Elementary Analysis through Examples and Exercises

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Elementary Analysis through Examples and Exercises

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Preface

It is hard to imagine that another elementary analysis book would contain material that in some vision could qualify as being new and needed for a discipline already abundantly endowed with literature. However, to understand analysis, beginning with the undergraduate calculus student through the sophisticated mathematically maturing graduate student, the need for examples and exercises seems to be a constant ingredient to foster deeper mathematical understanding. To a talented mathematical student, many elementary concepts seem clear on their first encounter.

However, it is the belief of the authors, this understanding can be deepened with a guided set of exercises leading from the so called "elementary" to the somewhat more "advanced" form. Insight is instilled into the material which can be drawn upon and implemented in later development. The first year graduate student attempting to enter into a research environment begins to search for some original unsolved area within the mathematical literature.

It is hard for the student to imagine that in many circumstances the advanced mathematical formulations of sophisticated problems require attacks that draw upon, what might be termed elementary techniques. However, if a student has been guided through a serious repertoire of examples and exercises, he/she should certainly see connections whenever they are encountered.

The seven chapters in this book contain, in the authors' opinion, a wide variety of problems, exercises and examples implemented by many instructors. The book can be used both by complete beginners in analysis, as well as by students that already have gone successfully through some calculus courses. The presented material is self-contained and the exposition is mostly deductive. Occasionally some notions are used before their formal definition, though, presumably, they will usually be known to the reader.

The content is in fact elementary but the strategy employed is to navigate through a wide assortment of problems connected to the nature of the chapter. A minimal amount of expository discussion is included at the start of each new section with a maximum emphasis placed on well selected examples and exercises capturing the essence of the material. It is the intention of the authors to have students assemble a very valuable collection of well-thought-out problems. We have separated the problems into examples and exercises. The examples contain a complete solution. In the exercises students are left to solve the problem. Often, answers only are provided. One should note that the included exercises in some cases are generally found in very advanced texts when they are needed to prove a more profound result. We feel that the students should study some of the pertinent advanced exercises in the early developmental stages, so that they can enhance their mathematical skills.

Chapter 1 introduces the field of real numbers from the traditional axiomatic presentation, as well as the R. Dedekind construction presented in 1872. Comparing the two methods is, in our opinion, vital to understanding new concepts. Contrast oftentimes invites the student to see why and where things differ and also where things go wrong. The induction principle is also included with a very complete set of examples and exercises implementing the process varying from the traditional problems to what we term the somewhat unusual problems.

Chapter 2 introduces the concepts of relations and functions. It includes a comprehensive review of the basic ideas and terminology implemented to express the different concepts. Various examples presented are oftentimes not included in a very elementary text, like, for instance, the Dirichlet function. Also included is the connection between rational functions and their representation using partial fraction decomposition.

Chapter 3 introduces the concept of a sequence of real numbers. The traditional theory surrounding sequences is presented such as limits, monotonicity and Cauchy sequences. However, it goes on to identify asymptotic behavior for sequences as well as proving some important estimates on the number e, which is the basis of what is probably the most important function in analysis, namely the exponential function. The discrete case is often drawn into the continuous case by examining such functions as the logarithmic, $f(x) = \ln x,\ x > 0$, and showing its connection to the discrete version, $f_n = \ln n,\ n \in \mathbf{N}$. Furthermore, iterative schemes are examined and how they can be implemented to generate converging sequences. Again the Landau notation commonly called "big oh" and "small oh" are included, showing how their asymptotic behavior definition can be reflected into sequences.

Chapter 4 introduces classical notion of limit to include the left and right limits. Special emphasis is placed on their connections to points of accumulation. Several examples illustrate the change of variable technique and how this process fosters good arithmetic behavior, so that a precise limit can be computed. The line asymptotes for the graphs of the functions are given as the geometrical application of limits. The Landau notation is again revisited and several examples and exercises show the students how these relationships control asymptotic behaviors.

Chapter 5 presents the classical notion of continuity and its relationship to points and points of accumulation. Left-side and right-side continuity, uniform continuity and theorems relating these properties are presented in a traditional manner. The delta–epsilon techniques together with their quantifiers are discussed in detail together with accompanying examples and exercises. Several somewhat unusual examples are presented to illustrate connections to the notion of the order of discontinuity at a point.

Chapter 6 introduces the derivative using a detailed discussion involving the left and right-hand derivatives. The traditional results regarding derivatives and their geometrical interpretations are included together with a rich assortment of examples and exercises. It continues to develop differentials and their connections to derivatives and the derivative remainders. Again examples illustrating these notions and calculated for some rather sophisticated functions are given. This chapter develops many applications for derivatives. The traditional results are included with several applications not necessarily found in many elementary texts. It identifies and proves the Laguerre, Hermite and Chebychev polynomials normally termed the special functions in mathematical physics have real roots. Several inequalities are proven using the mean value theorem for the differential calculus. The Taylor and Maclaurin formulas are included with a generous amount of examples and exercises. Of course, all of the standard information regarding preliminary curve sketching implementing first and second order derivative tests is included, together with a presentation of L'Hospital's rule.

Chapter 7 contains a detailed account for graphing functions. For this chapter the asymptotes to include slanted asymptotes, concavity, (local) maximums and minimums and points of inflection are presented again with some rather interesting functions. Let us note here each example is endowed with the appropriate graph of the given function. The figures were carefully produced, though they primarily serve as an illustration of the analytically achieved results. For such methodical reasons, the unit lengths on the $x-$ and $y-$axis are in some figures nonequal.

The extensive bibliography found at the end of the book is to provide the student with a wide range of resource material. It is also there to help indicate that this set of examples and exercises could benefit students studying from among a very broad spectrum of mathematical disciplines.

The origin of this book lies in several analysis and calculus courses that the authors gave to students on various levels and with, occasionally, quite different mathematical backgrounds. The experience we got through these lectures, is endowed, we hope successfully, in the given examples and exercises. We would like to thank many colleagues and students for their remarks, corrections, new problems (at least for us), their original, better or more precise solutions, and some other contribution(s) to this book. In particular, it is our pleasure to thank academician Dr. Olga Hadžić, from the Institute of Mathematics at the University of Novi Sad, for the careful reading of some early versions of the manuscript and her numerous improvements in the text. The hard job of preparing the figures was done by Mr. Milan Manojlović.

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Chapter 1

Real numbers

1.1 $(\mathbf{R}, +, \cdot, \leq)$ as a complete totally ordered field

1.1.1 Basic notions

Definition 1.1. *The* **field of real numbers** $\mathbf{R} = \{x, y, z, ...\}$ *is a set with two operations* addition *(+) and* multiplication *($\cdot$), and a* binary relation *termed less than or equal* ($\leq$) *defined with the following axioms.*

(R1) $(\forall x, y, z \in \mathbf{R}) \quad (x + y) + z = x + (y + z)$
(associative law for addition);

(R2) $(\exists 0 \in \mathbf{R})\, (\forall x \in \mathbf{R}) \quad x + 0 = 0 + x = x$
(existence of the additive identity element);

(R3) $(\forall x \in \mathbf{R})\, (\exists (-x) \in \mathbf{R}) \quad (-x) + x = x + (-x) = 0$
(existence of the additive inverse);

(R4) $(\forall x, y \in \mathbf{R}) \quad x + y = y + x$
(commutative law for addition);

(R5) $(\forall x, y, z \in \mathbf{R}) \quad (x \cdot y) \cdot z = x \cdot (y \cdot z)$
(associative law for multiplication);

(R6) $(\exists 1 \in \mathbf{R} \setminus \{0\})\, (\forall x \in \mathbf{R}) \quad x \cdot 1 = 1 \cdot x = x$
(existence of the multiplicative identity element);

(R7) $(\forall x \in \mathbf{R} \setminus \{0\})\, (\exists x^{-1} \in \mathbf{R} \setminus \{0\}) \quad x^{-1} \cdot x = x \cdot x^{-1} = 1$
(existence of the multiplicative inverse);

(R8) $(\forall x, y, z \in \mathbf{R}) \quad x \cdot (y + z) = x \cdot y + x \cdot z, \quad (x + y) \cdot z = x \cdot z + y \cdot z$
(distributive law of multiplication over addition);

(R9) $(\forall x, y \in \mathbf{R}) \quad x \cdot y = y \cdot x$
(commutative law for multiplication);

(R10) $(\forall x, y, z \in \mathbf{R}) \quad (x \leq y \wedge y \leq z) \Rightarrow x \leq z$
(transitivity of the binary relation $\leq$);

(R11) $(\forall x, y \in \mathbf{R}) \quad (x \leq y \wedge y \leq x) \Rightarrow x = y$
(antisymmetry of the binary relation $\leq$);

(R12) $(\forall x, y \in \mathbf{R}) \quad x \leq y \vee y \leq x$
(total ordering of the binary relation $\leq$);

(R13) $(\forall x, y, z \in \mathbf{R}) \quad x \leq y \Rightarrow (x + z \leq y + z)$
(compatibility of the binary relation $\leq$ with addition);

(R14) $(\forall x, y \in \mathbf{R}) \quad (0 \leq x \wedge 0 \leq y) \Rightarrow 0 \leq x \cdot y$
(compatibility of the binary relation $\leq$ with multiplication);

(R15) *Let X and Y be two nonempty subsets of the set $\mathbf{R}$ with the property*

$$(\forall x \in X)\,(\forall y \in Y) \quad x \leq y.$$

Then there exists an element $c \in \mathbf{R}$, such that

$$(\forall x \in X)\,(\forall y \in Y) \quad x \leq c \leq y$$

(completeness of the set $\mathbf{R}$).

From the axioms (R1), (R2) and (R3) it follows that $(\mathbf{R}, +)$ is a group, while by (R4) it becomes a commutative (or: Abelian) group. In the same manner, from the axioms (R5), (R6), (R7) and (R9) it follows that $(\mathbf{R} \setminus \{0\}, \cdot)$ is also a commutative group. The axiom (R8) gives the distributivity of multiplication over addition. The axioms (R10), (R11) and (R12) show that $\mathbf{R}$ is a totally ordered set with the binary relation $\leq$, while the axioms (R13) and (R14) give the connection of $\leq$ with the operations of addition and multiplication. The axioms (R1) - (R14) define $\mathbf{R}$ as a totally ordered field, and, finally, with (R15) the set $\mathbf{R}$ becomes a *complete totally ordered field.*

From now on we shall define $b - a := b + (-a)$ and call the operation " $-$ " *subtraction*, define $a/b := a \cdot b^{-1}$, $b \neq 0$, and call the operation "/" *division.* Instead of $x \cdot y$ we shall often write xy for multiplication.

Let x and y be two real numbers such that $x \leq y$. We define another binary relation " $\geq$ " by

$$(\forall x, y \in \mathbf{R}) \quad y \geq x \iff x \leq y.$$

If both $x \leq y$ and $x \neq y$, we shall simply write $x < y$. Also, we shall define $x > y \iff y < x$.

Definition 1.2. *The* **absolute value** *of a real number a is defined by*

$$|a| = \begin{cases} a, & \text{if} \quad a > 0; \\ 0, & \text{if} \quad a = 0; \\ -a, & \text{if} \quad a < 0. \end{cases}$$

A real number x is **positive** iff $x > 0$, resp. $x \in \mathbf{R}$ is **negative** iff $x < 0$.

We shall define the **distance** between two elements x and y from the set $\mathbf{R}$ as $|x - y|$. Clearly, the absolute value $|a|$ of a real number a gives its distance from the origin, i.e., from 0.

Definition 1.3. *An element $m \in \mathbf{R}$ is a* **lower bound** *(resp.* **upper bound***) of a nonempty set $X \subset \mathbf{R}$ if for every $x \in X$ it holds $m \leq x$ (resp. $m \geq x$).*

A set $X \subset \mathbf{R}$ is **bounded from below** *(resp.* **bounded from above***) if it has at least one lower (resp. upper) bound.*

A set $X \subset \mathbf{R}$ is **bounded** *if it is both bounded from below and from above. A set $X \subset \mathbf{R}$ is* **unbounded** *if it is not bounded.*

Definition 1.4. *An element $i \in \mathbf{R}$ is the* **infimum** *of a set $X \subset \mathbf{R}$ if the following two conditions hold.*
(i1) *$i \leq x$ for every $x \in X$ (i.e., i is a lower bound for X);*
(i2) *for every $i_1 > i$ there exists an element $x_1 \in X$ such that $x_1 < i_1$.*

Definition 1.5. *An element $s \in \mathbf{R}$ is the* **supremum** *of a set $X \subset \mathbf{R}$ if the following two conditions hold.*
(s1) *$s \geq x$ for every $x \in X$ (i.e., s is an upper bound for X);*
(s2) *for every $s_1 < s$ there exists an element $x_1 \in X$ such that $x_1 > s_1$.*

From Definition 1.4 (resp. Definition 1.5) it follows that the infimum is the *greatest lower bound* (resp. the supremum is the *smallest upper bound*) of a set. In Example 1.50, using axiom (R15), we shall prove that every subset of $\mathbf{R}$ bounded from below has an infimum, while every subset of $\mathbf{R}$ bounded from above has a supremum.

Definition 1.6. *The infimum (resp. supremum) of a set $X \subset \mathbf{R}$ is called* **minimum of** *X (resp.* **maximum of** *X) if it belongs to the set X.*

We shall denote by $\inf X$, $\sup X$, $\min X$ and $\max X$ the infimum, supremum, minimum and maximum of the set $X \subset \mathbf{R}$ respectively.

Bounded intervals. Let a and b be two real numbers such that $a < b$. Then the following subsets of $\mathbf{R}$ $(a,b), [a,b], (a,b]$ and $[a,b)$ are *bounded intervals* with endpoints a and b :

open interval: $(a,b) := \{x \in \mathbf{R} |\ a < x < b\}$;

closed interval: $[a,b] := \{x \in \mathbf{R} |\ a \leq x \leq b\}$;

other bounded intervals: $(a,b] := \{x \in \mathbf{R} |\ a < x \leq b\}$ and $[a,b) := \{x \in \mathbf{R} |\ a \leq x < b\}$.

Unbounded intervals. Let a and b be two real numbers. Then the sets $(a, +\infty)$, $[a, +\infty)$, $(-\infty, b)$, $(-\infty, b]$ and $(-\infty, +\infty)$ are *unbounded intervals,* defined as follows:

unbounded to the right: $(a, +\infty) := \{x \in \mathbf{R} |\ x > a\}$;
$[a, +\infty) := \{x \in \mathbf{R} |\ x \geq a\}$;

unbounded to the left: $(-\infty, b) := \{x \in \mathbf{R} |\ x < b\}$;
$(-\infty, b] := \{x \in \mathbf{R} |\ x \leq b\}$.

We also put $(-\infty, +\infty) := \{x \in \mathbf{R}\} = \mathbf{R}$, which is an interval both unbounded to the right and to the left.

Note that bounded (resp. unbounded) intervals are also bounded (resp. unbounded) sets in $\mathbf{R}$ in the sense of Definition 1.3.

Definition 1.7. *A nonempty set $X \subset \mathbf{R}$ is* **inductive** *if for every $x \in X$ it holds that the real number $x + 1$ is also in X.*

Definition 1.8. *The* **set of natural numbers***, denoted by* $\mathbf{N}$, *is the smallest inductive set containing the real number 1.*
Hence $\mathbf{N} = \{1, 2, 3, \dots\}$.

Quite often we shall use the set $\mathbf{N_0} := \mathbf{N} \bigcup \{0\}$.
The fundamental property of the set $\mathbf{N}$ gives the following statement, called the **mathematical induction principle**.

Theorem 1.9. *Assume a subset X of the set of natural numbers $\mathbf{N}$ has the following two properties.*

(mip1) $1 \in X$;

(mip2) *if* $n \in X$, *then* $n + 1$ *is also in* X.

Then necessarily $X = \mathbf{N}$.

Definition 1.10. *The* **set of integers***, denoted by* $\mathbf{Z}$, *includes the natural numbers, their additive inverses and the additive identity element, zero.*
Hence, $\mathbf{Z} = \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$.

The pair $(\mathbf{Z}, +)$ is an Abelian group, but $(\mathbf{Z} \setminus \{0\}, \cdot)$ is not.
The number $m \in \mathbf{Z}$ is a **divisor** of a number $n \in \mathbf{Z}$ if there exists an element $k \in \mathbf{Z}$ such that $n = km$.
An integer is **even** (resp. **odd**) if it is divisible (resp. not divisible) by 2.
A natural number p, $p > 1$, is **prime** if in the set $\mathbf{N}$ it has no divisors other than 1 and p itself.
The integers p and q are **relatively prime** if their only common divisors are the numbers 1 and -1.
If the largest common divisor of the integers p and q is the number 1, then we call the fraction $\frac{p}{q}$ **irreducible**.

Definition 1.11. *The* **set of rational numbers** *(or: fractions), denoted by* $\mathbf{Q}$, *make the real numbers of the form $p \cdot q^{-1}$, where $p, q \in Z$, $q \neq 0$.*
The **irrational numbers** *are those real numbers that are not rational.*

The set of rational numbers $\mathbf{Q}$ satisfies the first fourteen axioms from Definition 1.1, hence endowed with addition, multiplication and the binary relation $\leq$ it also becomes a totally ordered field. However, the set $\mathbf{Q}$ does not satisfy the completeness axiom (R15).

Theorem 1.12. The Archimedes theorem.
For every two real numbers $x > 0$ and y there exists a natural number n such that $nx > y$.

Theorem 1.13. The Cantor theorem.
Let for every $n \in \mathbf{N}$ be given a closed interval $[a_n, b_n]$ and assume that for $m > n$ it holds $[a_m, b_m] \subset [a_n, b_n]$, i.e., $a_n \leq a_m \leq b_m \leq b_n$. Then it holds

$$\bigcap_{n\in\mathbf{N}} [a_n, b_n] \neq \emptyset.$$

It is important to note that the Cantor theorem is *not true* in the set of rational numbers $\mathbf{Q}$.

In Example 1.51 we shall prove that axiom (R15) is equivalent with the conjunction of the Archimedes and the Cantor theorem. More precisely, the system of fifteen axioms (R1) - (R15) is equivalent to the system of fourteen axioms (R1) - (R14) plus the Archimedes and the Cantor theorem.

1.1.2 Examples and exercises

Example 1.14. *Using axioms (R1), (R2), (R3) and (R4) show that*

a) *in the set $\mathbf{R}$ there exists a unique additive identity;*

b) *in the set $\mathbf{R}$ every element has a unique additive inverse;*

c) *for given real numbers a and b the equation $a + x = b$ has a unique solution in $\mathbf{R}$.*

Solutions.

a) From the axiom (R2) it follows only the existence of an additive identity. Suppose there exist *two* such additive identities, say 0_1 and 0_2. Then from (R2) and (R4) it follows

$$0_1 = 0_1 + 0_2 = 0_2 + 0_1 = 0_2.$$

b) By axiom (R3) for a given element $x \in \mathbf{R}$ there exists an additive inverse. Suppose there exist two such inverse elements to x, say x_1 and x_2. Then from (R2), (R1) and (R4) it follows

$$x_1 = x_1 + 0 = x_1 + (x + x_2) = (x_1 + x) + x_2 = x_2 + (x_1 + x) = x_2 + 0 = x_2.$$

c) Using axioms (R4), (R1) , (R3) and (R2), let us check that the number $x := b + (-a)$ is a solution of the given equation:

$$a + x = a + (b + (-a)) = a + ((-a) + b) = (a + (-a)) + b = 0 + b = b + 0 = b.$$

Let us suppose that x_1 is some other solution of the given equation, i.e., it holds $b = a + x_1$. Then

$$(-a) + b = (-a) + (a + x_1) = ((-a) + a) + x_1 = 0 + x_1 = x_1.$$

Hence $x_1 = (-a) + b = b + (-a)$, which implies the uniqueness of the solution.

Example 1.15. *Using the axioms (R1), (R2), ..., (R14) and Example 1.14 show that for every $x, y, x', y' \in \mathbf{R}$ it holds*

a) $x \cdot 0 = 0$;	**b)** $-x = (-1) \cdot x$;
c) $-(-x) = x$;	**d)** $x(-y) = -(xy) = -(x)y$;
e) $(-x)(-y) = xy$;	**f)** $x \leq 0 \iff -x \geq 0$;
g) $x \leq y \iff -x \geq -y$;	**h)** $(x < 0 \wedge y < 0) \Rightarrow xy > 0$;
i) $(x \leq y \wedge x' \leq y') \Rightarrow x + x' \leq y + y'$;	**j)** $(x < 0 \wedge y > 0) \Rightarrow xy < 0$;
k) $0 < 1$;	**l)** $x > 0 \iff x^{-1} > 0$.

Solutions.

a) Using (R6), (R8) and (R2) we have

$$x + x \cdot 0 = x \cdot 1 + x \cdot 0 = x \cdot (1 + 0) = x \cdot 1 = x,$$

hence $x + x \cdot 0 = x$. Now from Example 1.14 c) it follows that $x \cdot 0$ is also an additive identity; from Example 1.14. a) then it follows $x \cdot 0 = 0$.

b) From axiom (R8) and a) it follows

$$x + (-1) \cdot x = (1 + (-1)) \cdot x = 0 \cdot x = x \cdot 0 = 0.$$

Thus $(-1) \cdot x$ is also an opposite number to x; from Example 1.14.b) then it follows $(-1) \cdot x = -x$.

c) By the definition of the element $-(-x)$ we can write $(-x) + (-(-x)) = 0$, while by the definition of the opposite element $-x$ it holds $(-x) + x = 0$. Using the uniqueness of the solution of the equation $a + x = b$ (see Example 1.14 c)) we obtain the statement.

d) From the relations

$$0 = x \cdot 0 = x(y + (-y)) = xy + x(-y), \quad \text{hence} \quad xy + x(-y) = 0$$

and $xy + (-(xy)) = 0$ (see c)), it follows $x(-y) = -(xy)$.

e) The statement follows from d) and c).

f) By axiom (R13) we have

$$x \leq 0 \Rightarrow (-x) + x \leq (-x) + 0 \Rightarrow 0 \leq -x.$$

g) $x \leq y \Rightarrow ((-x) + (-y)) + x \leq ((-x) + (-y)) + y \Rightarrow -y \leq -x.$

h) Using f), (R14) and e) we have

$$(x < 0 \wedge y < 0) \Rightarrow (-x > 0 \wedge -y > 0) \Rightarrow (-y)(-x) > 0 \iff xy > 0.$$

i) Since $x \leq y \Rightarrow x + x' \leq x' + y$ and $x' \leq y' \Rightarrow x' + y \leq y' + y$, from (R10) and (R4) the statement follows.

j) $(x < 0 \wedge 0 < y) \Rightarrow (0 < -x \wedge 0 < y) \Rightarrow (0 < (-x)y) \Rightarrow (0 < ((-1)x)y)$
$\Rightarrow (0 < (-1)(xy)) \Rightarrow (0 < -(xy)) \Rightarrow (xy < 0).$

k) The axiom (R6) implies $1 \neq 0$, hence by (R12) it is either $1 > 0$ or $1 < 0$. Suppose $1 < 0$; then from h) and $1 \neq 0$ it holds $(1 < 0 \wedge 1 < 0) \Rightarrow (0 < 1 \cdot 1) \Rightarrow 0 < 1$. This is a contradiction, hence necessarily $1 > 0$.

l) First of all, $x^{-1} \neq 0$. Suppose $x^{-1} < 0$. Then from j) it follows

$$(x^{-1} < 0 \wedge 0 < x) \Rightarrow (x \cdot x^{-1} < 0) \Rightarrow (1 < 0),$$

which is in contradiction with k).

Exercise 1.16. *Prove that*

a) $(x \cdot y = 0) \Rightarrow (x = 0 \vee y = 0)$;

b) $(\forall x \in \mathbf{R})(-1) \cdot (-x) = x$;

c) $(\forall x \in \mathbf{R})(-x) \cdot (-x) = x \cdot x =: x^2$;

d) $(\forall x, y, z \in \mathbf{R})(x < y \wedge y \leq z) \Rightarrow x < z$;

e) $(\forall x, y, z, w \in \mathbf{R}) \quad (x \leq y \wedge z < w) \Rightarrow (x + z < y + w)$.

Exercise 1.17. *Prove that*

a) *in the set* $\mathbf{R}$ *there exists a unique multiplicative identity;*

b) *for every* $x \neq 0$ *there exists a unique multiplicative inverse, denoted by* x^{-1};

c) *the equation* $a \cdot x = b$ *has a unique solution in* $\mathbf{R}$, *provided that* $a \in \mathbf{R}$, $a \neq 0$, *and* $b \in \mathbf{R}$.

Example 1.18. *Show that for every* $x \in \mathbf{R}$ *the following properties of the absolute value hold.*

a) $|x| = 0 \iff x = 0$; **b)** $|-x| = |x|$;
c) $-|x| \leq x \leq |x|$; **d)** $|x| \leq \varepsilon \iff -\varepsilon \leq x \leq \varepsilon$;
e) $|x| < \varepsilon \iff -\varepsilon < x < \varepsilon$.

Solutions.

a) Follows immediately from Definition 1.2.

b) For $x = 0$ this is trivial. For $x < 0$ it holds $-x > 0$ (see Example 1.15 f)), hence by Definition 1.2 $|x| = -x$ and $|-x| = -x$. Thus $|x| = |-x|$. The third case ($x > 0$) is handled similarly.

The equalities **c)** and **d)** follow from Examples 1.15 d), e), h) and j).

Example 1.19. *Prove that for every $x, y \in \mathbf{R}$ the following properties of the absolute value hold.*

a) $|x + y| \leq |x| + |y|$; **b)** $|x - y| \geq \big||x| - |y|\big|$;

c) $|x \cdot y| = |x| \cdot |y|$; **d)** $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$, $y \neq 0$.

Solutions.

a) Using (R13), from the inequalities

$$-|x| \leq x \leq |x| \quad \text{and} \quad -|y| \leq y \leq |y|,$$

(see Example 1.18 c)) it follows

$$-(|x| + |y|) \leq x + y \leq |x| + |y|, \quad \text{or} \quad |x + y| \leq |x| + |y|.$$

b) Using a), we obtain the following inequalities:

$$|x| = |x - y + y| \leq |x - y| + |y| \quad \text{and} \quad |y| = |y - x + x| \leq |-(x - y)| + |x|.$$

Hence

$$-|x - y| \leq |x| - |y| \leq |x - y| \Rightarrow \big||x| - |y|\big| \leq |x - y|.$$

Relations **c)** and **d)** follow immediately from Definition 1.2.

Exercise 1.20. *Show that for every $x_1, x_2, \ldots, x_n \in \mathbf{R}$ the following holds.*

$$\left|\sum_{k=1}^{n} x_k\right| := |x_1 + x_2 + \cdots + x_n| \leq \sum_{k=1}^{n} |x_k| \,.$$

Example 1.21. *Solve the following equation and inequation.*

a) $|x - 1| - |x + 2| = 1$; **b)** $|3x + 2| > 4|x - 1|$.

Solutions.

a) From the definition of the absolute value, for $x \in (-\infty, -2)$ ($\Longleftrightarrow$ $x+2<0$) the given equation can be written as $-(x-1)-(-(x+2))=1$, which is equivalent to $3=1$. Hence there is no solution in the interval $(-\infty,-2)$. For $-2<x\leq 1$ the given equation can be written as $-(x-1)-(x+2)=1$, so in the interval $(-2,1]$ the only solution is $x=-1$. For $x>1$ the equation has the form $x-1-(x+2)=1$, or $-3=1$, which means that no solutions exist in the interval $(1,+\infty)$.

b) Every point from the interval $\left(\frac{2}{7}, 6\right)$ is a solution of the given inequation.

Exercise 1.22. *Solve the following equations.*

a) $|x+3|=5$; b) $|x-1|-|2x-7|=2$;
c) $|x|-|x-1|+3|x-2|-2|x-3|=x+1$.

Answers. a) $x=-8 \vee x=2$. b) $x=4 \vee x=\frac{10}{3}$. c) $x=-1 \vee x\geq 3$.

Exercise 1.23. *Solve the following inequations.*

a) $|2x+1|\leq|4x-7|$; b) $\left|\frac{x+1}{x+5}\right|<1, \quad x\neq -5$;

c) $\left|\frac{5x+2}{2x-3}\right|\geq 2, \quad x\neq\frac{3}{2}$; d) $\left|\frac{x}{x+1}\right|>\frac{x}{x+1}, \quad x\neq -1$;

e) $|x^2-x|-|x|<1$; f) $|x^3-x^2|<|x^2+x|$;

g) $\big|1-|x-1|\big|<1$; h) $\big||x+1|-|x-1|\big|<1$.

Answers.

a) $x\in(-\infty,1]\bigcup[4,+\infty)$. b) $x\in(-3,+\infty)$.

c) $x\in(-\infty,-8]\bigcup\left(\frac{4}{9},\frac{3}{2}\right)\bigcup\left(\frac{3}{2},+\infty\right)$. d) $x\in(-1,0)$.

e) $x\in(-1,1+\sqrt{2})$. f) $x\in(1-\sqrt{2},0)\bigcup(0,1+\sqrt{2})$.

g) $x\in(-1,1)\bigcup(1,3)$. h) $x\in\left(-\frac{1}{2},\frac{1}{2}\right)$.

Example 1.24. *Prove that the sum of the first* n, $n\in\mathbf{N}$, *terms of a geometric sequence with quotient* $q\neq 1$ *is given by*

$$\sum_{k=1}^{n} q^{k-1}=\frac{q^n-1}{q-1}, \quad q\neq 1. \tag{1.1}$$

Solution. For $n = 1$ the formula reduces to $q^0 = \frac{q-1}{q-1}$, which is true. Assume that (1.1) is true for some $n = m$; then we must show that it is true also for $n = m+1$. To that end, we have

$$\sum_{k=1}^{m+1} q^{k-1} = \frac{q^m - 1}{q-1} + q^m = \frac{q^m - 1 + q^{m+1} - q^m}{q-1}.$$

Hence

$$\sum_{k=1}^{m+1} q^k = \frac{q^{m+1} - 1}{q-1}.$$

Thus we obtained formula (1.1) for $n = m+1$. By the mathematical induction principle it follows that formula (1.1) holds for all natural numbers n.

Example 1.25. *Prove the following formula for $n \in \mathbf{N}$:*

$$\underbrace{3 + 33 + 333 + \cdots + 33...3}_{n \text{ addends}} = \frac{10^{n+1} - 9n - 10}{27}. \tag{1.2}$$

Solution. Let us put $f(n) := \frac{10^{n+1} - 9n - 10}{27}$, $n \in \mathbf{N}$. For $n = 1$ formula (1.2) becomes

$$3 = \frac{10^2 - 9 \cdot 1 - 10}{27},$$

which is true. Assume that (1.2) holds for some $n = m$. Then we have using Example (1.24)

$$\underbrace{3 + 33 + 333 + \cdots + 33...3}_{m+1 \text{ addends}} = \frac{10^{m+1} - 9m - 10}{27} + \underbrace{33...3}_{m+1 \text{ ciphers}}$$

$$= \frac{1}{27} \cdot \left(10^{m+1} - 9m - 10 + 3 \cdot 27 \cdot (10^m + \cdots + 10^1 + 10^0)\right)$$

$$= \frac{1}{27} \cdot \left(10^{m+1} - 9m - 10 + 81 \cdot \frac{10^{m+1} - 1}{10 - 1}\right)$$

$$= \frac{1}{27 \cdot 9} \cdot \left(9 \cdot 10^{m+1} - 81m - 90 + 81 \cdot 10^{m+1} - 81\right)$$

$$= \frac{1}{27 \cdot 9} \cdot 9 \cdot \left(10^{m+2} - 9(m+1) - 10\right).$$

Simplifying the last expression on the right-hand side gives

$$\underbrace{3 + 33 + 333 + \cdots + 33...3}_{m+1 \text{ addends}} = f(m+1),$$

i.e., formula (1.2) for $n = m+1$. The mathematical induction principle gives the validity of formula (1.2) for every $n \in \mathbf{N}$.

Example 1.26. *Using the mathematical induction principle, prove the following formulas for $n \in \mathbf{N}$.*

a) $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$; b) $\displaystyle\sum_{k=1}^{n}(2k-1) = n^2$;

c) $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$; d) $\displaystyle\sum_{k=1}^{n}(2k-1)^2 = \frac{n(4n^2-1)}{3}$;

e) $\displaystyle\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$; f) $\displaystyle\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$.

Solutions.

a) For $n = 1$ the given formula reduces to $1 = \frac{1 \cdot 2}{2}$, which is true. If the formula is true for $n = m$, $m \in \mathbf{N}$, then

$$\sum_{k=1}^{m+1} k = \sum_{k=1}^{m} k + (m+1) = \frac{m(m+1)}{2} + (m+1) = \frac{(m+1)(m+2)}{2}.$$

Hence the formula is true for $n = m + 1$. By the mathematical induction principle, the set of all m for which the given formula is true coincides with the set of natural numbers $\mathbf{N}$.

f) For $n = 1$ the statement becomes the equality $\frac{1}{1 \cdot 3} = \frac{1}{2 \cdot 1 + 1}$. Let us assume that the given formula holds for $n = m$. Then we have

$$\sum_{k=1}^{m+1} \frac{1}{(2k-1)(2k+1)} = \frac{m}{2m+1} + \frac{1}{(2m+1)(2m+3)}$$

$$= \frac{m(2m+3)+1}{(2m+1)(2m+3)} = \frac{(m+1)(2m+1)}{(2m+1)(2m+3)} = \frac{m+1}{2m+3}.$$

Example 1.27. *Prove the following formulas for $n \in \mathbf{N}$.*

a) $\displaystyle\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$, *hence by Example 1.26 a) it follows*

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2;$$

b) $\displaystyle\sum_{k=1}^{n} k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}$.

Solution. **a)** The formula is true for $n = 1$. Supposing that the formula is true for $n = m$, (for some $m \in \mathbf{N}$), let us prove it for $n = m + 1$. Namely, from the inductive assumption and Example 1.26.a) it follows

$$\sum_{k=1}^{m+1} k^3 = \sum_{k=1}^{m} k^3 + (m+1)^3 = \left(\sum_{k=1}^{m} k\right)^2 + \left(2\left(\frac{m(m+1)}{2}\right)(m+1) + (m+1)^2\right)$$

$$= \left(\sum_{k=1}^{m} k\right)^2 + 2\left(\sum_{k=1}^{m} k\right)(m+1) + (m+1)^2 = \left(\sum_{k=1}^{m} k + (m+1)\right)^2 = \left(\sum_{k=1}^{m+1} k\right)^2.$$

Example 1.28. *The binomial coefficient* $\binom{n}{k}$, $n \in \mathbf{N}$, $k \in \mathbf{N_0}$, $0 \le k \le n$, *is defined by*

$$\binom{n}{k} := \frac{n!}{k! \cdot (n-k)!}.$$

(By definition, we put $0! = 1$.*)*

a) *Prove the following formula for the binomial coefficients.*

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}, \quad n \in \mathbf{N},\ k \in \mathbf{N_0},\ 0 \le k \le n-1.$$

b) *Prove the* **binomial formula**

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k, \quad a, b \in \mathbf{R},\ n \in \mathbf{N}. \tag{1.3}$$

Solution. Part **a)** is straightforward, so we go to
b) For $n = 1$ formula (1.3) reduces to the true formula

$$(a+b)^1 = \binom{1}{0} a^{1-0} b^{1-1} + \binom{1}{1} a^{1-1} b^{1-0}.$$

Assume (1.3) is true for some $n = m$. Then we have using a)

$$\begin{aligned}
(a+b)^{m+1} &= \left(\sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^k\right)(a+b) \\
&= \sum_{k=0}^{m} \binom{m}{k} a^{m-k+1} b^k + \sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^{k+1} \\
&= \binom{m}{0} a^{m+1} + \sum_{k=1}^{m} \left(\binom{m}{k} + \binom{m}{k-1}\right) a^{m-k+1} b^k + \binom{m}{m} b^{m+1} \\
&= a^{m+1} + \sum_{k=1}^{m} \binom{m+1}{k} a^{m-k+1} b^k + b^{m+1} = \sum_{k=0}^{m+1} \binom{m+1}{k} a^{m-k+1} b^k.
\end{aligned}$$

Example 1.29. *Prove the following formulas.*

$$\text{a)}\quad \sum_{k=1}^{n}\sin(k\alpha)=\frac{\sin\dfrac{n\alpha}{2}\cdot\sin\dfrac{(n+1)\alpha}{2}}{\sin\dfrac{\alpha}{2}},\qquad \alpha\notin\{2\pi\ell|\,\ell\in\mathbf{Z}\};$$

$$\text{b)}\quad \sum_{k=1}^{n}\cos(k\alpha)=\frac{\cos\dfrac{n\alpha}{2}\cdot\cos\dfrac{(n+1)\alpha}{2}}{\sin\dfrac{\alpha}{2}},\qquad \alpha\notin\{2\pi\ell|\,\ell\in\mathbf{Z}\}.$$

Solution. **a)** For $n=1$ the given formula becomes $\sin\alpha=\dfrac{\sin\dfrac{\alpha}{2}\sin\dfrac{2\alpha}{2}}{\sin\dfrac{\alpha}{2}}$, which is true. Supposing that the formula is true for $n=m$, let us prove it for $n=m+1$.

$$\begin{aligned}\sum_{k=1}^{m+1}\sin(k\alpha) &= \frac{\sin\dfrac{m\alpha}{2}\cdot\sin\dfrac{(m+1)\alpha}{2}}{\sin\dfrac{\alpha}{2}}+\sin((m+1)\alpha)\\ &= \sin\frac{(m+1)\alpha}{2}\left(\frac{\sin\dfrac{m\alpha}{2}}{\sin\dfrac{\alpha}{2}}+2\cos\frac{(m+1)\alpha}{2}\right)\\ &= \frac{\sin\dfrac{(m+1)\alpha}{2}}{\sin\dfrac{\alpha}{2}}\left(\sin\frac{(m+1)\alpha}{2}\cos\frac{\alpha}{2}+\cos\frac{(m+1)\alpha}{2}\sin\frac{\alpha}{2}\right)\\ &= \frac{\sin\dfrac{(m+1)\alpha}{2}\cdot\sin\dfrac{(m+2)\alpha}{2}}{\sin\dfrac{\alpha}{2}}.\end{aligned}$$

Example 1.30. *Prove the following formulas.*

$$\text{a)}\quad \prod_{k=1}^{n}\cos\frac{x}{2^k}=\frac{\sin x}{2^n\sin\dfrac{x}{2^n}},\qquad 0<x<\pi;$$

$$\text{b)}\quad \underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{n\ \textit{square roots}}=2\cos\frac{\pi}{2^{n+1}}.$$

Solutions.

a) For $n=1$ the given formula becomes $\cos\dfrac{x}{2}=\dfrac{\sin x}{2^1\sin\dfrac{x}{2^1}}$, which is equivalent with $\sin(2y)=2\sin y\cos y$ for $0<y<2\pi$. Next, we shall assume that the given formula holds for some $n = m$. Then we have

$$\begin{aligned}\prod_{k=1}^{m+1} \cos\frac{x}{2^k} &= \frac{\sin x}{2^m \sin\frac{x}{2^m}} \cdot \cos\frac{x}{2^{m+1}} \\ &= \frac{\sin x \cdot \cos\frac{x}{2^{m+1}}}{2^m \cdot 2\cos\frac{x}{2^{m+1}} \cdot \sin\frac{x}{2^{m+1}}} = \frac{\sin x}{2^{m+1} \cdot \sin\frac{x}{2^{m+1}}}.\end{aligned}$$

b) For $n = 1$ we have the correct equality $\sqrt{2} = 2\cos\frac{\pi}{4}$. Assume the formula is true for $n = m$. Then

$$\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{m+1 \text{ square roots}} = \sqrt{2 + 2\cos\frac{\pi}{2^{m+1}}}$$

$$= 2\sqrt{\frac{1+\cos\frac{\pi}{2^{m+1}}}{2}} = 2\cos\left(\frac{1}{2}\left(\frac{\pi}{2^{m+1}}\right)\right) = 2\cos\frac{\pi}{2^{m+2}},$$

which means that the formula is also true for $n = m + 1$.

Remark. We used the equality $\cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}}$ for $0 \le \alpha \le \pi$.

Example 1.31. *Prove the following inequalities for $n \in \mathbf{N}$.*

a) $(1 + x)^n \ge 1 + nx, \quad x > -1,$ **(the Bernoulli inequality)**;

b) $(1 + x_1)(1 + x_2)\cdots(1 + x_n) \ge 1 + x_1 + x_2 + \cdots + x_n,$
(the generalized Bernoulli inequality), where $x_k > -1,\ k = 1, \ldots, n,$ and all x_k are of the same sign;

c) $n! < \left(\frac{n+1}{2}\right)^n, \quad n > 1.$

Solutions.

a) For $n = 1$ we have $(1 + x)^1 \ge 1 + 1 \cdot x$, which is always true. Let us suppose now that the inequality is true for $n = m$, i.e., $(1 + x)^m \ge 1 + mx$. We shall prove next that it is then true for $n = m + 1$ and $x > -1$.

$$\begin{aligned}(1+x)^{m+1} &= (1+x)(1+x)^m \ge (1+x)(1+mx) \\ &= 1 + (m+1)x + mx^2 \ge 1 + (m+1)x, \quad \text{since} \quad mx^2 \ge 0.\end{aligned}$$

c) For $n=2$ the given inequality $2! < (\frac{2+1}{2})^2$ is clearly true. Assume the inequality is true for $n=m$, then it holds

$$\begin{aligned}(m+1)! &= (m+1)\cdot m! < (m+1)\left(\frac{m+1}{2}\right)^m \\ &= 2\cdot\left(\frac{m+2}{2}\right)^{m+1}\cdot\frac{1}{\left(1+\frac{1}{m+1}\right)^{m+1}}.\end{aligned}$$

From the Bernoulli inequality (see a)) it follows

$$\left(1+\frac{1}{m+1}\right)^{m+1} \geq 1+(m+1)\cdot\frac{1}{m+1} = 2,$$

hence

$$(m+1)! < 2\cdot\left(\frac{m+2}{2}\right)^{m+1}\cdot\frac{1}{2} = \left(\frac{m+2}{2}\right)^{m+1}.$$

Example 1.32. *Prove the following inequalities for $n \in \mathbf{N}$.*

a) $x_1+x_2+\cdots+x_n \geq n,\quad$ *where* $x_k>0,\ k=1,\ldots,n,$ *and* $x_1x_2\cdots x_n = 1;$

b) $\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n} > \frac{1}{2};$

c) $\frac{1}{n+1}+\frac{1}{n+2}+\cdots\frac{1}{3n+1} > 1;$

d) $\sqrt{n} < 1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}} < 2\sqrt{n},\quad n>1.$

Solutions.

a) We shall use the mathematical induction. For $n=1$ the statement is trivial. Assume it is true for $n=m$, i.e., that for any m positive numbers whose product is 1, the given inequality holds.

So let $x_1, x_2, \ldots, x_m, x_{m+1}$ be $m+1$ positive numbers whose product is equal to 1. Then either all x_k-s are equal to 1, in which case the left-hand side of the inequality becomes just $m+1$, or at least one of them is less than 1, say $x_m<1$ and, of course, at least one is greater than 1, say $x_{m+1}>1$. Then the m numbers $x_1, x_2, \ldots, x_{m-1}, x_m\cdot x_{m+1}$ satisfy the inductive assumption, hence

$$x_1+x_2+\cdots+x_{m-1}+x_m\cdot x_{m+1} \geq m \iff x_1+x_2+\cdots+x_{m-1} \geq m - x_m\cdot x_{m+1}.$$

Using the last inequality we obtain

$$x_1+x_2+\cdots+x_m+x_{m+1} \geq m - x_m\cdot x_{m+1} + x_m + x_{m+1}$$

$$= m+1+x_m(1-x_{m+1})+x_{m+1}-1 = m+1+(1-x_m)(x_{m+1}-1).$$

Since by assumption $1 - x_m > 0$ and $x_{m+1} - 1 > 0$, it follows that

$$x_1 + x_2 + \cdots + x_m + x_{m+1} \geq m + 1.$$

d) For $n = 2$ the given formula becomes

$$\sqrt{2} < 1 + \frac{1}{\sqrt{2}} < 2\sqrt{2},$$

which is true, since $1.41 < \sqrt{2} < 1.42$. Assume that the formula is true for some $n = m$. Then we have

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{m+1}} > \sqrt{m} + \frac{1}{\sqrt{m+1}}.$$

Once we prove that the expression

$$\sqrt{m} + \frac{1}{\sqrt{m+1}} - \sqrt{m+1} \tag{1.4}$$

is positive, the left-hand side of the given formula follows by the mathematical induction principle. To that end, the expression in (1.4) is equal to

$$\sqrt{m} + \frac{1}{\sqrt{m+1}} - \sqrt{m+1} = \frac{\sqrt{m(m+1)} + 1 - (m+1)}{\sqrt{m+1}} = \frac{\sqrt{m(m+1)} - m}{\sqrt{m+1}},$$

which is positive.

For the right-hand side inequality in the given formula, from the inductive assumption

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{m}} < 2\sqrt{m},$$

we have

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{m+1}} < 2\sqrt{m} + \frac{1}{\sqrt{m+1}}.$$

The right-hand side of the given formula will follow once we prove the following inequality.

$$\frac{2\sqrt{m(m+1)} + 1}{\sqrt{m+1}} - 2\sqrt{m+1} < 0.$$

In fact, we have

$$\begin{aligned} \frac{2\sqrt{m(m+1)} + 1}{\sqrt{m+1}} - 2\sqrt{m+1} &= \frac{2\sqrt{m(m+1)} + 1 - 2m - 2}{\sqrt{m+1}} \\ &= \frac{\sqrt{4m^2 + 4m} - \sqrt{4m^2 + 4m + 1}}{\sqrt{m+1}}, \end{aligned}$$

which is clearly less than zero.

Example 1.33. *The expressions*

$$\mathcal{A} := \mathcal{A}(x_1, x_2, \dots, x_n) := \frac{x_1 + x_2 + \cdots + x_n}{n}, \quad x_1, x_2, \dots, x_n \in \mathbf{R},$$

$$\mathcal{G} := \mathcal{G}(x_1, x_2, \dots, x_n) := \sqrt[n]{x_1 \cdot x_2 \cdots x_n}, \quad x_1, x_2, \dots, x_n \geq 0,$$

$$\mathcal{H} := \mathcal{H}(x_1, x_2, \dots, x_n) := \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}, \quad x_1, x_2, \dots, x_n > 0,$$

are respectively called the arithmetic, geometric and harmonic mean of the numbers $x_1, x_2, \dots, x_n$. *Prove the following inequalities.*

a) $$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdots x_n}, \quad x_1, x_2, \dots, x_n \geq 0;$$

b) $$\sqrt[n]{x_1 \cdot x_2 \cdots x_n} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}, \quad x_1, x_2, \dots, x_n > 0.$$

Remarks. The inequalities a) and b) mean that it holds

$$\mathcal{A} \geq \mathcal{G} \geq \mathcal{H}$$

for $x_1, x_2, \dots, x_n > 0$.

Note that the equalities in a) and b) occur if and only if $x_1 = x_2 = \ldots = x_n$.

We suggest to the reader to check that for $n = 2$ it holds

$$\mathcal{A} \cdot \mathcal{H} = \mathcal{G}^2.$$

Solutions.

a) For $n = 1$ the inequality reduces to a trivial equality. For $n = 2$ it becomes

$$\frac{x_1 + x_2}{2} - \sqrt{x_1 x_2} = \frac{(\sqrt{x_1} - \sqrt{x_2})^2}{2} \geq 0.$$

Suppose now that the inequality is true for $n = m > 2$. Then

$$\begin{aligned}\frac{x_1 + x_2 + \cdots + x_m + x_{m+1}}{m+1} &= \frac{m\frac{x_1 + x_2 + \cdots + x_m}{m} + x_{m+1}}{m+1} \\ &\geq \frac{m \sqrt[m]{x_1 \cdot x_2 \cdots x_m} + x_{m+1}}{m+1}.\end{aligned}$$

From the last inequality we have

$$\begin{aligned}&\frac{x_1 + x_2 + \cdots + x_m + x_{m+1}}{m+1} - \sqrt[m+1]{x_1 \cdot x_2 \cdots x_{m+1}} \\ &\geq \frac{m \sqrt[m]{x_1 \cdot x_2 \cdots x_m} + x_{m+1}}{m+1} - \sqrt[m+1]{x_1 \cdot x_2 \cdots x_{m+1}}\,.\end{aligned}$$

Putting $p^{m(m+1)} := x_1 \cdot x_2 \cdots x_m$ and $q^{m+1} := x_{m+1}$, we obtain

$$\begin{aligned}
&\frac{m\sqrt[m]{x_1\cdot x_2\cdots x_m}+x_{m+1}}{m+1} - \sqrt[m+1]{x_1\cdot x_2\cdots x_m\cdot x_{m+1}} \\
&= \frac{mp^{m+1}+q^{m+1}}{m+1} - p^m q = \frac{1}{m+1}\left(mp^m(p-q) - q(p^m - q^m)\right) \\
&= \frac{p-q}{m+1}\left(mp^m - qp^{m-1} - q^2p^{m-2} - \cdots - q^m\right) \\
&= \frac{p-q}{m+1}\left((p^m - qp^{m-1}) + (p^m - q^2p^{m-2}) + \cdots + (p^m - q^m)\right) \\
&= \frac{p-q}{m+1}\left(p^{m-1}(p-q) + p^{m-2}(p^2-q^2) + \cdots + (p^m - q^m)\right) \\
&= \frac{(p-q)^2}{m+1}\left(p^{m-1} + p^{m-2}(p+q) + \cdots + (p^{m-1} + p^{m-2}q + \cdots + q^{m-1})\right) \geq 0.
\end{aligned}$$

b) Applying the inequality a) to the numbers $\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}$, we obtain

$$\frac{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}}{n} \geq \sqrt[n]{\frac{1}{x_1}\cdot\frac{1}{x_2}\cdots\frac{1}{x_n}} = \frac{1}{\sqrt[n]{x_1\cdot x_2\cdots x_n}},$$

which is equivalent with the statement.

Exercise 1.34. *Prove that for $n \in \mathbf{N}$, $n > 2$, it holds*

a) $n! > 2^{n-1}$; **b)** $(n!)^2 > n^n$; **c)** $3^n n! > n^n > 2^n n!$;

d) $(2n-1)!! := (2n-1)(2n-3)\cdots 3\cdot 1 < 2^n \cdot n!$.

Example 1.35. *Show that the sum of two natural numbers is again a natural number.*

Solution. Let $m, n \in \mathbf{N}$; using the mathematical induction principle, we shall prove that $m+n$ is also a natural number. Let us denote by $\mathbf{N_1}$ the following set.

$$\mathbf{N_1} := \{n \in \mathbf{N} \mid (\forall m \in \mathbf{N})\; m+n \in \mathbf{N}\}.$$

The set $\mathbf{N}$ (of natural numbers) is an inductive one (see Definitions 1.7 and 1.8), which means that for every $m \in \mathbf{N}$ it follows $m+1 \in \mathbf{N}$, hence $1 \in \mathbf{N}_1$. Assuming that $n \in \mathbf{N}_1$, i.e., $m+n \in \mathbf{N}$ for every $m \in \mathbf{N}$, it follows from axiom (R1) that the number $m+(n+1) = (m+n)+1$ is also natural. Hence, by the mathematical induction principle $\mathbf{N_1} = \mathbf{N}$.

Example 1.36. *Using the mathematical induction principle, prove the following properties for the set of natural numbers.*

a) *If $m, n \in \mathbf{N}$ such that $m > n$, then $m - n$ is also a natural number;*

b) *the minimum of the set $\mathbf{N}$ is the number 1;*

c) *if $m, n \in \mathbf{N}$ such that $m > n$, then $m \geq n + 1$;*

d) *every nonempty subset A of $\mathbf{N}$ has a minimum.*

Solutions.

a) Let $n = 1$. Then from the assumption it follows $m > 1$, hence there exists an element $p \in \mathbf{N}$ such that $m = p + 1$. This is equivalent to $p = m - 1$, which means that the statement is true for $n = 1$. Suppose now that it is true for an element $k \in \mathbf{N}$; i.e., that it holds $m > k \Rightarrow m - k \in \mathbf{N}$. Assume $m > k + 1$. Then $m - k > 1$, so from the (already proved) statement for n=1, it follows

$$m - k > 1 \Rightarrow ((m - k) - 1) \in \mathbf{N} \Rightarrow (m - (k + 1)) \in \mathbf{N}.$$

Hence the statement is true for $k + 1 \in \mathbf{N}$, provided it is true for $k \in \mathbf{N}$.

b) Since by (R6) $1 \in \mathbf{N}$, we have to prove that for every $n \in \mathbf{N}$ it holds $1 \leq n$. For $n = 1$ this is trivial. Let us suppose that $1 \leq k$ for some $k \in \mathbf{N}$. Then from Examples 1.15 k) and 1.15 i) it follows

$$(1 \leq k \wedge 0 \leq 1) \Rightarrow 1 \leq k + 1.$$

c) Suppose $m, n \in \mathbf{N}$ and $m > n$. Then from a) it follows $(m - n) \in \mathbf{N}$ and from b) it follows $m - n \geq 1$. Hence $m \geq n + 1$.

d) Let us suppose the contrary, i.e., that there exists a nonempty set $A \subset \mathbf{N}$ which has no minimum. From b) it follows $1 \notin A$. Let us define a set B by

$$B = \{n \in \mathbf{N} |\ (\forall a \in A)\, n < a\}.$$

We shall prove that $B = \mathbf{N}$.

By b), for every $a \in A$ it holds $1 \leq a$; since $1 < a$, it follows that $1 \in B$.

Now assume that $n \in B$ and we shall prove $(n + 1) \in B$. Firstly, since A has no minimum, for every $a \in A$ there exists $a' \in A$ such that $a' < a$. This implies

$$n \in B \Rightarrow n < a' \Rightarrow (n + 1 < a' + 1 \leq a) \Rightarrow n + 1 < a,$$

which means that $(n + 1) \in B$. By the mathematical induction principle, it follows that $B = \mathbf{N}$. By the definition of the set B, it follows $A \cap B = \emptyset$, hence $A = A \cap \mathbf{N} = \emptyset$. This is in contradiction with the assumption that A is nonempty.

Exercise 1.37. *Let $m, n \in \mathbf{N}$. Prove that there exist unique numbers $q \in \mathbf{N_0}$ and $r \in \mathbf{N_0}$, $0 \le r < n$, such that $m = nq + r$.*

Exercise 1.38. *Prove that the product of natural numbers (resp. integers) is a natural number (resp. an integer).*

Exercise 1.39. *Prove that*

a) *a natural number can not be both even and odd;*

b) *every natural number is either even or odd;*

c) *if n is an even natural number, then so is n^2.*

Exercise 1.40. *Prove that $(\mathbf{Q}, +, \cdot, \le)$ is a totally ordered field, i.e., that $\mathbf{Q}$ satisfies the axioms (R1) - (R14).*

Example 1.41. *If a and b are two rational numbers, $a < b$, prove then that there exists an element $c \in \mathbf{Q}$, such that $a < c < b$.*

Solution. Let us put $c := \dfrac{a+b}{2}$. It holds

$$a = \frac{a}{2} + \frac{a}{2} < \frac{a}{2} + \frac{b}{2} = c < \frac{b}{2} + \frac{b}{2} = b.$$

(This procedure shows that actually there exist infinitely many rational numbers between any two given rational ones.)

Example 1.42. *Prove that the equation*

$$x^2 = 2 \tag{1.5}$$

has no solution *in the set of rational numbers* $\mathbf{Q}$.

Solution. Let us suppose that there exists a rational solution $x = r$ of equation (1.5); we may assume that $r > 0$. Then $r = \dfrac{p}{q}$, where p and q are natural numbers with largest common divisor 1, i.e., the fraction $\dfrac{p}{q}$ is irreducible. Since $r^2 = \dfrac{p^2}{q^2} = 2$, it follows $p^2 = 2q^2$. This implies that p^2 is an even number, hence p is even itself. We can write $p = 2k$ for some $k \in \mathbf{N}$, which gives us $q^2 = 2k^2$. Now q^2 is also an even number, and finally so is q. But then the fraction $\dfrac{p}{q}$ is reducible, since both its numerator and denominator are even. This is a contradiction, hence there is *no solution* of equation (1.5) in the set $\mathbf{Q}$.

Example 1.43. *Prove that the equation (1.5)* has a solution *in the set* $\mathbf{R}$.

Solution. Let us define two sets X and Y as follows.

$$X := \{x \in \mathbf{R}_+ |\, x^2 < 2\} \quad \text{and} \quad Y := \{y \in \mathbf{R}_+ |\, y^2 > 2\},$$

where $\mathbf{R}_+ = \{x \in \mathbf{R} |\ x > 0\}$ is the set of positive real numbers. Since $1^2 = 1 < 2$ and $2^2 = 4 > 2$, it follows that $1 \in X$ and $2 \in Y$, hence both X and Y are nonempty. Clearly, it holds $X \cap Y = \emptyset$. If x and y are positive numbers, in view of axiom (R14) it holds

$$x < y \iff x^2 < y^2.$$

This means that any element of X is smaller than any element from Y, hence the conditions of axiom (R15) are satisfied. By (R15), there exists a *real* number r such that

$$(\forall x \in X)\ (\forall y \in Y) \quad x \le r \le y. \tag{1.6}$$

Let us prove that $r^2 = 2$. By (R12) there are three possibilities, out of which exactly one is true, namely $r^2 < 2$ or $r^2 > 2$ or $r^2 = 2$. Of course, our goal is to prove that the last case is true; we shall show that the first two cases are impossible. Suppose first $r^2 < 2$. Then the real number $r_1 := 2\dfrac{r+1}{r+2}$ is in the set X, since

$$r_1^2 - 2 = 4 \cdot \frac{r^2 + 2r + 1}{(r+2)^2} - 2 = 2 \cdot \frac{r^2 - 2}{(r+2)^2} < 0.$$

However, it also holds

$$r - r_1 = r - 2\frac{r+1}{r+2} = \frac{r^2 - 2}{r+2} < 0,$$

which means that $r_1 \in Y$, hence $r_1 \in X \cap Y$ - a contradiction, since the last intersection is empty. The assumption $r^2 > 2$ gives a contradiction in an analogous way, so the third possibility remains, namely $r^2 = 2$.

Remark. By Example 1.42, any solution x of the equation (1.5) is not a rational number. In fact, there are two real numbers satisfying $x^2 = 2$, namely $\sqrt{2}$ and $-\sqrt{2}$.

Example 1.44. *Show that*

a) *the root* $\sqrt[m]{n}$ *for* $m, n \in \mathbf{N}$ *is either an integer or an irrational number;*

b) $\sqrt{n} + \sqrt{n+1}$ *is an irrational number for every* $n \in \mathbf{N}$;

c) $\sqrt{n + \sqrt{n}}$ *is an irrational number for every* $n \in \mathbf{N}$.

Solutions.

a) Let us assume the contrary, i.e., $\sqrt[m]{n} = \dfrac{p}{q}$, where p and q are integers with largest common divisor 1 and $|q| \neq 1$. Then $n = \dfrac{p^m}{q^m}$, which implies that p and q have a common divisor different from 1, which is a contradiction.

b) For every natural number n it holds

$$n^2 < n(n+1) < (n+1)^2,$$

hence $n(n+1)$ is not a complete square. Then by a), the root $\sqrt{n(n+1)}$ is an irrational number. Since

$$(\sqrt{n}+\sqrt{n+1})^2 = 2n+1+2\sqrt{n(n+1)},$$

it follows that the number $(\sqrt{n}+\sqrt{n+1})^2$ is irrational, hence so is the number $\sqrt{n}+\sqrt{n+1}$.

c) If $n = b^2$ for some $b \in \mathbf{N}$, then from the equality

$$\sqrt{n+\sqrt{n}} = \sqrt{b^2+b} = \sqrt{b(b+1)}$$

it follows that $\sqrt{n+\sqrt{n}}$ is not rational.

If n is not a square of some integer, then $\sqrt{n}$ is not rational, hence $\sqrt{n+\sqrt{n}}$ is neither a rational number.

Exercise 1.45. *Prove that*

a) *the number $\sqrt{3}$ is irrational;*

b) *the numbers $5\sqrt{2}$ and $3+\sqrt{2}$ are irrational, using that $\sqrt{2}$ is irrational;*

c) $\sqrt{\dfrac{n+1}{n}}$ *is an irrational number for every $n \in \mathbf{N}$.*

Example 1.46. *Prove that the Archimedes theorem (Theorem 1.12) holds in the set of rational numbers, i.e.,*

$$(\forall a \in \mathbf{Q},\ a > 0)\ (\forall b \in \mathbf{Q})\ (\exists n \in \mathbf{N})\quad na > b.$$

Solution. If $a > 0$ and $b < 0$, then we can take $n = 1$. So we can suppose that $a = \dfrac{m_1}{m_2}$, $b = \dfrac{n_1}{n_2}$, for some natural numbers m_1, m_2, n_1 and n_2. Put $n := m_2 n_1 + 1$; then

$$na = (m_2 n_1 + 1)\frac{m_1}{m_2} > m_1 n_1 \geq \frac{n_1}{n_2} = b,$$

since $m_1 n_2 \geq 1$.

Example 1.47. *Using the Archimedes theorem in* $\mathbf{R}$ *(see Theorem 1.12), show that*

a) *for every $x \in \mathbf{R}$, $x \neq 0$, there exists an $n \in \mathbf{N}$ such that $0 < \dfrac{1}{n} < |x|$;*

b) *if for a nonnegative number x it holds that for every $n \in \mathbf{N}$ $x < \dfrac{1}{n}$, then $x = 0$;*

c) *for every* $x \in \mathbf{R}$ *there exist numbers* $n_1, n_2 \in \mathbf{N}$ *such that* $-n_1 < x < n_2$;

d) *for every* $x \in \mathbf{R}$ *there exists a unique* $k_0 \in \mathbf{Z}$ *such that* $k_0 \leq x < k_0 + 1$;

e) *for every* $x, y \in \mathbf{R}$, $x < y$, *there exists a rational number* r, *such that* $x < r < y$. *(Compare with Example 1.41.)*

Solutions.

a) Assume $x \neq 0$, then $|x| > 0$. Thus from Theorem 1.12 it follows that for the real numbers 1 and $|x|$ there exists an $n \in \mathbf{N}$ such that $n|x| > 1$, or $\frac{1}{n} < |x|$.

b) If we suppose that $x > 0$, then from a) it follows that there exists an $n \in \mathbf{N}$ such that $x > \frac{1}{n}$. (Put $b = 1$ and $a = |x|$ in the Archimedes theorem.)

c) Since $1 > 0$ (see Example 1.15 k)), there exists an $n \in \mathbf{N}$ such that $n \cdot 1 > x$. Analogously, there exists an $n' \in \mathbf{N}$ such that $n' \cdot 1 > -x$ or $-n' < x$. Putting $n_1 = n'$ and $n_2 = n$, we get the statement.

d) Applying the Archimedes theorem, there exists a $k \in \mathbf{N}$ such that $k \cdot 1 > |x| \geq x$, hence $k > x$. Let k_0 be the largest among the numbers $0, \pm 1, \ldots, \pm k$ which are not greater than x. Then $k_0 \leq x < k_0 + 1$. Clearly, by the construction, the integer k_0 is unique.

Remark. The **greatest integer part** of a real number x denoted by $[x]$ is defined to be the largest integer less or equal to x.

e) If $x < y$, then $y - x > 0$, hence by a) there exists an $n \in \mathbf{N}$ such that $\frac{1}{n} < y - x$. By d), there exists a unique $p \in \mathbf{Z}$ such that $p = [nx]$, i.e., $p \leq nx < p + 1$. Then it holds $p + 1 < ny$, since otherwise $p + 1 \geq ny$ would imply

$$n(y - x) \leq p + 1 - p \Rightarrow y - x \leq \frac{1}{n},$$

a contradiction. Since $\frac{p+1}{n} \in \mathbf{Q}$ and $x < \frac{p+1}{n} < y$, for the rational number r we can take $\frac{p+1}{n}$.

Remark. From Example 1.47 e) it immediately follows that between any two real numbers a and b, $a < b$, there are infinitely many rational numbers and infinitely many irrational numbers.

Example 1.48. [1] *Let* $[a_n, b_n]$, $n \in \mathbf{N}$, *be a sequence of closed intervals with the following properties*

(i) $m < n \Rightarrow a_m \leq a_n \leq b_n \leq b_m$ *(every interval is contained the previous one);*

[1] In this example, we use the notion of the limit of a sequence - see Definition 3.1.

(ii) $\lim\limits_{n\to\infty}(b_n - a_n) = 0$ *(the lengths of intervals tend to zero).*

Prove that then there exists exactly one *real number* α *belonging to every interval* $[a_n, b_n]$.

Solution. By the Cantor theorem, the intersection $\bigcap\limits_{n\in\mathbf{N}}[a_n, b_n]$ is not empty. Hence, there exists a real number α which belongs to all intervals $[a_n, b_n]$, i.e., $a_n \leq \alpha \leq b_n$ for every $n \in \mathbf{N}$. We have to prove that α is the *only* such real number. Assume there exists another real number $\beta \neq \alpha$ such that $a_n \leq \beta \leq b_n$ for every $n \in \mathbf{N}$. Assume $\alpha < \beta$, then it holds $a_n \leq \alpha < \beta \leq b_n$ for every $n \in \mathbf{N}$. This implies $0 < \beta - \alpha < b_n - a_n$, a contradiction with the condition (ii). The assumption $\alpha > \beta$ is handled in an analogous way.

Example 1.49. *Is the Cantor theorem (Theorem 1.13) true when applied to a family of arbitrary intervals* $(I_n)_{n\in\mathbf{N}}$ *such that*

$$n > m \Rightarrow I_n \subset I_m? \tag{1.7}$$

Solution. In general, without the assumption that the intervals are *closed,* the Cantor theorem is not true. Namely, put $I_n = (0, \frac{1}{n})$, $n \in \mathbf{N}$, then the relation (1.7) holds. However, in view of Example 1.47 b), the intersection $\bigcap\limits_{n\in\mathbf{N}} I_n$ is empty.

Example 1.50. *Let us denote by (R15)' the following statement.*
(R15)' Every nonempty set $X \subset \mathbf{R}$ bounded from below has an infimum.

a) *Prove that every nonempty set* $X \subset \mathbf{R}$ *bounded from above has a supremum (resp. bounded from below has an infimum).*

b) *Prove that if in the system of axioms (R1) - (R15) one replaces the last one, namely (R15), with (R15)' then the system of axioms (R1) - (R14) and (R15)' becomes equivalent with the system (R1) - (R15).*

(Then we say that the axiom (R15) is equivalent with (R15)'.)

Solution.

a) We shall prove only the statement that if a nonempty set $X \subset \mathbf{R}$ is bounded from above, then it has a supremum. Let the set Y be defined by

$$Y = \{y \in \mathbf{R} | \, (\forall x \in X) \quad y \geq x\}.$$

By assumption, the set Y is nonempty and it holds

$$(\forall x \in X)\,(\forall y \in Y) \quad x \leq y,$$

hence, we can apply axiom (R15). It gives us the existence of a number $c \in \mathbf{R}$, such that

$$(\forall x \in X)\,(\forall y \in Y) \quad x \leq c \leq y. \tag{1.8}$$

By construction of the set Y, it holds $c = \min Y = \sup X$. Namely, if $c \notin Y$, then there exists an $x \in X$ such that $x > c$, which is in contradiction with (1.8).

b) In a) we proved that the system (R1) - (R15) implies (R15)'. We have yet to prove that if the axioms (R1) - (R14) and (R15)' hold, then the statement in (R15) (from Definition 1.1) follows.

Assume that X and Y are two nonempty sets from $\mathbf{R}$ such that

$$(\forall x \in X)\,(\forall y \in Y) \quad x \leq y. \tag{1.9}$$

The set Y is then bounded from below, hence by (R15)' the set Y has an infimum c. By (i1) from Definition 1.4, for every $y \in Y$ it holds $c \leq y$. Finally, let us show that $(\forall x \in X) \quad x \leq c$. Otherwise, there exists an element $x_1 \in X$ such that $x_1 > c$. But then the number x_1 is a lower bound for Y greater than c, which is in contradiction with (i2) from Definition 1.4.

Example 1.51. *Prove that the axiom (R15) is equivalent with the conjunction of the Archimedes and the Cantor theorem (see Theorems 1.12 and 1.13).*

Solution. We shall omit the proof of the statement that the system of axioms (R1) - (R14) plus (R15)', being equivalent to the system (R1) - (R15), implies the Archimedes and the Cantor theorem. We shall only prove that assuming axioms (R1) - (R14) and both the Archimedes and the Cantor theorem, every nonempty set bounded from below has an infimum. Then by Example 1.50 b), the axiom (R15) follows.

Assume that the nonempty set $B \subset \mathbf{R}$ is bounded from below with a number a_1, and let $b_1 \in B$. Then we claim that for every $m \in \mathbf{N}$ there exists the largest number $n_m \in \mathbf{N_0}$ such that

$$a_m := a_1 + \frac{n_m}{2^m}$$

is still a lower bound for B. In fact, if $a_1 = \inf B$, then $n_m = 0$. If $a_1 \neq \inf B$, then there exists $n_m \in \mathbf{N}$ such that

$$(\forall x \in B) \quad a_m = a_1 + \frac{n_m}{2^m} \leq x.$$

If this were not true, then for some $m \in \mathbf{N}$ there would be no such $n = n_m$, but rather it would hold $a_1 + \frac{n}{2^m} \leq x$ for every $n \in \mathbf{N}$ and every $x \in B$. But then $\frac{n}{2^m} \leq x - a_1$ for every $n \in \mathbf{N}$, which contradicts the Archimedes theorem.

Put now $b_m := a_m + \frac{1}{2^m} = a_1 + \frac{n_m + 1}{2^m}$; then $b_m \in B$ for every $m \in \mathbf{N}$. By construction,

$$a_1 \leq a_2 \leq \ldots \leq a_m \leq \ldots \leq b_m \leq \ldots \leq b_2 \leq b_1,$$

hence

$$[a_1, b_1] \supset [a_2, b_2] \supset \ldots [a_m, b_m] \supset \ldots .$$

By the Cantor theorem, there exists a real number b, such that

$$b \in \bigcap_{m \in \mathbf{N}} [a_m, b_m].$$

Then it holds $a_m \le b \le b_m$ for every $m \in \mathbf{N}$. Let us show next that for every $\varepsilon > 0$ there exists an $m \in \mathbf{N}$ such that $b - \varepsilon \le a_m$. In the contrary, there would exist an $\varepsilon > 0$ such that for every $m \in \mathbf{N}$ it holds $a_m < b - \varepsilon$. This would imply

$$a_m = a_1 + \frac{n_m}{2^m} < b - \varepsilon \le a_1 + \frac{n_m + 1}{2^m} - \varepsilon = a_m + \frac{1}{2^m} - \varepsilon.$$

Thus it follows

$$\varepsilon < \frac{1}{2^m} \iff 2^m < \frac{1}{\varepsilon}.$$

Using the Bernoulli inequality (see Example 1.31 a)) it would then follow

$$\frac{1}{\varepsilon} > 2^m = (1+1)^m \ge 1 + 1 \cdot m$$

for every $m \in \mathbf{N}$, which contradicts the Archimedes theorem.

So we obtained that for given $\varepsilon > 0$ there exists an $m \in \mathbf{N}$ such that $b - \varepsilon \le a_m$. In an analogous way, one can prove that for given $\varepsilon > 0$ there exists an $m \in \mathbf{N}$ such that $b \le b_m \le b + \varepsilon$ (do that!).

Putting these conclusions together, we obtain that for given $\varepsilon > 0$ there exists an $m \in \mathbf{N}$ such that

$$b - \varepsilon \le a_m \le b \le b_m \le b + \varepsilon. \tag{1.10}$$

Finally, let us show that $b = \inf B$. By the construction, for every $x \in B$ there exists an element $b_m \ge b$ such that $b_m \le x$, hence b is a lower bound for B. Further on, from relation (1.10) it follows that b is also the greatest lower bound for the set B.

Remark. This example shows that the Cantor theorem does not hold for the set of rational numbers $\mathbf{Q}$. Namely, as we shall see in Examples 3.33 and 3.36, the subset X of $\mathbf{Q}$ defined by

$$X := \left\{ \left(1 + \frac{1}{n}\right)^n \;\middle|\; n \in \mathbf{N} \right\}$$

is bounded from above, but has no supremum in $\mathbf{Q}$. In fact, it does have a supremum in $\mathbf{R}$ and it is the irrational number e.

Example 1.52. *Let the set A be given by* $A = \left\{ \frac{1}{3} \pm \frac{n}{3n+1} \;\middle|\; n \in \mathbf{N} \right\}$. *Prove that* $\inf A = 0$ *and* $\sup A = \frac{2}{3}$.

Solution. We have for $n \in \mathbf{N}$

$$\frac{1}{3} \pm \frac{n}{3n+1} \ge \frac{1}{3} - \frac{n}{3n+1} = \frac{1}{3(3n+1)} > 0.$$

This means that the set A is bounded from below with 0. By Example 1.50 it has an infimum. We prove next that just 0 is that infimum. If, however, a positive number ε would be the infimum of A, then clearly $\varepsilon \le \frac{1}{12}$ (put $n = 1$). Putting $n_0 := [\frac{1}{9\varepsilon}(1 - 3\varepsilon)] + 1$ it would then hold

$$n > n_0 \Rightarrow 0 < \frac{1}{3} - \frac{n}{3n+1} < \varepsilon,$$

a contradiction.

Let us prove now that the supremum of A is $\frac{2}{3}$. Firstly, $\frac{2}{3}$ is an upper bound of A, since it holds for every $n \in \mathbf{N}$

$$\frac{2}{3} - \left(\frac{1}{3} \pm \frac{n}{3n+1}\right) \geq \frac{1}{3(3n+1)} > 0.$$

Next, for every $\varepsilon > 0$ we choose n_0 as above; then for every $n > n_0$ it holds

$$\frac{2}{3} - \varepsilon < \frac{1}{3} + \frac{n}{3n+1} \leq \frac{2}{3}.$$

The last inequality implies that $\frac{2}{3}$ is the supremum of the set A, since it means that for every $\varepsilon > 0$ there exists an element x from A which belongs to the interval $\left(\frac{2}{3} - \varepsilon, \frac{2}{3}\right)$.

Example 1.53. *For a nonempty set $X \subset \mathbf{R}$ define*

$$-X := \{-x \mid x \in X\}.$$

Prove that

$$\textbf{a)}\ \inf(-X) = -\sup X; \qquad \textbf{b)}\ \sup(-X) = -\inf X,$$

provided that in a) (resp. in b)) X is bounded from above (resp. from below).

Solutions. We shall prove only part **a)**, since b) is quite analogous.

By Example 1.50 the set X has a supremum; let us denote it by M. Then since M is an upper bound for X, it holds

$$(\forall x \in X) \quad x \leq M \iff (\forall x \in X) \quad -x \geq -M$$

Thus it holds

$$(\forall y \in (-X)) \quad y \geq -M. \tag{1.11}$$

Since M is the smallest upper bound for X, it holds

$$(\forall \varepsilon > 0)\,(\exists x_1 \in X) \quad x_1 > M - \varepsilon \iff (\forall \varepsilon > 0)\,(\exists x_1 \in X) \quad -x_1 < -M + \varepsilon.$$

Thus it follows

$$(\forall \varepsilon > 0)\,(\exists x_2 \in (-X)) \quad x_2 < -M + \varepsilon. \tag{1.12}$$

The relations (1.11) and (1.12) mean that $-M$ is the infimum of the set $-X$.

Example 1.54. *Let X and Y be two nonempty subsets of $\mathbf{R}$ bounded from below (resp. from above). Put*

$$S := \{s = x + y \mid x \in X,\, y \in Y\}.$$

Prove that S has an infimum (resp. a supremum) and it holds

$$\inf S = \inf X + \inf Y \qquad (\textit{resp.}\ \ \sup S = \sup X + \sup Y).$$

Solution. In view of Example 1.50, if the sets X and Y are bounded from below, then they have an infimum; let us denote by $m_1 = \inf X$ and $m_2 = \inf Y$. Hence it holds

$$(\forall x \in X) \quad x \geq m_1 \quad \text{and} \quad (\forall y \in Y) \quad y \geq m_2.$$

Let s be an arbitrary element from S; then there exist $x \in X$ and $y \in Y$ such that $s = x + y$. Then we have

$$s = x + y \geq m_1 + m_2,$$

hence the set S is bounded from below by the sum $m_1 + m_2$. Let us show that the last number is also the greatest lower bound for S. For given ε, there exist $x_1 \in X$ and $y_1 \in Y$ such that

$$x_1 < m_1 + \frac{\varepsilon}{2} \quad \text{and} \quad y_1 < m_2 + \frac{\varepsilon}{2}.$$

This implies

$$(\forall \varepsilon > 0)\,(\exists s_1 := x_1 + y_1 \in S) \quad s_1 = x_1 + y_1 < m_1 + m_2 + \varepsilon,$$

and thus we obtain that $m_1 + m_2$ is the infimum of the set S.

Exercise 1.55. *Find, if any, the infimums and supremums of the following sets and check whether they are also their minimums or maximums.*

a) $X = \left\{ \frac{5n-1}{7n+2} \middle| \, n \in \mathbf{N} \right\}$; b) $X = \left\{ \frac{2}{n} \middle| \, n \in \mathbf{N} \right\}$;

c) $X = \left\{ 1 + \frac{2 \cdot (-1)^n}{n} \middle| \, n \in \mathbf{N} \right\}$; d) $X = \left\{ \sum_{k=1}^{n} \frac{1}{2^k} \middle| \, n \in \mathbf{N} \right\}$;

e) $X = \left\{ \frac{m}{n} \middle| \, m, n \in \mathbf{N}, \ m < n \right\}$.

Answers.

a) $\inf X = \min X = \frac{4}{9}$, $\sup X = \frac{5}{7}$. b) $\inf X = 0$, $\sup X = \max X = 2$.

c) $\inf X = \min X = -1$, $\sup X = 2$. d) $\inf X = \min X = \frac{1}{2}$, $\sup X = 1$.

e) $\inf X = 0$, $\sup X = 1$.

Exercise 1.56. *Let X and Y be two nonempty bounded subsets of $\mathbf{R}$, and put*

$$X - Y := \{z \in \mathbf{R} | \ (\exists x \in X)\,(\exists y \in Y) \ z = x - y\}.$$

Prove that $\sup(X - Y) = \sup X - \inf Y$.

Exercise 1.57. *Assume X and Y are two nonempty subsets of $\mathbf{R}_+ = (0, +\infty)$ bounded from below, and put*

$$Z := \{z \in \mathbf{R} | \ (\exists x \in X)\,(\exists y \in Y) \ z = x \cdot y\}.$$

Prove that

a) $\inf Z = \inf X \cdot \inf Y$ *if X and Y are bounded from below;*

a) $\sup Z = \sup X \cdot \sup Y$ *if X and Y are bounded from above.*

1.2 Cuts in Q

1.2.1 Basic notions

In Subsection 1.1 we gave the axioms (R1) - (R15) that defined the set of real numbers $\mathbf{R}$ as a complete totally ordered field, $(\mathbf{R}, +, \cdot, \leq)$. One can prove that, up to an isomorphism, there exists a unique totally ordered field $(\mathbf{Q}, +, \cdot, \leq)$ which satisfies the axioms (R1) - (R14). However, as one can see from Example 1.51, in Subsection 1.1.2, $\mathbf{Q}$ does not satisfy the last axiom (R15).

The goal of this section is to expose an effective construction of the set of real numbers $\mathbf{R}$ starting from the set of rational numbers $\mathbf{Q}$. This method, developed firstly by the German mathematician R. Dedekind in 1872, uses the so-called **cuts** in $\mathbf{Q}$. [2]

Definition 1.58. *A* **cut** α *in the set of rational numbers* $\mathbf{Q}$ *(shortly: cut) is a subset of* $\mathbf{Q}$ *with the following three properties.*

(c1) $\alpha \neq \emptyset \quad \wedge \quad \alpha \neq \mathbf{Q}$;

(c2) $(\forall a, b \in \mathbf{Q})\ (a \in \alpha) \quad b < a \Rightarrow b \in \alpha$;

(c3) α *has no maximum.*

The set of all cuts will be denoted by $\mathcal{R}$.

Every rational number defines a cut as follows.

Theorem 1.59. *Let* r *be a rational number and put*

$$r^* := \{p \in \mathbf{Q} |\ p < r\}. \tag{1.13}$$

Then r^* *is a cut, called* **rational cut**, *defined by the rational number* r.

The set of **rational cuts** will be denoted by $\mathcal{Q}$. In particular, 0^* (the *zero-cut*) and 1^* (the *one-cut*) are rational cuts. We shall see in Example 1.69 that $\mathcal{Q}$ is a proper subset of the set of all cuts $\mathcal{R}$; those cuts that are not rational, will be called **irrational cuts**.

In $\mathcal{R}$ we define the relation " $\prec$ " by

$$(\forall \alpha, \beta \in \mathcal{R}) \quad \alpha \prec \beta \iff (\exists p \in \mathbf{Q}) \quad p \in (\beta \setminus \alpha). \tag{1.14}$$

We also put $\alpha \preceq \beta$ if either $\alpha = \beta$ or $\alpha \prec \beta$. The notations " $\succ$ " and " $\succeq$ " are interpreted analogously. In Example 1.62 we shall prove that $\preceq$ is a total ordering in $\mathcal{R}$.

[2] Another important construction of the set $\mathbf{R}$ starting from $\mathbf{Q}$ uses the so-called Cauchy sequences (see Definition 3.8), and was elaborated by another German mathematician, namely G. Cantor. It is interesting to note that Cantor and Dedekind almost simultaneously announced their constructions to the mathematical community.

If a cut α satisfies $\alpha \succ 0^*$ (resp. $\alpha \succeq 0^*$) then it is called a **positive cut** (resp. **nonnegative cut**). The set of positive cuts will be denoted by $\mathcal{R}_+$. The negative and nonpositive cuts are defined analogously.

The **addition** in $\mathcal{R}$, denoted by $\oplus$, is defined by

$$\alpha \oplus \beta := \{x + y \mid x \in \alpha,\, y \in \beta\} \tag{1.15}$$

for arbitrary cuts $\alpha, \beta \in \mathcal{R}$. We shall show that $(\mathcal{R}, \oplus)$ is an Abelian group, with the additive identity 0^*.

The **absolute value** $|\alpha|$ of a cut α is defined by

$$|\alpha| := \begin{cases} \alpha, & \text{if} \quad \alpha \succeq 0^*; \\ \ominus\alpha, & \text{if} \quad \alpha \prec 0^*, \end{cases} \tag{1.16}$$

where $\ominus\alpha$ is the additive inverse of α for $\oplus$ (see Example 1.65 b)). Clearly, for every $\alpha \in \mathcal{R}$ it holds $|\alpha| \succeq 0^*$, and $|\alpha| = 0^*$ iff $\alpha = 0^*$.

In order to define the **multiplication** in $\mathcal{R}$, denoted by $\otimes$, we shall start with the multiplication in $\mathcal{R}_+$

$$\alpha \otimes \beta := \{xy \mid x \in \alpha,\, y \in \beta\} \bigcup \{x \in \mathbf{Q} \mid x \le 0\} \tag{1.17}$$

for arbitrary positive cuts $\alpha, \beta \in \mathcal{R}_+$. It is an easy task to show that $\alpha \otimes \beta$ is a positive cut when $\alpha, \beta \in \mathcal{R}_+$. Next, we put for every cut α

$$\alpha \otimes 0^* = 0^*.$$

Finally, if α and β are arbitrary elements in $\mathcal{R}$, then we put

$$\alpha \otimes \beta := \begin{cases} |\alpha| \otimes |\beta|, & \text{if either } \alpha \preceq 0^* \text{ and } \beta \preceq 0^*, \text{ or } \alpha \succeq 0^* \text{ and } \beta \succeq 0^*; \\ \ominus|\alpha| \otimes |\beta|, & \text{otherwise.} \end{cases} \tag{1.18}$$

We shall show in Example 1.66 that $(\mathcal{R} \setminus 0^*, \otimes)$ is an Abelian group, with the multiplicative identity 1^*.

Moreover, the distributive law holds in $\mathcal{R}$.

$$(\forall \alpha, \beta, \gamma \in \mathcal{R}) \quad (\alpha \oplus \beta) \otimes \gamma = (\alpha \otimes \gamma) \oplus (\beta \otimes \gamma). \tag{1.19}$$

So we obtain that $(\mathcal{R}, \oplus, \otimes, \preceq)$ is a totally ordered field. In Example 1.70 we shall show that it is isomorphic to the field $(\mathbf{R}, +, \cdot, \le)$. This means that there is a bijection[3] $\phi : \mathcal{R} \to \mathbf{R}$ such that for every $\alpha, \beta \in \mathcal{R}$ it holds

$$\alpha \preceq \beta \iff \phi(\alpha) \le \phi(\beta); \quad \phi(\alpha \oplus \beta) = \phi(\alpha) + \phi(\beta); \quad \phi(\alpha \otimes \beta) = \phi(\alpha) \cdot \phi(\beta).$$

Moreover, ϕ can be chosen with the property that its restriction to the set of rational cuts $\mathcal{Q}$ satisfies the condition

$$(\forall r \in \mathbf{Q}) \quad \phi(r^*) = r.$$

[3]See Subsection 2.1.1.

1.2.2 Examples and exercises

Example 1.60. *Show that if a rational number x does not belong to a cut α, then for every $y \in \alpha$ it holds $x > y$.*

Solution. Assume that there exists a $y \in \alpha$ such that $y \geq x$. Then by property (c2) from Definition 1.58 it follows that $x \in \alpha$, contradicting the assumption $x \notin \alpha$.

Example 1.61.

a) *Prove Theorem 1.59, i.e., that the set $r^* = \{p \in \mathbf{Q} | \, p < r\}$ is a cut for every rational number r.*

b) *Let α be a cut. Prove that the set $\mathbf{Q} \setminus \alpha$ has a minimum iff α is a rational cut.*

Solutions.

a) Take a rational number r and define the set r^* by (1.13). We have to check the three conditions (c1) - (c3) from Definition 1.58.

- The set r^* is nonempty, since it contains, for instance, the rational number $r - 1$, and since $r + 1 \notin r^*$, it is a proper subset of $\mathbf{Q}$.
- The second condition is satisfied by definition.
- Clearly, the supremum of r^* is r, which by definition is not in r^*.

Thus it follows that r^* is a cut.

b) By property (c1) from Definition 1.58, the set $\mathbf{Q} \setminus \alpha$ is nonempty. We shall analyze two cases. The first is when α is a rational cut and the second, when it is not.
Clearly, if $\alpha = r^*$ for some $r \in \mathbf{Q}$, then $r \notin \alpha$, which implies that $r \in (\mathbf{Q} \setminus \alpha)$. Thus $r = \inf(\mathbf{Q} \setminus \alpha) = \min(\mathbf{Q} \setminus \alpha)$. Let α be a cut from $\mathcal{R} \setminus \mathcal{Q}$. Assume that the set $\mathbf{Q} \setminus \alpha$ has a minimum r. Then by Example 1.60 for every $x \in \alpha$ it holds $x < r$, hence by Theorem 1.59 $\alpha = r^*$, which is a contradiction.

Example 1.62. *Prove that the binary relation $\preceq$ is a total ordering in the set of cuts $\mathcal{R}$.*

Solution. Let α, β be two arbitrary cuts. We have to prove that either $\alpha \preceq \beta$ or $\beta \preceq \alpha$.

If α and β are equal as sets, then by definition $\alpha \preceq \beta$ and $\beta \preceq \alpha$. Let $\alpha \neq \beta$. We have to prove that exactly one of the following two inclusions is true

$$\alpha \subset \beta \quad \text{or} \quad \beta \subset \alpha. \tag{1.20}$$

Assume the contrary to 1.20. Then there exist two rational numbers a and b such that $a \in (\alpha \setminus \beta)$ and $b \in (\beta \setminus \alpha)$.

We have

$$a \in (\alpha \setminus \beta) \iff (a \in \alpha \wedge a \notin \beta) \Rightarrow (\forall x \in \beta) \;\; x < a \tag{1.21}$$

(see Example 1.60). Analogously

$$b \in (\beta \setminus \alpha) \iff (b \in \beta \wedge b \notin \alpha) \Rightarrow (\forall y \in \alpha)\ \ y < b. \tag{1.22}$$

From (1.21) we obtain that $b \in \beta$ implies $b < a$, while $a \in \alpha$ implies $a < b$, hence we have a contradiction.

Example 1.63.

a) *If α and β are arbitrary cuts, then the set $\gamma := \alpha \oplus \beta$ defined by relation (1.15), is also a cut.*

b) *Prove that the product of two positive cuts given by relation (1.17) is again a positive cut.*
Hence by relation (1.18), the product of two arbitrary cuts is also a cut.

Solution. We shall prove only part
a) We have to prove that the set γ satisfies the properties (c1) - (c3) from Definition 1.58.

- Since α and β are cuts, then (c1) implies that there exist two rational numbers $a \in \alpha$ and $b \in \beta$, hence the rational number $c := a + b$ is in γ. On the other hand, again (c1) implies that there exist rational numbers $a' \notin \alpha$ and $b' \notin \beta$, hence the rational number $c' := a' + b'$ is not in γ. Thus γ is neither empty nor equals to the whole of $\mathbf{Q}$, which means that it satisfies (c1).

- Assume $c \in \gamma$ and let $x \in \mathbf{Q}$, $\ x < c$. We have to prove that $x \in \gamma$. From $c \in \gamma$ it follows that there exist $a \in \alpha$ and $b \in \beta$, such that $c = a + b$. Put $d := c - x > 0$. Then it holds

$$x = c - d = (a + b) - (\frac{d}{2} + \frac{d}{2}) = (a - \frac{d}{2}) + (b - \frac{d}{2}).$$

Now $a - \dfrac{d}{2}$ is in α and $b - \dfrac{d}{2}$ is in β by (c2), hence their sum x is in γ. Thus γ satisfies (c2).

- Finally, we have to prove that γ has no maximum. Assume the contrary, i.e., that there exists a maximum c_0 in γ. Then by the definition of $\oplus$ given in (1.15) there exist two rational numbers $a_0 \in \alpha$ and $b_0 \in \beta$ such that $c_0 = a_0 + b_0$. By (c3), the cuts α and β have no maximums, hence there exist $a_1 \in \alpha$ and $b_1 \in \beta$ such that $a_1 > a_0$ and $b_1 > b_0$. But then $a_1 + b_1$ is an element from γ such that

$$a_1 + b_1 > a_0 + b_0 = c_0,$$

which means that c_0 is not the maximum of γ. Hence γ has no maximum, and thus it satisfies (c3).

Exercise 1.64. *Prove that the operations* $\oplus$ *and* $\otimes$ *satisfy the commutative law,*

$$(\forall \alpha, \beta \in \mathcal{R}) \qquad \alpha \oplus \beta = \beta \oplus \alpha, \quad \alpha \otimes \beta = \beta \otimes \alpha,$$

the associative law,

$$(\forall \alpha, \beta, \gamma \in \mathcal{R}) \qquad (\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma), \quad (\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma),$$

and also the distributive law,

$$(\forall \alpha, \beta, \gamma \in \mathcal{R}) \qquad \alpha \otimes (\beta \oplus \gamma) = (\alpha \otimes \beta) \oplus (\alpha \otimes \gamma).$$

Example 1.65. *Prove that*

a) *the zero cut* 0^* *is the additive identity for* $\oplus$*;*

b) *the additive inverse for* $\alpha \in \mathcal{R}$ *for the operation* $\oplus$ *is the set* $\ominus\alpha$*, where*

$$\ominus\alpha := \{-x \in \mathbf{Q} \mid x \in (\mathbf{Q} \setminus \alpha),\ x \neq a\},$$

where a *is the minimum of the set* $\mathbf{Q} \setminus \alpha$*, provided this minimum exists.*

Solution. We shall prove only part
a) Let us put

$$\alpha_1 := \alpha \oplus 0^* = \{a + x \mid x \in \alpha,\ x < 0\}.$$

By Example 1.63, α_1 is a cut. We have to prove the set-equality

$$\alpha_1 = \alpha, \qquad \text{or equivalently} \qquad \alpha_1 \subset \alpha \wedge \alpha \subset \alpha_1.$$

- If $y \in \alpha_1$, then by (1.15) there exist rational numbers $a \in \alpha$ and $x < 0$ such that $y = a + x$. Hence $y < a$, which by (c2) means that $y \in \alpha$. Thus $\alpha_1 \subset \alpha$.

- If $a \in \alpha$, then by (c3) there exists $a' \in \alpha$, such that $a' > a$. Hence

$$a = a + (-a' + a') = (a - a') + a' = a' + (a - a').$$

  Since $a' \in \alpha$ and $(a - a') \in 0^*$, it follows that $a \in \alpha_1$. Thus $\alpha \subset \alpha_1$.
  We proved that $\alpha \oplus 0^* = \alpha$. By Example 1.64, it also holds

$$0^* \oplus \alpha = \alpha \oplus 0^* = \alpha.$$

Remark. From Examples 1.63 and 1.65 and Exercise 1.64 it follows that $(\mathcal{R}, \oplus)$ is an Abelian group.

Example 1.66. *Prove that*

a) *the one-cut* 1^* *is the multiplicative identity for* $\otimes$*;*

b) *for given $\alpha \in \mathcal{R}_+$, the set $\alpha^{\ominus 1^*}$ is the multiplicative inverse element of α for $\otimes$, where*

$$\alpha^{\ominus 1^*} := \{x \in \mathbf{Q} |\ x \leq 0\} \bigcup \left\{\frac{1}{x} |\ x \in (\mathbf{Q} \setminus \alpha),\ x \neq a, \right\} \tag{1.23}$$

if a is the minimum of $\mathbf{Q} \setminus \alpha$, provided this minimum exists.

For $\alpha \prec 0^*$, prove that its inverse is the cut $\ominus(|\alpha|^{\ominus 1^*})$.

Solution. We shall prove only part **b)** for the case $\alpha \succ 0^*$. First, we shall prove that the set $\alpha^{\ominus 1^*}$ is a cut.

- Since α is a cut, it follows from (c1) applied to α that $\alpha^{\ominus 1^*}$ is nonempty and is a proper subset of $\mathbf{Q}$.

- Let $x \in \alpha^{\ominus 1^*}$ and assume that the rational number y satisfies $y < x$; we have to prove that $y \in \alpha^{\ominus 1^*}$. If $y \leq 0$, then there is nothing left to prove. So we can assume that $0 < y < x$, which implies $0 < \frac{1}{x} < \frac{1}{y}$. Using the last inequality, it holds
$$x \in \alpha^{\ominus 1^*} \Rightarrow \frac{1}{x} \in (\mathbf{Q} \setminus \alpha) \Rightarrow \frac{1}{y} \in (\mathbf{Q} \setminus \alpha) \Rightarrow y \in \alpha^{\ominus 1^*}.$$

- Finally, we have to prove that $\alpha^{\ominus 1^*}$ has no maximal element. The set $\mathbf{Q} \setminus \alpha$ either has a minimum $a > 0$, or it has not.
In the first case, it holds by definition of $\alpha^{\ominus 1^*}$ (see relation (1.23)) that the positive rational number $1/a$ is not in $\alpha^{\ominus 1^*}$. Thus $\alpha^{\ominus 1^*}$ has then no maximum.
In the case when $\mathbf{Q} \setminus \alpha$ has no minimum, the following holds
$$(\forall x \in (\mathbf{Q} \setminus \alpha) \quad (\exists x' \in (\mathbf{Q} \setminus \alpha))\ x > x' > 0) \Rightarrow \frac{1}{x} < \frac{1}{x'}.$$
But $\frac{1}{x'}$ is in $\alpha^{\ominus 1^*}$.

Hence it follows that $\alpha^{\ominus 1^*}$ is a cut.
Using Example 1.63 it follows that for $\alpha \succ 0^*$ the set $\alpha \otimes \alpha^{\ominus 1^*}$ is also a cut.
We shall prove now that $\alpha \otimes \alpha^{\ominus 1^*} = 1^*$. This is equivalent to the conjunction of inclusions

$$(\alpha \otimes \alpha^{\ominus 1^*} \subset 1^*) \wedge (1^* \subset \alpha \otimes \alpha^{\ominus 1^*}).$$

- Let $z \in (\alpha \otimes \alpha^{\ominus 1^*})$. If $z \leq 0$, then $z \in 1^*$. Hence we can suppose $z > 0$; by definition there exist positive numbers $x \in \alpha$ and $y \in \alpha^{\ominus 1^*}$ such that $z = xy$. Hence
$$y \in \alpha^{\ominus 1^*} \Rightarrow \frac{1}{y} \in (\mathbf{Q} \setminus \alpha) \Rightarrow \frac{1}{y} > x \Rightarrow z = xy < 1 \Rightarrow z \in 1^*.$$

- Let now $z \in 1^*$. If $z \leq 0$, then $z \in \alpha \otimes \alpha^{\ominus 1^*}$ by the definition of the last product. So we can assume that $0 < z < 1$. Since α is a cut, there exists a positive rational number $x \in \alpha$ such that $\frac{x}{z} = y \notin \alpha$; otherwise the complement of α in $\mathbf{Q}$ would be empty. Put $y := \frac{x}{z} \in \mathbf{Q} \setminus \{\alpha\}$, then

$$\frac{1}{y} \in \alpha^{(-1)^*} \Rightarrow z = x \cdot \frac{1}{y} \Rightarrow z \in \alpha \otimes \alpha^{\ominus 1^*}.$$

Exercise 1.67. *Prove that $(\mathcal{Q}, \oplus, \otimes, \preceq)$ is a totally ordered field.*

Exercise 1.68. *Prove the following properties of rational cuts:*

a) *if α is a cut, then $r \in \mathbf{Q}$ is in α iff $r^* \prec \alpha$;*

b) *if α and β are cuts, then there exists a rational cut r^* such that $\alpha \prec r^* \prec \beta$;*

c) $(\forall r, p \in \mathbf{Q}) \quad r < p \iff r^* \prec p^*$;

d) $(\forall r, p, q \in \mathbf{Q}) \quad r + p = q \iff r^* \oplus p^* = q^*$;

e) $(\forall r, p, q \in \mathbf{Q}) \quad r \cdot p = q \iff r^* \otimes p^* = q^*$.

Example 1.69. *Prove that there exists a positive irrational cut β, i.e., a cut with the property that $\mathbf{Q} \setminus \beta$ has no minimum.*

Solution. Let us put

$$\beta := \{x \in \mathbf{Q}_+ |\ x^2 < 2\} \bigcup \{x \in \mathbf{Q} |\ x \leq 0\}.$$

(Of course, $\mathbf{Q}_+ = \{x \in \mathbf{Q} |\ x > 0\}$.) That the set β satisfies properties (c1) and (c2) is trivial. For (c3), assume that β has a maximum $y_0 \in \beta$. Then the rational number $y_1 := 2\frac{y_0 + 1}{y_0 + 2}$ is such that $y_1 > y_0$, but still $y_1^2 < 2$ (compare to Example 1.42). Hence β is a cut.

Finally, let us prove that β is an irrational cut. Assume the contrary, i.e., that there exists the smallest number x_0 in $\mathbf{Q} \setminus \beta$. But then the number $x_1 := 2\frac{x_0 + 1}{x_0 + 2}$ satisfies $x_1^2 > 2$, $x_1 > 0$ and $x_1 < x_0$. Hence x_1 is also in $\mathbf{Q} \setminus \beta$, but is smaller than x_0, a contradiction.

Example 1.70. The Dedekind theorem.
Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of cuts with the following properties.

