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Coherent-State Representation of Semiclassical Quantum Gravity

Sang Pyo Kim*

Department of Physics, Kunsan National University, Kunsan 573-701

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We elaborate the recently introduced asymptotically exact semiclassical quantum gravity derived from the Wheeler-DeWitt equation by finding a particular coherent-state representation of a quantum scalar field in which the back-reaction of the scalar field Hamiltonian exactly gives rise to the classical one. We show that classical spacetime emerges naturally from the coherent-state representation of the semiclassical quantum gravity.

The canonical quantum gravity based on the Wheeler-DeWitt (WDW) equation has been used, as one of its applications, to provide a self-consistent theory of quantum fields in a curved spacetime (for review and references, see Ref. 1). Most of the methods in this direction rely on the Born-Oppenheimer idea that the larger mass scale out of several mass scales of a quantum system becomes semiclassical first and the relatively smaller mass scales retain their quantum-mechanical nature. In this approach[1] to quantum field theory in curved spacetime, the classical spacetime emerges from the quantum domain, but the matter fields (typically scalar fields) keep their quantum-mechanical properties and satisfy the time-dependent functional Schrödinger equation. The advantage of this approach is that one may treat the semiclassical theory relatively simply, including some parts of gravitational quantum corrections to the matter fields and replacing the energy-momentum tensor by its quantum-mechanical expectation value. The disadvantage is that the quantum corrections of gravity to matter fields may not be achieved fully and the renormalization problem of wave functions may not be resolved in this approach to semiclassical quantum gravity. This approach to semiclassical quantum gravity, despite its shortcomings, proves quite useful when one considers the quantum gravity effects semiclassically, and especially in the context of cosmology.

A step has not been completely resolved in deriving classical gravity from the WDW equation. One usually assumes that classical gravity can be directly obtained from the WDW equation as the Hamilton-Jacobi equation. This could, of course, presumably sound physical in a certain domain, but the logical steps are not fulfilled, because considering the different mass scales of the gravitational fields and the matter fields, it would be more correct for the gravity to emerge first from the quantum domain and for the larger mass-scaled matter fields to emerge later. The scheme for deriving the semiclassical quantum gravity $G_{\mu\nu} = 8\pi < \hat{T}_{\mu\nu} >$ from the canonical quantum gravity $\hat{G}_{\mu\nu} = 8\pi\hat{T}_{\mu\nu}$ and then the classical gravity $G_{\mu\nu} = 8\pi T_{\mu\nu}$ from the semiclassical quantum gravity needs a rigorous foundation.

In previous papers [3], we have developed an asymptotic method to derive the quantum field theory of matter fields in a curved spacetime from the canonical quantum-gravity-based WDW equation[2]. In the case of the quantum Friedmann-Robertson-Walker (FRW) cosmology minimally coupled to a massive scalar field, we were able to construct a particular Fock space on which the expectation value of the scalar-field Hamiltonian operator has the same form in terms of a complex classical solution as that of the classical one [5].

In this paper, we elaborate further that scheme for deriving the semiclassical quantum gravity and the classical gravity from the canonical quantum gravity based on the WDW equation. It is found that the coherent states constructed on the particular Fock space [5] make matter fields emerge as classical variables. In this scheme, one is able to show how the semiclassical quantum gravity, a quantum field theory of matter fields, can be derived from canonical quantum gravity and also how classical gravity theory, the Einstein gravity theory of classical matter fields, can emerge from the semiclassical quantum theory of gravity. For simplicity, we shall consider the quantum FRW cosmology minimally coupled to a

*e-mail : sangkim@knusun1.kunsan.ac.kr

[1]We distinguish this approach, the so-called semiclassical quantum gravity, from the conventional quantum field theory in curved spacetimes. We refer to Ref. 2 for review of and references for the latter approach.

[2]For the path-integral approach, see Ref. 4, and references therein.

massive scalar field. One of the reasons for specifying the scalar field is that the coherent states of the massive scalar field in a curved spacetime can be constructed explicitly, as will be shown in this paper.

