Data Structures

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Today's material

- Prerequisites
- Sliding Window
- Heap
- Union-Find
- Precomputations like prefix sums
- Square root decomposition
- Segment trees
- Sparse tables

Prerequisites

We assume you know how to implement the following data structures using only fixed size arrays and pointers/objects:

- Dynamically sized arrays (like vector in C++)
- Singly/doubly linked lists (like list in C++)
- Queue and stack using either of the above

We also assume you have experience using `(unordered_){map,set}`

Sliding Window

A Sum Problem

Problem description
Write a program that, given an integer array of size N, finds the contiguous subarray of size K with the highest sum.

Input description
Input consist of two lines. The first line contains two space separated integers N, the size of the array, where $1 \leq N \leq 10^6$, and K, the size of the subarrays to consider, where $1 \leq K \leq N$. Then second line contains N space separated integers, the values of the array. Each value in the array is between -10^9 and 10^9.

Output description
Output one line, the sum of the highest valued contiguous subarray of size K.

A Sum Problem

Sample input	Sample output
10 4 17 20 0 1 5 24 8 2 4 1	39

Straightforward Solution

```python
n, k = map(int, input().split())
arr = list(map(int, input().split()))
highest = float('-inf')
for start in range(n-k+1):
    end = start + k
    total = 0
    for i in range(start, end):
        total += arr[i]
    highest = max((highest, total))
print(highest)
```

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- This solution constructs all size K contiguous subarrays.

Straightforward Solution

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n, k = map(int, input().split())
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for start in range(n-k+1):
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- This solution constructs all size K contiguous subarrays.
- What is the time complexity?

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- This solution constructs all size K contiguous subarrays.
- What is the time complexity?
- There are N starting points, each construction takes K steps, so $\mathcal{O}(NK)$.

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- This solution constructs all size K contiguous subarrays.
- What is the time complexity?
- There are N starting points, each construction takes K steps, so $\mathcal{O}(NK)$.
- Too slow!

Wasted Operations

- The subarray starting at index i has the sum $a_i + a_{i+1} + \cdots + a_{i+k-1}$.

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- We iterate over the indices $i+1, i+2, \ldots, i+k-1$ twice.

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- What changes between starting at i vs. starting at $i+1$?
- We subtract a_i.

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- We add a_{i+k}.

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- What changes between starting at i vs. starting at $i+1$?
- We subtract a_i.
- We add a_{i+k}.
- A shift from the subarray starting at i to the subarray starting at $i+1$ takes $\mathcal{O}(1)$ time.

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- We iterate over the indices $i + 1, i + 2, \ldots, i + k - 1$ twice.
- What changes between starting at i vs. starting at $i + 1$?
- We subtract a_i.
- We add a_{i+k}.
- A shift from the subarray starting at i to the subarray starting at $i + 1$ takes $\mathcal{O}(1)$ time.
- This is known as the sliding window technique, in this case with a fixed window size.

Sliding Window Solution

```python
n, k = map(int, input().split())
arr = list(map(int, input().split()))
total = 0
for i in range(k):
    total += arr[i]
highest = total
for i in range(n - k):
    total -= arr[i]
    total += arr[i+k]
    highest = max((highest, total))
print(highest)
```

Sliding Window Solution

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- What is the time complexity?
- This solution constructs the first size K contiguous subarray.

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- What is the time complexity?
- This solution constructs the first size K contiguous subarray.
- Then, $N - K$ times, an element is removed and another added.

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- This solution constructs the first size K contiguous subarray.
- Then, $N - K$ times, an element is removed and another added.
- Subtracting and adding numbers is constant time so $\mathcal{O}(N)$.

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- What is the time complexity?
- This solution constructs the first size K contiguous subarray.
- Then, $N - K$ times, an element is removed and another added.
- Subtracting and adding numbers is constant time so $\mathcal{O}(N)$.
- Fast enough!

A Substring Problem

Problem description
Write a program that, give a string of size N, finds the longest substring with K distinct elements.

Input description
Input consist of two lines. The first line contains two space separated integers N, the size of the string, where $1 \leq N \leq 10^6$, and K, the number of distinct elements the substring must have, where $1 \leq K \leq 26$. Then second line contains a string of length N consisting of English lowercase characters.

Output description
Output one line, the longest substring with K distinct elements. If no such string exists, output "`DOES NOT EXIST`", without quotations.

A Substring Problem

Sample input	Sample output
14 3 bacdcbcbcbabdb	cdcbcbcb

General Framework

```
from string import ascii_lowercase
n, k = map(int, input().split())
s = input()

best_ind, best_len = distinct_k(n, k, s)

if best_len == -1:
  print("DOES NOT EXIST")
else:
  print(s[best_ind:best_ind + best_len])
```

Straightforward Solution

```python
def distinct_k(n, k, s):
  best_ind, best_len = -1, -1
  for start in range(n):
    for end in range(start, n+1):
      substring = s[start:end]
      distinct = 0
      for symbol in ascii_lowercase:
        if symbol in substring:
          distinct += 1
      cur_len = len(substring)
      if distinct == k and cur_len > best_len:
        best_ind = start
        best_len = cur_len
  return best_ind, best_len
```

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def distinct_k(n, k, s):
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- There are $\mathcal{O}(N^2)$ substrings of the string

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- What is the time complexity?
- There are $\mathcal{O}(N^2)$ substrings of the string
- Checking each one takes us $\mathcal{O}(N)$ time, so $\mathcal{O}(N^3)$ in total.
- Way too slow!

Constant optimization

```python
def distinct_k(n, k, s):
  best_ind, best_len = -1, -1
  for start in range(n):
    for end in range(start, n+1):
      substring = s[start:end]
      present = [False for _ in range(26)]
      for symbol in substring:
        present[ord(symbol) - ord('a')] = True
      distinct = sum(present)
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- This is a little faster, by a factor of 26 approximately.

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- This is a little faster, by a factor of 26 approximately.
- Time complexity is the same.

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- Note that `counts` barely differs between adjacent values of `end`

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- This is a little faster, by a factor of 26 approximately.
- Time complexity is the same.
- Note that `counts` barely differs between adjacent values of `end`
- Build it as the substring grows.

Incremental

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def distinct_k(n, k, s):
  best_ind, best_len = -1, -1
  for start in range(n):
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      present[ord(s[end]) - ord('a')] = True
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- Now each substring is processed in constant time.

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- Now each substring is processed in constant time.
- Time complexity is $\mathcal{O}(N^2)$

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- Now each substring is processed in constant time.
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- For a given value of `ind`, adjacent `start` values have similar values of `counts`.

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- Now each substring is processed in constant time.
- Time complexity is $\mathcal{O}(N^2)$
- For a given value of `ind`, adjacent `start` values have similar values of `counts`.
- Note that adding characters will never decrease `distinct`.

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- Now each substring is processed in constant time.
- Time complexity is $\mathcal{O}(N^2)$
- For a given value of `ind`, adjacent `start` values have similar values of `counts`.
- Note that adding characters will never decrease `distinct`.
- However, removing elements from the front may reduce `distinct`.

Sliding Window

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def distinct_k(n, k, s):
  best_ind, best_len = -1, -1
  start, end, distinct = 0, 0, 0
  count = [0 for _ in range(26)]
  while start < n:
    while end < n:
      c = ord(s[end]) - ord('a')
      if distinct == k and count[c] == 0:
        break
      count[c] += 1
      end += 1
      distinct = sum(x > 0 for x in count)
    cur_len = end - start
    if distinct == k and cur_len > best_len:
      best_ind = start
      best_len = cur_len
    count[ord(s[start]) - ord('a')] -= 1
    start += 1
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- What is the time complexity? It may seem quadratic at first

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- Each element gets added and removed once, so $\mathcal{O}(N)$.

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- What is the time complexity? It may seem quadratic at first
- Each element gets added and removed once, so $\mathcal{O}(N)$.
- Lets introduce C, the number of different symbols possible.

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- Each element gets added and removed once, so $\mathcal{O}(N)$.
- Lets introduce C, the number of different symbols possible.
- The time complexity is actually $\mathcal{O}(NC)$, but we can do better!

Sliding Window Improved

```python
def distinct_k(n, k, s):
  best_ind, best_len = -1, -1
  start, end, distinct = 0, 0, 0
  count = [0 for _ in range(26)]
  while start < n:
    while end < n:
      c = ord(s[end]) - ord('a')
      if distinct == k and count[c] == 0:
        break
      if count[c] == 0:
        distinct += 1
      count[c] += 1
      end += 1
    cur_len = end - start
    if distinct == k and cur_len > best_len:
      best_ind = start
      best_len = cur_len
    c = ord(s[start]) - ord('a')
    count[c] -= 1
    if count[c] == 0:
      distinct -= 1
    start += 1
    distinct = sum(x > 0 for x in count)
  return best_ind, best_len
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- Now adding/removing an element is $\mathcal{O}(1)$.

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- Now adding/removing an element is $\mathcal{O}(1)$.
- The time complexity is now $\mathcal{O}(N + C)$.

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- Usually you want the maximal or the minimal window fulfilling a certain condition.

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- Step 2: Check if the current window is a better answer and possibly update. Then go to step 3.
- Step 3: Perform `remove` and go to step 1.
- Time complexity is $\mathcal{O}(N \cdot (X + Y))$ where X and Y are the cost of `add` and `remove`, respectively.

Heap

Heaps

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- Heaps are implemented in most standard libraries in the forms of priority queues.
- A heap is nothing but a binary tree satisfying *the heap condition*.
- The heap condition (for a min heap) says that the value of any given node is not greater than that of its chilren.