(D1) *a cut is in one and only one of these classes;*

(D2) *neither of these classes is empty;*

(D3) *for every cut $\alpha \in \mathcal{A}$ and every cut $\beta \in \mathcal{B}$ it holds $\alpha \preceq \beta$.*

Then there exists a unique cut γ such that

$$(\forall \alpha \in \mathcal{A})\ (\forall \beta \in \mathcal{B}) \qquad \alpha \preceq \gamma \preceq \beta. \tag{1.24}$$

Solution. Let us show the existence of the cut γ. The class $\mathcal{A}$ either has a maximum, or it has not. In the first case, put $c := \max \mathcal{A}$. Then the sought cut is $\gamma := c^*$. Next we assume that $\mathcal{A}$ has no maximum. Let us put

$$G = \{x \in \mathbf{Q} |\ x^* \in \mathcal{A}\}.$$

We shall prove that G is a cut (i.e., we shall check the properties (c1) - (c3) from Definition 1.58) and, moreover, show that G satisfies the inequalities in (1.24). This will then imply $G = \gamma$ is a cut.

- Since $\mathcal{A}$ is not empty, there exists a cut $\alpha \in \mathcal{A}$. By assumption, $\mathcal{A}$ has no maximum, hence there exists another cut $\alpha_1 \in \mathcal{A}$ such that $\alpha_1 \succ \alpha$. By Example 1.68 b), there exists a rational cut in $\mathbf{R}$ $r^* \in \mathcal{A}$ such that $\alpha \preceq r^* \preceq \alpha_1$. By the definition of G, r is in G.
  Since $\mathcal{B}$ is also nonempty, there exists a cut $\beta \in \mathcal{B}$. Since β is a cut, there exists a rational number p such that $p \notin \beta$. Hence

  $$\beta \prec p^* \Rightarrow p^* \in \mathcal{B} \Rightarrow p^* \notin \mathcal{A} \Rightarrow p \notin G.$$

  Thus the set G is neither empty nor does it equal the whole of $\mathbf{Q}$.

- Let $r \in G$ and take some rational number $r_1 < r$. Then it holds

  $$r^* \in \mathcal{A} \Rightarrow r_1^* \in \mathcal{A} \Rightarrow r_1 \in G.$$

  Thus (c2) holds for G.

- In order to show property (c3) for the set G, we have to prove that G has no maximum. Let us assume the contrary, i.e., that there exists a maximum $m \in \mathbf{Q}$ of G. We claim that then m^* is the maximum in $\mathcal{A}$, of course for the binary relation $\preceq$. Once we prove this, we shall get a contradiction with our assumption that the class $\mathcal{A}$ has no maximum. This will then mean that G can not have a maximum, i.e., (c3) holds for G. In order to prove that m^* is the maximum of $\mathcal{A}$, under the assumption that G has a maximum m, we shall suppose that m^* is not the maximum of $\mathcal{A}$. Then there exists a cut $\alpha \in \mathcal{A}$ such that $\alpha \succ m^*$. By Example 1.68 b), there exists a rational cut m_1^* such that $m^* \prec m_1^* \prec \alpha$. But then m can not be the maximum of G, since $m_1 > m$ and $m_1 \in G$.

Thus it follows that G is a cut. Let us take any $\alpha \in \mathcal{A}$. Since $\mathcal{A}$ has no maximum, there exists another cut $\alpha_1 \in \mathcal{A}$ such that $\alpha_1 \succ \alpha$. From Example 1.68 b) we find a rational cut r^* such that $\alpha \prec r^* \prec \alpha_1$. By definition, the rational number r is in G, hence $G \succ \alpha$. Let now $\beta \in \mathcal{B}$. Then either $G \preceq \beta$ (as we claim), or $G \succ \beta$. In the latter case there exists a rational cut $p^* \in \mathcal{B}$, such that $\beta \prec p^* \prec G$. But then

$p \in G$, hence $p^* \in \mathcal{A}$, which contradicts to the assumption that the classes $\mathcal{A}$ and $\mathcal{B}$ are disjoint.

So we obtained that the cut G has the property

$$(\forall \alpha \in \mathcal{A})\ (\forall \beta \in \mathcal{B}) \qquad \alpha \preceq G \preceq \beta.$$

Hence we can put $\gamma := G$.

Let us prove that this γ is unique. Assume that there exist two cuts γ_1 and γ_2 which satisfy relation (1.24) and let $\gamma_1 \prec \gamma_2$. There exists a rational number r such that $\gamma_1 \prec r^* \prec \gamma_2$. But then $\gamma_1 \prec r^*$ implies $r^* \in \mathcal{B}$, while $r^* \prec \gamma_2$ implies $r^* \in \mathcal{A}$. Thus the classes $\mathcal{A}$ and $\mathcal{B}$ are not disjoint, contrary to the assumption. The contradiction came from the assumption that there were two cuts which satisfy (1.24), hence the cut $\gamma = G$ is the unique cut satisfying (1.24).

Remark. Dedekind's theorem is, in fact, axiom (R15) from Definition 1.1 given at the beginning of Chapter 1. In the previous examples and exercises we either proved or just left to the reader to prove the properties of $\mathcal{R}$ analogous to those given by the first fourteen axioms (R1) - (R14) from Definition 1.1. With the last example, we thus proved that the set of rational cuts $\mathcal{R}$ is isomorphic to the set of real numbers $\mathbf{R}$ introduced in Definition 1.1.

Exercise 1.71. *Prove that every nonempty set $\mathcal{X} \subset \mathcal{R}$, bounded from above, has a supremum in $\mathcal{R}$, using Dedekind's theorem, Example 1.70 (compare to Example 1.50).*

Exercise 1.72. *Starting from the set of rational numbers $\mathbf{Q}$ given by axioms (R1) - (R14), the construction of cuts in $\mathbf{Q}$ gave the set of cuts. The last turned out to be isomorphic to the set $\mathbf{R}$ given by the axioms (R1) - (R15).*

Now, if one starts from the set of real numbers $\mathbf{R}$ given by axioms (R1) - (R15), and defines the cuts in $\mathbf{R}$ in the manner of Definition 1.58, does one get a set nonisomorphic to $\mathcal{R}$?

Answer. No. In fact, Dedekind's theorem shows that the cuts in $\mathbf{R}$ give again real numbers, hence no new elements arise from the above construction in $\mathbf{R}$.

1.3 The set R as a topological space

1.3.1 Basic notions

Definition 1.73. *A* **neighborhood of a point** $x_0 \in \mathbf{R}$ *is any set* $U(x_0) \subset \mathbf{R}$ *which contains an open interval* $(x_0 - \varepsilon, x_0 + \varepsilon)$ *for some* $\varepsilon > 0$.

Clearly, every neighborhood of $x_0 \in \mathbf{R}$ contains x_0, while every interval (a, b) that contains x_0 is also its neighborhood.

Definition 1.74. *A nonempty set* $A \subset \mathbf{R}$ *is an* **open set**, *if it is the neighborhood of each of its points.*

By definition, the empty set is also open.

Definition 1.75. *A set $A \subset \mathbf{R}$ is a* **closed set**, *if its complement in* $\mathbf{R}$, *the set* $\mathbf{R} \setminus A$, *is open.*

Any "open interval" (a, b) is an open set, while any "closed interval " $[a, b]$ is a closed set in the sense of the last two definitions (prove that!)

Definition 1.76. *Let A be a subset of the set of real numbers* $\mathbf{R}$.

a) *A point $x_0 \in \mathbf{R}$ is an* **interior point** *of the set A if A is a neighborhood of x_0.*

   The **interior** *of the set A, denoted by A°, is the set of interior points of A.*

b) *A point $x_0 \in \mathbf{R}$ is a* **point of closure** *of the set A if in every neighborhood of x_0 there exists at least one point from A.*

   The **closure** *of a set A, denoted by $\overline{A}$, is the set of points of closure of A.*

c) *The point $x_0 \in \mathbf{R}$ is an* **accumulation point** *of the set A if in every neighborhood of x_0 there exists at least one point from A different from x_0. The set of accumulation points of A will be denoted by A'.*

d) *The point $x_0 \in A$ is an* **isolated point** *of the set A if there exists a neighborhood of x_0 that contains no other points from A.*

e) *The point $x_0 \in \mathbf{R}$ is a* **boundary point** *of the set A if in every neighborhood of x_0 there exists at least one point from A and at least one point from its complement $\mathbf{R} \setminus A$. The set of boundary points of a set A is denoted by ∂A.*

A set is **bounded** if it is both bounded from below and from above (compare to Definition 1.3).

Definition 1.77. *A set $K \subset \mathbf{R}$ is* **compact** *if it is both bounded and closed.*

A set is **infinite** if there exists a bijection[4] that maps it onto one of its proper subsets. The number sets $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$ and $\mathbf{R}$ are all infinite sets. The sets that are not infinite are **finite**. An infinite set X is **countable** if there exists a bijection between X and the set of natural numbers $\mathbf{N}$. The sets $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{Q}$ are countable, however the set $\mathbf{R}$ is not.

Theorem 1.78. The Bolzano–Weierstrass theorem.
Every infinite and bounded set $A \subset \mathbf{R}$ has at least one accumulation point in $\mathbf{R}$ (which is not necessarily in A).

Definition 1.79. *A collection of sets $\{B_i | \ i \in I\}$ is a* **covering of a set** $A \subset \mathbf{R}$ *if for every $x \in A$ there exists an index i from the index set I such that $x \in B_i$. If, additionally, all the sets B_i are open, then the collection $\{B_i | \ i \in I\}$ is called* **open covering**.

[4]See Subsection 2.1.1.

Definition 1.80. *A set $A \subset \mathbf{R}$ has the* **Heine–Borel property** *if every open covering of A has a finite subcovering.*

In other words, if $\{O_i \mid i \in I\}$ is a collection of open sets whose union covers A, then the Heine–Borel property of A implies the existence of a finite set $I_1 \subset I$ such that

$$A \subset \bigcup_{i \in I_1} O_i.$$

Remark. In the Hausdorff topological spaces, (see Example 1.89) the Heine–Borel property is often used for the definition of compactness. In Example 1.96 we shall prove that a set in $\mathbf{R}$ is compact (= bounded and closed) iff it has the Heine–Borel property.[5]

1.3.2 Examples and exercises

Example 1.81. *Find the interior, the closure, all accumulation points, isolated points and boundary points for the following sets.*

a) $A = [0,1]$; **b)** $B = \left\{0, 1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots\right\}$;

c) $C = \left\{\frac{1}{2^n} \,\middle|\, n \in \mathbf{N}\right\}$; **d)** $D = \mathbf{N} = \{1, 2, \ldots\}$.

Answers.

a) $A^\circ = (0,1)$, $\overline{A} = [0,1]$, $A' = [0,1]$, no isolated points of A, $\partial A = \{0,1\}$.

b) $B^\circ = \emptyset$, $\overline{B} = B$, $B' = \{0\}$, all points from $B \setminus \{0\}$ are isolated, $\partial B = B$.

c) $C^\circ = \emptyset$, $\overline{C} = C \bigcup \{0\}$, $C' = \{0\}$, all points from C are isolated, $\partial C = C$.

d) $D^\circ = \emptyset$, $\overline{D} = D$, $D' = \emptyset$, all points from D are isolated, $\partial D = D$.

Example 1.82. *Show that if $\alpha \in \mathbf{R}$ is an accumulation point of a set $A \subset \mathbf{R}$, then in every neighborhood of α there exist infinitely many points from A.*

Solution. Let $U(\alpha)$ be a neighborhood of α, then by Definition 1.73 there exists an interval $(\alpha - \varepsilon_1, \alpha + \varepsilon_1) \subset U(\alpha)$. This interval is also a neighborhood of α. Now since α is an accumulation point of A, by Definition 1.76 c), there exists an element $\alpha_1 \neq \alpha$ such that

$$\alpha_1 \in (\alpha - \varepsilon_1, \alpha + \varepsilon_1) \bigcap A.$$

Put $\varepsilon_2 := (|\alpha - \alpha_1|)/2$. Then in the interval $(\alpha - \varepsilon_2, \alpha + \varepsilon_2)$ there exists an element $\alpha_2 \in A$ which is different both from α and α_1. Continuing this procedure ad infinitum, we get infinitely many points from A in the interval $(\alpha - \varepsilon_1, \alpha + \varepsilon_1)$, hence also in the given neighborhood $U(\alpha)$ of the point α.

[5] However, there exist topological spaces in which there exist sets that are bounded and closed, but still do not satisfy the Heine–Borel property!

Example 1.83. *Prove that every accumulation point of a set $A \subset \mathbf{R}$ is also in the closure of A. Is the opposite statement true?*

Solution. Let α be an accumulation point of the set A. Then in every neighborhood $U(\alpha)$ of α there exists a point $\beta \in A$, $\beta \neq \alpha$. By Definition 1.76 b), α is then in the closure of A.

The opposite statement is not true, as will be seen from the following example.

Let $A := (0,1) \bigcup \{2\}$. Then the closure of A is $\overline{A} = [0,1] \bigcup \{2\}$, while the set of accumulation points of A is $A' = [0,1]$. Namely, the point 2 is not an accumulation point of A, but rather its isolated point, since in the interval $(2 - \frac{1}{2}, 2 + \frac{1}{2})$ there are no points from A different from 2 itself.

Example 1.84. *Prove that for every set $A \subset \mathbf{R}$ it holds $\overline{A} = A \bigcup A'$, i.e., the closure of a set is equal to the union of that set and its accumulation points.*

Solution. By Definition 1.76 b) it holds $A \subset \overline{A}$, while by Example 1.83 $A' \subset \overline{A}$, hence $A \bigcup A' \subset \overline{A}$. For the opposite inclusion, assume $x \in \overline{A}$. Then either $x \in A$ (in which case there is nothing left to prove) or $x \notin A$. In the last case, we claim that x is an accumulation point of A. Otherwise, there would exist a neighborhood of x disjoint from A; however, then x could not be in the closure of A.

Example 1.85. *Prove that the closure of a set is closed.*

Solution. Let $\overline{A}$ be the closure of the set $A \subset \mathbf{R}$. We shall prove that the set $B := \mathbf{R} \setminus \overline{A}$ is open. Let $\beta \in B$. By Example 1.84,

$$\beta \notin \overline{A} \iff (\beta \notin A \wedge \beta \notin A').$$

In view of Definition 1.76 c), it follows that there exists a neighborhood $U(\beta)$ of β disjoint from A. By Definition 1.73, $U(\beta)$ contains an open interval $(\beta - \varepsilon, \beta + \varepsilon)$, which thus contains no accumulation points of A. Hence, the last interval is contained in B, which means that B is the neighborhood of each of its points. By Definition 1.74, the set B is then open, hence $\overline{A} = \mathbf{R} \setminus B$ is closed.

Exercise 1.86. *A set is closed iff it contains all of its accumulation points.*

Example 1.87. *The closure $\overline{A}$ of a set $A \subset \mathbf{R}$ is the smallest closed set that contains A.*

Solution. In Example 1.85 we proved that $\overline{A}$ is closed. Assume $B \subset \mathbf{R}$ is a closed set that contains A. We have to prove that $\overline{A} \subset B$. In the contrary, there would exist an element $\alpha \in \overline{A} \setminus B$. From Example 1.84 it would then follow that $\alpha \in \overline{A} \setminus A$ (since B contains A). Hence α is an accumulation point of A, which contradicts to Exercise 1.86.

Remark. This example justifies the term "closure".

Exercise 1.88. *The open sets in $\mathbf{R}$ satisfy the following three properties.*

(ts1) *The sets $\mathbf{R}$ and $\emptyset$ are open.*

(ts2) *An arbitrary union of open sets is open.*

(ts3) *A finite intersection of open sets is open.*

Remark. A family τ of sets contained in a set X, $\tau := \{O_i \mid i \in I\}$, where I is a set of indices, is called **topology** on X, if it satisfies the properties (ts1) - (ts3) on X, and the set X is called **topological space**. (Of course, in (ts1) one has to replace "$\mathbf{R}$" with "X".) The elements of τ are then called *open sets.*

Thus from Exercise 1.88 it follows that if one defines open sets in $\mathbf{R}$ as in Definition 1.74, the set $\mathbf{R}$ becomes a topological space.

Example 1.89. *Prove that for every two real points x and y, $x \neq y$, there exist two disjoint neighborhoods of these points.*

Solution. Let $r := |x - y| > 0$, then the intervals $(x - \frac{r}{3}, x + \frac{r}{3})$ and $(y - \frac{r}{3}, y + \frac{r}{3})$ are disjoint neighborhoods of x and y respectively.

Remark. A topological space with this property is called **Hausdorff space**. Thus we proved that the set $\mathbf{R}$ with the topology given with the open sets from Definition 1.74, is a Hausdorff space.

Exercise 1.90. *Check whether the sets from Example 1.81 are compact.*

Answers.

a) The set A is compact, since it is bounded and closed.

b) Firstly, the set B is bounded, since $B \subset [0, 1]$. The only accumulation point of B is 0, which belongs to B, hence it is also closed. Thus B is compact.

c) The set C is bounded, but does not contain its accumulation point 0. Hence C is not closed, which implies that it is not compact.

d) The set of natural numbers $\mathbf{N}$ has no accumulation points, hence $\overline{\mathbf{N}} = \mathbf{N}$. However, it is not bounded, hence $\mathbf{N}$ is not compact.

Example 1.91. *Show that every nonempty subset S of a compact set $K \subset \mathbf{R}$ has an infimum and a supremum, which both belong to K.*

Solution. By Definition 1.77 the set K is bounded, hence so is its subset S. By Example 1.50, there exist real numbers α and β such that $\inf S = \alpha$ and $\sup S = \beta$. We have to prove that $\alpha, \beta \in K$.

We shall only prove that $\alpha \in K$. So assume $\alpha \notin K$. Since K is closed, the set $\mathbf{R} \setminus K$ is open. This means that there exists an $\varepsilon > 0$ such that the interval $(\alpha - \varepsilon, \alpha + \varepsilon)$ is disjoint with K. Since $S \subset K$, it follows $(\alpha - \varepsilon, \alpha + \varepsilon) \bigcap S = \emptyset$. But then there is no element s_1 in S such that $s_1 < \alpha + \varepsilon$, which contradicts the assumption $\alpha = \inf S$.

Exercise 1.92. *Prove that*

a) *a finite set has no accumulation points;*

b) *every finite set is compact.*

Exercise 1.93. *Prove that a closed subset of a compact set is itself compact.*

Example 1.94. *A set $K \subset \mathbf{R}$ is compact if and only if its every infinite subset has an accumulation point which belongs to K.*

Solution. In view of Exercise 1.92, it is enough to observe the case when K is infinite.

- **The condition is necessary.** Assume that K is compact, i.e., that (by Definition 1.77) it is bounded and closed. Let S be an infinite subset of K. The set S is bounded, being a subset of a bounded set. By the Bolzano–Weierstrass theorem (Theorem 1.78), the set S has at least one accumulation point $\alpha \in \mathbf{R}$. The assumption $S \subset K$, implies that α is also an accumulation point for K. Now K is closed, hence by Example 1.86 it contains all of its accumulation points. This means that $\alpha \in K$.

- **The condition is sufficient.** Assume that every infinite subset of K has an accumulation point which belongs to K. We have to prove that K is both closed and bounded.

  Let us prove first that K contains all its accumulation points, which is, by Example 1.86, equivalent with the statement that K is closed. Let β be an accumulation point of K; we have to prove that β is in K. By Definition 1.76 c), for every $\varepsilon > 0$ there exists at least one point $a_1 \neq \beta$ in the set $(\beta-\varepsilon, \beta+\varepsilon) \bigcap K$. Put $d_1 := |\beta - a_1|$. Next, in the set $(\beta - \frac{d_1}{2}, \beta + \frac{d_1}{2}) \bigcap K$ there exists a point $a_2 \neq \beta$. Continuing this procedure ad infinitum, we construct an infinite set $S = \{a_1, a_2, \ldots\}$ contained in K. By construction, β is an accumulation point of S, hence $\beta \in K$.

  Finally, we have to prove that K is bounded. Assume that K is unbounded. We shall construct an infinite subset S of K which has no accumulation points in K. Let a_1 be an arbitrary element from K. By the Archimedes theorem, there exists a natural number n_1 such that $|a_1| < n_1$ (see Theorem 1.12). Since K is unbounded, there exists an element $a_2 \in K$ such that $n_1 < |a_2|$. Choose $n_2 \in \mathbf{N}$ such that $n_2 > |a_2|$; next, choose $a_3 > n_2$, etc. In this way we constructed an infinite set $S = \{a_1, a_2, \ldots\}$ contained in K, which has no accumulation points in K. This contradicts the assumption that K contains all the accumulation points of its infinite subsets.

Exercise 1.95. *Prove that every infinite sequence of elements from a compact set has a convergent subsequence (see Definition 3.44).*

Example 1.96. *Prove that a necessary and sufficient condition for the compactness of a set $K \subset \mathbf{R}$ is that it has the Heine–Borel property.*

Solution.

- **The condition is necessary.** Assume that K is a compact set in $\mathbf{R}$, i.e., that K is bounded and closed, but it does not satisfy the Heine–Borel property. This means that there exists an open covering $\{O_i \mid i \in I\}$ of K which has no finite subcovering. A compact set is bounded, thus there exists an interval $[a_1, b_1]$ containing K. Let us divide this interval on two equal parts and denote by $[a_2, b_2]$ that half of $[a_1, b_1]$ which has the property that there is no finite subcovering of the set $[a_2, b_2] \bigcap K$. Continuing this procedure, we come to an infinite sequence of closed intervals $[a_n, b_n]$, $n \in \mathbf{N}$, contained always in the previous one and having the property that their lengths tend to zero as $n \to \infty$. By Example 1.48, there exists a unique real number α contained in each interval $[a_n, b_n]$.

  We shall prove now that α is an accumulation point of K, i.e., that in every neighborhood of α there exist infinitely many elements from K. In fact, if $U(\alpha)$ is a neighborhood of α, then by Definition 1.73 there exists an $\varepsilon > 0$ such that $(\alpha - \varepsilon, \alpha + \varepsilon) \subset U(\alpha)$. Since the lengths of intervals $[a_n, b_n]$, $n \in \mathbf{N}$, tend to zero, there exists an integer n_0 such that for $n > n_0$ it holds

  $$[a_n, b_n] \subset (\alpha - \varepsilon, \alpha + \varepsilon) \subset U(\alpha).$$

  By construction, for $n > n_0$ every interval $[a_n, b_n]$ has infinitely many elements from K, thus it follows that α is an accumulation point of K. A closed set contains its accumulation points, hence $\alpha \in K$.

  The collection of sets $\{O_i \mid i \in I\}$ was assumed to be an open covering of K, hence there exists an index $i_1 \in I$ such that $\alpha \in O_{i_1}$. Moreover, there exists an integer n_1 with the property that for every $n > n_1$ it holds $[a_n, b_n] \subset O_{i_1}$. So we covered all these intervals with only one open set O_{i_1}, though we assumed that no finite subcovering could cover any of them.

- **The condition is sufficient.** Assume that K has the Heine–Borel property; we have to prove that K is compact. Let us assume the contrary, namely that K is not compact. By Example 1.94, there exists an infinite set $S \subset K$ with no accumulation point in K. Hence for every point $x \in (K \setminus S)$ there exists an open set O_x such that $S \bigcap O_x = \emptyset$. For every $y \in S$ there is an open set O_y from the open covering such that $O_y \bigcap S = \{y\}$; otherwise, y would be an accumulation point of S belonging to K.

  Clearly, it holds

  $$K \subset \Big(\bigcup_{x \in K \setminus S} O_x\Big) \bigcup \Big(\bigcup_{y \in S} O_y\Big),$$

  but since the set S is infinite, it is impossible to find a finite subcovering of K, which is a contradiction with the Heine–Borel property.

Example 1.97. *Does the set* $A = \left\{\frac{1}{n} \,\middle|\, n \in \mathbf{N}\right\}$ *have the Heine–Borel property?*

Solution. We shall show in two ways that A does not have the Heine–Borel property.

- **First method.** The number 0 is the only accumulation (cluster) point of the set A. Thus A is not a closed set implying that that it is neither compact. By Example 1.96 this is equivalent with the statement that A has no Heine–Borel property.

- **Second method.** Let us observe the open intervals

$$O_n = \left(\frac{1}{n} - \frac{1}{n^3}, \frac{1}{n} + \frac{1}{n^3}\right), \quad n \in \mathbf{N}.$$

Clearly $A \subset \bigcup_{n \in \mathbf{N}} O_n$, but the family $\{O_n \mid n \in \mathbf{N}\}$, has no finite subcovering of the set A. This follows from the following relations:

$$\frac{1}{n+1} = \frac{n^2-1}{(n+1)(n^2-1)} = \frac{n^2-1}{n^3+n^2-n-1} < \frac{n^2-1}{n^3} = \frac{1}{n} - \frac{1}{n^3},$$

for $n = 2, 3, \ldots$.

Exercise 1.98. *Let us define a function* $d : \mathbf{R} \times \mathbf{R} \to [0, +\infty)$ *called* **distance** *by the formula*

$$d(x,y) = |x-y|, \quad x, y \in \mathbf{R}.$$

Prove the following properties of the distance d.

(M1) $(\forall x, y \in \mathbf{R})$ $d(x,y) > 0$ *if* $x \neq y$ *and* $d(x,y) = 0$ *if* $x = y$;

(M2) $(\forall x, y \in \mathbf{R})$ $d(x,y) = d(y,x)$;

(M3) $(\forall x, y, z \in \mathbf{R})$ $d(x,z) \leq d(x,y) + d(y,z)$.

Remark. A set X with a function $d : X \times X \to [0, +\infty)$ satisfying the upper conditions (M1) - (M3) (with $\mathbf{R}$ replaced by X) is called a **metric space** .

Chapter 2

Functions

2.1 Real functions of one real variable

2.1.1 Basic notions

Let A and B be two nonempty sets. By definition, a **relation** f from A into B is a subset of the direct product $A \times B$.

Definition 2.1. *A relation f is a* **function** *which maps the set A into the set B if the following two conditions hold:*

(f1) *for every $x \in A$ there exists an element $y \in B$ such that the pair (x, y) is in f;*

(f2) *if the pairs (x, y_1) and (x, y_2) are in f, then necessarily $y_1 = y_2$.*

Then we write $f : A \to B$ and say that "f maps A into B". If the pair $(x, y) \in f$, we shall write $y = f(x)$. The set A is then called the **domain**, while the set B is called the **codomain** of the function f. The element x from the domain A is called the **independent variable**, while the element y from the codomain B is called the **dependent variable**.

The set

$$f(A) = \{y \in B | \ (\exists x \in A) \quad f(x) = y\}$$

is called the **range** of the function f. By Definition 2.1, $f(A) \subset B$ and, in general, the range is a proper subset of the codomain.

Clearly, a function f is determined with the triple (A, B, f). This also means that two functions $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$ are **equal** if and only if their domains are equal, i.e., $A_1 = A_2$, their codomains are equal, i.e., $B_1 = B_2$, and, of course, it holds $f_1(x) = f_2(x)$ for all $x \in A$.

A function $f : A \to B$ is a **one–to–one function** or **injective** if for every pair x_1 and x_2 from the domain A it holds

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2).$$

A function $f : A \to B$ is **surjective** if for every $y \in B$ there exists $x \in A$ such that $f(x) = y$. Clearly, for a surjective function its codomain and its range are the same, i.e., $f(A) = B$. It is usual to say for a surjective function $f : A \to B$ that it maps A *onto* B.

A function $f : A \to B$ is a **bijection** if it is both a one–to–one function and a surjection.

Suppose the function $f : A \to B$ is a bijection. Then, for every $y \in B$ there exists a unique element $x \in A$ such that $y = f(x)$. Now, the relation from B to A given by

$$f^{-1} := \{(y, x) \in B \times A |\, y = f(x)\}$$

is a function on B which will be called the **inverse function** for f. The inverse function is also a bijection and it holds

$$f^{-1}(y) = x \iff f(x) = y \quad \text{for every pair } (x, y) \in A \times B.$$

Let two functions $f : A \to B$ and $g : B \to C$ be given. Then the function $g \circ f : A \to C$ given by

$$g \circ f(x) := g(f(x)) \quad \text{for every } x \in A$$

is called the **composite function** of the functions g and f.

Let us remark that if a function $f : A \to B$ is a bijection and $f^{-1} : B \to A$ its inverse function, then

$$f^{-1} \circ f(x) = x \text{ for every } x \in A \quad \text{and} \quad f \circ f^{-1}(y) = y \text{ for every } y \in B.$$

In this book, we shall observe only those functions whose domains and codomains are some subsets of the set of real numbers $\mathbf{R}$. Such functions are called **real functions of the real variable**; shortly, we shall call them simply **functions**. Most often, a function is given with the analytical expression (formula) $y = f(x)$, where the sets A and B, i.e., the domain and the codomain of the function f, are not explicitly given. If not stated otherwise, in that case we shall always assume that for such a function the codomain is the whole set of real numbers $\mathbf{R}$, while the domain is the largest set for which the given formula has sense. This domain will be called **the natural domain** of the function f.

The **graph** G_f of a function $f : A \to B$ is the subset of the set $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ given by

$$G_f = \{(x, f(x))|\, x \in A\}.$$

Quite often, our main task will be to draw the graph of a function given with some formula. Some of the geometric properties of the graph are defined next.

A point x_0 from the domain of a function f is a **zero** (or: **root**) of a function f if it holds $f(x_0) = 0$. Geometrically, this means that the point $(x_0, 0)$ is the common point of the graph of f and the $x-$axis.

A function $f : A \to B$ is **monotonically increasing** (resp. **monotonically decreasing**) on the set $X \subset A$ if for every pair of elements x_1 and x_2 from the set X it holds

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \quad (\text{resp. } \; x_1 < x_2 \Rightarrow f(x_1) > f(x_2)).$$

A function $f : A \to B$ is **monotonically nondecreasing** (resp. **monotonically nonincreasing**) on the set $X \subset A$ if for every pair of elements x_1 and x_2 from the set X it holds

$$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2) \quad (\text{resp.} \quad x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)).$$

The importance of the monotonicity of a function is demonstrated in the following theorem.

Theorem 2.2. *If the function $f : A \to B$ is a surjection and is either monotonically increasing or monotonically decreasing on the whole domain A, then f is a bijection.*

A function $f : A \to B$ has a **local maximum** (resp. **local minimum**) in the point $x_0 \in A$ if there exists a number $\varepsilon > 0$ such that

$$(\forall x \in (x_0 - \varepsilon, x_0 + \varepsilon) \cap A) \quad f(x) \leq f(x_0) \quad (\text{resp.} \quad f(x) \geq f(x_0)\,).$$

A function $f : A \to B$ has a **global maximum** (resp. **global minimum**) in the point $x_0 \in A$ if

$$(\forall x \in A) \quad f(x) \leq f(x_0) \quad (\text{resp.} \quad f(x) \geq f(x_0)\,).$$

In the following, we usually omit the terms local and global and speak just about minimum or maximum.

A function $f : A \to B$ is **bounded** on the set $X \subset A$ if there exists a number $C > 0$ such that

$$(\forall x \in X) \quad |f(x)| \leq C.$$

Geometrically, this means that the graph of the function f over the set X is settled between two horizontal lines, namely $y = C$ and $y = -C$.

The set $X \subset \mathbf{R}$ is **symmetric** (to the origin) if for every point $x \in X$ it holds that the point $-x$ is also in X. (Notice that for $x \neq 0$ the points x and $-x$ are symmetric to the origin.)

Suppose that the domain A of a function $f : A \to B$ is *symmetric.* Then f is an

even function if for every $x \in A$ it holds $f(-x) = f(x)$;

odd function if for every $x \in A$ it holds $f(-x) = -f(x)$.

Geometrically, the graph of an even function is symmetric to the $y-$axis, while the graph of an odd function is symmetric to the origin.

A number $\tau \neq 0$ is called the **period** of the function $f : A \to B$ if for all $x \in A$ the points $x + \tau$ and $x - \tau$ are also in A and it holds

$$(\forall x \in A) \quad f(x + \tau) = f(x).$$

The smallest positive period, if it exists, is called the **basic period** of the function f. Clearly, if we know the basic period T of a function, then it is enough to draw its graph on any set $X \subset A$ of the length T.

A function $f : A \to B$ is called **concave upward** on the interval $(a,b) \subset A$ if for every pair $x_1, x_2 \in (a,b)$ and for every $\alpha \in (0,1)$ it holds

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2).$$

Geometrically, if a function f is *concave upward* on the interval (a,b), then the segment connecting any two points on its graph is *above* the graph.

A function $f : A \to B$ is called **concave downward** on the interval $(a,b) \subset A$ if for every pair $x_1, x_2 \in (a,b)$ and for every $\alpha \in (0,1)$ it holds

$$f(\alpha x_1 + (1-\alpha)x_2) \geq \alpha f(x_1) + (1-\alpha) f(x_2).$$

Geometrically, if a function f is *concave downward* on the interval (a,b), then the segment connecting any two points on its graph is *under* the graph. [1]

The **basic elementary functions** are the following ones:

- **power function**: $y = x^s$, $x \in \mathbf{R}$, for a fixed $s \in \mathbf{N}$ (in particular, for $s = 0$ this function is a constant one);
- **exponential function**: $y = a^x$, $x \in \mathbf{R}$, for $a > 0$ and $a \neq 1$;
- **logarithmic function**: $y = \log_a x$, $x \in (0, +\infty)$, for $a > 0$ and $a \neq 1$;
- **trigonometric functions**:

$$y = \sin x, \quad x \in \mathbf{R}; \qquad y = \cos x, \quad x \in \mathbf{R};$$

$$y = \tan x, \quad x \in \mathbf{R} \setminus \left\{ \frac{(2k+1)\pi}{2} \middle| \, k \in \mathbf{Z} \right\}; \qquad y = \cot x, \quad x \in \mathbf{R} \setminus \{ k\pi | \, k \in \mathbf{Z} \}.$$

- the **inverse functions** (with the possible restriction of the domains) of the up to now mentioned basic elementary functions.
  In particular, such are the so-called **inverse trigonometric functions**:

$$f(x) = \arcsin x, \quad x \in [-\pi/2, \pi/2]; \qquad f(x) = \arccos x, \quad x \in [0, \pi];$$

$$f(x) = \arctan x, \quad x \in \mathbf{R}; \qquad f(x) = \operatorname{arccot} x, \quad x \in \mathbf{R}.$$

In particular, if $a = \mathrm{e}$ (see Example 3.33), for the exponential and logarithmic function we shall write

$$y = \mathrm{e}^x = \exp(x), \quad x \in \mathbf{R}, \qquad \text{and} \qquad y = \ln x, \quad x > 0.$$

An **elementary function** is obtained by finitely many applications of the algebraic operations: addition, subtraction, multiplication and division, as well as the operations of composition of the basic elementary functions.

[1] Some authors use the words “convex” instead of “concave upward” and “concave ” instead of “concave downward”.

2.1.2 Examples and exercises

Example 2.3. *Let the sets A and B be given by $A = \{1, 2, ..., n\}$ and $B = \{1, 2, ..., m\}$ for some natural numbers n and m.*

a) *Determine the number of all functions $f : A \to B$, i.e., that map the set A into the set B.*

b) *What is the relation between n and m if the function $f : A \to B$ is a bijection? Determine then the number of* all *bijections from A onto B.*

Solutions.

a) The number of all functions $f : A \to B$ is equal to the number of all variations with repetition of m elements of the class n, i.e., it is m^n.

b) Clearly, the function $f : A \to B$ can be a bijection only if $n = m$. The number of all such bijections is equal to the number of all permutations without repetition of n elements, i.e., it is $n! := n \cdot (n-1) \cdots 2 \cdot 1$.

Example 2.4. *Assume $f : A \to B$ is a function. Prove that for arbitrary sets $X, Y \subset A$ the following holds.*

a) $X \subset Y \;\Rightarrow\; f(X) \subset f(Y)$ *(is the opposite implication always true?);*

b) $f(X \cup Y) = f(X) \cup f(Y)$;

c) $f(X \cap Y) \subset f(X) \cap f(Y)$ *(compare with b)).*

Solutions.

a) If the element y is in $f(X)$, then there exists an element $x \in X$ such that $f(x) = y$. Since $X \subset Y$, it follows that $x \in Y$ which implies $f(x) \in f(Y)$, i.e., $y \in f(Y)$.

That the opposite implication is in general false, shows the following counterexample.
Let the function $f : \mathbf{Z} \to \mathbf{Z}$ be given by $f(x) = x^4$, and let $X = \{1, -1\}$ and $Y = \{1, 2\}$. Then $f(X) = \{1\} \subset \{1, 16\} = f(Y)$, but it is not true that $X \subset Y$.

b) Let $y \in f(X \cup Y)$. Then there exists an element $x \in X \cup Y$, such that $y = f(x)$. This implies $x \in X$ or $x \in Y$, hence $y \in f(X)$ or $y \in f(Y)$. So we obtain that $y \in f(X) \cup f(Y)$. Thus we proved the inclusion $f(X \cup Y) \subset f(X) \cup f(Y)$.
The opposite inclusion $f(X) \cup f(Y) \subset f(X \cup Y)$ is proved in an analogous way (do it!).

c) If $y \in f(X \bigcap Y)$, then there exists an element $x \in X \bigcap Y$ such that $f(x) = y$. From $x \in X \cap Y$ it follows that both $x \in X$ and $x \in Y$, hence $y \in f(X)$ and $y \in f(Y)$. This means that $y \in f(X) \cap f(Y)$.
The following counterexample shows that in c) the equality does not hold always.

Let $A = B = \mathbf{R}$, and let $f(x) = x^2$, $x \in \mathbf{R}$, and take $X = \{-1, 2\}$, $Y = \{1, 2\}$. Then $f(X) = f(Y) = \{1, 4\}$. But then $f(X \cap Y) = f(\{2\}) = \{4\}$, while $f(X) \cap f(Y) = \{1, 4\}$, which shows that the sets $f(X \cap Y)$ and $f(X) \cap f(Y)$ are different.

Example 2.5. *Let $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$ be two functions. Determine whether the functions f and g given below are equal.*

a) $f(x) = \sqrt{x^6},\ A = \mathbf{R}$, $\qquad g(x) = x^3,\ B = \mathbf{R}$;

b) $f(x) = \sqrt{x^6},\ A = [0, +\infty)$, $\qquad g(x) = x^3,\ B = [0, +\infty)$;

c) $f(x) = \dfrac{2x}{x},\ A = \mathbf{R} \setminus \{0\}$, $\qquad g(x) = 2,\ B = \mathbf{R} \setminus \{0\}$;

d) $f(x) = \dfrac{2x}{x},\ A = \mathbf{R} \setminus \{0\}$, $\qquad g(x) = 2,\ B = \mathbf{R}$;

e) $f(x) = \sin^2 x + \cos^2 x,\ A = \mathbf{R}$, $\qquad g(x) = 1,\ B = \mathbf{R}$;

f) $f(x) = \ln x^2,\ A = (0, +\infty)$, $\qquad g(x) = 2\ln(x),\ B = (0, +\infty)$;

g) $f(x) = \ln x^2,\ A = \mathbf{R} \setminus \{0\}$, $\qquad g(x) = 2\ln(x),\ B = (0, +\infty)$.

Solutions.

b), c), e) and **f)** The functions f and g are equal.

a) The functions f and g are not equal, since $f(-1) = 1$, while $g(-1) = -1$. (However, the restriction of their domains from the whole $\mathbf{R}$ to the interval $[0, +\infty)$ makes them equal (see b).)

d) and **g)** The functions f and g are not equal, since their domains are different.

Example 2.6. *Determine the largest set $A \subset \mathbf{R}$ such that the following analytical expressions have sense. (We call then A the natural domain of the function given with that formula.)*

a) $f(x) = (x-3)\sqrt{\dfrac{x+2}{2-x}}$; b) $f(x) = x \cdot \sqrt{\cos\sqrt{x}}$;

c) $f(x) = \sqrt{\sin(x^2)}$; d) $f(x) = \ln^3\left(\sin(\frac{\pi}{x})\right)$;

e) $f(x) = \arccos\left(\dfrac{2x}{1+x^2}\right)$; f) $f(x) = \ln\left(\arcsin\left(\dfrac{x+2}{5-x}\right)\right)$;

g) $f(x) = \dfrac{\ln(x+4)}{\sqrt{|x|-x}}$; h) $f(x) = \ln\left(\sin\sqrt{-x^2}\right)$.

Solutions.

a) First, the given formula has no sense for $x = 2$, since for that value of x the denominator becomes zero. Further on, since the natural domain of the square root function $g(t) = \sqrt{t}$ is the interval $[0, +\infty)$, (or, equivalently, if $t \geq 0$), the given formula has sense if and only if

$$\frac{x+2}{2-x} \geq 0 \quad \Longleftrightarrow \quad \frac{(x+2)(2-x)}{(2-x)^2} \geq 0, \tag{2.1}$$

provided that $x \neq 2$.

The quadratic function $h(s) = (s+2)(2-s)$, $s \in \mathbf{R}$, has nonnegative values for $s \in [-2, 2]$ (draw the graph of the function h - it is a parabola!). Applying this to equation (2.1) gives that $A \subset [-2, 2]$; since we have to exclude the point $x = 2$, we finally have $A = [-2, 2)$.

The other method of finding the natural domain A of the function f is to use the equivalence

$$\frac{x+2}{2-x} \geq 0 \Longleftrightarrow \Big((x+2 \geq 0 \wedge 2-x > 0) \vee (x+2 \leq 0 \wedge 2-x < 0)\Big),$$

which, of course, gives the same result (check that!).

b) The natural domain of the function $g(t) = \sqrt{t}$ is the set $\{t \in \mathbf{R} |\ t \geq 0\}$, which means that $A \subset [0, +\infty)$. Further on,

$$\cos s \geq 0 \Longleftrightarrow s \in \bigcup_{k \in \mathbf{Z}} \left[(4k-1)\frac{\pi}{2}, (4k+1)\frac{\pi}{2}\right],$$

or, equivalently,

$$(4k-1)\frac{\pi}{2} \leq s \leq (4k+1)\frac{\pi}{2}, \ k \in \mathbf{Z}.$$

Using the condition $A \subset [0, +\infty)$ and putting in the last relation $s = \sqrt{x}$, $x \geq 0$, we obtain that if $x \geq 0$, then $\cos(\sqrt{x}) \geq 0$ holds if and only if

$$0 \leq x \leq \frac{\pi^2}{4}, \quad \text{or} \quad (4k-1)^2\frac{\pi^2}{4} \leq x \leq (4k+1)^2\frac{\pi^2}{4}$$

for some natural number k. This means that the natural domain of f is the set

$$A=\left[0,\frac{\pi^2}{4}\right]\cup\left(\bigcup_{k\in\mathbf{N}}\left[(4k-1)^2\frac{\pi^2}{4},(4k+1)^2\frac{\pi^2}{4}\right]\right).$$

c) The function $g(t)=\sin t$ is nonnegative iff $2k\pi\le t\le(2k+1)\pi,\ k\in\mathbf{Z}$, hence

$$\sin(x^2)\ge 0 \iff 2k\pi\le x^2\le(2k+1)\pi,\quad \text{for } k\in\mathbf{N_0}.$$

Since for every $x\in\mathbf{R}$ it holds $\sqrt{x^2}=|x|$, we obtain finally

$$\begin{aligned} A &= \{x\in\mathbf{R}|\ (\exists k\in\mathbf{N_0})\ \sqrt{2k\pi}\le|x|\le\sqrt{(2k+1)\pi}\ \} \\ &= \left(\bigcup_{k\in\mathbf{N_0}}\left[(\sqrt{2k\pi},\sqrt{(2k+1)\pi}\right]\right)\bigcup\left(\bigcup_{k\in\mathbf{N_0}}\left[-\sqrt{(2k+1)\pi},-\sqrt{2k\pi}\right]\right).\end{aligned}$$

d) The natural domain of the logarithmic function $g(t)=\ln t$ is the open interval $(0,+\infty)$, hence the natural domain A of the function f will be the set of all $x\in\mathbf{R}$ such that $\sin(\frac{\pi}{x})>0$. The last inequality is true iff there exists an integer k such that

$$2k\pi<\frac{\pi}{x}<(2k+1)\pi.$$

Solving these inequalities by x gives three cases.

- If $k=0$, then $0<\frac{1}{x}<1$, hence $x\in(1,+\infty)$;
- if $k\in\mathbf{Z}$ and $k>0$ (i.e., $k\in\mathbf{N}$), then $x\in\left(\frac{1}{2k+1},\frac{1}{2k}\right)$;
- if $k\in\mathbf{Z}$ and $k<0$, then $x\in\left(\frac{1}{2k+1},\frac{1}{2k}\right)$.

So we obtained that

$$A=\left(\bigcup_{k\in\mathbf{Z},k<0}\left(\frac{1}{2k+1},\frac{1}{2k}\right)\right)\cup\left(\bigcup_{k\in\mathbf{N}}\left(\frac{1}{2k+1},\frac{1}{2k}\right)\right)\bigcup(1,+\infty).$$

e) The formula $g(t)=\arccos t$ has sense iff $t\in[-1,1]$, hence the natural domain A of the given function f is determined by the inequality

$$-1\le\frac{2x}{1+x^2}\le 1 \iff -1-x^2\le 2x\le 1+x^2,$$

which is true for every $x\in\mathbf{R}$. So we obtained that $A=\mathbf{R}$.

f) The formula $g(u) = \ln u$ has sense for $u > 0$, which means that we must have

$$\arcsin\frac{x+2}{5-x} > 0 \iff \frac{x+2}{5-x} > 0 \iff x \in (-2,5).$$

As in e), we must also have

$$\left|\frac{x+2}{5-x}\right| \leq 1 \quad \text{i.e.,} \quad -1 \leq \frac{x+2}{5-x} \leq 1.$$

Thus we have

$$\left|\frac{x+2}{5-x}\right| \leq 1 \iff x \in (-\infty, 3/2]\,.$$

Hence, the natural domain of f is the interval $A = \left(-2, \frac{3}{2}\right]$.

g) The formula $g(u) = \ln(u+4)$ has sense for $u \in (-4,+\infty)$, while the inequality $|x| - x > 0$ is possible if and only if $x < 0$. Hence $A = (-4,0)$.

h) The natural domain of this function is the empty set.

Example 2.7. *The function f is defined on the closed interval $[0,1]$. Determine the set A on which the following composite functions can certainly be defined.*

a) $h(x) = f(x^4)$; **b)** $h(x) = f(\sin x)$;

c) $h(x) = f(x+3)$; **d)** $h(x) = f(\ln x)$.

Solutions.

a) If the range of the function $g(x) = x^4$ is the interval $[0,1]$, then the interval $[-1,1]$ is the (largest possible) domain of g. Hence, the domain A of the composite function $h = f \circ g$ is the interval $[-1,1]$.

b) The largest set for which $\sin(x) \in [0,1]$ is the union $\bigcup_{k\in\mathbf{Z}} [2k\pi, (2k+1)\pi]$.

c) The range of the function $g(x) = 3 + x$ is $[0,1]$. Hence

$$x \in [0,1] \iff x \in [-3,-2] =: A.$$

d) $\ln x \in [0,1]$ iff $x \in [1, \mathrm{e}]$.

Example 2.8. *Let the function $f : (0,+\infty) \to \mathbf{R}$ be given with the formula*

$$f\left(\frac{1}{x}\right) = x + \sqrt{1+x^2}.$$

Find the formula for f.

Solution. Since we have

$$f\left(\frac{1}{x}\right) = x + \sqrt{1+x^2} = x\left(1+\sqrt{\frac{1}{x^2}+1}\right) = \frac{1}{\frac{1}{x}}\left(1+\sqrt{\left(\frac{1}{x}\right)^2+1}\right),$$

it follows that

$$f(x) = \frac{1}{x}\left(1+\sqrt{1+x^2}\right), \quad x > 0.$$

Example 2.9. *Let the function* $f : \mathbf{R}\setminus\{-1\} \to \mathbf{R}$ *be given with the formula* $f(x) = \dfrac{x}{1+x}$. *Find the function* f_n, $n \in \mathbf{N}$, *where*

$$f_1 = f,\ f_2 = f\circ f_1 \ \textit{ and } \ f_n = f\circ f_{n-1} \ \textit{ for } \ n = 2,3,\ldots.$$

Determine also the natural domains of these composite functions.

Solution. The range of the function f is the set $\mathbf{R}\setminus\{-1\}$. Then it holds

$$f_2(x) = f\circ f_1(x) = f(f(x)) = f\left(\frac{x}{1+x}\right) = \frac{\dfrac{x}{1+x}}{1+\dfrac{x}{1+x}} = \frac{x}{1+2x}.$$

Clearly, the natural domain of the last formula is the set $\mathbf{R}\setminus\{-1/2\}$, though the definition of f_2 reduces its domain to the set $\mathbf{R}\setminus\{-1,-1/2\}$.

Let us prove by mathematical induction that for $n = 2,3,..,$ it holds

$$f_n(x) = \frac{x}{1+nx}, \quad x \in \mathbf{R}\setminus\left\{-1,-\frac{1}{2},\ldots,-\frac{1}{n}\right\}.$$

We proved already this formula for $n = 2$. Suppose it holds for $n = k$, for some natural number $k > 1$. Then

$$\begin{aligned} f_{k+1}(x) &= (f\circ f_k)(x) = f(f_k(x)) = f\left(\frac{x}{1+kx}\right) \\ &= \frac{\dfrac{x}{1+kx}}{1+\dfrac{x}{1+kx}} = \frac{x}{1+(k+1)x} \end{aligned}$$

for $x \in \mathbf{R}\setminus\left\{-1,-\frac{1}{2},\ldots,-\frac{1}{k},-\frac{1}{k+1}\right\}$.

Example 2.10. *The function* $f : \mathbf{R} \to \mathbf{R}$ *is given with the formula*

$$f(x) = \frac{x}{\sqrt{1+x^2}}.$$

Find the formula that gives the functions f_n *defined on* $\mathbf{R}$ *by*

$$f_n = \underbrace{f\circ f\circ\cdots\circ f}_{n}, \quad n = 2,3,\ldots.$$

Solution. Let us calculate f_2 first:

$$f_2(x) = f(f(x)) = f\left(\frac{x}{\sqrt{1+x^2}}\right) = \frac{\dfrac{x}{\sqrt{1+x^2}}}{\sqrt{1+\left(\dfrac{x}{\sqrt{1+x^2}}\right)^2}} = \frac{x}{\sqrt{1+2x^2}}.$$

Using the mathematical induction, the reader should check that for every $n = 2, 3, ..,$ it holds

$$f_n(x) = f(f_{n-1}(x)) = \frac{x}{\sqrt{1+nx^2}}, \quad x \in \mathbf{R}.$$

Example 2.11. *Find the function f given by the following formula.*

a) $f(x-2) = \dfrac{1}{x+3},\ x \neq -3;$ b) $f\left(\dfrac{1}{x}\right) = x^4 + 1;$

c) $f\left(x + \dfrac{1}{x}\right) = x^2 + \dfrac{1}{x^2},\ x \neq 0;$ d) $f(x^2) = 1 - x^3, x \geq 0.$

Solutions.

a) Putting $t := x - 2$ we get $x = t + 2$. Clearly if $x \in \mathbf{R} \setminus \{-3\}$, then $t \in \mathbf{R} \setminus \{-5\}$. Hence $f(t) = \dfrac{1}{t+5}$ for $t \neq -5$.

b) $f(x) = \dfrac{1+x^4}{x^4},\ x \neq 0.$

c) Since $x^2 + \dfrac{1}{x^2} = (x + \dfrac{1}{x})^2 - 2$, we shall use the change of variable $t := x + \dfrac{1}{x},\ x \neq 0$. So we obtain

$$f(t) = t^2 - 2,\ t \neq 0.$$

(Of course, the last formula is defined for $t = 0$; however, we have to omit this value because of the definition of the function f.)

d) $f(x) = 1 - x^{3/2},\ x \geq 0.$

Example 2.12. *Show that the function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = g(|x|),\ x \in \mathbf{R}$, is an even function for any function $g : \mathbf{R} \to \mathbf{R}$.*

Solution. For every $x \in \mathbf{R}$ it holds

$$f(-x) = g(|-x|) = g(|x|) = f(x),$$

which proves that f is an even function.

Example 2.13. *Prove that every function f whose domain A is a symmetric set can be written as a sum of an even and an odd function.*

Solution. The function f can be written as the sum

$$f(x) = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)),\ x \in A.$$

If we put $f_1(x) := \frac{1}{2}(f(x) + f(-x)),\ x \in A$, then

$$f_1(-x) = \frac{1}{2}(f(-x) + f(-(-x))) = f_1(x) \text{ for every } x \in A,$$

i.e., f_1 is an even function.

Analogously we show that the function $f_2(x) := \frac{1}{2}(f(x) - f(-x)),\ x \in A$, is odd.

Example 2.14. *Prove that*

a) *the sum of two even (resp. of two odd) functions is an even (resp. an odd) function;*

b) *both the product of two even or of two odd functions is an even function;*

c) *the product of an even and an odd function is an odd function.*

Solutions.

a) Suppose f and g are odd functions, both defined on a symmetric set $A \subset \mathbf{R}$. Then for their sum $f + g$ it holds

$$\begin{aligned}(f+g)(-x) &= f(-x) + g(-x) = -f(x) + (-g(x)) \\ &= -(f(x) + g(x)) = -((f+g)(x)),\end{aligned}$$

for every $x \in A$.

c) Suppose f is an even and g an odd function, both defined on a symmetric set $A \subset \mathbf{R}$. Then for their product fg it holds

$$(fg)(-x) = f(-x) \cdot g(-x) = f(x) \cdot (-g(x)) = -f(x)g(x) = -(fg)(x),$$

for every $x \in A$.

Remark. One must distinguish the product fg from the composition $f \circ g$ of two functions f and g.

Example 2.15. *If the number $T > 0$ is a period of a periodic function $f : A \to \mathbf{R}$, then the number kT, $k \in \mathbf{Z} \setminus \{0\}$, is also a period of f.*

Solution. We shall prove this statement first for positive integers k, using the principle of the mathematical induction. By assumption, the statement is true for $k = 1$. If it is true for $k \in \mathbf{N}$, then

$$f(x + (k+1)T) = f((x + kT) + T) = f(x + kT) = f(x),\ x \in A.$$

Hence by the principle of the mathematical induction, f is a periodic function with periods kT, $k \in \mathbf{N}$.

Using this, we show that $-kT$, $k \in \mathbf{N}$, is also a period of f :

$$f(x + (-kT)) = f(x - kT) = f((x - kT) + kT) = f(x), \ x \in A.$$

Thus every number of the form kT, $k \in \mathbf{Z} \setminus \{0\}$, is a period of f.

Example 2.16. *Show that if the function $f : \mathbf{R} \to \mathbf{R}$ is periodic with the period T, then the function $g : \mathbf{R} \to \mathbf{R}$ given by $g(x) = f(\alpha x + \beta)$, $x \in \mathbf{R}$, is periodic with the period T/α, provided that $\alpha \neq 0$.*

Solution. For every $x \in \mathbf{R}$ it holds

$$g\left(x + \frac{T}{\alpha}\right) = f\left(\alpha\left(x + \frac{T}{\alpha}\right) + \beta\right) = f\left((\alpha x + \beta) + T\right) = f(\alpha x + \beta) = g(x).$$

Example 2.17. *Check whether the following functions are periodic; if yes, find their basic periods T, if any.*

a) $f(x) = \sin^2(x)$, $x \in \mathbf{R}$;

b) $f(x) = \sin(x^2)$, $x \in \mathbf{R}$;

c) $f(x) = \sin|x|$, $x \in \mathbf{R}$;

d) $f(x) = \cos|x|$, $x \in \mathbf{R}$.

Solutions.

a) It holds that the $\sin^2 x = \dfrac{1}{2}(1 - \cos(2x))$, $x \in \mathbf{R}$. From Example 2.16 it follows that the function $g(x) = \cos(2x)$, $x \in \mathbf{R}$, is periodic with the basic period $2\pi/2 = \pi$. Hence, the function f is also periodic with the basic period π.

b) The zeros of the function f are of the form $\pm\sqrt{k\pi}$, $k \in \mathbf{N}$. Let us show that the distance between the zeros of f tends to zero as $k \to \infty$, see Subsection 4.1.1.

$$\lim_{k\to\infty}\left|\sqrt{(k+1)\pi} - \sqrt{k\pi}\right| = \lim_{k\to\infty} \frac{\pi}{\sqrt{(k+1)\pi} + \sqrt{k\pi}} = 0.$$

This implies that the function f is *not* periodic.

c) Only the numbers 2π and π are candidates for the basic period of f. However,

$$f\left(-\frac{\pi}{2}\right) = 1, \quad f\left(-\frac{\pi}{2} + 2\pi\right) = f\left(\frac{3\pi}{2}\right) = -1,$$

$$f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \quad f\left(\pi + \frac{\pi}{4}\right) = f\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}.$$

Hence, the function f is not periodic.

d) Periodic with basic period $T = 2\pi$ (check!).

Example 2.18. *Determine the basic periods of the following functions, if they exist.*

a) $f(x) = \cos\frac{x}{4} + \cos x + \frac{1}{2}\cos(3x) + \frac{1}{3}\cos(5x), \quad x \in \mathbf{R};$

b) $f(x) = \cos(x) + \cos(\sqrt{3}\,x), \quad x \in \mathbf{R};$

c) $f(x) = \sqrt{\tan x}, \quad x \in \bigcup_{k\in\mathbf{Z}} \left[k\pi, (2k+1)\frac{\pi}{2}\right);$

d) $f(x) = \tan\sqrt{x}, \quad x \in \left[0, \frac{\pi^2}{4}\right) \bigcup_{k\in\mathbf{N}} \left((2k-1)^2\frac{\pi^2}{4}, (2k+1)^2\frac{\pi^2}{4}\right).$

Solutions.

a) Firstly we have

$$\begin{aligned} f(x+T) - f(x) &= \left(\cos\frac{x+T}{4} - \cos\frac{x}{4}\right) + (\cos(x+T) - \cos x) \\ &\quad + \frac{1}{2}(\cos(3(x+T)) - \cos(3x)) + \frac{1}{3}(\cos(5(x+T)) - \cos(5x)) \\ &= -2\sin\frac{T}{8}\sin\frac{2x+T}{8} - 2\sin\frac{T}{2}\sin\frac{2x+T}{2} \\ &\quad - \sin\frac{3T}{2}\sin\frac{6x+3T}{2} - \frac{2}{3}\sin\frac{5T}{2}\sin\frac{10x+5T}{2}. \end{aligned}$$

Now $f(x+T) - f(x) \equiv 0$ if and only if for some m, n, p, q from $\mathbf{N}$ it holds

$$T = 8n\pi = 2m\pi, \quad T = \frac{2p\pi}{3} = \frac{2q\pi}{5}.$$

From the last equalities we have that $m = 4n,\ p = 12n,\ q = 20n$, and therefore the given function is periodic with basic period $T = 8\pi$.

b) The function $g(x) = \cos x,\ x \in \mathbf{R}$, is periodic with the basic period 2π, while the function $h(x) = \cos(\sqrt{3}\,x),\ x \in \mathbf{R}$, is periodic with the basic period $\frac{2\pi}{\sqrt{3}}$. However, the function $f = g+h$ is *not* periodic, because there exist no nonzero integers k and l such that $2\pi k = \frac{2\pi}{\sqrt{3}}l$.

c) Since for $T > 0$ it holds

$$\begin{aligned} \sqrt{\tan(x+T)} - \sqrt{\tan(x)} &= \frac{\tan(x+T) - \tan x}{\sqrt{\tan(x+T)} + \sqrt{\tan(x)}} \\ &= \frac{\sin T}{\cos(x+T)\cos x\left(\sqrt{\tan(x+T)} + \sqrt{\tan(x)}\right)}, \end{aligned}$$

the last expression is identically equal to 0 iff $T = k\pi$ for some $k \in \mathbf{Z}$. Therefore the given function is periodic with basic period $T = \pi$.

d) Not periodic.

Example 2.19. *Prove that the function f given with the formula*

$$f(x) = x - [x], \; x \in \mathbf{R},$$

is periodic and find its basic period (if it exists), where $[x]$ is the greatest integer part of x, see Example 1.47 d).

Solution. For $x = z + r$, where $z \in \mathbf{Z}$ and $0 \le r < 1$, it holds $[x] = z$. (Note that $[x] = x$ iff $x \in \mathbf{Z}$, otherwise $x - 1 < [x] < x$.) Then

$$f(x) = f(z + r) = z + r - z = r.$$

If $x \in \mathbf{Z}$, then $f(x) = 0$. Thus

$$(\forall x \in \mathbf{Z}) \;\; f(x) = f(x + 1).$$

If $x \in \mathbf{R} \setminus \mathbf{Z}$, then from the decomposition $x = z + r$, where $z \in \mathbf{Z}$ and $0 < r < 1$, it follows

$$f(x + 1) = (x + 1) - [x + 1] = x + 1 - (z + 1) = x - z = r = f(x).$$

Hence the basic period of f is 1.

Example 2.20. *Check whether the* **Dirichlet function** *given by the formula*

$$D(x) = \begin{cases} 1, & \textit{if } x \textit{ is a rational number;} \\ 0, & \textit{if } x \textit{ is an irrational number,} \end{cases}$$

is periodic.

Solution. Let us prove first that every rational number r is a period of the function D. Namely, since the sum of two rational numbers is again rational, it follows that $D(x + r) = 1 = D(x)$ for every rational number x, while the sum of a rational and an irrational number is an irrational number. Hence $D(x + r) = 0 = D(x)$ for every irrational number x. Thus we proved that every rational number r is the period of D. However, since there does not exist a *smallest positive* rational number, it follows that the Dirichlet function has *no* basic period.

(Let us add that no irrational number is the period of D - check that!)

Example 2.21. *Show that the following functions $f : A \to B$ are bijections and find their inverse functions.*

a) $f(x) = 3x + 4, \; A = B = \mathbf{R};$

b) $f(x) = \dfrac{x^2}{3}, \; A = (-\infty, 0], \; B = [0, +\infty);$

c) $f(x) = \dfrac{1 - x}{1 + x}, \; A = B = \left\{x \in \mathbf{R} \,\middle|\, |x| > 1\right\};$

d) $f(x) = \sinh x := \dfrac{1}{2}(\exp(x) - \exp(-x)), \; A = B = \mathbf{R}.$

Hint. Use Theorem 2.2.

Solutions.

a) Let us show first that f is a surjection. Namely, let $y \in \mathbf{R}$ given. Solving the equation $y = 3x + 4$ by x gives $x = \frac{1}{3}(y-4) \in \mathbf{R}$. Then for $x \in \mathbf{R}$ it holds

$$f(x) = 3\left(\frac{1}{3}(y-4)\right) + 4 = y,$$

which implies the surjectivity of f. Since for every pair $x_1, x_2 \in \mathbf{R}$ it holds

$$x_1 < x_2 \iff 3x_1 < 3x_2 \iff 3x_1 + 4 < 3x_2 + 4 \iff f(x_1) < f(x_2),$$

it follows that f is monotonically increasing on $\mathbf{R}$. Applying Theorem 2.2 we obtain that f is a bijection. Then we know that there exists an inverse function g. In order to find its analytical expression, we formally replace x and y in the given formula $y = 3x + 1$ and then attempt to solve the equation for y. This gives us

$$x = 3y + 4 \iff y = \frac{1}{3}(x-4).$$

Hence the inverse function $g : \mathbf{R} \to \mathbf{R}$ is given by the formula

$$g(x) = \frac{1}{3}(x-4).$$

b) Let $y \geq 0$ be given. From the equality $y = \dfrac{x^2}{3}$, $x \leq 0$, it follows $x = -\sqrt{3y}$. Then $f(x) = y$ and hence f is a surjection.

Next, for every pair $x_1, x_2 \in (-\infty, 0]$ it holds

$$x_1 < x_2 \iff x_1^2/3 > x_2^2/3 \iff f(x_1) > f(x_2).$$

Hence f is monotonically decreasing on $(-\infty, 0]$. From Theorem 2.2 it follows that f is a bijection from $(-\infty, 0]$ onto $[0, +\infty)$. Thus the inverse function $g : [0, +\infty) \to (-\infty, 0]$ exists and is given by the formula, $g(x) = -\sqrt{3x}$.

c) Let y such that $|y| > 1$ be given. Then solving the equation

$$y = \frac{1-x}{1+x}$$

by x it follows that a unique x such that $|x| > 1$ exists, namely $x = \dfrac{1-y}{1+y}$. So we showed that f is a surjection.

Let us prove now that f is monotonically decreasing on $(-\infty, -1)$. To that end, first we assume that $x_1, x_2 \in (-\infty, -1)$. Then we have

$$x_1 < x_2 \iff 1 - x_1 > 1 - x_2 \iff \frac{1-x_1}{1+x_1} > \frac{1-x_2}{1+x_2} \iff f(x_1) > f(x_2),$$

hence f is monotonically decreasing on $(-\infty, -1)$. Analogously we prove that f is also monotonically decreasing on $(1, +\infty)$. In view of Theorem 2.2, it follows that f is a bijection, thus it has an inverse function g. Exchanging x and y in the formula $y = \dfrac{1-x}{1+x}$, we get the equation $x = \dfrac{1-y}{1+y}$. Solving it by y we get

$$g(x) = \frac{1-x}{1+x}, \quad |x| > 1,$$

which means that the function f is equal to its inverse function g.

d) The function $f = \sinh x$ is monotonically increasing on $\mathbf{R}$, because it is a sum of two such functions. Solving the equation $y = \dfrac{1}{2}(\exp(x) - \exp(-x))$ by x gives

$$y = \frac{e^{2x} - 1}{2e^x} \iff 2ye^x - e^{2x} + 1 = 0 \iff e^x = y \pm \sqrt{y^2 + 1}.$$

Hence

$$x = \ln(y + \sqrt{y^2 + 1}).$$

Clearly, for given $y \in \mathbf{R}$ this $x \in \mathbf{R}$ has the property $y = f(x)$. Thus f is a surjection and from the monotonicity of f it follows that it is a bijection. The formula that gives the inverse function $g : \mathbf{R} \to \mathbf{R}$ is

$$g(x) = \ln(x + \sqrt{x^2 + 1}).$$

Example 2.22. *Assume the function $f : A \to B$ is a bijection, and denote by $f^{-1} : B \to A$ its inverse function. Prove that for arbitrary sets $X \subset A$ and $Y_1, Y_2 \subset B$ the following holds.*

a) $f(X) \neq \emptyset \Rightarrow X \neq \emptyset$;

b) $Y_1 \subset Y_2 \iff f^{-1}(Y_1) \subset f^{-1}(Y_2)$;

c) $f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$;

d) $f^{-1}(Y_1 \cup Y_2) = f^{-1}(Y_1) \cup f^{-1}(Y_2)$.

Solutions.

a) Since the function f is a bijection, it holds that its codomain B is equal to the range of f, the set $f(A)$. If the set $f(X)$ is nonempty, i.e., if there exists an element $y \in f(X)$, then it follows that there exists a unique element x from X such that $f(x) = y$.

b) Assume $Y_1 \subset Y_2$, and let x be an arbitrary element from the set $f^{-1}(Y_1)$. Denote $y = f(x)$; since f is a bijection, x is the unique element from the domain X whose image is y. By hypothesis y is also in Y_2, which implies $x \in f^{-1}(Y_2)$.

Assume now $f^{-1}(Y_1) \subset f^{-1}(Y_2)$, and take $y \in Y_1$. Then the element $x := f^{-1}(y)$ is in $f^{-1}(Y_1)$, hence $x \in f^{-1}(Y_2)$. But then it holds $y = f(x) \in Y_2$.

c) Using that f is a bijection, we have

$$x \in f^{-1}(Y_1 \cap Y_2) \iff (\exists_1\, y \in (Y_1 \cap Y_2))\ f(x) = y$$

$$\iff (y \in Y_1) \wedge (y \in Y_2) \iff (x \in f^{-1}(Y_1)) \wedge (x \in f^{-1}(Y_2))$$

$$\iff x \in (f^{-1}(Y_1) \cap f^{-1}(Y_2))\,.$$

d) Analogous to c).

Exercise 2.23. *Find the largest sets A and B such that the function $f : A \to B$ is a bijection, and then find its inverse function $g : B \to A$, if*

$$\textbf{a)}\ \ f(x) = \frac{1}{x^5}; \qquad\qquad \textbf{b)}\ \ f(x) = \frac{x}{x+1}.$$

Answers. .

a) $A = B = \mathbf{R} \setminus \{0\}, \quad g(x) = x^{-1/5}$.

b) $A = \mathbf{R} \setminus \{-1\}, \quad B = \mathbf{R} \setminus \{1\}, \quad g(x) = \dfrac{x}{1-x}$.

Example 2.24. *Using the graph of the function $g(x) = x^2$, $x \in \mathbf{R}$, sketch the graph of the following functions.*

$$\textbf{a)}\ \ f(x) = \frac{1}{1+x^2}; \qquad\qquad \textbf{b)}\ \ f(x) = e^{x^2};$$

$$\textbf{c)}\ \ f(x) = \sin x^2; \qquad\qquad \textbf{d)}\ \ f(x) = \ln x^2.$$

Solutions.

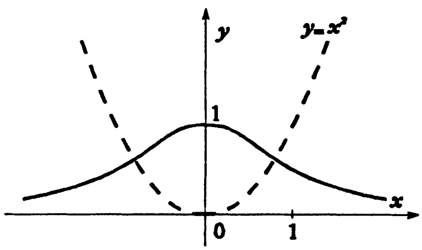

Fig. 2.1. $f(x) = \dfrac{1}{1+x^2}$

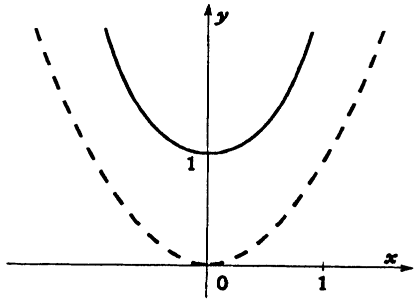

Fig. 2.2. $f(x) = e^{x^2}$

a) The function is defined on the interval $(-\infty, +\infty)$, is even and has no zeros. It has a maximum at $x = 0$ (both local and global), is increasing on $(-\infty, 0)$ and decreasing on $(0, +\infty)$, concave upward. (Figure 2.1.)

b) The function $f(x) = e^{x^2}$ is defined on the interval $(-\infty, +\infty)$, is even and has no zeros. It has a minimum at $x = 0$, is decreasing on $(-\infty, 0)$ and is increasing on $(0, +\infty)$, concave upward. (Figure 2.2.)

c) The function $f(x) = \sin x^2$ is defined on the interval $(-\infty, +\infty)$, is even, but not periodic (see Example 2.17 b)). The zeros of f are at $x = \pm\sqrt{k\pi}$, $k = 0, 1, \ldots$. This function has local maximums at $x = \pm\sqrt{\frac{\pi}{2} + 2k\pi}$, $k = 0, 1, \ldots$, and local minimums at $x = \pm\sqrt{\frac{3\pi}{2} + 2k\pi}$, $k = 0, 1, \ldots$. (Figure 2.3.)

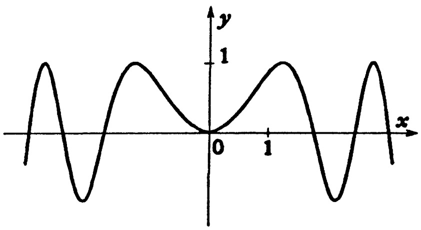

Fig. 2.3. $f(x) = \sin x^2$

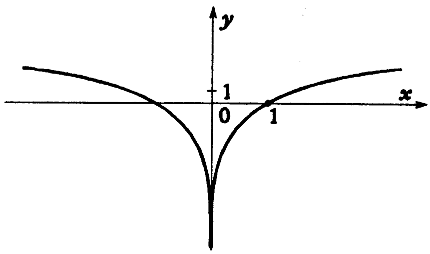

Fig. 2.4. $f(x) = \ln x^2$

d) The function $f(x) = \ln x^2$ is defined on $(-\infty, 0) \cup (0, +\infty)$, is even and has zeros at $x_1 = -1$, $x_2 = 1$. It has no extrema (i.e., no minimums or maximums) , is decreasing on $(-\infty, 0)$ and increasing on $(0, +\infty)$. (Figure 2.4.)

Example 2.25. *Using the graph of the function* $g(x) = \cos x$, $x \in \mathbf{R}$, *(see Figure 2.5), sketch the graph of the following functions.*

a) $f(x) = \dfrac{1}{\cos x}$;

b) $f(x) = \cos^n x$, $n \in \mathbf{N}$;

c) $f(x) = |\cos x|$;

d) $f(x) = \dfrac{1}{|\cos x|}$;

e) $f(x) = \ln|\cos x|$;

f) $f(x) = \dfrac{1}{\ln|\cos x|}$.

Solutions.

a) The function $f(x) = \dfrac{1}{\cos x}$ is defined on the union of the intervals

$$\left((2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2}\right), \quad k = 0, \pm 1, \pm 2, \ldots,$$

where it holds $\cos x \neq 0$. The function f is even, periodic with basic period 2π and has no zeros.

If $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then it holds

$$0 < \cos x < 1, \quad \text{i.e.,} \quad \frac{1}{\cos x} \geq 1,$$

meaning that on this interval the function has a minimum at $x = 0$. On the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2},\right)$ the function has a maximum at $x = \pi$. The vertical asymptotes of the graph are $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$. (Figure 2.6.)

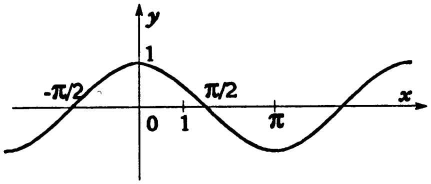

Fig. 2.5. $f(x) = \cos x$

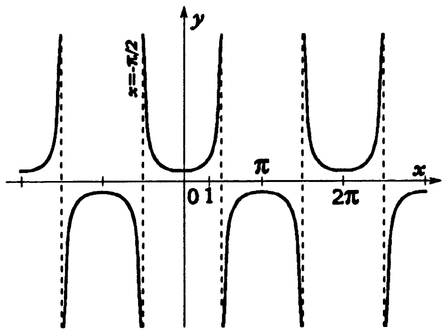

Fig. 2.6. $f(x) = \dfrac{1}{\cos x}$

b) The function $f(x) = \cos^n x$ is defined on the interval $(-\infty, +\infty)$ is even and its zeros are at the points $\frac{\pi}{2} + k\pi,\ k = 0, \pm 1, \pm 2, \ldots$.

If $n = 2k,\ k \in \mathbf{N}$, the function is periodic with basic period π. On the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the function has minimums at $-\frac{\pi}{2}, \frac{\pi}{2}$, and a maximum at 0. (Figure 2.7 for $n = 2$.)

If $n = 2k + 1,\ \ k \in \mathbf{N}$, the function is periodic with basic period 2π. On the interval $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$, the function has a minimum at π and a maximum at 0. (Figure 2.8 for $n = 3$.)

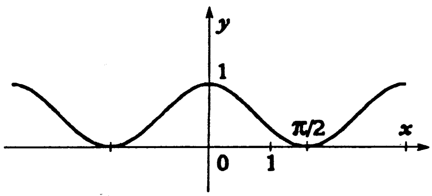

Fig. 2.7. $f(x) = \cos^2 x$

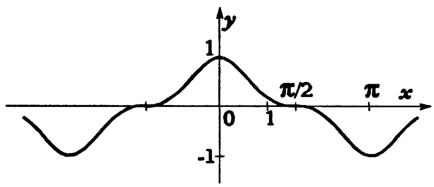

Fig. 2.8. $f(x) = \cos^3 x$

c) The function $f(x) = |\cos x|$ is defined on the interval $(-\infty, +\infty)$, is even and its zeros are at the points $\frac{\pi}{2} + k\pi,\ k = 0, \pm 1, \pm 2, \ldots$. This is a periodic function with basic period π.

On the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the function has minimums at the points $-\frac{\pi}{2}, \frac{\pi}{2}$, and a maximum at 0. (Figure 2.9.)

d) The function $f(x) = \frac{1}{|\cos x|}$ is defined on the intervals $\left((2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2}\right)$, $k = 0, \pm 1, \pm 2, \ldots$, where $\cos x \neq 0$. The function is even, periodic with basic period π and has no zeros.

On the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the function has a minimum at 0. It has vertical asymptotes $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$. (Figure 2.10.)

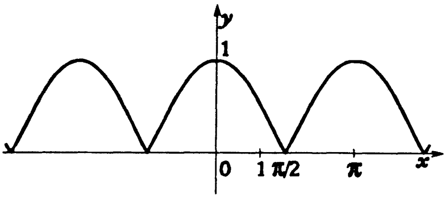

Fig. 2.9. $f(x) = |\cos x|$

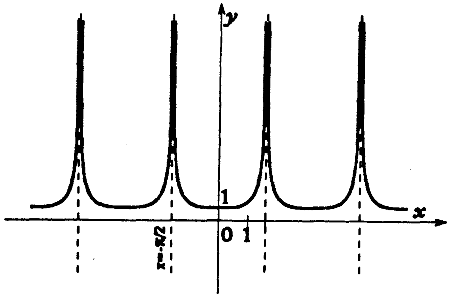

Fig. 2.10. $f(x) = \frac{1}{|\cos x|}$

e) The function $f(x) = \ln|\cos x|$ is defined on the intervals $\left((2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2}\right)$, $k = 0, \pm 1, \pm 2, \ldots$. The function is even, periodic with basic period π and has zeros at the points $k\pi,\ k = 0. \pm 1 \pm 2, \ldots$.

On the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the function has a maximum at $x = 0$. It has vertical asymptotes $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$. (Figure 2.11.)

f) The function $f(x) = \frac{1}{\ln|\cos x|}$ is defined on the intervals $\left(k \cdot \frac{\pi}{2}, (k+1) \cdot \frac{\pi}{2}\right)$, $k = 0, \pm 1, \pm 2, \ldots$. The function is even, periodic with basic period π and has no zeros. The function f has no extrema. The vertical asymptotes are $x = k\pi,\ k = 0, \pm 1, \pm 2, \ldots$. However, the lines $x = (2k+1)\frac{\pi}{2},\ k = 0, \pm 1, \pm 2, \ldots$, are not vertical asymptotes even though the function is not defined at these points. (Figure 2.12.)

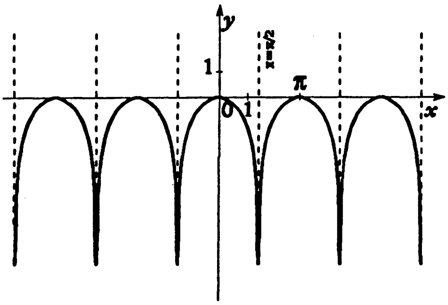

Fig. 2.11. $f(x) = \ln|\cos x|$

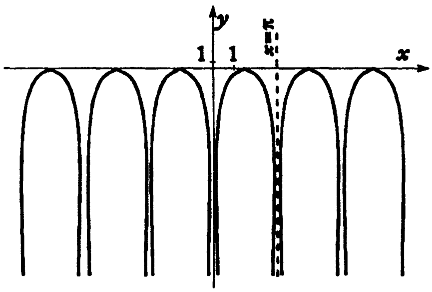

Fig. 2.12. $f(x) = \dfrac{1}{\ln|\cos x|}$

Example 2.26. *Sketch the graphs of the following functions.*

a) $f(x) = x + \cos x$; b) $f(x) = x\sin x$;

c) $f(x) = \sin\dfrac{1}{x}$; d) $f(x) = x\sin\dfrac{1}{x}$.

Solutions.

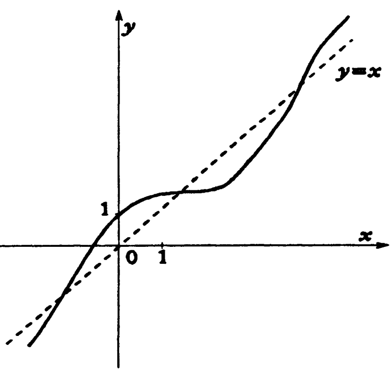

Fig. 2.13. $f(x) = x + \cos x$

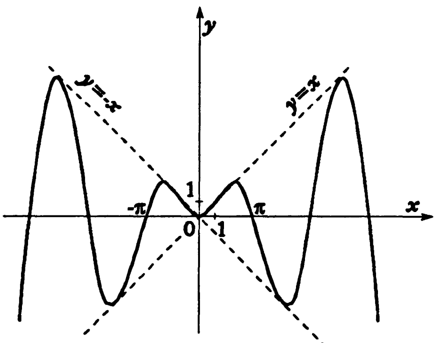

Fig. 2.14. $f(x) = x\sin x$

a) The function $f(x) = x + \cos x$ can be considered as the sum of the functions $f_1 = x$, $x \in \mathbf{R}$, and $f_2 = \cos x$, $x \in \mathbf{R}$.

This function is neither odd nor even, is not periodic and has a zero approximately at the point -0.739. It has no extrema. (Figure 2.13.)

b) The function $f(x) = x\sin x$ is defined for every $x \in \mathbf{R}$. This is an even not periodic function, which has zeros at the points $k\pi$, $k = 0, \pm1, \pm2, \ldots$. At the points $\dfrac{\pi}{2} + 2k\pi$, $k = 0, \pm1, \pm2, \ldots$, it holds $f(x) = x$ and at the points $\dfrac{3\pi}{2} + 2k\pi$, $k = 0, \pm1, \pm2, \ldots$, it holds $f(x) = -x$. (Figure 2.14.)

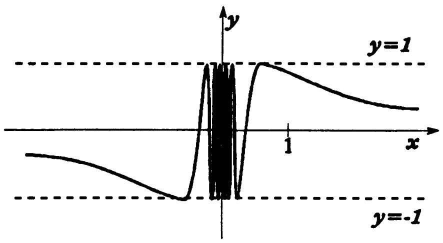

Fig. 2.15. $f(x) = \sin\dfrac{1}{x}$

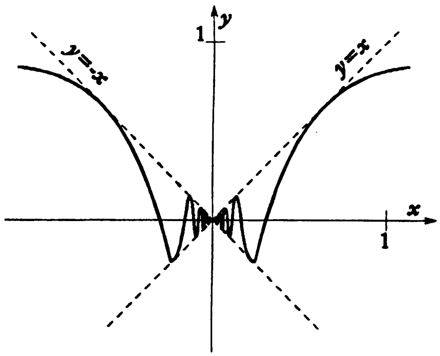

Fig. 2.16. $f(x) = x\sin\dfrac{1}{x}$

c) The function $f(x) = \sin\dfrac{1}{x}$ is defined for every $x \in \mathbf{R} \setminus \{0\}$. This is an odd, not periodic function with zeros at the points $\dfrac{1}{k\pi}$, $k = \pm 1, \pm 2, \ldots$. It is a bounded function, namely $-1 \le \sin\dfrac{1}{x} \le 1$.

The maximums of f are at the points $\dfrac{1}{(\pi/2) + 2k\pi}$, $k = 0, \pm 1, \pm 2, \ldots$, and minimums at the points $\dfrac{1}{(3\pi/2) + 2k\pi}$, $k = 0, \pm 1, \pm 2, \ldots$. (Figure 2.15.)

d) The function $f(x) = x \cdot \sin\dfrac{1}{x}$ is defined for every $x \in \mathbf{R} \setminus \{0\}$. This is an even, not periodic function with zeros $\dfrac{1}{k\pi}$, $k = \pm 1, \pm 2, \ldots$. (Figure 2.16.)

Example 2.27. *Sketch the graphs of the following functions.*

a) $f(x) = \arcsin x$; **b)** $f(x) = \arccos x$;

c) $f(x) = \arctan x$; **d)** $f(x) = \operatorname{arccot} x$.

Solutions.

a) The function $f : [-1, 1] \to \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ is monotonically increasing. (Figure 2.17.)

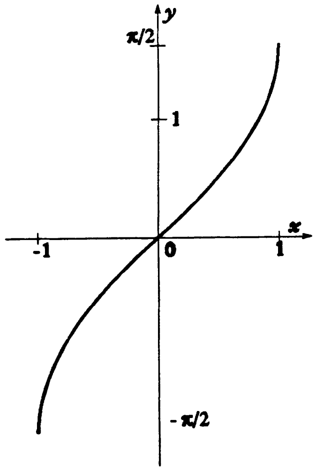

Fig. 2.17. $f(x) = \arcsin x$

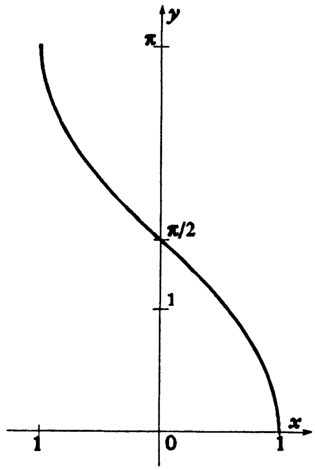

Fig. 2.18. $f(x) = \arccos x$

b) The function $f : [-1, 1] \to [0, \pi]$ is monotonically decreasing.

c) The function $f : (-\infty, +\infty) \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is monotonically increasing. The graph of f has two horizontal asymptotes $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$. (Figure 2.19.)

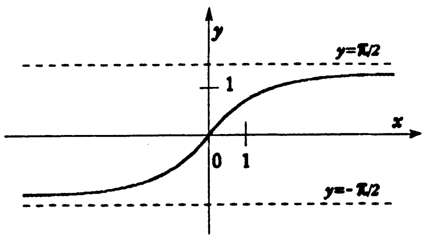

Fig. 2.19. $f(x) = \arctan x$

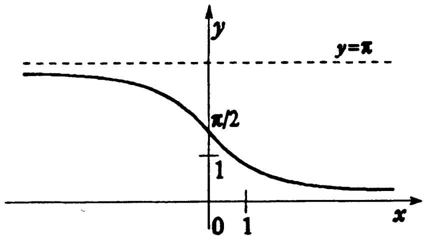

Fig. 2.20. $f(x) = \operatorname{arccot} x$

d) The function $f : (-\infty, +\infty) \to [0, \pi]$ is monotonically decreasing. The graph of f has two horizontal asymptotes $y = 0$ and $y = \pi$. (Figure 2.20.)

Example 2.28. *Show the following identities.*

a) $\arcsin x + \arccos x = \dfrac{\pi}{2}, \quad |x| \le 1;$

b) $\arcsin x = \arccos(\sqrt{1-x^2}), \quad 0 \le x \le 1;$

c) $\arcsin x \pm \arcsin y = \arcsin\left(x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right),$

$|\arcsin x \pm \arcsin y| \le \dfrac{\pi}{2}, \quad |x| \le 1, \ |y| \le 1;$

d) $\arccos x \pm \arccos y = \arccos(xy \pm \sqrt{1-x^2}\sqrt{1-y^2}),$

$0 \le \arccos x \pm \arccos y \le \pi, \quad |x| \le 1, \quad |y| \le 1;$

e) $\arctan x + \operatorname{arccot} x = \dfrac{\pi}{2}, \quad x \in \mathbf{R};$

f) $\arctan x \pm \arctan y = \arctan\left(\dfrac{x \pm y}{1 \mp xy}\right),$

$|\arctan x \pm \arctan y| < \dfrac{\pi}{2}.$

Solutions.

a) Let us denote $\alpha = \arcsin x$, hence for $x \in [-1, 1]$ it holds $\alpha \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$. Then we have

$$x = \sin\alpha = \cos\left(\frac{\pi}{2} - \alpha\right) \Rightarrow \frac{\pi}{2} - \alpha = \arccos x \Rightarrow \arcsin x + \arccos x = \frac{\pi}{2}.$$

b) Let us put $\alpha = \arcsin x$, $x \in [0, 1]$, hence $\alpha \in \left[0, \dfrac{\pi}{2}\right]$, then

$$x = \sin\alpha = \cos\left(\frac{\pi}{2} - \alpha\right) \Rightarrow x^2 - 1 = -\cos^2\alpha$$

$$\Rightarrow \cos\alpha = \sqrt{1-x^2} \Rightarrow \alpha = \arccos(\sqrt{1-x^2}).$$

c) If we put $\alpha = \arcsin x$ and $\beta = \arcsin y$, $\alpha, \beta, \alpha + \beta \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, then $x = \sin\alpha$ and $y = \sin\beta$, hence $\cos\alpha = \sqrt{1-x^2}$ and $\cos\beta = \sqrt{1-y^2}$. Applying the function arcsin to the formula

$$\sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \sin\beta \cdot \cos\alpha,$$

under the assumptions on x and y, we obtain the announced equality.

Example 2.29. *Sketch the graphs of the following functions.*

a) $f(x) = \sinh x = \dfrac{e^x - e^{-x}}{2}$;

b) $f(x) = \cosh x = \dfrac{e^x + e^{-x}}{2}$;

c) $f(x) = \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$;

d) $f(x) = \coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$.

Solutions.

a) The domain and codomain of this function is $\mathbf{R}$. The function is monotonically increasing and odd. (Figure 2.21.)

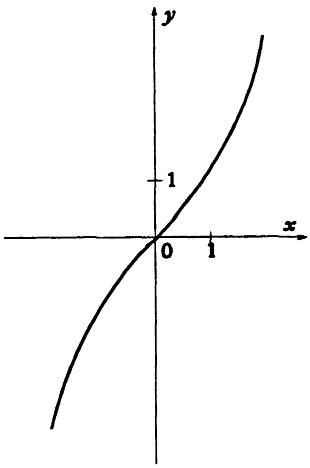

Fig. 2.21. $f(x) = \sinh x$

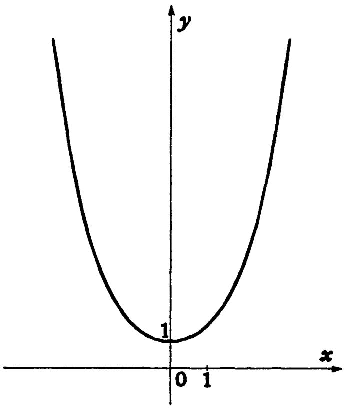

Fig. 2.22. $f(x) = \cosh x$

b) The function $f : \mathbf{R} \to [1, +\infty)$ is even and has a minimum at the point $x = 0$. (Figure 2.22.)

c) The function $f : (-\infty, +\infty) \to (-1, 1)$ is monotonically increasing and odd. (Figure 2.23.)

d) The function $f : (-\infty, 0) \cup (0, +\infty) \to \mathbf{R}$ is monotonically decreasing and odd. (Figure 2.24.)

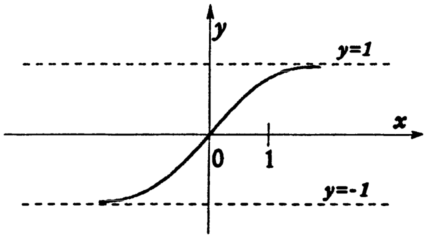

Fig. 2.23. $f(x) = \tanh x$

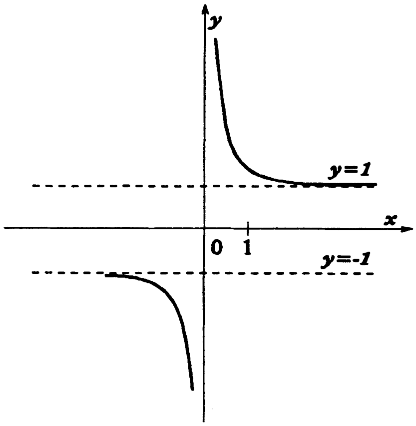

Fig. 2.24. $f(x) = \coth x$

Example 2.30. *Show for the functions*

$$\sinh x := \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x}), \quad \cosh x := \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x}) \quad \textit{and} \quad \tanh x := \frac{\sinh x}{\cosh x},$$

whose natural domains is the set of real numbers $\mathbf{R}$*, that the following identities hold for arbitrary real numbers* x *and* y.

a) $\cosh(x \pm y) = \cosh x \cdot \cosh y \pm \sinh x \cdot \sinh y;$

b) $\cosh(2x) = \cosh^2 x + \sinh^2 x;$

c) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y;$

d) $\sinh(2x) = 2 \sinh(x) \cdot \cosh(x);$

e) $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \cdot \tanh y};$

f) $\tanh(2x) = \dfrac{2 \tanh(x)}{1 + \tanh^2 x}.$

Solutions. Before we prove the given relations, let us note that by the definitions of the functions sinh and cosh, it follows

$$\mathrm{e}^x = \cosh x + \sinh x, \qquad \mathrm{e}^{-x} = \cosh x - \sinh x \tag{2.2}$$

for every $x \in \mathbf{R}$. Using the main property of the exponential function, namely $\mathrm{e}^{x+y} = \mathrm{e}^x \cdot \mathrm{e}^y$, $x \in \mathbf{R}$, and the equalities in (2.2), we obtain for every $x, y \in \mathbf{R}$

$$\mathrm{e}^{x+y} = \cosh(x+y) + \sinh(x+y) = (\cosh x + \sinh x)(\cosh y + \sinh y) \tag{2.3}$$

and

$$e^{-(x+y)} = \cosh(x+y) - \sinh(x+y) = (\cosh x - \sinh x)(\cosh y - \sinh y). \quad (2.4)$$

a) Summing (resp. subtracting) the equality (2.4) with (resp. from) the equality (2.3), we obtain the given equality.

b) Put $x = y$ in a).

e) Using the equalities a) and c), we obtain

$$\tanh(x \pm y) = \frac{\sinh(x \pm y)}{\cosh(x \pm y)}$$

$$= \frac{\sinh x \cdot \cosh y \pm \cosh x \cdot \sinh y}{\cosh x \cdot \cosh y \pm \sinh x \cdot \sinh y} = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \cdot \tanh y},$$

for every $x, y \in \mathbf{R}$.

Exercise 2.31. *Prove the following identities for every $x, y \in \mathbf{R}$.*

$$\textbf{a)} \quad \cosh x + \cosh y = 2\cosh\frac{x+y}{2}\cosh\frac{x-y}{2};$$

$$\textbf{b)} \quad \cosh x - \cosh y = 2\sinh\frac{x+y}{2}\sinh\frac{x-y}{2};$$

$$\textbf{c)} \quad \sinh x + \sinh y = 2\sinh\frac{x+y}{2}\cosh\frac{x-y}{2};$$

$$\textbf{d)} \quad \sinh x - \sinh y = 2\cosh\frac{x+y}{2}\sinh\frac{x-y}{2};$$

$$\textbf{e)} \quad \coth(x \pm y) = \frac{\coth x \cdot \coth y \pm 1}{\coth x \pm \coth y}.$$

2.2 Polynomials, rational and irrational functions

2.2.1 Basic notions

The function

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad x \in \mathbf{R}, \ (\text{or } x \in \mathbf{C}), \quad (2.5)$$

where the coefficients a_j, $j = 0, 1, \ldots, n$, are real numbers, is called **polynomial function** (shortly: **polynomial**) of degree $n \in \mathbf{N}$ if the coefficient $a_n \neq 0$.[2]

[2]In (2.5), the subscript n in $P_n(x)$ stands only to denote the degree of that polynomial.

By definition, the constant function is a polynomial of degree zero. Clearly, the natural domain of any polynomial is the whole set of real numbers $\mathbf{R}$. Occasionally, however, we shall even allow complex values for x in (2.5).

A number x_0 is a **zero** (or sometimes termed **root**) of the polynomial $P_n(x)$ from (2.5) if

$$P_n(x_0) = 0.$$

If a zero x_0 of a polynomial is a rational, real or respectively a complex number, then it will be called rational, real or respectively complex zero.

Theorem 2.32. Fundamental theorem of algebra.
Each polynomial of degree $n \in \mathbf{N}$ has exactly n (real and/or complex) zeros, among which some might be equal.

If x_0 is a real zero of the polynomial $P_n(x)$, then there exists a unique polynomial Q_{n-1} with real coefficients,

$$Q_{n-1}(x) \;=\; b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0, \quad x \in \mathbf{R}, \tag{2.6}$$

of degree $n-1$, such that

$$P_n(x) \;=\; (x - x_0)Q_{n-1}(x). \tag{2.7}$$

The method of quick determination of the polynomial $Q_{n-1}(x)$, the so-called **Horner scheme**, is explained below.

In view of Theorem 2.32 and the factorization (2.7), the number x_0 is called **zero of order** m, $m \in \mathbf{N}$, $1 \le m \le n$, of the polynomial $P_n(x)$ from (2.5) if there exists a polynomial $R_{n-m}(x)$ of degree $n-m$, such that $R_{n-m}(x_0) \neq 0$ and it holds

$$P_n(x) = (x - x_0)^m R_{n-m}(x).$$

Any polynomial $P_n(x)$ with real coefficients of the form (2.5), can be written in a unique way as the product

$$P_n(x) = a_n(x - x_1)^{m_1} \cdots (x - x_r)^{m_s}(x^2 + b_1 x + c_1)^{l_1} \cdots (x^2 + b_s x + c_s)^{l_s}, \tag{2.8}$$

where for the natural numbers m_j, $j = 1, \ldots, r$, and l_k, $k = 1, \ldots, s$, it holds

$$m_1 + \cdots + m_r + 2(l_1 + \cdots + l_s) = n.$$

In (2.8), the numbers x_j are mutually different zeros of order $m_j, j = 1, \ldots, r$, of the polynomial $P_n(x)$, while the zeros of the polynomials $x^2 + b_k x + c_k$, $k = 1, \ldots, s$, are conjugate complex numbers.

In general, it might be a tricky task to find the zeros of a polynomial. However, in the special (but quite often) case when the polynomial $P_n(x)$ from (2.5) has only *integer* coefficients, then the *candidates* for *rational zeros* can be found quickly. Namely, if the reduced fraction $\dfrac{p}{q}$, where $p \in \mathbf{Z}$ and $q \in \mathbf{N}$, is the zero of $P_n(x)$, then necessarily the numerator p is a divisor of a_0 (we write that shortly $p|a_0$), while the denominator q is a divisor of a_n $(q|a_n)$ (prove that!). In particular, if $a_n = 1$, then $q = 1$, hence a rational zero of $P_n(x)$ with integer coefficients is, in fact, an integer.

The **Horner scheme** allows a quick check whether a real number x_0 is or is not a zero of the polynomial $P_n(x)$. Namely, performing the division of the polynomial $P_n(x)$ with the polynomial $x - x_0$ (of degree one) gives

$$P_n(x) = (x - x_0)Q_{n-1}(x) + r. \tag{2.9}$$

where the polynomial $Q_{n-1}(x)$ of degree $(n - 1)$ is given by relation (2.6). Its coefficients $b_{n-1}, ..., b_0$ can be found by the formula

$$b_{n-1} = a_n, \quad b_{n-2} = x_0 b_{n-1} + a_{n-1}, ..., \quad b_0 = x_0 b_1 + a_1, \tag{2.10}$$

or

$$b_k = x_0 b_{k+1} + a_{k+1}, \quad k = 0, 1, \ldots, n-1,$$

and the remainder r is given by

$$r = x_0 b_0 + a_0.$$

Clearly, if $r = 0$, then the number x_0 **is** the zero of the polynomial $P_n(x)$ and relation (2.9) coincides with (2.7). However, if $r \neq 0$, then the number x_0 **is not** a zero of the polynomial $P_n(x)$.

The Horner scheme is most easily performed as follows.
First one writes in a row *all* the coefficients of the given polynomial function $P_n(x)$, including those that are equal to zero. Using the upper formulas for b_k, $k = 0, 1, .., n-1$, and the one for r, we make the following two–row scheme:

$$\begin{array}{cccccccc|c} a_n & a_{n-1} & a_{n-2} & \ldots & a_k & \ldots & a_0 & & x_0 \\ & b_{n-1} & b_{n-2} & \ldots & b_k & \ldots & b_0 & & r. \end{array}$$

All we have to do now is to apply the upper analysis, i.e., to check whether the remainder r is zero or not.

The **rational function** is the quotient of functions

$$R(x) = \frac{P_n(x)}{Q_m(x)}, \quad Q_m(x) \neq 0, \tag{2.11}$$

where $P_n(x)$ and $Q_m(x)$ are polynomials of degree n and m.

