NSCS

المدرسة الوطنية العليا في الأمن السيبراني
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Basic Training Cycle
Functions

Algebra 1 - Tutorial 3

Exercise 1

Consider the function Day: $E = \{$Days of the year 2024$\} \to F = \{$Saturday, Sunday,..., Friday$\}$ that associates each date of the year 2024 with the corresponding day of the week. Is the Day function injective, surjective, or bijective?

Exercise 2

Let E, F, and G be the sets defined as follows: $E = \{1,2,3\}$, $F = \{A,B,C,D\}$, and $G = \{+,*\}$.

1 Construct a function $f_1 : E \to F$ that is injective, and then a function $f_2 : E \to F$ that is non-injective. Can a surjective function from E to F be constructed?

2 Construct a function $g_1 : F \to G$ that is surjective, and then a function $g_2 : F \to G$ that is non-surjective. Can an injective function from F to G be constructed?

Exercise 3

Are the following functions injective?

1 $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^2 \end{cases}$

4 $j : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto |x| \end{cases}$

7 $m : \begin{cases} [0,\pi] \to [0,1] \\ x \mapsto |\cos(x)| \end{cases}$

2 $g : \begin{cases} \mathbb{R}_+ \to \mathbb{R} \\ x \mapsto x^2 \end{cases}$

5 $k : \begin{cases} \mathbb{R}_+^* \to \mathbb{R} \\ x \mapsto \ln(x^2) - \ln(3x) \end{cases}$

8 $n : \begin{cases} \mathbb{Z} \to \mathbb{Z} \\ x \mapsto 2x \end{cases}$

3 $h : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto 2x - 3 \end{cases}$

6 $l : \begin{cases} [0,\frac{\pi}{2}] \to [0,1] \\ x \mapsto \sqrt{\sin(x)} \end{cases}$

9 $p : \begin{cases} \mathbb{R} \to \mathbb{Z} \\ x \mapsto \lfloor x \rfloor \end{cases}$

Exercise 4

Are the following functions surjective?

1 $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^2 \end{cases}$

3 $h : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto 2x - 3 \end{cases}$

5 $k : \begin{cases} \mathbb{R} \to \mathbb{R}^+ \\ x \mapsto \sqrt{e^x} \end{cases}$

7 $m : \begin{cases} \mathbb{R} \to \mathbb{Z} \\ x \mapsto \lfloor x \rfloor \end{cases}$

2 $g : \begin{cases} \mathbb{R}_+ \to [0,1] \\ x \mapsto x^2 \end{cases}$

4 $j : \begin{cases} \mathbb{R} \to \mathbb{R}^+ \\ x \mapsto |x| \end{cases}$

6 $l : \begin{cases} \mathbb{Z} \to \mathbb{Z} \\ x \mapsto 2x \end{cases}$

Exercise 5

Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$z \mapsto z^3 - 1$$

1 Is the function f injective?

2 Justify that the function f is surjective.

3 What are the preimages of 7 under f?

Exercise 6

Prove that the following functions are bijective. For f, h and k, find their inverses.

$\boxed{1}$ $f : \begin{cases} [0; +\infty[\to [-5, +\infty[\\ x \mapsto x^2 - 5 \end{cases}$

$\boxed{3}$ $h : \begin{cases}]6, +\infty[\to \mathbb{R}^*_+ \\ x \mapsto \frac{1}{x-6} \end{cases}$

$\boxed{4}$ $k : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x, y) & \mapsto & (x+y, x-y). \end{array}$

$\boxed{2}$ $g : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \sin(x) + 2x \end{cases}$

Exercise 7

Consider the following function:
$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^2 + 3x - 4.$$

$\boxed{1}$ Is it injective? Surjective?

$\boxed{2}$ Determine I and J, two intervals not reduced to a point such that $g = f|_I^J$ is bijective.

$\boxed{3}$ Give an expression for g^{-1}.

Exercise 8

Consider the function:
$$f : \mathbb{C} \setminus \{2i\} \to \mathbb{C}, \quad z \mapsto \frac{z^2}{z - 2i}.$$

$\boxed{1}$ Find the preimages of $1 + i$ under f.

