2024/2025
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Algebra 2 - Tutorial 1

Basic Training Cycle
Matrices

Exercise 1

Compute the powers of the following matrices :

$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Exercise 2

Are the following matrices invertible ? If so, compute their inverse.

$$\begin{pmatrix} \cos a & 0 & -\sin a \\ 0 & 1 & 0 \\ \sin a & 0 & \cos a \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 \\ -2 & 3 & -1 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \ddots & 1 \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix}.$$

Exercise 3

Determine the rank of the following matrices depending on the parameters :

$$A = \begin{pmatrix} 1-a & 0 & 0 \\ -1 & 2-a & 1 \\ 2 & 0 & 3-a \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ bc & ca & ab \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix},$$

$$D = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 2 & 3 & 3 & \dots & n+1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n+1 & n+2 & \dots & 2n-1 \end{pmatrix}, \quad E = \begin{pmatrix} a & 1 & \dots & \dots & 1 \\ 1 & a & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & \dots & \dots & 1 & a \end{pmatrix}$$

Exercise 4

Let $A \in GL_n(\mathbb{K})$, $C \in GL_p(\mathbb{K})$, and $B \in \mathcal{M}_{n,p}(\mathbb{K})$.
Show that the block matrix (partitioned matrix) $\begin{pmatrix} A & B \\ 0_{p,n} & C \end{pmatrix} \in \mathcal{M}_{n+p}(\mathbb{K})$ is invertible, and determine its inverse.

Exercise 5

Let $A = \begin{pmatrix} 3 & 1 & -2 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}$.

$I)$ Define $P = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$.

1. Prove that P is invertible and find its inverse.
2. Compute $D = P^{-1}AP$, D^n then A^n.
3. Show that D is invertible and deduce that A is invertible.
4. Deduce the expression of A^{-n}.

$II)$

1. Let $B = A - 2I$. For $n \in \mathbb{N}^*$, compute B^n in terms of B.
2. Deduce A^n in terms of n, A and I.

$III)$

1. Show that $A^2 - 3A + 2I = 0$.
2. Prove by induction that there exist two sequences (a_n) and (b_n) such that, for all integers n,

$$A^n = a_n A + b_n I$$

   Give the recurrence relations satisfied by (a_n) and (b_n) and determine a_n and b_n in terms of n. Deduce the expression of A^n in terms of n, A, and I.
3. Justify that A is invertible and give its inverse.
4. Find the same expression of A^n by performing the Euclidean division of X^n by $X^2 - 3X + 2$.

Exercise 6

1. Show that any square matrix can be uniquely decomposed as the sum of a symmetric matrix and an antisymmetric matrix.
2. Let two symmetric matrices be given. Show that their product is symmetric if and only if the two matrices commute.
3. Show that, when it exists, the inverse of an antisymmetric matrix is also antisymmetric.

Exercise 7

Let $A \in GL_n(\mathbb{R})$ such that $A + A^{-1} = I_n$. Determine $A^k + A^{-k}$ for all $k \in \mathbb{N}$.

Exercise 8

Let $A = \begin{pmatrix} 1 & 1 & \lambda \\ \lambda - 1 & -1 & -1 \\ 0 & 2\lambda & 1 \end{pmatrix}$. Determine the values of $\lambda \in \mathbb{R}$ for which $A \in GL_3(\mathbb{R})$. For these values, compute A^{-1}. Do there exist complex values of λ that make A non-invertible in $M_3(\mathbb{C})$?

Exercise 9

Let $\text{tr}(A)$ denote the trace of a square matrix A, which is the sum of its diagonal entries.

1 Show that if A and B are two square matrices of the same size, and λ is a real number, the following relations hold :

- **a** $\text{tr}(\lambda A) = \lambda \text{tr}(A)$
- **b** $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$
- **c** $\text{tr}(AB) = \text{tr}(BA)$

2 Let $A \in \mathcal{M}_n(\mathbb{K})$. Show that the equation $AX - XA = I_n$, with unknown $X \in \mathcal{M}_n(\mathbb{K})$, has no solution.

Exercise 10

Let $n \in \mathbb{N}^*$. A matrix $A \in M_n(\mathbb{R})$ is said to be **nilpotent** if there exists $k \in \mathbb{N}$ such that $A^k = 0_n$. In this exercise, we consider a **nilpotent** matrix $A \in M_n(\mathbb{R})$.

1 Show that there exists $m \in \mathbb{N}^*$ such that $A^{m-1} \neq 0_n$ and $A^m = 0_n$. This integer m is called the **nilpotency index** of A.

2 Show that A is not invertible.

3 Show that for any nonzero natural number p, we have :

$$A^p - I_n = (A - I_n) \sum_{k=0}^{p-1} A^k.$$

Deduce that $I_n - A$ is invertible and compute its inverse.

4 Deduce the inverse of $M = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ (without using Gaussian elimination).

5 Show that $I_n - A^{-1}$ is nilpotent.

Exercise 11 : Proposed by the student Abderrahim Cherfaoui

Let A, B and C be $n \times n$ matrices with complex entries satisfying

$$A^2 = B^2 = C^2 \quad \text{and} \quad B^3 = ABC + 2I.$$

Prove that $A^6 = I$.

Exercise 12

Consider the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

1 Compute A^k for $k \in \mathbb{N}^*$.

2 Let $P \in \mathbb{R}_n[X]$, with $n \geq 2$. Prove that :

$$P(A) = \begin{pmatrix} P(2) & P'(2) & \frac{1}{2}P''(2) \\ 0 & P(2) & P'(2) \\ 0 & 0 & P(2) \end{pmatrix}.$$

Exercise 13

Let θ be a real number.

1 We define

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Compute R_θ^k for any natural number k.

2 Consider the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ defined by :

$$\begin{cases} x_0 = 1 \\ y_0 = 1 \\ \forall n \in \mathbb{N}, \quad x_{n+1} = \cos(\theta)x_n - \sin(\theta)y_n \\ \forall n \in \mathbb{N}, \quad y_{n+1} = \sin(\theta)x_n + \cos(\theta)y_n. \end{cases}$$

Using the previous question, find explicit expressions for the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$.