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Canonical Quantum Gravity and the Problem of Time [1] [2]

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Abstract

The aim of this paper is to provide a general introduction to the problem of time in quantum gravity. This problem originates in the fundamental conflict between the way the concept of 'time' is used in quantum theory, and the role it plays in a diffeomorphism-invariant theory like general relativity. Schemes for resolving this problem can be sub-divided into three main categories: (I) approaches in which time is identified before quantising; (II) approaches in which time is identified after quantising; and (III) approaches in which time plays no fundamental role at all. Ten different specific schemes are discussed in this paper which also contain an introduction to the relevant parts of the canonical decomposition of general relativity.

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Contents

1 INTRODUCTION

1.1 Preamble

These notes are based on a course of lectures given at the NATO Advanced Summer Institute “Recent Problems in Mathematical Physics”, Salamanca, June 15–27, 1992. The notes reflect part of an extensive investigation with Karel Kuchař into the problem of time in quantum gravity. An excellent recent review is Kuchař (1992*b*), to which the present article is complementary to some extent. In particular, my presentation is slanted towards the more conceptual aspects of the problem and, as this a set of lecture notes (rather than a review paper proper), I have also included a fairly substantial technical introduction to the canonical theory of general relativity. However, there is inevitably a strong overlap with many portions of Kuchař’s paper, and I am grateful to him for permission to include this material plus a number of ideas that have emerged in our joint discussions. Therefore, the credit for any good features in the present account should be shared between us; the credit for the mistakes I claim for myself alone.

1.2 Preliminary Remarks

The problem of ‘time’ is one of the deepest issues that must be addressed in the search for a coherent theory of quantum gravity. The major conceptual problems with which it is closely connected include:

- the status of the concept of *probability* and the extent to which it is *conserved*;
- the status of the associated concepts of *causality* and *unitarity*;
- the time-honoured debate about whether quantum gravity should be approached via a *canonical*, or a *covariant*, quantisation scheme;
- the extent to which *spacetime* is a meaningful concept;
- the extent to which classical *geometrical* concepts can, or should, be maintained in the quantum theory;
- the way in which our *classical world* emerged from some primordial quantum event at the big-bang;
- the whole question of the *interpretation* of quantum theory and, in particular, the domain of applicability of the conventional Copenhagen view.

The prime source of the problem of time in quantum gravity is the invariance of classical general relativity under the group $\mathrm{Diff}(\mathcal{M})$ of diffeomorphisms of the spacetime manifold $\mathcal{M}$. This stands against the simple Newtonian picture of a fixed time parameter, and tends to produce quantisation schemes that apparently lack any fundamental notion of time at all. From this perspective, the heart of the problem is contained in the following questions:

1. How should the notion of time be re-introduced into the quantum theory of gravity?
2. In particular, should attempts to identify time be made at the classical level, *i.e.,* before quantisation, or should the theory be quantised first?
3. Can 'time' still be regarded as a fundamental concept in a quantum theory of gravity, or is its status purely phenomenological? If the concept of time is not fundamental, should it be replaced by something that is: for example, the idea of a *history* of a system, or *process*, or an *ordering structure* that is more general than that afforded by the conventional idea of time?
4. If 'time' is only an approximate concept, how reliable is the rest of the quantum-mechanical formalism in those regimes where the normal notion of time is not applicable? In particular, how closely tied to the concept of time is the idea of probability? This is especially relevant in those approaches to quantum gravity in which the notion of time emerges only *after* the theory has been quantised.

In addition to these questions—which apply to quantum gravity in general—there is also the partly independent issue of the applicability of the concept of time (and, indeed, of quantum theory in general) in the context of quantum cosmology. Of particular relevance here are questions of (i) the status of the Copenhagen interpretation of quantum theory (with its emphasis on the role of measurements); and (ii) the way in which our present classical universe, including perhaps the notion of time, emerged from the quantum origination event.

A key ingredient in all these questions is the realisation that the notion of time used in conventional quantum theory is grounded firmly in Newtonian physics. Newtonian time is a fixed structure, external to the system: a concept that is manifestly incompatible with diffeomorphism-invariance and also with the idea of constructing a quantum theory of a truly closed system (such as the universe itself). Most approaches to the problem of time in quantum gravity [3] seek to address this central issue by identifying an *internal* time which is defined in terms of the system itself, using either the gravitational field or the matter variables that describe the material content of the universe. The various schemes differ in the way such an identification is made and the point in the procedure at which it is invoked. Some of these techniques inevitably require a significant reworking of the quantum formalism itself.

[3] A similar problem arises when discussing thermodynamics and statistical physics in a curved space-time. A recent interesting discussion is Rovelli (1991*d*) which makes specific connections between this problem of time and the one that arises in quantum gravity.

1.3 Current Research Programmes in Quantum Gravity

An important question that should be raised at this point is the relation of the problem of time to the various research programmes in quantum gravity that are currently active. The primary distinction is between approaches to quantum gravity that start with the *classical* theory of general relativity (or a simple extension of it) to which some quantisation algorithm is applied, and schemes whose starting point is a *quantum* theory from which classical general relativity emerges in some low-energy limit, even though that theory was not one of the initial ingredients. Most of the standard approaches to quantum gravity belong to the former category; superstring theory is the best-known example of the latter.

Some of the more prominent current research programmes in quantum gravity are as follows (for recent reviews see Isham (1985, 1987, 1992) and Alvarez (1989)).

- *Quantum Gravity and the Problem of Time*. This subject—the focus of the present paper—dates back to the earliest days of quantum gravity research. It has been studied extensively in recent years—mainly within the framework of the canonical quantisation of the classical theory of general relativity (plus matter)—and is now a significant research programme in its own right. A major recent review is Kuchař (1992*b*). Earlier reviews, from somewhat different perspectives, are Barbour & Smolin (1988) and Unruh & Wald (1989). There are also extensive discussions in Ashtekar & Stachel (1991) and in Halliwell, Perez-Mercander & Zurek (1992); other relevant literature will be cited at the appropriate points in our discussion.

- *The Ashtekar Programme*. A major technical development in the canonical formalism of general relativity was the discovery by Ashtekar (1986, 1987) of a new set of canonical variables that makes general relativity resemble Yang-Mills theory in several important respects, including the existence of non-local physical observables that are an analogue of the Wilson loop variables of non-abelian gauge theory (Rovelli & Smolin 1990, Rovelli 1991*a*, Smolin 1992); for a recent comprehensive review of the whole programme see Ashtekar (1991). This is one of the most promising of the non-superstring approaches to quantum gravity and holds out the possibility of novel non-perturbative techniques. There have also been suggestions that the Ashtekar variables may be helpful in resolving the problem of time (for example, in Ashtekar (1991)[pages 191-204]). The Ashtekar programme is not discussed in these notes, but it is a significant development and has important implications for quantum gravity research in general.

- *Quantum Cosmology*. This subject was much studied in the early days of quantum gravity and has enjoyed a renaissance in the last ten years, largely due to the work of Hartle & Hawking (1983) and Vilenkin (1988) on the possibility of constructing a quantum theory of the creation of the universe. The techniques employed have been mainly those of canonical quantisation, with particular emphasis on minisuperspace (*i.e.,* finite-dimensional) approximations. The problem of time is central to the subject of quantum cosmology and is often discussed within the context of these models, as are a number of other conceptual problems expected to arise in quantum gravity proper. A major recent review is Halliwell (1991*a*) which also contains a comprehensive bibliography (see also Halliwell (1990) and Halliwell (1992*a*)).

- *Low-Dimensional Quantum Gravity.* Studies of gravity in $1+1$ and $2+1$ dimensions have thrown valuable light on many of the difficult technical and conceptual issues in quantum gravity, including the problem of time (for example, Carlip (1990, 1991)). The reduction to lower dimensions produces major technical simplifications whilst maintaining enough of the flavour of the $3+1$-dimensional case to produce valuable insights into the full theory. Lower-dimensional gravity also has direct physical applications. For example, idealised cosmic strings involve the application of gravity in $2+1$ dimensions, and the theory in $1+1$ dimensions has applications in statistical mechanics. The subject of gravity in $2+1$ dimensions is reviewed in Jackiw (1992*b*) whilst recent developments in $1+1$ dimensional gravity are reported in Jackiw (1992*a*) and Teitelboim (1992) (in the proceedings of this Summer School).

- *Semi-Classical Quantum Gravity.* Early studies of this subject (for a review see Kibble (1981)) were centered on the equations

  $$G_{\alpha\beta}(X,\gamma] = \langle\psi|T_{\alpha\beta}(X;\gamma,\widehat{\phi}]|\psi\rangle \tag{1.3.1}$$

  where the source for the classical spacetime metric γ is an expectation value of the energy-momentum tensor of quantised matter $\widehat{\phi}$. More recently, equations of this type have been developed from a WKB-type approximation to the quantum equations of canonical quantum gravity, especially the Wheeler-DeWitt equation (§5.1.5). Another goal of this programme is to provide a coherent foundation for the construction of quantum field theory in a fixed background spacetime: a subject that has been of enduring interest since Hawking's discovery of quantum-induced radiation from a black hole.

  The WKB approach to solving the Wheeler-DeWitt equation is closely linked to a semi-classical theory of time and will be discussed in §5.4.

- *Spacetime Structure at the Planck Length.* There is a gnostic subculture of workers in quantum gravity who feel that the structure of space and time may undergo radical changes at scales of the Planck length. In particular, the idea surfaces repeatedly that the continuum spacetime picture of classical general relativity may break down in these regions. Theories of this type are highly speculative but could have significant implications for the question of 'time' which, as a classical concept, is grounded firmly in continuum mathematics.

- *Superstring Theory*. This is often claimed to be the 'correct' theory of quantum gravity, and offers many new perspectives on the intertwining of general relativity and quantum theory. Of particular interest is the recurrent suggestion that there exists a minimal length, with the implication that many normal spacetime concepts could break down at this scale. It is therefore unfortunate that the current approaches to superstring theory are mainly perturbative in character (involving, for example, graviton scattering amplitudes) and are difficult to apply directly to the problem of time. But the question of the implications of string theory is an intriguing one, not least because of the very different status assigned by the theory to important spacetime concepts such as the diffeomorphism group.

It is noteable that almost all studies of the problem of time have been performed in the framework of 'conventional' canonical quantum gravity in which attempts are made to apply quantisation algorithms to the field equations of classical general relativity. However, the resulting theory is well-known to be perturbatively non-renormalisable and, for this reason, much of the work on time has used finite-dimensional models that are free of ultraviolet divergences. Therefore, it must be emphasised that many of the specific problems of time are *not* connected with the pathological short-distance behaviour of the theory, and appear in an authentic way in these finite-dimensional model systems .

Nevertheless, the non-renormalisability is worrying and raises the general question of how seriously the results of the existing studies should be taken. The Ashtekar programme may lead eventually to a finite and well-defined 'conventional' quantum theory of gravity, but quite a lot is being taken on trust. For example, any addition of Riemann-curvature squared counter-terms to the normal Einstein Lagrangian would have a drastic effect on the canonical decomposition of the theory and could render irrelevant much of the discussion involving the Wheeler-DeWitt equation.

In practice, most of those who work in the field seem to believe that, whatever the final theory of quantum gravity may be (including superstrings), enough of the conceptual and geometrical structure of classical general relativity will survive to ensure the relevance of most of the general questions that have been asked about the meaning and significance of 'time'. However, it should not be forgotten that the question of what constitutes a conceptual problem—such as the nature of time—often cannot be decided in isolation from the technical framework within which it is posed. Therefore, it is feasible that when, for example, superstring theory is better understood, some of the conceptual issues that appear now to be genuine problems will be seen to be the result of asking ill-posed questions, rather than reflecting a fundamental problem in nature itself.

1.4 Outline of the Paper

We begin in §2 by discussing the status of time in general relativity and conventional quantum theory, and some of the *prima facie* problems that arise when attempts are made to unite these two, somewhat disparate, conceptual structures. As we shall see, approaches to the problem of time fall into three distinct categories—those in which time is identified before quantising, those in which time is identified after quantising, and 'timeless' schemes in which no fundamental notion of time is introduced at all. A brief description is given in §2 of the principle schemes together with the major difficulties that are encountered in their implementation.

Most approaches to the problem of time involve the canonical theory of general relativity, and §3 is devoted to this topic. Section §4 deals with the approaches to the problem of time that involve identifying time before quantising, while §5 addresses those schemes in which the gravitational field is quantised first. Then follows an account in §6 of the schemes that avoid invoking time as a fundamental category at all. As we shall see, attempts of this type become rapidly involved in general questions about the interpretation of quantum theory and, in particular, of the domain of applicability of the traditional Copenhagen approach. The paper concludes with a short summary of the current situation and some speculations about the future.

1.5 Conventions

The conventions and notations to be used are as follows. A Lorentzian metric γ on a four-dimensional spacetime manifold $\mathcal{M}$ is assumed to have signature $(-1,1,1,1)$. Lower case Greek indices refer to coordinates on $\mathcal{M}$ and take the values $0,1,2,3$. We adopt the modern differential-geometric view of coordinates as local *functions* on the manifold. Thus, if X^α, $\alpha = 0\ldots 3$ is a coordinate system on $\mathcal{M}$ the *values* of the coordinates of a point $Y \in \mathcal{M}$ are the four numbers $X^\alpha(Y)$, $\alpha = 0\ldots 3$. However, by a familiar abuse of notation, sometimes we shall simply write these numbers as Y^α. Lower case Latin indices refer to coordinates on the three-manifold Σ and range through the values $1,2,3$.

The symbol $f : X \to Y$ means that f is a map from the space X into the space Y. In the canonical theory of gravity there are a number of objects F that are functions simultaneously of (i) a point x in a finite-dimensional manifold (such as Σ or $\mathcal{M}$); and (ii) a function $G : X \to Y$ where X, Y can be a variety of spaces. It is useful to adopt a notation that reflects this property. Thus we write $F(x, G\,]$ to remind us that G is itself a function. If F depends on n points $x_1, \ldots x_n$ in a finite-dimensional manifold, and m functions $G_1, \ldots G_m$, we shall write $F(x_1, \ldots x_n; G_1, \ldots G_m\,]$.

2 QUANTUM GRAVITY AND THE PROBLEM OF TIME

2.1 Time in Conventional Quantum Theory

The problem of time in quantum gravity is deeply connected with the special role assigned to temporal concepts in standard theories of physics. In particular, in Newtonian physics, time—the parameter with respect to which change is manifest—is *external* to the system itself. This is reflected in the special status of time in conventional quantum theory:

1. Time is not a physical observable in the normal sense since it is not represented by an operator. Rather, it is treated as a background parameter which, as in classical physics, is used to mark the evolution of the system. In particular, it provides the parameter t in the time-dependent Schrödinger equation
$$i\hbar \frac{d\psi_t}{dt} = \widehat{H}\psi_t. \tag{2.1.1}$$
This special property of time applies both to non-relativistic quantum theory and to relativistic particle dynamics and quantum field theory. It is the reason why the meaning assigned to the time-energy uncertainty relation $\delta t\, \delta E \geq \frac{1}{2}\hbar$ is quite different from that pertaining to, for example, the position and the momentum of a particle.

2. This view of time is related to the difficulty of describing a truly *closed* system in quantum-mechanical terms. Indeed, it has been cogently argued that the only physical states in such a system are *eigenstates* of the Hamiltonian operator, whose time evolution is essentially trivial (Page & Wooters 1983). Of course, the ultimate closed system is the universe itself.

3. The idea of events happening at a single time plays a crucial role in the technical and conceptual foundations of quantum theory:

   - The notion of a *measurement* made at a particular time is a fundamental ingredient in the conventional Copenhagen interpretation. In particular, an *observable* is something whose value can be measured at a fixed time. On the other hand, a 'history' has no direct physical meaning except in so far as it refers to the outcome of a sequence of time-ordered measurements.
   - One of the central requirements of the *scalar product* on the Hilbert space of states is that it be conserved under the time evolution (2.1.1). This is closely connected to the unitarity requirement that probabilities always sum to one.
   - More generally, a key ingredient in the construction of the Hilbert space for a quantum system is the selection of a complete set of observables that are required to *commute* at a fixed value of time.

4. These ideas can be extended to systems that are compatible with special relativity: one simply replaces the unique time system of Newtonian physics with the set of relativistic inertial reference frames. The quantum theory can be made independent of a choice of frame if it carries a unitary representation of the Poincaré group. In the case of a relativistic quantum field theory, this is closely related to the requirement of microcausality, *i.e.,*

   $$[\,\widehat{\phi}(X), \widehat{\phi}(X')\,] = 0 \tag{2.1.2}$$

   for all spacetime points X and X' that are spacelike separated.

The background Newtonian time appears explicitly in the time-dependent Schrödinger equation (2.1.1), but it is pertinent to note that such a time is truly an abstraction in the sense that no *physical* clock can provide a precise measure of it (Unruh & Wald 1989). For suppose there is some quantum observable T that can serve as a 'perfect' physical clock in the sense that, for some initial state, its observed values increase monotonically with the abstract time parameter t. Since $\widehat{T}$ may have a continuous spectrum, let us decompose its eigenstates into a collection of normalisable vectors $|\tau_0\rangle, |\tau_1\rangle, |\tau_2\rangle \ldots$ such that $|\tau_n\rangle$ is an eigenstate of the projection operator onto the interval of the spectrum of $\widehat{T}$ centered on τ_n. Then to say that T is a perfect clock means:

1. For each m there exists an n with $n > m$ and $t > 0$ such that the probability amplitude for $|\tau_m\rangle$ to evolve to $|\tau_n\rangle$ in Newtonian time t is non-zero (*i.e.,* the clock has a non-zero probability of running forwards with respect to the abstract Newtonian time t). This means that

   $$f_{mn}(t) := \langle\tau_n|U(t)|\tau_m\rangle \neq 0 \tag{2.1.3}$$

   where $U(t) := e^{-it\widehat{H}/\hbar}$.

2. For each m and for all $t > 0$, the amplitude to evolve from $|\tau_m\rangle$ to $|\tau_n\rangle$ vanishes if $m > n$ (*i.e.,* the clock never runs backwards).

Unruh & Wald (1989) show that these conditions are incompatible with the physical requirement that the energy of the system be positive. This follows by studying the function $f_{mn}(t)$, $m > n$, in (2.1.3) for complex t and with $m > n$. Since $\widehat{H}$ is bounded below, f_{mn} is holomorphic in the lower-half plane and hence cannot vanish on any open real interval unless it vanishes identically for all t whose imaginary part is less than, or equal to, zero. However, the requirement that the clock never runs backwards is precisely that $f_{mn}(t) = 0$ for all $t > 0$, and hence $f_{mn}(t) = 0$ for all real t. But then, if $m < n$,

$$f_{mn}(t) = \langle\tau_n|U(t)|\tau_m\rangle = \langle\tau_m|U(t)^\dagger|\tau_n\rangle^* = f^*_{nm}(-t) \tag{2.1.4}$$

which, as we have just shown, vanishes for all $t > 0$. Hence the clock can never run forwards in time, and so perfect clocks do not exist. This means that any physical clock always has a small probability of sometimes running backwards in abstract Newtonian time.

An even stronger requirement on T is that it should be a 'Hamiltonian time observable' in the sense that

$$[\,\widehat{T}, \widehat{H}\,] = i\hbar. \tag{2.1.5}$$

This implies at once that $U(t)|T\rangle = |T+t\rangle$ where $\widehat{T}|T\rangle = T|T\rangle$, which is precisely the type of behaviour that is required for a perfect clock. However, it is well-known that self-adjoint operators satisfying (exponentiable) representations of (2.1.5) necessarily have spectra equal to the whole of $\mathbb{R}$, and hence (2.1.5) is manifestly incompatible with the requirement that $\widehat{H}$ be a positive operator.

This inability to represent abstract Newtonian time with any genuine physical observable is a fundamental property of quantum theory. As we shall see in §4.3, it is reflected in one of the major attempts to solve the problem of time in quantum gravity.

2.2 Time in a Diff($\mathcal{M}$)-invariant Theory

The special role of time in quantum theory suggests strongly that it (or, more generally, the system of inertial reference frames) should be regarded as part of the *a priori* classical background that plays such a crucial role in the Copenhagen interpretation of the theory. But this view becomes highly problematic once general relativity is introduced into the picture; indeed, it is one of the main sources of the problem of time in quantum gravity.

One way of seeing this is to focus on the idea that the equations of general relativity transform covariantly under changes of spacetime coordinates, and physical results are meant to be independent of choices of such coordinates. However, 'time' is frequently regarded as a coordinate on the spacetime manifold $\mathcal{M}$ and it might be expected therefore to play no fundamental role in the theory. This raises several important questions:

1. If time is indeed merely a coordinate on $\mathcal{M}$—and hence of no direct physical significance—how does the, all-too-real, property of change emerge from the formalism?

2. Like all coordinates, 'time' may be defined only in a local region of the manifold. This could mean that:

   - the time variable t is defined only for some finite range of values for t; or
   - the subset of points t=const may not be a complete three-dimensional submanifold of the spacetime.

   How are such global problems to be handled in the quantum theory?

3. Is this view of the coordinate nature of time compatible with normal quantum theory, based as it is on the existence of a universal, Newtonian time?

The global issues are interesting, but they are not central to the problem of time and therefore in what follows I shall assume the topology of the spacetime manifold $\mathcal{M}$ to be such that global time coordinates can exist. If $\mathcal{M}$ is equipped with a Lorentzian metric γ, such a global time coordinate can be regarded as the parameter in a foliation of $\mathcal{M}$ into a one-parameter family of spacelike hypersurfaces: the hypersurfaces of 'equal-time'. This useful picture will be employed frequently in our discussions.

A slightly different way of approaching these issues is to note that the equations of general relativity are covariant with respect to the action of the group $\mathrm{Diff}(\mathcal{M})$ of diffeomorphisms (*i.e.,* smooth and invertible point transformations) of the spacetime manifold $\mathcal{M}$. We shall restrict our attention to diffeomorphisms with compact support, by which I mean those that are equal to the unit map outside some closed and bounded region of $\mathcal{M}$. Thus, for example, a Poincaré-group transformation of Minkowski spacetime is not deemed to belong to $\mathrm{Diff}(\mathcal{M})$. This restriction is imposed because the role of transformations with a non-trivial action in the asymptotic regions of $\mathcal{M}$ is quite different from those that act trivially. It must also be emphasised that $\mathrm{Diff}(\mathcal{M})$ means the group of active point transformations of $\mathcal{M}$. This should not be confused with the pseudo-group which describes the relations between overlapping pairs of coordinate charts (although of course there is a connection between the two).

Viewed as an active group of transformations, the diffeomorphism group $\mathrm{Diff}(\mathcal{M})$ is analogous in certain respects to the gauge group of Yang-Mills theory. For example, in both cases the groups are associated with field variables that are non-dynamical, and with a canonical formalism that entails constraints on the canonical variables. However, in another, and very important sense the two groups are quite different. Yang-Mills transformations occur at a fixed spacetime point whereas the diffeomorphism group moves points around. Invariance under such an active group of transformations robs the individual points in $\mathcal{M}$ of any fundamental ontological significance. For example, if ϕ is a scalar field on $\mathcal{M}$ the value $\phi(X)$ at a particular point $X \in \mathcal{M}$ has no invariant meaning. This is one aspect of the Einstein 'hole' argument that has featured in several recent expositions (Earman & Norton 1987, Stachel 1989). It is closely related to the question of what constitutes an *observable* in general relativity—a surprisingly contentious issue that has generated much debate over the years and which is of particular relevance to the problem of time in quantum gravity. In the present context, the natural objects that are manifestly $\mathrm{Diff}(\mathcal{M})$-invariant are spacetime integrals like, for example,

$$F[\gamma] := \int_{\mathcal{M}} d^4X\,(-\det\gamma(X))^{\frac{1}{2}}\,R^{\alpha\beta\gamma\delta}(X,\gamma]\,R_{\alpha\beta\gamma\delta}(X,\gamma]. \tag{2.2.1}$$

Thus 'observables' of this type are intrinsically non-local.

These implications of $\mathrm{Diff}(\mathcal{M})$-invariance pose no real difficulty in the classical theory since once the field equations have been solved the Lorentzian metric on $\mathcal{M}$ can be used to give meaning to concepts like 'causality' and 'spacelike separated', even if these notions are not invariant under the action of $\mathrm{Diff}(\mathcal{M})$. However, the situation in the quantum theory is very different. For example, whether or not a hypersurface is spacelike depends on the spacetime metric γ. But in any quantum theory of gravity there will presumably be some sense in which γ is subject to quantum fluctuations. Thus causal relationships, and in particular the notion of 'spacelike', appear to depend on the quantum state. Does this mean that 'time' also is state dependent?

This is closely related to the problem of interpreting a microcausality condition like (2.1.2) in a quantum theory of gravity. As emphasised by Fredenhagen & Haag (1987), for most pairs of points $X, X' \in \mathcal{M}$ there exists at least one Lorentzian metric with respect to which they are *not* spacelike separated, and hence, in so far as all metrics are 'virtually present' (for example, by being summed over in a functional integral) the right hand side of (2.1.2) is never zero. This removes at a stroke one of the bedrocks of conventional quantum field theory!

The same argument throws doubts on the use of canonical commutation relations: with respect to what metric is the hypersurface meant to be spacelike? This applies in particular to the equation $[\widehat{g}_{ab}(x), \widehat{g}_{cd}(x')] = 0$ which, in a conventional reading, would imply that the canonical configuration variable g (a metric on a three-dimensional manifold Σ) can be measured simultaneously at two points x, x' in Σ. This difficulty in forming a meaningful interpretation of commutation relations also renders dubious any attempts to find a quantum gravity analogue of the powerful C^*-algebra approach to conventional quantum field theory.

These rather general problems concerning time and the diffeomorphism group might be resolved in several ways. One possibility is to restrict attention to spacetimes $(\mathcal{M}, \gamma)$ that are asymptotically flat. It is then possible to define *asymptotic* quantities, including time and space coordinates, using the values of fields in the asymptotic regions of $\mathcal{M}$. This works because the diffeomorphism group transformations have compact support and hence quantities of this type are manifestly invariant. In such a theory, time evolution would be meaningful only in these regions and could be associated with the appropriate generator of some sort of asymptotic Poincaré group. DeWitt used the concept of an asymptotic observable in his seminal investigations of covariant quantum gravity (DeWitt 1965, 1967a, 1967b, 1967c) .

However, it is not clear to what extent this really solves the problem of time. An asymptotic time might suffice for calculations of graviton-graviton scattering amplitudes, but it is difficult to see how it could be used in the quantum cosmological situations to which so much attention has been devoted in recent years. One can also question the extent to which such a notion of time can be related operationally to what could be measured with the aid of real physical clocks. This is particularly relevant in the light of the discussion below of the significance of internal time coordinates.

In any event, the present paper is concerned mainly with the problem of time in the context of compact three-spaces, or of non-compact spaces but with boundary data playing no fundamental role. In theories of this type, the problem of time and the spacetime diffeomorphism group might be approached in several different ways.

- General relativity could be forced into a Newtonian framework by assigning special status to some particular foliation of $\mathcal{M}$. For example, a special background metric γ could be introduced to act as a source of preferred reference frames and foliations. However, the action of a diffeomorphism on $\mathcal{M}$ generally maps a hypersurface into a different one, and hence insisting on preserving the special foliation generates a type of symmetry breaking in which the Diff$(\mathcal{M})$ invariance is reduced to the group of transformations that leave the foliation invariant. If the foliation is tied closely to the properties of the metric γ, this group is likely to be just the group of isometries (if any) of γ.

  There is an interesting point at issue at here. Some people (for example, certain process philosophers) might argue that, since we do in fact live in one particular universe, we should be free to exploit whatever special characteristics it might happen to have. Thus, for example, we could use the background 3^0K radiation to define some quasi-Newtonian universal time associated with the Robertson-Walker homogeneous cosmology with which it is naturally associated. [4] On the other hand, theoretical physicists tend to want to consider all *possible* universes under the umbrella of a single theoretical structure. Thus there is not much support for the idea of focussing on a special foliation associated with a contingent feature of the actual universe in which we happen to find ourselves.

- At the other extreme, one might seek a new interpretation of quantum theory in which the concept of time does not appear at all (or, at the very least, plays no fundamental role). Our familiar notion of temporal evolution will then have to 'emerge' from the formalism in some phenomenological way. As we shall see, some of the most interesting approaches to the problem of time are of this type.

- Events in $\mathcal{M}$ might be identified using the positions of physical particles. For example, if ϕ is a scalar field on $\mathcal{M}$ the value of the field where a particular particle *is*, is a Diff$(\mathcal{M})$-invariant number and is hence an observable in the sense discussed above. This idea can be generalised in several ways. In particular, the value of ϕ could be specified at that event where some collection of fields takes on a certain set of values (assuming there is a unique such event). These fields, or 'internal coordinates', might be specified using distributions of matter, or they could be part of the gravitional field complex itself. Almost all of the existing approaches to the problem of time involve ideas of this type.

This last point emphasises the important fact that a physical definition of time requires more than just saying that it is a local coordinate on the spacetime manifold. For example, in the real world, time is often measured with the aid of a spatially-localised physical clock, and this usually means the proper time along its worldline. This quantity is Diff$(\mathcal{M})$-invariant if the diffeomorphism group is viewed as acting simultaneously on

[4] An interesting recent suggestion by Valentini (1992) is that a preferred foliation of spacetime could arise from the existence of nonlocal hidden-variables.

the points in $\mathcal{M}$ (and hence on the world line) and the spacetime metric γ. More precisely, if the beginning and end-points of the world-line are labelled using internal coordinates, the proper time along the geodesic connecting them is an intrinsic property.

The idea of labelling spacetime events with the aid of physical clocks and spatial reference frames is of considerable importance in both the classical and the quantum theory of relativity. However, in the case of the quantum theory the example of proper time raises another important issue. The calculation of the value of an interval of proper time involves the spacetime metric γ, and therefore it has a meaning only *after* the equations of motion have been solved (and of course these equations include a contribution from the energy-momentum tensor of the matter from which the clock is made). This causes no problem in the classical theory, but difficulties arise in any theory in which the geometry of spacetime is subject to quantum fluctuations and therefore has no fixed value. There is an implication that time may become a quantum operator: a problematic concept that is not part of standard quantum theory .

2.3 Approaches to the Problem of Time

Most approaches to resolving the problem of time in quantum gravity agree that 'time' should be identified in terms of the internal structure of the system rather than being regarded as any sort of external parameter. Their differences lie in:

- the way in which such an identification is made;
- whether it is done before or after quantisation;
- the degree to which the resulting entity resembles the familiar time of conventional classical and quantum physics;
- the role it plays in the final interpretation of the theory.

In turn, these differences are closely related to the general question of how to handle the constraints on the canonical variables that are such an intrinsic feature of the canonical theory of general relativity (§3).

As discussed in Kuchař (1992*b*), the various approaches to the problem of time in quantum gravity can be organised into three broad categories:

I The first class of schemes are those in which an internal time is identified as a functional of the canonical variables, and the canonical constraints are solved, *before* the system is quantised. The aim is to reproduce something like a normal Schrödinger equation with respect to this choice of time. This category is the most conservative of the three since its central assumption is that the construction of *any* coherent quantum-theoretical structure requires something resembling the external, classical time of standard quantum theory.

II In the second type of scheme the procedure above is reversed and the constraints are imposed at a quantum level as restrictions on allowed state vectors and with time being identified only *after* this step. The states can be written as functionals $\Psi[g]$ of a three-geometry $g_{ab}(x)$ (the basic configuration variable in the theory) and the most important of the operator constraints is a functional differential equation for $\Psi[g]$ known as the Wheeler-DeWitt equation (5.1.17). The notion of time has to be recovered in some way from the solutions to this central equation. The key feature of approaches of this type is that the final probabilistic interpretation of the theory is made only *after* the identification of time. Thus the Hilbert space structure of the final theory may be related only very indirectly (if at all) to that of the quantum theory with which the construction starts.

III The third class of scheme embraces a variety of methods that aspire to maintain the timeless nature of general relativity by avoiding any specific conception of time in the quantum theory. Some start, as in II, by imposing the constraints at a quantum level, others proceed along somewhat different lines, but they all agree in espousing the view that it is possible to construct a technically coherent, and *conceptually complete*, quantum theory (including the probabilistic interpretation) without needing to make any direct reference to the concept of time which, at most, has a purely phenomenological status. It is this latter feature that separates these schemes from those of category II.

These three broad categories can be further sub-divided into the following ten specific approaches to the problem of time.

I Tempus ante quantum

1. *The internal Schrödinger interpretation.* Time and space coordinates are identified as specific functionals of the gravitational canonical variables, and are then separated from the dynamical degrees of freedom by a canonical transformation. The constraints are solved classically for the momenta conjugate to these variables, and the remaining physical (*i.e.,* 'non-gauge') modes of the gravitational field are then quantised in a conventional way, giving rise to a Schrödinger evolution equation for the physical states.
2. *Matter clocks and reference fluids.* This is an extension of the internal Schrödinger interpretation in which matter variables coupled to the geometry are used to label spacetime events. They are introduced in a special way aimed at facilitating the handling of the constraints that yield the Schrödinger equation.
3. *Unimodular gravity.* This is a modification of general relativity in which the cosmological constant λ is considered as a dynamical variable. A 'cosmological' time is identified as the variable conjugate to λ, and the constraints yield the Schrödinger equation with respect to this time. This approach can be treated as a special case of a reference fluid.

II Tempus post quantum

1. *The Klein-Gordon interpretation.* The Wheeler-DeWitt equation is considered as an infinite-dimensional analogue of the Klein-Gordon equation for a relativistic particle moving in a fixed background geometry. The probabilistic interpretation of the theory is based on the Klein-Gordon norm with the hope that it will be positive on some appropriate subspace of solutions to the Wheeler-DeWitt equation.

2. *Third quantisation.* The problems arising from the indefinite nature of the scalar product of the Klein-Gordon interpretation are addressed by suggesting that the solutions $\Psi[g]$ of the Wheeler-DeWitt equation are to be turned into operators. This is analogous to the second quantisation of a relativistic particle whose states are described by the Klein-Gordon equation.

3. *The semi-classical interpretation.* Time is deemed to be a meaningful concept only in some semi-classical limit of the quantum gravity theory based on the Wheeler-DeWitt equation. Using a form of WKB expansion, the Wheeler-DeWitt equation is approximated by a conventional Schrödinger equation in which the time variable is extracted from the state $\Psi[g]$. The probabilistic interpretation arises only at this level.

III Tempus nihil est

1. *The naïve Schrödinger interpretation.* The square $|\Psi[g]|^2$ of a solution $\Psi[g]$ of the Wheeler-DeWitt equation is interpreted as the probability density for 'finding' a spacelike hypersurface of $\mathcal{M}$ with the geometry g. Time enters as an internal coordinate function of the three-geometry, and is represented by an operator that is part of the quantisation of the complete three-geometry.

2. *The conditional probability interpretation.* This can be regarded as a sophisticated development of the naïve Schrödinger interpretation whose primary ingredient is the use of conditional probabilities for the results of a pair of observables A and B. This is deemed to be correct even in the absence of any proper notion of time; as such, it is a modification of the conventional quantum-theoretical formalism. In certain cases, one of the observables is regarded as defining an instant of time (*i.e.,* it represents a physical, and therefore imperfect, clock) at which the other variable is measured [5]. Dynamical evolution is then equated to the dependence of these conditional probabilities on the values of the internal clock variables.

3. *Consistent histories approach.* This is based on a far-reaching extension of normal quantum theory to a form that does not require the conventional

[5] Or, perhaps, 'has a value'; the language used reflects the extent to which one favours operational or realist interpretations of quantum theory. In practice, quantum schemes of type III are particularly prone to receive a 'many-worlds' interpretation.

Copenhagen interpretation. The main ingredient is a precise prescription from within the formalism itself that says when it is, or is not, meaningful to ascribe a probability to a *history* of the system. The extension to quantum gravity involves defining the notion of a 'history' in a way that avoids having to make any direct reference to the concept of time. In its current form, the scheme culminates in the hope that functional integrals over spacetime fields may be well-defined, even in the absence of a conventional Hilbert space structure.

4. *The frozen time formalism.* Observables in quantum gravity are declared to be operators that commute with all the constraints, and are therefore constants of the motion. Attempts are made to show that, although 'timeless', such observables can nevertheless be used to give a picture of dynamical evolution.

2.4 Technical Problems With Time

In the context of either of the first two types of scheme—I constrain before quantising, or II quantise before constraining—a number of potential technical problems can be anticipated. Some of the most troublesome are as follows (see Kuchař (1992*b*)).

- *The ultra-violet divergence problem.* Both schemes involve complicated classical functions of fields defined at the same spatial point. The perturbative non-renormalisability of quantum gravity suggests that the operator analogues of these expressions are very ill-defined. This pathology has no direct connection with most aspects of the problem of time but it throws a big question mark over some of the techniques used to tackle that problem.

- *The operator-ordering problem.* Horrendous operator-ordering difficulties arise when attempts are made to replace the classical constraints and Hamiltonians with operator equivalents. These difficulties cannot easily be separated from the ultra-violet divergence problem.

- *The global time problem.* Experience with non-abelian gauge theories suggests the existence of global obstructions to making the crucial canonical transformations that untangle the physical modes of the gravitational field from internal spacetime coordinates.

- *The multiple-choice problem.* The Schrödinger equation based on one particular choice of internal time may give a different quantum theory from that based on another. Which, if any, is correct? Can these different quantum theories be seen to be part of an overall scheme that *is* covariant? A similar problem can be anticipated with the identification of time in the solutions of the Wheeler-DeWitt equation.

- *The Hilbert space problem.* Schemes of type I have the big advantage of giving a natural inner product that is conserved with respect to the internal time variables. This leads to a straightforward interpretative framework. In the alternative constraint quantisation schemes the situation is quite different. The Wheeler-DeWitt equation is a second-order functional differential equation, and as such presents familiar problems if one tries to construct a genuine, positive-definite inner product on the space of its solutions. This is the 'Hilbert space problem'.

- *The spatial metric reconstruction problem.* The classical separation of the canonical variables into physical and non-physical parts can be inverted and, in particular, the metric g_{ab} can be expressed as a functional of the dynamical and non-dynamical modes. The 'spatial metric reconstruction problem' is whether something similar can be done at the quantum level. This is part of the general question of the extent to which classical geometrical properties can be, or should be, preserved in the quantum theory.

- *The spacetime problem.* If an internal space or time coordinate is to operate within a conventional spacetime context, it is necessary that, viewed as a function on $\mathcal{M}$, it be a *scalar* field; in particular, it must not depend on any background foliation of $\mathcal{M}$. However, the objects used in the canonical approach to general relativity are functionals of the *canonical* variables, and there is no *prima facie* reason for supposing they will satisfy this condition. The spacetime problem consists in finding functionals that do have this desirable property or, if this is not possible, understanding how to handle the situation and what it means in spacetime terms.

- *The problem of functional evolution.* This affects both approaches to the canonical theory of gravity. In the scheme in which time is identified before quantisation, the problem is the possible existence of anomalies in the algebra of the local Hamiltonians that are associated with the generalised Schrödinger equation. Any such anomaly would render this equation inconsistent. In quantisation schemes of type II (and in the appropriate schemes of type III), the worry is potential anomalies in the quantum version of the canonical analogue of the Lie algebra of the spacetime diffeomorphism group Diff$(\mathcal{M})$. In both cases, the consistency of the classical evolution (guaranteed by the closing properties of the appropriate algebras) is lost.

In theories of category III, time plays only a secondary role, and therefore, with the exception of the first two, most of the problems above are not directly relevant. However, analogues of several of them appear in type III schemes, which also have additional difficulties of their own. Further discussion is deferred to the appropriate sections of the notes.

3 CANONICAL GENERAL RELATIVITY

3.1 Introductory Remarks

Most of the discussion of the problem of time in quantum gravity has been within the framework of the canonical theory whose starting point is a three-dimensional manifold Σ which serves as a model for physical space. This is often contrasted with the so-called 'spacetime' (or 'covariant') approaches in which the basic entity is a four-dimensional spacetime manifold $\mathcal{M}$. The virtues and vices of these two different approaches have been the subject of intense debate over the years and the matter is still far from being settled. Not surprisingly, the problem of time looks very different in the two schemes. Some of the claimed advantages of the canonical approach to quantum gravity are as follows.

1. The spacetime approaches often employ formal functional-integral techniques in which a number of serious difficulties are swept under the carpet as problems concerned with the integration measure, the contour of integration *etc.* On the other hand, canonical quantisation is usually discussed in an operator-based framework, with the advantage that problems appear in a more explicit and, perhaps, tractable way.

2. The development of quantisation techniques that do not depend on a background metric seems to be easier in the canonical framework. This is particularly relevant to the problem of time.

3. For related reasons, canonical quantisation is better suited for discussing quantum cosmology, spacetime singularities and similar topics.

4. Canonical methods tend to place more emphasis on the geometrical structure of general relativity. In particular, it is easier to address the issue of the extent to which such structure is, or should be, maintained in the quantum theory.

5. Many of the deep conceptual problems in quantum gravity are more transparent in a canonical approach. This applies in particular to the problem of time and the, not unrelated, general question of the domain of applicability of the interpretative framework of conventional quantum theory.

It should be emphasised that some of these advantages arise only in comparison with weak-field perturbation theory, and this does not mean that canonical methods are intrinsically superior to those in which spacetime fields are used from the outset. For example, there could exist *bona fide* approaches to quantum gravity that involve the calculation of a functional integral like

$$Z = \int \mathcal{D}[g]\, e^{iS(g)} \tag{3.1.1}$$

using methods other than weak-field perturbation theory. A particularly interesting example is the consistent histories approach to the problem of time discussed in §6.2.