The most important examples of the rational functions are the following two types of **partial fractions**

$$\frac{A}{(x-a)^j}, \ j \in \mathbf{N}, \quad \text{and} \quad \frac{Bx + C}{(x^2 + bx + c)^k}, \ k \in \mathbf{N}, \tag{2.12}$$

for some natural numbers j and k. In relation (2.12), A, B, C, a, b and c are constants and the zeros of the polynomial $x^2 + bx + c$ are conjugate complex numbers.

Theorem 2.33. *A rational function of the form (2.11) can be written in a unique way as the sum of partial fractions of the form (2.12) and, if $n \geq m$, a polynomial of degree $n - m$. (If $n = m$, then the last polynomial reduces to a constant.)*

This theorem is essential when calculating the integrals of rational functions.

The **irrational function** is an elementary function which is not rational and can be defined as a composition of finitely many rational functions, power functions with rational exponents and arithmetic operations.

2.2.2 Examples and exercises

Example 2.34. *Find the rational zeros of the following polynomial functions.*

a) $P(x) = 2x^2 + 4x - 6;$

b) $P(x) = x^3 + 3x^2 - 4x - 12;$

c) $P(x) = 2x^4 - 13x^3 + 28x^2 - 23x + 6;$

d) $P(x) = x^5 - x^3 + x^2 - 1;$

e) $P(x) = x^7 + 6x^5 - 5x^4 + 9x^3 - 10x^2 + 4x - 5.$

Solutions.

a) The given polynomial $P(x)$ is of degree 2, so the roots can be found as the solutions of the quadratic equation and have the forms $x_1 = -3$, $x_2 = 1$. In view of Theorem 2.32 these are also the *only* zeros of $P(x)$; observe that they are both rational. This means that we have the factorization

$$2x^2 + 4x - 6 = 2(x - 1)(x + 3).$$

Remark. The zeros of the polynomial $P(x)$ could have been also found using Horner scheme.

b) For the polynomial $P(x)$ we shall find the divisors of 12, since $a_0 = 12$. The divisors are

$$\pm 1, \pm 2, \pm 3, \pm 6, \pm 12.$$

Further on, since $a_n = a_3 = 1$, the only possible rational zeros of the polynomial $P(x)$ are integers from the upper set of divisors.

Let us check whether $x = 1$ is a zero of $P(x)$

$$\begin{array}{cccc|c} 1 & 3 & -4 & -12 & \qquad x = 1 \\ & 1 & 4 & 0 & -12. \end{array}$$

Since the last number in the second row is -12 (i.e., $r = -12$), the number $x = 1$ is not a zero of $P(x)$.

Next, we check whether $x = 2$ is a zero of $P(x)$

$$\begin{array}{cccc|c} 1 & 3 & -4 & -12 & \qquad x = 2 \\ & 1 & 5 & 6 & 0. \end{array}$$

Now the last number in the second row is 0, (i.e., $r = 0$), hence $x = 2$ is a zero of the given polynomial. So we obtained the factorization

$$x^3 + 3x^2 - 4x - 12 = (x-2)(x^2 + 5x + 6).$$

We yet have to find the zeros of the polynomial $Q(x) := x^2 + 5x + 6$. We can find them either by solving the quadratic equation

$$x^2 + 5x + 6 = 0,$$

or as we shall do, using the Horner scheme.

Let us try the number $x = -2$

$$\begin{array}{cccl} 1 & 5 & 6 & | \quad x = -2 \\ & 1 & 3 & |0. \end{array}$$

Hence, the second zero of the polynomial $P_3(x)$ is the number $x = -2$. So we obtained the factorization

$$x^3 + 3x^2 - 4x - 12 = (x-2)(x+2)(x+3).$$

(The third zero is -3.)

c) Let us determine first the divisors of 6. They are

$$\pm 1, \pm 2, \pm 3, \pm 6.$$

The positive integers that are divisors of 2 are 1 and 2, hence the possible rational zeros of the given polynomial are

$$\pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{2}, \pm\frac{3}{2}.$$

Using the Horner scheme, we obtain that $x = 1, x = 2$ and $x = 3$ are the zeros of the given polynomial:

$$\begin{array}{cccccll} 2 & -13 & 28 & -23 & 6 & | & x = 1 \\ & 2 & -11 & 17 & -6 & |0 & x = 2 \\ & & 2 & -7 & 3 & |0 & x = 3 \\ & & & 2 & -1 & |0. & \end{array}$$

Now, the last row of this scheme shows that the fourth zero of the given polynomial is $x = \frac{1}{2}$. So we obtain the factorization

$$2x^4 - 13x^3 + 28x^2 - 23x + 6 = 2(x-1)(x-2)(x-3)(x-1/2).$$

d) The only possible rational zeros of the given polynomial are 1 and -1, hence the Horner scheme gives

$$\begin{array}{cccccccl} 1 & 0 & -1 & 1 & 0 & -1 & | & x=1 \\ & 1 & 1 & 0 & 1 & 1 & |0 & x=-1 \\ & & 1 & 0 & 0 & 1 & |0 & x=-1 \\ & & & 1 & -1 & 1 & |0. & \end{array}$$

This shows that

$$P(x) = x^5 - x^3 + x^2 - 1 = (x+1)^2(x-1)(x^2-x+1). \qquad (2.13)$$

The last factor $x^2 - x + 1$ has no real, therefore no rational zeros, but rather its zeros are conjugate complex numbers. This means that the formula (2.13) is the factorization of the form (2.8).

e) The possible rational zeros of $P(x)$ are $\pm 1, \pm 5$. The Horner scheme gives

$$\begin{array}{cccccccccl} 1 & 0 & 6 & -5 & 9 & -10 & 4 & -5 & | & x=1 \\ & 1 & 1 & 7 & 2 & 11 & 1 & 5 & |0 & x=1 \\ & & 1 & 2 & 9 & 11 & 22 & 23 & |28. & \end{array}$$

This shows that $x = 1$ is a zero of first order, and *not* of second order, since the last number in the third row is 28. We leave to the reader to check that the numbers $x = -1$, $x = 5$ and $x = -5$ are not the zeros of the given polynomial, hence it has no other rational zeros. Moreover, the polynomial $P(x)$ has no other real zeros, and the factorization of the form (2.8) is

$$\begin{aligned} P(x) &= x^7 + 6x^5 - 5x^4 + 9x^3 - 10x^2 + 4x - 5 \\ &= (x-1)(x^2+x+5)(x^2+1)^2. \end{aligned}$$

Example 2.35. *Decompose to partial fractions the following rational functions.*

a) $\dfrac{2}{x^2-1}$; b) $\dfrac{x^3-2x-35}{x^2-2x-15}$;

c) $\dfrac{x+1}{x^3-2x^2+x-2}$; d) $\dfrac{x+3}{x^4-5x^2+4}$;

e) $\dfrac{x^2+1}{(x-1)^3}$; f) $\dfrac{2x^2-4x+3}{x^4-6x^3+13x^2-12x+4}$.

Solutions.

a) The zeros of the polynomial $x^2 - 1$ (the denominator of the given rational function) are $x = 1$ and $x = -1$. Using Theorem 2.33, it follows that it can be decomposed as the sum of following partial fractions

$$\frac{2}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}.$$

From this equality we get the constants A and B in the following way. First, we add the last two fractions

$$\frac{A}{x-1}+\frac{B}{x+1}=\frac{A(x+1)+B(x-1)}{x^2-1}.$$

Since the denominator of the obtained fraction is the same as that of the given rational function, their numerators should be also the same, hence

$$2=x(A+B)+(A-B).$$

So we obtain the linear system of equations with unknown A and B

$$A+B=0, \quad A-B=2, \quad \text{which gives} \quad A=1, \quad B=-1.$$

This means that finally we have

$$\frac{2}{x^2-1}=\frac{1}{x-1}+\frac{-1}{x+1}.$$

b) Let us notice first that the degree of the numerator is greater than that of the denominator, so let us do the following transformations:

$$\begin{aligned}\frac{x^3-2x-35}{x^2-2x-15} &= \frac{x^3-2x^2-15x+2x^2+13x-35}{x^2-2x-15}\\ &= x+\frac{2x^2+13x-35}{x^2-2x-15}=x+2+\frac{17x-5}{x^2-2x-15}.\end{aligned}$$

Remark. Instead of these transformations, one can divide the polynomial $x^3-2x-35$ from the numerator with the polynomial $x^2-2x-15$ from the denominator, which gives the polynomial $x+2$ and the fraction

$$\frac{17x-5}{x^2-2x-15}.$$

Further on we have

$$\frac{17x-5}{x^2-2x-15}=\frac{A}{x-5}+\frac{B}{x+3}=\frac{A(x+3)+B(x-5)}{x^2-2x-15},$$

which gives us the system of linear equations

$$A+B=17, \quad 3A-5B=-5.$$

So we obtain $A=10$ and $B=7$ and finally

$$\frac{x^3-2x-35}{x^2-2x-15}=x+2+\frac{10}{x-5}+\frac{7}{x+3}.$$

c) The polynomial in the denominator can be written as

$$x^3 - 2x^2 + x - 2 = (x-2)(x^2+1).$$

Hence, there exist constants A, B and C, such that

$$\begin{aligned}\frac{x+1}{x^3-2x^2+x-2} &= \frac{A}{x-2} + \frac{Bx+C}{x^2+1}\\ &= \frac{Ax^2+A+Bx^2-2Bx+Cx-2C}{(x-2)(x^2+1)},\end{aligned}$$

or

$$\frac{x+1}{x^3-2x^2+x-2} = \frac{(A+B)x^2+(C-2B)x+A-2C}{x^3-2x^2+x-2}.$$

This gives us a system of linear equations

$$A+B=0, \quad C-2B=1, \quad A-2C=1,$$

whose solution is $A = 3/5, \quad B = -3/5, \quad C = -1/5.$

Finally we have

$$\frac{x+1}{x^3-2x^2+x-2} = \frac{3}{5(x-2)} - \frac{3x+1}{5(x^2+1)}.$$

d) The numbers $x = 1,\ x = 2,\ x = -1$ and $x = -2$ are the zeros of the polynomial in the denominator, hence

$$\frac{x+3}{x^4-5x^2+4} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-2} + \frac{D}{x+2}.$$

Adding together the last fractions gives us

$$\begin{aligned}\frac{x+3}{x^4-5x^2+4} &= \frac{Ax^3+Ax^2-4Ax-4A+Bx^3-Bx^2-4Bx+4B}{x^4-5x^2+4}\\ &+ \frac{Cx^3+2Cx^2-Cx-2C+Dx^3-2Dx^2-Dx+2D}{x^4-5x^2+4}\\ &= \frac{(A+B+C+D)x^3+(A-B+2C-2D)x^2}{x^4-5x^2+4}\\ &+ \frac{-(4A+4B+C+D)x-4A+4B-2C+2D}{x^4-5x^2+4}.\end{aligned}$$

Hence

$$A+B+C+D=0, \quad A-B+2C-2D=0,$$

$$-(4A+4B+C+D)=1, \quad -4A+4B-2C+2D=3,$$

which gives

$$A = -2/3,\ B = 1/3,\ C = 5/12,\ D = -1/12.$$

So we obtained the decomposition

$$\frac{x+3}{x^4-5x^2+4} = \frac{-2}{3(x-1)} + \frac{1}{3(x+1)} + \frac{5}{12(x-2)} - \frac{1}{12(x+2)}.$$

e) The number 1 is the zero of order 3 of the denominator. Thus we have

$$\begin{aligned}\frac{x^2+1}{(x-1)^3} &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} \\ &= \frac{Ax^2 - 2Ax + A + Bx - B + C}{(x-1)^3},\end{aligned}$$

which gives the system of linear equations

$$A = 1, \quad -2A + B = 0, \quad A - B + C = 1.$$

Its solution is $A = 1$, $B = 2$ and $C = 2$, hence we obtain the decomposition

$$\frac{x^2+1}{(x-1)^3} = \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{2}{(x-1)^3}.$$

f) The values $x = 1$ and $x = 2$ are zeros of order 2 of the polynomial in the denominator, hence

$$\begin{aligned}\frac{2x^2-4x+3}{x^4-6x^3+13x^2-12x+4} &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2} \\ &= \frac{Ax^3 - 5Ax^2 + 8Ax - 4A + Bx^2 - 4Bx + 4B}{x^4-6x^3+13x^2-12x+4} \\ &+ \frac{Cx^3 - 4Cx^2 + 5Cx - 2C + Dx^2 - 2Dx + D}{x^4-6x^3+13x^2-12x+4}.\end{aligned}$$

This gives the system

$$A + C = 0, \quad -5A + B - 4C + D = 2,$$

$$8A - 4B + 5C - 2D = -4, \quad -4A + 4B - 2C + D = 3,$$

whose solution is $A = 2$, $B = 1$, $C = -2$, $D = 3$.

Finally we have

$$\frac{2x^2-4x+3}{x^4-6x^3+13x^2-12x+4} = \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{-2}{x-2} + \frac{3}{(x-2)^2}.$$

Exercise 2.36. *Decompose to partial fractions the following rational functions.*

a) $\dfrac{3}{x^3+1}$; b) $\dfrac{4x^2+4x+26}{(2x-4)(x^2+1)^2}$;

c) $\dfrac{1}{x^4+1}$; d) $\dfrac{2}{x^4+x^2+1}$.

Solutions.

a) $$\frac{3}{x^3+1} = \frac{1}{x+1} - \frac{x-2}{x^2-x+1}.$$

b) $$\frac{4x^2+4x+26}{(2x-4)(x^2+1)^2} = \frac{1}{x-2} - \frac{x+2}{x^2+1} - \frac{3x+4}{(x^2+1)^2}.$$

c) $$\frac{1}{x^4+1} = \frac{\sqrt{2}x+2}{4(x^2+x\sqrt{2}+1)} - \frac{\sqrt{2}x-2}{4(x^2-x\sqrt{2}+1)}.$$

d) $$\frac{2}{x^4+x^2+1} = \frac{x+1}{x^2+x+1} - \frac{x-1}{x^2-x+1}.$$

Example 2.37. *Prove that the function* $f(x) = x^{1/2}$, $x \geq 0$, *is an irrational function.*[3]

Solution. Assume that there exist polynomials P and Q such that

$$x^{1/2} = \frac{P(x)}{Q(x)}, \quad x \geq 0.$$

Then

$$\frac{1}{\sqrt{x}} = \frac{P(x)}{xQ(x)}, \quad x > 0,$$

hence

$$\lim_{x\to+\infty} \frac{1}{\sqrt{x}} = \lim_{x\to+\infty} \frac{P(x)}{xQ(x)} = 0.$$

So the degree of $P(x)$ is less than the degree of $x \cdot Q(x)$ and the degree of P is $\leq$ than the degree of Q. Thus

$$\lim_{x\to+\infty} \sqrt{x} = \lim_{x\to+\infty} \frac{P(x)}{Q(x)} = K,$$

for some constant K, giving a contradiction to

$$\lim_{x\to+\infty} \sqrt{x} = +\infty.$$

Exercise 2.38. *Prove that the function* $f(x) = x^{p/q}$, $x > 0$, *is an irrational function, provided that* $q \in \mathbf{N}$, $q > 1$, *and the fraction* $\dfrac{p}{q}$ *is irreducible.*

Exercise 2.39. *Determine the irrational functions among the ones given or analyzed in this chapter.*

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[3]In the proof, we use the notion of the limit of a function at $+\infty$, given in Definition 4.4.

Chapter 3

Sequences

3.1 Introduction

3.1.1 Basic notions

A **sequence** is a function $a : \mathbf{N} \to \mathbf{R}$. It is usual to write

$$a_n := a(n),\ n \in \mathbf{N} \quad \text{and} \quad a = (a_n)_{n\in\mathbf{N}}.$$

Then a_n is called the **general term** of the sequence a.

Definition 3.1. *A real number ℓ is the* **limit of a sequence** $(a_n)_{n\in\mathbf{N}}$ *if for every $\varepsilon > 0$ there exists a natural number $n_0 = n_0(\varepsilon)$, such that for every $n > n_0$ it holds $| a_n - \ell |< \varepsilon$, i.e.,*

$$(\forall\varepsilon > 0)\,(\exists n_0 \in \mathbf{N})\,(\forall n \in \mathbf{N})\ \ n > n_0\ \Rightarrow | a_n - \ell |< \varepsilon. \tag{3.1}$$

Then we write $\lim\limits_{n\to\infty} a_n = \ell$.

If a sequence $(a_n)_{n\in\mathbf{N}}$ has the limit ℓ, we say that it **converges** to ℓ. We say that a sequence **diverges** if it does not converge. Let us consider two classes of *divergent* sequences.

Definition 3.2. *A sequence* $(a_n)_{n\in\mathbf{N}}$

- **tends to plus infinity**, *if for every positive number $M > 0$ there exists a natural number $n_0 = n_0(M)$, such that for every $n > n_0$ it holds $a_n > M$ (then we write $\lim\limits_{n\to\infty} a_n = +\infty$);*
- **tends to minus infinity** *if for every real number $M > 0$ there exists a natural number $n_0 = n_0(M)$, such that for every $n > n_0$ it holds $a_n < -M$ (then we write $\lim\limits_{n\to\infty} a_n = -\infty$).*

Definition 3.3. *A sequence $a = (a_n)_{n\in\mathbf{N}}$ is* **bounded** *if there exists a positive real number M such that for every $n \in \mathbf{N}$ it holds $|a_n| \leq M$.*

Definition 3.4. *A point $\ell \in \mathbf{R}$ is an* **accumulation point** *of a sequence $(a_n)_{n\in\mathbf{N}}$ if for every $\varepsilon > 0$ and every $n \in \mathbf{N}$, there exists a number $m \in \mathbf{N}$, $m > n$ (m depending on n and $\varepsilon > 0$), such that it holds $| a_m - \ell | < \varepsilon$.*

Theorem 3.5. *A sequence converges iff it is bounded and has exactly one accumulation point.*

Theorem 3.6. *If $(a_n)_{n\in\mathbf{N}}$ and $(b_n)_{n\in\mathbf{N}}$ are convergent sequences, then it holds*

- $\lim\limits_{n\to\infty}(a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$;
- $\lim\limits_{n\to\infty}(a_n \cdot b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$;
- $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$,
  where $b_n \neq 0$ for every $n \in \mathbf{N}$ and $\lim\limits_{n\to\infty} b_n \neq 0$.

Theorem 3.7.

- *Assume $(a_n)_{n\in\mathbf{N}}$ and $(b_n)_{n\in\mathbf{N}}$ are convergent sequences, with the property that there exists a natural number n_0 such that for every $n > n_0$ it holds $a_n \leq b_n$. Then*

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n.$$

- *If for the sequences $(a_n)_{n\in\mathbf{N}}$, $(b_n)_{n\in\mathbf{N}}$ and $(c_n)_{n\in\mathbf{N}}$ there exists a number n_0 such that for every natural number $n > n_0$ it holds*

$$a_n \leq b_n \leq c_n,$$

then the following implication holds

$$\left(\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = \ell\right) \Rightarrow \left(\lim_{n\to\infty} b_n = \ell\right).$$

Definition 3.8. *A sequence $(f_n)_{n\in\mathbf{N}}$ is a* **Cauchy sequence** *if it satisfies the following condition*

$$(\forall\varepsilon > 0)\,(\exists n_0 \in \mathbf{N})\,(\forall m, n \in \mathbf{N}) \quad m, n > n_0 \Rightarrow |f_m - f_n| < \varepsilon. \tag{3.2}$$

This condition is often replaced with the condition

$$(\forall\varepsilon > 0)\,(\exists n_0 \in \mathbf{N})\,(\forall n, p \in \mathbf{N}) \quad n > n_0 \Rightarrow |f_{n+p} - f_n| < \varepsilon. \tag{3.3}$$

Of course, both in (3.2) and (3.3) the natural number n_0 depends on ε.

Theorem 3.9. *A sequence of real numbers converges iff it is a Cauchy sequence.*

This property of the set **R** is called **completeness.** [1]

[1] One can prove that the axiom (R15) from Subsection 1.1 is equivalent with the last statement, which explains the term "completeness of **R**" in (R15).

3.1.2 Examples and exercises

Example 3.10. *Show by Definition 3.1 that the following sequences have a limit which is equal to zero.*

a) $a_n = \dfrac{1}{n}$; **b)** $b_n = \dfrac{n-1}{n^2+2}$;

c) $c_n = \dfrac{1}{\sqrt[3]{n+3}}$; **d)** $e_n = \dfrac{1}{n!}$;

e) $d_n = \dfrac{1}{n^\alpha}$, $\alpha > 0$.

Solutions.

a) We have to show that for every given positive number ε there exists a natural number $n_0 = n_0(\varepsilon)$ such that the following implication holds

$$(\forall n \in \mathbf{N})\ (n > n_0 \Rightarrow |a_n - 0| < \varepsilon), \quad \text{i.e.,} \quad (\forall n \in \mathbf{N})\ (n > n_0 \Rightarrow \frac{1}{n} < \varepsilon).$$

From the last relation it follows that for every n which has the property $n > \dfrac{1}{\varepsilon}$, it holds $\left|\dfrac{1}{n} - 0\right| = \dfrac{1}{n} < \varepsilon$. This means that we can take $n_0 = \left[\dfrac{1}{\varepsilon}\right] + 1$, where $[x]$ is the greatest integer part of x defined in Example 1.47.

Remark. It is sufficient to consider only the case when $\varepsilon > 0$ is "small" (say when $0 < \varepsilon < 1$). Namely, such a restriction does not loose on generality, because if there exists $n_0 = n_0(\varepsilon)$ such that for every $n \in \mathbf{N}$, $n > n_0$, it holds $|a_n - \ell| < \varepsilon$ (see relation (3.1)), then

$$(\forall \varepsilon_1 \geq \varepsilon)\ (\forall n \in \mathbf{N}) \quad n > n_0 \Rightarrow |a_n - \ell| < \varepsilon_1.$$

Further on, we shall consider only the case $0 < \varepsilon < 1$.

b) For every given $\varepsilon > 0$ there exists a natural number $n_0 = n_0(\varepsilon)$ such that

$$(\forall n \in \mathbf{N})(n > n_0 \Rightarrow |a_n - 0| < \varepsilon), \quad \text{i.e.,} \quad (\forall n \in \mathbf{N})(n > n_0 \Rightarrow \frac{n-1}{n^2+2} < \varepsilon).$$

Using the following relations

$$\frac{n-1}{n^2+2} < \frac{n}{n^2+2} = n^{-1}\frac{n^2}{n^2+2} < \frac{1}{n} < \varepsilon,$$

and the case a), we obtain that we can choose n_0 such that $n_0 = \left[\dfrac{1}{\varepsilon}\right] + 1.$

c) For given $\varepsilon > 0$ we shall find a natural number $n_0 = n_0(\varepsilon)$ such that for every $n > n_0$ it holds

$$\left|\frac{1}{\sqrt[3]{n+3}}\right| < \varepsilon.$$

From the implications

$$\left(\left|\frac{1}{\sqrt[3]{n+3}} - 0\right| = \frac{1}{\sqrt[3]{n+3}} < \varepsilon\right) \Rightarrow \left(\frac{1}{n+3} < \varepsilon^3\right) \Rightarrow \left(n+3 > \frac{1}{\varepsilon^3}\right),$$

it follows that we can take $n_0 = n_0(\varepsilon) = \left[\frac{1}{\varepsilon^3}\right] + 1.$

d) $\left(\left|\frac{1}{n!} - 0\right| = \frac{1}{n!} < \frac{1}{2^{n-1}} < \varepsilon\right) \Rightarrow \left(\ln\frac{1}{\varepsilon} < (n-1)\ln 2.\right)$

Thus we can take $n_0 = n_0(\varepsilon) = \left[\dfrac{\ln\left(\frac{1}{\varepsilon}\right)}{\ln 2}\right] + 1.$

e) If α is a positive rational number, then we have

$$\left|\frac{1}{n^\alpha} - 0\right| = \frac{1}{n^\alpha} < \varepsilon, \quad \text{hence} \quad \frac{1}{\varepsilon} < n^\alpha \quad \text{or} \quad \frac{1}{\sqrt[\alpha]{\varepsilon}} < n.$$

So we can take $n_0 = \left[\frac{1}{\sqrt[\alpha]{\varepsilon}}\right] + 1.$

If α is a positive irrational number, then there exists a rational number β such that it holds $0 < \beta < \alpha$ and $n^\beta < n^\alpha$. Hence

$$\left|\frac{1}{n^\alpha} - 0\right| = \frac{1}{n^\alpha} < \frac{1}{n^\beta}, \quad \text{and we can take} \quad n_0 = \left[\frac{1}{\sqrt[\beta]{\varepsilon}}\right] + 1.$$

Remark. The number $n_0 = n_0(\varepsilon)$ from Definition 3.1 is not uniquely determined. Namely, if for given $\varepsilon > 0$ we can find $n_0 = n_0(\varepsilon)$ such that the implication in (3.1) holds, then any natural number $n_1 > n_0$ could also give the implication in (3.1). This also means that in finding $n_0(\varepsilon)$ we can occasionally use rather rough approximations. In fact, it is important to understand that the existence of $n_0 = n_0(\varepsilon)$ for given $\varepsilon > 0$ is essential.

Example 3.11. *Show by Definition 3.1 that the following sequences have a limit which is equal to one.*

a) $a_n = \dfrac{n+1}{n+2}$; **b)** $b_n = \sqrt[n]{a}, \quad a > 0$; **c)** $c_n = \sqrt[n]{n}$.

Solutions.

a) From the relations
$$\left|\frac{n+1}{n+2}-1\right| = \frac{1}{n+2} < \varepsilon,$$
it follows that for each $\varepsilon > 0$, it holds
$$n+2 > \frac{1}{\varepsilon} \quad \text{or} \quad n > \frac{1}{\varepsilon} - 2.$$
Therefore we can take for n_0 any arbitrary natural number greater than the number $1/\varepsilon - 2$. For example, put $n_0 := n_0(\varepsilon) = \left[\frac{1}{\varepsilon}\right] + 1$. Then for every $n > n_0$ it holds
$$\left|\frac{n+1}{n+2}-1\right| < \varepsilon.$$

b) If we suppose that $a = 1$, then it is trivial that $\lim\limits_{n\to\infty} b_n = 1$.

If we suppose that $a > 1$, namely $a = 1+b$, $b > 0$, then for every $n \in \mathbf{N}$, $n > 1$, there exists a real number $c(n) > 0$ such that it holds
$$\sqrt[n]{1+b} = 1 + c(n), \quad \text{i.e.,} \quad 1+b = (1+c(n))^n.$$
Using the Bernoulli inequality
$$(1+k)^n \geq 1 + nk, \quad k > -1, \quad n \in \mathbf{N},$$
we obtain $1+b \geq 1 + n \cdot c(n)$, wherefrom we have $c(n) \leq \dfrac{b}{n}$. This means that for every $\varepsilon > 0$ it holds
$$|\sqrt[n]{1+b} - 1| = c(n) \leq \frac{b}{n} < \varepsilon,$$
provided that $n_0 := [b/\varepsilon] + 1$.

If we suppose that $0 < a < 1$, i.e., $a = \dfrac{1}{h}$, $h > 1$, then from Theorems 3.6 and 3.7 it follows that
$$\lim_{n\to\infty} \sqrt[n]{a} = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{h}} = 1.$$
Thus $\lim\limits_{n\to\infty} \sqrt[n]{a} = 1$ for any $a > 0$.

c) Using the binomial formula
$$\sqrt[n]{n} = 1 + d(n), \quad d(n) > 0, \quad \text{i.e.,}$$
$$n = (1+d(n))^n = 1 + nd(n) + \frac{n(n-1)}{2} d^2(n) + \cdots + d^n(n),$$
we obtain the estimation
$$n > \frac{n(n-1)}{2} d^2(n), \qquad \text{wherefrom we have} \quad d(n) < \sqrt{\frac{2}{n-1}}.$$

For every $\varepsilon > 0$, it holds

$$|\sqrt[n]{n} - 1| = d(n) < \sqrt{\frac{2}{n-1}} < \varepsilon, \text{ provided that } n > \left[\frac{2}{\varepsilon^2} + 1\right].$$

For given ε choose $n_0 = \left[\frac{2}{\varepsilon^2} + 1\right] + 1.$

Example 3.12. *Prove the following limits:*

a) $\lim\limits_{n\to\infty} q^n = \begin{cases} 0, & |q| < 1; \\ 1, & q = 1; \\ +\infty, & q > 1; \end{cases}$

b) $\lim\limits_{n\to\infty} n^b q^n = 0, \quad |q| < 1, \quad b \in \mathbf{R};$

c) $\lim\limits_{n\to\infty} \frac{a^n}{n!} = 0, \quad a \in \mathbf{R};$

d) $\lim\limits_{n\to\infty} \frac{n^a}{n!} = 0, \quad a \in \mathbf{R}.$

Solutions.

a) First of all, let us consider the case when $|q| < 1$. We can write then

$$|q| = \frac{1}{1+h}, \quad h > 0.$$

Then, for given $\varepsilon > 0$, using the Bernoulli inequality, the following estimation holds

$$|q^n - 0| = \frac{1}{(1+h)^n} \le \frac{1}{1+nh} \le \frac{1}{nh} < \varepsilon, \quad \text{for every} \quad n > \frac{1}{h\varepsilon}.$$

So we can take $n_0(\varepsilon) = \left[\frac{1}{h\varepsilon}\right] + 1$. Thus it follows for $|q| < 1$ that $\lim\limits_{n\to\infty} q^n = 0$.

If $q = 1$, then it is obvious that $\lim\limits_{n\to\infty} q^n = 1$.

If $q > 1$, then we have

$$q^n = \frac{1}{r^n}, \quad 0 < r < 1, \quad \text{hence} \quad \lim_{n\to\infty} q^n = +\infty,$$

because $\lim\limits_{n\to\infty} r^n = 0$.

Remark. In the case $q \le -1$, the sequence q^n does not converge. If in particular $q = -1$, then it has two accumulation points 1 and -1.

b) In the case when $b < 0$, the statement is trivial, while for $b = 0$ it reduces to a).

If $b > 0$, then we can choose k, $k \in \mathbf{N}$, such that $k > b$. For $h > 0$ and $n > 2k$ then it holds

$$(1+h)^n > \binom{n}{k} h^k = \frac{n(n-1)\cdots(n-k+1)}{k!} h^k > \left(\frac{n}{2}\right)^k h^k \frac{1}{k!},$$

because

$$n > 2k \Rightarrow n > 2k - 2 \Rightarrow 2n > n + 2k - 2 \Rightarrow n - k + 1 > \frac{n}{2}.$$

If we take $h > 0$, such that $|q| = \dfrac{1}{1+h} < 1$, then from the previous estimations we have

$$|n^b q^n - 0| = n^b |q|^n = \frac{n^b}{(1+h)^n} < \frac{n^b 2^k k!}{n^k h^k} = \frac{2^k k!}{h^k} n^{b-k} < \frac{2^k k!}{h^k} n^{b_1 - k} < \varepsilon,$$

for every

$$n > \left[\sqrt[k-b_1]{\frac{\varepsilon h^k}{2^k k!}} \right] + 1,$$

where $b_1 \in \mathbf{Q}$ such that $b < b_1 < k$.

c) If it holds that $|a| < 1$, then from a) and Example 3.10 d) it follows that $\lim\limits_{n\to\infty} \dfrac{a^n}{n!} = 0$.

If $|a| > 1$, then it holds

$$\left|\frac{a^n}{n!}\right| = \frac{|a^n|}{n!} = \frac{|a|}{1} \cdot \frac{|a|}{2} \cdot \frac{|a|}{3} \cdots \frac{|a|}{k} \cdot \frac{|a|}{k+1} \cdots \frac{|a|}{n},$$

where $k \in \mathbf{N}$ is chosen sufficiently big in order to satisfy the following estimations

$$\frac{|a|}{k} \geq 1 \quad \text{and} \quad \frac{|a|}{k+1} < 1.$$

Therefore we can write

$$0 \leq \left|\frac{a}{n!}\right| < \frac{|a|}{1} \cdot \frac{|a|}{2} \cdot \frac{|a|}{3} \cdots \frac{|a|}{k} \cdot \left(\frac{|a|}{k+1}\right)^{n-k} = M \cdot \left(\frac{|a|}{k+1}\right)^{n-k},$$

where M is a constant. From a) it follows that

$$\lim_{n\to\infty} M \cdot \left(\frac{|a|}{k+1}\right)^{n-k} = 0.$$

From the last two relations and Theorem 3.7 it holds that $\lim\limits_{n\to\infty} \dfrac{a^n}{n!}$ exists and is equal to zero.

d) Let us suppose that $a > 0$ and let the natural numbers k and n satisfy the following inequalities

$$k > a \quad \text{and} \quad n > 2k, \quad \text{hence} \quad n - k + 1 > \frac{n}{2}, \ \ldots, \ n > \frac{n}{2}.$$

Then from the estimations

$$\begin{aligned}\frac{n^a}{n!} &< \frac{n^k}{n!} = \frac{n^k}{(n-k)!\,(n-k+1)(n-k+2)\cdots n} \\ &< \frac{1}{(n-k)!}\cdot\frac{n^k}{\left(\frac{n}{2}\right)^k} = \frac{2^k}{(n-k)!},\end{aligned}$$

it follows that for every $\varepsilon > 0$, there exists a natural number $n_0 = n_0(\varepsilon)$ such that it holds

$$\left|\frac{n^a}{n!}\right| < \frac{2^k}{(n-k)!} < \frac{2^k}{n-k} < \varepsilon,$$

for every $n > n_0$. For example, $n_0 = k + 1 + \left[2^k/\varepsilon\right]$.

Exercise 3.13. *Show by definition that the following sequences converge to zero.*

a) $f_n = \dfrac{(-1)^n}{n^{1995}}$; **b)** $f_n = \dfrac{1995}{\sqrt{n}}$;

c) $f_n = \dfrac{1995^n}{n!}$; **d)** $f_n = \dfrac{n^{1995}}{n!}$.

Exercise 3.14. *Prove the following.*

a) $\lim\limits_{n\to\infty} n^5 = +\infty$; **b)** $\lim\limits_{n\to\infty} \left(\dfrac{n}{3}\right)^2 = +\infty$;

c) $\lim\limits_{n\to\infty} \dfrac{n!}{12^n} = +\infty$; **d)** $\lim\limits_{n\to\infty} \left(\dfrac{20}{n} - n\right) = -\infty$;

e) $\lim\limits_{n\to\infty} n^{1/k} = +\infty,\quad k \in \mathbf{N}$; **f)** $\lim\limits_{n\to\infty} \sqrt[n]{n!} = +\infty$.

Example 3.15. *Determine the following limits.*

a) $\lim\limits_{n\to\infty} \dfrac{2n^5 - 3n^2 + 1}{n^5 + 3n + 2}$; **b)** $\lim\limits_{n\to\infty} \dfrac{3n^4 + 2n^2 + 1}{n^3 + 1}$;

c) $\lim\limits_{n\to\infty} \dfrac{8n^2 + 3n + 1}{n^3 + 1}$; **d)** $\lim\limits_{n\to\infty} \dfrac{(2n+1)^3 - 8n^3}{n^2 + 1}$;

e) $\lim\limits_{n\to\infty} \left(\dfrac{n^2 + 2n + 3}{n+1} - \dfrac{n^3 + 5}{n^2 + 2n + 1}\right)$; **f)** $\lim\limits_{n\to\infty} (\sqrt{n+2} - \sqrt{n})$;

g) $\lim\limits_{n\to\infty} (\sqrt{n^2 + 2n} - n)$; **h)** $\lim\limits_{n\to\infty} \dfrac{\sqrt{n+1}}{\sqrt{n+2} + \sqrt{n+3}}$.

Solutions. We shall repeatedly use Theorem 3.6.

a) $$\lim_{n\to\infty}\frac{2n^5-3n^2+1}{n^5+3n+2}=\lim_{n\to\infty}\frac{n^5\left(2-\dfrac{3}{n^3}+\dfrac{1}{n^5}\right)}{n^5\left(1+\dfrac{3}{n^4}+\dfrac{2}{n^5}\right)}=\frac{\lim\limits_{n\to\infty}\left(2-\dfrac{3}{n^3}+\dfrac{1}{n^5}\right)}{\lim\limits_{n\to\infty}\left(1+\dfrac{3}{n^4}+\dfrac{2}{n^5}\right)}=2.$$

b) $$\lim_{n\to\infty}\frac{3n^4+2n^2+1}{n^3+1}=\lim_{n\to\infty}\frac{n^4\left(3+\dfrac{2}{n^2}+\dfrac{1}{n^4}\right)}{n^3\left(1+\dfrac{1}{n^3}\right)}=+\infty.$$

c) $$\lim_{n\to\infty}\frac{8n^2+3n+1}{n^3+1}=\lim_{n\to\infty}\frac{n^2\left(8+\dfrac{3}{n}+\dfrac{1}{n^2}\right)}{n^3\left(1+\dfrac{1}{n^3}\right)}=\lim_{n\to\infty}\frac{1}{n}\cdot\frac{8+\dfrac{3}{n}+\dfrac{1}{n^2}}{1+\dfrac{1}{n^3}}=0\cdot 8=0.$$

d) $$\lim_{n\to\infty}\frac{(2n+1)^3-8n^3}{n^2+1}=\lim_{n\to\infty}\frac{8n^3+12n^2+6n+1-8n^3}{n^2+1}=12.$$

e) $$\lim_{n\to\infty}\left(\frac{n^2+2n+3}{n+1}-\frac{n^3+5}{n^2+2n+1}\right)=\lim_{n\to\infty}\frac{n^3+3n^2+5n+3-n^3-5}{(n+1)^2}$$
$$=\lim_{n\to\infty}\frac{3n^2+5n-2}{(n+1)^2}=\lim_{n\to\infty}\frac{n^2\left(3+\dfrac{5}{n}-\dfrac{2}{n^2}\right)}{n^2\left(1+\dfrac{1}{n}\right)^2}=3.$$

f) $$\lim_{n\to\infty}(\sqrt{n+2}-\sqrt{n})=\lim_{n\to\infty}\frac{(\sqrt{n+2}-\sqrt{n})(\sqrt{n+2}+\sqrt{n})}{(\sqrt{n+2}+\sqrt{n})}$$
$$=\lim_{n\to\infty}\frac{2}{(\sqrt{n+2}+\sqrt{n})}=0.$$

g) $$\lim_{n\to\infty}(\sqrt{n^2+2n}-n)=\lim_{n\to\infty}\frac{(\sqrt{n^2+2n}-n)(\sqrt{n^2+2n}+n)}{\sqrt{n^2+2n}+n}$$
$$=\lim_{n\to\infty}\frac{2n}{n\left(\sqrt{1+\dfrac{2}{n}}+1\right)}=1.$$

h) $$\lim_{n\to\infty}\frac{\sqrt{n+1}}{\sqrt{n+2}+\sqrt{n+3}}=\lim_{n\to\infty}\frac{\sqrt{n}\cdot\sqrt{1+\dfrac{1}{n}}}{\sqrt{n}\cdot\left(\sqrt{1+\dfrac{2}{n}}+\sqrt{1+\dfrac{3}{n}}\right)}=\frac{1}{2}.$$

Example 3.16. *Determine the limit of each of the following sequences.*

a) $\dfrac{1+2+3+\cdots+n}{n^2}$; **b)** $\dfrac{1^2+2^2+3^2+\cdots+n^2}{n^3}$;

c) $\dfrac{1\cdot 2+2\cdot 3+\cdots+n(n+1)}{n^3}$; **d)** $\dfrac{1+3+5+\cdots+2n-1}{n+1}-\dfrac{2n+1}{2}$;

e) $\dfrac{1}{2}+\dfrac{3}{2^2}+\dfrac{5}{2^3}+\cdots+\dfrac{2n-1}{2^n}$; **f)** $\dfrac{1+a+\cdots+a^n}{1+b+\cdots+b^n}$, $0<|a|,|b|<1$.

Solutions.

a) Using mathematical induction we have (see Example 1.26 a))

$$\lim_{n\to\infty}\frac{1+2+3+\cdots+n}{n^2}=\lim_{n\to\infty}\frac{n(n+1)}{2n^2}=\frac{1}{2}.$$

b) Using mathematical induction we have (see Example 1.26 c))

$$\lim_{n\to\infty}\frac{1^2+2^2+3^2+\cdots+n^2}{n^3}=\lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}=\frac{1}{3}.$$

c) In this case we can write

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1\cdot 2+2\cdot 3+3\cdot 4+\cdots+n(n+1)}{n^3}\\
&=\lim_{n\to\infty}\frac{1(1+1)+2(2+1)+3(3+1)+\cdots+n(n+1)}{n^3}\\
&=\lim_{n\to\infty}\frac{1^2+2^2+3^2+\cdots+n^2+1+2+3+\cdots+n}{n^3}\\
&=\lim_{n\to\infty}\frac{\dfrac{n(n+1)(2n+1)}{6n^3}+\dfrac{n(n+1)}{2}}{n^3}=\frac{1}{3}.
\end{aligned}$$

d) In this case we have (see Example 1.26 b))

$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{1+3+5+\cdots+2n-1}{n+1}-\frac{2n+1}{2}\right)\\
&=\lim_{n\to\infty}\left(\frac{n^2}{n+1}-\frac{2n+1}{2}\right)=\lim_{n\to\infty}\frac{-3n-1}{2(n+1)}=-\frac{3}{2}.
\end{aligned}$$

e) Putting $f_n = \frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + \cdots + \frac{2n-1}{2^n}$, (using Example 1.24) we have

$$\begin{aligned}\frac{f_n}{2} &= f_n - \frac{f_n}{2} = \frac{1}{2} + \left(\frac{3}{2^2} - \frac{1}{2^2}\right) + \cdots + \left(\frac{2n-1}{2^n} - \frac{2n-3}{2^n}\right) - \frac{2n-1}{2^{n+1}} \\ &= \frac{1}{2} + \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}\right) - \frac{2n-1}{2^{n+1}} = \frac{1}{2} + \frac{1}{2} \cdot \frac{1 - \frac{1}{2^{n-1}}}{1 - \frac{1}{2}} - \frac{2n-1}{2^{n+1}}.\end{aligned}$$

Thus we obtain

$$\begin{aligned}\lim_{n\to\infty} f_n &= \lim_{n\to\infty} \left(1 + 2\left(1 - \frac{1}{2^{n-1}}\right) - \frac{2n-1}{2^n}\right) \\ &= 3 - \lim_{n\to\infty} \frac{1}{2^{n-2}} - 2\lim_{n\to\infty} \frac{n}{2^n} + \lim_{n\to\infty} \frac{1}{2^n} = 3.\end{aligned}$$

f) In the numerator and the denominator we have geometric sums and therefore we can write

$$\lim_{n\to\infty} \frac{1 + a + \cdots + a^n}{1 + b + \cdots + b^n} = \lim_{n\to\infty} \frac{\dfrac{1 - a^{n+1}}{1-a}}{\dfrac{1 - b^{n+1}}{1-b}} = \frac{1-b}{1-a} \cdot \lim_{n\to\infty} \frac{1 - a^{n+1}}{1 - b^{n+1}} = \frac{1-b}{1-a}.$$

Exercise 3.17. *Determine the limits of the sequences given below.*

a) $x_n = \sqrt{16 + \frac{1}{n^2}}$;

b) $x_n = \left(16 + \frac{1}{2n}\right)^{-1/3}$;

c) $x_n = \frac{2n+3}{\sqrt{4n^2+3}}$;

d) $x_n = \frac{\sqrt{n^2+1} + \sqrt{n+1}}{\sqrt[3]{n^3+2n+12} + n}$;

e) $x_n = \sqrt{n^2+n} - \sqrt{n^2-n}$;

f) $x_n = \sqrt[3]{n+1} - \sqrt[3]{n-3}$.

Answers.

a) 4. b) $\frac{1}{2\sqrt[3]{2}}$. c) 1. d) 1. e) 1. f) 0.

Exercise 3.18. *Determine the limits of the sequences given below.*

a) $x_n = \dfrac{\sqrt[n]{3}-3}{1+\sqrt[n]{1995}}$; b) $x_n = \dfrac{\sqrt[n]{64}-1}{\sqrt[n]{8}-1}$;

c) $x_n = \dfrac{1}{1-\sqrt[n]{3}} - \dfrac{2}{1-\sqrt[n]{9}}$; d) $x_n = \dfrac{\sqrt[n]{n^5}+\sqrt[n]{5}}{5\sqrt[n]{n^2}+\sqrt[n]{5n}}$;

e) $x_n = \dfrac{n^4+4^n}{n+4^{n+1}}$; f) $x_n = \left(\dfrac{1995}{n}\right)^n$;

g) $x_n = \dfrac{16^n+n!}{4^n+(n+1)!}$; h) $x_n = \dfrac{3^{n^2-2n}+n^3}{(n^3)!}$.

Answers.

a) -1. b) 2. c) $-\dfrac{1}{2}$. d) $\dfrac{1}{3}$.

e) $\dfrac{1}{4}$. f) 0. g) 0. h) 0.

Exercise 3.19. *Determine the limits of the following sequences.*

a) $x_n = \dfrac{1-2+3-\cdots+(2n-1)-2n}{\sqrt{n^2+1}}$;

b) $y_n = \dfrac{1^2+3^2+\cdots+(2n-1)^2}{n^3}$;

c) $z_n = \dfrac{1}{\sqrt{n}}\left(\dfrac{1}{\sqrt{2}+\sqrt{4}} + \dfrac{1}{\sqrt{4}+\sqrt{6}} + \cdots + \dfrac{1}{\sqrt{2n}+\sqrt{2n+2}}\right)$.

Answers.

a) -1. b) $\dfrac{4}{3}$. c) $\dfrac{\sqrt{2}}{2}$.

Example 3.20. *Determine the limit of the following sequences.*

a) $x_n = \dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \cdots + \dfrac{1}{(2n-1)\cdot(2n+1)}$;

b) $y_n = \left(1-\dfrac{1}{2^2}\right)\cdot\left(1-\dfrac{1}{3^2}\right)\cdots\left(1-\dfrac{1}{n^2}\right)$;

c) $z_n = \left(1-\dfrac{1}{3}\right)\cdot\left(1-\dfrac{1}{6}\right)\cdots\left(1-\dfrac{1}{\dfrac{n(n+1)}{2}}\right)$;

d) $w_n = \dfrac{2^3-1}{2^3+1}\cdot\dfrac{3^3-1}{3^3+1}\cdots\dfrac{n^3-1}{n^3+1}$.

Solutions.

a) In this case we have

$$x_n = \frac{1}{2}\cdot\left(1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\frac{1}{5}-\cdots+\frac{1}{2n-1}-\frac{1}{2n+1}\right)=\frac{1}{2}\cdot\left(1-\frac{1}{2n+1}\right),$$

and $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty}\frac{1}{2}\left(1-\frac{1}{2n+1}\right)=\frac{1}{2}.$

b) Using the following transformation $1-\frac{1}{j^2}=\frac{(j-1)(j+1)}{j^2}$, $j=2,3,...,n$, we obtain

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)$$

$$=\lim_{n\to\infty}\left(\frac{1\cdot 3}{2^2}\cdot\frac{2\cdot 4}{3^2}\cdot\frac{3\cdot 5}{4^2}\cdots\frac{(n-1)(n+1)}{n^2}\right)=\lim_{n\to\infty}\frac{1}{2}\cdot\frac{n+1}{n}=\frac{1}{2}.$$

c) Since it holds

$$1-\frac{1}{\frac{j(j+1)}{2}}=\frac{(j-1)(j+2)}{j(j+1)},\quad j=2,3,...,n,$$

we have

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}\left(1-\frac{1}{3}\right)\left(1-\frac{1}{6}\right)\cdots\left(1-\frac{1}{\frac{n(n+1)}{2}}\right)$$

$$=\lim_{n\to\infty}\left(\frac{1\cdot 4}{2\cdot 3}\cdot\frac{2\cdot 5}{3\cdot 4}\cdots\frac{(n-1)(n+2)}{n(n+1)}\right)=\lim_{n\to\infty}\frac{1}{3}\cdot\frac{n+2}{n}=\frac{1}{3}.$$

d) Taking for $j=2,3,...,n$

$$\frac{j^3-1}{j^3+1}=\frac{(j-1)(j^2+j+1)}{(j+1)(j^2-j+1)}=\frac{(j-1)(j^2+j+1)}{(j+1)((j-1)^2+(j-1)+1)},$$

we obtain

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty}\frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdots\frac{n^3-1}{n^3+1}$$

$$=\lim_{n\to\infty}\left(\frac{1\cdot 7}{3\cdot 3}\cdot\frac{2\cdot 13}{4\cdot 7}\cdot\frac{3\cdot 21}{5\cdot 13}\cdot\frac{4\cdot 34}{6\cdot 21}\cdots\frac{(n-1)(n^2+n+1)}{(n+1)((n-1)^2+(n-1)+1)}\right)$$

$$=\lim_{n\to\infty}\frac{2}{3}\cdot\frac{n^2+n+1}{n(n+1)}=\frac{2}{3}.$$

Exercise 3.21. *Determine the limits of the following sequences.*

a) $f_n = \dfrac{a_1}{a_0 S_1} + \dfrac{a_2}{S_1 S_2} + \cdots + \dfrac{a_n}{S_{n-1} S_n}$,

b) $g_n = \ln\left(1 - \dfrac{1}{2^2}\right) + \ln\left(1 - \dfrac{1}{3^2}\right) + \cdots + \ln\left(1 - \dfrac{1}{n^2}\right) = \sum_{j=2}^{n} \ln\left(1 - \dfrac{1}{j^2}\right)$,

where in a) $(a_n)_{n\in\mathbb{N}}$ is an increasing sequence of positive numbers and the sum $S_n := \sum_{k=0}^{n} a_k$ diverges to $+\infty$.

Answers.

a) $\lim_{n\to\infty} f_n = 1/a_0$. **Hint.** Use the equalities

$$\frac{a_1}{a_0 S_1} = \frac{1}{a_0} - \frac{1}{S_1}, \quad \frac{a_2}{S_1 S_2} = \frac{1}{S_1} - \frac{1}{S_2}, \quad \ldots, \quad \frac{a_n}{S_{n-1} S_n} = \frac{1}{S_{n-1}} - \frac{1}{S_n}.$$

b) **Hint.** Use the following transformation

$$\sum_{j=2}^{n} \ln\left(1 - \frac{1}{j^2}\right) = \ln \prod_{j=2}^{n} \left(1 - \frac{1}{j^2}\right).$$

The logarithmic function is a continuous one and therefore we have

$$\lim_{n\to\infty} \sum_{j=2}^{n} \ln\left(1 - \frac{1}{j^2}\right) = \ln \lim_{n\to\infty} \prod_{j=2}^{n} \left(1 - \frac{1}{j^2}\right) = \ln \frac{1}{2}.$$

Example 3.22. *Evaluate*

a) $\lim_{n\to\infty} \left(\dfrac{1}{\sqrt{n^2+1}} + \dfrac{1}{\sqrt{n^2+2}} + \cdots + \dfrac{1}{\sqrt{n^2+n}}\right)$;

b) $\lim_{n\to\infty} \dfrac{a^n}{(1+a)(1+a^2)\cdots(1+a^n)}$, $\quad a > 0$;

c) $\lim_{n\to\infty} \left(3 - \dfrac{1}{n}\right)^{(-1)^n/n}$

Solutions.

a) Using the inequalities

$$\frac{n}{\sqrt{n^2+n}} \le \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}} \le \frac{n}{\sqrt{n^2+1}}$$

and

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = 1 = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}},$$

from Theorem 3.7 we obtain

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}\right) = 1.$$

b) If $0 < a < 1$, we have

$$0 < \frac{a^n}{(1+a)(1+a^2)\cdots(1+a^n)} < a^n,$$

and using Example 3.12 a) and Theorem 3.7 we obtain

$$\lim_{n\to\infty} \frac{a^n}{(1+a)(1+a^2)\cdots(1+a^n)} = 0.$$

If $a > 1$ we have

$$0 < \frac{a^n}{(1+a)(1+a^2)\cdots(1+a^n)} < \frac{a^n}{a\cdot a^2\cdots a^n} = \frac{a^n}{a^{n(n+1)/2}},$$

thus

$$\lim_{n\to\infty} \frac{a^n}{(1+a)(1+a^2)\cdots(1+a^n)} = 0.$$

If $a = 1$, then

$$\lim_{n\to\infty} \frac{1}{\underbrace{2\cdot 2\cdots 2}_{n-\text{terms}}} \cdot \lim_{n\to\infty} \frac{1}{2^n} = 0.$$

Therefore we can write for all $a > 0$

$$\lim_{n\to\infty} \frac{a^n}{(1+a)(1+a^2)\cdots(1+a^n)} = 0.$$

c) Denoting the general term of the given sequence by h_n, we have

$$h_n = \begin{cases} \left(3 - \dfrac{1}{n}\right)^{-1/n}, & n = 1, 3, 5, \ldots; \\[2ex] \left(3 - \dfrac{1}{n}\right)^{1/n}, & n = 2, 4, 6, \ldots. \end{cases}$$

Thus we obtain the inequality

$$3^{-1/n} < h_n < 3^{1/n}.$$

Since by Example 3.11 b) it holds

$$\lim_{n\to\infty} 3^{-1/n} = \lim_{n\to\infty} 3^{1/n} = 1,$$

we have $\lim\limits_{n\to\infty} h_n = 1.$

Example 3.23. *Prove the implication*

$$\lim_{n\to\infty} a_n = a \Rightarrow \lim_{n\to\infty} |a_n| = |a|.$$

Is the opposite statement true?

Solution. From Definition 3.1,

$$(\forall \varepsilon > 0)\,(\exists n_0 \in \mathbf{N})\,(\forall n \in \mathbf{N}) \quad n > n_0 \;\Rightarrow\; |a_n - a| < \varepsilon,$$

it follows

$$(\forall \varepsilon > 0)\,(\exists n_0 \in \mathbf{N})\,(\forall n \in \mathbf{N})\; n > n_0 \;\Rightarrow\; \big||a_n| - |a|\big| \leq |a_n - a| < \varepsilon,$$

and this means that $\lim\limits_{n\to\infty} |a_n| = |a|$.

The opposite is not always true. For instance, the sequence given by

$$a_n = \frac{(n+1)(-1)^n}{n}, \quad n \in \mathbf{N},$$

does not converge, though the sequence $(|a_n|)_{n\in\mathbf{N}}$ does converge to 1.

Exercise 3.24. *Examine whether the following implications are true.*

$$\textbf{a)}\quad \lim_{n\to\infty} f_n^2 = f^2 \Rightarrow \lim_{n\to\infty} f_n = f;$$

$$\textbf{b)}\quad \lim_{n\to\infty} f_n^3 = f^3 \Rightarrow \lim_{n\to\infty} f_n = f.$$

Answers.

a) No. For example, the sequence $f_n = (-1)^n$, $n \in \mathbf{N}$, does not converge, but the stationary sequence $f_n^2 = ((-1)^n)^2 = 1$, $n \in \mathbf{N}$, converges to 1.

b) Yes.

Exercise 3.25. *Show that there exist sequences $(a_n)_{n\in\mathbf{N}}$ and $(b_n)_{n\in\mathbf{N}}$ such that $\lim\limits_{n\to\infty} a_n = +\infty$ and $\lim\limits_{n\to\infty} b_n = +\infty$, but*

a) *$(a_n + b_n)_{n\in\mathbf{N}}$ converges;* **b)** $\lim\limits_{n\to\infty}(a_n + b_n) = -\infty$; **c)** $\lim\limits_{n\to\infty}(a_n + b_n) = +\infty$;

d) *the sequence $(a_n + b_n)_{n\in\mathbf{N}}$ neither converges nor diverges to $+\infty$ or to $-\infty$.*

Answers. For example, the following sequences can be used.

a) $a_n = \dfrac{1}{n} + n, \quad b_n = \dfrac{1}{n} - n; \qquad \lim\limits_{n\to\infty}(a_n + b_n) = \lim\limits_{n\to\infty} \dfrac{2}{n} = 0.$

b) $a_n = \dfrac{1}{n} + n, \quad b_n = \dfrac{1}{n} - 2n; \qquad \lim\limits_{n\to\infty}(a_n + b_n) = \lim\limits_{n\to\infty} \left(\dfrac{2}{n} - n\right) = -\infty.$

c) $a_n = \dfrac{1}{n} + 2n, \quad b_n = \dfrac{1}{n} - n; \qquad \lim\limits_{n\to\infty}(a_n + b_n) = \lim\limits_{n\to\infty} \left(\dfrac{2}{n} + n\right) = +\infty.$

d) $a_n = (-1)^n + n$, $\quad b_n = -n$; the sequence $(a_n + b_n)_{n\in\mathbf{N}}$ neither converges nor diverges to $+\infty$ or to $-\infty$.

Exercise 3.26. *Show that there exist sequences $(a_n)_{n\in\mathbf{N}}$ and $(b_n)_{n\in\mathbf{N}}$ such that $\lim\limits_{n\to\infty} a_n = +\infty$ and $\lim\limits_{n\to\infty} b_n = +\infty$, but*

a) $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$; **b)** $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = A,\ A > 0$; **c)** $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = +\infty$;

d) *the sequence* $\left(\frac{a_n}{b_n}\right)_{n\in\mathbf{N}}$ *neither converges nor diverges to* $+\infty$ *or to* $-\infty$.

Answers. For example, the following sequences can be used.

a) $a_n = n,\quad b_n = n^2$; $\quad \lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{1}{n} = 0.$

b) $a_n = 2n,\quad b_n = n$; $\quad \lim\limits_{n\to\infty} \frac{a_n}{b_n} = 2.$

c) $a_n = n^2,\quad b_n = n$; $\quad \lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} n = +\infty.$

d) $a_n = (3 + (-1)^n)\, n,\quad b_n = n$; the sequence $\left(\frac{a_n}{b_n}\right)_{n\in\mathbf{N}}$ neither converges nor diverges to $+\infty$ or to $-\infty$.

Exercise 3.27. *Show that there exist sequences* $(a_n)_{n\in\mathbf{N}}$ *and* $(b_n)_{n\in\mathbf{N}}$ *such that* $\lim\limits_{n\to\infty} a_n = 0$ *and* $\lim\limits_{n\to\infty} b_n = +\infty$, *but*

a) $\lim\limits_{n\to\infty} a_n \cdot b_n = 0$; **b)** $\lim\limits_{n\to\infty} a_n \cdot b_n = A,\ A \neq 0$; **c)** $\lim\limits_{n\to\infty} a_n \cdot b_n = +\infty$;

d) *the sequence* $(a_n \cdot b_n)_{n\in\mathbf{N}}$ *neither converges nor diverges to* ∞ *or to* $-\infty$.

Answers. For example, the following sequences can be used.

a) $a_n = \frac{1}{n^2},\quad b_n = n$; $\quad \lim\limits_{n\to\infty} (a_n \cdot b_n) = \lim\limits_{n\to\infty} \frac{1}{n} = 0.$

b) $a_n = \frac{1}{n},\quad b_n = 3n$; $\quad \lim\limits_{n\to\infty} (a_n \cdot b_n) = 3.$

c) $a_n = \frac{1}{n},\quad b_n = n^2$; $\quad \lim\limits_{n\to\infty} (a_n \cdot b_n) = +\infty.$

d) $a_n = \frac{(-1)^n}{n^2},\quad b_n = n^2$; clearly, the sequence $(a_n \cdot b_n)_{n\in\mathbf{N}}$ neither converges nor diverges to $+\infty$ or to $-\infty$.

Exercise 3.28.

- *Assume that the sequences* $(a_n)_{n\in\mathbf{N}}$ *and* $(b_n)_{n\in\mathbf{N}}$ *satisfy either* $\lim\limits_{n\to\infty} a_n = +\infty$ *and* $\lim\limits_{n\to\infty} b_n = +\infty$, *or* $\lim\limits_{n\to\infty} a_n = -\infty$ *and* $\lim\limits_{n\to\infty} b_n = -\infty$. *Prove then*

$$\lim_{n\to\infty} a_n b_n = +\infty.$$

- *Assume that the sequences* $(a_n)_{n\in\mathbf{N}}$ *and* $(b_n)_{n\in\mathbf{N}}$ *satisfy either* $\lim\limits_{n\to\infty} a_n = -\infty$ *and* $\lim\limits_{n\to\infty} b_n = +\infty$, *or* $\lim\limits_{n\to\infty} a_n = +\infty$ *and* $\lim\limits_{n\to\infty} b_n = -\infty$. *Prove then*
$$\lim_{n\to\infty} a_n b_n = -\infty.$$

Exercise 3.29. *Show that there exist sequences* $(a_n)_{n\in\mathbf{N}}$ *and* $(b_n)_{n\in\mathbf{N}}$ *such that* $\lim\limits_{n\to\infty} a_n = 0$ *and* $\lim\limits_{n\to\infty} b_n = 0$, *but*

a) $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$; **b)** $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 1$; **c)** $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = +\infty$;

d) *the sequence* $\left(\dfrac{a_n}{b_n}\right)_{n\in\mathbf{N}}$ *neither converges nor diverges to* $+\infty$ *or to* $-\infty$.

Example 3.30. *Examine whether the following are Cauchy sequences.*

a) $a_n = 1 + \dfrac{1}{4} + \dfrac{1}{9} + \cdots + \dfrac{1}{n^2}$;

b) $b_n = \dfrac{\sin 1}{2} + \dfrac{\sin 2}{2^2} + \cdots + \dfrac{\sin n}{2^n}$;

c) $c_n = \dfrac{\cos 1!}{1\cdot 2} + \dfrac{\cos 2!}{2\cdot 3} + \cdots + \dfrac{\cos n!}{n\cdot(n+1)}$;

d) $d_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n}$ (*the* **harmonic sequence**) ;

e) $e_n = 1 + \dfrac{1}{\ln 2} + \dfrac{1}{\ln 3} + \cdots + \dfrac{1}{\ln n}$.

Solutions.

a) Using Definition 3.8, we shall show that for every $\varepsilon > 0$, we can find $n_0 = n_0(\varepsilon)$ such that for every $n \geq n_0$ and for every $p \in \mathbf{N}$ it holds $|a_{n+p} - a_n| < \varepsilon$. First we have

$$|a_{n+p} - a_n| = \left|1 + \frac{1}{4} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(n+p)^2} - 1 - \frac{1}{4} - \cdots - \frac{1}{n^2}\right|$$

$$= \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+p)^2}$$

$$< \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots + \frac{1}{(n+p-1)(n+p)}$$

$$= \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+2} - \frac{1}{n+3} + \cdots + \frac{1}{n+p-1} - \frac{1}{n+p}$$

$$= \frac{1}{n} - \frac{1}{n+p} < \frac{1}{n}.$$

Then from the inequalities

$$|a_{n+p} - a_n| < \frac{1}{n} < \varepsilon, \quad \text{for every} \quad n > [1/\varepsilon] + 1,$$

it follows that we can take $n_0 = n_0(\varepsilon) = [1/\varepsilon] + 1$. The sequence $(a_n)_{n\in\mathbf{N}}$ is Cauchy's, hence from Theorem 3.9 it follows that it is a convergent sequence.

b) For given $\varepsilon > 0$ and arbitrary $p \in \mathbf{N}$, we have

$$\begin{aligned} |b_{n+p} - b_n| &= \left|\frac{\sin(n+1)}{2^{n+1}} + \cdots + \frac{\sin(n+p)}{2^{n+p}}\right| \\ &\leq \frac{|\sin(n+1)|}{2^{n+1}} + \cdots + \frac{|\sin(n+p)|}{2^{n+p}} \\ &\leq \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+p}} \leq \frac{1}{2^{n+1}} \frac{1}{1 - \frac{1}{2}} = \frac{1}{2^n} < \varepsilon \end{aligned}$$

for every $n > \left[\frac{-\ln\varepsilon}{\ln 2}\right] + 1$. This means that the sequence $(b_n)_{n\in\mathbf{N}}$ is Cauchy's.

c) The sequence $(c_n)_{n\in\mathbf{N}}$ is Cauchy's, because for given $\varepsilon > 0$ it holds

$$\begin{aligned} |c_{n+p} - c_n| &= \left|\frac{\cos(n+1)!}{(n+1)(n+2)} + \frac{\cos(n+2)!}{(n+2)(n+3)} + \cdots + \frac{\cos(n+p)!}{(n+p)(n+p+1)}\right| \\ &\leq \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)} + \cdots + \frac{1}{(n+p)(n+p+1)} \\ &= \frac{1}{(n+1)} - \frac{1}{(n+p+1)} < \frac{1}{(n+1)} < \varepsilon, \end{aligned}$$

provided that $n > [1/\varepsilon] + 1$.

d) In this case we shall show that the sequence $(d_n)_{n\in\mathbf{N}}$ is not Cauchy's. This means that there exists $\varepsilon > 0$, such that for every n_0, there exist $n > n_0$ and $p \in \mathbf{N}$ satisfying $|d_{n+p} - d_n| > \varepsilon$.

Let us take $\varepsilon = 1/4$. From

$$|d_{n+p} - d_n| = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p} > \frac{p}{n+p},$$

it follows that for $n = p$ we have

$$|d_{n+p} - d_n| = |d_{2n} - d_n| \geq \frac{1}{2} > \frac{1}{4}, \quad \text{for every} \quad n \in \mathbf{N}.$$

So the sequence $(d_n)_{n\in\mathbf{N}}$ is not Cauchy's and does not converge.

e) The sequence $(e_n)_{n\in\mathbf{N}}$ diverges, because from the inequality $\ln x < x$, which holds for $x > 1$, it follows that for $\varepsilon = 1/3$ it holds

$$|e_{n+p} - e_n| = \frac{1}{\ln(n+1)} + \frac{1}{\ln(n+2)} + \cdots + \frac{1}{\ln(n+p)} > \frac{p}{\ln(n+p)} > \frac{p}{n+p}.$$

For $p = n$ we obtain

$$|e_{n+p} - e_n| > \frac{1}{2} > \frac{1}{3}.$$

Therefore the sequence $(e_n)_{n\in\mathbf{N}}$ is not Cauchy's, hence it diverges.

3.2 Monotone sequences

3.2.1 Basic notions

Definition 3.31. *A sequence $(a_n)_{n\in\mathbf{N}}$ is*

- **monotonically increasing** *(resp.* **monotonically nondecreasing***) if for every $n \in \mathbf{N}$ it holds $a_n < a_{n+1}$ (resp. $a_n \leq a_{n+1}$);*
- **monotonically decreasing** *(resp.* **monotonically nonincreasing***) if for every $n \in \mathbf{N}$ it holds $a_n > a_{n+1}$ (resp. $a_n \geq a_{n+1}$).*

Theorem 3.32. *A monotonically nondecreasing (resp. nonincreasing) sequence bounded from above (resp. from below) converges.*

3.2.2 Examples and exercises

Example 3.33. *Show that the sequence*

$$a_n = \left(1 + \frac{1}{n}\right)^n, \tag{3.4}$$

a) *is monotonically increasing;*

b) *is bounded from above with the number 3.*

Solution.

a) Let us show that the sequence given by relation (3.4) is monotonically increasing in two ways.

• **First method.** We shall consider the quotient $\dfrac{a_n}{a_{n+1}}$ and show that it is less than 1.

$$\frac{a_n}{a_{n+1}} = \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n+1}\right)^{n+1}} = \left(\frac{\frac{n+1}{n}}{\frac{n+2}{n+1}}\right)^n \cdot \frac{1}{\frac{n+2}{n+1}} = \frac{1}{\left(\frac{n(n+2)}{(n+1)^2}\right)^n} \cdot \frac{n+1}{n+2}.$$

Using the Bernoulli inequality

$$(1+h)^n \geq 1+nh, \quad h>-1,$$

it holds

$$\left(\frac{n(n+2)}{(n+1)^2}\right)^n = \left(1-\frac{1}{(n+1)^2}\right)^n \geq 1-\frac{n}{(n+1)^2}$$

and

$$\frac{a_n}{a_{n+1}} \leq \frac{1}{1-\dfrac{n}{(n+1)^2}} \cdot \frac{n+1}{n+2} = \frac{n^3+3n^2+3n+1}{n^3+3n^2+3n+2} < 1.$$

• **Second method.** Next we consider the difference $a_{n+1}-a_n$ and show that it is positive. We have

$$a_{n+1}-a_n = \left(1+\frac{1}{n+1}\right)^{n+1} - \left(1+\frac{1}{n}\right)^n$$

$$= \sum_{j=0}^{n}\left(\binom{n+1}{j}\frac{1}{(n+1)^j} - \binom{n}{j}\frac{1}{n^j}\right) + \frac{1}{(n+1)^{n+1}}$$

$$= \sum_{j=1}^{n}\left(\frac{(n+1)n\cdots(n-j+2)}{j!(n+1)^j} - \frac{n(n-1)\cdots(n-j+1)}{j!\,(n)^j}\right) + \frac{1}{(n+1)^{n+1}}$$

$$= \sum_{j=1}^{n}\frac{1}{j!}\left\{1\cdot\left(1-\frac{1}{n+1}\right)\cdot\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{j-1}{n+1}\right)\right.$$

$$\left. - \left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{j-1}{n}\right)\right\} + \frac{1}{(n+1)^{n+1}} > 0,$$

because $1-\dfrac{j}{n+1} > 1-\dfrac{j}{n}$ and $\dfrac{1}{(n+1)^{n+1}} > 0, \quad j,n \in \mathbf{N}$.

b) Using the binomial formula, we have

$$\left(1+\frac{1}{n}\right)^n = 1+1+\frac{n(n-1)}{2!\,n^2}+\cdots+\frac{n(n-1)(n-2)\cdots(n-(n-1))}{n!\,n^n}.$$

From the following inequalities

$$\frac{n(n-1)}{2n^2} = \frac{1}{2}\left(1-\frac{1}{n}\right) < \frac{1}{2},$$

$$\frac{n(n-1)(n-2)}{3!\cdot n^3} = \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) < \frac{1}{3!},$$

$$\cdots\cdots$$

$$\frac{n(n-1)\cdots(n-(n-1))}{n!\cdot n^n} = \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right) < \frac{1}{n!},$$

we obtain

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &< 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} < 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}\\ &= 1+\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} < 1+2=3.\end{aligned}$$

Note that since $(a_n)_{n\in\mathbf{N}}$ is monotonically increasing, it also holds

$$a_n = \left(1+\frac{1}{n}\right)^n \geq \left(1+\frac{1}{1}\right)^1 = 2, \quad n \in \mathbf{N}.$$

Remark. From Theorem 3.32 it follows that the given sequence converges. Its limit is the irrational number e (see Example 3.36).

Example 3.34. *Determine the following limits.*

a) $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{3n}$;

b) $\lim\limits_{n\to\infty}\left(1-\frac{1}{n}\right)^{n}$;

c) $\lim\limits_{n\to\infty}\left(\frac{2n+3}{2n}\right)^{3n+2}$;

d) $\lim\limits_{n\to\infty}\left(\frac{n+1}{n-1}\right)^{n}$;

e) $\lim\limits_{n\to\infty}\left(\frac{n^2-1}{n^2+1}\right)^{n^2}$;

f) $\lim\limits_{n\to\infty}\left(\frac{n-1}{n+1}\right)^{n^2}$;

g) $\lim\limits_{n\to\infty}\frac{\ln\sqrt{n+1}-\ln\sqrt{n}}{n}$;

h) $\lim\limits_{n\to\infty} n\cdot(\ln\sqrt{n+1}-\ln\sqrt{n})$.

Solutions.

a) $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{3n} = \left(\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\right)^3 = \mathrm{e}^3.$

b)
$$\begin{aligned}\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n} &= \lim_{n\to\infty}\frac{1}{\left(\frac{n}{n-1}\right)^n} = \frac{1}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n-1}\right)^{n-1}}\cdot\lim_{n\to\infty}\frac{1}{1+\frac{1}{n-1}}\\ &= \frac{1}{\mathrm{e}}\cdot 1 = \frac{1}{\mathrm{e}}.\end{aligned}$$

c)
$$\begin{aligned}\lim_{n\to\infty}\left(\frac{2n+3}{2n}\right)^{3n+2} &= \lim_{n\to\infty}\left(1+\frac{3}{2n}\right)^{3n}\cdot\lim_{n\to\infty}\left(1+\frac{3}{2n}\right)^2 = \lim_{n\to\infty}\left(1+\frac{1}{\frac{2n}{3}}\right)^{\frac{2n}{3}\cdot\frac{9}{2}}\\ &= \left(\lim_{n\to\infty}\left(1+\frac{1}{\frac{2n}{3}}\right)^{2n/3}\right)^{9/2}\cdot 1 = \mathrm{e}^{9/2}.\end{aligned}$$

d) $$\lim_{n\to\infty}\left(\frac{n+1}{n-1}\right)^n = \lim_{n\to\infty}\left(\frac{n\left(1+\dfrac{1}{n}\right)}{n\left(1-\dfrac{1}{n}\right)}\right)^n = \lim_{n\to\infty}\frac{\left(1+\dfrac{1}{n}\right)^n}{\left(1-\dfrac{1}{n}\right)^n}$$
$$= \frac{\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n}{\lim\limits_{n\to\infty}\left(1-\dfrac{1}{n}\right)^{(-n)(-1)}} = \mathrm{e}^2.$$

The last example can be done also as follows.

$$\lim_{n\to\infty}\left(\frac{n+1}{n-1}\right)^n = \lim_{n\to\infty}\left(1+\frac{2}{n-1}\right)^{n-1}\cdot\lim_{n\to\infty}\left(1+\frac{2}{n-1}\right)$$
$$= \lim_{n\to\infty}\left(1+\frac{2}{n-1}\right)^{2(n-1)/2} = \left(\lim_{n\to\infty}\left(1+\frac{2}{n-1}\right)^{(n-1)/2}\right)^2 = \mathrm{e}^2.$$

e) $$\lim_{n\to\infty}\left(\frac{n^2-1}{n^2+1}\right)^{n^2} = \lim_{n\to\infty}\left(\frac{\left(1-\dfrac{1}{n^2}\right)^{n^2}}{\left(1+\dfrac{1}{n^2}\right)^{n^2}}\right) = \mathrm{e}^{-2}.$$

f) $$\lim_{n\to\infty}\left(\frac{n-1}{n+1}\right)^{n^2} = \lim_{n\to\infty}\left(\frac{\left(1-\dfrac{1}{n}\right)^n}{\left(1+\dfrac{1}{n}\right)^n}\right)^n = \lim_{n\to\infty}(e^{-2})^n = 0.$$

g) $$\lim_{n\to\infty}\frac{\ln\sqrt{n+1}-\ln\sqrt{n}}{n} = \lim_{n\to\infty}\frac{1}{2}\ln\left(\frac{n+1}{n}\right)^{1/n} = \frac{1}{2}\ln\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{1/n}$$
$$= \frac{1}{2}\ln 1 = 0.$$

h) $$\lim_{n\to\infty} n\cdot(\ln\sqrt{n+1}-\ln\sqrt{n}) = \lim_{n\to\infty}\frac{1}{2}\ln\left(\frac{n+1}{n}\right)^n = 1/2.$$

Example 3.35. *Prove the following estimations.*

$$0 < \mathrm{e} - \sum_{j=0}^{k}\frac{1}{j!} < \frac{1}{k\cdot k!},\quad k\in\mathbf{N}. \tag{3.5}$$

Solution. First we can write

$$\left(1+\frac{1}{n}\right)^n = \sum_{j=0}^{k}\binom{n}{j}\frac{1}{n^j} + \sum_{j=k+1}^{n}\binom{n}{j}\frac{1}{n^j},\quad 0<k<n. \tag{3.6}$$

The second addend can be estimated as follows.

$$\sum_{j=k+1}^{n}\binom{n}{j}\frac{1}{n^j} = \sum_{j=k+1}^{n}\frac{n(n-1)(n-2)\cdots(n-j+1)}{j!}\cdot\frac{1}{n^j}$$
$$= \sum_{j=k+1}^{n}\frac{1}{j!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{j-1}{n}\right).$$

The last expression is less or equal than

$$\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\cdot$$

$$\cdot\ \left\{\frac{1}{k+1}\left(1-\frac{k}{n}\right)+\frac{1}{(k+1)(k+2)}\left(1-\frac{k}{n}\right)\left(1-\frac{k+1}{n}\right)+\cdots\right.$$

$$\left.+\ \frac{1}{(k+1)(k+2)\cdots(k+n)}\cdot\left(1-\frac{k}{n}\right)\left(1-\frac{k+1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)\right\}.$$

Thus we obtain

$$\sum_{j=k+1}^{n}\binom{n}{j}\frac{1}{n^j}=\sum_{j=k+1}^{n}\frac{n(n-1)(n-2)\cdots(n-j+1)}{j!}\cdot\frac{1}{n^j}$$

$$<\ \frac{1}{k!}\left(\frac{1}{k+1}+\frac{1}{(k+1)^2}+\cdots+\frac{1}{(k+1)^{n-k}}\right)<\frac{1}{k!}\cdot\frac{1}{k+1}\cdot\frac{1}{1-\dfrac{1}{k+1}}.$$

So it holds

$$\sum_{j=k+1}^{n}\binom{n}{j}\frac{1}{n^j}<\frac{1}{k\cdot k!}. \tag{3.7}$$

From relations (3.6) and (3.7) we obtain for $0<k<n$:

$$0<\left(1+\frac{1}{n}\right)^n-\sum_{j=0}^{k}\binom{n}{j}\frac{1}{n^j}=\sum_{j=k+1}^{n}\binom{n}{j}\frac{1}{n^j}<\frac{1}{k\cdot k!}. \tag{3.8}$$

The last estimations imply

$$0\le\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n-\lim_{n\to\infty}\sum_{j=0}^{k}\binom{n}{j}\frac{1}{n^j}\le\frac{1}{k\cdot k!}.$$

In order to obtain relation (3.5), let us show

$$\lim_{n\to\infty}\sum_{j=0}^{k}\binom{n}{j}\frac{1}{n^j}=\sum_{j=0}^{k}\frac{1}{j!}. \tag{3.9}$$

From the relations

$$\begin{aligned}\binom{n}{j}\frac{1}{n^j}&=\frac{n(n-1)(n-2)\cdots(n-j+1)}{j!\,n^j}\\&=\frac{1}{j!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{j-1}{n}\right),\end{aligned}$$

it follows that for fixed $j\in\mathbf{N}$ it holds

$$\lim_{n\to\infty}\binom{n}{j}\frac{1}{n^j}=\frac{1}{j!},$$

wherefrom we obtain relation (3.9).

Remark. From the estimations given by relation (3.5), it follows that

$$\lim_{n\to\infty}\sum_{j=0}^{n}\frac{1}{j!}=\lim_{n\to\infty}\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)=\mathrm{e}.$$

Example 3.36. *Prove that the real number* e *is not rational.*

Solution. Let us suppose the opposite, i.e. that e is a rational number. Then it can be written as $\mathrm{e}=\dfrac{p}{q}$, where $p,q\in\mathbf{N}$, $2<\dfrac{p}{q}<3$ and $\operatorname{lcd}(p,q)=1$. As usual, $\operatorname{lcd}(p,q)$ stands for the *largest common divisor* of the numbers p and q. If in relation (3.5) we put $k=q$, then we obtain

$$0<\mathrm{e}-\left(1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{q!}\right)<\frac{1}{q\cdot q!},$$

or

$$0<q\left[\mathrm{e}\,q!-\left(q!+q!+\frac{q!}{2!}+\frac{q!}{3!}+\cdots+\frac{q!}{q!}\right)\right]<1.$$

Taking $\mathrm{e}=\dfrac{p}{q}$, we have

$$0<q\cdot\left[p(q-1)!-\left(q!+q!+\frac{q!}{2!}+\frac{q!}{3!}+\cdots+\frac{q!}{q!}\right)\right]<1. \tag{3.10}$$

It is obvious that the numbers

$$p(q-1)!,\quad q!\quad\text{and}\quad q!+q!+\frac{q!}{2!}+\frac{q!}{3!}+\cdots+\frac{q!}{q!}$$

are integers and therefore the expression

$$p(q-1)!-\left(q!+q!+\frac{q!}{2!}+\frac{q!}{3!}+\cdots+\frac{q!}{q!}\right)$$

represents an integer. The supposition $q>1$ is in contradiction with relation (3.10).

Remark. $\mathrm{e}\approx 2.718281828459045$.

Example 3.37. *Prove that* $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt[n]{n!}}=0.$

Solution. Using mathematical induction, let us show first

$$n!>\left(\frac{n}{3}\right)^{n},\quad n\in\mathbf{N}. \tag{3.11}$$

For $n = 1$ the inequality is correct. Let us suppose that it is correct for $n = k$, i.e., $k! > \left(\frac{k}{3}\right)^k$. For $n = k+1$ we have

$$(k+1)! = k!(k+1) > \left(\frac{k}{3}\right)^k (k+1) = \left(\frac{k+1}{3}\right)^{k+1} \cdot \frac{3}{\left(1+\frac{1}{k}\right)^k}. \tag{3.12}$$

In Example 3.33 it is shown that $\left(1+\frac{1}{k}\right)^k < 3$, so from relation (3.12) it follows that

$$(k+1)! > \left(\frac{k+1}{3}\right)^{k+1}.$$

So the relation (3.11) is correct. Therefore for given $\varepsilon > 0$ one can find n_0 such that it holds

$$\left|\frac{1}{\sqrt[n]{n!}}\right| = \frac{1}{\sqrt[n]{n!}} < \frac{1}{\sqrt[n]{\left(\frac{n}{3}\right)^n}} = \frac{3}{n} < \varepsilon,$$

for every $n > n_0(\varepsilon) = [3/\varepsilon] + 1$.

Example 3.38. *Let us consider the sequence given by*

$$a_n = \left(1+\frac{1}{n}\right)^{n+1}, \quad n \in \mathbf{N}.$$

a) *Show that the sequence is monotonically decreasing.*

b) *Determine* $\lim\limits_{n\to\infty} a_n$.

c) *Prove the inequality* $\frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n}$.

d) *Prove the equality*

$$\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{2n}\right) = \ln 2.$$

e) *Prove the equality* $\lim\limits_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{2n}+\cdots+\frac{1}{kn}\right) = \ln k, \quad k \in \mathbf{N}.$

Solutions.

a) Using the Bernoulli inequality we have

$$\begin{aligned}\frac{a_n}{a_{n+1}} &= \frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)^{n+2}} = \left(1+\frac{1}{n(n+2)}\right)^{n+1} \cdot \frac{n+1}{n+2} \\ &> \left(1+\frac{n+1}{n(n+2)}\right) \cdot \frac{n+1}{n+2} = \frac{n^3+4n^2+4n+1}{n^3+4n^2+4n} > 1.\end{aligned}$$

This means that $a_n > a_{n+1}$, so the sequence $(a_n)_{n\in\mathbf{N}}$ is monotonically decreasing.

b) The sequence $(a_n)_{n\in\mathbf{N}}$ is monotonically decreasing and bounded from below (for example with zero), and therefore from Theorem 3.32 it follows that this is a convergent sequence and its limit can be ordered as

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\cdot\left(1+\frac{1}{n}\right) = \mathrm{e}\,.$$

c) From a), b) and Example 3.33 it follows

$$\left(1+\frac{1}{n}\right)^{n} < \mathrm{e} < \left(1+\frac{1}{n}\right)^{n+1}.$$

Since a logarithmic function is monotonically increasing, it holds

$$n\ln\left(1+\frac{1}{n}\right) < 1 < (n+1)\ln\left(1+\frac{1}{n}\right),$$

wherefrom we have

$$\ln\left(1+\frac{1}{n}\right) < \frac{1}{n} \quad\text{and}\quad \ln\left(1+\frac{1}{n}\right) > \frac{1}{n+1}.$$

So we can write

$$\frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n}.$$

d) Using the inequality proved in c), we obtain the following inequalities for $n > 1$.

$$\begin{aligned}
\ln\left(1+\frac{1}{n}\right) &< \frac{1}{n} < \ln\left(1+\frac{1}{n-1}\right),\\
\ln\left(1+\frac{1}{n+1}\right) &< \frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right);\\
&\cdots\cdots\\
\ln\left(\frac{2n+1}{2n}\right) &< \frac{1}{2n} < \ln\left(\frac{2n}{2n-1}\right).
\end{aligned}$$

Summing the previous inequalities, we get

$$\sum_{k=n}^{2n}\ln\left(1+\frac{1}{k}\right) < \sum_{k=n}^{2n}\frac{1}{k} < \sum_{k=n}^{2n}\ln\left(1+\frac{1}{k-1}\right).$$

Using the transformations

$$\sum_{k=n}^{2n}\ln\left(1+\frac{1}{k}\right) = \ln\prod_{k=n}^{2n}\left(1+\frac{1}{k}\right) = \ln\left(\frac{n+1}{n}\cdot\frac{n+2}{n+1}\cdots\frac{2n+1}{2n}\right),$$

we obtain

$$\ln\left(\frac{2n+1}{n}\right) < \sum_{k=n}^{2n}\frac{1}{k} < \ln\left(\frac{2n}{n-1}\right).$$

The logarithmic function is continuous on the interval $(0,+\infty)$ and therefore we have

$$\lim_{n\to\infty}\ln\left(\frac{2n+1}{n}\right) = \lim_{n\to\infty}\ln\left(\frac{2n}{n-1}\right) = \ln 2.$$

Using Theorem 3.7 we obtain finally

$$\lim_{n\to\infty}\sum_{k=n}^{2n}\frac{1}{k} = \ln 2.$$

e) The proof is similar to the one in d), so we omit it.

Example 3.39. *Prove that the sequence*

$$a_n = \sum_{k=1}^{n}\frac{1}{k} - \ln n$$

is monotonically decreasing and bounded from below.

Solution. From Example 3.38 c) it follows

$$a_n - a_{n+1} = \ln\left(1+\frac{1}{n}\right) - \frac{1}{n+1} > 0,$$

implying that $a_n > a_{n+1}$, i.e., $(a_n)_{n\in\mathbf{N}}$ is a monotonically decreasing sequence.
The sequence is bounded from below, because (see Example 3.38 b))

$$\begin{aligned} a_n &= \sum_{k=1}^{n}\frac{1}{k} - \ln n > \sum_{k=1}^{n}\ln\left(1+\frac{1}{k}\right) - \ln n \\ &= \ln\left(2\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{n+1}{n}\cdot\frac{1}{n}\right) = \ln\frac{n+1}{n} > 0\,. \end{aligned}$$

From Theorem 3.32 it follows that the sequence $(a_n)_{n\in\mathbf{N}}$ converges and has a limit $\lim\limits_{n\to\infty} a_n =: \gamma$. The limit γ is called **Euler's constant**.

Example 3.40. *Prove the following two inequalities.*

a) $\left(\dfrac{n}{\mathrm{e}}\right)^n < n! < \mathrm{e}\left(\dfrac{n}{2}\right)^n, \quad n\in\mathbf{N};$

b) $\dfrac{r}{1+r} < \ln(1+r) < r, \quad r\in\mathbf{Q}, \quad r>-1.$

Solutions.

a) The proof of the left-hand side inequality is essentially the same to the proof of relation (3.11) in Example 3.37. So let us prove the right-hand side inequality. To that end, using the mathematical induction and the fact that $\left(1+\frac{1}{n}\right)^n > 2$ for $n > 1$, it follows

$$n! < \left(\frac{n+1}{2}\right)^n .$$

Further on we have

$$\left(\frac{n+1}{2}\right)^n = \mathrm{e}\left(\frac{n}{2}\right)^n \cdot \frac{\left(\frac{n+1}{2}\right)^n}{\mathrm{e}\left(\frac{n}{2}\right)^n} = \mathrm{e}\left(\frac{n}{2}\right)^n \cdot \frac{\left(1+\frac{1}{n}\right)^n}{\mathrm{e}} < \mathrm{e}\left(\frac{n}{2}\right)^n .$$

b) Let us take $r = p/q,\ p, q \in \mathbf{N}$. Then we obtain using Example 3.38 c)

$$\begin{aligned}
\ln(1+r) &= \ln\left(1+\frac{p}{q}\right) = \ln\left(\frac{q+1}{q}\cdot\frac{q+2}{q+1}\cdots\frac{q+p}{q+p-1}\right)\\
&= \ln\left(1+\frac{1}{q}\right) + \ln\left(1+\frac{1}{q+1}\right) + \cdots + \ln\left(1+\frac{1}{q+p-1}\right)\\
&< \frac{1}{q} + \frac{1}{q+1} + \cdots + \frac{1}{q+p-1} < \frac{p}{q}.
\end{aligned}$$

Thus

$$\ln(1+r) < r. \tag{3.13}$$

Also, from Example 3.38 c) it follows

$$\ln(1+r) > \frac{1}{q+1} + \frac{1}{q+2} + \cdots + \frac{1}{q+p} > \frac{p}{q+p} = \frac{\frac{p}{q}}{\frac{p}{q}+1} = \frac{r}{r+1}. \tag{3.14}$$

If we take $-1 < r < 0$ and $r \in \mathbf{Q}$, then we can put $r = -r_1$, where $0 < r_1 < 1$. Denoting by $r_2 = \frac{r_1}{1-r_1}$, it follows that $r_2 \in \mathbf{Q}$ and $r_2 > 0$. From relations (3.13) and (3.14) we obtain

$$\frac{r_2}{1+r_2} < \ln(1+r_2) < r_2.$$

Putting $r_2 = \frac{r_1}{1-r_1}$ in the last relation, we get

$$r_1 < \ln\left(1+\frac{r_1}{1-r_1}\right) < \frac{r_1}{1-r_1}.$$

Then it follows

$$\frac{r}{1+r} = \frac{-r_1}{1-r_1} < -\ln\left(1+\frac{r_1}{1-r_1}\right) = \ln(1-r_1) = \ln(1+r) < -r_1 = r\,,$$

and these relations immediately give the inequalities in b).

Example 3.41. *Show that the following sequences converge.*

a) $f_n = \sqrt{c + \sqrt{c + \sqrt{c + \cdots \sqrt{c}}}} = \sqrt{c + f_{n-1}}, \quad f_1 = \sqrt{c}, \quad c > 0;$

b) $f_{n+1} = \frac{1}{2}\left(a + f_n^2\right), \quad f_1 = \frac{a}{2}, \quad 0 < a \leq 1;$

c) $f_{n+1} = \frac{1}{2}\left(f_n + \frac{b}{f_n}\right), \quad f_1 \neq 0, \quad b \geq 0;$

d) $f_{n+1} = \frac{1}{m}\left((m-1)f_n + \frac{b}{f_n^m - 1}\right), \quad f_1 \neq 0, \quad b > 0,\ m \in \mathbf{N},\ n > 2;$

e) $f_{n+1} = f_n - \sin f_n, \quad 0 \leq f_1 < \pi.$

Solutions.

a) Using mathematical induction, we shall prove that this sequence is monotonically increasing and bounded from above.

It is obvious that it holds $f_2 = \sqrt{c + \sqrt{c}} > f_1$. Assume $f_n > f_{n-1}$. Then we have

$$\sqrt{c + f_n} > \sqrt{c + f_{n-1}} \Rightarrow f_{n+1} > f_n.$$

Next we show that for all $n \in \mathbf{N}$ it holds $f_n < \sqrt{c} + 1$. Firstly we have $f_1 = \sqrt{c} < \sqrt{c} + 1$. If we suppose $f_n = \sqrt{c} < \sqrt{c} + 1$ for some natural number n, then it follows

$$f_{n+1} = \sqrt{c + f_n} < \sqrt{c + \sqrt{c} + 1} < \sqrt{(\sqrt{c} + 1)^2} = \sqrt{c} + 1.$$

Thus we showed that $(f_n)_{n\in\mathbf{N}}$ monotonically increases and is a bounded sequence. From Theorem 3.32 it follows that there exists a real number ℓ such that it holds

$$\lim_{n\to\infty} f_n = \ell \quad \text{and also} \quad \lim_{n\to\infty} f_{n-1} = \ell.$$

The limit ℓ can be ordered from

$$f_n^2 = c + f_{n-1} \Rightarrow \lim_{n\to\infty} f_n^2 = c + \lim_{n\to\infty} f_{n-1} \Rightarrow \ell^2 = c + \ell,$$

and it has the form

$$\ell_{1,2} = \frac{1 \pm \sqrt{1 + 4c}}{2}.$$

All terms of our sequence are positive and therefore the number

$$\ell_1 = \frac{1 - \sqrt{1 + 4c}}{2} < 0$$

can not be the searched limit. Hence the limit of the sequence $(f_n)_{n\in\mathbf{N}}$ is

$$\ell = \ell_2 = \frac{1 + \sqrt{1 + 4c}}{2}.$$

b) We shall show that the sequence $(f_n)_{n\in\mathbf{N}}$ is monotonically increasing and bounded from above. It holds that $f_n > 0$ for every $n \in \mathbf{N}$ and

$$f_2 = \frac{1}{2}\left(a + \left(\frac{a^2}{4}\right)\right) = \frac{a}{2} + \frac{1}{2}\left(\frac{a}{2}\right)^2 > \frac{a}{2} = f_1.$$

If we suppose that $f_n > f_{n-1}$ or $f_n - f_{n-1} > 0$, then we have

$$2(f_{n+1} - f_n) = f_n^2 - f_{n-1}^2 = (f_n + f_{n-1})(f_n - f_{n-1}) > 0.$$

Further on, we have $f_1 = \dfrac{a}{2} \leq \dfrac{1}{2} < 1$. Assuming that $f_n < 1$, we obtain

$$f_{n+1} = \frac{1}{2}\left(a + f_n^2\right) = \frac{a}{2} + \frac{f_n^2}{2} < 1.$$

Thus the sequence is convergent and there exists an $\ell \in \mathbf{R}$ such that

$$\lim_{n\to\infty} f_n = \ell = \lim_{n\to\infty} f_{n-1}.$$

The limit ℓ can be ordered from

$$\ell = \frac{1}{2}\left(a + \ell^2\right) \Rightarrow \ell_{1,2} = \frac{2 \pm \sqrt{4 - 4a}}{2}.$$

So the limit of $(f_n)_{n\in\mathbf{N}}$ is

$$\ell = \ell_2 = 1 - \sqrt{1-a}.$$

In this case we could not accept the number $\ell_1 = 1 + \sqrt{1-a} > 1$, (for $0 < a < 1$) as the limit of $(f_n)_{n\in\mathbf{N}}$, because we have shown $f_n < 1$, for every $n \in \mathbf{N}$.

c) We have to analyze tree cases, depending on b and f_1.

- **Case 1.** If $b = 0$, then it holds

$$f_n = \frac{f_1}{2^{n-1}} \quad \text{and} \quad \lim_{n\to\infty} f_n = 0.$$

- **Case 2.** If $b > 0$ and $f_1 > 0$, then $f_n > 0$ for some $n \in \mathbf{N}$ implies

$$f_{n+1} = \frac{1}{2}\left(f_n + \frac{b}{f_n}\right) > 0.$$

So we have $f_n > 0$ for every n. From the transformations

$$2f_{n+1} = f_n + \frac{b}{f_n} \quad \Rightarrow \quad 2f_{n+1} - 2\sqrt{b} = f_n - 2\sqrt{b} + \frac{b}{f_n}$$

$$\Rightarrow \quad 2(f_{n+1} - \sqrt{b}) = \left(f_n - \sqrt{\frac{b}{f_n}}\right)^2 > 0,$$

we can conclude that $f_n > \sqrt{b}$ for $n > 1$. Using the last inequality we get

$$f_{n+1} - f_n = \frac{1}{2}\left(\frac{b}{f_n} - f_n\right) = \frac{1}{2} \cdot \frac{b - f_n^2}{f_n} < 0,$$

wherefrom we obtain that the given sequence monotonically decreases (at least for $n \geq 2$). Therefore this sequence is convergent and there exists an $\ell \in \mathbf{R}$ such that

$$\lim_{n\to\infty} f_{n+1} = \ell = \lim_{n\to\infty} f_n.$$

The number ℓ can be ordered from the given recurrence relation

$$f_{n+1} = \frac{1}{2}\left(f_n + \frac{b}{f_n}\right).$$

Thus $\ell = \dfrac{1}{2}\left(\ell + \dfrac{b}{\ell}\right) > 0$. Since $f_n > 0$ for all $n \in \mathbf{N}$, we get finally $\ell = \sqrt{b}$.

- **Case 3.** If we have $b > 0$ and $f_1 < 0$, analogously it can be shown that the sequence is monotonically increasing and it holds that $f_n < 0$ for every $n \in \mathbf{N}$. This means that the considered sequence converges with the limit $x = -\sqrt{b}$.

Remark. In the Cases 2 and 3, the sequence is not decreasing (resp. increasing), but it is rather *eventually* decreasing (resp. eventually increasing); i.e., the sequence becomes monotone starting from an index n_0. This situation happens, for instance, when in Case 2 it is assumed $0 < f_1 < \sqrt{b}$.

d) In Example 1.33 we have shown that the arithmetic mean is not smaller than the geometric mean of any finitely many positive numbers. Thus

$$f_{n+1} = \frac{(m-1)f_n + \dfrac{b}{f_n^{m-1}}}{m} \geq \sqrt[m]{f_n^{m-1} \cdot \frac{b}{f_n^{m-1}}} = \sqrt[m]{b}$$

for every $n \in \mathbf{N}$. The sequence $(f_n)_{n\in\mathbf{N}}$ is monotonically decreasing because

$$f_{n+1} - f_n = \frac{1}{m}\left(\frac{b}{f_n^{m-1}} - f_n\right) = \frac{b - f_n^m}{m f_n^{m-1}} \leq 0.$$

Similarly as in c), we can show that $\lim_{n\to\infty} f_n = \sqrt[m]{b}$.

e) Let us show that the given sequence is bounded. To that end, let us show $0 \leq f_n < \pi$ for every $n \in \mathbf{N}$. This is true for f_1. Now if we suppose that $0 \leq f_n < \pi$, then $0 \leq \sin(f_n) < f_n$, which implies

$$0 \leq f_{n+1} = f_n - \sin(f_n) \leq f_n < \pi.$$

From the last relation it also follows that the sequence is monotonically decreasing and therefore it converges. The function $\sin x$ is continuous and the limit ℓ can be ordered from the equation

$$\ell = \ell - \sin \ell, \tag{3.15}$$

which follows from

$$\lim_{n\to\infty} f_{n+1} = \ell = \lim_{n\to\infty} \sin(f_n) = \sin \ell.$$

Equation (3.15) has two solutions on the interval $[0, \pi]$, namely $\ell_1 = 0$ and $\ell_2 = \pi$. From the inequalities $f_n \leq f_1 < \pi$ it follows that the limit is $\ell = \ell_1 = 0$.

Exercise 3.42. *Prove that the sequences given by the following recurrence formulas converge and determine their limits.*

a) $a_{n+1} = \sqrt{2 + a_n}, \quad a_1 > 0;$ **b)** $b_{n+1} = \dfrac{1}{3}\left(2b_n + \dfrac{125}{b_n^2}\right), \quad b_1 \neq 0;$

c) $c_{n+1} = 1 - c_n^2, \quad c_1 = \dfrac{1}{2};$ **d)** $d_{n+1} = d_n(1 - d_n), \quad 0 < d_1 < 1.$

Answers.

a) 2. **b)** 5. **c)** $\dfrac{\sqrt{5}-1}{2}$. **d)** 0.

Example 3.43. *Prove that the sequences* $(f_n)_{n\in\mathbf{N}}$ *and* $(g_n)_{n\in\mathbf{N}}$ *given by*

$$f_{n+1} = \frac{f_n + g_n}{2}, \qquad g_{n+1} = \frac{2f_n g_n}{f_n + g_n}, \quad f_1 > 0, \quad g_1 > 0,$$

converge to the same limit.

Solution.

- **First method.** It is obvious that $f_n > 0$ and $g_n > 0$ for every $n \in \mathbf{N}$. Also it holds that $f_{n+1} \geq g_{n+1}$ for every $n \in \mathbf{N}$ (comparison between the arithmetic and the harmonic mean - see Example 1.33). Therefore we have

$$f_{n+1} - f_n = \frac{f_n + g_n}{2} - f_n = \frac{g_n - f_n}{2} \leq 0, \quad \text{for every } \ n > 2, n \in \mathbf{N}.$$

Thus the sequence $(f_n)_{n\in\mathbf{N}}$ is monotonically decreasing, while from

$$\frac{g_{n+1}}{g_n} = \frac{2f_n}{f_n + g_n} > 1, \quad \text{for } \ n = 2, 3, \ldots ,$$

it follows that the sequence $(g_n)_{n\in\mathbf{N}}$ is monotonically increasing.

