Ministry of Higher Education and Scientific Research
National Higher School of Cybersecurity
1st Year Integrated Preparatory Cycle **Academic year : 2024 -2025**
Mathematical Analysis 1

Worksheet 2: Limits and continuity

Exercise 1. *Find the domain of each function using interval notation*

1. $h(x) = \dfrac{1}{\sqrt{x} - \sqrt{2-x}}$ 2*. $h(x) = \sqrt{\dfrac{1-|x|}{2-|x|}}$ 3*. $h(x) = \ln \ln \ln x$

4. $h(x) = \dfrac{1}{[x]}$ 5. $h(x) = (1+\ln x)^{\frac{1}{x}}$ 6. $h(x) = \dfrac{1}{\sin x}$

7. $h(x) = \sqrt{\dfrac{x^2-1}{x^3-1}}$ 8. $h(x) = \ln(1-\cos 2x)$ 9. $h(x) = \ln\left(\dfrac{2-|x|}{|x|-1}\right)$

Exercise 2. *Let $a \in \mathbb{R}$. Determine for each function ,(according to the values of a) the domain of definition of*

1. $h_1(x) = \sqrt{a^2 - |x| + x^2}$ 2. $h_2(x) = \ln\left(\dfrac{1-ax}{1+ax}\right)$

and study its parity

Exercise 3. *Find the function f given by the following formula.*

1. $f(x-2) = \dfrac{1}{x+3}; x \neq 3$ 2. $f\left(\dfrac{1}{x}\right) = x^4 + 1$ 3. $f\left(x + \dfrac{1}{x}\right) = x^2 + \dfrac{1}{x^2}; x \neq 0$

Exercise 4. *The function f is defined on the closed interval $[0,1]$. Determine the set A on which the following composite functions can certainly be defined*

1. $h(x) = f(x^4)$ 2. $h(x) = f(\sin x)$ 3. $h(x) = f(x+3)$ 4. $h(x) = f(\ln x)$

Exercise 5. *Let the function $f : \mathbb{R} \to \mathbb{R}$ given with the formula*

$$f(x) = \frac{x}{\sqrt{1+x^2}}$$

Find the formula that gives the functions f_n defined on $\mathbb{R}$ by

$$f_n = \underbrace{f \circ f \overset{n}{\circ} f \circ ... f}, n \geq 2$$

Exercise 6. *Let*

$$f(x) = \frac{x}{1+|x|}$$

1. *Show that f is a bijection and find its inverse function.*

2. *Using the basic definition of monotony, show that f is strictly increasing function*

Exercise 7. *Check whether the following functions are periodic; if yes, find their basic periods T, if any.*

$$1.\ h(x) = \sin^2 x,\ \ x \in \mathbb{R} \qquad 2.\ h(x) = \sin\left(x^2\right),\ \ x \in \mathbb{R} \qquad 3.\ h(x) = \sin\left(|x|\right)$$

Exercise 8. *Study the parity of*

$$1.\ h(x) = \frac{\tan x - x}{x^3 \cos x} \qquad 2.\ h(x) = \frac{\sin^2(2x) - \cos(3x)}{\tan x}, \qquad 3.\ h(x) = \sin\left(\frac{1}{x}\right)$$

Exercise 9. *Say if the following functions are periodic*

$1.h_1(x) = \cos\left(\dfrac{x}{4}\right) + \cos(x) + \frac{1}{2}\cos(3x) + \dfrac{1}{3}\cos(5x), x \in \mathbb{R}$

$2^*.h_2(x) = \sin\left(\dfrac{1}{x}\right), x \in \mathbb{R}$

$3^*.h_3(x) = \sin(5x) + \cos(3x+1), x \in \mathbb{R}$

Exercise 10. *Let $f : \mathbb{R} \to \mathbb{R}^*$ such that*

$$\forall x \in \mathbb{R} : f(x) = f(x-1)\, f(x+1) \tag{0.1}$$

1. *Show that f is a periodic function*

2. *Check that the functions $x \mapsto e^{\cos\left(\frac{\pi x}{3}\right)}$ and $x \mapsto e^{\sin\left(\frac{\pi x}{3}\right)}$ are solutions to the equation (0.1)*

