21-01-26
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Analysis II - W.S 2

Basic Training Cycle
On classical Inverse functions

Exercise 1

1. Let $x \in \left]0, \frac{\pi}{2}\right[$. Show that

$$\sin x = \frac{\tan x}{\sqrt{1+\tan^2 x}} \qquad \cos x = \frac{1}{\sqrt{1+\tan^2 x}}$$

2. Show that

$$0 < \arctan\left(\frac{3}{4}\right) + \arctan\left(\frac{5}{12}\right) < \frac{\pi}{2}$$

3. Solve

$$\arcsin(x) = \arctan(\frac{3}{4}) + \arctan(\frac{5}{12})$$

Exercise 2

Let $x, y \in \mathbb{R}$ such that

$$x = \ln\left(\tan\left(\frac{y}{2} + \frac{\pi}{4}\right)\right)$$

compute $\cosh x$ and $\sinh x$.

Exercise 3

1. Compute

$$\cosh\left(\frac{1}{2}\ln 3\right) \text{ and } \sinh\left(\frac{1}{2}\ln 3\right)$$

2. Show

$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y).$$

3. Deduce the solution of the equation

$$2\cosh(x) + \sinh(x) = \sqrt{3}\cosh(5x)$$

Exercise 4

Solve the following equation

$$\ln(\cosh(x)) = 2.$$

Exercise 5

Let us consider the real function

$$f(x) = \arcsin\left(\frac{1-x^2}{1+x^2}\right)$$

Show that f is defined and continuous on $\mathbb{R}$.

Compute the derivative of f on $\mathbb{R}^*$, and deduce on which set the function f is differentiable.
Compute

$$\lim_{x\to\pm\infty} f(x).$$

Draw up the table of variations of f and sketch the graph of f.

Exercise 6

Study the variations and plot (or draw) the graph of the functions defined by the following equations

$$f(x) = \arctan\left(\frac{x}{1-x^2}\right) \qquad f(x) = \tanh\left(\frac{1}{x}\right)$$

Exercise 7

Solve the following equations

$$\cosh(x) = \sqrt{5} \qquad \arcsin(x) = \arccos\left(\frac{1}{3}\right) - \arccos\left(\frac{1}{3}\right) \qquad \arctan\left(\frac{x}{2}\right) = \pi$$

Exercise 8

1. Show that for all $x \in \mathbb{R}$

$$\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$$

2. Solve in the equation

$$\arccos\left(\frac{4}{5}\right) = 2\arctan\left(\frac{1}{3}\right)$$

Exercise 9

Exercise 10

The aim of this exercise is only to show that the function $\arccos(x)$, this function is not even in an orthonormal cartesian coordinate system whose origin is $(0,0)$, this assertion is not true if we make a translation towards the point $\left(0,\frac{\pi}{2}\right)$

$$\forall x \in [-1,1] : \arccos(x) + \arccos(-x) = \pi$$

Exercise 11

1. Show that for all $x, y \in \mathbb{R}$ such that $0 < x < y$:

$$\frac{x-y}{\ln y - \ln x} < \frac{x+y}{2}$$

2. Deduce that for all $n \in \mathbb{N}^*$:

$$\sum_{k=1}^{n} \frac{k}{\ln\left(1+\frac{1}{k}\right)} < \frac{n(n+1)(4n+5)}{12}$$

Exercise 12

Show that for all $x > 0$:

$$\ln\left(1+\frac{1}{x}\right) \leq \frac{1}{\sqrt{x(x+1)}}$$

Exercise 13

$\boxed{1}$ Study and Sketch the graph of the function

$$f(x) = \arcsin\left(2x^2 - 1\right)$$

$\boxed{2}$ Study and Sketch the graph of the function

$$f(x) = \arctan\left(\sqrt{\frac{1-\sin x}{1+\sin x}}\right)$$

$\boxed{3}$ Study and Sketch the graph of the function

$$f(x) = \arcsin\left(\frac{1-x^2}{1+x^2}\right)$$

Exercise 14

Show that

$\boxed{1}$ $\arctan x + \arctan 2x = \dfrac{\pi}{4}$

$\boxed{2}$ $2\arctan x = \arctan\left(\dfrac{2x}{1-x^2}\right) + \pi sgn(x)$

$\boxed{3}$ $\frac{\pi}{4} + \arctan x = \arctan\left(\dfrac{1+x}{1-x}\right)$

Exercise 15

$\boxed{1}$ Show that for $a, b \in [0,1]$: $\arctan a + \arctan b = \arctan\left(\dfrac{a+b}{1-ab}\right)$

$\boxed{2}$ Show that $1 + \cosh x + \cosh 2x + \cosh 3x + \ldots + \cosh nx = \dfrac{1}{2} + \dfrac{\cosh nx - \cosh(n+1)x}{2(1-\cosh x)}$

Exercise 16

Consider the function

$$f(x) = \frac{x}{2} - \arcsin\left(\sqrt{\frac{1+\sin x}{2}}\right)$$

$\boxed{1}$ Find the domain of definition of f denoted by D_f.

$\boxed{2}$ Show that

$$\forall x \in D_f : f(x+2\pi) = f(x) + \pi$$

$\boxed{3}$ Show that

$$\forall x \in D_f : f(x) + f(-x) = -\frac{\pi}{2}$$

$\boxed{4}$ Simpify the expression of f and draw its curve

Exercise 17

Let

$$f(x) = \arctan\left(\sqrt{\frac{1-x}{1+x}}\right)$$

$\boxed{1}$ Find the domain of definition of this function and study its differentiability

$\boxed{2}$ Simplify the expression of this function

Exercise 18

Show that

$$\arctan\left(2\sqrt{2}\right) + 2\arctan\left(\sqrt{2}\right) = \pi.$$

Exercise 19

Compute

$$\sin\left(\frac{1}{2}\arcsin\left(\frac{3}{4}\right)\right)$$