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Basic Training Cycle

Anal 2 - Tutorial 7

Real Sequences

Exercise 1

(NHSM 25) For each of the following sequences (u_n) whose $n-$th term is given below, calculate the three first terms and determine expressions of u_{2n} and u_{n+1}.

$$u_n = \frac{1^2+2^2+3^2+\cdots+n^2}{n^3}, \quad u_n = \frac{1}{(n-2)(n+1)}, \; u_n = \frac{1.3.5...(2n-1)}{2.4...(2n)}, \quad u_n = \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}.$$

Exercise 2

(NHSM 25) Study the monotony of the following sequences:

$$u_n = \frac{2^n}{n!}, \quad U_n = n+(-1)^n, \quad u_n = 2n+(-1)^n, \quad u_n = n^2-2(-1)^n.$$

Exercise 3

Show that the following sequences are convergent and determine their limits

$$(1)\;\; u_n = \frac{\sum\limits_{k=1}^{n} E\left(kx\right)}{n^2}; \; x \in \mathbb{R}; \qquad (2)\;\; u_n = \sum_{k=0}^{2n} \frac{k}{k+n^2}; \qquad (3)\;\; u_n = \sum_{k=0}^{n} \frac{1}{\mathscr{C}_n^k}.$$

Exercise 4

(NHSM 25) By using (ε, N) definition of the limit, show that:

$\boxed{1}$ $\lim\limits_{n\to+\infty} na^n = 0 \; (|a|<1), \quad \lim\limits_{n\to+\infty} \sqrt[n]{a} = 1 \; (a>1), \quad \lim\limits_{n\to+\infty} \sqrt[n]{n} = 1.$

$\boxed{2}$ $\lim\limits_{n\to+\infty} \dfrac{\ln(n+1)}{\ln n} = 1, \quad \lim\limits_{n\to+\infty} \dfrac{\ln\left(1+e^n\right)}{2n} = \frac{1}{2}, \quad \lim\limits_{n\to+\infty} 3^{2n+1} = +\infty.$

Exercise 5

(NHSM 25) Determine the limit of each of the following sequences.

$\boxed{1}$ $u_n = \dfrac{\sin(n)}{n}, \quad u_n = \dfrac{n^n}{n!}, \quad u_n = \dfrac{E\left(n^2\sin\left(\frac{1}{n}\right)\right)}{2n+1}, \quad u_n = \dfrac{1+a+a^2+\ldots+a^n}{1+b+b^2+\ldots+b^n} (|a|<1, |b|<1),$

$\boxed{2}$ $u_n = \dfrac{3^n-7^n}{3^n+7^n}, \quad u_n = \sum_{k=1}^{n} \dfrac{1}{\sqrt{n^2+1}}, \quad u_n = \sum\limits_{k=0}^{k=n} \dfrac{k!}{(n+2)!}$

$\boxed{3}$ $u_n = \sum_{k=1}^{n} \dfrac{1}{\sqrt{k}}$, ***Hint.*** See that $\frac{1}{2\sqrt{k}} \geq \sqrt{k+1}-\sqrt{k}$.

Exercise 6

Study the convergence of the complex sequence defined by $u_0 \in \mathbb{C}$ and

$$\forall n \in \mathbb{N}; u_{n+1} = \frac{2u_n - \overline{u_n}}{3}.$$

Exercise 7

Let $(a;b) \in \mathbb{C}^2$ and $(z_n)_{n\in\mathbb{N}}$ a complex sequence such that

$$z_{2n} \to a \text{ and } z_{2n+1} \to b$$

Show that the sequence $(z_n z_{n+1})_{n\in\mathbb{N}}$ is convergent and determine its limit.

Exercise 8

Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ a real sequences in $\mathbb{R}_+^*$. Set

$$w_n = \frac{(u_n)^3 + (v_n)^3}{(u_n)^2 + (v_n)^2}$$

Show that

$$(u_n \to 0 \text{ and } u_n \to 0) \Rightarrow w_n \to 0.$$

Exercise 9

Let $a \in \mathbb{R}$ and $(u_n)_{n\in\mathbb{N}}$; $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ three real sequences such that

$$u_n + v_n + w_n \to 3a$$

and

$$(u_n)^2 + (v_n)^2 + (w_n)^2 \to 3a^2$$

Show that

$$u_n \to a \ ; \ v_n \to a; w_n \to a.$$

Exercise 10

Show that

$$u_n = \prod_{k=1}^{n} \left(1 + \frac{1}{k.k!}\right) \quad \text{and} \quad v_n = \left(1 + \frac{1}{n.n!}\right) u_n$$

are adjacent.

Exercise 11

Let $(u_n)_{n\in\mathbb{N}}$ the sequence defined by

$$\begin{cases} u_0 = 0 & u_1 = 1 \\ \forall n \in \mathbb{N} & u_{n+2} = u_{n+1} + u_n \end{cases}$$

$\boxed{1}$ Give the explicit expression of this sequence

$\boxed{2}$ Show that for all $n \in \mathbb{N}$

$$(u_{n+1})^2 - u_{n+2}u_n = (-1)^n$$

$\boxed{3}$ Deduce that $\left(\dfrac{u_{n+1}}{u_n}\right)_{n\in\mathbb{N}}$ converges

4 Show that for all $n \in \mathbb{N}$

$$\sum_{k=0}^{n} C_n^k u_k = u_{2n} \text{ and } \sum_{k=0}^{n} (-1)^k C_n^k u_k = -u_n.$$

Exercise 12

Let $(u_n)_{n\in\mathbb{N}}$ the sequence defined by

$$\begin{cases} u_0 = 0 & u_1 = 1 \\ \forall n \in \mathbb{N} & u_{n+2} = 10u_{n+1} - 21u_n + 12n \end{cases}$$

Calculate u_n.