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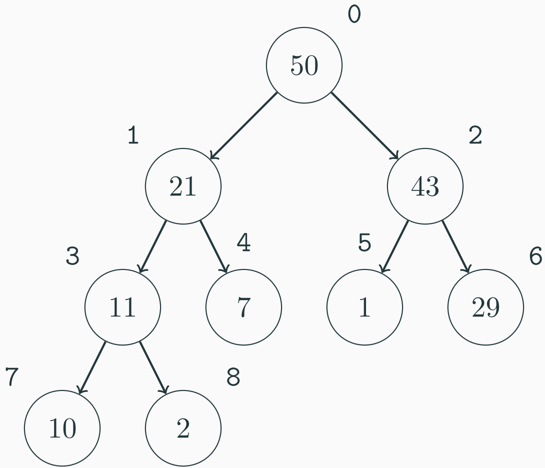
- We can do this by putting the root at index 1. Then the children of item at index i are simply at $2i$ and $2i+1$. The parent of any item $i > 1$ is then $\left\lfloor \frac{i}{2} \right\rfloor$.
- We could do this using raw arrays (then index 0 can be used to store its size), but the examples will be given in C++ using vectors.

Heaps




```
ARRAY: [SIZE, 50, 21, 43, 11, 7, 1, 29, 10, 2]
```

Heaps

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- Items can be deleted by replacing the smallest value with a leaf and then fixing the heap condition downwards.
- Let us see how this would look in C++.

C++ implementation (min-heap)

```
template<typename T> struct Heap {
    vector<T> h; Heap() : h(1) { }
    constexpr size_t size() { return h.size() - 1; }
    constexpr T peek() { return h[1]; }
    void swim(size_t i) {
        while(i != 1 && h[i] < h[i / 2]) {
            swap(h[i], h[i / 2]);
            i /= 2; } }
    void sink(size_t i) {
        while(true) {
            size_t mn = i;
            if(2 * i + 1 < h.size() && h[mn] > h[2 * i + 1]) mn = 2 * i + 1;
            if(2 * i  < h.size() && h[mn] > h[2 * i]) mn = 2 * i;
            if(mn != i) swap(h[i], h[mn]), i = mn;
            else break; } }
    void pop() {
        h[1] = h.back();
        h.pop_back(); sink(1); }
    void push(T x) {
        h.push_back(x);
        swim(h.size() - 1); } };
```

Heaps

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- We provide it for demonstration of representing binary trees with an array.

Union-Find

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- Operation `union(x, y)` unions the sets of which x and y are members.

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- `join(1, 4)` finally results in $\{\{1, 2, 3, 4, 5\}\}$.
- At any given point `find(x)` returns some value in the same set as x.
- The important bit is that `find(x)` returns the same value for all elements of the same set, the representative.

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- To get the representative of x we go to the parent of our current item (starting at x) until the item has no parent.
- Then to unite x, y we simply make the representative of x the parent of the representative of y.

Naïve Union-Find implementation

```
struct union_find {
    vector<int> parent;
    union_find(int n) {
        parent = vector<int>(n);
        for(int i = 0; i < n; i++) {
            parent[i] = i;
        }
    }
    int find(int x) {
        return parent[x] == x ? x : find(parent[x]);
    }
    void unite(int x, int y) {
        parent[find(x)] = find(y);
    }
};
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- We can also do this by flattening the chain each time we query `find`, so the amortized complexity becomes good.
- Here the worst case is still $\mathcal{O}(n)$ but the amortized complexity is $\mathcal{O}(\alpha(n))$ which may as well be a constant, as it is < 5 for n equal to the number of atoms in the observable universe.

Path compressed Union-Find implementation

```
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    union_find(int n) {
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        for (int i = 0; i < n; i++) {
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    }
    int find(int x) {
        if(parent[x] == x) return x;
        return parent[x] = find(parent[x]);
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    void unite(int x, int y) {
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- By modifying the data structure it can also contain more queryable data
  - Number of different sets currently
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  - An iterable list of all elements of the set containing x
- When tracking size you can use it to always perform small-to-large merges for $\mathcal{O}(\log n)$ time complexity.

Example problem: Skolavslutningen

- https://open.kattis.com/problems/skolavslutningen

Range Queries

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- We want to answer these queries efficiently, or in other words, without looking through all elements.
- Sometimes we also want to update elements.

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- How do we support these queries efficiently?
- Simplification: only support queries of the form $\mathrm{sum}(0, j)$
- Notice that $\mathrm{sum}(i, j) = \mathrm{sum}(0, j) - \mathrm{sum}(0, i - 1)$

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- Can we support updating efficiently? No, at least not without modification

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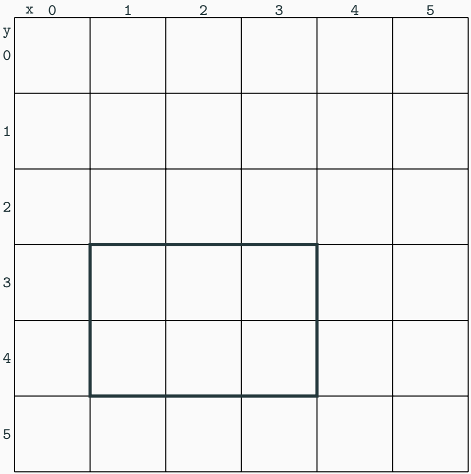
- Then the formula becomes

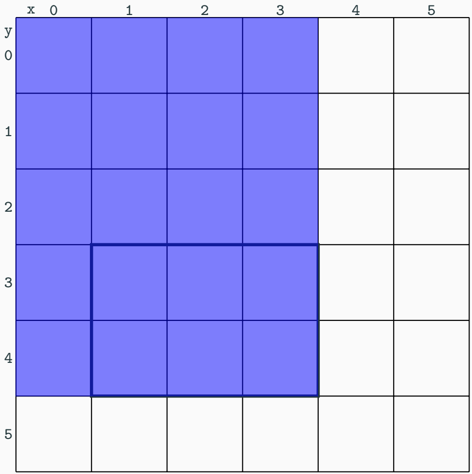
$$\begin{aligned}\mathrm{sum}(x_i, x_j, y_i, y_j) &= \mathrm{sum}(0, x_j, 0, y_j)\\ &- \mathrm{sum}(0, x_{i-1}, 0, y_j)\\ &- \mathrm{sum}(0, x_j, 0, y_{i-1})\\ &+ \mathrm{sum}(0, x_{i-1}, 0, y_{i-1})\end{aligned}$$

2D sum



query(1, 3, 3, 4)

2D sum



`query(1, 3, 3, 4)`

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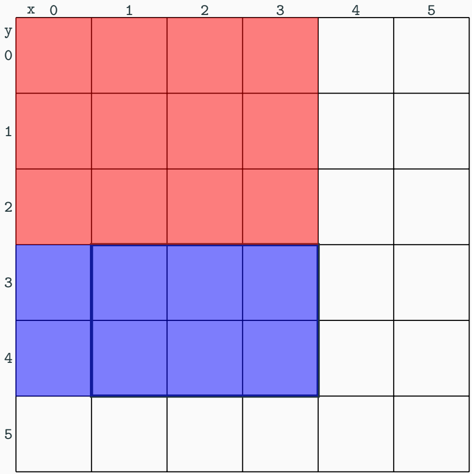

query(1, 3, 3, 4)

query(0, 3, 0, 4)

query(0, 4, 0, 2)

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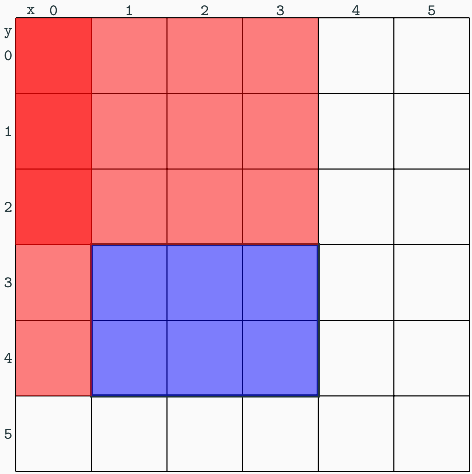


```
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query(0, 3, 0, 4)

query(0, 4, 0, 2)

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```

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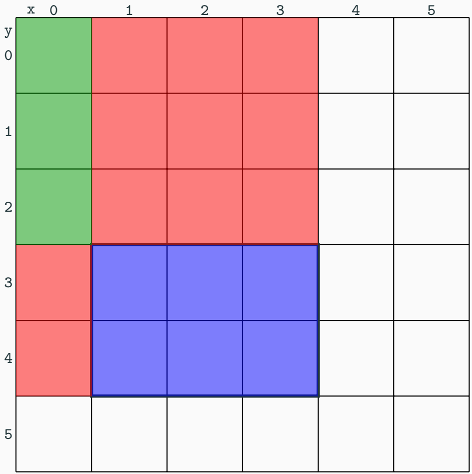


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  - Updating in $O(\sqrt{n})$
  - Querying in $O(n/\sqrt{n} + \sqrt{n}) = O(\sqrt{n})$
- Also known as square root decomposition, and is a very powerful technique

Example problem: Supercomputer

- https://open.kattis.com/problems/supercomputer

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- Can we do better?

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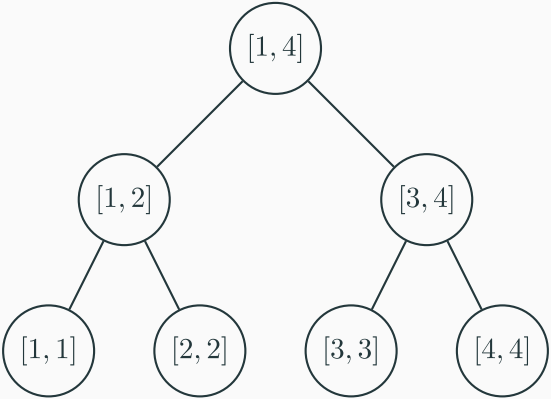
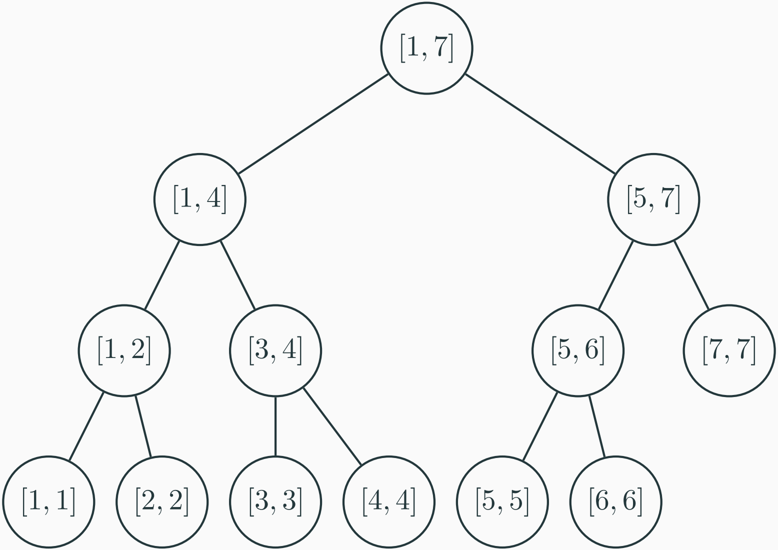
- We travel down the tree looking for the left and right end points, adding intervals that are completely inside our query range.
- When we update a value we only need to update the parents of that node up to the root, at most $\mathcal{O}(\log(n))$ nodes.

Drawn Segment Tree, $n = 4$



Drawn Segment Tree, $n = 7$

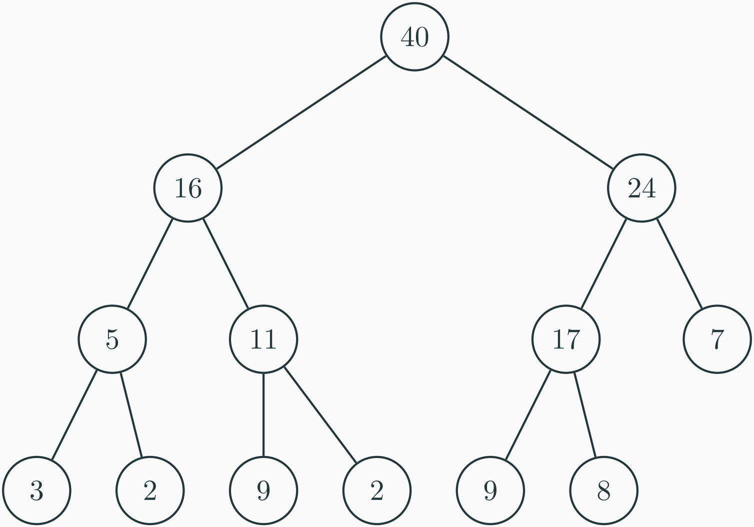


Segment Tree - Code

