Graph Theory

– Course 2 –

Chapter 2: FUNDAMENTAL CONCEPTS(1/1)

Chapter outline

- BRIEF HISTORY
- APPLICATION OF GT
- DEFINITIONS
  - Directed/undirected graph (vertices, arcs/edges, degree, order, adjacency, path/chain, cycle/circuit) (see chapter 1)
  - Special graphs (symmetric, antisymmetric, transitive, regular)
- REPRESENTATION OF A GRAPH
- CONNEXITY AND STRONG CONNEXION
- EULERIAN AND HAMILTONIAN PROBLEMS
- BI-PARTIAL, ADJOINT AND K- COLORING GRAPHS

Section 1:
a bit of history

history

- 1736, Euler: the bridges of Königsberg … *mathematical recreations… …chemistry, electricity…*
- 1852, De Morgan (Guthrie): four colors
- 1946, Kuhn, Ford and Fulkerson, Roy, etc.
  … *operational research* …
- Since 1960, applications… (computer science)

Section 2:
Areas of application of the graph theory

Application of graph theory

Graphs: what are they for?

Graph theory has many applications and is used in various fields such as:

- Social Sciences
- The Industry
- Geography
- Architecture
- Chemistry
- Physics
- Biology
- Computer science
- Mathematics

Application of graph theory

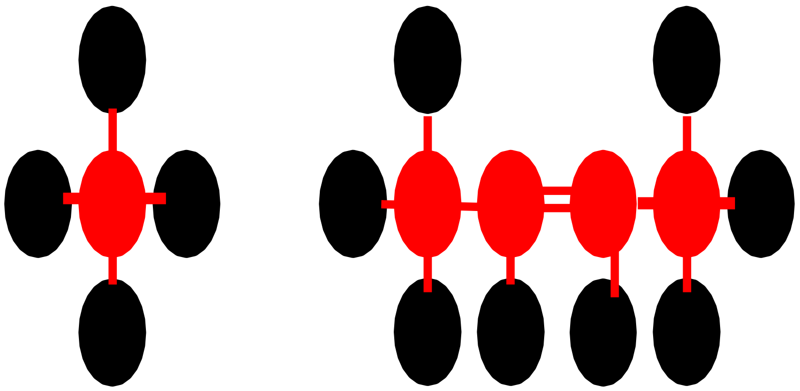
Graphs: what are they for?

Other areas of application

- Geography (cartography)
- Architecture (plans),
- linguistics (semantics).
- The WEB (link graph, calculation of relevance in search engines, etc.
- "Small Worlds" Graphs (Kevin Bacon)
- Optical networks
- Databases (dependencies)
- Knowledge Bases
- Compilation techniques
- Digital imaging (scenes, compression)
- Graph grammars (dynamic aspects)

Molecule modeling

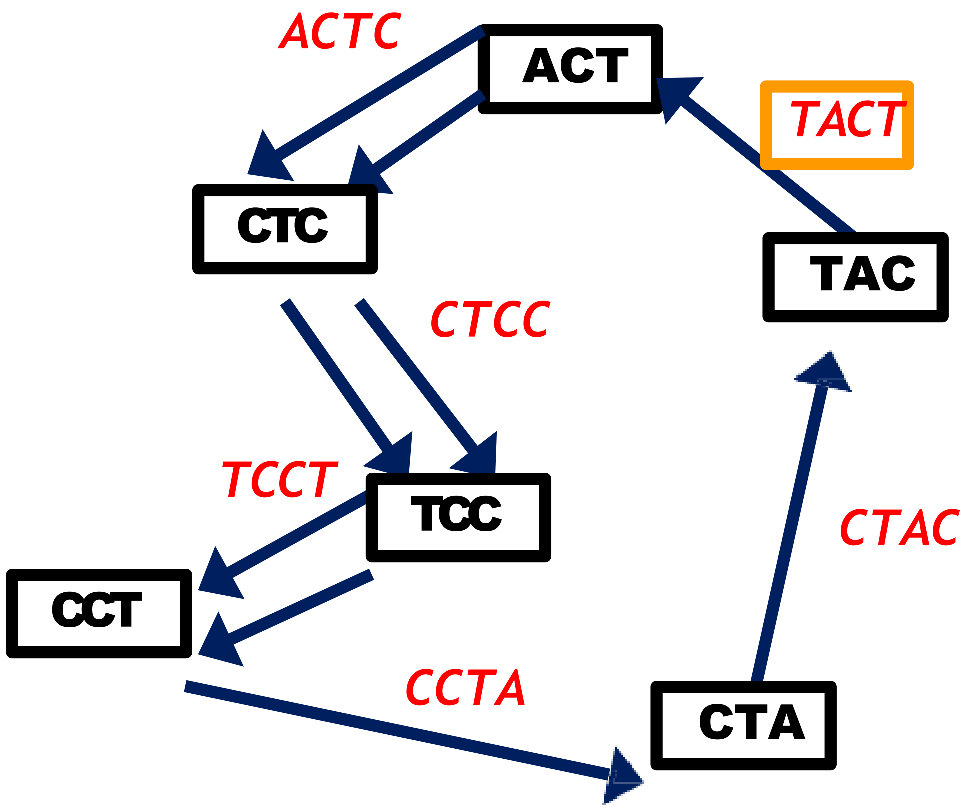
- **Graphs (multigraphs) with constraints on the degrees of the vertices according to the vertex type**