As a simple quantum cosmological model, we consider a FRW cosmology with the metric

$$ds^2 = -N^2 dt^2 + a^2 d\sigma_k^2. \tag{1}$$

The action for the FRW cosmology minimally coupled to a homogeneous and isotropic massive scalar field is

$$I = \int dt \Bigg[-\frac{3m_P^2}{8\pi} a^3 \left(\frac{1}{N} \left(\frac{\dot{a}}{a}\right)^2 - N\frac{k}{a^2} \right) + a^3 \left(\frac{\dot{\phi}^2}{2N} - \frac{Nm^2}{2}\phi^2 \right) \Bigg] \tag{2}$$

where $k = 1, 0, -1$ corresponds to a closed, flat, and open universe, respectively. In the above action, we dropped the surface term. We used the unit system such that $c = \hbar = 1$ and $\frac{1}{G} = m_P^2$. The conjugate momenta are

$$\pi_a = -\frac{3m_P^2}{4\pi}\frac{a\dot{a}}{N}, \quad \pi_\phi = \frac{a^3\dot{\phi}}{N}. \tag{3}$$

From the super-Hamiltonian constraint of the ADM formulation,

$$\mathcal{H} = -\frac{2\pi}{3m_P^2}\frac{1}{a}\pi_a^2 - \frac{3m_P^2}{8\pi}ka + \frac{1}{2a^3}\pi_\phi^2 + \frac{a^3m^2}{2}\phi^2 = 0 \tag{4}$$

one obtains the WDW equation

$$\left[\frac{2\pi}{3m_P^2}\frac{1}{a}\frac{\partial^2}{\partial a^2} - \frac{3m_P^2}{8\pi}ka - \frac{1}{2a^3}\frac{\partial^2}{\partial\phi^2} + \frac{a^3m^2}{2}\phi^2 \right]\Psi(a,\phi) = 0. \tag{5}$$

One of the questions closely related with the correspondence between quantum and classical theory in general is to understand how and when one may recover the classical gravity theory

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi}{3m_P^2}\left(\frac{\dot{\phi}^2}{2} + \frac{m^2}{2}\phi^2\right), \tag{6}$$

which is obtained by varying the action of Eq. (2) with respect to the lapse function N and by fixing the temporal gauge by $N = 1$. The other scalar-field equation that constitutes the classical gravity is obtained from the variation of the action with respect to ϕ:

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + m^2\phi = 0. \tag{7}$$

One has frequently used the WKB wave function $\Psi(a,\phi) = F(a,\phi)e^{iS(a,\phi)}$, where F is a slowly varying function. The rapidly changing phase factor satisfies the Hamilton-Jacobi equation

$$-\frac{2\pi}{3m_P^2}\frac{1}{a}\left(\frac{\partial S}{\partial a}\right)^2 - \frac{3m_P^2}{8\pi}ka + \frac{1}{2a^3}\left(\frac{\partial S}{\partial\phi}\right)^2 + \frac{a^3m^2}{2}\phi^2 = 0. \tag{8}$$

By identifying $\frac{\partial S}{\partial a} = \pi_a$ and $\frac{\partial S}{\partial \phi} = \pi_\phi$ in Eq. (3), we recover the classical equation, Eq. (6). However, in this approach to classical gravity, there remains the one unexplained problem that the large-mass scale difference between the gravitational field and the matter field in a later stage of cosmological evolution makes the gravitational field classical but keeps the matter field quantum mechanical following the Born-Oppenheimer idea. This is the main conceptual idea behind the semiclassical quantum gravity theory.

Below we shall develop an alternative approach, in which the classical gravity can be derived from the semiclassical quantum gravity which in turn can be derived from the quantum gravity based on the WDW equation. It was shown [3] that the semiclassical quantum gravity derived from the WDW equation consists of the gravitational field equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi}{3m_P^2}\frac{1}{a^3}\left\langle \hat{H}_m \right\rangle \tag{9}$$

and the time-dependent Schrödinger equation of the scalar field

$$i\frac{\partial}{\partial t}\Phi(\phi,t) = \hat{H}_m\Phi(\phi,t) \tag{10}$$

where

$$\hat{H}_m = \frac{1}{2a^3}\hat{\pi}_\phi^2 + \frac{a^3m^2}{2}\hat{\phi}^2 \tag{11}$$

is the scalar-field Hamiltonian. As was shown explicitly and fully in Ref. 3, the semiclassical quantum-gravity equations are asymptotically exact in the sense of $\frac{1}{m_P^2} \to 0$, *i.e*, $O(\frac{1}{m_P^2})$, provided that one chooses the quantum states of the scalar field as the eigenstates of the invariant $\hat{I}_m$ satisfying [6]

$$\frac{\partial \hat{I}_m}{\partial t} - i\big[\hat{I}_m, \hat{H}_m\big] = 0. \tag{12}$$

The equivalence between different approaches to semiclassical quantum gravity was recently shown in Ref. 7. We shall, however, use the approach of Ref. 3 in which it is relatively easier to construct the coherent-state representation compared with other approaches.