```
struct segment_tree {
    segment_tree *left, *right;
    int from, to, value;
    segment_tree(int from, int to)
        : from(from), to(to), left(NULL), right(NULL), value(0) { }
};

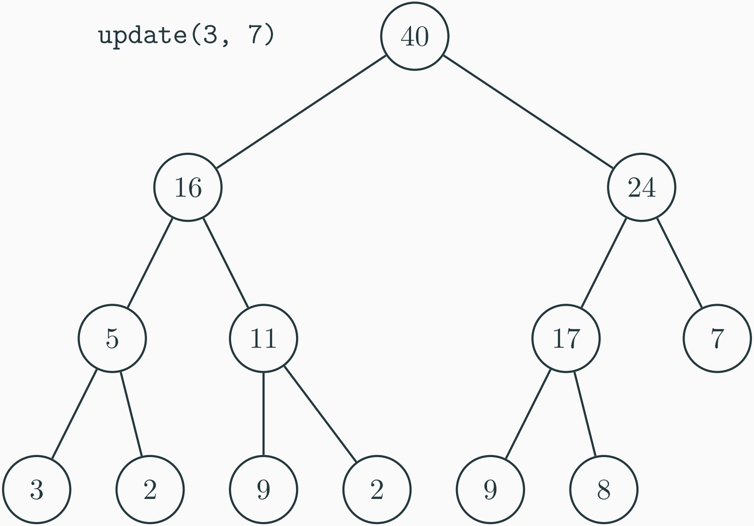
segment_tree* build(const vector<int> &arr, int l, int r) {
    if (l > r) return NULL;
    segment_tree *res = new segment_tree(l, r);
    if (l == r) {
        res->value = arr[l];
    } else {
        int m = (l + r) / 2;
        res->left = build(arr, l, m);
        res->right = build(arr, m + 1, r);
        if (res->left != NULL) res->value += res->left->value;
        if (res->right != NULL) res->value += res->right->value;
    }
    return res;
}
```

Updates

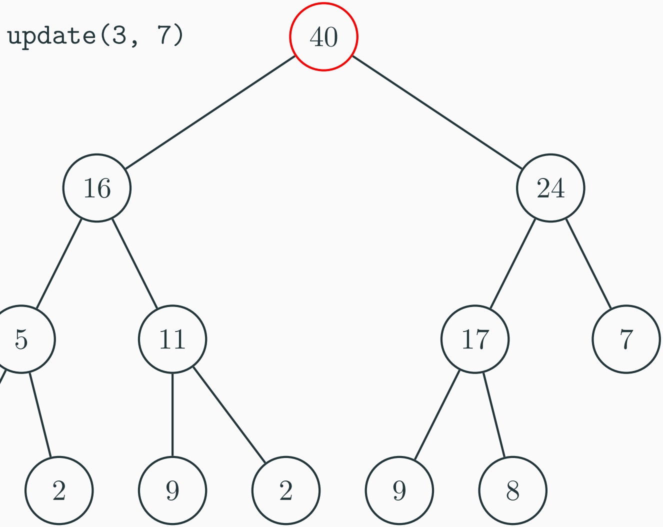


Updates

`update(3, 7)`



Updates



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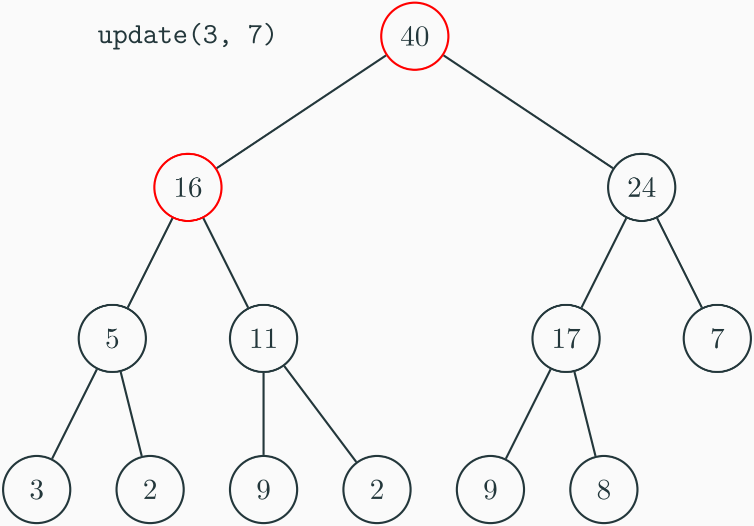

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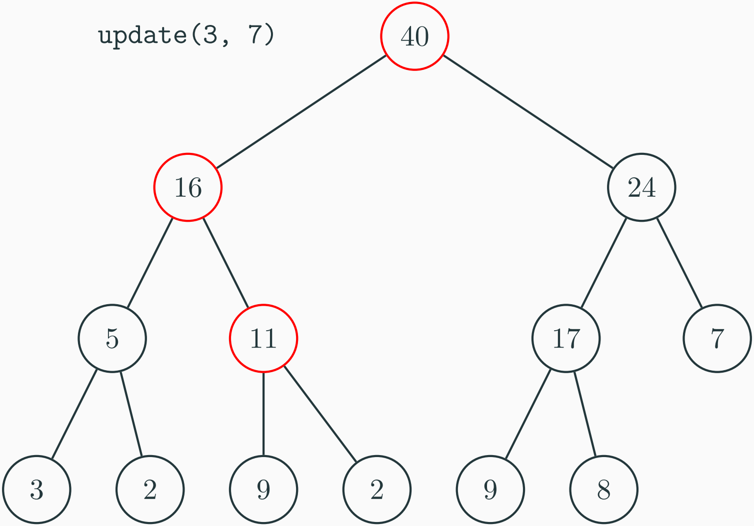

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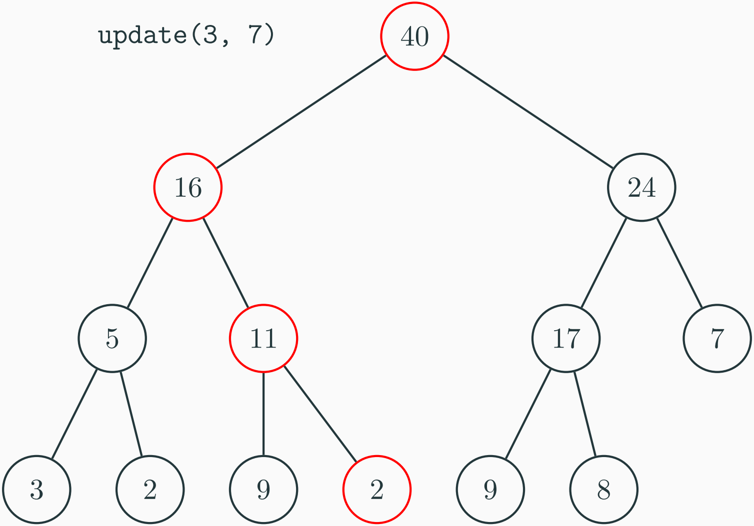

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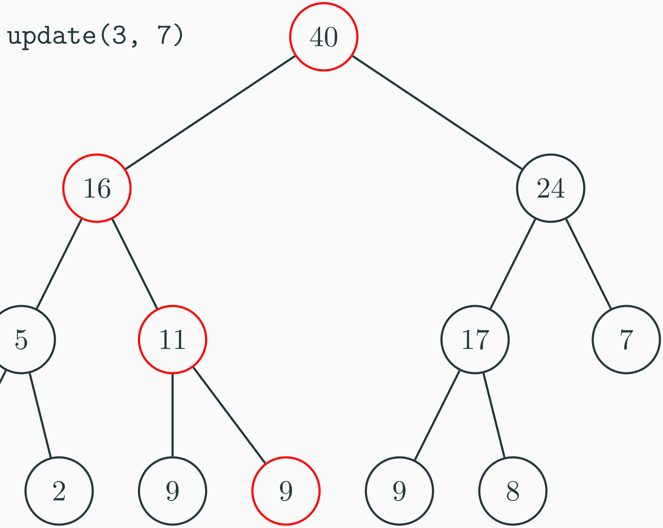

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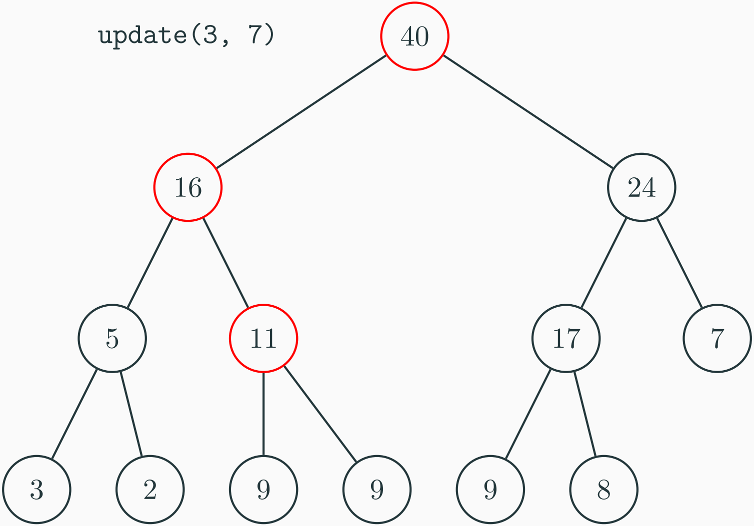

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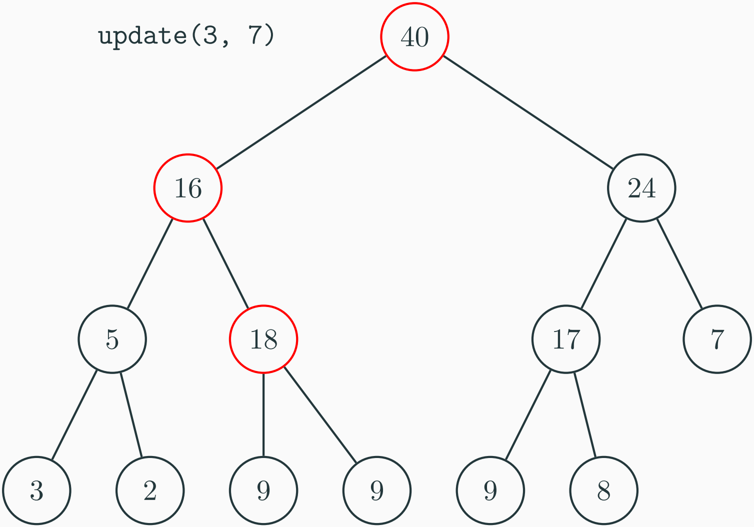

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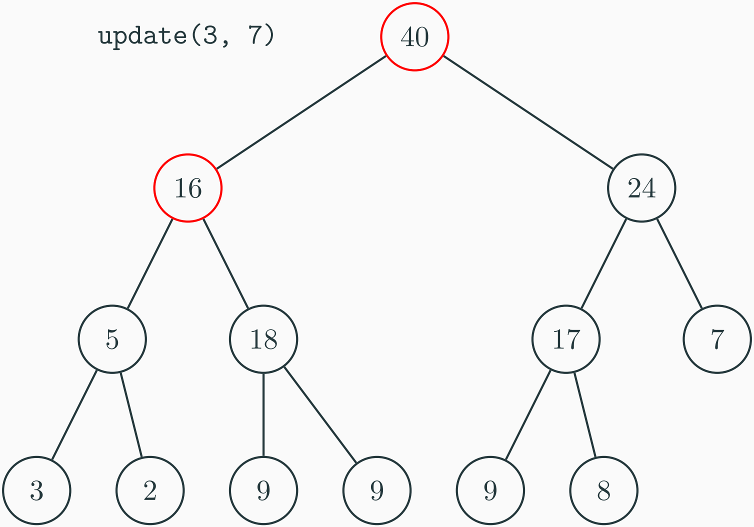

