Prefix Sum

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- We want to answer these queries efficiently, or in other words, without looking through all elements.
- Sometimes we also want to update elements.

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- Notice that $\mathrm{sum}(i, j) = \mathrm{sum}(0, j) - \mathrm{sum}(0, i-1)$

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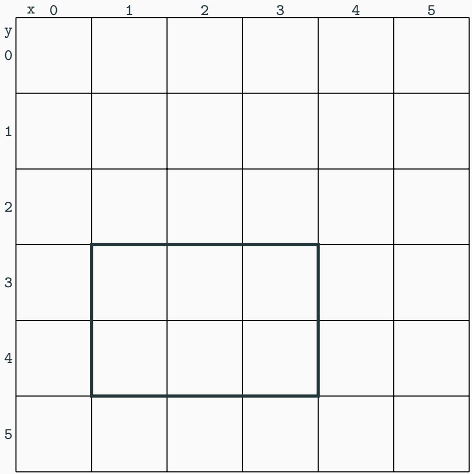
- We let $\mathrm{sum}(x_i, x_j, y_i, y_j)$ denote the query for the sum from x_i to x_j along the x-dimension, and the same for y.
- Then the formula becomes

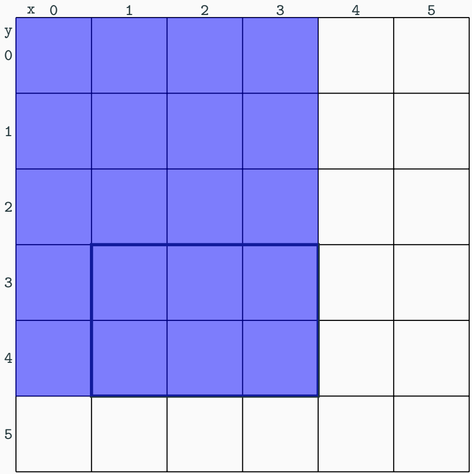
$$\begin{aligned}\mathrm{sum}(x_i, x_j, y_i, y_j) &= \mathrm{sum}(0, x_j, 0, y_j)\\ &- \mathrm{sum}(0, x_{i-1}, 0, y_j)\\ &- \mathrm{sum}(0, x_j, 0, y_{i-1})\\ &+ \mathrm{sum}(0, x_{i-1}, 0, y_{i-1})\end{aligned}$$

2D sum



`query(1, 3, 3, 4)`

2D sum



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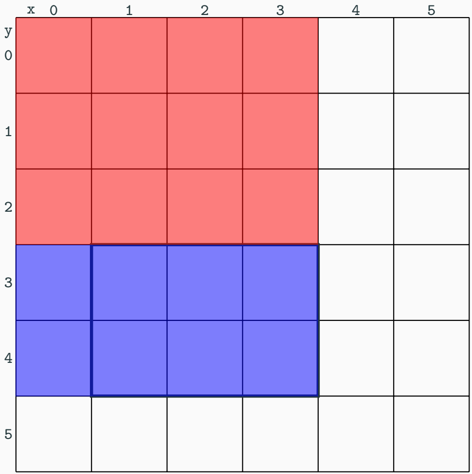

`query(1, 3, 3, 4)`

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`query(0, 4, 0, 2)`

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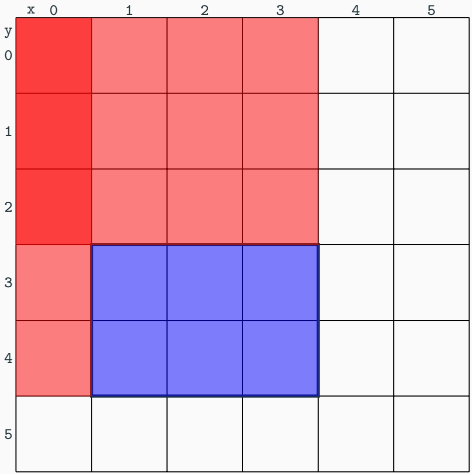

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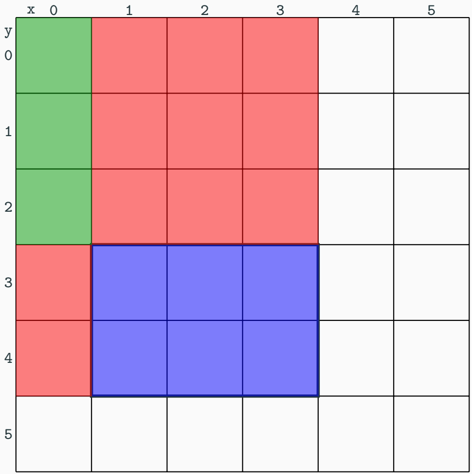


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