Number Theory

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Mathematical introduction

Important point

$$\text{Computer Science} \subset \text{Mathematics}$$

- Problems often require mathematical analysis to be solved efficiently.
- Using a bit of math before coding can also shorten and simplify code.

Pattern finding

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  - See if the solutions form a pattern.
- Does the pattern involve some overlapping subproblem?
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- Knowing reoccurring identities and sequences can be helpful.

Arithmetic progression

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$$2, 5, 8, 11, 14, 17, 20, \ldots$$

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- This is called an arithmetic progression.

$$a_n = a_{n-1} + c$$

Arithmetic progression

- Depending on the situation we may want to get the n-th element

$$a_n = a_1 + (n-1)c$$

- Or the sum over a finite portion of the progression

$$S_n = \frac{n(a_1 + a_n)}{2}$$

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- Remember this one?

$$1 + 2 + 3 + 4 + 5 + \ldots + n = \frac{n(n+1)}{2}$$

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$$1, 2, 4, 8, 16, 32, 64, 128, \ldots$$

- More generally

$$s, sr, sr^2, sr^3, sr^4, sr^5, sr^6, \ldots$$

$$a_n = sr^{n-1}$$

Geometric progression

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$$\sum_{i=0}^{n} sr^i = \frac{s(1 - r^n)}{(1 - r)}$$

- Or from the m-th element to the n-th

$$\sum_{i=m}^{n} sr^i = \frac{s(r^m - r^{n+1})}{(1 - r)}$$

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- And also the exponential

```
double exp(double x);
```

Example

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- What if $k = 500$ ($\sim 1.7 \cdot 10^{615}$), or something larger?
- Impossible to work with the numbers in a normal fashion.
- Why not $\log$?

Example

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- But how do we do this with only $\ln$ or $\log_{10}$?
- Change base!

$$\log_b(a) = \frac{\log_d(a)}{\log_d(b)} = \frac{\ln(a)}{\ln(b)}$$

- Now we can at least count the length without converting bases

Example

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- More generally

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

- For division

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

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- Using this identity and the ones we've covered, we get

$$x = \left\lceil (k-1) \cdot \frac{\ln(10)}{\ln(17)} \right\rceil$$

Base conversion

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- What if we actually need to use base conversion?
- Simple algorithm

```
vector<int> toBase(int base, int val) {
    vector<int> res;
    while(val) {
        res.push_back(val % base);
        val /= base;
    }
    return val;
}
```

- Starts from the 0-th digit, and calculates the multiple of each power.

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- What else can we do if we are working with real numbers?
- We compare them to a certain degree of precision like in binary search.
- Two numbers are deemed equal if their difference is less than some small epsilon.

```
const double EPS = 1e-9;

if(abs(a - b) < EPS) {
...
}
```

Working with doubles

- Less than operator:

```
if(a < b - EPS) {
...
}
```

- Less than or equal:

```
if(a  < b + EPS) {
...
}
```

- The rest of the operators follow.

Primes and factorization

Definitions that everybody should know

- Prime number is a positive integer greater than 1 that has no positive divisor other than 1 and itself.
- Greatest Common Divisor of two integers a and b is the largest number that divides both a and b.
- Least Common Multiple of two integers a and b is the smallest integer that both a and b divide.

Primality checking

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- Naive method: Iterate over all $1 < i < n$ and check if i divides n.
  - $O(N)$
- Better: If n is not a prime, it has a divisor $\leq \sqrt{n}$.
  - Iterate up to $\sqrt{n}$ instead.
  - $O(\sqrt{N})$

$\mathcal{O}(\sqrt{n})$ check

```
bool is_prime(ll x) {
    if(x <= 1) return 0;
    for(ll i = 2; i * i <= x; ++i)
        if(x % i == 0)
            return false;
    return true;
}
```

Modular arithmetic

Modulo

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- This implies that we can do all the computation with integers *modulo* n.
- But what does this mean? Taking an integer modulo n means taking the remainder of it when we divide by n.
- Thus if we do everything modulo n we consider every number a multiple of n apart the same. So modulo 7 the numbers $1, 8, -6, 15, -13, \ldots$ are all the same, and so are $5, -2, 12, -9, 19, \ldots$.

Modulo

- In this system it's simplest if we pick one of the infinite set of equivalent numbers to be the one we use to represent them. We usually choose the representatives $0, 1, \ldots, n-1$ if we're working modulo n.

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- Then to do addition, subtraction and multiplication we just do it as usual, but add or subtract multiples of n afterwards so we end up back in $\{0, 1, \ldots, n-1\}$.
- This means this set, which we denote $\mathbb{Z}_n$, is a ring, for those familiar with that terminology.

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- This implies that we can do all the computation in $\mathbb{Z}_n$.
- This is often very useful, since the numbers never get too big and we don't generally have to worry about over/underflow.