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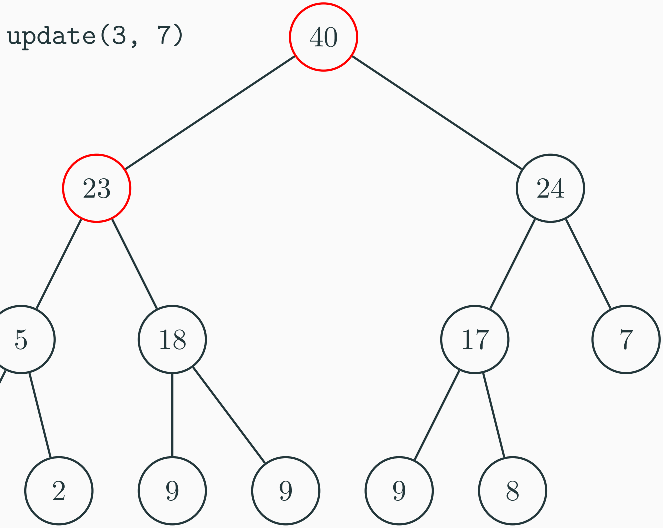

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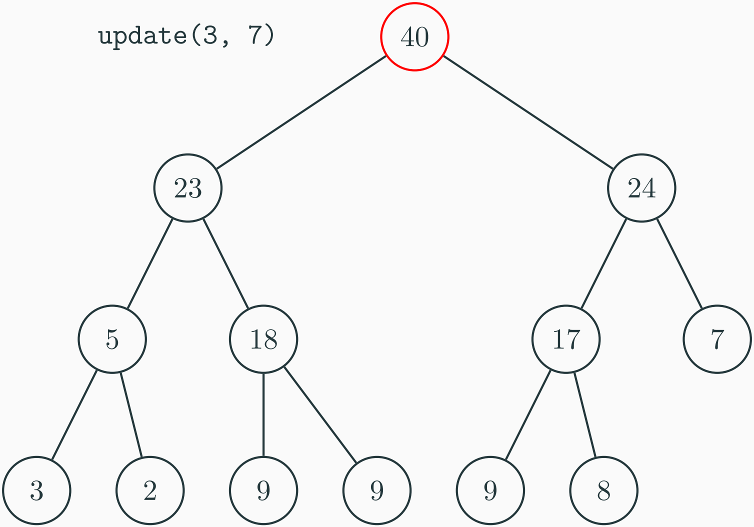

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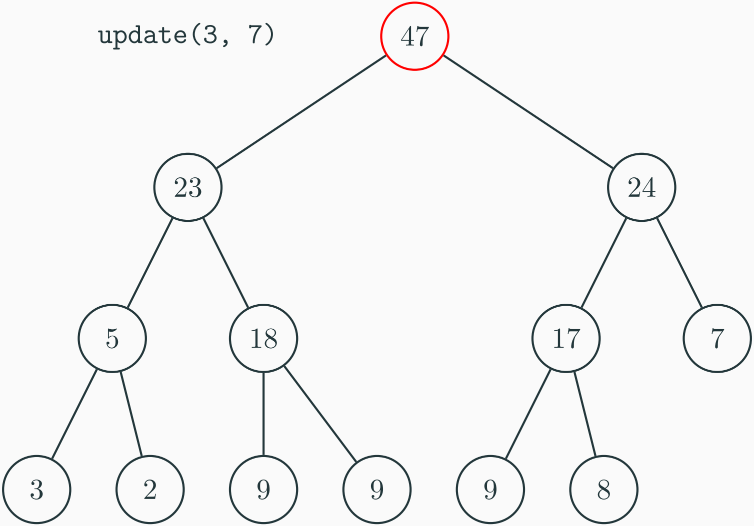

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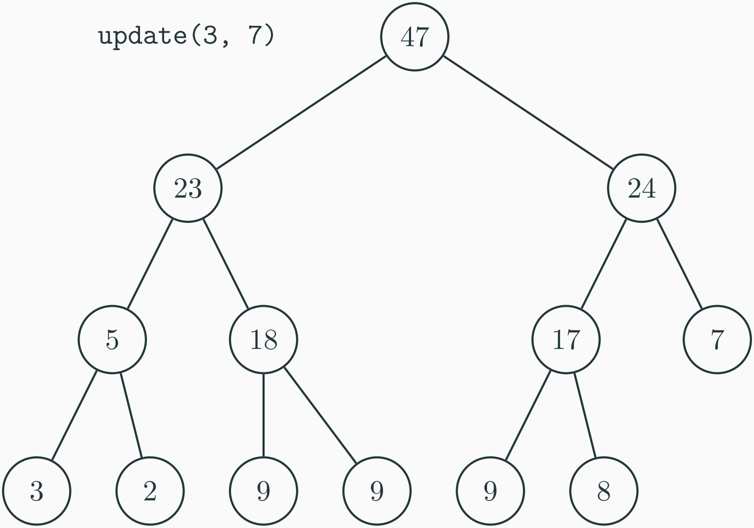

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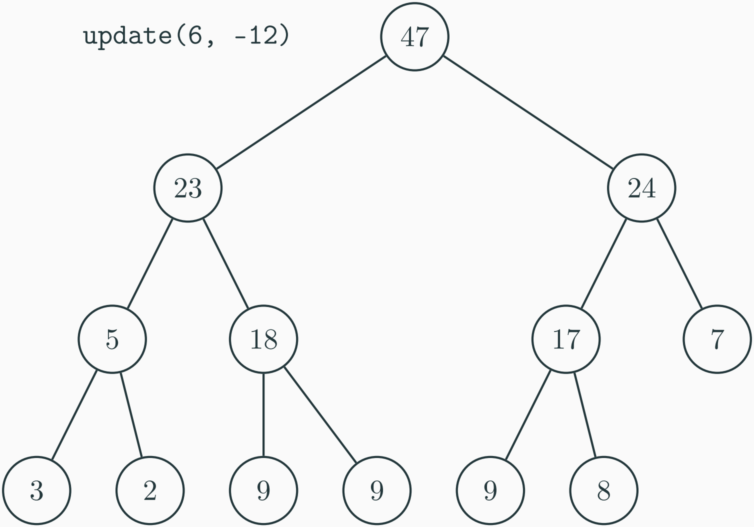

Updates

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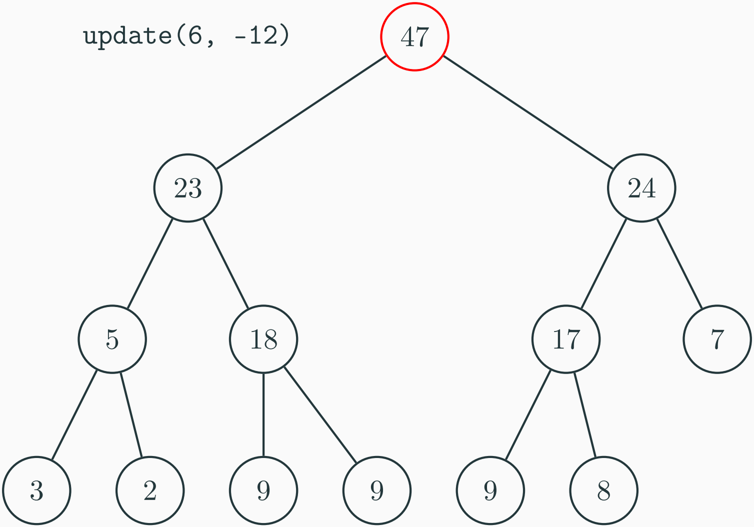

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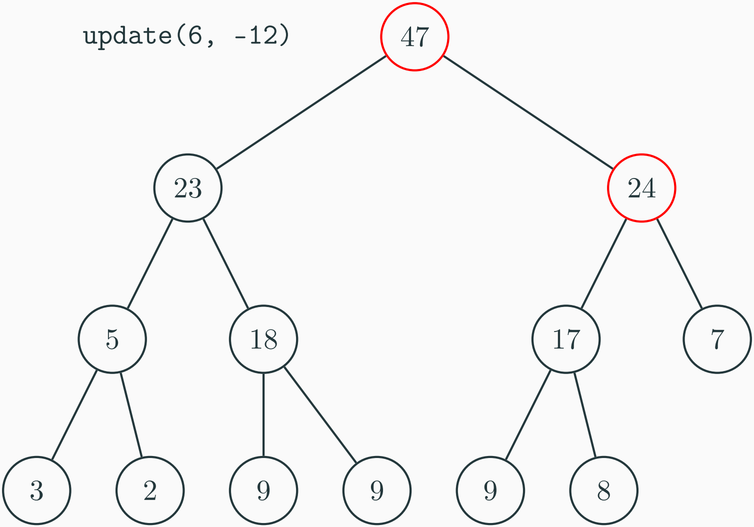

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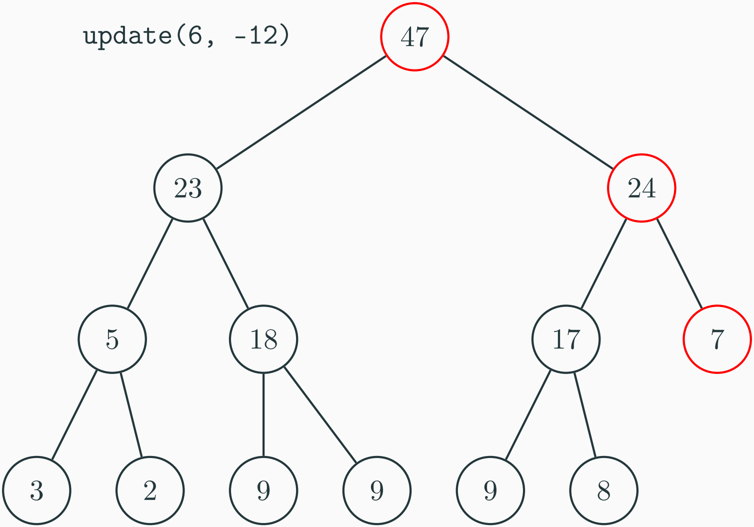

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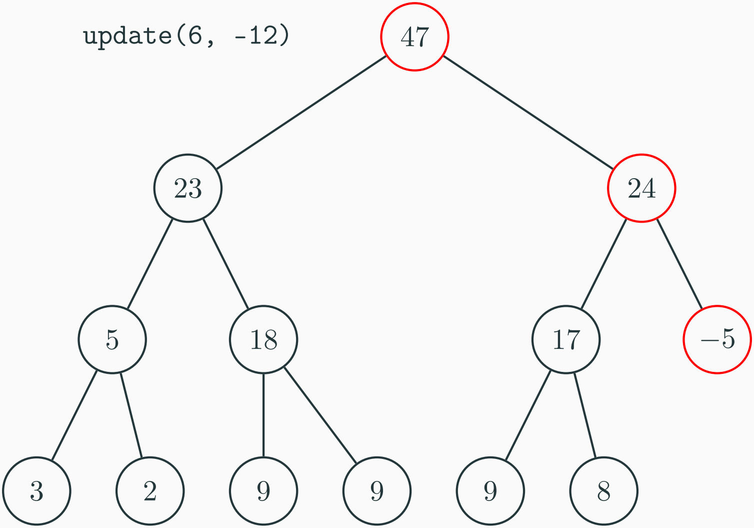

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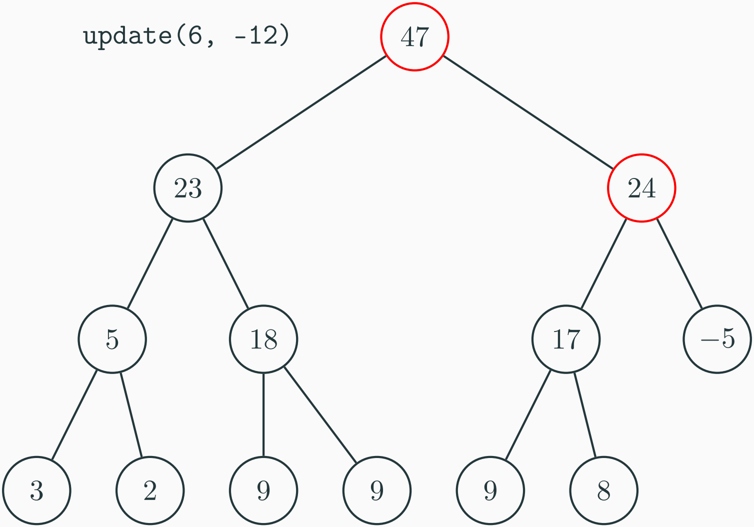

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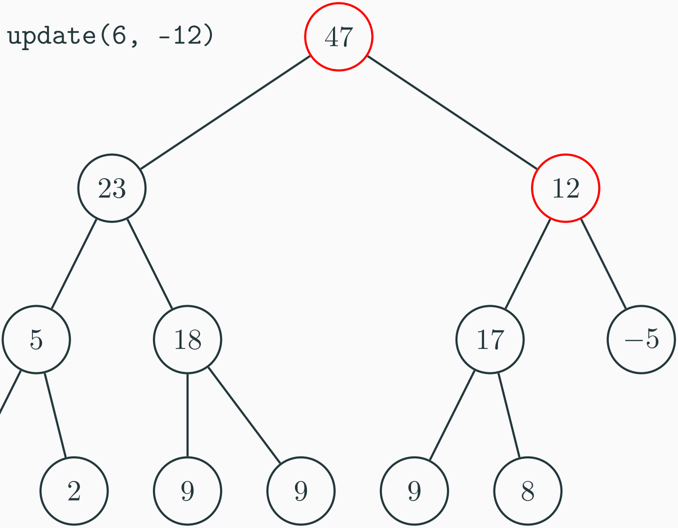

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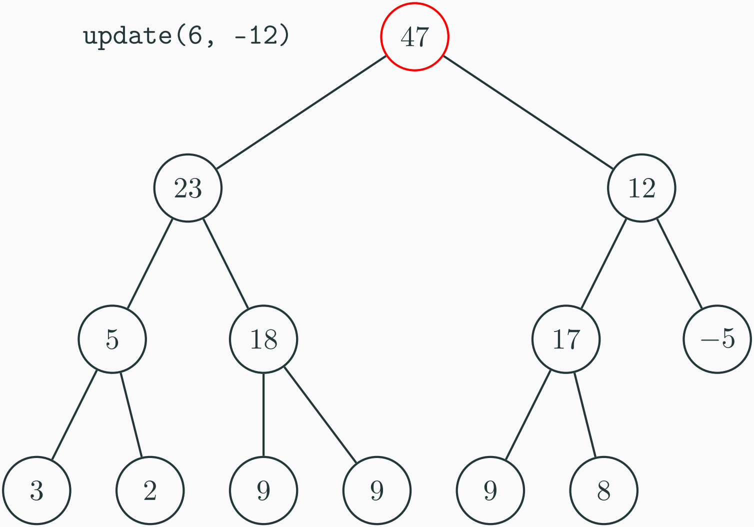

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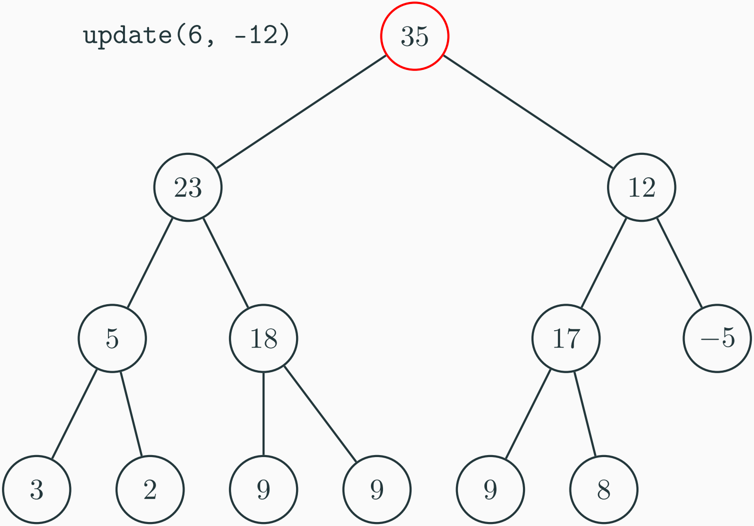

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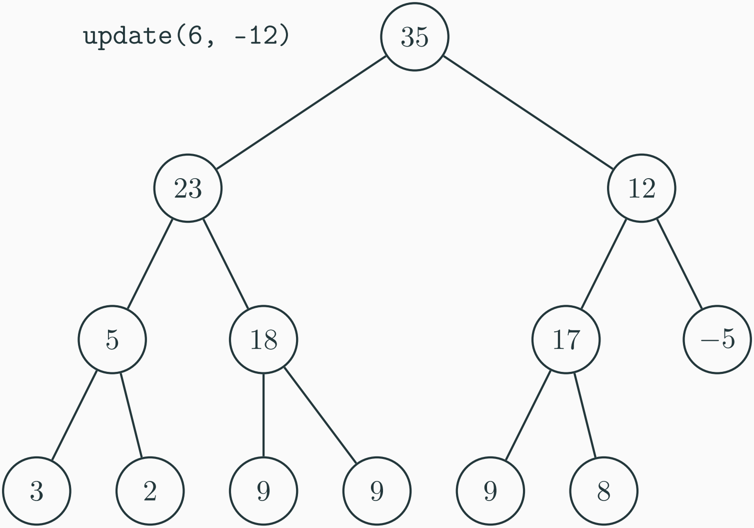

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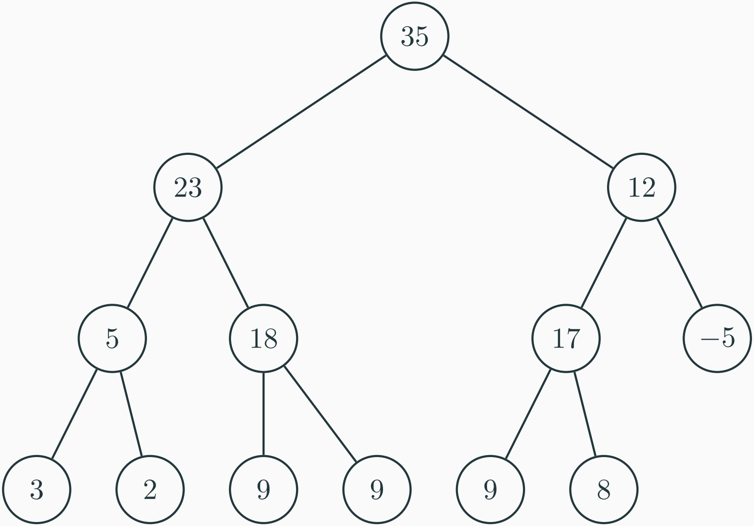
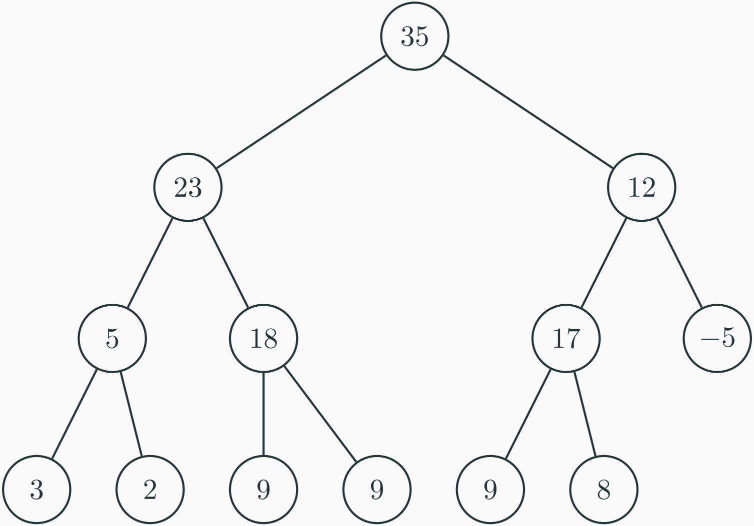


Updating a Segment Tree - Code