Of many invariants, it was found that two particular invariants are very useful and convenient in constructing the Fock space [5]:

$$\begin{aligned} \hat{A}^\dagger(t) &= \phi_c(t)\hat{\pi}_\phi - a^3(t)\dot{\phi}_c(t)\hat{\phi}, \\ \hat{A}(t) &= \phi_c^*(t)\hat{\pi}_\phi - a^3(t)\dot{\phi}_c^*(t)\hat{\phi} \end{aligned} \tag{13}$$

where ϕ_c is a complex solution of Eq. (7) with the boundary conditions

$$\begin{aligned} a^3(t)\Big(\phi_c(t)\dot{\phi}_c^*(t) - \phi_c^*(t)\dot{\phi}_c(t)\Big) &= i, \\ \mathrm{Im}\left(\frac{\dot{\phi}_c(t)}{\phi_c(t)}\right) &> 0. \end{aligned} \tag{14}$$

The second boundary condition ensures the vacuum state is constructed coincident with the adiabatic vacuum. In fact, $\hat{A}^\dagger(t)$ acts as the creation operator and $\hat{A}(t)$ as the annihilation operator on the Fock space of number states:

$$\hat{A}^\dagger(t)\hat{A}(t)|n,t>= n|n,t> . \tag{15}$$

The exact quantum states are given by

$$\Phi_n(\phi,t) = e^{-i\omega_n(t)}|n,t> \tag{16}$$

where

$$\omega_n(t) = \int < n,t|\hat{H}_m - i\frac{\partial}{\partial t}|n,t> \tag{17}$$

is a time-dependent phase factor.

The Bogoliubov transformation between two different times is given by

$$\begin{aligned}\hat{A}^\dagger(t) &= u(t)\hat{A}^\dagger(t_0) + v(t)\hat{A}(t_0),\\ \hat{A}(t) &= v^*(t)\hat{A}^\dagger(t_0) + u^*(t)\hat{A}(t_0)\end{aligned} \tag{18}$$

where

$$\begin{aligned}u(t,t_0) &= ia^3\Big(\dot{\phi}_c(t)\phi_c^*(t_0) - \phi_c(t)\dot{\phi}_c^*(t_0)\Big),\\ v(t,t_0) &= ia^3\Big(\phi_c(t)\dot{\phi}_c(t_0) - \dot{\phi}_c(t)\phi_c(t_0)\Big).\end{aligned} \tag{19}$$

The relation

$$|u(t,t_0)|^2 - |v(t,t_0)|^2 = 1 \tag{20}$$

can be shown by direct substitution. The above relation can be parameterized as

$$\begin{aligned}u(t,t_0) &= \cosh\nu e^{-i\theta_u},\\ v(t,t_0) &= \sinh\nu e^{-i\theta_v}.\end{aligned} \tag{21}$$

Then, we find a unitary transformation of the creation operators between two different times,

$$\hat{A}^\dagger(t) = \hat{S}^\dagger(t,t_0)\hat{A}^\dagger(t_0)\hat{S}(t,t_0), \tag{22}$$

in terms of the squeeze operator

$$\begin{aligned}&\hat{S}(t,t_0) \;=\\ &\exp\Big(i\theta_u\hat{A}^\dagger(t_0)\hat{A}(t_0)\Big)\exp\Big(\frac{\nu}{2}e^{-i(\theta_u-\theta_v)}\hat{A}^{\dagger 2}(t_0) - \text{h.c.}\Big)\end{aligned} \tag{23}$$

The unitary transformation of the annihilation operators can also be found similarly by taking the hermitian conjugate of that of the creation operators. Note that the squeeze operator is a unitary operator.