3. *Try to another solution to equation (0.1)*

Exercise 11. *Let $f : \mathbb{R} \to \mathbb{R}$ the function defined by*

$$f(x) = \frac{2x}{|x|+1}.$$

1. *Show that f is bounded is bounded on $\mathbb{R}$.*

2. *Show that f is an odd function .*

Exercise 12. *Using the abstract definitions of limits, show that*

$$\lim_{x\to 4}(2x-1) = 7 \qquad \lim_{x\to+\infty}\frac{3x-1}{2x+1} = \frac{3}{2} \qquad \lim_{x\to 1}\frac{\sqrt{x}-1}{\sqrt[3]{x}+1} = \frac{3}{2}$$

$$\lim_{x\to a}\sqrt[n]{x} = \sqrt[n]{a}, a > 0 \qquad \lim_{x\to -3^+}\frac{4}{x+3} = +\infty \qquad \lim_{x\to 0}\frac{\sqrt{x^2+x+1}-1}{x} = \tfrac{1}{2}$$

Exercise 13. *Let $f :]-a, a[\to \mathbb{R}^*$ be a real function satisfying*

$$\lim_{x\to 0}\left(f(x)+\frac{1}{f(x)}\right)=2$$

Show that

$$\lim_{x\to 0} f(x) = 1$$

Exercise 14. *Give an example of function f satisfying*

$$\lim_{x\to 0} f(x)f(2x) = 0$$

and the limit $\lim\limits_{x\to 0} f(x)$ does not exist

Exercise 15. *Show that the following limits do not exist*

$$\lim_{x\to 0}\cos\left(\frac{1}{x}\right) \qquad \lim_{x\to 0}\cos(\ln x) \qquad \lim_{x\to +\infty}\sin\frac{x^3}{x^2+1}$$

Exercise 16. ***Let $f, g : \mathbb{R} \to \mathbb{R}$ be periodic functions of periods a and b such that*

$$\lim_{x\to 0}\frac{f(x)}{x} = u \in \mathbb{R} \text{ and } \lim_{x\to 0}\frac{f(x)}{x} = v \in \mathbb{R}^*$$

Compute

$$\lim_{n\to +\infty}\frac{f\left((3+\sqrt{7})^n a\right)}{f\left((2+\sqrt{2})^n b\right)}$$

Exercise 17. *Let f a real function given by the following formula*

$$f(x) = \begin{cases} -2x & if \quad x \le -1 \\ x^2+1 & if \quad -1 < x \le 0 \\ \sin\dfrac{x}{4} & if \quad x > 0 \end{cases}$$

1. *Compute the following limits*

$$\begin{array}{ccc} \lim\limits_{x\to -1^-} f(x) & \lim\limits_{x\to 0^-} f(x) & \lim\limits_{x\to \pi^-} f(x) \\ \lim\limits_{x\to -1^+} f(x) & \lim\limits_{x\to 0^+} f(x) & \lim\limits_{x\to \pi^+} f(x) \end{array}$$

2. *What about*

$$\lim_{x\to -1} f(x) \qquad \lim_{x\to 0} f(x) \qquad \lim_{x\to \pi} f(x)$$

Exercise 18. *Compute the following limits*

1. $$\lim_{x\to 0}\frac{x^8+12x^6+3x^3}{x^7+4x^6+x^5+x^3} \qquad \lim_{x\to 2}\frac{x^3-8}{x^2-4} \qquad \lim_{x\to 1}\frac{x^{101}-101x+100}{x^2-2x+1}$$

Exercise 19. *Compute the following limits*

1. $\lim\limits_{x\to 2}\dfrac{x-2}{\sqrt{x+2}-2}$ $\quad\lim\limits_{x\to 8}\dfrac{\sqrt{8+x}-4}{\sqrt[3]{x}-2}$ $\quad\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+x+1}-x-1}{x}$

2. $\lim\limits_{x\to a^+}\dfrac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}, a>0$ $\quad\lim\limits_{x\to 0}\dfrac{\sqrt[n]{1+x}-1}{x}, n\in\mathbb{Z}^*$

*3**.* $\lim\limits_{x\to 0}\dfrac{\sqrt[n]{1+ax}.\sqrt[m]{1+bx}-1}{x}$ *with* $a,b\in\mathbb{R}^*, m,n\in\mathbb{Z}^*$

Exercise 20. *Compute the following limits*

1. $\lim\limits_{x\to 0}\dfrac{\sin(ax)}{\sin(bx)}, a,b\in\mathbb{R}^*$ $\quad\lim\limits_{x\to 0}\dfrac{2}{\sin(2x)\sin x}-\dfrac{1}{\sin^2 x}$

2. $\lim\limits_{x\to 0}\dfrac{\cos(2x^3)-1}{\sin^6(2x)}$ $\quad\lim\limits_{x\to 0}\dfrac{x^2}{\sqrt{1+x\sin x}-\sqrt{\cos x}},$