methane CH4 butene C4H8

Decoding DNA chains

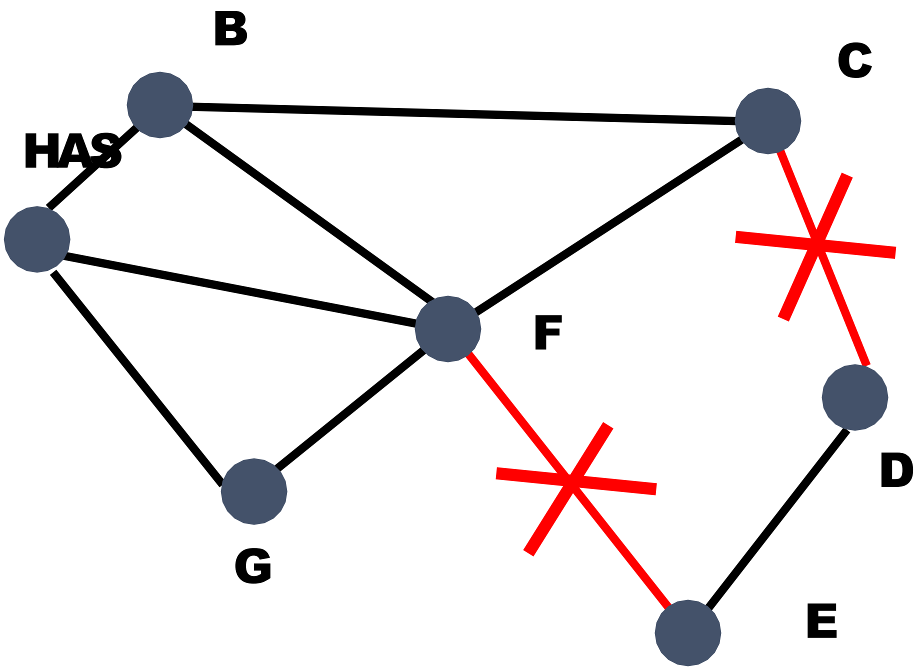
Hybridized probes: TCCT, ACTC, CTAC, TCCT, ACTC, CTCC, TACT, CCTA, CTCC