Division

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$$a \cdot a^{-1} = 1 \pmod{n}$$

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- Such an a^{-1} does not always exist, let's see how we can find it when it does exist though.

Euclidean algorithm

- The Euclidean algorithm is a recursive algorithm that computes the GCD of two numbers.

```
template<typename T>
T gcd(T a, T b){
    return b == T(0) ? a : gcd(b, a % b);
}
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- Runs in $O(\log N)$.

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- Runs in $O(\log N)$.
- Notice that this can also compute LCM

$$\text{lcm}(a, b) = \frac{a \cdot b}{\gcd(a, b)}$$

- See Wikipedia to see how it works and for proofs.

Extended Euclidean algorithm

- Reversing the steps of the Euclidean algorithm we get the Bézout's identity

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  which simply states that there always exist x and y such that the equation above holds.
- The extended Euclidean algorithm computes the GCD and the coefficients x and y.
- Each iteration it add up how much of b we subtracted from a and vice versa.

Extended Euclidean algorithm

```
template <typename T>
T egcd(T a, T b, T& x, T& y) {
    if (b == 0) {
        x = T(1);
        y = T(0);
        return a;
    } else {
        T d = egcd(b, a % b, x, y);
        x -= a / b * y;
        swap(x, y);
        return d;
    }
}
```

Applications

- Essential step in the RSA algorithm.
- Essential step in many factorization algorithms.
- Can be generalized to other algebraic structures.
- Fundamental tool for proofs in number theory.
- Many other algorithms for GCD

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- Working modulo n often requires division (multiplication by inverse).
- Given some $a \pmod{n}$, then the multiplicative inverse $a^{-1} \pmod{n}$ exists iff. a and n are coprime.
- It so happens that when we have from EGCD algorithm

$$ax + ny = \gcd(a, n) = 1$$

then

$$a^{-1} \equiv x \pmod{n}$$

Modular inverse

```
template<typename T>
T mod_inv(T a, T m) {
    T x, y, d = egcd(a, m, x, y);
    return d == T(1) ? (x%m+m)%m : T(-1);
}
```

Discrete logarithm

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Discrete logarithm

- What about logarithm? YES!
  - But difficult.
  - Basis for some cryptography such as elliptic curve, Diffie-Hellmann.
- Google “Discrete Logarithm” if you want to know more.

Chinese remainder theorem

What is the lowest number n such that when divided by

... 3 it leaves 2 in remainder.

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When stated mathematically, find n where

$$n \equiv 2 \pmod{3}$$
$$n \equiv 3 \pmod{5}$$
$$n \equiv 2 \pmod{7}$$

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Let $n_1, n_2, \ldots, n_k$ be pairwise coprime positive integers, and let x be the solution to the system of linear congruences

$$\begin{aligned} x &\equiv b_1 \pmod{n_1} \\ x &\equiv b_2 \pmod{n_2} \\ &\vdots \\ x &\equiv b_k \pmod{n_k} \end{aligned}$$

Chinese remainder theorem

- The Chinese remainder theorem only states that there exists a solution and it is unique modulus the product of the moduli.
- To obtain the solution x

$$x \equiv b_1 c_1 \frac{N}{n_1} + \ldots + b_k c_k \frac{N}{n_k}$$

  where $N = n_1 n_2 \cdots n_k$.
- The coefficients c_i are determined from

$$c_i \frac{N}{n_i} \equiv 1 \pmod{n_i}$$

  (the multiplicative inverse of $\frac{N}{n_i}$ modulus n_i)
- Use EGCD to compute c_i.

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- The probability of the program being wrong every time is so vanishingly small.
- You would spend your time better worrying about space rays flipping your bits while you run the program.

Miller-Rabin concept

- Let us first note that if $x^2 = 1\ (mod\ p)$ this can be factored as $(x-1)(x+1) = 0\ (mod\ p)$ and since p is prime this means $x = \pm 1\ (mod\ p)$.

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- Now take some $p > 2$ and $a < p$. Write $p - 1 = 2^s d$ s.t. d is odd. Then by taking the square root on each side of the equation $a^{p-1} = 1\ (mod\ p)$ (which we know is true) then either the right side will at some point equal -1 and we have to stop, or we eventually divide out all powers of two in a. This either $a^d = 1\ (mod\ p)$ or $a^{2^r d} = -1\ (mod\ p)$ for some $0 \leq r \leq s - 1$.

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- Thus to prove that n is not prime we try to find $a < n$ s.t. $a^d \neq 1 \ (mod\ n)$ and $a^{2^r d} \neq -1 \ (mod\ n)$ for all $0 \leq r \leq s - 1$.

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- Thus the algorithm is such that if it says p is not prime, this is definitely true. If it says p is a prime, it really means "I couldn't exclude the possibility that p is prime, but it could be non-prime".
- If we test many a the odds are in our favor. Thus we let the program take a variable `k` saying how often it should run. This runs in $\mathcal{O}(k \log(n)^3)$ for large n.

Miller-Rabin implementation

