Chapter 1 (Part 1): Mathematical Logic

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Outlines of this talk

- Statements
- Predicate, Quantifiers
- Methods of Proof

Statements

Statements

A Statement (or proposition) is a sentence that is definitely true or definitely false

- Examples:
  1- Every number is divisible by 2.
  2- Every even number is divisible by 2.
  3- Oran is the capital of Algeria.
  4- "A" comes after "B" in the English alphabetic.
  5- Mahrez scored 572 goals during his career.

- Non-Examples:
  1- x is even.
  2- She is smart.
  3- He is a good student.
  4- $x + 3 \geq 2$
- More examples of statements:

  P: For every integer $n > 1$, the number $2^n - 1$ is prime.
  Q: Every polynomial of degree n has at most n roots.

Statements

If P is a statement and Q is another statement, we will define new statements constructed from P and Q

A logical conjunction

The conjunction of P and Q (read: P and Q) is the statement $P \wedge Q$ which asserts that P and Q are both true. The "P and Q" statement is false otherwise.

We summarize this in a truth table:

p	q	p ∧ q	q ∧ p
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

Figure: Truth table of $P \wedge Q$

Example:

1- $(2 + 1 = 5) \wedge (3 + 6 = 1)$

2- P: The number 1 is even and the number 3 is odd.

3- Q : The number 3 is even and the number 2 is odd.

4- R:The number 2 is even and the number 4 is odd.

Statements

A logical disjunction

The disjunction of P and Q(read: P or Q) is the statement $\vee$ which asserts that either P is true, or Q is true, or both are true, and is false otherwise.

We summarize this in a truth table:

p	q	p ∨ q	q ∨ p
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

Figure: Truth table of $P \vee Q$

Examples: P : $\sqrt{2}$ is rational and Q : $\frac{1+\sqrt{5}}{2} < 2$.

The statement $P \vee Q$ is true because Q is true.

P: The number 2 is even or the number 3 is odd.

Q: The number 1 is even or the number 3 is odd.

R:The number 2 is even or the number 4 is odd.

S:The number 3is even or the number 2 is odd.

Statements

The negation "Not"

A negation is an operator on the logical value of a proposition that sends true to false and false to true. The negation (or logical Not) of P, denoted $\neg P$.

p	¬ p
T	F
F	T

Figure: Truth table of $\neg P$

Statements

The implication (conditional statement)

The statement $\neg P \vee Q$ is denoted by $P \Rightarrow Q$. We say :

1)P implies Q

2)if P, then Q

- The statement P is called the hypothesis of the implication, and the statement Q is called the conclusion of the implication.
- The truth table is:

P	Q	$\neg P$	P$\Rightarrow Q\,(\neg P \vee Q)$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Statements

Remark

1) $Q \to P$, this is the converse of $P \to Q$
2) $\neg P \to \neg Q$, this is the inverse of $P \to Q$,
3) $\neg Q \to \neg P$, this is the contrapositive of $P \to Q$

Examples:

1) P: $5^2 < 0$ and Q : $1 < 2$. ($P \to Q$ is true)
2) "$2 + 2 = 5 \to \sqrt{2} = 2$" (true)(if P is false, the statement $P \to Q$ is always true".

Statements

The biconditional or double implication

(read: P if and only if Q) is defined by

$$P \iff Q\text{" is the statement " } (P \rightarrow Q) \text{ and } (Q \rightarrow P)\text{"}$$

This statement is true when P and Q are true or when P and Q are false.

The truth table is:

p	q	p $\Rightarrow$ q	q $\Rightarrow$ p	p $\Leftrightarrow$ q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Figure: Truth table of $P \iff Q$

Example: P: "$1 = 2$" , Q: "the number of primes is finite"
$P \iff Q$ is true because both P and Q are false.