The sequences $(f_n)_{n\in\mathbf{N}}$ and $(g_n)_{n\in\mathbf{N}}$ are bounded from below, resp. above, because

$$f_{n+1} \geq g_{n+1} \geq \dots \geq g_2 \quad \text{and} \quad g_{n+1} \leq f_{n+1} \leq \dots \leq f_2,$$

for every $n \in \mathbf{N}$. So these two sequences converge. Let us denote

$$f = \lim_{n\to\infty} f_n \quad \text{and} \quad g = \lim_{n\to\infty} g_n.$$

Then we have

$$\lim_{n\to\infty} f_{n+1} = \frac{1}{2}\left(\lim_{n\to\infty} f_n + \lim_{n\to\infty} g_n\right) \Rightarrow f = \frac{f+g}{2} \Rightarrow f = g.$$

From the equality

$$g_{n+1} = \frac{2f_n g_n}{f_n + g_n},$$

it follows

$$f_{n+1}g_{n+1} = f_n g_n = f_{n-1}g_{n-1} = \dots = f_1 g_1,$$

and we obtain

$$f_n g_n = f_1 g_1, \quad n = 1, 2, \dots, \text{ hence } \quad f \cdot g = f_1 g_1 .$$

Finally we have

$$f = g = \sqrt{f_1 g_1}.$$

- **Second method.** From the equalities

$$f_{n+1}g_{n+1} = f_n g_n, \qquad f_n g_n = f_1 g_1,$$

it follows

$$f_{n+1} = \frac{f_n}{2} + \frac{f_1 g_1}{2f_n} = \frac{1}{2}\left(f_n + \frac{f_1 g_1}{f_n}\right).$$

So we obtain a sequence of the same form as in Example 3.41 c).

3.3 Accumulation points and subsequences

3.3.1 Basic notions

Definition 3.44. *Assume that a sequence $(a_n)_{n\in\mathbf{N}}$ is defined with the function a, i.e., $a(n) = a_n$, $n \in \mathbf{N}$. A* **subsequence** *$(a_{n_k})_{k\in\mathbf{N}}$ of the sequence $(a_n)_{n\in\mathbf{N}}$ is a restriction of the function a to an infinite subset $\{n_k |\ k \in \mathbf{N}\}$ of the set $\mathbf{N}$, satisfying the condition $n_1 < n_2 < \dots < n_k < \dots$.*

Theorem 3.45. *Any subsequence of a convergent sequence converges and they both have the same limit.*

A subsequence $(f_{n_k})_{\in\mathbf{N}}$ of a sequence $(f_n)_{n\in\mathbf{N}}$ which has the property $f_{n_k} = \ell$ for some number ℓ and for all $k \in \mathbf{N}$ is called **stationary subsequence.** Clearly, then ℓ is an accumulation point of the sequence $(f_n)_{n\in\mathbf{N}}$.

Definition 3.46.

- *The* **limes inferior** *of a sequence* $(a_n)_{n\in N}$ *is the limit*

$$\mathcal{I} = \lim_{n\to\infty}(\inf\{a_m |\ m \geq n\}).$$

- *The* **limes superior** *of a sequence of real numbers* $(a_n)_{n\in N}$ *is the limit*

$$\mathcal{S} = \lim_{n\to\infty}(\sup\{f_m |\ m \geq n\}).$$

Each of the last two limits either exist in the (usual) sense of Definition 3.1, or is $+\infty$ or $-\infty$ (see Definition 3.2). In any of these cases we shall write

$$\mathcal{I} = \liminf_{n\to\infty} a_n \quad \text{and} \quad \mathcal{S} = \limsup_{n\to\infty} a_n.$$

Let us note that the *limes inferior* and the *limes superior* of a sequence are resp. the smallest and the largest accumulation point of that sequence.

3.3.2 Examples and exercises

Example 3.47. *If at least one of the following conditions is fulfilled:*

(ap1) *the set* $\{n|\ f_n = \ell\}$ *is infinite;*

(ap2) *for every* $\varepsilon > 0$, *the set* $(\ell-\varepsilon, \ell+\varepsilon) \cap \{f_n|\ n \in \mathbf{N}\}$ *is infinite,*

then the point ℓ *is an accumulation point for the sequence* $(f_n)_{n\in\mathbf{N}}$. *Prove.*

Solution. If the set $M_1 = \{n|\ f_n = \ell\}$ is infinite, then its elements can be ordered into a monotonically increasing sequence which diverges to $+\infty$. So we can put $f_{n_k} = \ell$ for $k \in \mathbf{N}$ and $n_1 < n_2 < \dots < n_k < \dots \to +\infty$. This means that for every $m \in \mathbf{N}$ there exist $n_k \in M_1$, such that it holds $n_k > m$. Then, for arbitrary $\varepsilon > 0$, we have

$$|f_{n_k} - \ell| = |\ell - \ell| < \varepsilon.$$

We obtain that from condition (ap1) it follows that ℓ is an accumulation point for the sequence $(f_n)_{n\in\mathbf{N}}$.

If for every $\varepsilon > 0$, the set $(\ell-\varepsilon, \ell+\varepsilon) \cap \{f_n|\ n \in \mathbf{N}\}$ is infinite, then the set $M_2 = \{n|\ |f_n - \ell| < \varepsilon\}$ can be ordered into a monotonically increasing sequence $n_1 < n_2 < \dots$ which diverges to $+\infty$. This means that, for every $m \in \mathbf{N}$, there exists $k \in \mathbf{N}$, such that $n_k \in M_2$ and $n_k > m$. So for arbitrary $\varepsilon > 0$ we have

$$|f_{n_k} - \ell| < \varepsilon.$$

We obtain that from condition (ap2) it follows that ℓ is an accumulation point for the sequence $(f_n)_{n\in\mathbf{N}}$.

Example 3.48. *If the point ℓ is an accumulation point of the sequence $(f_n)_{n\in\mathbf{N}}$, then at least one of the conditions (ap1) or (ap2) from Example 3.47 are fulfilled. Prove.*

Solution. Let us suppose that ℓ is an accumulation point for the sequence $(f_n)_{n\in\mathbf{N}}$ and that the condition (ap1) is not fulfilled (this means that the sequence $(f_n)_{n\in\mathbf{N}}$ has no stationary subsequence $(f_{n_k})_{n\in\mathbf{N}}$ with the property $(f_{n_k})_{k\in\mathbf{N}} = \ell$, for every $k \in \mathbf{N}$). We have to show that then condition (ap2) is fulfilled.

Let us suppose that this is not true, namely that there exists an $\varepsilon > 0$ such that the set $(\ell - \varepsilon, \ell + \varepsilon) \cap \{f_n |\ n \in \mathbf{N}\}$ is finite. If n_1 is the smallest natural number such that $n > n_1 \Rightarrow f_n \neq \ell$, then the supposition means that the finite set $(\ell - \varepsilon, \ell + \varepsilon) \cap \{f_n |\ n \in \mathbf{N}\}$ is either empty, or there exists a natural number $n_2 > n_1$ such that for every $n > n_2 \Rightarrow f_n \notin (\ell - \varepsilon, \ell + \varepsilon)$. Hence this means that ℓ is not an accumulation point for the sequence $(f_n)_{n\in\mathbf{N}}$, a contradiction.

Remark. The statements in Examples 3.47 and 3.48 can be together expressed as follows.
A real number ℓ is an accumulation point for the sequence $(f_n)_{n\in\mathbf{N}}$ if and only if either the sequence $(f_n)_{n\in\mathbf{N}}$ has a stationary subsequence whose every element is equal to ℓ, or every interval containing ℓ has an infinite number of terms of the sequence $(f_n)_{n\in\mathbf{N}}$ (or both).

Example 3.49. *Find a sequence with*

a) *two accumulation points;*

b) *three accumulation points;*

c) *infinitely many accumulation points;*

d) *only one accumulation point, but the sequence diverges.*

Solutions.

a) The sequence $(f_n)_{n\in\mathbf{N}}$, where $f_n = (-1)^n$, has two accumulation points $\ell_1 = 1$ and $\ell_2 = -1$, because $f_{2k} = 1$ and $f_{2k+1} = -1$ for $k = 0, 1, 2, \ldots$ (see condition (ap1) from Example 3.47).

b) The sequence $(g_n)_{n\in\mathbf{N}}$, where $g_n = \dfrac{n}{n+1} \sin \dfrac{2\pi n}{3}$, $n \in \mathbf{N}$, has three accumulation points. Namely, its stationary subsequence

$$g_{3k} = \frac{3k}{3k+1} \sin(2\pi k) = 0$$

converges to 0 and this means that 0 is an accumulation point for the sequence $(g_n)_{n\in\mathbf{N}}$. Further on, we have

$$\lim_{k\to\infty} g_{3k+1} = \lim_{k\to\infty} \frac{3k+1}{3k+2} \sin\left(2k\pi + \frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}.$$

So $\dfrac{\sqrt{3}}{2}$ is another accumulation point for the sequence $(g_n)_{n\in\mathbf{N}}$. Finally from

$$g_{3k+2} = \frac{3k+2}{3k+3}\sin\left(2k\pi + \frac{4\pi}{3}\right)$$

it follows that $-\dfrac{\sqrt{3}}{2}$ is the third accumulation point for the sequence $(g_n)_{n\in\mathbf{N}}$.

As another example of a sequence with three accumulation points, one can take the sequence with the general term $h_n = \cos\dfrac{n\pi}{2}$, $n \in \mathbf{N}$.

c) We shall use the sequence whose accumulation point is every positive rational number given by the following scheme

$$\begin{array}{ccccccccc}
1/1 & \to & 1/2 & & 1/3 & \to & 1/4 & \dots & 1/n\dots \\
 & \swarrow & & \nearrow & & \swarrow & & & \\
2/1 & & 2/2 & & 2/3 & & 2/4 & \dots & 2/n\dots \\
\downarrow & \nearrow & & \swarrow & & & & & \\
3/1 & & 3/2 & & 3/3 & & 3/4 & \dots & 3/n\dots \\
\dots & & \dots & & \dots & & \dots & \dots & \dots \\
k/1 & & k/2 & & k/3 & & k/4 & \dots & k/n\dots,
\end{array}$$

or

$$f_1 = 1, \quad f_2 = 1/2, \quad f_3 = 2/1, \quad f_4 = 3/1, \quad f_5 = 2/2, \dots .$$

d) The sequence $(a_n)_{n\in\mathbf{N}}$, where $a_n = n + (-1)^n n$, $n \in \mathbf{N}$, has only one accumulation point, because it holds that $a_{2k+1} = 0$, for $k = 1, \dots$ (see Example 3.47 (ap1)).

But the sequence $(a_n)_{n\in\mathbf{N}}$ does not converge, because

$$a_{2k} = 4k, \quad \text{hence} \quad \lim_{n\to\infty} a_{2k+1} = +\infty.$$

Exercise 3.50. *Show that the sequence $(f_n)_{n\in\mathbf{N}}$, given by*

$$f_{2n} = 1 \quad \text{and} \quad f_{2n+1} = \frac{2n}{2n+1},$$

satisfies both conditions of Example 3.47.

Example 3.51. *Assume that $\mathcal{I} = \liminf\limits_{n\to\infty} a_n$ is a real number. Prove that then $\mathcal{I}$ is the* smallest *accumulation point of the sequence $(a_n)_{n\in N}$.*

Solution. The sequence

$$f_n = \inf\{a_m \mid m > n\}$$

is a monotonically nondecreasing one, and from Definition 3.46 it follows $\lim\limits_{n\to\infty} f_n = \mathcal{I}$. Then, for given $\varepsilon > 0$, there exists a natural number $n_0 = n_0(\varepsilon)$, such that it holds

$$n > n_0(\varepsilon) \Rightarrow \mathcal{I} - \varepsilon < f_n \leq \mathcal{I} + \varepsilon.$$

From the definition of the sequence $(f_n)_{n\in\mathbf{N}}$, it follows that for every $n \in \mathbf{N}$ there exists a number $m_n = m_n(\varepsilon) \geq n_0$ satisfying $f_n \leq a_{m_n} \leq f_n + \varepsilon$. The numbers m_n can be chosen so that the sequence $(m_n)_{n\in\mathbf{N}}$ becomes monotonically increasing. Then for every m_n it holds

$$\mathcal{I} - \varepsilon \leq a_{m_n} \leq \mathcal{I} + \varepsilon.$$

So $\mathcal{I}$ is an accumulation point of the sequence $(a_n)_{n\in N}$; we have yet to show that it is also its smallest accumulation point. From the inequalities

$$n > n_0 \wedge m > n \ \Rightarrow\ \mathcal{I} - \varepsilon \leq f_n \leq a_m$$

and since $\varepsilon > 0$ is arbitrary, it follows that $\mathcal{I}$ is the smallest accumulation point for the sequence $(a_n)_{n\in N}$.

Remarks.

1. If the sequence $(a_n)_{n\in N}$ is not bounded from below, it is obvious that

$$\lim_{n\to\infty}(\inf\{a_m \mid m \leq n\}) = -\infty.$$

In this case we usually write

$$\liminf_{n\to\infty} a_n = -\infty.$$

But it is easy to find an example of the sequence $(a_n)_{n\in\mathbf{N}}$ such that

$$\liminf_{n\to\infty} a_n = +\infty.$$

For an example, take simply $a_n = n,\ \ n \in \mathbf{N}$.

2. In the previous example it was shown that the point $\mathcal{I}$ satisfying $\mathcal{I} = \liminf\limits_{n\to\infty} a_n$ is the smallest accumulation point of the sequence $(a_n)_{n\in\mathbf{N}}$. So we can prove that in this case for every $\varepsilon > 0$ it holds that

- there are infinitely many terms a_n such that $a_n < \mathcal{I} + \varepsilon$;
- there are at most finitely many terms a_n such that $a_n < \mathcal{I} - \varepsilon$.

3. Similarly as in Example 3.51, it can be shown that the point S satisfying that $\mathcal{S} = \limsup\limits_{n\to\infty} a_n$ is the biggest accumulation point of the sequence $(a_n)_{n\in\mathbf{N}}$. So we can show that in this case for every $\varepsilon > 0$ it holds that

- there are infinitely many terms a_n such that $a_n > \mathcal{S} - \varepsilon$;
- there are at most finitely many terms a_n such that $a_n > \mathcal{S} + \varepsilon$.

Example 3.52. *Determine* $\limsup\limits_{n\to\infty} f_n$ *and* $\liminf\limits_{n\to\infty} f_n$ *for the following sequences.*

$$\text{a)}\quad f_n = [(-1)^n - 1]n + \frac{1}{n}, \quad n \in \mathbf{N};$$

$$\text{b)}\quad f_n = \frac{a + bn^{(-1)^n}}{1 + n^{(-1)^n}}, \quad a < b, \quad n \in \mathbf{N};$$

$$\text{c)}\quad f_n = \frac{n^2}{1+n^2}\sin\frac{2\pi n}{3}, \quad n \in \mathbf{N};$$

$$\text{d)}\quad f_n = 1 + (-1)^{n+1} + 3(-1)^{n(n-1)/2}, \qquad n \in \mathbf{N}.$$

Solutions.

a) From $f_{2k} = [(-1)^{2k} - 1]\cdot 2k + \dfrac{1}{2k} = \dfrac{1}{2k}$, it follows $\lim\limits_{k\to\infty} f_{2k} = 0$; further on, from $f_{2k+1} = [(-1)^{2k+1} - 1]\cdot(2k+1) + \dfrac{1}{2k+1}$, it follows $\lim\limits_{k\to\infty} f_{2k} = -\infty$. Thus it holds

$$\limsup_{n\to\infty} f_n = 0 \quad \text{and} \quad \liminf_{n\to\infty} f_n = -\infty.$$

b)
$$f_{2k} = \frac{a + b\cdot 2k}{1+2k}, \quad k = 1,2,..., \quad \lim_{k\to\infty} f_{2k} = b = \limsup_{n\to\infty} f_n;$$
$$f_{2k-1} = \frac{a + \dfrac{b}{2k-1}}{1 + \dfrac{1}{2k-1}}, \quad k = 1,2,..., \quad \lim_{k\to\infty} f_{2k-1} = a = \liminf_{n\to\infty} f_n.$$

c)
$$f_{3k-2} = \frac{\sqrt{3}(3k-2)^2}{2(1+(3k-2)^2)}, \quad k = 1,2,..., \quad \lim_{k\to\infty} f_{3k-2} = \frac{\sqrt{3}}{2} = \limsup_{n\to\infty} f_n;$$
$$f_{3k-1} = \frac{-\sqrt{3}(3k-1)^2}{2(1+(3k-1)^2)}, \quad k = 1,2,..., \quad \lim_{k\to\infty} f_{3k-1} = \frac{-\sqrt{3}}{2} = \liminf_{n\to\infty} f_n;$$
$$f_{3k} = \frac{(3k)^2}{1+(3k)^2}\cdot 0, \quad k = 1,2,\dots, \lim_{k\to\infty} f_{3k} = 0.$$

d)
$$f_{4k} = 1 + (-1)^{4k+1} + 3(-1)^{4k(4k-1)/2} = 3,$$
$$f_{4k+1} = 5 = \limsup_{n\to\infty} f_n, \qquad f_{4k+2} = -3 = \liminf_{n\to\infty} f_n \quad \text{and} \quad f_{4k+3} = -1.$$

Exercise 3.53. *Prove that a necessary and sufficient condition for a sequence* $(f_n)_{n\in\mathrm{N}}$ *to converge is*

$$\limsup_{n\to\infty} f_n = \liminf_{n\to\infty} f_n = \lim_{n\to\infty} f_n = \ell. \tag{3.16}$$

Answer. The statement immediately follows from Theorem 3.5, which claims that a sequence converges iff it is both bounded and has exactly one accumulation point.

Exercise 3.54. *If $(f_{n_k})_{k\in\mathbf{N}}$ is a subsequence of a convergent sequence $(f_n)_{n\in\mathbf{N}}$, prove then*

$$\liminf_{n\to\infty} f_n \le \liminf_{k\to\infty} f_{n_k} \le \limsup_{k\to\infty} f_{n_k} \le \limsup_{n\to\infty} f_n.$$

Answer. Clearly, every accumulation point of a subsequence is the accumulation point of the sequence, but not every accumulation point of the sequence is necessarily an accumulation point of its subsequence.

Exercise 3.55. *Prove Theorem 3.45, i.e., the following.*
Any subsequence of a convergent sequence has the same limit as the sequence.

Example 3.56. *Let us suppose that $(a_n)_{n\in\mathbf{N}}$ and $(b_n)_{n\in\mathbf{N}}$ are bounded sequences and satisfy the following condition:*

$$a_n \le b_n \quad \textit{for} \quad n \ge n_0.$$

Prove that then it holds

$$\liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} b_n \quad \textit{and} \quad \limsup_{n\to\infty} a_n \le \limsup_{n\to\infty} b_n.$$

Solution. We shall prove only the first inequality.

Let us denote by

$$a = \liminf_{n\to\infty} a_n \quad \text{and} \quad b = \liminf_{n\to\infty} b_n,$$

and suppose the opposite, i.e., $a > b$. This means that there exists a real number $d > 0$ such that $d = a - b$.

Since a is the smallest accumulation point of the sequence $(a_n)_{n\in\mathbf{N}}$, then there exists $n_1 \in \mathbf{N}$ such that it holds

$$a_n > a - \frac{d}{3} \quad \text{for all} \quad n > n_1.$$

Since b is the smallest accumulation point of the sequence $(b_n)_{n\in\mathbf{N}}$, then for infinitely many indices $n \in \mathbf{N}$ it holds

$$b_n < b + \frac{d}{3}.$$

Hence for infinitely many indices $n \in \mathbf{N}$ it holds

$$b_n < b + \frac{d}{3} < a - \frac{d}{3} < a_n, \quad \text{hence} \quad b_n < a_n.$$

This is the opposite of the assumption $a_n \le b_n$, $n \in \mathbf{N}$, $n > n_0$.

Exercise 3.57. *The sequence $(x_n)_{n\in N}$ is given as follows.*

a) $x_n = (-1)^{n+1}\left(3 + \dfrac{2}{n}\right);$ **b)** $x_n = 1 + \dfrac{n}{n+2}\cos\dfrac{n\pi}{2}.$

Determine $\inf\{x_n |\ n \in N\}$, $\sup\{x_n |\ n \in N\}$, $\liminf_{n\to\infty} x_n$ *and* $\limsup_{n\to\infty} x_n$, *and then compare them.*

Answers.

a) $\inf\{x_n |\, n \in N\} = -4,\ \liminf\limits_{n\to\infty} x_n = -3,\ \limsup\limits_{n\to\infty} x_n = 3,\ \sup\{x_n |\, n \in N\} = 5.$

b) $\inf\{x_n |\, n \in N\} = \liminf\limits_{n\to\infty} x_n = 0, \qquad \sup\{x_n |\, n \in N\} = \limsup\limits_{n\to\infty} x_n = 2.$

Example 3.58. *If* $(f_n)_{n\in\mathbf{N}}$ *is a sequence of positive numbers, prove then the following relations.*

a) $\limsup\limits_{n\to\infty} f_n = -\liminf\limits_{n\to\infty}(-f_n);$ **b)** $\liminf\limits_{n\to\infty} \dfrac{1}{f_n} = \dfrac{1}{\limsup\limits_{n\to\infty} f_n}.$

Solutions.

a) Let us denote

$$\limsup_{n\to\infty} f_n = L, \quad L \in \mathbf{R}. \tag{3.17}$$

Then for every $\varepsilon > 0$, there are

- infinitely many terms f_n such that $f_n > L - \varepsilon$;
- at most finitely many terms f_n such that $f_n > L + \varepsilon$. (3.18)

So from relations (3.18) it follows that for every $\varepsilon > 0$ there are

- infinitely many terms $-f_n$ such that $-f_n < -L + \varepsilon$;
- at most finitely many terms $-f_n$ such that $-f_n < -L - \varepsilon$.

The terms $-f_n$ belong to the sequence $(-f_n)_{n\in\mathbf{N}}$. Thus

$$\liminf_{n\to\infty}(-f_n) = -L.$$

Remark. If $\limsup\limits_{n\to\infty} f_n = +\infty$, then $\liminf\limits_{n\to\infty}(-f_n) = -\infty$.

b) Let us take L from relation (3.17) and assume $L > 0$. Then from relations (3.18), for every $\varepsilon > 0$, $\varepsilon < L$, there are

- infinitely many terms $\dfrac{1}{f_n}$ such that $\dfrac{1}{f_n} < \dfrac{1}{L - \varepsilon} =: \dfrac{1}{L} + \varepsilon_1$;
- at most finitely many terms $\dfrac{1}{f_n}$ such that $\dfrac{1}{f_n} < \dfrac{1}{L + \varepsilon} =: \dfrac{1}{L} - \varepsilon_2$,

where $\varepsilon_1, \varepsilon_2 > 0$. The terms $1/f_n$ belong to the sequence $(1/f_n)_{n\in\mathbf{N}}$. This means that

$$\liminf_{n\to\infty} f_n = \frac{1}{L} = \frac{1}{\limsup\limits_{n\to\infty} f_n}.$$

If, however, $\limsup\limits_{n\to\infty} f_n = 0$, then for every $\varepsilon > 0$, there are

- infinitely many terms f_n such that $f_n < \varepsilon$
- at most finitely many terms f_n such that $f_n > \varepsilon$.

This means that for every $\varepsilon > 0$ there are

- infinitely many terms $\frac{1}{f_n}$ such that $\frac{1}{f_n} > \frac{1}{\varepsilon}$,
- at most finitely many terms $\frac{1}{f_n}$ such that $\frac{1}{f_n} < \frac{1}{\varepsilon}$.

So we have

$$\liminf_{n\to\infty} \frac{1}{f_n} = +\infty = \frac{1}{\limsup\limits_{n\to\infty} f_n}.$$

Example 3.59. *If $(f_n)_{n\in\mathbf{N}}$ and $(g_n)_{n\in\mathbf{N}}$ are two sequences of real numbers, then the following inequalities hold.*

a) $$\liminf_{n\to\infty} f_n + \liminf_{n\to\infty} g_n \le \liminf_{n\to\infty}(f_n+g_n) \le \begin{cases} \liminf\limits_{n\to\infty} f_n + \limsup\limits_{n\to\infty} g_n \\ \\ \limsup\limits_{n\to\infty} f_n + \liminf\limits_{n\to\infty} g_n \end{cases}$$

$$\le \limsup_{n\to\infty}(f_n+g_n) \le \limsup_{n\to\infty} f_n + \limsup_{n\to\infty} g_n\,;$$

b) $$\liminf_{n\to\infty} f_n \cdot \liminf_{n\to\infty} g_n \le \liminf_{n\to\infty}(f_n\cdot g_n) \le \begin{cases} \liminf\limits_{n\to\infty} f_n \cdot \limsup\limits_{n\to\infty} g_n \\ \\ \limsup\limits_{n\to\infty} f_n \cdot \liminf\limits_{n\to\infty} g_n \end{cases}$$

$$\le \limsup_{n\to\infty}(f_n\cdot g_n) \le \limsup_{n\to\infty} f_n \cdot \limsup_{n\to\infty} g_n.$$

In b) we additionally assume that $f_n > 0$ and $g_n > 0$, for all $n \in \mathbf{N}$.

Solutions.

a) Let us denote by

$$f = \liminf_{n\to\infty} f_n, \quad g = \liminf_{n\to\infty} g_n \quad \text{and} \quad \ell = \liminf_{n\to\infty}(f_n+g_n);$$

$$F = \limsup_{n\to\infty} f_n, \quad G = \limsup_{n\to\infty} g_n \quad \text{and} \quad L = \limsup_{n\to\infty}(f_n+g_n).$$

1. First we shall prove that $f+g \le \ell$. If we suppose the opposite, i.e., $\ell < f+g$, then it holds $d := (f+g) - \ell > 0$. For every $\varepsilon > 0$, there are

- at most finitely many terms $f_n < f - \varepsilon$;
- at most finitely many terms $g_n < g - \varepsilon$.

So it follows that, for every $\varepsilon > 0$, there are

$$\bullet \text{ at most finitely many terms} \quad f_n + g_n < f + g - 2\varepsilon. \tag{3.19}$$

This holds also for $\varepsilon := \frac{d}{3}$. Since ℓ is an accumulation point of the sequence $(f_n + g_n)_{n\in\mathbf{N}}$, there are infinitely many terms $f_n + g_n < \ell + \frac{d}{3}$. But this is in contradiction with relation (3.19). So we have proved the first inequality.

2. Now we shall prove that $\ell \le f + G$. For every $\varepsilon > 0$ there are

- infinitely many terms $f_n < f + \varepsilon$;
- infinitely many terms $g_n < G+\varepsilon$, but only finitely many terms $g_n > G+\varepsilon$.

So it follows that for every $\varepsilon > 0$ there are

$$\bullet \text{ infinitely many terms} \quad f_n + g_n < f + G + 2\varepsilon. \tag{3.20}$$

Since ℓ is the smallest accumulation point of the sequence $(f_n + g_n)_{n\in\mathbf{N}}$, it follows that $\ell \le f + G$.

3. Let us prove that $f + G \le L$. Assume the opposite, i.e., $L < f + G$. Then it holds $(f + G) - L = d_2 > 0$. For every $\varepsilon > 0$ there are

- infinitely many terms $f_n > f - \varepsilon$;
- infinitely many terms $g_n > G-\varepsilon$, but only finitely many terms $g_n > G+\varepsilon$,

So it follows that for every $\varepsilon > 0$, there are

$$\bullet \text{ infinitely many terms} \quad f_n + g_n > f + G - 2\varepsilon. \tag{3.21}$$

Since L is the greatest accumulation point of the sequence $(f_n + g_n)_{n\in\mathbf{N}}$, there are

- at most finitely many terms $f_n + g_n > L + \dfrac{d_2}{3}$

and this relation is in contradiction with relation (3.21) for $\varepsilon = \dfrac{d_2}{3}$.

4. Finally, we shall prove $L \le F + G$. Let us suppose the opposite, i.e., $L > F + G$, then it holds $L - (F + G) = d_3 > 0$. For every $\varepsilon > 0$ there are

- at most finitely many terms $f_n > F + \varepsilon$;
- at most finitely many terms $g_n > G + \varepsilon$.

So it follows that for every $\varepsilon > 0$ there are

$$\bullet \text{ at most finitely many terms} \quad f_n + g_n > F + G + 2\varepsilon. \tag{3.22}$$

Since L is an accumulation point of the sequence $(f_n + g_n)_{n\in\mathbf{N}}$, there are

- infinitely many terms $f_n + g_n > L - \dfrac{d_3}{3}$.

But this relation is in contradiction with relation (3.22), for $\varepsilon = d_3/3$.

b) We shall only prove that $f \cdot g \le \ell$, where we used the notations

$$f = \liminf_{n\to\infty} f_n \quad \text{and} \quad g = \liminf_{n\to\infty} g_n, \quad \ell = \liminf_{n\to\infty}(f_n \cdot g_n).$$

If at least one of the real numbers f or g is zero, then the inequality $fg \le \ell$ is trivial. Assume $f > 0$ and $g > 0$ and suppose that it holds $\ell < f \cdot g$. Then, for every $\varepsilon > 0$, $\varepsilon < \min\{f, g\}/2$, there are

- at most finitely many terms $f_n < f - \varepsilon$;
- at most finitely many terms $g_n < g - \varepsilon$.

So it follows that, for every $\varepsilon > 0$, there are

- at most finitely many terms $f_n \cdot g_n < f \cdot g - \varepsilon(f + g - \varepsilon)$.

This is in contradiction with the fact that ℓ is an accumulation point of the sequence $(f_n g_n)_{n\in\mathbf{N}}$. In fact, there exist infinitely many terms in every interval surrounding ℓ.

The other relations can be proved similarly as in a).

Exercise 3.60. *The following sequences are given*

a) $f_n = (-1)^{n(n+1)/2} \sin^2 \dfrac{n\pi}{2}$; $\qquad g_n = (-1)^{n(n+1)/2} \cos^2 \dfrac{n\pi}{2}$;

b) $f_n = 2 + (-1)^n$; $\qquad g_n = 2 - (-1)^n + \dfrac{1}{2}(-1)^{n(n+1)/2}$.

Determine the smallest and the largest accumulation point for every sequence and determine

$$\limsup_{n\to\infty}(f_n + g_n), \quad \liminf_{n\to\infty}(f_n + g_n), \quad \limsup_{n\to\infty}(f_n \cdot g_n), \quad \liminf_{n\to\infty}(f_n \cdot g_n).$$

Answers.

a) $\liminf\limits_{n\to\infty} f_n = -1$, $\qquad \liminf\limits_{n\to\infty} g_n = -1$, $\qquad \liminf\limits_{n\to\infty}(f_n + g_n) = -1$,

$\limsup\limits_{n\to\infty} f_n = 1$, $\qquad \limsup\limits_{n\to\infty} g_n = 1$, $\qquad \limsup\limits_{n\to\infty}(f_n + g_n) = 1$.

b) $\liminf\limits_{n\to\infty} f_n = 1$, $\qquad \liminf\limits_{n\to\infty} g_n = \dfrac{1}{2}$, $\qquad \liminf\limits_{n\to\infty}(f_n \cdot g_n) = \dfrac{3}{2}$

$\limsup\limits_{n\to\infty} f_n = 3$, $\qquad \limsup\limits_{n\to\infty} g_n = \dfrac{7}{2}$, $\qquad \limsup\limits_{n\to\infty}(f_n \cdot g_n) = \dfrac{9}{2}$.

Example 3.61. *Two sequences* $(x_n)_{n\in N}$ *and* $(y_n)_{n\in N}$ *are given and the first one converges, i.e., there exists an* x, *such that* $x = \lim_{n\to\infty} x_n$. *Then it holds*

a) $\liminf_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \liminf_{n\to\infty} y_n$;

b) *if additionally* $x_n > 0$ *and* $y_n > 0$ *for all* $n \in \mathbf{N}$, *then*

$$\liminf_{n\to\infty}(x_n \cdot y_n) = \lim_{n\to\infty} x_n \cdot \liminf_{n\to\infty} y_n.$$

Solutions.

a) From Example 3.59 we have

$$\lim_{n\to\infty} \inf x_n + \lim_{n\to\infty} \inf y_n \le \lim_{n\to\infty} \inf(x_n + y_n) \le \lim_{n\to\infty} \sup x_n + \lim_{n\to\infty} \inf y_n$$

and from the convergence of the sequence $(x_n)_{n\in N}$, i.e.,

$$\lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$$

(see Example 3.53), we obtain the given equality.

b) Left to the reader.

Example 3.62. *Prove the following relations:*

$$\inf\{x_n \mid n \in \mathbf{N}\} \le \liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n \le \sup\{x_n \mid n \in \mathbf{N}\}.$$

Solution. We shall prove only the first relation. Since $i := \inf\{x_n \mid n \in \mathbf{N}\}$ is the largest lower bound of the last set, it holds that $i \le x_n$ for every $n \in \mathbf{N}$. Therefore we have

$$i \le \lim_{n\to\infty}(\inf\{x_m \mid m \ge n\}) = \liminf_{n\to\infty} x_n.$$

Remark. See Exercise 3.57 a).

Example 3.63. *Let us suppose that the sequence* $(f_n)_{n\in\mathbf{N}}$ *has two accumulation points* a *and* b *and assume* $\liminf_{n\to\infty} f_n = -\infty$ *and* $\limsup_{n\to\infty} f_n = +\infty$. *Prove that then there exist subsequences* $(n_l)_{l\in\mathbf{N}}$, $(n_k)_{k\in\mathbf{N}}$, $(n_p)_{p\in\mathbf{N}}$ *and* $(n_q)_{q\in\mathbf{N}}$ *of the set* $\mathbf{N}$ *such that*

$$\lim_{l\to\infty} f_{n_l} = -\infty, \quad \lim_{k\to\infty} f_{n_k} = a,$$

$$\lim_{p\to\infty} f_{n_p} = b \quad \textit{and} \quad \lim_{q\to\infty} f_{n_q} = +\infty.$$

Solution. We shall construct only two subsequences of $(f_n)_{n\in\mathbf{N}}$ converging to a and $+\infty$ respectively. First put

$$M = \{n \in \mathbf{N} \mid f_n = a\}.$$

If the set M is infinite, then it can be written in a unique way as an increasing sequence $(n_k)_{k\in\mathbf{N}}$ which diverges to $+\infty$. Clearly, $(f_{n_k})_{k\in\mathbf{N}}$ is a stationary subsequence of $(f_n)_{n\in\mathbf{N}}$ which converges to a.

If, however, the set M is finite or empty, then there exists an $n_0 \in \mathbf{N}$ such that

$$(\forall n \in \mathbf{N}) \quad n > n_0 \Rightarrow f_n \neq a.$$

Put $n_1 := n_0 + 1$ and $\varepsilon_1 := |f_{n_1} - a|/2$. Since a is an accumulation point of $(f_n)_{n\in\mathbf{N}}$, there exists a natural number $n_2 > n_1$ such that

$$f_{n_2} \in (a - \varepsilon_1, a + \varepsilon_1).$$

Putting $\varepsilon_2 := |f_{n_2} - a|/2$, there exists a term f_{n_3} in the interval $(a - \varepsilon_2, a + \varepsilon_2)$, etc. Continuing this procedure ad infinitum, we obtain a subsequence $(f_{n_k})_{k\in\mathbf{N}}$ of the sequence $(f_n)_{n\in\mathbf{N}}$ which converges to the point a.

A subsequence $(f_{n_q})_{q\in\mathbf{N}}$ which diverges to $+\infty$ can be obtained from the fact that the sequence $(f_n)_{n\in\mathbf{N}}$ is not bounded from above. Namely, denote by n_1 the smallest natural number such that $f_{n_1} > 1$. Next, put n_2 for the smallest integer greater than $f_{n_1} + n_1$. Continuing in this manner we get a monotonically increasing sequence $(n_q)_{q\in\mathbf{N}}$ which diverges to infinity. This sequence is the set of indices of a monotonically increasing subsequence $(f_{n_q})_{q\in\mathbf{N}}$ which diverges to $+\infty$.

Exercise 3.64. *If a sequence $(x_n)_{n\in N}$ is bounded from below, then there exists*

$$\lim_{n\to\infty}(\inf\{x_m|\ m \geq n\}).$$

If a sequence $(x_n)_{n\in N}$ is bounded from above, then there exists

$$\lim_{n\to\infty}(\sup\{x_m|\ m \geq n\}).$$

Prove.

Example 3.65. *Let $(f_n)_{n\in\mathbf{N}}$ be a sequence and denote by*

$$F_n := \frac{f_1 + f_2 + \cdots + f_n}{n}, \quad n \in \mathbf{N}, \tag{3.23}$$

its sequence of arithmetic means. Prove then the following implication.

$$\lim_{n\to\infty} f_n = f \ \Rightarrow\ \lim_{n\to\infty} F_n = f.$$

Solution. From the condition $\lim\limits_{n\to\infty} f_n = f$, it follows that

$$(\forall\varepsilon > 0)\ (\exists n_0 \in \mathbf{N})\ (\forall n \in \mathbf{N}) \quad n > n_0 \ \Rightarrow\ |f_n - f| < \varepsilon/2.$$

Then we can write

$$\begin{aligned}
|F_n - f| &= \left|\frac{f_1 + f_2 + \cdots + f_{n_0} + \cdots + f_n}{n} - f\right| \\
&\leq \frac{1}{n}(|f_1 - f| + |f_2 - f| + \cdots + |f_{n_0-1} - f| + |f_{n_0} - f| + \cdots + |f_n - f|) \\
&< \frac{A}{n} + \frac{n - n_0}{n}\cdot\frac{\varepsilon}{2} < \frac{A}{n} + \frac{\varepsilon}{2},
\end{aligned}$$

where $A = |f_1 - f| + |f_2 - f| + \cdots + |f_{n_0} - f|$. There exists a natural number $n_1 = n_1(\varepsilon)$ such that

$$n > n_1 \Rightarrow \frac{A}{n} < \frac{\varepsilon}{2}.$$

Put $n_2 = n_2(\varepsilon) := \max\{n_0, n_1\}$. Then we obtain

$$(\forall\varepsilon > 0)\ (\exists n_2 \in \mathbf{N})\ (\forall n \in \mathbf{N})\quad n > n_2 \Rightarrow |F_n - f| < \varepsilon.$$

Remark. The opposite statement does not necessarily hold. For example, the sequence $f_n = (-1)^n, n \in \mathbf{N}$ does not converge, but $\lim\limits_{n\to\infty} F_n = 0$, if $F_n, n \in \mathbf{N}$ are defined as in relation (3.23).

Example 3.66. *Let $(g_n)_{n\in\mathbf{N}}$ be a sequence of positive numbers and denote by*

$$G_n := \lim_{n\to\infty} \sqrt[n]{g_1 g_2 \cdots g_n} \tag{3.24}$$

its sequence of geometric means. Prove then the following implication

$$\lim_{n\to\infty} g_n = g \Rightarrow \lim_{n\to\infty} G_n = g,$$

provided that the limit $g > 0$.

Solution. The logarithmic function is continuous, hence from

$$\lim_{n\to\infty} g_n = g \quad \text{it follows that} \quad \lim_{n\to\infty} \ln g_n = \ln \lim_{n\to\infty} g_n = \ln g.$$

Since it holds

$$\ln G_n = \frac{1}{n}(\ln g_1 + \ln g_2 + \cdots + \ln g_n),$$

we can apply the previous example and obtain

$$\lim_{n\to\infty} \ln G_n = \ln g, \quad \text{hence finally} \quad \lim_{n\to\infty} \sqrt{g_1 \cdots g_n} = g.$$

Remark. The opposite does not necessarily hold. Find an example!

Example 3.67. *Prove that if a sequence $(g_n)_{n\in\mathbf{N}}$ has the property*

$$\lim_{n\to\infty} (g_n - g_{n-1}) = g, \quad \textit{then} \quad \lim_{n\to\infty} \frac{g_n}{n} = g.$$

Solution. Let us denote by $f_n = g_n - g_{n-1}$, $g_0 := 0$. Then

$$\lim_{n\to\infty} f_n = \lim_{n\to\infty} (g_n - g_{n-1}) = g$$

and let us calculate the sequence of arithmetic means of the sequence $(f_n)_{n\in\mathbf{N}}$. We have

$$F_n = \frac{f_1 + f_2 + \cdots + f_n}{n} = \frac{g_1 - g_0 + g_2 - g_1 + g_3 - g_2 + \cdots + g_n - g_{n-1}}{n} = \frac{g_n}{n}.$$

From Example 3.65, it follows that

$$\lim_{n\to\infty} \frac{g_n}{n} = \lim_{n\to\infty} F_n = g.$$

Example 3.68. *Prove that if a sequence $(h_n)_{n\in\mathbf{N}}$ of positive numbers has the property*

$$\lim_{n\to\infty}\frac{h_n}{h_{n-1}}=h, \quad \textit{then} \quad \lim_{n\to\infty}\sqrt[n]{h_n}=h.$$

Solution. Let us denote $g_n=\dfrac{h_n}{h_{n-1}}$, $h_0:=1$. Then from

$$G_n=\sqrt[n]{g_1\cdots g_n}=\sqrt[n]{\frac{h_1}{h_0}\cdots\frac{h_n}{h_{n-1}}}=\sqrt[n]{h_n}$$

and from Example 3.66, it follows that

$$\lim_{n\to\infty}\sqrt[n]{h_n}=h.$$

Example 3.69. *Determine the limits of the following sequences, given with their general terms.*

a) $f_n=\dfrac{\sqrt[n]{n!}}{n}$; **b)** $g_n=\dfrac{\sqrt[n]{(kn)!}}{n^k}$, $k\in N$;

c) $h_n=\dfrac{1}{n}\sqrt[n]{(n+1)(n+2)\cdots 2n}.$

Solutions.

a) If we denote by $x_n:=\dfrac{n!}{n^n}$, then

$$\frac{x_n}{x_{n-1}}=\frac{\dfrac{n!}{n^n}}{\dfrac{(n-1)!}{(n-1)^{n-1}}}=\frac{n(n-1)^{n-1}}{n^n}=\left(\frac{n-1}{n}\right)^{n-1}=\frac{1}{\left(1+\dfrac{1}{n-1}\right)^{n-1}}.$$

So we have $\lim\limits_{n\to\infty}\dfrac{x_n}{x_{n-1}}=\dfrac{1}{\mathrm{e}}$ and from the previous example it follows

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{\mathrm{e}}.$$

b) In this case we put $y_n=\dfrac{(kn)!}{n^{kn}}$ and obtain

$$\frac{y_n}{y_{n-1}}=\frac{\dfrac{(kn)!}{n^{kn}}}{\dfrac{(k(n-1))!}{(n-1)^{k(n-1)}}}=\frac{(n-1)^{kn}}{n^{kn}}\cdot\frac{kn\cdot(kn-1)\cdots(k(n-1)+1)}{(n-1)^k}$$

$$=\frac{1}{\left(\left(1+\dfrac{1}{n-1}\right)^{n-1}\right)^k}\cdot\frac{kn\cdot(kn-1)\cdots(k(n-1)+1)}{n^k}.$$

Thus we have $\lim_{n\to\infty} \frac{y_n}{y_{n-1}} = \frac{k^k}{e^k}$ and finally

$$\lim_{n\to\infty} \frac{\sqrt[n]{(kn)!}}{n^k} = \frac{k^k}{e^k}, \quad k \in \mathbf{N}.$$

c) $\lim_{n\to\infty} h_n = \frac{4}{e}$.

Example 3.70. *Let the sequence $(F_n)_{n\in\mathbf{N}}$ be the sequence of arithmetic means of a sequence $(f_n)_{n\in\mathbf{N}}$. Prove that then it holds*

$$\liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} F_n \le \limsup_{n\to\infty} F_n \le \limsup_{n\to\infty} f_n.$$

Solution. We shall prove only that

$$\limsup_{n\to\infty} F_n \le \limsup_{n\to\infty} f_n.$$

Let us denote by $f = \limsup_{n\to\infty} f_n$ and assume that $f \ge 0$. Then for every $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$ such that for every $n > n_0$ it holds $f_n < f + \varepsilon$. So for $n \ge n_0$ we have

$$\begin{aligned} F_n &= \frac{f_1 + f_2 + \cdots + f_{n_0} + \cdots + f_n}{n} \\ &< \frac{f_1 + f_2 + \cdots + f_{n_0}}{n} + \frac{(n-n_0)(f+\varepsilon)}{n} \le \frac{B}{n} + f + \varepsilon, \end{aligned}$$

where we put $B := f_1 + f_2 + \cdots + f_{n_0}$. For given $\varepsilon_1 > 0$ there exists $n_1 \in \mathbf{N}$ such that $\frac{|B|}{n} < \varepsilon_1$ for every $n > n_1$. Let $\varepsilon_2 > 0$ be given. Then choose $\varepsilon := \varepsilon_2/2$ and also $\varepsilon_1 := \varepsilon_2/2$. Let us put $n_2 := \max\{n_0, n_1\}$. Then it holds

$$(\forall n \in \mathbf{N}) \quad n > n_2 \Rightarrow F_n < f + \varepsilon_2,$$

and this means that $\limsup_{n\to\infty} F_n \le f$.

The case $f < 0$ is similar and left to the reader.

Exercise 3.71. *Let us suppose that $(p_n)_{n\in\mathbf{N}}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \sum_{k=1}^{n} p_k = +\infty$, and let $(a_n)_{n\in\mathbf{N}}$ be a sequence of real numbers. Put*

$$F_n := \frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_1 + \cdots + p_n}, \quad n \in \mathbf{N}.$$

(F_n is called the sequence of generalized arithmetic means *of the sequence $(a_n)_{n\in\mathbf{N}}$.) Prove then the following.*

$$\liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} F_n \le \limsup_{n\to\infty} F_n \le \limsup_{n\to\infty} a_n.$$

Thus if the sequence $(a_n)_{n\in\mathbf{N}}$ converges, then its sequence of generalized arithmetic means converges to the same limit.

Example 3.72. The Stolz theorem.
Let us suppose that the sequence of positive numbers $(P_n)_{n\in\mathbf{N}}$ satisfies the following two conditions

$$\lim_{n\to\infty} P_n = +\infty \quad \textit{and} \quad P_{n+1} > P_n.$$

If there exists $\displaystyle\lim_{n\to\infty} \frac{Q_n - Q_{n-1}}{P_n - P_{n-1}}$, *prove then the following equality.*

$$\lim_{n\to\infty} \frac{Q_n}{P_n} = \lim_{n\to\infty} \frac{Q_n - Q_{n-1}}{P_n - P_{n-1}}.$$

Solution. Using the conditions on the sequence P_n, we can construct the following two sequences

$$P_n = p_1 + p_2 + \cdots + p_n \quad \text{and} \quad Q_n = p_1 a_1 + p_2 a_2 + \cdots + p_n a_n.$$

So we have

$$\frac{Q_n - Q_{n-1}}{P_n - P_{n-1}} = \frac{p_n a_n}{p_n} = a_n.$$

This means that

$$\lim_{n\to\infty} \frac{Q_n - Q_{n-1}}{P_n - P_{n-1}} = \lim_{n\to\infty} a_n.$$

If there exists $\lim_{n\to\infty} a_n = \ell$, then from the previous exercise it follows that the sequence of generalized arithmetic means converges to the same limit, i.e.,

$$\lim_{n\to\infty} \frac{Q_n}{P_n} = \ell.$$

Example 3.73. *Prove that the statement opposite to the one from the Stolz theorem is not necessarily true. In other words, even if the limit*

$$\lim_{n\to\infty} \frac{Q_n}{P_n} = \ell \qquad \textit{exists, then the limit}$$

$$\lim_{n\to\infty} \frac{Q_n - Q_{n-1}}{P_n - P_{n-1}} \qquad \textit{might not exist.}$$

Solution. If we take, for example,

$$P_n = n \quad \text{and} \quad Q_n = \sum_{k=1}^{n} \sin\frac{k\pi}{3}, \quad n \in \mathbf{N},$$

then there exists

$$\lim_{n\to\infty} \frac{Q_n}{P_n} = \lim_{n\to\infty} \frac{\sin\frac{\pi}{3} + \sin\frac{2\pi}{3} + \cdots + \sin\frac{n\pi}{3}}{n} = 0.$$

However, the limit $\lim\limits_{n\to\infty} \dfrac{Q_n - Q_{n-1}}{P_n - P_{n-1}}$ does not exist, because

$$\liminf_{n\to\infty} \frac{Q_n - Q_{n-1}}{P_n - P_{n-1}} = \liminf_{n\to\infty} \sin\frac{n\pi}{3} = -\frac{\sqrt{3}}{2},$$

$$\limsup_{n\to\infty} \frac{Q_n - Q_{n-1}}{P_n - P_{n-1}} = \limsup_{n\to\infty} \sin\frac{n\pi}{3} = \frac{\sqrt{3}}{2}.$$

Example 3.74. *Determine the following limits by using the Stolz theorem.*

a) $f_n = \dfrac{1^k + 2^k + \cdots + n^k}{n^{k+1}}, \quad k \in \mathbf{N};$ **b)** $g_n = \dfrac{1 + \dfrac{1}{\sqrt{2}} + \cdots + \dfrac{1}{\sqrt{n}}}{n};$

c) $h_n = \dfrac{1^p + 3^p + \cdots + (2n+1)^p}{n^{p+1}}, \quad p \in \mathbf{Q};$ **d)** $x_n = \dfrac{\ln n}{n^{1/k}}, \quad k \in \mathbf{N};$

e) $y_n = \dfrac{1^k + 2^k + \cdots + n^k}{n^k} - \dfrac{n}{k+1}, \quad k \in \mathbf{N}.$

Solutions.

a) Let us denote by $Q_n = 1^k + 2^k + \cdots + n^k$ and $P_n = n^{k+1}$, $n \in \mathbf{N}$. Then from the equalities

$$\lim_{n\to\infty} \frac{Q_n - Q_{n-1}}{P_n - P_{n-1}} = \lim_{n\to\infty} \frac{n^k}{n^{k+1} - (n-1)^{k+1}}$$

$$= \lim_{n\to\infty} \frac{n^k}{n^{k+1} - n^{k+1} + (k+1)n^k - \cdots - (-1)^{k+1}} = \frac{1}{k+1}$$

we obtain that $\lim\limits_{n\to\infty} \dfrac{Q_n}{P_n} = \lim\limits_{n\to\infty} f_n = \dfrac{1}{k+1}.$

b) If we put $Q_n = 1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{n}}$ and $P_n = n$, $n \in \mathbf{N}$, then we have

$$\lim_{n\to\infty} g_n = \lim_{n\to\infty} \frac{Q_n}{P_n} = \lim_{n\to\infty} \frac{Q_n - Q_{n-1}}{P_n - P_{n-1}} = \frac{\frac{1}{\sqrt{n}}}{1} = 0.$$

c) $\lim\limits_{n\to\infty} h_n = \dfrac{2^p}{p+1}.$

d) Putting $Q_n = \ln n$ and $P_n = n^{1/k}$, $n \in \mathbf{N}$, we have

$$\lim_{n\to\infty}\frac{Q_n - Q_{n-1}}{P_n - P_{n-1}} = \lim_{n\to\infty}\frac{\ln n - \ln(n-1)}{n^{1/k} - (n-1)^{1/k}} = \lim_{n\to\infty}\frac{\ln\left(1+\dfrac{1}{n-1}\right)}{n^{1/k}\left(1-\left(1-\dfrac{1}{n}\right)^{1/k}\right)}$$

$$\leq \lim_{n\to\infty}\frac{1}{n-1}\cdot\frac{1}{n^{1/k}\left(1-\left(1-\dfrac{1}{n}\right)^{1/k}\right)}$$

$$= \lim_{n\to\infty}\frac{1}{n-1}\cdot\frac{1}{n^{(1/k)-1}}\sum_{j=0}^{k-1}\left(1-\frac{1}{n}\right)^{j/k} = 0.$$

Thus from Theorem 3.7 it follows that $\lim\limits_{n\to\infty}\dfrac{Q_n - Q_{n-1}}{P_n - P_{n-1}}$ exists and

$$\lim_{n\to\infty}\frac{Q_n - Q_{n-1}}{P_n - P_{n-1}} = 0 \Rightarrow \lim_{n\to\infty}\frac{Q_n}{P_n} = 0.$$

e) Let us denote $Q_n = (k+1)(1^k+2^k+\cdots+n^k)-n^{k+1}$ and $P_n = (k+1)n^k$, $n \in \mathbf{N}$. Then we have

$$\lim_{n\to\infty}\frac{Q_n}{P_n} = \lim_{n\to\infty}\frac{(k+1)(1^k+2^k+\cdots+n^k)-n^{k+1}}{(k+1)n^k}$$

$$= \lim_{n\to\infty}\frac{Q_n - Q_{n-1}}{P_n - P_{n-1}} = \lim_{n\to\infty}\frac{(k+1)n^k - (n^{k+1}-(n-1)^{k+1})}{(k+1)(n^k-(n-1)^k)} = \frac{1}{2}.$$

Example 3.75. *Let two sequences $(a_n)_{n\in\mathbf{N}}$ and $(b_n)_{n\in\mathbf{N}}$ be given and let us define the sequence $(c_n)_{n\in\mathbf{N}}$ by*

$$c_n = \frac{a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1}{n}, \quad n = 1, 2, \dots .$$

a) *If $\lim\limits_{n\to\infty} a_n = 0$ and $|b_n| \leq B$ for every $n \in \mathbf{N}$, then it holds $\lim\limits_{n\to\infty} c_n = 0$.*

b) *If $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$, then it holds $\lim\limits_{n\to\infty} c_n = a\cdot b$.*

Prove.

Solutions.

a) From the implication

$$\lim_{n\to\infty} a_n = 0 \Rightarrow \lim_{n\to\infty}|a_n| = 0,$$

and the Stolz theorem, it follows

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|a_k| = \lim_{n\to\infty}|a_n| = 0.$$

Since the sequence $(b_n)_{n\in\mathbf{N}}$ is bounded, i.e., $|b_n| \le B$ for every $n \in \mathbf{N}$, we have

$$|c_n| \le \frac{|a_1| + |a_2| + \cdots + |a_n|}{n} B \quad \text{and} \quad \lim_{n\to\infty} |c_n| \le \lim_{n\to\infty} \frac{B}{n} \sum_{k=1}^{n} |a_k| = 0.$$

This means that $\lim\limits_{n\to\infty} c_n = 0$.

b) Let us put

$$\lim_{n\to\infty} a_n = a \quad \text{and} \quad x_n = a_n - a \quad \text{for every} \quad n \in \mathbf{N}.$$

Then it holds $\lim\limits_{n\to\infty} x_n = 0$ and

$$\begin{aligned} c_n &= \frac{(x_1 + a)b_n + \cdots + (x_n + a)b_1}{n} \\ &= \frac{x_1 b_n + \cdots + x_n b_1}{n} + a \cdot \frac{b_n + \cdots + b_1}{n} = f_n + g_n. \end{aligned}$$

From a) it follows that

$$\lim_{n\to\infty} f_n = \lim_{n\to\infty} \frac{x_1 b_n + \cdots + x_n b_1}{n} = 0,$$

while

$$\lim_{n\to\infty} g_n = \lim_{n\to\infty} a \cdot \frac{b_n + \cdots + b_1}{n} = a \cdot b.$$

Therefore $\lim\limits_{n\to\infty} c_n = a \cdot b$.

3.4 Asymptotic relations

3.4.1 Basic notions

Definition 3.76. *We say that a sequence $(f_n)_{n\in N}$ is* **asymptotically equivalent** *to a sequence $(g_n)_{n\in N}$ as n tends to ∞ if $g_n \neq 0$ for every $n \in \mathbf{N}$ and it holds*

$$\lim_{n\to\infty} \frac{f_n}{g_n} = 1.$$

Then we write

$$f_n \sim g_n \quad \textit{as} \quad n \to \infty.$$

Definition 3.77. *Assume that $(g_n)_{n\in N}$ is a sequence with positive terms. We say that*

- *a sequence $(f_n)_{n\in N}$ is* **big oh** *of $(g_n)_{n\in N}$ as $n \to \infty$ if there exist an $n_0 \in \mathbf{N}$ and a constant $K > 0$ such that*

  $$|f_n| \le K \cdot g_n, \quad \textit{for every } n > n_0,$$

  and we write $f_n = O(g_n)$ as $n \to \infty$;

- *a sequence* $(f_n)_{n\in N}$ *is* **small oh** *of* $(g_n)_{n\in N}$ *as* $n \to \infty$ *if*

$$\lim_{n\to\infty} \frac{f_n}{g_n} = 0,$$

and we write $f_n = o(g_n)$ *as* $n \to \infty$.

In particular, if the sequences $(f_n)_{n\in \mathrm{N}}$ and $(g_n)_{n\in \mathrm{N}}$ diverge to $+\infty$ and satisfy the condition

$$f_n = o(g_n) \ \text{ as } n \to \infty \iff \lim_{n\to\infty} \frac{f_n}{g_n} = 0,$$

then we say that the sequence $(g_n)_{n\in \mathrm{N}}$ diverges faster to infinity then the sequence $(f_n)_{n\in \mathrm{N}}$ and we write

$$f_n \prec g_n, \quad \text{as } n \to \infty.$$

3.4.2 Examples and exercises

Example 3.78. *Let us give the following two sequences* $(f_n)_{n\in \mathrm{N}}$ *and* $(g_n)_{n\in \mathrm{N}}$ *by*

$$\textbf{a)} \quad f_n = n^4 + 3n^2 + 2 \quad \textit{and} \quad g_n = n^4;$$

$$\textbf{b)} \quad f_n = \sqrt[n]{n!} \quad \textit{and} \quad g_n = \frac{n}{\mathrm{e}};$$

$$\textbf{c)} \quad f_n = \ln n! \quad \textit{and} \quad g_n = n \ln n\,.$$

Prove that f_n *is asymptotically equivalent to* g_n, *denoted by* $f_n \sim g_n$ *as* $n \to \infty$.

Solutions. By Definition 3.76 we can write

$$f_n \sim g_n \ \text{ as } \ n \to \infty \iff \lim_{n\to\infty} \frac{f_n}{g_n} = 1$$

and we have to check the last limit in all three cases.

a) $\displaystyle\lim_{n\to\infty} \frac{n^4 + 3n^2 + 2}{n^4} = 1.$

b) We shall prove the given asymptotic equivalence in two ways.

- **First method.** Use Example 3.69 a).
- **Second method.** We shall use the **Stirling formula**

$$n! = n^n \mathrm{e}^{-n} \sqrt{2\pi n} \cdot \mathrm{e}^{\theta/(12n)}, \quad 0 < \theta < 1,$$

thus

$$n! \sim n^n \mathrm{e}^{-n} \sqrt{2\pi n} \quad \text{as } n \to \infty. \tag{3.25}$$

This implies

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{\dfrac{n}{\mathrm{e}}} = \lim_{n\to\infty} \frac{\sqrt[n]{n^n \mathrm{e}^{-n} \sqrt{2\pi n}\, \mathrm{e}^{\theta/(12n)}}}{\dfrac{n}{\mathrm{e}}} = 1.$$

c) Using the Stirling formula we have

$$\lim_{n\to\infty}\frac{\ln n!}{n\ln n}=\lim_{n\to\infty}\frac{n\ln n-n+\dfrac{1}{2}\ln(2\pi n)+\dfrac{\theta}{12n}}{n\ln n}=1.$$

Example 3.79. *Prove the following asymptotic relations.*

a) $n^a \prec n^b$ *for* $0<a<b$;

b) $p^n \prec q^n$ *for* $0<p<q$;

c) $n^a \prec q^n$ *for* $a>0,\, q>1$;

d) $q^n \prec n!$ *for* $q>1$;

e) $\ln n \prec n^a$ *for* $a>0$;

f) $n! \prec n^n$.

Solutions.

e) There exists $q\in\mathbf{N}$, such that $a>1/q$. Then it holds

$$\frac{\ln n}{n^a}<\frac{\ln n}{n^{1/q}}.$$

From the relations

$$\frac{\ln(n+1)-\ln n}{(n+1)^{1/q}-n^{1/q}}=\frac{\ln\left(1+\dfrac{1}{n}\right)^n}{n((n+1)^{1/q}-n^{1/q})}\le\frac{q(n+1)^{(q-1)/q}}{n}\ln\left(1+\frac{1}{n}\right)^n,$$

it follows

$$\lim_{n\to\infty}\frac{\ln(n+1)-\ln n}{(n+1)^{1/q}-n^{1/q}}=0.$$

The last inequality we obtained from the following identity

$$1=(\sqrt[q]{n+1}-\sqrt[q]{n})\left(\sqrt[q]{(n+1)^{q-1}}+\sqrt[q]{(n+1)^{q-2}n}+\cdots+\sqrt[q]{n^{q-1}}\right),$$

which implies

$$\sqrt[q]{n+1}-\sqrt[q]{n}\ge\frac{1}{q\sqrt[q]{(n+1)^{q-1}}}.$$

From the Stolz theorem we have

$$\lim_{n\to\infty}\frac{\ln n}{n^{1/q}}=0,\quad\text{and this implies}\quad\lim_{n\to\infty}\frac{\ln n}{n^a}=0.$$

Remark. Using this example, we can form a (partial) scale of growths for sequences diverging to infinity as follows.

$$\ln n\prec n^a\prec n^b\prec q^n\prec n!\prec n^n,\quad 0<a<b,\quad q>1.$$

Example 3.80. *If* $(f_n)_{n\in\mathbf{N}}$ *is a given sequence such that* $f_n>1$ *and* $\lim_{n\to\infty}f_n=+\infty$, *then it holds* $q^{f_n}\succ f_n$, *for* $q>1$.

Solution. Since $[f_n] \le f_n < [f_n] + 1$ and $q^{f_n} \ge q^{[f_n]}$, then

$$\frac{f_n}{q^{f_n}} < \frac{[f_n]+1}{q^{[f_n]}} = \frac{n_k+1}{q^{n_k}}.$$

The sequence given by $\dfrac{n_k+1}{q^{n_k}}$ is a subsequence of the sequence given by $\dfrac{n+1}{q^n}$. From the limit

$$\lim_{n\to\infty} \frac{n+1}{q^n} = 0$$

it follows that

$$\lim_{n\to\infty} \frac{n_k+1}{q^{n_k}} = 0, \quad \text{i.e.,} \quad \lim_{n\to\infty} \frac{q^{f_n}}{f_n} = 0.$$

From this example we can form another scale of the growth of sequences which diverge to infinity,

$$n^a \prec q^{n^a} \prec q^{q^{n^a}} \prec ..., \quad a > 0,\ q > 1.$$

Example 3.81. *If it holds $q > 1$ and $a_n \succ b_n, n \in \mathbf{N}$, prove then*

$$q^{a_n} \succ q^{b_n}.$$

Solution. From the following equalities

$$\lim_{n\to\infty} \frac{q^{a_n}}{q^{b_n}} = \lim_{n\to\infty} q^{a_n - b_n} = q^{a_n(1-b_n/a_n)} = +\infty,$$

because $\displaystyle\lim_{n\to\infty} \frac{b_n}{a_n} = 0, \quad \lim_{n\to\infty} a_n = +\infty$, we obtain

$$\lim_{n\to\infty} \frac{q^{b_n}}{q^{a_n}} = 0.$$

From this example we can again form the following scale of growth for the sequences which diverge to infinity

$$q^{\sqrt{n}} \prec q^n \prec q^{n^2} \prec ..., \quad q > 1.$$

Example 3.82. *If $(f_n)_{n\in\mathbf{N}}$ is a given sequence such that $f_n > 1$ and $\lim_{n\to\infty} f_n = +\infty$, prove then*

$$\ln f_n \prec f_n.$$

Solution.

$$\frac{\ln f_n}{f_n} < \frac{\ln([f_n]+1)}{[f_n]} = \frac{\ln(n_k+1)}{n_k}.$$

The sequence given by $\dfrac{\ln(n_k+1)}{n_k}$ is a subsequence of the sequence given by $\dfrac{\ln(n+1)}{n}$. From Example 3.79 e) it follows that

$$\lim_{n\to\infty} \frac{\ln(n+1)}{n} = 0, \quad \text{i.e.,} \quad \lim_{n\to\infty} \frac{\ln f_n}{f_n} = 0.$$

Example 3.83. *If the sequences $(f_n)_{n\in\mathbf{N}}$ and $(f_n)_{n\in\mathbf{N}}$ satisfy $f_n > 0$, $g_n > 0$, prove then*

a) $O(1) + O(1) = O(1)$; **b)** $o(1) + o(1) = o(1)$;

c) $o(1) = O(1)$; **d)** $O(f_n) + O(g_n) = O(f_n + g_n)$;

e) $o(f_n) + o(g_n) = o(f_n + g_n)$,

where $f_n = O(1)$ means that the sequence $(f_n)_{n\in\mathbf{N}}$ is bounded and $f_n = o(1)$ means that the sequence $(f_n)_{n\in\mathbf{N}}$ tends to 0, and $(f_n + g_n)_{n\in\mathbf{N}} = (f_n)_{n\in\mathbf{N}} + (g_n)_{n\in\mathbf{N}}$.

Remark. The asymptotic relations in all cases a) - e) are supposed to be read only from left to the right.

Solutions.

a) The sum of two bounded sequences is a bounded sequence.

b) The sum of two sequences converging to zero is a sequence converging to zero too.

c) A convergent sequence is a bounded one.

d) By Definition 3.77 we have

$$f_n = O(g_n) \iff ((\exists M > 0)\, (\exists n_0 \in \mathbf{N})\, (n > n_0) \quad |f_n| \le M g_n).$$

Therefore, the expressions

$$\Psi_n = O(f_n), \quad U_n = O(g_n) \quad \text{mean} \quad |\Psi_n| \le K f_n, \quad |U_n| \le M g_n,$$

for some $K > 0$, $M > 0$ and $n > n_0$, for some $n_0 \in \mathbf{N}$. Thus it holds for $n > n_0$

$$|\Psi_n + U_n| \le K f_n + M g_n \le L(f_n + g_n),$$

where $L = \max\{K, M\}$. This means that $\Psi_n + U_n = O(f_n + g_n)$.

e) The expressions

$$a_n = o(f_n), \quad b_n = o(g_n) \quad \text{mean} \quad \lim_{n\to\infty} \frac{a_n}{f_n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{b_n}{g_n} = 0.$$

It holds $\left|\dfrac{a_n}{f_n + g_n}\right| \le \left|\dfrac{a_n}{f_n}\right|$ and $\left|\dfrac{b_n}{f_n + g_n}\right| \le \left|\dfrac{b_n}{g_n}\right|$. Then we have

$$\lim_{n\to\infty} \frac{a_n + b_n}{f_n + g_n} = \lim_{n\to\infty} \left(\frac{a_n}{f_n + g_n} + \frac{b_n}{f_n + g_n}\right) = 0.$$

So we obtain $o(f_n) + o(g_n) = o(f_n + g_n)$.

Exercise 3.84. *If the sequences* $(f_n)_{n\in\mathbf{N}}$ *and* $(g_n)_{n\in\mathbf{N}}$ *satisfy* $f_n > 0$ *and* $g_n > 0$, $n \in \mathbf{N}$, *prove then*

a) $O(f_n) + o(g_n) = O(f_n + g_n)$; **b)** $O(f_n)\cdot O(g_n) = O(f_n\cdot g_n)$;

c) $O(f_n)\cdot o(g_n) = o(f_n\cdot g_n)$; **d)** $o(f_n)\cdot o(g_n) = o(f_n\cdot g_n)$.

Example 3.85. *If* $f_n \sim g_n$, $n\to\infty$, *then it holds* $f_n - g_n = o(g_n)$. *Prove.*

Solution. The limit $\lim\limits_{n\to\infty}\dfrac{f_n}{g_n} = 1$, means that for every $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$ such that it holds

$$\left|\frac{f_n}{g_n} - 1\right| < \varepsilon, \quad \text{for every} \quad n > n_0.$$

Thus we have

$$-\varepsilon < \frac{f_n - g_n}{g_n} < \varepsilon, \quad \text{meaning} \quad \lim_{n\to\infty}\frac{f_n - g_n}{g_n} = 0, \quad \text{hence} \quad f_n - g_n = o(g_n).$$

Exercise 3.86. *Prove that* $f_n \sim g_n$ *as* $n\to\infty$, *if the sequences* $(f_n)_{n\in\mathbf{N}}$ *and* $(g_n)_{n\in\mathbf{N}}$ *are given as follows.*

a) $f_n = \dfrac{2n^3 + 3n^2 + n + 1}{3n^2 + 1}$, $g_n = \dfrac{2}{3}n$;

b) $f_n = 1 + 2 + 3 + \cdots + n$, $g_n = \dfrac{n^2}{2}$;

c) $f_n = \sqrt[3]{n^4} + n + 2$, $g_n = n\sqrt[3]{n}$;

d) $f_n = 1^p + 3^p + \cdots + (2n+1)^p$, $g_n = \dfrac{2^p n^{p+1}}{p+1}$, $p \in \mathbf{Q}$.

Exercise 3.87.

a) *Does a "fastest" sequence* $(g_n)_{n\in\mathbf{N}}$ *exist, i.e., such that*

$$\lim_{n\to\infty}\frac{f_n}{g_n} = 0, \quad \textit{for every} \quad (f_n)_{n\in\mathbf{N}}?$$

b) *Does a "slowest" sequence exist?*

c) *If for two sequences* f_n *and* g_n *it holds that* $f_n \prec g_n$ *as* $n\to\infty$, *prove that then there exists a sequence* $(c_n)_{n\in\mathbf{N}}$ *such that*

$$f_n \prec c_n \prec g_n \quad \textit{as } n\to\infty.$$

Answers.

a) No, because it always holds $f_n \prec f_n^n$.

b) No, because it always holds $\ln f_n \prec f_n$.

c) If the given sequences have positive terms, then, for an example, one can take $c_n = \sqrt{f_n g_n}$.

Chapter 4

Limits of functions

4.1 Limits

4.1.1 Basic notions

For the definition of the limit of a function at a point, we need the notion of the *accumulation point of a set,* see Definition 1.76 c).

Definition 4.1. *Let x_0 be an accumulation point of the domain $A \subset \mathbf{R}$ of a function $f : A \to \mathbf{R}$. Then the number L is the* **limit of the function f as x approaches** *x_0 iff for every $\varepsilon > 0$, there exists a $\delta > 0$, $\delta = \delta(\varepsilon)$, such that for every $x \in A$ with the property*

$$0 < |x - x_0| < \delta \quad \textit{it holds} \quad |f(x) - L| < \varepsilon.$$

Then we write $f(x) \to L$ when $x \to x_0$, $x \in A$, or rather

$$\lim_{x \to x_0,\, x \in A} f(x) = L.$$

Note that we do not need the function f to be defined in the point x_0. Even if $x_0 \in A$, the value of f at x_0 is irrelevant.

Using logical symbols, Definition 4.1 can be expressed as follows.

$$\lim_{x \to x_0,\, x \in A} f(x) = L$$

$$\iff (\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\forall x \in A) \quad 0 < \mid x - x_0 \mid < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Definition 4.1 is equivalent to the following.

Definition 4.2. *Let x_0 be an accumulation point of the domain A of a function $f : A \to \mathbf{R}$. Then the number L is the* **limit of the function f as x approaches** *x_0 if for every sequence $(x_n)_{n \in \mathbf{N}}$ from $A \setminus \{x_0\}$ converging to x_0 it holds*

$$\lim_{n \to \infty} f(x_n) = L.$$

If in Definition 4.1 we take only the values $x \in A$ greater (resp. smaller) than x_0, we get the definition of the **right-hand limit** (resp. **left-hand limit**) of a function $f : A \to \mathbf{R}$ at the point x_0. The right-hand limit of f at x_0 is denoted by

$$\lim_{x \to x_0+,\, x \in A} f(x),$$

while the left-hand limit of f at x_0 is denoted by

$$\lim_{x \to x_0-,\, x \in A} f(x).$$

Theorem 4.3. *Let x_0 be an accumulation point of the domain of a function $f : A \to \mathbf{R}$. If both the left-hand and the right-hand limit of f at the point x_0 exist, then a necessary and sufficient condition for the existence of the limit of f at x_0 is the equality*

$$\lim_{x \to x_0-,\, x \in A} f(x) = \lim_{x \to x_0+,\, x \in A} f(x) =: L. \tag{4.1}$$

Clearly, (4.1) is equivalent to $\lim\limits_{x \to x_0,\, x \in A} f(x) = L$.

Most often, the domain A of the function f will be its the natural domain. In that case, we shall simply write

$$\lim_{x \to x_0} f(x), \quad \lim_{x \to x_0+} f(x) \quad \text{and} \quad \lim_{x \to x_0-} f(x)$$

for the limit, right-hand limit and the left-hand limit of the function f at the point x_0.

Definition 4.4.

- *Assume the domain A of the function $f : A \to \mathbf{R}$ contains the interval $(a, +\infty)$ for some $a \in \mathbf{R}$. The number L is the* **limit of f at $+\infty$** *if for every $\varepsilon > 0$ there exists a number $T > a$, $T = T(\varepsilon)$, such that for every $x > T$ it holds $| f(x) - L |< \varepsilon$. Then we write*

$$\lim_{x \to +\infty} f(x) = L. \tag{4.2}$$

- *Assume the domain A of the function $f : A \to \mathbf{R}$ contains the interval $(-\infty, b)$ for some $b \in \mathbf{R}$. The number L is the* **limit of f at $-\infty$** *if for every $\varepsilon > 0$ there exists a number $T < b$, $T = T(\varepsilon)$, such that for every $x < T$ it holds $| f(x) - L |< \varepsilon$. Then we write*

$$\lim_{x \to -\infty} f(x) = L. \tag{4.3}$$

In Definition 4.4 x went in the first case to *plus infinity* through increasing positive values, while in the second case it went to *minus infinity* through decreasing negative values.

Definition 4.5. *Assume the domain A of the function f contains the interval (x_0, b) and assume that for every $T > 0$, there exists a $\delta > 0$, $\delta = \delta(T)$, such that for every $x \in A$ and $x \in (x_0, x_0 + \delta)$ it holds $f(x) > T$. Then we say that the function f* **tends to plus infinity** *when $x \to x_0+$, and write*

$$\lim_{x \to x_0+} f(x) = +\infty. \tag{4.4}$$

Analogous meanings have the following notations:

$$\lim_{x \to x_0+} f(x) = -\infty, \quad \lim_{x \to x_0-} f(x) = +\infty \quad \text{and} \quad \lim_{x \to x_0-} f(x) = -\infty.$$

Relation of the limit of a function with the basic operations and inequalities gives the following statement.

Theorem 4.6. *Assume the functions f and g are defined on a set $A \subset \mathbf{R}$ and let x_0 be an accumulation point of A. Moreover, assume that the following two limits exist:*

$$\lim_{x \to x_0,\, x \in A} f(x) = L \quad \text{and} \quad \lim_{x \to x_0,\, x \in A} g(x) = K.$$

Then it holds

- $\displaystyle\lim_{x \to x_0,\, x \in A} (f(x) \pm g(x)) = L \pm K;$
- $\displaystyle\lim_{x \to x_0,\, x \in A} (f(x) \cdot g(x)) = L \cdot K;$
- $\displaystyle\lim_{x \to x_0,\, x \in A} \frac{f(x)}{g(x)} = \frac{L}{K},$
  where we additionally assume that there exists a $\delta > 0$ such that $g(x) \neq 0$, for all x in the set $(x_0 - \delta,\ x_0 + \delta) \cap A$ and, moreover, $K \neq 0$;
- $(\exists \delta > 0)\ (\forall x \in A) \quad (0 < |x - x_0| < \delta \Rightarrow f(x) \le g(x)) \quad \Rightarrow \quad L \le K.$

The last equalities and inequalities remain true if the point x_0 is replaced with one of the symbols $+\infty$ or $-\infty$.

4.1.2 Examples and exercises

Example 4.7. *Using Definitions 4.1 and 4.2, show that*

$$\lim_{x \to 2} \frac{x^2 - 4}{x^2 - 2x} = 2.$$

Solutions.

- First, let us use Definition 4.1. For arbitrary $\varepsilon > 0$ and $x \in (1, 3)$, $x \neq 2$, we can write
$$\left| \frac{x^2 - 4}{x^2 - 2x} - 2 \right| = \left| \frac{x + 2}{x} - 2 \right| = \frac{|x - 2|}{x}.$$

So we can take $\delta := \varepsilon$, and for $x \in (1,3)$ we have

$$(0 < |x-2| < \delta) \Rightarrow \left|\frac{x^2-4}{x^2-2x} - 2\right| = \frac{|x-2|}{x} < \frac{\delta}{1} = \varepsilon.$$

Therefore, we have

$$(\forall\varepsilon > 0)\ (\exists\delta = \varepsilon)\ (\forall x \in (1,3))\quad 0 < |x-2| < \delta \Rightarrow \left|\frac{x^2-4}{x^2-2x} - 2\right| < \varepsilon.$$

- In order to apply Definition 4.2, let us consider a sequence $(x_n)_{n\in\mathbf{N}}$ such that for every $n \in \mathbf{N}$, $1 < x_n < 3$, $x_n \neq 2$ and

$$\lim_{n\to\infty} x_n = 2.$$

Then we have

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{x_n^2-4}{x_n^2-2x_n} = \lim_{n\to\infty} \frac{x_n+2}{x_n} = \frac{\lim\limits_{n\to\infty} x_n + 2}{\lim\limits_{n\to\infty} x_n} = 2.$$

Example 4.8. *Show that the function* $f(x) = \sin\dfrac{\pi}{x}$, $x \in R\setminus\{0\}$, *has no limit at the point* $x_0 = 0$.

Solution. Let us consider two sequences given by $x_n = \dfrac{1}{n}$ and $y_n = \dfrac{2}{4n+1}$, $n = 1,2,\ldots$, which have the same limit 0, i.e., $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = 0$. But it holds

$$\lim_{n\to\infty} f(x_n) = \sin\pi n = 0, \quad \lim_{n\to\infty} f(y_n) = \sin\pi\frac{4n+1}{2} = 1.$$

Therefore from Definition 4.2 it follows that there is no $\lim\limits_{x\to 0}\sin\dfrac{\pi}{x}$.

Remark. Neither the limit $\lim\limits_{x\to 0+} f(x)$ nor $\lim\limits_{x\to 0-} f(x)$ exists.

Exercise 4.9. *Prove that* $\lim\limits_{x\to 0} f(x)$ *does not exist for the following functions.*

a) $f(x) = \cos\operatorname{sgn}\left(\dfrac{1}{x}\right)$; **b)** $f(x) = \operatorname{sgn}\cos\left(\dfrac{1}{x}\right)$.

Exercise 4.10. *Assume the functions* f *and* g *have no limits at the point* $x = x_0$. *Does this imply that the following limits also do not exist:*

$$\lim_{x\to x_0}(f(x)+g(x)) \quad \textit{and} \quad \lim_{x\to x_0}(f(x)\cdot g(x))\ ?$$

Answer. Not necessarily. The limits $\lim\limits_{x\to 0}\dfrac{1}{x}$ and $\lim\limits_{x\to 0}\left(-\dfrac{1}{x}\right)$ do not exist, but it holds

$$\lim_{x\to 0}\left(\frac{1}{x} + \left(-\frac{1}{x}\right)\right) = \lim_{x\to 0} 0 = 0.$$

Example 4.11. *Show that* $\lim\limits_{x\to 2}\dfrac{1}{(x-2)^2}=+\infty.$

Solution. Let $T>0$ be given. Then from the relations

$$\frac{1}{(x-2)^2}>T, \quad (x-2)^2<\frac{1}{T}$$

it follows that we can find $\delta=\dfrac{1}{\sqrt{T}}$ such that it holds

$$0<|x-2|<\frac{1}{\sqrt{T}}=\delta \quad\Rightarrow\quad \frac{1}{(x-2)^2}>T.$$

From Definition 4.5 it follows that $\lim\limits_{x\to 2}\dfrac{1}{(x-2)^2}=+\infty.$

Example 4.12. *Determine the following limits.*

a) $\lim\limits_{x\to 2}\dfrac{2x^3-3x^2-x-2}{x-2}$;

b) $\lim\limits_{x\to 0}\dfrac{x^8+12x^6+3x^3}{x^7+4x^6+x^5+x^3}$;

c) $\lim\limits_{x\to 5}\dfrac{3x^2-13x-10}{2x^2-7x-15}$;

d) $\lim\limits_{x\to 0}\left(\dfrac{2x+3}{x^2-x}-3\dfrac{x+1}{x^3-x}\right)$;

e) $\lim\limits_{x\to 2}\dfrac{x^3-8}{x^2-4}$;

f) $\lim\limits_{x\to 1}\dfrac{x^7+12x^5-13x^4+5x^2+4x-9}{x^5-4x^4+3x^3-2x^2+x+1}$;

g) $\lim\limits_{x\to 2}\dfrac{x^5-3x^4+5x^3-7x^2+4}{x^5-5x^4+8x^3+x^2-12x+4}$.

Solutions.

a) $\lim\limits_{x\to 2}\dfrac{2x^3-3x^2-x-2}{x-2}=\lim\limits_{x\to 2}\dfrac{(x-2)(2x^2+x+1)}{x-2}=\lim\limits_{x\to 2}(2x^2+x+1)=11.$

b) $\lim\limits_{x\to 0}\dfrac{x^8+12x^6+3x^3}{x^7+4x^6+x^5+x^3}=\lim\limits_{x\to 0}\dfrac{x^3(x^5+12x^3+3)}{x^3(x^4+4x^3+x^2+1)}=3.$

c) $\lim\limits_{x\to 5}\dfrac{3x^2-13x-10}{2x^2-7x-15}=\lim\limits_{x\to 5}\dfrac{(x-5)(3x+2)}{(x-5)(2x+3)}=\lim\limits_{x\to 5}\dfrac{3x+2}{2x+3}=\dfrac{17}{13}.$

d) $\lim\limits_{x\to 0}\left(\dfrac{2x+3}{x^2-x}-3\dfrac{x+1}{x^3-x}\right)=\lim\limits_{x\to 0}\dfrac{2x^2+2x}{x(x-1)(x+1)}=-2.$

e) $\lim\limits_{x\to 2}\dfrac{x^3-8}{x^2-4}=\lim\limits_{x\to 2}\dfrac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}=\dfrac{12}{4}=3.$

f) If the value $x=x_0$ is a zero of the polynomial $P_n(x)$, then it holds

$$P_n(x)=(x-x_0)Q_{n-1}(x),$$

where the coefficients of the polynomial $Q_{n-1}(x)$ can be ordered by using the Horner scheme (see Subsection 2.2.1). For the polynomial in the numerator it holds

$$\begin{array}{cccccccc|l} 1 & 0 & 12 & -13 & 0 & 5 & 4 & -9 & x=1 \\ & 1 & 1 & 13 & 0 & 0 & 5 & 9 & 0. \end{array}$$

So we have

$$x^7+12x^5-13x^4+5x^2+4x-9=(x-1)(x^6+x^5+13x^4+5x+9).$$

For the polynomial in the denominator we have

$$\begin{array}{cccccc|l} 1 & -4 & 3 & -2 & 1 & 1 & x=1 \\ & 1 & -3 & 0 & -2 & -1 & 0, \end{array}$$

and it follows

$$x^5-4x^4+3x^3-2x^2+x+1=(x-1)(x^4-3x^3-2x-1).$$

So we obtain

$$\begin{aligned} &\lim_{x\to 1}\frac{x^7+12x^5-13x^4+5x^2+4x-9}{x^5-4x^4+3x^3-2x^2+x+1} \\ = &\lim_{x\to 1}\frac{(x-1)(x^6+x^5+13x^4+5x+9)}{(x-1)(x^4-3x^3-2x-1)}=-\frac{29}{5}. \end{aligned}$$

g) In this case we have the following Horner scheme

$$\begin{array}{cccccc|l} 1 & -3 & 5 & -7 & 0 & 4 & x=2 \\ & 1 & -1 & 3 & -1 & -2 & 0. \end{array}$$

Therefore, it holds

$$x^5-3x^4+5x^3-7x^2+4=(x-2)(x^4-x^3+3x^2-x-2).$$

Also, we have

$$\begin{array}{cccccc|l} 1 & -5 & 8 & 1 & -12 & 4 & x=2 \\ & 1 & -3 & 2 & 5 & -2 & 0. \end{array}$$

So, we can write

$$x^5-5x^4+8x^3+x^2-12x+4=(x-2)(x^4-3x^3+2x^2+5x-2).$$

Therefore it holds

$$\lim_{x\to 2}\frac{x^5-3x^4+5x^3-7x^2+4}{x^5-5x^4+8x^3+x^2-12x+4}=\lim_{x\to 2}\frac{(x-2)(x^4-x^3+3x^2-x-2)}{(x-2)(x^4-3x^3+2x^2+5x-2)}=2.$$

Exercise 4.13. *Determine the following limits.*

a) $\lim\limits_{x\to 2}\dfrac{x^2-3x+2}{x^2-4}$; b) $\lim\limits_{x\to+\infty}\dfrac{x^2-3x+2}{x^2-4}$;

c) $\lim\limits_{x\to -2}\dfrac{x^2-3x+2}{(x+2)^2}$; d) $\lim\limits_{x\to 1}\dfrac{x^7+5x^6-4x^4-7x^2+5}{x^6-3x^5+x^3+2x-1}$.

Answers.

a) $\dfrac{1}{4}$. b) 1. c) $+\infty$. d) $-\dfrac{7}{4}$.

Example 4.14. *Determine the following limits.*

a) $\lim\limits_{x\to 1}\dfrac{x^{101}-101x+100}{x^2-2x+1}$;

b) $\lim\limits_{x\to 1}\left(\dfrac{m}{1-x^m}-\dfrac{n}{1-x^n}\right)$, $m,n\in\mathbf{N}$;

c) $\lim\limits_{x\to 1}\dfrac{x^{m+1}-x^{n+1}+x^n-mx+m-1}{(x-1)^2}$, $m,n\in\mathbf{N}$;

d) $\lim\limits_{x\to 0}\dfrac{(1+mx)^n-(1+nx)^m}{x^2}$, $m,n\in\mathbf{N}$.

Solutions.

a) $\lim\limits_{x\to 1}\dfrac{x^{101}-101x+100}{x^2-2x+1}=\lim\limits_{x\to 1}\dfrac{(x-1)^2(x^{99}+2x^{98}+\cdots+99x+100)}{(x-1)^2}=5050.$

b) Assume $m>n>3$. If we introduce $t=x-1$ ($t\to 0$ when $x\to 1$), then we obtain

$$\lim_{x\to 1}\left(\frac{m}{1-x^m}-\frac{n}{1-x^n}\right)=\lim_{t\to 0}\left(\frac{m}{1-(1+t)^m}-\frac{n}{1-(1+t)^n}\right)$$

$$=\lim_{t\to 0}\left(\frac{m}{-t^m-\binom{m}{1}t^{m-1}-\cdots-\binom{m}{m-1}t}-\frac{n}{-t^n-\binom{n}{1}t^{n-1}-\cdots-\binom{n}{n-1}t}\right)$$

$$= \lim_{t\to 0} \frac{\left(-m\frac{n(n-1)}{2} + n\frac{m(m-1)}{2}\right)t + t^2\left(-m\binom{n}{n-3} + n\binom{m}{m-3}\right) + \cdots}{t(t^{m-1} + mt^{m-2} + \cdots + m)(t^{n-1} + nt^{n-2} + \cdots + n)}$$

$$+ \lim_{t\to 0} \frac{t^{n-1}\left(-m + n\binom{m}{m-n}\right) + \cdots + nt^{m-1}}{t(t^{m-1} + mt^{m-2} + \cdots + m)(t^{n-1} + nt^{n-2} + \cdots + n)}$$

$$= \lim_{t\to 0} \frac{m\cdot n\left(\frac{m-1}{2} - \frac{n-1}{2}\right) + o(t)}{(t^{m-1} + mt^{m-2} + \cdots + m)(t^{n-1} + nt^{n-2} + \cdots + n)} = \frac{m-n}{2}.$$

The other cases are left to the reader.

c) Taking $t = x - 1$ ($t \to 0$ when $x \to 1$), we obtain

$$\lim_{x\to 1} \frac{x^{m+1} - x^{n+1} + x^n - mx + m - 1}{(x-1)^2}$$

$$= \lim_{t\to 0} \frac{(t+1)^{m+1} - (t+1)^{n+1} + (t+1)^n - m(t+1) + m - 1}{t^2}$$

$$= \lim_{t\to 0} \frac{t^2\left(\frac{(m+1)m}{2} - \frac{n(n+1)}{2} + \frac{n(n-1)}{2}\right) + o(t^2)}{t^2} = \frac{m^2 + m - 2n}{2}.$$

Here $o(t^2)$ stands for a function $\phi(t)$ with the property $\lim_{t\to 0} \frac{\phi(t)}{t^2} = 0$ (see Definition 4.54).

d) Similarly as in the previous case we have

$$\lim_{x\to 0} \frac{(1+mx)^n - (1+nx)^m}{x^2} = \frac{n(n-1)}{2}m^2 - \frac{m(m-1)}{2}n^2 = \frac{mn(n-m)}{2}.$$

Exercise 4.15. *Determine the following limits.*

a) $\lim_{x\to 1}\left(\frac{3}{1-x^3} + \frac{1}{x-1}\right)$; **b)** $\lim_{x\to 1} \frac{x^m - 1}{x^n - 1}$, $m, n \in \mathbf{N}$;

c) $\lim_{x\to 1} \frac{(x^n - 1)(x^{n-1} - 1)\cdots(x^{n-m-1} - 1)}{(x-1)(x^2-1)\cdots(x^m - 1)}$, $m, n \in \mathbf{N}$, $m < n$.

Answers.

a) 1. **b)** $\frac{m}{n}$. **c)** $\frac{n(n-1)\cdots(n-m-1)}{1\cdot 2\cdots m}$.

Example 4.16. *Prove that* $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$, $a > 0$, *and* $\lim_{x\to 0+} \sqrt[n]{x} = 0$, $n \in \mathbf{N}$.

Solution. Let us first suppose that $a > 0$. Then for arbitrary $\varepsilon > 0$, from

$$|\sqrt[n]{x} - \sqrt[n]{a}| = \frac{|x-a|}{\sqrt[n]{x^{n-1}} + \sqrt[n]{x^{n-2}a} + \cdots + \sqrt[n]{a^{n-1}}} < \frac{|x-a|}{\sqrt[n]{a^{n-1}}} < \varepsilon,$$

it follows that we can order δ such that it holds

$$|x-a| < \delta := \varepsilon \sqrt[n]{a^{n-1}} \Rightarrow |\sqrt[n]{x} - \sqrt[n]{a}| < \varepsilon.$$

If $a = 0$, then, for $x > 0$ and arbitrary $\varepsilon > 0$, it follows that there exists $\delta := \varepsilon^n$ such that it holds

$$0 < x < \delta \Rightarrow |\sqrt[n]{x}| < \varepsilon,$$

and this means that $\lim_{x\to 0+} \sqrt[n]{x} = 0$.

Remark. It holds that $\lim_{x\to+\infty} \sqrt[n]{x} = +\infty$, because for every $T > 0$ it holds $\sqrt[n]{x} > T$, provided that $x > T^n$.

Example 4.17. *Determine the following limits.*

a) $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$; b) $\lim_{x\to 3} \frac{\sqrt{x+6}-3}{x-3}$;

c) $\lim_{x\to 2} \frac{x-2}{\sqrt{x+2}-2}$; d) $\lim_{x\to 0} \frac{\sqrt{x^2+x+1}-x-1}{x}$;

e) $\lim_{x\to 8} \frac{\sqrt{8+x}-4}{\sqrt[3]{x}-2}$; f) $\lim_{x\to a+} \frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}$; $a > 0$.

Solutions.

a) $$\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} = \lim_{x\to 9} \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)} = \lim_{x\to 9} \frac{x-9}{(x-9)(\sqrt{x}+3)} = 1/6.$$

b) $$\lim_{x\to 3} \frac{\sqrt{x+6}-3}{x-3} = \lim_{x\to 3} \frac{(\sqrt{x+6}-3)(\sqrt{x+6}+3)}{(x-3)(\sqrt{x+6}+3)} = 1/6.$$

c) $$\lim_{x\to 2} \frac{x-2}{\sqrt{x+2}-2} = \lim_{x\to 2} \frac{(x-2)(\sqrt{x+2}+2)}{(\sqrt{x+2}-2)(\sqrt{x+2}+2)}$$
$$= \lim_{x\to 2} \frac{(\sqrt{x+2}+2)(x-2)}{x-2} = 4.$$

d) $$\lim_{x\to 0} \frac{\sqrt{x^2+x+1}-x-1}{x} = \lim_{x\to 0} \frac{(\sqrt{x^2+x+1}-x-1)(\sqrt{x^2+x+1}+x+1)}{x(\sqrt{x^2+x+1}+x+1)}$$
$$= \lim_{x\to 0} \frac{-x}{x(\sqrt{x^2+x+1}+x+1)} = \frac{-1}{1+1} = -1/2.$$

e) $$\lim_{x\to 8}\frac{\sqrt{8+x}-4}{\sqrt[3]{x}-2}=\lim_{x\to 8}\frac{(\sqrt{8+x}-4)(\sqrt{8+x}+4)(\sqrt[3]{x^2}+2\sqrt[3]{x}+4)}{(\sqrt[3]{x}-2)(\sqrt[3]{x^2}+2\sqrt[3]{x}+4)(\sqrt{8+x}+4)}$$
$$=\lim_{x\to 8}\frac{(x-8)(\sqrt[3]{x^2}+2\sqrt[3]{x}+4)}{(x-8)(\sqrt{8+x}+4)}=\frac{12}{8}=\frac{3}{2}.$$

f) $$\lim_{x\to a+}\frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}=\lim_{x\to a+}\left(\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x^2-a^2}}+\frac{\sqrt{x-a}}{\sqrt{x^2-a^2}}\right)$$
$$=\lim_{x\to a+}\left(\frac{x-a}{(\sqrt{x}+\sqrt{a})\sqrt{x^2-a^2}}+\frac{1}{\sqrt{x+a}}\right)$$
$$=\lim_{x\to a+}\left(\frac{1}{\sqrt{x}+\sqrt{a}}\cdot\sqrt{\frac{x-a}{x+a}}+\frac{1}{\sqrt{x+a}}\right)=\frac{1}{\sqrt{2a}}.$$

Example 4.18. *Determine*

a) $\displaystyle\lim_{x\to 7}\frac{\sqrt{x+2}-\sqrt[3]{x+20}}{\sqrt[4]{x+9}-2}$;

b) $\displaystyle\lim_{x\to 0}\frac{\sqrt[n]{1+x}-1}{x}$, $\quad n\in\mathbf{Z}\setminus\{0\}$;

c) $\displaystyle\lim_{x\to 0}\frac{\sqrt[n]{1+ax}\cdot\sqrt[m]{1+bx}-1}{x}$, $\quad m,n\in\mathbf{Z}\setminus\{0\},\ \ a,b\in\mathbf{R}$;

d) $\displaystyle\lim_{x\to 0}\frac{\sqrt[m]{1+P(x)}-1}{x}$, $\quad$ *where* $P(x)=a_1x+a_2x^2+\cdots+a_nx^n\not\equiv 0,\quad m\in\mathbf{Z}\setminus\{($

e) $\displaystyle\lim_{x\to 1}\frac{(1-\sqrt{x})(1-\sqrt[3]{x})\cdots(1-\sqrt[n]{x})}{(1-x)^{n-1}}$ $\quad n\in\mathbf{N}$.

Solutions.

a) $$\lim_{x\to 7}\frac{\sqrt{x+2}-\sqrt[3]{x+20}}{\sqrt[4]{x+9}-2}=\lim_{x\to 7}\frac{\dfrac{\sqrt{x+2}-3}{x-7}+\dfrac{3-\sqrt[3]{x+20}}{x-7}}{\dfrac{\sqrt[4]{x+9}-2}{x-7}}$$
$$=\frac{\lim\limits_{x\to 7}\dfrac{\sqrt{x+2}-3}{x-7}+\lim\limits_{x\to 7}\dfrac{3-\sqrt[3]{x+20}}{x-7}}{\lim\limits_{x\to 7}\dfrac{\sqrt[4]{x+9}-2}{x-7}}=\frac{\dfrac{1}{6}-\dfrac{1}{27}}{\dfrac{1}{32}}=\frac{112}{27}.$$

b) If we change the variables $t=\sqrt[n]{1+x}-1$, then from Example 4.16 it follows that $\lim\limits_{x\to 0}\sqrt[n]{1+x}=1$, and we obtain that $t\to 0$ when $x\to 0$. So we have

$$\lim_{x\to 0}\frac{\sqrt[n]{1+x}-1}{x}=\lim_{t\to 0}\frac{t}{(1+t)^n-1}=\lim_{t\to 0}\frac{t}{nt+\binom{n}{2}t^2+\cdots+t^n}=\frac{1}{n}.$$

c) We assume $a, b \neq 0$, the other cases are left to the reader. Using b), we obtain

$$\lim_{x\to 0}\frac{\sqrt[n]{1+ax}\cdot\sqrt[m]{1+bx}-1}{x}=\lim_{x\to 0}\frac{\sqrt[m]{1+bx}(\sqrt[n]{1+ax}-1)+\sqrt[m]{1+bx}-1}{x}$$

$$=\lim_{x\to 0}\sqrt[m]{1+bx}\cdot a\cdot\lim_{x\to 0}\frac{\sqrt[n]{1+ax}-1}{ax}+b\cdot\lim_{x\to 0}\frac{\sqrt[m]{1+bx}-1}{bx}=\frac{a}{n}+\frac{b}{m}.$$

d) We assume that $m\in\mathbf{N}$, the case $-m\in\mathbf{N}$ is left to the reader.

$$\lim_{x\to 0}\frac{\sqrt[m]{1+P(x)}-1}{x}=\lim_{x\to 0}\frac{\sqrt[m]{1+P(x)}-1}{P(x)}\cdot\frac{P(x)}{x}$$

$$=\lim_{x\to 0}\frac{\sqrt[m]{1+P(x)}-1}{P(x)}\cdot\lim_{x\to 0}\left(a_1+a_2x+\cdots+a_nx^{n-1}\right)$$

$$=a_1\cdot\lim_{x\to 0}\frac{1}{\sqrt[m]{(1+P(x))^{m-1}}+\sqrt[m]{(1+P(x))^{m-2}}+\cdots+1}=\frac{a_1}{m}.$$

e) After changing of variables $t=1-x$, where $t\to 0$ when $x\to 1$, we obtain

$$\lim_{x\to 1}\frac{(1-\sqrt{x})(1-\sqrt[3]{x})\cdots(1-\sqrt[n]{x})}{(1-x)^{n-1}}$$

$$=\lim_{t\to 0}\left(\frac{1-\sqrt{1-t}}{t}\cdot\frac{1-\sqrt[3]{1-t}}{t}\cdots\frac{1-\sqrt[n]{1-t}}{t}\right)=\frac{1}{2}\cdot\frac{1}{3}\cdots\frac{1}{n}=\frac{1}{n!}.$$

Exercise 4.19. *Determine the limits of the functions given below.*

a) $\displaystyle\lim_{x\to 0}\frac{x}{\sqrt[3]{8+x}-2}$; **b)** $\displaystyle\lim_{x\to 2}\frac{\sqrt{9+2x-x^2}-\sqrt{3+x+x^2}}{2x-x^2}$;

c) $\displaystyle\lim_{x\to -5}\frac{\sqrt[3]{6+x}+x+4}{\sqrt[3]{9+2x}+1}$; **d)** $\displaystyle\lim_{x\to 1}\frac{1-\sqrt[3]{x}}{1-\sqrt[7]{x}}$;

e) $\displaystyle\lim_{x\to 1}\frac{\sqrt[m]{x}-1}{\sqrt[n]{x}-1}$, $m,n\in\mathbf{N}$; **f)** $\displaystyle\lim_{x\to 0}\frac{\sqrt[n]{c+x}-\sqrt[n]{c-x}}{x}$, $n\in\mathbf{N}$, $c>0$;

g) $\displaystyle\lim_{x\to 0}\frac{\sqrt[m]{1+ax}-\sqrt[n]{1+bx}}{x}$, $m,n\in\mathbf{N}$, $a,b\in\mathbf{R}$.