3.2 Quantum Field Theory in a Curved Background

3.2.1 The Canonical Formalism

In approaching the canonical theory of general relativity it is helpful to begin by considering briefly the canonical quantisation of a scalar field ϕ propagating in a background Lorentzian metric γ on a spacetime manifold $\mathcal{M}$. The action for the system is

$$S[\phi] = -\int_{\mathcal{M}} d^4X\,(-\det\gamma(X))^{\frac{1}{2}}\left(\frac{1}{2}\gamma^{\alpha\beta}(X)\,\partial_\alpha\phi(X)\,\partial_\beta\phi(X) + V(\phi(X))\right) \qquad (3.2.1)$$

where $V(\phi)$ is an interaction potential for ϕ (which could include a mass term).

This system has a well-posed classical Cauchy problem only if $(\mathcal{M},\gamma)$ is a globally-hyperbolic pair (Hawking & Ellis 1973). In particular, this means that $\mathcal{M}$ is topologically equivalent to the Cartesian product $\Sigma\times\mathbb{R}$ where Σ is some spatial three-manifold and $\mathbb{R}$ is a global time direction. To acquire the notion of dynamical evolution we need to *foliate* $\mathcal{M}$ into a one-parameter family of embeddings $\mathcal{F}_t : \Sigma \to \mathcal{M}$, $t\in\mathbb{R}$, of Σ in $\mathcal{M}$ that are spacelike with respect to the background Lorentzian metric $\gamma_{\alpha\beta}$; the real number t can then serve as a time parameter. To say that an embedding $\mathcal{E}:\Sigma\to\mathcal{M}$ is *spacelike* means that the pull-back $\mathcal{E}^*(\gamma)$—a symmetric rank-two covariant tensor field on Σ—has signature $(1,1,1)$ and is positive definite. We recall that the components of $\mathcal{E}^*(\gamma)$ are [6]

$$(\mathcal{E}^*(\gamma))_{ab}(x) := \gamma_{\alpha\beta}(\mathcal{E}(x))\,\mathcal{E}^\alpha{}_{,a}(x)\,\mathcal{E}^\beta{}_{,b}(x) \qquad (3.2.2)$$

on the three-manifold Σ.

The canonical variables are defined on Σ and consist of the scalar field ϕ and its conjugate variable π (a scalar density) which is essentially the time-derivative of ϕ. Classically these constitute a well-defined set of Cauchy data and satisfy the Poisson bracket relations

$$\{\phi(x),\phi(x')\} = 0 \qquad (3.2.3)$$
$$\{\pi(x),\pi(x')\} = 0 \qquad (3.2.4)$$
$$\{\phi(x),\pi(x')\} = \delta(x,x'). \qquad (3.2.5)$$

The Dirac δ-function is defined such that the smeared fields satisfy

$$\{\phi(f),\pi(h)\} = \int_\Sigma d^3x\,f(x)\,h(x) \qquad (3.2.6)$$

[6] The symbol $\mathcal{E}^\alpha(x)$ means $X^\alpha(\mathcal{E}(x))$ where X^α, $\alpha=0,1,2,3$ is a coordinate system on $\mathcal{M}$. The quantity defined by the left hand side of (3.2.2) is independent of the choice of such a system.

where the test-functions f and h are respectively a scalar density and a scalar [7] on the three-manifold Σ.

The dynamical evolution of the system is obtained by constructing the Hamiltonian $H(t)$ in the usual way from the action (3.2.1) and the given foliation of $\mathcal{M}$. The resulting equations of motion are

$$\frac{\partial\phi(x,t)}{\partial t} = \{\phi(x), H(t)\} \tag{3.2.7}$$
$$\frac{\partial\pi(x,t)}{\partial t} = \{\pi(x), H(t)\}. \tag{3.2.8}$$

Note that these give the evolution with respect to the time parameter associated with the specified foliation. However, the physical fields can be evaluated on any spacelike hypersurface, and this should not depend on the way the hypersurface happens to be included in a particular foliation. Thus it should be possible to write the physical fields as functions $\phi(x,\mathcal{E}\,]$, $\pi(x,\mathcal{E}\,]$ of $x \in \Sigma$ and functionals of the embedding functions $\mathcal{E}$. Indeed, the Hamiltonian equations of motion (3.2.7–3.2.8)) are valid for all foliations and imply the existence of four functions $h_\alpha(x,\mathcal{E}\,]$ of the canonical variables such that $\phi(x,\mathcal{E}\,]$ and $\pi(x,\mathcal{E}\,]$ satisfy the functional differential equations

$$\frac{\delta\phi(x,\mathcal{E}\,]}{\delta\mathcal{E}^\alpha(x')} = \{\phi(x,\mathcal{E}\,], h_\alpha(x',\mathcal{E}\,]\} \tag{3.2.9}$$
$$\frac{\delta\pi(x,\mathcal{E}\,]}{\delta\mathcal{E}^\alpha(x')} = \{\pi(x,\mathcal{E}\,], h_\alpha(x',\mathcal{E}\,]\} \tag{3.2.10}$$

that describe how these fields change under an infinitesimal deformation of the embedding $\mathcal{E}$.

3.2.2 Quantisation of the System

The formal canonical quantisation of this system follows the usual rule of replacing Poisson brackets with operator commutators. Thus (3.2.3–3.2.5) become

$$[\,\widehat{\phi}(x), \widehat{\phi}(x')\,] = 0 \tag{3.2.11}$$
$$[\,\widehat{\pi}(x), \widehat{\pi}(x')\,] = 0 \tag{3.2.12}$$
$$[\,\widehat{\phi}(x), \widehat{\pi}(x')\,] = i\hbar\,\delta(x,x') \tag{3.2.13}$$

which can be made rigorous by smearing and exponentiating in the standard way.

The next step is to choose an appropriate representation of this operator algebra. One might try to emulate elementary wave mechanics by taking the state space of the

[7] We are using a definition of the Dirac delta function $\delta(x,x')$ that is a scalar in x and a scalar density in x'. Thus, if f is a scalar function on Σ we have, formally, $f(x) = \int_\Sigma d^3x'\,\delta(x,x')f(x')$.

quantum field theory to be a set of functionals Ψ on the topological vector space E of all classical fields, with the canonical operators defined as

$$
\begin{aligned}
(\widehat{\phi}(x)\Psi)[\phi] &:= \phi(x)\Psi[\phi] && (3.2.14)\\
(\widehat{\pi}(x)\Psi)[\phi] &:= -i\hbar\frac{\delta\Psi[\phi]}{\delta\phi(x)}. && (3.2.15)
\end{aligned}
$$

Thus the inner product would be

$$\langle\Psi|\Phi\rangle = \int_E d\mu[\phi]\,\Psi^*[\phi]\,\Phi[\phi] \tag{3.2.16}$$

and, for a normalised function Ψ,

$$\mathrm{Prob}(\phi \in B; \Psi) = \int_B d\mu[\phi]\,|\Psi[\phi]|^2 \tag{3.2.17}$$

is the probability that a measurement of the field on Σ will find it in the subset B of E.

This analysis can be made rigorous after smearing the fields with functions from an appropriate test function space. It transpires that the support of a state functional Ψ is typically on *distributions*, rather than smooth functions, so that the inner product is really

$$\langle\Psi|\Phi\rangle = \int_{E'} d\mu[\phi]\,\Psi^*[\phi]\,\Phi[\phi] \tag{3.2.18}$$

where E' denotes the topological dual of E. Furthermore, there is no infinite-dimensional version of Lebesgue measure, and hence if (3.2.15) is to give a self-adjoint operator it must be modified to read

$$(\widehat{\pi}(x)\Psi)[\phi] := -i\hbar\frac{\delta\Psi[\phi]}{\delta\phi(x)} + i\rho(x)\Psi[\phi] \tag{3.2.19}$$

where $\rho(x)$ is a function that compensates for the weight factor in the measure $d\mu$ used in the construction of the Hilbert space of states $L^2(E', d\mu)$.

The dynamical evolution of the system can be expressed in the Heisenberg picture as the commutator analogue of the Poisson bracket relations (3.2.7–3.2.8). Alternatively, one can adopt the Schrödinger picture in which the time evolution of state vectors is given by [8]

$$i\hbar\frac{\partial\Psi(t,\phi]}{\partial t} = H(t;\widehat{\phi},\widehat{\pi}\,]\,\Psi(t,\phi] \tag{3.2.20}$$

or, if the classical theory is described by the functional equations (3.2.9–3.2.10), by the functional differential equations

$$i\hbar\frac{\delta\Psi[\mathcal{E},\phi]}{\delta\mathcal{E}^\alpha(x)} = h_\alpha(x;\mathcal{E},\widehat{\phi},\widehat{\pi}\,]\,\Psi[\mathcal{E},\phi]. \tag{3.2.21}$$

[8]As usual, the notation $H(t;\widehat{\phi},\widehat{\pi}\,]$ must be taken with a pinch of salt. The classical fields $\phi(x)$ and $\pi(x)$ can be replaced in the Hamiltonian $H(t;\phi,\pi\,]$ with their operator equivalents $\widehat{\phi}(x)$ and $\widehat{\pi}(x)$ only after a careful consideration of operator ordering and regularisation.

However, note that the steps leading to (3.2.20) or (3.2.21) are valid only if the inner product on the Hilbert space of states is t (resp. $\mathcal{E}$) independent. If not, a compensating term must be added to (3.2.20) (resp. (3.2.21)) if the scalar product $\langle\Psi_t|\Phi_t\rangle_t$ (resp. $\langle\Psi_{\mathcal{E}}|\Phi_{\mathcal{E}}\rangle_{\mathcal{E}}$) is to be independent of t (resp. the embedding $\mathcal{E}$). This is analogous to the quantum theory of a particle moving on an n-dimensional Riemannian manifold Q with a time-dependent background metric $g(t)$. The natural inner product

$$\langle\psi_t|\phi_t\rangle_t := \int_Q d^n q\,(\det g(q,t))^{\frac{1}{2}}\,\psi_t^*(q)\,\phi_t(q) \tag{3.2.22}$$

is not preserved by the naïve Schrödinger equation

$$i\hbar\frac{d\psi_t}{dt} = \widehat{H}(t)\psi_t \tag{3.2.23}$$

because the time derivative of the right hand side of (3.2.22) acquires an extra term coming from the time-dependence of the metric $g(t)$. In the Heisenberg picture the weight function $\rho(q)$ becomes time-dependent.

The major technical problem in quantum field theory on a curved background $(\mathcal{M},\gamma)$ is the existence of infinitely many unitarily inequivalent representations of the canonical commutation relations (3.2.11–3.2.13). If the background metric γ is static, the obvious step is to use the timelike Killing vector to select the representation. This gives rise to a consistent one-particle picture of the quantum field theory. In other special, but non-static, cases (for example, a Robertson-Walker metric) there may be a 'natural' choice for the representations that gives rise to a picture of particle creation by the background metric, and for genuine astrophysical applications this may be perfectly adequate.

The real problems arise if one is presented with a generic metric γ, in which case it is not at all clear how to proceed. A minimum requirement is that the Hamiltonians $\widehat{H}(t)$, or the Hamiltonian densities $\widehat{h}_\alpha(x,\mathcal{E}\,]$, should be well-defined. However, there is an unpleasant possibility that the representations could be t (resp. $\mathcal{E}$) dependent, and in such a way that those corresponding to different values of t (resp. $\mathcal{E}$) are unitarily inequivalent, in which case the dynamical equations (3.2.20) (resp. (3.2.21)) are not meaningful. This particular difficulty can be overcome by using a C^*-algebra approach, but the identification of physically-meaningful representations remains a major problem.

3.3 The Arnowitt-Deser-Misner Formalism

3.3.1 Introduction of the Foliation

Let us consider now how these ideas extend to the canonical formalism of general relativity itself. The early history of this subject was grounded in the seminal work of Dirac (1958a,1958b) and culminated in the investigations by Arnowitt, Deser and Misner (1959a, 1959b, 1960a, 1960b, 1960c, 1960d, 1961a, 1961b, 1962). These original studies involved the selection of a specific coordinate system on the spacetime manifold, and some of the global issues were thereby obscured. A more geometrical, global approach was developed by Kuchař in a series of papers Kuchař (1972, 1976a, 1976b, 1976c, 1977, 1981a) and this will be adopted here. The treatment will be fairly cursory since the main aim of this course is to develop the conceptual and structural aspects of the problem of time. A more detailed account of the technical issues can be found in the forthcoming work Isham & Kuchař (1994).

The starting point is a four-dimensional manifold $\mathcal{M}$ and a three-dimensional manifold Σ that play the roles of physical spacetime and three-space respectively. The space Σ is assumed to be compact; if not, some of the expressions that follow must be augmented by surface terms. Furthermore, the topology of $\mathcal{M}$ is assumed to be such that it can be foliated by a one-parameter family of embeddings $\mathcal{F}_t : \Sigma \to \mathcal{M}$, $t \in \mathbb{R}$, of Σ in $\mathcal{M}$ (of course, then there will be many such foliations). As in the case of the scalar field theory, this requirement imposes a significant *a priori* topological limitation on $\mathcal{M}$ since (by the definition of a foliation) the map $\mathcal{F} : \Sigma \times \mathbb{R} \to \mathcal{M}$, defined by $(x,t) \mapsto \mathcal{F}(x,t) := \mathcal{F}_t(x)$, is a diffeomorphism of $\Sigma \times \mathbb{R}$ with $\mathcal{M}$.

Since $\mathcal{F}$ is a diffeomorphism from $\Sigma \times \mathbb{R}$ to $\mathcal{M}$, its inverse $\mathcal{F}^{-1} : \mathcal{M} \to \Sigma \times \mathbb{R}$ is also a diffeomorphism and can be written in the form

$$\mathcal{F}^{-1}(X) = (\sigma(X), \tau(X)) \in \Sigma \times \mathbb{R} \tag{3.3.1}$$

where $\sigma : \mathcal{M} \to \Sigma$ and $\tau : \mathcal{M} \to \mathbb{R}$. The map τ is a global *time function* and gives the natural time parameter associated with the foliation in the sense that $\tau(\mathcal{F}_t(x)) = t$ for all $x \in \Sigma$. However, from a physical point of view such a definition of 'time' is rather artificial (how would it be measured?) and is a far cry from the notion of time employed in the construction of real clocks. This point is not trivial and we shall return to it later.

Note that for each $x \in \Sigma$ the map $\mathcal{F}_x : \mathbb{R} \to \mathcal{M}$ defined by $t \mapsto \mathcal{F}(x,t)$ is a curve in $\mathcal{M}$ and therefore has a one-parameter family of tangent vectors on $\mathcal{M}$, denoted $\dot{\mathcal{F}}_x(t)$, whose components are $\dot{\mathcal{F}}^\alpha_x(t) = \partial \mathcal{F}^\alpha(x,t)/\partial t$. The flow lines of the ensuing vector field (known as the *deformation* vector [9] of the foliation) are illustrated in Figure 1 which also shows the normal vector n_{t_1} on the hypersurface $\mathcal{F}_{t_1}(\Sigma) \subset \mathcal{M}$. In general, if $\mathcal{E} : \Sigma \to \mathcal{M}$ is a spacelike embedding, the *normal* vector field n to $\mathcal{E}$ is defined by the equations

$$\begin{aligned} n_\alpha(x, \mathcal{E}\,]\, \mathcal{E}^\alpha{}_{,a}(x) &= 0 && (3.3.2)\\ \gamma^{\alpha\beta}(\mathcal{E}(x))\, n_\alpha(x, \mathcal{E}\,]\, n_\beta(x, \mathcal{E}\,] &= -1 && (3.3.3)\end{aligned}$$

for all $x \in \Sigma$. Equation (3.3.2) defines what it means to say that n is *normal* to the hypersurface $\mathcal{E}(\Sigma)$, while (3.3.3) is a normalisation condition on n and emphasises that

[9] The deformation vector is a hybrid object in the sense that if x is a point in the three-space Σ the vector $\dot{\mathcal{F}}_x(t)$ lies in the tangent space $T_{\mathcal{F}(x,t)}\mathcal{M}$ at the point $\mathcal{F}(x,t)$ in the four-manifold $\mathcal{M}$. Such an object is best regarded as an element of the space $T_{\mathcal{F}_t}\mathrm{Emb}(\Sigma, \mathcal{M})$ of vectors tangent to the infinite-dimensional manifold $\mathrm{Emb}(\Sigma, \mathcal{M})$ of embeddings of Σ in $\mathcal{M}$ at the particular embedding $\mathcal{F}_t$. This way of viewing things is quite useful technically but I shall not develop it further here.

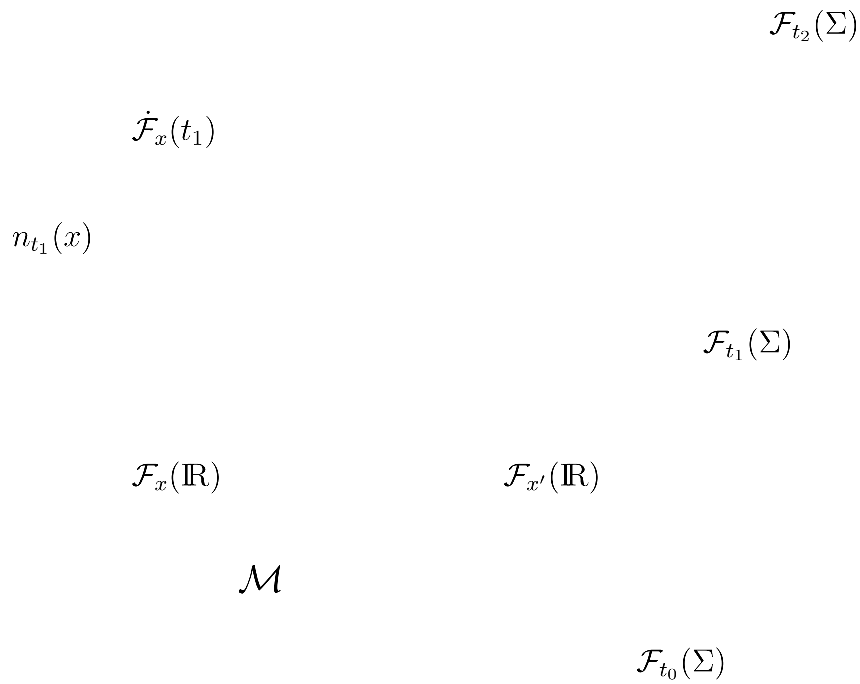

Figure 1: The flow lines of the foliation of $\mathcal{M}$

this vector is timelike with respect to the Lorentzian metric on $\mathcal{M}$. The minus sign on the right hand side of (3.3.3) reflects our choice of signature on $(\mathcal{M}, \gamma)$ as $(-1, 1, 1, 1)$.

In considering the dynamical evolution, a particularly interesting quantity is the functional derivative of $g_{ab}(x, \mathcal{E}\,]$ with respect to $\mathcal{E}$ projected along the normal vector n. A direct calculation (Hojman, Kuchař & Teitelboim 1976, Kuchař 1974) shows that

$$n^\alpha(x, \mathcal{E}\,] \frac{\delta}{\delta \mathcal{E}^\alpha(x)} g_{ab}(x', \mathcal{E}\,] = -2K_{ab}(x, \mathcal{E}\,]\, \delta(x, x') \tag{3.3.4}$$

where K is the *extrinsic curvature* of the hypersurface $\mathcal{E}(\Sigma)$ defined by

$$K_{ab}(x, \mathcal{E}\,] := -{}^4\nabla_\alpha n_\beta(x, \mathcal{E}\,]\, \mathcal{E}^\alpha{}_{,a}(x)\, \mathcal{E}^\beta{}_{,b}(x) \tag{3.3.5}$$

where ${}^4\nabla_\alpha n_\beta(x, \mathcal{E}\,]$ denotes the covariant derivative obtained by parallel transporting the cotangent vector $n(x, \mathcal{E}\,] \in T^*_{\mathcal{E}(x)}\mathcal{M}$ along the hypersurface $\mathcal{E}(\Sigma)$ using the Lorentzian four-metric γ on $\mathcal{M}$. It is straightforward to show that K is a symmetric tensor, *i.e.,* $K_{ab}(x, \mathcal{E}\,] {=} K_{ba}(x, \mathcal{E}\,]$.

3.3.2 The Lapse Function and Shift Vector

The next step is to decompose the deformation vector into two components, one of which lies along the hypersurface $\mathcal{F}_t(\Sigma)$ and the other of which is parallel to n_t. In particular we can write

$$\dot{\mathcal{F}}^\alpha(x,t) = N(x,t)\,\gamma^{\alpha\beta}(\mathcal{F}(x,t))\,n_\beta(x,t) + N^a(x,t)\,\mathcal{F}^\alpha{}_{,a}(x,t) \tag{3.3.6}$$

where we have used $n_\beta(x,t)$ instead of the more clumsy notation $n_\beta(x,\mathcal{F}_t\,]$. The quantities $N(t)$ and $N^a(t)$ are known respectively as the *lapse* function and the *shift* vector associated with the embedding $\mathcal{F}_t$.

Note that, like the normal vector, the lapse and shift depend on both the spacetime metric γ and the foliation. This relationship can be partially inverted with, for a fixed foliation, the lapse and shift functions being identified as parts of the metric tensor γ. This can be seen most clearly by studying the 'pull-back' $\mathcal{F}^*(\gamma)$ of γ by the foliation $\mathcal{F} : \Sigma \times \mathrm{I\!R} \to \mathcal{M}$ in coordinates X^α, $\alpha = 0\ldots 3$, on $\Sigma \times \mathrm{I\!R}$ that are adapted to the product structure in the sense that $X^{\alpha=0}(x,t) = t$, and $X^{\alpha=1,2,3}(x,t) = x^{a=1,2,3}(x)$ where x^a, $a = 1,2,3$ is some coordinate system on Σ. The components of $\mathcal{F}^*(\gamma)$ are

$$(\mathcal{F}^*\gamma)_{00}(x,t) = N^a(x,t)\,N^b(x,t)\,g_{ab}(x,t) - (N(x,t))^2 \tag{3.3.7}$$
$$(\mathcal{F}^*\gamma)_{0a}(x,t) = N^b(x,t)\,g_{ab}(x,t) \tag{3.3.8}$$
$$(\mathcal{F}^*\gamma)_{ab}(x,t) = g_{ab}(x,t) \tag{3.3.9}$$

where $g_{ab}(x,t)$ is shorthand for $g_{ab}(x,\mathcal{F}_t\,]$ and is given by

$$g_{ab}(x,t) := (\mathcal{F}_t^*\,\gamma)_{ab}(x) = \gamma_{\alpha\beta}(\mathcal{F}(x,t))\,\mathcal{F}^\alpha{}_{,a}(x,t)\,\mathcal{F}^\beta{}_{,b}(x,t). \tag{3.3.10}$$

We see from (3.3.6) that the lapse function and shift vector provide information on how a hypersurface with constant time parameter t is related to the displaced hypersurface with constant parameter $t+\delta t$ as seen from the perspective of an enveloping spacetime. More precisely, for a given Lorentzian metric γ on $\mathcal{M}$ and foliation $\mathcal{F} : \Sigma \times \mathrm{I\!R} \to \mathcal{M}$, the lapse function specifies the proper time separation $\delta_\perp\tau$ between the hypersurfaces $\mathcal{F}_t(\Sigma)$ and $\mathcal{F}_{t+\delta t}(\Sigma)$ measured in the direction normal to the first hypersurface:

$$\delta_\perp\tau(x) = N(x,t)\delta t. \tag{3.3.11}$$

The shift vector $\vec{N}(x)$ determines how, for each $x \in \Sigma$, the point $\mathcal{F}_{t+\delta t}(x)$ in $\mathcal{M}$ is displaced with respect to the intersection of the hypersurface $\mathcal{F}_{t+\delta t}(\Sigma)$ with the normal geodesic drawn from the point $\mathcal{F}_t(x) \equiv \mathcal{F}_x(t)$. If this intersection point can be obtained by evaluating $\mathcal{F}_{t+\delta t}$ at a point on Σ with coordinates $x^a + \delta x^a$ then

$$\delta x^a(x) = -N^a(x,t)\delta t \tag{3.3.12}$$

as illustrated in Figure 2.

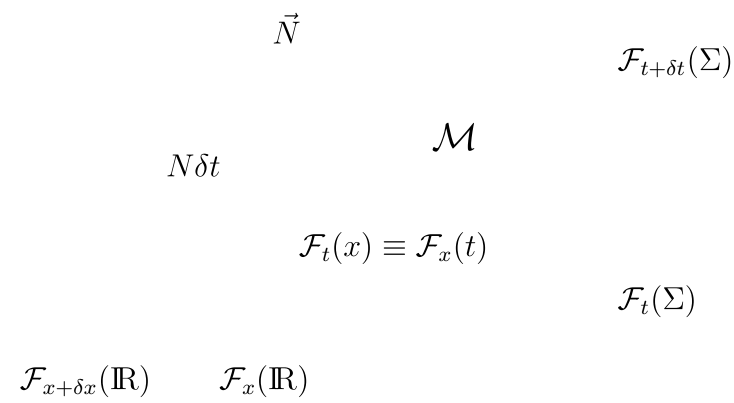

Figure 2: The lapse function and shift vector of a foliation

It is clear from (3.3.7–3.3.8) that the shift vector and lapse function are essentially the γ_{0a} and γ_{00} parts of the spacetime metric. It was in terms of this coordinate language that the original ADM approach was formulated. As we shall see shortly, the Einstein field equations do not lead to any dynamical development for these variables.

3.3.3 The Canonical Form of General Relativity

The canonical analysis of general relativity proceeds as follows. We choose some reference foliation $\mathcal{F}^{\mathrm{ref}} : \Sigma \times \mathrm{I\!R} \approx \mathcal{M}$ and consider first the situation in which $\mathcal{M}$ carries a *given* Lorentzian metric γ that satisfies the vacuum Einstein field equations $G_{\alpha\beta}(X, \gamma] = 0$ and is such that each leaf $\mathcal{F}^{\mathrm{ref}}_t(\Sigma)$ is a hypersurface in $\mathcal{M}$ that is spacelike with respect to γ. The problem is to find a set of canonical variables for this system and an associated set of first-order differential equations that determine how the variables evolve from one leaf to another and whose solution recovers the given metric γ. Of course, in practice this procedure is used in the situation in which γ is unknown and is to be determined by solving the Cauchy problem using given Cauchy data.

The Riemannian metric on Σ defined by (3.3.10) plays the role of the basic configuration variable. The rate of change of g_{ab} with respect to the label time t is related to the extrinsic curvature (3.3.5) K for the embeddings $\mathcal{F}^{\mathrm{ref}}_t : \Sigma \to \mathcal{M}$ by

$$K_{ab}(x,t) = \frac{1}{2N(x,t)}\big(- \dot{g}_{ab}(x,t) + L_{\vec{N}} g_{ab}(x,t)\big) \tag{3.3.13}$$

where $L_{\vec{N}} g_{ab}$ denotes the Lie derivative [10] of g_{ab} along the shift vector field $\vec{N}$ on Σ.

The key step in deriving the canonical form of the action principle is to pull-back the Einstein Lagrangian density by the foliation $\mathcal{F}^{\rm ref} : \Sigma \times \mathrm{I\!R} \to \mathcal{M}$ and express the result as a function of the extrinsic curvature, the metric g, and the lapse vector and shift function. This gives

$$\begin{aligned}(\mathcal{F}^{\rm ref})^*(|\gamma|^{\frac{1}{2}} R[\gamma]) &= N|g|^{\frac{1}{2}}(K_{ab}K^{ab} - (K^a_a)^2 + R[g]) \\ &\quad -2\frac{d}{dt}(|g|^{\frac{1}{2}} K_a{}^a) - (|g|^{\frac{1}{2}}(K_a{}^a N^b - g^{ab} N_{,a}))_{,b}\end{aligned} \tag{3.3.14}$$

where $R[g]$ and $|g|$ denote respectively the curvature scalar and the determinant of the metric on Σ.

The spatial-divergence term vanishes because Σ is assumed to be compact. However, the term with the total time-derivative must be removed by hand to produce a genuine action principle. The resulting 'ADM' action for matter-free gravity can be written as

$$S[g, N, \vec{N}] = \frac{1}{\kappa^2} \int dt \int_\Sigma d^3x\, N|g|^{\frac{1}{2}}(K_{ab}K^{ab} - (K_a{}^a)^2 + R[g]) \tag{3.3.15}$$

where $\kappa^2 := 8\pi G/c^2$. This action is to be varied with respect to the Lorentzian metric $(\mathcal{F}^{\rm ref})^*(\gamma)$ on $\Sigma \times \mathrm{I\!R}$, and hence with respect to the paths of spatial geometries, lapse functions, and shift vectors associated via (3.3.7–3.3.9) with $(\mathcal{F}^{\rm ref})^*(\gamma)$. The extrinsic curvature K is to be thought of as the explicit functional of these variables given by (3.3.13).

The canonical analysis now proceeds in the standard way except that, since (3.3.15) does not depend on the time derivatives of N or $\vec{N}$, in performing the Legendre transformation to the canonical action, the time derivative $\partial g_{ab}/\partial t$ is replaced by the conjugate variable p^{ab} but N and $\vec{N}$ are left untouched. The variable conjugate to g_{ab} is

$$p^{ab} := \frac{\delta S}{\delta \dot{g}_{ab}} = -\frac{|g|^{\frac{1}{2}}}{\kappa^2}(K^{ab} - g^{ab} K_c{}^c) \tag{3.3.16}$$

which can be inverted in the form

$$K^{ab} = -\kappa^2 |g|^{-\frac{1}{2}}(p^{ab} - \frac{1}{2} g^{ab} p_c{}^c). \tag{3.3.17}$$

The action obtained by performing the Legendre transformation is

$$S[g, p, N, \vec{N}] = \int dt \int_\Sigma d^3x\, (p^{ab}\dot{g}_{ab} - N\mathcal{H}_\perp - N^a \mathcal{H}_a) \tag{3.3.18}$$

where

$$\mathcal{H}_a(x; g, p) \;:=\; -2 p_a{}^b{}_{|b}(x) \tag{3.3.19}$$

$$\mathcal{H}_\perp(x; g, p) \;:=\; \kappa^2 \mathcal{G}_{ab\,cd}(x, g)\, p^{ab}(x)\, p^{cd}(x) - \frac{|g|^{\frac{1}{2}}(x)}{\kappa^2} R(x, g) \tag{3.3.20}$$

[10] This quantity is equal to $N_{a|b} + N_{b|a}$ where $|$ means covariant differentiation using the induced metric g_{ab} on Σ.

in which

$$\mathcal{G}_{ab\,cd}(x,g] := \frac{1}{2}|g|^{-\frac{1}{2}}(x)(g_{ac}(x)g_{bd}(x) + g_{bc}(x)g_{ad}(x) - g_{ab}(x)g_{cd}(x)) \tag{3.3.21}$$

is the 'DeWitt supermetric' on the space of three-metrics (DeWitt 1967a). The functions $\mathcal{H}_a$ and $\mathcal{H}_\perp$ of the canonical variables (g,p) play a key role in the theory and are known as the *supermomentum* and *super-Hamiltonian* respectively.

It must be emphasised that the variables g, p, N and $\vec{N}$ are treated as independent in varying the action (3.3.18). In particular, $g_{ab}(x,t)$ is no longer viewed as the restriction of a spacetime metric γ to a particular hypersurface $\mathcal{F}_t(\Sigma)$ of $\mathcal{M}$. Rather, $t \mapsto g_{ab}(x,t)$ is any path in the space $\mathrm{Riem}(\Sigma)$ of Riemannian metrics on Σ, and similarly $t \mapsto p^{ab}(x,t)$ (resp. $N(x,t)$, $N^a(x,t)$) is any path in the space of all contravariant, rank-2 symmetric tensor densities (resp. functions, vector fields) on Σ. Consistency with the earlier discussion is ensured by noting that the equation obtained by varying the action (3.3.18) with respect to p^{ab}

$$\dot{g}_{ab}(x,t) = \frac{\delta H[N,\vec{N}]}{\delta p^{ab}(x,t)}, \tag{3.3.22}$$

can be solved algebraically for p^{ab} and, with the aid of (3.3.17), reproduces the relation (3.3.13) between the time-derivative of g_{ab} and the extrinsic curvature. Here, the functional

$$H[N,\vec{N}](t) := \int_\Sigma d^3x\,(N\mathcal{H}_\perp + N^a\mathcal{H}_a) \tag{3.3.23}$$

of the canonical variables (g,p) acts as the Hamiltonian of the system.

Varying the action (3.3.18) with respect to g_{ab} gives the dynamical equations

$$\dot{p}^{ab}(x,t) = -\frac{\delta H[N,\vec{N}]}{\delta g_{ab}(x,t)} \tag{3.3.24}$$

while varying N^a and N leads respectively to

$$\mathcal{H}_a(x;g,p] = 0 \tag{3.3.25}$$

and

$$\mathcal{H}_\perp(x;g,p] = 0 \tag{3.3.26}$$

which are *constraints* on the canonical variables (g,p).

Note that the action (3.3.18) contains no time derivatives of the fields N or $\vec{N}$ which appear manifestly as Lagrange multipliers that enforce the constraints (3.3.26) and (3.3.25). This confirms the status of the lapse function and shift vector as non-dynamical variables that can be specified as *arbitrary* functions on $\Sigma \times \mathbb{R}$; indeed, this must be done in some way before the dynamical equations can be solved (see §3.3.6).

The set of equations (3.3.22), (3.3.24), (3.3.25) and (3.3.26) are completely equivalent to Einstein's equations $G_{\alpha\beta} = 0$ in the following sense (Fischer & Marsden 1979):

- Let γ by any Lorentzian metric on $\mathcal{M}$ that satisfies the vacuum Einstein equations $G_{\alpha\beta}(X,\gamma] = 0$ and let $t \mapsto \mathcal{F}_t : \Sigma \to \mathcal{M}$ be a one-parameter family of spacelike embeddings of Σ in $\mathcal{M}$ that foliates $\mathcal{M}$ with associated lapse function and shift vector given by (3.3.6). Then the family of induced metrics $t \mapsto g(t)$ and momenta $t \mapsto p(t)$ (where $p(t)$ is computed from γ using (3.3.13) and (3.3.16)) satisfies the equations (3.3.22), (3.3.24), (3.3.25) and (3.3.26).
- Conversely, if $t \mapsto \mathcal{F}_t$ is a spacelike foliation of $(\mathcal{M},\gamma)$ such that the evolution and constraint equations above hold, then γ satisfies the vacuum field equations.

In effect, equations (3.3.22) and (3.3.24) reproduce the projections of the Einstein equations that are tangent to the hypersurfaces $t =$*const.*, while (3.3.25) and (3.3.26) reproduce the normal projections of these equations, $n^\alpha\, n^\beta\, G_{\alpha\beta} = 0$ and $n^\alpha\, G_{\alpha a} = 0$.

3.3.4 The Constraint Algebra

The classical canonical algebra of the system is expressed by the basic Poisson brackets

$$\begin{aligned}
\{g_{ab}(x), g_{cd}(x')\} &= 0 && (3.3.27)\\
\{p^{ab}(x), p^{cd}(x')\} &= 0 && (3.3.28)\\
\{g_{ab}(x), p^{cd}(x')\} &= \delta^c_{(a}\delta^d_{b)}\,\delta(x,x') && (3.3.29)
\end{aligned}$$

which can be used to cast the dynamical equations into a canonical form in which the right hand sides of (3.3.22) and (3.3.24) are $\{g_{ab}(x), H[N,\vec{N}]\}$ and $\{p^{ab}(x), H[N,\vec{N}]\}$ respectively. However, the constraints (3.3.25) and (3.3.26) imply that not all the variables $g_{ab}(x)$ and $p^{cd}(x)$ are independent, and so, as in all systems with constraints, the Poisson bracket relations (3.3.27–3.3.29) need to be used with care.

A crucial property of the canonical formalism is the closure of the Poisson brackets of the super-Hamiltonian and supermomentum, computed using (3.3.27–3.3.29); *i.e.,* the set of constraints is *first class*. This is contained in the fundamental relations (Dirac 1965)

$$\begin{aligned}
\{\mathcal{H}_a(x), \mathcal{H}_b(x')\} &= -\mathcal{H}_b(x)\,\partial^{x'}_a\delta(x,x') + \mathcal{H}_a(x')\,\partial^x_b\delta(x,x') && (3.3.30)\\
\{\mathcal{H}_a(x), \mathcal{H}_\perp(x')\} &= \mathcal{H}_\perp(x)\,\partial^x_a\delta(x,x') && (3.3.31)\\
\{\mathcal{H}_\perp(x), \mathcal{H}_\perp(x')\} &= g^{ab}(x)\,\mathcal{H}_a(x)\,\partial^{x'}_b\delta(x,x') - \\
&\quad\; g^{ab}(x')\,\mathcal{H}_a(x')\,\partial^x_b\delta(x,x'). && (3.3.32)
\end{aligned}$$

Using the smeared variables [11]

$$H[N] := \int_\Sigma d^3x\, N(x)\,\mathcal{H}_\perp(x), \quad H[\vec{N}] := \int_\Sigma d^3x\, N^a(x)\,\mathcal{H}_a(x), \qquad (3.3.33)$$

[11] The integrals are well-defined since both $\mathcal{H}_a$ and $\mathcal{H}_\perp$ are densities on Σ.

where N and $\vec{N}$ are any scalar function and vector field on Σ, these equations can be written as

$$\begin{aligned}
\{H[\vec{N}_1], H[\vec{N}_2]\,\} &= H([\,\vec{N}_1, \vec{N}_2\,]) && (3.3.34)\\
\{H[\vec{N}], H[N]\,\} &= H[L_{\vec{N}}\, N] && (3.3.35)\\
\{H[N_1], H[N_2]\,\} &= H[\vec{N}] && (3.3.36)
\end{aligned}$$

where, in (3.3.36), $N^a(x) := g^{ab}(x)(N_1(x)N_2(x),_b - N_2(x)N_1(x),_b\,)$.

The geometrical interpretation of these expressions is as follows.

1. The Lie algebra of the diffeomorphism group $\mathrm{Diff}(\Sigma)$ is generated by the vector fields on Σ with minus the commutator of a pair of vector fields

$$[\,N_1, N_2\,]^a := N_1^b\, N_2^a,_b - N_2^b\, N_1^a,_b \qquad (3.3.37)$$

   playing the role of the Lie bracket. Thus (3.3.34) shows that the map $\vec{N} \to -H[\vec{N}]$ is a homomorphism of the Lie algebra of $\mathrm{Diff}(\Sigma)$ into the Poisson bracket algebra of the theory. A direct calculation of the Poisson brackets of $H[\vec{N}]$ with $g_{ab}(x)$ and $p^{cd}(x)$ confirms the interpretation of $-H[\vec{N}]$ as a generator of spatial diffeomorphisms.

2. Similarly, $H[N]$ can be interpreted as generating deformations of a hypersurface *normal* to itself as embedded in $\mathcal{M}$. However, note that, unlike the analogous statement for $H[\vec{N}]$, this interpretation applies only *after* the field equations have been solved.

Two important things should be noted about the 'gauge' algebra (3.3.34–3.3.36):

1. It is *not* the Lie algebra of $\mathrm{Diff}(\mathcal{M})$ even though this was the invariance group of the original theory.

2. The presence of the g^{ab} factor in the right hand side of (3.3.36) means it is *not* a genuine Lie algebra at all.

These two features are closely related since the Dirac algebra (3.3.34–3.3.36) is essentially the Lie algebra of $\mathrm{Diff}(\mathcal{M})$ *projected* along, and normal to, a spacelike hypersurface. The significance of this in the quantum theory will emerge later.

Even though the Dirac algebra is not a genuine Lie algebra, it still generates an action on the canonical variables (g, p). This is obtained by integrating the infinitesimal changes of the form

$$\delta_{(N,\vec{N})}\, g_{ab}(x) := \{g_{ab}(x), H[N] + H[\vec{N}]\} \qquad (3.3.38)$$

$$\delta_{(N,\vec{N})}\, p^{ab}(x) := \{p^{ab}(x), H[N] + H[\vec{N}]\} \qquad (3.3.39)$$

for arbitrary infinitesimal smearing functions N and $\vec{N}$. We shall refer to the set of all such trajectories in the phase space $\mathcal{S}$ of pairs (g,p) as the *orbits* of the Dirac algebra. Note that, because of the first-class nature of the constraints $\mathcal{H}_a(x) = 0 = \mathcal{H}_\perp(x)$, the subspace $\mathcal{S}_\mathcal{C} \subset \mathcal{S}$ on which the constraints are satisfied is mapped into itself by the transformations (3.3.38–3.3.39). On this constraint surface, an orbit consists of the set of all pairs (g,p) that can be obtained from spacelike slices of some specific Lorentzian metric γ on $\mathcal{M}$ that satisfies the vacuum Einstein equations.

A peculiar feature of general relativity is that an orbit on the constraint surface includes the *dynamical evolution* of a pair (g,p) with respect to any choice of lapse function and shift vector. Indeed, the dynamical equations (3.3.22) and (3.3.24) are simply a special case of the transformations above. This has an important implication for the notion of an 'observable'. In a system with first-class constraints, an observable is normally defined to be any function on the phase space of the system whose Poisson bracket with the constraints vanishes weakly, *i.e.,* it vanishes on the constraint surface. In the present case, this means that A is an observable if and only if

$$\{\, A, H[N,\vec{N}]\,\} \approx 0 \tag{3.3.40}$$

for all N and $\vec{N}$, where $H[N,\vec{N}] := H[N] + H[\vec{N}]$. Thus any such quantity is a *constant* of the motion with respect to evolution along the foliation associated with N and $\vec{N}$. We shall deal later with the quantum analogue of this situation which is deeply connected with the problem of time in quantum gravity.

3.3.5 The Role of the Constraints

The constraints (3.3.25–3.3.26) are of major importance in both the classical and the quantum theories of gravity, and it is useful at this point to gather together various results concerning them.

1. The constraints are consistent with the equations of motion (3.3.22) and (3.3.24) in the sense of being automatically maintained in time. For example,

$$\frac{d\mathcal{H}_\perp(x)}{dt} = \{\mathcal{H}_\perp(x), H[N,\vec{N}]\} \tag{3.3.41}$$

   and the right hand side vanishes on the constraint surface $\mathcal{S}_C$ in phase space by virtue of the closing nature of the algebra (3.3.30–3.3.32).

