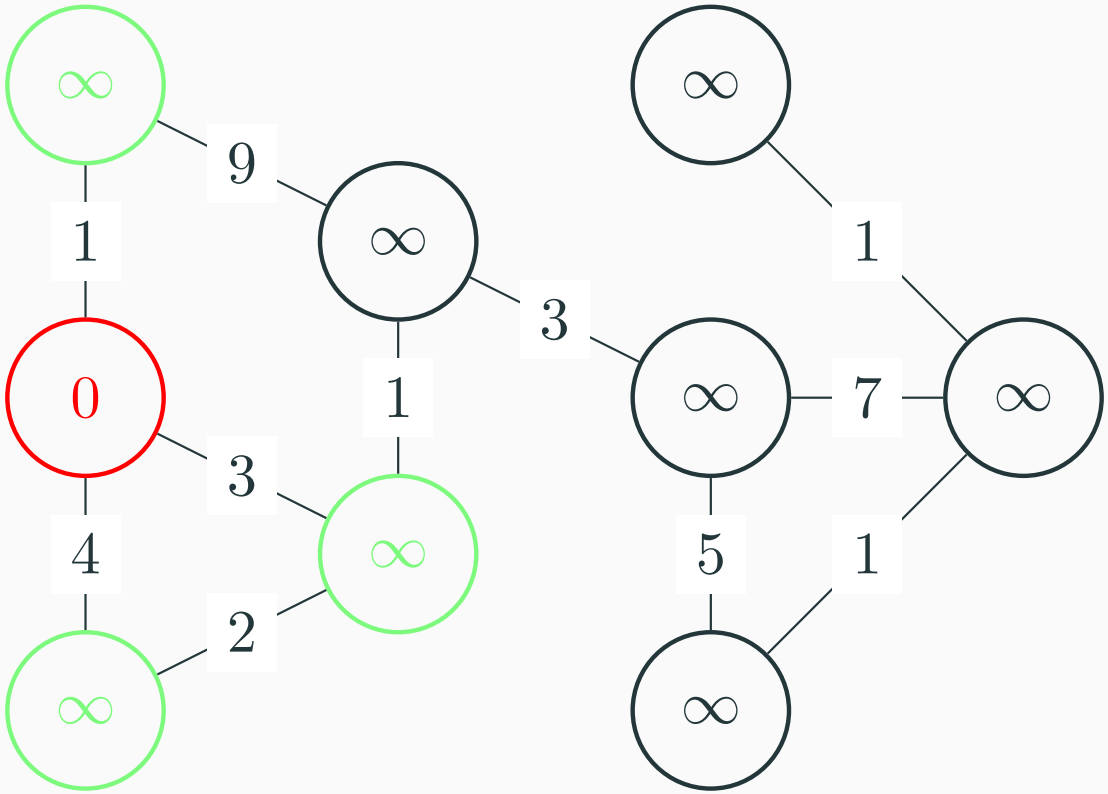

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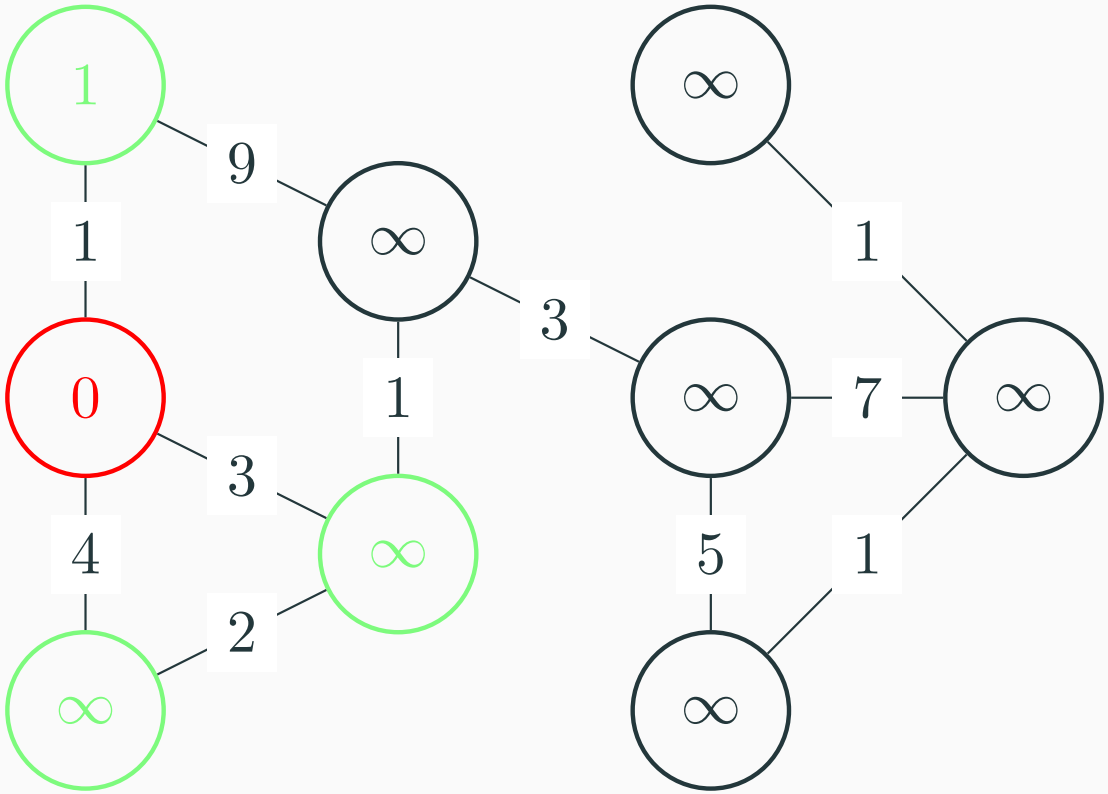

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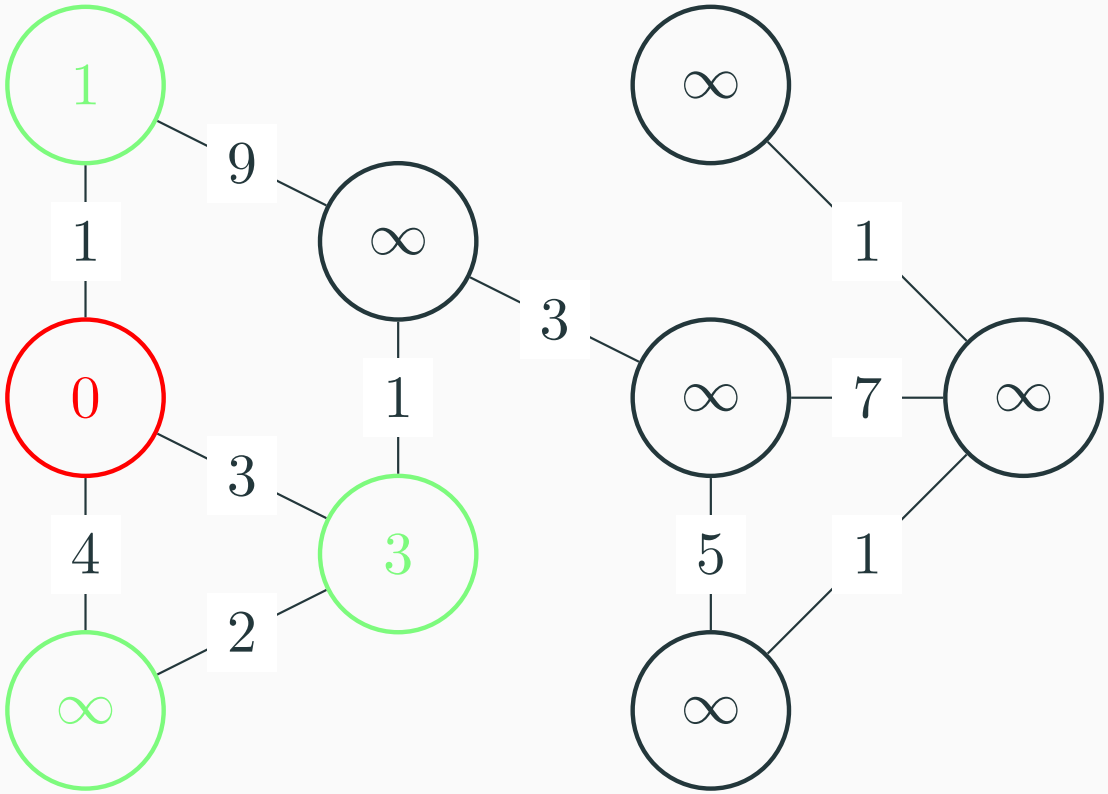

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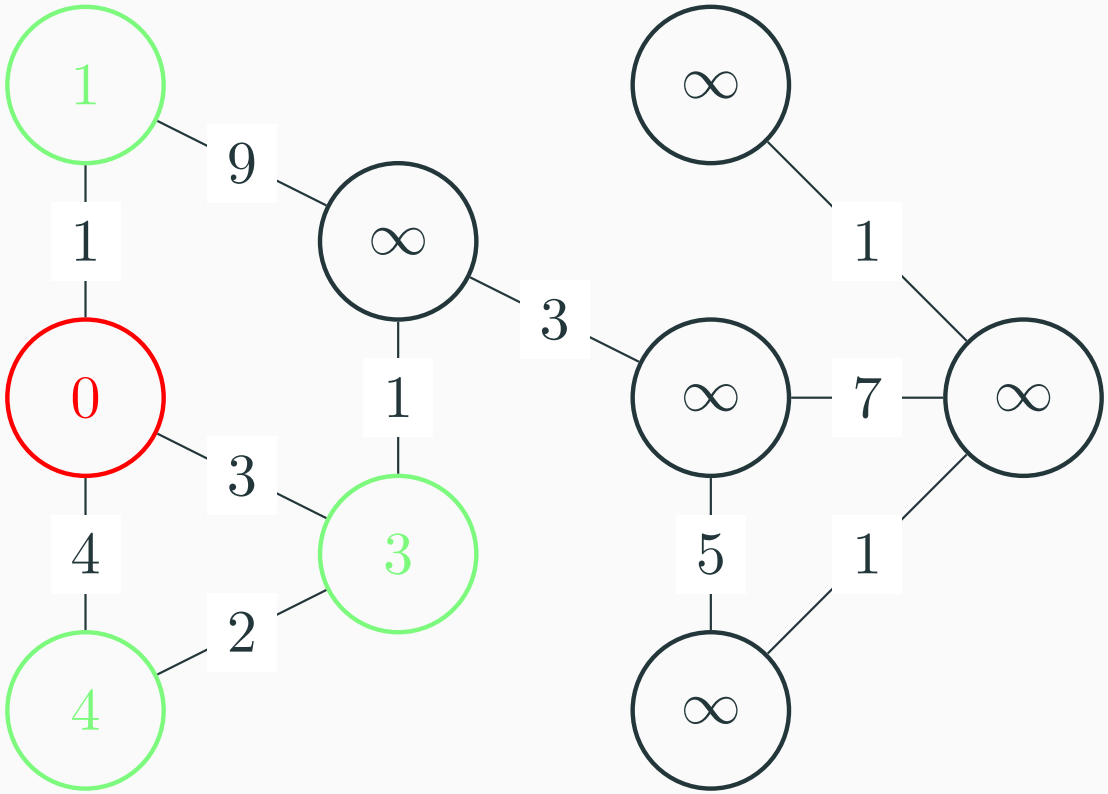

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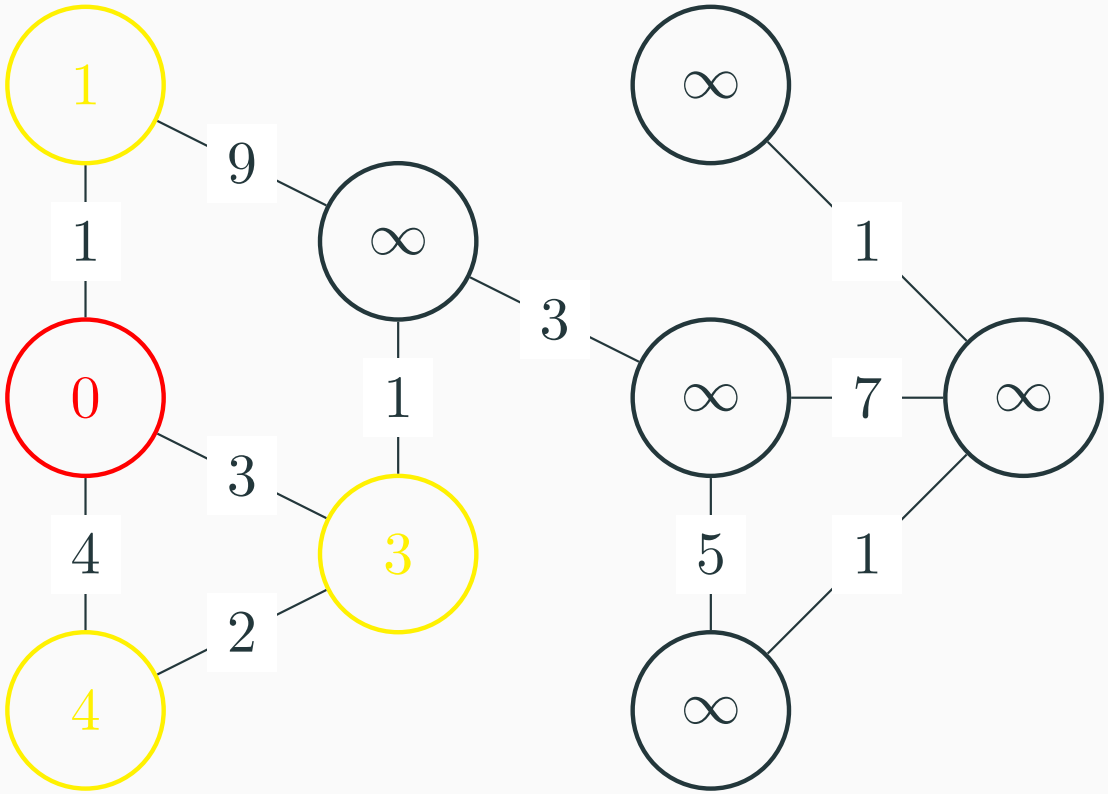

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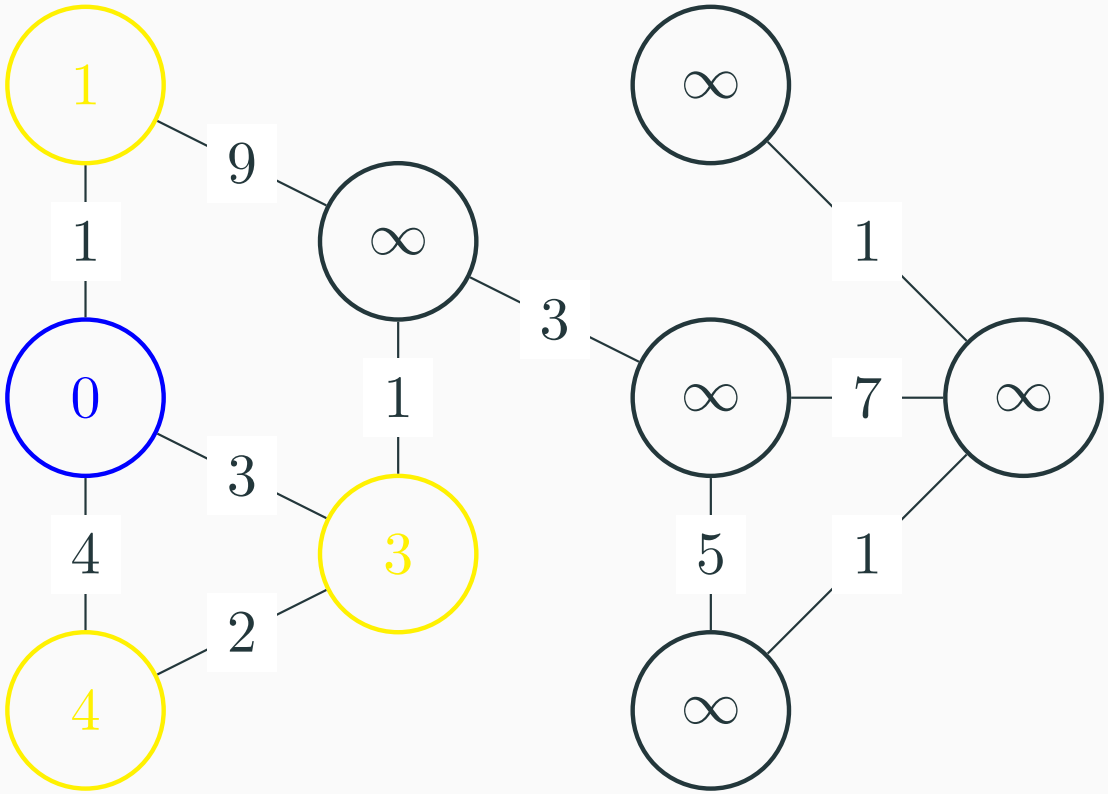