```c
int update(segment_tree *tree, int i, int val) {
    if (tree == NULL) return 0;
    if (tree->to < i) return tree->value;
    if (i < tree->from) return tree->value;
    if (tree->from == tree->to && tree->from == i) {
        tree->value = val;
    } else {
        tree->value = update(tree->left, i, val) + update(tree->right, i, val);
    }
    return tree->value;
}
```

Querying



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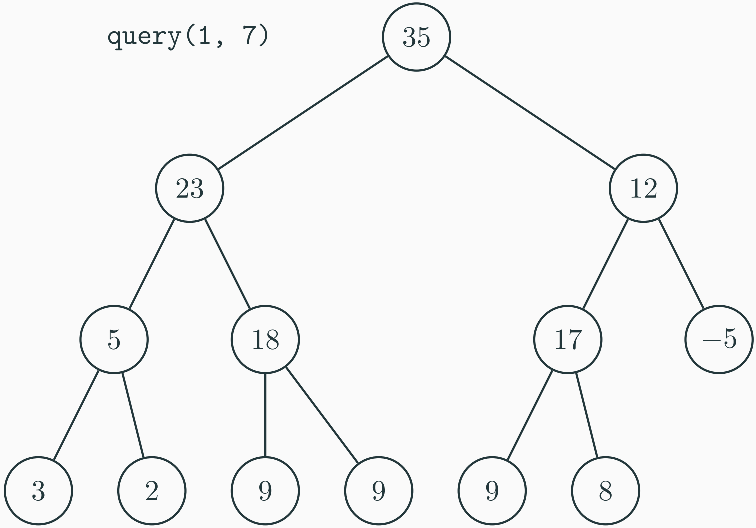

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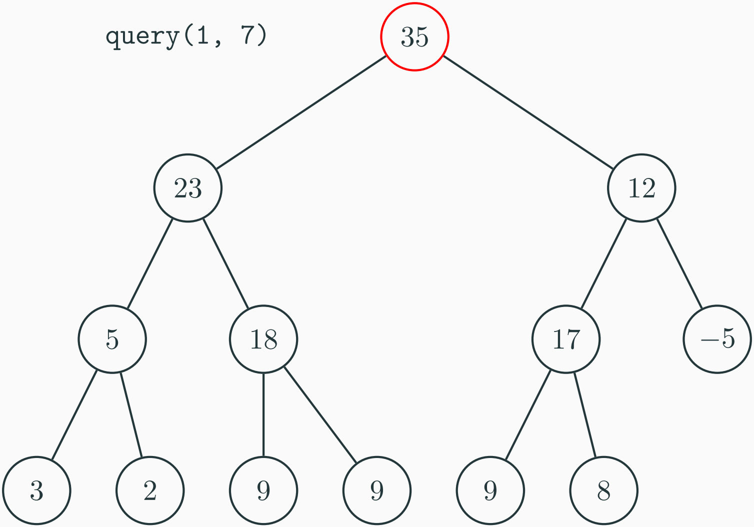

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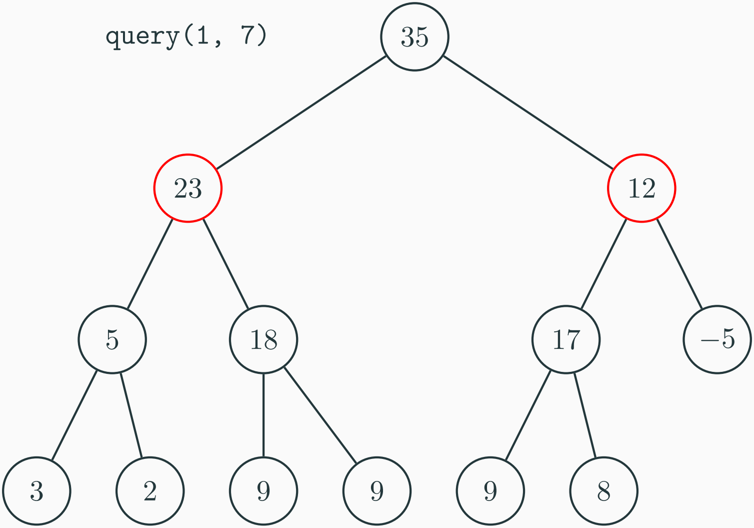
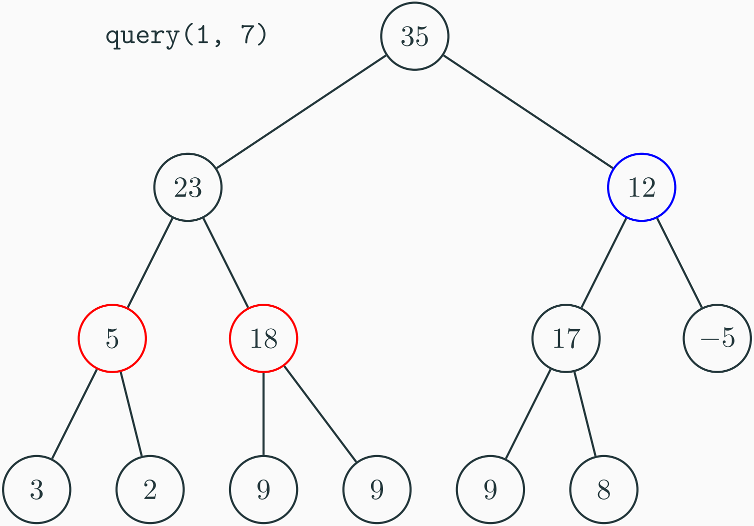


Querying

`query(1, 7)`



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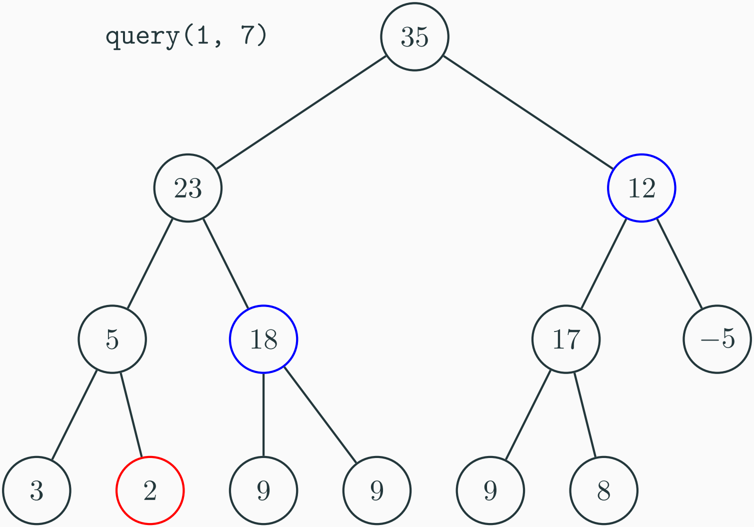

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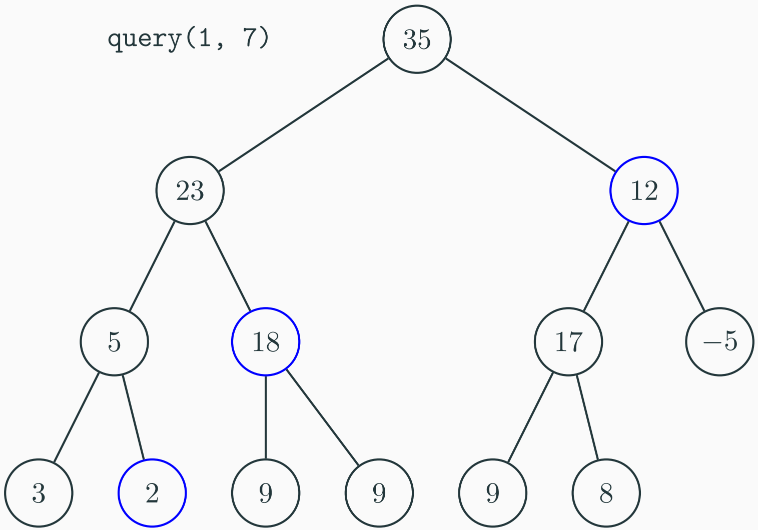
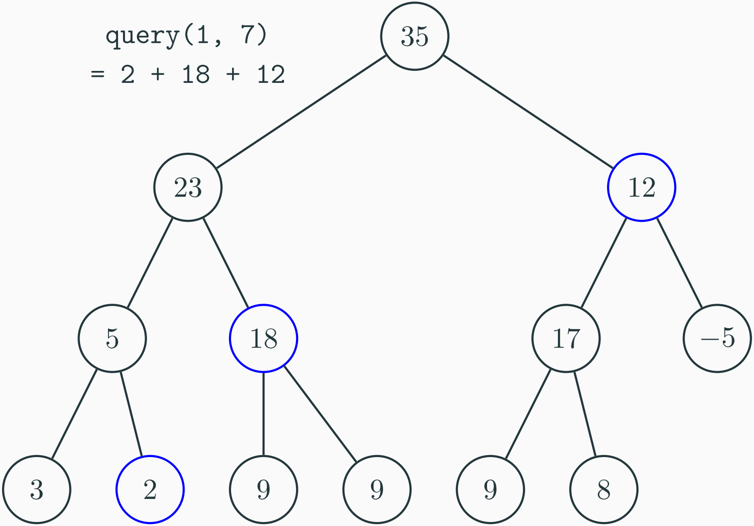


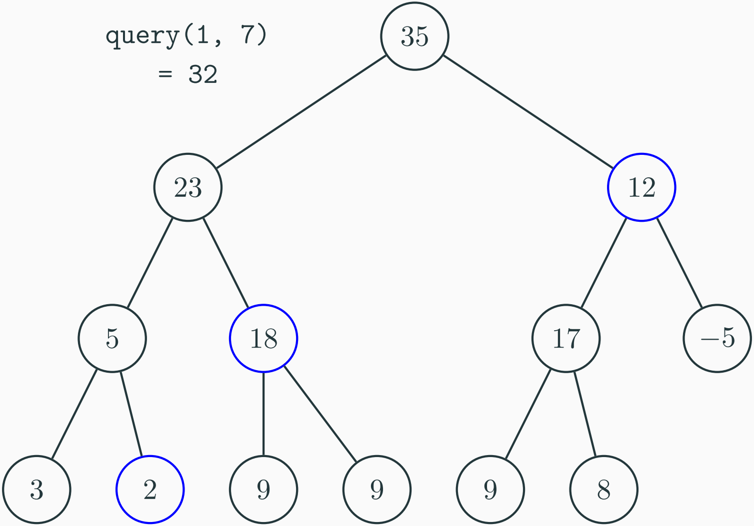
Querying