Application of graph theory

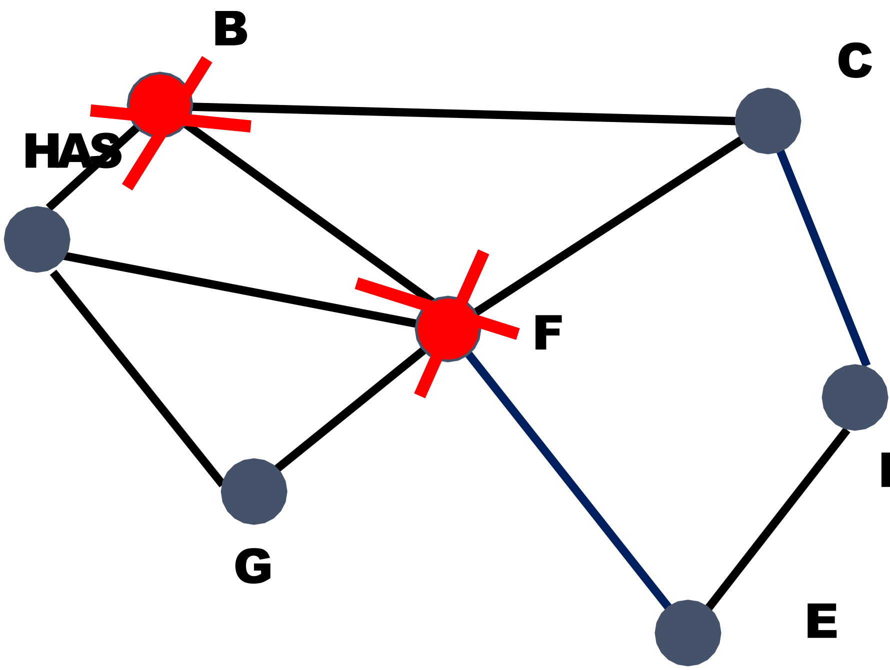
Example

Network reliability



Solve the problem of the breakdown channels of communication

Network reliability

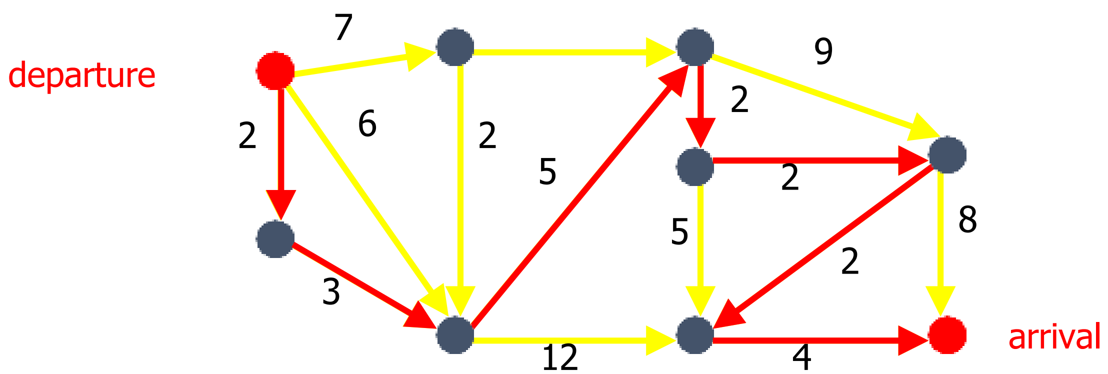


Solve the problem of the breakdown of relay summits

Example

- Best route

city map, arcs are valued by the travel time for each section



Shortest path problem

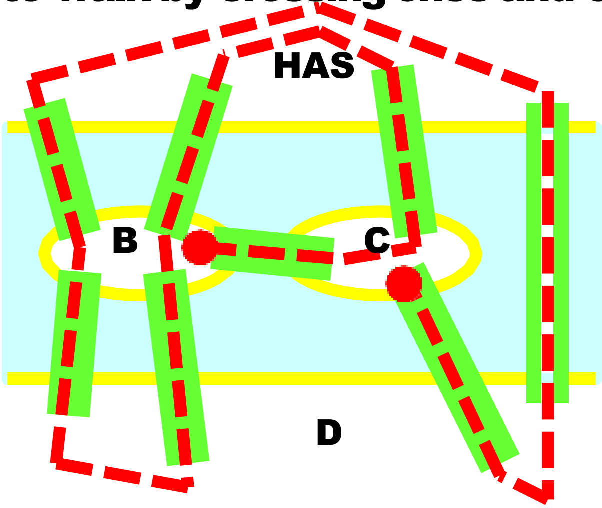
...