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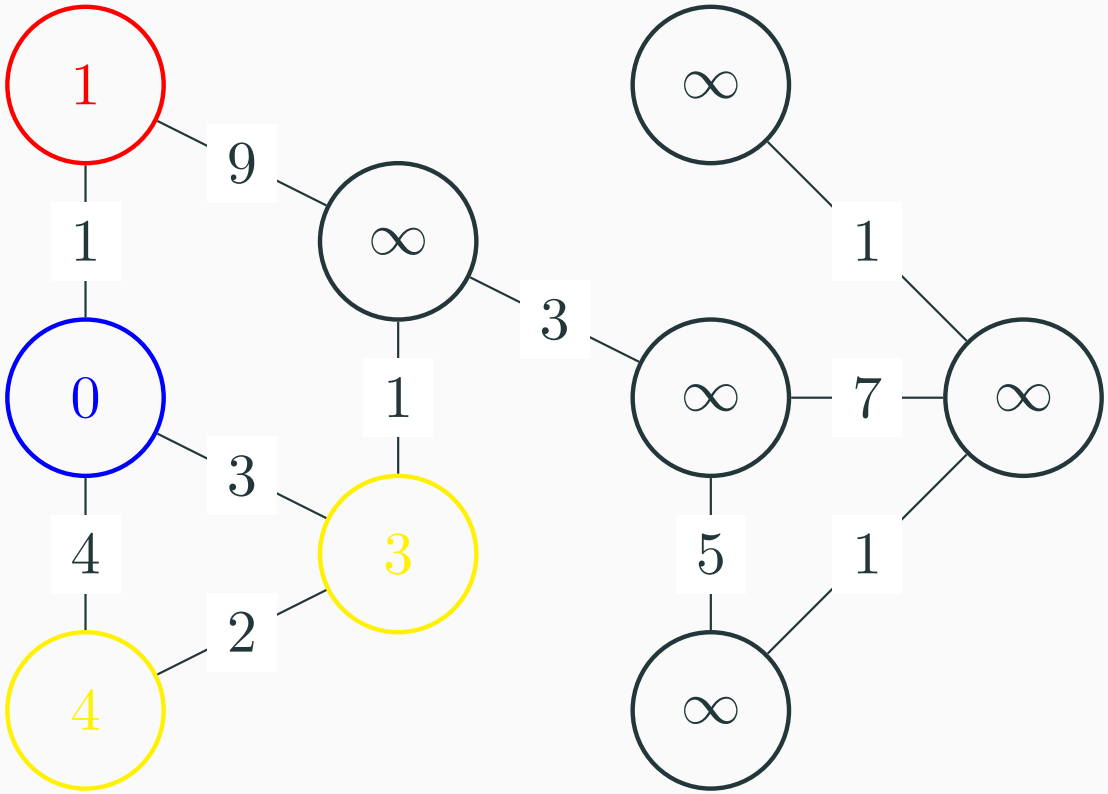

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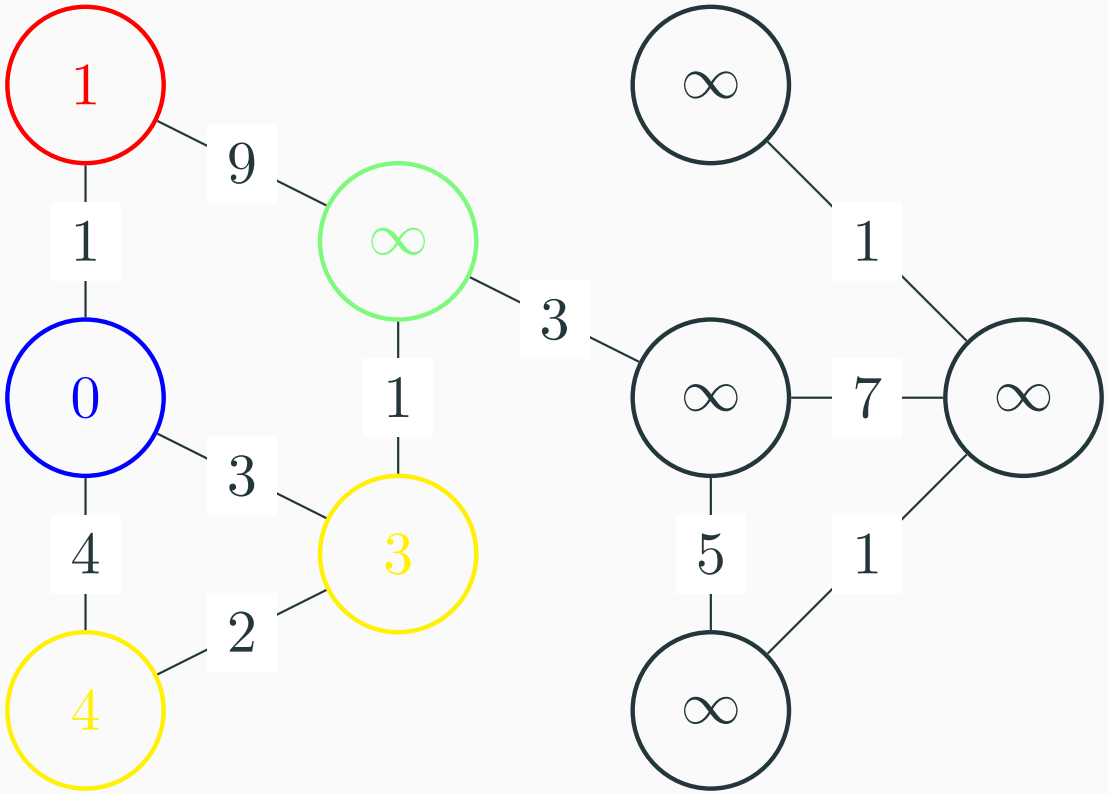

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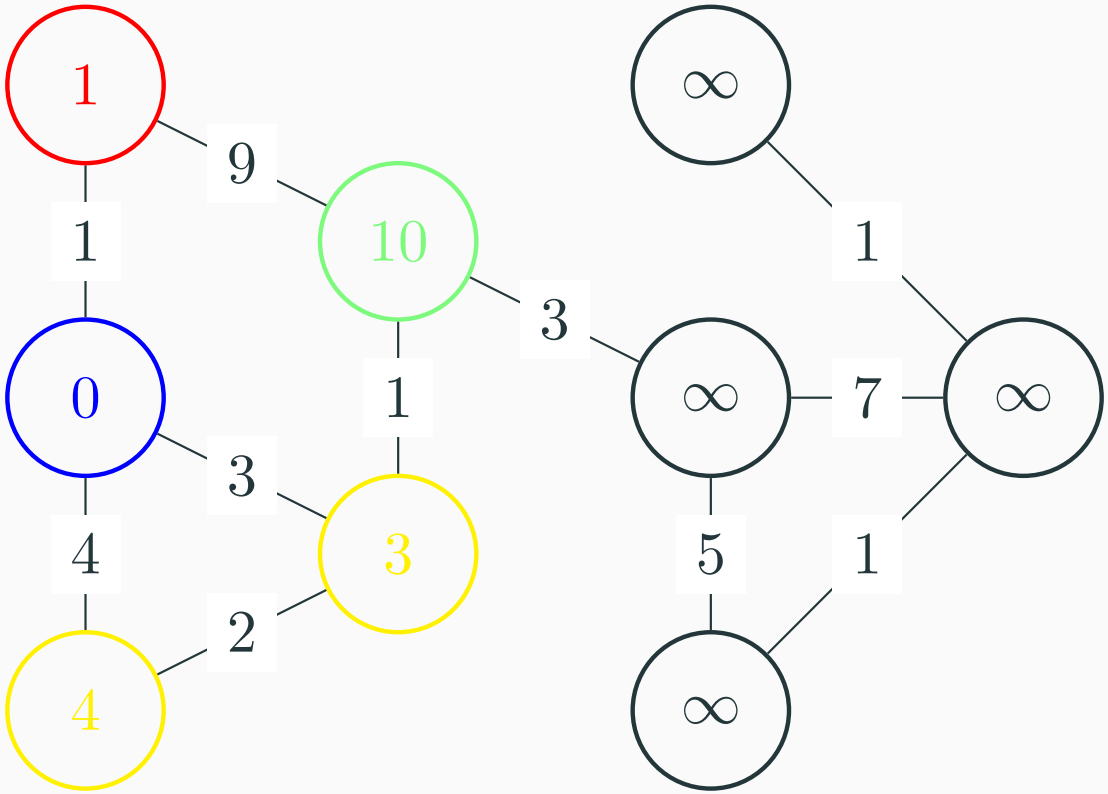

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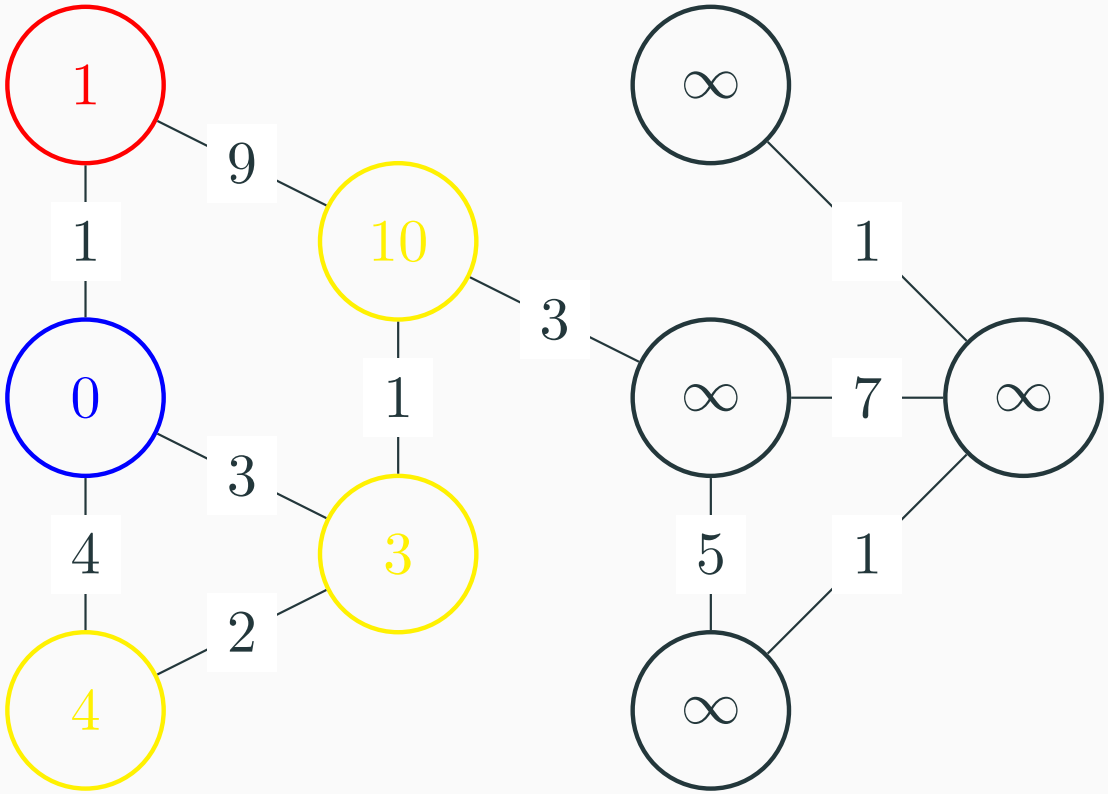

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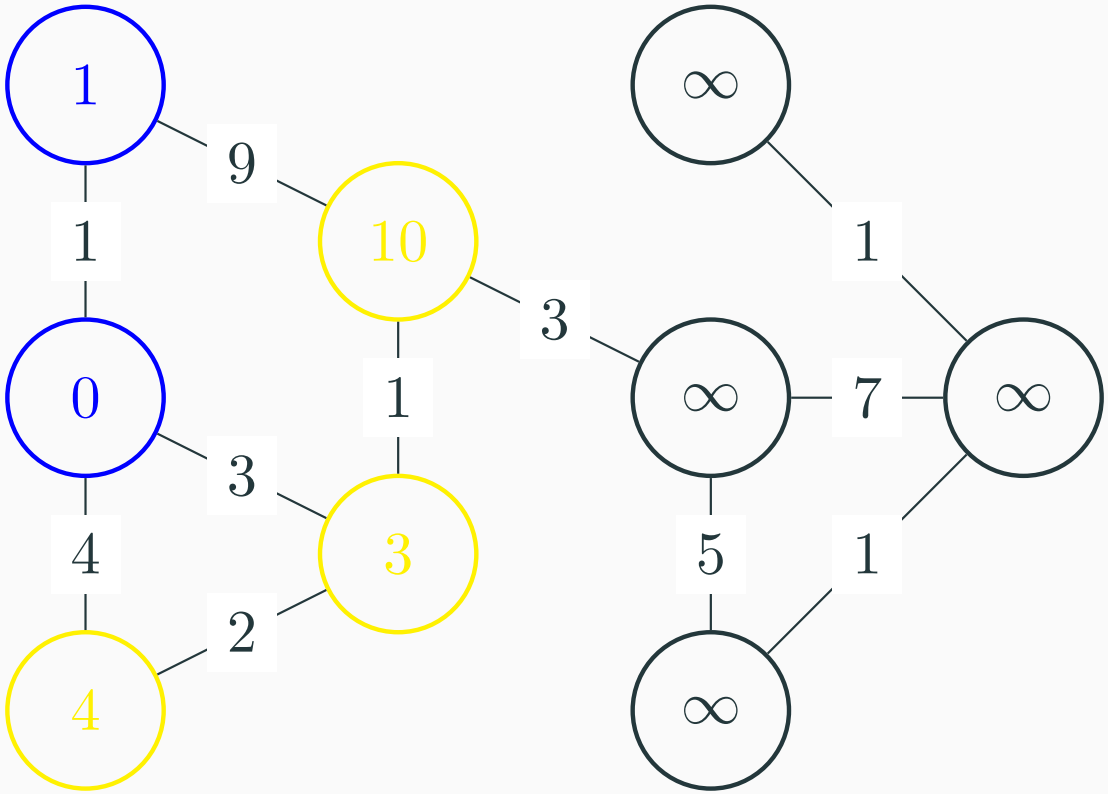

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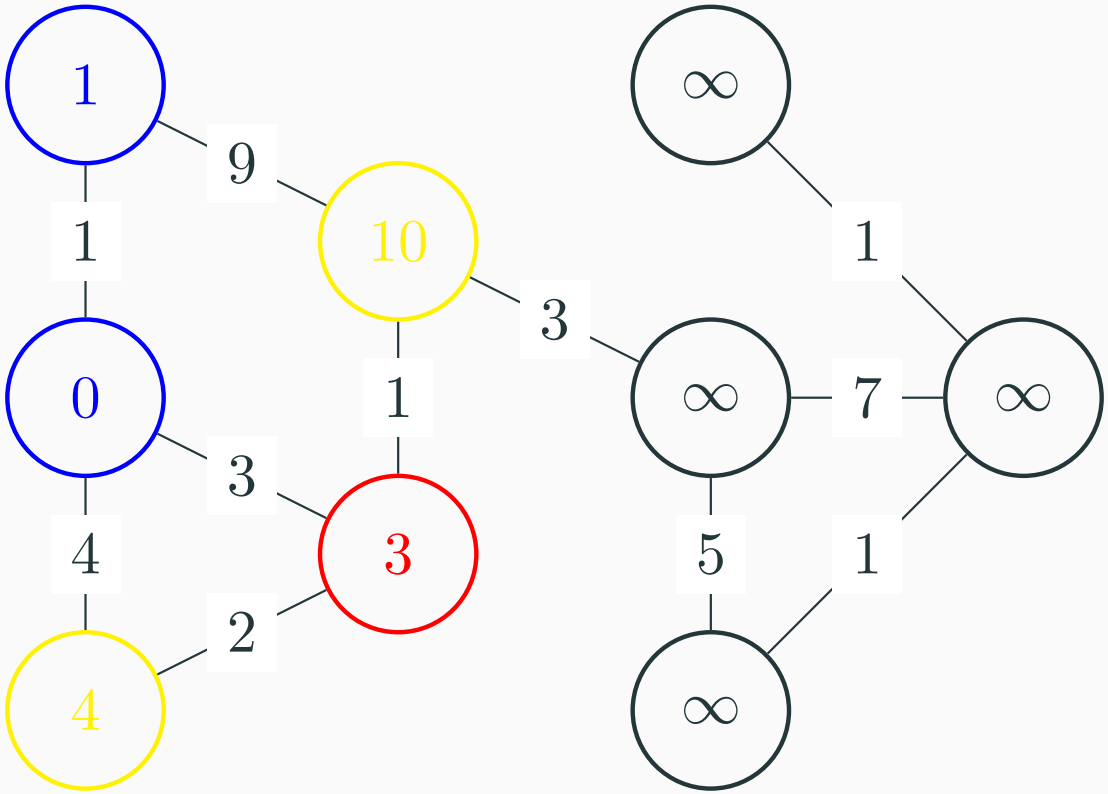

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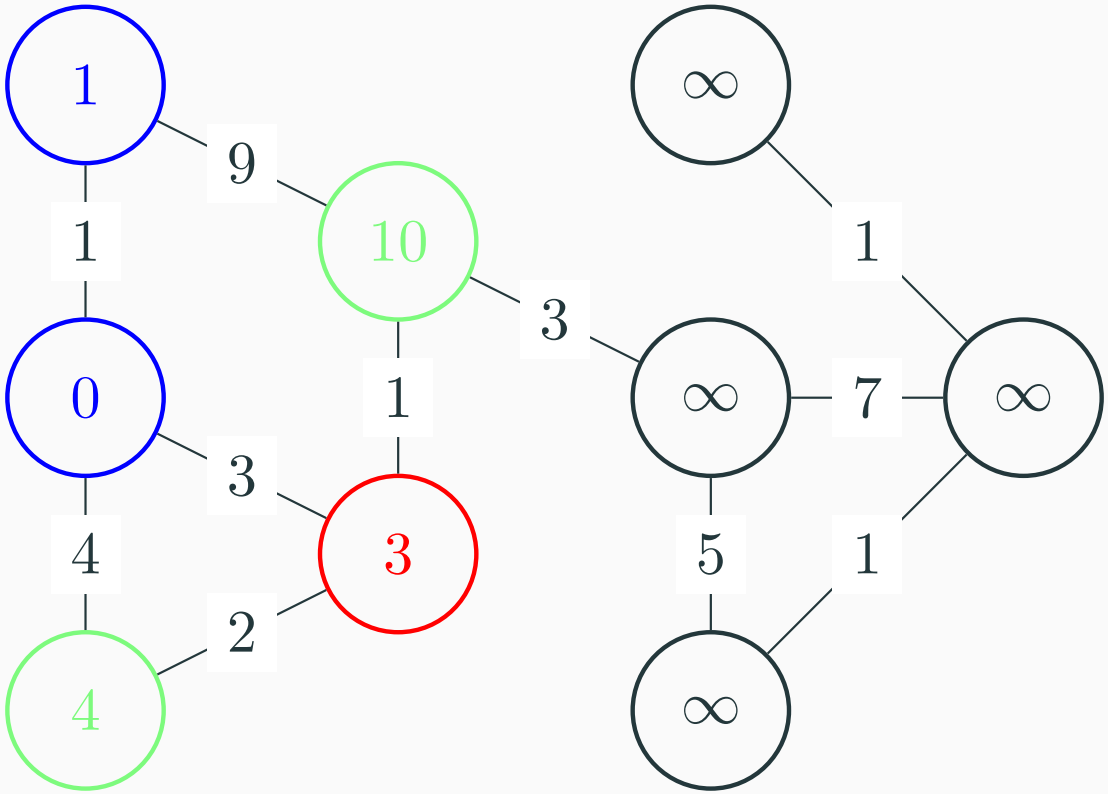