$\boxed{2}$ Is f injective? Surjective? Bijective? Justify your answer.

Exercise 9

Let E and F be two sets and $f : E \to F$ a function.

$\boxed{1}$ Let $(A, B) \in \mathscr{P}(E)^2$.

    (a) Show that $A \subseteq B \Rightarrow f(A) \subseteq f(B)$. Is the converse true?

    (b) Show that $f(A \cup B) = f(A) \cup f(B)$.

    (c) Show that $f(A \cap B) \subseteq f(A) \cap f(B)$. Is there equality?

$\boxed{2}$ Let $(C, D) \in \mathscr{P}(F)^2$.

    (a) Show that $C \subseteq D \Rightarrow f^{-1}(C) \subseteq f^{-1}(D)$. Is the converse true?

    (b) Show that $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

    (c) Show that $f^{-1}(C \cup D) \subseteq f^{-1}(C) \cup f^{-1}(D)$. Is there equality?

Exercise 10

Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by $x \mapsto x^2$.
Find the following sets :

$$f^{-1}(\{1\}), f([1,4]), \quad f^{-1}([-1,4]), \quad f(f^{-1}([-1,4])), \quad f^{-1}(f([1,4])),$$

$$f([-3,-1] \cap [-2,1]), \quad f^{-1}(]-\infty, 2] \cap [1, +\infty[)$$

Exercise 11

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x - \lfloor x \rfloor$. Give the image of $\mathbb{R}$ under f.

Exercise 12

$\boxed{1}$ Give two functions f and g such that $h = f \circ g$ in the following cases:

    (a) $h(x) = \sqrt{x^2 + 3}$

    (b) $h(x) = \cos(\ln(x))$

    (c) $h(x) = (3x + e^x)^5$

$\boxed{2}$ Let f and g be two functions from $\mathbb{R}$ to $\mathbb{R}$ defined by:

$$f(x) = 3x + 1 \quad \text{and} \quad g(x) = x^2 - 1.$$

    Say whether $g \circ f$ and $f \circ g$ exist.

$\boxed{3}$ Do the same for the functions $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto e^x$, and $g : \mathbb{R}^* \to \mathbb{R}^*$, $x \mapsto \frac{1}{x}$.

$\boxed{4}$ Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be given by $x \mapsto \frac{x}{x+1}$, and let $n \in \mathbb{N}^*$. Determine $f \circ f \circ \cdots \circ f(x)$ (applied n times).

Exercise 13

Consider the functions:

$$f : \mathbb{N} \to \mathbb{N}, \quad n \mapsto 2n,$$

$$g : \mathbb{N} \to \mathbb{N}, \quad n \mapsto \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

$\boxed{1}$ Are f and g injective? Surjective?

$\boxed{2}$ Determine $g \circ f$ and $f \circ g$.

Exercise 14

Let f be defined as: $f : \mathbb{Z} \to \mathbb{Z}, \quad n \mapsto n + (-1)^n$.

$\boxed{1}$ Calculate $f \circ f$. What can be deduced about f?

$\boxed{2}$ Solve the equation: $347 = n + (-1)^n$ where $n \in \mathbb{Z}$.

Exercise 15

Let a be a real number. Consider the following function: $f_a : \mathbb{R} \setminus \{a\} \to \mathbb{R}, \quad x \mapsto \frac{1}{x-a} + a$.

$\boxed{1}$ Show that the image of the function f_a is included in $\mathbb{R} \setminus \{a\}$.

$\boxed{2}$ Compute $f_a \circ f_a$. What can we conclude? Illustrate the result graphically.

Exercise 16

Let E, F, G be three sets, and let $f : E \to F$ and $g : F \to G$ be two functions. Show that:

$\boxed{1}$ $g \circ f$ is injective $\implies$ f is injective.

$\boxed{2}$ $g \circ f$ is surjective $\implies$ g is surjective.

$\boxed{3}$ ($g \circ f$ is injective and f is surjective) $\implies$ g is injective.

$\boxed{4}$ ($g \circ f$ is surjective and g is injective) $\implies$ f is surjective.