Statements

Proposition

Let P, Q, R be three statements. We have the following (true) equivalence:

- $P \iff \neg(\neg P)$
- $P \wedge Q \iff Q \wedge P$ (commutative law for conjunction)
- $P \vee Q \iff Q \vee P$ (commutative law for disjunction)
- $(P \wedge Q) \wedge R \iff P \wedge (Q \wedge R)$ (associative law for conjunction)
- $(P \vee Q) \vee R \iff P \vee (Q \vee R)$ (associative law for disjunction)
- $\neg(P \wedge Q) \iff (\neg P) \vee (\neg Q)$ (De Morgan's law)
- $\neg(P \vee Q) \iff (\neg P) \wedge (\neg Q)$ (De Morgan's law)
- $(P \wedge (Q \vee R)) \iff (P \wedge Q) \vee (P \wedge R)$ (distributive law)
- $(P \vee (Q \wedge R)) \iff (P \vee Q) \wedge (P \vee R)$ (distributive law)
- $P \to Q \iff \neg Q \to \neg P$ (law of the contrapositive)

Tautology, Contradiction,contingency

1) A tautology, if it is always true. Example:$p \vee \neg p$.
2) A contradiction, if it always false. Example:$p \wedge \neg p$.
3) A contingency, if it is neither a tautology nor a contradiction. Example:p.

Predicates and Quantifiers

Predicates

A predicate is a statement that contains variables. A predicate may be true or false depending on the values of these variables.A predicate is noted by $P(x), Q(x), ..$ where x is in the set D.

Example:

1- $P(x)$: x is human.

2- $Q(x, y)$: x is the parent of y.

3- $R(x)$:$x + 2 = x^2$

Quantifiers

The universal quantifier

$$\forall x \,(\text{for all x})\,, P(x)$$

means that the predicate $P(x)$ is true for all possible values of x. An element for which $P(x)$ is false is called a counterexample of $\forall x, P(x)$.

Example:

- "For all integers n, the integer $n(n+1)$ is even".
- We could take a first step towards a symbolic representation of this statement by writing

$$"\forall n \in \mathbb{Z},\;\; n(n+1) \text{ is even}$$

Quantifiers

The existential quantifier

$$\exists x \, (\text{exists x}) \, , P(x)$$

Means that there exists an x where $P(x)$is true.
Sometimes, we will use also

$$\exists ! x \, (\textbf{exists x}) \, , P(x)$$

It means that there exists a unique x where $P(x)$ is true.

- Synonyms for "there exists" include "there is, "there are, "some", and "at least one".

  Example:

- " there exists an integer n such that $n^2 - n + 1 = 0$.
- A symbolic representation of this statement is obtained by writing

$$\exists n \in \mathbb{Z}, n^2 - n + 1 = 0,$$

Quantifiers

Negating statements involving quantifiers:

1) The negating of "$\forall x \in E \ p(x)$" is $\exists x \in E \neg p(x)$.

2) The negating of "$\forall x \in E \ p(x)$" is $\exists x \in E \neg p(x)$.

Example:

1) The negation of "$\forall x \in [1, +\infty[\ (x^2 \geq 1)$" is the assertion "$\exists x \in [1, +\infty[\ (x^2 < 1)$".

2) The negation of $\exists \ z \in \mathbb{C} \ (z^2 + z + 1 = 0)$ is $\forall \ z \in \mathbb{C} \ (z^2 + z + 1 \neq 0)$.

It is not more difficult to write the negation of complex sentences. For the statement "$\forall x \in \mathbb{R}, \exists \ y > 0 \ (x + y > 10)$" its negation is "$\exists x \in \mathbb{R}, \forall \ y > 0 \ (x + y \leq 10)$".