2. The constraints lead to a well-posed Cauchy problem for the dynamical equations (Hawking & Ellis 1973, Fischer & Marsden 1979) once the undetermined quantities N and $\vec{N}$ have been fixed in some way (see below).

3. If a Lorentzian metric γ on the spacetime $\mathcal{M}$ satisfies the vacuum Einstein equations $G_{\alpha\beta}(X,\gamma] = 0$ then the constraint equations (3.3.25–3.3.26) are satisfied on

all spacelike hypersurfaces of $\mathcal{M}$. (It is understood that p^{ab} is to be computed from the given spacetime geometry using the definition (3.3.5) of the extrinsic curvature K_{ab} and the relation (3.3.17) between p^{ab} and K^{ab}.)

4. Conversely, let $(\mathcal{M}, \gamma)$ be a Lorentzian spacetime with the property that the constraint equations (3.3.25–3.3.26) are satisfied on every spacelike hypersurface. Then γ necessarily satisfies all ten Einstein field equations $G_{\alpha\beta}(X, \gamma] = 0$.

This last result is highly significant since it means the dynamical aspects of the Einstein equations are already contained in the constraints alone. This plays a crucial role in the Dirac quantisation programme (see §5.1) and also in the Hamilton-Jacobi approach to the classical theory.

The proof is rather simple. For any given foliation of $\mathcal{M}$, the Hamiltonian constraints $\mathcal{H}_\perp(x; g(t), p(t)] = 0$ are equivalent to the spacetime equation

$$n^\alpha(x,t)\, n^\beta(x,t)\, G_{\alpha\beta}(\mathcal{F}(x,t), \gamma] = 0 \tag{3.3.42}$$

where n^α is the vector normal to the hypersurface $\mathcal{F}_t(\Sigma)$ at the point $\mathcal{F}(x,t)$ in $\mathcal{M}$. Now, at any given point $X \in \mathcal{M}$, and for any (normalised) timelike vector m in the tangent space $T_X\mathcal{M}$, the foliation can be chosen so that m is the normal vector at that point. Thus, by ranging over all possible foliations, we find that for all $X \in \mathcal{M}$ and for all timelike vectors m at X

$$m^\alpha(X)\, m^\beta(X)\, G_{\alpha\beta}(X, \gamma] = 0. \tag{3.3.43}$$

Now suppose m_1, m_2 is any pair of timelike vectors. Then $m_1 + m_2$ is also timelike and hence $(m_1 + m_2)^\alpha (m_1 + m_2)^\beta\, G_{\alpha\beta} = 0$. However, $m_1^\alpha\, m_1^\beta\, G_{\alpha\beta} = m_2^\alpha\, m_2^\beta\, G_{\alpha\beta} = 0$ and hence, since $G_{\alpha\beta} = G_{\beta\alpha}$, we see that

$$m_1^\alpha\, m_2^\beta\, G_{\alpha\beta} = 0. \tag{3.3.44}$$

Now let σ be any spacelike vector. Then there exist timelike vectors m_1, m_2 such that $\sigma = m_1 - m_2$. It follows from (3.3.44) that for any timelike m,

$$m^\alpha\, \sigma^\beta\, G_{\alpha\beta} = 0. \tag{3.3.45}$$

Finally, a similar result shows that for any pair of spacelike vectors σ_1, σ_2,

$$\sigma_1^\alpha\, \sigma_2^\beta\, G_{\alpha\beta} = 0. \tag{3.3.46}$$

However, any vector $u \in T_p\mathcal{M}$ can be written as the sum of a spacelike and a timelike vector, and hence for all $X \in \mathcal{M}$ and $u, v \in T_X\mathcal{M}$ we have

$$u^\alpha\, v^\beta\, G_{\alpha\beta}[\gamma] = 0, \tag{3.3.47}$$

which means precisely that $G_{\alpha\beta}[\gamma] = 0$.

Note that:

1. The super-Hamiltonian constraints [12] $\mathcal{H}_\perp(x, \mathcal{E}] = 0$ for all spacelike embeddings $\mathcal{E}$ are sufficient by themselves: it is not necessary to impose the supermomentum constraints $\mathcal{H}_a(x, \mathcal{E}] = 0$ in addition.

2. There is a similar result for electromagnetism: if the initial-value equations $\mathrm{div}\vec{E} = 0$ hold in every inertial frame (*i.e.,* on every spacelike hyperplane), the electromagnetic field must necessarily evolve according to the dynamical Maxwell equations. [13]

3.3.6 Eliminating the Non-Dynamical Variables

I wish now to return to the canonical equations of motion (3.3.22–3.3.26) and the status of the non-dynamical degrees of freedom. These variables must be removed in some way before the equations of motion can be solved. As emphasised already, these redundant variables include N and $\vec{N}$ which must be fixed before the abstract label time t in the foliation can be related to a physical quantity like proper time. However, it is important to note that these are not the only variables in the theory that are non-dynamical. This is clear from a count of degrees of freedom. The physical modes of the gravitational field should correspond to $2 \times \infty^3$ configuration [14] variables (the two circular polarisations of a gravitational wave; the two helicity states of a graviton), or $4 \times \infty^3$ phase space variables. However, if a specific coordinate system [15] is chosen on Σ then, after eliminating N and $\vec{N}$, we have the $12 \times \infty^3$ variables $(g_{ab}(x), p^{cd}(x))$. In principle, the four constraint equations (3.3.25) and (3.3.26) can be used to remove a further $4 \times \infty^3$ variables, but this still leaves $8 \times \infty^3$, which is $4 \times \infty^3$ too many.

The physical origin of these spurious variables lies in the $\mathrm{Diff}(\mathcal{M})$-invariance of the original Einstein action. The orbits of the Dirac algebra on the phase space of pairs (g, p) are similar in many respects to the orbits of the Yang-Mills gauge group on the space of vector-potentials and, as in that case, it is necessary to impose some sort of 'gauge'. This will have the desired effect of eliminating the remaining $4 \times \infty^3$ non-physical degrees of freedom.

The exact way in which these additional non-physical variables are to be identified and removed depends on how the lapse function and shift vector are fixed. The main possibilities are as follows.

1. Set N and $\vec{N}$ equal to specific functions [16] on $\Sigma \times \mathbb{R}$. If these functions are

[12] I have written $\mathcal{H}_\perp(x, \mathcal{E}]$ to emphasise that the variables g_{ab} and p^{ab} that appear in $\mathcal{H}_\perp$ are computed from the given Lorentzian four-metric γ using the spacelike embedding $\mathcal{E} : \Sigma \to \mathcal{M}$.

[13] Karel Kuchař, private communication.

[14] The notation $n \times \infty^m$ refers to a field complex with n components defined on an m-dimensional manifold.

[15] In most of the discussion in this paper I shall neglect possible global issues such as, for example, the fact that a three-manifold Σ that is compact cannot be covered by a single coordinate system.

[16] To correspond to a proper foliation it is necessary that, for all $(x, t) \in \Sigma \times \mathbb{R}$, $N(x, t) \neq 0$ and,

substituted into the dynamical equations (3.3.22) and (3.3.24) they yield four equations of the form $\dot{F}^A(x,t;g(t),p(t)] = 0$, $A = 0,1,2,3$, whose solutions are $F^A(x,t;g(t),p(t)]{=}f^A(x)$ for any arbitrary set of functions f^A on Σ. Solving these equations enables an extra $4 \times \infty^3$ variables to be removed, as was required.

This technique can be generalised in various ways. For example, N and $\vec{N}$ can be set equal to specific functionals $N(x,t;g,p]$ and $N^a(x,t;g,p]$ of the canonical variables. Another possibility is to impose conditions on the time derivatives of N and $\vec{N}$ rather than on N and $\vec{N}$ themselves. This is necessary in the context of the so-called 'covariant gauges'.

2. Another option is to start by imposing four conditions

$$F^A(x,t;g(t),p(t)] = 0 \tag{3.3.48}$$

which restrict the paths $t \mapsto (g(t),p(t))$ in the phase space. This method is popular in path-integral approaches to canonical quantisation where it is implemented using a (formal) functional δ-function (see §4.1). The main requirement is that each orbit of the Dirac group contains one, and only one, of these restricted paths on the subspace of the phase space defined by the initial value constraints (3.3.25–3.3.26). As we shall discuss later, there can be global obstructions (in the phase space) to the existence of such functions F^A. Care must also be taken to avoid eliminating genuine physical degrees of freedom in addition to the non-dynamical modes. This potential problem has been discussed at length by Teitelboim (1982, 1983a, 1983b, 1983c, 1983d) .

Since the equations $F^A(x,t;g(t),p(t)] = 0$ are valid for all times, the time-derivative must also vanish. If the ensuing equations $\dot{F}^A(x,t;g(t),p(t)] = 0$ are substituted into the dynamical equations (3.3.22) and (3.3.24), there results a set of four elliptic partial-differential equations for N and $\vec{N}$ that can be solved (in principle) to eliminate these variables as specific functionals of g and p.

3. The third approach to eliminating the non-dynamical variables will play a major role in our discussions of time. This involves the parametrisation of spacetime points by 'internal' space and time coordinates, defined as the values of various functionals $\mathcal{X}^A(x;g,p]$, $A = 0,1,2,3$ of the canonical variables, which can be set equal to some fixed functions $\chi^A_t(x)$. In effect, this leads to an equation of the type (3.3.48) and has the same technical consequences although the physical interpretation is more transparent.

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by convention, we require the continuous function N to satisfy $N(x,t) > 0$. This means that proper time increases in the same direction as the foliation time label t; changing the sign of N corresponds to reversing the sign of t.

3.4 Internal Time

3.4.1 The Main Ideas

The idea of specifying spacetime points by values of the fields was emphasised in the canonical theory by Baierlein, Sharp & Wheeler (1962), and is of sufficient importance to warrant a subsection in its own right. As we shall see, this is related to the problem of rewriting (3.3.18) as a true canonical action for just the physical modes of the gravitational field.

The key idea is to introduce quantities $\mathcal{T}(x; g, p]$ and $\mathcal{Z}^a(x; g, p]$, $a = 1, 2, 3$, that can serve as the time and spatial coordinates for a spacetime event that is associated (in a way yet to be specified) with the point $x \in \Sigma$ and the pair of canonical variables (g, p); the collection of four functions $(\mathcal{T}, \mathcal{Z}^a)$ will be written as $\mathcal{X}^A$, $A = 0, 1, 2, 3$. Strictly speaking, we should worry about the fact that the three-manifold Σ is compact and hence cannot be covered by a single system of coordinates. The same will be true for the spacetime manifold $\Sigma \times \mathbb{R}$, and therefore we should allow for more than one set of the internal spatial functions $\mathcal{Z}^a$. This could be handled in various ways. One option is to work directly with collections of spatial coordinate functions that can cover Σ globally. But this is complicated since the number of coordinate charts used on a manifold is arbitrary provided only that it is greater than, or equal to, the mininum number determined by the manifold topology.

A slightly more elegant (although ultimately equivalent) approach is to realise that, since $\mathcal{M}$ is diffeomorphic to $\Sigma \times \mathbb{R}$, a spacetime event can be labelled by specifying (i) a value for a time variable, and (ii) a *point* in the three-manifold Σ. Thus we could try to extend the scheme above to include functions $\mathcal{Z}(x; g, p]$ whose values lie in Σ, rather than in some region of $\mathbb{R}^3$. When used in this way, I shall refer to Σ as the spatial *label* space and denote it by Σ_l. A particular set of functions $\mathcal{Z}^a(x; g, p]$ can then be associated with each set of local coordinate functions x^a on Σ_l according to the identification $\mathcal{Z}^a(x; g, p] := x^a(\mathcal{Z}(x; g, p])$. I shall not become too involved in this subtlety here since we are mainly concerned with broad principles rather than technical details. However, if desired, the discussion that follows can be generalised to include this concept of a label space.

To see how the idea of internal coordinates is used let us consider first the case of a given Einstein spacetime $(\mathcal{M}, \gamma)$ where we wish to use the functionals to locate an event X in $\mathcal{M}$. The key steps are as follows.

1. Fix a reference foliation $\mathcal{F}^{\mathrm{ref}} : \Sigma \times \mathbb{R} \to \mathcal{M}$ with spacelike leaves and write its inverse $(\mathcal{F}^{\mathrm{ref}})^{-1} : \mathcal{M} \to \Sigma \times \mathbb{R}$ as
$$(\mathcal{F}^{\mathrm{ref}})^{-1}(X) := (\sigma^{\mathrm{ref}}(X), \tau^{\mathrm{ref}}(X)) \in \Sigma \times \mathbb{R} \tag{3.4.1}$$
where $\sigma^{\mathrm{ref}} : \mathcal{M} \to \Sigma$, and where $\tau^{\mathrm{ref}} : \mathcal{M} \to \mathbb{R}$ is a global reference-time function on $\mathcal{M}$. This illustrates the general sense in which space and time are not intrinsic

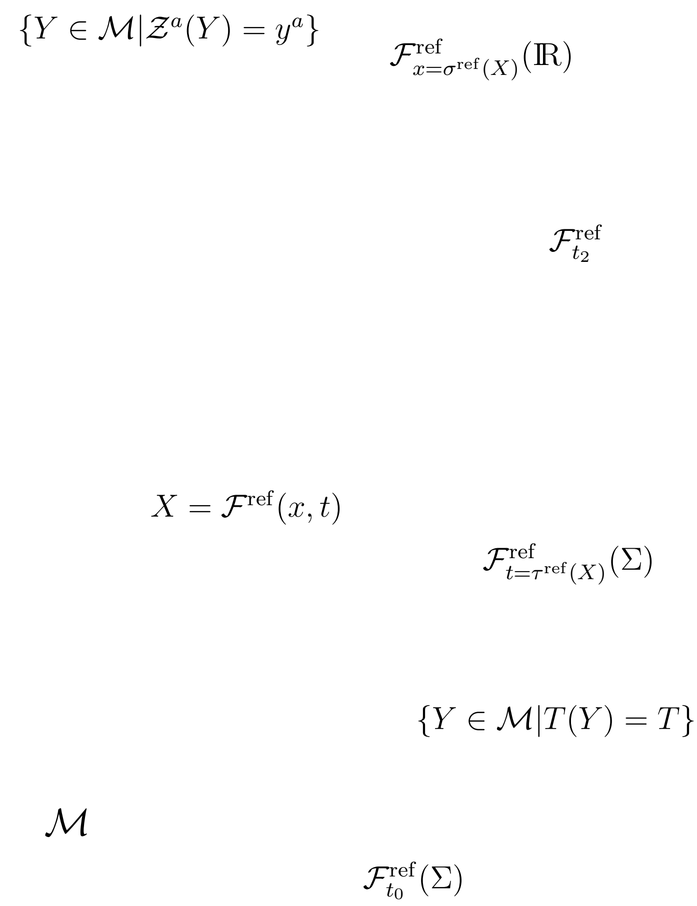

Figure 3: The foliation of $\mathcal{M}$ and the hypersurface $T(Y) = T$ of constant internal time.

spacetime structures but are put into $\mathcal{M}$ by hand. Thus an instant of time, labelled by the number t, is the hypersurface

$$\mathcal{F}^{\rm ref}_t(\Sigma) := \{\mathcal{F}^{\rm ref}_t(x) | x \in \Sigma\} = (\tau^{\rm ref})^{-1}\{t\}, \tag{3.4.2}$$

and a point in space x is the worldline

$$\mathcal{F}^{\rm ref}_x(\rm I\!R) := \{\mathcal{F}^{\rm ref}_x(t) | t \in \rm I\!R\} = (\sigma^{\rm ref})^{-1}\{x\} \tag{3.4.3}$$

where we recall that $\mathcal{F}^{\rm ref}_t : \Sigma \to \mathcal{M}$ and $\mathcal{F}^{\rm ref}_x : \rm I\!R \to \mathcal{M}$ are defined by $\mathcal{F}^{\rm ref}_t(x) := \mathcal{F}^{\rm ref}(x,t)$ and $\mathcal{F}^{\rm ref}_x(t) := \mathcal{F}^{\rm ref}(x,t)$ respectively. This is illustrated in Figure 3.

2. Construct the particular leaf of the reference foliation that passes through the given event X. According to the notation above, this embedding of Σ in $\mathcal{M}$ is $\mathcal{F}^{\rm ref}_{t=\tau^{\rm ref}(X)}$.

Let $x \in \Sigma$ be such that $\mathcal{F}^{\text{ref}}_{t=\tau^{\text{ref}}(X)}(x) \equiv \mathcal{F}^{\text{ref}}_x(\tau^{\text{ref}}(X)) = X$, *i.e.*, $x = \sigma^{\text{ref}}(X)$. Compute:

(a) the induced metric $g_{ab}(x, \mathcal{F}^{\text{ref}}_{t=\tau^{\text{ref}}(X)}]$ on Σ using (3.2.2);

(b) the induced momentum $p^{cd}(x, \mathcal{F}^{\text{ref}}_{t=\tau^{\text{ref}}(X)}]$, using the definition (3.3.5) of the extrinsic curvature $K_{ab}(x, \mathcal{F}^{\text{ref}}_{t=\tau^{\text{ref}}(X)}]$ and the relation (3.3.16) between p and K.

3. Use these values of g and p in the definitions of $\mathcal{T}$ and $\mathcal{Z}^a$ to ascribe time and space coordinates to the event X.

In summary, the 'internal time' coordinate of the point $X \in \mathcal{M}$ is defined to be

$$T(X) := \mathcal{T}(\sigma^{\text{ref}}(X); g[\mathcal{F}^{\text{ref}}_{t=\tau^{\text{ref}}(X)}], p[\mathcal{F}^{\text{ref}}_{t=\tau^{\text{ref}}(X)}]], \tag{3.4.4}$$

while the spatial labels are

$$Z^a(X) := \mathcal{Z}^a(\sigma^{\text{ref}}(X); g[\mathcal{F}^{\text{ref}}_{t=\tau^{\text{ref}}(X)}], p[\mathcal{F}^{\text{ref}}_{t=\tau^{\text{ref}}(X)}]]. \tag{3.4.5}$$

Alternatively, one can write

$$T(\mathcal{F}^{\text{ref}}(x,t)) = \mathcal{T}(x; g(t), p(t)] \tag{3.4.6}$$
$$Z^a(\mathcal{F}^{\text{ref}}(x,t)) = \mathcal{Z}^a(x; g(t), p(t)] \tag{3.4.7}$$

where $g(t)$ and $p(t)$ denote the metric and momentum induced from γ on the hypersurface $\mathcal{F}^{\text{ref}}_t(\Sigma)$ of $\mathcal{M}$.

Note that to be consistent such an identification of internal coordinates requires the functions $\mathcal{T}$ and $\mathcal{Z}^a$ to be chosen such that the following two conditions are satisfied with respect to the given Lorentzian metric γ on $\mathcal{M}$:

1. For all values of the constant number $T \in \mathbb{R}$, the 'instant of time' $\{Y \in \mathcal{M} | T(Y) = T\}$ must be a *spacelike* subspace of $(\mathcal{M}, \gamma)$.

2. For all $\vec{y} \in \mathbb{R}^3$, the set $\{Y \in \mathcal{M} | Z^a(Y) = y^a\}$ must be a *timelike* subspace [17] of $(\mathcal{M}, \gamma)$.

3.4.2 Reduction to True Canonical Form

In the context of the general canonical theory—where there is no fixed spacetime metric—the internal time and space functionals are used to specify a set of non-dynamical degrees

[17] Strictly speaking, this will be true only locally if the Z^a are local coordinate functionals. This problem can be overcome by using the labelling space Σ_l and requiring that, for all points $y \in \Sigma_l$, the set $\{Y \in \mathcal{M} | \mathcal{Z}(Y) = y\}$ is timelike.

of freedom. This is part of the programme to reduce the theory to true canonical form. The key steps are as follows.

1. Perform a canonical transformation

$$(g_{ab}(x), p^{cd}(x)) \mapsto (\mathcal{X}^A(x), \mathcal{P}_B(x); \phi^r(x), \pi_s(x)) \tag{3.4.8}$$

in which the $12 \times \infty^3$ variables $(g_{ab}(x), p^{cd}(x))$ are mapped into

- the four functions $\mathcal{X}^A(x)$ specifying a particular choice of internal space and time coordinates;
- their four conjugate momenta $\mathcal{P}_B(x)$;
- the two modes $\phi^r(x)$, $r = 1, 2$, which represent the physical degrees of freedom of the gravitational field;
- their conjugate momenta $\pi_s(x)$, $s = 1, 2$.

The statement that $\mathcal{P}_B$ are the momenta conjugate to the four internal coordinate variables $\mathcal{X}^A$ means they satisfy the Poisson bracket relations (computed using the basic Poisson brackets (3.3.27–3.3.29))

$$\{\mathcal{X}^A(x), \mathcal{X}^B(x')\} = 0 \tag{3.4.9}$$
$$\{\mathcal{P}_A(x), \mathcal{P}_B(x')\} = 0 \tag{3.4.10}$$
$$\{\mathcal{X}^A(x), \mathcal{P}_B(x')\} = \delta^A_B\, \delta(x, x') \tag{3.4.11}$$

while the corresponding relations for the 'physical' [18] variables (ϕ^r, π_s) are

$$\{\phi^r(x), \phi^s(x')\} = 0 \tag{3.4.12}$$
$$\{\pi_r(x), \pi_s(x')\} = 0 \tag{3.4.13}$$
$$\{\phi^r(x), \pi_s(x')\} = \delta^r_s\, \delta(x, x'). \tag{3.4.14}$$

It is also assumed that all cross brackets of $\mathcal{X}^A$ and $\mathcal{P}_B$ with ϕ^r and π_s vanish:

$$\begin{aligned} 0 = \{\phi^r(x), \mathcal{X}^A(x')\} = \{\phi^r(x), \mathcal{P}_B(x')\} = \\ \{\pi_s(x), \mathcal{X}^A(x')\} = \{\pi_s(x), \mathcal{P}_B(x')\}. \end{aligned} \tag{3.4.15}$$

Note that some or all of these relations may need to be generalised to take account of the global topological properties of the function spaces concerned. There may also be global obstructions to some of the steps.

[18] To avoid confusion it should be emphasised that ϕ^r and π_s may *not* be observables in the sense of satisfying (3.3.40).

2. Express the super-Hamiltonian and supermomentum as functionals of these new canonical variables and write the canonical action (3.3.18) as

$$S[\phi, \pi, N, \vec{N}, \mathcal{X}, \mathcal{P}] = \int dt \int_\Sigma d^3x (\mathcal{P}_A \dot{\mathcal{X}}^A + \pi_r \dot{\phi}^r - N\mathcal{H}_\perp - N^a \mathcal{H}_a) \tag{3.4.16}$$

where all fields are functions of x and t, and where $\mathcal{H}_\perp$ and $\mathcal{H}_a$ are rewritten as functionals of $\mathcal{X}_A$, $\mathcal{P}_B$, ϕ^r and π_s. Note that ϕ^r, π_s, $\mathcal{X}^A$, $\mathcal{P}_B$, N and $\vec{N}$ are all to be varied in (3.4.16) as independent functions. The fields $\mathcal{X}^A$ are interpreted geometrically as defining an embedding of Σ in $\mathcal{M}$ via the parametric equations

$$T = \mathcal{T}(x) \tag{3.4.17}$$
$$Z^a = \mathcal{Z}^a(x) \tag{3.4.18}$$

where $\mathcal{T}(x) := \mathcal{X}^0(x)$ and $\mathcal{Z}^a(x) := \mathcal{X}^a(x)$, $a = 1, 2, 3$. The conjugate variables $P_B(x)$ can be viewed as the energy and momentum densities of the gravitational field measured on this hypersurface. Note however that the spacetime picture that lies behind these interpretations is defined only *after* solving the equations of motion and reconstructing a Lorentzian metric on $\mathcal{M}$.

3. Remove $4 \times \infty^3$ of the $8 \times \infty^3$ non-dynamical variables by solving [19] the constraints $\mathcal{H}_\perp(x) = 0$ and $\mathcal{H}_a(x) = 0$ for the variables $\mathcal{P}_A(x)$ in the form

$$\mathcal{P}_A(x) + h_A(x; \mathcal{X}, \phi, \pi] = 0. \tag{3.4.19}$$

4. Remove the remaining $4 \times \infty^3$ non-dynamical variables by 'deparametrising' the canonical action functional (3.4.16) by substituting into it the solution (3.4.19) of the initial value equations. This gives

$$S[\phi, \pi] = \int dt \int_\Sigma d^3x \, \{\pi_r(x,t)\dot{\phi}^r(x,t) - h_A(x; \chi_t, \phi(t), \pi(t)]\dot{\chi}_t^A(x)\} \tag{3.4.20}$$

which can be shown to give the correct field equations for the physical fields ϕ^r and π_s. Note that, in (3.4.20), the four quantities $\mathcal{X}^A$ are no longer to be varied but have instead been set equal to some *prescribed* functions χ_t^A of x. This is valid since, after solving the constraints, the remaining dynamical equations of motion give no information about how the variables $\mathcal{X}^A$ evolve in parameter time t. In effect, in terms of the original internal coordinate functionals, we have chosen a set of conditions

$$\mathcal{X}^A(x; g(t), p(t)] = \chi_t^A(x) \tag{3.4.21}$$

that restrict the phase-space paths $t \mapsto (g(t), p(t))$ over which the action is to be varied. In this way of looking at things, (3.4.21) can be regarded as additional constraints which, when added to the original constraints, make the entire set second-class.

[19] There may be global obstructions to this step.

The lapse function and shift vector play no part in this reduced variational principle. However, it will be necessary to reintroduce them if one wishes to return to a genuine spacetime picture. As discussed in §3.3.6, this can be done by solving the additional set of Einstein equations that are missing from the set generated by the reduced action. These are elliptic partial-differential equations for N and $\vec{N}$.

The equations of motion derived from the reduced action (3.4.20) are those corresponding to the Hamiltonian

$$H_{\rm true}(t) := \int_\Sigma d^3x\, \dot{\chi}_t^A(x)\, h_A(x;\chi_t,\phi(t),\pi(t)] \tag{3.4.22}$$

and can be written in the form

$$\begin{aligned}\frac{\partial\phi^r(x,t)}{\partial t} &= \{\phi^r(x,t), H_{\rm true}(t)\}_{\rm red} &(3.4.23)\\ \frac{\partial\pi_s(x,t)}{\partial t} &= \{\pi_s(x,t), H_{\rm true}(t)\}_{\rm red} &(3.4.24)\end{aligned}$$

where $\{\,,\,\}_{\rm red}$ denotes the Poisson bracket evaluated using only the physical modes ϕ^r and π_s. Thus, at least formally, the system has been reduced to one that looks like the conventional field theory discussed earlier in section §3.2 and epitomised by the dynamical equations (3.2.7–3.2.8). A particular choice for the four functions χ_t^A (using some coordinate system x^a on Σ) is

$$\begin{aligned}\chi_t^0(x) &= t &(3.4.25)\\ \chi_t^a(x) &= x^a &(3.4.26)\end{aligned}$$

for which the true Hamiltonian (3.4.22) is just the integral over Σ of h_0. Thus we arrive at the fully-reduced form of canonical general relativity as derived in the work of Arnowitt, Deser and Misner.

3.4.3 The Multi-time Formalism

Although a particular time label t (corresponding to some reference foliation of $\mathcal{M}$ into a one-parameter family of hypersurfaces) has been used in the above, this label is in fact quite arbitrary. This is reflected in the arbitrary choice of the functions χ_t^A in the 'gauge conditions' (3.4.21). Indeed, the physical fields (ϕ^r,π_s) ought to depend only on the hypersurface in $\mathcal{M}$ on which they are evaluated, not on the way in which that hypersurface happens to be included in a particular foliation. Thus it should be possible to write the physical fields $\phi^r(x,\mathcal{X}]$, $\pi_s(x,\mathcal{X}]$ as functions of $x\in\Sigma$ and functionals of the internal coordinates $\mathcal{X}^A$ which are now regarded as *arbitrary* functions of $x\in\Sigma$ rather than being set equal to some fixed set.

This is indeed the case. More precisely, the Hamiltonian equations of motion (3.4.23–3.4.24), which hold for all choices of the functions χ^A, imply that there exist Hamiltonian densities $h_A(x, \mathcal{X}\,]$ such that $\phi^r(x, \mathcal{X}\,]$ and $\pi_s(x, \mathcal{X}\,]$ satisfy the functional differential equations

$$\frac{\delta\phi^r(x, \mathcal{X}\,]}{\delta\mathcal{X}^A(x')} = \{\phi^r(x, \mathcal{X}\,], h_A(x', \mathcal{X}\,]\}_{\rm red} \tag{3.4.27}$$

$$\frac{\delta\pi_s(x, \mathcal{X}\,]}{\delta\mathcal{X}^A(x')} = \{\pi_s(x, \mathcal{X}\,], h_A(x', \mathcal{X}\,]\}_{\rm red} \tag{3.4.28}$$

which is a gravitational analogue [20] of the scalar field equations (3.2.9–3.2.10).

This so-called 'bubble-time' or 'multi-time' canonical formalism is of considerable significance and will play an important role in our discussions of the problem of time in quantum gravity (Kuchař 1972).

4 IDENTIFY TIME BEFORE QUANTISATION

4.1 Canonical Quantum Gravity: Constrain Before Quantising

4.1.1 Basic Ideas

We shall now discuss the first of those interpretations of type I in which time is identified as a functional of the geometric canonical variables *before* the application of any quantisation algorithm. These approaches to the problem of time are conservative in the sense that the quantum theory is constructed in an essentially standard way. In particular, 'time' is regarded as part of the *a priori* background structure used in the formulation of the quantum theory. There are two variants of this approach. The first starts with the fully-reduced formalism, and culminates in a quantum version of the Poisson-bracket dynamical equations (3.4.23–3.4.24); the second starts with the multi-time dynamical equations (3.4.27–3.4.28) and ends with a multi-time Schrödinger equation.

In the first approach, the problem is to quantise the first-order action principle (3.3.18) in which the canonical variables $g_{ab}(x)$ and $p^{cd}(x)$ are subject to the constraints $\mathcal{H}_a(x; g, p] = 0 = \mathcal{H}_\perp(x; g, p]$ where $\mathcal{H}_a$ and $\mathcal{H}_\perp$ are given by (3.3.19) and (3.3.20) respectively, and where all non-dynamical variables are removed before quantisation. The account that follows is very brief and does not address the very difficult technical question of whether it is really feasible to construct a consistent quantum theory of gravity in this way by applying some quantisation algorithm to the classical theory of general relativity.

The main steps are as follows.

1. Impose a suitable 'gauge'. As explained in §3.3.6, this can be done in several ways:

[20] However, note that the quantities $\mathcal{E}^\alpha$ in (3.2.9–3.2.10) are the components of an embedding map $\mathcal{E} : \Sigma \to \mathcal{M}$ with respect some (unspecified) coordinate system on $\mathcal{M}$, whereas the quantities $\mathcal{X}^A$ in (3.4.27–3.4.28) are coordinate functions on $\mathcal{M}$.

- Set the lapse function N and shift vector $\vec{N}$ equal to specific functions (or functionals) of the remaining canonical variables.
- *Or*, impose a set of conditions $F^A(x,t;g,p] = 0,\ A = 0,1,2,3$.
- *Or*, identify a set $\mathcal{X}^A(x;g,p],\ A = 0,1,2,3$ of internal spacetime coordinates and set these equal to some set of *fixed* functions $\chi_t^A(x)$.

In all three cases, the outcome is that N and $\vec{N}$ drop out of the formalism, as do $8\times\infty^3$ of the $12\times\infty^3$ canonical variables.

2. Construct a canonical action (*cf* (3.4.20)) to reproduce the dynamical equations (*cf* (3.4.23–3.4.23)) of the remaining $4\times\infty^3$ 'true' canonical variables $(\phi^r(x), \pi_s(x))$.

3. Impose canonical commutation relations on these variables and then proceed as in any standard [21] quantum theory. In particular, we obtain a Schrödinger equation of the form

$$i\hbar\frac{\partial\Psi_t}{\partial t} = \widehat{H}_{\rm true}\Psi_t \tag{4.1.1}$$

where $\widehat{H}_{\rm true}$ is the quantised version of the Hamiltonian (3.4.22) for the physical modes introduced in the classical Poisson bracket evolution equations (3.4.23–3.4.24).

A variant of this operator approach uses a formal, canonical path-integral to compute expressions like [22]

$$Z = \int \Pi_{x,t}\mathcal{D}g_{ab}(x,t)\,\mathcal{D}p^{cd}(x,t)\,\mathcal{D}N(x,t)\,\mathcal{D}N^a(x,t)\,\delta[F^A]\,|\det\{F^A,\mathcal{H}_B\}|\,e^{\frac{i}{\hbar}S[g,p,N,\vec{N}]} \tag{4.1.2}$$

where $S[g,p,N,\vec{N}\,]$ is given by (3.3.18), $\mathcal{H}_{A=0}$ denotes $\mathcal{H}_\perp$, and where the gauge conditions $F^A(x,t;g,p\,] = 0$ are enforced by the functional Dirac delta-function. Note that because S is a linear functional of N and $\vec{N}$, the functional integrals over these variables in (4.1.2) formally generate functional delta-functions [23] which impose the constraints $\mathcal{H}_\perp(x) = 0 = \mathcal{H}_a(x)$.

4.1.2 Problems With the Formalism

Neither the operator nor the functional-integral programmes are easy to implement, and both are unattractive for a number of reasons. For example:

[21] This last step is rather problemetic since the reduced phase space obtained by the construction above is topologically non-trivial, and hence the fields $\phi^r(x)$ and $\pi_s(x)$ are only *local* coordinates on this space. This means the naive canonical commutation relations should be replaced by an operator algebra that respects the global structure of the system (Isham 1984).

[22] This is only intended as an example. As defined, Z is just a number, and to produce physical predictions it is necessary to include external sources, or background fields, in the standard way.

[23] There is a subtlety here, since the classical lapse function N is always positive. If the same condition is imposed on the variable N in the functional integral it will not lead to a delta function.

- The constraints $\mathcal{H}_a(x; g, p] = 0 = \mathcal{H}_\perp(x; g, p]$ cannot be solved in a closed form [24]. Weak-field perturbative methods could be used but these throw little light on the problem of time and inevitably founder on the problem of non-renormalisability.

- The programme violates the geometrical structure of general relativity by removing parts of the metric tensor. This makes it very difficult to explore the significance of spacetime geometry in the quantum theory.

- Choosing a gauge breaks the gauge invariance of the theory and, as usual, the final quantum results must be shown to be independent of the choice. In the functional-integral approach, this is the reason for the functional Jacobian $|\det\{F^A, \mathcal{H}_B\}|$ in (4.1.2). However, this integral is so ill-defined that a proper demonstration of gauge invariance is as hard as it is in the operator approaches. The issue of gauge invariance is discussed at length by Barvinsky (1991).

4.2 The Internal Schrödinger Interpretation

4.2.1 The Main Ideas

The classical dynamical evolution described above employs the single-time Poisson bracket equations of the form (3.4.23–3.4.24) and, in the operator form, leads to the gravitational analogue (4.1.1) of the Schrödinger equation (3.2.20). However, the resulting description of the evolution is in terms of a fixed foliation of spacetime, whereas a picture that is more in keeping with the spirit of general relativity would be one that involves an analogue of the functional-differential equations (3.2.9–3.2.10) and the associated multi-time Schrödinger equation (3.2.21) that describe evolution along the deformation of arbitrary hypersurfaces of $\mathcal{M}$.

The internal Schrödinger interpretation refers to such a generalisation in which one quantises the multi-time version of the canonical formalism of classical general relativity. As always in canonical approaches to general relativity, the first step is to introduce a reference foliation $\mathcal{F}^{\rm ref} : \Sigma \times \mathrm{I\!R} \to \mathcal{M}$ which is used to define the canonical variables. However, it is important to appreciate that this should serve only as an intermediate tool in the development of the theory and, like a basis set in a vector space theory, it should not appear in the final results.

The key steps in constructing the internal Schrödinger interpretation are as follows:

1. Pick a set of classical functions $\mathcal{T}(x; g, p], \mathcal{Z}^a(x; g, p]$ that can serve as internal time and space coordinates.

[24] The Ashtekar variables might change this situation as they lead to powerful new techniques for tackling the constraints (Ashtekar 1986, Ashtekar 1987). For two recent developments in this direction see Newman & Rovelli (1992) and Manojlović & Miković (1992).

2. Perform the canonical transformation (3.4.8)

$$(g_{ab}(x), p^{cd}(x)) \mapsto (\mathcal{X}^A(x), \mathcal{P}_B(x); \phi^r(x), \pi_s(x)) \tag{4.2.1}$$

   in which $(\phi^r(x), \pi_s(x))$ are identified with the $4 \times \infty^3$ physical canonical modes of the gravitational field.

3. Solve the super-Hamiltonian constraint $\mathcal{H}_\perp(x) = 0$ for the variables $\mathcal{P}_A(x)$ in the form (3.4.19)

$$\mathcal{P}_A(x) + h_A(x; \mathcal{X}, \phi, \pi] = 0. \tag{4.2.2}$$

4. Identify $h_A(x; \mathcal{X}, \phi, \pi]$ as the set of Hamiltonian densities appropriate to a multi-time version of the classical dynamical equations as in (3.4.27–3.4.28).

5. Quantise the system in the following steps:

   (a) Replace the Poisson bracket relations (3.4.12–3.4.14) with the canonical commutation relations

$$\begin{aligned} [\,\widehat{\phi}^r(x), \widehat{\phi}^s(x')\,] &= 0 && (4.2.3)\\ [\,\widehat{\pi}_r(x), \widehat{\pi}_s(x')\,] &= 0 && (4.2.4)\\ [\,\widehat{\phi}^r(x), \widehat{\pi}_s(x')\,] &= i\hbar\delta^r_s\,\delta(x, x'). && (4.2.5) \end{aligned}$$

   (b) Try to construct quantum Hamiltonian densities $\widehat{h}_A(x)$ by the usual substitution rule in which the fields ϕ^r and π_s in the classical expression $h_A(x; \mathcal{X}, \phi, \pi]$ are replaced by their operator analogues $\widehat{\phi}^r$ and $\widehat{\pi}_s$.

   (c) The classical multi-time evolution equations (3.4.27–3.4.28) can be quantised in one of two ways. The first is to replace the classical Poisson brackets expressions with Heisenberg-picture operator relations. The second is to use a Schrödinger picture in which the constraint equation (4.2.2) becomes the multi-time Schrödinger equation

$$i\hbar \frac{\delta \Psi_{\mathcal{X}}}{\delta \mathcal{X}^A(x)} = h_A(x; \mathcal{X}, \widehat{\phi}, \widehat{\pi}\,]\Psi_{\mathcal{X}} \tag{4.2.6}$$

   for the state vector $\Psi_{\mathcal{X}}$.

This fundamental equation can be justified from two slightly different points of view. The first is to claim that, since the classical Hamiltonian associated with the internal coordinates $\mathcal{X}^A$ is h_A, the Schrödinger equation (4.2.6) follows from the basic time-evolution axiom of quantum theory, suitably generalised to a multi-time situation. However, note that this step is valid only if the inner product on the Hilbert space of states is $\mathcal{X}$-independent. If not, a compensating term must be added to (4.2.6) if the scalar product $\langle \Psi_{\mathcal{X}}, \Phi_{\mathcal{X}} \rangle_{\mathcal{X}}$ is to be independent of the internal coordinate functions. This is analogous to the situation discussed earlier for the functional Schrödinger equation (3.2.21) for a scalar field theory in a background spacetime.

Another approach to deriving (4.2.6) is to start with the classical constraint equation (4.2.2) and then impose it at the quantum level as a constraint on the allowed state vectors,

$$(\widehat{\mathcal{P}}_A(x) + h_A(x; \widehat{X}, \widehat{\phi}, \widehat{\pi}\,])\Psi = 0. \tag{4.2.7}$$

The Schrödinger equation (4.2.6) is reproduced if one makes the formal substitution

$$\widehat{\mathcal{P}}_A(x) = -i\hbar \frac{\delta}{\delta \mathcal{X}^A(x)}. \tag{4.2.8}$$

This approach has the advantage of making the derivation of the Schrödinger equation look somewhat similar to that of the Wheeler-DeWitt equation. However, once again this is valid only if the scalar product is $\mathcal{X}$-independent. If not, it is necessary to replace (4.2.8) with

$$\widehat{\mathcal{P}}_A(x) = -i\hbar \frac{\delta}{\delta \mathcal{X}^A(x)} + F_A(x, \mathcal{X}] \tag{4.2.9}$$

where the extra term $F_A(x, \mathcal{X}]$ is chosen to compensate the $\mathcal{X}$-dependence of the scalar product.

Considerable care must be taken in interpreting (4.2.8) or (4.2.9). These equations might appear to come from requiring the existence of self-adjoint operators $\widehat{\mathcal{X}^A}(x)$ and $\widehat{P}_B(x')$ that satisfy canonical commutation relations

$$\begin{aligned}
[\,\widehat{\mathcal{X}^A}(x), \widehat{\mathcal{X}^B}(x')\,] &= 0 && (4.2.10)\\
[\,\widehat{\mathcal{P}}_A(x), \widehat{\mathcal{P}}_B(x')\,] &= 0 && (4.2.11)\\
[\,\widehat{\mathcal{X}^A}(x), \widehat{\mathcal{P}}_B(x')\,] &= i\hbar\delta^A_B\,\delta(x, x') && (4.2.12)
\end{aligned}$$

which are the quantised versions of the classical Poisson-bracket relations (3.4.9–3.4.11). However, the example of elementary wave mechanics shows that such an interpretation is misleading. In particular, the $i\hbar\partial/\partial t$ that appears in the time-dependent Schrödinger equation cannot be viewed as an operator defined on the physical Hilbert space of the theory.

4.2.2 The Main Advantages of the Scheme

Many problems arise in the actual implementation of this programme, but let us first list the main advantages.