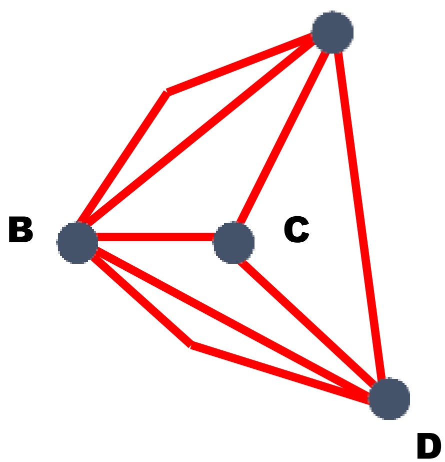
Example

- The bridges of Königsberg

the town of KOENIGSBERG (KALININGRAD) watered by the PREGEL river and having seven bridges

Is it possible to walk by crossing once and only once?HASeach of the bridges?

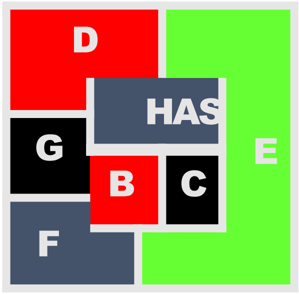


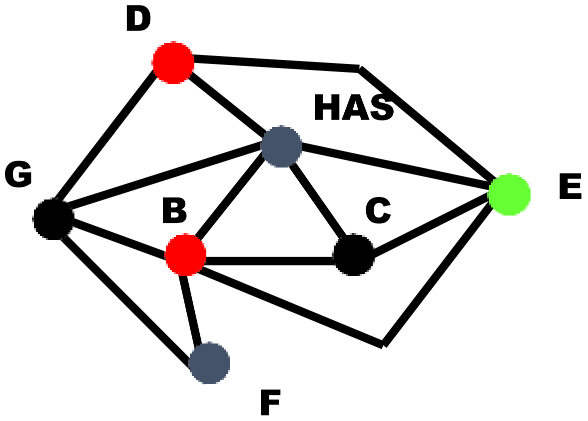


There exists an "Eulerian" path if and only if 0 or 2 vertices are of odd degree…

Example

- Four Color Problem





Two adjacent vertices do not receive the same color

Section 3: GT Basics

Basics

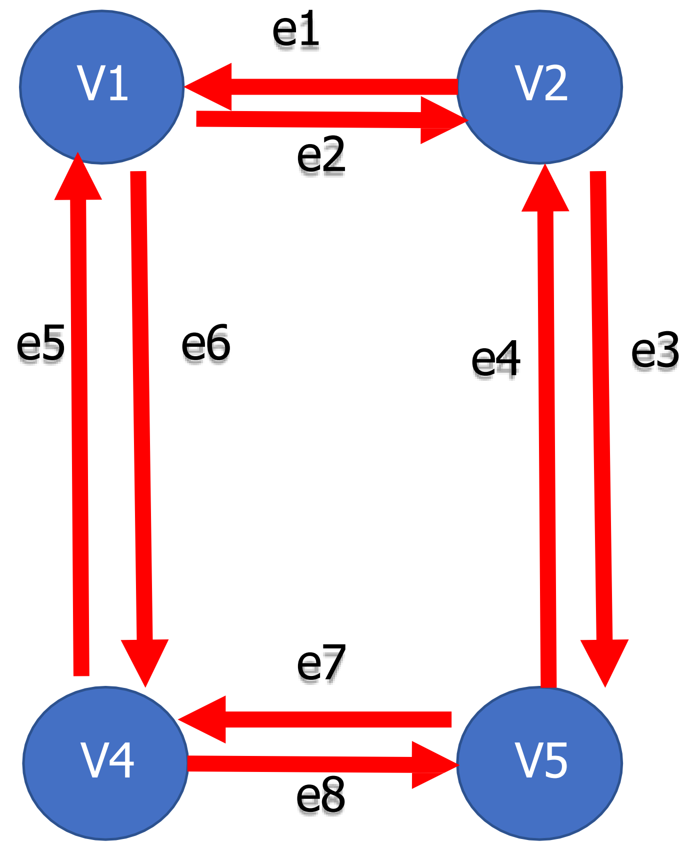
- See chapter 1
- **DEFINITIONS**
  - Special graphs (symmetric, antisymmetric, transitive, regular)
  - Representation of a graph
    - Incidence matrix at arcs
    - Adjacency matrix
  - Connectivity and strong connectivity
- **EULERIAN AND HAMILTONIAN PROBLEMS**
- **BI-PARTIAL, ADJOINT AND K-COLORING GRAPHS**

Symmetric graph (*symmetric graph*)

- It is a graph G =(V,E)

such that: $(xy) \in E \Rightarrow (yx) \in E$.