Quantifiers

Remarks :

1) The order of quantifiers is very important. For example the two logical sentences are different.

$$"\forall x \in A,\ \exists y \in B\ ,\quad P(x,y)"$$

$$"\exists y \in B,\ \forall x \in B\ ,\quad P(x,y)"$$

Example: " In every universty there is a library "
$\forall U \quad university, \exists l \quad library : \quad l \in U$ (Proposition true)
" There is a library which is located in every university "
$\exists l, \forall U \quad : l \in U$

Remark

1) $\forall x \in A, \forall y \in B \Leftrightarrow \forall y \in B, \forall x \in A$

2) $\exists x \in A, \exists y \in B \Leftrightarrow \exists y \in B, \exists x \in A$

A proof is an argument that demonstrates why a conclusion is true. A mathematical proof is an argument that demonstrates why a mathematical statement is true, following the rules of mathematics.

Methods of Proofs

Direct proof

To prove a statment of the form

$$If \quad P, then \quad Q$$

Assum that P is true, then show that Q must be true as well.

Example: For each $x \in \mathbb{Z}$, if x is even, then $5x + 3$ is odd.
Proof:(exercise)

Remark

1) Let $n \in \mathbb{Z}$. We say n is an even integer if $n = 2k$ for some $k \in \mathbb{Z}$.

2) Let $n \in \mathbb{Z}$. We say that $n$is an odd integer if $n = 2k + 1$ for some $k \in \mathbb{Z}$.

Contrapositive proof

Let P and Q be statements. Then

$$P \Rightarrow Q \Leftrightarrow \neg Q \Rightarrow \neg P$$

Example1: Given $x \in \mathbb{Z}$, if $3x - 7$ is even, then x is odd.
Proof:(exercise)

Example 2

If the integer n^2 is even, then n is even.

Proof: Suppose that the integer n is not even, that is, it is odd. We want to show that n^2 is odd. Since n is odd, there exists an an integer k so that $n = 2k + 1$. Then,

$$n^2 = (2k+1)^2 = 4k^2 + 1 + 4k = 4k^2 +' k + 1 = 2(2k^2 + 2k) + 1.$$

Since $2k^2 + 2k$ is an integer, n^2 is odd.

proof by cases looks as follows:

Proof by cases

proof $P \Rightarrow Q$ by cases:

-1 Write $P \Leftrightarrow r_1 \vee r_2 \vee r_3 \vee .... \vee r_k$

-2 Separately prove $r_i \Rightarrow Q$ for each i, using any method.

-3 Conclude that $P \Rightarrow Q$, since $P \Rightarrow Q \Leftrightarrow (r_1 \Rightarrow Q)(r_2 \Rightarrow Q)...(r_k \Rightarrow Q)$.

Example

If the integer n^2 is a multiple of 3, then n is a multiple of 3.

Remark In the following example we make use the fact that every integer n can be uniquely written in the form $3k + r$, where k is an integer and r equals $0, 1, 2$

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Example

If the integer n^2 is a multiple of 3, then n is a multiple of 3.

Remark In the following example we make use the fact that every integer n can be uniquely written in the form $3k + r$, where k is an integer and r equals $0, 1, 2$ The integer r is the remainder on dividing n by 3. When the remainder equals 0 we have $n = 3k$, so that n is a multiple of 3.

Proof We prove the contrapositive: if n is not a multiple of 3, then n^2 is not a multiple of 3. Suppose n is not a multiple of 3. Then the remainder when n is divided by 3 equals 1 or 2. This leads to two cases:

- Case 1. The remainder on dividing n by 3 equals 1. Then, there exists an integer k so that $n = 3k + 1$. Hence $n^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$. Since $(3k^2 + 2k)$ is an integer, the remainder on dividing n^2 by 3 equals 1. Therefore n^2 is not a multiple of 3.

- Case 2. The remainder on dividing n by 3 equals 2. Then, there exists an integer k so that $n = 3k + 2$. Hence $n^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$. Since $(3k^2 + 4k + 1)$ is an integer, the remainder on dividing n^2 by 3 equals 1. Therefore n^2 is not a multiple of 3.

Both cases have now been considered. In each of them, we have shown that n^2 is not a multiple of 3. It now follows that if n is not a multiple of 3, then n^2 is not a multiple of 3. This completes the proof.

Exercise

If the $x \in \mathbb{Z}$, then $x^2 + x$ is even.