```
query(1, 7)
= 2 + 18 + 12
```



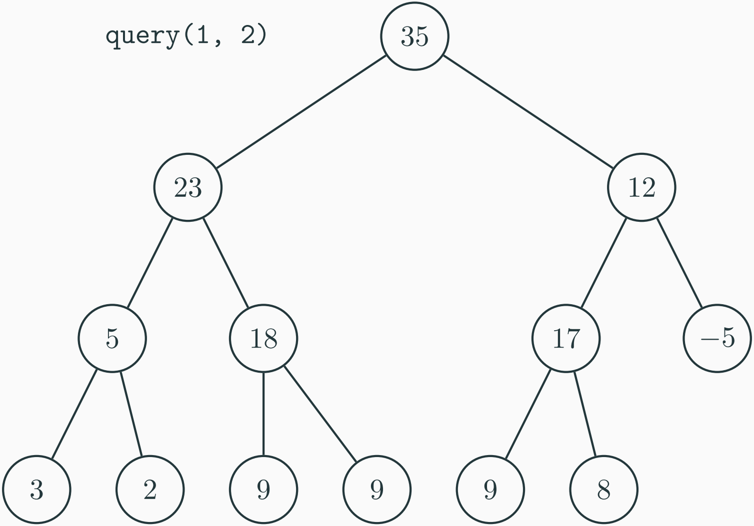
Querying