*3**.* $\lim\limits_{x\to 0}\dfrac{\cos(a+2x)-2\cos(a+x)+\cos(a)}{x^2}, a\in\mathbb{R}^*$

Exercise 21. *Compute the following limits*

1. $\lim\limits_{x\to 0}\dfrac{2\sin\left(\sqrt{x+1}-1\right)}{x},$ $\quad\lim\limits_{x\to 0}\dfrac{x^2}{\sqrt{1+x\sin x}-\sqrt{\cos x}}$

*2**.* $\lim\limits_{x\to 0}\dfrac{\sqrt[m]{\cos(ax)}-\sqrt[n]{\cos(bx)}}{x^2}, a,b\in\mathbb{R}, m,n\in\mathbb{N}$ $\quad\lim\limits_{x\to+\infty}\sin\sqrt{x+1}-\sin\sqrt{x},$

Exercise 22. *Compute the following limits*

1**. $\lim\limits_{x\to 0^+}\dfrac{x}{a}\left[\dfrac{b}{x}\right], b,c\in\mathbb{R}_+^*$ $\quad$ 2. $\lim\limits_{x\to 0}x\left[x-\dfrac{1}{x}\right]$ $\quad$ 3**. $\lim\limits_{x\to 0}\dfrac{[\ln x]}{x}$ $\quad$ 4. $\lim\limits_{x\to 0}\dfrac{[\ln\sqrt{x}]}{x}$

Exercise 23. *Compute the following limits*

1. $\lim\limits_{x\to 0^+}\left(x^2\left(1+2+3++....+\left[\dfrac{1}{x}\right]\right)\right)$

2**. $\lim\limits_{x\to 0^+}\left(\left[\dfrac{1}{x}\right]+2\left[\dfrac{2}{x}\right]+3\left[\dfrac{3}{x}\right]++....+\left[\dfrac{k}{x}\right]\right), k\in\mathbb{N}^*$

Exercise 24. ***Let* a *a real parameter and* $f:\mathbb{R}\to\mathbb{R}$ *the function defined by :*

$$f(x)=\begin{cases} x^2+x+1 & \text{if } x\geq 1\\ ax^3+bx+2 & \text{if } x<1.\end{cases}$$

Study the continuity of f *on* $\mathbb{R}$

Exercise 25. *Let* λ *a real parameter and* $f:\mathbb{R}\to\mathbb{R}$ *the function defined by :*

$$f(x)=\begin{cases} \dfrac{1}{1-x} & \text{if } x\leq\frac{1}{2}\\ \sqrt{x^2+\lambda x} & \text{if } x>\frac{1}{2}.\end{cases}$$

Exercise 26. ***Let $a > 0$ a real parameter and $f : \mathbb{R} \to \mathbb{R}$ the function defined by :*

$$f(x) = \begin{cases} x^2 + x + \dfrac{1}{a} & \text{if } x \leq 0 \\ \dfrac{\sin ax}{x} + (x-a)[x] - \sqrt{x} & \text{if } 0 < x \leq a. \end{cases}$$

Exercise 27. ***Let $f : \mathbb{R} \to \mathbb{R}$ the function defined by :*

$$f(x) = \begin{cases} \frac{x}{1+e^{\frac{1}{x}}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Study the continuity of this function on its domain of definition.

Exercise 28. *Let $f : \mathbb{R} \to \mathbb{R}$ the function defined by :*

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Study the continuity of this function on its domain of definition.

Exercise 29. ***Let $f : \mathbb{R} \to \mathbb{R}$ the function defined by :*

$$f(x) = \begin{cases} e^{\frac{1}{x}} & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x^2 \ln\left(1 + \frac{1}{x}\right) & \text{if } x > 0 \end{cases}$$

Study the continuity of this function on its domain of definition.

Exercise 30. ***Let $f : \mathbb{R} \to \mathbb{R}$ the function defined by :*

$$f(x) = \begin{cases} 1 + x\sqrt{x} & \text{if } x \geq 0 \\ 1 + \ln(1 + x^2) & \text{if } x < 0 \end{cases}$$

Study the continuity of this function on its domain of definition.

Exercise 31. *Let $f : \mathbb{R} \to \mathbb{R}$ the function defined by :*

$$f(x) = \begin{cases} \dfrac{\ln(1 + 2x^2)}{x} - 1 & x < 0 \\ b & x = 0 \\ x^2 + x - a & x > 0 \end{cases}$$

Determine the real numbers $a; b$ so that f is continuous on $\mathbb{R}$; in particular at the point $x = 0$.