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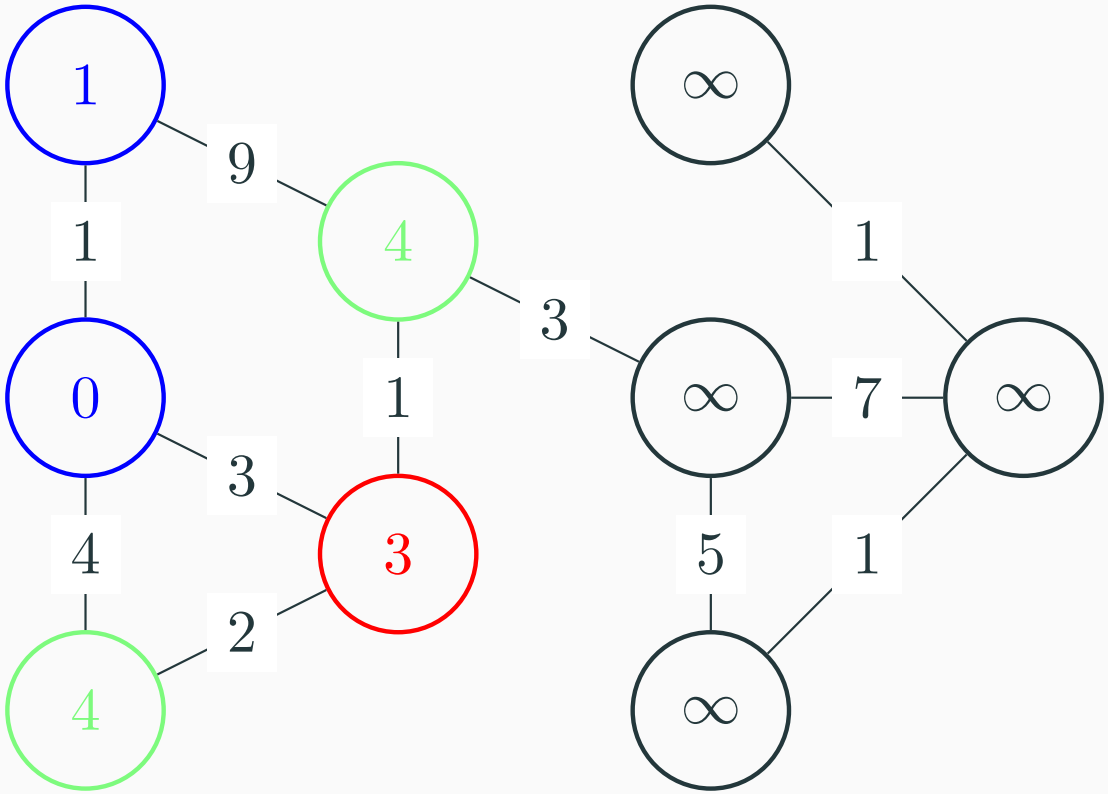

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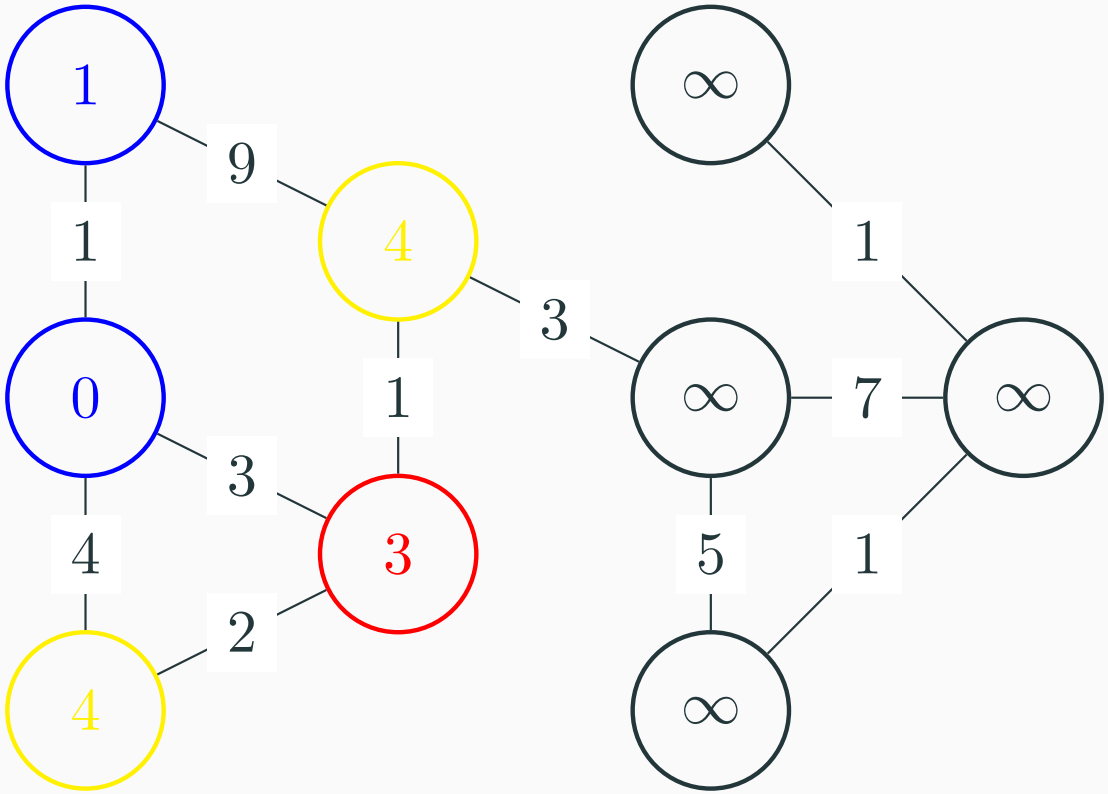

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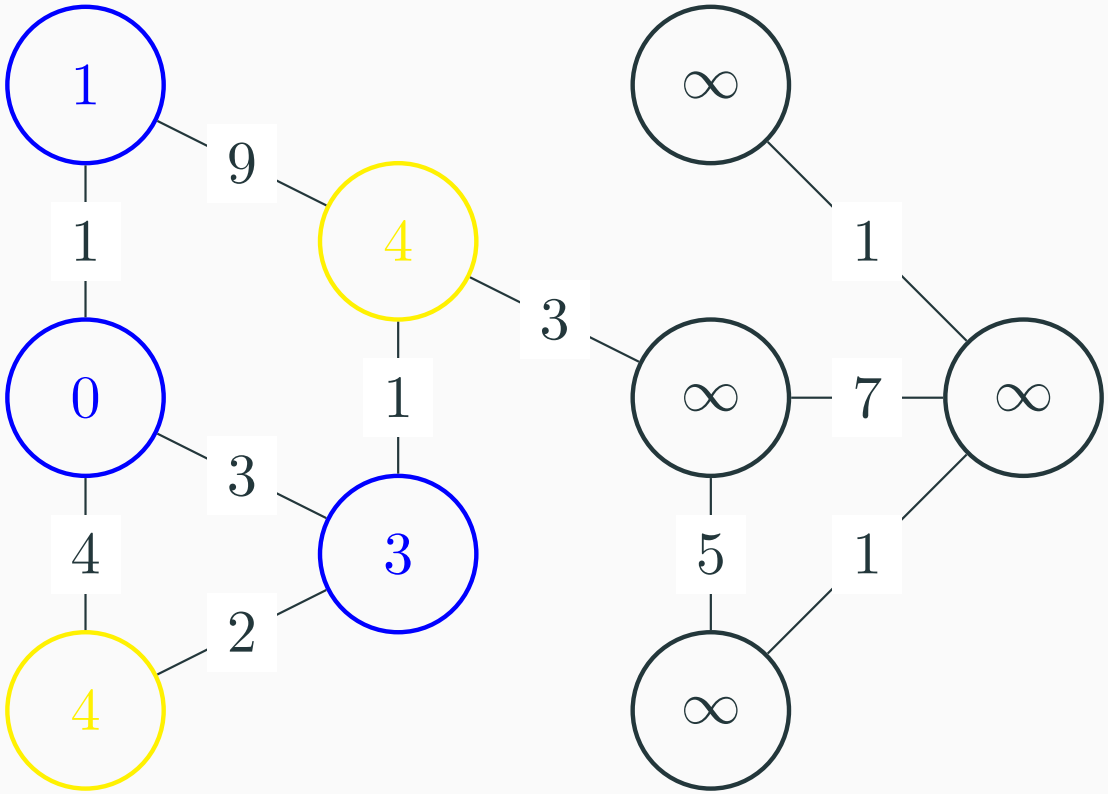

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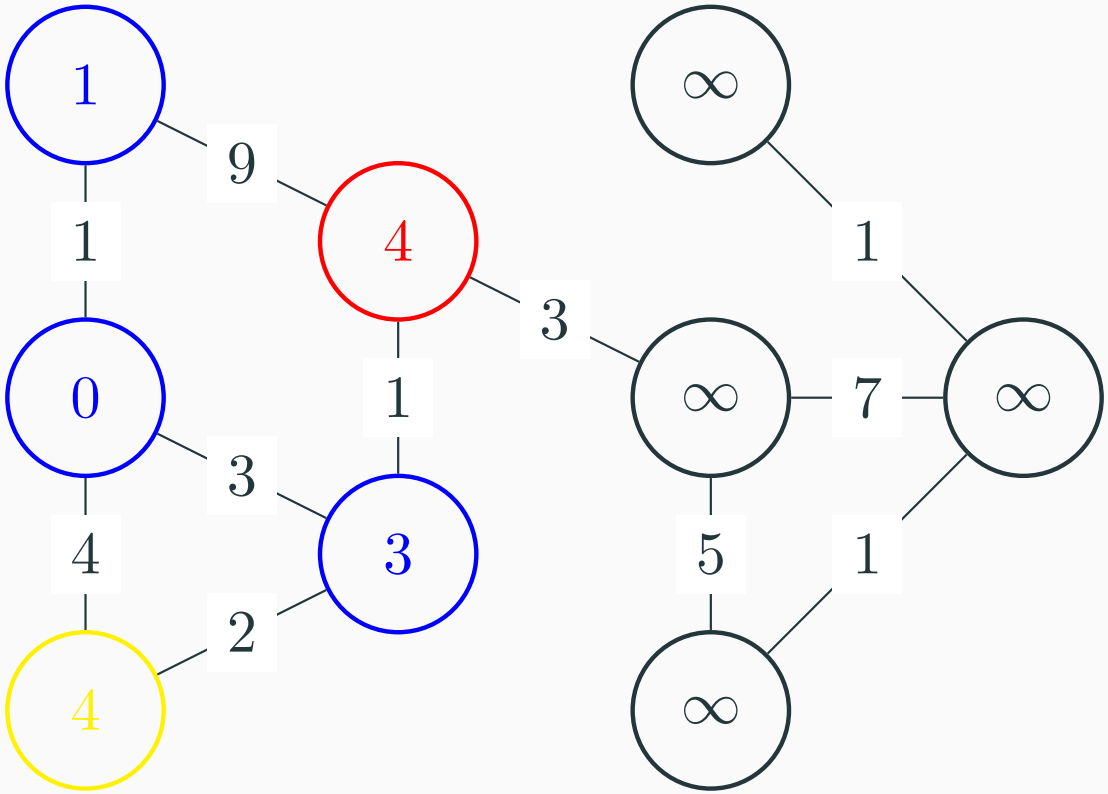

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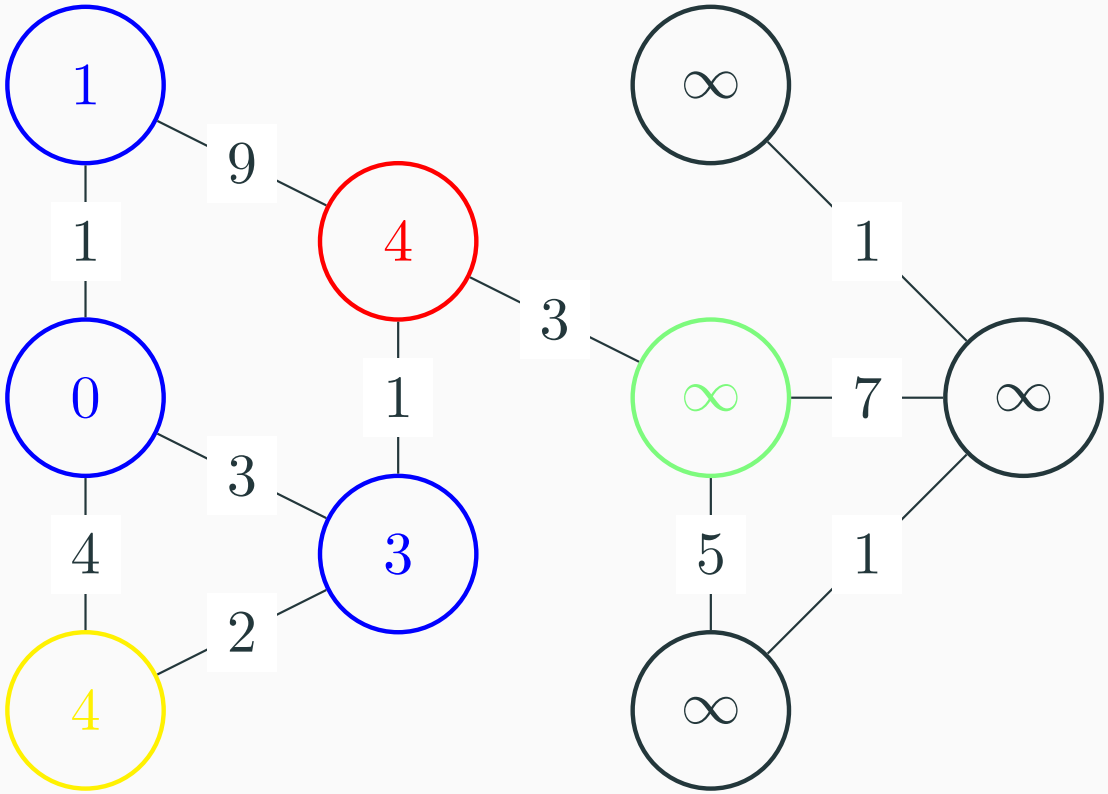
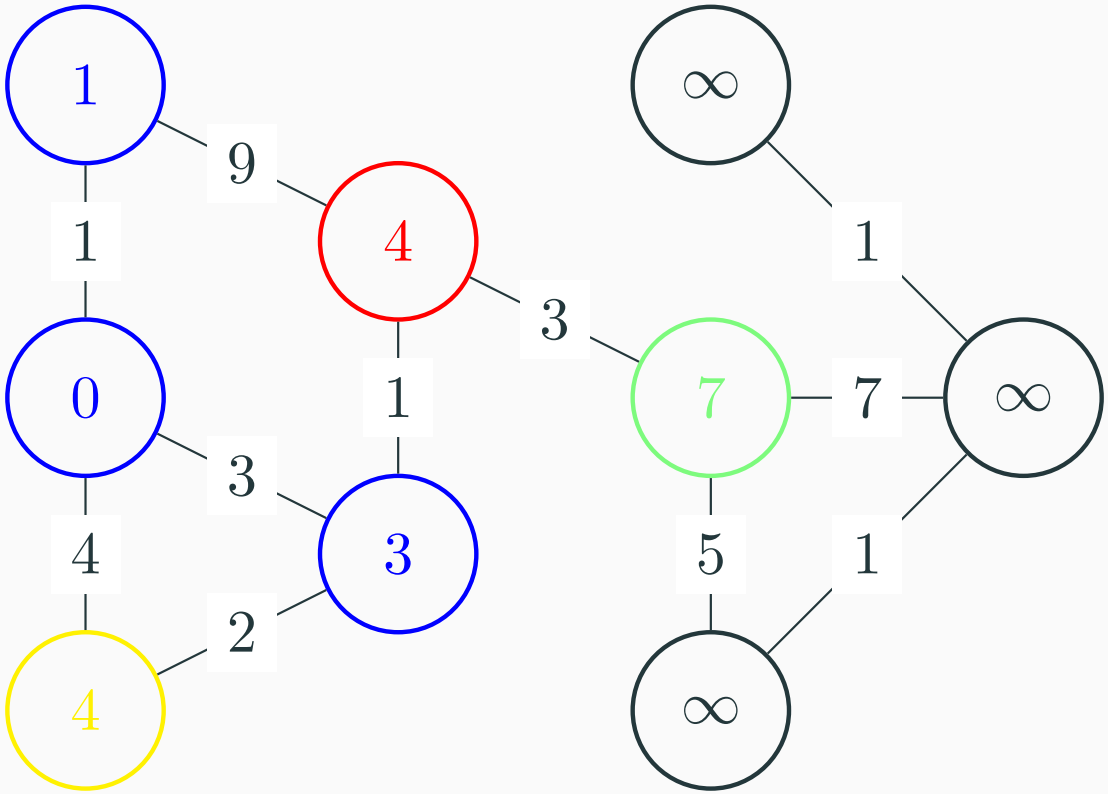