```
query(1, 7)
    = 32
```

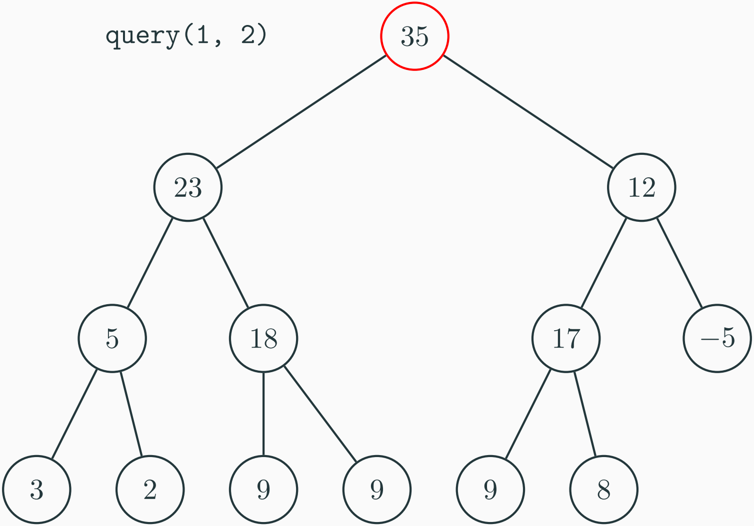


Querying

`query(1, 2)`



Querying



Querying

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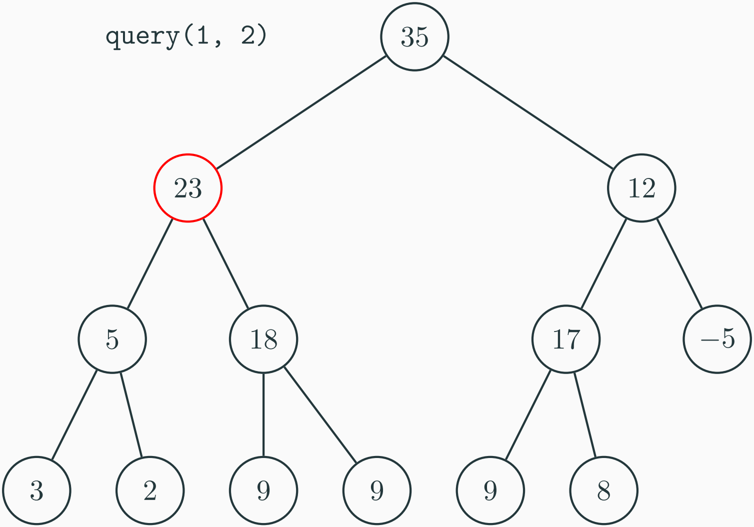

Querying

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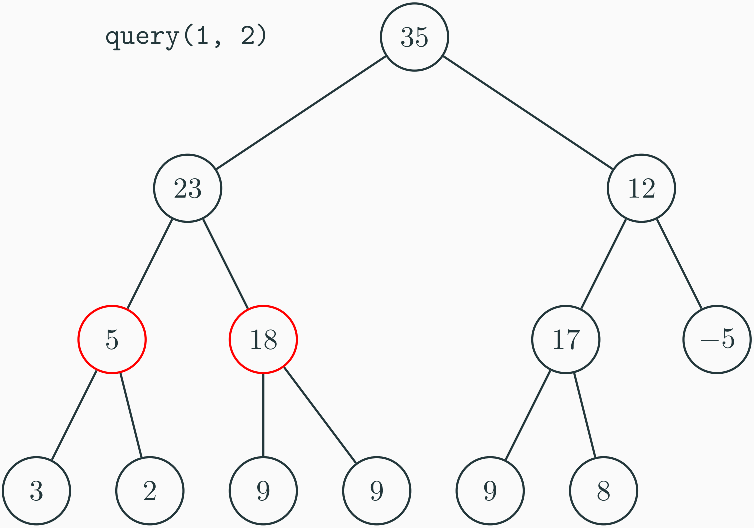

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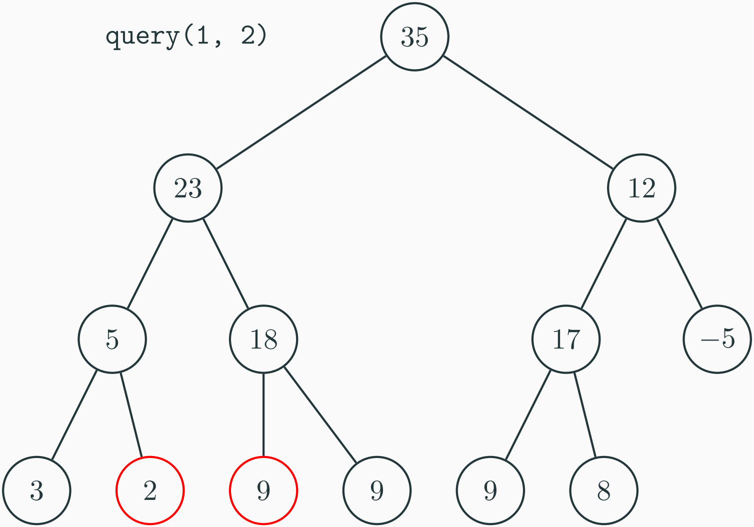

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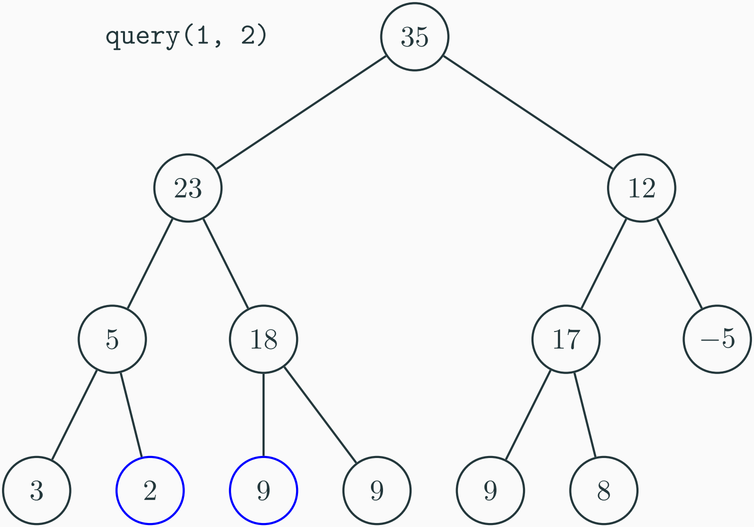

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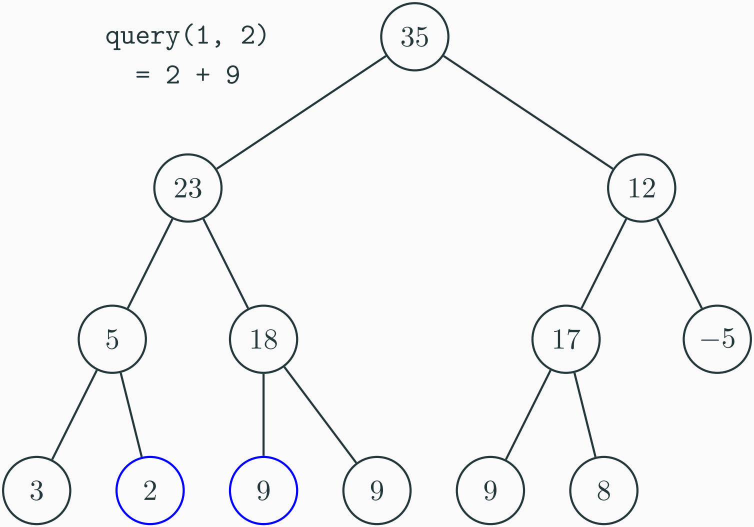
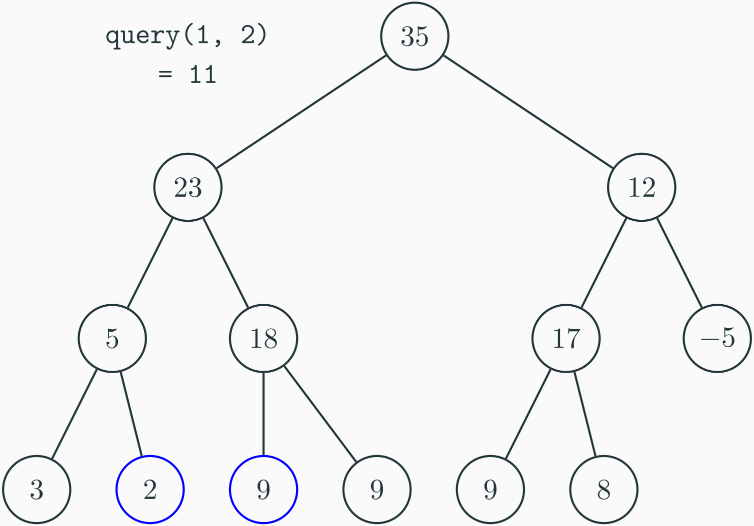


Querying

`query(1, 2)`
`= 11`



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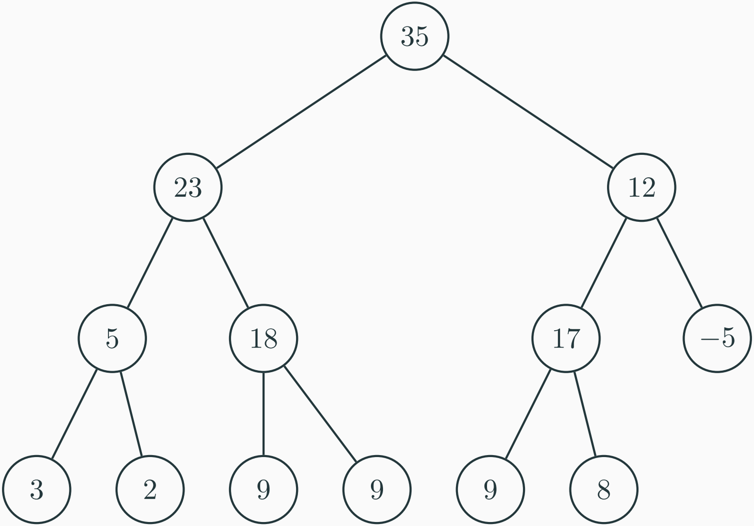

Querying a Segment Tree - Code