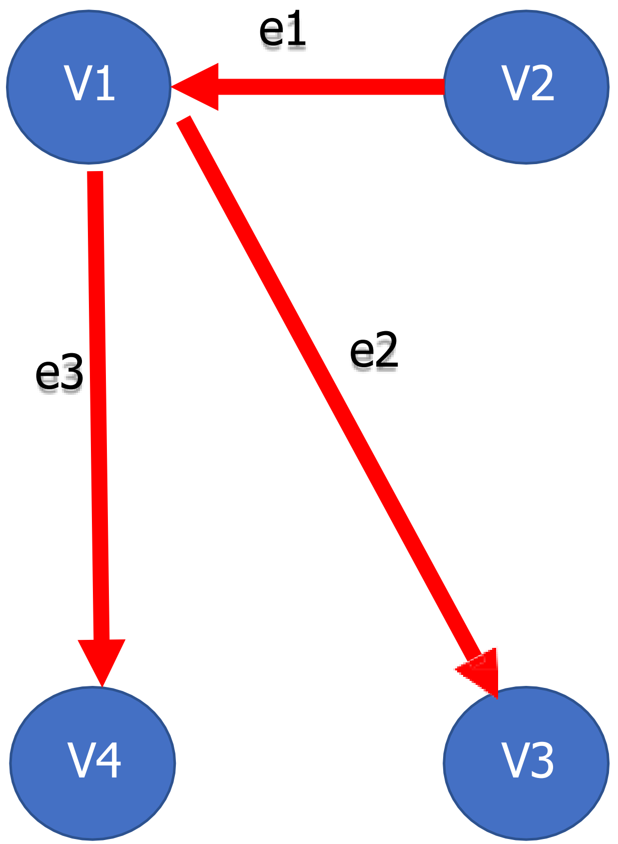
- Any pair of adjacent vertices is connected in both directions.



Antisymmetric graph (*antisymmetric graph*)

- It is a graph $G = (V,E)$

such that: $(xy) \in E \Rightarrow (yx) \notin E$.

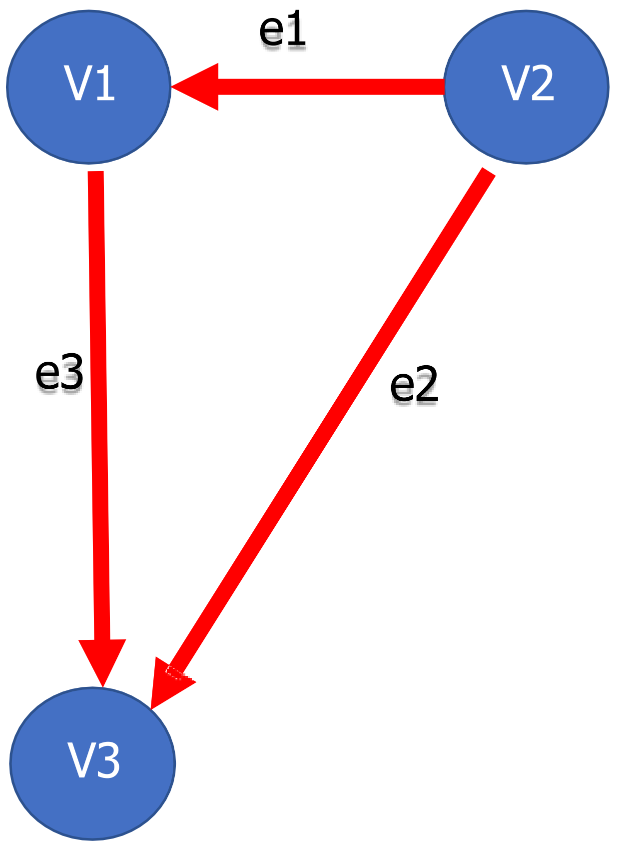


Transitive graph(*transitive graph*)

- This is a graph $G = (V,E)$

such that:

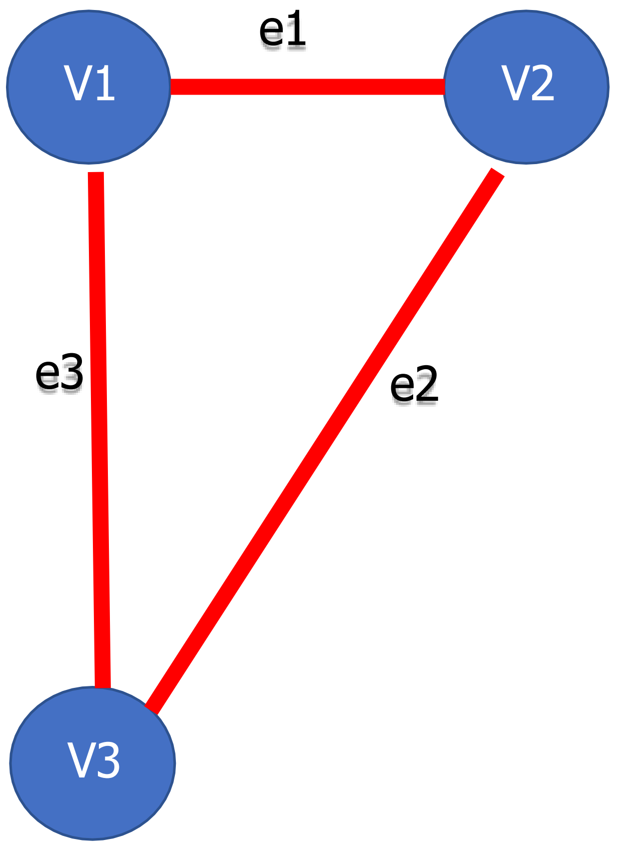
$(xy) \in E$ and $(yz) \in E \Rightarrow (xz) \in E$.



Regular graph(*regular graph*)

A graph G =(V,E)

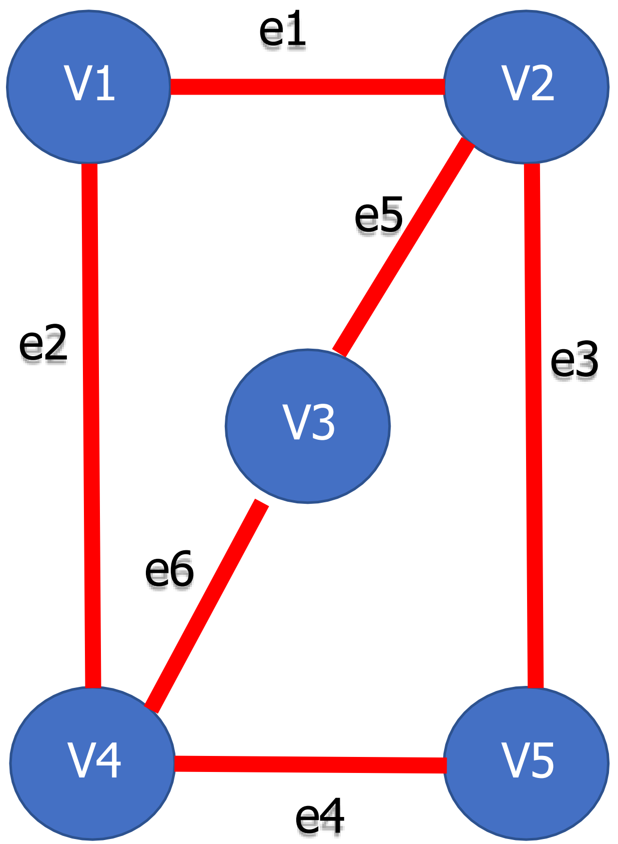
is said to be regular if the degrees of all its vertices are equal.