```
template <typename T>
bool is_probably_prime(T n, int k) {
    if (n % 2 == 0) return n == T(2);
    if (n <= 3) return n == T(3);
    T d = n - 1, r = T(0);
    while (d % 2 == 0) d >>= 1, r++;
    for (int i = 0; i < k; ++i) {
        T a = (n - 3) * rand() / RAND_MAX + 2;
        T x = modpow(a, d, n);
        if(x == T(1) || x == T(n - 1)) continue;
        bool ok = false;
        for(T j = 0; j < r - 1; ++j) {
            x = (x * x % n + n) % n;
            if(x == T(1)) return false;
            if(x == T(n - 1)) { ok = true; break; }
        }
        if(!ok) return false;
    }
    return true;
}
```

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- Instead, our preferred method of prime generation is the sieve of Eratosthenes.
  - For all numbers from 2 to $\sqrt{n}$:
  - If the number is not marked, iterate over every multiple of the number up to n and mark them.
  - The unmarked numbers are those that are not a multiple of any smaller number.
  - $O(N \log \log N)$

Eratosthenes example

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

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50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

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40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
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		2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
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Sieve of Eratosthenes

```
template <typename T>
vector<T> eratosthenes(T n){
    vector<bool> isMarked(n+1, false);
    vector<T> primes;
    T i = T(2);
    for(; i*i <= n; i++)
        if (!isMarked[i]) {
            primes.push_back(i);
            for(T j = i; j <= n; j += i)
                isMarked[j] = true;
        }
    for (; i <= n; i++)
        if (!isMarked[i])
            primes.push_back(i);
    return primes;
}
```

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To factor an integer n:

- Use the sieve of Eratosthenes to generate all the primes up $\sqrt{n}$
- Iterate over all the primes generated and check if they divide n, and determine the largest power that divides n.

Factoring code

```
template <typename T>
map<T, T> factor(T N) {
    vector<T> primes;
    primes = eratosthenes(static_cast<T>(sqrt(N+1)));
    map<T, T> factors;
    for(const auto prime : primes)
        T power = 0;
        while(N % prime == T(0)){
            power++;
            N /= prime;
        }
        if(power > T(0)){
            factors[prime] = power;
        }
    }
    if (N > T(1)) {
        factors[N] = T(1);
    }
    return factors;
}
```

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- We can use the birthday paradox to our advantage.
- If we have n items we are expected to receive a duplicate once we have picked $\mathcal{O}(\sqrt{n})$ from the collection at random.
- We will use this to factor n. But first a small side step.

Floyd cycle finding

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- The trick is that i is a multiple of the cycle length of f iff $f^{[i]}(x) = f^{[2i]}(x)$.
- Thus we only have to consider that equation when trying to find the cycle length. When that is done we can go back to find where the cycle began and check its size.

Floyd implementation

```
#include <bits/stdc++.h>
using namespace std;

template <typename T, typename F>
pair<int, int> floyd(F&& f, T x0) {
    T t = f(x0), h = f(f(x0));
    while(t != h) {
        t = f(t);
        h = f(f(h));
    }
    int length = 0;
    t = x0;
    while(t != h) {
        t = f(t);
        h = f(h);
        length++;
    }
    int start = 1;
    T h = f(t);
    while(t != h) {
        h = f(h);
        start++;
    }
    return {start, length}
}
```

Pollard rho factorization

- Let $g(x) = x^2 + 1 \ (mod\ n)$ and create the sequence $x_1 = x, x_2 = g(x), x_3 = g(g(x)), \ldots$ where x is chosen randomly.

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- Slow for primes, but much faster for composite numbers.
- Checking for primality first using Miller-Rabin can be useful.

Pollard rho implementation

```
template <typename T>
T rho(T n) {
    vector<T> seed = {
        T(2), T(3), T(4), T(5), T(7), T(11), T(13), T(1031)
    };
    for(auto s : seed) {
        T x = s, y = x, d = T(1);
        while(d == T(1)) {
            x = ((x * x + 1) % n + n) % n;
            y = ((y * y + 1) % n + n) % n;
            y = ((y * y + 1) % n + n) % n;
            d = gcd(abs(x - y), n);
        }
        if(d == n) continue;
        return d;
    }
    return -1;
}
```

Number theory functions

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- The sum of all positive divisors in x-th power

$$\sigma_x(n) = \prod_{i=1}^{k} \frac{(p_i^{(e_i+1)x} - 1)}{(p_i - 1)}$$

More number theory functions

- The Euler's totient function

$$\phi(n) = n \cdot \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)$$

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- Euler's theorem, if a and n are coprime

$$a^{\phi(n)} = 1 \pmod{n}$$

  Fermat's theorem is a special case when n is a prime.