Answers.

a) 12. **b)** $\frac{7}{12}$. **c)** 2. **d)** $\frac{7}{3}$.

e) $\frac{n}{m}$ (change the variable x to $(1+t)^{mn}$).

f) $\frac{2\sqrt[n]{c}}{nc}$. **g)** $\frac{a}{m}-\frac{b}{n}$.

Example 4.20. *Prove that*

$$\lim_{x\to+\infty}\frac{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0}{b_mx^m+b_{m-1}x^{m-1}+\cdots+b_0}=\begin{cases}(\operatorname{sgn}\frac{a_n}{b_m})\cdot\infty, & n>m;\\ \frac{a_n}{b_m}, & n=m;\\ 0, & n<m,\end{cases}$$

whenever $a_n\neq 0$ *and* $b_m\neq 0$.

Solution. The given expression can be written as

$$\frac{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0}{b_mx^m+b_{m-1}x^{m-1}+\cdots+b_0}$$

$$=\frac{a_n}{b_m}\cdot x^{n-m}\cdot\frac{1+\frac{a_{n-1}}{a_n}x^{-1}+\cdots+\frac{a_0}{a_n}x^{-n}}{1+\frac{b_{m-1}}{b_m}x^{-1}+\cdots+\frac{b_0}{b_m}x^{-m}}. \tag{4.5}$$

Let us suppose that $n>m$. Since,

$$\lim_{x\to+\infty}\left(\frac{a_{n-1}}{a_n}x^{-1}+\cdots+\frac{a_0}{a_n}x^{-n}\right)=0$$

and

$$\lim_{x\to+\infty}\left(\frac{b_{m-1}}{b_m}x^{-1}+\cdots+\frac{b_0}{b_m}x^{-m}\right)=0,$$

there exist positive numbers X_1 and X_2 such that the following inequalities hold.

$$\left|\frac{a_{n-1}}{a_n}x^{-1}+\cdots+\frac{a_0}{a_n}x^{-n}\right|\leq\frac{1}{2},\quad\text{for}\quad x>X_1,$$

and

$$\left|\frac{b_{m-1}}{b_m}x^{-1}+\cdots+\frac{b_0}{b_m}x^{-m}\right|\leq\frac{1}{2}\quad\text{for}\quad x>X_2.$$

So for $x\geq\max\{X_1,X_2\}$, we obtain

$$x^{n-m}\left|\frac{1+\frac{a_{n-1}}{a_n}x^{-1}+\cdots+\frac{a_0}{a_n}x^{-n}}{1+\frac{b_{m-1}}{b_m}x^{-1}+\cdots+\frac{b_0}{b_m}x^{-m}}\right|$$

$$\geq\ x^{n-m}\cdot\frac{1-\left|\frac{a_{n-1}}{a_n}x^{-1}+\cdots+\frac{a_0}{a_n}x^{-n}\right|}{1+\left|\frac{b_{m-1}}{b_m}x^{-1}+\cdots+\frac{b_0}{b_m}x^{-m}\right|}\geq x^{n-m}\frac{1-\frac{1}{2}}{1+\frac{1}{2}}=\frac{x^{n-m}}{3}.$$

Since $\lim\limits_{x\to+\infty} x^{m-n} = +\infty$, we have

$$\lim_{x\to+\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0} = \left(\operatorname{sgn}\frac{a_n}{b_m}\right) \cdot \infty.$$

If $m = n$, then from relation (4.5) it follows

$$\lim_{x\to+\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0} = \frac{a_n}{b_m}.$$

At last, if $n < m$, then from $\lim\limits_{x\to+\infty} x^{m-n} = 0$ and relation (4.5) we have

$$\lim_{x\to+\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0}$$

$$= \frac{a_n}{b_m} \cdot \lim_{x\to+\infty} x^{n-m} \cdot \lim_{x\to+\infty} \frac{1 + \dfrac{a_{n-1}}{a_n}x^{-1} + \cdots + \dfrac{a_0}{a_n}x^{-n}}{1 + \dfrac{b_{m-1}}{b_m}x^{-1} + \cdots + \dfrac{b_0}{b_m}x^{-m}} = \frac{a_n}{b_m} \cdot 0 \cdot 1 = 0.$$

Example 4.21. *Determine the following limits.*

a) $\lim\limits_{x\to+\infty} \dfrac{5x^3 + 3x^2 + 2x + 5}{5x^3 + x^2 + x + 3}$; **b)** $\lim\limits_{x\to+\infty} \dfrac{5x^3 + 3x^2 + 2x + 5}{5x^4 + x^2 + x + 3}$;

c) $\lim\limits_{x\to+\infty} \dfrac{5x^4 + 3x^2 + 2x + 5}{5x^3 + x^2 + x + 3}$; **d)** $\lim\limits_{x\to+\infty} \dfrac{\sqrt{x+1}}{\sqrt{x + \sqrt{x + \sqrt{x}}}}$;

e) $\lim\limits_{x\to+\infty} \left(\sqrt{x^2 - 1} - \sqrt{x^2 + 1}\right)$; **f)** $\lim\limits_{x\to+\infty} \left(\sqrt{16x^2 + x - 1} - 4x\right)$;

g) $\lim\limits_{x\to+\infty} \left(\sqrt[3]{x^3 + 3x^2} - \sqrt{x^2 - 2x}\right)$; **h)** $\lim\limits_{x\to+\infty} x\left(\sqrt{x^2 + 2x} - 2\sqrt{x^2 + x} + x\right)$;

i) $\lim\limits_{x\to+\infty} \left(\sqrt[n]{(x + a_1)(x + a_2)\cdots(x + a_n)} - x\right)$.

Solutions.

a) $$\lim_{x\to+\infty} \frac{5x^3 + 3x^2 + 2x + 5}{5x^3 + x^2 + x + 3} = \lim_{x\to+\infty} \frac{x^3\left(5 + \dfrac{3}{x} + \dfrac{2}{x^2} + \dfrac{5}{x^3}\right)}{x^3\left(5 + \dfrac{1}{x} + \dfrac{1}{x^2} + \dfrac{3}{x^3}\right)} = 1.$$

b) $$\lim_{x\to+\infty} \frac{5x^3 + 3x^2 + 2x + 5}{5x^4 + x^2 + x + 3} = \lim_{x\to+\infty} \frac{x^3\left(5 + \dfrac{3}{x} + \dfrac{2}{x^2} + \dfrac{5}{x^3}\right)}{x^4\left(5 + \dfrac{1}{x^2} + \dfrac{1}{x^3} + \dfrac{3}{x^4}\right)} = 0.$$

c) $$\lim_{x\to+\infty}\frac{5x^4+3x^2+2x+5}{5x^3+x^2+x+3}=\lim_{x\to+\infty}\frac{x^4\left(5+\dfrac{3}{x}+\dfrac{2}{x^2}+\dfrac{5}{x^3}\right)}{x^3\left(5+\dfrac{1}{x}+\dfrac{1}{x^2}+\dfrac{3}{x^3}\right)}=+\infty.$$

d) $$\lim_{x\to+\infty}\frac{\sqrt{x+1}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}=\lim_{x\to+\infty}\frac{\sqrt{x}\cdot\sqrt{1+\dfrac{1}{x}}}{\sqrt{x}\cdot\sqrt{1+\sqrt{\dfrac{1}{x}+\sqrt{\dfrac{1}{x^3}}}}}=1.$$

e) $$\lim_{x\to+\infty}(\sqrt{x^2-1}-\sqrt{x^2+1})=\lim_{x\to+\infty}\frac{(\sqrt{x^2-1}-\sqrt{x^2+1})(\sqrt{x^2-1}+\sqrt{x^2+1})}{\sqrt{x^2-1}+\sqrt{x^2+1}}$$
$$=\lim_{x\to+\infty}\frac{-2}{\sqrt{x^2-1}+\sqrt{x^2+1}}=0.$$

f) $$\lim_{x\to+\infty}\left(\sqrt{16x^2+x-1}-4x\right)=\lim_{x\to+\infty}\frac{\left(\sqrt{16x^2+x-1}-4x\right)\left(\sqrt{16x^2+x-1}+4x\right)}{\sqrt{16x^2+x-1}+4x}$$
$$=\lim_{x\to+\infty}\frac{x-1}{\sqrt{16x^2+x-1}+4x}=\lim_{x\to+\infty}\frac{x-1}{4x\cdot\sqrt{1+\dfrac{1}{16x}-\dfrac{1}{16x^2}}+4x}=\frac{1}{8}.$$

g) $$\lim_{x\to+\infty}\left(\sqrt[3]{x^3+3x^2}-\sqrt{x^2-2x}\right)=\lim_{x\to+\infty}\left(\sqrt[3]{x^3+3x^2}-x\right)+\lim_{x\to+\infty}\left(x-\sqrt{x^2-2x}\right)$$
$$=\lim_{x\to+\infty}\frac{3x^2}{\sqrt[3]{(x^3+3x^2)^2}+x\sqrt[3]{x^3+3x^2}+x^2}+\lim_{x\to+\infty}\frac{2x}{x+\sqrt{x^2-2x}}=2.$$

h) $$\lim_{x\to+\infty}x\left(\sqrt{x^2+2x}-2\sqrt{x^2+x}+x\right)=\lim_{x\to+\infty}x\frac{2x\left(\sqrt{x^2+2x}-x-1\right)}{\sqrt{x^2+2x}+x+2\sqrt{x^2+x}}$$
$$=\lim_{x\to+\infty}\frac{-2x^2}{\left(\sqrt{x^2+2x}+x+2\sqrt{x^2+x}\right)\left(\sqrt{x^2+2x}+x+1\right)}=-\frac{1}{4}.$$

i) Taking $x=\dfrac{1}{t}$, where $t\to 0+$ when $x\to+\infty$, we obtain

$$\sqrt[n]{(x+a_1)(x+a_2)\cdots(x+a_n)}-x=\frac{\sqrt[n]{1+P(t)}-1}{t},$$

where

$P(t)=(a_1+a_2+\cdots+a_n)t+(a_1a_2+a_1a_3+\cdots+a_{n-1}a_n)t^2+\cdots+a_1a_2\cdots a_nt^n.$

From Example 4.18 d) it follows

$$\lim_{x\to+\infty}\left(\sqrt[n]{(x+a_1)(x+a_2)\cdots(x+a_n)}-x\right)$$

$$=\lim_{t\to0+}\frac{\sqrt[n]{1+P(t)}-1}{t}=\frac{a_1+a_2+\cdots+a_n}{n}.$$

Exercise 4.22. *Determine the following limits of the functions.*

a) $\lim\limits_{x\to+\infty} \dfrac{(x+2)(3-5x)}{(2x+1)^2}$; b) $\lim\limits_{x\to+\infty} \dfrac{(3x^3+2x-1)^6}{(3x^6+8x^2+1)^3}$;

c) $\lim\limits_{x\to+\infty} \left(\dfrac{x^4+2x^3}{1+x^3} - x\right)$; d) $\lim\limits_{x\to+\infty} \left(\dfrac{x^2}{2x+1} + \dfrac{x^3+4x^2-2}{1-2x^2}\right)$;

e) $\lim\limits_{x\to+\infty} \dfrac{3x^6-1}{\sqrt{x^{12}+2x^4+2}}$; f) $\lim\limits_{x\to+\infty} \left(\sqrt{x^4+2x^2-1} - \sqrt{x^4-2x^2-1}\right)$;

g) $\lim\limits_{x\to+\infty} \left(\sqrt{x^2+\sqrt{x^2+\sqrt{x^2+\sqrt{x^2}}}} - x\right)$;

h) $\lim\limits_{x\to+\infty} x^2 \left(\sqrt{x^4+x^2\sqrt{x^4+1}} - \sqrt{2x^4}\right)$.

Answers.

a) $-\dfrac{5}{4}$. b) 27. c) 2. d) $-\dfrac{9}{4}$.

e) 3. f) 2. g) $\dfrac{1}{2}$. h) $\dfrac{\sqrt{2}}{8}$.

Example 4.23. *Knowing that* $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$, *determine the following limits.*

a) $\lim\limits_{x\to 0} \dfrac{\tan x}{x}$; b) $\lim\limits_{x\to 0} \dfrac{\sin ax}{\sin bx}$, $0 \neq a, b \in \mathbf{R}$;

c) $\lim\limits_{x\to 0} \left(\dfrac{2}{\sin 2x \sin x} - \dfrac{1}{\sin^2 x}\right)$; d) $\lim\limits_{x\to 0} \dfrac{\cos 2x^3 - 1}{\sin^6 2x}$;

e) $\lim\limits_{x\to 0} \dfrac{x^2}{\sqrt{1+x\sin x} - \sqrt{\cos x}}$; f) $\lim\limits_{x\to 0} \dfrac{\tan(a+x)\tan(a-x) - \tan^2 a}{x^2}$;

g) $\lim\limits_{x\to 0} \dfrac{\cos(a+2x) - 2\cos(a+x) + \cos a}{x^2}$.

In f) and g), we assume $a \neq 0$.

Solution.

a) The function $f(x) = \cos x$ is continuous at the point $x = 0$ and therefore we have

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{\cos x} = 1.$$

b) $$\lim_{x\to 0}\frac{\sin ax}{\sin bx}=\lim_{x\to 0}\frac{a\cdot\dfrac{\sin ax}{ax}}{b\cdot\dfrac{\sin bx}{bx}}=\frac{a}{b}.$$

c) $$\lim_{x\to 0}\left(\frac{2}{\sin 2x\sin x}-\frac{1}{\sin^2 x}\right)=\lim_{x\to 0}\frac{2\sin x-2\sin x\cos x}{\sin 2x\sin^2 x}$$

$$=\lim_{x\to 0}2\frac{1-\cos x}{\sin 2x\sin x}=\lim_{x\to 0}\frac{\dfrac{1-\cos x}{x^2}}{\dfrac{\sin 2x\sin x}{2x^2}}=\lim_{x\to 0}\frac{\dfrac{2\sin^2(x/2)}{4(x/2)^2}}{\dfrac{\sin 2x}{2x}\cdot\dfrac{\sin x}{x}}=\frac{1}{2}.$$

d) $$\lim_{x\to 0}\frac{\cos 2x^3-1}{\sin^6 2x}=\lim_{x\to 0}\frac{-2\sin^2 x^3}{\sin^6 2x}=-2\lim_{x\to 0}\left(\frac{\left(\dfrac{\sin x^3}{x^3}\right)^2}{2^6\left(\dfrac{\sin 2x}{2x}\right)^6}\right)=-\frac{1}{32}.$$

e) $$\lim_{x\to 0}\frac{x^2}{\sqrt{1+x\sin x}-\sqrt{\cos x}}=\lim_{x\to 0}\frac{x^2(\sqrt{1+x\sin x}+\sqrt{\cos x})}{1+x\sin x-\cos x}$$

$$=\lim_{x\to 0}\frac{\sqrt{1+x\sin x}+\sqrt{\cos x}}{\dfrac{1-\cos x}{x^2}+\dfrac{\sin x}{x}}=\frac{2}{\dfrac{1}{2}+1}=\frac{4}{3}.$$

f) $$\lim_{x\to 0}\frac{\tan(a+x)\tan(a-x)-\tan^2 a}{x^2}=\lim_{x\to 0}\frac{1}{x^2}\left(\frac{\tan^2 a-\tan^2 x}{1-\tan^2 a\tan^2 x}-\tan^2 a\right)$$

$$=\lim_{x\to 0}\frac{\tan^2 x}{x^2}(\tan^4 a-1)=-\frac{\cos 2a}{\cos^4 a}.$$

g) $$\lim_{x\to 0}\frac{\cos(a+2x)-2\cos(a+x)+\cos a}{x^2}$$

$$=\lim_{x\to 0}\frac{1}{x^2}((\cos(a+2x)-\cos(a+x))-(\cos(a+x)-\cos a))$$

$$=\lim_{x\to 0}\frac{1}{x^2}\left(-2\sin\frac{x}{2}\cdot\sin\left(a+\frac{3x}{2}\right)+2\sin\frac{x}{2}\cdot\sin\left(a+\frac{x}{2}\right)\right)$$

$$=\lim_{x\to 0}\frac{-2\sin\frac{x}{2}}{x^2}\left(\sin\left(a+\frac{3x}{2}\right)-\sin\left(a+\frac{x}{2}\right)\right)$$

$$=\lim_{x\to 0}\frac{-2\sin\frac{x}{2}\cdot 2\sin\frac{x}{2}\cdot\cos(a+x)}{x^2}=-\cos a.$$

Exercise 4.24. *Determine the following limits.*

a) $\lim\limits_{x\to 0}\dfrac{\sin 5x}{\sin x}$; b) $\lim\limits_{x\to 0} 3x\cot 3x$;

c) $\lim\limits_{x\to 0}\dfrac{\sin x}{\sin 7x-\sin 9x}$; d) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$;

e) $\lim\limits_{x\to 0}\dfrac{1+\sin x-\cos x}{1+\sin ax-\cos ax}$; f) $\lim\limits_{x\to 0}\dfrac{\cot(a+2x)-2\cot(a+x)+\cot a}{x^2}$.

In e) and f), we assume $a\neq 0$.

Answers.

a) 5. b) 1. c) $-\dfrac{1}{2}$.

d) $\dfrac{1}{2}$. e) $\dfrac{1}{a}$. f) $\dfrac{2\cos a}{\sin^3 a}$.

Example 4.25. *Determine*

a) $\lim\limits_{x\to 1}\dfrac{\sin 4\pi x}{\sin 5\pi x}$; b) $\lim\limits_{x\to a}\dfrac{\sin x-\sin a}{x-a}$;

c) $\lim\limits_{x\to a}\dfrac{\cot x-\cot a}{x-a}$, $a\neq k\pi$; d) $\lim\limits_{x\to\pi/3}\dfrac{\tan^3 x-3\tan x}{\cos(x+\frac{\pi}{6})}$;

e) $\lim\limits_{x\to+\infty} x\sin\dfrac{2\pi}{x}$; f) $\lim\limits_{x\to+\infty} 3x^2\left(\cos\dfrac{1}{x}-\cos\dfrac{3}{x}\right)$.

Solutions.

a) After changing of variables $t=x-1$, $t\to 0$ when $x\to 1$, we obtain

$$\lim_{x\to 1}\frac{\sin 4\pi x}{\sin 5\pi x}=\lim_{t\to 0}\frac{\sin(4\pi(t+1))}{\sin(5\pi(t+1))}=\lim_{t\to 0}\frac{\sin(4\pi t)}{-\sin(5\pi t)}=-\frac{4}{5}.$$

b) $$\lim_{x\to a}\frac{\sin x-\sin a}{x-a}=\lim_{x\to a}\frac{2\sin\dfrac{x-a}{2}\cos\dfrac{x+a}{2}}{x-a}=\cos a.$$

c) $$\lim_{x\to a}\frac{\cot x-\cot a}{x-a}=\lim_{x\to a}\frac{\sin(a-x)}{\sin x\sin a}\cdot\frac{1}{x-a}=-\frac{1}{\sin^2 a}.$$

d) $$\lim_{x\to\pi/3}\frac{\tan^3 x-3\tan x}{\cos(x+\dfrac{\pi}{6})}=\lim_{x\to\pi/3}\frac{\tan x\left(\tan^2 x-\tan^2\dfrac{\pi}{3}\right)}{\cos(x+\dfrac{\pi}{6})}$$

$$=-\lim_{x\to\pi/3}\tan x\left(\tan x+\tan\frac{\pi}{3}\right)\frac{\sin(x-\dfrac{\pi}{3})}{\cos x\cos\dfrac{\pi}{3}\sin\left(\pi/2-(x+\dfrac{\pi}{6})\right)}=-24.$$

e) After changing of variables $t = \dfrac{2\pi}{x}$, $t \to 0+$ when $x \to +\infty$, we obtain

$$\lim_{x\to+\infty} x\sin\frac{2\pi}{x} = \lim_{t\to0+} 2\pi\frac{\sin t}{t} = 2\pi.$$

f) $$\lim_{x\to+\infty} 3x^2\left(\cos\frac{1}{x} - \cos\frac{3}{x}\right) = \lim_{x\to+\infty} 3x^2\left(2\sin\frac{1}{x}\sin\frac{2}{x}\right)$$

$$= \lim_{x\to+\infty} 3\cdot2\cdot2\frac{\sin\frac{1}{x}}{\frac{1}{x}}\cdot\frac{\sin\frac{2}{x}}{\frac{2}{x}} = 12.$$

Exercise 4.26. *Determine the following limits.*

a) $\displaystyle\lim_{x\to\pi}\frac{\sin x}{\pi^2 - x^2}$;

b) $\displaystyle\lim_{x\to1}\frac{\sin(x-1)}{1-x^3}$;

c) $\displaystyle\lim_{x\to\pi/3}\frac{2\cos^2 x + \cos x - 1}{2\cos^2 x - 3\cos x + 1}$;

d) $\displaystyle\lim_{x\to\pi}\frac{\cos(x+\pi) - \cos(3x+\pi)}{(x-\pi)^2}$.

Answers.

a) $\dfrac{1}{2\pi}$. **b)** $-\dfrac{1}{3}$. **c)** -3; **d)** 4.

Example 4.27. *Determine the following limits.*

a) $\displaystyle\lim_{x\to0}\frac{2\sin(\sqrt{x+1}-1)}{x}$;

b) $\displaystyle\lim_{x\to0}\frac{x^2}{\sqrt{1+x\sin x}-\sqrt{\cos x}}$;

c) $\displaystyle\lim_{x\to0}\frac{\sqrt[m]{\cos ax}-\sqrt[n]{\cos bx}}{x^2}$;

d) $\displaystyle\lim_{x\to+\infty}(\sin\sqrt{x+1}-\sin\sqrt{x})$,

where in c) we assume $m, n \in \mathbf{N}$ *and* $a, b, \in \mathbf{R}$.

Solutions.

a) $$\lim_{x\to0}\frac{2\sin(\sqrt{x+1}-1)}{x} = 2\lim_{x\to0}\frac{\sin\dfrac{x}{\sqrt{x+1}+1}}{\dfrac{x}{\sqrt{x+1}+1}\left(\sqrt{x+1}+1\right)} = 1.$$

b) $$\lim_{x\to0}\frac{x^2}{\sqrt{1+x\sin x}-\sqrt{\cos x}} = \lim_{x\to0}\frac{x^2\left(\sqrt{1+x\sin x}+\sqrt{\cos x}\right)}{1+x\sin x-\cos x}$$

$$= \lim_{x\to0}\frac{\sqrt{1+x\sin x}+\sqrt{\cos x}}{\dfrac{1-\cos x}{x^2}+\dfrac{\sin x}{x}} = \frac{4}{3}.$$

c) $\displaystyle\lim_{x\to 0}\frac{\sqrt[m]{\cos ax}-\sqrt[n]{\cos bx}}{x^2}=\lim_{x\to 0}\left(\frac{\sqrt[m]{\cos ax}-1}{x^2}-\frac{\sqrt[n]{\cos bx}-1}{x^2}\right)=\frac{1}{2}\left(\frac{b^2}{n}-\frac{a^2}{m}\right).$

d) $\displaystyle\lim_{x\to+\infty}\left(\sin\sqrt{x+1}-\sin\sqrt{x}\right)=2\lim_{x\to+\infty}\left(\sin\frac{\sqrt{x+1}-\sqrt{x}}{2}\cdot\cos\frac{\sqrt{x+1}+\sqrt{x}}{2}\right)$

$\displaystyle=\lim_{x\to+\infty}2\sin\frac{1}{2(\sqrt{x+1}+\sqrt{x})}\cos\frac{\sqrt{x+1}+\sqrt{x}}{2}.$

Since $\displaystyle|\cos\frac{\sqrt{x+1}+\sqrt{x}}{2}|\le 1$ for all $x\in\mathbf{R}$, we have

$\displaystyle|\lim_{x\to+\infty}(\sin\sqrt{x+1}-\sin\sqrt{x})|\le\lim_{x\to+\infty}|2\sin\frac{1}{2(\sqrt{x+1}+\sqrt{x})}|\cdot 1=0.$ Thus $\displaystyle\lim_{x\to+\infty}\left(\sin\sqrt{x+1}-\sin\sqrt{x}\right)=0.$

Exercise 4.28. *Determine the following limits.*

a) $\displaystyle\lim_{x\to 0}\frac{\tan^2 x}{\sqrt{3}-\sqrt{2+\cos x}};$ **b)** $\displaystyle\lim_{x\to 0}\frac{\sqrt{\cos x}-\sqrt[3]{\cos x}}{\sin^2 x};$

c) $\displaystyle\lim_{x\to\pi/2}\frac{\sqrt[4]{\sin x}-\sqrt[3]{\sin x}}{\cos^2 x};$ **d)** $\displaystyle\lim_{x\to+\infty}\left(\cos\sqrt{x+1}-\cos\sqrt{x}\right).$

Answers.

a) $4\sqrt{3}$. **b)** $-\dfrac{1}{12}$. **c)** $\dfrac{1}{24}$. **d)** 0.

Example 4.29. *Determine*

a) $\displaystyle\lim_{x\to -2}\frac{\arcsin(x+2)}{x^2+2x};$ **b)** $\displaystyle\lim_{x\to 0}\frac{\pi-4\arctan\dfrac{1}{1+x}}{x}.$

Solutions.

a) After changing a variable $t=\arcsin(x+2)$, i.e., $\sin t=x+2$, where $t\to 0$ when $x\to -2$, we obtain

$$\lim_{x\to -2}\frac{\arcsin(x+2)}{x^2+2x}=\lim_{x\to -2}\frac{1}{x}\cdot\frac{\arcsin(x+2)}{x+2}=-\frac{1}{2}\cdot\lim_{t\to 0}\frac{t}{\sin t}=-\frac{1}{2}.$$

b) After changing a variable $t=\pi-4\arctan\dfrac{1}{x+1}$, i.e., $x=\cot\dfrac{\pi-t}{4}-1$, where $t\to 0$ when $x\to 0$, we obtain

$$\lim_{x\to 0}\frac{\pi-4\arctan\dfrac{1}{1+x}}{x}=\lim_{t\to 0}\frac{t}{\cot\dfrac{\pi-t}{4}-1}$$

$$=\lim_{t\to 0}\frac{t\sin\dfrac{\pi-t}{4}\sin\dfrac{\pi}{4}}{\sin\left(\dfrac{\pi}{4}-\dfrac{\pi-t}{4}\right)}=\lim_{t\to 0}\frac{t\cdot\dfrac{1}{\sqrt{2}}\cdot\dfrac{1}{\sqrt{2}}}{\sin\dfrac{t}{4}}=2.$$

Exercise 4.30. *Determine the following limits.*

a) $\displaystyle\lim_{x\to 0}\frac{\arcsin x}{3x}$; b) $\displaystyle\lim_{x\to 0}\frac{\arctan 2x}{x}$.

Answers.

a) $\dfrac{1}{3}$. b) 2.

Exercise 4.31. *Let us suppose that the following inequalities*

$$g_1(x)\le f(x)\le g_2(x)$$

are satisfied on the interval $(\alpha,\beta)\setminus\{x_0\}$, $x_0\in(\alpha,\beta)$. *Additionally, if there exists a number* L *such that*

$$\lim_{x\to x_0} g_1(x)=\lim_{x\to x_0} g_2(x)=L,$$

then, using Theorem 4.6, prove

$$\lim_{x\to x_0} f(x)=L.$$

Example 4.32. *Show that*

a) $\displaystyle\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x=\mathrm{e}$; b) $\displaystyle\lim_{x\to 0}(1+x)^{1/x}=\mathrm{e}$.

Solutions.

a) Let x take the values of the sequence $(x_k)_{k\in\mathbf{N}}$ with the property $\lim\limits_{k\to+\infty}x_k=+\infty$. Then for every monotonically increasing sequence $(n_k)_{k\in\mathbf{N}}$ such that $n_k\in\mathbf{N}$ and $n_k\to\infty$ when $k\to+\infty$, it holds $n_k\le x_k\le n_k+1$ and

$$\frac{1}{n_k+1}\le\frac{1}{x_k}\le\frac{1}{n_k},\quad \text{hence}\quad 1+\frac{1}{n_k+1}\le 1+\frac{1}{x_k}\le 1+\frac{1}{n_k}.$$

Wherefrom we have

$$\left(1+\frac{1}{n_k+1}\right)^{n_k}\le\left(1+\frac{1}{x_k}\right)^{x_k}\le\left(1+\frac{1}{n_k}\right)^{n_k+1},$$

or

$$\left(1+\frac{1}{n_k+1}\right)^{n_k+1}\left(1+\frac{1}{n_k+1}\right)^{-1}\le\left(1+\frac{1}{x_k}\right)^{x_k}\le\left(1+\frac{1}{n_k}\right)^{n_k}\left(1+\frac{1}{n_k}\right).$$

Since

$$\lim_{k\to+\infty}\left(1+\frac{1}{n_k}\right)^{n_k}=\lim_{k\to+\infty}\left(1+\frac{1}{n_k+1}\right)^{n_k+1}=\mathrm{e},$$

we obtain

$$\lim_{x\to+\infty}\left(1+\frac{1}{x_k}\right)^{x_k}=\mathrm{e},\quad \text{i.e.,}\quad \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x=\mathrm{e}.$$

b) If $x \to 0+$, then for $t = \dfrac{1}{x}$ we have $t \to +\infty$ and from a) we obtain

$$\lim_{x\to 0+}(1+x)^{1/x} = \lim_{t\to+\infty}\left(1+\frac{1}{t}\right)^t = \mathrm{e}.$$

If $x \to 0-$, then for $s = -\dfrac{1}{x}$ it holds $s \to +\infty$ and we have

$$\lim_{x\to 0-}(1+x)^{1/x} = \lim_{s\to+\infty}\left(1-\frac{1}{s}\right)^{-s} = \lim_{s\to+\infty}\left(\frac{s}{s-1}\right)^s$$

$$= \lim_{s\to+\infty}\left(1+\frac{1}{s-1}\right)^{s-1}\cdot\lim_{s\to+\infty}\left(1+\frac{1}{s-1}\right) = \mathrm{e}\cdot 1 = \mathrm{e}.$$

Using Theorem 4.3, from the equalities

$$\lim_{x\to 0-}(1+x)^{1/x} = \lim_{x\to 0+}(1+x)^{1/x} = \mathrm{e},$$

it follows that

$$\lim_{x\to 0}(1+x)^{1/x} = \mathrm{e}.$$

Example 4.33. *Show that*

a) $\lim\limits_{x\to x_0} a^x = a^{x_0},\ \ a>0,\ x_0\in\mathbf{R};$ **b)** $\lim\limits_{x\to x_0}\ln x = \ln x_0,\ \ x_0>0;$

c) $\lim\limits_{x\to x_0}(u(x))^{v(x)} = a^b,\ \ x_0\in\mathbf{R}.$

In c), we suppose that $u(x)>0$ and that there exist $a>0$ and b such that

$$\lim_{x\to x_0}u(x) = a \quad \textit{and} \quad \lim_{x\to x_0}v(x) = b.$$

Solution.

a) Assume first $a>1$. In Example 3.11 b) it was shown that

$$\lim_{n\to\infty}a^{1/n} = \lim_{n\to\infty}a^{-1/n} = 1,$$

and therefore, for given $\varepsilon>0$, there exists $n_0\in\mathbf{N}$ such that for $a>1$ it holds

$$1-\frac{\varepsilon}{a^{x_0}} < a^{-1/n_0} < a^{1/n_0} < 1+\frac{\varepsilon}{a^{x_0}}.$$

Note that we used the fact that the function a^x is monotonically increasing on the whole $\mathbf{R}$. If we suppose that $|x-x_0|<\dfrac{1}{n_0}$, then from $-\dfrac{1}{n_0}<x-x_0<\dfrac{1}{n_0}$, it follows

$$1-\frac{\varepsilon}{a^{x_0}} < a^{-1/n_0} < a^{x-x_0} < a^{1/n_0} < 1+\frac{\varepsilon}{a^{x_0}},$$

wherefrom we have

$$-\frac{\varepsilon}{a^{x_0}} < a^{x-x_0}-1 < \frac{\varepsilon}{a^{x_0}}.$$

So for $\delta := \frac{1}{n_0}$ and $|x - x_0| < \delta$, we have

$$|a^x - a^{x_0}| = a^{x_0}|a^{x-x_0} - 1| < \varepsilon.$$

Thus for $a > 1$, it follows $\lim\limits_{x \to x_0} a^x = a^{x_0}$ for any $x_0 \in \mathbf{R}$. In the case when $0 < a < 1$, i.e., $a = \frac{1}{b}$, $b > 1$, it holds

$$\lim_{x \to x_0} a^x = \lim_{x \to x_0} \frac{1}{b^x} = \frac{1}{b^{x_0}} = a^{x_0}.$$

b) We proved the following inequalities (see Example 3.38 c))

$$\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n} \quad \text{and} \quad -\frac{1}{n-1} < \ln\left(1 - \frac{1}{n}\right) < -\frac{1}{n}, \; n > 1.$$

Since the ln function is monotonically increasing, we can write

$$-\frac{1}{n-1} < \ln\left(1 - \frac{1}{n}\right) < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}.$$

For given $\varepsilon > 0$ and $\varepsilon \leq \frac{1}{2}$, there exists n_0 such that

$$-\varepsilon < -\frac{1}{n_0 - 1} < \ln\left(1 - \frac{1}{n_0}\right) < \ln\left(1 + \frac{1}{n_0}\right) < \frac{1}{n_0} < \varepsilon. \tag{4.6}$$

If we suppose that $|x - x_0| < \frac{x_0}{n_0}$, then we have

$$-\frac{1}{n_0} < \frac{x - x_0}{x_0} < \frac{1}{n_0}. \tag{4.7}$$

From relations (4.6) and (4.7) we obtain

$$-\varepsilon < \ln\left(1 - \frac{1}{n_0}\right) < \ln\left(1 + \frac{x - x_0}{x_0}\right) < \ln\left(1 + \frac{1}{n_0}\right) < \varepsilon.$$

This means that

$$\left|\ln\left(1 + \frac{x - x_0}{x_0}\right)\right| < \varepsilon, \quad \text{hence} \quad |\ln x - \ln x_0| < \varepsilon,$$

under the condition $|x - x_0| < \delta$ with $\delta := \frac{x_0}{n_0}$. So we get for $x_0 > 0$

$$\lim_{x \to x_0} \ln x = \ln x_0.$$

c) From a) and b) we obtain

$$\begin{aligned} \lim_{x \to x_0} (u(x))^{v(x)} &= \lim_{x \to x_0} \exp\left(v(x) \ln u(x)\right) = \exp\left(\lim_{x \to x_0} v(x) \cdot \lim_{x \to x_0} \ln u(x)\right) \\ &= e^{b \ln a} = a^b. \end{aligned}$$

Example 4.34. *If the functions $u(x)$ and $v(x)$ satisfy*

$$\lim_{x\to x_0} u(x) = 1 \quad \textit{and} \quad \lim_{x\to x_0} v(x) = \infty,$$

then we have

$$\lim_{x\to x_0} u(x)^{v(x)} = \exp\left(\lim_{x\to x_0}(u(x)-1)\,v(x)\right), \tag{4.8}$$

provided the last limit exists.

Solution. We have

$$\begin{aligned}\lim_{x\to x_0} u(x)^{v(x)} &= \lim_{x\to x_0}\left(1+(u(x)-1))^{1/(u(x)-1)}\right)^{(u(x)-1)v(x)}\\ &= \lim_{x\to x_0}\exp\left((u(x)-1)v(x)\ln(1+(u(x)-1))^{1/(u(x)-1)}\right)\\ &= \exp\left(\lim_{x\to x_0}(u(x)-1)\,v(x)\right).\end{aligned}$$

Example 4.35. *Determine the following limits.*

a) $\displaystyle\lim_{x\to+\infty}\left(\frac{x+1}{x-1}\right)^x$; **b)** $\displaystyle\lim_{x\to+\infty}\left(\frac{2x+3}{2x+2}\right)^{x+2}$;

c) $\displaystyle\lim_{x\to 0}(\sqrt{1+x}-x)^{2/x}$; **d)** $\displaystyle\lim_{x\to 0}(\cos x)^{-\cot^2 x}$;

e) $\displaystyle\lim_{x\to 0}\left(\frac{1+\tan x}{1+\sin x}\right)^{1/\sin^3 x}$ **f)** $\displaystyle\lim_{x\to 0}\left(\frac{1+\tan x}{1+\sin x}\right)^{1/\sin x}$

Solutions. We shall repeatedly use Example 4.34.

a) In this case we put $u(x) = \dfrac{x+1}{x-1}$ and $v(x) = x$, hence

$$(u(x)-1)\,v(x) = \left(\frac{x+1}{x-1}-1\right)x = \frac{2x}{x-1}.$$

Thus we obtain

$$\lim_{x\to+\infty}\left(\frac{x+1}{x-1}\right)^x = \exp\left(\lim_{x\to+\infty}\frac{2x}{x-1}\right) = \mathrm{e}^2.$$

b) Using the previous method we have

$$\lim_{x\to+\infty}\left(\frac{2x+3}{2x+2}\right)^{x+2} = \exp\left(\lim_{x\to+\infty}\frac{x+2}{2x+2}\right) = \mathrm{e}^{1/2} = \sqrt{\mathrm{e}}.$$

Also, this example can be done as follows.

$$\lim_{x\to+\infty}\left(\frac{2x+3}{2x+2}\right)^{x+2} = \lim_{x\to+\infty}\left(\left(1+\frac{1}{2x+2}\right)^{2x+2}\right)^{1/2}\cdot\lim_{x\to+\infty}\left(1+\frac{1}{2x+2}\right) = \sqrt{\mathrm{e}}.$$

c) If $u(x) = \sqrt{1+x} - x$ and $v(x) = \dfrac{2}{x}$, then

$$\lim_{x\to 0}(\sqrt{1+x} - x) = 1 \text{ and } \lim_{x\to 0+} v(x) = \lim_{x\to 0+} \frac{2}{x} = +\infty.$$

Thus we have

$$(u(x)-1)\, v(x) = \left(\sqrt{1+x} - (x+1)\right) \frac{2}{x}.$$

From the formula (4.8) we get

$$\lim_{x\to 0+} (\sqrt{1+x} - x)^{2/x} = \exp\left(\lim_{x\to 0+} \frac{2}{x}(\sqrt{1+x} - (x+1))\right) = \mathrm{e}^{-1}.$$

We leave to the reader to show that $\lim\limits_{x\to 0-} (\sqrt{1+x} - x)^{2/x} = \dfrac{1}{\mathrm{e}}$, thus

$$\lim_{x\to 0}(\sqrt{1+x} - x)^{2/x} = \frac{1}{\mathrm{e}}.$$

d) $\lim\limits_{x\to 0}(\cos x)^{-\cot^2 x} = \lim\limits_{x\to 0}\left(1 - \sin^2 x\right)^{-\cos^2 x/(2\sin^2 x)} = \sqrt{\mathrm{e}}.$

e) If we denote by $u(x) = \dfrac{1+\tan x}{1+\sin x}$ and $v(x) = \dfrac{1}{\sin^3 x}$, then we have

$$\begin{aligned}(u(x)-1)\, v(x) &= \frac{\tan x - \sin x}{1+\sin x}\cdot\frac{1}{\sin^3 x} = \frac{\tan x(1-\cos x)}{1+\sin x}\cdot\frac{1}{\sin^3 x}\\ &= \frac{2\sin^2\frac{x}{2}}{\cos x(1+\sin x)}\cdot\frac{1}{\sin^2 x}.\end{aligned}$$

Therefore we have

$$\lim_{x\to 0}\left(\frac{1+\tan x}{1+\sin x}\right)^{1/\sin^3 x} = \exp\left(\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{\cos x(1+\sin x)\sin^2 x}\right) = \sqrt{\mathrm{e}}.$$

f) $\lim\limits_{x\to 0}\left(\dfrac{1+\tan x}{1+\sin x}\right)^{1/\sin x} = 1.$

Exercise 4.36. *Determine*

a) $\lim\limits_{x\to+\infty}\left(\dfrac{x}{x+1}\right)^{x}$;

b) $\lim\limits_{x\to+\infty}\left(\dfrac{x^2+2}{x^2-2}\right)^{x^2-1}$;

c) $\lim\limits_{x\to\pi/2}(1+\cot x)^{\tan x}$;

d) $\lim\limits_{x\to 0}(1+x^4)^{\cot^4 x}$.

Answers.

a) $\dfrac{1}{\mathrm{e}}$. b) e^4. c) e. d) e.

Exercise 4.37. *Prove that*

a) $\lim\limits_{x\to x_0} \log_a x = \log_a x_0,\ a>0, x_0>0$; b) $\lim\limits_{x\to x_0} \arccos x = \arccos x_0,\ |x_0|<1$;

c) $\lim\limits_{x\to 0+} \arctan\dfrac{1}{x} = \dfrac{\pi}{2}$; d) $\lim\limits_{x\to -\infty} \arctan x = -\dfrac{\pi}{2}$.

Example 4.38. *Determine the following limits.*

a) $\lim\limits_{x\to 5} \dfrac{\log_5 -1}{x-5}$; b) $\lim\limits_{x\to+\infty} x\log_3\dfrac{x+5}{x+3}$;

c) $\lim\limits_{x\to 0} \dfrac{\ln(1+4x+x^2)+\ln(1-4x+x^2)}{2x^2}$; d) $\lim\limits_{x\to 0} \dfrac{\sqrt{4+x}-2-\sin x}{\ln(1+x)}$;

e) $\lim\limits_{x\to 0} \dfrac{\ln\cos 4x}{\ln\cos 3x}$; f) $\lim\limits_{x\to\infty} x^2\ln\left(\sin\left(\dfrac{\pi}{2}-\dfrac{\pi}{x}\right)\right)$;

g) $\lim\limits_{x\to-\infty} \dfrac{\ln(1+3^x)}{\ln(1+2^x)}$; h) $\lim\limits_{x\to+\infty} \dfrac{\ln(1+3^x)}{\ln(1+2^x)}$.

Solutions.

a)
$$\lim_{x\to 5}\frac{\log_5 x-1}{x-5} = \lim_{x\to 5}\frac{\log_5 x-\log_5 5}{x-5} = \lim_{x\to 5}\log_5\left(\frac{x}{5}\right)^{1/(x-5)}$$
$$= \lim_{x\to 5}\log_5\exp\left(\ln\left(1+\frac{x-5}{5}\right)^{(5/(x-5))\cdot 1/5}\right) = \frac{1}{5}\log_5 \mathrm{e} = \frac{1}{5\ln 5}.$$

b)
$$\lim_{x\to+\infty} x\log_3\frac{x+5}{x+3} = \frac{1}{\ln 3}\cdot\lim_{x\to+\infty}\ln\left(1+\frac{2}{x+3}\right)^x = \frac{2}{\ln 3}.$$

c)
$$\lim_{x\to 0}\frac{\ln(1+4x+x^2)+\ln(1-4x+x^2)}{2x^2} = \lim_{x\to 0}\ln(1-14x^2+x^4)^{1/(2x^2)}$$
$$= \lim_{x\to 0}\left(\ln(1-14x^2+x^4)^{1/(x^2(x^2-14))}\right)^{(x^2-14)/2} = -7.$$

d)
$$\lim_{x\to 0}\frac{\sqrt{4+x}-2-\sin x}{\ln(1+x)} = \lim_{x\to 0}\left(\frac{\sqrt{4+x}-2-\sin x}{x}\cdot(\ln(1+x)^{1/x})^{-1}\right)$$
$$= \left(\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x} - \lim_{x\to 0}\frac{\sin x}{x}\right)\cdot\lim_{x\to 0}\left(\ln(1+x)^{1/x}\right)^{-1} = -\frac{3}{4}.$$

e)
$$\lim_{x\to 0}\frac{\ln\cos 4x}{\ln\cos 3x} = \lim_{x\to 0}\frac{\sin^2 4x}{\sin^2 3x}\cdot\frac{\ln(1-\sin^2 4x)^{-1/\sin^2 4x}}{\ln(1-\sin^2 3x)^{-1/\sin^2 3x}} = \frac{16}{9}.$$

f) $\lim\limits_{x\to+\infty} x^2 \ln\left(\sin\left(\frac{\pi}{2} - \frac{\pi}{x}\right)\right) = \frac{1}{2} \lim\limits_{x\to+\infty} \ln\left(1 - \sin^2\frac{\pi}{x}\right)^{x^2} = -\frac{\pi^2}{2}.$

g) $\lim\limits_{x\to-\infty} \frac{\ln(1+3^x)}{\ln(1+2^x)} = \lim\limits_{t\to+\infty} \frac{\ln\left(1+3^{-t}\right)}{\ln\left(1+2^{-t}\right)} = \lim\limits_{t\to+\infty} \frac{2^t}{3^t} \cdot \frac{\ln\left(1+3^{-t}\right)^{3^t}}{\ln\left(1+2^{-t}\right)^{2^t}} = 0.$

h) $\lim\limits_{x\to+\infty} \frac{\ln(1+3^x)}{\ln(1+2^x)} = \lim\limits_{x\to+\infty} \frac{x\ln 3 + \ln\left(1+\frac{1}{3^x}\right)}{x\ln 2 + \ln\left(1+\frac{1}{2^x}\right)} = \frac{\ln 3}{\ln 2}.$

Example 4.39. *Determine*

a) $\lim\limits_{x\to 0} \frac{e^x - 1}{x}$; **b)** $\lim\limits_{x\to 0} \frac{(1+x)^b - 1}{x}$, $b \in \mathbf{R}$;

c) $\lim\limits_{x\to 0} \frac{1-\cos^b x}{x^2}$, $b \in \mathbf{R}$; **d)** $\lim\limits_{x\to 0} \frac{e^{x^2} - (\cos x)^{2\sqrt{3}}}{x^2}$.

Solutions.

a) Changing the variables $t = e^x - 1$, i.e., $x = \ln(t+1)$, where $t \to 0$ when $x \to 0$, we obtain

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{t\to 0} \frac{t}{\ln(t+1)} = \lim_{t\to 0} \left(\ln(t+1)\right)^{\frac{1}{t}}\Big)^{-1} = 1.$$

b) For $b = 0$ it is trivial that the given limit is equal to $0 = b$. For $b \neq 0$, we start with the following transformation

$$\frac{(1+x)^b - 1}{x} = \frac{e^{b\ln(1+x)} - 1}{b\ln(1+x)} \cdot \frac{b\ln(1+x)}{x}.$$

Since $\lim\limits_{x\to 0} \frac{e^{b\ln(1+x)} - 1}{b\ln(1+x)} = 1$ and $\lim\limits_{x\to 0} \frac{b\ln(1+x)}{x} = b$, we obtain finally

$$\lim_{x\to 0} \frac{(1+x)^b - 1}{x} = b.$$

c) For $b = 0$ the limit is zero. For $b \neq 0$, similarly as in b), we can write

$$\lim_{x\to 0} \frac{1-\cos^b x}{x^2} = \lim_{x\to 0} \frac{1 - e^{b\ln(\cos x)}}{b\ln(\cos x)} \cdot \lim_{x\to 0} \frac{b\ln(\cos x)}{x^2} = 1 \cdot b \cdot \frac{1}{2} = \frac{b}{2}.$$

d) From a) and c) we obtain

$$\lim_{x\to 0} \frac{e^{x^2} - (\cos x)^{2\sqrt{3}}}{x^2} = \lim_{x\to 0} \frac{e^{x^2} - 1}{x^2} + \lim_{x\to 0} \frac{1 - (\cos x)^{2\sqrt{3}}}{x^2} = 1 + \sqrt{3}.$$

Example 4.40. *Assume $a > 0$ and determine*

$$\text{a) } \lim_{x\to a}\frac{a^x - x^a}{x-a}; \qquad \text{b) } \lim_{x\to a}\frac{x^x - a^a}{x-a}; \qquad \text{c) } \lim_{x\to a}\frac{a^{a^x} - a^{x^a}}{a^x - x^a}.$$

Solutions.

a) Using Example 4.39 b), we have

$$\begin{aligned}\lim_{x\to a}\frac{a^x - x^a}{x-a} &= a^a \lim_{x\to a}\frac{a^{x-a}-1}{x-a} - \lim_{x\to a}\frac{\left(1+\dfrac{x-a}{a}\right)^a - 1}{\dfrac{x-a}{a}}\cdot a^{a-1}\\ &= a^a\ln a - a\cdot a^{a-1} = a^a\ln(a/\mathrm{e}).\end{aligned}$$

b) From the transformation

$$\frac{x^x - a^a}{x-a} = \frac{x^x - x^a}{x-a} + \frac{x^a - a^a}{x-a}$$

and from $\lim\limits_{x\to a}(x-a)\ln x = 0$, it follows that

$$\lim_{x\to a}\frac{x^x - x^a}{x-a} = \lim_{x\to a}\frac{\mathrm{e}^{a\ln x}\left(\mathrm{e}^{(x-a)\ln x} - 1\right)}{(x-a)\ln x}\cdot\ln x = a^a\ln a.$$

Using Example 4.39 b), we have

$$\lim_{x\to a}\frac{x^a - a^a}{x-a} = \lim_{x\to a}\frac{a^{a-1}\left(\left(1+\dfrac{x-a}{a}\right)^a - 1\right)}{\dfrac{x-a}{a}} = a^a,$$

and therefore we get

$$\lim_{x\to a}\frac{x^x - a^a}{x-a} = a^a(\ln a + 1) = a^a\ln(a\mathrm{e}).$$

c)
$$\lim_{x\to a}\frac{a^{a^x} - a^{x^a}}{a^x - x^a} = \lim_{x\to a}\frac{a^{a^x - x^a} - 1}{a^x - x^a}\cdot\lim_{x\to a}a^{x^a} = \lim_{t\to 0}\frac{a^t - 1}{t}a^{a^a} = a^{a^a}\ln a,$$

because we put $t = a^x - x^a$ and $t \to 0$ when $x \to a$.

Exercise 4.41. *Determine*

a) $\lim\limits_{x\to 0+}(2+\sqrt{x})$; b) $\lim\limits_{x\to 0-}(2+\sqrt{-x})$;

c) $\lim\limits_{x\to 0-}\dfrac{1}{x}$; d) $\lim\limits_{x\to 0+}\dfrac{1}{x}$;

e) $\lim\limits_{x\to 0-}\dfrac{x+|x|}{2x}$; f) $\lim\limits_{x\to 0+}\dfrac{x+|x|}{2x}$;

g) $\lim\limits_{x\to 0-}\arcsin(x+1)$; h) $\lim\limits_{x\to 0+}\arcsin(x+1)$;

i) $\lim\limits_{x\to 0-}\mathrm{e}^{\cot x}$; j) $\lim\limits_{x\to 0+}\mathrm{e}^{\cot x}$;

k) $\lim\limits_{x\to 2-}\dfrac{1}{x^2+\mathrm{e}^{\frac{1}{2-x}}}$; l) $\lim\limits_{x\to 2+}\dfrac{1}{x^2+\mathrm{e}^{\frac{1}{2-x}}}$.

Answers.

a) 2.	b) 2.	c) $-\infty$.	d) $+\infty$.
e) 0.	f) 1.	g) $\pi/2$.	h) Senseless.
i) 0.	j) $+\infty$.	k) 0.	l) $1/4$.

Example 4.42. *Determine*

a) $\lim\limits_{x\to 0-}\dfrac{\sinh x}{x}$; b) $\lim\limits_{x\to 0-}\dfrac{\cosh x-1}{x^2}$; c) $\lim\limits_{x\to 0-}\dfrac{\tanh x}{2x}$.

Solutions. For the convenience of the reader, we rewrite the definition of the hyperbolic functions.

$$\sinh x=\frac{\mathrm{e}^x-\mathrm{e}^{-x}}{2},\qquad \cosh x=\frac{\mathrm{e}^x+\mathrm{e}^{-x}}{2}\quad\text{and}\quad \tanh x=\frac{\sinh x}{\cosh x}=\frac{\mathrm{e}^x-\mathrm{e}^{-x}}{\mathrm{e}^x+\mathrm{e}^{-x}}.$$

a) $$\lim_{x\to 0}\frac{\sinh x}{x}=\lim_{x\to 0}\frac{\mathrm{e}^x-\mathrm{e}^{-x}}{2x}=\lim_{x\to 0}\mathrm{e}^{-x}\cdot\frac{\mathrm{e}^{2x}-1}{2x}=1.$$

b) $$\lim_{x\to 0}\frac{\cosh x-1}{x^2}=\lim_{x\to 0}\frac{2\sinh^2\frac{x}{2}}{x^2}=\frac{1}{2}\lim_{x\to 0}\left(\frac{\sinh\frac{x}{2}}{\frac{x}{2}}\right)^2=\frac{1}{2}.$$

c) $$\lim_{x\to 0}\frac{\tanh x}{2x}=\lim_{x\to 0}\frac{1}{2}\cdot\frac{\sinh x}{x}\cdot\frac{1}{\cosh x}=\frac{1}{2}.$$

Exercise 4.43. *Determine*

a) $\lim\limits_{x\to 1-} \arctan \dfrac{1}{1-x}$, $\qquad \lim\limits_{x\to 1+} \arctan \dfrac{1}{1-x}$;

b) $\lim\limits_{x\to 0-} \dfrac{1}{1+e^{1/x}}$, $\qquad \lim\limits_{x\to 0+} \dfrac{1}{1+e^{1/x}}$;

c) $\lim\limits_{x\to -\infty} \left(1+\dfrac{1}{|x|}\right)^x$, $\qquad \lim\limits_{x\to +\infty} \left(1+\dfrac{1}{|x|}\right)^x$;

d) $\lim\limits_{x\to -\infty} \left(\sqrt{x^2+5x}-\sqrt{x^2+2x+1}\right)$, $\qquad \lim\limits_{x\to +\infty} \left(\sqrt{x^2+5x}-\sqrt{x^2+2x+1}\right)$.

Answers.

a) $\dfrac{\pi}{2}$, $-\dfrac{\pi}{2}$. $\qquad$ b) 1, 0. $\qquad$ c) e^{-1}, e. $\qquad$ d) $-\dfrac{3}{2}$, $\dfrac{3}{2}$.

Example 4.44. *Show for $a>1$, $k>0$:*

$$\text{a)} \lim_{x\to+\infty} \frac{x^k}{a^x} = 0; \qquad \text{b)} \lim_{x\to+\infty} \frac{\log_a x}{x^k} = 0.$$

Solutions.

a) Using Example 3.12 b), we have

$$\lim_{n\to\infty} \frac{n^k}{a^n} = 0 \quad \text{hence} \quad \lim_{n\to\infty} \frac{(n+1)^k}{a^n} = 0.$$

This means that there exists $n_0 \in \mathbf{N}$ such that, for given $\varepsilon > 0$, it holds

$$\frac{(n+1)^k}{a^n} < \varepsilon, \quad \text{for every} \quad n > n_0.$$

If we denote $n = [x]$, then we have $n \le x < n+1$ and

$$0 < \frac{x^k}{a^x} < \frac{(n+1)^k}{a^n} < \varepsilon, \quad \text{for} \quad n > n_0.$$

b) Taking $t = x^k$, $t \to +\infty$ when $x \to +\infty$, we get

$$\lim_{x\to+\infty} \frac{\log_a x}{x^k} = \frac{1}{k} \cdot \lim_{t\to+\infty} \frac{\log_a t}{t}.$$

From Example 3.79 e) it follows

$$\lim_{n\to\infty} \frac{\log_a n}{n} = 0, \quad \text{hence} \quad \lim_{n\to\infty} \frac{\log_a (n+1)}{n} = 0,$$

and this means that for given $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$, such that for $n > n_0$ it holds

$$0 \le \frac{\log_a (n+1)}{n} \le \varepsilon.$$

Taking $t > n_0 + 1$ and $n = [t]$ (thus $n > n_0$), we can write

$$0 \le \frac{\log_a t}{t} \le \frac{\log_a (n+1)}{n} \le \varepsilon, \quad \text{i.e.,} \quad \lim_{x\to+\infty} \frac{\log_a x}{x^k} = 0.$$

Example 4.45. *Prove* $\lim_{x\to 0}\left(x\cdot\sin\frac{1}{x}\right)=0.$

Solution. For arbitrary $\varepsilon>0$, it follows from relations

$$\left|x\cdot\sin\frac{1}{x}-0\right|=\left|x\cdot\sin\frac{1}{x}\right|<|x|,$$

that there exists $\delta=\varepsilon$, such that the following implication holds

$$0<|x|<\delta\Rightarrow\left|x\cdot\sin\frac{1}{x}\right|<\varepsilon.$$

Remark. We leave to the reader to show for $\alpha>0$

$$\lim_{x\to 0+}\left(x^{\alpha}\cdot\sin\frac{1}{x}\right)=0.$$

Example 4.46. *Determine the following limits.*

a) $\lim_{x\to+\infty}\left((x+3)\ln(x+3)-2(x+2)\ln(x+2)+(x+1)\ln(x+1)\right);$

b) $\lim_{x\to 0+}\ln(x\ln b)\ln\left(\dfrac{\ln bx}{\ln\frac{x}{b}}\right),\quad b>1.$

Solutions.

a) $\lim_{x\to+\infty}\left((x+3)\ln(x+3)-2(x+2)\ln(x+2)+(x+1)\ln(x+1)\right)$

$$=\lim_{x\to+\infty}\ln\left(\frac{(x+3)^{x+3}(x+1)^{x+1}}{(x+2)^{2(x+2)}}\right)=\lim_{x\to+\infty}\ln\left(\frac{\left(\dfrac{x+3}{x+2}\right)^{x+2}(x+3)}{\left(\dfrac{x+2}{x+1}\right)^{x+1}(x+2)}\right)=0.$$

b) $$\lim_{x\to 0+}\ln(x\ln b)\ln\left(\frac{\ln bx}{\ln\frac{x}{b}}\right)=\lim_{x\to 0+}\ln(x\ln b)\cdot\ln\left(1+\frac{\ln bx-\ln\frac{x}{b}}{\ln\frac{x}{b}}\right)$$

$$=\lim_{x\to 0+}\frac{\ln(x\ln b)\ln b^2}{\ln\frac{x}{b}}\cdot\ln\left(1+\frac{\ln b^2}{\ln\frac{x}{b}}\right)^{\ln(x/b)/\ln b^2}=\ln b^2.$$

4.2 Asymptotes for the graphs of functions

4.2.1 Basic notions

Definition 4.47. *A line* $x=x_0$ *is called* **vertical asymptote** *for the graph of a function* $f:A\to\mathbf{R}$ *if at least one of the following limits*

$$\lim_{x\to x_0+}f(x)\quad or\quad\lim_{x\to x_0-}f(x)$$

is equal either to $+\infty$ *or to* $-\infty$.

Definition 4.48. *A line* $y = n$ *is called* **horizontal asymptote** *when* $x \to +\infty$ *(resp. when* $x \to -\infty$*) for the graph of a function* $f : A \to \mathbf{R}$ *if it holds*

$$\lim_{x\to+\infty} f(x) = n \quad (\textit{resp.} \lim_{x\to-\infty} f(x) = n).$$

Definition 4.49. *A line* $y = kx + n$, $k \neq 0$, *is called* **slanted asymptote** *when* $x \to +\infty$ *(resp. when* $x \to -\infty$*) of the graph of a function* $f : A \to \mathbf{R}$ *if it holds*

$$\lim_{x\to+\infty}(f(x) - (kx + n)) = 0 \quad (\textit{resp.} \lim_{x\to-\infty}(f(x) - (kx + n)) = 0).$$

The numbers k and n can be ordered from the following.

$$\begin{aligned} k &= \lim_{x\to+\infty} \frac{f(x)}{x} \quad (\text{resp. } k = \lim_{x\to-\infty} \frac{f(x)}{x}) \\ n &= \lim_{x\to+\infty}(f(x) - kx) \quad (\text{resp } n = \lim_{x\to-\infty}(f(x) - kx)), \end{aligned} \tag{4.9}$$

provided that both limits in (4.9) when $x \to +\infty$ (resp. when $x \to -\infty$) exist.

4.2.2 Examples and exercises

Example 4.50 *Determine the asymptotes for the graphs of the following functions.*

a) $f(x) = \dfrac{2x}{\sqrt{4 - x^2}}$; **b)** $f(x) = \ln\dfrac{1 - x}{1 + x}$;

c) $f(x) = \dfrac{x}{1 + x^2}$; **d)** $f(x) = \sqrt{x + 1} - \sqrt{x - 1}$;

e) $f(x) = \sqrt{x} + \sqrt{4 - x}$; **f)** $f(x) = \arcsin e^x$.

Solutions.

a) The function f is defined on the interval $(-2, 2)$. From

$$\lim_{x\to-2+} \frac{2x}{\sqrt{4 - x^2}} = -\infty, \qquad \lim_{x\to2-} \frac{2x}{\sqrt{4 - x^2}} = +\infty,$$

it follows that the graph of f has vertical asymptotes $x = 2$ and $x = -2$.

b) The function f is defined on the interval $(-1, 1)$. From

$$\lim_{x\to-1+} \ln\frac{1 - x}{1 + x} = +\infty, \qquad \lim_{x\to1-} \ln\frac{1 - x}{1 + x} = -\infty,$$

it follows that the graph of f has vertical asymptotes $x = 1$ and $x = -1$.

c) The graph of f has a horizontal asymptote $y = 0$ when both $x \to +\infty$ and $x \to -\infty$, because

$$\lim_{x\to+\infty} \frac{x}{1 + x^2} = 0$$

and

$$\lim_{x\to-\infty} \frac{x}{1 + x^2} = 0.$$

d) The function f is defined on the interval $[1, +\infty)$. The graph of f has no vertical asymptotes, because

$$\lim_{x\to 1+} (\sqrt{x+1} - \sqrt{x-1}) = \sqrt{2}.$$

It has a horizontal asymptote $y = 0$ when $x \to +\infty$, because

$$\lim_{x\to +\infty} (\sqrt{x+1} - \sqrt{x-1}) = 0.$$

e) The function f is defined on the interval $[0, 4]$. Its graph has no asymptotes.

f) The function f is defined on the interval $(-\infty, 0]$. It has no vertical asymptotes, because

$$\lim_{x\to 0-} \arcsin \mathrm{e}^x = \frac{\pi}{2}.$$

However, from

$$\lim_{x\to -\infty} \arcsin \mathrm{e}^x = 0,$$

it follows that the graph of f has a horizontal asymptote $y = 0$ when $x \to -\infty$.

Example 4.51 *Determine the asymptotes for graphs the following functions.*

a) $f(x) = \dfrac{1}{1-x^2}$; **b)** $f(x) = x - \dfrac{1}{x}$;

c) $f(x) = \arctan x$; **d)** $f(x) = \sqrt{x^2+3}$;

e) $f(x) = \ln(1 + \mathrm{e}^x)$; **f)** $f(x) = \dfrac{x^2+1}{x-2}$.

Solutions.

a) The natural domain of the function f is the set $\mathbf{R} \setminus \{1, -1\}$. Its graph has vertical asymptotes $x = 1$ and $x = -1$, because

$$\lim_{x\to 1+} \frac{1}{1-x^2} = -\infty \quad \text{and} \quad \lim_{x\to -1+} \frac{1}{1-x^2} = +\infty.$$

(Note that $\displaystyle\lim_{x\to 1-} \frac{1}{1-x^2} = +\infty$, $\displaystyle\lim_{x\to -1-} \frac{1}{1-x^2} = -\infty$.) Its graph has a horizontal asymptote $y = 0$ when $x \to +\infty$ and when $x \to -\infty$, because

$$\lim_{x\to +\infty} \frac{1}{1-x^2} = 0 \quad \text{and} \quad \lim_{x\to -\infty} \frac{1}{1-x^2} = 0.$$

b) The graph of the function f has a vertical asymptote $x = 0$ and a slanted asymptote $y = x$ when $x \to +\infty$ and when $x \to -\infty$, because

$$k = \lim_{x\to \pm\infty} \frac{x - \dfrac{1}{x}}{x} = 1 \quad \text{and} \quad n = \lim_{x\to \pm\infty} \left(x - \frac{1}{x} - x\right) = 0.$$

c) The graph of f has two horizontal asymptotes, namely $y = -\pi/2$ when $x \to -\infty$ and $y = \pi/2$ when $x \to +\infty$.

d) The graph of f has two slanted asymptotes, namely $y = -x$ when $x \to -\infty$ and $y = x$ when $x \to +\infty$.

e) The graph of f has a horizontal asymptote $y = 0$ when $x \to -\infty$, because

$$\lim_{x\to-\infty} \ln(1 + e^x) = 0.$$

Note that

$$\lim_{x\to+\infty} \ln(1 + e^x) = +\infty,$$

hence it has no horizontal asymptote when $x \to +\infty$. However, it has a slanted asymptote $y = x$ when $x \to +\infty$, because

$$\lim_{x\to+\infty} \frac{\ln(1 + e^x)}{x} = 1 \quad \text{and} \quad \lim_{x\to+\infty} (\ln(1 + e^x) - x) = 0.$$

f) The graph of f has a vertical asymptote $x = 2$ and a slanted asymptote $y = x+2$ when $x \to +\infty$ and when $x \to -\infty$.

Exercise 4.52 *Determine the asymptotes for the graphs of the following functions.*

a) $f(x) = x^2 e^x$; **b)** $f(x) = e^{-x^2}$;

c) $f(x) = e^{1/x}$; **d)** $f(x) = e^{-1/x^2}$;

e) $f(x) = e^{1/x^2}$; **f)** $f(x) = xe^{-1/x}$.

Answers. The graph of f has a

a) horizontal asymptote $y = 0$ when $x \to -\infty$;

b) horizontal asymptote $y = 0$ when $x \to +\infty$, and also when $x \to -\infty$;

c) vertical asymptote $x = 0$ and a horizontal asymptote $y = 1$ both when $x \to -\infty$ and when $x \to +\infty$;

d) horizontal asymptote $y = 1$ both when $x \to -\infty$ and $x \to +\infty$, because

$$\lim_{x\to-\infty} e^{-1/x^2} = 1 \quad \text{and} \quad \lim_{x\to+\infty} e^{-1/x^2} = 1,$$

but has no vertical asymptotes, because

$$\lim_{x\to0-} e^{-1/x^2} = 0 \quad \text{and} \quad \lim_{x\to0+} e^{-1/x^2} = 0;$$

e) vertical asymptote $x = 0$ and a horizontal asymptote $y = 1$ when $x \to -\infty$ and also when $x \to +\infty$;

f) vertical asymptote $x = 0$ and a slanted asymptote $y = x - 1$ when $x \to -\infty$ and also when $x \to +\infty$.

4.3 Asymptotic relations

4.3.1 Basic notions

In this subsection we shall suppose that the domains of the functions f and g contain a set $(a, x_0) \cup (x_0, b)$.

Definition 4.53. *We say that a function f is* **asymptotically equivalent** *to the function g as x approaches x_0 if there exists a function ϕ such that*

$$f(x) = \phi(x) \cdot g(x), \ x \in (a, b), \ x \neq x_0,$$

and it holds $\lim\limits_{x \to x_0} \phi(x) = 1.$

Then we write $f(x) \sim g(x)$ as $x \to x_0$. A sufficient condition for the asymptotic equivalence $f(x) \sim g(x)$ as $x \to x_0$ is the equality $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = 1.$

Definition 4.54. *We say that a function f is* **small oh** *of the function g* **as** *x* **approaches** *x_0 if there exists a function ϕ such that*

$$f(x) = \phi(x) \cdot g(x), \ x \in (a, b), \ x \neq x_0,$$

and it holds $\lim\limits_{x \to x_0} \phi(x) = 0.$

Then we write $f = o(g)$ as $x \to x_0$. If $g(x) \neq 0$ for every $x \neq x_0$, then a necessary and sufficient condition for the asymptotic relation $f(x) = o(g(x))$ as $x \to x_0$ is the equality $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = 0.$

In particular, if $g(x) \equiv 1$, then $f(x) = o(1)$ as $x \to x_0$. This means that f tends to zero as $x \to x_0$.

Definition 4.55. *We say that a function f is* **big oh** *of the function g as x approaches x_0 if there exists a constant $K > 0$ such that*

$$|f(x)| \leq K \cdot |g(x)|, \ x \in (a, b), \ x \neq x_0.$$

Then we write $f(x) = O(g(x))$ as $x \to x_0$. In particular, if $g(x) \equiv 1, \ x \in (a, b)$, then $f(x) = O(1)$ as $x \to x_0$. This means that, for some $\delta > 0$, the function f is bounded on the set $(x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$.

These tree notions, namely the asymptotic equivalence, "big oh" and "small oh", can also be defined in the cases $x \to x_0 + 0$, $x \to x_0 - 0$, $x \to +\infty$ and $x \to -\infty$.

4.3.2 Examples and exercises

Example 4.56. *Show that*

a) $\mathrm{e}^x - 1 \sim \dfrac{1}{2} \sin 2x, \ x \to 0;$ **b)** $\sinh x \sim x, \ x \to 0;$

c) $\left(\dfrac{x+1}{x-1}\right)^x \sim \mathrm{e}^2, \ x \to +\infty;$ **d)** $\ln(1 + 3^x) \sim \dfrac{\ln 3}{\ln 2} \ln(1 + 2^x), \ x \to +\infty.$

Solutions.

a) From the equality $\lim\limits_{x\to 0}\dfrac{e^x-1}{\sin 2x}=1/2$, by Definition 4.53 it follows

$$e^x-1\sim\frac{1}{2}\sin 2x,\quad x\to 0.$$

b) Follows from Example 4.42 a).

c) Follows from Example 4.35 a).

d) Follows from Example 4.38 h).

Example 4.57. *Prove that* $f(x)\sim g(x),\ x\to 0$, *where*

$$f(x)=\begin{cases}\sin x, & x\in\mathbf{Q};\\ 0, & x\in\mathbf{R}\setminus\mathbf{Q},\end{cases}\qquad \textit{and}\qquad g(x)=\begin{cases}x, & x\in\mathbf{Q};\\ 0, & x\in\mathbf{R}\setminus\mathbf{Q},\end{cases}$$

with $\mathbf{Q}$ *denoting the set of rational numbers.*

Solution. Let us put $\psi(x)=\dfrac{\sin x}{x}$, $x\neq 0$. Then it holds

$$f(x)=\psi(x)g(x)\quad\text{and}\quad\lim_{x\to 0}\psi(x)=1,$$

and by Definition 4.53 this means that $f(x)\sim g(x)$, when $x\to 0$.

Remark. Note that in this case the limit $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)}$ does *not* exist.

Example 4.58. *Determine the following limits.*

a) $\lim\limits_{x\to 0}\dfrac{\sin 3x+2\arctan 2x+3x^2}{\ln(1+2x+\sin^2 x)+xe^x}$;

b) $\lim\limits_{x\to 0}\dfrac{\ln\cos x}{\tan x^2}$;

c) $\lim\limits_{x\to+\infty}x\left(\ln\left(1+\dfrac{x}{2}\right)-\ln\dfrac{x}{2}\right)$;

d) $\lim\limits_{x\to 0}(1-x^3)^{\cot x}$.

Solutions.

a) From the following asymptotic behaviors

$$\sin 3x\sim 3x,\quad \arctan 2x\sim 2x,\quad xe^x\sim x,\quad \ln(1+2x+\sin^2 x)\sim 2x+\sin^2 x\sim 2x,$$

when $x\to 0$, we have

$$\lim_{x\to 0}\frac{\sin 3x+2\arctan 2x+3x^2}{\ln(1+2x+\sin^2 x)+xe^x}=\lim_{x\to 0}\frac{7x}{3x}=\frac{7}{3}.$$

b) Since

$$2\ln\cos x = \ln(1-\sin^2 x) \sim -\sin^2 x \sim -x^2, \quad \tan x^2 \sim x^2 \quad \text{when} \quad x \to 0,$$

we have

$$\lim_{x\to 0}\frac{\ln\cos x}{\tan x^2} = \frac{-1}{2}\lim_{x\to 0}\frac{x^2}{x^2} = -\frac{1}{2}.$$

c) $\displaystyle\lim_{x\to+\infty} x\left(\ln\left(1+\frac{x}{2}\right)-\ln\frac{x}{2}\right) = \lim_{x\to+\infty} x\ln\left(1+\frac{2}{x}\right)$

$$= \lim_{x\to+\infty} x\left(\frac{2}{x}+o\left(\frac{2}{x}\right)\right) = \lim_{x\to+\infty}\left(2+\frac{o\left(\frac{1}{x}\right)}{\frac{1}{x}}\right) = 2.$$

d) From the expressions

$$\lim_{x\to 0}(1-x^3)^{\cot x} = e^{\ell}, \quad \text{where} \quad \ell = \lim_{x\to 0}\frac{\ln(1-x^3)}{\tan x} = \lim_{x\to 0}\frac{-x^3}{x} = 0,$$

it follows

$$\lim_{x\to 0}(1-x^3)^{\cot x} = 1.$$

Exercise 4.59. *Prove the following asymptotic relations.*

a) $\displaystyle\frac{x^m}{1+x^n} \sim x^m,\ x \to 0;$

b) $\displaystyle\frac{x^m}{1+x^n} \sim x^{m-n},\ x \to +\infty;$

c) $2-2\cos^b x \sim bx^2,\ x \to 0,\ b \neq 0;$

d) $\arctan 2x \sim 2x,\ x \to 0;$

e) $\arctan x \sim \dfrac{\pi}{2},\ x \to +\infty;$

f) $\displaystyle\frac{1}{\cos^m x} \sim \frac{1}{\left(\frac{\pi}{2}-x\right)^m},\ x \to \frac{\pi}{2}.$

In this case, m and n are natural numbers.

Exercise 4.60. *Determine the numerical constants a and b such that $f(x) \sim ax^b$, when $x \to x_0$, if*

a) $f(x) = \sqrt{3x+\sqrt{x+\sqrt{x}}}, \quad x_0 = 0;$

b) $f(x) = \sqrt{3x+\sqrt{x+\sqrt{x}}}, \quad x_0 = +\infty;$

c) $f(x) = 5e^{x^4} + (\cos x - 1)^2 + x^6 - 5, \quad x_0 = 0;$

d) $f(x) = \sin^2 3x + \arcsin^2 x + 2\arctan x^2, \quad x_0 = 0.$

Answers.

a) $a = 1,\ b = \dfrac{1}{8}.$ b) $a = \sqrt{3},\ b = \dfrac{1}{2}.$

c) $a = \dfrac{21}{4},\ b = 4.$ d) $a = 12,\ b = 2.$

Example 4.61. *Show the following asymptotic equalities.*

a) $x^2 \sin \sqrt[3]{x} = \sqrt[3]{x^7} + o(\sqrt[3]{x^7}),\ x \to 0;$ b) $\mathrm{e}^x - 1 = x + o(x),\ x \to 0;$

c) $\log_a x = o(x^c),\ a > 1,\ c > 0,\ x \to 0+.$

Solutions.

a) From the following limits

$$\lim_{x\to 0} \frac{x^2 \sin \sqrt[3]{x}}{\sqrt[3]{x^7}} = 1 \quad \text{and} \quad \lim_{x\to 0} \left(\frac{x^2 \sin \sqrt[3]{x} - \sqrt[3]{x^7}}{\sqrt[3]{x^7}} \right) = 0,$$

we obtain the given asymptotic equality.

b) Follows from Example 4.39 a).

c) Follows from Example 4.44 b).

Example 4.62. *Determine the following limits.*

a) $\lim\limits_{x\to 0} \left(2\mathrm{e}^{x/(x+1)} - 1\right)^{(x^2+1)/x};$ b) $\lim\limits_{x\to 0} \dfrac{\ln \cos ax}{\ln \cos bx},\ a, b \neq 0;$

c) $\lim\limits_{x\to 0} \left(\dfrac{1 + \sin x \cos ax}{1 + \sin x \cos bx} \right)^{\cot^3 x}.$

Solutions.

a) From the following expression $\mathrm{e}^{x/(x+1)} - 1 = \dfrac{x}{x+1} + o(x),\ x \to 0$, and Example 4.34 we can write

$$\lim_{x\to 0} \left(2\mathrm{e}^{x/(x+1)} - 1\right)^{(x^2+1)/x} = \exp\left(\lim_{x\to 0} 2 \left(\mathrm{e}^{x/(x+1)} - 1\right) \cdot \frac{x^2+1}{x} \right)$$

$$= \exp\left(\lim_{x\to 0} 2 \left(\frac{x}{x+1} + o(x) \right) \cdot \frac{x^2+1}{x} \right)$$

$$= \exp\left(\lim_{x\to 0} 2 \left(\frac{x^2+1}{x+1} + \frac{(x^2+1)o(x)}{x} \right) \right) = \mathrm{e}^2.$$

b) Using the asymptotic equality $\ln(1+x) = x + o(x)$, $x \to 0$, we have

$$\lim_{x\to 0}\frac{\ln\cos ax}{\ln\cos bx} = \lim_{x\to 0}\frac{\ln\left(1-\dfrac{a^2x^2}{2}+o(x^2)\right)}{\ln\left(1-\dfrac{b^2x^2}{2}+o(x^2)\right)} = \lim_{x\to 0}\frac{-\dfrac{a^2x^2}{2}+o(x^2)}{-\dfrac{b^2x^2}{2}+o(x^2)} = \frac{a^2}{b^2}.$$

c) Let us first consider the following transformations when $x \to 0$.

$$\left(\frac{1+\sin x\cos ax}{1+\sin x\cos bx}-1\right)\cdot\cot^3 x = \frac{\cos ax-\cos bx}{1+\sin x\cos bx}\cdot\frac{\cos^3 x}{\sin^2 x}$$

$$= \frac{\left(\dfrac{b^2-a^2}{2}x^2+o(x^2)\right)}{1+(x+o(x))\cdot\left(1-\dfrac{b^2x^2}{2}+o(x^2)\right)}\cdot\frac{\left(1-\dfrac{x^2}{2}+o(x^2)\right)^3}{(x^2+o(x^2))}$$

$$= \frac{\dfrac{b^2-a^2}{2}x^2+o(x^2)}{x^2+o(x^2)}.$$

Then we have

$$\lim_{x\to 0}\left(\frac{1+\sin x\cos ax}{1+\sin x\cos bx}\right)^{\cot^3 x}$$

$$= \exp\left(\lim_{x\to 0}\cot^3 x\cdot\left(\frac{1+\sin x\cos ax}{1+\sin x\cos bx}-1\right)\right) = \exp\left((b^2-a^2)/2\right).$$

Example 4.63. *Are the following asymptotic equalities correct?*

a) $1995x + x\cos x = O(x)$, $x \to +\infty$; **b)** $x = O(1995x + x\cos x)$, $x \to +\infty$;

c) $x = O(x + x\cos x)$, $x \to +\infty$; **d)** $\sqrt{x^2+3} - x = O\left(\dfrac{1}{x}\right)$, $x \to +\infty$.

Solutions.

a) Since it holds $|1995x + x\cos x| \le C|x|$, $x \in \mathbf{R}$, (for example for $x \ge 1$ one can take $C = 1996$), it follows that the given statement is true.

b) The statement is true, because for $x \ge 1$ it holds

$$|x| \le |x + (1994+\cos x)x| = 1\cdot|1995x + x\cos x|.$$

c) The statement is *not* true, because for every $x_0 > 0$ and for every $C > 0$, there exists $x_1 > x_0$ such that it holds

$$1+\cos x_1 < \frac{1}{C} \quad\text{or}\quad x_1 > C(x_1 + x_1\cos x_1).$$

d) The statement is true, because it holds

$$|\sqrt{x^2+3}-x| < 3\cdot\frac{1}{x}, \quad x \geq 1.$$

Exercise 4.64. *Show*

a) $1-\cos x = \dfrac{x^2}{2} + o(x),\ x \to 0;$ **b)** $a^x - 1 = x\ln a + o(x \ln a),\ x \to 0;$

c) $x + x\cos x = O(x),\ x \to 0;$ **d)** $\dfrac{1}{x} = O\left(\sqrt{x^2+1} - |x|\right),\ x \to +\infty.$

Example 4.65. *Let f be a positive function on some neighbourhood of the point x_0. Prove the following asymptotic relations when $x \to x_0$.*

a) $O(O(f)) = O(f);$ **b)** $o(O(f)) = o(f);$

c) $O(o(f)) = o(f);$ **d)** $o(f) + O(f) = O(f).$

Solutions. We shall prove only parts a) and d), the others are left to the reader.

a) Put $g = O(f)$ and $h = O(g)$. Then by Definition 4.55 there exist constants $K_1 > 0$ and $K_2 > 0$ such that in some neighbourhood U of x_0 it holds for $x \neq x_0$

$$|g(x)| \leq K_1 f(x) \text{ and } |h(x)| \leq K_2 g(x)$$

Then on the set $U \setminus \{x_0\}$ it holds

$$|h(x)| \leq K_2 K_1 f(x),$$

which gives the statement.

d) Put $g = o(f)$ and $h = O(f)$. Then by Definitions 4.54 and 4.55 there exist a function ϕ, a constant $K > 0$ and a neighbourhood U of x_0 such that

$$g(x) = \phi(x) f(x), \quad \lim_{x\to x_0} \phi(x) = 0 \quad \text{and} \quad |h(x)| \leq K f(x), \quad x \in U \setminus \{x_0\}.$$

Then the sum of the functions g and h on the set $U \setminus \{x_0\}$ can be written as

$$g(x) + h(x) = \phi(x) f(x) + h(x).$$

Since $\phi(x)$ tends to zero as $x \to x_0$, there exists a neighbourhood $U_1 \subset U$ of the point x_0, such that on the set $U_1 \setminus \{x_0\}$ it holds $|\phi(x)| \leq 1$. Thus

$$(\forall x \in U_1 \setminus \{x_0\}) \quad |g(x) + h(x)| \leq 1 \cdot f(x) + K \cdot f(x) \leq K_1 f(x)$$

for $K_1 := 1 + K$.

Exercise 4.66. *Show that if the functions f and g have the properties $f(x) \neq 0$ and $g(x) \neq 0$ for $x \neq x_0$, then as $x \to x_0$ it holds*

$$f(x) \sim g(x) \iff g(x) - f(x) = o(g).$$

Hint. From the equality $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = 1$ it follows $\lim\limits_{x\to x_0} \left(1 - \dfrac{f(x)}{g(x)}\right) = 0$, wherefrom we obtain

$$\lim_{x\to x_0} \frac{g(x) - f(x)}{g(x)} = 0, \quad \text{meaning} \quad g(x) - f(x) = o(g), \quad x \to x_0.$$

Example 4.67. *Let us suppose that $f(x) \sim f_1(x)$ and $g(x) \sim g_1(x)$, when $x \to x_0$. If there exists $\lim\limits_{x\to x_0} \dfrac{f_1(x)}{g_1(x)}$, then there exists $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)}$ satisfying*

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f_1(x)}{g_1(x)}.$$

Prove.

Solution. From Example 4.66 it follows $f(x) = f_1(x) + o(f_1(x))$ and $g(x) = g_1(x) + o(g_1(x))$ when $x \to x_0$. Therefore we can write

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f_1(x) + o(f_1(x))}{g_1(x) + o(g_1(x))} = \lim_{x\to x_0} \frac{f_1(x)}{g_1(x)} \cdot \lim_{x\to x_0} \frac{1 + \dfrac{o(f_1(x))}{f_1(x)}}{1 + \dfrac{o(g_1(x))}{g_1(x)}} = \lim_{x\to x_0} \frac{f_1(x)}{g_1(x)}.$$

Chapter 5

Continuity

5.1 Continuity at a point

5.1.1 Basic notions

Definition 5.1. *A function* $f : A \subset \mathbf{R} \to \mathbf{R}$ *is* **continuous at a point** $x_0 \in A$ *if for every* $\varepsilon > 0$ *there exists a number* $\delta > 0$, *depending on* ε *and on the point* x_0, *such that for every* $x \in A$ *with the property* $|x - x_0| < \delta$ *it holds* $|f(x) - f(x_0)| < \varepsilon$.

Using logical symbols, the last definition can be expressed as follows.
A function $f : A \subset \mathbf{R} \to \mathbf{R}$ *is* **continuous at a point** $x_0 \in A$ *iff*

$$(\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall x \in A) \quad |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon. \tag{5.1}$$

If the point $x_0 \in A$ is an accumulation point of the set A, then the following two, mutually equivalent, definitions can also be used.

Definition 5.2. *A function* $f : A \subset \mathbf{R} \to \mathbf{R}$ *is* **continuous at a point** $x_0 \in A$, *where* x_0 *is an accumulation point of the domain* A, *if it holds*

$$\lim_{x \to x_0,\, x \in A} f(x) = f(x_0).$$

Definition 5.3. *A function* $f : A \subset \mathbf{R} \to \mathbf{R}$ *is* **continuous at a point** $x_0 \in A$, *where* x_0 *is an accumulation point of the domain* A, *if for every sequence* $(x_n)_{n \in \mathrm{N}}$ *of elements from* A *it holds that*

$$\lim_{n \to \infty} f(x_n) = f(x_0).$$

Definition 5.4.

- *Assume that the domain of a function* f *contains an interval* $(a, x_0]$. *Then the function* $f : A \to \mathbf{R}$ *is* **continuous from the left side** *at the point* x_0 *if*

$$\lim_{x \to x_0-} f(x) = f(x_0).$$

- *Assume that the domain of a function f contains an interval $[x_0, b)$. Then the function $f : A \to \mathbf{R}$ is* **continuous from the right side** *at the point x_0 if*

$$\lim_{x \to x_0+} f(x) = f(x_0).$$

Theorem 5.5. *A function $f : (a, b) \to \mathbf{R}$ is continuous at a point $x_0 \in (a, b)$ if and only if both one sided limits of f at x_0 exist, the limit of the function f at x_0 also exists, and all these three limits are equal to $f(x_0)$, i.e.,*

$$\lim_{x \to x_0-} f(x) = \lim_{x \to x_0+} f(x) = \lim_{x \to x_0} f(x) = f(x_0).$$

In view of this theorem, one can classify the discontinuities of a function f defined on an open interval. Namely, if a function $f : (a, b) \subset \mathbf{R} \to \mathbf{R}$ is **discontinuous at a point** $x_0 \in (a, b)$, then there are three possibilities.

- If the limit $\lim\limits_{x \to x_0} f(x)$ exists and equals to some number $L \neq f(x_0)$, then f has a **removable discontinuity** at the point x_0;

- if both one sided limits

$$\lim_{x \to x_0-} f(x) =: L_1 \quad \text{and} \quad \lim_{x \to x_0+} f(x) =: L_2 \tag{5.2}$$

  exist, but are unequal, $L_1 \neq L_2$, then f has a **first order discontinuity** at the point x_0;

- if at least one of the two one sided limits in (5.2) does not exist, then f has a **second order discontinuity** at the point x_0.

A function $f : A \subset \mathbf{R} \to \mathbf{R}$ is **continuous on a set** $B \subset A$ if it is continuous at every point from B. It is important to note that all basic elementary functions are continuous on their natural domains.

If the functions $f : A_1 \subset \mathbf{R} \to \mathbf{R}$ and $g : A_2 \subset \mathbf{R} \to \mathbf{R}$ are continuous at a point $x_0 \in A_1 \cap A_2$ (resp. on a set $B \subset A_1 \cap A_2$), then the following functions are also continuous at the point $x_0 \in A_1 \cap A_2$ (resp. on the set $B \subset A_1 \cap A_2$) :

- $f + g$ (sum of f and g),
- $f \cdot g$ (product of f and g),
- $\dfrac{f}{g}$ (quotient of f and g),

Of course, for the quotient $\dfrac{f}{g}$ this holds only when $g(x_0) \neq 0$ (resp. when $g(x) \neq 0$ for every $x \in B$).

Assume that a function $g : A \subset \mathbf{R} \to \mathbf{R}$ is continuous at a point $x_0 \in A$ and that a function $f : B \subset g(A) \to \mathbf{R}$ is continuous at the point $g(x_0)$. Then the composite function $h = f \circ g$ is also continuous at the point x_0. Shortly we say that the

- composition of continuous functions is continuous.

5.1.2 Examples and Exercises

Example 5.6. *Using Definition 5.1, check the continuity of the following functions at an arbitrary point x_0 from their domains.*

a) $f(x) = 2x + 1, \quad x \in \mathbf{R};$ **b)** $f(x) = x^2, \quad x \in \mathbf{R};$

c) $f(x) = \sqrt{x}, \quad x \in [0, +\infty);$ **d)** $f(x) = \sqrt[3]{x}, \quad x \in \mathbf{R};$

e) $f(x) = \arctan x, \quad x \in \mathbf{R}.$

Solutions.

a) Fix $x_0 \in \mathbf{R}$. Let $\varepsilon > 0$ be given. From the inequality

$$|f(x) - f(x_0)| = 2|x - x_0| < 2\delta, \quad x \in \mathbf{R},$$

it follows that choosing $\delta := \varepsilon/2$ the implication (5.1) holds, i.e.,

$$(\forall x \in \mathbf{R}) \quad |x - x_0| < \delta = \varepsilon/2 \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

b) Fix $x_0 \in \mathbf{R}$. Let $\varepsilon > 0$ be given. Then it holds

$$|f(x) - f(x_0)| = |x^2 - x_0^2| = |(x - x_0)^2 + 2x_0(x - x_0)| \leq |x - x_0|^2 + 2|x_0|\,|x - x_0|.$$

In order to determine $\delta > 0$, we assume $|x - x_0| < \delta$ and $\delta < 1$, hence

$$|f(x) - f(x_0)| < \delta^2 + 2|x_0|\delta < \delta(1 + 2|x_0|),$$

and assume that the last expression is less than ε. Then choosing

$$\delta := \frac{1}{2} \min\left\{1, \frac{\varepsilon}{1 + 2|x_0|}\right\},$$

it follows that

$$(\forall x \in \mathbf{R}) \quad |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

Remark. The number δ from Definition 5.1 (hence also in the last example) is not uniquely determined, Namely, if for given ε, we find a δ such that for every $x \in (x_0 - \delta, x_0 + \delta)$, it holds $|f(x) - f(x_0)| < \varepsilon$, then for every other δ_1, $0 < \delta_1 < \delta$, it holds

$$|x - x_0| < \delta_1 \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

Again, it is important that for given $\varepsilon > 0$ a $\delta > 0$ *exists.*

c) Fix first $x_0 > 0$. Let $\varepsilon > 0$ be given. From the relations

$$|\sqrt{x} - \sqrt{x_0}| = \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}} < \frac{|x - x_0|}{\sqrt{x_0}},$$

assuming that $|x - x_0| < \delta$, it follows

$$|f(x) - f(x_0)| < \frac{\delta}{\sqrt{x_0}} < \varepsilon.$$

Hence we can determine $\delta := \varepsilon\sqrt{x_0}$, which then gives the implication (5.1). The case $x_0 = 0$ is left to the reader.

d) Fix $x_0 > 0$; the case $x_0 = 0$, being somewhat easier, is omitted.

Let $\varepsilon > 0$ be given. If $|x - x_0| < \delta$, then

$$\begin{aligned} |\sqrt[3]{x} - \sqrt[3]{x_0}| &= \frac{|x - x_0|}{\sqrt[3]{x^2} + \sqrt[3]{xx_0} + \sqrt[3]{x_0^2}} \\ &= \frac{|x - x_0|}{\left(\sqrt[3]{x} + \frac{\sqrt[3]{x_0}}{2}\right)^2 + \frac{3}{4}\sqrt[3]{x_0^2}} < \frac{\delta}{\frac{3}{4}\sqrt[3]{x_0^2}}. \end{aligned}$$

Hence δ can be determined as $\delta := \varepsilon \cdot \frac{3}{4}\sqrt[3]{x_0^2}$.

e) Let $\varepsilon > 0$ be given. Let first $x_0 = 0$. Then for all $x \in \mathbf{R}$ it holds

$$|f(x) - f(0)| = |\arctan x - \arctan 0| = |\arctan x| \leq |x|,$$

see Example 6.89 b). Clearly, choosing $\delta := \frac{\varepsilon}{2}$, it follows

$$|x - 0| < \delta \Rightarrow |f(x) - f(0)| < \varepsilon.$$

Assume now $x_0 \neq 0$ and let $x > x_0$; i.e., $\delta_1 := x - x_0 > 0$. Using Example 2.28 f), it follows that

$$\begin{aligned} |\arctan x - \arctan x_0| &= \arctan\frac{x - x_0}{1 + xx_0} \\ &\leq \frac{x - x_0}{1 + xx_0} = \frac{\delta_1}{1 + x_0^2 + \delta_1 x_0} \leq \frac{\delta_1}{1 + x_0^2 - \delta_1 x_0} < \varepsilon, \end{aligned}$$

provided that $\delta_1 < \dfrac{1 + x_0^2}{1 + \varepsilon x_0}\varepsilon$.

Similarly, if $x < x_0$, i.e., $\delta_2 := -(x - x_0) > 0$, then

$$|\arctan x - \arctan x_0| \leq \frac{\delta_2}{1 + x_0^2 - \delta_2 x_0} < \varepsilon, \tag{5.3}$$

provided that again $\delta_2 < \dfrac{1 + x_0^2}{1 + \varepsilon x_0}\varepsilon$.

Hence, putting

$$\delta := \frac{1}{2} \cdot \min\left\{\frac{1 + x_0^2}{1 + \varepsilon x_0}\varepsilon, \frac{|x_0|}{2}\right\}$$

(the latter in order to keep the intervals $(x_0 - \delta_j, x_0 + \delta_j)$, $j = 1, 2$, away from the point 0) we get the implication in (5.1).

Exercise 5.7. *Check the continuity of the following functions at an arbitrary point x_0 from their domains.*

a) $f(x) = \sin x, \quad x \in \mathbf{R};$ b) $f(x) = \cos x, \quad x \in \mathbf{R};$

c) $f(x) = \arcsin x, \quad x \in [-1, 1];$ d) $f(x) = \arccos x, \quad x \in [-1, 1];$

e) $f(x) = \tan x, \quad x \in \mathbf{R} \setminus \{(2k+1)\pi/2 \big| \; k \in \mathbf{Z}\};$

f) $f(x) = \cot x, \quad x \in \mathbf{R} \setminus \{k\pi | \; k \in \mathbf{Z}\}.$

Exercise 5.8. *Check the continuity of the following functions at an arbitrary point x_0 from their domains.*

a) $f(x) = \mathrm{e}^x, \quad x \in \mathbf{R};$ b) $f(x) = a^x, \quad x \in \mathbf{R}, \; a > 0;$

c) $f(x) = \ln x, \quad x \in (0, +\infty);$ d) $f(x) = \log_a x, \quad x \in (0, +\infty), \; a > 0;$

e) $f(x) = \sinh x, \quad x \in \mathbf{R};$ f) $f(x) = \cosh x, \quad x \in \mathbf{R};$

g) $f(x) = \tanh x, \quad x \in \mathbf{R};$ h) $f(x) = \coth x, \quad x \in \mathbf{R} \setminus \{0\}.$

Exercise 5.9. *Check the continuity of the following functions at an arbitrary point x_0 from their domains.*

a) $f(x) = \cos(ax + b), \quad x \in \mathbf{R};$ b) $f(x) = x^2 + 2x + 2, \quad x \in \mathbf{R};$

c) $f(x) = x^3 + 2\cos x, \quad x \in \mathbf{R};$ d) $f(x) = \sqrt[3]{x} \sin 3x, \quad x \in \mathbf{R};$

e) $f(x) = x \cot x, \quad x \in \mathbf{R} \setminus \{k\pi | \; k \in \mathbf{Z}\};$ f) $f(x) = \dfrac{2 - 2\cos x}{x^2}, \quad x \in \mathbf{R} \setminus \{0\}.$

Example 5.10. *Using Definition 5.1, check the continuity of the following functions at an arbitrary point x_0 from their domains.*

a) $f(x) = \begin{cases} \dfrac{x}{|x|}, & x \neq 0; \\ -1, & x = 0; \end{cases}$ b) $D(x) = \begin{cases} 0, & x \in \mathbf{I} := \mathbf{R} \setminus \mathbf{Q}; \\ 1, & x \in \mathbf{Q}. \end{cases}$

Here $\mathbf{Q}$ *and* $\mathbf{I}$ *stand for the (mutually disjoint) sets of rational and irrational numbers, which together make the set of real numbers* $\mathbf{R}$.

Remark. The function in b), called *Dirichlet's function,* was already defined in Example 2.20.

Solutions.

a) From the definition of the absolute value, it follows that $f(x) = 1$ if $x > 0$, and $f(x) = -1$ if $x \leq 0$. Let us show first that f is continuous at every point $x_0 \neq 0$. Firstly, if $|x - x_0| < \frac{1}{2}|x_0|$, then x and x_0 are of the same sign. Hence

$$|f(x) - f(x_0)| = \left| \frac{x}{|x|} - \frac{x_0}{|x_0|} \right| = 0.$$

Let $\varepsilon > 0$ be given; putting $\delta = \frac{|x_0|}{2}$, the following implication holds.

$$(\forall x \in \mathbf{R}) \quad |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

Next we show that f is discontinuous at $x_0 = 0$. Since

$$\lim_{x \to 0+} f(x) = 1 \quad \text{and} \quad \lim_{x \to 0-} f(x) = -1 = f(-1),$$

the function f is only continuous from the left at 0, but it is not continuous at the point 0. In fact, f has a first order discontinuity at 0.

b) We claim that Dirichlet's function is discontinuous at every real point x_0, rational or irrational. We shall prove this for $x_0 \in \mathbf{Q}$. (The other case, namely $x_0 \in \mathbf{I}$, is handled in a completely analogous way.)

In order to prove the discontinuity of the function D at x_0, we must show that there exists an $\varepsilon > 0$ such that for every $\delta > 0$ there exists an $x \in \mathbf{R}$ (x depending on δ) with the property

$$|x - x_0| < \delta \Rightarrow |D(x) - D(x_0)| \geq \varepsilon.$$

Let us put $\varepsilon := 1/2$. Then for every $\delta > 0$ there exists an irrational number x_δ such that $|x_\delta - x_0| < \delta$. However, then it holds

$$|D(x_\delta) - D(x_0)| = |0 - 1| = 1 > 1/2.$$

Here we used the fact that the set $\mathbf{I}$ of irrational numbers is dense in the set $\mathbf{Q}$ of rational numbers; i.e., in every neighborhood of any $x_0 \in \mathbf{Q}$ there exists at least one irrational number. In fact, there are infinitely many irrational numbers in any neighborhood of an arbitrary number $x_0 \in \mathbf{Q}$. (See Example 1.47 e), from which the last statement easily follows.)

Exercise 5.11. *Prove that the function f is discontinuous at every $x \in \mathbf{R}$, if*

$$f(x) = \begin{cases} -1, & x \in \mathbf{Q}; \\ 1, & x \in \mathbf{R} \setminus \mathbf{Q}. \end{cases}$$

Exercise 5.12. *If the function f is discontinuous at a point $x_0 \in \mathbf{R}$, is then the function f^2 also discontinuous at that point?*

Answer. Not necessarily. (See the previous exercise.)

Example 5.13. *Assume that the function $f : (a, b) \to \mathbf{R}$ has the following property at a point $x_0 \in (a, b)$.*

a) $(\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall x \in (a, b))\quad |f(x) - f(x_0)| < \varepsilon \Rightarrow |x - x_0| < \delta;$

b) $(\forall \delta > 0)\ (\exists \varepsilon > 0)\ (\forall x \in (a, b))\quad |f(x) - f(x_0)| < \varepsilon \Rightarrow |x - x_0| < \delta;$

c) $(\forall \delta > 0)\ (\exists \varepsilon > 0)\ (\forall x \in (a, b))\quad |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$

What can one say about the continuity of f at x_0?