Example



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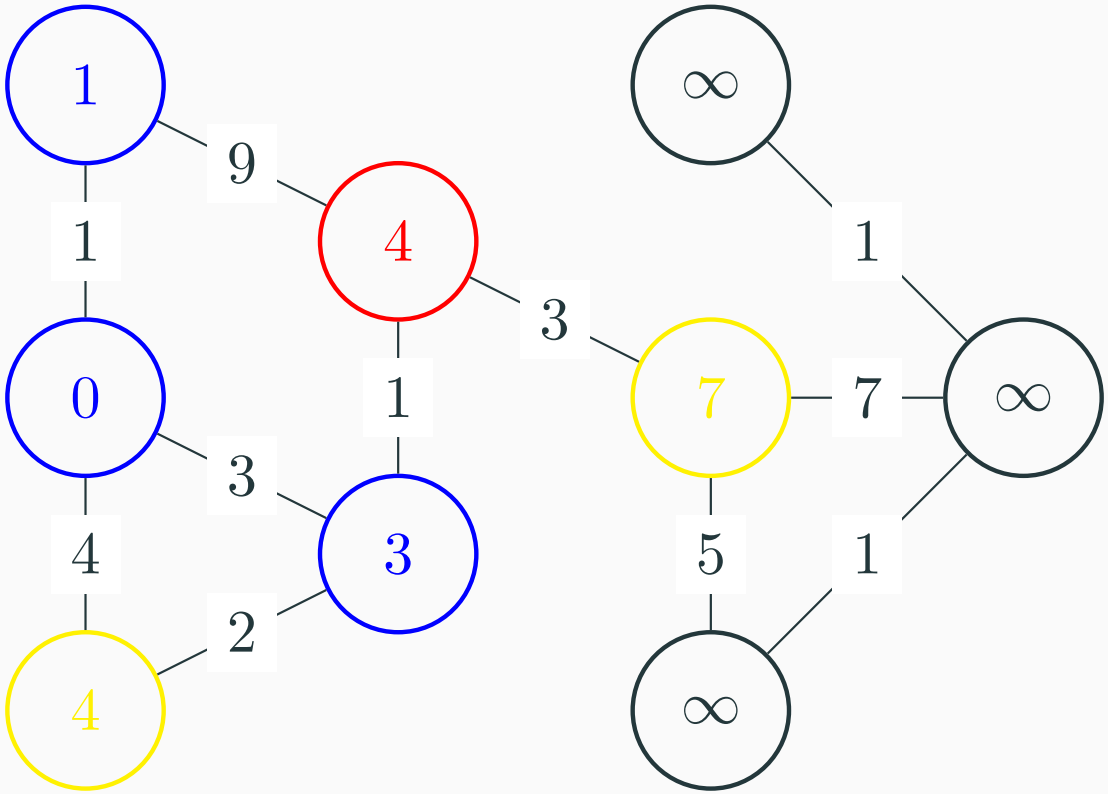

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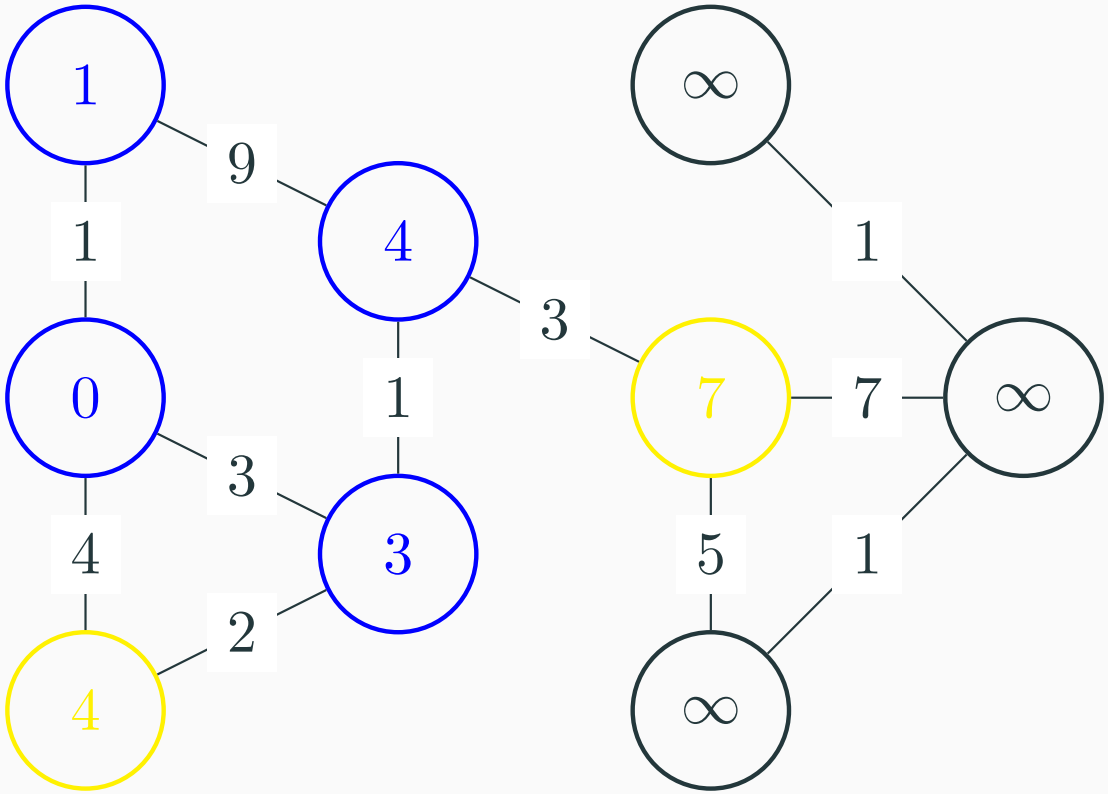
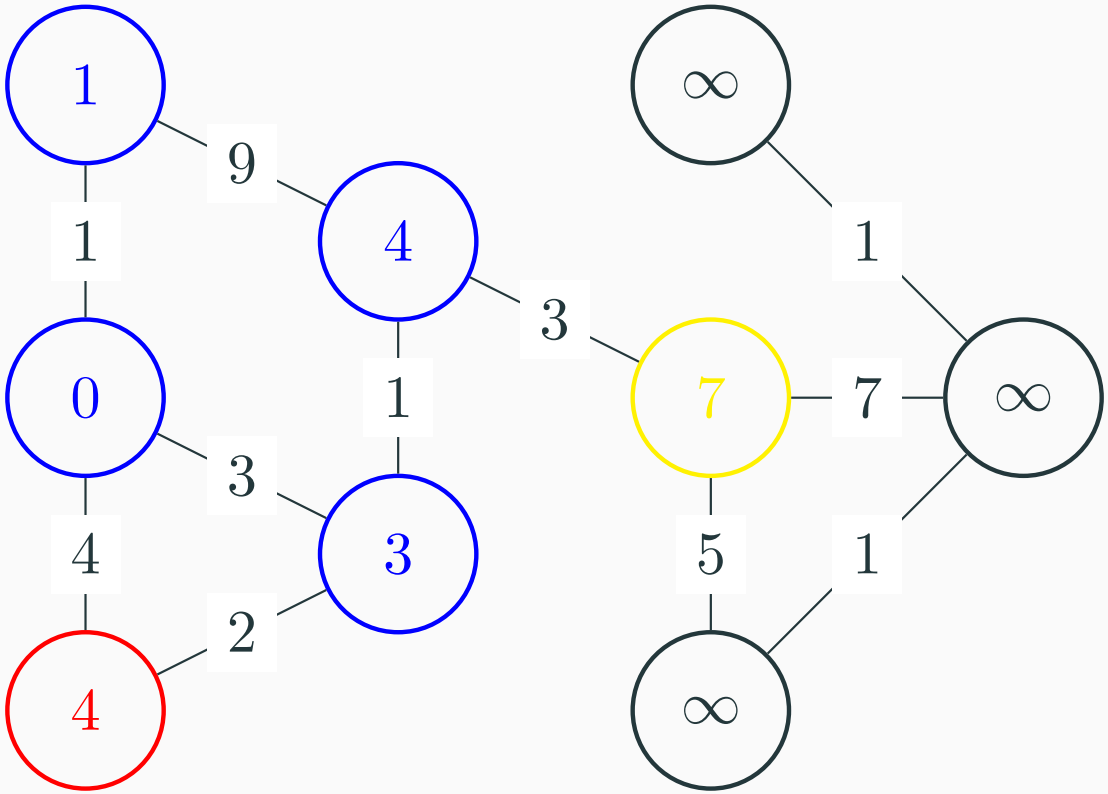


Example



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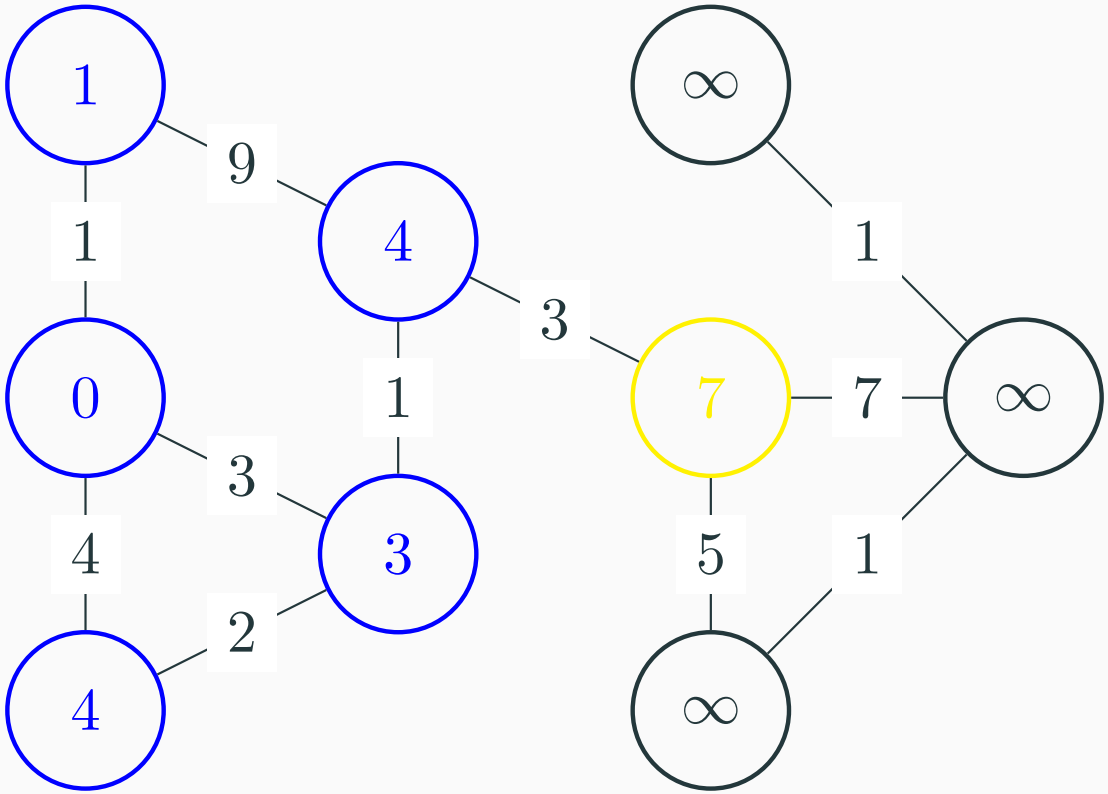

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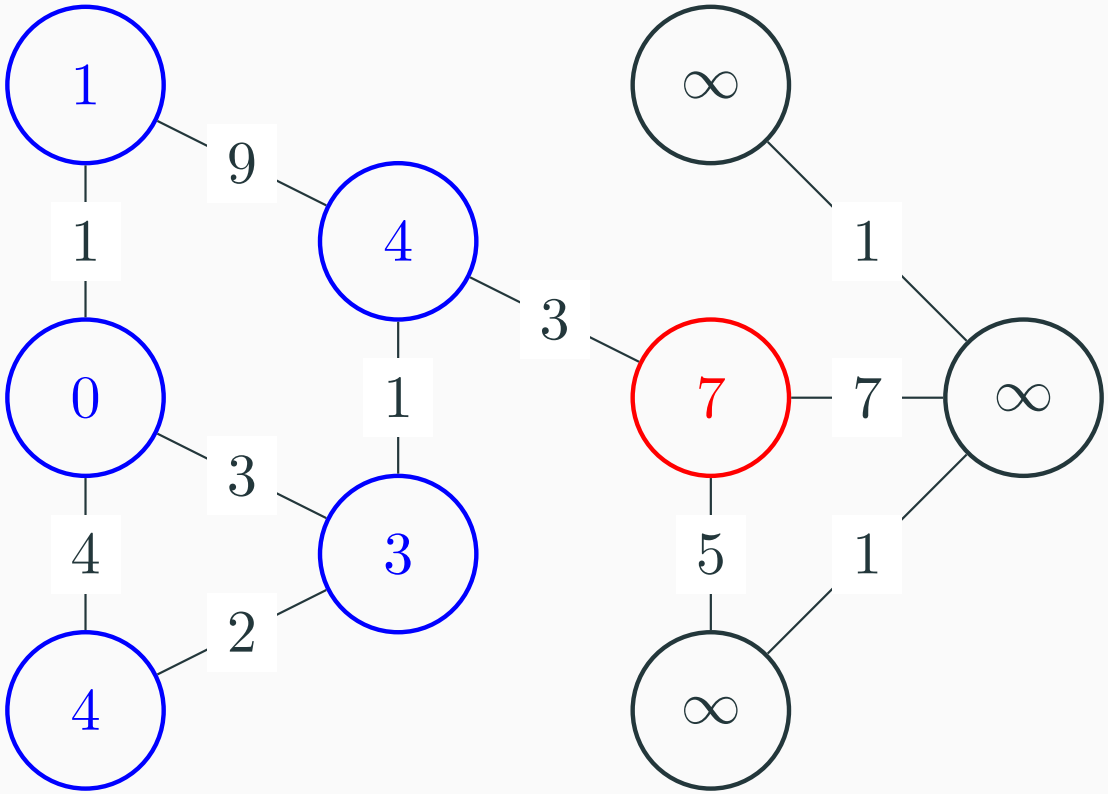

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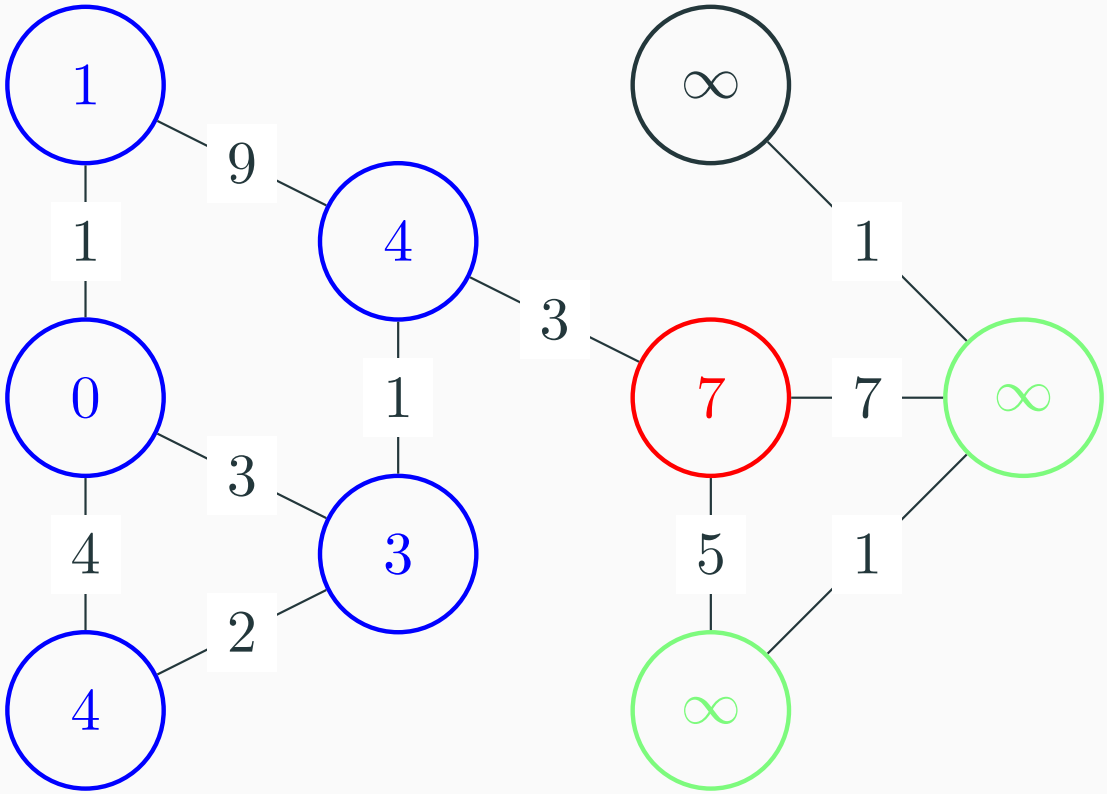
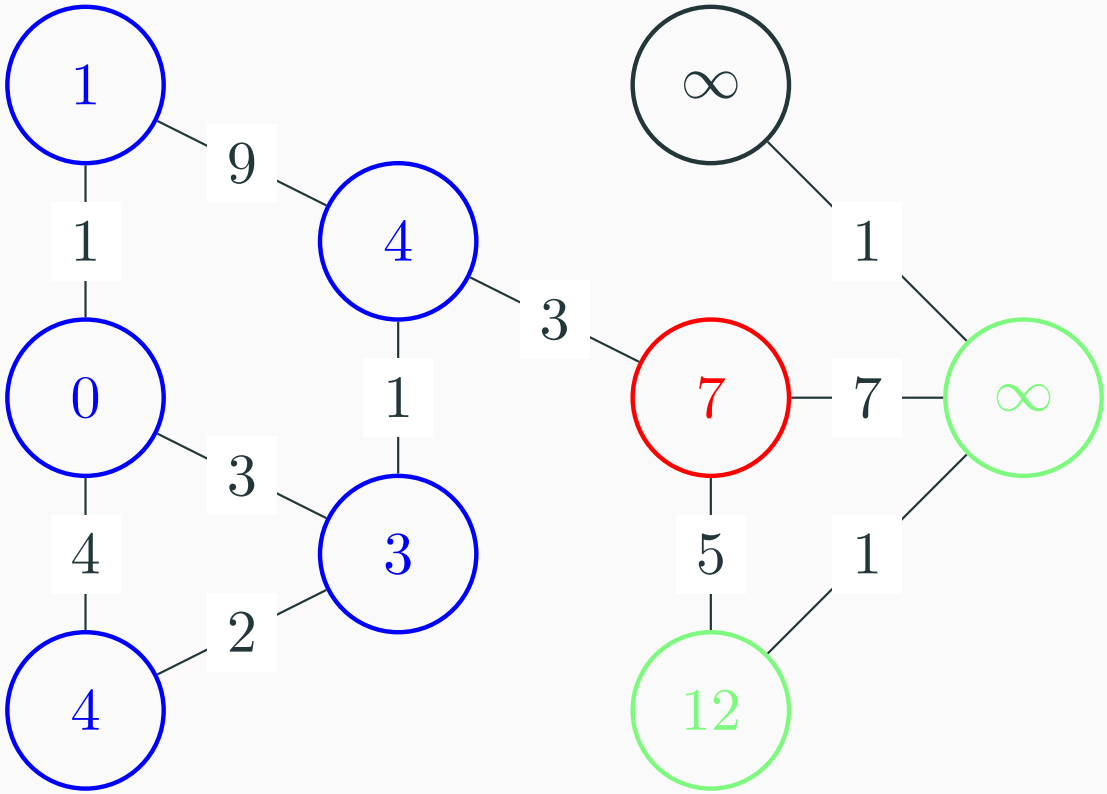