1. We have argued in the introductory sections that one of the main sources of the problem of time in quantum gravity is the special role played by time in normal physics; in particular, its position as something that is external to the system and part of the *a priori* classical background of the Copenhagen interpretation of quantum theory. The canonical transformation (4.2.1) applied *before* quantisation could be said to cast quantum gravity into the same mould.

2. The relative freedom from interpretational problems is epitomised by the ease with which the theory can be equipped (at least formally) with a Hilbert space structure. In particular, since $(\phi^r(x), \pi_s(x))$ are genuine physical modes of the gravitational field, it is fully consistent with conventional quantum theory to insist that (once suitable smeared) the corresponding operators are *self-adjoint*. This is certainly not the case for the analogous statement in the Dirac, constraint-quantisation, scheme in which all $12 \times \infty^3$ variables $(g_{ab}(x), p^{cd}(x))$ are assigned operator status. It is then consistent to follow the earlier discussion in §3.2 of quantum field theory in a background spacetime and define the quantum states as functionals $\Psi[\phi]$ of the configuration variables $\phi^r(x)$ with the canonical operators $\widehat{\phi}^r(x)$ and $\widehat{\pi}_s(x)$ defined as the analogue of (3.2.14) and (3.2.15) respectively. Of course, the cautionary remarks made earlier about measures on infinite-dimensional spaces apply here too.

3. This Hilbert space structure provides an unambiguous probabilistic interpretation of the theory. Namely, if the normalised state is Ψ, and if a measurement is made of the true gravitational degrees of freedom ϕ^r on the hypersurface represented by the internal coordinates $\mathcal{X}^A$, the probability of the result lying in the subset B of field configurations is (*cf* (3.2.17))

$$\mathrm{Prob}(\Psi; \phi \in B) = \int_B d\mu(\phi)\, |\Psi[\mathcal{X}, \phi]|^2. \tag{4.2.13}$$

4. The associated inner product also allows the construction of a meaningful notion of a quantum *observable*. This is simply any well-defined operator functional $\widehat{F} = F[\mathcal{X}, \widehat{\phi}, \widehat{\pi}]$ of the true gravitational variables $\widehat{\phi}^r(x)$, $\widehat{\pi}_s(x)$ and of the internal spacetime coordinates $\mathcal{X}^A(x)$ that is self-adjoint in the inner product. The usual rules of quantum theory allow one to ask, and in principle to answer, questions about the spectrum of $\widehat{F}$ and the probabilistic distributions of the values of the observable F on the hypersurface defined by $\mathcal{X}$.

4.2.3 A Minisuperspace Model

One of the basic questions in this programme is how to select the internal embedding variables and, in particular, the internal time variable $\mathcal{T}(x)$. Three general options have been explored:

- *Intrinsic time* $\mathcal{T}(x, g]$. The internal-time value of a spacetime event is constructed entirely from the internal metric $g_{ab}(x)$ carried by the hypersurface of the reference foliation that passes through the event.

- *Extrinsic time* $\mathcal{T}(x; g, p]$. To identify the time coordinate of an event, one needs to know not only the intrinsic metric on the hypersurface, but also its extrinsic curvature, and hence how the hypersurface would appear from the perspective of a classical spacetime constructed from the pair (g, p).

- *Matter time* $\mathcal{T}(x; g, p, \phi, p_\phi]$. Time is not constructed from geometric data alone but from matter fields ϕ coupled to gravity, or from a combination of matter and gravity fields.

Very little is known about how to choose an internal time in the full theory of quantum gravity although it is noteable that the use of an extrinsic time is natural in the linearised theory and was discussed in some of the original series of ADM papers (Arnowitt, Deser & Misner 1960*a*, Arnowitt, Deser & Misner 1962), and in Kuchař (1970). This type of time appears naturally also in the theory of quantised cylindrical gravitational waves (Kuchař 1971, Kuchař 1973, Kuchař 1992*b*), and in the Ashtekar formalism (Ashtekar 1991).

The full theory is very complicated and most studies have been in the context of so-called *minisuperspace* models in which only a finite number of the gravitational degrees of freedom are invoked in the quantum theory, the remainder being eliminated by the imposition of symmetries on the spatial metric. These models represent homogeneous (but, in general, anisotropic) cosmologies, of which the various Bianchi-type vacuum spacetimes (in which the spatial three-manifolds are assorted three-dimensional Lie groups) have been the most widely studied. The requirement of homogeneity limits the allowed hypersurfaces to a priviledged foliation in which each leaf can be labelled by a single time variable.

Restricting things in this way has the big advantage that the worst quantum field theory problems—in particular, non-renormalisability—do not arise, which means that some of the conceptual issues can be discussed in a technical framework that is mathematically well-defined. The main limitation of these models is the absence of any proper perturbative scheme into which they can be made to fit. At the classical level, models of this type correspond to exact solutions of Einstein's field equations, but this not true in the corresponding quantum theory since, for example, the infinite set of neglected modes presumably possess zero-point fluctuations. Some of these issues have been discussed in detail by Kuchař & Ryan (1989) to which the reader is referred for further information. However, we shall not get too involved with these particular issues here since our main use of minisuperspace models is to illustrate various specific features of the problem of time, and this they do quite well.

A simple example is when the spacetime geometries are restricted to be one of the familiar homogeneous and isotropic Robertson-Walker metrics

$$g_{\alpha\beta}(X)\, dX^\alpha \otimes dX^\beta := -N(t)^2\, dt \otimes dt + a(t)^2 \omega_{ab}(x)\, dx^a \otimes dx^b \tag{4.2.14}$$

where $\omega_{ab}(x)$ is the metric for a three-space of constant curvature k. The case $k = 1$ is associated with Σ being a three-sphere, while $k = 0$ and $k = -1$ correspond to the flat and hyperbolic cases respectively. Note that the lapse vector $\vec{N}$ does not appear in this expression.

Spacetimes of this type produce a non-vanishing Einstein tensor $G_{\alpha\beta}$ and therefore require a source of matter. Much work in recent times has employed a scalar field ϕ for this purpose, and we shall use this here. It can be shown (Kaup & Vitello 1974, Blyth & Isham 1975) that the coupled equations [25] for the variables a, ϕ and N can all be derived from the first-order action [26]

$$S[a, p_a, \phi, p_\phi, N] = \int dt\, (p_a \dot{a} + p_\phi \dot{\phi} - N H_{\mathrm{RW}}) \qquad (4.2.15)$$

where p_a, a, p_ϕ, ϕ and N are functions of t and are to be varied independently of each other. The super-Hamiltonian in this model is

$$H_{\mathrm{RW}} := \frac{-p_a^2}{24a} - 6ka + \frac{p_\phi^2}{2a^3} + a^3 V(\phi). \qquad (4.2.16)$$

A natural choice for an intrinsic time in the $k = 1$ model is the radius a of the universe. Thus, according to the discussion above, the reduced Hamiltonian for this system can be found by solving the constraint

$$\frac{-p_a^2}{24a} - 6a + \frac{p_\phi^2}{2a^3} + a^3 V(\phi) = 0 \qquad (4.2.17)$$

for the variable p_a. Hence

$$h_a = \pm\sqrt{2}4\Big(- 6t^2 + \frac{p_\phi^2}{2t^2} + t^4 V(\phi)\Big)^{\frac{1}{2}}. \qquad (4.2.18)$$

A simple extrinsic time is $t := p_a$ which (in the simple case $V(\phi) = 0$) leads to the Hamiltonian

$$h_{p_a} = \pm\Big(\frac{-\frac{1}{12}t^2 \pm (\frac{1}{144}t^4 + 48p_\phi^2)^{\frac{1}{2}}}{24}\Big)^{\frac{1}{2}}, \qquad (4.2.19)$$

while a natural definition using the matter field is $t := \phi$, which yields

$$h_\phi = \pm(\frac{a^2 p_a^2}{12} + 12a^4 - 2a^6 V(t))^{\frac{1}{2}}. \qquad (4.2.20)$$

4.2.4 Mean Extrinsic Curvature Time

One of the few definitions of internal time that has been studied in depth is one in which spacetime is foliated by hypersurfaces of constant mean intrinsic curvature. The $\mathcal{T}$ functional is

$$\mathcal{T}(x; g, p] := \tfrac{2}{3}|g|^{-\frac{1}{2}}(x)\, p^a{}_a(x) \qquad (4.2.21)$$

[25] There is no specific information in these notes about how the canonical formalism for general relativity is extended to include matter but, generally speaking, nothing very dramatic happens. In particular, although the supermomentum and super-Hamiltonian constraints acquire contributions from the matter variables, the crucial Dirac algebra is still satisfied. However, the details can be quite complicated, especially for systems with internal degrees of freedom; the subject as a whole has been analysed comprehensively in a series of papers by Kuchař (1976a, 1976b, 1976c, 1977).

[26] For simplicity, I have chosen units in which $8\pi G/c^4 = 1$.

which is conjugate to

$$\mathcal{P}_{\mathcal{T}}(x;g,p] := -|g|^{\frac{1}{2}}(x) \tag{4.2.22}$$

in the sense that

$$\{\mathcal{T}(x), \mathcal{P}_{\mathcal{T}}(x')\} = \delta(x,x'). \tag{4.2.23}$$

The use of (4.2.21) as an internal time variable in the spatially compact case was developed in depth by York and his collaborators (York 1972*b*, York 1972*a*, Smarr & York 1978, York 1979, Isenberg & Marsden 1976). An interesting account of its use in $2+1$ dimensions is Carlip (1990, 1991).

The first step in implementing the quantisation programme is to extend the definitions (4.2.21–4.2.22) to a full canonical transformation of the type (4.2.1). One possible example is

$$(g_{ab}(x), p^{cd}(x)) \mapsto (\mathcal{T}(x), \sigma_{ab}(x), \mathcal{P}_{\mathcal{T}}(x), \pi^{cd}(x)) \tag{4.2.24}$$

where the 'conformal metric' $\sigma_{ab}(x)$ and its 'conjugate' momentum $\pi^{cd}(x)$ are defined by

$$\sigma_{ab} \;:=\; |g|^{-\frac{1}{3}} g_{ab}, \tag{4.2.25}$$

$$\pi^{ab} \;:=\; |g|^{\frac{1}{3}}(p^{ab} - \tfrac{1}{3} p\, g^{ab}). \tag{4.2.26}$$

Note that $\det\sigma_{ab} = 1$ and π^{ab} is traceless, and so each field corresponds to $5\times\infty^3$ independent variables. [27] The next step therefore should be to remove $3\times\infty^3$ variables from each by identifying the spatial parts $\mathcal{Z}^a(x;g,p]$ of the embedding variables plus their conjugate momenta. However, since we are concerned primarily with the problem of time, we shall not perform this (rather complex) step but concentrate instead on what is already entailed by the use of the preliminary canonical transformation (4.2.24). The complete analysis is contained in the papers by York *et al* cited above.

In terms of these new variables $\mathcal{T}$, $\mathcal{P}_{\mathcal{T}}$, σ_{ab} and π^{ab}, the super-Hamiltonian constraint $\mathcal{H}_\perp = 0$ becomes the equation for $\Phi := (-\mathcal{P}_{\mathcal{T}})^{\frac{1}{6}}$

$$\frac{1}{\kappa^2}\left(\triangle_\sigma \Phi - \tfrac{1}{8} R[g]\right) + \kappa^2\left(\tfrac{1}{8}\pi^{ab}\pi_{ab}\Phi^{-7} - \tfrac{3}{64}\mathcal{T}^2\Phi^5\right) = 0 \tag{4.2.27}$$

where $\triangle_\sigma$ is the Laplacian operator constructed from the metric σ_{ab}. This is a non-linear, elliptic, partial-differential equation which, in principle, can be solved for Φ (and hence for $\mathcal{P}_{\mathcal{T}} = -|g|^{\frac{1}{2}}$) as a functional of the remaining canonical variables

$$\mathcal{P}_{\mathcal{T}}(x) + h(x;\mathcal{T},\sigma,\pi] = 0. \tag{4.2.28}$$

In the quantum theory this leads at once to the desired functional Schrödinger equation

$$i\hbar\frac{\delta\Psi[\mathcal{T},\sigma]}{\delta\mathcal{T}(x)} = (h(x;\mathcal{T},\widehat{\sigma},\widehat{\pi}\,]\Psi)[\mathcal{T},\sigma]. \tag{4.2.29}$$

[27] This is reflected in the fact that the Poisson bracket of $\sigma_{ab}(x)$ with $\pi^{cd}(x')$ is proportional to the *traceless* version of the Kronecker-delta. Thus these variables are not a conjugate pair in the strict sense: to get such variables it would be necessary first to solve the second-class constraints $\det\sigma_{ab} = 1$ and $\pi^c_c = 0$. Alternatively, these variables could be employed as part of an over-complete set in a group-theoretical approach to the quantum theory.

4.2.5 The Major Problems

As remarked earlier, considerable conceptual advantages accrue from using the internal Schrödinger interpretation of time, and it is therefore unfortunate that the scheme is riddled with severe technical difficulties. These fall into two categories. The first are those quantum-field theoretic problems that are expected to arise in any theory that we know from weak-field perturbative analysis to be non-renormalisable. In particular, the Hamiltonian densities $h_A(x; \mathcal{X}, \phi, \pi]$ are likely to be highly non-linear functions of the fields $\phi^r(x), \pi_s(x)$ and we shall encounter the usual problems when trying to define products of quantum fields evaluated at the same point.

The second class of problems are those that more closely involve the question of time itself and are not directly linked to the non-renormalisability of the theory. Indeed, many such problems arise already in minisuperspace models which, having only a finite number of degrees of freedom, are free from the worst difficulties of quantum field theory. I shall concentrate here mainly on this second type of problem, using the classification mentioned briefly at the end of §2.4; see also Isham (1992) and Kuchař (1992*b*).

The Global Time Problem. It is far from obvious that it is possible to perform a canonical transformation (4.2.1) with the desired characteristics. This raises the following questions:

1. What properties must the functionals $\mathcal{X}^A(x; g, p]$ possess in order to serve as internal spacetime coordinates?

2. If such functions do exist, is it possible to perform a canonical transformation of the type (4.2.1) which is such that:

   (a) the constraints $\mathcal{H}_\perp(x; g, p] = 0$ and $\mathcal{H}_a(x; g, p] = 0$ can be solved globally (on the phase space) for $\mathcal{P}_A$ in the form (4.2.2); and

   (b) there is a unique such solution?

One example of the first point is the requirement hat the internal space and tim coordinates produce a genuine foliation of the spacetime. In particular, if $t \mapsto (g(t), p(t))$ is any curve in the Cauchy data of a solution to the vacuum Einstein equations, we require $d\mathcal{T}(x; g(t), p(t)]/dt > 0$. Of course, if γ is static, this condition can never be satisfied, which means that spacetimes of this type do not admit internal time functions. This is hardly surprising, but it does show that we are unlikely to find conditions on $\mathcal{T}$ that are to be satisfied on *all* spacetimes.

The existence of functionals satisfying these conditions in various minisuperspace models has been studied by Hájíček (1986, 1988, 1989, 1990a, 1990b). He showed that these finite-dimensional models generically exhibit global obstructions to the construction of such functionals, and the evidence suggests that the problem may get worse as the number of degrees of freedom increases. See also the comments in Torre (1992).

This is not really surprising since, if χ^A are to serve as proper 'gauge functions', the subspace of the space $\mathcal{S}$ of all (g,p) satisfying the equations $\mathcal{X}^A(x,g,p] = \chi^A(x;g,p]$ (and also the constraints $\mathcal{H}_\perp(x) = 0 = \mathcal{H}_a(x)$) must intersect each orbit of the Dirac algebra just once. This is analogous to gauge-fixing in Yang-Mills theory and therefore one anticipates the presence of a Gribov phenomenon in the form of obstructions to the construction of such a global gauge (Singer 1978). This possibility arises because the topological structure of the space of physical configurations is non-trivial. However, this structure seems not to have been studied systematically in the full theory of general relativity, and so the subject warrants further investigation.

The Spatial Metric Reconstruction Problem. At a classical level the canonical transformation (3.4.8) can be inverted and, in particular, the metric $g_{ab}(x)$ can be expressed as a functional $g_{ab}(x;\mathcal{X},\mathcal{P},\phi,\pi]$ of the embedding variables $(\mathcal{X}^A,\mathcal{P}_B)$ and the physical degrees of freedom (ϕ^r,π_s). The question is whether something similar can be done at the quantum level. In particular:

- is it possible to make sense of an expression like $g_{ab}(x;\mathcal{X},\widehat{\mathcal{P}},\widehat{\phi},\widehat{\pi}\,]$ with $\widehat{\mathcal{P}}_A$ being replaced by $-\widehat{h}_A$?;
- if so, is there any sense in which this operator looks like an operator version of a Riemannian *metric*?

The first question is exceptionally difficult to answer. Even classically, $g_{ab}(x)$ is likely to be a highly non-linear and non-local function of the physical modes (ϕ^r,π_s), and therefore intractable problems of operator ordering and infinite operator-products can be expected in the quantum theory. One might attempt to answer the second question by showing that, if they can be defined at all, the operators $g_{ab}(x;\mathcal{X},\widehat{\mathcal{P}},\widehat{\phi},\widehat{\pi}\,]$ and ${p_a}^b(x;\mathcal{X},\widehat{\mathcal{P}},\widehat{\phi},\widehat{\pi}\,]$ satisfy the *affine* commutation relations (5.1.6–5.1.8) that arise in the quantisation schemes of type II where all $6\times\infty^3$ modes $g_{ab}(x)$ are afforded operator status (see §5.1).

The Definition of the Operators $\widehat{h}_A(x)$. Even apart from the question of ultra-violet divergences, many problems appear when trying to construct operator equivalents of the Hamiltonian/momentum densities $h_A(x;\mathcal{X},\phi,\pi]$ that arise as the solutions (4.2.2) of the initial-value constraints for the conjugate variables $\mathcal{P}_A(x)$. For example:

1. As mentioned earlier, the solution for $\mathcal{P}_A(x)$ may exist only locally in phase space, and there may be more than one such solution. In the latter case it might be possible to select a particular solution on 'physical grounds' (such as the requirement that the physical Hamiltonian is a positive functional of the canonical variables) but the status of such a step is not clear since it means that certain classical solutions to the field equations are deliberately excluded.

Simple examples of these phenomena can be seen in the Robertson-Walker model (4.2.18–4.2.20). The constraint (4.2.17) is quadratic in the conjugate variable p_a and therefore has the *two* solutions in (4.2.18). Note also that for a number of typical potential functions $V(\phi)$ the expression under the square root in (4.2.18) is negative for sufficiently large t, and the range of such values depends on the values of the canonical variables. Thus even classically the constraint can be solved by a *real* p_a only in a restricted region of phase space and values of t.

2. Even if it does exist, the classical solution for $h_A(x; \mathcal{X}, \phi, \pi]$ is likely be a very complicated expression of the canonical variables. Indeed, it may exist only in some implicit sense: a good example is the solution Φ of the elliptic partial-differential equation (4.2.27) that determines $\mathcal{P}_{\mathcal{T}}$ in the case of the mean extrinsic curvature time. The solution may also be a very non-local function of the canonical variables.

These properties of the classical solution pose various problems at the quantum level. For example:

- Operator ordering is likely to be a major difficulty. This is particularly relevant to the problem of functional evolution discussed below.

- The operator that represents a physical quantum Hamiltonian is required to be self-adjoint and positive. In a simple model, the positivity requirement may involve just selecting a particular solution to the constraints, but self-adjointness and positivity are very difficult to check in a situation in which even the classical expression is only an implicit function of the canonical variables.

- As remarked already, the constraint equations may well be algebraic equations for $\mathcal{P}_A$ whose solution therefore involves taking roots of some operator $\widehat{K}$. This can be done with the aid of the spectral theorem provided $\widehat{K}$ is a positive, self-adjoint operator—which takes us back to the preceding problem. If $\widehat{K}$ is *not* positive then even if the quantum Hamiltonian exists it is not self-adjoint, and so the time evolution becomes non-unitary.

3. The solutions to the constraints are likely to be explicit functionals of the internal spacetime coordinate functions, which gives rise to a time-dependent Hamiltonian. This effect, which can be seen clearly in the minisuperspace example, has several implications:

- A time-dependent Hamiltonian means that energy can be fed into, or taken out of, the quantum system. In normal physics, this happens whenever the system is not closed, with the time dependence of the Hamiltonian being determined by the environment to which the system couples. However, a compact three-manifold Σ (the 'universe') has no external environment, and so the time-dependence seems a little odd.

- If $h(t)$ is time-dependent, the Schrödinger equation

$$i\hbar \frac{d\psi_t}{dt} = \widehat{h}(t)\psi_t \tag{4.2.30}$$

does not lead to the simple second-order equation

$$-\hbar^2 \frac{d^2\psi_t}{dt^2} = (\widehat{h}(t))^2 \psi_t \tag{4.2.31}$$

because of the extra term involving the time derivative of $\widehat{h}(t)$. In the case of quantum gravity this means that the functional Schrödinger equation does *not* imply the Wheeler-DeWitt equation. This is not necessarily a bad thing but it does illustrates the potential inequivalence of different approaches to the canonical quantisation of gravity.

- As remarked already, roots of operators can be handled if the object whose root is being taken is self-adjoint and positive. However, in general this will be true only for certain ranges of t (which can depend on the values of the true canonical variables). Thus a proper Hamiltonian operator may exist only for a limited range of time values. A clear example of this is the Robertson-Walker model (4.2.18).

These difficulties in defining the operators $\widehat{h}_A(x)$ may seem collectively to constitute a major objection to the internal Schrödinger interpretation. However, the apparent failure to identify a completely satisfactory set of internal spacetime coordinates is not necessarily a disaster: it might reflect something of genuine physical significance. For example, much work has been done in recent years on quantum theories of the creation of the universe and, in any such theory, something peculiar must necessarily happen to time near the origination point. A good example is the schemes of Hartle & Hawking (1983) and Vilenkin (1988), in both of which there is a sense in which time becomes imaginary. Perhaps the tendency to produce a non-unitary evolution in the internal time variables is a reflection of this effect.

The Multiple Choice Problem. Generically, there is no geometrically natural choice for the internal spacetime coordinates and, classically, all have an equal standing. However, this classical cornucopia becomes a real problem at the quantum level since there is no reason to suppose that the theories corresponding to different choices of time will agree.

The crucial point is that two different choices of internal coordinates are related by a canonical transformation and, in this sense, are classically of equal validity. However, one of the central properties/problems of the quantisation of any non-linear system is that, because of the well-known Van-Hove phenomenon (Groenwold 1946, Van Hove 1951), most classical canonical transformations *cannot* be represented by unitary operators while, at the same time, maintaining the irreducibility of the canonical commutation relations. This means that in quantising a system it is always necessary to select some preferred sub-algebra of classical observables which is to be quantised (Isham 1984). One analogue of this situation in quantum gravity is precisely the dependence of the theory on the choice of internal time. It must be emphasised that this phenomenon is not related to ultra-violet divergences, or other pathologies peculiar to quantum field theories, but arises already in finite-dimensional systems. A good example is the Robertson-Walker model used above to illustrate various types of internal time. The three Hamiltonians (4.2.18), (4.2.19) and (4.2.20) are associated with different quantum theories of the same classical system.

The Problem of Functional Evolution. The problem of functional evolution is concerned with the key question of the consistency of the dynamical evolution generated by the constraints; in particular, the preservation of the constraints as the system evolves in time. Much more than most of the other problems of time, the issue of functional evolution depends critically on our ability to construct quantum gravity as a consistent quantum *field* theory by giving sense to the infinite collection of Hamiltonians and their commutation relations.

At the classical level, there is no problem. The consistency of the original constraints (3.3.25), (3.3.26) with the dynamical equations (3.3.22), (3.3.24) follows from the first-class nature of the constraints, *i.e.,* their Poisson brackets vanish on the constraint subspace by virtue of the Dirac algebra (3.3.30–3.3.32). Similarly, the consistency of the internal time dynamical equations (3.4.27–3.4.28) is ensured by the Poisson bracket relations between $h_A(x)$ with $h_B(x')$. These results mean that the classical evolution of the system from one initial hypersurface to another is independent of the family of hypersurfaces chosen to interpolate between them.

In the internal Schrödinger interpretation, the analogous requirement on the quantum operators is

$$\frac{\delta\widehat{h}_A(x,\mathcal{X}\,]}{\delta\mathcal{X}^B(x')} - \frac{\delta\widehat{h}_B(x,\mathcal{X}\,]}{\delta\mathcal{X}^A(x')} + \frac{1}{i\hbar}[\,\widehat{h}_A(x,\mathcal{X}\,],\widehat{h}_B(x',\mathcal{X}\,]\,] = 0. \tag{4.2.32}$$

If this fails, the functional Schrödinger equation (4.2.6) breaks down.

It is clear that these conditions can be checked only if the operators concerned are defined properly, which raises the entire gamut of problems in quantum gravity, including those of operator-ordering and ultra-violet divergences. Not surprisingly, very little can be said about this problem in the full theory. The best that can be done is to elucidate some of the issues on sufficiently simple (and hence almost trivial) systems. It is essential however that these systems have an *infinite* number of degrees of freedom, *i.e.,* we have to deal with a genuine *field* theory for the phenomenon to arise at all. In particular, minisuperspace models tell us nothing about this particular problem of time.

One useful example is the functional evolution of a parametrised, massless scalar field propagating on a 1+1-dimensional flat cylindrical Minkowskian background [28] spacetime: a system that has been studied [29] in some detail by Kuchař (1988, 1989a, 1989b). There is a canonical formulation of this theory which casts the super-Hamiltonian and

[28]It should be possible to generalise these results to any parametrised free field theory on a flat four-dimensional background, but this does not seem to have been done.

[29]Similarly, there exists a canonical transformation which casts the super-Hamiltonian and super-momentum constraints of a bosonic string moving in a d-dimensional target space into those of a parametrised theory of $d-2$ independent scalar fields propagating on a two-dimensional Minkowskian

supermomentum constraints into the same form as those of the midisuperspace model of cylindrical gravitational waves. This links the functional evolution problem in the parametrised field theory with that in quantum gravity proper (Torre 1991).

In this particular case it has been shown that with the aid of careful regularisation and renormalisation the apparent anomaly in the commutator of the $\widehat{h}_A(x)$ operators can be removed, and hence a consistent quantum evolution attained. However, in the full theory of quantum gravity the problem of functional evolution is particularly difficult to disentangle from the ambiguities generated by ultra-violet divergences, and very little is known about its solution.

The Spacetime Problem. We have seen how in the canonical version of general relativity 'time' is to be viewed as a function variable rather than the single parameter of Newtonian physics. In classical geometrodynamics we are required to locate an event X in an Einstein spacetime $(\mathcal{M}, \gamma)$ using the canonical data on an embedding that passes through X. Given a particular choice $(\mathcal{T}, \mathcal{Z}^a)$ of internal time and space functions, the coordinates associated with the point X were given in (3.4.4) and (3.4.5) as

$$T(X) := \mathcal{T}(\sigma^{\mathrm{ref}}(X); g[\mathcal{F}^{\mathrm{ref}}_{t=\tau^{\mathrm{ref}}(X)}\,], p[\mathcal{F}^{\mathrm{ref}}_{t=\tau^{\mathrm{ref}}(X)}\,]\,], \tag{4.2.33}$$

and

$$Z^a(X) := \mathcal{Z}^a(\sigma^{\mathrm{ref}}(X); g[\mathcal{F}^{\mathrm{ref}}_{t=\tau^{\mathrm{ref}}(X)}\,], p[\mathcal{F}^{\mathrm{ref}}_{t=\tau^{\mathrm{ref}}(X)}\,]\,]. \tag{4.2.34}$$

These internal embedding variables have some peculiar properties. In particular, if a different reference foliation is used, so that the hypersurface $\mathcal{F}^{\mathrm{ref}}_{t=\tau^{\mathrm{ref}}(X)} : \Sigma \to \mathcal{M}$ passing through the same point X in $\mathcal{M}$ is different, then the 'time' value $T(X)$ alloted to the event will generally not be the same. Thus the time of an event depends not just on the event itself, but also on a choice of spatial hypersurface passing through the event.

Viewed from a spacetime perspective, this feature is pathological since coordinates on a manifold are local scalar functions. It is physically highly undesirable since it means that the results and the interpretation of the theory depend on the choice of the reference foliation $\mathcal{F}^{\mathrm{ref}}$. Having to choose a specific background foliation is equivalent to introducing a Newtonian-type universal time parameter, which is something we wish to avoid since there is no natural place in general relativity for such a field-independent reference system. Indeed, our goal is to construct a formalism in which $\mathcal{F}^{\mathrm{ref}}$ drops out entirely from the final result.

This undesirable behaviour of the internal time $\mathcal{T}(x; g, p]$ can be avoided only if it has a vanishing Poisson bracket with the generator of 'tilts' or 'bends' of the hypersurface. In particular, we require (Kuchař 1976*c*, Kuchař 1982, Kuchař 1991*b*)

$$\{\mathcal{T}(x), H[N]\,\} = 0 \tag{4.2.35}$$

background. This enables the functional evolution problem to be posed, and solved, for the bosonic string (Kuchař & Torre 1989, Kuchař & Torre 1991*c*).

for all test functions N that vanish at a point $x \in \Sigma$. Of course, the same limitation should also be imposed on the internal spatial coordinates $\mathcal{Z}^a(x; g, p]$. The search for internal spacetime coordinates that satisfy (4.2.35) constitutes the *spacetime problem*.

The requirement (4.2.35) is rather strong. It clearly excludes any intrinsic time like $R(x, g]$ that is a local functional of the three-geometry alone. It also excludes many obvious choices of an extrinsic time. For example, the time function (4.2.21) is certainly not a spacetime scalar: when a hypersurface is bent around a given event in a vacuum Einstein spacetime, the mean extrinsic curvature $\frac{2}{3}|g|(x)^{-\frac{1}{2}}\, g^{ab}(x)\, p_{ab}(x)$ changes, even if the event remains the same. Thus the canonical coordinate (4.2.21) cannot be turned into a coordinate on spacetime.

There do exist functionals $\mathcal{T}(x; g, p]$ of the canonical data that *are* local spacetime scalars. For example, take the square ${}^{(4)}R_{\alpha\beta\gamma\delta}(X, \gamma]\, {}^{(4)}R^{\alpha\beta\gamma\delta}(X, \gamma]$ of the Riemann curvature tensor of a vacuum Einstein spacetime, and reexpress it in terms of the canonical data on a spacelike hypersurface. But note that not all functionals $\mathcal{T}(x; g, p]$ of this type can serve as time functions. Two necessary conditions are:

1. $\{\mathcal{T}(x), \mathcal{T}(x')\} = 0$;
2. for any given Lorentzian metric γ on $\mathcal{M}$ that satisfies the vacuum Einstein field equations, the hypersurfaces of equal $\mathcal{T}$ time must be *spacelike*. (These hypersurfaces can be evaluated using (4.2.33–4.2.34) with any convenient reference foliation $\mathcal{F}^{\mathrm{ref}}$; the answer will be independent of the choice of $\mathcal{F}^{\mathrm{ref}}$ for internal spacetime functionals that are compatible with the spacetime problem.)

However, even if these conditions are satisfied, it is still necessary to split off $\mathcal{T}$ from the rest of the canonical variables by a canonical transformation (3.4.8), and this is by no means a trivial task. In fact, Kuchař and I are not aware of a single concrete example of a decomposition of the canonical variables based on a local scalar time function $\mathcal{T}(x; g, p]$. Of course, another possibility is to construct scalars from *matter* fields in the theory, and this is the topic of the next subsection.

4.3 Matter Clocks and Reference Fluids

4.3.1 The Basic Ideas

We have seen that it is difficult to produce a satisfactory definition of time using only the canonical variables of the gravitational field. In particular, there is nothing in the canonical formalism itself to provide insight into how the internal spacetime functionals $\mathcal{T}(x; g, p]$ and $\mathcal{Z}^a(x; g, p]$ are to be selected. However, in practice, location in time and in space is not performed in this way. Real physical clocks are made of matter with definite properties—an observation that has generated a recent flurry of interest in the idea of 'quantum clocks' with the hope that they may lead to a more tractable approach to the problem of time.

In a sense, the idea of matter clocks has already been implicit in what has been said so far. For example, one of the dynamical variables in the simple minisuperspace model §4.2.3 is the spatially-homogeneous scalar field ϕ, and this can be used to define time. Indeed, (4.2.20) is the Hamiltonian obtained by selecting $t = \phi$ as a classical time variable. However, a 'quantum clock' does not mean an arbitrary collection of particle or matter-field variables, but rather a device whose self-interaction and coupling to the gravitational field are deliberately optimised to serve as a measure of time. The important question is the extent to which the ability to measure (or, more precisely, to define) time using one of these systems is compatible with its realisation as a real physical entity. In particular, it must have a positive energy: a property that, as emphasised in §2, is far from being trivial

Two different types of system are feasible. The first is a point-particle clock that can measure time only at points along its worldline. The second is a cloud of such clocks that fills the space Σ and which can therefore provide a global measure of time and spatial position. The origin of this idea lies in the old classical notion of a reference fluid, and was first applied to quantum gravity in a major way by DeWitt (1962) (see also DeWitt (1967*a*)) who discussed a gravitational analogue of the famous Bohr-Rosenfeld analysis of the measurability of the quantised electromagnetic field (Bohr & Rosenfeld 1933, Bohr & Rosenfeld 1978).

In schemes of this type, the cloud of clocks is regarded as a realistic material medium with a Lagrangian that accurately describes its physical properties. An important question is the precise sense in which the ensuing structure corresponds to a genuine coordinate system on the spacetime manifold. A different approach is to *start* with a fixed set of coordinate conditions imposed on the spacetime metric γ, and then implement them by appending the conditions to the action with a family of Lagrange multipliers. These extra terms in the action are then parametrised (*i.e.,* made invariant under the action of $\mathrm{Diff}(\mathcal{M})$) and interpreted as the source terms for a special type of matter. Schemes of this type have their origin in the general problem of understanding if, and how, the full spacetime diffeomorphism group $\mathrm{Diff}(\mathcal{M})$ (rather than its projections in the form of the Dirac algebra) should be represented in the canonical theory of gravity (Bergmann & Komar 1972, Salisbury & Sundermeyer 1983, Isham & Kuchař 1985*a*, Isham & Kuchař 1985*b*, Kuchař 1986*a*, Lee & Wald 1990).

4.3.2 The Gaussian Reference Fluid

The work of Kuchař and his collaborators has been especially significant and I shall illustrate it here with the example discussed in Kuchař & Torre (1991*a*) of a Gaussian reference fluid. Other examples are harmonic coordinate conditions (Kuchař & Torre 1991*b*, Stone & Kuchař 1992) and the $K = const$ slicing condition (Kuchař 1992*a*).

Kuchař & Torre (1989) handle the conformal, harmonic and light-cone gauges in the bosonic string in the same way, and the canonical treatment of two-dimensional induced quantum gravity is discussed in Torre (1989).

The main steps in the development of the Gaussian reference fluid are as follows.

1. The Gaussian coordinate conditions are $\gamma^{00}(X) = -1$ and $\gamma^{0a}(X) = 0$, $a = 1,2,3$, but a more covariant-looking expression can be obtained by introducing a set of spacetime functions $\mathcal{T}$, $\mathcal{Z}^a$, $a = 1,2,3$ and imposing the Gaussian conditions in the form

$$\gamma^{\alpha\beta}(X)\,\mathcal{T}_{,\alpha}(X)\,\mathcal{T}_{,\beta}(X) \;=\; -1 \tag{4.3.1}$$
$$\gamma^{\alpha\beta}(X)\,\mathcal{T}_{,\alpha}(X)\,\mathcal{Z}^a{}_{,\beta}(X) \;=\; 0. \tag{4.3.2}$$

These should be viewed as a (spacetime-coordinate independent) set of partial differential equations for the functions $(\mathcal{T}, \mathcal{Z}^a)$, a set of whose solutions (there is some arbitrariness) are the Gaussian coordinate functions on $\mathcal{M}$.

2. The conditions (4.3.1–4.3.2) are now added to the dynamical system with the aid of Lagrange multipliers M and M_a, $a = 1,2,3$. The extra term in the action is

$$S_F[\gamma, M, \vec{M}, \mathcal{T}, \vec{\mathcal{Z}}] := \int_{\mathcal{M}} d^4X\,(-\det\gamma)^{\frac{1}{2}}\big(-\tfrac{1}{2}M\left(\gamma^{\alpha\beta}\mathcal{T}_{,\alpha}\,\mathcal{T}_{,\beta}+1\right) + M_a\,\gamma^{\alpha\beta}T_{,\alpha}\,\mathcal{Z}^a{}_{,\beta}\big). \tag{4.3.3}$$

Note that this action is invariant under coordinate transformations on the a superscript of $\mathcal{Z}^a$ provided that M_a transforms accordingly. This is consistent with the idea advanced in §3.4 that objects like $\mathcal{Z}$ are best thought of as taking their values in a *labelling* three-manifold Σ_l, in which case the superscript a refers to a coordinate system on Σ_l. The Lagrange multiplier M_a is then a hybrid object whose domain is the spacetime manifold $\mathcal{M}$ but whose values lie in the cotangent bundle to Σ_l. [30]

3. The action (4.3.3) is defined covariantly on the spacetime manifold $\mathcal{M}$, and is in 'parametrised' form in the sense that $(\mathcal{T}, \mathcal{Z}^a)$ are regarded as functions that can be varied freely. These four extra variables can be interpreted as describing a material system—the 'Gaussian reference fluid'—that interacts with the gravitational field and has its own energy-momentum tensor. Variation of the action with respect to $\mathcal{T}$ and $\mathcal{Z}^a$ gives the Euler hydrodynamic equations of the reference fluid, which turns out to be a heat-conducting fluid (Kuchař & Torre 1991*a*).

4. A complete canonical analysis shows that the constraints of the full theory have the form

$$\Pi_a(x) + \tilde{\mathcal{H}}_a(x; \mathcal{T}, \mathcal{Z}, g, p\,] \;=\; 0 \tag{4.3.4}$$
$$\Pi_{\mathcal{T}}(x) + \tilde{\mathcal{H}}_\perp(x; \mathcal{T}, \mathcal{Z}, g, g\,] \;=\; 0 \tag{4.3.5}$$

[30]Thus $\vec{M} : \mathcal{M} \to T^*\Sigma_l$ with $\vec{M}(X) \in T^*_{\mathcal{Z}(X)}\Sigma_l$. In general, if v belongs to the cotangent space $T^*_{\mathcal{Z}(X)}\Sigma_l$ of Σ_l, we interpret the symbol $\mathcal{Z}^a{}_{,\alpha}(X)\,v_a$, $\alpha = 0\ldots3$ as the coordinate representation on $\mathcal{M}$ of the pull-back $(\mathcal{Z}^*v)$ of v to $\mathcal{M}$ by the map $\mathcal{Z} : \mathcal{M} \to \Sigma_l$.

where $\tilde{\mathcal{H}}_a$ and $\tilde{\mathcal{H}}_\perp$ are linear combinations of the usual gravitational supermomentum and super-Hamiltonian (but with coefficients that depend on $\mathcal{T}$, $\mathcal{Z}$ and g), and where $\Pi_a(x)$ and $\Pi_{\mathcal{T}}(x)$ are the momenta conjugate to the variables $\mathcal{Z}^a(x)$ and $\mathcal{T}(x)$ respectively.

4.3.3 Advantages and Problems

The use of the Gaussian reference fluid has several big advantages over a purely geometrical set of internal coordinate functions. In particular:

- The most obvious property of the constraints (4.3.4–4.3.5) is that they are *linear* in the momenta Π_a and $\Pi_{\mathcal{T}}$ of the matter-field coordinate functions. In the quantum theory, this leads at once to a functional Schrödinger equation of the type (4.2.6), and hence to an uncontentious intepretation of a state $\Psi[\,\mathcal{T}, \mathcal{Z}, g\,]$ as the probability amplitude for measuring the three-metric $g_{ab}(x)$ on the hypersurface in $\mathcal{M}$ associated with $(\mathcal{T}, \mathcal{Z}^a)$. Since the constraints are manifestly linear in the reference-fluid momenta, the Hamiltonian densities that appear in the Schrödinger equation are local, explicit functions of the canonical variables. In addition, the Hamiltonian is only quadratic in $p^{ab}(x)$. These are significant advantages over the situation that arises when purely geometrical time and space functions are used.
- The spacetime problem is non-existent because the variables $(\mathcal{T}, \mathcal{Z}^a)$ in the action (4.3.3) are defined from the very start as genuine spacetime scalar fields.

These gains are attractive, and it is therefore unfortunate that they are offset to some extent by several basic problems:

- The energy-momentum tensor of the matter fluid does not satisfy the famous energy-conditions of general relativity, and therefore the system cannot be regarded as physically realisable. On the other hand, if one starts with a physically correct matter system (such as, for example, the scalar field ϕ in the minisuperspace model in §4.2.3) the simple linear dependence on $\Pi_{\mathcal{T}}$ is lost. This suspension between Charybdis and Scylla is arguably an inevitable consequence of the general problem of the non-existence of physical Hamiltonian clocks (§2).
- In classical general relativity, it is well-known that Gaussian coordinate conditions almost invariably breakdown somewhere, and therefore Gaussian coordinates are defined only *locally* on the spacetime manifold $\mathcal{M}$. This should be reflected in the quantum theory at some point, but it is not clear where, or how.
- Even if the technical problems above can be overcome, there still remains the issue of how fundamental are the Gaussian-type coordinate conditions, and with what entities in the real world the associated matter variables are to be identified. This is particularly appropriate in discussions of early universe quantum cosmology where there seems to be no room for a simple 'phenomenological' type of analysis.

4.4 Unimodular Gravity

One rather special example of a reference fluid is associated with the unimodular coordinate condition

$$\det \gamma_{\alpha\beta}(X) = 1 \tag{4.4.1}$$

that has often been used in discussions of classical general relativity. If the procedure outlined above is applied to this particular condition, the ensuing parametrised theory corresponds to the usual theory of general relativity but with a cosmological 'constant' that is a dynamical variable, rather than a fixed constant. The use of this theory as a possible solution to the problem of time has been discussed in several recent papers, and especially by Unruh and Wald (Henneaux & Teitelboim 1989, Unruh 1988, Unruh 1989, Unruh & Wald 1989, Brown & York 1989, Kuchař 1991*a*).