We introduce the coherent states on the Fock space of the eigenstates of the invariant. At an arbitrary initial time, we define a coherent state by [8]

$$\hat{A}(t_0)|\alpha,t_0>= \alpha|\alpha,t_0> \tag{24}$$

where α is a complex number. In terms of the creation operator acting on the vacuum state at that time, they can be read as

$$|\alpha,t_0>= e^{-\frac{|\alpha|^2}{2}}\sum_0^\infty \frac{\alpha^n}{\sqrt{n!}}|n,t_0> . \tag{25}$$

It can be shown that the coherent states also transform unitarily,

$$|\alpha,t>= \hat{S}^\dagger(t,t_0)|\alpha,t_0> . \tag{26}$$

Thus, it follows that

$$\hat{A}(t)|\alpha,t>= \alpha|\alpha,t> . \tag{27}$$

The action of the creation operator is the hermitian conjugate of the annihilation operator $< \alpha,t|\hat{A}^\dagger(t) = \alpha^* < \alpha,t|$. The above coherent state is a particular case of the coherent and squeezed states of a time-dependent (an-)harmonic oscillator which were constructed group-theoretically in Ref. 9.

From the relations

$$\begin{aligned}\hat{\phi} &= (-i)\Big(\phi_c\hat{A}^\dagger(t) - \phi_c^*\hat{A}(t)\Big),\\ \hat{\pi}_\phi &= (-ia^3)\Big(\dot{\phi}_c\hat{A}^\dagger(t) - \dot{\phi}_c^*\hat{A}(t)\Big),\end{aligned} \tag{28}$$

we can show that $|\alpha,t>$ really describes the classical trajectory

$$< \alpha,t|\hat{\phi}|\alpha,t>= \varphi_c(t) \tag{29}$$

where

$$\varphi_c = \frac{\alpha^*\phi_c - \alpha\phi_c^*}{i} \tag{30}$$

is a real classical solution. The expectation value of the position operator gives, indeed, the real classical orbit. The expectation value of the scalar-field Hamiltonian taken with respect to the coherent state has the simple form

$$\begin{aligned}< \alpha,t|\hat{H}_m|\alpha,t> \;&=\; \frac{a^3}{2}\left(\dot{\varphi}_c^2(t) + m^2\varphi_c^2(t)\right)\\ &\quad+\frac{a^3}{2}\left(\dot{\phi}_c^*(t)\dot{\phi}_c(t) + m^2\phi_c^*(t)\phi_c(t)\right).\end{aligned} \tag{31}$$

Note that the last two terms of the expectation value come from the quantum fluctuation of vacuum and can be removed by the normal ordering of the operators:

$$< \alpha,t| : \hat{H}_m : |\alpha,t>= \frac{a^3}{2}\left(\dot{\varphi}_c^2(t) + m^2\varphi_c^2(t)\right). \tag{32}$$

Remembering that a coherent state is a superposition of the eigenstates of a particular invariant, we see that the decoupling theorem of Lewis and Riesenfeld [6] between off-diagonal terms still holds and that the semiclassical quantum gravity in the coherent-state representation is asymptotically exact.

In summary we have elaborated a previous scheme in which the canonical quantum gravity based on the Wheeler-DeWitt equation led to the semiclassical quantum gravity and the classical gravity [3,5]. The coherent-state representation is found to make the expectation value of the quantum energy-momentum tensor reduce to the classical one. Even though we showed the coherent-state representation for the quantum Friedmann-Robertson-Walker cosmology minimally coupled to a particular free massive scalar field, we put forth a *conjecture that there may exist coherent-state representations of the semiclassical quantum gravity for a generic geometry coupled to fundamental fields such as scalar fields and fermionic fields.*

The result of this paper may have an important implication for and application to cosmology. Assuming that the Wheeler-DeWitt equation is valid just below the Planck scale, we can investigate the conditions under which the semiclassical quantum and the classical gravities coupled to the inflaton emerge from the canonical quantum gravity. We can also study recent hot issues, such as the preheating due to the parametric resonance of a quantum scalar field coupled to the inflaton which executes a coherent oscillation, and, more fundamentally, whether such a mechanism for abundant particle creation and entropy production can be robust in the semiclassical quantum gravity. The applications of this formalism to cosmology will be addressed in a future publication.

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