Contradiction:

Another technique is proof by contradiction.

Proof by Contradiction

Let R and S be statements.

Contradiction:

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Proof by Contradiction

Let R and S be statements.If $(\neg R) \Rightarrow S$ is true and S is false, then R must be true.

Example

$\sqrt{2}$ is irrational.

Proof: Assume $\sqrt{2}$ is rational , so $\sqrt{2} = \frac{a}{b}$ with a, b have no common factors other than 1 and $b \neq 0$.

Contradiction:

Another technique is proof by contradiction.

Proof by Contradiction

Let R and S be statements.If $(\neg R) \Rightarrow S$ is true and S is false, then R must be true.

Example

$\sqrt{2}$ is irrational.

Proof: Assume $\sqrt{2}$ is rational , so $\sqrt{2} = \frac{a}{b}$ with a, b have no common factors other than 1 and $b \neq 0$.
$\sqrt{2} = \frac{a}{b} \Rightarrow 2b^2 = a^2$
so b^2 is even
so by By Example 2,,a is even .
$\exists k \in \mathbb{Z} | a = 2k$ .It now follows that $2b^2 = a^2 = (2k)^2 = 4k^2$, so that $b^2 = 2k^2$.
Therefore b^2 is even so By Example 2, b is even. We have now derived the contradiction (a and b have no common factors) and (a and b are both even).
Therefore, $\sqrt{2}$ is not rational.

Counterexample:

Proof by Counterexample

To show that a proposition $P : \forall x \in D, P(x)$ is false, we must show that $\neg P$ is true.

Example

Let $f : \mathbb{R} \to \mathbb{R}$ a function defined by $f(x) = x^2$
Show that the proposition :

$$\forall (x, y) \in \mathbb{R}^2, f(x) = f(y) \Rightarrow x = y$$

is false.

Proof by Induction:

Proof by induction is a way of proving that a certain statement is true for every positive integer. Proof by induction has four steps:

1) Prove the base case: this means proving that the statement is true for the initial value, normally $n = 1$ or $n = 0$

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2) Assume that the statement is true for the value $n = k$. This is called the inductive hypothesis.
3) Prove the inductive step: prove that if the assumption that the statement is true for $n = k$, it will also be true for $n = k + 1$.
4) Write a conclusion to explain the proof, saying: "If the statement is true for $n = k$, the statement is also true for $n = k + 1$. Since the statement is true for $n = 1$, it must also be true for $n = 2$, $n = 3$ and for any other positive integer."

Example

Prove that for all positive integers n, $3^{2n+2} + 8n - 9$ is divisible by 8.

Proof by Induction:

Proof: First define $P(n) = 3^{2n+2} + 8n - 9$ is divisible by 8, for $n \geq 1$

- **Base Case**. For $n = 1$ we have $P(1) = 3^{2+2} + 8 - 9 = 3^4 - 1 = 81 - 1 = 80$. Then, 80 is clearly divisible by 8, hence $P(1)$ is true.

- **Inductive Step**. Let us set $n \geq 1$. Suppose $P(n)$ is true. We will show that $P(n+1)$ is true.

$$\begin{aligned} P(n+1) &= 3^{2(n+1)+2} + 8(n+1) - 9 \\ &= 3^{2n+4} + 8n + 8 - 9 = 3^{2n+4} + 8n - 9 + 8 \\ &= 9 \times 3^{2n+2} + 8n - 9 + 8 \\ &= 3^{2n+2} + 8 \times 3^{2n+2} + 8n - 9 + 8 = 3^{2n+2} + 8n - 9 + 8 \times 3^{2n+2} + 8 \\ &= P(n) + 8 \times 3^{2n+2} + 8 \end{aligned}$$

Since this is the sum of different terms that are all divisible by 8, $P(n+1)$ must also be divisible by 8 too.

- **Conclusion**. Hence, by the principle of mathematical induction, $P(n)$ is true for all integers $n \geq 1$.