Example



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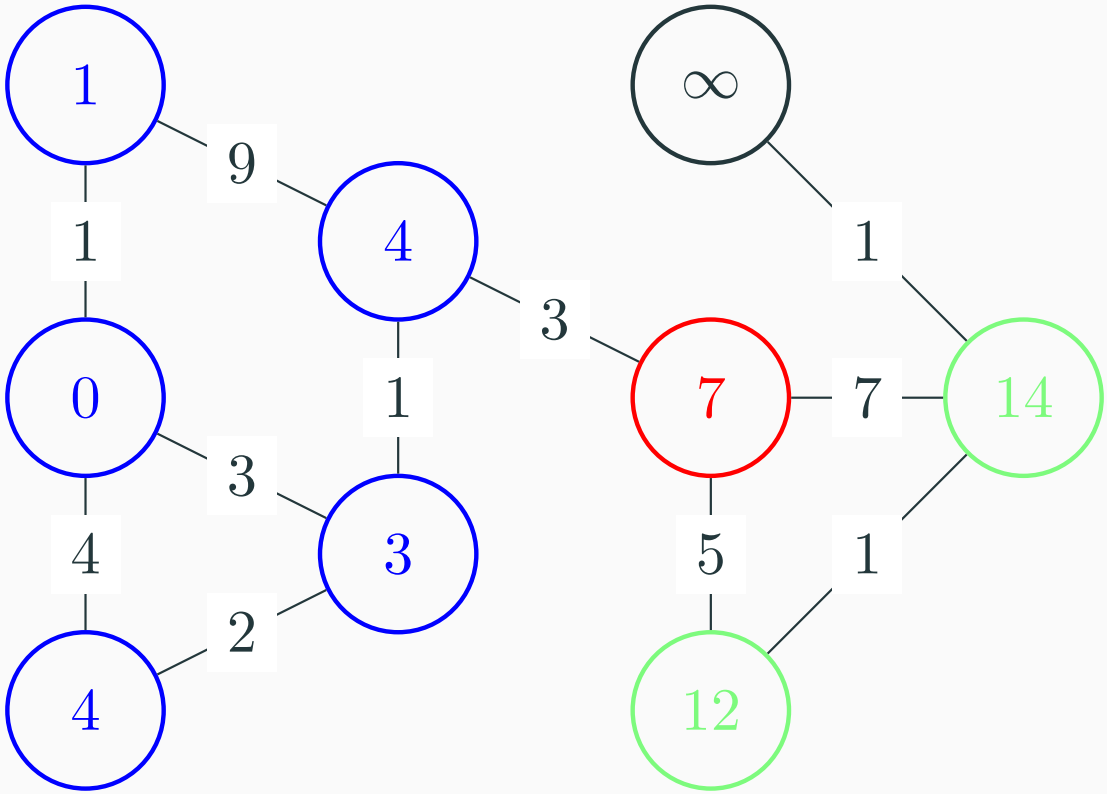

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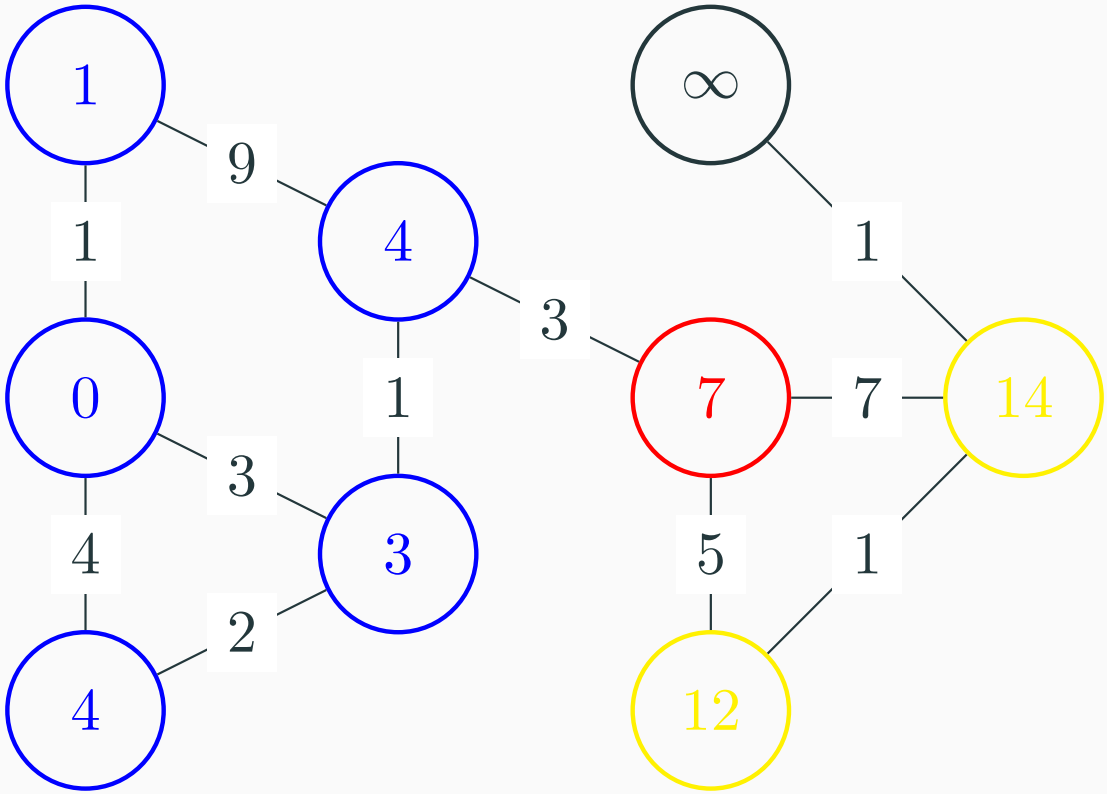

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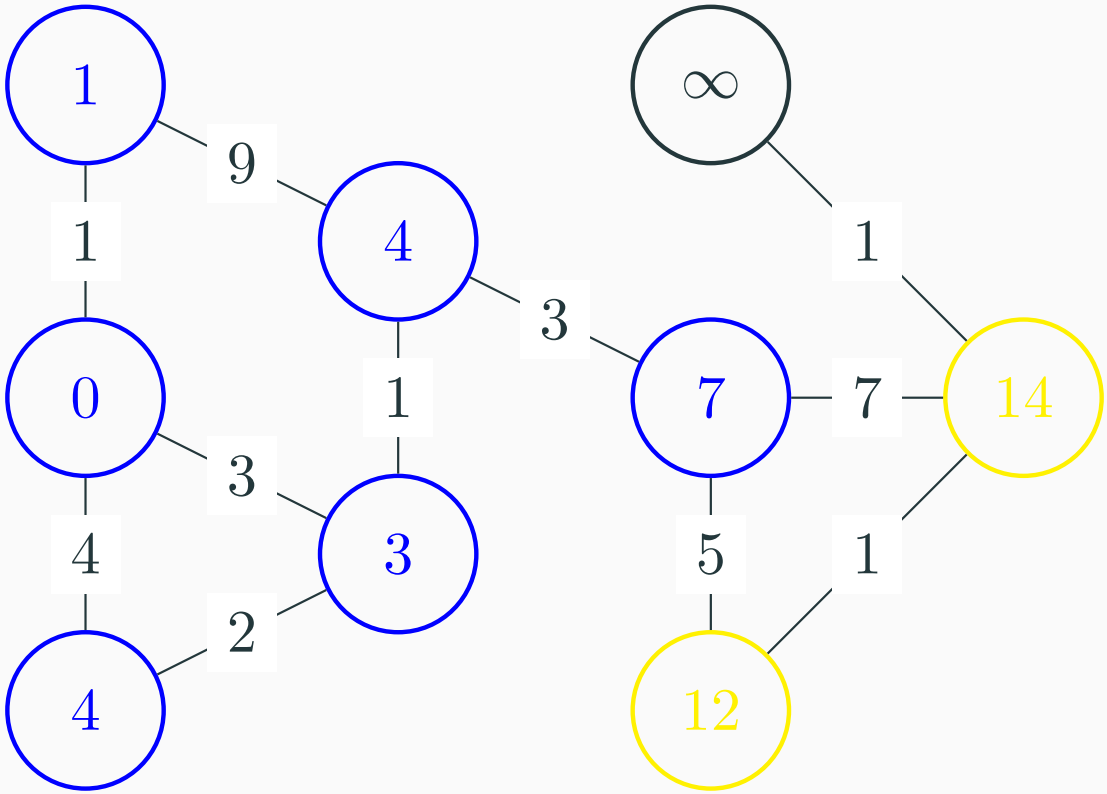

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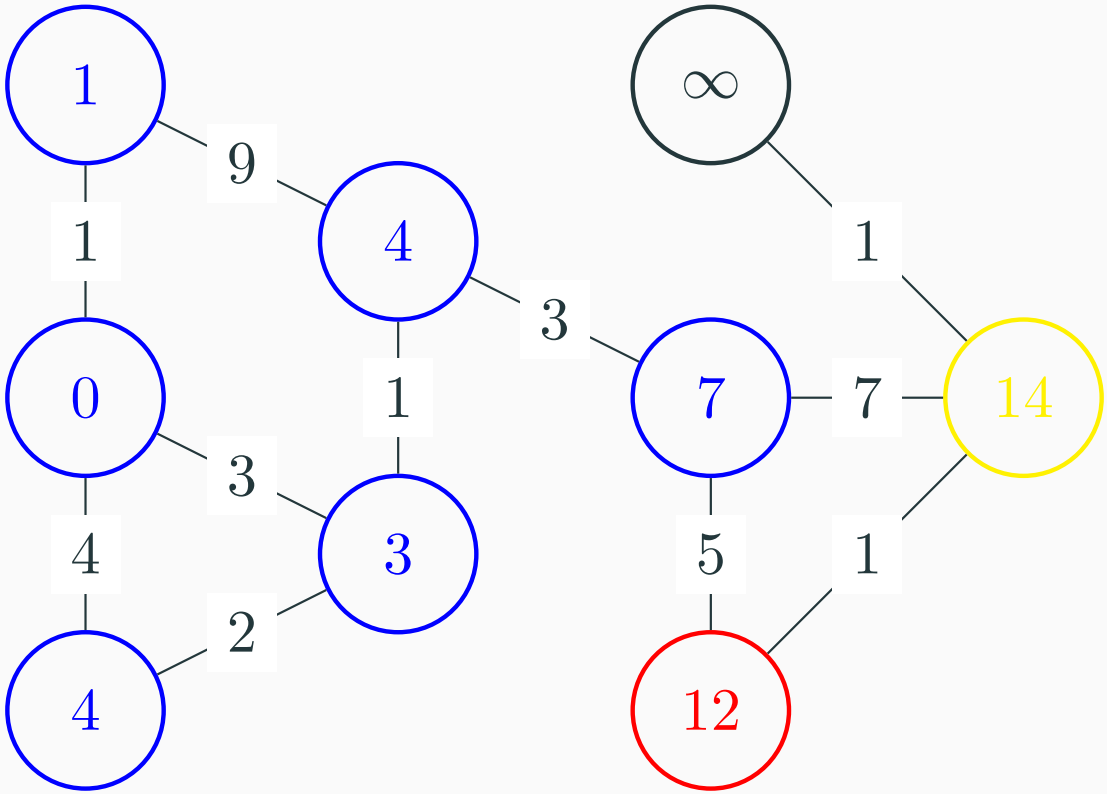

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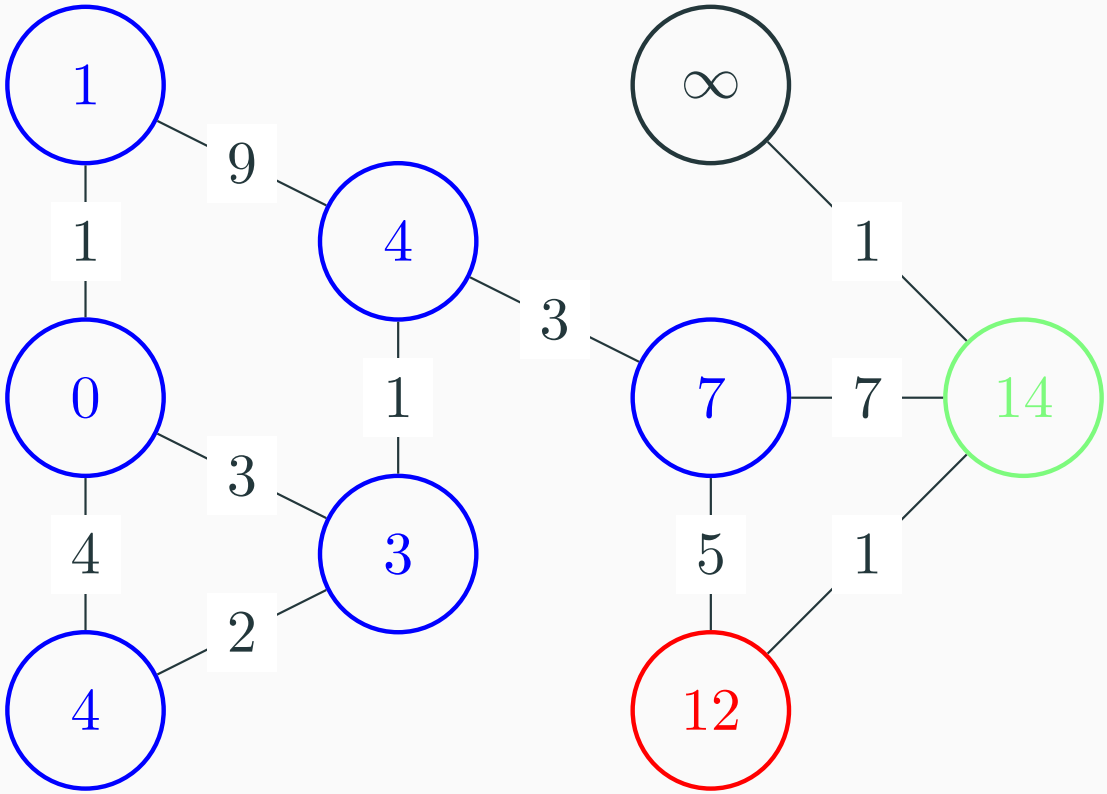