The super-Hamiltonian constraint of this modified theory is

$$\lambda + |g(x)|^{-\frac{1}{2}}\,\mathcal{H}_\perp(x) = 0 \tag{4.4.2}$$

in which what would normally be the cosmological constant λ appears as the momentum conjugate to a variable τ that is identified as a 'cosmological time'. The implication in the quantum theory is that dynamical evolution with respect to τ is described by the family of ordinary Schrödinger equations

$$i\hbar\frac{\partial\Psi(\tau,g\,]}{\partial\tau} = |g(x)|^{-\frac{1}{2}}\,(\widehat{\mathcal{H}}_\perp(x)\Psi)(\tau,g\,] \tag{4.4.3}$$

parametrised by the point x in Σ.

The problem is how to interpret such a family of equations. Kuchař studied this question with great care (Kuchař 1991*a*, Kuchař 1992*b*) and showed that the correct dynamical Schrödinger equation is

$$i\hbar\frac{\partial\Psi(\tau,g\,]}{\partial\tau} = \Big(\int_\Sigma d^3x\,|g|^{\frac{1}{2}}\Big)^{-\frac{1}{2}}\int_\Sigma d^3x\,(\widehat{\mathcal{H}}_\perp(x)\Psi)(\tau,g\,]. \tag{4.4.4}$$

If this was the only equation satisfied by $\Psi(\tau,g\,]$ it would be viable to interpret this function as the probability density for the three-metric g at a given value of the time τ. However, the effect of the existence of the family of equations (4.4.3) is that the state vector Ψ must also satisfy the collection of constraints

$$|g(x)|^{-\frac{1}{2}}\,\widehat{\mathcal{H}}_\perp(x))_{,a}\;\Psi(\tau,g\,] = 0 \tag{4.4.5}$$

where $x \in \Sigma$. But the three-geometry operator does not commute with these constraints, and therefore the interpretation of $\Psi(\tau,g]$ as a probability distribution for g is not tenable.

The geometrical origin of this problem is that the cosmological time measures the four-volume enclosed between two embeddings of the associated internal time functional

$\mathcal{T}(x)$ but given one of the embeddings the second is not determined uniquely by the value of τ: two embeddings that differ by a zero four-volume (something that can happen easily in a spacetime with a *Lorentzian* signature) cannot be separated in this way. The extra constraints (4.4.5) are to be interpreted as saying that the theory is independent of this arbitrariness. For further details see the cited papers by Kuchař.

5 IDENTIFY TIME AFTER QUANTISATION

5.1 Canonical Quantum Gravity: Quantise Before Constraining

5.1.1 The Canonical Commutation Relations for Gravity

In approaches to the problem of time of category II, a quantum theory is constructed without solving the constraints, which are then imposed at the quantum level. The identification of 'time' is made *after* this process, and is used to give the final physical interpretation of the theory, particularly the probabilistic aspects. This final structure may be related only loosely to the quantum structure with which the construction started. As we shall see, this leads to a picture of quantum gravity that is radically different from that afforded by the internal Schrödinger interpretation.

The starting point is the operator version of the Poisson-bracket algebra (3.3.27–3.3.29) in the form of the canonical commutation relations

$$
\begin{aligned}
[\,\widehat{g}_{ab}(x), \widehat{g}_{cd}(x')\,] &= 0 && (5.1.1)\\
[\,\widehat{p}^{ab}(x), \widehat{p}^{cd}(x')\,] &= 0 && (5.1.2)\\
[\,\widehat{g}_{ab}(x), \widehat{p}^{cd}(x')\,] &= i\hbar\, \delta^c_{(a}\delta^d_{b)}\, \delta(x, x') && (5.1.3)
\end{aligned}
$$

of operators defined on the three-manifold Σ. Several things should be said about this algebra:

- The classical object $g_{ab}(x)$ is not merely a symmetric covariant tensor: it is also a *metric* tensor, *i.e.,* at each point $x \in \Sigma$ the matrix $g_{ab}(x)$ is invertible with signature $(1, 1, 1)$. In particular, for any non-vanishing vector-density field v^a (of an appropriate weight) we have

  $$g(v \otimes v) := \int_\Sigma d^3x\, v^a(x)\, v^b(x)\, g_{ab}(x) > 0, \tag{5.1.4}$$

  and it is reasonable to require the corresponding quantum operators $\widehat{g}(v \otimes v)$ to satisfy the analogous equations

  $$\widehat{g}(v \otimes v) > 0. \tag{5.1.5}$$

It is noteworthy that the canonical commutation relations (5.1.1–5.1.3) are *incompatible* with these geometrical properties of $g_{ab}(x)$ provided that the smeared versions of the operators $\widehat{p}^{cd}(x)$ are self-adjoint. In that case they can be exponentiated to give unitary operators which when acting on $\widehat{g}(v\otimes v)$ show that the spectrum of $\widehat{g}(v\otimes v)$ can take on negative values.

This problem can be partly remedied by replacing (5.1.1–5.1.3) with a set of *affine* relations (Klauder 1970, Pilati 1982, Pilati 1983, Isham 1984, Isham & Kakas 1984*a*, Isham & Kakas 1984*b*, Isham 1992),

$$\begin{aligned}
[\,\widehat{g}_{ab}(x),\widehat{g}_{cd}(x')\,] &= 0 && (5.1.6)\\
[\,\widehat{p}_a{}^b(x),\widehat{p}_c{}^d(x')\,] &= i\hbar(\delta_a^d\,\widehat{p}_c{}^b(x)-\delta_c^b\,\widehat{p}_a{}^d(x))\,\delta(x,x') && (5.1.7)\\
[\,\widehat{g}_{ab}(x),\widehat{p}_c{}^d(x')\,] &= i\hbar\,\delta^d_{(a}\widehat{g}_{b)c}(x)\,\delta(x,x'). && (5.1.8)
\end{aligned}$$

At a classical level, the corresponding Poisson brackets are equivalent to the standard canonical relations (3.3.27–3.3.29) with $p_c{}^d(x) := g_{cb}(x)\,p^{bd}(x)$. However, the situation at the quantum level is very different and, for example, there exist many representations of th affine relations (5.1.6–5.1.8) in which the spectrum of the smeared metric operator 'almost' satisfies the operator inequality (5.1.5); 'almost' in the sense that the right hand side is ≥ 0 rather than a strict inequality. This result is helpful, but it provides only a partial resolution of the general question of the extent to which the classical geometrical properties of $g_{ab}(x)$ can be, or should be, captured in the quantum theory. This is how the *spatial metric reconstruction* problem appears in this approach to quantum gravity.

- We are working in the 'Schrödinger representation' in which no time dependence is carried by the canonical variables. As we shall see, there is more to this than meets the eye.

- Equation (5.1.1) (or the affine analogue (5.1.6)) is a form of microcausality. However, the functional form of the constraints is independent of any foliation of spacetime, and therefore it is not clear what this 'microcausal' property means in terms of the usual ideas of an 'equal-time' hypersurface, or indeed how the notion of spacetime structure (as opposed to spatial structure) appears at all. At this stage, the most that can be said is that, whatever the final spacetime interpretation may be, (5.1.1) implies that the points of Σ are to be regarded as spacelike separated.

5.1.2 The Imposition of the Constraints

The key question is how the constraint equations (3.3.25–3.3.26) are to be handled. The essence of the Dirac approach is to impose them as constraints on the physically allowed states in the form

$$\begin{aligned}
\mathcal{H}_a(x;\widehat{g},\widehat{p})\Psi &= 0 && (5.1.9)\\
\mathcal{H}_\perp(x;\widehat{g},\widehat{p})\Psi &= 0. && (5.1.10)
\end{aligned}$$

We recall that, in the classical theory, the constraints are *equivalent* to the dynamical equations in the sense that if they are satisfied on all spatial hypersurfaces of a Lorentzian metric γ, then γ necessarily satisfies the Einstein vacuum field equations. This is reflected in the quantum theory by the assumption that the operator constraints (5.1.9–5.1.10) are the *sole* technical content of the theory, *i.e.,* the dynamical evolution equations are *not* imposed as well.

This is closely related to the following fundamental observation. According to the first-order action (3.3.18), the canonical Hamiltonian (3.3.23) associated with general relativity is

$$H[N, \vec{N}](t) = \int_\Sigma d^3x(N(x)\mathcal{H}_\perp(x) + N^a(x)\mathcal{H}_a(x)) \tag{5.1.11}$$

where N and $\vec{N}$ are regarded as external, c-number functions. However, (5.1.11) has a rather remarkable implication for the putative Schrödinger equation

$$i\hbar\frac{d}{dt}\Psi_t = \widehat{H}[N, \vec{N}](t)\Psi_t \tag{5.1.12}$$

since if the state Ψ_t satisfies the constraint equations (5.1.9–5.1.10), we see that it has no time dependence at all! Similarly, it is not meaningful to speak of a 'Schrödinger' or a 'Heisenberg' picture since the matrix elements between physical states of a Heisenberg-picture field will be the same as for the field in the Schrödinger picture.

This so-called 'frozen formalism' caused much confusion when it was first discovered since it seems to imply that nothing happens in a quantum theory of gravity. Clearly this is some sort of quantum analogue of the fact that classical observables (*i.e.,* those satisfying (3.3.40)) are constants of the motion. These days, this situation is understood to reflect the absence of any external time parameter in general relativity, and therefore, in particular, the need to discuss the measurement of time with the aid of functionals of the internal variables in the theory. This perspective dominates almost all work on the problem of time in quantum gravity.

5.1.3 Problems with the Dirac Approach

Many problems arise when attempting to implement the Dirac scheme. For example:

1. To what extent can, or should, the classical Poisson-bracket algebra (3.3.30–3.3.32) be maintained in the quantum theory? The constraints (3.3.19) and (3.3.20) are highly non-linear functions of the canonical variables and involve non-polynomial products of field operators evaluated at the same point. Thus we are lead inevitably to the problems of regularisation, renormalisation, operator ordering, and potential anomalies. This is the form taken by the *functional evolution problem* in this approach to quantisation.

2. It is not clear what properties are expected of the constraint operators $\widehat{\mathcal{H}_\perp}(x)$ and $\widehat{\mathcal{H}_a}(x)$; in particular, should they be self-adjoint? Since one presumably starts with self-adjoint representations of the canonical commutation representations (5.1.1–5.1.3), it is perhaps natural to require self-adjointness for the super-Hamiltonian and supermomentum operators. However, this has been challenged several times and the issue is clearly of significance in discussing the operator-ordering problem (Komar 1979*b*, Komar 1979*a*, Kuchař 1986*c*, Kuchař 1986*b*, Kuchař 1987, Hajicek & Kuchař 1990*a*, Hajicek & Kuchař 1990*b*). The possibility of using a non-hermitian operator can be partly justified by noting that the Hilbert space structure on the space that carries the representation of the canonical algebra may be only distantly related to the Hilbert space structure that ought to be imposed on the *physical* states (*i.e.,* those that satisfy the constraints).

3. More generally, what is the relation between these two Hilbert spaces? This question has important implications for the problem of time.

4. What is meant by an 'observable' in a quantum theory of this type? By analogy with the classical result (3.3.40), one might be tempted to postulate that an operator $\widehat{A}$ defined on the starting Hilbert space corresponds to an observable if

$$[\,\widehat{A}, \widehat{H}[N, \vec{N}]\,] = 0 \tag{5.1.13}$$

for all test functions N and $\vec{N}$. However, it can be argued that it is sufficient for (5.1.13) to be satisfied on the subspace of *solutions* to the Dirac constraints (5.1.9–5.1.10). This is the quantum analogue of the fact that the classical condition (3.3.40) for an observable is a *weak* equality, *i.e.,* it holds only on the subspace of the classical phase space given by the solutions (g,p) to the classical constraints $\mathcal{H}_a(x; g,p] = 0 = \mathcal{H}_\perp(x; g,p]$.

5.1.4 Representations on Functionals $\Psi[g]$

In attempting to find concrete representations of the canonical algebra (5.1.1–5.1.3) it is natural to try an analogue of the quantum scalar field representations (3.2.14) and (3.2.15). Thus the state vectors are taken to be functionals $\Psi[g]$ of Riemannian metrics g on Σ, and the canonical operators are defined as

$$(\widehat{g}_{ab}(x)\Psi)[g] \;:=\; g_{ab}(x)\Psi[g] \tag{5.1.14}$$

$$(\widehat{p}^{cd}(x)\Psi)[g] \;:=\; -i\hbar\frac{\delta\Psi[g]}{\delta g_{cd}(x)}. \tag{5.1.15}$$

These equations have been used widely in the canonical approach to quantum gravity although, even when suitably smeared, they do not define proper self-adjoint operators because of the absence of any Lebesgue measure on Riem(Σ) (*e.g.,* Isham (1992)) and, as remarked earlier, they are also incompatible with the positivity requirement (5.1.5).

Let us consider the Dirac constraints (5.1.9–5.1.10) in this representation. From a physical perspective it is easy to see the need for them. Formally, the domain space of the state functionals is $\mathrm{Riem}(\Sigma)$, and to specify a metric $g_{ab}(x)$ at a point $x \in \Sigma$ requires six numbers (the components of the metric in some coordinate system). However, the true gravitational system should have only *two* degrees of freedom per spatial point, and therefore four of the six degrees of freedom need to be lost. This is precisely what is achieved by the imposition of the constraints (5.1.9–5.1.10). As we saw in §4.1, the same counting argument applies if the system is reduced to true canonical form before quantising.

The easiest constraints to handle are those in the first set

$$(\widehat{H}[\vec{N}]\Psi)[g] = 0. \tag{5.1.16}$$

We saw earlier that the classical functions $H[\vec{N}]$ are the infinitesimal generators of the diffeomorphism group of Σ, and the same might be expected to apply here. The key question is whether the operator-ordering problems can be solved so that the algebra (3.3.34) is preserved at the quantum level. In practice, this is fairly straightforward; indeed, one powerful way of *solving* the operator-ordering problem for the $\widehat{H}[\vec{N}]$ generators is to insist that they form a self-adjoint representation of the Lie algebra of $\mathrm{Diff}(\Sigma)$.

The implications of (5.1.16) are then a straightforward analogue of those in conventional Yang-Mills gauge theories. The group $\mathrm{Diff}(\Sigma)$ acts as a group of transformations on the space $\mathrm{Riem}(\Sigma)$ of Riemannian metrics on Σ, with $f \in \mathrm{Diff}(\Sigma)$ sending $g \in \mathrm{Riem}(\Sigma)$ to f^*g. Apart from certain technical niceties, this leads to a picture in which $\mathrm{Riem}(\Sigma)$ is fibered by the orbits of the $\mathrm{Diff}(\Sigma)$ action. Then (5.1.16) implies that the state functional Ψ is constant (modulo possible θ-vacuua effects) on the orbits of $\mathrm{Diff}(\Sigma)$, and therefore passes to a function on the *superspace* $\mathrm{Riem}(\Sigma)/\mathrm{Diff}(\Sigma)$ of $\mathrm{Diff}(\Sigma)$ orbits (Misner 1957, Higgs 1958).

5.1.5 The Wheeler-DeWitt Equation

We must consider now the final constraint $\widehat{\mathcal{H}}_\perp(x)\Psi = 0$. Unlike the constraint $H[\vec{N}]\Psi = 0$, this has no simple group-theoretic interpretation sinc as remarked earlier, the presence of the explicit $g^{ab}(x)$ factor on the right hand side of (3.3.32) means that (3.3.30–3.3.32) is not a genuine Lie algebra. Thus the operator-ordering problem becomes much harder. If we choose as a simple example the ordering in which all the p^{cd} variables are placed to the right of the g_{ab} variables, the constraint (5.1.10) becomes

$$-\hbar^2\kappa^2\mathcal{G}_{ab\,cd}(x,g)\frac{\delta^2\Psi[g]}{\delta g_{ab}(x)\,\delta g_{cd}(x)} - \frac{|g|^{\frac{1}{2}}(x)}{\kappa^2}\,R(x,g)\Psi[g] = 0 \tag{5.1.17}$$

where $\mathcal{G}_{ab\,cd}$ is the DeWitt metric defined in (3.3.21).

Equation (5.1.17) is the famous Wheeler-DeWitt equation (Wheeler 1962, Wheeler 1964, DeWitt 1967*a*, Wheeler 1968). It is the heart of the Dirac constraint quantisation approach to the canonical theory of quantum gravity, and everything must be extracted from it. Needless to say, there are a number of problems and questions that need to be considered. For example:

1. The ordering chosen in (5.1.17) is a simple one but there is no particular reason why it should be correct. One popular alternative is to write the 'kinetic-energy' term as a covariant functional-Laplacian using the DeWitt metric (3.3.21). This part of the operator is then invariant under redefinitions of coordinates on Riem(Σ). Of course, a key issue in discussing the ordering of $\widehat{\mathcal{H}_\perp}(x)$, $x \in \Sigma$, is whether or not these operators are expected to be self-adjoint (the ordering chosen in (5.1.17) is certainly *not* self-adjoint in the scalar product formally associated with the choice (5.1.14–5.1.15) for the canonical operators).

2. The Wheeler-DeWitt equation contains products of functional differential operators evaluated at the same spatial point and is therefore likely to produce $\delta(0)$ singularities when acting on a wide variety of possible state functionals. Thus regularisation will almost certainly be needed.

3. A major question is how to approach the problem of solving the Wheeler-DeWitt equation. One obvious tactic is to deal with it as a functional differential equation *per se*. However, whether or not this is valid depends on the general interpretation of the constraint equations $\widehat{\mathcal{H}_\perp}(x)\Psi = 0$, $x \in \Sigma$. If these equations mean that Ψ is a simultaneous eigenvector of self-adjoint operators $\widehat{\mathcal{H}_\perp}(x)$, $x \in \Sigma$, with eigenvalue 0 then, as with eigenfunction problems for ordinary differential operators, some sort of boundary value conditions need to be imposed on Ψ, and the theory itself is not too informative about what these might be. In practice, there has been a tendency to solve the constraint equation as a functional differential equation without checking that 0 is a genuine eigenvalue (*i.e.,* without worrying about boundary conditions). This can lead to highly misleading results—a fact that is often overlooked, especially in discussions of minisuperspace approximations to the theory.

4. One of the hardest problems is to decide what the Wheeler-DeWitt equation means in physical terms. In particular, the notions of 'time' and 'time-evolution' must be introduced in some way. The central idea is one we have mentioned several times already: time must be defined as an *internal* property of the gravitational system (plus matter) rather than being identified with some external parameter in the universe. We shall return frequently to this important topic in our analysis of the various approaches to the problem of time in quantum gravity.

5.1.6 A Minisuperspace Example

Further discussion of the Wheeler-DeWitt equation is assisted by having access to the minisuperspace model discussed in §4.2.3. To derive the Wheeler-DeWitt equation for this system we need first to confront some of the problems mentioned above. In particular:

- The classical variable a satisfies the inequality $a \geq 0$. How is this inequality to be implemented in the quantum theory? This is a very simple example of the spatial metric reconstruction problem.
- The classical expression $p_a^2/24a$ in the super-Hamiltonian (4.2.16) will lead to operator-ordering problems in the quantum theory.

The first problem can be tackled in several different ways:

1. Ignore the problem and impose standard commutation relations $[\,\widehat{a}, \widehat{p}_a\,] = i\hbar$ even though we know this leads to a spectrum for $\widehat{a}$ which is the entire real line. This is the minisuperspace analogue of taking the 'naïve ' commutation relations (5.1.1–5.1.3). The Hilbert space [31] will be $L^2(\mathbb{R}, da)$ with the operators defined in the usual way as

$$(\widehat{a}\psi)(a) \;:=\; a\psi(a) \tag{5.1.18}$$
$$(\widehat{p}_a\psi)(a) \;:=\; -i\hbar\frac{d\psi(a)}{da}. \tag{5.1.19}$$

   The problem in this approach is to give some physical meaning to the negative values of a.

2. Insist on using the Hilbert space $L^2(\mathbb{R}_+, da)$ of functions that are concentrated on $\mathbb{R}_+$ but keep the definitions (5.1.18–5.1.19). The conjugate momentum $\widehat{p}_a$ is no longer self-adjoint but, nevertheless, it is possible to arrange for the super-Hamiltonian to be a self-adjoint function of $\widehat{a}$, $\widehat{p}_a$, $\widehat{p}_a^\dagger$, $\widehat{\phi}$ and $\widehat{p}_\phi$.

3. Perform a canonical transformation at the classical level to a new variable Ω defined by $a = e^\Omega$. This new variable ranges freely over the entire real line and can therefore be quantised as part of a conventional set of commutation relations using the Hilbert space $L^2(\mathbb{R}, d\Omega)$. The conjugate variable is $p_\Omega := e^{-\Omega}p_a$ and the super-Hamiltonian is

$$H_{\mathrm{RW}} := e^{-3\Omega}\left(\frac{-p_\Omega^2}{24} + \frac{p_\phi^2}{2}\right) - 6ke^\Omega + e^{3\Omega}V(\phi). \tag{5.1.20}$$

4. Use the affine relation $[\,\widehat{a}, \widehat{\pi}_a\,] = i\hbar\widehat{a}$ which is the minisuperspace analogue of the full set of affine relations (5.1.6–5.1.8). The affine momentum π_a is related classically to the canonical momentum p_a by $\pi_a := ap_a$, and the super-Hamiltonian becomes

$$H_{\mathrm{RW}} := \frac{1}{a^3}\left(\frac{-\pi_a^2}{24} + \frac{p_\phi^2}{2}\right) - 6ka + a^3V(\phi). \tag{5.1.21}$$

[31] This is only that part of the Hilbert space which refers to the a-variable. The full Hilbert space is $L^2(\mathbb{R}^2, da\, d\phi)$.

A self-adjoint representation of the affine commutation relations can be defined on the Hilbert space $L^2(\mathbb{R}_+, da/a)$ by

$$(\widehat{a}\psi)(a) \;:=\; a\psi(a) \tag{5.1.22}$$

$$(\widehat{\pi}_a\psi)(a) \;:=\; -i\hbar\, a\frac{d\psi(a)}{da}. \tag{5.1.23}$$

Note that the transformation $a := e^{\Omega}$ sets up an equivalence between this approach and the previous one.

To get a sample Wheeler-DeWitt equation, let us choose method three with the Ω variable satisfying standard commutation relations. Then, ignoring the operator-ordering problem, we get

$$\left(\hbar^2 e^{-3\Omega}\Big(\frac{1}{24}\frac{\partial^2}{\partial\Omega^2} - \frac{1}{2}\frac{\partial^2}{\partial\phi^2}\Big) - 6ke^{\Omega} + e^{3\Omega}V(\phi)\right)\psi(\Omega,\phi) = 0. \tag{5.1.24}$$

This simple model illustrates many of the features of the full canonical theory of general relativity and will be very useful in what follows. For a recent treatment using path-integral techniques see Linden & Perry (1991).

5.2 The Klein-Gordon Interpretation for Quantum Gravity

5.2.1 The Analogue of a Point Particle Moving in a Curved Spacetime

The key issue now is how the solutions to the Wheeler-DeWitt equation (5.1.17) are to be interpreted. This involves two related questions:

1. What inner product should be placed on the solutions?
2. How is the notion of time evolution to be extracted from the Wheeler-DeWitt equation?

One natural inner product might seem to be [32]

$$\langle\Psi|\Phi\rangle := \int_{\mathrm{Riem}(\Sigma)} \mathcal{D}g\,\Psi^*[g]\,\Phi[g] \tag{5.2.1}$$

since this is the scalar product with respect to which, for example, the canonical operators (5.1.14) and (5.1.15) are self-adjoint (at least formally). Indeed, this is precisely what is used in the so-called 'naïve Schrödinger interpretation' discussed in §6.1. However, the definition (5.2.1) can be applied to *any* functional of the three-metric g, which

[32]Here $\mathcal{D}g$ denotes the formal analogue of the Lebesgue measure. No such measure really exists on $\mathrm{Riem}(\Sigma)$ and a more careful discussion would need to take this into account.

as we shall see in §6.1 is the cause of considerable difficulties. In the present section we are concerned rather with finding a scalar product that applies only to *solutions* to the Wheeler-DeWitt equation.

The central idea is to explore the analogy between the Wheeler-DeWitt equation (5.1.17) and the Klein-Gordon equation of a particle moving in a curved space with an arbitrary, time-dependent potential (DeWitt 1967*a*). The validity of this analogy can be seen especially clearly in the simple minisuperspace model (5.1.24).

The point-particle model has been discussed in great depth in recent years by Kuchař; here I shall sketch only the main ideas (Kuchař 1991*b*, Kuchař 1992*b*). Consider a relativistic particle of mass M moving in a four-dimensional spacetime $(\mathcal{M}, \gamma)$ where γ is a fixed Lorentzian metric. The classical trajectories of the particle in $\mathcal{M}$ are parametrised by an arbitrary real number τ, and the theory is invariant under the reparametrisation $\tau \mapsto \tau'(\tau)$. This invariance leads to the constraint $H(X, P) = 0$ with the super-Hamiltonian

$$H(X,P) := \frac{1}{2M}\gamma^{\alpha\beta}(X)\, P_\alpha\, P_\beta + V(X) \tag{5.2.2}$$

where $V(X)$ is the (positive) potential-energy term.

In the quantum theory, this constraint becomes the Klein-Gordon equation

$$(\gamma^{\alpha\beta}(X)\nabla_\alpha\nabla_\beta + V(X))\Psi(X) = 0 \tag{5.2.3}$$

where a convenient choice has been made for the operator ordering of the kinetic-energy term. The standard interpretation of this equation is based on the pairing between any pair of solutions Ψ, Φ defined by

$$\langle\Psi,\Phi\rangle_{\mathrm{KG}} := \int_{\mathcal{E}(\Sigma)} d\Sigma_\alpha(X)\, \frac{1}{2i}\gamma^{\alpha\beta}(X)(\Psi(X)^*\overrightarrow{\partial}_\beta\Phi - \Psi(X)^*\overleftarrow{\partial}_\beta\,\Phi(X)) \tag{5.2.4}$$

where the integral is taken over the hypersurface $\mathcal{E}(\Sigma)$ of $\mathcal{M}$ defined by an embedding $\mathcal{E} : \Sigma \to \mathcal{M}$ that is spacelike with respect to the background metric γ on $\mathcal{M}$. Note that $d\Sigma_\alpha(X)$ is the directed hypersurface volume-element in $\mathcal{M}$ defined by

$$d\Sigma_\alpha(X) := \epsilon_{\alpha\beta\gamma\delta}\, dX^\beta \wedge dX^\gamma \wedge dX^\delta. \tag{5.2.5}$$

It follows from the Klein-Gordon equation that $\langle\Psi,\Phi\rangle_{\mathrm{KG}}$ is independent of $\mathcal{E}$, which suggests that $\langle\Psi,\Phi\rangle_{\mathrm{KG}}$ might be a suitable choice for a scalar product. However, as things stand, this is not viable since the pairing is *not* positive definite. Indeed

- $\langle\Psi,\Psi\rangle_{\mathrm{KG}} = 0$ for all *real* functions Ψ;
- complex solutions to the Klein-Gordon equation exist for which $\langle\Psi,\Psi\rangle_{\mathrm{KG}} < 0$.

The standard way of resolving this problem is to look for a timelike vector field U on $\mathcal{M}$ that is a Killing vector for the spacetime metric γ and is also such that the potential

V is constant along its flow lines. A natural choice of time function $\tau(X)$ is then the parameter along these flow lines, defined as a solution to the partial differential equation

$$U^\alpha(X)\partial_\alpha\tau(X) = 1. \tag{5.2.6}$$

If such a Killing vector exists, it follows at once that the energy $E(X,P) = -P_\tau := -U^\alpha(X)\,P_\alpha$ of the particle is a constant of the motion with $\{E,H\} = 0$. On quantisation this becomes

$$[\,\widehat{E},\widehat{H}\,] = 0 \tag{5.2.7}$$

which, if all the operators are self-adjoint with respect to the original inner product (5.2.1), means it is possible to find simultaneous eigenstates of

$$\widehat{E} := i\hbar U^\alpha\partial_\alpha. \tag{5.2.8}$$

and the super-Hamiltonian (5.2.2). It is therefore meaningful to select those solutions of the Klein-Gordon equation that have positive energy, and it is straightforward to see that the inner product (5.2.4) is *positive* on such solutions. Furthermore, restricted to such solutions, the Klein-Gordon equation can be shown to be equivalent to a conventional Schrödinger equation using the chosen time parameter. This equation is obtained by factorising the super-Hamiltonian in the form

$$H = (P_\tau + h)(P_\tau - h). \tag{5.2.9}$$

which is equivalent to the constraint

$$P_\tau + h = 0 \tag{5.2.10}$$

on the subspace of the phase space of positive-energy solutions. In effect, the Schrödinger equation is obtained by imposing this second constraint as a constraint on allowed state vectors.

The construction above forms the basis for a physically meaningful interpretation of the quantum theory. However, if no suitable Killing vector U exists there is no consistent one-particle quantisation of this theory.

5.2.2 Applying the Idea to Quantum Gravity

In trying to apply these ideas to the Wheeler-DeWitt equation, the key observation is that the DeWitt metric (3.3.21) on Riem(Σ) has a hyperbolic character in which the conformal modes of the metric play the role of time-like directions, *i.e.,* the transformation $\gamma_{ab}(x) \mapsto F(x)\gamma_{ab}(x)$, $F(x) > 0$, is a 'time-like' displacement in Riem(Σ). This suggests that it may be possible to choose some internal time functional $\mathcal{T}(x,g]$ so that the Wheeler-DeWitt equation can be written in the form

$$\begin{aligned}-\hbar^2\kappa^2\Big(\frac{\delta^2}{\delta\mathcal{T}^2(x)} - \mathcal{F}^{R_1\,R_2}(x;\mathcal{T},\sigma]\frac{\delta^2}{\delta\sigma^{R_1}(x)\,\delta\sigma^{R_2}(x)}\Big)\Psi[\mathcal{T},\sigma]&\\ -\frac{|g|^{\frac{1}{2}}(x;\mathcal{T},\sigma]}{\kappa^2}\,R(x;\mathcal{T},\sigma]\Psi[\mathcal{T},\sigma] = 0&\end{aligned} \tag{5.2.11}$$

where $\sigma^R(x, g]$, $R = 1, \ldots, 5$ denotes the $5 \times \infty^3$ modes of the metric variables $g_{ab}(x)$ that remain after identifying the $1 \times \infty^3$ internal time modes $\mathcal{T}(x)$.

The starting point is the formal pairing (the analogue of the point-particle expression (5.2.4))

$$\langle \Psi, \Phi \rangle := i \prod_x \int_\Sigma d\Sigma^{ab}(x)\, \Psi^*[\gamma] \Big(\mathcal{G}_{ab\,cd}(x, g] \frac{\overrightarrow{\delta}}{\delta g_{cd}(x)} - \frac{\overleftarrow{\delta}}{\delta g_{cd}(x)} \mathcal{G}_{ab\,cd}(x, g] \Big) \Phi[\gamma] \qquad (5.2.12)$$

between solutions Ψ and Φ of the Wheeler-DeWitt equation. The functional integral is over some surface in Riem(Σ) that is spacelike with respect to the DeWitt metric (3.3.21), and $d\Sigma^{ab}(x)$ is the directed surface-element in Riem(Σ) at the point $x \in \Sigma$. Of course, considerable care would be needed to make this expression rigorous. For example, it is necessary to take account of the Diff(Σ)-invariance and then project the inner product down to Riem(Σ)/Diff(Σ), *i.e.,* we must also include the action of the supermomentum constraints $\widehat{\mathcal{H}}_a(x)\Psi = 0$. Note also that the precise form of the scalar product depends on how the operator-ordering problem in the Wheeler-DeWitt equation is solved. However, the essential idea is clear. In particular, the expression (5.2.12) has the important property of being invariant under deformations of the 'spatial' hypersurface in Riem(Σ). This is the quantum-gravity analogue of the requirement in the normal Klein-Gordon equation that the scalar product (5.2.4) be time independent.

In the minisuperspace example, the Wheeler-DeWitt equation (5.1.24) can be simplified by multiplying [33] both sides by $e^{3\Omega}$ to give

$$\left(\hbar^2 \Big(\frac{1}{24} \frac{\partial^2}{\partial \Omega^2} - \frac{1}{2} \frac{\partial^2}{\partial \phi^2} \Big) - 6k e^{4\Omega} + e^{6\Omega} V(\phi) \right) \psi(\Omega, \phi) = 0. \qquad (5.2.13)$$

The associated scalar product is simply

$$\langle \psi, \phi \rangle := i \int_{\Omega = \text{const}} d\phi \, \left(\psi^* \frac{\partial \phi}{\partial \Omega} - \phi \frac{\partial \psi^*}{\partial \Omega} \right) \qquad (5.2.14)$$

which is conserved in Ω-time by virtue of (5.2.13). Note that (5.2.13) has the anticipated form (5.2.11) for the Wheeler-DeWitt equation expressed using an appropriate internal time $\mathcal{T}(x, g]$.

Unfortunately, the right hand side of (5.2.12) cannot serve as a genuine Hilbert space inner product because, as in the analogous case of the point particle, it is not positive definite. Guided by the point-particle example, one natural way of trying to resolve this problem is to look for a vector field on Riem(Σ) that is a Killing vector for the DeWitt metric (3.3.21) and that scales the potential term $|g|^{\frac{1}{2}}(x)\, R(x, g]$ in an appropriate way.

[33] This step is contentious. It is true that the constraint equation $\widehat{\mathcal{H}}_\perp \Psi = 0$ is not formally affected by multiplying on the left by any invertible operator, but this 'renormalisation' of $\widehat{\mathcal{H}}_\perp$ affects the total constraint algebra, and the implications of this need to be considered at some point. Of course, it also affects the hermiticity properties of the original constraint.

Sadly, Kuchař has shown that Riem(Σ) admits no such vector (Kuchař 1981*a*, Kuchař 1991*b*), and hence there is no possibility of defining physical states as an analogue of the positive-frequency solutions of the normal Klein-Gordon equation (but see Friedman & Higuchi (1989) for the asymptotically-flat case). However, even if such a Killing vector did exist, there are other problems. For example (Kuchař 1992*b*):

- The potential term $|g|^{\frac{1}{2}}(x)\, R(x, g]$ can take on both negative and positive values, whereas the potential $V(X)$ in the point-particle super-Hamiltonian (5.2.2) was required to be positive. Without this condition it is not possible to prove the positivity of the Klein-Gordon scalar product restricted to positive-energy solutions.

- There are good physical reasons for selecting just the positive-energy solutions for the point-particle, but the justification for the analogous step in the gravitational case is not clear. In particular, it is quite legitimate for the geometries along a path in superspace to both expand and contract in volume, and this means a classical solution to Einstein's equations can have either sign of E. Therefore, there is no justification for picking just the positive-frequency modes. Note that this objection applies already to the simple minisuperspace model discussed above with the scalar product (5.2.14).

- The attempt to construct a Klein-Gordon interpretation of the Wheeler-DeWitt equation entails the selection of some intrinsic time functional $\mathcal{T}(x, g]$. However, as discussed earlier, any such choice will necessarily fall foul of the *spacetime problem* whose resolution requires an internal time to be a functional of the conjugate momenta $p^{cd}(x)$ as well as the metric variables $g_{ab}(x)$. Hence $\mathcal{T}(x, g]$ cannot be interpreted as a genuine spacetime coordinate.

- The feasability of a Klein-Gordon interpretation is dependent on the fact that the classical super-Hamiltonian $\mathcal{H}_\perp(x; g, p]$ is *quadratic* in the momentum variables $p^{ab}(x)$. This property is lost if any powers of the Riemann curvature $R_{\alpha\beta\gamma\delta}(X, \gamma]$ are added to the classical spacetime action of the theory. Expressions of this type are likely to arise as counter-terms in almost any attempt to construct a proper quantisation of the gravitational field (including superstring theory, but excepting the Ashtekar programme in its current form) and it is important to have some feel for how they change the situation. Several of the approaches to the problem of time are sensitive to the precise form of $\mathcal{H}_\perp$, but this is particularly so of the Klein-Gordon interpretation.

The problems above, plus the non-existence of a suitable Killing vector on Riem(Σ), seem to form an immovable block to resolving the Hilbert space problem of how to turn the solutions to the Wheeler-DeWitt equation into a genuine Hilbert space.

5.3 Third Quantisation

There have been several different reactions to the failure of the Klein-Gordon approach to the Wheeler-DeWitt equation. In the case of a relativistic particle with an external spacetime-dependent metric or potential, the impossibility of isolating positive-frequency solutions is connected with a breakdown of the one-particle interpretation of the theory, and the standard resolution is to second-quantise the system by turning the Klein-Gordon wave function into a quantum field.

It has been suggested several times that a similar process might be needed in quantum gravity with $\Psi[g]$ becoming an operator $\widehat{\Psi}[g]$ in some new Hilbert space. This procedure is usually called 'third quantisation' since the original Wheeler-DeWitt equation is already the result of a quantum field theory (Kuchař 1981*b*, Coleman 1988, Giddings & Strominger 1988, McGuigan 1988, McGuigan 1989). However, it is unclear what this means, or if the problem of time can really be solved in this way. Some of the many difficulties that arise when trying to implement this programme are as follows.

1. The approach to third quantisation that is closest to conventional quantum field theory involves constructing a Fock space whose 'one-particle' sector is associated with the functionals $\Psi[g]$. But this raises several difficulties:

- What is the analogue of the one-particle Hilbert space which is to form the basis for the Fock-space construction? The difficulty is that the problem of time is closely related to the Hilbert space problem, so where do we start? In the case of a particle in the presence of a spacetime-dependent potential, one common way of resolving this issue is to begin with a well-defined free theory with a proper Hilbert space, and then to regard the interactions with the background as a perturbation that can annihilate and create the quanta of this theory. However, this procedure is dubious if the spacetime dependence comes from the spacetime metric γ itself unless there is some sense in which one can usefully write $\gamma_{\alpha\beta}(X)$ as the sum $\eta_{\alpha\beta} + h_{\alpha\beta}(X)$ of the fixed Minkowskian metric $\eta_{\alpha\beta}$ plus a small perturbation $h_{\alpha\beta}(X)$. This is even more inappropriate in the full quantum-gravity theory since there is no obvious way of writing the DeWitt metric (3.3.21) on Riem(Σ) as a sum of this type.

- If a one-particle Hilbert space *can* be constructed, what is the interpretation of states that are tensor products of the vectors in such a space? In the case of particles, if $|x_1\rangle$ and $|x_2\rangle$ are states corresponding to a particle localised at points x_1 and x_2 respectively, then the tensor product $|x_1\rangle|x_2\rangle$ describes a pair of particles, both of which move in the *same* physical three-space. However, if $|g_1\rangle$ and $|g_2\rangle$ are eigenstates of the metric operator, it is not clear to what the product state $|g_1\rangle|g_2\rangle$ refers. The simplest thing might be to say that g_1 and g_2 are both metrics on the same space Σ, but this has no obvious physical interpretation.

2. A more meaningful interpretation is that g_1 and g_2 are metrics on *different* copies of Σ or, equivalently, a single metric on the disjoint union of two copies of Σ. However

this also raises a number of problems:

- The transition from a state $|g\rangle$ to a state $|g_1\rangle|g_2\rangle$ corresponds to a topology change in which Σ bifurcates into two copies of itself. But this is unlikely to be compatible with the Wheeler-DeWitt equation. Are we to look for some non-linear interaction between the operators $\widehat{\Psi}[g]$ that describes such a process; analogous perhaps to what is done in string field theory? Any such term would signify a radical departure from the usual field equations of general relativity.

- A normal Fock-space construction involves Bose statistics but it is not clear what it means physically to say that $|g_1, g_2\rangle$ is *symmetric* in g_1 and g_2. The two copies of Σ are disjoint, and presumably no causal connection can be made between them. So what is the operational significance of a Bose structure? The use of Fermi statistics would be even more bizarre!

- Once the original space Σ has been allowed to bifurcate into a pair of copies of itself it seems logical to extend the topology change to include an *arbitrary* final three-manifold. Thus the scope of the theory is increased enormously.

- It is difficult to see how the bifurcation of space helps with the problem of time. Presumably the Wheeler-DeWitt equation will continue to hold in each disconnected piece of the universe, and then the problem of internal time reappears in each.

3. If $\Psi[g]$ becomes a self-adjoint operator it should correspond to some sort of observable. But what can that be, and how could it be measured?

The idea of third quantisation is intriguing and could lead to a radical change in the way in which quantum gravity is perceived. But, in the light of the comments above, it is difficult to see how it can resolve the problem of time.

5.4 The Semiclassical Approximation to Quantum Gravity

5.4.1 The Early Ideas

Studies of the semiclassical approach to quantum gravity date back to the work of Møller (1962) which was based on the idea that a consistent unification of general relativity and quantum theory might not require quantising the gravitational field itself but only the matter to which it couples. The suggested implementation of this scheme was the system of equations

$$G_{\alpha\beta}(X,\gamma] = \langle\psi|T_{\alpha\beta}(X;\gamma,\widehat{\phi}]|\psi\rangle \tag{5.4.1}$$

and

$$i\hbar\frac{d\widehat{\phi}}{dt} = [\,\widehat{H}[\gamma],\widehat{\phi}\,] \tag{5.4.2}$$

in which the source of the gravitational field is the expectation value in some state $|\psi\rangle$ of the energy-momentum tensor of the quantised matter ϕ. The time label in the Heisenberg-picture equation of motion (5.4.2) must be related to the time coordinate used in (5.4.1). If ϕ is field, (5.4.2) would probably be replaced with a set of relativistically-covariant field equations. The hope is that the pair of equations (5.4.1–5.4.2) is *exact* and comprises a consistent, complete solution to the problem of quantum gravity.