Solutions. We shall find three functions that will satisfy the stated implications in a), b) and c) respectively, but neither of them will turn out to be continuous at the point x_0. This means that the order of the quantifiers as well as the direction of the implication in Definition 5.1 are essential.

a) Let us define the function f by (see Figure 5.1).

$$f(x) = \begin{cases} x, & x \leq 1; \\ x + 1, & x > 1. \end{cases}$$

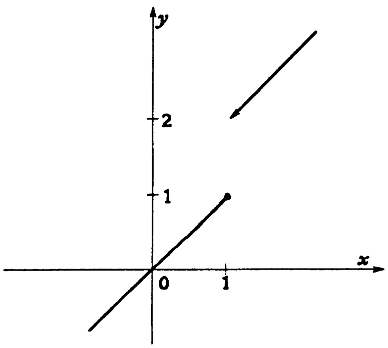

Figure 5.1.

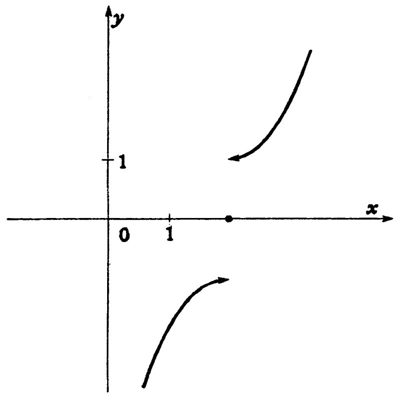

Figure 5.2.

Clearly, this function is continuous on the set $\mathbf{R} \setminus \{1\}$, while it has a first order discontinuity at the point $x_0 = 1$. Let us show, however, that f does have the stated property at the point $x_0 = 1$. To that end, for $\varepsilon > 1$ we put $\delta := \varepsilon - 1$. Then for every $x \in \mathbf{R}$ it holds

$$|f(x) - f(1)| < \varepsilon \Rightarrow |x - 1| < \delta. \tag{5.4}$$

For $\varepsilon \leq 1$, we put simply $\delta := \varepsilon$. Then the set of points $x > 1$ that satisfy the inequality $|f(x) - f(1)| < \varepsilon$ is empty, hence the implication (5.4) is true. For $x < 1$ it holds

$$|f(x) - f(1)| < \varepsilon \iff |x - 1| < \varepsilon.$$

b) The function f given by

$$f(x) = \begin{cases} x^2, & x \neq 0; \\ 1, & x = 0, \end{cases}$$

has a removable discontinuity at $x_0 = 0$, while it is continuous on the set $\mathbf{R} \setminus \{0\}$. Let us show that it satisfies the stated condition at the point $x_0 = 0$. For given $\delta > 0$ let us choose ε such that $1 < \varepsilon < 1 + \delta^2$ (say $\varepsilon := 1 + \frac{\delta^2}{2}$). Then for every $x \in \mathbf{R} \setminus \{0\}$ it holds

$$(|f(x) - f(0)| = |x^2 - 1| < x^2 + 1 < \varepsilon) \Rightarrow (|x - 0| = \sqrt{|x|^2 + 1 - 1} < \sqrt{\varepsilon - 1} < \delta).$$

c) Let the function f given by

$$f(x) = \begin{cases} x^2 - 4x + 5, & x > 2; \\ 0, & x = 2; \\ -x^2 + 4x - 5, & x < 2 \end{cases}$$

(see Figure 5.2). This function has a first order discontinuity at the point $x_0 = 2$, and is continuous on the set $\mathbf{R} \setminus \{2\}$. Let us show that it satisfies the stated condition at the point $x_0 = 2$. Namely, for given $\delta > 0$ let us define $\varepsilon := 1 + \delta^2$. Then for every $x \in \mathbf{R}$ it holds

$$(|x - 2| < \delta) \Rightarrow (|f(x) - f(2)| = 1 + (x - 2)^2 < 1 + \delta^2 = \varepsilon).$$

In fact, the condition stated in c) gives only the boundedness of the function f in some neighborhood of the point x_0.

Example 5.14. *Prove that if the function $f : (a, b) \to \mathbf{R}$ is continuous at a point $x_0 \in (a, b)$ and it holds $f(x_0) \neq 0$, then there exists a $\delta > 0$ with the following property.*

a) $(\forall x \in (a, b))\ \ |x - x_0| < \delta \Rightarrow f(x) > \dfrac{f(x_0)}{2}$, *provided that* $f(x_0) > 0$;

b) $(\forall x \in (a, b))\ \ |x - x_0| < \delta \Rightarrow f(x) < \dfrac{f(x_0)}{2}$, *provided that* $f(x_0) < 0$.

Solution. We shall prove only the case **a)**, the other one being analogous. By the definition of the continuity of f at x_0, for $\varepsilon := \frac{f(x_0)}{2}$, there exists a $\delta > 0$ such that the following implication holds.

$$(\forall x \in (a, b))\ \ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \frac{f(x_0)}{2}.$$

Hence for all $x \in (a, b)$, such that $x_0 - \delta < x < x_0 + \delta$, it holds

$$\left(-\frac{f(x_0)}{2} < f(x) - f(x_0) < \frac{f(x_0)}{2}\right) \Rightarrow \left(\frac{f(x_0)}{2} < f(x) < \frac{3f(x_0)}{2}\right).$$

Remark. This example gives an important property of a continuous function f at some point x_0 where $f(x_0) \neq 0$. Namely, then there exists an interval $(x_0 - \delta, x_0 + \delta)$ on which f has the same sign as the number $f(x_0)$.

Example 5.15. *Check the continuity at* 0 *of the following functions.*

a) $f(x) = \begin{cases} \left|\dfrac{\sin x}{x}\right|, & x \neq 0; \\ 1, & x = 0; \end{cases}$ **b)** $f(x) = \begin{cases} \dfrac{\sin x}{|x|}, & x \neq 0; \\ 1, & x = 0; \end{cases}$

c) $f(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right), & x \neq 0; \\ C, & x = 0; \end{cases}$ **d)** $f(x) = \begin{cases} x\sin\left(\dfrac{1}{x}\right), & x \neq 0; \\ 0, & x = 0. \end{cases}$

In c), C stands for some real number. We suggest to the reader to prove the continuity of these four functions on the set $\mathbf{R} \setminus \{0\}$.

Solutions.

a) Let us show that f is continuous at the point 0. Using the limit $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$, see Example 4.23, we have

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \left|\frac{\sin x}{x}\right| = \lim_{x\to 0+} \frac{\sin x}{x} = 1 \qquad \text{and}$$

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \left|\frac{\sin x}{x}\right| = \lim_{x\to 0-} \frac{-\sin x}{-x} = 1.$$

Since also $f(0) = 1$, in view of Theorem 5.5, f is continuous at 0.

b) Since

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{\sin x}{|x|} = \lim_{x\to 0+} \frac{\sin x}{x} = 1 \qquad \text{and}$$

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{\sin x}{|x|} = \lim_{x\to 0-} \frac{\sin x}{-x} = -1,$$

the function f has a first order discontinuity at the point 0.

c) Let us prove first that f has no limit at 0. If $x_n = \dfrac{1}{n\pi}$, $n \in \mathbf{N}$, then $\lim\limits_{n\to\infty} x_n = 0$, and if $x_n' = \dfrac{2}{(4n+1)\pi}$, $n \in \mathbf{N}$, then also $\lim\limits_{n\to\infty} x_n' = 0$. However, it holds

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \sin(n\pi) = 0 \quad \text{and} \quad \lim_{n\to\infty} f(x_n') = \lim_{n\to\infty} \sin\left((4n+1)\frac{\pi}{2}\right) = 1.$$

In view of Definition 4.2, it follows that f has no limit at 0. Hence, from Theorem 5.5 we obtain that regardless of the choice of the number C, the function f has a second order discontinuity at 0. (See Figure 2.15.)

d) Let us prove that f is continuous at 0. For given $\varepsilon > 0$ let us choose $\delta := \varepsilon$. Then for every $x \in \mathbf{R}$ such that $0 < |x - 0| < \delta$ it holds

$$|f(x) - f(0)| = |x \sin \frac{1}{x} - 0| \leq |x| = |x - 0| < \delta = \varepsilon.$$

Thus $\lim\limits_{x \to 0} f(x) = 0 = f(0)$, and from Theorem 5.5 follows the continuity of f at the point 0. (See Figure 2.16.)

Example 5.16. *Find the points of discontinuities and their kind for the following functions.*

a) $\operatorname{sgn} x := \begin{cases} 1, & x > 0; \\ 0, & x = 0; \\ -1, & x < 0; \end{cases}$ **b)** $f(x) = \operatorname{sgn}^2 x, \quad x \in \mathbf{R}.$

Solutions.

a) Since

$$\lim_{x \to 0+} \operatorname{sgn} x = 1 \quad \text{and} \quad \lim_{x \to 0-} \operatorname{sgn} x = -1,$$

f has a first order discontinuity at 0. (Figure 5.3.)

b) Since $\operatorname{sgn}^2 x = 1$ for $x \neq 0$, it follows that f has a removable discontinuity at 0.

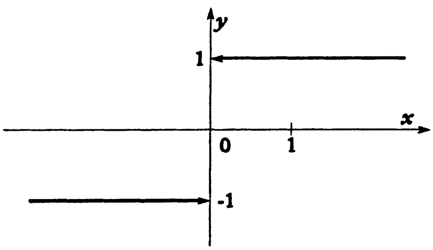

Figure 5.3.

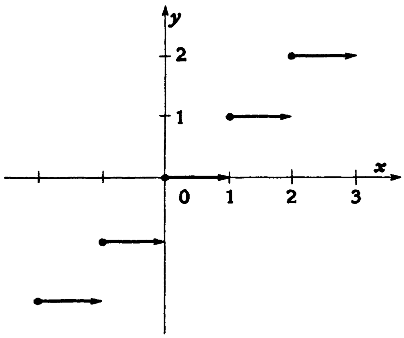

Figure 5.4.

Example 5.17. *Find the points of discontinuities and their order for the following functions.*

a) $f(x) = [x], \quad x \in \mathbf{R};$ **b)** $f(x) = ax + b[x], \quad x \in \mathbf{R},\ b \neq 0;$

c) $f(x) = x[x], \quad x \in \mathbf{R};$ **d)** $f(x) = [x] \cdot \sin(\pi x), \quad x \in \mathbf{R}.$

Solutions. Clearly, each of these four functions is continuous on the set $\mathbf{R} \setminus \mathbf{Z}$. We shall next examine what happens at the integer points.

a) From the definition of the function $f(x) = [x]$, i.e., the one that assigns to a real number x its greatest integer part $[x]$, it follows that f has a first order discontinuity at every integer k. (Figure 5.4.)

b) First we find the right- and left-hand limits of f at an arbitrary integer k.

$$\lim_{x \to k+} (ax + b[x]) = ak + bk = (a+b)k \qquad \text{and}$$

$$\lim_{x \to k-} (ax + b[x]) = ak + b(k-1) = (a+b)k - b.$$

Since $b \neq 0$, it follows that f has first order discontinuities at every integer k.

c) At the point $x_0 = 0$ the function f is continuous, since $f(0) = 0$, and it holds

$$\lim_{x \to 0+} x[x] = \lim_{x \to 0+} x \cdot \lim_{x \to 0+} [x] = 0 \cdot 0 = 0 \qquad \text{and}$$

$$\lim_{x \to 0-} x[x] = \lim_{x \to 0-} x \cdot \lim_{x \to 0-} [x] = 0 \cdot (-1) = 0.$$

At an arbitrary integer $k \neq 0$ we have

$$\lim_{x \to k+} x[x] = \lim_{x \to k+} x \cdot \lim_{x \to k+} [x] = k \cdot k = k^2 \quad \text{and}$$

$$\lim_{x \to k-} x[x] = \lim_{x \to k-} x \cdot \lim_{x \to k-} [x] = k \cdot (k-1) = k^2 - k.$$

Hence f has a first order discontinuity at every integer $k \neq 0$.

d) The right and left side limits of f at k are both equal to zero, since

$$\lim_{x \to k+} [x] \cdot \sin(\pi x) = k \lim_{x \to k+} \sin(\pi x) = 0 \qquad \text{and}$$

$$\lim_{x \to k-} [x] \cdot \sin(\pi x) = (k-1) \lim_{x \to k-} \sin(\pi x) = 0.$$

Since also $f(k) = 0$, from Theorem 5.5 it follows that f is continuous at every integer point.

Example 5.18. *Find the points of discontinuity and their order for the following functions.*

a) $f(x) = \begin{cases} \left[\dfrac{1}{x}\right], & x \neq 0; \\ 0, & x = 0; \end{cases}$ **b)** $f(x) = \begin{cases} x \cdot \left[\dfrac{1}{x}\right], & x \neq 0; \\ 0, & x = 0; \end{cases}$

c) $f(x) = \begin{cases} \left[\dfrac{1}{x^2}\right] \cdot \operatorname{sgn}\left(\sin(\frac{1}{x^2})\right), & x \neq 0; \\ 0, & x = 0. \end{cases}$

Solutions.

a) Putting $x = \frac{1}{t}$ and $t = [t] + r(t)$, where $0 \leq r(t) < 1$, we obtain

$$\lim_{x\to 0+}\left[\frac{1}{x}\right] = \lim_{t\to+\infty}(t - r(t)) = +\infty.$$

Analogously it holds

$$\lim_{x\to 0-}\left[\frac{1}{x}\right] = -\infty.$$

This means that f has a second order discontinuity at 0.

For $x \neq 0$, putting again $x = \frac{1}{t}$, we obtain that f has first order discontinuities at the points $\frac{1}{k}$, $k \in \mathbf{Z}$.

b) Since it holds $\lim_{x\to 0} x \cdot [1/x] = 1$, the function f has a removable discontinuity at 0. At the point $x = 1/k$, $k \in \mathbf{Z} \setminus \{0\}$, it holds

$$\lim_{x\to 1/k+} x \cdot \left[\frac{1}{x}\right] = \lim_{\varepsilon\to 0+}\left(\frac{1}{k} + \varepsilon\right) \cdot \left[\frac{1}{1/k+\varepsilon}\right] = \frac{1}{k} \cdot \lim_{\varepsilon\to 0+}\left[\frac{k}{1+k\varepsilon}\right] = \frac{k-1}{k}.$$

Similarly we can get

$$\lim_{x\to 1/k-} x \cdot \left[\frac{1}{x}\right] = \frac{1}{k} \cdot k = 1.$$

Hence f has first order discontinuities at the points $\frac{1}{k}$, $k \in \mathbf{Z} \setminus \{0\}$.

c) Let us check first the continuity of f at 0. We start from the following two sequences that converge to zero.

$$x_n = \sqrt{\frac{2}{\pi(1+4n)}},\quad n \in \mathbf{N}, \quad \text{and} \quad x'_n = \sqrt{\frac{2}{\pi(3+4n)}},\quad n \in \mathbf{N}.$$

It holds

$$\lim_{n\to\infty} f(x_n) = +\infty \quad \text{and} \quad \lim_{n\to\infty} f(x'_n) = -\infty.$$

Thus f has no limit or one sided limits at 0, hence f has a second order discontinuity at 0.

We have yet to examine the points $\pm\frac{1}{\sqrt{|k|}}$, $k \in \mathbf{Z} \setminus \{0\}$. We claim that f has first order discontinuities at these points. To that end, we shall find only the right- and left-hand side limits of f at the points $\frac{1}{\sqrt{k}}$, $k \in \mathbf{N}$, since the other cases are handled similarly. We have

$$\lim_{x\to 1/\sqrt{k}+}\left[\frac{1}{x^2}\right]\cdot\operatorname{sgn}\left(\sin(\frac{1}{x^2})\right)=\lim_{\varepsilon\to 0+}\left[\frac{1}{\left(\frac{1}{\sqrt{k}}+\varepsilon\right)^2}\right]\cdot\operatorname{sgn}\left(\sin\frac{1}{\left(\frac{1}{\sqrt{k}}+\varepsilon\right)^2}\right)$$

$$=\lim_{\varepsilon\to 0+}\left[\frac{k}{(1+\varepsilon\sqrt{k})^2}\right]\cdot\operatorname{sgn}\left(\sin\frac{k}{(1+\varepsilon\sqrt{k})^2}\right)=(k-1)\cdot\operatorname{sgn}(\sin k)$$

and

$$\lim_{x\to 1/\sqrt{k}-}\left[\frac{1}{x^2}\right]\cdot\operatorname{sgn}\left(\sin(\frac{1}{x^2})\right)=\lim_{\varepsilon\to 0+}\left[\frac{1}{\left(\frac{1}{\sqrt{k}}-\varepsilon\right)^2}\right]\cdot\operatorname{sgn}\left(\sin\frac{1}{\left(\frac{1}{\sqrt{k}}-\varepsilon\right)^2}\right)$$

$$=\lim_{\varepsilon\to 0+}\left[\frac{k}{(1-\varepsilon\sqrt{k})^2}\right]\cdot\operatorname{sgn}\left(\sin\frac{k}{(1-\varepsilon\sqrt{k})^2}\right)=k\cdot\operatorname{sgn}(\sin k).$$

Since the left- and the right-hand side limits of f at $\frac{1}{\sqrt{k}}$ are different, the function f has first order discontinuities at the points $\frac{1}{\sqrt{k}}$, $k\in\mathbf{N}$.

Example 5.19. *Determine the constant C, if possible, in order to obtain (at least one sided) continuity of the functions given below at the given point x_0.*

$$\textbf{a)}\quad f(x)=\begin{cases}\dfrac{|x|-x}{x^2}, & x\neq 0;\\[2mm] C, & x=0,\end{cases}\qquad x_0=0;$$

$$\textbf{b)}\quad f(x)=\begin{cases}\dfrac{|x+1|}{x+1}, & x\neq -1;\\ C, & x=-1,\end{cases}\qquad x_0=-1;$$

$$\textbf{c)}\quad f(x)=\begin{cases}\exp\left(x+\dfrac{1}{x}\right), & x\neq 0;\\ C, & x=0,\end{cases}\qquad x_0=0;$$

$$\textbf{d)}\quad f(x)=\begin{cases}\cos^2\left(\dfrac{1}{x}\right), & x\neq 0;\\ C, & x=0,\end{cases}\qquad x_0=0;$$

$$\textbf{e)}\quad f(x)=\begin{cases}x\ln x^2, & x\neq 0;\\ C, & x=0;\end{cases}\qquad x_0=0.$$

Solutions.

a) The function f can be written as

$$f(x) = \begin{cases} 0, & x > 0; \\ -\dfrac{2}{x}, & x < 0; \\ C, & x = 0. \end{cases}$$

Hence

$$\lim_{x \to 0-} f(x) = +\infty \quad \text{and} \quad \lim_{x \to 0+} f(x) = 0,$$

which means that f has a second order discontinuity at 0. If $C = 0$, then f turns out to be continuous from the right at 0, but clearly for no values of C can f be continuous.

b) Since f is equal to

$$f(x) = \begin{cases} -1, & x < -1; \\ 1, & x > -1; \\ C, & x = -1, \end{cases}$$

it has a first order discontinuity at $x = -1$. For $C = -1$, f is continuous from the left side, while for $C = 1$ it is continuous from the right side.

c) Since the following relations hold,

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \exp\left(x + \frac{1}{x}\right) = 0, \quad \lim_{x \to 0+} f(x) = \lim_{x \to 0+} \exp\left(x + \frac{1}{x}\right) = +\infty,$$

the function f has a second order discontinuity at $x = 0$. For $C = 0$ the function f becomes continuous from the left side at 0.

d) Since the limit of the function

$$f(x) = \cos^2\left(\frac{1}{x^2}\right), \quad x \neq 0,$$

does not exist at 0, f has a second order discontinuity at $x = 0$. Obviously, for no values of C can f be continuous at least from one side.

e) Since from Example 4.44 b) we have

$$\lim_{x \to 0} x \ln x^2 = \lim_{x \to 0} 2x \ln |x| = \lim_{t \to +\infty} \frac{2 \ln t}{t} = 0,$$

it follows that the function f is continuous at 0 if $C = 0$, resp. has a removable discontinuity at 0 if $C \neq 0$.

Exercise 5.20. *Determine the constant C, if possible, in order to make the following functions continuous at the given point x_0.*

$$\text{a)}\quad f(x) = \begin{cases} \dfrac{x-2}{x^2-4}, & x \neq 2; \\ C, & x = 2, \end{cases} \qquad x_0 = 2;$$

$$\text{b)}\quad f(x) = \begin{cases} Cx^3+1, & x \geq 0; \\ x, & x < 0, \end{cases} \qquad x_0 = 0;$$

$$\text{c)}\quad f(x) = \begin{cases} \sin 2x, & x \leq \dfrac{\pi}{2}; \\ C(x+1), & x > \dfrac{\pi}{2}, \end{cases} \qquad x_0 = \frac{\pi}{2};$$

$$\text{d)}\quad f(x) = \begin{cases} \dfrac{1}{\ln(x-1)}, & x > 1; \\ C+1, & x \leq 1, \end{cases} \qquad x_0 = 1.$$

Answers.

a) The function is continuous at the point $x_0 = 2$ if $C = \dfrac{1}{4}$.

b) The function has a first order discontinuity at the point $x_0 = 0$, and for no value of the constant C can f be made continuous.

c) The function is continuous at the point $x_0 = \pi/2$ if $C = 0$.

d) The function is continuous at the point $x_0 = 1$ if $C = -1$.

Exercise 5.21. *Determine the constants a and b, if possible, in order to make the following functions f continuous on $\mathbf{R}$.*

$$\text{a)}\quad f(x) = \begin{cases} x-2, & x \leq 2; \\ ax+b, & 2 < x < 3; \\ \sqrt{x-2}, & x \geq 3; \end{cases}$$

$$\text{b)}\quad f(x) = \begin{cases} \dfrac{1}{x^2-4}, & x \neq 2,\ x \neq -2; \\ a, & x = 2; \\ b, & x = -2. \end{cases}$$

Answers.

a) $a = 1, \quad b = -2.$

b) The function has second order discontinuities at the points $x_1 = 2$ and $x_2 = -2$, and therefore we can not order the constants a and b to make f continuous on $\mathbf{R}$.

Example 5.22. *Determine the constants a, b and c, if possible, in order to obtain continuity of the functions given below on their domains.*

a) $$f(x) = \begin{cases} 2x, & |x| \leq 1; \\ x^2 + ax + b, & |x| > 1; \end{cases}$$

b) $$f(x) = \begin{cases} \dfrac{(x-2)^2}{x^2-4}, & x \notin \{-2, 2\}; \\ a, & x = -2; \\ b, & x = 2; \end{cases}$$

c) $$f(x) = \begin{cases} \dfrac{\dfrac{1}{x} - \dfrac{1}{x+1}}{\dfrac{1}{x-1} - \dfrac{1}{x}}, & x \notin \{-1, 0, 1\}; \\ a, & x = -1; \\ b, & x = 0; \\ c, & x = 1; \end{cases}$$

d) $$f(x) = \begin{cases} \dfrac{x \cos\left(\dfrac{x}{2}\right)}{\sin x}, & x \in \left[\dfrac{-\pi}{2}, \dfrac{3\pi}{2}\right] \setminus \{0, \pi\}; \\ a, & x = 0; \\ b, & x = \pi. \end{cases}$$

Solutions.

a) Since it holds

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} 2x = 2, \qquad \lim_{x\to 1+} f(x) = \lim_{x\to 1+} (x^2 + ax + b) = 1 + a + b,$$

$$\lim_{x\to -1-} f(x) = \lim_{x\to -1-} (x^2 + ax + b) = 1 - a + b, \qquad \lim_{x\to -1+} f(x) = \lim_{x\to -1+} 2x = -2,$$

the function f is continuous on $\mathbf{R}$ only if

$$2 = 1 + a + b \quad \text{and} \quad -2 = 1 - a + b.$$

Hence we must choose $a = 2,\ b = -1$.

b) Since $\lim\limits_{x\to -2+} f(x) = -\infty$, f has a second order discontinuity at $x = -2$, regardless of the choice of a. However, since

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{(x-2)^2}{x^2-4} = \lim_{x\to 2} \frac{x-2}{x+2} = 0,$$

f is continuous at the point 2 if $b = 0$.

c) It holds

$$\lim_{x\to -1-} f(x) = +\infty, \qquad \lim_{x\to 0} f(x) = -1, \qquad \lim_{x\to 1} f(x) = 0.$$

This means that f has a second order discontinuity at $x=-1$, regardless of a; for $b=-1$ it becomes continuous at 0, and finally for $c=0$ the function f becomes continuous at the point 1.

d) We leave to the reader to show that

$$\lim_{x\to 0} f(x) = 1 \quad \text{and} \quad \lim_{x\to \pi} f(x) = \frac{\pi}{2},$$

hence for $a=1$ and $b=\dfrac{\pi}{2}$ the function f becomes continuous on its domain.

Example 5.23. *Examine the continuity of the compositions of the functions $f \circ g$ and $g \circ f$ on* $\mathbf{R}$ *if*

a) $f(x) = \text{sgn}x, \qquad g(x) = 1 + x^2;$

b) $f(x) = \text{sgn}x, \qquad g(x) = x(1+x);$

a) $f(x) = \text{sgn}x, \qquad g(x) = 1 + x - [x].$

Solutions.

a) The function

$$f(g(x)) = \text{sgn}(1+x^2) = 1$$

is continuous on $\mathbf{R}$. However, the function

$$g(f(x)) = 1 + (\text{sgn}x)^2 = \begin{cases} 2, & x<0; \\ 1, & x=0; \\ 2, & x>0, \end{cases}$$

has a removable discontinuity at $x_0 = 0$.

b) In this case we have

$$f(g(x)) = \text{sgn}(x(1+x)) = \begin{cases} -1, & -1<x<0; \\ 0, & x=-1, \;\; x=0; \\ 1, & x<-1, \; x>0, \end{cases}$$

$$g(f(x)) = \text{sgn}\,(x(1+\text{sgn}\,x)) = \begin{cases} 0, & x\le 0; \\ 2, & x>0. \end{cases}$$

Both functions have first order discontinuities at the point $x=0$, while the function $f\circ g$ has the same kind discontinuity at the point $x=-1$.

c) Both functions are continuous because

$$f(g(x)) = \text{sgn}(1+x-[x]) = 1, \qquad g(f(x)) = 1 + \text{sgn}x - [\text{sgn}x] = 1.$$

Exercise 5.24 *Prove that a monotone function can have only first order discontinuities.*

Example 5.25. *If the function $f : A \subset \mathbf{R} \to \mathbf{R}$ is continuous on the compact set $K \subset A$, then its range,*

$$f(K) := \{y \in \mathbf{R} |\ (\exists x \in K)\, y = f(x)\},$$

is also a compact set.

Solution. The set $f(K)$ can be either finite or infinite. If $f(K)$ is finite, then it is both bounded and closed, i.e., by Definition 1.77 $f(K)$ is then compact.

If $f(K)$ has infinitely many elements, then for the compactness of $f(K)$ it is enough to prove that every infinite subset $S \subset f(K)$ has an accumulation point that belongs to $f(K)$ (see Example 1.94). To that end, let us put

$$T := \{x \in K |\ (\exists y \in S)\, y = f(x)\}.$$

Since f is a function, the last set is also infinite; being a subset of the compact set K, T has an accumulation point x_0 that belongs to K. We shall show that $f(x_0)$ is an accumulation point of the set S. To that end, let $(x_n)_{n \in \mathbf{N}}$ be any sequence from T that converges to x_0, such that for every $n \in \mathbf{N}$ it holds $x_n \neq x_0$. Since f is continuous at x_0, using Definition 5.3, we have

$$\lim_{n\to\infty} f(x_n) = f(x_0),$$

which implies that $f(x_0)$ is an accumulation point of the set S.

Example 5.26. *Discuss the continuity of the following functions, and sketch their graphs.*

a) $f(x) = \lim\limits_{n\to\infty} \dfrac{1}{2 + x^n}, \quad x \geq 0;$

b) $f(x) = \lim\limits_{n\to\infty} \dfrac{n^x - n^{-x}}{n^x + n^{-x}}, \quad x \in \mathbf{R};$

c) $f(x) = \lim\limits_{n\to\infty} \sqrt[n]{3 + x^{4n}}, \quad x \in \mathbf{R};$

d) $f(x) = \lim\limits_{n\to\infty} (1 - \sin^{2n} x), \quad x \in \mathbf{R};$

e) $f(x) = \lim\limits_{n\to\infty} \dfrac{x}{2 + (2\sin x)^{4n}}, \quad x \in \mathbf{R}.$

Solutions.

a) It holds

$$\lim_{n\to\infty} x^n = \begin{cases} 0, & x \in [0,1); \\ 1, & x = 1; \\ +\infty, & x > 1. \end{cases}$$

Hence the function f is equal to

$$f(x) = \begin{cases} 1/2, & 0 \leq x < 1; \\ 1/3, & x = 1; \\ 0 & x > 1, \end{cases}$$

and the function f has a discontinuity of first order at the point $x = 1$.

Remark. This example shows that a function obtained as a limit of a sequence of continuous functions is not necessarily continuous.

b) We have $f(x) = \operatorname{sgn} x$, hence by Example 5.16 a) the function f has a discontinuity of first order at the point $x = 0$.

c) We shall prove that

$$f(x) = \begin{cases} 1, & |x| \leq 1; \\ x^4, & |x| > 1. \end{cases}$$

For $|x| \leq 1$ it holds

$$\sqrt[n]{3} \leq \sqrt[n]{3 + x^{4n}} \leq \sqrt[n]{3+1} = \sqrt[n]{4},$$

hence by Example 3.11 b) and Theorem 3.7 we obtain

$$f(x) = \lim_{n\to\infty} \sqrt[n]{3 + x^{4n}} = 1.$$

For $|x| > 1$ it holds

$$f(x) = \lim_{n\to\infty} \sqrt[n]{3 + x^{4n}} = \lim_{n\to\infty} x^4 \cdot \sqrt[n]{1 + \frac{3}{x^{4n}}} = x^4.$$

Observe that the limit function f is continuous on $\mathbf{R}$.

d) The function f has removable discontinuities at the points $(2k+1)\pi/2,\ k \in \mathbf{Z}$, since it equals to

$$f(x) = \begin{cases} 1, & x \neq (2k+1)\pi/2,\ k \in \mathbf{Z}; \\ 0, & x = (2k+1)\pi/2,\ k \in \mathbf{Z}. \end{cases}$$

e) We start from the following relations.

$$|2\sin x| < 1 \iff (\exists k \in \mathbf{Z})\ |x - k\pi| < \pi/6,$$

$$|2\sin x| > 1 \iff (\exists k \in \mathbf{Z})\ \frac{\pi}{6} < |x - k\pi| < \frac{5\pi}{6}.$$

Hence the limit function f is

$$f(x) = \begin{cases} x/2, & |\sin x| < 1/2, & \text{i.e., } |x - k\pi| < \pi/6,\ k \in \mathbf{Z}; \\ x/3, & |\sin x| = 1/2, & \text{i.e., } x = k\pi \pm \pi/6,\ k \in \mathbf{Z}; \\ 0, & |\sin x| > 1/2, & \text{i.e., } \pi/6 < |x - k\pi| < 5\pi/6,\ k \in \mathbf{Z}. \end{cases}$$

At the points $x_k = k\pi \pm \pi/6,\ k \in \mathbf{Z}$, f has discontinuities of the first order.

Example 5.27. *Discuss the continuity of the following functions.*

a) Dirichlet's function: $D(x) = \lim_{m\to\infty}\left(\lim_{n\to\infty} \cos^n(\pi m!\, x)\right),\ x \in \mathbf{R};$

b) $f(x) = x \cdot D(x)$, *where the function D was defined in a);*

c) Riemann's function:

$$R(x) = \begin{cases} \dfrac{1}{n}, & x = \dfrac{m}{n}, \ m \in \mathbf{Z}, \ n \in \mathbf{N}, \ \mathrm{lcd}(m,n) = 1; \\ 0, & x \in \mathbf{R} \setminus \mathbf{Q}. \end{cases}$$

Compare a) to Example 5.10 b). In c), $\mathrm{lcd}(m,n)$ stands for the *largest common divisor* of the numbers m and n. When $\mathrm{lcd}(m,n) = 1$, the fraction $\dfrac{m}{n}$ is irreducible.

Solutions.

a) • **First method.** Assume first that x is a rational number, i.e., $x = \dfrac{p}{q}$ for some $p \in \mathbf{Z}$ and $q \in \mathbf{N}$. Then for $m > q$ it holds

$$m!\,x = m!\,\frac{p}{q} = 1 \cdot 2 \cdots (q-1)(q+1) \cdots m \cdot p,$$

hence $m!\,x$ is an even number. This implies

$$\cos(\pi m!\,x) = 1, \quad \text{hence} \quad D(x) = 1 \ \text{ for every } x \in \mathbf{Q}.$$

If x is an irrational number, then for no m can the number $m!\,x$ be an integer. But then

$$|\cos(\pi m!\,x)| < 1, \ \text{which implies} \ \lim_{n \to \infty} \cos^n(\pi m!\,x) = 0.$$

Passing to the limit in m, the last equality gives

$$(\forall x \in \mathbf{R} \setminus \mathbf{Q}) \quad D(x) = 0.$$

Thus we obtained the same function as in Example 5.10 b). We proved there that D is discontinuous at every real point x.

• **Second method.** Let us prove the same fact in another way, namely using Definition 5.3.

If x_0 is an arbitrary real number, then there exist two sequences, one of rational, and the other of irrational numbers, both converging to x_0 (see the Remark after Example 1.47). Denoting the first by $(r_n)_{n \in \mathbf{N}}$ and the second by $(i_n)_{n \in \mathbf{N}}$, we have

$$\lim_{n \to \infty} r_n = x_0 \Rightarrow \lim_{n \to \infty} D(r_n) = 1 \qquad \text{and}$$

$$\lim_{n \to \infty} i_n = x_0 \Rightarrow \lim_{n \to \infty} D(i_n) = 0.$$

Hence for no $x \in \mathbf{R}$ does the limit $\lim\limits_{x \to x_0} D(x)$ exist, which means that D has a second order discontinuity at every real point x_0.

b) If $x_0 = 0$, then it holds for every $\varepsilon > 0$

$$|f(x) - f(0)| = |x \cdot D(x) - 0| \leq |x| < \varepsilon,$$

provided that $|x - 0| < \delta := \varepsilon$. Thus f is continuous at 0.

We shall prove next that 0 is the only real number at which f is continuous. To that end, let $x_0 \in \mathbf{R} \setminus \{0\}$ and let $(r_n)_{n\in\mathbf{N}}$ and $(i_n)_{n\in\mathbf{N}}$ be two sequences of rational, resp. irrational numbers, such that

$$\lim_{n\to\infty} r_n = x_0 \quad \text{and} \quad \lim_{n\to\infty} i_n = x_0.$$

Then it holds

$$\lim_{n\to\infty} f(r_n) = x_0 \cdot 1 = x_0 \quad \text{and} \quad \lim_{n\to\infty} f(i_n) = x_0 \cdot 0 = 0.$$

This means that f has a second order discontinuity at every real point different of zero.

c) We shall prove that Riemann's function is continuous at every irrational, but is discontinuous at every rational number.

Assume first $x_0 \in \mathbf{R}\setminus\mathbf{Q}$. Let us observe a sequence of rational numbers $(r_k)_{k\in\mathbf{N}}$ converging to x_0, where $r_k = \dfrac{m_k}{n_k}$, and assume that $\operatorname{lcd}(m_k, n_k) = 1$ for every $k \in \mathbf{N}$. But then $\lim_{k\to\infty} n_k = +\infty$, hence

$$\lim_{k\to\infty} R(r_k) = \lim_{k\to\infty} 0 = \lim_{k\to\infty} R\left(\frac{m_k}{n_k}\right) = \lim_{k\to\infty} \frac{1}{n_k} = 0 = R(x_0). \tag{5.5}$$

It is trivial that for every sequence of irrational numbers $(i_k)_{k\in\mathbf{N}}$ converging to x_0 it holds

$$\lim_{k\to\infty} R(i_k) = 0 = R(x_0). \tag{5.6}$$

Putting the results from (5.5) and (5.6) together, it follows that for every sequence of *real* numbers the same remains true. But this means that R is continuous at every irrational number.

Assume now that x_0 is a rational number of the form $x_0 = \dfrac{m}{n}$, where $\operatorname{lcd}(m,n) = 1$. By the definition of the function R, it holds $R(x_0) = \dfrac{1}{n}$. Let us put now

$$r_k := \frac{mk+1}{nk}, \quad k \in \mathbf{N}.$$

This sequence of rational numbers converges to x_0 as $k \to \infty$; however, it holds

$$\lim_{k\to\infty} R(r_k) = \lim_{k\to\infty} \frac{1}{nk} = 0 \neq \frac{1}{n} = R(x_0).$$

Hence the function R has a discontinuity at every rational point.

Example 5.28. *The function $f : \mathbf{R} \to \mathbf{R}$ is given by*

$$f(x) = \begin{cases} 0, & x = 0; \\ 1/x, & x \in \mathbf{Q} \setminus \{0\}; \\ x, & x \in \mathbf{I} := \mathbf{R} \setminus \mathbf{Q}. \end{cases}$$

a) *Prove that f is a bijection.*

b) *Discuss the continuity of the function f at the points $x' = 1$, $x'' = -1$ and $x''' = 0$.*

c) *Discuss the continuity of the function f at the point $x_0 \in \mathbf{R} \setminus \{-1, 0, 1\}$.*

Solutions.

a) We have to prove that f is both an injection,

$$(\forall x_1, x_2 \in \mathbf{R}) \quad x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2),$$

and a surjection,

$$(\forall y \in \mathbf{R})\ (\exists x \in \mathbf{R}) \quad y = f(x).$$

To that end, let us put

$$g : \mathbf{Q} \setminus \{0\} \to \mathbf{Q} \setminus \{0\}, \ \ g(x) = 1/x. \tag{5.7}$$

and

$$h : \mathbf{I} \to \mathbf{I}, \ \ h(x) = x. \tag{5.8}$$

Note that $f = g$ on the set $\mathbf{Q} \setminus \{0\}$ and $f = h$ on the set I. It is an easy exercise to check that both g and h are injective and surjective functions. The ranges of g and h are disjoint, while the union of their ranges is the set

$$(\mathbf{Q} \setminus \{0\}) \bigcup I = \mathbf{R} \setminus \{0\}.$$

Thus f is also a bijection on the last set.

Finally, since

$$(\forall x \in \mathbf{R}) \quad f(x) = 0 \iff x = 0$$

it follows that f is a bijection on the whole $\mathbf{R}$.

b) We shall prove the continuity of f at the point $x' = 1$ by using Definition 5.3. Let $(x_k)_{k \in \mathbf{N}}$ be an arbitrary sequence of positive real numbers converging to $x' = 1$; it is no restriction to assume $x_k \neq 0$ for every $k \in \mathbf{N}$. Our task is to show

$$\lim_{k \to \infty} f(x_k) = 1 \ (= f(1)). \tag{5.9}$$

Let $\varepsilon >$ be given. Let us put

$$A_1 = \{k \in \mathbf{N} |\ x_k \in \mathbf{Q} \setminus \{0\}\} \ \text{ and } \ A_2 = \{k \in \mathbf{N} |\ x_k \in \mathbf{R} \setminus \mathbf{Q}\}.$$

Then exactly one of the following two possibilities is true.

(A) Both sets A_1 and A_2 are infinite;

(B) one of the sets A_1 or A_2 is finite, while the other one is infinite.

In the case **(A)**, for $j = 1, 2$, it holds

$$\lim_{k\to\infty,\, k\in A_j} x_k = 1. \tag{5.10}$$

Let us define the number $\mathbf{i}$ as the following infimum:

$$\mathbf{i} := \inf\{|x_k| \;\big|\; k \in A_1\}.$$

Clearly, either $\mathbf{i} = 0$, or, as we claim, $\mathbf{i}$ is positive. Namely, if $\mathbf{i} = 0$, then there exists a subsequence $(x_{k_l})_{l\in\mathbf{N}}$ of the sequence $(x_k)_{k\in A_1}$ which converges to 0. (Observe that each x_{k_l} is nonzero.) But the existence of the last subsequence is in contradiction with (5.10). Hence $\mathbf{i} > 0$.

By the definition of f, it holds $f(x_k) = \dfrac{1}{x_k}$, $k \in A_1$, and using the inequality $\mathbf{i} > 0$, it follows that

$$\begin{aligned} |f(x_k) - 1| &= \frac{|x_k - 1|}{|x_k|} \\ &\leq (\inf\{|x_k| \;\big|\; k \in A_1\})^{-1} \cdot |x_k - 1| = \mathbf{i}^{-1} \cdot |x_k - 1|. \end{aligned} \tag{5.11}$$

For given $\varepsilon > 0$, we choose $k_1 \in \mathbf{N}$ such that

$$(\forall k \in A_1)\;\; k > k_1 \Rightarrow (|x_k - 1| < \mathbf{i} \cdot \varepsilon).$$

This implies

$$(\forall k \in A_1)\quad k > k_1 \Rightarrow |f(x_k) - f(1)| < \mathbf{i}^{-1} \cdot (\mathbf{i} \cdot \varepsilon) = \varepsilon. \tag{5.12}$$

Further on, since $f(x_k) = x_k$ for every $k \in A_2$, for given $\varepsilon > 0$ there exists $k_2 \in \mathbf{N}$ such that

$$(\forall k \in A_2)\;\; k > k_2 \Rightarrow (|f(x_k) - f(1)| = |x_k - 1| < \varepsilon). \tag{5.13}$$

Thus from (5.12) and (5.13) it follows that the following implication holds.

$$(\forall k \in N)\quad k > \max\{k_1, k_2\} \Rightarrow |f(x_k) - f(1)| < \varepsilon, \tag{5.14}$$

which means that f is continuous under the assumption **(A)**.

The case **(B)** is somewhat simpler and is left to the reader as an exercise.

Also, the proof of the continuity of f at the point $x'' = -1$ is completely analogous to the previous proof, hence omitted.

Finally, we shall prove that f has a discontinuity at $x''' = 0$. Clearly

$$\lim_{x\to 0,\, x\in\mathbf{I}} f(x) = \lim_{x\to 0,\, x\in\mathbf{I}} x = 0 = f(0).$$

For a rational number x such that $|x| < 1$ it holds

$$|f(x) - f(0)| = |f(x)| = \frac{1}{|x|} > 1.$$

Choose $\varepsilon := \frac{1}{2}$. Then for every $\delta > 0$ there exists a nonzero element $x_\delta \in \mathbf{Q}$ such that

$$|x_\delta| < \delta \Rightarrow |f(x_\delta) - f(0)| > 1 > \varepsilon.$$

Hence the limit $\lim\limits_{x \to 0,\, x \in \mathbf{Q}} f(x)$ is certainly not 0 (in fact, it does not exist), thus in view of Theorem 5.5, f can not be continuous at 0.

c) We shall prove that f is discontinuous in each point of the set $\mathbf{R} \setminus \{-1, 0, 1\}$. Let first $x_0 \in \mathbf{Q} \setminus \{-1, 0, 1\}$. Then for every $k \in \mathbf{N}$ there exists an irrational number x_k, $x_k \neq x_0$, in the interval $(x_0 - \frac{1}{k}, x_0 + \frac{1}{k})$, thus $\lim\limits_{k \to \infty} x_k = x_0$. But then

$$\lim_{k \to \infty} f(x_k) = \lim_{k \to \infty} x_k = x_0 \neq \frac{1}{x_0} = f(x_0),$$

meaning that f is discontinuous at x_0.

The other case, namely when $x_0 \in \mathbf{I}$, is analogous and left to the reader.

5.2 Uniform continuity

5.2.1 Basic notions

Definition 5.29. *A function $f : A \subset \mathbf{R} \to \mathbf{R}$ is* **uniformly continuous** *on the set $X \subset A$ if for every $\varepsilon > 0$ there exists a number $\delta > 0$, depending* only *on ε, such that for every pair $x_1, x_2 \in X$ with the property $|x_1 - x_2| < \delta$ it holds $|f(x_1) - f(x_2)| < \varepsilon$.*

Using logical symbols, the Definition 5.29 can be expressed as follows.
A function $f : A \subset \mathbf{R} \to \mathbf{R}$ is **uniformly continuous on a set** $X \subset A$ iff

$$(\forall \varepsilon > 0)\ (\exists \delta > 0)\ (\forall x_1, x_2 \in X) \quad |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon. \tag{5.15}$$

Clearly, a uniformly continuous function on a set X is also continuous at every point of that set. As will be seen from Example 5.34 below, the opposite is not true in general. However, the following statement holds.

Theorem 5.30. *If a function $f : A \subset \mathbf{R} \to \mathbf{R}$ is continuous on a compact set $K \subset A$, then it is also uniformly continuous on K.*

The following two theorems give two important properties of a continuous function on a compact set, resp. on a closed bounded interval (which is thus also a compact set).

Theorem 5.31. *If a function $f : A \subset \mathbf{R} \to \mathbf{R}$ is continuous on a compact set $K \subset A$, then it attains its minimum and maximum on the set K.*

In other words, there exist two numbers x_m and x_M in K such that for every $x \in K$ it holds

$$f(x_m) \leq f(x) \leq f(x_M).$$

Theorem 5.32. *Assume that a function $f : A \subset \mathbf{R} \to \mathbf{R}$ is continuous on an interval $[a, b] \subset A$, and assume, for instance, that $f(a) < f(b)$. Then for every y such that $f(a) < y < f(b)$, there exists an $x \in (a, b)$ such that $f(x) = y$.*

In other words, a continuous function maps an interval onto another interval.

5.2.2 Examples and exercises

Example 5.33. *Prove the uniform continuity of the following functions.*

a) $f(x) = x^2 - 3x - 1, \quad x \in [3, 6];$ **b)** $f(x) = 2x - 1, \quad x \in \mathbf{R}.$

Solutions.

a) Of course, the uniform continuity of f follows at once from Theorem 5.30, since f is continuous on the compact set $[3, 6]$. We shall, however, show that directly. For every pair $x_1, x_2 \in [3, 6]$ it follows

$$|f(x_1) - f(x_2)| = |(x_1^2 - 3x_1 - 1) - (x_2^2 - 3x_2 - 1)| = |x_1 - x_2| \cdot |x_1 + x_2 - 3| \leq 9 \cdot |x_1 - x_2|.$$

So for given $\varepsilon > 0$ we can choose $\delta := \varepsilon/9$, in order to obtain the implication

$$(\forall x_1, x_2 \in [3, 6]) \quad |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon.$$

b) Since for every pair $x_1, x_2 \in \mathbf{R}$ it holds

$$|f(x_1) - f(x_2)| = 2 \cdot |x_1 - x_2|,$$

for given $\varepsilon > 0$ we can choose $\delta := \varepsilon/2$. Then the following implication holds

$$(\forall x_1, x_2 \in \mathbf{R}) \quad |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon.$$

Notice that this function is uniformly continuous on the *noncompact* set $\mathbf{R}$ (since $\mathbf{R}$ is not bounded).

Example 5.34. *Prove that the function* $f(x) = \dfrac{1}{x}, \quad x > 0,$ *is*

a) *uniformly continuous on the set* $(c, 1)$, *where* $0 < c < 1$;

b) *not uniformly continuous, though continuous, on the set* $(0, 1]$.

Solutions.

a) Let $\varepsilon > 0$ be given. For every pair x_1, x_2 from the interval $(c, 1)$ it holds

$$|f(x_1) - f(x_2)| = \frac{|x_1 - x_2|}{x_1 x_2} \leq \frac{|x_1 - x_2|}{c^2}.$$

This allows us to choose $\delta := c^2\varepsilon$, since then it holds

$$(\forall x_1, x_2 \in (c, 1)) \; (|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon).$$

Question: Why could we not simply apply Theorem 5.30 ?

b) In order to prove that f is not uniformly continuous on the open interval $(0, 1)$, we shall use the sequences $x_n = \dfrac{1}{n}$, $n \in \mathbf{N}$, and $x'_n = \dfrac{1}{n+1}$, $n \in \mathbf{N}$. Obviously both sequences converge to 0, i.e.,

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} x'_n = 0.$$

However, it holds

$$|f(x_n) - f(x'_n)| = |f\left(\frac{1}{n}\right) - f\left(\frac{1}{n+1}\right)| = |n - (n+1)| = 1.$$

If we put $\varepsilon := \dfrac{1}{2}$, then for every $0 < \delta < 1$ there exists an $n \in \mathbf{N}$ (depending on δ) such that

$$|x_n - x'_n| = \frac{1}{n(n+1)} < \delta.$$

But then we have

$$|f(x_n - f(x'_n)| = 1 > \frac{1}{2} = \varepsilon.$$

Finally, let us prove the continuity of the function f at every point $x_0 \in (0, 1]$. For given $\varepsilon > 0$ let us choose

$$\delta := \min\left\{\frac{x_0}{2}, \frac{\varepsilon}{2x_0^2}\right\}. \tag{5.16}$$

Then the following implication holds:

$$|f(x) - f(x_0)| = \frac{|x - x_0|}{x \cdot x_0} < \frac{2|x - x_0|}{x_0^2} < \varepsilon$$

provided that $|x - x_0| < \delta$, proving the continuity of f at every point $x_0 \in (0, 1]$.

Remark. It is important to understand that the number δ in (5.16) was depending not only on ε, but also on the observed point x_0. In fact, in this case it was impossible to choose δ *uniformly* in x_0.

Example 5.35. *Discuss the uniform continuity of the following functions on their domains.*

a) $f(x) = \ln x, \ x \in (0,1)$;

b) $f(x) = x \cdot \sin x, \ x \in [0,+\infty)$;

c) $f(x) = \begin{cases} x \cdot \sin \dfrac{1}{x}, & x \in (0,\pi); \\ 0, & x = 0; \end{cases}$

d) $f(x) = \mathrm{e}^x \cdot \cos \dfrac{1}{x}, \ x \in (0,1)$.

Solutions.

a) We shall prove that the function $f(x) = \ln x$, $x \in (0,1)$, is continuous at every point $x_0 \in (0,1)$, but is not uniformly continuous on the open interval $(0,1)$.

Fix $x_0 \in (0,1)$ and let $\varepsilon \in (0,1)$ be given. The inequality

$$|f(x) - f(x_0)| = |\ln x - \ln x_0| < \varepsilon$$

is equivalent to

$$-\varepsilon < \ln \frac{x}{x_0} < \varepsilon \iff x_0\mathrm{e}^{-\varepsilon} < x < x_0\mathrm{e}^{\varepsilon}.$$

Hence we can choose $\delta := \min\{x_0 - x_0\mathrm{e}^{-\varepsilon}, x_0\mathrm{e}^{\varepsilon} - x_0\}$, so that the following implication holds:

$$(\forall x \in (0,1)) \ \ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

In the same manner one can prove the continuity of the logarithmic function on the whole interval $(0,+\infty)$.

In order to prove that f is not uniformly continuous on $(0,1)$, we shall start from two sequences converging to zero, $(x_n)_{n\in\mathbf{N}}$ and $(x'_n)_{n\in\mathbf{N}}$, where

$$x_n := \frac{1}{\mathrm{e}^n} \quad \text{and} \quad x'_n := \frac{1}{\mathrm{e}^{n+1}}, \quad \text{for} \quad n \in \mathbf{N}.$$

Put $\varepsilon := 1/2$. Then for every $\delta > 0$ there exists an $n = n_\delta \in \mathbf{N}$ such that

$$|x_n - x'_n| = \frac{\mathrm{e} - 1}{\mathrm{e}^{n+1}} < \delta.$$

For such n it holds, however,

$$|f(x_n) - f(x'_n)| = |(-n) - (-(n+1))| = 1 > \varepsilon,$$

contradicting the uniform continuity of f on $(0,1)$.

b) We shall prove that the given function f is not uniformly continuous on the interval $[0,+\infty)$.

Let $x_n = n\pi$ and $x'_n = n\pi + \dfrac{1}{n}$. Then for given $\delta > 0$, there exists $n = n_\delta$ such that

$$|x_n - x'_n| = \frac{1}{n} < \delta.$$

Put $\varepsilon = 1$; then it holds for $n > 2$:

$$|f(x_n) - f(x_n')| = \left|n\pi \sin(n\pi) - \left(n\pi + \frac{1}{n}\right)\sin\left(n\pi + \frac{1}{n}\right)\right|$$

$$= |(-1)^n \sin\frac{1}{n}| \left(n\pi + \frac{1}{n}\right) > \left(\frac{2}{\pi}\cdot\frac{1}{n}\right)(n\pi + \frac{1}{n}) > 2 > \varepsilon.$$

We used the inequality

$$\sin x > \frac{2}{\pi}x, \quad 0 < x < \frac{\pi}{2}.$$

Hence f is not uniformly continuous on $[0, +\infty)$. Notice, however, that f is a product of two uniformly continuous functions on $[0, +\infty)$ (prove that).

c) Let us observe first that $|f(x)| \leq |x|$ for all $x \in [0, \pi)$.

Let $\varepsilon > 0$ be given; we shall assume that $0 < \varepsilon < \pi$ (this is no restriction). If $x_1, x_2 \in [0, \frac{\varepsilon}{2}]$, then it holds for $\delta_1 := \varepsilon/2$:

$$|x_1 - x_2| < \delta_1 \Rightarrow |f(x_1) - f(x_2)| < x_1 + x_2 < \varepsilon, \tag{5.17}$$

If $x_1, x_2 \in [\frac{\varepsilon}{2}, \pi]$, then

$$|f(x_1) - f(x_2)| = \left|x_1 \sin\frac{1}{x_1} - x_2 \sin\frac{1}{x_2}\right|$$

$$= \left|x_1 \sin\frac{1}{x_1} - x_2 \sin\frac{1}{x_2} + x_2 \sin\frac{1}{x_1} - x_2 \sin\frac{1}{x_1}\right|$$

$$\leq |x_1 - x_2|\left|\sin\frac{1}{x_1}\right| + x_2\left|2\sin\frac{x_1 - x_2}{2x_1x_2}\cos\frac{x_1 + x_2}{2x_1x_2}\right|$$

$$\leq |x_1 - x_2| + 2x_2\frac{|x_1 - x_2|}{2x_1x_2} \leq |x_1 - x_2| + \frac{1}{x_1}|x_1 - x_2|.$$

Thus $|f(x_1) - f(x_2)| < |x_1 - x_2|\left(1 + \frac{2}{\varepsilon}\right)$. This shows that choosing $\delta_2 := \frac{\varepsilon^2}{2 + \varepsilon}$ the following implication holds

$$(\forall x_1, x_2 \in [\varepsilon, \pi)) \quad |x_1 - x_2| < \delta_2 \Rightarrow |f(x_1) - f(x_2)| < \varepsilon. \tag{5.18}$$

Putting relations (5.17) and (5.18) together, we obtain that for given $\varepsilon > 0$ we can choose δ as

$$\delta := \min\left\{\frac{\varepsilon}{2}, \frac{\varepsilon^2}{2 + \varepsilon}\right\},$$

in order to get the implication

$$(\forall x_1, x_2 \in [0, \pi)) \quad |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon.$$

Thus we proved the uniform continuity of f on $(0, \pi)$.

d) Since $0 \notin (0,1)$, the given function is continuous in each point of its domain. We shall prove next that it is not uniformly continuous on $(0,1)$.

Let $x_n = \dfrac{1}{2n\pi}$ and $x'_n = \dfrac{1}{(2n+1)\pi}$ for $n \in \mathbf{N}$, hence it holds

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x'_n = 0.$$

Now, for given $\delta > 0$ there exists $n = n_\delta$ such that

$$|x_n - x'_n| = \frac{1}{2n(2n+1)\pi} < \delta.$$

However, for every n it holds

$$\begin{aligned} |f(x_n) - f(x'_n)| &= \left|\exp\left(\frac{1}{2n\pi}\right)\cos 2n\pi - \exp\left(\frac{1}{(2n+1)\pi}\right)\cos(2n+1)\pi\right| \\ &= \exp\left(\frac{1}{2n\pi}\right) + \exp\left(\frac{1}{2n+1)\pi}\right) > 2, \end{aligned}$$

contradicting thus the uniform continuity of f on $(0,1)$.

Example 5.36. *Prove that the function* $f(x) = \dfrac{|\sin x|}{x}$, $x \neq 0$, *is uniformly continuous on each of the intervals* $(-1,0)$ *and* $(0,1)$, *but is not uniformly continuous on their union* $(-1,0)\bigcup(0,1)$.

Solution. The function f equals on the interval $(0,1)$ to the function

$$F_1(x) = \begin{cases} 1, & x = 0; \\ \dfrac{\sin x}{x}, & 0 < x < 1; \\ \sin 1, & x = 1. \end{cases}$$

Clearly, F_1 is continuous on the closed interval $[0,1]$, hence it is uniformly continuous there. This also gives the uniform continuity of f on the interval $(0,1)$. Analogously, the function F_2 given by

$$F_2(x) = \begin{cases} -\sin 1, & x = -1; \\ -\dfrac{\sin x}{x}, & -1 < x < 0; \\ -1, & x = 0. \end{cases}$$

is uniformly continuous on the closed interval $[-1,0]$, and equals to f on $(-1,0)$. Hence f is uniformly continuous on $(-1,0)$.

However, we shall prove next that f is not uniformly continuous on the union

$$A := (-1,0)\bigcup(0,1).$$

Let $x_n = -\dfrac{1}{n+1}$ and $x'_n = \dfrac{1}{n+1}$ for $n \in \mathbf{N}$. These two sequences are from A and both tend to zero as $n \to \infty$. For given $\delta > 0$, there exits $n = n_\delta$ such that

$$|x_n - x'_n| = \frac{2}{n+1} < \delta.$$

But for every $n \in \mathbf{N}$ it holds

$$\begin{aligned}
|f(x_n) - f(x'_n)| &= \left| \frac{\left|\sin\left(\frac{-1}{n+1}\right)\right|}{-\frac{1}{n+1}} - \frac{\left|\sin\left(\frac{1}{n+1}\right)\right|}{\frac{1}{n+1}} \right| \\
&= 2(n+1)\left|\sin\frac{1}{n+1}\right| \geq 2(n+1)\cdot\left(\frac{2}{\pi}\cdot\frac{1}{n+1}\right) = \frac{4}{\pi}.
\end{aligned}$$

Hence for (say) $\varepsilon = 1$, it holds $|f(x_n) - f(x'_n)| > \varepsilon$, proving that f is not uniformly continuous on the set $(-1,0)\bigcup(0,1)$.

Example 5.37. *If the function $f : I \to \mathbf{R}$ is bounded, monotone and continuous on the open interval I, then f is also uniformly continuous on I if*

a) $I = (a,b)$; **b)** $I = (a,+\infty)$.

Solution.

a) Let us prove first that a bounded, monotone and continuous function on the open interval (a,b) has a right-hand limit at a. Assume, say, that f is monotonically increasing on (a,b). Since f is bounded on (a,b), it follows that the range of the function f,

$$B := f((a,b)) = \{y \in \mathbf{R} |\ (\exists x \in (a,b))\ y = f(x)\}$$

is bounded from below. By Example 1.50 a), it follows that the set B has an infimum, which we shall denote by L. Let us show that

$$\lim_{x\to a+} f(x) = L. \tag{5.19}$$

Let $(x_n)_{n\in\mathbf{N}}$ be an arbitrary sequence of numbers from (a,b) such that

$$\lim_{n\to\infty} x_n = a.$$

By passing (if necessary) to a subsequence, we can suppose that $(x_n)_{n\in\mathbf{N}}$ is a monotonically decreasing sequence. By assumption, then the sequence

$(f(x_n))_{n\in\mathbf{N}}$ is also a monotonically decreasing sequence. Since $f(x_n) \geq L$ for every $n \in \mathbf{N}$, there exists the limit

$$L' := \lim_{n\to\infty} f(x_n). \tag{5.20}$$

Clearly, either $L' > L$ or, as we claim, $L' = L$.

If $L' > L$, then there exists an element $y_0 \in B$ such that $L' > y_0 > L$ (because L is the infimum of B). Since f is continuous on the interval (a, b), we can apply Theorem 5.32, giving us the existence of a unique number $x_0 \in (a, b)$ such that $f(x_0) = y_0$. From $\lim\limits_{n\to\infty} x_n = a$ it follows that there exists an n_0 such that it holds

$$n > n_0 \Rightarrow a < x_n < x_0.$$

This implies

$$n > n_0 \Rightarrow f(x_n) \leq f(x_0) = y_0.$$

Since $y_0 < L'$, it follows that for infinitely many indices n it holds $f(x_n) < L'$, which in view of the decrease of $f(x_n)$ is contradicting the definition of L' in (5.20). Hence necessarily $L = L'$, implying $L = \lim\limits_{n\to\infty} f(x_n)$. Since $(x_n)_{n\in\mathbf{N}}$ was an arbitrary sequence tending to a, it follows that (5.19) holds.

In an analogous way one can prove that the left-hand limit of f at the point b exists as well; let us put

$$K := \lim_{x\to b-} f(x). \tag{5.21}$$

Let us introduce next the following function:

$$F(x) := \begin{cases} L, & x = a; \\ f(x), & a < x < b; \\ K, & x = b. \end{cases} \tag{5.22}$$

By the choice of L and K, the function F is continuous on the compact set $[a, b]$, hence by Theorem 5.30 it is uniformly continuous on $[a, b]$. But then trivially follows that f is uniformly continuous on (a, b).

b) In the same way as in a), one proves that there exists the

$$\lim_{x\to a+} f(x) = L.$$

Let us assume, say, that f is monotonically increasing. Then from the boundedness of f it follows that the set

$$B := f\left((a, +\infty)\right) = \{y \in \mathbf{R} |\ (\exists x \in (a, +\infty))\ y = f(x)\}$$

is bounded from above. Hence by Example 1.50 a), it follows that B has a supremum, which we shall denote by K. Similarly as was done in a), one can prove that

$$\lim_{x\to\infty} f(x) = K.$$

In order to prove the uniform continuity of f on $(a,+\infty)$, let us put

$$F(x) := \begin{cases} L, & x = a; \\ f(x), & x > 0. \end{cases}$$

Let $\varepsilon > 0$ be given. Then there exists a number $T > a$, depending on ε, such that

$$\forall x > T \;\Rightarrow\; |F(x) - K| < \frac{\varepsilon}{4}. \tag{5.23}$$

In view of a), F is uniformly continuous on the set $[a,T]$, hence f is uniformly continuous on $(a,T]$. This means that there exists a $\delta > 0$ such that

$$(\forall x_1, x_2 \in (a,T]) \;\left(|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \frac{\varepsilon}{2} \right). \tag{5.24}$$

We have still to prove that f is uniformly continuous on the set $[T,+\infty)$ as well. To that end, from (5.23) it follows that for $x_1 > T$ and $x_2 > T$ it holds

$$\begin{aligned} |f(x_1) - f(x_2)| &< |f(x_1) - K + K - f(x_2)| \\ &\le\; |f(x_1) - K| + |K - f(x_2)| \le \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \end{aligned} \tag{5.25}$$

Hence $|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon/2$, and from relations (5.24) and (5.25) it follows that f is uniformly continuous on the set $[T,+\infty)$.

Remark.

- The function F from (5.22), continuous on $[a,b]$ and equal to f on (a,b), is called the **continuous extension** of f from (a,b) to $[a,b]$.
- The assumption on monotonicity of f in Example 5.37 is essential. Namely, the function

$$g(x) = \sin(1/x), \;\; x \in (0,1)$$

is bounded and continuous on $(0,1)$, but can not be continuously extended to $[0,1]$ (see Example 5.15 c)).

Exercise 5.38. *Show that the function f is uniformly continuous on the set X given below.*

a) $f(x) = \sin x, \;\; X = \mathbf{R}$; **b)** $f(x) = \cos x, \;\; X = \mathbf{R}$;

c) $f(x) = x^2, \;\; X = (-2,2)$. **d)** $f(x) = \sqrt{x}, \;\; X = [1,+\infty)$.

Exercise 5.39 *Examine the uniform continuity of the following functions on their domains.*

a) $f(x) = \dfrac{\sin x}{x}, \;\; x \in (0,\pi)$; **b)** $f(x) = x \sin x, \;\; x \in [0,+\infty)$;

c) $f(x) = \dfrac{x}{9 - x^2}, \;\; x \in (-2,2)$; **d)** $f(x) = \dfrac{1}{\sqrt{x}}, \;\; x > 0$.

Answers. In **a)**, **b)** and **c)**, the functions are uniformly continuous on their domains, while in **d)**, f is not uniformly continuous, but only continuous.

Chapter 6

Derivatives

6.1 Introduction

6.1.1 Basic notions

Definition 6.1. *Let f be a real function defined on an open interval (a, b) and let $x_0 \in (a, b)$. Then the following limit*

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{6.1}$$

(provided it exists) is called the **first derivative of f at the point x_0**.

Besides the notation $f'(x_0)$ for the first derivative of the function f at x_0 one also uses the notation $\frac{df}{dx}(x_0)$.

Occasionally, instead of $f'(x_0)$ we shall write $f'_x(x_0)$ to emphasize the variable x in which the derivative is found.

The number h in (6.1) is called the **increment of the independent variable** x at the point x_0, while the difference $f(x_0 + h) - f(x_0)$ is called the **increment of the dependent variable** at the point x_0. Thus relation (6.1) can be interpreted as the limit of the quotient of the increments of the dependent and the independent variable, when the latter tends to zero.

One can also define the **one sided derivative** of a function at some point x_0 from its domain as the one sided limit of the quotient $\frac{f(x_0 + h) - f(x_0)}{h}$. The right-hand side and the left-hand side derivative of f at x_0 are denoted by $f'_+(x_0)$, and $f'_-(x_0)$. In the next table let us give the first derivatives of the most commonly used elementary functions, together with the largest subsets of $\mathbf{R}$ on which they exist.

Table of first derivatives

1. $(x^n)' = nx^{n-1}, \quad n \in \mathbf{Z}, \ x \in \mathbf{R}.$

2. $(x^\alpha)' = \alpha x^{\alpha-1}, \quad \alpha \neq 0, \ x > 0.$

3. $(\sin x)' = \cos x, \quad x \in \mathbf{R}.$

4. $(\cos x)' = -\sin x, \quad x \in \mathbf{R}.$

5. $(\tan x)' = \dfrac{1}{\cos^2 x}, \quad x \in \mathbf{R} \setminus \{(2k+1)\frac{\pi}{2} \mid k \in \mathbf{Z}\}.$

6. $(\cot x)' = \dfrac{-1}{\sin^2 x}, \quad x \in \mathbf{R} \setminus \{k\pi \mid k \in \mathbf{Z}\}.$

7. $(a^x)' = a^x \cdot \ln a, \quad a > 0, \ x \in \mathbf{R}, \qquad (\mathrm{e}^x)' = \mathrm{e}^x, \quad x \in \mathbf{R}.$

8. $(\log_a x)' = \dfrac{1}{x \cdot \ln a}, \quad a > 0, \ a \neq 1, \ x > 0, \qquad (\ln x)' = \dfrac{1}{x}, \quad x > 0.$

9. $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}, \quad |x| < 1.$

10. $(\arccos x)' = \dfrac{-1}{\sqrt{1-x^2}}, \quad |x| < 1.$

11. $(\arctan x)' = \dfrac{1}{1+x^2}, \quad x \in \mathbf{R}.$

12. $(\mathrm{arccot}\, x)' = \dfrac{-1}{1+x^2}, \quad x \in \mathbf{R}.$

If the functions f and g are defined on an interval (a, b) and have first derivatives at the point $x \in (a, b)$, then

- the **first derivative of the sum**, resp. **difference of functions** f and g is

$$(f(x) \pm g(x))' = f'(x) \pm g'(x);$$

- the **first derivative of the product of functions** f and g is

$$(f(x) \cdot g(x))' = f'(x)g(x) + f(x)g'(x),$$

  and, in particular,

$$(A \cdot f(x))' = A \cdot f'(x),$$

  where A is an arbitrary real constant;

- the **first derivative of the quotient of functions** f and g is

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}, \quad \text{provided that } g(x) \neq 0.$$

Assume that the function $g : (a, b) \to (c, d)$ has a first derivative at $x_0 \in (a, b)$ and assume that the function $f : (c, d) \to \mathbf{R}$ has a first derivative at the point $g(x_0) \in (c, d)$. Then the **first derivative at x_0 of the composite function**

$$h = f \circ g, \quad h : (a, b) \to \mathbf{R},$$

exists and the so called **chain rule** holds:

$$h'(x_0) = f'_g(g(x_0)) \cdot g'(x_0). \tag{6.2}$$

If the surjective function $f : (a, b) \to (c, d)$ satisfies the following three conditions

(i) the function f has a first derivative at the point $x_0 \in (a, b)$;

(ii) the function f is monotone on the interval (a, b);

(iii) the number $f'(x_0)$ is different from zero,

then there exists an **inverse function**

$$f^{-1} : (c, d) \to (a, b)$$

to the function f.

Then the **first derivative of the inverse function** at the point $y_0 = f(x_0)$ is

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} \quad \text{or} \quad (f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}. \tag{6.3}$$

Assume that the equation

$$F(x, y) = 0 \tag{6.4}$$

defines uniquely a function

$$y = f(x), \quad x \in (a, b).$$

Then we say that f is given implicitly with equation (6.4).

Assume that for some $x_0 \in (a, b)$, the derivatives of F from (6.4) exist and, moreover, $F'_x(x_0, y_0) \neq 0$, with $y_0 = f(x_0)$. Then from the chain rule (6.2) it follows that the **first derivative of the implicitly given function** f at the point x_0 is

$$f'(x_0) = -\frac{F'_y(x_0, y_0)}{F'_x(x_0, y_0)} \tag{6.5}$$

Assume that two differentiable functions $y = y(t)$ and $x = x(t)$ of the same variable ("parameter") $t \in (\alpha, \beta)$ are given. Then the function $y = f(x)$, determined as the mapping f that assigns to $x = x(t)$ the number $y = y(t)$ is called a **parametric function**.

The **first derivative of the parametric function** $y = f(x)$ at the point x_0 is given by

$$y'(x_0) = \frac{y'_t(t_0)}{x'_t(t_0)}, \quad t_0 \in (\alpha, \beta) \tag{6.6}$$

where $x(t_0) = x_0$. Clearly, formula (6.6) has meaning only at those points t_0, where x and y have derivatives and it holds $x'_t(t_0) \neq 0$.

Definition 6.2. *If the increment* Δy *of the function* $f:(a,b)\to\mathbf{R}$ *at the point* $x_0\in(a,b)$ *can be written in the form*

$$\Delta y = f(x_0+h)-f(x_0) = D\cdot h + r(h)\cdot h,$$

for some number D *(independent from* h*), and it holds*

$$\lim_{h\to 0} r(h) = 0,$$

then for the function f *we say that it is* **differentiable at the point** x_0.

Theorem 6.3. *A function* $f:(a,b)\to\mathbf{R}$ *is differentiable at the point* $x_0\in(a,b)$ *if and only if it has a first derivative at that point.*[1]

Hence from Definition 6.2 it follows $D=f'(x_0)$.

The **differential** of a differentiable function f at a point x is the *linear function* $df(x):\mathbf{R}\to\mathbf{R}$ of the increment h,

$$df(x)(h) = D\cdot h,\quad h\in\mathbf{R},$$

or, in view of Theorem 6.3, and by omitting h,

$$df(x) = f'(x)\,dx. \tag{6.7}$$

Using Definition 6.2 and Theorem 6.3, we have the approximate formula

$$f(x_0+h)\approx f(x_0)+f'(x_0)\cdot h, \tag{6.8}$$

provided that f is differentiable at x_0 and h is "small".

Assume that a function f has a first derivative at every point $x\in(a,b)$. Then the function $f':(a,b)\to\mathbf{R}$, is called the **first derivative** of f. If the first derivative of the function f' at the point $x_0\in(a,b)$ exists, then it is called the **second derivative of the function** f **at the point** x_0 and will be denoted by $f''(x_0)$.

One defines analogously the third, fourth, ..., $n-$th derivative of a function f at the point x_0, and they are denoted respectively by $f'''(x_0), f^{(4)}(x_0),\dots, f^{(n)}(x_0)$.

If a function $f:(a,b)\to\mathbf{R}$ has a first derivative at the point $x_0\in(a,b)$, then the line

$$y-y_0 = f'(x_0)(x-x_0), \tag{6.9}$$

where $y_0=f(x_0)$, is called the **tangent line of the graph** of the function f at the point $T\left(x_0,f(x_0)\right)$. If, additionally, it holds $f'(x_0)\neq 0$, the line

$$y-y_0 = -\frac{1}{f'(x_0)}(x-x_0) \tag{6.10}$$

is the **perpendicular line of the graph** of the function f at the point $T\left(x_0,f(x_0)\right)$.

Assume a function f has a first derivative at a point x_0. If $0\le\alpha<\pi$ is the angle between the tangent line at the point x_0 and the positive direction of the $x-$axis, then it holds

$$\tan\alpha = f'(x_0). \tag{6.11}$$

In other words, the slope of the tangent line of the graph f at some point is exactly the value of the first derivative of f at that point.

[1]This theorem is *not true* for functions of two or more variables.

6.1.2 Examples and exercises

Example 6.4. *Find by definition the first derivatives of the following functions at the given points.*

a) $f(x) = x(x-1)^2(x-2)^3, \;\; x \in \mathbf{R},$ *at the points* $x_0 = 0, \; x_1 = 1, \; x_2 = 2;$

b) $f(x) = \sqrt{1+x}, \;\; x \geq -1,$ *at the points* $x_0 = 1, \; x_1 = 0,$ *and also the right-hand side derivative at the point* $x_2 = -1;$

c) $f(x) = x + (x-1)\arcsin\sqrt{\dfrac{x}{x+1}}, \;\; x > 0,$ *at the point* $x_0 = 1;$

d) $f(x) = \sqrt[5]{x+1}, \;\; x \in \mathbf{R},$ *at the point* $x_0 = -1.$

Solutions.

a) Using Definition (6.1), we have

$$f'(0) = \lim_{h\to 0} \frac{h(h-1)^2(h-2)^3 - 0}{h} = -8,$$

$$f'(1) = \lim_{h\to 0} \frac{(1+h)(1+h-1)^2(1+h-2)^3 - 0}{h} = 0,$$

$$f'(2) = \lim_{h\to 0} \frac{(2+h)(2+h-1)^2(2+h-2)^3 - 0}{h} = 0.$$

b) We have

$$f'(1) = \lim_{h\to 0} \frac{\sqrt{1+1+h} - \sqrt{2}}{h} = \lim_{h\to 0} \frac{1}{\sqrt{2+h} + \sqrt{2}} = \frac{1}{2\sqrt{2}},$$

$$f'(0) = \lim_{h\to 0} \frac{\sqrt{1+h} - \sqrt{1}}{h} = \frac{1}{2},$$

$$f'_+(-1) = \lim_{h\to 0+} \frac{\sqrt{1+(-1+h)} - \sqrt{1+(-1)}}{h} = \lim_{h\to 0+} \frac{1}{\sqrt{h}} = +\infty.$$

Geometrically, the last equality means that the vertical line $x = -1$ is the tangent line to the graph of f at the point $(-1, 0)$.

c) It holds

$$f'(1) = \lim_{h\to 0} \frac{1 + h + (1+h-1)\arcsin\sqrt{\dfrac{1+h}{1+h+1}} - 1}{h} = 1 + \frac{\pi}{4}.$$

d) In this case we have

$$f'(-1) = \lim_{h\to 0} \frac{\sqrt[5]{-1+h+1} - 0}{h} = +\infty.$$

Exercise 6.5. *Find by definition the first derivatives of the following functions.*

a) $f(x) = \dfrac{1}{x^3}$, *at the point* $x_0 = 1$;

b) $f(x) = \sqrt[3]{(1+x)^2}$, *at the points* $x_0 = 0$, $x_1 = -1$;

c) $f(x) = 3 \cdot |2+x|$, *at the point* $x_0 = -3$.

Answers.

a) $f'(1) = -3$. b) $f'(0) = \dfrac{2}{3}$; $f'(-1)$ does not exist.

c) $f'(-3) = -3$.

Example 6.6. *Find by definition the first derivatives of the following functions.*

a) $f(x) = x^2 + 2x$, $x \in \mathbf{R}$; **b)** $f(x) = \sqrt[3]{x-1}$, $x \in \mathbf{R}$;

c) $f(x) = \ln(x+1)$, $x > -1$; **d)** $f(x) = \cos(2x)$, $x \in \mathbf{R}$,

at the point x *from its domain.*

Solutions.

a) $f'(x) = \lim\limits_{h\to 0} \dfrac{(x+h)^2 + 2(x+h) - x^2 - 2x}{h} = 2x + 2.$

b) For $x \neq 1$, it holds

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sqrt[3]{x+h-1} - \sqrt[3]{x-1}}{h} \\ &= \lim_{h\to 0} \frac{1}{\sqrt[3]{(x-1+h)^2} + \sqrt[3]{(x-1)(x-1+h)} + \sqrt[3]{(x-1)^2}} \\ &= \frac{1}{3\sqrt[3]{(x-1)^2}}. \end{aligned}$$

At the point $x = 1$ it holds

$$f'_+(1) = \lim_{h\to 0+} \frac{\sqrt[3]{h} - \sqrt[3]{0}}{h} = +\infty,$$

$$f'_-(1) = \lim_{h\to 0-} \frac{\sqrt[3]{h} - \sqrt[3]{0}}{h} = +\infty,$$

which means that the tangent line of the graph of f at $x = 1$ is parallel with the y-axis.

c) It holds for $x > -1$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\ln(x+h+1) - \ln(x+1)}{h} = \lim_{h\to 0} \ln\left(1 + \frac{h}{x+1}\right)^{1/h} \\ &= \frac{1}{x+1} \ln \lim_{h\to 0}\left(1 + \frac{h}{x+1}\right)^{(x+1)/h} = \frac{1}{x+1}. \end{aligned}$$

d) We have

$$f'(x) = \lim_{h\to 0} \frac{\cos(2(x+h)) - \cos(2x)}{h} = \lim_{h\to 0} \frac{-2\sin(2x+h)\cdot \sin h}{h} = -2\sin(2x).$$

Exercise 6.7. *Find by definition the first derivatives of the following functions. at the point $x \in \mathbf{R}$.*

a) $f(x) = x\cos 2x$; **b)** $f(x) = \sqrt[3]{(1+x)^2}$; **c)** $f(x) = 3^{2+x}$.

Example 6.8. *Find the first derivatives of the following functions.*

a) $f(x) = \sqrt[3]{1+x^4}, \quad x \in \mathbf{R};$

b) $f(x) = \sqrt{1 + \sqrt[3]{1 + \sqrt[4]{1+x^4}}}, \quad x \in \mathbf{R};$

c) $f(x) = 3^{2\tan^3 x}, \quad x \neq (2k+1)\frac{\pi}{2}, \ k \in \mathbf{Z};$

d) $f(x) = \ln \sin \operatorname{arccot}\, \mathrm{e}^x, \quad x \in \mathbf{R};$

e) $f(x) = \ln \sqrt[4]{\dfrac{2\mathrm{e}^{2x}}{1+\cos x}}, \quad x \in (0, \pi/2);$

f) $f(x) = \log_3 \log_5 \log_7 x, \quad x > 7.$

Solutions. We leave to the reader to find the natural domains of the given functions and the largest subsets of $\mathbf{R}$ on which their first derivatives exist.

a) Let us put $u(x) = 1 + x^4$. Then from the chain rule it follows that

$$f'(x) = \frac{1}{3}(u(x))^{-2/3}u'(x) = \frac{4x^3}{3\sqrt[3]{(1+x^4)^2}}, \quad x \in \mathbf{R}.$$

b) $$f'(x) = \frac{x^3}{6\sqrt{1+\sqrt[3]{1+\sqrt[4]{1+x^4}}}} \cdot \frac{1}{\sqrt[3]{(1+\sqrt[4]{1+x^4})^2}} \cdot \frac{1}{(\sqrt[4]{1+x^4})^3}, \quad x \in \mathbf{R}.$$

c) $f'(x) = 3^{2\tan^3 x} \ln 3 \cdot (2\tan^3 x)' = 6 \ln 3 \cdot 3^{2\tan^3 x} \dfrac{\tan^2 x}{\cos^2 x}, \quad x \neq (2k+1)\dfrac{\pi}{2},\ k \in \mathbf{Z}.$

d) In this case we have

$$\begin{aligned} f'(x) &= \frac{1}{\sin \operatorname{arccot}\ \mathrm{e}^x}(\sin \operatorname{arccot}\ \mathrm{e}^x)' \\ &= \frac{1}{\sin \operatorname{arccot}\ \mathrm{e}^x} \cos \operatorname{arccot}\ \mathrm{e}^x \cdot (\operatorname{arccot}\ \mathrm{e}^x)' \\ &= \cot(\operatorname{arccot}\ (\mathrm{e}^x)) \frac{-\mathrm{e}^x}{1+\mathrm{e}^{2x}} = -\frac{\mathrm{e}^{2x}}{1+\mathrm{e}^{2x}}, \quad x \in \mathbf{R}. \end{aligned}$$

e) Since it holds

$$f(x) = \frac{1}{4}\ln(\mathrm{e}^{2x}) - \frac{1}{4}\ln \cos^2 \frac{x}{2} = \frac{x}{2} - \frac{1}{2}\ln \cos \frac{x}{2},$$

we obtain

$$f'(x) = \frac{1}{2} + \frac{1}{4}\tan\frac{x}{2}, \quad x \in (0, \pi/2).$$

f) Since $\log_a x = \dfrac{\ln x}{\ln a}$, for $x > 7$, $a > 0$ and $a \neq 1$ (prove that!), it follows that

$$\begin{aligned} f'(x) &= \frac{1}{\ln 3 \log_5 \log_7 x}(\log_5 \log_7 x)' \\ &= \frac{1}{\log_5 \log_7 x} \cdot \frac{1}{\log_7 x} \cdot \frac{1}{x} \cdot \frac{1}{\ln 3 \ln 5 \ln 7} = \frac{1}{x \ln 3 \ln x \ln \log_7 x}. \end{aligned}$$

Exercise 6.9. *Find the first derivatives of the following functions.*

a) $f(x) = \sinh x, \quad x \in \mathbf{R}$;

b) $f(x) = \cosh x, \quad x \in \mathbf{R}$;

c) $f(x) = \tanh x, \quad x \in \mathbf{R}$;

d) $f(x) = \coth x, \quad x \neq 0$;

e) $f(x) = \ln(x + \sqrt{x^2+1}), \quad x \in \mathbf{R}$;

f) $f(x) = \ln(x + \sqrt{x^2-1}), \quad x \in \mathbf{R}$;

Answers.

a) $f'(x) = \cosh x.$

b) $f'(x) = \sinh x.$

c) $f'(x) = \dfrac{1}{\cosh^2 x}.$

d) $f'(x) = \dfrac{-1}{\sinh^2 x}.$

e) $f'(x) = \dfrac{1}{\sqrt{x^2+1}}.$

f) $f'(x) = \dfrac{1}{\sqrt{x^2-1}}.$

Exercise 6.10. *Find the first derivatives of the following functions.*

a) $f(x) = \left(\sqrt[5]{x} - \dfrac{1}{\sqrt[5]{x}}\right)^9, \quad x \neq 0;$

b) $f(x) = \dfrac{\sqrt{x^2+1} - x}{\sqrt{x^2+1} + x} + \dfrac{\sqrt{x^2+1} + x}{\sqrt{x^2+1} - x}, \quad x \in \mathbf{R};$

c) $f(x) = \cos\cos\cos\cos 2x, \quad x \in \mathbf{R};$

d) $f(x) = \dfrac{1}{2}\arctan\dfrac{2x}{1-x^2}, \quad x \in (-1, 1);$

e) $f(x) = \mathrm{e}^x + \mathrm{e}^{\mathrm{e}^x} + \mathrm{e}^{\mathrm{e}^{\mathrm{e}^x}}, \quad x \in \mathbf{R};$

f) $f(x) = \ln(\ln^2(\ln^3(x^2))), \quad x \neq 0;$

g) $f(x) = \dfrac{1}{\sqrt{a^2-b^2}}\arcsin\dfrac{a\sin x + b}{a + b\sin x}, \quad |b| < a;\ x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right);$

h) $f(x) = \dfrac{1}{\sqrt{b^2-a^2}}\ln\dfrac{b + a\sin x - \sqrt{b^2-a^2}\cdot\cos x}{a + b\sin x}, \quad |a| < |b|.$

Answers. **a)** $f'(x) = \dfrac{9}{5}\left(\sqrt[5]{x} - \dfrac{1}{\sqrt[5]{x}}\right)^8\left(\dfrac{1}{\sqrt[5]{x^4}} + \dfrac{1}{\sqrt[5]{x^6}}\right).$ **b)** $f'(x) = 8x.$

c) $f'(x) = 2\sin\cos\cos\cos 2x \cdot \sin\cos\cos 2x \cdot \sin\cos 2x \cdot \sin 2x.$

d) $f'(x) = \dfrac{1}{1+x^2}.$ **e)** $f'(x) = \mathrm{e}^x + \mathrm{e}^x\mathrm{e}^{\mathrm{e}^x} + \mathrm{e}^x\mathrm{e}^{\mathrm{e}^x}\mathrm{e}^{\mathrm{e}^{\mathrm{e}^x}}.$

f) $f'(x) = \dfrac{12}{|x|\cdot\ln x^2\cdot\ln(\ln^3(x^2))}.$ **g)** $f'(x) = \dfrac{1}{a + b\sin x}.$

h) $f'(x) = \dfrac{1}{a + b\sin x}$ (compare with g)).

Example 6.11. *Find the first derivatives of the following functions.*

a) $f(x) = x^x,\ x > 0;$ **b)** $f(x) = x^{\sqrt{x}},\ x > 0;$

c) $f(x) = (\sqrt{x})^x,\ x > 0;$ **d)** $f(x) = x^{\sin x},\ x > 0;$

e) $f(x) = (\sin x)^{\cos x},\ \sin x > 0;$ **f)** $f(x) = (2 + \sin x)^x,\ x \in \mathbf{R}.$

Solutions.

a) The function f can be written as $f(x) = \mathrm{e}^{x\ln x}$, $x > 0$, whose first derivative is

$$f'(x) = \mathrm{e}^{x\ln x}(\ln x + 1) = x^x(\ln x + 1).$$

This first derivative can be found also by taking the logarithm of the function $y = x^x$ and then differentiating the obtained expression.

In fact, from $\ln y = x \ln x$ it follows

$$\frac{y'}{y} = \ln x + 1 \Rightarrow y' = y(\ln x + 1).$$

Putting now $y = f(x) = x^x$ in the last equality gives again

$$y' = f'(x) = x^x(\ln x + 1).$$

b) From the expression

$$f(x) = \mathrm{e}^{\sqrt{x}\ln x}, \quad x > 0,$$

it follows

$$f'(x) = \mathrm{e}^{\sqrt{x}\ln x}\left(\frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x}\right) = x^{\sqrt{x}} \cdot \frac{\ln x + 2}{2\sqrt{x}}.$$

c) Since it holds $f(x) = \mathrm{e}^{(x \ln x)/2}$, it follows that

$$f'(x) = \mathrm{e}^{(x\ln x)/2}\left(\frac{1 + \ln x}{2}\right).$$

d) The first derivative of f is

$$f'(x) = \mathrm{e}^{\sin x \ln x}\left(\cos x \ln x + \frac{\sin x}{x}\right).$$

e) From $f(x) = \mathrm{e}^{\cos x \ln \sin x}$ it follows that

$$f'(x) = \mathrm{e}^{\cos x \ln \sin x}\left(-\sin x \ln \sin x + \frac{\cos^2 x}{\sin x}\right).$$

f) From $f(x) = \mathrm{e}^{x \ln(2+\sin x)}$ we obtain

$$f'(x) = \mathrm{e}^{x\ln(2+\sin x)}\left(\ln(2 + \sin x) + \frac{x \cos x}{2 + \sin x}\right).$$

Example 6.12. *Prove that a differentiable function $f : (a,b) \to \mathbf{R}$ at the point $x_0 \in (a,b)$ is also continuous at that point.*

Solution. By Definition 6.2 the increment of f at x_0 can be written in the form

$$f(x_0 + h) - f(x_0) = D \cdot h + r(h) \cdot h,$$

where D does not depend on h, while the remainder $r(h)$ satisfies the condition $\lim\limits_{h\to 0} r(h) = 0$. Hence $r(h)$ is bounded by for example $|D| + 1$ for $|h|$ sufficiently small. Thus there exists $\delta_1 > 0$ such that the following implication holds

$$(\forall\, h \in \mathbf{R}) \quad |h| < \delta_1 \Rightarrow |r(h)| < |D| + 1.$$

But then for given $\varepsilon > 0$ there exists $\delta := \frac{1}{2}\min\left\{\frac{\varepsilon}{2|D|+1}, \delta_1\right\}$ such that for $|h| < \delta$ it holds

$$|f(x_0 + h) - f(x_0)| \leq (|D| + |r(h)|)|h| < (2|D| + 1)\frac{\varepsilon}{2|D| + 1} = \varepsilon.$$

Example 6.13. *The function f is defined by $f(x) = |x|$, $x \in \mathbf{R}$. Check whether*

a) *f is continuous at the point $x = 0$;*

b) *f has a first derivative at the point $x = 0$;*

c) *f is differentiable at the point $x = 0$.*

Solutions.

a) Let us prove that f is continuous at 0. Namely, for given $\varepsilon > 0$ it holds

$$|f(x) - f(0)| = \Big||x| - |0|\Big| = |x - 0| < \varepsilon,$$

provided that $|x - 0| < \delta := \varepsilon$.

b) The right and left derivative of f at 0 are

$$f'_+(0) = \lim_{h \to 0+} \frac{|h| - 0}{h} = \frac{h}{h} = 1, \quad f'_-(0) = \lim_{h \to 0-} \frac{|h| - 0}{h} = \frac{-h}{h} = -1.$$

Since these one sided derivatives are different, f has no first derivative at 0.

Remark. This example shows that a continuous function at some point does not necessarily have a first derivative at that point.

c) Once we have proved in b) that f has no first derivative at the point 0, it follows at once from Theorem 6.3 that f is not differentiable at 0. Still, we shall prove that f is not differentiable at 0 without using that theorem.

To that end let us write the increment of f at the point $x = 0$ in the form

$$f(h) - f(0) = D \cdot h + r(h) \cdot h \iff |h| = (D + r(h))h,$$

where h is the increment of the independent variable. We shall show that $\lim_{h \to 0} r(h)$ does not exist, whatever value one chooses for the constant D. In fact, it holds

$$h > 0 \Rightarrow r(h) = 1 - D$$

and

$$h < 0 \Rightarrow r(h) = -1 - D.$$

Since $1 - D \neq -1 - D$ for every D, the limit at zero of the remainder $r(h)$ does not exist, giving us the nondifferentiability of f at 0.

Example 6.14. *Find the largest sets on which the first derivative of the function f is*

(i) *continuous;*

(ii) *differentiable;*

(iii) *continuously differentiable.*

a) $f(x) = x \cdot |x|, \quad x \in \mathbf{R};$

b) $f(x) = \ln|x|, \quad x \neq 0;$

c) $f(x) = |(x+2)^2(x-1)^3|, \quad x \in \mathbf{R};$

d) $f(x) = |\cos^3 x|, \quad x \in \mathbf{R};$

e) $f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0; \\ 0, & x = 0; \end{cases}$

f) $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0; \\ 0, & x = 0; \end{cases}$

g) $f(x) = \begin{cases} x^\alpha \sin \dfrac{1}{x}, & x \neq 0; \\ 0, & x = 0, \end{cases}$ *where* α *is a positive parameter.*

Solutions.

a) **(i)** The function f is a product of two continuous functions on $\mathbf{R}$, hence it is continuous on the whole set $\mathbf{R}$,

(ii) Since $|x| = x \cdot \operatorname{sgn} x, \quad x \in \mathbf{R}$, we can write $x \cdot |x| = x^2 \cdot \operatorname{sgn} x, \quad x \in \mathbf{R}$. The last function clearly has first derivative in each $x \neq 0$, and for such x it holds

$$f'(x) = 2x \cdot \operatorname{sgn} x = 2|x|.$$

Let us show next that the first derivative of f at 0 also exists.

$$f'(0) = \lim_{h \to 0} \frac{h^2 \cdot \operatorname{sgn} h - 0}{h} = 0.$$

(iii) Moreover, the function $f' : \mathbf{R} \to \mathbf{R}$ is also continuous at 0, since

$$\lim_{x \to 0} f'(x) = \lim_{x \to 0} (2|x|) = 0 = f'(0).$$

b) **(i)** The function f is continuous on the set $\mathbf{R} \setminus \{0\}$. Note that one can not discuss the continuity of f at 0, since f is not defined there.

(ii) It holds $f(x) = \ln(x \operatorname{sgn} x)$ for $x \neq 0$, hence

$$f'(x) = \frac{1}{x \cdot \operatorname{sgn} x} \operatorname{sgn} x = \frac{1}{x}, \; x \neq 0.$$

(iii) Clearly, the function f' defined by

$$f'(x) = \frac{1}{x}, \ x \neq 0$$

is continuous on its domain. (In view of Example 5.34, it is not uniformly continuous there.)

c) **(i)** The function is continuous on $\mathbf{R}$.

(ii) Since $f(x) = |(x+2)^2(x-1)^3| = (x+2)^2(x-1)^3\text{sgn}(x-1), \ x \in \mathbf{R}$, the first derivative of f for $x \neq 1$ is

$$\begin{aligned} f'(x) &= 2(x+2)(x-1)^3\text{sgn}(x-1) + (x+2)^2 3(x-1)^2\text{sgn}(x-1) \\ &= (x+2)(5x^2 - x - 4)|x-1|. \end{aligned}$$

The function f has also a first derivative at the point $x = 1$. Namely, its left-hand side derivative at 1 is given by

$$f'_-(1) = \lim_{h \to 0-} \frac{|((1+h)+2)^2((1+h)-1)^3| - 0}{h} = \lim_{h \to 0-} \frac{(3+h)^2|h|^3}{h} = 0,$$

and, similarly, the right-hand side derivative at the point 1 is also 0. This means that $f'(1) = 0$.

(iii) The function f' is continuous at the whole set $\mathbf{R}$. In particular, it holds

$$\lim_{x \to 1} f'(x) = \lim_{x \to 1}(x+2)(5x^2 - x - 4)|x-1| = 0 = f'(1).$$

d) **(i)** The continuity of f on $\mathbf{R}$ follows from the following statement.

If a function $f : A \subset \mathbf{R} \to \mathbf{R}$ is continuous on its domain, then so is the function $|f|$ given by $|f|(x) := |f(x)|, \ x \in A$.

We suggest to the reader to prove this theorem, and also to find a counterexample that will show that the opposite is not true in general.

(ii) As in c), we have

$$f'(x) = 3\cos^2 x(-\sin x) \cdot \text{sgn}(\cos x) = -\frac{3}{2}\sin(2x)|\cos x|,$$

for every $x \in \mathbf{R}$.

(iii) The function f' is continuous on $\mathbf{R}$.

e) **(i)** In Example 5.15 d) we showed that f is continuous on $\mathbf{R}$.

(ii) and **(iii)** For $x \neq 0$ it holds

$$f'(x) = \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}.$$

However, since the limit

$$\lim_{h\to 0}\frac{h\sin\frac{1}{h} - 0}{h} = \lim_{h\to 0}\sin\frac{1}{h}$$

does not exist, f has no first derivative at the point 0.

(iii) The first derivative is continuous on the set $\mathbf{R}\setminus\{0\}$.

f) **(i)** The function f is continuous on $\mathbf{R}$.

(ii) It holds for $x \neq 0$

$$f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x},$$

while for $x = 0$ we have

$$f'(0) = \lim_{h\to 0}\frac{h^2\sin\frac{1}{h} - 0}{h} = \lim_{h\to 0} h\sin\frac{1}{h} = 0, \tag{6.12}$$

which means that f' exists on the whole set $\mathbf{R}$.

(iii) The function f' is continuous on $\mathbf{R}\setminus\{0\}$. Since the limit

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0}\left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)$$

does not exist, the function f' has a second order discontinuity at the point 0.

Remark. The function f' obtained as a derivative of some function f, can never have first order discontinuities, although, as we just saw, it might have second order discontinuities.

g) Similarly as in e) and f), it follows that f is continuous for $\alpha > 0$ on $\mathbf{R}$, the derivative f' exists for $\alpha > 1$ on $\mathbf{R}$, and, finally, f' is continuous on $\mathbf{R}$ for $\alpha > 2$.

Example 6.15. *Find the largest sets on which the first derivatives of the following functions exist.*

$$\text{a)}\quad f(x) = \begin{cases} 2-x, & x<2; \\ (2-x)(3-x), & 2 \le x \le 3; \\ -(3-x), & x>3; \end{cases}$$

$$\text{b)}\quad f(x) = \begin{cases} \arctan x, & |x| \le 1; \\ \dfrac{\pi}{4}\operatorname{sgn} x + \dfrac{|x|-1}{2}, & |x|>1; \end{cases}$$

$$\text{c)}\quad f(x) = \begin{cases} x^3+1, & x \le 0; \\ e^{-1/x}+1, & x>0; \end{cases}$$

$$\text{d)}\quad f(x) = \begin{cases} x^2\left|\sin\dfrac{\pi}{x}\right|, & x \ne 0; \\ 0, & x=0; \end{cases}$$

$$\text{e)}\quad f(x) = \begin{cases} x, & x \in \mathbf{Q} \\ 0, & x \in \mathbf{R}\setminus\mathbf{Q}. \end{cases}$$

Solutions.

a) From the limits

$$\lim_{x\to 2-} f(x) = \lim_{x\to 2+} f(x) = 0 = f(2)$$

and

$$\lim_{x\to 3-} f(x) = \lim_{x\to 3+} f(x) = 0 = f(3)$$

it follows that f is continuous at the points 2 and 3 respectively, thus it is continuous on $\mathbf{R}$. Let us see what happens with the first derivatives of f at the points 2 and 3. We have

$$f'_-(2) = \lim_{h\to 0-}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0-}\frac{2-(h+2)-0}{h} = -1;$$

$$f'_+(2) = \lim_{h\to 0+}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0+}\frac{(2-(2+h))(3-(2+h))-0)}{h} = -1,$$

implying that $f'(2)$ exists and equals to -1. Similarly we prove that

$$f'_-(3) = f'_+(3) = 1,$$

hence $f'(3)=1$. Thus the function f' (the first derivative of f) exists on the whole $\mathbf{R}$ and it holds

$$f'(x) = \begin{cases} -1, & x \le 2; \\ 2x-5, & 2<x<3; \\ 1, & x \ge 3. \end{cases}$$

b) We leave to the reader to check that f is continuous on $\mathbf{R}$.

Clearly, the derivatives of f at the points -1 and 1 are to be examined. We have

$$\begin{aligned}
f'_+(-1) &= \lim_{h\to 0+} \frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0+} \frac{\arctan(-1+h)+\frac{\pi}{4}}{h} \\
&= \lim_{h\to 0+} \frac{1}{h} \arctan \frac{-1+h+1}{1-(-1+h)} = \frac{1}{2}, \\
f'_-(-1) &= \lim_{h\to 0-} \frac{f(-1+h)-f(-1)}{h} \\
&= \lim_{h\to 0-} \frac{\frac{\pi}{4}\operatorname{sgn}(-1+h) + \frac{|-1+h|-1}{2} - (-\frac{\pi}{4})}{h} = -\frac{1}{2},
\end{aligned}$$

hence f has no first derivative at -1. In a similar way we obtain

$$f'_-(1) = f'_+(1) = \frac{1}{2},$$

hence $f'(1)$ exists and equals $\frac{1}{2}$.