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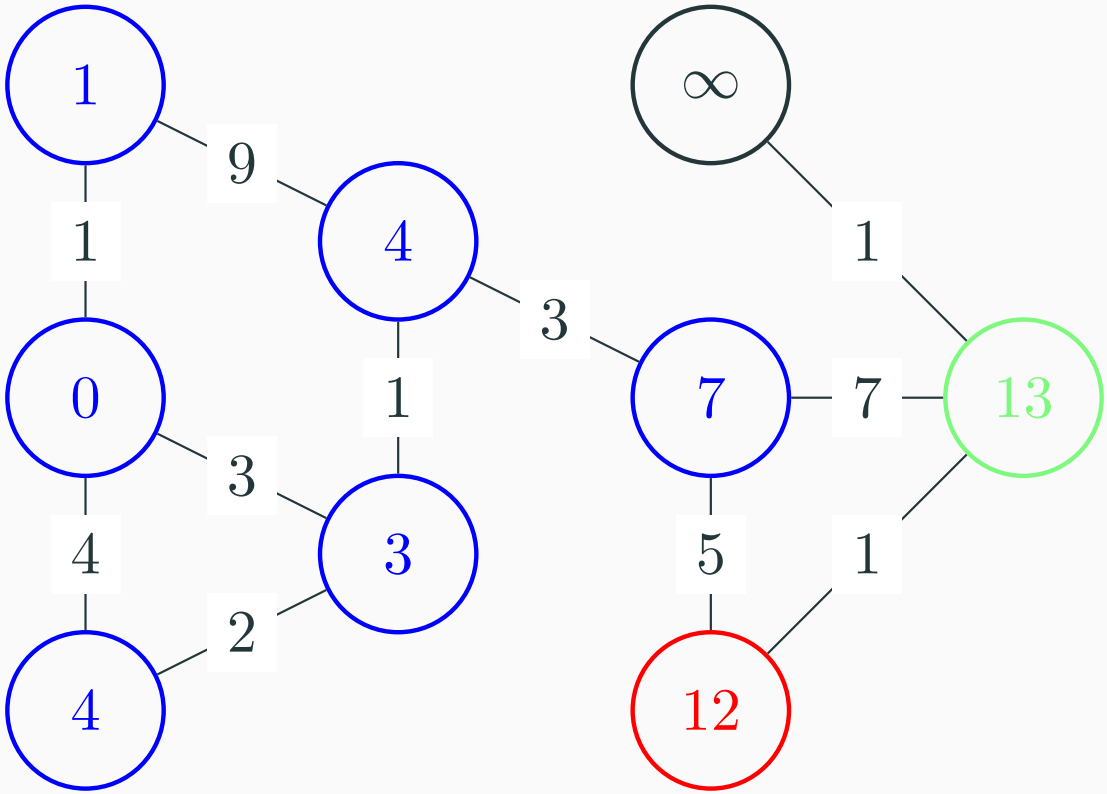
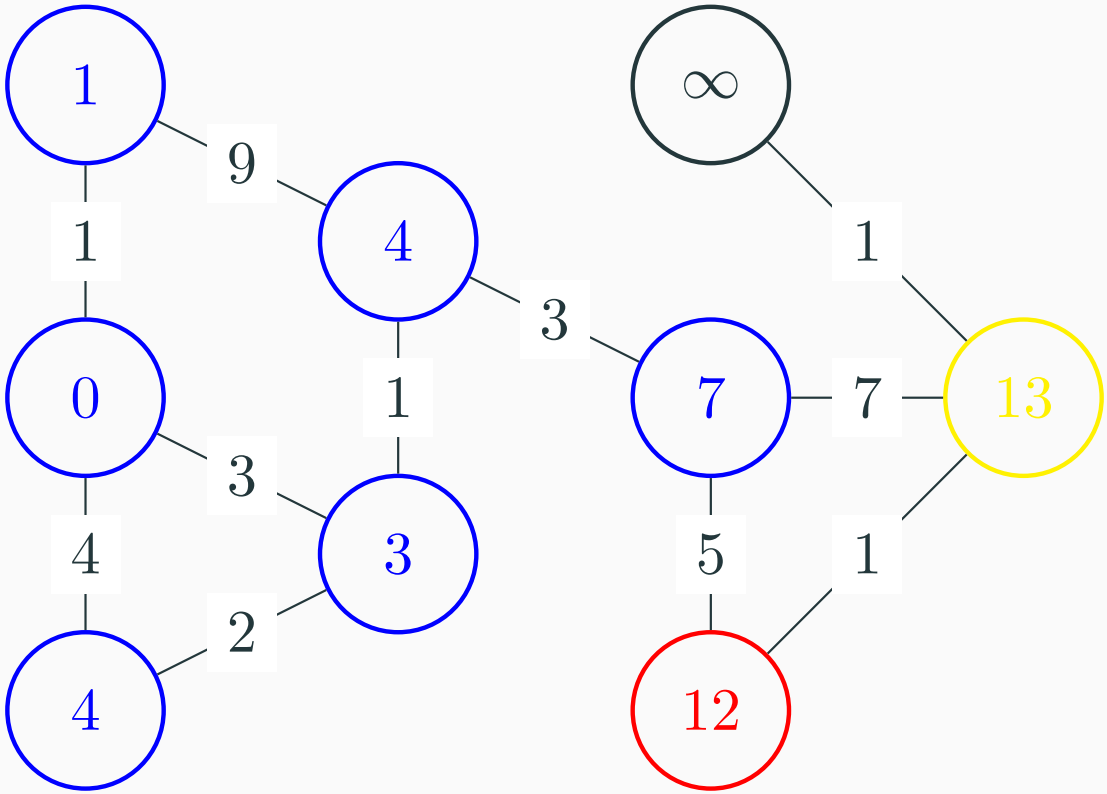
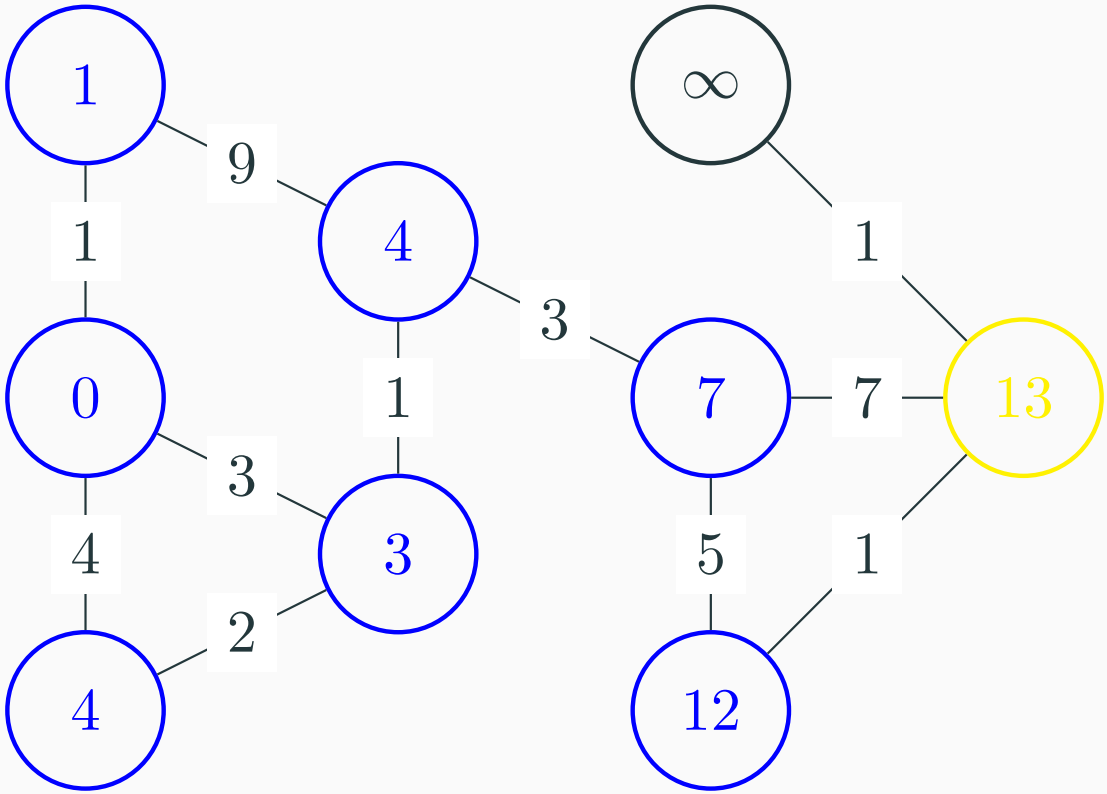


Example



Example



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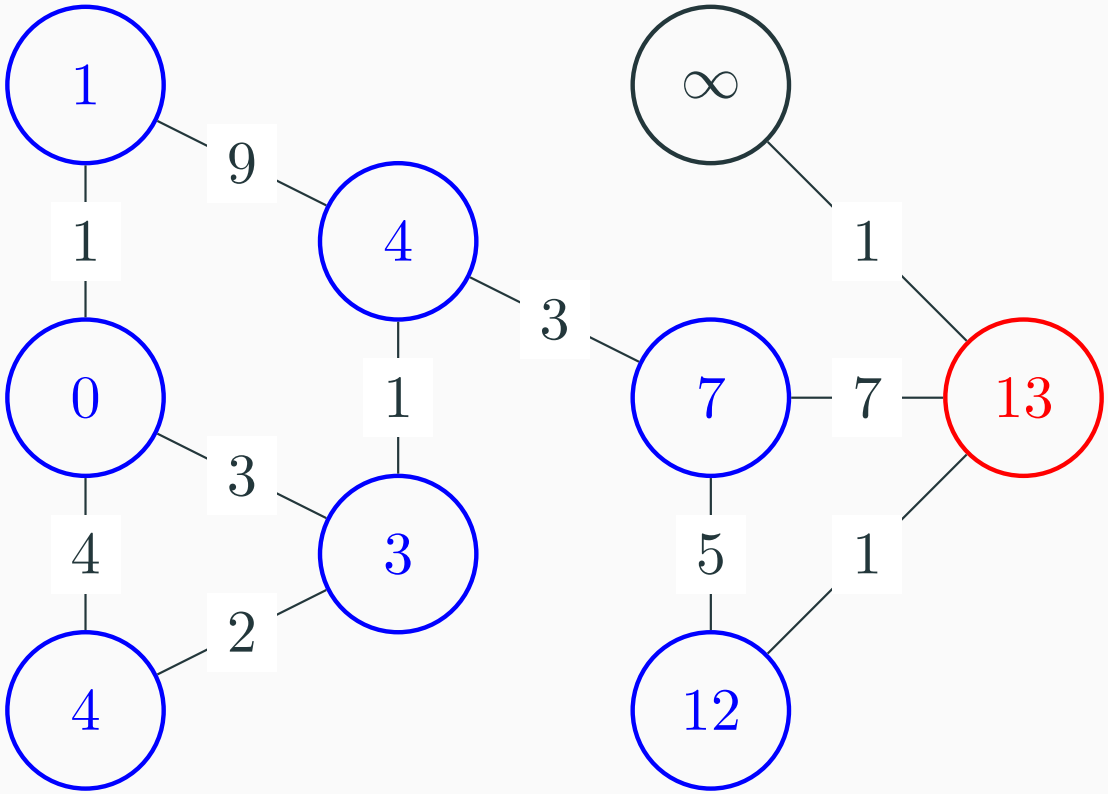

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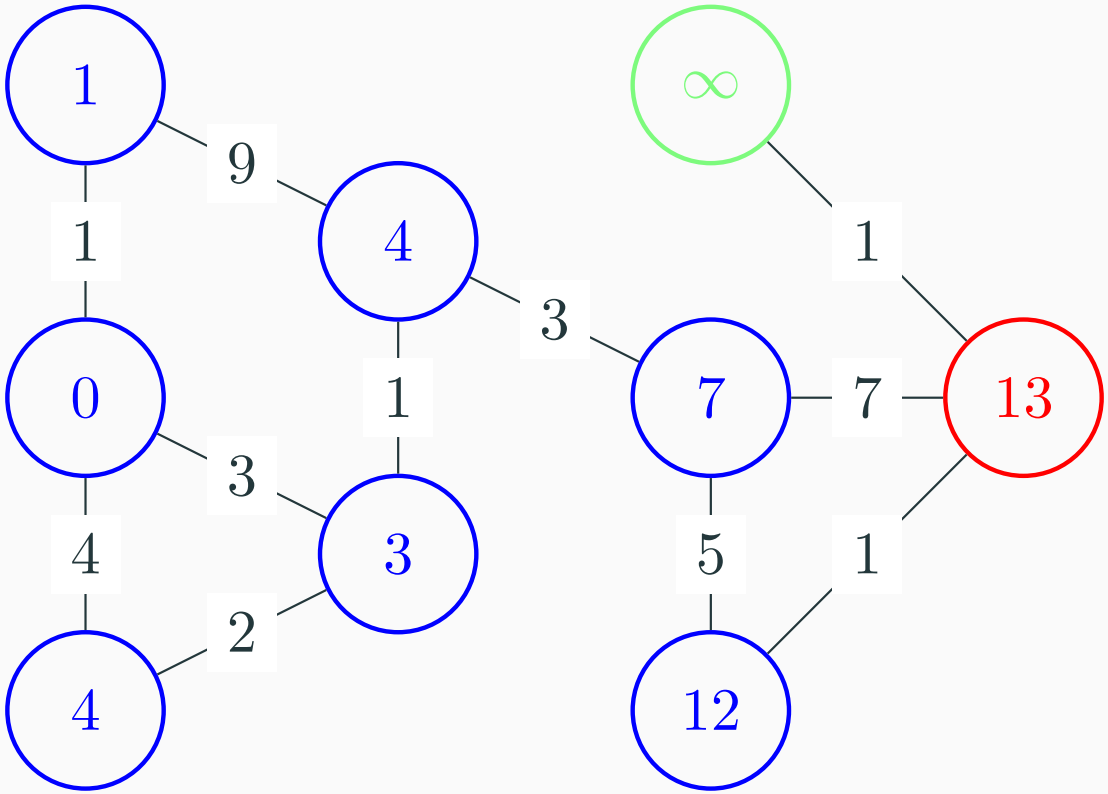

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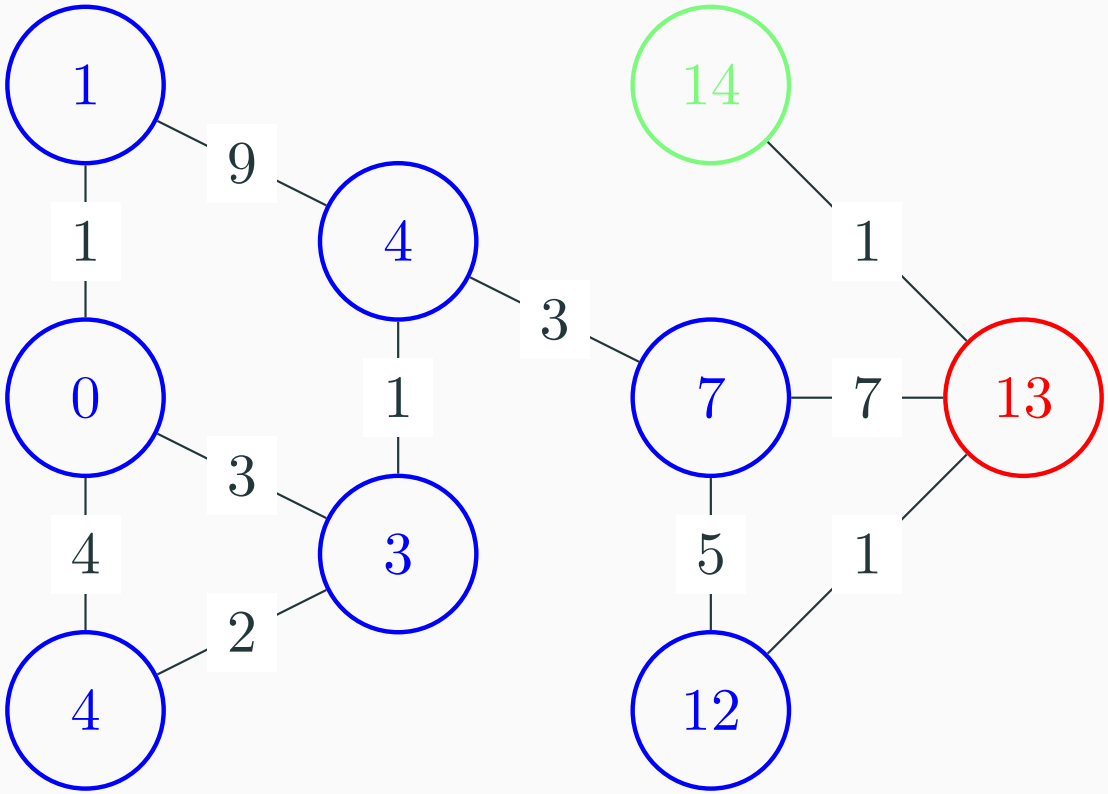

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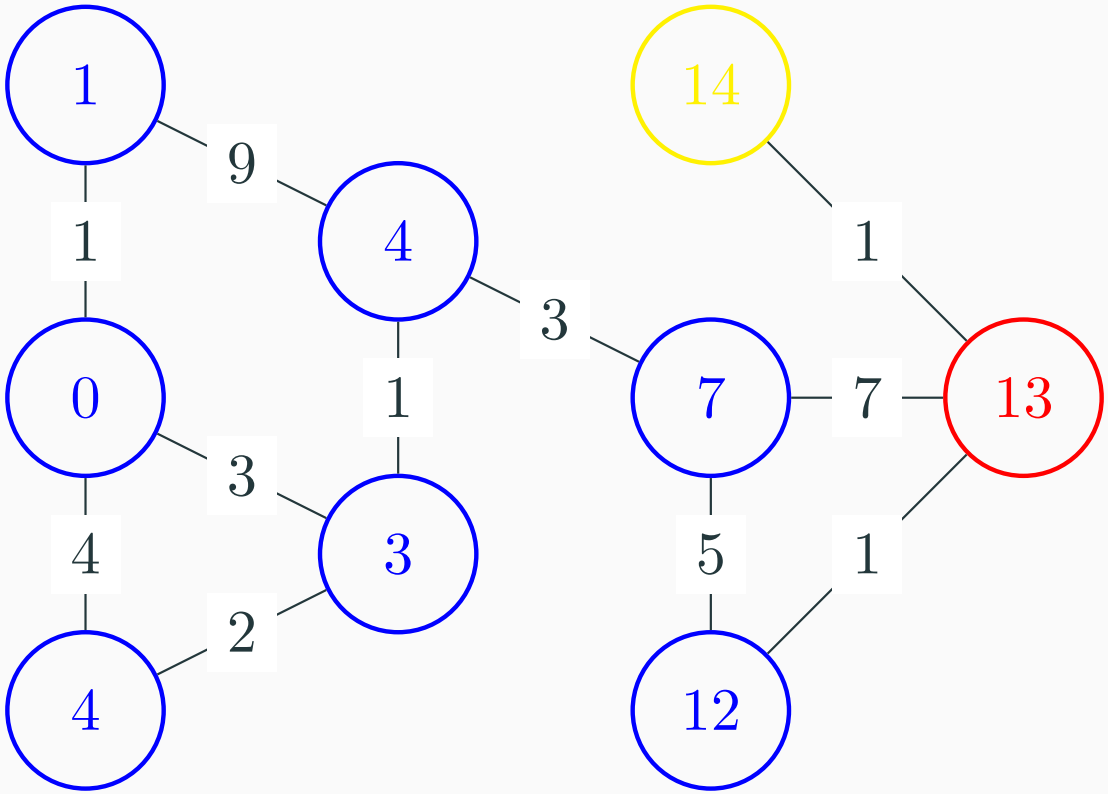