The early 1980s saw a renewal of interest in this approach, although the results were not conclusive (see Kibble (1981) for a survey of the situation at that time, and Duff (1981) for a general criticism). One problem is that if the matter is chosen to be a quantised field, the right hand side of (5.4.1) can be defined only after the quantum energy-momentum tensor has been regularised and renormalised (Randjbar-Daemi, Kay & Kibble 1980). This procedure has been much studied in the general context of quantum field theory in a curved background spacetime and is widely agreed to be ambiguous if the metric is non-stationary. Counter-terms arise that involve higher powers of the Riemann curvature and which have a significant effect on the Einstein field equations. For example, there have been claims, and counter claims, that the system is intrinsically unstable (Horowitz & Wald 1978, Horowitz 1980, Horowitz & Wald 1980, Horowitz 1981, Suen 1989*a*, Suen 1989*b*, Simon 1991, Fulling 1990).

From the perspective of conventional quantum theory, the coupled equations (5.4.1) and (5.4.2) are rather peculiar. In particular, different states $|\psi_1\rangle$ and $|\psi_2\rangle$ give rise to different background spacetime geometries γ_1 and γ_2, and so it is difficult to make much sense of the superposition principle. The theory is therefore difficult to interpret since none of the standard quantum-mechanical rules are applicable. Another question is the status of different states $|\psi\rangle$ and metrics γ that satisfy the coupled equations (5.4.1) and (5.4.2). Is one particular state to be selected via, for example, some quantum cosmological theory of the initial state? Or do all possible solutions have some physical meaning?

One might wonder if the whole idea of quantising everything but the gravitational field is simply inconsistent—perhaps along the lines of the famous argument in Bohr & Rosenfeld (1933) which showed that the electromagnetic field *has* to be quantised if it is to couple consistently to the current generated by quantised matter. However, as Rosenfeld himself pointed out, there is no direct analogue for gravity since the proof for electromagnetism involves taking to infinity the ratio e/m of the charge e to the inertial mass m of a test particle—a procedure that is impossible in the gravitational case since the analogue of e is the gravitational mass whose ratio to the inertial mass is fixed by the equivalence principle (Rosenfeld 1963). There have been several attempts since then (*e.g.,* Eppley & Hannah (1977), Page & Geilker (1981)) to clarify the situation but it is still somewhat unclear.

5.4.2 The WKB Approximation to Pure Quantum Gravity

The semi-classical approach has reappeared in recent years in the guise of a Born-Oppenheimer/WKB approximation to quantum gravity, and has been particularly discussed in the context of the problem of time. The starting point is no longer the equations (5.4.1–5.4.2) describing a purely classical spacetime metric coupled to quantum matter. Instead, one begins with the full quantum-gravity theory in the form of the Wheeler-DeWitt equation augmented to incorporate the matter degrees of freedom. A solution $\Psi[g, \phi]$ to this equation is then subject to a WKB-type expansion with the aim of showing that the lowest-order term satisfies a functional *Schrödinger* equation with respect to an internal time function that is determined by the state function Ψ. Thus, to this order of approximation, the second-order Wheeler-DeWitt equation is replaced by a first-order Schrödinger equation, and hence by a system that can be given a probabilistic interpretation using the associated inner product. This is therefore a good example of a type-II scheme: the physical interpretation appears only *after* a time variable has been identified in a preliminary quantum theory.

One aim of this approach is to provide a framework that interpolates between quantum gravity proper and the, better understood, subject of quantum field theory in a fixed background spacetime. For example, there have been several discussions of how the original semiclassical equations (5.4.1) and (5.4.2) arise in this framework. However, since these equations (or, rather, their analogues) are now only *approximate*, some of the problems discussed above disappear or, more precisely, appear in a different light.

As far as the general problem of time in quantum gravity is concerned, the main idea can be summarised by saying that 'time' is only a meaningful concept in a quantum state that has some semi-classical component which can serve to define it. Thus time is an *approximate*, semi-classical concept, and its definition depends on the quantum *state* of the system. In particular, time would have no meaning in a quantum cosmology in which the universe never 'emerges' into a semi-classical region. Thus this approach pays some deference to the general idea that time is part of the classical background assumed in the Copenhagen interpretation of quantum theory.

At a technical level, the starting point is a WKB technique for obtaining an approximate solution to the Wheeler-DeWitt equation for pure geometrodynamics (see Singh & Padmanabhan (1989) for a comprehensive review, and Kuchař (1992*b*) for a recent, and very careful, discussion). This involves looking for a solution in the form

$$\Psi[g] = A[g]e^{iS[g]/\hbar\kappa^2} \tag{5.4.3}$$

where $S[g]$ is real, and where $A[g]$ is a positive, real function of g that is 'slowly varying' in the sense that

$$\hbar\kappa^2 \left|\frac{\delta A[g]}{\delta g_{ab}}\right| \ll \left|A[g]\frac{\delta S[g]}{\delta g_{ab}}\right|. \tag{5.4.4}$$

Inserting (5.4.3–5.4.4) into the Wheeler-DeWitt equation shows that, to lowest order in an expansion in powers of $\hbar\kappa^2 \simeq (L_P)^2$ (where $L_P := (G\hbar/c^3)^{\frac{1}{2}}$ is the Planck length), the phase S satisfies the Hamilton-Jacobi equation

$$\mathcal{G}_{ab\,cd}(x,g)\frac{\delta S[g]}{\delta g_{ab}(x)}\frac{\delta S[g]}{\delta g_{cd}(x)} - |g|^{\frac{1}{2}}(x)R(x,g] = 0 \tag{5.4.5}$$

of classical general relativity (the supermomentum constraints $\widehat{\mathcal{H}}_a(x)\Psi = 0$ have the same implication as before). The amplitude factor A obeys the 'conservation law'

$$\mathcal{G}_{ab\,cd}(x,g]\frac{\delta}{\delta g_{ab}(x)}\left(A^2[g]\frac{\delta S[g]}{\delta g_{cd}(x)}\right) = 0. \tag{5.4.6}$$

It should be noted that the precise form of (5.4.5) and (5.4.6) depends on the choice of operator ordering in the Wheeler-DeWitt equation. I have used a simple ordering in (5.1.17) in which the functional derivatives stand to the right of the DeWitt metric, but one might prefer, for example, a version in which the kinetic energy term is formally invariant under transformations of coordinates on Riem(Σ). This will lead to minor changes in the Hamilton-Jacobi equation (5.4.5) and the conservation law (5.4.6) (to illustrate this point see the analogous equations in Kuchař (1992*b*)).

5.4.3 Semiclassical Quantum Gravity and the Problem of Time

The recent surge of interest in the application of WKB methods to the problem of time began with the important work of Banks (1985). The basic idea is a type of Born-Oppenheimer approach and has been developed further in a number of papers; for example Hartle (1986), Zeh (1986), Zeh (1988), Brout (1987), Brout, Horowitz & Wiel (1987), Brout & Venturi (1989), Englert (1989), Halliwell (1987), Singh & Padmanabhan (1989), Padmanabhan (1989*b*), Kiefer & Singh (1991) and Halliwell (1991*a*).

Banks considered a system of matter fields coupled to gravity for which the Wheeler-DeWitt equation for the combined system can be written as (*cf* (5.1.17))

$$-\hbar^2\kappa^2\mathcal{G}_{ab\,cd}(x,g]\frac{\delta^2\Psi}{\delta g_{ab}(x)\,\delta g_{cd}(x)}[g] - \left(\frac{|g|^{\frac{1}{2}}(x)}{\kappa^2}R(x,g] - \widehat{\mathcal{H}}_m(x,g]\right)\Psi[g] = 0 \tag{5.4.7}$$

where $\mathcal{H}_m(x,g]$ is the Hamiltonian density for the matter. The next step is to find solutions to (5.4.7) of the special form

$$\Psi[g,\phi] = A[g]\,\Phi[g,\phi]\,e^{iS[g]/\hbar\kappa^2} \tag{5.4.8}$$

where ϕ denotes the collection of matter variables. The function $\Phi[g,\phi]$ is expanded as a power series [34] in Newton's constant G

$$\Phi[g,\phi] = \psi[g,\phi] + \sum_{n=1}^{\infty} G^n\psi_{(n)}[g,\phi] \tag{5.4.9}$$

[34] As always, an expansion in a dimensioned constant needs to be handled carefully. The physical expansion parameter will be a dimensionless parameter constructed from G and, for example, some energy scale in the theory.

which, together with the ansatz (5.4.8), is inserted into the Wheeler-DeWitt equation (5.4.7). Keeping just the lowest-order terms in the expansion shows that the phase factor S satisfies the Hamilton-Jacobi equation (5.4.5) as before. Furthermore, the amplitude factor A can be selected to satisfy the conservation equation (5.4.6) (there is clearly some ambiguity in writing the overall amplitude as a product $A[g]\Phi[g,\phi]$). Finally, the lowest-order term $\psi[g,\phi]$ in (5.4.9) satisfies the *first-order* functional differential equation

$$-2i\hbar\mathcal{G}_{ab\,cd}(x,g)\frac{\delta S[g]}{\delta g_{ab}(x)}\frac{\delta\psi[g,\phi]}{\delta g_{cd}(x)}+(\widehat{\mathcal{H}}_m(x,g]\psi)[g,\phi]=0. \tag{5.4.10}$$

The key step now is to find a functional $\mathcal{T}(x,g]$ such that

$$2\mathcal{G}_{ab\,cd}(x,g]\frac{\delta S[g]}{\delta g_{ab}(x)}\frac{\delta\mathcal{T}(x',g]}{\delta g_{cd}(x)}=\delta(x,x') \tag{5.4.11}$$

which can serve as an intrinsic time functional for the system. This must be augmented by a set $\sigma^R(x,g]$, $R=1\ldots 5$, of functions on Riem(Σ) that are 'comoving' along the flow lines generated by $\mathcal{T}$ in Riem(Σ) in the sense that

$$2\mathcal{G}_{ab\,cd}(x,g]\frac{\delta S[g]}{\delta g_{ab}(x)}\frac{\delta\sigma^R(x',g]}{\delta g_{cd}(x)}=0. \tag{5.4.12}$$

Then, using the functions $(\mathcal{T},\sigma^R)$ as coordinates on Riem(Σ), (5.4.10) becomes the functional Schrödinger equation

$$i\hbar\frac{\delta\psi[\mathcal{T},\sigma,\phi]}{\delta\mathcal{T}(x)}=(\widehat{\mathcal{H}}_m(x;\mathcal{T},\sigma]\psi)[\mathcal{T},\sigma,\phi], \tag{5.4.13}$$

which is the desired result.

There are various things to note about this construction.

1. The equation (5.4.11) shows clearly how the definition of the time function $\mathcal{T}(x,g]$ depends on the quantum state via the choice of the solution S to the Hamilton-Jacobi equation (5.4.5). This approach to the problem of time in quantum gravity can be traced back to the early ideas of DeWitt (1967*a*) and Misner (1992). It was developed further in Lapchinski & Rubakov (1979) and Banks (1985), and formed a central ingredient in the interpretation of quantum cosmology given in Vilenkin (1989).

2. Padmanabhan (1990) and Greensite (1990, 1991a, 1991b) have suggested an extension of the idea above in which time is defined with respect to the phase of *any* solution of the Wheeler-DeWitt equation, not just one that is in WKB form. Their construction can be regarded as a result of requiring quantum gravity to satisfy the Ehrenfest principle (see also Vink (1992) and Squires (1991)).

3. Only the gravitational field is treated in this semiclassical way. The matter fields are fully quantised, although the probability associated with (5.4.13) is conserved only to the same order in the expansion to which (5.4.13) is valid.

4. The scheme can be extended to include genuine quantum fluctuations of the gravitational field itself (*i.e.,* fluctuations around the background metrics associated with the solution to the Hamilton-Jacobi equation (5.4.5)). Typical examples are the work by Halliwell and Hawking on inhomogeneous perturbations of a homogeneous universe (Halliwell & Hawking 1985, Halliwell 1991*a*), and Vilenkin (1989) who writes the spatial metric $g_{ab}(x)$ as the sum of a classical background $g_{ab}^{\rm class}(x)$ and a quantum part $\widehat{h}_{ab}(x)$.

5. If S is a *real* solution to the Hamilton-Jacobi equation (5.4.5), the function $\Psi[g] = A[g]\, e^{iS[g]/\hbar\kappa^2}$ oscillates rapidly along paths in superspace. However, imaginary S solutions can also exist, and these produce an exponential type of behaviour. This is often interpreted as indicating a Riemannian rather than Lorentzian spacetime picture and has been much studied in quantum cosmology with the idea that the universe tunnels from an 'imaginary-time' region (Halliwell 1987, Halliwell 1990).

5.4.4 The Major Problems

The WKB approach to the definition of time is interesting, but it raises a number of difficult questions and problems. For example:

1. In so far as one starts with the Wheeler-DeWitt equation, the problems discussed earlier in §5.1.5 apply here too. These include singular operator-products, factor ordering and, in particular, the issue of whether or not the Wheeler-DeWitt equation is to be regarded as a genuine *eigenvalue* equation for a set of self-adjoint operators $\widehat{\mathcal{H}_\perp}(x)$, $x \in \Sigma$, defined on the Hilbert space that carries the representation of the canonical operators $(\widehat{g}_{ab}(x), \widehat{p}^{cd}(x))$: *i.e.,* the question of the boundary conditions in $\mathrm{Riem}(\Sigma)$ that are to be used in solving the Wheeler-DeWitt equation.

2. It is not obvious that there exist global solutions on superspace to the defining equation (5.4.11) for the internal time. Indeed, the analysis in Hájíček (1986) of a minisuperspace model suggests otherwise. This is related to the *global time problem* that arises in the internal Schrödinger and Klein-Gordon interpretations of time.

3. The internal time $\mathcal{T}$ defined by (5.4.11) is a functional of the metric $g_{ab}(x)$ only. However, as we argued earlier in the context of the Klein-Gordon interpretation, an intrinsic function of this type will not resolve the *spacetime problem* of constructing a scheme that is independent of the initial choice of a reference foliation.

4. The simple WKB approximation breaks down at the turning points of S, and a more careful treatment is needed near such regions.

5. The elegant first-order form of the Schrödinger equation (5.4.10) is lost at the next order in the WKB approximation. This makes it difficult to assess the proper status of (5.4.10) and to understand how the physics of time changes as one gets nearer to the Planck regime.

6. The WKB ansatz (5.4.8) is only one of a very large number of possible types of solution to the Wheeler-DeWitt equation. Why should it have such a preferred status?

   In particular, why should one not consider *superpositions* of WKB solutions to the Wheeler-DeWitt equation? Indeed, in some of the the original discussions it was assumed that the aim was to construct a coherent superposition of such solutions to produce a quantum state that approximates a single classical spacetime manifold (Gerlach 1969). However, if the state vector is a sum

   $$\Psi[g] := \sum_j A_j[g]\, \Phi_j[g,\phi]\, e^{iS_j[g]/\hbar\kappa^2} \tag{5.4.14}$$

   then each solution S_j of the Hamilton-Jacobi equation will lead to its *own* definition of time. A particularly relevant example is a wave function of the form $e^{iS[g]} + e^{-iS[g]}$ which is real, and is therefore a natural type of semiclassical solution to the Hartle-Hawking ansatz for the wave function of the universe. The two parts of this function are usually said to correspond to expanding and contracting universes respectively (this can mean, for example, their behaviour with respect to an extrinsic time variable like ${p^a}_a(x)$), and it is very difficult to see what such a quantum superposition could mean. The situation is reminiscent of the Schrödinger-cat problem in ordinary quantum theory where a single value for a macroscopic property has to be extracted from a quantum state that is a linear superposition of eigenstates.

7. The alternative is to keep just a single WKB function, but this also raises several difficulties:

   (a) As remarked earlier when discussing the Klein-Gordon inner product (5.2.12), the Wheeler-DeWitt equation is real, and therefore naturally admits real solutions. On the other hand, the time-dependent Schrödinger equation (5.4.13) is intrinsically complex because of the $i = \sqrt{-1}$ in the left hand side. As emphasised by Barbour & Smolin (1988) and Barbour (1990), this means that any attempt to derive the latter from the former will necessarily entail the imposition by hand of some correlation between the real and imaginary parts of Ψ. Keeping to just a single WKB function is an example of this, essentially ad hoc, procedure.

   (b) A quantum cosmologist might respond that this is precisely what is to be expected in a theory that describes the quantum creation of the universe via the medium of a unique solution to the Wheeler-DeWitt equation, and the

Vilenkin scheme does indeed produce a solution that is naturally complex (Vilenkin 1989). On the other hand, the Hartle-Hawking ansatz leads to a *real* solution of the equation (although this issue is clouded by ambiguities in deciding what the ansatz really means).

   (c) With the aid of Wigner functions and other techniques, Halliwell interprets a single WKB solution as describing a whole *family* of classical spacetimes (Halliwell 1987, Halliwell 1990). In effect, these different classical trajectories in superspace are labelled by the $\sigma^R(x, g]$ functions in (5.4.13). But this again raises the 'Schrödinger-cat' problem in the guise of having to decide how any specific spacetime arises.

8. We mentioned earlier that the WKB scheme has been extended to include quantum fluctuations in the metric. The extreme example of such an extension is to keep classical only those gravitational degrees of freedom that can be used to define internal clocks and spatial reference frames. This requires a split of the gravitational modes akin to that employed in the internal Schrödinger interpretation, and similar difficulties are encountered. In general there is a serious difficulty in deciding which modes of the gravitational field are to remain classical and which are to be subject to quantum fluctuations. There is no obvious physical reason for making such a selection.

9. If time is only a semi-classical concept, the notion of probability—as, for example, in the usual interpretation associated with a Schrödinger equation like (5.4.13)—is also likely to be valid only in some approximate sense. At best, this leaves open the question of how to interpolate between the Schrödinger equation (5.4.13) and the starting, and as yet uninterpreted, Wheeler-DeWitt equation (5.4.7); at worse, it throws doubt on the utility of the entire quantum programme.

5.5 Decoherence of WKB Solutions

5.5.1 The Main Idea in Conventional Quantum Theory

Of the problems listed above, the two that are particularly awkward at the conceptual level are:

1. A single $e^{iS[g]}$ corresponds to *many* classical spacetimes.

2. The lack of any *prima facie* reason for excluding a combination $\sum_i A_i[g]\Phi_i[g, \phi]e^{iS_i[g]}$ of WKB solutions, each term of which gives rise to its *own* intrinsic time function satisfying (5.4.11).

There have been a number of recent claims that these, and related, problems can be solved by invoking the notion of *decoherence*. A particularly useful general review is Zurek (1991).

The idea of decoherence has been developed as part of the general investigation into the foundations of quantum theory that has been growing steadily during the last decade. One reason for this activity has been an increasing awareness of the inadequacy of the traditional Copenhagen interpretation of quantum theory, especially the posited dualism between the classical and quantum worlds, the emphasis on measurement as a primary interpretative category, and the associated invocation of 'reduction of the state vector' as a descriptive process that lies outside the deterministic evolution afforded by the Schrödinger equation. Considerations of this type have been enhanced by attempts to develop physical devices (*e.g.,* SQUIDS) that are of macroscopic size but which are nevertheless expected to exhibit genuine quantum properties.

The other major motivation for the renewed interest in the foundations of quantum theory is the many advances that have taken place in cosmology, especially the realisation that the quantum state of the very early universe could have been responsible for the large-scale properties of the universe we see around us today. But the universe itself is the ultimate closed system, and there can be no external observer to make measurements. In particular, the notion of state-vector reduction is an anathema to most people who work in quantum cosmology.

To see how the idea of decoherence arises, consider a quantum-mechanical system $\mathcal{S}$ and an observable A with eigenstates [35] $|a_1\rangle_{\mathcal{S}}, \ldots |a_N\rangle_{\mathcal{S}}$ corresponding to the eigenvalues $a_1 \ldots a_N$ of the associated self-adjoint operator $\widehat{A}$. Any (normalised) state $|\psi\rangle_{\mathcal{S}}$ in the Hilbert space of the system can be expanded as

$$|\psi\rangle_{\mathcal{S}} = \sum_{i=1}^{N} \psi_i |a_i\rangle_{\mathcal{S}} \tag{5.5.1}$$

where the complex expansion-coefficients ψ_i are given by $\psi_i = {}_{\mathcal{S}}\langle a_i|\psi\rangle_{\mathcal{S}}$. The standard interpretation is that if a measurement is made of A, then (i) the result will necessarily be one of the eigenvalues of $\widehat{A}$; and (ii) the probability of getting a particular value a_i is $|\psi_i|^2$. However, prior to the measurement, the state $|\psi\rangle_{\mathcal{S}}$ has no direct ontological interpretation vis-a-vis the observable A .

Such a view of $|\psi\rangle_{\mathcal{S}}$ may be acceptable when applied to sub-atomic systems, but it seems problematic if the system concerned is Schrödinger's unfortunate cat or, even worse, if the components in (5.5.1) represent the different terms in the sum (5.4.14) of WKB solutions to the Wheeler-DeWitt equation. One might then prefer to interpret $|\psi\rangle_{\mathcal{S}}$ as describing, for example, an ensemble of systems in which every element *possesses* a value for A, and the fraction having the value a_i is $|\psi_i|^2$; *i.e.,* an essentially classical probabilistic interpretation of the results of measuring A. However, such a situation is described quantum mechanically by the density matrix

$$\rho_{\mathrm{mix}} = \sum_{i=1}^{N} |\psi_i|^2 \, |a_i\rangle_{\mathcal{S}} \, {}_{\mathcal{S}}\langle a_i| \tag{5.5.2}$$

[35] The number of linearly-independent eigenstates N can be finite or infinite, but for simplicity I have assumed that the spectrum of $\widehat{A}$ is discrete and non-degenerate.

whereas the density matrix associated with the pure state $|\psi\rangle_{\mathcal{S}}$ is

$$\rho_\psi := |\psi\rangle_{\mathcal{S}\,\mathcal{S}}\langle\psi| = \sum_{i=1}^{N}\sum_{j=1}^{N}\psi_i\psi_j^*\,|a_i\rangle_{\mathcal{S}\,\mathcal{S}}\langle a_j| \tag{5.5.3}$$

which differs from (5.5.2) by off-diagonal terms. Of course, in the conventional interpretation of quantum theory, after the act of measurement the appropriate state *is* the mixed state ρ_{mix}, and the transformation

$$|\psi\rangle_{\mathcal{S}\,\mathcal{S}}\langle\psi| \to \rho_{\mathrm{mix}} \tag{5.5.4}$$

is what is meant by the 'reduction of the state vector'. Such a transformation can never arise from the effect of a unitary operator and hence, for example, cannot be described as the outcome of applying the Schrödinger equation to the combined system of object plus apparatus. The 'measurement problem' consists in reconciling this statement with the need to regard the constituents of an actual piece of equipment as being quantum mechanical.

The goal of decoherence is to show that there are many situations in which the replacement of the pure state $|\psi\rangle_{\mathcal{S}}$ by the mixed state ρ_{mix} can be understood as a viable consequence of the theory itself *without* the need to invoke measurement acts as a primary concept. As emphasised by Zurek, no actual quantum system is really isolated: there is always some environment $\mathcal{E}$ to which it couples and which, presumably, can also be described quantum mechanically (Zurek 1981, Zurek 1982, Zurek 1983, Zurek 1986, Zurek 1991). Suppose the environment starts in a state $|\phi\rangle_{\mathcal{E}}$ and with a coupling between the system and environment such that the environment states become correlated with those of the system, *i.e.,* $|a_i\rangle_{\mathcal{S}}|\phi\rangle_{\mathcal{E}}$ evolves to $|a_i\rangle_{\mathcal{S}}|\phi_i\rangle_{\mathcal{E}}$ (thus the environment performs an 'ideal measurement' of A). Then, by the superposition principle, if the initial state of $\mathcal{S}$ is the vector $|\psi\rangle_{\mathcal{S}}$ in (5.5.1), the state for the composite system $\mathcal{S}+\mathcal{E}$ will evolve as

$$\left(\sum_{i=1}^{N}\psi_i|a_i\rangle_{\mathcal{S}}\right)|\phi\rangle_{\mathcal{E}} \to \sum_{i=1}^{N}\psi_i|a_i\rangle_{\mathcal{S}}|\phi_i\rangle_{\mathcal{E}}. \tag{5.5.5}$$

This state is thoroughly entangled. However, if one is interested only in the properties of the system $\mathcal{S}$, the relevant state is the reduced density-matrix $\rho_{\mathcal{S}}$ obtained by summing over (*i.e.,* tracing out) the environment states. The result is

$$\rho_{\mathcal{S}} = \sum_{i=1}^{N}\sum_{j=1}^{N}\psi_i\psi_j^*|a_i\rangle_{\mathcal{S}\,\mathcal{E}}\langle\phi_i|\phi_j\rangle_{\mathcal{E}\,\mathcal{E}}\langle a_j| \tag{5.5.6}$$

which, if the environment states are approximately orthogonal (*i.e.,* ${}_{\mathcal{E}}\langle\phi_i|\phi_j\rangle_{\mathcal{E}} \simeq \delta_{ij}$), becomes

$$\rho_{\mathcal{S}} \simeq \sum_{i=1}^{N}|\psi_i|^2\,|a_i\rangle_{\mathcal{S}\,\mathcal{S}}\langle a_i|. \tag{5.5.7}$$

Thus the system behaves 'as if' a state reduction has take place to the level of accuracy reflected in the $\simeq$ sign. This is what is meant by decoherence.

The mechanism works because the evolution of the traced-out density matrix satisfies a master equation rather than being induced by a unitary evolution (Caldeira & Leggett 1983, Joos & Zeh 1985). In effect, the environment 'continuously measures' the system, and hence gives rise to a continuous process of state reduction. This, it is claimed, is the reason why Schrödinger's cat is in fact always seen to be either dead or alive, never a superposition of the two. Numerical studies have shown that the reduction of real physical systems can take place very quickly. For example, the gas molecules surrounding a piece of equipment in the laboratory will serve very well, and even the background 3^0K radiation is sufficient to produce the desired effect in a very short time: a typical figure for a macroscopic object is $10^{-23}s$ (Joos & Zeh 1985, Zurek 1986, Unruh & Zurek 1989, Unruh 1991).

Note that the actual collection of states into which the initially pure state collapses (*i.e.,* whether they are eigenstates of this or that particular observable of $\mathcal{S}$) is determined by the coupling of the system to the environment. The essential requirement is that the operator concerned should commute with the interaction Hamiltonian describing the system-environment coupling (Zurek 1981).

5.5.2 Applications to Quantum Cosmology

For the purpose of this paper, the main interest in the above is the possibility of applying these techniques to the semiclassical solutions to the Wheeler-DeWitt equation discussed in the previous section. However, some modification of these ideas is clearly required if they are to be applied in the cosmological context: by definition, the universe in its entirety is the one system which has no environment. In practice, this is done by supposing that certain modes of the gravitational or matter fields can serve as an environment for the rest. For a brief, recent review see Kiefer (1992).

The idea that classical spacetime might 'emerge' in this way was discussed in Joos (1986) and Joos (1987). The first quantum-cosmology calculation seems to have been that of Kiefer (1987) who considered a scalar field coupled to gravity in a situation in which the inhomogeneous modes serve to decohere the homogeneous modes of both fields. Later papers by this author are Kiefer (1989*a*) and Kiefer (1989*b*); for a recent review see Kiefer (1992). Halliwell (1989) considered a homogeneous, DeSitter-space metric coupled to a scalar field. The idea was that some of the inhomogeneous scalar modes can act as the environment for the metric mode. Halliwell showed that decoherence occurred for a single $e^{iS[g]}$ solution and also that, in a Hartle-Hawking semiclassical solution of the form $e^{iS[g]} + e^{-iS[g]}$, the two terms $e^{iS[g]}$ and $e^{-iS[g]}$ decohered by the same mechanism; see also Morikawa (1989). Other relevant work is Mellor (1991), who discusses decoherence in the context of a Klein-Kaluza model, and Padmanabhan (1989*a*) who studied the possibility of defining decoherence between different three-geometries in Riem(Σ).

The concept of decoherence is very interesting and could be of considerable importance in the context of quantum cosmology and the problem of time. However, several non-trivial problems arise that are peculiar to the use of this concept in quantum gravity and which need more study (see Kuchař (1992*b*) for a further critique). In particular:

1. There does not seem to be any general way of deciding how to separate the modes into those that are to be kept and those that are to serve as an environment.

2. In ordinary quantum theory, the transformation into the mixed state (5.5.6) is only valid so long as the environment modes are deliberately ignored: the 'true' density matrix is the one associated with the pure state (5.5.5) and still contains the off-diagonal interference terms. The transformation from pure to mixed is therefore only 'as if' and, although this may be more than adequate for normal purposes, it is not really clear what is going on in the case of quantum cosmology. There may be situations in which information is genuinely lost: for example down an event horizon, as in the suggestion by Hawking (1982) that the presence of a black hole can transform a pure state into a mixed state, and some of the models discussed in the literature involve a similar process using a cosmological event horizon (*e.g.,* Halliwell (1987)). However, most of the calculations that have been performed are not of this type, and their meaning remains unclear.

3. A rather serious technical problem arises when trying to adapt the ideas of decoherence to quantum gravity. The process of tracing-out modes requires a Hilbert space in which to take the traces. But, as we have seen, the Hilbert space problem for the Wheeler-DeWitt equation is still unsolved except, perhaps, in so far as the semi-classical Schrödinger equation (5.4.13) comes associated with a natural inner product. However, this equation arises only *after* the decision has been made to select a single $e^{iS[g]}$ solution, and so the corresponding inner product cannot be used to perform operator tracing in the calculation of a claimed decoherence effect. In practice, many people resort to the simple inner product (5.2.1) but, as we shall see in §6.1, this brings problems of its own and is hard to justify.

4. This problem is related to various opaque features that arise when decoherence is applied in discussions of the problem of time. In particular, there is a tendency to show (or try to show) that time itself is something that decoheres—in sharp contradistinction to normal quantum theory where decoherence is a *process* that happens *in* time. This is a natural consequence of the fact that time is not an external parameter in quantum gravity but rather something that must be constructed in some internal way. However, it is by no means clear if, or in what sense, time is represented by an operator in quantum gravity as is suggested by the idea that it decoheres. Indeed, this is one of the main distinctions between the internal Schrödinger interpretation and those interpretations based on the Wheeler-DeWitt equation. This difficult concept of decohering time is deeply connected with the Hilbert space problem for the Wheeler-DeWitt equation, and is an area in which the consistent histories interpretation (to be discussed in §6.3) may have something to offer.

6 TIMELESS INTERPRETATIONS OF QUANTUM GRAVITY

6.1 The Naïve Schrödinger Interpretation

The general philosophy behind all 'timeless' interpretations of quantum gravity is the belief that it should be possible to construct a well-defined quantum formalism without the need to make a specific identification of time at any stage. Approaches of this type invariably employ some sort of 'internal clock' to measure the passage of time, but such a notion of time is understood to be purely phenomenological, and hence of no fundamental conceptual or technical significance. In particular, the choice of time plays no basic role in the construction of the theory. It is fully accepted that such a phenomenological time may only approximate the external time of Newtonian physics and that as a consequence a Schrödinger equation may arise at best as an approximate description of dynamical evolution. However, it is affirmed that the theory nonetheless admits a precise probabilistic interpretation with a well-defined Hilbert space structure.

The simplest example of such a scheme is the 'naïve [36] Schrödinger interpretation' whose central claim is that quantum gravity should be approached by quantising before constraining, and that the physically-correct inner product is (5.2.1)

$$\langle\Psi|\Phi\rangle := \int_{\mathrm{Riem}(\Sigma)} \mathcal{D}g\, \Psi^*[g]\, \Phi[g] \tag{6.1.1}$$

in which the measure $\mathcal{D}g$ is defined [37] on the space $\mathrm{Riem}(\Sigma)$ of Riemannian metrics on the three-manifold Σ.

At a first glance, the use of the scalar product (6.1.1) seems rather natural. After all, $L^2(\mathrm{Riem}(\Sigma), \mathcal{D}g)$ is the Hilbert space on which the canonical operators (5.1.14) and (5.1.15) are formally self-adjoint. Indeed, by starting with the canonical commutation relations (5.1.1–5.1.3) (or their affine generalisations (5.1.6–5.1.8)), the spectral theory associated with the abelian algebra generated by the commuting operators $\widehat{g}_{ab}(x)$ means the (rigorous version of the) scalar product (6.1.1) is bound to enter the theory somewhere. The same Hilbert space is also used (sometimes implicitly) in discussions of whether or not the constraint operators $\widehat{\mathcal{H}_\perp}(x)$ and $\widehat{\mathcal{H}_a}(x)$ are self-adjoint.

The scalar product (6.1.1) defines a large class of square-integrable functions of $g_{ab}(x)$, but many of these are deemed to be unphysical. More precisely, the constraints (5.1.9–5.1.10)

$$\mathcal{H}_a(x;\widehat{g},\widehat{p}\,)\Psi = 0 \tag{6.1.2}$$

$$\mathcal{H}_\perp(x;\widehat{g},\widehat{p}\,)\Psi = 0 \tag{6.1.3}$$

[36] The appellation 'naïve' was coined by Unruh & Wald (1989).

[37] But recall my earlier caveats about the need to use distributional metrics, the problem of constructing a proper measure, *etc.*

serve to project out the 'physical' subspace of the Hilbert space $L^2(\mathrm{Riem}(\Sigma), \mathcal{D}g)$ of such functions. Of course, (6.1.3) reproduces the Wheeler-DeWitt equation (5.1.17). However, and unlike—for example—in the Klein-Gordon interpretation, the scalar product (6.1.1) is assumed to have a direct physical meaning with no specific reference to the Wheeler-DeWitt equation. This basic interpretation of $\Psi[g]$ is that the probability of 'finding' a hypersurface in $\mathcal{M}$ on which the three-metric g lies in the measurable subset B of $\mathrm{Riem}(\Sigma)$ is

$$\mathrm{Prob}(g \in B; \Psi) = \int_B \mathcal{D}g \, |\Psi[g]|^2. \tag{6.1.4}$$

This interpretation has often been used in studies of quantum cosmology, especially by Hawking and collaborators; for example Hartle & Hawking (1983), Hawking (1984*b*), Hawking (1984*a*), Hawking & Page (1986) and Hawking & Page (1988) (see also Castagnino (1988)). It has some attractive properties, not least of which is its simplicity and the fact that, unlike the Klein-Gordon pairing (5.2.12), (6.1.1) defines a genuine, positive-definite scalar product (modulo mathematical problems in constructing a proper measure theory). This interpretation also gives a clear 'wave-packet' picture of how a classical spacetime arises: one can say that the state $\Psi[g]$ is related to a specific Lorentzian spacetime γ if $\Psi[g]$ vanishes on almost every metric g except those that correspond to the restriction of γ to some spacelike hypersurface.

Notwithstanding these advantages, the naïve Schrödinger interpretation has some peculiar features that stem from the 'timeless' nature of the description. This is illustrated by the example above of how to recover a classical spacetime: it is the whole *spacetime* that is described by Ψ, not just the configuration of the physical variables on a single time slice. In general, a typical application of the naïve Schrödinger interpretation is to pose questions of the type 'What is the probability of finding this or that universe?' rather than questions dealing with this or that *evolution* of the same universe.

The issue becomes clearer if we think more carefully about what is intended by the statement that (6.1.4) is the probability of 'finding' g in some subset of $\mathrm{Riem}(\Sigma)$. By analogy with normal quantum theory, one would expect to talk about 'measuring' the three-metric, but it is hard to see what this means. As we have emphasised several times, measurements are usually made at a single value of a time parameter, and the results of time-ordered sequences of such measurements provide the dynamical evolution of the system. But such language is inappropriate here: the time parameter cannot be fixed since, in effect, it is *part* of the metric $g_{ab}(x)$. We might drop the use of measurement language and adopt a more realist stance by saying that (6.1.4) is the probability of g *being in* the subset B of $\mathrm{Riem}(\Sigma)$, although, to make sense, this probably needs to be augmented with some sort of many-worlds interpretation of the quantum theory.

However, the structure is still peculiar. This is partly because, as it stands, the interpretation given above could apply to *any* function $\Psi[g]$ of the $6\times\infty^3$ variables $g_{ab}(x)$, $x \in \Sigma$, that are needed to specify a three-metric. The supermomentum constraints

$\widehat{\mathcal{H}}_a(x)\Psi = 0$ remove $3 \times \infty^3$ variables [38], which leaves $3 \times \infty^3$. However, to specify a physical configuration of the gravitational field requires $2 \times \infty^3$ variables, and so the functions $\Psi[g]$ depend on an extra $1 \times \infty^3$ variables which, of course, correspond to an internal time function $\mathcal{T}(x, g]$. Thus the time variable is part of the configuration space $\mathrm{Riem}(\Sigma)$ and, in effect, is represented by an operator; that is why we get a timeless interpretation. For example, in the simple minisuperspace model discussed in §5.1.6, the wave function $\psi(\Omega, \phi)$ is interpreted as the probability amplitude for finding a matter configuration ϕ *and* a radius $a = \ln \Omega$. Note that the imposition of the super-Hamiltonian constraint $\widehat{\mathcal{H}}_\perp \Psi = 0$ does not remove this extra configuration variable (as it would if the constraint function $\mathcal{H}_\perp$ was linear, rather than quadratic, in $p^{ab}(x)$) but leads instead to the Wheeler-DeWitt equation on $\Psi[g]$.

The analogue in ordinary wave mechanics would be to interpret $|\psi(x,t)|^2$ as the probability density of 'finding the particle at point x *and* time to be t', in which t is regarded as an eigenvalue of some time operator $\widehat{T}$. The Schrödinger equation then seems to follow from imposing the constraint

$$(\widehat{p}_T + H(t, \widehat{x}, \widehat{p})\psi)(x,t) = 0 \tag{6.1.5}$$

on allowed state vectors with $\widehat{p}_T := i\hbar\partial/\partial t$. However, the conventional interpretation of such a state is that, for *fixed* t, $|\psi(x,t)|^2$ is the probability distribution in x. Thus the naïve Schrödinger interpretation of quantum gravity is based on an idea that involves a significant change in the quantum formalism. Note also that any solution to (6.1.5) will not be square-integrable in x and t, and hence, at best, it is possible to talk about *relative* probabilities only. This arises because the spectrum of the operator $\widehat{p}_T + \widehat{H}$ on $L^2(\mathbb{R}^2, dx\, dt)$ is *continuous*, and hence its eigenstates are not normalisable.

A notable property of this construction is that the time operator $\widehat{T}$ does not commute with the constraint operator $\widehat{p}_T + H(t, \widehat{x}, \widehat{p})$. In the quantum gravity case this is reflected in the fact that the projection operator onto a measurable subset B of $\mathrm{Riem}(\Sigma)$ does not commute with $\widehat{\mathcal{H}}_\perp(x)$ (since $[\,\widehat{g}_{ab}(x'), \widehat{\mathcal{H}}_\perp(x)\,] \neq 0$). In this sense, the naïve Schrödinger interpretation of probability is inconsistent with the Wheeler-DeWitt equation unless one studiously avoids asking about the state function *after* a hypersurface has been found with a given three-metric. What is at stake here is the crucial question of what is meant by an 'observable' $\widehat{A}$, and what role the concept plays in the construction of the physical Hilbert space. There is no problem if this means only that $[\,\widehat{A}, \widehat{\mathcal{H}}_a(x)\,] = 0$: one merely passes to the version of the formalism in which the states are defined on $\mathrm{Riem}(\Sigma)/\mathrm{Diff}(\Sigma)$. The difficulty arises if it is required in addition that $\widehat{A}$ commutes with the super-Hamiltonian operators $\widehat{\mathcal{H}}_\perp(x)$. It does not help to impose the weaker condition that $[\,\widehat{A}, \widehat{\mathcal{H}}_\perp(x)\,] = 0$ only on physical states Ψ that satisfy $\widehat{\mathcal{H}}_\perp(x)\Psi = 0$:

[38] The classical constraint functions $\mathcal{H}_a(x)$ generate the $\mathrm{Diff}(\Sigma)$ action on $\mathrm{Riem}(\Sigma)$ and fibre it into orbits. Then, modulo θ-vacuum effects, the constraints $\widehat{\mathcal{H}}_a(x)\Psi = 0$ can be interpreted as saying that the theory is really defined on the superspace $\mathrm{Riem}(\Sigma)/\mathrm{Diff}(\Sigma)$ of such orbits, in which case the inner product (6.1.1) must be replaced with an integral over $\mathrm{Riem}(\Sigma)/\mathrm{Diff}(\Sigma)$.

the operator $\widehat{g}_{ab}(x)$ maps such a state into one that is not annihilated by the super-Hamiltonian operator, and this is incompatible even with the weak condition.

Another problem is that, analogously to the operator $\widehat{p}_T + H(t, \widehat{x}, \widehat{p})$ in (6.1.5), the operators $\widehat{\mathcal{H}_\perp}(x)$, $x \in \Sigma$, defined on the Hilbert space $L^2(\mathrm{Riem}(\Sigma), \mathcal{D}g)$ can be expected to have continuous spectra, and so the solutions to the Wheeler-DeWitt equation are not normalisable. Thus the inner product (6.1.1) does not induce an inner product on the physical states. This is a serious difficulty for the naïve Schrödinger interpretation and is one of the main attractions of the Klein-Gordon programme in which the aim is to define a scalar product only on *solutions* to the Wheeler-DeWitt equation, not on a general function of $g_{ab}(x)$.