Exercise 13

Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ a real sequences such that

$$u_0 = v_0 = 0$$

and

$$\forall n \in \mathbb{N} : \begin{cases} u_{n+1} = -u_n + 2v_n + 1 \\ v_{n+1} = -4u_n + 5v_n + 2^n \end{cases}$$

Give the explicit expression of these sequences.

Exercise 14

Give a complete study of the following sequences

$$\begin{cases} u_0 = 1 \\ u_{n+1} = \dfrac{(u_n)^2}{u_n + 1} \end{cases} ; \begin{cases} u_0 = 2 \\ u_{n+1} = \sqrt{u_n + 1} \end{cases} ; \begin{cases} u_0 \in \left[\frac{1}{3}; +\infty\right[\\ u_{n+1} = \sqrt{u_n - \dfrac{2}{9}} \end{cases}$$

Exercise 15

Give a complete study of the following real sequence

$$\begin{cases} u_1 > 0 \\ u_{n+1} = \dfrac{\sqrt{nu_n}}{u_n + 1}; n \in \mathbb{N}^* \end{cases}$$

Exercise 16

Let $(u_n)_{n\in\mathbb{N}}$ complex sequence such that the follwing . Assume that subsequences

$$(u_{2p})_{p\in\mathbb{N}};\ (u_{2p+1})_{p\in\mathbb{N}};\ (u_{3p})_{p\in\mathbb{N}}$$

are convergent. Show that $(u_n)_{n\in\mathbb{N}}$ is a convergent sequence.

Exercise 17

Let $(u_n)_{n\in\mathbb{N}}$ a $\mathbb{Z}-$valued sequence such that the follwing . Assume that subsequences Show that

$$(u_n)_{n\in\mathbb{N}} \text{ converges iff } (u_n)_{n\in\mathbb{N}} \text{ is constant.}$$

Exercise 18

Let $(u_n)_{n\in\mathbb{N}}$ a complex bounded sequence such that

$$u_0 = v_0 = 0$$

and

$$\forall n \in \mathbb{N} : u_{2n} = 2u_n - 1$$

Show that $(u_n)_{n\in\mathbb{N}}$ is constant.

Exercise 19

Let $(u_n)_{n\in\mathbb{N}}$ a real sequence such that

$$u_0 = u_1 = u_2 = 1$$

and

$$\forall n \in \mathbb{N} : u_{n+3} = \frac{u_{n+2}u_{n+1} + 1}{u_n}$$

$\boxed{1}$ Prove that

$$\forall n \in \mathbb{N} : u_{n+4} = 4u_{n+2} - u_n$$

$\boxed{2}$ Deduce that

$$\forall n \in \mathbb{N} : (u_n)_{n\in\mathbb{N}} \in \mathbb{N}^*$$

Exercise 20

Let $a < b \in]0;1[$ and Let $(u_n)_{n\in\mathbb{N}}$; $(v_n)_{n\in\mathbb{N}}$ real sequences such that

$$u_0 = a;\ v_0 = b$$

such that

$$u_0 = v_0 = 0$$

and

$$\forall n \in \mathbb{N} : u_{n+1} = (u_n)^{v_n} \text{ and } v_{n+1} = (v_n)^{u_n}$$

Show that $(u_n)_{n\in\mathbb{N}}$ converges and $(v_n)_{n\in\mathbb{N}}$ converges to 1

Exercise 21

Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be the sequences defined for $n \in \mathbb{N}$ by

$$\begin{cases} u_0 > 0 \\ u_{n+1} = \dfrac{u_n + v_n}{2} \end{cases} ; \quad \begin{cases} v_0 > 0 \\ v_{n+1} = \dfrac{u_n + \sqrt{u_n.v_n} + v_n}{2} \end{cases} ;$$

Show that $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ converge to the same limit ℓ and

$$v_1 < l < u_1.$$

Exercise 22

(NHSM 25) Consider the sequence defined recursively by

$$x_1 = \sqrt{2}, \quad x_n = \sqrt{2 + x_{n-1}}.$$

$\boxed{1}$ Show, by induction, that $x_n < 2$ for all n.

$\boxed{2}$ Show, by induction, that $x_n < x_{n+1}$ for all n.

$\boxed{3}$ Find the limit of $\{x_n\}$.

Exercise 23

(NHSM 25) Study the nature of the sequences of the n-th term

$$\left\{ \begin{cases} u_0 = \alpha \in \mathbb{R}, \\ u_{n+1} = u_n + 1, \ \forall n \in \mathbb{N}, \end{cases} \begin{cases} u_0 = \alpha \in \mathbb{R}, \\ u_{n+1} = \frac{u_n}{2} + 1, \ \forall n \in \mathbb{N}, \end{cases} \begin{cases} u_0 = 1, \\ u_{n+1} = \sqrt{u_n + 2}, \ \forall n \in \mathbb{N}. \end{cases} \right\}$$