```
int query(segment_tree *tree, int l, int r) {
    if (tree == NULL) return 0;
    if (l <= tree->from && tree->to <= r) return tree->value;
    if (tree->to < l) return 0;
    if (r < tree->from) return 0;
    return query(tree->left, l, r) + query(tree->right, l, r);
}
```

Segment Tree

- Simple to use Segment Trees for $\min$, $\max$, $\gcd$, and other similar operators, basically the same code.

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- Any associative operator will work.
- So any operator f such that $f(a, f(b, c)) = f(f(a, b), c)$ for all a, b, c.

Example problem: Movie Collection

- https://open.kattis.com/problems/moviecollection

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- Idea: Be lazy and procrastinate changes until they are needed!

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- Reset the lazy value.
- Traverse to child nodes if needed.

Code example

See implementation example, for example here.

Sparse Table

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- Then we continue until we reach j, moving i along and collecting the results.
- This is what is known as a sparse table.

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- Querying takes $\mathcal{O}(\log(n))$, however updating is slow and difficult.
- Why would we then ever use this instead of segment trees?

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- How might we use sparse tables to do better?

Binary lifting ctd.

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- Then we can compute these in increasing order of j, calculating $j = 1$ using f itself and then for larger j letting $f^{[2^j]}(x) = f^{[2^{j-1}]}(f^{[2^{j-1}]}(x))$
- Thus we can precompute the table in $\mathcal{O}(n(\mathcal{O}(f) + \log(n)))$ and each query takes $\mathcal{O}(\log(m))$, a much better time complexity

Sparse table example

7	1	6	4	8	0	9	2	2	7	1	6

$j = 0$

Sparse table example

												$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

8												$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

8	7											$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

8	7	10										$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

8	7	10	12									$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

8	7	10	12	8								$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

8	7	10	12	8	9							$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

8	7	10	12	8	9	11						$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

8	7	10	12	8	9	11	4					$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

8	7	10	12	8	9	11	4	9				$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

8	7	10	12	8	9	11	4	9	8			$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

8	7	10	12	8	9	11	4	9	8	7		$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

8	7	10	12	8	9	11	4	9	8	7	6	$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

18	19	18	21	19	13	20	12	16	14	7	6	$j = 2$
8	7	10	12	8	9	11	4	9	8	7	6	$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

37	32	38	33	35	27	27	18	16	14	7	6	$j = 3$
18	19	18	21	19	13	20	12	16	14	7	6	$j = 2$
8	7	10	12	8	9	11	4	9	8	7	6	$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Sparse table example

`query(1, 8) = 19 + 9 + 2`

37	32	38	33	35	27	27	18	16	14	7	6	$j = 3$
18	$\boxed{19}$	18	21	19	13	20	12	16	14	7	6	$j = 2$
8	7	10	12	8	$\boxed{9}$	11	4	9	8	7	6	$j = 1$
7	1	6	4	8	0	9	$\boxed{2}$	2	7	1	6	$j = 0$

Sparse table example

```
query(0, 9) = 37 + 9
```

37	32	38	33	35	27	27	18	16	14	7	6	$j = 3$
18	19	18	21	19	13	20	12	16	14	7	6	$j = 2$
8	7	10	12	8	9	11	4	9	8	7	6	$j = 1$
7	1	6	4	8	0	9	2	2	7	1	6	$j = 0$

Example problem: Stikl

- https://open.kattis.com/problems/stikl