6.2 The Conditional Probability Interpretation

6.2.1 The Main Ideas

The conditional probability interpretation is a development of the naïve Schrödinger approach that has been studied especially by Page and Wooters (Page & Wooters 1983, Hawking 1984*a*, Wooters 1984, Page 1986*b*, Page 1986*a*, Page 1989, Page 1991, Page & Hotke-Page 1992); see also Englert (1989), Deutsch (1990), Squires (1991) and Collins & Squires (1992). The Hilbert space is the one used before, *i.e.,* the space of all functionals $\Psi[g]$ that are square-integrable with respect to the inner product (6.1.1). However, the interpretation is different: $|\Psi[g]|^2$ is no longer regarded as the absolute probability density of finding a three-metric g but is instead thought of as the probability of finding the $2 \times \infty^3$ physical modes of g *conditional* on the remaining $1 \times \infty^3$ variables—the internal-time part of g—being equal to some specific function (I am assuming that the constraints $\widehat{\mathcal{H}_a}(x)\Psi = 0$ have already been solved). The claim or hope is that this does not require any specific split of g into physical parts and an internal time function; indeed, the interpretation is supposed to be correct for any such choice. Thus in the minisuperspace model in §5.1.6, the (suitably-normalised) wave-function $|\psi(\Omega, \phi)|^2$ can be regarded equally as the probability density in ϕ at fixed Ω (*i.e.,* regarding Ω as an internal metric-field time) or as the probability density in $\Omega = \ln a$ at fixed ϕ (*i.e.,* regarding ϕ as a matter field time).

The original work of Page and Wooters was not aimed at quantum cosmology alone but at the more general problem of the quantisation of any closed system. They argued, *pace* Bohr, that in the normal Schrödinger equation the time parameter t is an external parameter, and hence has no place in the quantum theory if the system is truly closed. Instead, time must be measured with a physical clock that is part of the system itself. Their interpretation of $|\psi(x, t)|^2$ is that it gives the probability distribution in x conditional on the value of the internal clock being t. The normal Schrödinger time-dependent equation is replaced by an eigenvalue equation

$$\widehat{H}_{\mathrm{tot}}\psi = E\psi \tag{6.2.1}$$

where H_{tot} is the Hamiltonian of the total system, which includes the physical clock and its interaction with the rest of the system. Whatever can be said about 'time development' has to be extracted from this equation. This is done by studying the dependence of conditional probabilities on the value of the internal-clock variable on which the probabilities are conditioned.

The relevance for quantum gravity of these ideas should be clear. The eigenvalue equation (6.2.1) becomes the super-Hamiltonian constraints (6.1.3), and the conditioning is on the internal time functional $\mathcal{T}(x, g]$ being equal to a specific function $\tau(x)$; *i.e.,* the internal clock is defined by a configuration of the gravitational field. Of course, if matter fields are present, they also can serve to define an internal time. Note that, as in the naïve Schrödinger interpretation, the internal clock is represented by a genuine operator on the Hilbert space of the total system.

6.2.2 Conditional Probabilities in Conventional Quantum Theory

To understand the novel features of this idea is it useful to recall briefly how the notion of conditional probability enters conventional quantum theory; this will also be helpful in our discussion in §6.3 of the consistent histories interpretation.

Let the (mixed) state of a quantum system at some time $t = 0$ be ρ_0. In the Schrödinger representation, the state ρ_t at time t is related to the $t = 0$ state by the unitary transformation

$$\rho_t = U(t)\,\rho_0\,U(t)^{-1} \tag{6.2.2}$$

where $U(t) := \exp(-it\widehat{H}/\hbar)$. Therefore, if a measurement of an observable A is made at time t_1, the probability that the result will lie in some subset α of the eigenvalue spectrum of the operator $\widehat{A}$ is

$$\begin{aligned}\mathrm{Prob}(A \in \alpha, t_1; \rho_0) &= \mathrm{tr}(P^A_\alpha\,\rho_{t_1}) \\ &= \mathrm{tr}(P^A_\alpha(t_1)\,\rho_0)\end{aligned} \tag{6.2.3}$$

where $P^A_\alpha(t_1)$ is the Heisenberg-picture operator defined by

$$P^A_\alpha(t_1) := U(t_1)^{-1}\,P^A_\alpha\,U(t_1) \tag{6.2.4}$$

(with the reference time chosen to be $t = 0$) and P^A_α is the operator that projects onto the subset α; for example, if the spectrum of $\widehat{A}$ is a set $a_1, \ldots a_N$ of non-degenerate discrete eigenvalues, then

$$P^A_\alpha := \sum_{a_i \in \alpha} |a_i\rangle\langle a_i|. \tag{6.2.5}$$

If the measurement of A yields a result lying in α, any further predictions must be made using the density matrix

$$\rho_\alpha := \frac{P^A_\alpha(t_1)\,\rho_0\,P^A_\alpha(t_1)}{\mathrm{tr}(P^A_\alpha(t_1)\,\rho_0)} \tag{6.2.6}$$

and the transformation

$$\rho_{t_1} \to \rho_\alpha = \frac{P^A_\alpha(t_1)\,\rho_0\,P^A_\alpha(t_1)}{\mathrm{tr}(P^A_\alpha(t_1)\,\rho_0)} \tag{6.2.7}$$

is the analogue for density matrices of the familiar reduction of the state vector.

Now let the system evolve until time t_2 when a measurement of an observable B is made. According to the discussion above, the probability of finding B in a range β, given that (*i.e.,* conditional on) A was found to be in α at time t_1, is

$$\mathrm{Prob}(B \in \beta, t_2 \,|\, A \in \alpha, t_1; \rho_0) = \mathrm{tr}(P^B_\beta(t_2)\rho_\alpha) = \frac{\mathrm{tr}(P^B_\beta(t_2)\,P^A_\alpha(t_1)\,\rho_0\,P^A_\alpha(t_1))}{\mathrm{tr}(P^A_\alpha(t_1)\,\rho_0)}. \tag{6.2.8}$$

6.2.3 The Timeless Extension

The extension of the ideas above to the situation in which there is no external time parameter proceeds as follows. The physical observables in the theory are regarded as operators that commute with the total Hamiltonian H_{tot}. As a consequence, there is no difference between the Heisenberg and the Schrödinger pictures of time evolution. Indeed, there is no time evolution at all in the sense of a change with respect to any external parameter t; in particular, the density matrix ρ of the system satisfies $[\,H_{\mathrm{tot}}, \rho\,] = 0$: a truly 'frozen' formalism. Nevertheless, it is assumed that much of the framework of conventional quantum mechanics is still applicable. In particular, it is deemed meaningful to talk about the conditional probability of finding B in the range β, given that A lies in α, and to assert that, if the state of the system is ρ, the value of this quantity is

$$\mathrm{Prob}(B \in \beta \,|\, A \in \alpha; \rho) = \frac{\mathrm{tr}(P^B_\beta\,P^A_\alpha\,\rho\,P^A_\alpha)}{\mathrm{tr}(P^A_\alpha\,\rho)}. \tag{6.2.9}$$

The extension to the quantum gravity situation is obvious and, again, there is no external time parameter.

The suggested form (6.2.9) should be compared carefully with the expression (6.2.8) of conventional quantum theory. There are no t-labels in (6.2.9) and therefore, in particular, no sense in which the quantities B and A are time-ordered (as they are in (6.2.8), with $t_2 > t_1 > t_0$). Furthermore, and unlike the case in conventional quantum theory, the expression (6.2.9) is not obtained via any process of state reduction. Rather it is simply *postulated* as one of the fundamental interpretative rules of the theory. Correspondingly, the concept of 'measurement' is only a secondary one: like time, it is not something that comes from 'outside' but is instead only a way of talking about a particular type of interaction between certain sub-elements of the closed system. As a consequence we are confronted almost inevitably with a many-worlds interpretation of the theory; indeed, supporters of the conditional probability interpretation of quantum gravity are almost always strong advocates of a post-Everett view of quantum mechanics.

Thus we have a timeless picture of quantum theory. This does not mean that the notion of time-evolution is devoid of any content, but the challenge is to recover it in some way from the conditional probability expression. This is done as follows. Let T be a quantity that we wish to use as an internal clock to measure the change in another quantity A. Then we study the probability $\mathrm{Prob}(A \in \alpha | T = \tau; \rho)$ and see how it varies with τ. This is the dynamical evolution in the theory.

The conditional probability interpretation is certainly attractive. It captures nicely the idea that the passage of time should be identified with correlations inside the system rather than reflecting changes with respect to an external parameter. In this respect it seems well-suited for application to the problem of time in quantum gravity, and it is certainly an improvement on the naïve Schrödinger interpretation. However, this new interpretation gives rise to various problems of its own, and these deserve careful consideration.

1. The central difficulty is that the probabilistic rule (6.2.9) constitutes a significant departure from conventional quantum theory, and the resulting structure may not be self-consistent. In particular, we need clear guidelines for deciding when a variable has the property that it is appropriate to condition on its values. For example, if applied to conventional quantum theory where there *is* an external time parameter t, the formalism makes sense only if T is a 'good' clock in the sense discussed in §2. If T is a bad clock, and hence can take on the same value τ at two different values of t, the probability of A conditional on $T = \tau$ is not well-defined. For a closed system, this raises the general question of how we know whether or not a particular quantity affords a consistent choice for an internal time variable.

2. Another feature of the conditional probability interpretation is the peculiar lack of any sense of history. That is, there is no way of directly comparing things at different times. All statements are of the form 'the probability of A is this, when B is that' and, in that sense, always refer to the single 'now' at which the statement is made. Page defends the situation by citing the general philosophical position that statements about the past are really counter-factual claims that certain consistency conditions would be met if *present* records are examined. However, not everyone is convinced by this argument; in particular see the critical analysis in Kuchař (1992*b*) and the discussion between Page and Kuchař reported in Page & Hotke-Page (1992)

3. There is also a potential problem with the constraint $\widehat{H}_{\rm tot}\psi = E\psi$ or, in the gravitational case, the super-Hamiltonian constraint $\widehat{\mathcal{H}}_\perp(x)\Psi = 0$. As emphasised in Kuchař (1992*b*), if an internal clock is to function as such, it *cannot* commute with the constraint operator, and—in that sense—it is not a physical observable. In the gravitational context, this means that any derivation of (6.2.9) based on a state reduction with an internal time $\mathcal{T}(x, g]$ taking the value $\tau(x)$,

$$\rho \to P_\tau^{\mathcal{T}}\, \rho\, P_\tau^{\mathcal{T}} / \mathrm{tr}(P_\tau^{\mathcal{T}}\, \rho) \tag{6.2.10}$$

would be incompatible with the Wheeler-DeWitt equation for density matrices (which is $[\,\mathcal{H}_\perp(x), \rho\,] = 0$) since the reduced ρ violates this condition. In so far as (6.2.9) is simply *postulated*, this may not be a problem—indeed, expressions of this type are used frequently in the many-worlds interpretation of standard quantum theory without invoking the idea of a collapse—but, together with the absence of any time labels, it does show the extent to which the conditional probability interpretation deviates from conventional quantum ideas.

4. Finally, we still have the *spacetime* problem since in the form above the programme uses an internal time $\mathcal{T}(x, g]$ which, as has been mentioned several times, cannot lead to a local scalar function on the spacetime manifold $\mathcal{M}$. However, this problem might be addressed by applying the conditional probability interpretation to a system that includes the type of matter reference fluids discussed in §4.3.

6.3 The Consistent Histories Interpretation

6.3.1 Preamble

In some respects, the consistent histories approach to the problem of time can be regarded as a development of the conditional probability interpretation, with the particular advantage of enabling questions about the history of the system to be addressed directly. In particular, it takes account of the fact that decoherence is really a *process* that develops in time and that, for example, a system that has decohered could in principle recohere at some later time. The scheme culminates in a suggestion that gravity should be quantised with something like a functional integral over spacetime geometries.

In itself, this does not seem so novel: formal quantisation schemes of this type have been considered for many years and inevitably founder on intractable technical problems. Nor is it obvious how such an approach solves the problem of time, or thows any light on the related question of constructing the Hilbert space of states. For example, one way of defining a real-time functional integral is to start with the canonical Hilbert space quantum theory and then define the integral as the limit of a time-sliced approximation to some matrix element of the unitary evolution operator (*i.e.,* using the Trotter product formula). But this is no help in a situation in which the Hilbert space structure is one of the things we are trying to discover. The rival euclidean approach to quantum gravity does not help much either. For example, in the Hartle-Hawking ansatz the central object is the functional integral over all euclidean-signature metrics γ on a four-manifold $\mathcal{M}$ with a single three-boundary Σ

$$\Psi[g] := \int \mathcal{D}\gamma\, e^{-iS_E[\gamma]/\hbar}, \tag{6.3.1}$$

where S_E is the euclidean action, and where γ restricted to Σ is the given three-metric g (Hartle & Hawking 1983). The function $\Psi[g]$ can be shown formally to satisfy the Wheeler-Dewitt equation (Halliwell 1988, Halliwell 1991*b*), and this particular state is then regarded as the 'wave-function of the universe'. But this does not help with the problem of time since we are simply faced once more with the difficult question of how to interpret solutions of the Wheeler-DeWitt equation.

However, there is far more to the idea of consistent histories than simply a call to return to spacetime functional integrals. It stems from what is in fact a radical revision of the formalism of quantum theory in general. I must admit, when I first came across the idea I was not very enthusiastic. But my feelings have undergone a major change recently and I am inclined now [39] to rate it as one of the most significant developments in quantum theory during the last 25 years.

The consistent histories interpretation is a thorough-going post-Everett scheme in the sense that measurement is not a fundamental category; instead the theory itself prescribes when it is meaningful to say that a measurement has taken place. In particular, there is no external reduction of the state vector. In the original formalism, 'time' appears in the standard way as an external parameter. However, the relegation of measurement to an internal property gives rise to the hope that time may be treated likewise; indeed, for our present purposes, this is one of the main attractions of the approach.

6.3.2 Consistent Histories in Conventional Quantum Theory

Let us start by summarising the idea in the context of non-gravitational quantum physics. The seminal papers are Griffiths (1984), Omnès (1988a, 1988b, 1988c, 1989, 1990), Gell-Mann (1987) and Gell-Mann &Hartle (1990a, 1990b, 1990c). Comprehensive recent reviews of both the gravitational and the non-gravitational case are Hartle (1991*a*) and Omnès (1992); see also Alberich (1990, 1991, 1992), Blencowe (1991), Dowker & Halliwell (1992) and Halliwell (1992*b*).

Consider first the description in conventional quantum theory of the process of making a series of measurements separated in time. Each measurement can be regarded as asking a set of questions—the answer to which is either yes or no—and each such question is represented by a hermitian projection operator; typically the projection onto a subset of the spectrum of some operator $\widehat{A}$ (so that the question is 'does the value of A lie in the given subset?'). Let Q denote the set of all questions pertaining to a particular measurement. Then if q is one such question, the associated projection operator will be written P^Q_q. We want the set of questions to be mutually exclusive (*i.e.,* the answer to at most one question is 'yes') and exhaustive (*i.e.,* the answer to at least one question is 'yes'), which means the projection operators must satisfy

$$P^Q_q\, P^Q_{q'} = \delta_{q\,q'}\, P^Q_q \tag{6.3.2}$$

and

$$\sum_{q\in Q} P^Q_q = 1. \tag{6.3.3}$$

[39] This enantiadromia was entirely the result of gentle pressure from Jim Hartle encouraging me to read the original papers and to think about them carefully. I most grateful to him for his efforts in this direction!

Now consider making a series of measurements at times $t_1 < t_2 \ldots < t_N$ with corresponding sets $Q_1, Q_2, \ldots, Q_N$ of yes-no questions. The quantity of interest is the absolute probability $\mathrm{Prob}(q_N\, t_N, q_{N-1}\, t_{N-1}, \ldots, q_1\, t_1; \rho_0)$ of obtaining 'yes' to questions $q_1 \in Q_1$ at time t_1, $q_2 \in Q_2$ at time t_2, …, $q_N \in Q_n$ at time t_N, given that the state at time $0 \leq t_1$ was the density matrix ρ_0. The discussion of conditional probabilities in §6.2 leading to (6.2.9) can be extended to show that

$$\begin{aligned}&\mathrm{Prob}(q_N\, t_N, q_{N-1}\, t_{N-1}, \ldots q_1\, t_1; \rho_0) = \\ &\qquad \mathrm{tr}(P^{Q_N}_{q_N}(t_N)\, P^{Q_{N-1}}_{q_{N-1}}(t_{N-1}) \ldots P^{Q_1}_{q_1}(t_1)\, \rho_0\, P^{Q_1}_{q_1}(t_1)\ \ldots P^{Q_{N-1}}_{q_{N-1}}(t_{N-1})\, P^{Q_N}_{q_N}(t_N))\end{aligned} \tag{6.3.4}$$

where the projection operators are in the Heisenberg picture as defined by (6.2.2).

It must be emphasised that (6.3.4) is derived using conventional ideas of sequences of measurements and associated state-vector reductions like (6.2.7). However, the intention of the consistent histories interpretation of quantum theory is to sidestep this language completely by talking directly about the probability of a history; the idea of 'measurement' is then regarded as a secondary concept that can be described using the history language applied to the entire system (*i.e.,* including what used to be regarded as an observer). For this reason, following the example of John Bell, I shall talk about a hermitian operator $\widehat{A}$ representing a 'beable' rather than an 'observable'. Used in this way, a 'history' means any sequence of projection operators

$$P^{Q_N}_{q_N}(t_N)\, P^{Q_{N-1}}_{q_{N-1}}(t_{N-1}) \ldots P^{Q_1}_{q_1}(t_1) \tag{6.3.5}$$

satisfying the conditions (6.3.2–6.3.3). Note that this is a considerable generalisation of the notion of history as used in a standard path integral where it usually means a path in the configuration space of the system. A history of this particular type can regarded as a limit of a sequence of histories (6.3.5) of a special type in which the projection operators project onto vanishingly small regions of the configuration space, and the separation between time points tends to zero.

The desire to assign probabilities to histories is initially frustrated by the fact that this is precisely what *cannot* be done in conventional quantum theory. All that is possible there is to give a probability *amplitude* for a history, but then the passage to the probability itself introduces interference terms between different histories. This is seen most clearly in the Feynman path-integral approach. The amplitudes for paths $a(t)$, $b(t)$ in configuration space are $A[a] := e^{iS[a]/\hbar}$ and $A[b] := e^{iS[b]/\hbar}$ respectively, where $S[a]$ denotes the classical action evaluated on the path a. But then, generally speaking, $|\, A[a] + A[b]\,|^2 \neq |\, A[a]\,|^2 + |\, A[b]\,|^2$ because of the interference term $|\, A[a]\, A[b]\,|$. The classic example is the two-slit experiment.

The central idea in the consistent histories approach is that, although generic histories cannot be assigned probabilities, this may be possible for certain special families of histories: the so-called 'consistent" families. The key technical ingredient is the *decoherence functional* $D(h', h)$ which is a function of pairs of histories h', h associated to the same collection of questions $Q_1, Q_2, \ldots, Q_N$. If

$$h' := P^{Q_N}_{q'_N}(t_N)\, P^{Q_{N-1}}_{q'_{N-1}}(t_{N-1}) \ldots P^{Q_1}_{q'_1}(t_1) \tag{6.3.6}$$

and

$$h := P^{Q_N}_{q_N}(t_N)\, P^{Q_{N-1}}_{q_{N-1}}(t_{N-1}) \ldots P^{Q_1}_{q_1}(t_1) \tag{6.3.7}$$

then $D(h', h)$ is defined by

$$D(h', h) := \mathrm{tr}(P^{Q_N}_{q'_N}(t_N)\, P^{Q_{N-1}}_{q'_{N-1}}(t_{N-1}) \ldots P^{Q_1}_{q'_1}(t_1)\, \rho_0\, P^{Q_1}_{q_1}(t_1)\, \ldots P^{Q_{N-1}}_{q_{N-1}}(t_{N-1})\, P^{Q_N}_{q_N}(t_N)) \tag{6.3.8}$$

which provides a good measure of the size of the interference terms between the two histories. [40] The family of histories is said to be *consistent* if $D(h', h) = 0$ for all pairs h, h' for which $h \neq h'$. If this is so, we *assign* the probability to h given by (6.3.4). Thus, in this approach to quantum theory, the fundamental interpretative rule is

$$D(h', h) = \delta_{h'\,h}\, \mathrm{Prob}(q_N\, t_N, q_{N-1}\, t_{N-1}, \ldots q_1\, t_1; \rho_0) \tag{6.3.9}$$

where $\mathrm{Prob}(q_N\, t_N, q_{N-1}\, t_{N-1}, \ldots q_1\, t_1; \rho_0)$ is *defined* by (6.3.4); but note again that this probability is assigned to the history *only* if the consistency conditions (6.3.9) are satisfied [41] for all histories in the family under consideration. It is straightforward to show that probabilities arrived at in this way obey all the basic rules of classical probability theory. It must be emphasised that the decoherence functional is *computed* using the mathematical techniques of standard quantum theory: it is only the probability *interpretation* that is new.

The main task is to find families of consistent histories. In practice, one may decide that exact consistency is not needed: it may be sufficient if (6.3.9) is *approximately* true [42], although the degree of approximation that is deemed appropriate will depend on the physical situation involved. However, even with this weakened requirement most families of histories will not satisfy the consistency condition. The most discriminating sets of projection operators $\{P^Q_q \mid q \in Q\}$ are those in which each operator P^Q_q projects onto a one-dimensional range. This would happen if $\{P^Q_q \mid q \in Q\}$ is the set of spectral projection operators for a complete set of commuting 'beables' with discrete eigenvalue spectra. Families of histories associated with collections $Q_1, Q_2, \ldots, Q_N$ of sets Q_i of questions of this type are least likely to be consistent. To gain consistency starting with such a family it will be necessary to *coarse-grain* the histories—another important concept in the general programme.

To coarse-grain a set Q of questions means to partition Q into subsets of less precise questions. If $\bar{Q}$ denotes the new set of questions, and if $\bar{q}$ is one of the partitions, then

[40] This is most easily seen by considering the special case where ρ is a pure state $|\psi\rangle\langle\psi|$.

[41] In his original paper, Griffiths showed that it is sufficient if the *real* part of the off-diagonal parts of $D(h', h)$ vanish.

[42] This raises the intriguing notion of 'approximate probabilities'.

the projection operator corresponding to the new question $\bar{q}$ ('do any of the questions in the set $\{q \in \bar{q} \subset Q\}$ have the answer 'yes'?') is

$$P^{\bar{Q}}_{\bar{q}} = \sum_{q \in \bar{q}} P^{Q}_{q}, \tag{6.3.10}$$

and the associated decoherence functional is

$$D(\bar{h}', \bar{h}) = \sum_{h' \in \bar{h}'} \sum_{h \in \bar{h}} D(h', h). \tag{6.3.11}$$

The idea is that with an appropriate coarse-graining this new set of histories may be consistent. One extreme act of coarse-graining is to choose a single partition for one of the questions, at time t_j say. This results in the trivial question whose answer is always 'yes' and is represented by the unit operator; in effect that particular time t_j is removed from the sequence. Of course, there is a converse in which a family of histories can be 'fine-grained' by inserting a set of questions at a time intermediate between a consecutive pair in the original family.

The conventional interpretation of quantum physics can be recovered using the idea of a 'quasi-classical domain'. A quantum theory is said to have a quasi-classical domain if there exists a consistent family of histories with the property that the values of certain, sufficiently coarse-grained 'beables' are correlated in time in a way that reproduces the equations of some piece of classical physics. Such variables could include the coarse-grained features of actual pieces of measuring equipment, with the histories involved describing, for example, the production of persistent records: a property that has frequently been seen as the signature of a successful 'measurement'.

It must be emphasised that consistency is a property of a complete *family* of histories. Many different such families may exist, giving different perspectives on the picture of reality portrayed by quantum theory. Properties like complementarity arise from the existence of families that are mutually incompatible. In a situation like this, a 'many-worlds' (or, 'many-histories') interpretation of quantum theory seems inevitable. But note that the 'many histories' involved come not only from the different histories associated with a fixed collection of questions $Q_1, Q_2, \ldots Q_N$; the collections of questions themselves are also variable—any collection leading to a consistent family of histories is admissible.

6.3.3 The Application to Quantum Gravity

The consistency condition (6.3.9) depends on the state ρ_0; in particular, this is true of the existence of quasi-classical domains. As emphasised by Gell-Mann and Hartle, this implies that the manifest existence in our current world of a quasi-classical domain depends ultimately on the initial state ρ_0 that existed shortly after the 'initial' big-bang. From this perspective, the classical features of our present-day world must be seen as a contingent property of the big-bang: they could have been otherwise. Indeed, for all we know, the quantum theory of our universe may admit other consistent families of histories with no quasi-classical domains at all; a valid concept in the context of a post-Everett interpretation of quantum theory.

Many discussions of this type can be carried out within the framework of a non-quantised background metric. However, problems arise when we come to quantum gravity itself. The very concept of a 'history' rests on the notion of a time parameter and, as we have seen, this is an elusive entity. Gell-Mann and Hartle address this problem by proposing an extension of the formalism in which the notion of 'history' becomes a primary one with no *a priori* reference to sequences of questions or beables ordered in any external time. The basic ingredients are:

1. families of mathematical objects called 'histories';
2. a notion of 'coarse-graining' whereby families of histories are partitioned into exclusive and exhaustive sub-families;
3. a 'decoherence functional' $D(h', h)$ defined on pairs of histories.

The decoherence functional must have the following properties:

- *Hermiticity*: $D(h', h) = D^*(h, h')$
- *Positivity*: $D(h, h) \geq 0$
- *Normalisation*: $\sum_{h',h} D(h', h) = 1$
- *The principle of superposition*: $D(\bar{h}', \bar{h}) := \sum_{h' \in \bar{h}'} \sum_{h \in \bar{h}} D(h', h)$ where $\bar{h}'$ and $\bar{h}$ are coarse-grained histories.

A particular family of histories is said to be *consistent* if $D(h', h) = 0$ unless $h' = h$, and then a probability $\mathrm{Prob}(h)$ is assigned to each member of such a family by the rule

$$D(h', h) = \delta_{h'\,h}\,\mathrm{Prob}(h). \tag{6.3.12}$$

As before, the strict equality might be replaced with an approximate equality where the approximation reflects the physical situation to which the formalism is applied.

These rules constitute the entire theory. In particular, there is no *prima facie* Hilbert space structure, although a 'phenomenological' one may 'emerge' in some domains of the theory. However, this absence of the standard mathematical formalism can cause problems when trying to implement the scheme. For example, finite or countably-infinite sums are used in (6.3.2–6.3.3) because the underlying Hilbert space is assumed to be separable. But it is not obvious that sums are sufficient in the absence of any such structure. Some of the collections of histories could well be non-countably infinite, which suggests that integrals are more appropriate, and this is likely to produce major technical problems. It also places in doubt the notion of a 'most-discriminative' set of histories from which all others can be obtained by coarse-graining. This happens already in the Hilbert space theory for an operator with a continuous spectrum, but the spectral theory for such operators enables one to avoid using integrals and to keep to the well-defined sums in (6.3.2–6.3.3). As we shall see, this problem is relevant to the application of these ideas in quantum gravity.

Several attempts have been made to apply this generalised formalism to general relativity. The first involves an extension of the conventional sum-over-histories formalism to situations in which, although there are paths in the configuration space, the theory is invariant under reparametrisations of these paths; in this sense the paths are a generalised form of a 'history'. Theories of this type were studied in some depth by Teitelboim (1982, 1983a, 1983b, 1983c, 1983d), but their development in the context of the consistent histories formalism has been mainly at the hands of Hartle who has emphasised the significance of the existence of many types of spacetime-oriented coarse-graining that have no analogue in a conventional Hamiltonian quantum theory (Hartle 1988*a*, Hartle 1991*b*); in particular, there may be no notion of a state being associated with a *spacelike* hypersurface of spacetime. Hartle (1988*b*) has also shown in a simple model how the notion of time, and conventional Hamiltonian quantum mechanics, can emerge from the formalism as a reading on a physical clock—the same general philosophy that underlies the conditional probability interpretation discussed earlier. These path-integral constructions are the subject of a careful critique in Kuchař (1992*b*).

More recently, Hartle (1991*a*) has proposed that the generalised consistent histories interpretation be extended to quantum gravity by defining a (most-discriminative) 'history' to be a Lorentzian metric γ on the spacetime manifold $\mathcal{M}$ plus a specification of the values of a set of spacetime fields ϕ. The decoherence functional is then defined as

$$D(h', h) := \int_{h'} \mathcal{D}\gamma' \,\mathcal{D}\phi' \int_h \mathcal{D}\gamma \,\mathcal{D}\phi \, e^{i(S[\gamma',\phi'] - S[\gamma,\phi])/\hbar} \tag{6.3.13}$$

where $S[\gamma, \phi]$ is the classical action, and where the integral is over the constituents of the coarse-grained histories h and h'. The motivation for this expression is that the analogous object in the path-integral version of normal Hamiltonian quantum mechanics is the correct choice to reproduce the conventional theory. Note that (6.3.13) apparently contains no reference to a state ρ. However, if desired, this can be thought of as boundary conditions that could be imposed on the spacetime geometries and matter field configurations appearing in the integrals. In particular, quantum cosmological considerations are coded into the behaviour of these fields near the big-bang region.

The development of the theory now proceeds as discussed earlier. Thus one seeks consistent families of such histories from which the probabilistic interpretation can be extracted. In particular, the problem of time reduces to studying the classical correlations between the various variables, including actual physical clocks, in a quasi-classical domain. Hence the view taken of 'time' is essentially the same as in the conditional probability interpretation, but the structural framework of the theory is better defined.

6.3.4 Problems With the Formalism

The consistent histories approach has many attractive features, but also some difficult problems and challenges that need to be taken seriously.

1. The expression (6.3.13) illustrates the problem mentioned earlier about the need to use integrals rather than sums. Functional integrals can provide valuable heuristic insights into the structure of a would-be quantum theory, but they are rarely well-defined mathematically. On the contrary, in the case of general relativity the theory is known to be perturbatively non-renormalisable, and hence the chances of making proper mathematical sense of (6.3.13) are not high. One might adopt a semi-classical approximation (*e.g.,* Kiefer (1991), Hartle (1991*a*)) but this is not terribly satisfactory given the lack of the proper theory that is supposedly being approximated. In particular, in the absence of a non-perturbative evaluation, the functional integral (6.3.13) is at best a low-energy phenomenological description that must be cut-off at energies where the effects of the more basic theory (superstrings?) become significant. This may well be the best way of justifying the use of the WKB approach (via a saddle-point approximation to the functional integral), but it leaves unanswered the question of what happens at the Planck length and, in particular, the problem of time at that scale.

2. Even at a formal level, there is a problem attached to (6.3.13) that arises from the presumed Diff($\mathcal{M}$) invariance of the theory. This could be implemented by requiring the elements being integrated over to be Diff($\mathcal{M}$)-invariant equivalence classes of fields, but it is notoriously difficult to construct the Diff($\mathcal{M}$)-invariant measures needed to facilitate this process. The conventional, heuristic approach is to choose a gauge, define the gauge-fixed functional integrals in the standard way, and then to show that the theory is independent of the choice of gauge. However, in the case of gravity, fixing a gauge means making a choice of internal time *etc*, and then we must confront once more all the problems discussed in earlier sections. In other words, to construct the decoherence functional it may be necessary first to solve the problem of time, and so we are in danger of going round in circles. Thus further study is needed into the possibility of performing a functional integral like (6.3.13) *without* having to invoke a conventional Hilbert-space formalism. [43] Indeed, if something like (6.3.13) *could* be defined properly, it would be consistent with the general Gell-Mann-Hartle philosophy to expect the conventional, Hilbert-space structure to emerge only in some coarse-grained limit of the theory.

3. A major challenge is to find what type of coarse-graining is needed to produce a consistent family of histories using the spacetime fields γ and ϕ. At the very least, we need families that are consistent up to the approximations that may be inherent in pretending that (6.3.13) is a fundamental expression rather than a phenomenological reflection of a more basic theory.

4. This issue is connected to one of the major questions of quantum cosmology: 'What

[43] Of course, this does not rule out the possibility that the functional integral may be defined using *some* Hilbert space, but one that is not that of the conventional Hamiltonian formalism.

types of consistent families of histories give rise to a quasi-classical domain, and how is this related to the conditions in the early universe?' (*e.g.,* Gell-Mann & Hartle (1992)). A related issue is the extent to which the *only* relevant Planck-length era is that of the very early universe. More precisely, what physics would we find at the Planck scale if we could probe it here and now? The question is whether spacetime has some type of foam structure, and if so how this affects, or is reflected in, the consistent histories approach to quantum gravity.

Let me conclude by reaffirming my belief that the Gell-Mann-Hartle axioms constitute a significant generalisation of quantum theory. Their suggested implementation via the decoherence functional (6.3.13) represents a rather conservative approach to quantum gravity and runs into the difficulties mentioned above. But one can imagine more radical attempts involving, for example, some notion of generalised causal sets. The entire scheme certainly deserves very serious further study.

6.4 The Frozen Formalism: Evolving Constants of Motion

Rovelli has advocated recently an interesting approach to the problem of time that shares the central philosophy of the other 'timeless' schemes discussed earlier (Rovelli 1990, 1991a, 1991b, 1991c). Thus the main claim is that it is possible to construct a coherent quantum gravity scheme— including a probabilistic interpretation—without making any specific identification of time, which will rather emerge as a phenomenological concept associated with physical clocks and internal time variables.

Rovelli's starting point is his affirmation that, in the canonical version of classical general relativity, an *observable* is any functional $A[\,g,p\,]$ of the canonical variables $(g_{ab}(x), p^{cd}(x))$ whose Poisson bracket (computed using the basic relations (3.3.27–3.3.29)) with all the constraint functions vanishes:

$$\begin{aligned} \{A, \mathcal{H}_a(x)\} &= 0 && (6.4.1)\\ \{A, \mathcal{H}_\perp(x)\} &= 0. && (6.4.2) \end{aligned}$$

Properly speaking, it is probably more correct to require these Poisson brackets to vanish only *weakly* (as in the right hand side of (3.3.40)), but this point is not addressed in the original papers and I shall not go into it here (it is not of any great significance).

Since the Hamiltonian (3.3.23) for the canonical theory is $H[\,N,\vec{N}\,] := \int_\Sigma d^3x\,(\vec{N}\mathcal{H}_\perp + N^a\mathcal{H}_a)$, these conditions imply that

$$\frac{dA}{dt}(g(t),p(t)) = 0. \qquad (6.4.3)$$

Thus, as emphasised in §3.3.4, an observable is automatically a constant of motion with respect to evolution along the foliation associated with any choice of lapse function N and shift vector $\vec{N}$. This is the 'frozen formalism' of classical, canonical general relativity.

There are two different approaches to the construction of the quantum theory of this system. The first uses the group-theoretical scheme advocated inIsham (1984) with the aim of finding a self-adjoint operator representation of the classical Poisson-bracket algebra of all observables (or, perhaps, some selected subset of them) obeying (6.4.1–6.4.2). The feasibility of adopting such an approach lies in the observation that if $A[\,g, p\,]$ and $B[\,g, p\,]$ are a pair of functions which satisfy (6.4.1–6.4.2) then the Jacobi identity implies that $\{A, B\}$ also satisfies these conditions. Thus the set of all observables is closed under the Poisson bracket operation. If the resulting algebra is a genuine Lie algebra, a self-adjoint operator representation can be found by looking for unitary representations of the associated Lie group. Note that, by constructing the physical Hilbert space in this way one arrives at a probabilistic interpretation without making any specific identification of time.

Unfortunately, sets of observables that generate a true Lie algebra are rather rare, and the algebra seems more likely to be one in which the coefficient of the Poisson bracket of two generators is a non-trivial function of the canonical variables. It is difficult to find self-adjoint representations of algebras of this type because of awkward problems involving the ordering of the generators and their q-number coefficients. Certainly, no one has succeeded in constructing a proper quantum gravity scheme in this way, although this is partly due to the difficulty in finding *classical* functions that satisfy (6.4.1–6.4.2). An interesting model calculation is given in Rovelli (1990) (but note the criticism in Hájíček (1991) and the response of Rovelli (1991*c*)). For a comprehensive analysis of schemes of this type applied to finite-dimensional examples see Tate (1992).

An alternative approach is to start with the scheme employed in §5 which is based on an operator representation of the canonical commutation relations (5.1.1–5.1.3) (or their affine generalisation (5.1.6–5.1.8)) on some Hilbert space $\mathcal{H}$. The physical state space $\mathcal{H}_{\rm phys}$ is deemed to be all vectors in $\mathcal{H}$ that satisfy the operator constraints (5.1.9–5.1.10)

$$\mathcal{H}_a(x; \widehat{g}, \widehat{p}]\Psi \;=\; 0 \tag{6.4.4}$$

$$\mathcal{H}_\perp(x; \widehat{g}, \widehat{p}]\Psi \;=\; 0, \tag{6.4.5}$$

and a physical observable is then defined to be any operator $A[\,\widehat{g}, \widehat{p}\,]$ that satisfies the operator analogue of (6.4.1–6.4.2)

$$[\,\widehat{A}, \widehat{\mathcal{H}_\alpha}(x)\,] \;=\; 0 \tag{6.4.6}$$

$$[\,\widehat{A}, \widehat{\mathcal{H}_\perp}(x)\,] \;=\; 0. \tag{6.4.7}$$

These equations are compatible with (6.4.4–6.4.5) in the sense that any operator satisfying them maps the physical subspace $\mathcal{H}_{\rm phys}$ into itself. A weaker version of (6.4.6–6.4.7) is to require the commutators to vanish only on the physical subspace.

The next step is to place a suitable scalar product on $\mathcal{H}_{\rm phys}$. As emphasised earlier, this cannot be done simply by regarding $\mathcal{H}_{\rm phys}$ as a subspace of the original Hilbert space $\mathcal{H}$: the continuous nature of the spectra of the constraint operators means that the vectors in $\mathcal{H}_{\rm phys}$ all have an infinite $\mathcal{H}$-norm. It is by no means clear how to set about finding the correct scalar product but presumably a minimal requirement is that the physical observables satisfying (6.4.6–6.4.7) should be self-adjoint in the new Hilbert space structure.

This issue raises the general question of how physical observables are actually to be constructed (this is also very relevant for the first approach). Rovelli claims that a particularly important class is formed by the so-called 'evolving constants of motion', which serve also to introduce some notion of dynamical evolution. The basic idea is best explained in a simple model with a single super-Hamiltonian constraint $H(q,p)$ defined on a finite-dimensional phase space $\mathcal{S}$. A classical physical observable is then defined to be any function $A(q,p)$ such that $\{A,H\} = 0$ (or, perhaps, $\{A,H\} \approx 0$). The next step is to introduce some internal time function $\mathcal{T}(q,p)$ with the property that, for any $t \in \mathbb{R}$, the hypersurface

$$\mathcal{S}_t := \{(q,p) \in \mathcal{S} | \mathcal{T}(q,p) = t\} \tag{6.4.8}$$

intersects each dynamical trajectory (on the constraint surface) generated by H once and only once (of course, there may be global obstructions to finding such a function). Note that this requirement means that $\{\mathcal{T}, H\} \neq 0$, and hence the internal time function $\mathcal{T}$ is *not* a physical observable in the sense above.

The key idea is to associate with each function F on $\mathcal{S}$ a one-parameter family of observables (*i.e.,* constants of motion) F_t, $t \in \mathbb{R}$, defined by the two conditions

$$\begin{aligned} \{F_t, H\} &= 0 && (6.4.9)\\ F_t|_{\mathcal{S}_t} &= F|_{\mathcal{S}_t} && (6.4.10) \end{aligned}$$

i.e., the observable F_t is equal to F on the subspace $\mathcal{S}_t$ of the phase space $\mathcal{S}$. Dynamical evolution with respect to the internal time is then described by saying how the *family* of observables F_t, $t \in \mathbb{R}$, depends on t. This is therefore a classical analogue of the Heisenberg picture of time development in quantum theory.

A direct Poisson-bracket calculation shows that

$$\{\mathcal{T}, H\} \frac{dF_t(q,p)}{dt} = \{F, H\}. \tag{6.4.11}$$

Note that if $\mathcal{T}$ is a 'perfect Hamiltonian clock' then, by definition, we have

$$H = p_{\mathcal{T}} + h \tag{6.4.12}$$

where the clock Hamiltonian is $p_{\mathcal{T}}$—the conjugate to the internal time function, so that $\{\mathcal{T}, H\} = 1$—and the Hamiltonian h describing the rest of the system is independent of $p_{\mathcal{T}}$. It follows that $\{F, h\} = \{F_t, h\}$, so that (6.4.11) becomes

$$\frac{dF_t(q,p)}{dt} = \{F_t, h\}, \tag{6.4.13}$$

which is the usual equation of motion. Thus (6.4.11) can be viewed as a *bona fide* generalisation of conventional mechanics to the situation where the only time variable is an internal one.

Rovelli's suggestion is that these evolving constants of motion should form the basis for a quantisation of the system. Thus, in the group-theoretical approach, the key algebra to be represented is the Poisson-bracket algebra generated by the classical quantities F_t, $t \in \mathbb{R}$. The main problem here will be to decide whether or not the set of all such objects forms a genuine Lie algebra. If it does, a unitary representation of the associated Lie group will yield the desired quantum observables. If—as seems more likely—it forms only a function algebra (*i.e.,* with q-number coefficients), it will be necessary to think again about how to find self-adjoint operator representations.

In the alternative, constraint-quantisation approach one needs operator equivalents of the defining equations (6.4.9–6.4.10). The hope is that the inner product on the physical states $\mathcal{H}_{\text{phys}}$ can then be determined by the requirement that all operators $\widehat{F}_t$ are self-adjoint. A number of severe technical problems arise in this version of the programme and are articulated in Kuchař (1992*b*). For example:

- The operator form of (6.4.9) is ill-defined and ambiguous. Neither is it clear that, even if they could be defined properly, the conditions (6.4.9–6.4.10) are sufficient to yield a unique operator $\widehat{F}_t$ from a given operator $\widehat{F}$. This particular problem can be avoided by starting with the classical versions F_t, which *are* well-defined, and then trying to make them into operators. But severe operator-ordering problems will inevitably enter at this point and are likely to be intractable. This is because the classical object F_t can be obtained only by *solving* the classical equations of motion, and hence it is likely to be, at best, an implicit function of the starting function F.
- The global time problem means that no globally-defined internal time function exists. In this circumstance there is a good case for arguing that the associated operators $\widehat{F}_t$ should *not* be self-adjoint. This is the basis of the objection to Rovelli's procedure in Hájíček (1991).
- It seems most unlikely that a single Hilbert space can be used for all possible choices of an internal time function $\mathcal{T}$. Thus the multiple choice and Hilbert space problems appear once more.