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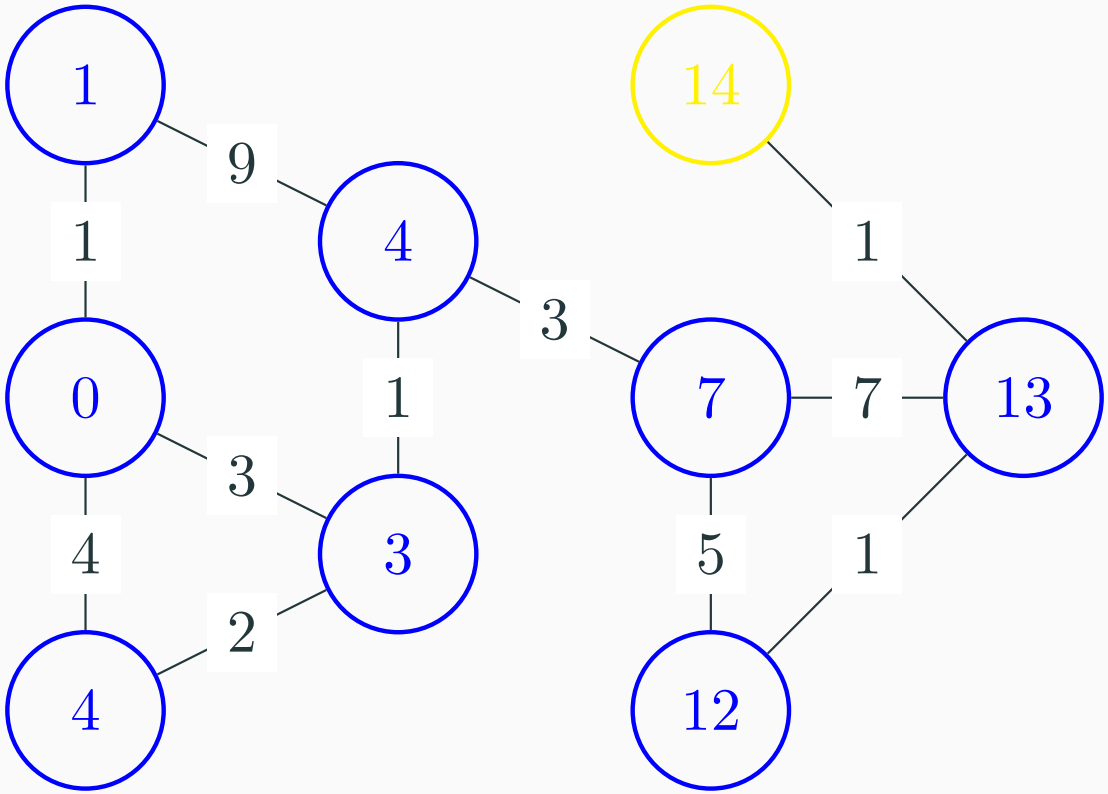

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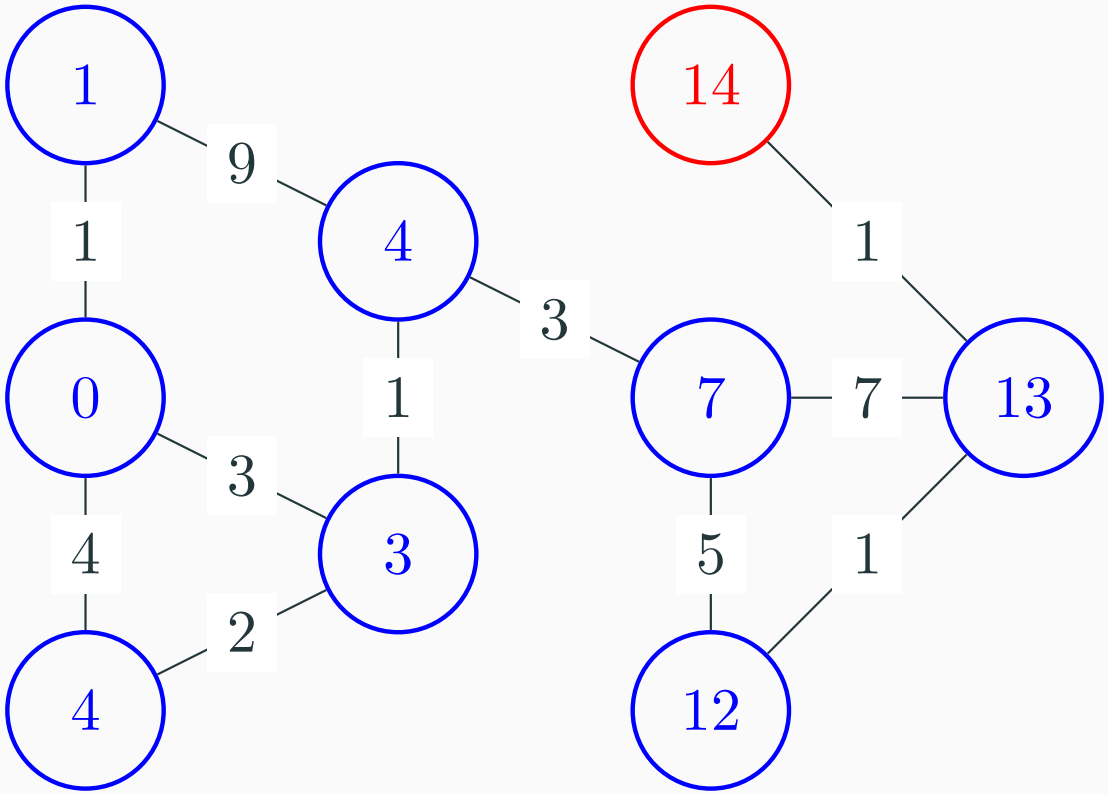

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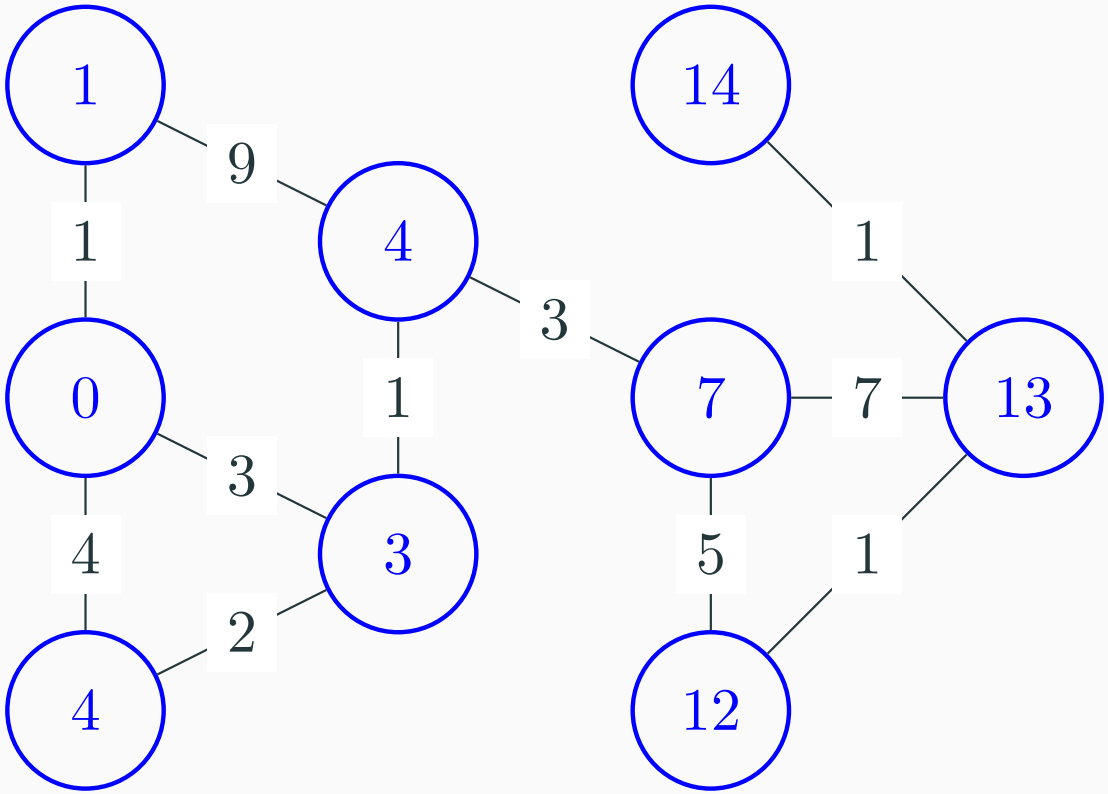


Implementation detail

- There is one thing to watch out for
- The original version of the algorithm uses a heap where you can update the values as you change them
- This is not possible in most standard libraries, so we use a trick instead
- We push a new entry into the heap each time we update the weight to a lower one
- It is **very important** to only process an entry on the heap if the weight is correct to avoid repeated calculation when doing this!

Dijkstra's algorithm

```
vector<edge> adj[100];
vector<int> dist(100, INF);

void dijkstra(int start) {
    dist[start] = 0;
    priority_queue<pair<int, int>, vector<pair<int, int>>, greater<pair<int, int>>> pq;
    pq.push(make_pair(dist[start], start));

    while (!pq.empty()) {
        int u = pq.top().second;
        int w = pq.top().first;
        pq.pop();
        if(dist[u] != w) continue;

        for (int i = 0; i < adj[u].size(); i++) {
            int v = adj[u][i].v;
            int w = adj[u][i].weight;

            if (w + dist[u] < dist[v]) {
                dist[v] = w + dist[u];
                pq.push(make_pair(dist[v], v));
            }
        }
    }
}
```

Dijkstra's algorithm

- For each edge in the graph we might have to push to the heap
- We visit each vertex at most once and iterate over its neighbours
- Thus the time complexity is $O((V + E) \log E)$

- Note that this only works for non-negative weights
- The correctness relies on the fact that once we are done with a vertex we never have to think about it again, which is not true when we have negative weights

Dealing with negative weights

Negative weights

- What do we do then when we have negative weights?
- We can use the Bellman-Ford algorithm, at the cost of a worse time complexity

Bellman-Ford

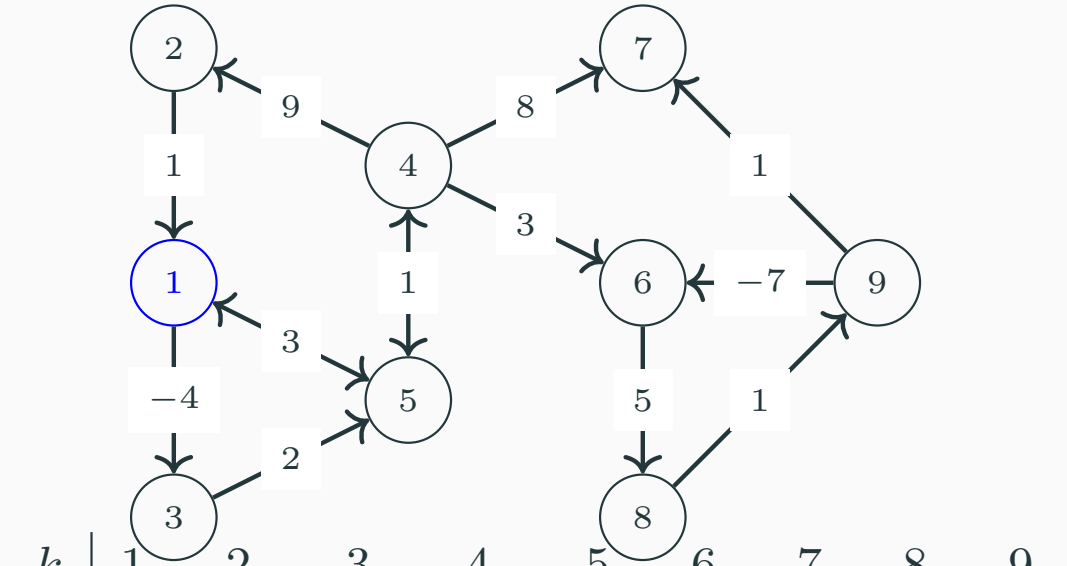
- It is based on dynamic programming, let $f(v, k)$ be the shortest path from the source u to a vertex v that doesn't use more than k vertices to get there
- Then as base cases we get $f(u, 0) = 0$ and $f(v, 0) = \infty$ for $v \neq u$
- Otherwise we can look at $f(w, k-1)$ for some w, and add a single extra edge, giving the recurrence

$$f(v, k) = \min\left(f(v, k-1), \min_{x \in N^-(v)} (w(x, v) + f(x, k-1))\right)$$

Bellman-Ford

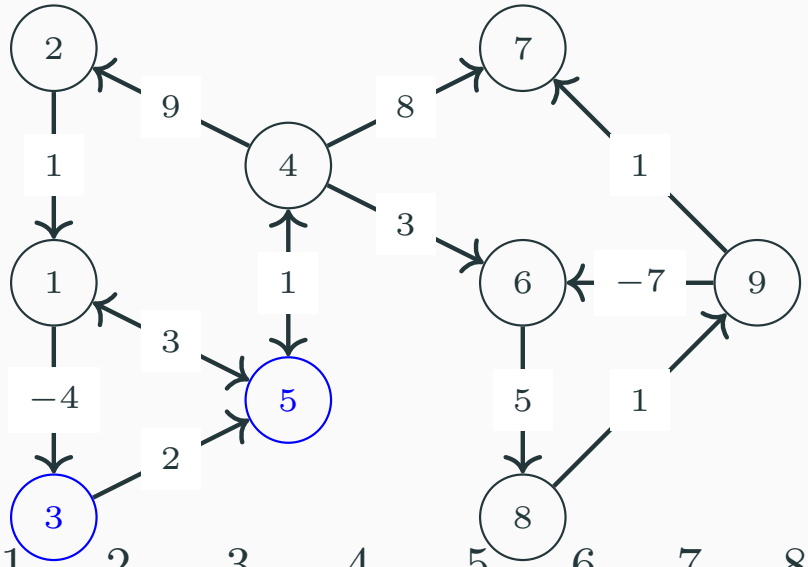
- To implement this efficiently we will make some changes from just a straight-forward DP implementation (but that would work!)
- Each $f(v, k)$ depends only on $f(w, k-1)$ for some different w, so we only have to store our current row k and the last one
- This essentially gets us the usual implementation of Bellman-Ford

Example



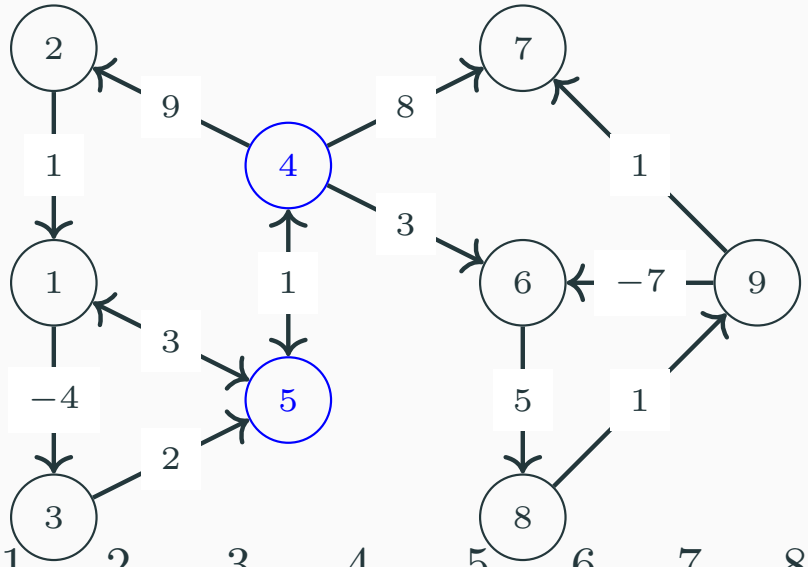
k	1	2	3	4	5	6	7	8	9
0	0	∞	∞	∞	∞	∞	∞	∞	∞

Example



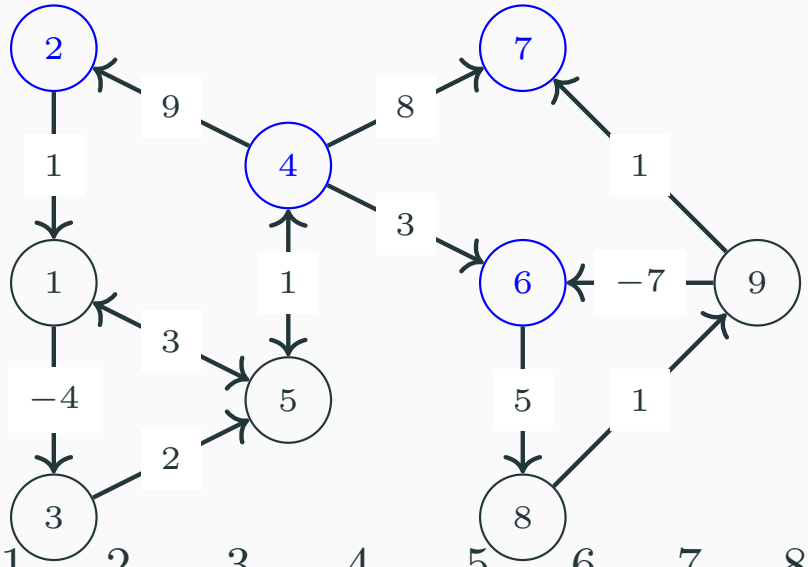
k	1	2	3	4	5	6	7	8	9
0	0	∞	∞	∞	∞	∞	∞	∞	∞
1	0	∞	-4	∞	3	∞	∞	∞	∞