So we get

$$f'(x) = \begin{cases} \dfrac{1}{1+x^2}, & -1 < x \le 1; \\[2ex] -\dfrac{1}{2}, & x < -1; \\[2ex] \dfrac{1}{2}, & x > 1. \end{cases}$$

c) For later purposes, let us show first that

$$\lim_{x\to 0+} \frac{e^{-1/x}}{x^m} = 0$$

for every $m = 0, 1, \ldots$. Putting $t := \frac{1}{x}$ gives that $t \to +\infty$ when $x \to 0+$. Hence

$$\lim_{x\to 0+} \frac{e^{-1/x}}{x^m} = \lim_{t\to +\infty} t^m e^{-t} = 0. \tag{6.13}$$

(Note that the easiest way to prove the last equality is to apply m–times the L'Hospital's rule, see Section 6.4.)

On the set $\mathbf{R} \setminus \{0\}$ the function f is continuous. From (6.13) for $m = 0$ it follows that

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} e^{-1/x} + 1 = 1,$$

and since $\lim\limits_{x\to 0-} f(x) = 1 = f(0)$, it follows that f is continuous also at the point $x = 0$.

Let us find the right-hand side first derivative of f at zero

$$f'_+(0) = \lim_{h\to 0+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0+} \frac{(\mathrm{e}^{-1/h}+1)-1}{h} = \lim_{h\to 0+} \frac{\mathrm{e}^{-1/h}}{h} = 0,$$

where we used (6.13) for $m = 1$. Since also

$$f'_-(0) = 0,$$

f has a first derivative in 0. Thus we obtain

$$f'(x) = \begin{cases} 3x^2, & x \le 0; \\ \dfrac{\mathrm{e}^{-1/x}}{x^2}, & x > 0. \end{cases}$$

Notice that f' is a continuous function on $\mathbf{R}$.

d) It is easy to show that f is continuous on $\mathbf{R}$ (do that). Its first derivative is

$$f'(x) = \begin{cases} 2x\left|\sin\dfrac{\pi}{x}\right| - \pi\cos\dfrac{\pi}{x}\,\mathrm{sgn}\left(\sin\dfrac{\pi}{x}\right), & x \ne 0,\quad x \ne 1/k,\ k \in \mathbf{Z}\setminus\{0\}; \\ 0, & x = 0. \end{cases}$$

At the points $x = 1/k$, $k \in \mathbf{Z}\setminus\{0\}$, f has no first derivative.

e) This function is continuous at the point 0, since it holds $|f(x)| \le |x|$ for every $x \in \mathbf{R}$. In other real points f is not continuous (one can use the method from Example 5.10 b)).

Clearly, the only point where f might be differentiable is the point 0. In fact, we shall prove that f is not differentiable at 0. Let us examine the quotient

$$Q(h) := \frac{f(h)-f(0)}{h} = \frac{f(h)}{h}.$$

We have

$$Q(h) = \begin{cases} 1, & h \in \mathbf{Q}; \\ 0, & h \in \mathbf{R}\setminus\mathbf{Q}. \end{cases}$$

Hence

$$f'(0) = \lim_{h\to 0} Q(h)$$

does not exist. This means that the set on which f' exists is empty.

Exercise 6.16. *Assume that the domain A of the function f is a symmetric neighbourhood of zero. Then*

- *If f is an odd continuous function, then it holds $f(0) = 0$.*
- *The first derivative of an odd (resp. even) function is even (resp. odd).*

Example 6.17. *Find the first derivative of the inverse function f^{-1} for the given function f, where*

a) $f(x) = 2x + 1,\ x \in \mathbf{R};$

b) $f(x) = \sqrt{x} + 2,\ x > 0;$

c) $f(x) = x^2 - 2x,\ x > -1;$

d) $f(x) = \cos x,\ x \in (0, \pi);$

e) $f(x) = \sinh x,\ x \in \mathbf{R};$

f) $f(x) = \cosh x,\ x > 0.$

Solutions.

a)
- **First method.** Putting $y = f(x) = 2x + 1$, it follows $x = \dfrac{y-1}{2}$. Thus the function $f^{-1}(x) = \dfrac{x-1}{2}$, $x \in \mathbf{R}$, is the inverse function for the given function f. The first derivative of f^{-1} is

$$(f^{-1})'(x) = \frac{1}{2}, \quad x \in \mathbf{R}.$$

- **Second method.** Since $f'(x) = 2$ and $f'(f^{-1}(x)) = 2$ from the formula (6.3) it follows

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{2}, \quad x \in \mathbf{R}.$$

b) In this case we shall use the Second method from a). To that end, we have

$$f'(x) = \frac{1}{2\sqrt{x}}, \quad x > 0 \quad \text{and} \quad f^{-1}(x) = (x-2)^2,$$

hence the first derivative of the inverse function f^{-1} to the given function f is

$$(f^{-1})'(x) = 2\sqrt{f^{-1}(x)} = 2(x-2), \quad x > 2.$$

c) From the equation $x = f(y) = y^2 - 2y$, it follows $f'(y) = 2y - 2$. Thus the first derivative of the inverse function f^{-1} to f in variable y is

$$(f^{-1})'(y) = \frac{1}{2y-2} = \frac{1}{2(y-1)}.$$

Since $(y-1)^2 = x + 1$, we get finally

$$(f^{-1})'(x) = \frac{1}{2\sqrt{x+1}}, \quad x > -1.$$

d) From $f'(x) = -\sin x, \;\; x \in (0, \pi)$, it follows

$$\begin{aligned}(f^{-1})'(x) &= (\arccos x)' = -\frac{1}{\sin(f^{-1}(x))} = -\frac{1}{\sin(\arccos x)} \\ &= -\frac{1}{\sqrt{1-\cos^2(\arccos x)}} = -\frac{1}{\sqrt{1-x^2}}, \quad x \in (-1,1).\end{aligned}$$

e) From $f'(x) = \cosh x$ it follows for $x \in \mathbf{R}$

$$\begin{aligned}(f^{-1})' = (\text{arcsinh} x)' &= \frac{1}{\cosh(f^{-1}(x))} = \frac{1}{\cosh(\text{arcsinh } x)} \\ &= \frac{1}{\sqrt{1+\sinh^2(\text{arcsinh } x)}} = \frac{1}{\sqrt{1+x^2}}.\end{aligned}$$

f) $(\text{arccosh } x)' = \dfrac{1}{\sqrt{x^2-1}}, \quad x > 1.$

Example 6.18. *Find the domains and the first derivatives of the inverse functions of the following functions.*

$$\textbf{a)}\;\; f(x) = x + \ln x, \; x > 0; \qquad \textbf{b)}\;\; f(x) = \frac{x^2}{1+x^2}, \; x < 0.$$

Solutions.

a) For $x > 0$ it holds $f'(x) = y'_x = \dfrac{x+1}{x} > 0$, hence there exists a unique function f^{-1} whose domain B is the range of f, and is inverse to f. Clearly, $B = \mathbf{R}$; unfortunately, it is impossible to find the analytic formula for f^{-1} explicitly. Still, we shall find its first derivative. Namely by relation (6.3) it follows:

$$(f^{-1})'(y) = \frac{1}{y'_x} = \frac{1}{\dfrac{x+1}{x}} = \frac{x}{x+1},$$

where $x = x(y)$ is the solution in x of the equation $y = x + \ln x$.

b) Since for $x < 0$ it holds $f'(x) = \dfrac{2x}{(1+x^2)^2}$, it follows that there exists a unique inverse function f^{-1} to f with domain $(0,1) = f\left((-\infty, 0)\right)$. Its derivative is

$$(f^{-1})'(y) = \frac{(1+x^2)^2}{2x} = \frac{x^3}{2y^2},$$

where $x = x(y)$ in the solution in x of the equation $y = \dfrac{x^2}{1+x^2}$.

Example 6.19. *Prove that there is but one differentiable function on* $\mathbf{R}$ *such that*

$$y^3 + 2y = x, \quad y(3) = 1, \tag{6.14}$$

and find its derivative y'_x.

Solution. The existence of the function $y = y(x)$ follows from the Implicit function theorem, which is somewhat out of the scope of this book. However, in this case we can prove its existence in the following way.
The function

$$F(x,y) = y^3 + 2y - x$$

has all possible partial derivatives.

- We differentiate the equation $F(x,y) = 0$, with respect to y, giving us

  $$x'_y = 2 + 3y^2 > 0.$$

  Notice that we observed x as a function of y. Now the function $x = x(y)$ is monotone for $y \in \mathbf{R}$. Hence the inverse function of $x = x(y)$ exists and is also differentiable on $\mathbf{R}$. Then it holds (see (6.3))

  $$y'_x = \frac{1}{x'_y} = \frac{1}{2+3y^2}.$$

  Let us assume that there are two solutions of equation (6.14) and let us denote them by $y_1(x)$ and $y_2(x)$. Then it holds

  $$y_1^3 + 2y_1 = x \quad \text{and} \quad y_2^3 + 2y_2 = x.$$

  This implies

  $$y_1^3 - y_2^3 + 2(y_1 - y_2) = 0 \Rightarrow (y_1 - y_2)(y_1^2 + y_1y_2 + y_2^2 + 2) = 0.$$

  Since $y_1^2 + y_1y_2 + y_2^2 + 2 > 0$ for any y_1 and y_2, it follows that $y_1(x) = y_2(x)$ for all $x \in \mathbf{R}$.

- The other method of finding y'_x was explained in the introduction.
  Let us differentiate in x the equation (6.14), assuming that $y = y(x)$. Then we obtain

  $$3y^2 \cdot y'_x + 2 \cdot y'_x = 1.$$

  Solving by y'_x, we again obtain

  $$y'_x = \frac{1}{2+3y^2}.$$

Example 6.20. *Find the first derivatives of the following functions $y = y(x)$ given implicitly.*

a) $x^2 + y^2 = 4;$ b) $2x - 3y + 3 = x^2 + 2y - 6x;$

c) $\sqrt{x} + \sqrt{y} = 5x;$ d) $x^4 + 4x^2y^2 - 3xy^3 + 3x = 0;$

e) $(y^2 - 9)^3 = (2x^3 + 3x - 1)^2;$ f) $(2 + xy)^2 = 3x^2 - 7.$

Solutions.

a) Observe that $(y^2)'_x = 2yy'$, hence differentiating the given equation by x it follows:

$$2x + 2yy' = 0 \Rightarrow y' = -x/y.$$

b) From the equation $2 - 3y' = 2x + 2y' - 6$ it follows $y' = \dfrac{-2x + 8}{5}.$

c) It holds that $\dfrac{1}{2\sqrt{x}} + \dfrac{y'}{2\sqrt{y}} = 5$, hence

$$y' = 10\sqrt{y} - \frac{\sqrt{y}}{\sqrt{x}}.$$

d) Firstly we have

$$4x^3 + 8xy^2 + 8x^2yy' - 3y^3 - 9xy^2y' + 3 = 0,$$

which implies

$$y' = \frac{-4x^3 - 8xy^2 + 3y^3 - 3}{8x^2y - 9xy^2}.$$

e) In this case we have

$$6(y^2 - 9)^2yy' = 2(2x^3 + 3x - 1)(6x^2 + 3),$$

which implies

$$y' = \frac{(2x^3 + 3x - 1)(2x^2 + 1)}{y(y^2 - 9)^2}.$$

f) Since $2(2 + xy)(y + xy') = 6x$, it follows

$$y' = \frac{3x - 2y - xy^2}{x(2 + xy)}.$$

Example 6.21. *Find the first derivatives of the following parametric functions.*

a) $x = t^2 + 2t,\ y = 2t^3 - 6t,\quad t \in \mathbf{R};$

b) $x = 2(t - \sin t),\ y = 2(1 - \cos t),\quad t \in \mathbf{R};$

c) $x = 2\cos^3 t,\ y = \sin^3 t,\quad t \in \left(0, \dfrac{\pi}{2}\right).$

Solutions.

a) Since it holds $x'_t = 2t+2$, $y'_t = 6t^2 - 6$, it follows from relation (6.6)

$$y'_t = \frac{6t^2-6}{2t+2} = \frac{3t^2-3}{t+1}, \quad t \neq -1.$$

b) From the relations $x'_t = 2(1-\cos t)$, $y'_t = 2\sin t$, $t \in \mathbf{R}$, it follows

$$y' = \frac{\sin t}{(1-\cos t)}, \quad t \in \mathbf{R} \setminus \{2k\pi |\ k \in \mathbf{Z}\}.$$

c) We have $x'_t = 6\cos^2 t(-\sin t)$, $y'_t = 3\sin^2 t \cos t$, thus

$$y' = \frac{3\sin^2 t\cos t}{-6\cos^2 t\sin t} = \frac{-1}{2\cot t}, \quad t \in (0, \pi/2).$$

Example 6.22. *Find the differentials of the following functions given below.*

a) $f(x) = 2\sqrt{\cos\dfrac{1}{x}} + \ln(x^2+1);$

b) $f(x) = 5\mathrm{arctg}(2x+7)\mathrm{e}^{3x+1} + 12^x;$

c) $f(x) = \dfrac{\sin x + \mathrm{tg} x}{x^3 + 3^x + 3x};$

d) $y = \arcsin(5x^4+2) + \dfrac{5^x + 4x^2}{\exp(\sin x + \cos x)}.$

Solutions.

a) Since the first derivative of f is

$$f'(x) = \frac{\sin\dfrac{1}{x}}{x^2\sqrt{\cos\dfrac{1}{x}}} + \frac{2x}{1+x^2},$$

it follows (see relation (6.7)) that the differential of f at the point x from the domain of f is

$$df(x) = f'(x)\,dx = \left(\frac{\sin\dfrac{1}{x}}{x^2\sqrt{\cos\dfrac{1}{x}}} + \frac{2x}{1+x^2}\right) dx.$$

b) Using relation (6.7), it follows

$$df(x) = \left(5 \cdot \frac{2}{1+(2x+7)^2} e^{3x+1} + 15 \cdot \operatorname{arctg}(2x+7) e^{3x+1} + 12^x \ln 12\right) dx.$$

c) It holds

$$df(x) = \frac{(\cos x + 1/\cos^2 x)(x^3 + 3^x + 3x) - (\sin x + \operatorname{tg} x)(3x^2 + 3^x \ln 3 + 3)}{(x^3 + 3^x + 3x)^2}\, dx.$$

d) The differential of f at the point x from the domain of f is

$$df(x) = \left(\frac{20x^3}{\sqrt{1-(5x^4+2)^2}} + \frac{(5^x \ln 5 + 8x) - (5^x + 4x^2)(\cos x - \sin x)}{\exp(\sin x + \cos x)}\right) dx.$$

Example 6.23. *If* $f(x) = \sqrt[m]{u^2+v^2}$, *where* $m \in \mathbf{N}$, $m > 1$, *while* $u = u(x)$ *and* $v = v(x)$ *are differentiable functions of the variable* x, *find the differential* $df(x)$.

Solution. Using relation (6.7), we have

$$\begin{aligned} df(x) &= d\left(\sqrt[m]{u^2+v^2}\right) = \frac{1}{m}(u^2+v^2)^{(1-m)/m} d(u^2+v^2) \\ &= \frac{2}{m}(u^2+v^2)^{(1-m)/m}(u\,du + v\,dv). \end{aligned}$$

Example 6.24. *Find the approximate value of the number* A, *whose exact value is*

a) $A = \sqrt{4.0003}$; **b)** $A = \ln(1.001)$; **c)** $A = \sqrt[3]{1.0003}$.

Solutions.

a) For the function $f(x) = \sqrt{x}$, $x \geq 0$, its first derivative is $f'(x) = \dfrac{1}{2\sqrt{x}}$, $x > 0$, hence for $x = 4$, $h = 0.0003$, it follows from formula (6.8)

$$\sqrt{4.0003} \approx \sqrt{4} + \frac{1}{2\sqrt{4}} 0.0003 = 2 + \frac{0.0003}{4} = 2.000075\,.$$

b) Let us put $h = 0.001$, $x = 1$; then it holds

$$\ln(1.001) \approx \ln 1 + \frac{1}{1} 0.001 = 0.001\,.$$

c) In this case we have $h = 0.0003$, $x = 1$, hence

$$\sqrt[3]{1.0003} \approx \sqrt[3]{1} + \frac{1}{3\sqrt[3]{1}} 0.0003 = 1.0001\,.$$

Example 6.25. *Prove the approximate formula*

$$\sqrt[n]{a^n + x} \approx a + \frac{x}{n\, a^{n-1}}, \tag{6.15}$$

where $a > 0$, $n \in \mathbf{N}$, *while the number* x *satisfies* $|x| \ll a^n$ *(i.e.,* $|x|$ *is much smaller than* a^n*).*

Solution. If we put $f(x) := \sqrt[n]{a^n + x}$, then we have

$$f'(0) = \frac{1}{n}(a^n + x)^{(1-n)/n}\Big|_{x=0} = \frac{1}{n\, a^{n-1}}.$$

Since f is differentiable at 0, it follows

$$f(x) - f(0) = \frac{x}{n\, a^{n-1}} + x \cdot r(x), \tag{6.16}$$

where the remainder r satisfies $\lim\limits_{x\to 0} r(x) = 0$, see Definition 6.2 and Theorem 6.3. Thus neglecting the addend $x \cdot r(x)$ in (6.16), we obtain the approximate formula (6.15), provided that $|x|$ is much smaller than a^n.

Exercise 6.26. *Prove the following approximate formulas for "small"* $|x|$.

a) $\dfrac{1}{\sqrt[3]{1-x}} \approx 1 + \dfrac{x}{3}$; **b)** $\sin x \approx x$; **c)** $\cos x \approx 1$.

Example 6.27. *Assume that the functions* $u = u(x)$ *and* $v = v(x)$ *are* n *times differentiable on an interval* (a, b)*, where* $n \in \mathbf{N}$. *Then using the usual convention* $u^{(0)}(x) := u(x)$*, the following, so-called* **Leibniz formula**, *holds:*

$$(u(x) \cdot v(x))^{(n)} = \sum_{j=0}^{n} \binom{n}{j} u^{(j)}(x) v^{(n-j)}(x), \quad x \in (a, b). \tag{6.17}$$

Solution. For $n = 1$ the relation (6.17) reduces to the well known formula of the product of differentiable functions.

Assume that relation (6.17) is true for $n = k$; we have to show that then it is true for $n = k + 1$. It holds for $x \in (a, b)$

$$(u(x) \cdot v(x))^{(k+1)} = \left((u(x) \cdot v(x))^{(k)}\right)' = \left(\sum_{j=0}^{k} \binom{k}{j} u^{(j)}(x) v^{(k-j)}(x)\right)'$$

$$= \sum_{j=0}^{k} \binom{k}{j} u^{(j+1)}(x) v^{(k-j)}(x) + \sum_{j=0}^{k} \binom{k}{j} u^{(j)}(x) v^{(k-j+1)}(x)$$

$$= \binom{k}{0} u'(x) v^{(k)}(x) + \binom{k}{1} u''(x) v^{(k-1)}(x) + \cdots + \binom{k}{k-1} u^{(k)}(x) v'(x) + \binom{k}{k} u^{(k+1)}(x) v(x)$$

$$+ \binom{k}{0} u(x) v^{(k+1)}(x) + \binom{k}{1} u'(x) v^{(k)}(x) + \cdots + \binom{k}{k-1} u^{(k-1)}(x) v''(x) + \binom{k}{k} u^{(k)}(x) v'(x)$$

$$= \sum_{j=0}^{k+1} \binom{k+1}{j} u^{(j)}(x) v^{(k+1-j)}(x).$$

Thus we obtained relation (6.17) for $n = k+1$, once we assumed its correctness for $n = k$. By the principle of the mathematical induction, it follows that (6.17) is correct for all $n \in \mathbf{N}$.

The reader should note that we used the equality

$$\binom{k}{j} + \binom{k}{j+1} = \binom{k+1}{j+1},$$

which is true for all $k \in \mathbf{N}$ and $0 \le j \le k-1$, see Example 1.28 a).

Example 6.28. *Assume that the function f is three times differentiable on its domain. Find g'' and g''', if the function g is given by:*

a) $g(x) = f(x^2)$; **b)** $g(x) = f(1/x)$;

c) $g(x) = f(\mathrm{e}^x)$; **d)** $g(x) = f(\ln x)$.

Solutions.

a) Using the rule for the derivative of the composite function, it follows that

$$g'(x) = 2xf'(x^2), \qquad g''(x) = 4x^2 f''(x^2) + 2f'(x^2),$$
$$g'''(x) = 8x^3 f'''(x^2) + 12x f''(x^2).$$

b) We have

$$g'(x) = -\frac{1}{x^2} f'(1/x), \qquad g''(x) = \frac{2}{x^3} f'(1/x) + \frac{1}{x^4} f''(1/x),$$
$$g'''(x) = -\frac{1}{x^6} f'''(1/x) - \frac{6}{x^5} f''(1/x) - \frac{6}{x^4} f'(1/x).$$

c) $g''(x) = \mathrm{e}^{2x} f''(\mathrm{e}^x) + \mathrm{e}^x f'(\mathrm{e}^x)$,

$$g'''(x) = \mathrm{e}^{3x} f'''(\mathrm{e}^x) + 3\mathrm{e}^{2x} f''(\mathrm{e}^x) + \mathrm{e}^x f'(\mathrm{e}^x).$$

d) $g''(x) = \dfrac{1}{x^2}\left(f''(\ln x) - f'(\ln x)\right)$,

$$g'''(x) = \frac{1}{x^3}\left(f'''(\ln x) - 3f''(\ln x) + 2f'(\ln x)\right).$$

Exercise 6.29. *Prove that the coefficients a_j, $0 \le j \le n$, of the polynomial*

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad x \in \mathbf{R},$$

satisfy the following condition:

$$a_j = \frac{P_n^{(j)}(0)}{j!}, \quad 0 \le j \le n. \tag{6.18}$$

Hint. Putting $x = 0$ in the equality

$P_n^{(j)}(x) = a_n n(n-1)\cdots(n-j+1)x^{n-j} + a_{n-1}(n-1)(n-2)\cdots(n-j)x^{n-j-1} + \cdots + a_j j!,$

we obtain (6.18) for $0 < j \le n$.

Example 6.30. *Find the j–th derivative ($j \in \mathbf{N}$) of the functions given below.*

a) $f(x) = \mathrm{e}^x, \ \ x \in \mathbf{R};$ **b)** $f(x) = 2^x, \ \ x \in \mathbf{R};$

c) $f(x) = \sin x, \ \ x \in \mathbf{R};$ **d)** $f(x) = \cos x, \ \ x \in \mathbf{R};$

e) $f(x) = \dfrac{1}{1-x}, \ |x| < 1;$ **f)** $f(x) = \ln(1+x); \ |x| < 1.$

In particular, find $f^{(j)}(0)$ for $j \in \mathbf{N}$.

Solutions.

a) Since $f'(x) = \mathrm{e}^x$ and $f''(x) = \mathrm{e}^x$ for every $x \in \mathbf{R}$ and using the mathematical induction on $j \in \mathbf{N}$ it follows that for all $j \in \mathbf{N}$, $f^{(j)}(x) = \mathrm{e}^x$. Hence $f^{(j)}(0) = 1$, for all $j \in \mathbf{N}$.

b) Similarly as in a), we have $f^{(j)}(x) = 2^x \ln^j 2$, for every $x \in \mathbf{R}$ and $j \in \mathbf{N}$. Thus $f^{(j)}(0) = \ln^j 2, \ j \in \mathbf{N}$.

c) If $f(x) = \sin x, \ x \in \mathbf{R}$, then $f'(x) = \cos x, \ x \in \mathbf{R}, \ f''(x) = -\sin x, \ x \in \mathbf{R}$, $f'''(x) = -\cos x, \ x \in \mathbf{R}$ and $f^{(4)}(x) = \sin x, \ x \in \mathbf{R}$. Since

$$f(x) = f^{(4)}(x) = \sin x, \ x \in \mathbf{R},$$

it follows for every $x \in \mathbf{R}$ and every $j \in \mathbf{N_0}$ that

$$f^{(4j)}(x) = \sin x, \qquad f^{(4j+1)}(x) = \cos x,$$

$$f^{(4j+2)}(x) = -\sin x, \qquad f^{(4j+3)}(x) = -\cos x,$$

or we can write $f^{(n)}(x) = \sin\left(x + \dfrac{n\pi}{2}\right)$ $n \in \mathbf{N}$. Thus we have for $j \in \mathbf{N_0}$

$$f^{(4j)}(0) = f^{(4j+2)}(0) = 0, \qquad f^{(4j+1)}(x) = 1, \quad f^{(4j+3)}(x) = -1.$$

d) Analogously as in c) we have for every $x \in \mathbf{R}$ and every $j \in \mathbf{N_0}$

$$f^{(4j)}(x) = \cos x, \qquad f^{(4j+1)}(x) = -\sin x,$$

$$f^{(4j+2)}(x) = -\cos x, \qquad f^{(4j+3)}(x) = \sin x,$$

or we can write $f^{(n)}(x) = \cos\left(x + \dfrac{n\pi}{2}\right)$ $n \in \mathbf{N}$. In particular,

$$f^{(4j)}(0) = 1, \quad f^{(4j+2)}(0) = -1, \qquad f^{(4j+1)}(0) = f^{(4j+3)}(0) = 0.$$

e) We leave to the reader to check that for every $j \in \mathbf{N}$, it holds

$$f^{(j)}(x) = \frac{j!}{(1-x)^{j+1}}, \quad |x| < 1.$$

Hence for every $j \in \mathbf{N}$ it holds $f^{(j)}(0) = j!$.

f) The first and the second derivative of the function $f(x) = \ln(1+x)$, $|x| < 1$, are

$$f'(x) = \frac{1}{1+x}, \ |x| < 1, \quad \text{and} \quad f''(x) = \frac{-1}{(1+x)^2}, \ |x| < 1.$$

By the mathematical induction it follows then for every $j \in \mathbf{N}$

$$f^{(j)}(x) = \frac{(-1)^{j-1}(j-1)!}{(1+x)^j}, \ |x| < 1$$

(check!) and thus $f^{(j)}(0) = (-1)^{j-1}(j-1)!$.

Example 6.31. *Find the n–th derivative of the following functions.*

a) $f(x) = \dfrac{1}{x^2 - 3x - 2}, \ x \notin \{1, 2\}$; **b)** $f(x) = \sin^2 x, \ x \in \mathbf{R}$;

c) $f(x) = \sin^4 x + \cos^4 x, \ x \in \mathbf{R}$; **d)** $f(x) = \ln \dfrac{a+bx}{a-bx}, \ a^2 - b^2x^2 > 0$.

Solutions.

a) Since it holds for $x \neq 1$, $x \neq 2$,

$$f(x) = \frac{1}{x^2 - 3x - 2} = \frac{1}{x-2} - \frac{1}{x-1},$$

it follows that

$$f^{(n)}(x) = (-1)^n n! \left(\frac{1}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}} \right).$$

b) Since we have $\cos^2 x = \dfrac{1}{2} + \dfrac{1}{2}\cos 2x$, it follows that

$$f^{(n)}(x) = 2^{n-1}\cos(2x + n\pi/2)$$

Compare to Example 6.30 c), d).

c) Since $f(x) = \dfrac{3}{4} + \dfrac{1}{4}\cos 4x$, it follows for $n \in \mathbf{N}$.

$$f^{(n)}(x) = 4^{n-1}\cos(4x + n\pi/2).$$

d) From $f'(x) = \dfrac{b}{a+bx} + \dfrac{b}{a-bx}$ we get

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!\,b^n}{(a+bx)^n} + \frac{(n-1)!\,b^n}{(a-bx)^n}.$$

Exercise 6.32. *Find the n−th derivative of the following functions.*

a) $f(x) = \dfrac{1}{1+x}, \quad |x| < 1;$ **b)** $f(x) = \ln(1-x), \quad |x| < 1;$

c) $f(x) = \sin^3 x, \ x \in \mathbf{R};$ **d)** $f(x) = \sin^2 2x \cos 3x, \quad x \in \mathbf{R}.$

Answers.

a) $f^{(n)}(x) = \dfrac{(-1)^n n!}{(1+x)^{n+1}}.$ **b)** $f^{(n)}(x) = \dfrac{-(n-1)!}{(1-x)^n}.$

c) $f(x) = \dfrac{3}{4}\sin x - \dfrac{1}{4}\sin 3x, \quad f^{(n)}(x) = \dfrac{3}{4}\sin\left(x + \dfrac{n\pi}{2}\right) - \dfrac{3^n}{4}\sin\left(3x + \dfrac{n\pi}{2}\right).$

d) $f(x) = \dfrac{1-\cos 4x}{2}\cos 3x = \dfrac{\cos 3x}{2} - \dfrac{1}{4}(\cos 7x + \cos x),$

$$f^{(n)}(x) = \frac{3^n}{2}\cos\left(3x + \frac{n\pi}{2}\right) - \frac{1}{4}\left(7^n\cos\left(7x + \frac{n\pi}{2}\right) + \cos\left(x + \frac{n\pi}{2}\right)\right).$$

Example 6.33. *Prove that the function*

$$f(x) = \begin{cases} \mathrm{e}^{-1/x}, & x > 0; \\ 0, & x \le 0. \end{cases}$$

is infinitely differentiable on $\mathbf{R}$*, and, in particular, it holds* $f^{(n)}(0) = 0$ *for all* $n \in \mathbf{N}$.

Solution. Clearly, only the point $x = 0$ is to be discussed. Firstly, f is continuous at 0, because it holds

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \mathrm{e}^{-1/x} = \lim_{t\to+\infty} \mathrm{e}^{-t} = 0$$

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} 0 = 0 = f(0).$$

For $x > 0$ we have

$$f'(x) = \frac{1}{x^2}\mathrm{e}^{-1/x}, \qquad f''(x) = \left(\frac{1}{x^4} - \frac{2}{x^3}\right)\mathrm{e}^{-1/x},$$

$$f'''(x) = \left(\frac{1}{x^6} - \frac{6}{x^5} + \frac{6}{x^4}\right)\mathrm{e}^{-1/x}.$$

We leave to the reader to check that it holds

$$f^{(n)}(x) = Q_{2n}(1/x)\mathrm{e}^{-1/x},$$

where $Q_{2n}(x)$ is a polynomial of degree $2n$ such that $Q_{2n}(0) = 0$. Thus (see Section 6.4) we have

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} Q_{2n}(1/x)\mathrm{e}^{-1/x} = \lim_{t\to +\infty} Q_{2n}(t)\mathrm{e}^{-t} = 0.$$

Since obviously $f^{(n)}(x) = 0$ for all $x < 0$, it follows that

$$\lim_{x\to 0-} f^{(n)}(x) = \lim_{x\to 0-} 0 = 0.$$

Thus the right and the left-hand side limits of $f^{(n)}$ at 0 are both equal to zero.

We still have to prove the existence of $f^{(n)}(0)$. Of course, $f_-^{(n)}(0) = 0$, for all $n \in \mathbf{N}$. Further on we have

$$f'_+(0) = \lim_{h\to 0+} \frac{f(0+h) - f(0)}{h} = \frac{\mathrm{e}^{-1/h}}{h} = 0.$$

If we assume that $f_+^{(n)}(0) = 0$ for some $n \in \mathbf{N}$, then it holds

$$\begin{aligned} f_+^{(n+1)}(0) &= \lim_{h\to 0+} \frac{f^{(n)}(0+h) - f^{(n)}(0)}{h} \\ &= \lim_{h\to 0+} \frac{1}{h} Q_{2n}(1/h)\mathrm{e}^{-1/h} = 0. \end{aligned}$$

Hence by the principle of the mathematical induction it follows that $f_+^{(n)}(0) = 0$ for all $n \in \mathbf{N}$, which implies the existence of $f^{(n)}$ on $\mathbf{R}$.

Remark. This example shows that there exist infinitely differentiable functions whose Maclaurin's polynomial of any degree is identically equal to zero, even though the function is not identically equal to zero.

Example 6.34. *Find the equations of the tangent and perpendicular lines at the given point $T(x_0, f(x_0))$ on the graph of the following functions.*

a) $f(x) = \sqrt{x}$, $x_0 = 4$;

b) $f(x) = \mathrm{e}^{x^2-1}$, $x_0 = 1$;

c) $f(x) = \arctan x^2$, $x_0 = 0$;

d) $f(x) = \arcsin\left(\dfrac{x+2}{2}\right)$, $x_0 = 0$.

Solutions.

a) The equation of a line with slope k passing through the point $T(x_0, y_0)$ has the form

$$y - y_0 = k \cdot (x - x_0).$$

The geometric interpretation of the first derivative at the point x_0 is the tangent line of the graph of f at its point $T(x_0, y_0)$, i.e., $k = f'(x_0)$ (see relation (6.11)) together with $f(x_0) = y_0$.

The first derivative of f at the point x is

$$f'(x) = \frac{1}{2\sqrt{x}},$$

hence putting $x_0 = 4$ gives us $f'(4) = \frac{1}{4}$. The value of f at $x_0 = 4$ is $y_0 = f(4) = 2$. Thus the equation of the sought tangent line has the form

$$y - 2 = \frac{1}{4}(x - 4), \quad \text{or} \quad 4y - x = 4.$$

The equation of the perpendicular line of the graph f at its point $T(x_0, y_0)$ is of the form (see equation (6.10))

$$y - y_0 = k_1 \cdot (x - x_0),$$

where $k_1 = -\frac{1}{k}$, provided, that the slope $k(= f'(x_0))$ is nonzero. Hence $k_1 = -4$, which gives the equation of the perpendicular line at the point $T(4,2)$,

$$y + 4x = 18.$$

b) From the first derivative $f'(x) = 2xe^{x^2-1}$ it follows that the slope k of the tangent line is $k = f'(1) = 2$. Hence the equation of the sought tangent line has the form

$$y - 1 = k(x - 1) \Rightarrow y - 1 = 2(x - 1),$$

or finally

$$y - 2x + 1 = 0.$$

The equation of the perpendicular line of the graph f at its point $T(1,1)$ is of the form

$$2y + x - 3 = 0.$$

c) From $y' = \dfrac{2x}{1 + x^4}$ it follows that $y'(0) = 0$. Thus the tangent line has the equation

$$y - 0 = 0 \cdot (x - 0) \quad \text{hence } y = 0.$$

The sought tangent line is, in fact, the $x-$axis. Clearly, the perpendicular line of the graph f at the point $T(0,0)$ is the $y-$axis, whose equation is $x = 0$.

d) The function f has for its natural domain the interval $[-4, 0)$, the absolute value of the argument of the arcsin function must be less than or equal to one. It is easy to show that f is continuous from the left-hand side at the point 0. However, f has no left sided derivative at the point 0, since

$$\lim_{h \to 0-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0-} \frac{\arcsin\left(\dfrac{h+2}{2}\right) - \arcsin 1}{h}$$

$$= \lim_{t \to \pi/2-} \frac{t - \pi/2}{2(\sin t - 1)} = \lim_{z \to 0-} \frac{z}{2(\cos z - 1)} = \lim_{z \to 0-} \frac{z}{-4(\sin^2(z/2))} = +\infty.$$

This shows that the sought tangent line is the $y-$axis, whose equation is $x = 0$. The equation of the sought perpendicular line at the point $T(0, \frac{\pi}{2})$ is $y = 0$, which is, in fact, the $x-$axis.

Example 6.35. *Determine the parameter k so that the line $y = kx + 1$ becomes the tangent line of the parabola*

$$\{(x, y)|\ y^2 = 4x,\ x \geq 0\},$$

and find their common point.

Solution. The given curve is the union of graphs of two functions, namely of

$$f_1(x) = 2\sqrt{x},\ x \geq 0, \quad \text{and} \quad f_2(x) = -2\sqrt{x},\ x \geq 0.$$

Let us denote by $T(x_0, y_0)$, $x_0 > 0$, the common point of the tangent line and the parabola. Then

$$y_0 = \pm 2\sqrt{x_0} \ \text{ and } \ k = \pm\frac{1}{\sqrt{x_0}}.$$

This gives

$$y_0 = kx_0 + 1 \Rightarrow 2\sqrt{x_0} = \frac{1}{\sqrt{x_0}}x_0 + 1.$$

So we obtain $x_0 = 1$ and its common point with the parabola and the graph of f_1 is $T(1, 2)$. The equation of the sought tangent line is $y = x + 1$.

Notice that there is no tangent line to the graph of f_2, since

$$-2\sqrt{x_0} = -\frac{1}{\sqrt{x_0}}x_0 + 1 \Rightarrow \sqrt{x_0} = -1 < 0,$$

a contradiction. In fact, at the point $x = 0$, the graph of the function f_2 and also the given parabola have a vertical tangent line.

Example 6.36. *Find the tangent lines to the parabola $f(x) = x^2 - 3x + 1$ through the point $A(2, -2)$ and determine the common point of each tangent line and the parabola.*

Solution. Let us denote by $(x_0, f(x_0))$ the common point of the tangent line and the given parabola. The slope of the tangent line is $k = f'(x_0) = 2x_0 - 3$, while its equation is

$$y - f(x_0) = k(x - x_0), \quad \text{or}$$
$$-2 - (x_0^2 - 3x_0 + 1) = (2x_0 - 3)(2 - x_0).$$

This implies the quadratic equation

$$x_0^2 - 4x_0 + 3 = 0, \quad \text{whose solutions are} \quad x_0' = 1, \quad x_0'' = 3.$$

Thus the common points are $B(1, -1)$ and $C(3, 1)$, and the slopes of tangent lines through the point A are $k_1 = -1$ and $k_2 = 3$. The equations of the tangent lines are

$$y + x = 0 \quad \text{and} \quad y = 3x - 8.$$

Example 6.37. *The parametric representation of the central ellipse is*

$$x = a\cos t, \quad y = b\sin t, \quad 0 \le t \le 2\pi,$$

where $a > 0$ and $b > 0$ are the so-called half-axes of the ellipse. Prove that the area of the triangle AOB is not less than the product ab, where A and B are those points on the $x-$ and $y-$axis in which the tangent line at an arbitrary point (x, y) of the ellipsis intersects the $x-$ and the $y-$axis respectively, while $O(0,0)$ is the origin.

Solution. Since the ellipse is symmetric both to $x-$ and to the $y-$axis, it is enough to analyze the case $0 < t < \pi/2$; clearly in our case the values $t = 0$ and $t = \pi/2$ have no geometric meaning. The tangent line of the ellipse at the point $(x(t), y(t))$ has the equation

$$y - b\sin t = \left(-\frac{b}{a}\cot t\right)(x - a\cos t).$$

The intersection of this line with the coordinate axes are the points $A\left(\dfrac{a}{\cos t}, 0\right)$ and $B\left(0, \dfrac{b}{\sin t}\right)$. Hence the area $\mathcal{A}_{\triangle AOB}$ of the triangle AOB satisfies

$$\mathcal{A}_{\triangle AOB} = \frac{1}{2}\frac{a}{\cos t}\frac{b}{\sin t} = \frac{ab}{2\sin 2t} \ge ab.$$

6.2 Mean value theorems

6.2.1 Basic notions

Theorem 6.38. Rolle's Theorem.
If a function $f : [a, b] \to \mathbf{R}$ is

- *continuous on a closed interval $[a, b]$,*
- *differentiable on the open interval (a, b),*
- *$f(a) = f(b)$,*

then there exists at least one number $\xi \in (a, b)$ such that $f'(\xi) = 0$.

Theorem 6.39. The Lagrange Theorem.
If a function $f : [a, b] \to \mathbf{R}$ is

- *continuous on a closed interval $[a, b]$,*
- *differentiable on the open interval (a, b),*

then there exists at least one number $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}. \tag{6.19}$$

Geometrically, Theorem 6.38 (resp. 6.39) means that the tangent line at the point ξ is parallel to $x-$axis (resp. to the line $y - f(a) = \dfrac{f(b) - f(a)}{b - a}(x - a)$).

6.2.2 Examples and exercises

Example 6.40. *Show that the function $f(x) = x(x-1)(x-2)$ satisfies the conditions of Rolle's theorem on the intervals $[0,1]$, $[1,2]$ and $[0,2]$ and determine the corresponding values of ξ.*

Solution. The given function is polynomial, hence it is continuous and has first derivatives at every point of these intervals. Since it holds $f(0) = f(1) = f(2) = 0$, the function f satisfies the conditions of Rolle's theorem on all three intervals $[0,1]$, $[1,2]$ and $[0,2]$.

We have $f'(x) = 3x^2 - 6x + 2$, and

$$f'(x) = 0 \quad \text{for} \quad x_{1,2} = 1 \pm \frac{\sqrt{3}}{3}.$$

So the points in which the first derivative of the considered function is equal to zero are

$$\xi_1 = 1 - \frac{\sqrt{3}}{3} \in [0,1], \quad \text{and} \quad \xi_2 = 1 + \frac{\sqrt{3}}{3} \in [1,2].$$

On the interval $[0,2]$ the function f' has two real zeros, ξ_1 and ξ_2.

Example 6.41. *On the intervals $(-1,1)$ and $(1,2)$ find the points in which the tangent lines of the graph of the function $f(x) = (x^2-1)(x-2)$ are horizontal.*

Solution. The considered function is continuous and has a first derivative at every point of these two intervals and it holds that $f(-1) = f(1) = f(2) = 0$. Hence f satisfies the conditions of Rolle's theorem on the intervals $[-1,1]$ and $[1,2]$, and also on the interval $[-1,2]$.

From $f'(x) = 3x^2 - 4x - 1$ it follows that $f'(x) = 0$ for

$$\xi_1 = \frac{2-\sqrt{7}}{3} \in [-1,1], \qquad \xi_2 = \frac{2+\sqrt{7}}{3} \in [1,2].$$

The tangent lines of the graph of the function $f(x) = (x^2-1)(x-2)$ in the points ξ_1 and ξ_2 are horizontal ($f'(\xi_1) = f'(\xi_2) = 0$).

Example 6.42. *Check whether the following functions satisfy the conditions of Rolle's theorem.*

a) $f(x) = \dfrac{1-\sqrt[3]{x^2}}{2}$ *on the interval* $[-1,1]$;

b) $f(x) = |x-1|$ *on the interval* $[0,2]$;

c) $f(x) = \displaystyle\sum_{i=1}^{n} \frac{a_i}{i}\sin(ix)$ *on the interval* $[0,\pi]$;

d) $f(x) = \displaystyle\sum_{i=1}^{n} \frac{a_i}{i}\sin(ix)$ *on the interval* $[0,\pi/2]$.

Solutions.

a) No, because the function f has no first derivative at the point $x = 0$, as seen from either of the following one-sided derivatives at 0:

$$\lim_{h\to 0+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0+} \frac{\frac{1}{2} - \frac{\sqrt[3]{h^2}}{2} - \frac{1}{2}}{h} = -\infty;$$

$$\lim_{h\to 0-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0-} \frac{-\sqrt[3]{h^2}}{2h} = +\infty.$$

b) No, because the function f has no first derivative at the point $x = 1$ (see Example 6.13).

c) Yes, and from Rolle's theorem we obtain that the equation

$$\sum_{i=1}^{n} a_i \cos(ix) = 0 \tag{6.20}$$

has a solution on interval $[0, \pi]$. Namely, the first derivative of the considered function is just the left side of the equation (6.20).

d) No, because it is not true that $f(0) = f(\pi/2)$. Namely, the value $f(\pi/2)$ depends on n and on the coefficients a_i, $i = 1, 2, ..., n$.

Example 6.43. *If the polynomial*

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0,$$

with real coefficients has only real roots, then its derivatives, $P_n', P_n'', ..., P_n^{(n-1)}$, *have real roots as well.*

Solution. Let us suppose first that all roots are simple. Then from Rolle's theorem it follows that there exist $n-1$ real roots of the polynomial $P_n'(x)$. Every root of the polynomial $P_n'(x)$ is located between the two roots of $P_n(x)$. Similarly we obtain that the polynomial $P_n''(x)$ has $n-2$ real roots, and so on. Note that the $(n-1)-$th derivative

$$P_n^{(n-1)}(x) = a_n n!\, x + a_{n-1}(n-1)!$$

has one real root while, the $n-$th derivative

$$P_n^{(n)}(x) = a_n n!$$

is a constant.

If a number x_0 is a multiple real root of order $m > 1$ of a polynomial $P_n(x)$, then it is also a root of the polynomial's derivative, as follows immediately from the representation

$$P_n(x) = a_n(x - x_0)^m Q_{n-m}(x).$$

Example 6.44. *Prove that the roots of the Legendre polynomial,*

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left((x^2-1)^n\right), \tag{6.21}$$

are real and located in the interval $(-1,1)$.

Solution. The polynomial $R_n(x) = (x^2-1)^n$ has on the interval $[-1,1]$ exactly $2n$ real roots, namely

$$x_1 = x_2 = \ldots = x_n = 1 \quad \text{and} \quad x_{n+1} = x_{n+2} = \ldots = x_{2n} = -1.$$

Since

$$R_n^{(n)}(x) = 2^n n! P_n(x), \tag{6.22}$$

it follows from the previous example and Rolle's theorem that on the interval $(-1,1)$ there exist n real roots of the $n-$th derivative of the polynomial $R_n(x)$. Hence, from (6.22) it follows that there exist n real roots of the Legendre polynomial $P_n(x)$.

Example 6.45. *If a function f has a finite derivative f' at every point of the finite or infinite interval (a,b) and it holds*

$$\lim_{x\to a+} f(x) = \lim_{x\to b-} f(x), \tag{6.23}$$

then there exists at least one point $c \in (a,b)$ satisfying $f'(c) = 0$. Prove.

Solution. First we shall assume that (a,b) is a finite interval and let us denote by C the limits in relation (6.23). Then the function

$$F(x) = \begin{cases} f(x), & x \in (a,b); \\ C, & x \in \{a,b\}, \end{cases}$$

satisfies the conditions of Rolle's theorem, since it is continuous on the closed interval [a,b], has finite first derivative on the open interval (a,b) and it holds $F(a) = F(b)$. Hence there exists a point $c \in (a,b)$ such that $F'(c) = f'(c) = 0$.

If the interval (a,b) is infinite, but the limits in (6.23) are finite,

$$\lim_{x\to a+} f(x) = \lim_{x\to b-} f(x) = C,$$

then we consider the lines $y = C+\varepsilon$, $y = C-\varepsilon$. At least one of them intersects the curve f in (at least) two points with apscisses a_1, a_2, provided that ε is sufficiently small. The function f satisfies the conditions of Rolle's theorem on $[a_1,a_2] \subset (a,b)$ and this implies that there exists a point $c \in (a,b)$ such that $f'(c) = 0$.

Similarly one can treat the case when the limits in relation (6.23) are $+\infty$ or $-\infty$.

Example 6.46. *Prove that*

a) *Laguerre's polynomial,*

$$L_n(x) = \mathrm{e}^x \frac{d^n}{dx^n}(x^n \mathrm{e}^{-x}),$$

has n positive roots;

b) *the Chebychev-Hermite polynomial,*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}),$$

has n real roots;

c) *the equation,*

$$(1+x^2)^n \frac{d^n}{dx^n}\left((1+x^2)^{-1}\right) = 0,$$

has n real roots.

Solutions.

a) The function $f(x) = x^n e^{-x}$ satisfies

$$f(0) = \lim_{x\to+\infty} f(x) = 0.$$

From the previous example there exists a point $\xi_1 \in (0, +\infty)$ such that $f'(\xi_1) = 0$ and also $\lim_{x\to+\infty} f'(x) = 0$. Applying again the same procedure one obtains two points $\xi_2 \in (0, \xi_1)$ and $\xi_3 \in (\xi_2, +\infty)$ such that $f''(\xi_2) = f''(\xi_3) = 0$, and also $f''(0) = 0$, and $\lim_{x\to+\infty} f''(x) = 0$. Proceeding with this procedure we obtain that the function $f^{(n-1)}(x)$ has n real roots including the point $x = 0$, while $f^{(n)}(x)$ has n roots also, because $f^{(n)}(0) \neq 0$. Therefore the polynomial

$$L_n(x) = e^x f^{(n)}(x)$$

has n positive zeros.

b) Let us consider the function $g(x) = e^{-x^2}$ on the interval $(-\infty, +\infty)$ and similarly as in the previous example we obtain that $g^{(n)}(x)$ has n real roots. So the polynomial

$$H_n(x) = (-1)^n e^{x^2} g^{(n)}(x)$$

has n real roots.

c) In this case we consider the function $h(x) = \dfrac{1}{1+x^2}$, $x \in \mathbf{R}$, and apply the same procedure as in b).

Example 6.47. *A differentiable function f is a constant function on the interval $[a,b]$ if and only if its derivative f' is equal to zero at each point of the open interval (a,b).*

Solution. It is trivial that if a function is a constant on the interval $[a,b]$, then its derivative is zero on (a,b).

So we have to prove the opposite, i.e., if $f'(x) = 0$ for every $x \in (a,b)$, then it holds that $f(x_1) = f(x_2)$ for each $x_1, x_2 \in [a,b]$.

From Lagrange's theorem it follows that there exists a point $\xi \in (x_1, x_2)$, hence $\xi \in (a, b)$, such that
$$f(x_1) - f(x_2) = f'(\xi)(x_1 - x_2).$$
From the supposition $f'(x) = 0$ and the last relation we obtain $f(x_1) = f(x_2)$ for each pair $x_1, x_2 \in [a, b]$.

Example 6.48. *If the derivatives of two functions $F_1(x)$ and $F_2(x)$ are equal on some interval, i.e., it holds*

$$F_1'(x) = F_2'(x), \quad x_1, x_2 \in (a, b),$$

then the difference of these functions is a constant.

Solution. If we denote by $f(x) = F_1(x) - F_2(x)$, then it holds that $f'(x) = 0$ on the interval (a, b). So from Example 6.47 we obtain that on the same interval $f(x)$ is a constant.

Example 6.49.

a) *If the first derivative of a function f is positive (resp. nonnegative) on an interval (a, b), then this function is monotonically increasing (resp. nondecreasing) on (a, b).*

b) *If the first derivative of the function g is negative (resp. nonpositive) on an interval (a, b), then this function is monotonically decreasing (resp. nonincreasing) on (a, b).*

Solutions.

a) Assume that $f'(x) > 0$ for all $x \in (a, b)$, (resp. $f'(x) \geq 0$) for all $x \in (a, b)$. If x_1, x_2 are two points from the given interval and $x_1 < x_2$, then by Lagrange's theorem there exists a point $\xi \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1).$$

Since by assumption the first derivative of the function f is positive (resp. nonnegative) on the whole interval (a, b), from the last relation we obtain

$$f(x_2) - f(x_1) > 0 \quad (\text{resp. } f(x_1) - f(x_2) \geq 0).$$

This means that the function f is monotonically increasing (resp. nondecreasing) on (a, b).

b) Analogous to a).

Example 6.50. The Cauchy theorem.
Assume that f and g are

- *continuous on a closed interval $[a, b]$,*

- *differentiable on the open interval* (a, b).

If, additionally, $g'(x) \neq 0$ *for all* $x \in (a, b)$, *then there exists at least one number* $\xi \in (a, b)$ *such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}. \tag{6.24}$$

(If $g(a) \neq g(b)$, *then the condition* $g'(x) \neq 0$, $x \in (a, b)$, *can be replaced by the weaker condition* $f(x)'^2 + g(x)'^2 \neq 0$, $x \in (a, b)$.*)*

Solution. If $g'(x) \neq 0$, $x \in (a, b)$, then by Example 6.49 it follows that the function g is monotone on (a, b). Assume that g is monotonically increasing on (a, b). Then there exists an inverse function $x = x(t)$, defined and differentiable on the interval $[\alpha, \beta]$, where $\alpha = g(a)$, $\beta = g(b)$. Note that $g(a) < g(b)$.

The function $f(x)$ can be written as the function $f(x) = f(x(t))$, depending on $t \in [\alpha, \beta]$. Thus f is a differentiable function, as the composition of differentiable functions. Applying the Lagrange's theorem and the chain rule we get

$$\frac{f(x(\beta)) - f(x(\alpha))}{\beta - \alpha} = f_t'(x(c))x_t'(c)$$

for some $c \in (\alpha, \beta)$, or

$$\frac{f(b) - f(a)}{(g(b) - g(a))} = \frac{f'(x(c))}{g'(x(c))}.$$

Taking $x(c) = \xi$, we obtain the statement of the theorem.

The other cases are left to the reader.

Remarks. The Lagrange's theorem is a special case of Cauchy's theorem, when the function g is given by $g(x) = x$.

Example 6.51. *Check can one apply Cauchy's theorem to the following functions:*

a) $f(x) = x^2$, $g(x) = x^3$, $x \in [-1, 1]$;

b) $f(x) = x^2 + 2x$, $g(x) = x^3$, $x \in [-1, 1]$.

Solutions.

a) No, we can not apply Cauchy's theorem to these functions. Namely, we have $g(-1) = -1 \neq 1 = g(1)$, but at the point $x = 0$ it holds $f'(0)^2 + g'(0)^2 = 0$. Hence the condition $f'(x)^2 + g'(x)^2 \neq 0$, for all $x \in [-1, 1]$, is not satisfied.

b) Yes, because in this case we have $g(-1) = -1 \neq 1 = g(1)$, $f'(x) = 2x + 2$ and $g'(x) = 3x^2$, so it holds $(f'(x))^2 + (g'(x))^2 \neq 0$, for all $x \in [-1, 1]$.
Let us find the point ξ satisfying

$$\frac{f(1) - f(-1)}{g(1) - g(-1)} = \frac{f'(\xi)}{g'(\xi)}, \quad \xi \in (-1, 1).$$

From

$$\frac{3+1}{1+1} = \frac{2\xi + 2}{3\xi^2}, \quad \text{or} \quad 3\xi^2 - \xi - 1 = 0,$$

we obtain $\xi_1 = \dfrac{1 + \sqrt{13}}{6} \in (-1, 1)$ and $\xi_2 = \dfrac{1 - \sqrt{13}}{6} \in (-1, 1)$.

Example 6.52. *Using Lagrange's theorem show the following inequalities.*

a) $|\sin ax - \sin ay| \leq |a| \cdot |x-y|, \quad a \neq 0;$

b) $|\arctan x - \arctan y| \leq |x-y|, \quad x, y \in \mathbf{R};$

c) $\dfrac{x-y}{x} < \ln \dfrac{x}{y} < \dfrac{x-y}{y}, \quad 0 < y < x;$

d) $\dfrac{x}{1+x} < \ln(1+x) < x, \quad x > 0;$

e) $\mathrm{e}^x > 1 + x, \quad x \in \mathbf{R};$

f) $\mathrm{e}^x > \mathrm{e}x, \quad x > 1;$

g) $n(b-a)a^{n-1} < b^n - a^n < n(b-a)b^{n-1}, \quad 0 < a < b, \ n \in \mathbf{N};$

h) $\dfrac{\alpha}{n^{\alpha+1}} < \left(\dfrac{1}{(n-1)^\alpha} - \dfrac{1}{n^\alpha}\right), \quad n \in \mathbf{N}, \ \alpha > 0.$

Solutions.

a) Applying Lagrange's theorem to the function $f(t) = \sin at$, which is continuous and differentiable on any interval $[y, x]$, we obtain that there exists a point $\xi \in (y, x)$, such that it holds

$$\sin ax - \sin ay = a(x-y)\cos a\xi, \quad \text{wherefrom}$$

$$|\sin ax - \sin ay| = |a| \cdot |\cos a\xi| \cdot |x-y| \leq |a| \cdot |x-y|.$$

b) The function $f(t) = \arctan t$ is continuous and differentiable on the interval $[y, x]$, so there exists a point $\xi \in (y, x)$ with the property

$$\arctan x - \arctan y = \frac{x-y}{1+\xi^2}.$$

So we obtain

$$|\arctan x - \arctan y| = \frac{|x-y|}{1+(\xi)^2} \leq |x-y|.$$

c) The function $f(x) = \ln x$ is continuous and differentiable on the interval $[y, x]$, $0 < y < x$, hence, there exists a point $\xi \in (y, x)$ such that

$$\ln x - \ln y = \frac{x-y}{\xi}.$$

From the relations

$$|\ln x - \ln y| = \frac{|x-y|}{\xi} \quad \text{and} \quad \frac{1}{x} < \frac{1}{\xi} < \frac{1}{y},$$

we obtain the given inequality.

d) Using the function $f(t) = \ln(1+t)$ on the interval $[0, x]$, and the relation

$$\frac{\ln(1+x) - \ln(1+0)}{x} = \frac{1}{1+\xi}, \quad \text{for } \xi \in (0, x), \quad \text{hence } \frac{1}{1+x} < \frac{1}{1+\xi} < 1,$$

we obtain the inequality.

e) The function $f(x) = \mathrm{e}^x$ on the interval $[0, x]$, $x > 0$ (resp. on $[x, 0]$, $x < 0$), satisfies the conditions of Lagrange's theorem,

$$\frac{\mathrm{e}^x - \mathrm{e}^0}{x} = \mathrm{e}^\xi \Rightarrow \mathrm{e}^x - 1 = x\mathrm{e}^\xi.$$

- If $x > 0$, hence $\xi > 0$, then $\mathrm{e}^\xi > 1$ and $\mathrm{e}^x > 1 + x$.
- If $x < 0$, hence $\xi < 0$, then $\mathrm{e}^\xi < 1$, but $x\mathrm{e}^\xi > x$, thus the inequality is true.

f) Taking the same function as in the previous case and the interval $[1, x]$, we have

$$\frac{\mathrm{e}^x - \mathrm{e}^1}{x - 1} = \mathrm{e}^\xi > \mathrm{e}, \text{ for some } \xi \in (1, x).$$

g) **Hint.** Use the function $y = t^n$ and the interval $[a, b]$.

h) Taking the function $f(x) = \dfrac{1}{x^\alpha}$ on the interval $[n-1, n]$, we obtain the given inequality.

Example 6.53. *If a derivative of a function f is bounded on an interval (a, b), then f is uniformly continuous on (a, b).*

Solution. Assume that there exists $M > 0$ such that $|f'(x)| \le M$. Then applying the Lagrange theorem, for every pair of points $x_1, x_2 \in (a, b)$ we can write

$$|f(x_1) - f(x_2)| = |f'(\xi)||x_1 - x_2| \le M|x_1 - x_2|, \quad \xi \in (x_1, x_2).$$

Hence for given $\varepsilon > 0$, we put $\delta := \dfrac{\varepsilon}{M}$ and obtain

$$(\forall x_1, x_2 \in (a, b)) \quad |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon.$$

6.3 Taylor's formula

6.3.1 Basic notions

Theorem 6.54. Taylor's theorem.
Let f be a continuous function and n a positive integer such that all derivatives up to the n-th are continuous on an interval $[a, b]$ and there exists a finite $f^{(n+1)}$ on the

interval (a,b). *Then there exists* $\xi \in (a,b)$ *such that for every* $x \in (a,b)$ *the following* **Taylor's formula** *holds*

$$\begin{aligned} f(x) &= f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) \\ &\quad + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi). \end{aligned}$$

The following polynomial is called **Taylor's polynomial** of order n at the point a for a function f.

$$P_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a).$$

Thus the Taylor's formula can be written as

$$f(x) = P_n(x) + R_n(x), \tag{6.25}$$

where the **remainder** R_n is given by

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi), \quad \xi = a + \theta\cdot(x-a), \;\; 0 < \theta = \theta(x) < 1.$$

Using Definition 4.54 we can write relation (6.25) as

$$f(x) = P_n(x) + o\left((x-a)^n\right), \quad x \to a.$$

For $a = 0$ we obtain **Maclaurin's formula**

$$\begin{aligned} f(x) = &\; f(0) + xf'(0) + \frac{x^2}{2!}f''(x) + \frac{x^3}{3!}f'''(0) \\ &+ \cdots + \frac{x^n}{n!}f^{(n)}(0) + \frac{x^{n+1}}{(n+1)!}f^{(n+1)}(\xi), \quad \xi \in (a,b). \end{aligned}$$

The polynomial

$$P_n(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0),$$

is called **Maclaurin's polynomial** of order n for the function f.

6.3.2 Examples and exercises

Exercise 6.55. *Show that the Taylor's polynomial for a polynomial* $P_n(x)$ *of degree* n *is just the polynomial itself.*

Example 6.56. *Develop the polynomial*

$$f(x) = x^4 - 5x^3 - 3x^2 + 7x + 6$$

into powers of $x - 2$.

Solution. From $f(2) = -16$ and

$$\begin{array}{llll} f'(x) = 4x^3 - 15x^2 - 6x + 7, & & f'(2) = -33 & \\ f''(x) = 12x^2 - 30x - 6 & & f''(2) = -18; & \\ f'''(x) = 24x - 30, & f'''(2) = 18, & f^{(4)}(x) = 24, & f^{(4)}(2) = 24, \end{array}$$

it follows

$$\begin{aligned} x^4 - 5x^3 - 3x^2 + 7x + 6 &= -16 - 33(x-2) - 18\frac{(x-2)^2}{2} \\ &+ 18\frac{(x-2)^3}{3!} + 24\frac{(x-2)^4}{4!}. \end{aligned}$$

Thus

$$\begin{aligned} x^4 - 5x^3 - 3x^2 + 7x + 6 &= -16 - 33(x-2) - 9(x-2)^2 \\ &+ 3(x-2)^3 + (x-2)^4. \end{aligned}$$

In this case the remainder is equal to zero, because the fifth derivative of f is identically equal to zero.

Example 6.57. *Approximate the function* $f(x) = \mathrm{e}^x$ *using the Maclaurin's polynomial of the*

a) *first degree;* **b)** *second degree;* **c)** *third degree,*

and determine the corresponding remainder.

Solutions. From $f'(x) = \mathrm{e}^x = f''(x) = f'''(x) = f^{(4)}(x)$, and $f(0) = f'(0) = f''(0) = f'''(0) = 1$, it follows that

a) $\mathrm{e}^x = 1 + x + o(x),\ x \to 0$ and $R_1 = \dfrac{x^2}{2}\mathrm{e}^\xi.$

b) $\mathrm{e}^x = 1 + x + \dfrac{x^2}{2} + o(x^2),\ x \to 0$ and $R_2 = \dfrac{x^3}{3!}\mathrm{e}^\xi.$

c) $\mathrm{e}^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3!} + o(x^3),\ x \to 0$ and $R_3 = \dfrac{x^4}{4!}\mathrm{e}^\xi,$

where in all three cases $\xi = \theta x,\ 0 < \theta < 1$, and $1 < \mathrm{e}^\xi < \mathrm{e}$.

Note that for $x \in [0, 1]$, the remainders can be estimated by

$$R_n \le \frac{x^{n+1}}{(n+1)!}\mathrm{e} \le \frac{\mathrm{e}}{(n+1)!}, \quad n = 1, 2, 3.$$

Example 6.58. *Apply the Maclaurin's formula to the functions given below.*

a) $f(x) = \mathrm{e}^x$; **b)** $f(x) = \sin x$; **c)** $f(x) = \cos x$;

d) $f(x) = \dfrac{1}{1-x}$; **e)** $f(x) = \ln(1-x)$; **f)** $(1+x)^\alpha,\ \alpha \in \mathbf{Q}$.

Solutions. Using Example 6.30 we obtain, when $x \to 0$:

a) $\mathrm{e}^x = \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n)$;

b) $\sin x = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + o(x^{2n+2})$;

c) $\cos x = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} + o(x^{2n+1})$;

d) $\frac{1}{1-x} = \sum_{k=0}^{n} x^k + o(x^n)$;

e) $\ln(1-x) = -\sum_{k=1}^{n} \frac{x^k}{k} + o(x^n)$;

f) $(1+x)^\alpha = \sum_{k=0}^{n} \binom{\alpha}{k} x^k + o(x^n)$.

In f), $\alpha \in \mathbf{R} \setminus \{0\}$ and by definition

$$\binom{\alpha}{0} = 1, \quad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-(k-1))}{k!}, \ k \in \mathbf{N}.$$

Example 6.59. *Find Maclaurin's formula for the functions given below.*

a) $f(x) = \mathrm{e}^{5x+1}$;

b) $f(x) = \frac{1}{\sqrt{1-x}}$;

c) $f(x) = \sin(2x + \pi/3)$;

d) $f(x) = \ln(x + \mathrm{e})$.

Solutions. In this example we use the following simple statement. If $b \neq 0$ and

$$f(x) = \sum_{k=0}^{n} a_k x^k + o(x^n), \quad \text{then} \quad f(bx) = \sum_{k=0}^{n} a_k b^k x^k + o(x^n), \quad x \to 0.$$

a) From $\mathrm{e}^{5x+1} = \mathrm{e} \cdot \mathrm{e}^{5x}$ we obtain

$$\mathrm{e}^{5x+1} = \mathrm{e} \sum_{k=0}^{n} \frac{5^k x^k}{k!} + o(x^n), \quad x \to 0.$$

b) From Example 6.58 f) we have

$$\frac{1}{\sqrt{1-x}} = (1 + (-1)x)^{-1/2},$$

hence

$$\frac{1}{\sqrt{1-x}} = \sum_{k=0}^{n} (-1)^k \binom{-1/2}{k} x^k + o(x^n), \quad x \to 0, \quad \text{where for} \quad k \in \mathbf{N}$$

$$\binom{-1/2}{k} = \frac{\left(-\frac{1}{2}\right) \cdot \left(-\frac{1}{2} - 1\right) \cdots \left(-\frac{1}{2} - (k-1)\right)}{k!} = (-1)^k \frac{(2k-1)!!}{2^k k!}.$$

Thus

$$\frac{1}{\sqrt{1-x}} = 1 + \sum_{k=1}^{n} \frac{(2k-1)!!}{2^k k!} x^k + o(x^n), \quad x \to 0.$$

c) From $f^{(k)}(x) = 2^k \sin(2x + \pi/3 + k\pi/2)$ and $f^{(k)}(0) = 2^k \sin\left(\frac{\pi}{6}(2+3k)\right)$, $k \in \mathbf{N}$, and from Example 6.58 b) we have

$$\sin(2x+\pi/3) = \sum_{k=0}^{n}(-1)^k \frac{2^{2k+1}x^{2k+1}}{(2k+1)!}\sin\left(\frac{\pi}{6}(2+3k)\right) + o(x^{2n+2}), \quad x \to 0.$$

d) From $\ln(x+\mathrm{e}) = 1 + \ln(1 + x/\mathrm{e})$ and Example 6.58 e) we get

$$\ln(x+\mathrm{e}) = 1 + \sum_{k=1}^{n}\frac{(-1)^k x^k}{\mathrm{e}^k k} + o(x^n), \quad x \to 0.$$

Example 6.60. *Apply the Maclaurin's formula to the following functions.*

a) $f(x) = (x^2+5)\mathrm{e}^{3x}$;

b) $f(x) = \dfrac{(x^2+3\mathrm{e}^x)}{\mathrm{e}^{2x}}$;

c) $f(x) = (x+2)\sqrt{1+x}$;

d) $f(x) = \cos x + |x|^5$;

e) $f(x) = \ln\dfrac{2-3x}{3+2x}$;

f) $f(x) = \ln(6 + 11x + 6x^2 + x^3)$;

g) $f(x) = \dfrac{1-2x^2}{2+x-x^2}$;

h) $f(x) = \dfrac{x^2+2}{x^3+x^2+x+1}$.

Solutions.

a) Using the equality $(x^2+5)\mathrm{e}^{3x} = x^2\mathrm{e}^{3x} + 5\mathrm{e}^{3x}$ we have

$$\begin{aligned}
f(x) &= x^2\sum_{k=0}^{n-2}\frac{3^k x^k}{k!} + o(x^n) + 5\sum_{k=0}^{n}\frac{3^k x^k}{k!} + o(x^n)\\
&= \sum_{k=2}^{n}\frac{3^{k-2}x^k}{(k-2)!} + 5\sum_{k=2}^{n}\frac{3^k x^k}{k!} + 5 + 15x + o(x^n)\\
&= 5 + 15x + \sum_{k=2}^{n}\left(\frac{3^{k-2}}{(k-2)!} + 5\frac{3^k}{k!}\right)x^k + o(x^n)\\
&= 5 + 15x + \sum_{k=2}^{n}\left(\frac{3^{k-2}}{k!}(k(k-1)+5\cdot 9)\right)x^k + o(x^n), \quad x \to 0.
\end{aligned}$$

b) In this case we have

$$\begin{aligned} f(x) &= x^2 \mathrm{e}^{-2x} + 3\mathrm{e}^{-x} = x^2 \left(\sum_{k=0}^{n-2} \frac{(-2x)^k}{k!} + o(x^{n-2}) \right) + 3 \sum_{k=0}^{n} \frac{(-x)^k}{k!} + o(x^n) \\ &= \sum_{k=2}^{n} \frac{(-2)^{k-2} x^k}{(k-2)!} + 3 - 3x + 3 \sum_{k=2}^{n} \frac{(-1)^k x^k}{k!} + o(x^n) \\ &= 3 - 3x + \sum_{k=2}^{n} (3 + k(k-1)2^{k-2}) \frac{(-1)^k x^k}{k!} + o(x^n), \quad x \to 0. \end{aligned}$$

c) Using Example 6.58 f) for $\alpha = 1/2$, we get

$$\sqrt{1-x} = 1 - \frac{x}{2} - \sum_{k=2}^{n} \frac{(2k-3)!!}{2^k k!} x^k + o(x^n), \quad x \to 0.$$

Hence

$$\begin{aligned} (x+2)\sqrt{1-x} &= x - \frac{x^2}{2} - \sum_{k=3}^{n} \frac{(2k-5)!!}{2^{k-1}(k-1)!} x^k + o(x^n) \\ &\quad + 2 - x - 2 \sum_{k=2}^{n} \frac{(2k-3)!!}{2^k k!} x^k + o(x^n) \\ &= 2 - \frac{3x^2}{4} - \sum_{k=3}^{n} \frac{6(2k-5)!!(k-1)}{2^k k!} x^k + o(x^n), \quad x \to 0. \end{aligned}$$

d) Let us consider the function $h(x) = |x|^5$. It has only four derivatives on $\mathbf{R}$ and it holds

$$h(0) = h'(0) = h''(0) = h'''(0) = h^{(4)}(0) = 0.$$

Now the function h has a fifth derivative on the set $\mathbf{R} \setminus \{0\}$, so we can write

$$h(x) = P_4(x) + o(x^4) = o(x^4), \quad x \to 0,$$

since $P_4(x) \equiv 0$. Thus we have

$$\cos x + |x|^5 = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4), \quad x \to 0.$$

e) Since

$$\ln \frac{2-3x}{3+2x} = \ln 2 - \ln 3 + \ln\left(1 - \frac{3x}{2}\right) - \ln\left(1 + \frac{2x}{3}\right),$$

it follows from Example 6.58 e) that

$$\begin{aligned} f(x) &= \ln 2 - \ln 3 - \sum_{k=1}^{n} \frac{3^k x^k}{2^k k} + o(x^n) + \sum_{k=1}^{n} \frac{(-2)^k x^k}{3^k k} + o(x^n) \\ &= \ln 2 - \ln 3 + \sum_{k=1}^{n} \frac{((-4)^k - 9^k) x^k}{6^k k} + o(x^n), \quad x \to 0. \end{aligned}$$

f) From
$$\ln(6+11x+6x^2+x^3)=\ln(3+x)+\ln(2+x)+\ln(1+x),$$
it follows
$$\begin{aligned} f(x) &= \ln 6+\sum_{k=1}^{n}\frac{(-1)^{k-1}x^k}{3^k k}+\sum_{k=1}^{n}\frac{(-1)^{k-1}x^k}{2^k k}+\sum_{k=1}^{n}\frac{(-1)^{k-1}x^k}{k}+o(x^n)\\ &= \ln 6+\sum_{k=1}^{n}\frac{(-1)^{k-1}x^k}{6^k k}(2^k+3^k+6^k)+o(x^n),\quad x\to 0.\end{aligned}$$

g) From
$$\frac{1-2x^2}{2+x-x^2}=2+\frac{-7}{3(2-x)}+\frac{-1}{3(1+x)},$$
it follows for $x\to 0$
$$f(x)=2-\frac{7}{6}\cdot\sum_{k=0}^{n}\frac{x^k}{2^k}-\frac{1}{3}\cdot\sum_{k=0}^{n}(-x)^k+o(x^n)=\frac{1}{2}+\sum_{k=1}^{n}\frac{\left((-2)^{k+1}-7\right)x^k}{3\cdot 2^{k+1}}+o(x^n).$$

h) In this case we have
$$\begin{aligned} f(x) &= \frac{x^2+2}{x^3+x^2+x+1}=\frac{1}{2}\left(\frac{1-x}{1+x^2}+\frac{3}{1+x}\right)\\ &= \frac{1}{2}\left(\sum_{k=0}^{n}(-1)^k x^{2k}-\sum_{k=0}^{n-1}(-1)^k x^{2k+1}+3\sum_{k=0}^{2n}(-1)^k x^k\right)+o(x^{2n})\\ &= \sum_{k=0}^{n}\frac{(3+(-1)^k)x^{2k}}{2}+\sum_{k=0}^{n-1}\frac{((-1)^{k+1}-3)x^{2k+1}}{2}+o(x^{2n}),\quad x\to 0.\end{aligned}$$

Example 6.61. *Apply the Maclaurin's formula to the functions below.*

a) $f(x)=x^2\cos^2 x$; **b)** $f(x)=\sin^2 x\cos^2 x$;

c) $f(x)=\sin^4+\cos^4 x$; **d)** $f(x)=\arctan x$;

e) $f(x)=\arcsin x$; **f)** $f(x)=\sinh x$;

g) $f(x)=\cosh x$; **h)** $f(x)=\sinh x\cdot\cosh 2x$.

Solutions.

a) Since $x^2\cos^2 x=x^2\dfrac{1+\cos 2x}{2}$, it follows from Example 6.58 c)
$$\begin{aligned} f(x) &= \frac{x^2}{2}+\frac{1}{2}\sum_{k=0}^{n-1}\frac{(-1)^k 2^{2k}x^{2k+2}}{(2k)!}+o(x^{2n+1})\\ &= x^2+\sum_{k=2}^{n}\frac{(-1)^{k-1}2^{2k-3}x^{2k}}{(2k-2)!}+o(x^{2n+1}),\quad x\to 0.\end{aligned}$$

b) Since it holds
$$\sin^2 x\cos^2 x = \frac{1}{8}(1-\cos 4x),$$
we have
$$f(x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}2^{4k-3}x^{2k}}{(2k)!} + o(x^{2n+1}), \quad x \to 0.$$

c) From
$$\sin^4 x + \cos^4 x = \frac{1}{4}(3+\cos 4x),$$
we have
$$f(x) = 1 + \sum_{k=1}^{n} \frac{(-1)^{k}2^{4k-2}x^{2k}}{(2k)!} + o(x^{2n+1}), \quad x \to 0.$$

d) Since
$$(\arctan x)' = \frac{1}{1+x^2} = \sum_{k=0}^{n}(-1)^k x^{2k} + o(x^{2n+1}),$$
it follows
$$\arctan x = \sum_{k=0}^{n}(-1)^k \frac{x^{2k+1}}{2k+1} + o(x^{2n+2}), \quad x \to 0.$$

e) Since
$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} = 1 + \sum_{k=1}^{n} \frac{(2k-1)!!}{2^k k!}x^{2k} + o(x^{2n+1}),$$
it follows
$$\arcsin x = x + \sum_{k=1}^{n} \frac{(2k-1)!!}{2^k k!(2k+1)}x^{2k+1} + o(x^{2n+2}).$$

f) From $\sinh x = \dfrac{e^x - e^{-x}}{2}$, we obtain
$$\sinh x = \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!} + o(x^{2n+2}), \quad x \to 0.$$

g) From $\cosh x = \dfrac{e^x + e^{-x}}{2}$, we obtain
$$\cosh x = \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!} + o(x^{2n+1}), \quad x \to 0.$$

h) Since $\sinh x \cdot \cosh 2x = \dfrac{e^{3x} - e^{x} + e^{-x} - e^{-3x}}{4} = \dfrac{1}{2}\sinh 3x - \dfrac{1}{2}\sinh x$, it follows
$$\begin{aligned}\sinh x \cdot \cosh 2x &= \frac{1}{2}\left(\sum_{k=0}^{n} \frac{3^{2k+1}x^{2k+1}}{(2k+1)!} - \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!}\right) + o(x^{2n+2}) \\ &= \sum_{k=0}^{n} \frac{x^{2k+1}}{2(2k+1)!}(3^{2k+1}-1) + o(x^{2n+2}), \quad x \to 0.\end{aligned}$$

Example 6.62. *Determine*

$$\text{a)}\quad \lim_{x\to 0}\frac{2\sqrt{1+2\tan x}-2\mathrm{e}^x+2x^2}{\arcsin x-\sin x};$$

$$\text{b)}\quad \lim_{x\to 0}\frac{x+\dfrac{3x^2}{2}-\dfrac{2}{1-x}+\exp(\arctan x)+1}{\ln\dfrac{1-x}{1+x}+2x};$$

$$\text{c)}\quad \lim_{x\to 0}\frac{\cosh(\sin x)-\sqrt[5]{1-\dfrac{x^2}{2}-\dfrac{3x^2}{5}}}{\tanh x-x}.$$

Solutions.

a) In this case we shall consider Maclaurin's formulas for the corresponding functions. Since

$$\sqrt{1+t}=1+\frac{t}{2}-\frac{t^2}{8}+\frac{t^3}{16}+o(x^3),\quad x\to 0,\qquad t=\tan x=x+\frac{x^3}{3}+o(x^3),\quad x\to 0,$$

we get

$$\begin{aligned}\sqrt{1+2\tan x} &= 1+\frac{2\tan x}{2}-\frac{(2\tan x)^2}{8}+\frac{(2\tan x)^3}{16}+o(\tan^3 x),\\ &= 1+x-\frac{x^2}{2}+\frac{5x^3}{6}+o(x^3),\quad x\to 0.\end{aligned}$$

Also, from the equalities

$$\mathrm{e}^x=1+x+\frac{x^2}{2}+\frac{x^3}{6}+o(x^3),$$

$$\sin x=x-\frac{x^3}{6}+o(x^3),\quad x\to 0$$

$$\arcsin x=x+\frac{x^3}{6}+o(x^3),\quad x\to 0,\quad \text{hence}$$

$$\arcsin x-\sin x=\frac{x^3}{3}+o(x^3),\quad x\to 0,$$

we obtain

$$\lim_{x\to 0}\frac{2\sqrt{1+2\tan x}-2\mathrm{e}^x+2x^2}{\arcsin x-\sin x}=\lim_{x\to 0}\frac{\frac{4}{3}x^2+o(x^3)}{\frac{1}{3}x^3+o(x^3)}=4.$$

b) Using the following relations as $x \to 0$:

$$\arctan x = x - \frac{x^3}{3} + o(x^3),$$

$$\exp(\arctan x) = 1 + \left(x - \frac{x^3}{3}\right) + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3) = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} + o(x^3),$$

$$\ln\frac{1-x}{1+x} = -2x - \frac{2}{3}x^3 + o(x^3),$$

$$\frac{2}{1-x} = 2 + 2x + 2x^2 + 2x^3 + o(x^3),$$

we obtain

$$\lim_{x\to 0}\frac{x + \dfrac{3x^2}{2} - \dfrac{2}{1-x} + \exp(\arctan x) + 1}{\ln\dfrac{1-x}{1+x} + 2x} = \lim_{x\to 0}\frac{-\dfrac{13}{6}x^3 + o(x^3)}{-\dfrac{2}{3}x^3 + o(x^3)} = 13/4.$$

c) From the relations

$$\begin{aligned}
\cosh(\sin x) &= 1 + \frac{1}{2}\left(x - \frac{x^3}{6}\right)^2 + \frac{1}{24}\left(x - \frac{x^3}{6}\right)^4 + o(x^4),\\
&= 1 + \frac{x^2}{2} - \frac{x^4}{8} + o(x^4), \quad x \to 0;\\
\left(1 - \frac{x^2}{2}\right)^{1/5} &= 1 - \frac{x^2}{10} - \frac{x^4}{50} + o(x^4), \quad x \to 0;\\
\tanh x &= x - \frac{x^3}{3} + o(x^4) \quad x \to 0,
\end{aligned}$$

we obtain

$$\lim_{x\to 0}\frac{\cosh(\sin x) - \sqrt[5]{1 - \dfrac{x^2}{2}} - \dfrac{3x^2}{5}}{\tanh x - x} = \lim_{x\to 0}\frac{\dfrac{-21x^4}{200} + o(x^4)}{-\dfrac{x^3}{3} + o(x^4)} = 0.$$

Example 6.63. *Determine*

a) $\lim\limits_{x\to 0}(\cos(xe^x) - \ln(1-x) - x)^{1/x^3}$;

b) $\lim\limits_{x\to 0}\left(\cos(\sin x) + \dfrac{x^2}{2}\right)^{1/(x^2(\sqrt{1+2x}-1))}$;

c) $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin x \arctan x} - \dfrac{1}{\tan x \arcsin x}\right)$.

Solutions.

a) In this case we use the following Maclaurin's formulas:

$$xe^x = x + x^2 + o(x^2), \qquad x \to 0, \quad \text{hence}$$

$$\cos(xe^x) = 1 - x^2/2 - x^3 + o(x^3), \qquad x \to 0,$$

$$\cos(xe^x) - \ln(1-x) = 1 + x - 2x^3/3 + o(x^3), \qquad x \to 0$$

and we obtain

$$\lim_{x\to 0}(\cos(xe^x) - \ln(1-x) - x)^{1/x^3} = \lim_{x\to 0}\left(1 - 2x^3/3 + o(x^3)\right)^{1/x^3} = e^{-2/3}.$$

b) From

$$\sin x = x - x^3/6 + o(x^4), \qquad \cos(\sin x) = 1 - \frac{x^2}{2} + \frac{5}{24}x^4 + o(x^3), \quad x \to 0,$$

$$\sqrt{1+2x} = 1 + x + o(x), \qquad \frac{1}{x^2(\sqrt{1+2x}-1)} = \frac{1}{x^3 + o(x^3)}, \quad x \to 0,$$

we obtain

$$\lim_{x\to 0}\left(\cos(\sin x) + \frac{x^2}{2}\right)^{1/(x^2(\sqrt{1+2x}-1))} = \lim_{x\to 0}\left(1 + 5x^4/4! + o(x^4)\right)^{1/(x^3+o(x^3))} = 1.$$

c) In this case we have

$$\lim_{x\to 0}\left(\frac{1}{\sin x \arctan x} - \frac{1}{\tan x \arcsin x}\right) = \lim_{x\to 0}\frac{x^4 + o(x^4)}{x^4 + o(x^4)} = 1.$$

6.4 L'Hospital's Rule

6.4.1 Basic notions

Let c be a real number, or one of the symbols $+\infty$ or $-\infty$.

The expression $\dfrac{f(x)}{g(x)}$ is said to have **undetermined form** "0/0" at $x = c$, if

$$\lim_{x\to c} f(x) = 0, \quad \lim_{x\to c} g(x) = 0.$$

The expression $\dfrac{f(x)}{g(x)}$ is said to have **undetermined form** "∞/∞" at $x = c$ if

$$\lim_{x\to c} f(x) = \infty, \quad \lim_{x\to c} g(x) = \infty.$$

Theorem 6.64. L'Hospital's Rule
Suppose the functions f and g are differentiable at every point, except possibly at c in an interval (a, b). If $g'(x) \neq 0$ for $x \neq c$, and if $\dfrac{f(x)}{g(x)}$ has either the undetermined form "$\dfrac{0}{0}$" or "$\dfrac{\infty}{\infty}$" at $x = c$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}, \tag{6.26}$$

provided $\dfrac{f'(x)}{g'(x)}$ has a limit as x approaches c.

The statement of Theorem 6.64 remains true also if $\lim\limits_{x\to c} \dfrac{f'(x)}{g'(x)} = \infty$, and then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \infty.$$

Moreover, the point c can be replaced with

6.4.2 Examples and Exercises

Example 6.65. *Determine the following limits by using L'Hospital's rule.*

a) $\displaystyle\lim_{x\to 0} \frac{\cos x + 3x - 1}{2x}$;

b) $\displaystyle\lim_{x\to 0} \frac{e^x + e^{-x} - 2\cos x}{x \sin 2x}$;

c) $\displaystyle\lim_{x\to 0} \frac{\ln \cos ax}{\ln \cos bx}$, $a, b \neq 0$;

d) $\displaystyle\lim_{x\to a} \frac{x^a - a^x}{a^x - a^a}$, $a > 0$, $a \neq 1$;

e) $\displaystyle\lim_{x\to 0} \frac{(a + x)^x - a^x}{x^2}$, $a > 0$;

f) $\displaystyle\lim_{x\to 1} \frac{x^x - x}{\ln x - x + 1}$.

Solutions.

a) The functions $f(x) = \cos x + 3x - 1$ and $g(x) = 2x$ are differentiable in an interval containing the point 0. The given expression has the undetermined form "$\dfrac{0}{0}$" when $x \to 0$, and

$$\lim_{x\to 0} \frac{\cos x + 3x - 1}{2x} = \lim_{x\to 0} \frac{-\sin x + 3}{2} = 3/2.$$

Clearly the conditions of the L'Hospital rule are fulfilled in the cases b)–f), and therefore we have the following.

b) $$\lim_{x\to 0} \frac{e^x + e^{-x} - 2\cos x}{x \sin 2x} = \lim_{x\to 0} \frac{e^x - e^{-x} + 2\sin x}{\sin 2x + 2x\cos 2x}$$
$$= \lim_{x\to 0} \frac{e^x + e^{-x} + 2\cos x}{4\cos 2x - 4x\sin 2x} = 4/4 = 1.$$

c) $$\lim_{x\to 0}\frac{\ln\cos ax}{\ln\cos bx}=\lim_{x\to 0}\frac{\dfrac{-a\sin ax}{\cos ax}}{\dfrac{-b\sin bx}{\cos bx}}=\lim_{x\to 0}\frac{a\sin ax\cdot\cos bx}{b\sin bx\cdot\cos ax}$$
$$=\lim_{x\to 0}\frac{a^2\dfrac{\sin ax}{ax}\cdot\cos bx}{b^2\dfrac{\sin bx}{bx}\cdot\cos ax}=\frac{a^2}{b^2}.$$

d) $$\lim_{x\to a}\frac{x^a-a^x}{a^x-a^a}=\lim_{x\to a}\frac{ax^{a-1}-a^x\ln a}{a^x\ln a}=\frac{1-\ln a}{\ln a}.$$

e) $$\lim_{x\to 0}\frac{(a+x)^x-a^x}{x^2}=\lim_{x\to 0}\frac{e^{x\ln(a+x)}\left(\ln(a+x)+\dfrac{x}{a+x}\right)-a^x\ln a}{2x}$$
$$=\lim_{x\to 0}\frac{e^{x\ln(a+x)}\left(\left(\ln(a+x)+\dfrac{x}{a+x}\right)^2+\dfrac{1}{a+x}+\dfrac{a}{(a+x)^2}\right)-a^x\ln^2 a}{2}=\frac{1}{a}.$$

f) $$\lim_{x\to 1}\frac{x^x-x}{\ln x-x+1}=\lim_{x\to 1}\frac{x^x(\ln x+1)-1}{\dfrac{1}{x}-1}=\lim_{x\to 1}\frac{x^{x+1}(\ln x+1)-x}{1-x}$$
$$=\lim_{x\to 1}\frac{x^{x+1}(\ln x+1)\left(1+\dfrac{1}{x}+\ln x\right)+x^x-1}{-1}=-2.$$

Example 6.66. *Determine*

a) $\displaystyle\lim_{x\to+\infty}\frac{3x^2+2x-2}{x^2-1}$; b) $\displaystyle\lim_{x\to+\infty}\frac{2\ln x}{x^b},\ b>0$;

c) $\displaystyle\lim_{x\to+\infty}\frac{x^b}{e^{ax}},\ b>0,\ a\in\mathbf{R}$; d) $\displaystyle\lim_{x\to+\infty}\frac{\ln(\ln x)}{\ln x}$.

Solutions. The expressions in this example have the undetermined form "$\dfrac{\infty}{\infty}$", and the conditions of the L'Hospital rule are fulfilled. So we have the following.

a) $$\lim_{x\to+\infty}\frac{3x^2+2x-2}{x^2-1}=\lim_{x\to+\infty}\frac{6x+2}{2x}=\frac{6}{2}=3.$$

b) $$\lim_{x\to+\infty}\frac{2\ln x}{x^b}=\lim_{x\to+\infty}\frac{\dfrac{2}{x}}{bx^{b-1}}=\lim_{x\to+\infty}\frac{2}{bx^b}=0.$$

c) Clearly, if $a\le 0$, then $\displaystyle\lim_{x\to+\infty}\frac{x^b}{e^{ax}}=+\infty$.

Let us assume that $a>0$. Then we can find $k\in\mathbf{N}$ such that $k\ge b$ and applying the L'Hospital rule k–times we obtain

$$\lim_{x\to+\infty}\frac{x^k}{e^{ax}}=\lim_{x\to+\infty}\frac{k!}{a^ke^{ax}}=0.$$

Since $0 < \dfrac{x^b}{e^{ax}} \le \dfrac{x^k}{e^{ax}}$, $x > 0$, using Exercise 4.31 it follows that

$$\lim_{x\to+\infty} \frac{x^b}{e^{ax}} = 0.$$

d) $$\lim_{x\to+\infty} \frac{\ln(\ln x)}{\ln x} = \lim_{x\to+\infty} \frac{\dfrac{1}{x\ln x}}{\dfrac{1}{x}} = \lim_{x\to+\infty} \frac{1}{\ln x} = 0.$$

Example 6.67. *Determine*

a) $\lim\limits_{x\to+\infty} x(e^{1/x} - 1)$;

b) $\lim\limits_{x\to 0+} \sin x \ln \cot x$;

c) $\lim\limits_{x\to 0+} x^\alpha \ln^\beta\left(\dfrac{1}{x}\right)$, $\alpha, \beta > 0$;

d) $\lim\limits_{x\to+\infty} x\left(\pi - 2\arcsin\left(\dfrac{x}{\sqrt{x^2+1}}\right)\right)$.

Solutions.

a) $$\lim_{x\to+\infty} x(e^{1/x} - 1) = \lim_{x\to+\infty} \frac{e^{1/x} - 1}{\dfrac{1}{x}} = \lim_{x\to+\infty} \frac{e^{1/x}\left(\dfrac{-1}{x^2}\right)}{\dfrac{-1}{x^2}} = \lim_{x\to+\infty} e^{1/x} = 1.$$

b) $$\lim_{x\to 0+} \sin x \ln \cot x = \lim_{x\to 0+} \frac{\ln\cot x}{\dfrac{1}{\sin x}} = \lim_{x\to 0+} \frac{\dfrac{-1}{\cot x \sin^2 x}}{\dfrac{-\cos x}{\sin^2 x}} = \lim_{x\to 0+} \frac{\sin x}{\cos^2 x} = 0.$$

c) Let us find first for $k \in \mathbf{N}$ the limit

$$\lim_{x\to 0+} x^{\alpha/k} \ln\left(\frac{1}{x}\right).$$

Applying the L'Hospital rule, we get

$$\lim_{x\to 0+} x^{\alpha/k} \ln\left(\frac{1}{x}\right) = \lim_{x\to 0+} \frac{-\dfrac{1}{x}}{-\dfrac{\alpha}{k}x^{-\alpha/k-1}} = \frac{k}{\alpha} \lim_{x\to 0+} x^{\alpha/k} = 0. \tag{6.27}$$

Thus also

$$\lim_{x\to 0+} x^\alpha \ln^k\left(\frac{1}{x}\right) = 0.$$

Choosing $k \in \mathbf{N}$ such that $k > \beta$, it holds

$$0 \le x^\alpha \left|\ln^\beta x\right| \le x^\alpha \left|\ln^k x\right|, \quad x > 0,$$

which, in view of (6.27), implies finally

$$\lim_{x\to 0+} x^\alpha \ln^\beta\left(\frac{1}{x}\right) = 0.$$

d) $$\lim_{x\to+\infty} x\left(\pi - 2\arcsin\left(\frac{x}{\sqrt{x^2+1}}\right)\right) = \lim_{x\to+\infty} \frac{\pi - 2\arcsin\left(\dfrac{x}{\sqrt{x^2+1}}\right)}{\dfrac{1}{x}}$$

$$= \lim_{x\to+\infty} \frac{-2\dfrac{1}{\sqrt{1-\frac{x^2}{x^2+1}}} \cdot \dfrac{\sqrt{x^2+1} - \dfrac{2x^2}{2\sqrt{x^2+1}}}{x^2+1}}{-\dfrac{1}{x^2}} = \lim_{x\to+\infty} \frac{-2\dfrac{1}{x^2+1}}{-\dfrac{1}{x^2}} = 2.$$

Example 6.68. *Determine*

a) $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x}\right)$; b) $\lim\limits_{x\to 0}\left(\dfrac{1}{e^x - 1} - \dfrac{1}{x}\right)$.

Solutions. In these two examples the undetermined forms "$\infty - \infty$" appear, so firstly, we have to perform some transformations in order to obtain limits which can be found by using the L'Hospital rule.

a) $$\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right) = \lim_{x\to 0} \frac{\sin^2 x - x^2}{x^2 \sin^2 x}$$

$$= \lim_{x\to 0} \frac{-\sin^2 x + \cos^2 x - 1}{\sin^2 x + 4x\sin x\cos x + x^2(\cos^2 x - \sin^2 x)}$$

$$= \lim_{x\to 0} \frac{-2\sin^2 x}{\sin^2 x + 4x\sin x\cos x + x^2(\cos^2 x - \sin^2 x)}$$

$$= \lim_{x\to 0} \frac{-2}{1 + 4\dfrac{x}{\sin x}\cos x + \dfrac{x^2}{\sin^2 x}(\cos^2 x - \sin^2 x)} = -\frac{1}{3}.$$

b) $$\lim_{x\to 0}\left(\frac{1}{e^x - 1} - \frac{1}{x}\right) = \lim_{x\to 0} \frac{x - e^x + 1}{xe^x - x} = \lim_{x\to 0} \frac{1 - e^x}{xe^x + e^x - 1}$$

$$= \lim_{x\to 0} \frac{-e^x}{xe^x + 2e^x} = -1/2.$$

Example 6.69. *Determine*

a) $\lim\limits_{x\to 0}(1+2x)^{1/x}$; b) $\lim\limits_{x\to 1} x^{1/(1-x)}$;

c) $\lim\limits_{x\to 0}(\cos x)^{1/x^2}$; d) $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{1/x^2}$;

e) $\lim\limits_{x\to 0}(\dfrac{2}{\pi}\arccos x)^{1/x}$; f) $\lim\limits_{x\to 0}\left(\dfrac{(1+x)^{1/x}}{e}\right)^{1/x}$;

g) $\lim\limits_{x\to+\infty}\left(\dfrac{2}{\pi}\arctan x\right)^{x}$; h) $\lim\limits_{x\to 0}(1+\tanh x)^{1/x}$.

Solutions. In these cases we have the undetermined forms "1^∞".

a) The function $y = (1+2x)^{1/x}$ can be written in a form

$$\ln y = \frac{1}{x}\ln(1+2x), \quad \text{wherefrom it holds}$$

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln(1+2x)}{x} = \lim_{x\to 0}\frac{\dfrac{2}{1+2x}}{1} = 2.$$

Therefore we have

$$\lim_{x\to 0}(1+2x)^{1/x} = \exp\left(\lim_{x\to 0}\ln y\right) = \mathrm{e}^2.$$

b) Let us consider the function $y = \exp(\ln u)$, where $u = x^{1/(1-x)}$. It holds

$$\lim_{x\to 1}\exp(\ln u) = \exp\left(\lim_{x\to 1}\ln u\right) = \exp\left(\lim_{x\to 1}\frac{1}{1-x}\ln x\right).$$

So we obtain

$$\left(\lim_{x\to 1}\frac{\ln x}{1-x} = \lim_{x\to 1}\frac{\dfrac{1}{x}}{-1} = -1\right) \Rightarrow \lim_{x\to 0} x^{1/(1-x)} = \mathrm{e}^{-1}.$$

c) In this case we have

$$\lim_{x\to 0}(\cos x)^{1/x^2} = \exp\left(\lim_{x\to 0}\frac{\ln\cos x}{x^2}\right) = \exp\left(\lim_{x\to 0}\frac{-\dfrac{\sin x}{\cos x}}{2x}\right) = \mathrm{e}^{-1/2}.$$

d) $$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2} = \exp\left(\lim_{x\to 0}\frac{\ln\dfrac{\sin x}{x}}{x^2}\right) = \exp\left(\lim_{x\to 0}\frac{\dfrac{x}{\sin x}\cdot\dfrac{x\cos x - \sin x}{x^2}}{2x}\right)$$

$$= \exp\left(\frac{1}{2}\lim_{x\to 0}\frac{x\cos x - \sin x}{x^2\sin x}\right) = \exp\left(\frac{1}{2}\lim_{x\to 0}\frac{-x\sin x}{2x\sin x + x^2\cos x}\right) = \mathrm{e}^{-1/6}.$$

e) $$\lim_{x\to 0}\left(\frac{2}{\pi}\arccos x\right)^{1/x} = \exp\left(\lim_{x\to 0}\frac{\ln\left(\dfrac{2}{\pi}\arccos x\right)}{x}\right)$$

$$= \exp\left(\lim_{x\to 0}\left(-\frac{1}{\arccos x}\cdot\frac{1}{\sqrt{1-x^2}}\right)\right) = \mathrm{e}^{-2/\pi}.$$

f) $$\lim_{x\to 0}\left(\frac{(1+x)^{1/x}}{\mathrm{e}}\right)^{1/x} = \exp\left(\lim_{x\to 0}\frac{1}{x}\ln\left(\frac{(1+x)^{1/x}}{\mathrm{e}}\right)\right)$$

$$= \exp\left(\lim_{x\to 0}\frac{\ln(1+x) - x}{x^2}\right) = \exp\left(\lim_{x\to 0}\frac{\dfrac{1}{x+1} - 1}{2x}\right) = \mathrm{e}^{-1/2}.$$

g) $$\lim_{x\to+\infty}\left(\frac{2}{\pi}\arctan x\right)^x = \exp\left(\lim_{x\to+\infty} x\ln\left(\frac{2}{\pi}\arctan x\right)\right)$$
$$= \exp\left(\lim_{x\to+\infty}\frac{\dfrac{1}{1+x^2}\cdot\dfrac{1}{\arctan x}}{-\dfrac{1}{x^2}}\right) = \mathrm{e}^{-2/\pi}.$$

h) $$\lim_{x\to 0}(1+\tanh x)^{1/x} = \exp\left(\lim_{x\to 0}\frac{\dfrac{1}{1+\tanh x}\cdot\dfrac{1}{\cosh^2 x}}{1}\right) = \mathrm{e}.$$

Example 6.70. *Determine*

a) $\lim\limits_{x\to 0+}(\mathrm{e}^x-1)^x$; **b)** $\lim\limits_{x\to 0+} x^x$;

c) $\lim\limits_{x\to 0+} x^{1/\ln\sinh x}$; **d)** $\lim\limits_{x\to 0+}(\arcsin x)^{\tan x}$.

Solutions. These examples are of the undetermined form "0^0".

a) For the function $y=(\mathrm{e}^x-1)^x$ we have $\ln y = x\ln(\mathrm{e}^x-1)$. Hence

$$\lim_{x\to 0+}\ln y = \lim_{x\to 0+}\frac{\ln(\mathrm{e}^x-1)}{\dfrac{1}{x}} = \lim_{x\to 0+}\frac{\dfrac{\mathrm{e}^x}{\mathrm{e}^x-1}}{\dfrac{-1}{x^2}}$$
$$= \lim_{x\to 0+}\frac{-x^2\mathrm{e}^x}{\mathrm{e}^x-1} = \lim_{x\to 0+}\frac{-(2x\mathrm{e}^x+x^2\mathrm{e}^x)}{\mathrm{e}^x} = 0.$$

Therefore we have

$$\lim_{x\to 0+}(\mathrm{e}^x-1)^x = \mathrm{e}^{\lim\limits_{x\to 0}\ln y} = \mathrm{e}^0 = 1.$$

b) From the equalities

$$\lim_{x\to 0+}\frac{\ln x}{\dfrac{1}{x}} = \lim_{x\to 0+}\frac{\dfrac{1}{x}}{\dfrac{-1}{x^2}} = \lim_{x\to 0+}(-x) = 0,$$

we obtain

$$\lim_{x\to 0+} x^x = \mathrm{e}^{\lim\limits_{x\to 0}\ln y} = \mathrm{e}^0 = 1.$$

c) In this case we have

$$\lim_{x\to 0+}\frac{\ln x}{\ln\sinh x} = \lim_{x\to 0+}\frac{\dfrac{1}{x}}{\dfrac{1}{\sinh x}\cosh x} = 1, \qquad \text{hence} \qquad \lim_{x\to 0+} x^{1/\ln\sinh x} = \mathrm{e}.$$

d) From

$$\lim_{x\to 0+} \frac{\ln(\arcsin x)}{\cot x} = \lim_{x\to 0+} \frac{\dfrac{1}{\arcsin x}\cdot\dfrac{1}{\sqrt{1-x^2}}}{-\dfrac{1}{\sin^2 x}} = 0,$$

we obtain $\lim\limits_{x\to 0+} (\arcsin x)^{\tan x} = 1.$

Example 6.71. *Determine*

a) $\lim\limits_{x\to +\infty} x^{1/x}$; b) $\lim\limits_{x\to +\infty} (3x^2 + 3^x)^{1/x}$.