These are real difficulties and need to be taken seriously. However, they are no worse than those that arise in any of the other approaches to the problem of time and Rovelli's scheme deserves serious attention, not least because it emphasises once again the importance of the still-debated question of what is to be regarded as an observable in a quantum theory of gravity.

7 CONCLUSIONS

We have discussed three main ways of approaching the central question of how time should be introduced into a quantum theory of gravity. In theories of type I, time is defined internally at a classical level: a procedure that is associated with the removal of all redundant variables before quantisation and which culminates in the production of a standard Schrödinger time-evolution equation for the physical modes of the gravitational and matter fields. This approach is relatively uncontroversial at a conceptual level but it runs into severe technical problems including obstructions to global existence, and local non-uniqueness. It also seems rather *ad hoc* and it is aesthetically unattractive.

In approaches of type II, all the canonical variables are quantised and the constraints are imposed at the quantum level *à la* Dirac as constraints on allowed state vectors. Unfortunately, there is no universally-agreed way of interpreting the ensuing Wheeler-DeWitt equation; certainly none of the ideas produced so far is satisfactory. However, it must be emphasised that there is no real justification for extending the Dirac approach to constraint generators that are *quadratic* functions of the momentum variables. Therefore, although it may be heretical to suggest it, the Wheeler-DeWitt equation—elegant though it be—may be completely the wrong way of formulating a quantum theory of gravity.

Approaches of type III differ from types I and II in ascribing to the concept of 'time' only a secondary, phenomenological status: a move that is inevitably associated with some change in the quantum formalism itself. Techniques of this sort are particularly well-suited for handling the deep philosophical issues that arise in quantum cosmology when quantum theory is applied to the universe as a whole. To my mind, the consistent-histories approach is the most far-reaching in its implications, but it needs further development, especially in the direction of finding a more adventurous definition of what is meant by a 'history' in the context of quantum gravity.

Let me emphasise once more that most of the problems of time in quantum gravity are *not* associated with the existence of ultraviolet divergences in the weak-field perturbative quantisation; in particular, many interpretative difficulties arise already in infinity-free, minisuperspace models. Therefore, I feel it is correct to say that the problems encountered in unravelling the concept of time in quantum gravity are grounded in a fundamental inconsistency between the basic conceptual frameworks of quantum theory and general relativity. In responding to this situation the main task is to decide whether 'time' should preserve the basic role it plays in classical general relativity—something that is most naturally achieved by incorporating it into the quantum formalism by the application of a quantization algorithm to the classical theory—or if it is a concept that should emerge phenomenologicaly from a theoretical framework based on something very different from 'quantising' classical general relativity.

If the former is true, which suggests a type I approach to the problem, the best bet could be some 'natural' choice of internal time dictated by the technical requirements of mathematical consistency in a quantisation scheme; for example the programme currently being pursued by Abhay Ashtekar and collaborators.

If the latter is true, two key questions arise: (i) what is this new framework?, and (ii) how, if at all, does it relate to the existing approaches to quantum gravity, especially the semi-classical scheme? In particular, how does the framework yield conventional quantum theory and our normal ideas of space and time in their appropriate domains?

The most widely-studied scheme of this sort is superstring theory but, in its current manifestation, this is not well-suited for addressing these basic questions. The idea of strings moving in a spacetime already presupposes a great deal about the structure of space and time; and the quantisation techniques employed presuppose most of structure of standard quantum theory, particularly at a conceptual level. It may well be that a new, non-perturbative approach to superstring theory will involve a radical reappraisal of the ideas of space, time and quantum theory; but this remains a task for the future. Perhaps the answer is to find a superstring version of Ashtekar's formalism (or an Ashtekarisation of superstring theory), and with the conceptual aspects of quantum theory being handled by a consistent-histories formalism. A nice challenge for the next few years!

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References

Albrecht, A. (1990), 'Identifying decohering paths in closed quantum systems'. preprint.

Albrecht, A. (1991), 'Investigating decoherence in a simple system'. preprint.

Albrecht, A. (1992), Two perspectives on a decohering spin, *in* J. Halliwell, J. Perez-Mercader & W. Zurek, eds, 'Physical Origins of Time Asymmetry', Cambridge University Press, Cambridge.

Alvarez, E. (1989), 'Quantum gravity: an introduction to some recent results', *Rev. Mod. Phys.* **61**, 561–604.

Arnowitt, R., Deser, S. & Misner, C. (1959*a*), 'Dynamical structure and definition of energy in general relativity', *Phys. Rev.* **116**, 1322–1330.

Arnowitt, R., Deser, S. & Misner, C. (1959*b*), 'Quantum theory of gravitation: General formalism and linearized theory', *Phys. Rev.* **113**, 745–750.

Arnowitt, R., Deser, S. & Misner, C. (1960*a*), 'Canonical variables for general relativity', *Phys. Rev.* **117**, 1595–1602.

Arnowitt, R., Deser, S. & Misner, C. (1960*b*), 'Consistency of the canonical reduction of general relativity', *J. Math. Phys.* **1**, 434–439.

Arnowitt, R., Deser, S. & Misner, C. (1960*c*), 'Energy and the criteria for radiation in general relativity', *Phys. Rev.* **118**, 1100–1104.

Arnowitt, R., Deser, S. & Misner, C. (1960*d*), 'Finite self-energy of classical point particles', *Phys. Rev. Lett.* **4**, 375–377.

Arnowitt, R., Deser, S. & Misner, C. (1961*a*), 'Coordinate invariance and energy expressions in general relativity', *Phys. Rev.* **122**, 997–1006.

Arnowitt, R., Deser, S. & Misner, C. (1961*b*), 'Wave zone in general relativity', *Phys. Rev.* **121**, 1556–1566.

Arnowitt, R., Deser, S. & Misner, C. (1962), The dynamics of general relativity, *in* L. Witten, ed., 'Gravitation: An Introduction to Current Research', Wiley, New York, pp. 227–265.

Ashtekar, A. (1986), 'New variables for classical and quantum gravity', *Phys. Rev. Lett.* **57**, 2244–2247.

Ashtekar, A. (1987), 'New Hamiltonian formulation of general relativity', *Phys. Rev.* **D36**, 1587–1602.

Ashtekar, A. (1991), *Lectures on Non-Perturbative Canonical Gravity*, World Scientific Press, Singapore.

Ashtekar, A. & Stachel, J., eds (1991), *Conceptual Problems of Quantum Gravity*, Birkhäuser, Boston.

Baierlein, R., Sharp, D. & Wheeler, J. (1962), 'Three-dimensional geometry as carrier of information about time', *Phys. Rev.* **126**, 1864–1865.

Banks, T. (1985), 'TCP, quantum gravity, the cosmological constant and all that', *Nucl. Phys.* **B249**, 332–360.

Barbour, J. (1990), 'Time, gauge fields, and the Schrödinger equation in quantum cosmology'. Preprint.

Barbour, J. & Smolin, L. (1988), 'Can quantum mechanics be sensibly applied to the universe as a whole?'. Yale University preprint.

Barvinsky, A. (1991), 'Unitarity approach to quantum cosmology'. University of Alberta preprint.

Bergmann, P. & Komar, A. (1972), 'The coordinate group of symmetries of general relativity', *Int. J. Mod. Phys.* **5**, 15–28.

Blencowe, M. (1991), 'The consistent histories interpretation of quantum fields in curved spacetime', *Ann. Phys. (NY)* **211**, 87–111.

Blyth, W. & Isham, C. (1975), 'Quantisation of a Friedman universe filled with a scalar field', *Phys. Rev.* **D11**, 768–778.

Bohr, N. & Rosenfeld, L. (1933), 'Zur frage der messbarkeit der elektromagnetischen feldgrossen', *Kgl. Danek Vidensk. Selsk. Math.-fys. Medd.* **12**, 8.

Bohr, N. & Rosenfeld, L. (1978), On the question of the measurability of electromagnetic field quantities (English translation), *in* R. Cohen & J. Stachel, eds, 'Selected Papers by Léon Rosenfeld', Reidel, Dordrecht.

Brout, R. (1987), 'On the concept of time and the origin of the cosmological temperature', *Found. Phys.* **17**, 603–619.

Brout, R. & Venturi, G. (1989), 'Time in semiclassical gravity', *Phys. Rev.* **D39**, 2436–2439.

Brout, R., Horowitz, G. & Wiel, D. (1987), 'On the onset of time and temperature in cosmology', *Phys. Lett.* **B192**, 318–322.

Brown, J. & York, J. (1989), 'Jacobi's action and the recovery of time in general relativity', *Phys. Rev.* **D40**, 3312–3318.

Caldeira, A. & Leggett, A. (1983), 'Path integral approach to quantum Brownian motion', *Physica* **A121**, 587–616.

Carlip, S. (1990), 'Observables, gauge invariance, and time in 2+1 dimensional quantum gravity', *Phys. Rev.* **D42**, 2647–2654.

Carlip, S. (1991), Time in $2 + 1$ dimensional quantum gravity, *in* 'Proceedings of the Banff Conference on Gravitation, August 1990'.

Castagnino, M. (1988), 'Probabilistic time in quantum gravity', *Phys. Rev.* **D39**, 2216–2228.

Coleman, S. (1988), 'Why is there something rather than nothing: A theory of the cosmological constant', *Nucl. Phys.* **B310**, 643–668.

Collins, P. & Squires, E. (1992), 'Time in a quantum universe'. In press, Foundations of Physics.

Deutsch, D. (1990), 'A measurement process in a stationary quantum system'. Oxford University preprint.

DeWitt, B. (1962), The quantization of geometry, *in* L. Witten, ed., 'Gravitation: An Introduction to Current Research', Wiley, New York.

DeWitt, B. (1965), *Dynamical Theory of Groups and Fields*, Wiley, New York.

DeWitt, B. (1967*a*), 'Quantum theory of gravity. I. The canonical theory', *Phys. Rev.* **160**, 1113–1148.

DeWitt, B. (1967*b*), 'Quantum theory of gravity. II. The manifestly covariant theory', *Phys. Rev.* **160**, 1195–1238.

DeWitt, B. (1967*c*), 'Quantum theory of gravity. III. Applications of the covariant theory', *Phys. Rev.* **160**, 1239–1256.

Dirac, P. (1958*a*), 'Generalized Hamiltonian dynamics', *Proc. Royal Soc. of London* **A246**, 326–332.

Dirac, P. (1958*b*), 'The theory of gravitation in Hamiltonian form', *Proc. Royal Soc. of London* **A246**, 333–343.

Dirac, P. (1965), *Lectures on Quantum Mechanics*, Academic Press, New York.

Dowker, H. & Halliwell, J. (1992), 'The quantum mechanics of history: The decoherence functional in quantum mechanics', *Phys. Rev.*

Duff, M. (1981), Inconsistency of quantum field theory in a curved spacetime, *in* C. Isham, R. Penrose & D. Sciama, eds, 'Quantum Gravity 2: A Second Oxford Symposium', Clarendon Press, Oxford, pp. 81–105.

Earman, J. & Norton, J. (1987), 'What price spacetime substantialism?', *Brit. Jour. Phil. Science* **38**, 515–525.

Englert, F. (1989), 'Quantum physics without time', *Phys. Lett.* **B228**, 111–114.

Eppley, K. & Hannah, E. (1977), 'The necessity of quantizing the gravitational field', *Found. Phys.* **7**, 51–68.

Fischer, A. & Marsden, J. (1979), The initial value problem and the dynamical formulation of general relativity, *in* S. Hawking & W. Israel, eds, 'General Relativity: An Einstein Centenary Survey', Cambridge University Press, Cambridge, pp. 138–211.

Fredenhagen, K. & Haag, R. (1987), 'Generally covariant quantum field theory and scaling limits', *Comm. Math. Phys* **108**, 91–115.

Friedman, J. & Higuchi, A. (1989), 'Symmetry and internal time on the superspace of asymptotically flat geometries', *Phys. Rev.* **D41**, 2479–2486.

Fulling, S. (1990), 'When is stability in the eye of the beholder? Comments on a singular initial value problem for a nonlinear differential equation arising in semiclassical cosmology', *Phys. Rev.* **D42**, 4248–4250.

Gell-Mann, M. (1987), 'Superstring theory—closing talk at the 2nd Nobel Symposium on Particle Physics', *Physica Scripta* **T15**, 202–209.

Gell-Mann, M. & Hartle, J. (1990*a*), Alternative decohering histories in quantum mechanics, *in* K. Phua & Y. Yamaguchi, eds, 'Proceedings of the 25th International Conference on High Energy Physics, Singapore, August, 2–8, 1990', World Scientific, Singapore.

Gell-Mann, M. & Hartle, J. (1990*b*), Quantum mechanics in the light of quantum cosmology, *in* S. Kobayashi, H. Ezawa, Y. Murayama & S. Nomura, eds, 'Proceedings of the Third International Symposium on the Foundations of Quantum Mechanics in the Light of New Technology', Physical Society of Japan, Tokyo, pp. 321–343.

Gell-Mann, M. & Hartle, J. (1990*c*), Quantum mechanics in the light of quantum cosmology, *in* W. Zurek, ed., 'Complexity, Entropy and the Physics of Information, SFI Studies in the Science of Complexity, Vol. VIII', Addison-Wesley, Reading, pp. 425–458.

Gell-Mann, M. & Hartle, J. (1992), 'Classical equations for quantum systems'. UCSB preprint UCSBTH-91-15.

Gerlach, U. (1969), 'Derivation of the ten Einstein field equations from the semiclassical approximation to quantum geometrodynamics', *Phys. Rev.* **117**, 1929–1941.

Giddings, S. & Strominger, A. (1988), 'Baby universes, third quantization and the cosmological constant', *Nucl. Phys.* **B231**, 481–508.

Greensite, J. (1990), 'Time and probability in quantum cosmology', *Nucl. Phys.* **B342**, 409–429.

Greensite, J. (1991*a*), 'Ehrenfests' principle in quantum-gravity', *Nucl. Phys.* **B351**, 749–766.

Greensite, J. (1991*b*), 'A model of quantum gravitational collapse', *Int. J. Mod. Phys.* **A6**, 2693–2706.

Griffiths, R. (1984), 'Consistent histories and the interpretation of quantum mechanics', *J. Stat. Phys.* **36**, 219–272.

Groenwold, H. (1946), 'On the principles of elementary quantum mechanics', *Physica* **12**, 405–460.

Hajicek, P. & Kuchař, K. (1990*a*), 'Constraint quantization of parametrized relativistic gauge systems in curved spacetimes', *Phys. Rev.* **D41**, 1091–1104.

Hajicek, P. & Kuchař, K. (1990*b*), 'Transversal affine connection and quantisation of constrained systems', *J. Math. Phys.* **31**, 1723–1732.

Hájíček, P. (1986), 'Origin of nonunitarity in quantum gravity', *Phys. Rev.* **D34**, 1040–1048.

Hájíček, P. (1988), 'Reducibility of parametrised systems', *Phys. Rev.* **D38**, 3639–3647.

Hájíček, P. (1989), 'Topology of parametrised systems', *J. Math. Phys.* **20**, 2488–2497.

Hájíček, P. (1990*a*), 'Dirac quantisation of systems with quadratic constraints', *Class. Quan. Grav.* **7**, 871–886.

Hájíček, P. (1990*b*), 'Topology of quadratic super-Hamiltonians', *Class. Quan. Grav.* **7**, 861–870.

Hájíček, P. (1991), 'Comment on 'Time in quantum gravity: An hypothesis'', *Phys. Rev.* **D44**, 1337–1338.

Halliwell, J. (1987), 'Correlations in the wave function of the universe', *Phys. Rev.* **D36**, 3626–3640.

Halliwell, J. (1988), 'Derivation of the Wheeler-DeWitt equation from a path integral for minisuperspace models', *Phys. Rev.* **D38**, 2468–2481.

Halliwell, J. (1989), 'Decoherence in quantum cosmology', *Phys. Rev.* **D39**, 2912–2923.

Halliwell, J. (1990), 'A bibliography of papers on quantum cosmology', *Int. J. Mod. Phys.* **A5**, 2473–2494.

Halliwell, J. (1991*a*), Introductory lectures on quantum cosmology, *in* S. Coleman, J. Hartle, T. Piran & S. Weinberg, eds, 'Proceedings of the Jerusalem Winter School on Quantum Cosmology and Baby Universes', World Scientific, Singapore.

Halliwell, J. (1991*b*), The Wheeler-DeWitt equation and the path integral in minisuperspace quantum cosmology, *in* A. Ashtekar & J. Stachel, eds, 'Conceptual Problems of Quantum Gravity', Birkhäuser, Boston, pp. 75–115.

Halliwell, J. (1992*a*), The interpretation of quantum cosmology models, *in* 'Proceedings of the 13th International Conference on General Relativity and Gravitation, Cordoba, Argentina'.

Halliwell, J. (1992*b*), 'Smeared Wigner functions and quantum-mechanical histories', *Phys. Rev.*

Halliwell, J. & Hawking, S. (1985), 'Origin of structure in the universe', *Phys. Rev.* **D31**, 1777–1791.

Halliwell, J., Perez-Mercander, J. & Zurek, W., eds (1992), *Physical Origins of Time Asymmetry*, Cambridge University Press, Cambridge.

Hartle, J. (1986), Prediction and observation in quantum cosmology, *in* B. Carter & J. Hartle, eds, 'Gravitation and Astrophysics, Cargese, 1986', Plenum, New York.

Hartle, J. (1988*a*), 'Quantum kinematics of spacetime. I. Nonrelativistic theory', *Phys. Rev.* **D37**, 2818–2832.

Hartle, J. (1988*b*), 'Quantum kinematics of spacetime. II. A model quantum cosmology with real clocks', *Phys. Rev.* **D38**, 2985–2999.

Hartle, J. (1991*a*), The quantum mechanics of cosmology, *in* S. Coleman, P. Hartle, T. Piran & S. Weinberg, eds, 'Quantum Cosmology and Baby Universes', World Scientific, Singapore.

Hartle, J. (1991*b*), 'Spacetime grainings in nonrelativistic quantum mechanics', *Phys. Rev.* **D44**, 3173–3195.

Hartle, J. & Hawking, S. (1983), 'Wave function of the universe', *Phys. Rev.* **D28**, 2960–2975.

Hawking, S. (1982), 'The unpredictability of quantum gravity', *Comm. Math. Phys* **87**, 395–416.

Hawking, S. (1984*a*), Lectures in quantum cosmology, *in* B. DeWitt & R. Stora, eds, 'Relativity, Groups and Topology II', North-Holland, Amsterdam, pp. 333–379.

Hawking, S. (1984*b*), 'The quantum state of the universe', *Nucl. Phys.* **B239**, 257–276.

Hawking, S. & Ellis, G. (1973), *The Large Scale Structure of Space-Time*, Cambridge University Press, Cambridge.

Hawking, S. & Page, D. (1986), 'Operator ordering and the flatness of the universe', *Nucl. Phys.* **B264**, 185–196.

Hawking, S. & Page, D. (1988), 'How probable is inflation?', *Nucl. Phys.* **B298**, 789–809.

Henneaux, M. & Teitelboim, C. (1989), 'The cosmological constant and general covariance', *Phys. Lett.* **B222**, 195–199.

Higgs, P. (1958), 'Integration of secondary constraints in quantized general relativity', *Phys. Rev. Lett.* **1**, 373–375.

Hojman, S., Kuchař, K. & Teitelboim, C. (1976), 'Geometrodynamics regained', *Ann. Phys. (NY)* **96**, 88–135.

Horowitz, G. (1980), 'Semiclassical relativity: The weak-field limit', *Phys. Rev.* **D21**, 1445–1461.

Horowitz, G. (1981), Is flat space-time unstable?, *in* C. Isham, R. Penrose & D. Sciama, eds, 'Quantum Gravity 2: A Second Oxford Symposium', Clarendon Press, Oxford, pp. 106–130.

Horowitz, G. & Wald, R. (1978), 'Dynamics of Einstein's equations modified by a higher-derivative term', *Phys. Rev.* **D17**, 414–416.

Horowitz, G. & Wald, R. (1980), 'Quantum stress energy in nearly conformally flat spacetimes', *Phys. Rev.* **D21**, 1462–1465.

Isenberg, J. & Marsden, J. (1976), 'The York map is a canonical transformation', *Ann. Phys. (NY)* **96**, 88–135.

Isham, C. (1984), Topological and global aspects of quantum theory, *in* B. DeWitt & R. Stora, eds, 'Relativity, Groups and Topology II', North-Holland, Amsterdam, pp. 1062–1290.

Isham, C. (1985), Aspects of quantum gravity, *in* A. Davies & D. Sutherland, eds, 'Superstrings and Supergravity: Proceedings of the 28th Scottish Universities Summer School in Physics, 1985', SUSSP Publications, Edinburgh, pp. 1–94.

Isham, C. (1987), Quantum gravity—grg11 review talk, *in* M. MacCallum, ed., 'General Relativity and Gravitation: Proceedings of the 11th International Conference on General Relativity and Gravitation', Cambridge University Press, Cambridge.

Isham, C. (1992), Conceptual and geometrical problems in quantum gravity, *in* H. Mitter & H. Gausterer, eds, 'Recent Aspects of Quantum Fields', Springer-Verlag, Berlin, pp. 123–230.

Isham, C. & Kakas, A. (1984*a*), 'A group theoretical approach to the canonical quantisation of gravity: I. Construction of the canonical group', *Class. Quan. Grav.* **1**, 621–632.

Isham, C. & Kakas, A. (1984*b*), 'A group theoretical approach to the canonical quantisation of gravity: II. Unitary representations of the canonical group', *Class. Quan. Grav.* **1**, 633–650.

Isham, C. & Kuchař, K. (1985*a*), 'Representations of spacetime diffeomorphisms. I. Canonical parametrised spacetime theories', *Ann. Phys. (NY)* **164**, 288–315.

Isham, C. & Kuchař, K. (1985*b*), 'Representations of spacetime diffeomorphisms. II. Canonical geometrodynamics', *Ann. Phys. (NY)* **164**, 316–333.

Isham, C. & Kuchař, K. (1994), *Quantum Gravity and the Problem of Time.* In preparation.

Jackiw, R. (1992*a*), Gauge theories for gravity on a line, *in* 'Proceedings of NATO Advanced Study Institute 'Recent Problems in Mathematical Physics, Salamanca, Spain, 1992'', Kluwer, The Netherlands.

Jackiw, R. (1992*b*), Update on planar gravity ($\approx$ physics of infinite cosmic strings), *in* H. Sato, ed., 'Proceedings of the Sixth Marcel Grossman Meeting on General Relativity', World Scientific, Singapore.

Joos, E. (1986), 'Why do we observe a classical spacetime?', *Phys. Lett.* **A116**, 6–8.

Joos, E. (1987), Quantum theory and the emergence of a classical world, *in* D. Greenberger, ed., 'New Techniques and Ideas in Quantum Measurement', New York Academic of Sciences, New York.

Joos, E. & Zeh, H. (1985), 'The emergence of classical properties through interaction with the environment', *Zeitschrift für Physik* **B59**, 223–243.

Kaup, D. & Vitello, A. (1974), 'Solvable quantum cosmological models and the importance of quantizing in a special canonical frame', *Phys. Rev.* **D9**, 1648–1655.

Kibble, T. (1981), Is a semi-classical theory of gravity viable?, *in* C. Isham, R. Penrose & D. Sciama, eds, 'Quantum Gravity 2: A Second Oxford Symposium', Clarendon Press, Oxford, pp. 63–80.

Kiefer, C. (1987), 'Continuous measurement of mini-superspace variables by higher multipoles', *Class. Quan. Grav.* **4**, 1369–1382.

Kiefer, C. (1989*a*), 'Continuous measurement of intrinsic time by fermions', *Class. Quan. Grav.* **6**, 561–566.

Kiefer, C. (1989*b*), 'Quantum gravity and Brownian motion', *Phys. Lett.* **A139**, 201–203.

Kiefer, C. (1991), 'Interpretation of the decoherence functional in quantum cosmology', *Class. Quan. Grav.* **8**, 379–392.

Kiefer, C. (1992), Decoherence in quantum cosmology, *in* 'Proceedings of the Tenth Seminar on Relativistic Astrophysics and Gravitation, Postdam, 1991', World Scientific, Singapore.

Kiefer, C. & Singh, T. (1991), 'Quantum gravitational corrections to the functional Schrödinger equation', *Phys. Rev.* **D44**, 1067–1076.

Klauder, J. (1970), Soluble models of guantum gravitation, *in* M. Carmeli, S. Flicker & Witten, eds, 'Relativity', Plenum, New York.

Komar, A. (1979*a*), 'Consistent factoring ordering of general-relativistic constraints', *Phys. Rev.* **D19**, 830–833.

Komar, A. (1979*b*), 'Constraints, hermiticity, and correspondence', *Phys. Rev.* **D19**, 2908–2912.

Kuchař, K. (1970), 'Ground state functional of the linearized gravitational field', *J. Math. Phys.* **11**, 3322–3344.

Kuchař, K. (1971), 'Canonical quantization of cylindrical gravitational waves', *Phys. Rev.* **D4**, 955–985.

Kuchař, K. (1972), 'A bubble-time canonical formalism for geometrodynamics', *J. Math. Phys.* **13**, 768–781.

Kuchař, K. (1973), Canonical quantization of gravity, *in* 'Relativity, Astrophysics and Cosmology', Reidel, Dordrecht, pp. 237–288.

Kuchař, K. (1974), 'Geometrodynamics regained: A Lagrangian approach', *J. Math. Phys.* **15**, 708–715.

Kuchař, K. (1976*a*), 'Dynamics of tensor fields in hyperspace III.', *J. Math. Phys.* **17**, 801–820.

Kuchař, K. (1976*b*), 'Geometry of hyperspace I.', *J. Math. Phys.* **17**, 777–791.

Kuchař, K. (1976*c*), 'Kinematics of tensor fields in hyperspace II.', *J. Math. Phys.* **17**, 792–800.

Kuchař, K. (1977), 'Geometrodynamics with tensor sources IV', *J. Math. Phys.* **18**, 1589–1597.

Kuchař, K. (1981*a*), Canonical methods of quantisation, *in* C. Isham, R. Penrose & D. Sciama, eds, 'Quantum Gravity 2: A Second Oxford Symposium', Clarendon Press, Oxford, pp. 329–374.

Kuchař, K. (1981*b*), 'General relativity: Dynamics without symmetry', *J. Math. Phys.* **22**, 2640–2654.

Kuchař, K. (1982), 'Conditional symmetries in parametrized field theories', *J. Math. Phys.* **25**, 1647–1661.

Kuchař, K. (1986*a*), 'Canonical geometrodynamics and general covariance', *Found. Phys.* **16**, 193–208.

Kuchař, K. (1986*b*), 'Covariant factor ordering for gauge systems', *Phys. Rev.* **D34**, 3044–3057.

Kuchař, K. (1986*c*), 'Hamiltonian dynamics of gauge systems', *Phys. Rev.* **D34**, 3031–3043.

Kuchař, K. (1987), 'Covariant factor ordering of constraints may be ambiguous', *Phys. Rev.* **D35**, 596–599.

Kuchař, K. (1991*a*), 'Does an unspecified cosmological constant solve the problem of time in quantum gravity?', *Phys. Rev.* **D43**, 3332–3344.

Kuchař, K. (1991*b*), The problem of time in canonical quantization, *in* A. Ashtekar & J. Stachel, eds, 'Conceptual Problems of Quantum Gravity', Birkhäuser, Boston, pp. 141–171.

Kuchař, K. (1992*a*), 'Extrinsic curvature as a reference fluid in canonical gravity', *Phys. Rev.* **D45**, 4443–4457.

Kuchař, K. (1992*b*), Time and interpretations of quantum gravity, *in* 'Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics', World Scientific, Singapore.

Kuchař, K. & Ryan, M. (1989), 'Is minisuperspace quantization valid? Taub in Mixmaster', *Phys. Rev.* **D40**, 3982–3996.

Kuchař, K. & Torre, C. (1989), 'World sheet diffeomorphisms and the cylindrical string', *J. Math. Phys.* **30**, 1769–1793.

Kuchař, K. & Torre, C. (1991*a*), 'Gaussian reference fluid and the interpretation of geometrodynamics', *Phys. Rev.* **D43**, 419–441.

Kuchař, K. & Torre, C. (1991*b*), 'Harmonic gauge in canonical gravity', *Phys. Rev.* **D44**, 3116–3123.

Kuchař, K. & Torre, C. (1991*c*), Strings as poor relatives of general relativity, *in* A. Ashtekar & J. Stachel, eds, 'Conceptual Problems of Quantum Gravity', Birkhäuser, Boston, pp. 326–348.

Lapchinski, V. & Rubakov, V. (1979), 'Canonical quantization of gravity and quantum field theory in curved spacetime', *Acta. Phys. Pol.* **B10**, 1041.

Lee, J. & Wald, R. (1990), 'Local symmetries and constraints', *J. Math. Phys.* **31**, 725–743.

Linden, N. & Perry, M. (1991), 'Path integrals and unitarity in quantum cosmology', *Nucl. Phys.* **B357**, 289–307.

Manojlović & Miković (1992), 'Gauge fixing and independent canonical variables in the ashtekar formulation of general relativity'.

McGuigan, M. (1988), 'Third quantization and the Wheeler-DeWitt equation', *Phys. Rev.* **D38**, 3031–3051.

McGuigan, M. (1989), 'Universe creation from the third quantized vacuum', *Phys. Rev.* **D39**, 2229–2233.

Mellor, F. (1991), 'Decoherence in quantum Kaluza-Klein theories', *Nucl. Phys.* **B353**, 291–301.

Misner, C. (1957), 'Feynman quantization of general relativity', *Rev. Mod. Phys.* **29**, 497–509.

Misner, C. (1992), Minisuperspace, *in* J. Klauder, ed., 'Magic without Magic: John Archibald Wheeler, A Collection of Essays in Honor of his 60th Birthday', Freeman, San Francisco.

Møller, C. (1962), The energy-momentum complex in general relativity and related problems, *in* A. Lichnerowicz & M. Tonnelat, eds, 'Les Théories Relativistes de la Gravitation', CNRS, Paris.

Morikawa, M. (1989), 'Evolution of the cosmic density matrix', *Phys. Rev.* **D40**, 4023–4027.

Newman, E. & Rovelli, C. (1992), 'Generalized lines of force as the gauge invariant degrees of freedom for general relativity and Yang-Mills theory'. University of Pittsburgh preprint.

Omnès, R. (1988*a*), 'Logical reformulation of quantum mechanics. I. Foundations', *J. Stat. Phys.* **53**, 893–932.

Omnès, R. (1988*b*), 'Logical reformulation of quantum mechanics. II. Interferences and the Einstein-Podolsky-Rosen experiment', *J. Stat. Phys.* **53**, 933–955.

Omnès, R. (1988*c*), 'Logical reformulation of quantum mechanics. III. Classical limit and irreversibility', *J. Stat. Phys.* **53**, 957–975.

Omnès, R. (1989), 'Logical reformulation of quantum mechanics. III. Projectors in semi-classical physics', *J. Stat. Phys.* **57**, 357–382.

Omnès, R. (1990), 'From Hilbert space to common sense: A synthesis of recent progress in the interpretation of quantum mechanics', *Ann. Phys. (NY)* **201**, 354–447.

Omnès, R. (1992), 'Consistent interpretations of quantum mechanics', *Rev. Mod. Phys.* **64**, 339–382.

Padmanabhan, T. (1989*a*), 'Decoherence in the density matrix describing quantum three-geometries and the emergence of classical spacetime', *Phys. Rev.* **D39**, 2924–2932.

Padmanabhan, T. (1989*b*), 'Semiclassical approximations to gravity and the issue of back-reaction', *Class. Quan. Grav.* **6**, 533–555.

Padmanabhan, T. (1990), 'A definition for time in quantum cosmology', *Pramana Jour. Phys.* **35**, L199–L204.

Page, D. (1986*a*), 'Density matrix of the universe', *Phys. Rev.* **D34**, 2267–2271.

Page, D. (1986*b*), Hawking's wave function for the universe, *in* R. Penrose & C. Isham, eds, 'Quantum Concepts in Space and Time', Clarendon Press, Oxford, pp. 274–285.

Page, D. (1989), 'Time as an inaccessible observable'. ITP preprint NSF-ITP-89-18.

Page, D. (1991), Intepreting the density matrix of the universe, *in* A. Ashtekar & J. Stachel, eds, 'Conceptual Problems of Quantum Gravity', Birkhäuser, Boston, pp. 116–121.

Page, D. & Geilker, C. (1981), 'Indirect evidence for quantum gravity', *Phys. Rev. Lett.* **47**, 979–982.

Page, D. & Hotke-Page, C. (1992), Clock time and entropy, *in* J. Halliwell, J. Perez-Mercander & W. Zurek, eds, 'Physical Origins of Time Asymmetry', Cambridge University Press, Cambridge.

Page, D. & Wooters, W. (1983), 'Evolution without evolution: Dynamics described by stationary observables', *Phys. Rev.* **D27**, 2885–2892.

Pilati, M. (1982), 'Strong coupling quantum gravity I: solution in a particular gauge', *Phys. Rev.* **D26**, 2645–2663.

Pilati, M. (1983), 'Strong coupling quantum gravity I: solution without gauge fixing', *Phys. Rev.* **D28**, 729–744.

Randjbar-Daemi, S., Kay, B. & Kibble, T. (1980), 'Renormalization of semi-classical field theories', *Phys. Rev. Lett.* **91B**, 417–420.

Rosenfeld, L. (1963), 'On quantization of fields', *Nucl. Phys.* **40**, 353–356.

Rovelli, C. (1990), 'Quantum mechanics without time: A model', *Phys. Rev.* **D42**, 2638–2646.

Rovelli, C. (1991*a*), 'Ashtekar formulation of general relativity and loop-space non-perturbative quantum gravity: A report.'. Pittsburgh University preprint.

Rovelli, C. (1991*b*), Is there incompatibility between the ways time is treated in general relativity and in standard quantum theory?, *in* A. Ashtekar & J. Stachel, eds, 'Conceptual Problems of Quantum Gravity', Birkhäuser, Boston, pp. 126–140.

Rovelli, C. (1991*c*), 'Quantum evolving constants. Reply to 'Comments on time in quantum gravity: An hypothesis'', *Phys. Rev.* **D44**, 1339–1341.

Rovelli, C. (1991*d*), 'Statistical mechanics of gravity and the problem of time'. University of Pittsburgh preprint.

Rovelli, C. (1991*e*), 'Time in quantum gravity: An hypothesis', *Phys. Rev.* **D43**, 442–456.

Rovelli, C. & Smolin, L. (1990), 'Loop space representation of quantum general relativity', *Nucl. Phys.* **B331**, 80–152.

Salisbury, D. & Sundermeyer, K. (1983), 'Realization in phase space of general coordinate transformations', *Phys. Rev.* **D27**, 740–756.

Simon, J. (1991), 'The stability of flat space, semiclassical gravity and higher derivatives', *Phys. Rev.* **D43**, 3308–3316.

Singer, I. (1978), 'Some remarks on the Gribov ambiguity', *Comm. Math. Phys* **60**, 7–12.

Singh, T. & Padmanabhan, T. (1989), 'Notes on semiclassical gravity', *Ann. Phys. (NY)* **196**, 296–344.

Smarr, L. & York, J. (1978), 'Kinematical conditions in the construction of spacetime', *Phys. Rev.* **D17**, 2529–2551.

Smolin, L. (1992), 'Recent developments in nonperturbative quantum gravity'. Syracuse University Preprint.

Squires, E. (1991), 'The dynamical role of time in quantum cosmology', *Phys. Lett.* **A155**, 357–360.

Stachel, J. (1989), Einstein's search for general covariance, 1912-1915, *in* D. Howard & J. Stachel, eds, 'Einstein and the History of General Relativity: Vol I', Birkhäuser, Boston, pp. 63–100.

Stone, C. & Kuchař, K. (1992), 'Representation of spacetime diffeomorphisms in canonical geometrodynamics under harmonic coordinate conditions', *Class. Quan. Grav.* **9**, 757–776.

Suen, W. (1989*a*), 'Minkowski spacetime is unstable in semi-classical gravity', *Phys. Rev. Lett.* **62**, 2217–2226.

Suen, W. (1989*b*), 'Stability of the semiclassical Einstein equation', *Phys. Rev.* **D40**, 315–326.

Tate, R. (1992), 'An algebraic approach to the quantization of constrained systems: Finite dimensional examples'. PhD. Dissertation.

Teitelboim, C. (1982), 'Quantum mechanics of the gravitational field', *Phys. Rev.* **D25**, 3159–3179.

Teitelboim, C. (1983*a*), 'Causality versus gauge invariance in quantum gravity and supergravity', *Phys. Rev. Lett.* **50**, 705–708.

Teitelboim, C. (1983*b*), 'Proper time gauge in the quantum theory of gravitation', *Phys. Rev.* **D28**, 297–309.

Teitelboim, C. (1983*c*), 'Quantum mechanics of the gravitational field', *Phys. Rev.* **D25**, 3159–3179.

Teitelboim, C. (1983*d*), 'Quantum mechanics of the gravitational field in asymptotically flat space', *Phys. Rev.* **D28**, 310–316.

Teitelboim, C. (1992), *in* 'Proceedings of NATO Advanced Study Institute 'Recent Problems in Mathematical Physics, Salamanca, Spain, 1992'', Kluwer, The Netherlands.

Torre, C. (1989), 'Hamiltonian formulation of induced gravity in two dimensions', *Phys. Rev.* **D40**, 2588–2597.

Torre, C. (1991), 'A complete set of observables for cylindrically symmetric gravitational fields', *Class. Quan. Grav.* **8**, 1895–1911.

Torre, C. (1992), 'Is general relativity an "already parametrised" theory?'.

Unruh, W. (1988), Time and quantum gravity, *in* M. Markov, V. Berezin & V. Frolov, eds, 'Proceedings of the Fourth Seminar on Quantum Gravity', World Scientific, Singapore, pp. 252–268.

Unruh, W. (1989), 'Unimodular theory of canonical quantum gravity', *Phys. Rev.* **D40**, 1048–1052.

Unruh, W. (1991), Loss of quantum coherence for a damped oscillator, *in* A. Ashtekar & J. Stachel, eds, 'Conceptual Problems of Quantum Gravity', Birkhäuser, Boston, pp. 67–74.

Unruh, W. & Wald, R. (1989), 'Time and the interpretation of quantum gravity', *Phys. Rev.* **D40**, 2598–2614.

Unruh, W. & Zurek, W. (1989), 'Reduction of the wave-packet in quantum Brownian motion', *Phys. Rev.* **D40**, 1071–1094.

Valentini, A. (1992), 'Non-local hidden-variables and quantum gravity'. SISSA preprint 105/92/A.

Van Hove, L. (1951), 'On the problem of the relations between the unitary transformations of quantum mechanics and the canonical transformations of classical mechanics', *Acad. Roy. Belg.* **37**, 610–620.

Vilenkin, A. (1988), 'Quantum cosmology and the initial state of the universe', *Phys. Rev.* **D39**, 888–897.

Vilenkin, A. (1989), 'Interpretation of the wave function of the universe', *Phys. Rev.* **D39**, 1116–1122.

Vink, J. (1992), 'Quantum potential interpretation of the wavefunction of the universe', *Nucl. Phys.* **B369**, 707–728.

Wheeler, J. (1962), Neutrinos, gravitation and geometry, *in* 'Topics of Modern Physics. Vol 1', Academic Press, New York, pp. 1–130.

Wheeler, J. (1964), Geometrodynamics and the issue of the final state, *in* C. DeWitt & B. DeWitt, eds, 'Relativity, Groups and Topology', Gordon and Breach, New York and London, pp. 316–520.

Wheeler, J. (1968), Superspace and the nature of quantum geometrodynamics, *in* C. DeWitt & J. Wheeler, eds, 'Batelle Rencontres: 1967 Lectures in Mathematics and Physics', Benjamin, New York, pp. 242–307.

Wooters, W. (1984), 'Time replaced by quantum corrections', *Int. J. Theor. Phys.* **23**, 701–711.

York, J. (1972*a*), 'Mapping onto solutions of the gravitational initial value problem', *J. Math. Phys.* **13**, 125–130.

York, J. (1972*b*), 'Role of conformal three-geometry in the dynamics of gravitation', *Phys. Rev. Lett.* **28**, 1082–1085.

York, J. (1979), Kinematics and dynamics of general relativity, *in* L. Smarr, ed., 'Sources of Gravitational Radiation', Cambridge University Press, Cambridge, pp. 83–126.

Zeh, H. (1986), 'Emergence of classical time from a universal wave function', *Phys. Lett.* **A116**, 9–12.

Zeh, H. (1988), 'Time in quantum gravity', *Phys. Lett.* **A126**, 311–317.

Zurek, W. (1981), 'Pointer basis of quantum apparatus: Into what mixture does the wave packet collapse?', *Phys. Rev.* **D24**, 1516–1525.

Zurek, W. (1982), 'Environment-induced superselection rules', *Phys. Rev.* **D26**, 1862–1880.

Zurek, W. (1983), Information transer in quantum measurements:Irreversibility and amplification, *in* P. Meystre & M. Scully, eds, 'Quantum Optics, Experimental Gravity, and Measurement Theory', Plenum, New York, pp. 87–116.

Zurek, W. (1986), Reduction of the wave-packet: How long does it take?, *in* G. Moore & M. Scully, eds, 'Frontiers of Nonequilibrium Statistical Physics', Plenum, New York, pp. 145–149.

Zurek, W. (1991), Quantum measurements and the environment-induced transition from quantum to classical, *in* A. Ashtekar & J. Stachel, eds, 'Conceptual Problems of Quantum Gravity', Birkhäuser, Boston, pp. 43–66.