Example



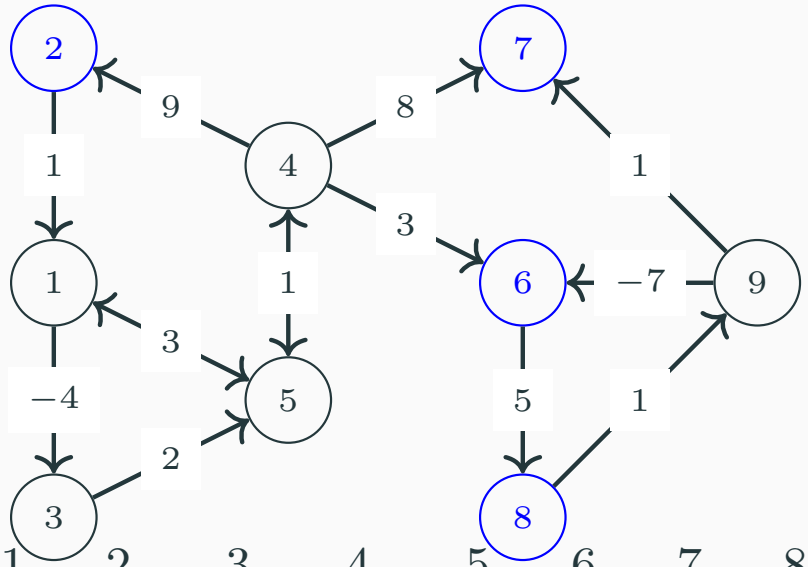
k	1	2	3	4	5	6	7	8	9
0	0	∞	∞	∞	∞	∞	∞	∞	∞
1	0	∞	-4	∞	3	∞	∞	∞	∞
2	0	∞	-4	4	-2	∞	∞	∞	∞

Example



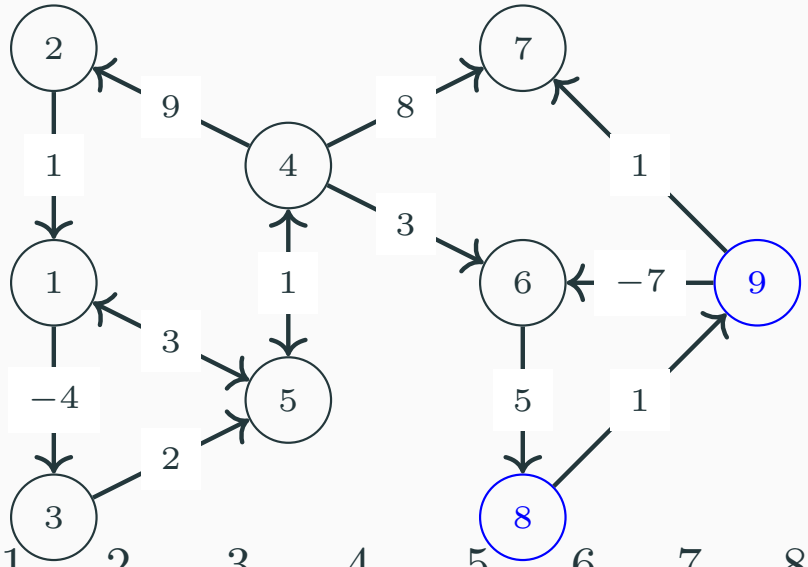
k	1	2	3	4	5	6	7	8	9
0	0	∞	∞	∞	∞	∞	∞	∞	∞
1	0	∞	-4	∞	3	∞	∞	∞	∞
2	0	∞	-4	4	-2	∞	∞	∞	∞
3	0	13	-4	-1	-2	7	12	∞	∞

Example



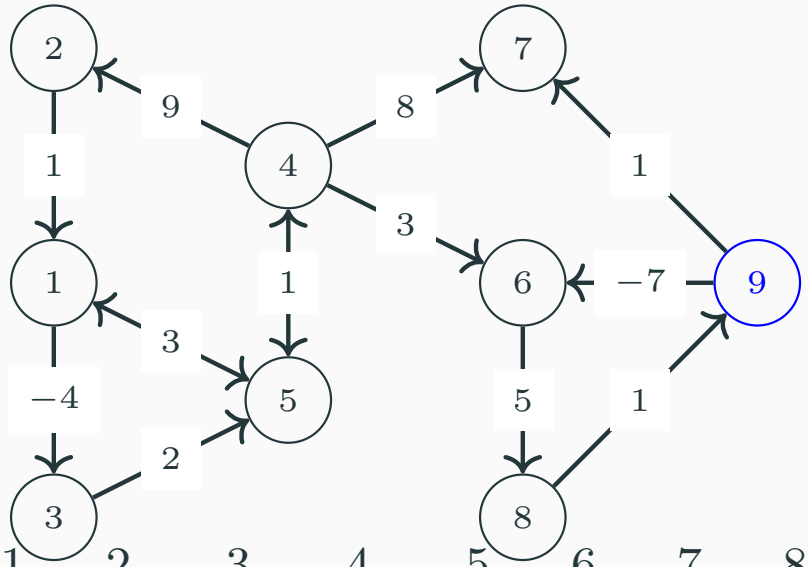
k	1	2	3	4	5	6	7	8	9
0	0	∞	∞	∞	∞	∞	∞	∞	∞
1	0	∞	-4	∞	3	∞	∞	∞	∞
2	0	∞	-4	4	-2	∞	∞	∞	∞
3	0	13	-4	-1	-2	7	12	∞	∞
4	0	8	-4	-1	-2	2	7	12	∞

Example



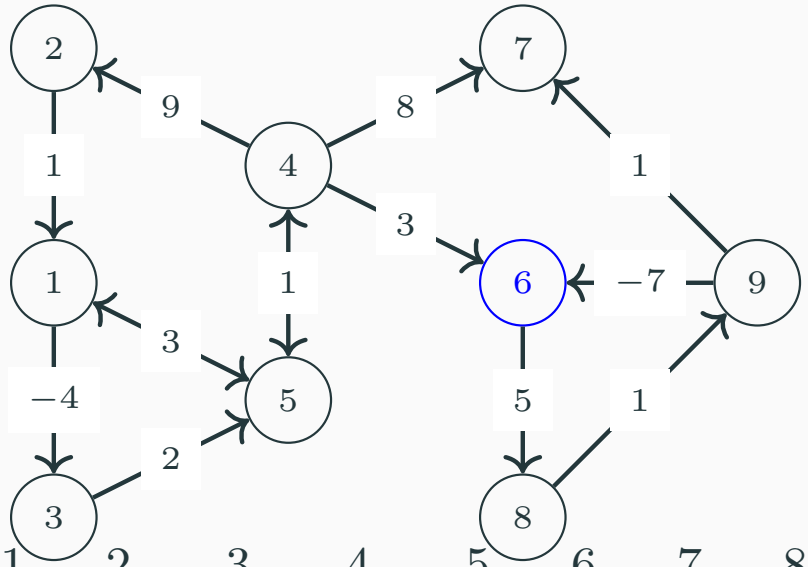
k	1	2	3	4	5	6	7	8	9
0	0	∞	∞	∞	∞	∞	∞	∞	∞
1	0	∞	-4	∞	3	∞	∞	∞	∞
2	0	∞	-4	4	-2	∞	∞	∞	∞
3	0	13	-4	-1	-2	7	12	∞	∞
4	0	8	-4	-1	-2	2	7	12	∞
5	0	8	-4	-1	-2	2	7	7	13

Example



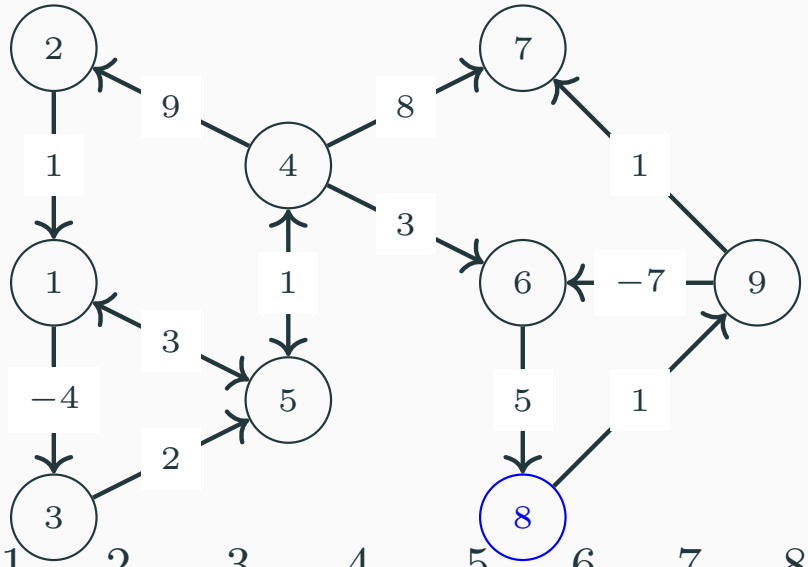
k	1	2	3	4	5	6	7	8	9
0	0	∞	∞	∞	∞	∞	∞	∞	∞
1	0	∞	-4	∞	3	∞	∞	∞	∞
2	0	∞	-4	4	-2	∞	∞	∞	∞
3	0	13	-4	-1	-2	7	12	∞	∞
4	0	8	-4	-1	-2	2	7	12	∞
5	0	8	-4	-1	-2	2	7	7	13
6	0	8	-4	-1	-2	2	7	7	8

Example



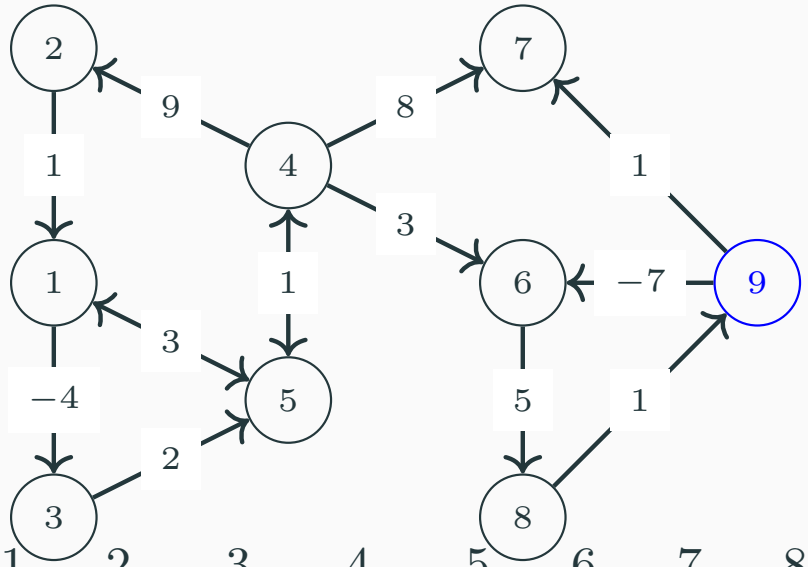
k	1	2	3	4	5	6	7	8	9
0	0	∞	∞	∞	∞	∞	∞	∞	∞
1	0	∞	-4	∞	3	∞	∞	∞	∞
2	0	∞	-4	4	-2	∞	∞	∞	∞
3	0	13	-4	-1	-2	7	12	∞	∞
4	0	8	-4	-1	-2	2	7	12	∞
5	0	8	-4	-1	-2	2	7	7	13
6	0	8	-4	-1	-2	2	7	7	8
7	0	8	-4	-1	-2	1	7	7	8

Example



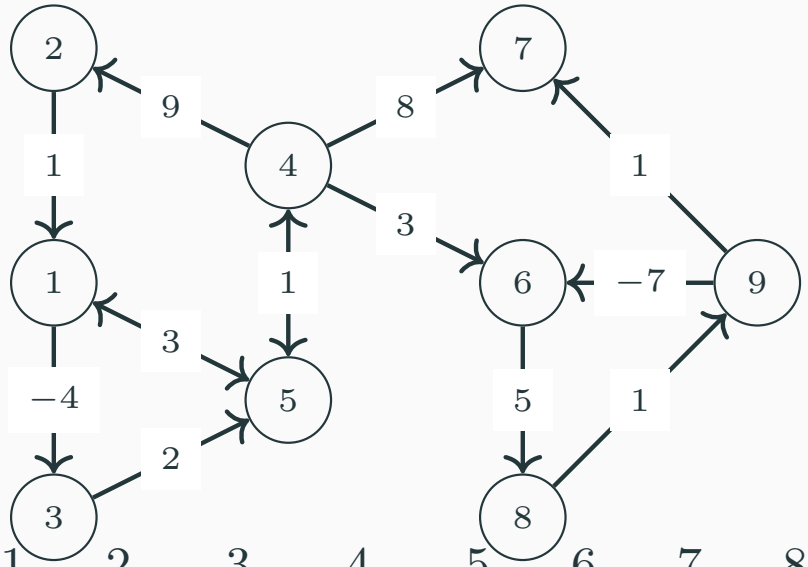
k	1	2	3	4	5	6	7	8	9
0	0	∞	∞	∞	∞	∞	∞	∞	∞
1	0	∞	-4	∞	3	∞	∞	∞	∞
2	0	∞	-4	4	-2	∞	∞	∞	∞
3	0	13	-4	-1	-2	7	12	∞	∞
4	0	8	-4	-1	-2	2	7	12	∞
5	0	8	-4	-1	-2	2	7	7	13
6	0	8	-4	-1	-2	2	7	7	8
7	0	8	-4	-1	-2	1	7	7	8
8	0	8	-4	-1	-2	1	7	6	8

Example



k	1	2	3	4	5	6	7	8	9
0	0	∞	∞	∞	∞	∞	∞	∞	∞
1	0	∞	-4	∞	3	∞	∞	∞	∞
2	0	∞	-4	4	-2	∞	∞	∞	∞
3	0	13	-4	-1	-2	7	12	∞	∞
4	0	8	-4	-1	-2	2	7	12	∞
5	0	8	-4	-1	-2	2	7	7	13
6	0	8	-4	-1	-2	2	7	7	8
7	0	8	-4	-1	-2	1	7	7	8
8	0	8	-4	-1	-2	1	7	6	8
9	0	8	-4	-1	-2	1	7	6	7

Example



k	1	2	3	4	5	6	7	8	9
0	0	∞	∞	∞	∞	∞	∞	∞	∞
1	0	∞	-4	∞	3	∞	∞	∞	∞
2	0	∞	-4	4	-2	∞	∞	∞	∞
3	0	13	-4	-1	-2	7	12	∞	∞
4	0	8	-4	-1	-2	2	7	12	∞
5	0	8	-4	-1	-2	2	7	7	13
6	0	8	-4	-1	-2	2	7	7	8
7	0	8	-4	-1	-2	1	7	7	8
8	0	8	-4	-1	-2	1	7	6	8
9	0	8	-4	-1	-2	1	7	6	7

Bellman-Ford

- How do we read negative cycles from this though?
- In a graph with no negative cycles a shortest path visits each vertex at most once
- Thus we can simply do $V - 1$ more values of k to see if anything gets shorter
- If the distance to v gets shorter then there is a negative cycle on the way to v

Bellman-Ford algorithm

```
vector<edge> adj[100];
vector<int> dist(100, INF);

void bellman_ford(int n, int start) {

    dist[start] = 0;

    for (int i = 0; i < n - 1; i++) {
        for (int u = 0; u < n; u++) {
            for (int j = 0; j < adj[u].size(); j++) {
                int v = adj[u][j].v;
                int w = adj[u][j].weight;
                dist[v] = min(dist[v], w + dist[u]);
            }
        }
    }
}
```

Bellman-Ford

- This gets us a $\mathcal{O}(VE)$ algorithm, which is quite a bit slower than Dijkstra
- A possible improvement is to store every vertex that received an update for the last k value in a queue, and only checking things they could affect at the next k
- This is still worst case $\mathcal{O}(VE)$, but in the average case much much faster
- This is generally known as the SPFA algorithm (Shortest Path Faster Algorithm)

All pairs shortest path

Bellman-Ford

- What if we want to get the distance from every vertex to every vertex?
- If there are no negative cycles running Dijkstra's from every vertex works pretty well
- But for negative cycles running Bellman-Ford V times is pretty slow
- Luckily we can get a bulk discount in this case!
- This is known as the Floyd-Warshall algorithm

Floyd-Warshall algorithm

- We use dynamic programming again.
- Let $\mathrm{sp}(k, i, j)$ be the shortest path from i to j if we're only allowed to travel through the vertices $0, \ldots, k$
- Base case: $\mathrm{sp}(k, i, j) = 0$ if $i = j$
- Base case: $\mathrm{sp}(-1, i, j) = \mathrm{weight}[i][j]$ if $(i, j) \in E$
- Base case: $\mathrm{sp}(-1, i, j) = \infty$
- $\mathrm{sp}(k, i, j) = \min \begin{cases} \mathrm{sp}(k-1, i, k) + \mathrm{sp}(k-1, k, j) \\ \mathrm{sp}(k-1, i, j) \end{cases}$

Floyd-Warshall algorithm

```
int dist[1000][1000];
int weight[1000][1000];

void floyd_warshall(int n) {
    for (int i = 0; i < n; i++) {
        for (int j = 0; j < n; j++) {
            dist[i][j] = i == j ? 0 : weight[i][j];
        }
    }

    for (int k = 0; k < n; k++) {
        for (int i = 0; i < n; i++) {
            for (int j = 0; j < n; j++) {
                dist[i][j] = min(dist[i][j], dist[i][k] + dist[k][j]);
            }
        }
    }
}
```

Floyd-Warshall algorithm

- Computes all-pairs shortest paths
- Time complexity is clearly $O(n^3)$
- Very simple to code
- Though we can do better! By combining everything we did so far and mixing in an algorithm known as Johnson's algorithm, Floyd-Warshall can be beat. But this won't be in the homework (maybe the bonus problems!)