Solutions. These examples are of the undetermined form "∞^0".

a) From

$$\lim_{x\to+\infty} \frac{\ln x}{x} = \lim_{x\to+\infty} \frac{1}{x} = 0,$$

we get $\lim\limits_{x\to+\infty} x^{1/x} = \mathrm{e}^0 = 1.$

b) From

$$\lim_{x\to+\infty} \frac{\ln(3x^2+3^x)}{x} = \lim_{x\to+\infty} \frac{6x + 3^x \ln 3}{3x^2+3^x} = \ln 3,$$

we obtain $\lim\limits_{x\to+\infty} (3x^2+3^x)^{1/x} = 3.$

Example 6.72. *Determine*

a) $\lim\limits_{x\to+\infty} \dfrac{2x - \sin x}{2x + \sin x}$; b) $\lim\limits_{x\to 0} \dfrac{x^3 \sin \frac{1}{x}}{\sin^2 x}$.

Solutions.

a) For $x > 2$, we have

$$\frac{2x-1}{2x+1} \le \frac{2x - \sin x}{2x + \sin x} \le \frac{2x+1}{2x-1} \quad \text{and} \quad \lim_{x\to+\infty} \frac{2x-1}{2x+1} = \lim_{x\to+\infty} \frac{2x+1}{2x-1} = 1,$$

hence

$$\lim_{x\to+\infty} \frac{2x - \sin x}{2x + \sin x} = 1.$$

In this case we can not apply L'Hospital's rule, because the following limit

$$\lim_{x\to+\infty} \frac{(2x - \sin x)'}{(2x + \sin x)'} = \lim_{x\to+\infty} \frac{2 - \cos x}{2 + \cos x},$$

does not exist, namely there is no $\lim\limits_{x\to+\infty} \cos x$.

b) Again, the following limit of the quotient of the derivatives

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)} \quad \text{does not exist,}$$

so we can not apply L'Hospital's rule. Still, we can calculate the limit:

$$\lim_{x\to 0}\frac{x^3\sin\dfrac{1}{x}}{\sin^2 x} = \lim_{x\to 0}\frac{x\sin\dfrac{1}{x}}{\dfrac{\sin^2 x}{x^2}} = 0.$$

6.5 Local extrema and monotonicity of functions

6.5.1 Basic notions

Theorem 6.73. *Let f be a function that is continuous on a closed interval $[a,b]$ and differentiable on the open interval (a,b).*

- *If $f'(x) > 0$ for all $x \in (a,b)$, then f is increasing on $[a,b]$.*
- *If $f'(x) < 0$ for all $x \in (a,b)$, then f is decreasing on $[a,b]$.*

For a proof of this theorem, see Example 6.49.

Theorem 6.74. *Let f be a function that is continuous on a closed interval $[a,b]$ and differentiable on the open interval (a,b).*

- *If f is increasing on $[a,b]$, then $f'(x) \geq 0$ for all $x \in (a,b)$,*
- *If f is decreasing on $[a,b]$, then $f'(x) \leq 0$ for all $x \in (a,b)$.*

Definition 6.75. *A number $c \in (a,b)$ is a* **critical number** *(or:* **critical point***) of a function $f : [a,b] \to \mathbf{R}$ if either $f'(c) = 0$ or $f'(c)$ does not exist.*

A necessary condition for the existence of extrema gives the following.

Theorem 6.76. *If a function $f : [a,b] \to \mathbf{R}$ is continuous on $[a,b]$ and has a local extremum (maximum or minimum) at a number $c \in (a,b)$, then c is a critical number of f.*

Sufficient conditions for extrema are given in the following two statements.

Theorem 6.77. The first derivative test.
Suppose c is a critical number of a function f and (a,b) is an open interval containing c. Suppose further that f is continuous on $[a,b]$ and differentiable on (a,b), except possibly at c.

- *If $f'(x) > 0$, for $x \in (a,c)$ and $f'(x) < 0$ for $x \in (c,b)$, then $f(c)$ is a local maximum of f.*

- *If $f'(x) < 0$, for $x \in (a, c)$ and $f'(x) > 0$ for $x \in (c, b)$ then $f(c)$ is a local minimum of f.*

- *If either $f'(x) < 0$, or $f'(x) > 0$ for all $\in (a, b)$, except possibly in c, then $f(c)$ is not a local extremum of a function f.*

Theorem 6.78. The second derivative test.
Suppose that f is a two times differentiable function on (a, b), containing c and let $f'(c) = 0$.

- *If $f''(c) < 0$, then f has a local maximum at c.*

- *If $f''(c) > 0$, then f has a local minimum at c.*

Theorem 6.79. *Assume a function f has at the point c all derivatives up to the order $n > 2$, and it holds*

$$f'(c) = f''(c) = \ldots = f^{(n-1)}(c) = 0, \quad \textit{but} \quad f^{(n)}(c) \neq 0.$$

- *If n is an* even number, *then c* **is** *an extremum point.*

- *If n is an* odd number, *then c* **is not** *an extremum point.*

6.5.2 Examples and exercises

Example 6.80. *Determine the critical numbers of the function*

$$f(x) = (x+5)^2 \sqrt[3]{x-4}, \quad x \in \mathbf{R}.$$

Solution. The first derivative of f for $x \neq 4$ is

$$\begin{aligned} f'(x) &= 2(x+5)(x-4)^{1/3} + (x+5)^2 \frac{1}{3(x-4)^{2/3}} \\ &= \frac{(x+5)^2 + 6(x+5)(x-4)}{3(x-4)^{2/3}} = \frac{(x+5)(7x-19)}{3(x-4)^{2/3}}. \end{aligned}$$

Clearly we see that $f'(x) = 0$ for $x = -5$ and $x = 19/7$. Also the first derivative does not exist for $x = 4$, hence the given function has three critical numbers, namely $x_1 = -5$, $x_2 = 19/7$, $x_3 = 4$.

Example 6.81. *Determine the local extrema of the following functions and the intervals on which they are monotonically increasing or decreasing.*

a) $f(x) = \dfrac{x}{x^2 - 6x + 16}$, $x \in \mathbf{R}$; **b)** $f(x) = \dfrac{x}{3} - \sqrt[3]{x}$, $x \in \mathbf{R}$;

c) $f(x) = x^{2/3}(x^2 - 8)$, $x \in \mathbf{R}$; **d)** $f(x) = x - \sin x$, $x \in \mathbf{R}$;

e) $f(x) = \sin^2 x$, $x \in \mathbf{R}$; **f)** $f(x) = x + \ln x$, $x > 0$.

Solutions.

a) From $f'(x) = \dfrac{16-x^2}{(x^2-6x+16)^2}$, it follows that $f'(x) = 0$ for $x_1 = 4$ and $x_2 = -4$, and the critical points are $A(4, 1/2)$, $B(-4, -1/19)$. Using Theorem 6.78, from

$$f''(x) = \frac{2x^3 - 96x + 192}{(x^2-6x+16)^3}, \quad f''(4) < 0, \; f''(-4) > 0,$$

it follows that the function f has a maximum at the point A, and it has a minimum at B. Further on, we have

$$f'(x) > 0 \iff 16 - x^2 > 0,$$

$$(\text{resp. } \; f'(x) < 0 \iff 16 - x^2 < 0),$$

hence

- the function f increases for $x \in (-4, 4)$;
- the function f decreases for $x \in (-\infty, -4) \cup (4, +\infty)$.

b) In this case we have

$$\begin{aligned} f'(x) &= \frac{1}{3} - \frac{1}{3}x^{-2/3} = \frac{1}{3}\left(1 - \frac{1}{x^{2/3}}\right) \\ &= \frac{x^{2/3}-1}{3x^{2/3}} = \frac{x^2-1}{3x^{2/3}(x^{4/3} + x^{2/3} + 1)}. \end{aligned}$$

From $f'(x) = 0$, for $x_1 = -1$ and $x_2 = 1$ we obtain that the function has a maximum at the point $A(-1, 2/3)$, and a minimum at $B(1, -2/3)$.

- The function decreases for $x^2 - 1 < 0$, i.e., for $x \in (-1, 1)$.
- The function f increases for $x \in (-\infty, -1) \cup (1, +\infty)$.

Note that $x = 0$ is a critical point of f, but it has no extremum there.

c) From

$$f'(x) = \frac{8(x^2-2)}{3x^{1/3}}, \quad x \neq 0,$$

it follows that the critical points of f are $x_1 = \sqrt{2}$, $x_2 = -\sqrt{2}$ and $x_3 = 0$. From the second derivative of f it follows that f has local minimums at x_1 and at x_2.

From the sign of f' it follows that f has a local maximum at the point $x_3 = 0$, even though the first derivative does not exist at 0.

Thus we obtain

- the function f increases when $f'(x) > 0$, i.e., for

$$x \in \left(-\sqrt{2}, 0\right) \cup \left(\sqrt{2}, +\infty\right);$$

- the function f decreases for $x \in \left(-\infty, -\sqrt{2}\right) \cup \left(0, \sqrt{2}\right)$.

d) Since $f'(x) = 1 - \cos x$, the function f increases on $\mathbf{R} \setminus \{2k\pi | \, k \in \mathbf{Z}\}$. Also, from

$$f''(2k\pi) = \sin(2k\pi) = 0, \text{ and } f'''(2k\pi) = \cos(2k\pi) = 1 \neq 0,$$

it follows that the function f increases on the whole $\mathbf{R}$. The function has no extrema points.

e) From $f'(x) = 2\sin x \cos x = \sin 2x$, it follows that $f'(x) = 0$ for $x_k = \dfrac{k\pi}{2}$, $k \in \mathbf{Z}$. Since $f''(x) = 2\cos(2x)$ and $f''(x_k) = 2(-1)^k$, we obtain that at the points x_{2j}, the function f has minimums, and at the points x_{2j+1}, the function f has maximums, for $j \in \mathbf{Z}$. Further on,

- the function f increases if $x \in \displaystyle\bigcup_{k\in\mathbf{Z}} \left(k\pi, \frac{(2k+1)\pi}{2}\right)$;
- the function f decreases if $x \in \displaystyle\bigcup_{k\in\mathbf{Z}} \left(\frac{(2k+1)\pi}{2}, (k+1)\pi\right)$.

f) From $f'(x) = 1 + \dfrac{1}{x} = \dfrac{1+x}{x}$, it follows that the function has no extrema points. The function f increases on its whole domain, i.e., on $(0, +\infty)$.

Example 6.82. *Examine the extrema point of the following functions.*

a) $f(x) = \cosh x + \cos x$; **b)** $f(x) = x^3 + x^4$.

Solutions.

a) The function is differentiable for every $x \in \mathbf{R}$. We also have $f'(x) = \sinh x - \sin x$ and it holds

$$f'(x) = 0 \quad \text{for} \quad x = 0.$$

Thus

$$f''(x) = \cosh x - \cos x, \quad f''(0) = 0, \quad \text{and} \quad f'''(x) = \sinh x + \sin x, \quad f'''(0) = 0,$$

but

$$f^{(4)}(x) = \cosh x + \cos x, \quad f^{(4)}(0) = 2 \neq 0.$$

This means that the given function has an extremum point at $x = 0$. Since $f^{(4)}(0) > 0$, at the point $x = 0$ f has a minimum.

b) The critical points are $x_1 = 0$ and $x_2 = -3/4$, since $f'(x) = 3x^2 + 4x^3$.

From

$$f''(x) = 6x + 12x^2, \quad f''(0) = 0, \quad \text{and} \quad f'''(x) = 6 + 24x, \quad f'''(0) = 6,$$

it follows that this function has no extremum at $x_1 = 0$. Actually, $A(0,0)$ is a point of inflection, see Subsection 6.6.1.

The function has a minimum at x_2.

Example 6.83. *Examine the monotonicity and find the extrema points for the function $y = f(x)$ given parametrically by*

$$x = \frac{t^3}{t^2+1}, \quad y = \frac{t^3 - 2t^2}{t^2+1}, \quad t \in \mathbf{R}.$$

Solution. From

$$x'_t = \frac{t^2(t^2+3)}{(t^2+1)^2} \quad \text{and} \quad y'_t = \frac{t(t-1)(t^2+t+4)}{(t^2+1)^2}$$

and the derivative of a parametric function (see Subsection 6.1.1), we have

$$y'_x = \frac{y'_t}{x'_t} = \frac{(t-1)(t^2+t+4)}{t(t^2+3)}, \quad t \neq 0.$$

Thus it follows that $y'_x = 0$ for $t = 1$, i.e., for $x = 1/2$. In fact the function f has two critical points. Namely, $x_1 = 0$ is obtained from $t = 0$, and the other, $x_2 = 1/2$, from $t = 1$.

- If $t < 0$, then $x < 0$, and from $y'_x > 0$ it follows that the function increases for $x \in (-\infty, 0)$.
- If $t \in (0,1)$, then $x \in (0,1/2)$, and from $y'_x < 0$ it follows that the function decreases for $x \in (0,1/2)$. Therefore, at the point $x_1 = 0$ the function has a maximum.
- If $t \in (1,+\infty)$, then $x \in (1/2,+\infty)$ and from $y'_x > 0$ it follows that the function increases for $x \in (1/2,+\infty)$. Therefore at the point $x = 1/2$ the function has a minimum.

Example 6.84. *Let us consider the following functions.*

$$f(x) = \begin{cases} \mathrm{e}^{-1/x^2}, & x \neq 0; \\ 0, & x = 0, \end{cases} \qquad g(x) = \begin{cases} x\mathrm{e}^{-1/x^2}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

Prove the following.

a) $f^{(n)}(0) = g^{(n)}(0) = 0.$

b) *The function f has a minimum at $x = 0$, but the function g has no extrema at $x = 0$.*

Solutions.

a) It is clear that

$$f'(x) = \frac{2}{x^3}\mathrm{e}^{-1/x^2},\quad f''(x) = \left(\frac{4}{x^6} - \frac{6}{x^4}\right)\mathrm{e}^{-1/x^2}, \ldots,\quad f^{(n)}(x) = R_{3n}\left(\frac{1}{x}\right)\mathrm{e}^{-1/x^2},$$

where $R_{3n}\left(\frac{1}{x}\right)$ is a polynomial of $\frac{1}{x}$ and of order $3n$. Applying the L'Hospital rule k-times, we get

$$\lim_{x\to 0}\frac{\mathrm{e}^{-1/x^2}}{x^k} = 0,\quad k \in \mathbf{N}.$$

So we obtain that $f'(0) = 0$. Analogously, we can show that $g'(0) = 0$.

b) The function f has a minimum at $x = 0$, because it holds $f(x) > 0,\ x \neq 0$. The function g has no extrema at $x = 0$, because we have

$$g(x) > 0,\ x > 0, \quad \text{and} \quad g(x) < 0,\ x < 0.$$

Example 6.85. *Determine the biggest term of the sequence given by*

a) $f_n = \dfrac{\sqrt{n}}{n + 1995}$; **b)** $f_n = \dfrac{n^2}{n^3 + 200}$;

c) $f_n = \dfrac{n^{12}}{\mathrm{e}^n}$; **d)** $f_n = \sqrt[n]{n}$.

Solutions.

a) The function

$$f(x) = \frac{\sqrt{x}}{x + 1995} \quad \text{with} \quad f'(x) = \frac{1995 - x}{2\sqrt{x}(x + 1995)^2}$$

has its maximum at the point $x = 1995$, because $f'(x) > 0$ for $x < 1995$ (the function is increasing), and $f'(x) < 0$, for $x > 1995$ (the function is decreasing). So n=1995.

b) The function $f(x) = \dfrac{x^2}{x^3 + 200}$ has a critical point at $x = \sqrt[3]{400} \approx 7.37$. Since $f(7) \approx 9.02 \cdot 10^{-2} > f(8) \approx 8.99 \cdot 10^{-2}$, it follows that $n = 7$.

c) From $f(x) = \dfrac{x^{12}}{\mathrm{e}^x}$, $f'(x) = \dfrac{12x^{11} - x^{12}}{\mathrm{e}^x}$, it follows that $n = 12$.

d) In this case we consider the function $f(x) = x^{1/x}$, whose derivative is

$$f'(x) = x^{(1/x)-2}(1 - \ln x).$$

Thus the maximum of this function is for $x = \mathrm{e}$. Since $f_2 = \sqrt{2} \approx 1.41$, $f_3 = \sqrt[3]{3} \approx 1.44$, it follows that $n = 3$.

Example 6.86. *If a function ϕ is a monotonically increasing differentiable function and $|f'(x)| \leq \phi'(x)$ for $x \geq x_0$, then it holds*

$$|f(x) - f(x_0)| \leq \phi(x) - \phi(x_0) \quad \textit{for} \quad x \geq x_0.$$

Solution. The functions f and ϕ satisfy the conditions of the Cauchy theorem 6.50, so we can write

$$\left|\frac{f(x) - f(x_0)}{\phi(x) - \phi(x_0)}\right| = \left|\frac{f'(\xi)}{\phi'(\xi)}\right| \leq 1, \quad \xi \in (x_0, x).$$

Since the function ϕ is increasing we obtain

$$|f(x) - f(x_0)| \leq |\phi(x) - \phi(x_0)| = \phi(x) - \phi(x_0) \quad \text{for} \quad x \geq x_0.$$

Example 6.87. *If*

(i) *the functions f and g are n-times differentiable;*

(ii) $f^{(k)}(x_0) = g^{(k)}(x_0)$ *for* $k = 0, 1, \ldots, n-1$;

(iii) $f^{(n)}(x) > g^{(n)}(x)$ *for* $x > x_0$,

then prove $\quad f(x) > g(x), \quad x > x_0.$

Solutions. The function given by $\phi^{(n-1)}(x) = f^{(n-1)}(x) - g^{(n-1)}(x)$ satisfies the conditions

$$\phi^{(n-1)}(x_0) = 0 \quad \text{and} \quad \phi^{(n)}(x) > 0, \quad \text{for } x > x_0.$$

Applying the Lagrange theorem to the function $\phi^{(n-1)}(x)$ on the interval $[x_0, x]$, we get

$$\phi^{(n-1)}(x) - \phi^{(n-1)}(x_0) = \phi^{(n)}(\xi)(x - x_0), \quad \xi \in (x_0, x),$$

hence $\phi^{(n-1)}(x) > 0$. This implies

$$f^{(n-1)}(x) > g^{(n-1)}(x) \quad \text{for} \quad x > x_0.$$

Similarly we obtain

$$f^{(n-2)}(x) > g^{(n-2)}(x) \quad \text{for} \quad x > x_0,$$

and continuing this procedure finally we get

$$f(x) > g(x) \quad \text{for} \quad x > x_0.$$

Example 6.88. *Prove the following inequalities:*

a) $x - \frac{x^2}{2} < \ln(1+x) < x, \;\; x > 0;$ **b)** $\ln(1+x) > \frac{x}{x+1}, \;\; x > 0;$

c) $x^a - 1 > a(x-1), \;\; a > 2, \; x > 1;$ **d)** $\sqrt[n]{x} - \sqrt[n]{a} < \sqrt[n]{x-a}, \;\; n > 1, \; x > a > 0.$

Solutions.

a) Let us denote by $f(x) = x - \frac{x^2}{2}$, $g(x) = \ln(1+x)$ and $h(x) = x$. It holds that $f(0) = g(0) = h(0) = 0$ and

$$f'(x) = 1 - x < g'(x) = \frac{1}{1+x} < h'(x) = 1, \quad x > 0.$$

From Example 6.87 we get $f(x) < g(x) < h(x), \;\; x > 0$, i.e.,

$$x - \frac{x^2}{2} < \ln(1+x) < x, \quad x > 0.$$

b) In this case we denote $f(x) = \ln(1+x)$ and $g(x) = \frac{x}{x+1}$. Then from $f(0) = g(0) = 0$ and

$$f'(x) = \frac{1}{1+x} > g'(x) = \frac{1}{(1+x)^2},$$

we get

$$f(x) > g(x), \quad \text{i.e.,} \quad \ln(1+x) > \frac{x}{x+1} \quad \text{for} \quad x > 0.$$

c) Denoting by $f(x) = x^a - 1$ and $g(x) = a(x-1)$, in view of $f(1) = 0 = g(1)$ and

$$f'(x) = ax^{a-1} > a = g'(x), \;\; x > 1,$$

we obtain the given inequality.

d) If we take $f(x) = \sqrt[n]{x} - \sqrt[n]{a}$ and $g(x) = \sqrt[n]{x-a}$, then this functions satisfy the conditions of Example 6.87 and the given inequality follows.

Example 6.89. *Prove the following inequalities.*

a) $\cos x \geq 1 - \frac{x^2}{2}, \quad x \in \mathbf{R};$ **b)** $\arctan x \leq x, \quad x \geq 0;$

c) $\frac{2}{\pi}x < \sin x, \quad 0 < x < \frac{\pi}{2};$ **d)** $\sin x + \tan x > 2x, \quad 0 < x < \frac{\pi}{2}.$

Solutions.

a) Taking $f(x) = \cos x$, $g(x) = 1 - \dfrac{x^2}{2}$, we have $f(0) = g(0) = 1$ and $f'(x) = -\sin x$, $g'(x) = -x$. The function $h(x) = x - \sin x$ was considered in Example 6.81 d). Since this function is a monotonically increasing one and $h(0) = 0$, it holds

$$x - \sin x > 0, \quad x > 0, \quad \text{i.e.,} \quad x > \sin x, \quad x > 0.$$

From $f'(x) > g'(x)$, $x > 0$, it follows that

$$f(x) > g(x), \quad x > 0, \quad \text{i.e.,} \quad \cos x > 1 - \frac{x^2}{2}, \quad x > 0.$$

The functions f and g are even, which means that the last inequality holds for $x < 0$, also.

b) Putting $f(x) = x - \arctan x$, $x \geq 0$, we get $f'(x) = 1 - \dfrac{1}{1+x^2} > 0$, hence $f(x) \geq f(0)$ for $x \geq 0$, which implies the given inequality.

c) It is enough to show that the function f given by

$$f(x) = \sin x - \frac{2x}{\pi}, \quad x \in \left[0, \frac{\pi}{2}\right],$$

has exactly two zeros on $\left[0, \dfrac{\pi}{2}\right]$. Let us suppose that f has a zero $x_1 \in \left(0, \dfrac{\pi}{2}\right)$, besides the zeros $x_0 = 0$ and $x_2 = \dfrac{\pi}{2}$. Then from Rolle's theorem it follows that there exist at least two points $x_3 \in (0, x_1)$ and $x_4 \in \left(x_1, \dfrac{\pi}{2}\right)$, such that $f'(x_3) = f'(x_4) = 0$. But the equation $f'(x) = 0$, i.e., $\cos x - \dfrac{2}{\pi} = 0$, has only one solution on the interval $\left(0, \dfrac{\pi}{2}\right)$. So the function f is of the same sign on this interval. Since it holds (for instance)

$$f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} - \frac{2}{3} > 0,$$

we get $\sin x > \dfrac{2x}{\pi}$ for $x \in \left(0, \dfrac{\pi}{2}\right)$.

d) The functions $f(x) = \sin x + \tan x$ and $g(x) = 2x$ satisfy the conditions of the Example 6.87, since $f(0) = g(0) = 0$, $f'(0) = 2 = g'(0)$, and

$$f''(x) = -\sin x + 2\frac{\sin x}{\cos^3 x} = \frac{\sin x(2 - \cos^3 x)}{\cos^3 x} > 0 = g''(x), \quad x \in \left(0, \frac{\pi}{2}\right).$$

Therefore it holds $f(x) > g(x)$, $x \in \left(0, \dfrac{\pi}{2}\right)$.

6.6 Concavity

6.6.1 Basic notions

Definition 6.90.

- *A function* $f : (a,b) \to \mathbf{R}$ *is* **concave upward** *on* (a,b) *if for every pair* $x_1, x_2 \in (a,b)$ *and for every* $\alpha \in (0,1)$ *it holds*

$$f(\alpha x_1 + (1-\alpha)x_2) < \alpha f(x_1) + (1-\alpha) f(x_2).$$

- *A function* $f : (a,b) \to \mathbf{R}$ *is* **concave downward** *on* (a,b) *if for every pair* $x_1, x_2 \in (a,b)$ *and for every* $\alpha \in (0,1)$ *it holds*

$$f(\alpha x_1 + (1-\alpha)x_2) > \alpha f(x_1) + (1-\alpha) f(x_2).$$

Theorem 6.91. *If a function* f *is two times differentiable on an open interval* (a,b), *then* f *is*

- *concave upward if for all* $x \in (a,b)$ *it holds* $f''(x) > 0$;
- *concave downward if for all* $x \in (a,b)$ *it holds* $f''(x) < 0$.

Theorem 6.92. *Let* f *be a differentiable function on an interval* (a,b).

Then the function f *is concave upward iff the graph of* f *is above the tangent line through every point of an interval* (a,b);

The function f *is concave downward iff the graph of* f *is below the tangent line through every point of an interval* (a,b).

Definition 6.93. *Let* f *be a continuous function on an interval* (a,b), *differentiable at the point* $c \in (a,b)$. *If the function* f *changes its concavity at* c, *then* c *is called* **inflection point** *of* f.

Then the point $A(c, f(c))$ *is called* **inflection point of the graph** *of* f.

A necessary condition for the existence of the point of inflection gives the following theorem.

Theorem 6.94. *If* c *is a point of inflection of a function* f, *then either*

$$f''(c) = 0, \quad \text{or} \quad f''(c) \quad \text{does not exist.}$$

Sufficient conditions for the existence of the point of inflection give the following two theorems.

Theorem 6.95. *Let a function* f *be differentiable at a point* c, *and two times differentiable on an open interval* (a,b) *containing* c, *except possibly at* c. *Then* c *a point of inflection of* f, *if one of the following assumptions holds:*

a) $f''(x) > 0$ *if* $a < x < c$ *and* $f''(x) < 0$ *if* $c < x < b$; *or*

b) $f''(x) < 0$ *if* $a < x < c$ *and* $f''(x) > 0$ *if* $c < x < b$.

Theorem 6.96. *If a function f has at a point c all derivatives up to the order n, $n > 2$, and it holds*

$$f''(c) = f'''(c) = \ldots = f^{(n-1)}(c) = 0, \quad \textit{while} \quad f^{(n)}(c) \neq 0,$$

then if n is an odd number, *c* is *a point of inflection, while if n is an* even number, *c* is not *a point of inflection.*

6.6.2 Examples and exercises

Example 6.97. *Find the points of inflection and the intervals of concavity upward and downward of the following functions.*

a) $f(x) = x^4 - 6x^2 + 5x + 3,\ x \in \mathbf{R}$; **b)** $f(x) = \dfrac{|5x-5|}{x\sqrt{x}},\ x > 0$;

c) $f(x) = 3\exp(\sqrt[3]{x}),\ x \in \mathbf{R}$; **d)** $f(x) = x\sin(\ln x),\ x > 0$.

Solutions.

a) From $f''(x) = 12(x^2-1)$, it follows that $f''(x) > 0$ for $x \in (-\infty, -1) \bigcup (1, +\infty)$, where the function f is concave upward, and for $x \in (-1, 1)$ the function is concave downward.
The points $A(1,3)$, $B(-1,-7)$ are the points of inflection, because $f''(x) = 0$, for $x_{1,2} = \pm 1$ and from $f'''(x) = 24x$, it follows that $f'''(x) \neq 0$, for $x = \pm 1$.

b) The function f is defined on the interval $(0, +\infty)$ and can be written as

$$f(x) = 5 \cdot \begin{cases} \dfrac{x-1}{x\sqrt{x}}, & x \geq 1; \\[2ex] \dfrac{1-x}{x\sqrt{x}}, & 0 < x < 1, \end{cases}$$

and is differentiable at every point $x > 0$ except at $x = 1$. The second derivative has the form

$$f''(x) = \frac{15}{4} \cdot \begin{cases} \dfrac{x-5}{x^3\sqrt{x}}, & x > 1; \\[2ex] \dfrac{5-x}{x^3\sqrt{x}}, & 0 < x < 1, \end{cases}$$

and it holds $f''(5) = 0$, and

$$f''(x) > 0,\ x \in (0,1), \qquad f''(x) < 0,\ x \in (1,5), \qquad f''(x) > 0,\ x \in (5, +\infty).$$

Therefore, on the intervals $(0,1)$ and $(5,+\infty)$ the function is concave upward and on the interval $(1,5)$ the function is concave downward. The point $A(5,4/\sqrt{5})$ is a point of inflection, but the point $B(1,0)$ is not, because the function f has no first derivative at $x=1$.

c) In this case we have

$$f'(x)=\frac{\exp(\sqrt[3]{x})}{\sqrt[3]{x^2}},\qquad f''(x)=\exp(\sqrt[3]{x})\cdot\frac{\sqrt[3]{x}(\sqrt[3]{x}-2)}{3x^2},$$

for $x\neq 0$. The function f satisfies $\lim\limits_{x\to 0} f'(x)=+\infty$, but

$$\lim_{x\to 0-} f''(x)=+\infty \quad\text{and}\quad \lim_{x\to 0+} f''(x)=-\infty.$$

Also, we have $f''(x)=0$, for $x=8$. From

$$f''(x)>0,\ \ x<0,\quad f''(x)<0,\ \ 0<x<8,\quad f''(x)>0,\ \ x>8,$$

we get that the function f has two points of inflection, $A(0,3)$ and $B(8,3\mathrm{e}^2)$, and it is concave upward for $x\in(-\infty,0)\bigcup(8,+\infty)$ and concave downward for $x\in(0,8)$.

d) The domain of this function is the interval $(0,+\infty)$ and

$$f''(x)=(\sin(\ln x)+\cos(\ln x))'=\frac{1}{x}(\cos(\ln x)-\sin(\ln x))=\frac{\sqrt{2}}{x}\cos\left(\ln x+\frac{\pi}{4}\right).$$

The function is concave upward (i.e., $f''(x)>0$) if

$$-\frac{3\pi}{4}+2k\pi<\ln x<\frac{\pi}{4}+2k\pi,\quad k=0,\pm1,\pm2,...,$$

or

$$\exp(\frac{-3\pi}{4}+2k\pi)<x<\exp(\frac{\pi}{4}+2k\pi),$$

and is concave downward if

$$\frac{\pi}{4}+2k\pi<\ln x<\frac{5\pi}{4}+2k\pi,\quad k=0,\pm1,\pm2,...,$$

or

$$\exp(\frac{\pi}{4}+2k\pi)<x<\exp(\frac{5\pi}{4}+2k\pi).$$

The inflection points are

$$A_k=\left(\exp\left(\frac{\pi}{4}+2k\pi\right),\frac{\exp\left(\frac{\pi}{4}+2k\pi\right)}{\sqrt{2}}\right),\quad k=0,\pm1,\pm2,...,$$

$$B_k=\left(\exp\left(\frac{5\pi}{4}+2k\pi\right),\frac{-\exp\left(\frac{5\pi}{4}+2k\pi\right)}{\sqrt{2}}\right),\quad k=0,\pm1,\pm2,....$$

Example 6.98. *Determine the inflection points of the function*

$$x = 3 + \cot t, \quad y = -2\sin t + \frac{1}{\sin t}, \quad 0 < t < \pi.$$

Solution. From

$$x'_t = -\frac{1}{\sin^2 t}, \quad y'_t = -\frac{\cos t(2\sin^2 t + 1)}{\sin^2 t},$$

and

$$y'_x = \frac{y'_t}{x'_t}, \quad y''_{xx} = (y'_x)'_t \cdot \frac{1}{x'_t},$$

we obtain

$$y'_x = \cos t(2\sin^2 t + 1), \quad (y'_x)'_t = 3\cos 2t \sin t,$$

and finally

$$y''_{xx} = -3\sin^3 t \cos 2t.$$

Therefore $y''_{xx} = 0$ for $t = \frac{\pi}{4}$, and $t = \frac{3\pi}{4}$. Since

- $y''_{xx} < 0$ for $t \in (0, \frac{\pi}{4})$, the function is concave downward there;
- $y''_{xx} > 0$ for $t \in (\frac{\pi}{4}, \frac{3\pi}{4})$, the function is concave upward there;
- $y''_{xx} < 0$ for $t \in (\frac{3\pi}{4}, \pi)$, the function is concave downward there,

it follows that the points $t = \frac{\pi}{4}$ and $t = \frac{3\pi}{4}$, or $A(4, 0)$ and $B(2, 0)$, respectively, are the points of inflection for the function $y = y(x)$.

Example 6.99. *Prove that if $f''(x) > 0$ for all $x \in (a, b)$, then f is concave upward on (a, b). I.e., prove then that for every pair $x_1, x_2 \in (a, b)$ and arbitrary numbers α_1, α_2 such that*

$$\alpha_1 > 0, \quad \alpha_2 > 0, \quad \alpha_1 + \alpha_2 = 1 \tag{6.28}$$

the following inequality holds

$$f(\alpha_1 x_1 + \alpha_2 x_2) < \alpha_1 f(x_1) + \alpha_2 f(x_2). \tag{6.29}$$

(See Theorem 6.91.)

Solution. Let us suppose that $f''(x) > 0$ for all $x \in (a, b)$ and let the arbitrary numbers α_1, α_2, satisfy the conditions given by relation (6.28). If x_1, x_2 belong to the interval (a, b), and $x_1 < x_2$, then $\alpha_1 x_1 + \alpha_2 x_2 \in (a, b)$ also. Using Lagrange's theorem we obtain

$$f(\alpha_1 x_1 + \alpha_2 x_2) - f(x_1) = f'(\xi_1)\alpha_2(x_2 - x_1),$$

$$f(x_2) - f(\alpha_1 x_1 + \alpha_2 x_2) = f'(\xi_2)\alpha_1(x_2 - x_1),$$

where $x_1 < \xi_1 < \alpha_1 x_1 + \alpha_2 x_2$, and $\alpha_1 x_1 + \alpha_2 x_2 < \xi_2 < x_2$. Multiplying the last two equalities with α_1, α_2 respectively, we get

$$\alpha_1 f(\alpha_1 x_1 + \alpha_2 x_2) - \alpha_1 f(x_1) = f'(\xi_1)\alpha_1\alpha_2(x_2 - x_1),$$

$$\alpha_2 f(x_2) - \alpha_2 f(\alpha_1 x_1 + \alpha_2 x_2) = f'(\xi_2)\alpha_1\alpha_2(x_2 - x_1),$$

wherefrom

$$\alpha_1 f(x_1) + \alpha_2 f(x_2) = (\alpha_1 + \alpha_2) f(\alpha_1 x_1 + \alpha_2 x_2) + \alpha_1\alpha_2(x_2 - x_1)(f'(\xi_2) - f'(\xi_1)).$$

The function f has a second derivative on the interval (a, b) so we can apply Lagrange's theorem again, giving us

$$f'(\xi_2) - f'(\xi_1) = (\xi_2 - \xi_1) f''(\xi_3), \quad \xi_3 \in (\xi_1, \xi_2).$$

Therefore, we can write

$$\alpha_1 f(x_1) + \alpha_2 f(x_2) = f(\alpha_1 x_1 + \alpha_2 x_2) + \alpha_1\alpha_2(x_2 - x_1)(\xi_2 - \xi_1) f''(\xi_3)).$$

From the last relation, in view of $f''(x) > 0$, follows relation (6.29).

Example 6.100. *Prove the following inequalities.*

a) $$\frac{a^n + b^n}{2} > \left(\frac{a+b}{2}\right)^n, \quad a > 0,\ b > 0,\ a \neq b,\ \ n \in \mathbf{N};$$

b) $$\frac{e^x + e^y}{2} > e^{(x+y)/2}, \quad x \neq y.$$

Solutions. If $\alpha_1 = \alpha_2 = \dfrac{1}{2}$, then from Definition 6.90 we obtain the following inequalities for $x_1, x_2 \in (a, b)$:

$$\frac{f(x_1) + f(x_2)}{2} > f\left(\frac{x_1 + x_2}{2}\right). \tag{6.30}$$

for the concave upward function on the interval (a, b), and

$$\frac{f(x_1) + f(x_2)}{2} < f\left(\frac{x_1 + x_2}{2}\right), \tag{6.31}$$

for the concave downward function on the interval (a, b).

Since the functions

a) $f(x) = x^n, \quad x > 0, \ \ n \in \mathbf{N};$ **b)** $f(x) = e^x, \quad x \in \mathbf{R},$

are concave upward ($f''(x) > 0$, for x in their domains), relation (6.30) holds.

Example 6.101. *If f is a concave upward function on the interval (a,b) and the points $x_1, x_2, \ldots, x_n$, belong to (a,b) and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are nonnegative numbers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, then the so-called* **Iensen's inequality** *holds:*

$$f(\alpha_1 x_1 + \cdots + \alpha_n x_n) \le \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n). \tag{6.32}$$

Prove.

Solution. If $n = 2$, then we have inequality (6.29). In this relation we have equality if one of α_1 or α_2 is equal to zero. If $n > 2$ then we apply mathematical induction. We can denote by $\beta = \alpha_2 + \cdots + \alpha_n > 0$ and it holds $\frac{\alpha_2}{\beta} + \cdots + \frac{\alpha_n}{\beta} = 1$.

Let us suppose that

$$f(\frac{\alpha_2}{\beta}x_2 + \cdots + \frac{\alpha_n}{\beta}x_n) \le \frac{\alpha_2}{\beta}f(x_2) + \cdots + \frac{\alpha_n}{\beta}f(x_n).$$

Then we have

$$f\left(\alpha_1 x_1 + \cdots + \alpha_n x_n\right) = f\left(\alpha_1 x_1 + \beta\left(\frac{\alpha_2}{\beta}x_2 + \cdots + \frac{\alpha_n}{\beta}x_n\right)\right)$$

$$\le \alpha_1 f(x_1) + \beta\left(\frac{\alpha_2}{\beta}f(x_2) + \cdots + \frac{\alpha_n}{\beta}f(x_n)\right) \le \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n).$$

Example 6.102. *Prove*

a) $\alpha_1 \ln x_1 + \cdots + \alpha_n \ln x_n \le \ln(\alpha_1 x_1 + \cdots + \alpha_n x_n)$,

$$x_i \ge 0, \quad \alpha_i \ge 0, \; i = 1,2,...n, \quad \sum_{i=1}^{n} \alpha_i = 1;$$

b) $\sqrt[n]{x_1 \cdots x_n} \le \dfrac{x_1 + \cdots + x_n}{n}, \quad x_i \ge 0, \; i = 1,...,n.$

Solutions.

a) The function $f(x) = \ln x$ is concave downward on $(0,+\infty)$ and for this function it holds

$$f(\alpha_1 x_1 + \cdots + \alpha_n x_n) \ge \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n). \tag{6.33}$$

Therefore we can write

$$\alpha_1 \ln(x_1) + \cdots + \alpha_n \ln(x_n) \le \ln(\alpha_1 x_1 + \cdots + \alpha_n x_n). \tag{6.34}$$

b) From relation (6.34) we obtain

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} \le \alpha_1 x_1 + \cdots + \alpha_n x_n,$$

wherefrom for $\alpha_1 = \ldots = \alpha_n = \frac{1}{n}$ we obtain

$$\sqrt[n]{x_1 \cdots x_n} \le \frac{x_1 + \cdots + x_n}{n}, \quad x_i \ge 0, \quad i = 1,...,n.$$

The last relation is the well known connection between the arithmetic and the geometric mean, see Example 1.33.

Chapter 7

Graphs of functions

Example 7.1. *Determine and sketch the graph of the function*

$$f(x) = (x+2)^2(x-1)^3.$$

Solution. The graph of f is sketched in Figure 7.1.

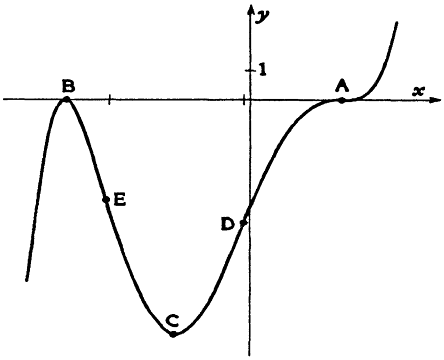

Fig. 7.1. $f(x) = (x+2)^2(x-1)^3$

- The domain of the given function is $\mathbf{R}$.
- This is neither an odd nor an even function.
- The zeros of the function f are found from $f(x) = 0$, hence

$$(x+2)^2(x-1)^3 = 0 \iff x \in \{1, -2\}.$$

- The first derivative of f is the function

$$f'(x) = (x+2)(x-1)^2(5x+4), \quad x \in \mathbf{R},$$

with zeros $x_1 = 1,\ x_2 = -2$ and $x_3 = -\dfrac{4}{5}$.

- The critical points of f are $A(1,0)$, $B(-2,0)$, $C(-0.8,-8.40)$.
- The function f is increasing when $f'(x)>0$, namely when

$$(x+2)(5x+4)>0 \iff x\in(-\infty,-2)\bigcup(-\frac{4}{5},+\infty).$$

- The function f is decreasing when

$$(x+2)(5x+4)<0, \quad \text{i.e., for} \quad x\in\left(-2,-\frac{4}{5}\right).$$

- The second derivative of the function f is

$$f''(x)=2(x-1)(10x^2+16x+1),$$

hence $f''(x)=0$ for $x_1=1$, $x_4\approx -0.07$ and $x_5\approx -1.53$.

  - Since $f''(-2)=-54<0$, the critical point $B(-2,0)$ is a maximum.
  - From $f''(-0.8)>0$, it follows that the function f has a minimum at the point $C(-0.8,-8.40)$.

    Let us remark that the first derivative does not change its sign at $x=1$, and though it holds $f'(1)=0$, it has no extremum at $x=1$.
  - The function f is concave upward if $f''(x)=2(x-1)(10x^2+16x+1)>0$, i.e., for $x\in(x_5,x_4)\bigcup(1,+\infty)$.
  - The function f is concave downward if $f''(x)=2(x-1)(10x^2+16x+1)<0$, i.e., for $x\in(-\infty,x_5)\bigcup(x_4,1)$.
  - From the previous conclusions it follows that the function f has three points of inflection $A(1,0)$ $D(x_4,f(x_4))$ $E(x_5,f(x_5))$. It holds $f(x_4)\approx -4.56$ and $f(x_5)\approx -3.58$.

- The given function has no line asymptotes.

Example 7.2. *Determine and sketch the graph of the function*

$$f(x)=\frac{x+1}{x^2+1}.$$

Solution. The graph of f is sketched in Figure 7.2.

- The domain of the given function is $\mathbf{R}$.
- This is neither an odd nor an even function.
- The function f is equal to zero for $x=-1$.
- The first derivative of f is the function

$$f'(x)=\frac{-x^2-2x+1}{(x^2+1)^2}.$$

- The zeros of the first derivative are $x_1 = -1 + \sqrt{2}$, $x_2 = -1 - \sqrt{2}$, so the critical points of f are

$$A\left(-1+\sqrt{2}, \frac{\sqrt{2}+1}{2}\right) \quad \text{and} \quad B\left(-1-\sqrt{2}, -\frac{\sqrt{2}-1}{2}\right).$$

- The function f is increasing for $-x^2 - 2x + 1 > 0$, i.e., for

$$x \in (-1-\sqrt{2}, -1+\sqrt{2}).$$

- The function f is decreasing for $-x^2 - 2x + 1 < 0$, i.e., for

$$x \in (-\infty, -1-\sqrt{2}) \bigcup (-1+\sqrt{2}, +\infty).$$

- The second derivative of the function f is

$$f''(x) = \frac{2x^3 + 6x^2 - 6x - 2}{(x^2+1)^3} = 2\frac{(x-1)(x^2+4x+1)}{(x^2+1)^3}.$$

  - Since $f''(-1+\sqrt{2}) < 0$, the function has a maximum at the critical point $A\left(-1+\sqrt{2}, \frac{\sqrt{2}+1}{2}\right)$.
  - Since $f''(-1-\sqrt{2}) > 0$ the function has a minimum at the critical point $B\left(-1-\sqrt{2}, -\frac{\sqrt{2}-1}{2}\right)$.
  - The function f is concave upward if $(x-1)(x^2+4x+1) > 0$, and this is satisfied for $x \in (-2-\sqrt{3}, -2+\sqrt{3}) \bigcup (1, +\infty)$.
  - The function f is concave downward if $f''(x) < 0$, i.e.,

$$x \in (-\infty, -2-\sqrt{3}) \bigcup (-2+\sqrt{3}, 1).$$

  - From the previous conclusions it follows that the function f has three points of inflection, namely

$$C(1,1), \quad D\left(-2+\sqrt{3}, \frac{1+\sqrt{3}}{4}\right) \quad \text{and} \quad E\left(-2-\sqrt{3}, \frac{1-\sqrt{3}}{4}\right).$$

- Asymptotes.

  - The function has no vertical asymptotes.
  - The function has a horizontal asymptote $y = 0$ both when $x \to +\infty$ and $x \to -\infty$, because

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x+1}{x^2+1} = 0.$$

– The function has no slanted asymptote.

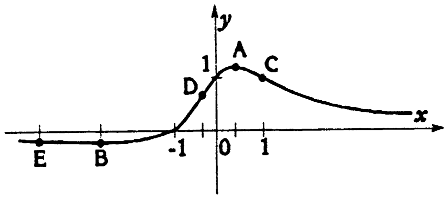

Fig. 7.2. $f(x) = \dfrac{x+1}{x^2+1}$

Example 7.3. *Determine and sketch the graph of the function*

$$f(x) = \frac{x-5}{x^2-9}.$$

Solution. The graph of f is sketched in Figure 7.3.

- The domain of the given function is $(-\infty, -3) \cup (-3, 3) \cup (3, +\infty)$.
- This is neither an odd nor an even function.
- The function f is equal to zero for $x = 5$.
- The first derivative of f is the function

$$f'(x) = \frac{-x^2 + 10x - 9}{(x^2-9)^2}.$$

  – The zeros of the first derivative are $x = 1$ and $x = 9$. The critical points of f are $A(1, 1/2)$ and $B(9, 1/18)$.
  – The function f is increasing for $f'(x) > 0$, i.e., for $x \in (1, 3) \cup (3, 9)$.
  – The function f is decreasing for $x \in (-\infty, -3) \cup (-3, 1) \cup (9, +\infty)$.

- The second derivative of the function f is

$$f''(x) = \frac{2(x^3 - 15x^2 + 27x - 45)}{(x^2-9)^3}.$$

  – Since $f''(1) > 0$, the function has a minimum at the critical point $A(1, 1/2)$.
  – Since $f''(9) < 0$, the function has a maximum at the critical point $B(9, 1/18)$.
  – In this case we can not order directly the zeros of the numerator of $f''(x)$. From $f''(13) < 0$ and $f''(14) > 0$ it follows that for some $x_0 \in (13, 14)$ it holds $f''(x_0) = 0$.

– The function f is concave upward if

$$f''(x) > 0 \iff x \in (-3,3) \cup (x_0, +\infty),$$

because

$$g(x) = 2(x^3 - 15x^2 + 27x - 45) < 0 \quad \text{for} \quad x < x_0$$

and $\quad x^2 - 9 < 0 \quad$ for $\quad x \in (-3,3)$. Also, we have

$$g(x) > 0 \quad \text{for} \quad x > x_0, \quad \text{and} \quad x^2 - 9 > 0 \quad \text{for} \quad x \in (x_0, +\infty).$$

– The function is concave downward for $x \in (-\infty, -3) \cup (3, x_0)$.

- Asymptotes.

– The function has two vertical asymptotes $x = -3$ and $x = 3$, since

$$\lim_{x \to -3-} \frac{x-5}{x^2-9} = -\infty; \qquad \lim_{x \to -3+} \frac{x-5}{x^2-9} = +\infty;$$

$$\lim_{x \to 3-} \frac{x-5}{x^2-9} = +\infty; \qquad \lim_{x \to 3+} \frac{x-5}{x^2-9} = -\infty.$$

– Function has a horizontal asymptote $y = 0$ both when $x \to +\infty$ and $x \to -\infty$, because

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{x-5}{x^2-9} = 0.$$

– The function has no slanted asymptote.

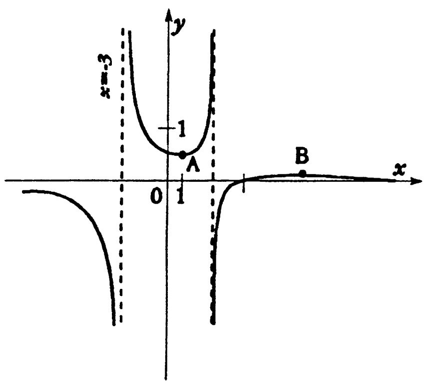

Fig. 7.3. $f(x) = \dfrac{x-5}{x^2-9}$

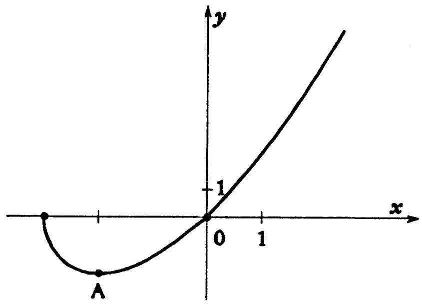

Fig. 7.4. $f(x) = x\sqrt{x+3}$

Example 7.4. *Determine and sketch the graph of the function*

$$f(x) = x\sqrt{x+3}.$$

Solution. The graph of f is sketched in Figure 7.4.

- The domain of the given function is $[-3, +\infty)$.
- The zeros of the function f are $x_1 = 0, \quad x_2 = -3$.
- The first derivative of f is the function

$$f'(x) = \frac{x}{2\sqrt{x+3}} + \sqrt{x+3} = \frac{3x+6}{2\sqrt{x+3}}.$$

  - The first derivative has a zero for $3x + 6 = 0$, i.e., $x = -2$, so the critical point of f is $A(-2, -2)$.
  - The function f is increasing for $3x + 6 > 0$, i.e., for $x > -2$, because $2\sqrt{x+3} > 0$ for every x from the domain.
  - The function f is decreasing for $-3 < x < -2$.

- The second derivative of the function f is

$$f''(x) = \frac{3(x+4)}{4(x+3)^{3/2}}.$$

  - From $f''(-2) > 0$ it follows that $A(-2, -2)$ is a minimum for the function.
  - Since $f''(x) > 0$ for every x from domain, the function is concave upward.

- There are no line asymptotes.

Example 7.5. *Determine and sketch the graph of the function*

$$f(x) = \frac{x}{\sqrt[3]{x^2 - 1}}.$$

Solution. The graph of f is sketched in Figure 7.5.

- The domain of the given function is $(-\infty, -1) \bigcup (-1, 1) \bigcup (1, +\infty)$.
- This is an odd function, because

$$f(-x) = \frac{-x}{\sqrt[3]{(-x)^2 - 1}} = -f(x).$$

- The zero of the function f is $x = 0$.
- The first derivative of f is the function

$$f'(x) = \frac{x^2 - 3}{3\sqrt[3]{(x^2 - 1)^4}}.$$

- The zeros of the first derivative are $x_1 = \sqrt{3}$ and $x_2 = -\sqrt{3}$. The critical points of f are $A\left(\sqrt{3}, \frac{\sqrt{3}}{\sqrt[3]{2}}\right)$, $B\left(-\sqrt{3}, -\frac{\sqrt{3}}{\sqrt[3]{2}}\right)$.
- The function f is increasing for $x \in (-\infty, -\sqrt{3}) \cup (\sqrt{3}, +\infty)$.
- The function f is decreasing for $x \in (-\sqrt{3}, -1) \cup (-1, 1) \cup (1, \sqrt{3})$.

• The second derivative of the function f is

$$f''(x) = \frac{2x(9 - x^2)}{9\sqrt[3]{(x^2-1)^7}},$$

hence $f''(x) = 0$ for $x_3 = 0$, $x_4 = 3$ and $x_5 = -3$.

- Since $f''(\sqrt{3}) > 0$, the function has a minimum at $A\left(\sqrt{3}, \frac{\sqrt{3}}{\sqrt[3]{2}}\right)$.
- Since $f''(-\sqrt{3}) < 0$, the function has a maximum at $B\left(-\sqrt{3}, -\frac{\sqrt{3}}{\sqrt[3]{2}}\right)$.
- The function f is concave upward if $x \in (-\infty, -3) \cup (-1, 0) \cup (1, 3)$.
- The function is concave downward for $x \in (-3, -1) \cup (0, 1) \cup (3, +\infty)$.
- The points of inflection are $O(0,0)$, $C(3, 3/2)$ and $D(-3, -3/2)$.

• Asymptotes.

- The function has two vertical asymptotes, namely $x = 1$ and $x = -1$, because

$$\lim_{x \to -1-} f(x) = \lim_{x \to -1-} \frac{x}{\sqrt[3]{x^2-1}} = -\infty, \qquad \lim_{x \to -1+} \frac{x}{\sqrt[3]{x^2-1}} = +\infty,$$

$$\lim_{x \to 1-} \frac{x}{\sqrt[3]{x^2-1}} = -\infty, \qquad \lim_{x \to 1+} \frac{x}{\sqrt[3]{x^2-1}} = +\infty.$$

- The function has no slanted asymptote, because

$$\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{\frac{x}{\sqrt[3]{x^2-1}}}{x} = 0.$$

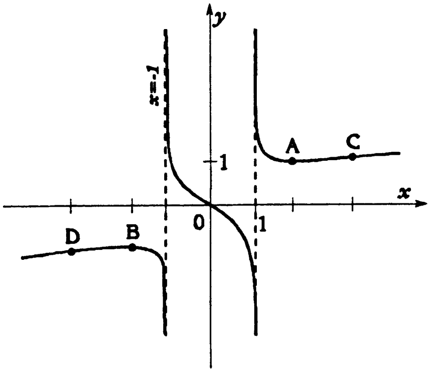

Fig. 7.5. $f(x) = \dfrac{x}{\sqrt[3]{x^2-1}}$

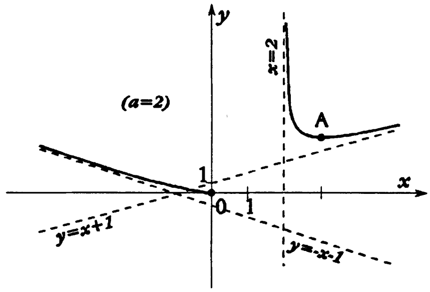

Fig. 7.6. $f(x) = \sqrt{\dfrac{x^3}{x-a}}$

Example 7.6. *Determine and sketch the graph of the function*

$$f(x) = \sqrt{\frac{x^3}{x-a}}, \quad a > 0.$$

Solution. The graph of f is sketched in Figure 7.6, for $a = 2$.

- The domain of the given function is $(-\infty, 0] \bigcup (a, +\infty)$, because

$$f(x) = \sqrt{\frac{x^3}{x-a}} = |x| \cdot \sqrt{\frac{x}{x-a}}, \quad \text{hence} \quad x(x-a) > 0 \vee x = 0.$$

- The function f is equal to zero for $x = 0$, where it has a minimum.
- The first derivative of f is the function

$$f'(x) = \left(x - \frac{3a}{2}\right)\sqrt{\frac{x}{(x-a)^3}}.$$

  - The zero of the first derivative is $x = \dfrac{3a}{2}$. The critical point of f is $A\left(\dfrac{3a}{2}, \dfrac{3a\sqrt{3}}{2}\right)$.
  - The function f is increasing for $x \in (\dfrac{3a}{2}, +\infty)$.
  - The function f is decreasing for $x \in (-\infty, 0) \bigcup (a, \dfrac{3a}{2})$.

- The second derivative of the function f is

$$f''(x) = \sqrt{\frac{x-a}{x^3}} \cdot \frac{3a^2x}{4(x-a)^3} = \frac{3a^2}{4\sqrt{x(x-a)^5}},$$

and it has no zeros.

  – Since $f''(\frac{3a}{2}) > 0$, the function has a minimum at the point

$$A\left(\frac{3a}{2}, \frac{3\sqrt{3}a}{2}\right).$$

  – The function f is concave upward on the whole domain.

- Asymptotes.

  – The function has a vertical asymptote $x = a$, but not at $x = 0$, because

$$\lim_{x\to a+} \sqrt{\frac{x^3}{x-a}} = +\infty, \qquad \lim_{x\to 0-} f(x) = \lim_{x\to 0-} \sqrt{\frac{x^3}{x-a}} = 0.$$

  – The function has slanted asymptotes $y = x + \frac{a}{2}$ when $x \to +\infty$ and $y = -x - \frac{a}{2}$ when $x \to -\infty$, because

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{\sqrt{\frac{x^3}{x-a}}}{x} = 1, \quad \lim_{x\to-\infty} \frac{\sqrt{\frac{x^3}{x-a}}}{x} = -1,$$

$$\lim_{x\to+\infty} (f(x) - x) = \lim_{x\to+\infty} \left(\sqrt{\frac{x^3}{x-a}} - x\right) = \frac{a}{2},$$

$$\lim_{x\to-\infty} \left(\sqrt{\frac{x^3}{x-a}} + x\right) = -\frac{a}{2}.$$

Example 7.7. *Determine and sketch the graph of the function*

$$f(x) = \sqrt[3]{x^2-1} - \sqrt[3]{x^2}.$$

Solution. The graph of f is sketched in Figure 7.7.

- The domain of the given function is $\mathbf{R}$.
- This is an even function.
- There are no zeros for this function.

- The first derivative is

$$f'(x) = \frac{2}{3}\frac{\sqrt[3]{x^4} - \sqrt[3]{(x^2-1)^2}}{\sqrt[3]{x}\sqrt[3]{(x^2-1)^2}}.$$

  - The critical points are

$$A(-1,-1),\ B\left(-\frac{1}{\sqrt{2}}, -\sqrt[3]{4}\right),\quad C(0,-1),\ D\left(\frac{1}{\sqrt{2}}, -\sqrt[3]{4}\right) \text{ and } E(1,-1),$$

  because the first derivative has two zeros $x_{1,2} = \frac{1}{\sqrt{2}}$, but is not defined at the points $x_3 = 0,\ x_4 = -1$ and $x_5 = 1$.

  - The function f is increasing for $x \in (-\frac{\sqrt{2}}{2}, 0) \bigcup (\frac{\sqrt{2}}{2}, \infty)$.
  - The function f is decreasing for $x \in (-\infty, -\frac{\sqrt{2}}{2}) \bigcup (0, \frac{\sqrt{2}}{2})$.
  - The function has minimums at the points

$$B\left(-\frac{1}{\sqrt{2}}, -\sqrt[3]{4}\right),\ D\left(\frac{1}{\sqrt{2}}, -\sqrt[3]{4}\right).$$

  - The function has a maximum at the point $C(0,-1)$, even though it does not have a horizontal tangent line at $x = 0$. Namely, we have

$$\lim_{x\to 0-} f'(x) = \lim_{x\to 0-} \frac{2}{3}\frac{\sqrt[3]{x^4} - \sqrt[3]{(x^2-1)^2}}{\sqrt[3]{x}\sqrt[3]{(x^2-1)^2}} = +\infty,$$

  and

$$\lim_{x\to 0+} f'(x) \lim_{x\to 0+} \frac{2}{3}\frac{\sqrt[3]{x^4} - \sqrt[3]{(x^2-1)^2}}{\sqrt[3]{x}\sqrt[3]{(x^2-1)^2}} = -\infty.$$

Note that the tangent lines through the points A and E are parallel with the $y-$axis.

- The second derivative is

$$f''(x) = \frac{2}{9}\left(\frac{1}{\sqrt[3]{x^4}} - \frac{3+x^2}{\sqrt[3]{(x^2-1)^5}}\right).$$

  - The function is concave upward for $x \in (-1, 0) \cup (0, 1)$.
  - The function is concave downward for $x \in (-\infty, -1) \cup (1, +\infty)$.
  - From

$$\lim_{x\to -1+} f''(x) = \lim_{x\to -1+} \frac{2}{9}\left(\frac{1}{\sqrt[3]{x^4}} - \frac{3+x^2}{\sqrt[3]{(x^2-1)^5}}\right) = +\infty,$$

$$\lim_{x\to -1-} f''(x) = -\infty,\quad \lim_{x\to 1-} f''(x) = +\infty \text{ and } \lim_{x\to 1+} f''(x) = -\infty,$$

  it follows that the function changes its concavity for $x = 1$ and $x = -1$.

- Asymptotes.
  - The function has no vertical asymptote.
  - It has a horizontal asymptote $y = 0$ both when $x \to +\infty$ and $x \to -\infty$, because

$$\lim_{x\to\pm\infty}(\sqrt[3]{x^2-1}-\sqrt[3]{x^2}) = \lim_{x\to\pm\infty}\frac{-1}{\sqrt[3]{(x^2-1)^2}+\sqrt[3]{x^2(x^2-1)}+\sqrt[3]{x^4}} = 0.$$

  - There is no slanted asymptote.

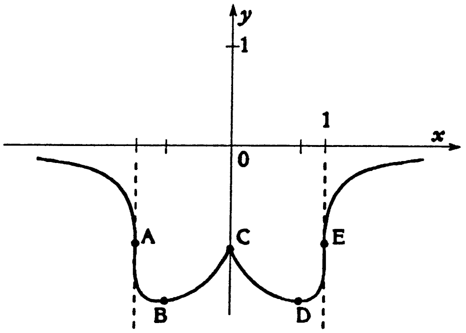

Fig. 7.7. $f(x) = \sqrt[3]{x^2-1}-\sqrt[3]{x^2}$

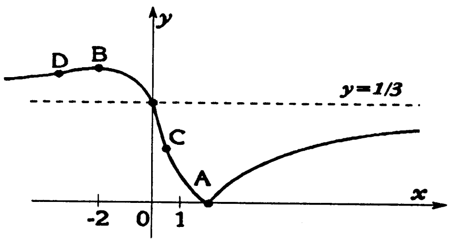

Fig. 7.8. $f(x) = \dfrac{|2-x|}{3\sqrt{x^2+4}}$

Example 7.8. *Determine and sketch the graph of the function*

$$f(x) = \frac{|2-x|}{3\sqrt{x^2+4}}.$$

Solution. The graph of f is sketched in Figure 7.8. The given function can be considered as the following two functions

$$f_1(x) = \frac{2-x}{3\sqrt{x^2+4}}, \qquad \text{for} \quad x \le 2,$$

$$f_2(x) = \frac{x-2}{3\sqrt{x^2+4}}, \qquad \text{for} \quad x > 2.$$

Let us remark that it holds $f_1(2) = \lim_{x\to 2+} f_2(x)$ and therefore this is a continuous function on $\mathbf{R}$.

- The domain of the given function is $\mathbf{R}$.
- The function $f_1(x)$ has a zero at the point $A(2, 0)$, while the function $f_2(x)$ has no zeros.

- The first derivatives are

$$f_1'(x) = -\frac{2}{3}\frac{x+2}{(x^2+4)^{3/2}}, \quad x \le 2, \quad \text{and} \quad f_2'(x) = \frac{2}{3}\frac{x+2}{(x^2+4)^{3/2}}, \quad x > 2.$$

  - The first derivative of f_1 has a zero at $x = -2$.
  - The first derivative of the function f_2 has no zeros.
  - The function f_1 is increasing in $x \in (-\infty, -2)$.
  - The function f_1 is decreasing in $x \in (-2, 2)$.
  - The function f_1 has a maximum at $x_B = -2$.
  - The function f_1 has a minimum at $x_A = 2$.
  - The function f_2 is increasing in $x \in (2, +\infty)$.

- The second derivatives are

$$f_1''(x) = \frac{4}{3} \cdot \frac{x^2+3x-2}{(x^2+4)^{5/2}}, \quad x \le 2, \quad \text{and} \quad f_2''(x) = -\frac{4}{3} \cdot \frac{x^2+3x-2}{(x^2+4)^{5/2}}, \quad x > 2.$$

  - The function f_1 has two points of inflection, namely

$$C\left(\frac{\sqrt{17}-3}{2}, 0.41\right) \quad \text{and} \quad D\left(-\frac{3+\sqrt{17}}{2}, 0.45\right).$$

  - The function f_1 is concave upward if

$$f_1''(x) > 0, \quad \text{i.e., for} \quad x \in (-\infty, x_D) \cup (x_C, 2).$$

  - The function f_1 is concave downward for $x \in (x_D, x_C)$.
  - The function f_2 has no points of inflection.
  - The function f_2 is concave downward if $f_2''(x) < 0$, i.e., for every $x \in (2, +\infty)$.

- Asymptotes.
  - The function has no vertical asymptote.
  - The function has a horizontal asymptote $y = 1/3$ both when $x \to -\infty$ and $x \to +\infty$, because

$$\lim_{x \to -\infty} \frac{2-x}{3\sqrt{x^2+4}} = 1/3 \qquad \lim_{x \to +\infty} \frac{x-2}{3\sqrt{x^2+4}} = 1/3.$$

  - There is no slanted asymptote.

Example 7.9. *Determine and sketch the graph of the function*

$$f(x) = 2\sin x + \cos 2x.$$

Solution. The graph of f is sketched in Figure 7.9.

- The domain is $(-\infty, +\infty)$.
- This function is periodic with basic period 2π, and therefore it is enough to examine this function on the interval $[0, 2\pi]$.
- This is neither an odd nor an even function.
- The zeros are calculated from

$$f(x) = 0, \quad \text{i.e.,} \quad 2\sin x + \cos 2x = 0.$$

After the transformation $\cos 2x = \cos^2 x - \sin^2 x$, we obtain

$$2\sin x + \cos^2 x - \sin^2 x = 0, \quad \text{i.e.,} \quad 1 + 2\sin x - 2\sin^2 x = 0.$$

Taking $t = \sin x$, we obtain the equation $1 + 2t - 2t^2 = 0$, with the solutions

$$t_{1,2} = \frac{1 \pm \sqrt{3}}{2}.$$

The value $t_1 = \dfrac{1+\sqrt{3}}{2} > 1$ can not be equal to $\sin x$ for any $x \in \mathbf{R}$, and therefore we are taking $t_2 = \dfrac{1-\sqrt{3}}{2}$.
Thus the zeros of f are $x_1 \approx 3.42$, and $x_2 \approx 5.91$.

- The first derivative is

$$f'(x) = 2\cos x - 2\sin 2x = 2\cos x - 4\sin x \cos x = 2\cos x(1 - 2\sin x),$$

and it has zeros when

$$\cos x = 0 \quad \text{or} \quad 1 - 2\sin x = 0.$$

Thus the critical points of f on the interval $[0, 2\pi]$ are

$$A\left(\frac{\pi}{2}, 1\right), \quad B(\frac{3\pi}{2}, -3), \quad C\left(\frac{5\pi}{6}, 3/2\right) \quad \text{and} \quad D\left(\frac{\pi}{6}, 3/2\right).$$

- The second derivative of the function is $f'' = -2\sin x - 4\cos 2x$ and it holds

$$f''(\frac{\pi}{2}) = 2, \quad f''(\frac{3\pi}{2}) = 6, \quad f''(\frac{\pi}{6}) = -3, \quad f''(\frac{5\pi}{6}) = -3,$$

wherefrom

  – the point $A\left(\dfrac{\pi}{2}, 1\right)$ is a local minimum;

  – the point $B(\dfrac{3\pi}{2}, -3)$ is a local minimum;

– the point $C\left(\frac{\pi}{6}, 3/2\right)$ is a local a maximum;

– the point $D(\frac{\pi}{6}, 3/2)$ is a local maximum.

- The given function has no asymptotes.

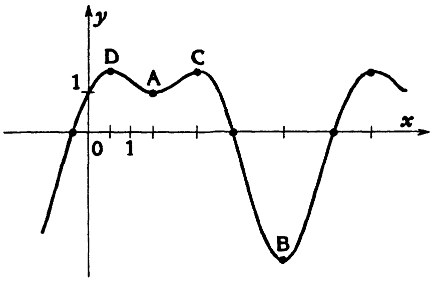

Fig. 7.9. $f(x) = 2\sin x + \cos 2x$

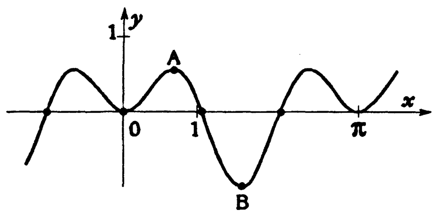

Fig. 7.10. $f(x) = \sin x \sin 3x$

Example 7.10. *Determine and sketch the graph of the function*

$$f(x) = \sin x \sin 3x.$$

Solution. The graph of f is sketched in Figure 7.10. The function can be written as

$$f(x) = \frac{\cos 2x - \cos 4x}{2} = \sin^2 x(3 - 4\sin^2 x).$$

- The domain is $(-\infty, +\infty)$.
- This function is periodic with basic period π.
- This is an even function.
- The zeros of f over $[0, \pi]$ are obtained from

$$\sin x \sin 3x = 0, \quad \text{which holds for} \quad x_1 = 0, \quad x_2 = \frac{\pi}{3} \quad \text{and} \quad x_3 = \frac{2\pi}{3}.$$

- The first derivative is

$$f'(x) = \sin 2x(4\cos 2x - 1).$$

  – The point $O(0,0)$ is a local minimum.
  – The point $B(\pi/2, -1)$ is a local minimum.
  – The point $A(\arccos(1/4)/2, 9/16)$ is one of the two local maximums.

- The second derivative of the function is $f'' = 16\cos^2 2x - 2\cos 2x - 8$.

- The inflection points are at

$$x_4 = \pm\frac{1}{2}\arccos\frac{\sqrt{129}+1}{16}, \quad x_5 = \pm\frac{1}{2}\left(\pi - \arccos\frac{\sqrt{129}-1}{16}\right).$$

- The given function has no line asymptotes.

Example 7.11. *Determine and sketch the graph of the function*

$$f(x) = \sin^3 x + \cos^3 x.$$

Solution. The graph of f is sketched in Figure 7.11.

- The domain is $(-\infty, +\infty)$.
- This function is periodic with basic period 2π.
- This is an even function.
- The zeros are obtained from the equation

$$\sin^3 x + \cos^3 x = (\sin x + \cos x)\left(1 - \frac{\sin 2x}{2}\right) = 0,$$

  whose solutions are $x_1 = \dfrac{3\pi}{4}$ and $x_2 = \dfrac{7\pi}{4}$.

- The first derivative is

$$f'(x) = \frac{3}{2}\sin 2x(\sin x - \cos x).$$

  - The points $A(0,1)$, $B\left(\frac{\pi}{2},1\right)$ and $C\left(\frac{5\pi}{4}, -\frac{\sqrt{2}}{2}\right)$ are local maximums.
  - The points $D(\frac{\pi}{4}, \frac{\sqrt{2}}{2})$, $E(\pi,-1)$, and $F(\frac{3\pi}{2},-1)$ are local minimums.

- The second derivative of the function is $f''(x) = \dfrac{9}{2}(\sin x + \cos x)\left(\sin 2x - \dfrac{2}{3}\right)$.
- The inflection points are for

$$x_1 = \frac{3\pi}{4}, \quad x_2 = \frac{7\pi}{4} \text{ and } x_3 = \frac{\arcsin(3/2)}{2} \approx 0.36,$$

  and also for $x_4 \approx 1.21$, $x_5 \approx 3.51$ and $x_6 \approx 4.35$.

- The given function has no asymptotes.

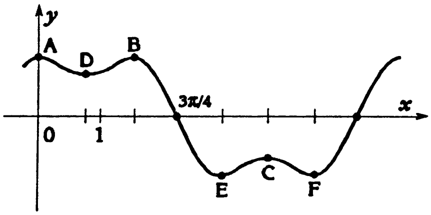

Fig. 7.11. $f(x) = \sin^3 x + \cos^3 x$

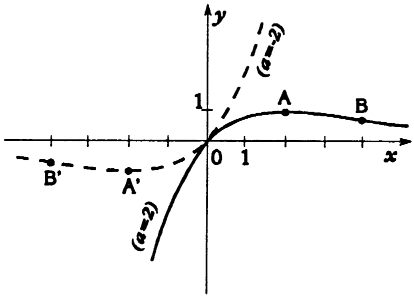

Fig. 7.12. $f(x) = x\mathrm{e}^{-x/a}$

Example 7.12. *Determine and sketch the graph of the function*

$$f(x) = x\mathrm{e}^{-x/a}, \quad a \neq 0.$$

Solution. The graph of f is sketched in Figure 7.12, both for $a = 2$ and $a = -2$.

- The domain is $(-\infty, +\infty)$.
- This function is neither odd nor even.
- The zero of the function is at $x = 0$.
- The first derivative is
$$f'(x) = \mathrm{e}^{-x/a}\left(1 - \frac{x}{a}\right).$$
  - If $a > 0$, then the point $A\left(a, \frac{a}{\mathrm{e}}\right)$ is a maximum, and if $a < 0$, then the point $A\left(a, \frac{a}{\mathrm{e}}\right)$ is a minimum of the function.
  - If $a > 0$, then the function is increasing in $(-\infty, a)$ and if $a < 0$, then the function is increasing in $(a, +\infty)$.
  - If $a > 0$, then the function is decreasing in $(a, +\infty)$ and if $a < 0$, then the function is decreasing in $(-\infty, a)$.
- The second derivative is
$$f''(x) = x\mathrm{e}^{-x/a}\left(\frac{x}{a^2} - \frac{2}{a}\right).$$
  - The point of inflection is $B(2a, 2a\mathrm{e}^{-2})$, for every $a \neq 0$.
  - The function is concave upward in $(2a, +\infty)$.
  - The function is concave downward in $(-\infty, 2a)$.

- Asymptotes.
  - The function has a horizontal asymptote $y = 0$ when $x \to +\infty$ if $a > 0$ (resp. when $x \to -\infty$ if $a < 0$), because

$$\lim_{x\to+\infty} x\mathrm{e}^{-x/a} = 0 \quad \text{for } a > 0, \quad \text{and} \quad \lim_{x\to-\infty} x\mathrm{e}^{-x/a} = 0, \quad \text{for } a < 0.$$

Example 7.13. *Determine and sketch the graph of the function*

$$f(x) = (1 - x^2)\mathrm{e}^{-2x}.$$

Solution. The graph of f is sketched in Figure 7.13.

- The domain of the function is the set $\mathbf{R}$.
- The function is neither odd nor even.
- The zeros of the function are $x_1 = 1,\ x_1 = -1$.
- The first derivative is

$$f'(x) = (-2 - 2x + 2x^2)\mathrm{e}^{-2x}.$$

  - The critical points are

$$A\left(\frac{1-\sqrt{5}}{2}, \frac{(-1+\sqrt{5})\mathrm{e}^{-(1-\sqrt{5})}}{2}\right) \quad \text{and} \quad B\left(\frac{1+\sqrt{5}}{2}, \frac{(-1-\sqrt{5})\mathrm{e}^{-(1+\sqrt{5})}}{2}\right).$$

  - The function f is increasing for $x \in \left(-\infty, \dfrac{1-\sqrt{5}}{2}\right) \cup \left(\dfrac{1+\sqrt{5}}{2}, +\infty\right)$.
  - The function f is decreasing for $x \in \left(\dfrac{1-\sqrt{5}}{2}, \dfrac{1+\sqrt{5}}{2}\right)$.
- The second derivative is $f''(x) = (2 + 8x - 4x^2)\mathrm{e}^{-2x}$.
  - Since $f''\left(\dfrac{1+\sqrt{5}}{2}\right) > 0$, it follows that the minimum is at the point

$$B\left(\frac{1+\sqrt{5}}{2}, \frac{(-1-\sqrt{5})\mathrm{e}^{-(1+\sqrt{5})}}{2}\right).$$

  - Since $f''\left(\dfrac{1-\sqrt{5}}{2}\right) < 0$, it follows that the maximum is at the point

$$A\left(\frac{1-\sqrt{5}}{2}, \frac{(-1+\sqrt{5})\mathrm{e}^{-(1-\sqrt{5})}}{2}\right).$$

– The inflection points are

$$C\left(1-\frac{\sqrt{6}}{2},\frac{-3+2\sqrt{6}}{2}\mathrm{e}^{-2+\sqrt{6}}\right),\qquad D\left(1+\frac{\sqrt{6}}{2},\frac{-3-2\sqrt{6}}{2}\mathrm{e}^{-2-\sqrt{6}}\right).$$

– The function is concave upward for

$$x\in\left(1-\frac{\sqrt{6}}{2},1+\frac{\sqrt{6}}{2}\right).$$

– The function is concave downward for

$$x\in\left(-\infty,1-\frac{\sqrt{6}}{2}\right)\bigcup\left(1+\frac{\sqrt{6}}{2},+\infty\right).$$

- Asymptotes.

  – The function has a horizontal asymptote when $x\to+\infty$, because

$$\lim_{x\to+\infty}(1-x^2)\mathrm{e}^{-2x}=0.$$

Note that

$$\lim_{x\to-\infty}(1-x^2)e^{-2x}=-\infty.$$

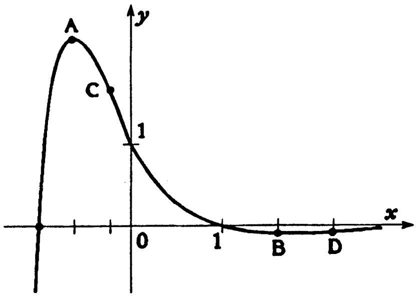

Fig. 7.13. $f(x)=(1-x^2)\mathrm{e}^{-2x}$.

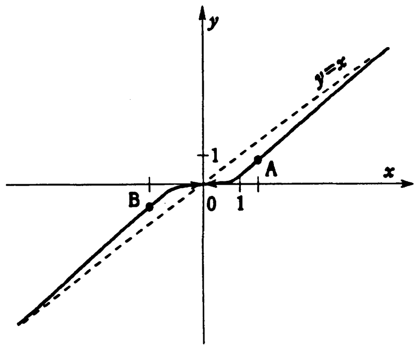

Fig. 7.14. $f(x)=x\mathrm{e}^{-1/x^2}$

Example 7.14. *Determine and sketch the graph of the function*

$$f(x)=x\mathrm{e}^{-1/x^2}.$$

Solution. The graph of f is sketched in Figure 7.14.

- The domain of the function is $(-\infty,0)\bigcup(0,+\infty)$.
- This is an odd function.

- It has no zeros.
- The first derivative is

$$f'(x) = \frac{(x^2+2)e^{-1/x^2}}{x^2}.$$

  - Since f' has no real zeros, the given function does not have any critical points.
  - This is an increasing function for every $x \in (-\infty, 0) \bigcup (0, +\infty)$.

- The second derivative is

$$f''(x) = \frac{-2x^2+4}{x^5}e^{-1/x^2}.$$

  - The inflection points are

$$A(\sqrt{2}, \sqrt{2}e^{-1/2}) \quad \text{and} \quad B(-\sqrt{2}, -\sqrt{2}e^{-1/2}).$$

  - The function f is concave upward for $x \in (-\infty, -\sqrt{2}) \cup (0, \sqrt{2})$.
  - The function f is concave downward for

$$x \in (-\sqrt{2}, 0) \cup (\sqrt{2}, +\infty).$$

- Asymptotes.
  - Though the function is not defined at $x = 0$, it has no vertical asymptote there, because

$$\lim_{x\to 0-} xe^{-1/x^2} = 0, \qquad \lim_{x\to 0+} xe^{-1/x^2} = 0.$$

  - It has no horizontal asymptote.
  - It has a slanted asymptote $y = x$ both when $x \to +\infty$ and when $x \to -\infty$, because

$$k = \lim_{x\to\pm\infty} \frac{xe^{-1/x^2}}{x} = 1, \quad \text{and}$$

$$n = \lim_{x\to\pm\infty} (xe^{-1/x^2} - x) = \lim_{x\to\pm\infty} x(e^{-1/x^2} - 1) = \lim_{x\to\pm\infty} \frac{e^{-1/x^2} - 1}{x \cdot \dfrac{1}{x^2}} = 0.$$

Remark. This function can be continuously extended to the point $x = 0$, if one defines $f(0) = 0$.

Example 7.15. *Determine and sketch the graph of the function*

$$f(x) = -(x+2)e^{1/x}.$$

Solution. The graph of f is sketched in Figure 7.15.

- The given function is defined for $x \in (-\infty, 0) \bigcup (0, +\infty)$.
- The function is neither an odd, nor an even function.
- The only zero is $x = -2$.
- The first derivative is

$$f'(x) = \frac{x+2}{x^2}\mathrm{e}^{1/x} - \mathrm{e}^{1/x} = \frac{2+x-x^2}{x^2}\mathrm{e}^{1/x}.$$

- The critical points are $A(2, -4\mathrm{e}^{1/2})$, $B(-1, -\mathrm{e}^{-1})$.
  - The function f is decreasing for $x \in (-\infty, -1) \bigcup (2, +\infty)$.
  - The function f is increasing for $x \in (-1, 0) \bigcup (0, 2)$.
- The second derivative of the function f is

$$f''(x) = -\frac{5x+2}{x^4}\mathrm{e}^{1/x}.$$

  Since

$$f''(2) = -\frac{12}{2^4}\mathrm{e}^{1/2} < 0 \quad \text{and} \quad f''(-1) = 3\mathrm{e}^{-1} > 0,$$

  the following holds.
  - The function f has a maximum at $A(2, -4\mathrm{e}^{1/2})$.
  - It has a minimum at $B(-1, -\mathrm{e}^{-1})$.
  - The inflection point is $C\left(-\frac{2}{5}, -\frac{8}{5}\mathrm{e}^{-5/2}\right)$.
  - The function is concave downward for $x \in (-2/5, 0) \cup (0, +\infty)$.
  - The function is concave upward for $x \in (-\infty, -2/5)$.
- Asymptotes.
  - The vertical asymptote is $x = 0$, and it holds

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+}(-x-2)\mathrm{e}^{1/x} = -\infty; \quad \lim_{x\to 0-} f(x) = \lim_{x\to 0-}(-x-2)\mathrm{e}^{1/x} = 0.$$

  - Since

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}(-x-2)\mathrm{e}^{1/x} = -\infty,$$

    the function has no horizontal asymptotes.

– The slanted asymptote is $y = -x - 3$ when $x \to \pm\infty$, because

$$k = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{-x-2}{x} e^{1/x} = -1, \quad \text{and}$$

$$\begin{aligned} n = & \lim_{x\to\pm\infty} (f(x) - kx) = \lim_{x\to\pm\infty} \left((-x-2)e^{1/x} + x\right) \\ = & \lim_{x\to\pm\infty} (x(1 - e^{1/x})) - \lim_{x\to\pm\infty} 2e^{1/x} \\ = & \lim_{x\to\pm\infty} \frac{1 - e^{1/x}}{\frac{1}{x}} - 2 = \lim_{t\to 0\pm} \frac{1-e^t}{t} - 2 = -3. \end{aligned}$$

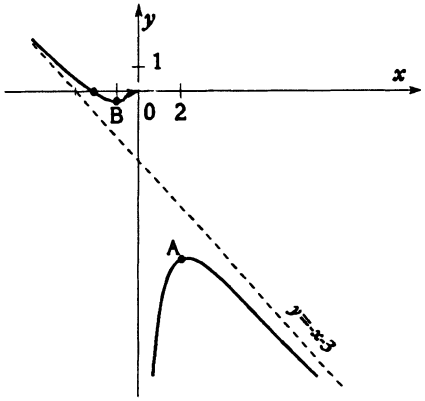

Fig. 7.15. $f(x) = -(x+2)e^{1/x}$

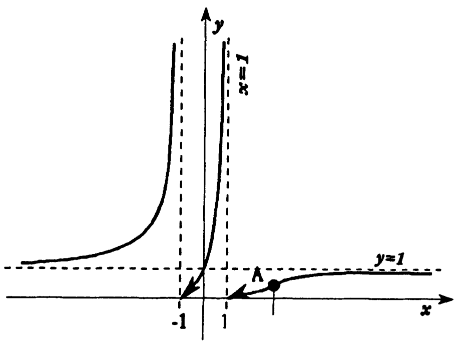

Fig. 7.16. $f(x) = e^{2x/(1-x^2)}$

Example 7.16. *Determine and sketch the graph of the function*

$$f(x) = e^{2x/(1-x^2)}.$$

Solution. The graph of f is sketched in Figure 7.16.

- The domain of the function is $(-\infty, -1) \bigcup (-1, 1) \bigcup (1, +\infty)$.
- The function is neither odd nor even.
- There are no zeros for the function.
- The first derivative is

$$f'(x) = \frac{2(x^2+1)e^{2x/(1-x^2)}}{(1-x^2)^2}.$$

  – The first derivative has no zeros.
  – The function f is increasing for every x from the domain.

- The second derivative is

$$f''(x) = 4e^{2x/(1-x^2)}\frac{-x^5 + x^4 - 2x^3 + 2x^2 + 3x + 1}{(1-x^2)^4}.$$

  - It has a zero $x_A \in (1,2)$, and there it has an inflection point.
  - The function is concave upward for $x \in (-\infty,-1) \bigcup (-1,1) \bigcup (1,x_0)$ and concave downward for $x \in (x_0,+\infty)$.

- Asymptotes.
  - The vertical asymptotes are $x = -1$ and $x = 1$, because

$$\lim_{x\to -1-} e^{2x/(1-x^2)} = +\infty, \qquad \lim_{x\to -1+} e^{2x/(1-x^2)} = 0,$$

$$\lim_{x\to 1-} e^{2x/(1-x^2)} = +\infty, \qquad \lim_{x\to 1+} e^{2x/(1-x^2)} = 0.$$

  - The horizontal asymptote is $y = 1$ when $x \to \pm\infty$, because

$$\lim_{x\to\pm\infty} e^{2x/(1-x^2)} = 1.$$

  - There are no slanted asymptotes.

Example 7.17. *Determine and sketch the graph of the function*

$$f(x) = \frac{\ln x}{x}.$$

Solution. The graph of f is sketched in Figure 7.17.

- The domain of the function is $(0,+\infty)$.
- The point $x = 1$ is a zero of the function.
- The first derivative is

$$f'(x) = \frac{1-\ln x}{x^2}, \quad \text{hence} \quad f'(x) = 0 \text{ for } x = e.$$

  - The function f is increasing for $x \in (0,e)$.
  - The function f is decreasing for $x \in (e,+\infty)$.
  - The point $A(e,1/e)$ is a maximum of this function.

- The second derivative is

$$f''(x) = \frac{2\ln x - 3}{x^3}.$$

- The point of inflection is $B(e^{3/2}, 3e^{-3/2}/2)$.
- The function is concave upward for $x \in (e^{3/2},+\infty)$, and concave downward for $x \in (0,e^{3/2})$.

- Asymptotes.
  - The vertical asymptote is $x = 0$, because $\lim\limits_{x \to 0+} \dfrac{\ln x}{x} = -\infty$.
  - The horizontal asymptote is $y = 0$ when $x \to +\infty$, because

$$\lim_{x \to +\infty} \frac{\ln x}{x} = 0.$$

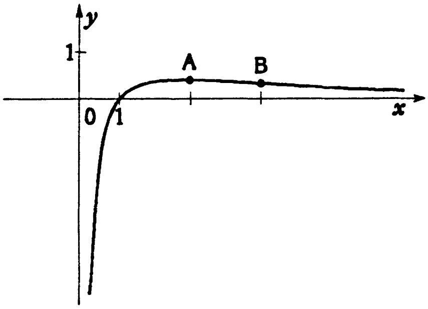

Fig. 7.17. $f(x) = \dfrac{\ln x}{x}$

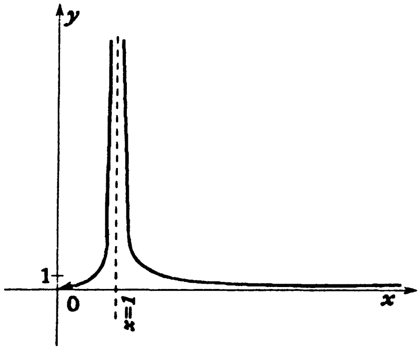

Fig. 7.18. $f(x) = \dfrac{\sqrt{x}}{\ln^2 x}$

Example 7.18. *Determine and sketch the graph of the function*

$$f(x) = \frac{\sqrt{x}}{\ln^2 x}.$$

Solution. The graph of f is sketched in Figure 7.18.

- The domain of the function is $(0,1) \bigcup (1,+\infty)$.
- The first derivative is

$$f'(x) = \frac{\ln x - 4}{2\sqrt{x} \ln^3 x}, \quad f'(x) = 0, \quad \text{for} \quad x = \mathrm{e}^4.$$

  - The function f is increasing for $x \in (0,1) \bigcup (\mathrm{e}^4, +\infty)$.
  - The function f is decreasing for $x \in (1, \mathrm{e}^4)$.
  - The point $A(\mathrm{e}^4, \mathrm{e}^2/8)$ is a minimum of this function.
- The second derivative is

$$f''(x) = \frac{24 - \ln^2 x}{4\sqrt{x^3} \ln^4 x}.$$

- The points of inflection are at $x_1 = \mathrm{e}^{\sqrt{24}}, \quad x_2 = \mathrm{e}^{-\sqrt{24}}$.

- The function is concave upward for $x \in (e^{-\sqrt{24}}, 1) \bigcup (1, e^{\sqrt{24}})$, and concave downward for $x \in (0, e^{-\sqrt{24}}) \bigcup (e^{\sqrt{24}}, +\infty)$.

- Asymptotes.
  - The vertical asymptote is $x = 1$, because $\lim\limits_{x \to 1\pm} \dfrac{\sqrt{x}}{\ln^2 x} = +\infty$.
  - There are no horizontal or slanted asymptotes.

Example 7.19. *Determine and sketch the graph of the function*

$$f(x) = (1+x)^{1/x} = exp\left(\frac{\ln(1+x)}{x}\right).$$

Solution. The graph of f is sketched in Figure 7.19.

- The domain of the function is $(-1, 0) \bigcup (0, +\infty)$.

- The first derivative is

$$f'(x) = (1+x)^{1/x}\left(\frac{1}{x(x+1)} - \frac{\ln(1+x)}{x^2}\right).$$

  - From the inequality
$$\frac{x}{x+1} < \ln(1+x) < x,$$
we obtain that the function f is decreasing on its entire domain.

- The second derivative is

$$f''(x) = (1+x)^{1/x}\left(\left(\frac{1}{x(x+1)} - \frac{\ln(1+x)}{x^2}\right)^2 + \frac{1}{x^3}\left(2\ln(x+1) - \frac{2x+3x^2}{(1+x)^2}\right)\right).$$

  The function
$$\phi(x) = 2\ln(x+1) - \frac{2x+3x^2}{(1+x)^2}$$
  has the first derivative
$$\phi'(x) = \frac{2x^2}{(1+x)^3} > 0, \quad \text{for} \quad -1 < x < +\infty.$$
  From $\phi(0) = 0$, it follows that
$$\phi(x) < 0 \text{ if } -1 < x < 0, \quad \text{and} \quad \phi(x) > 0 \text{ for } x > 0,$$
  wherefrom we get
$$\frac{\phi(x)}{x^3} > 0 \text{ for } x \in (-1, 0) \bigcup (0, +\infty).$$
  The first addend in the second derivative is grater than zero, and since the second is also grater then zero, it follows $f''(x) > 0$ for every x from the domain. This means that

– the function is concave upward on the whole domain.

- Asymptotes.
  – The vertical asymptote is $x = -1$, because

$$\lim_{x\to -1+}(1+x)^{1/x} = +\infty \quad \text{and} \quad \lim_{x\to 0-}(1+x)^{1/x} = \lim_{x\to 0+}(1+x)^{1/x} = \mathrm{e}.$$

  – The function has a horizontal asymptote $y = 1$ when $x \to +\infty$, because

$$\lim_{x\to +\infty}(1+x)^{1/x} = 1.$$

Remark. The point $A(0, \mathrm{e})$ does not belong to the graph of f, since the point 0 is not in its natural domain. However, f can be continuously extended to $x = 0$ if one defines $f(0) = \mathrm{e}$ (see the Remark after Example 5.37).

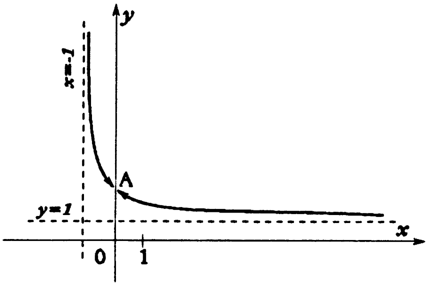

Fig. 7.19. $f(x) = (1+x)^{1/x}$

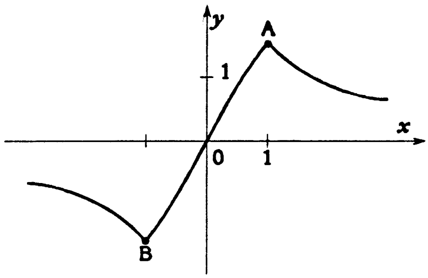

Fig. 7.20. $f(x) = \arcsin \dfrac{2x}{1+x^2}$

Example 7.20. *Determine and sketch the graph of the function*

$$f(x) = \arcsin \frac{2x}{1+x^2}.$$

Solution. The graph of f is sketched in Figure 7.20.

- The domain of the function is $(-\infty, +\infty)$, because $(x-1)^2 \geq 0$ and thus

$$-1 \leq \frac{2x}{1+x^2} \leq 1.$$

- This is an odd function.
- The function has a zero at $x = 0$.
- The first derivative is

$$f'(x) = \frac{2\operatorname{sgn}(1-x^2)}{1+x^2}, \quad x \neq \pm 1.$$

- It has no zeros and it holds

$$\lim_{x\to -1-} f'(x) = -1, \qquad \lim_{x\to -1+} f'(x) = 1,$$

$$\lim_{x\to 1-} f'(x) = 1, \qquad \lim_{x\to 1+} f'(x) = -1.$$

- The function f is increasing for $x \in (-1, 1)$.
- The function f is decreasing for $x \in (-\infty, -1) \bigcup (1, +\infty)$.
- It has a minimum at $B\left(-1, -\dfrac{\pi}{2}\right)$.
- It has a maximum at $A\left(1, \dfrac{\pi}{2}\right)$.

• The second derivative is

$$f''(x) = \frac{-4x}{(1+x^2)^2}\operatorname{sgn}(1-x^2), \quad x \neq \pm 1$$

and it has a zero at $x = 0$ and this is the inflection point $O(0, 0)$.

• The function is concave upward for $x \in (-1, 0) \bigcup (1, +\infty)$ and concave downward for $x \in (-\infty, -1) \bigcup (0, 1)$.

• From the previous conclusions it follows that the critical points

$$A(1, \pi/2) \quad \text{and} \quad B(-1, -\pi/2)$$

are extrema points and also points in which the graph changes its concavity.

• Asymptotes.

- There is no vertical asymptote.
- The horizontal asymptote is $y = 0$ when $x \to \pm\infty$, because

$$\lim_{x\to\pm\infty} \arcsin\frac{2x}{1+x^2} = 0.$$

- There is no slanted asymptote.

Example 7.21. *Determine and sketch the graph of the function*

$$f(x) = \arccos\frac{1-x^2}{1+x^2}.$$

Solution. The graph of f is sketched in Figure 7.21.

• The domain of the function is $(-\infty, +\infty)$, because it holds

$$-1 < -1 + \frac{2}{1+x^2} = \frac{1-x^2}{1+x^2} < 1. \tag{7.1}$$

- This is an even function.
- The function has no zeros, as follows from relation (7.1).
- The first derivative is $f'(x) = \dfrac{2 \cdot \operatorname{sgn} x}{1+x^2}, \quad x \neq 0.$
  - It has no zeros and it holds

$$\lim_{x \to 0-} f'(x) = -2, \quad \lim_{x \to 0+} f'(x) = 2.$$

  - So the function has a minimum at $O(0,0)$.
  - The function f is increasing for $x \in (0, +\infty)$.
  - The function f is decreasing for $x \in (-\infty, 0)$.
- The second derivative is

$$f''(x) = \frac{-4|x|}{(1+x^2)^2}, \quad x \neq 0.$$

- The function has no inflection points, because it holds $f''(x) < 0$, for $x \neq 0$.
- The function is concave downward for every x in the domain.
- Asymptotes.
  - There is no vertical asymptote.
  - The horizontal asymptote is $y = \pi$ when $x \to \pm\infty$, because

$$\lim_{x \to \pm\infty} \arccos \frac{1-x^2}{1+x^2} = \pi.$$

  - There is no slanted asymptote.

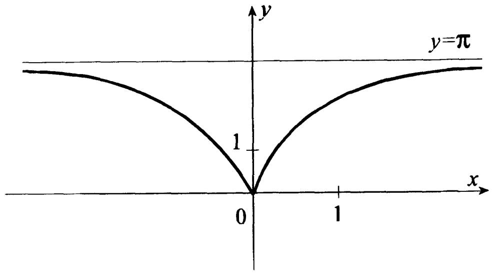

Fig. 7.21. $f(x) = \arccos \dfrac{1-x^2}{1+x^2}$

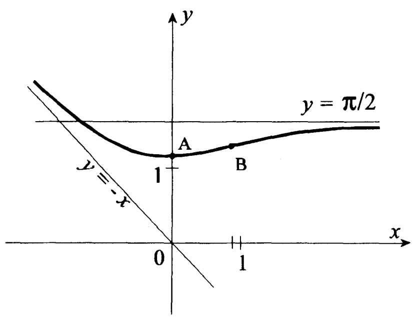

Fig. 7.22. $\arctan e^x - \ln\left(\sqrt{\dfrac{e^{2x}}{1+e^{2x}}}\right)$

Example 7.22. *Determine and sketch the graph of the function*

$$f(x) = \arctan \mathrm{e}^x - \ln \left(\sqrt{\frac{\mathrm{e}^{2x}}{1+\mathrm{e}^{2x}}} \right).$$

Solution. The graph of f is sketched in Figure 7.22.

- The domain of the function is $(-\infty, +\infty)$.
- The function has no zeros.
- The first derivative is
$$f'(x) = \frac{\mathrm{e}^x - 1}{1+\mathrm{e}^{2x}}.$$
- The critical point is $A\left(0, \frac{\pi}{4} + \frac{1}{2}\ln 2\right)$.
  - The function f is increasing for $x \in (0, +\infty)$.
  - The function f is decreasing for $x \in (-\infty, 0)$.
  - So the function has a minimum at $A\left(0, \frac{\pi}{4} + \frac{1}{2}\ln 2\right)$.
- The second derivative is
$$f''(x) = \frac{\mathrm{e}^x(-\mathrm{e}^{2x} + 2\mathrm{e}^x + 1)}{(1+\mathrm{e}^{2x})^2}.$$
- It has a zero at $x_B = \ln(1+\sqrt{2})$, which is the point of inflection.
- The function is concave downward for $x \in (-\infty, \ln(1+\sqrt{2}))$,
- The function is concave upward for $x \in \left(\ln(1+\sqrt{2}), +\infty\right)$.
- Asymptotes.
  - The horizontal asymptote is $y = \pi/2$ when $x \to +\infty$, because
$$\lim_{x\to+\infty} \left(\arctan \mathrm{e}^x - \ln \left(\sqrt{\frac{\mathrm{e}^{2x}}{1+\mathrm{e}^{2x}}} \right) \right) = \frac{\pi}{2}.$$
  - The slanted asymptote is $y = -x$ when $x \to -\infty$.

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