Representation of a graph

- <u>Incidence matrix at the arcs:</u>

- N vertices *numbered from 1 to n* and m arcs *numbered from 1 to m* so a matrix n*m



$A_{i,j}$ **= +1 If-an arc numbered j admits vertex i as origin**

$A_{i,j}$ **= -1 If-a numbered arc j admits The vertex i as arrival**

$A_{i,j}$ **= 0 Otherwise.**

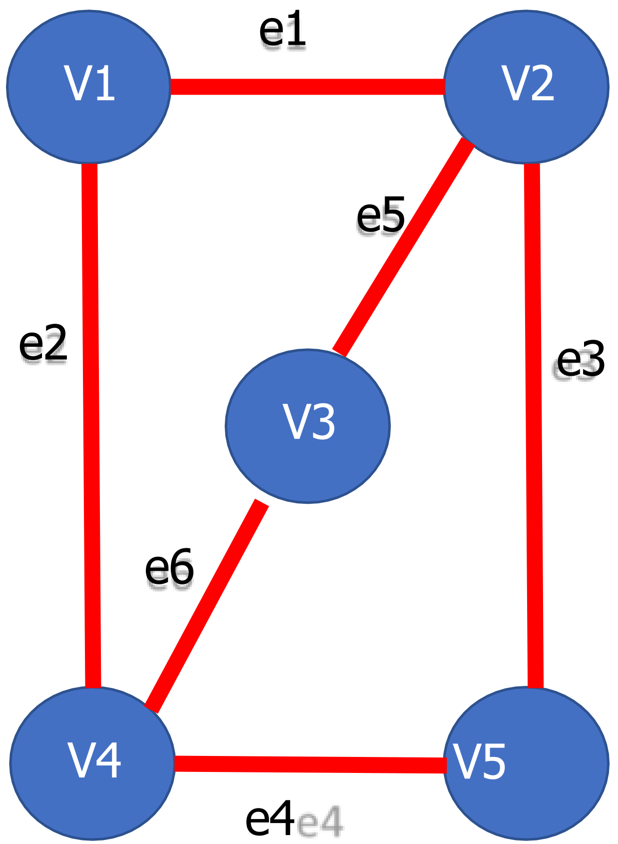
Representation of a graph

- **<u>Incidence matrix at the arcs:</u>**
- **<u>Undirected graph</u>**

Let's take the case of the graph opposite: It has 5 vertices and 6 edges, the incidence matrix will therefore have 5 rows and 6 columns

	e1	e2	e3	e4	e5	e6
v1	1	1	0	0	0	0
v2	1	0	1	0	1	0
v3	0	0	0	0	1	1
v4	0	1	0	1	0	1
v5	0	0	1	1	0	0

V1 is incident to e1 and e2



If the graph is not directed there is no notion of origin or arrival

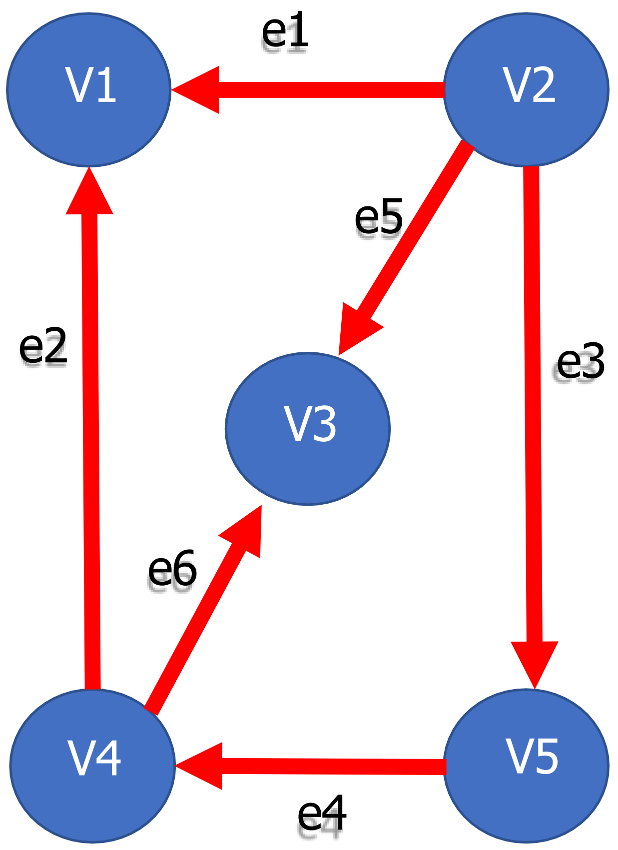
Representation of a graph

- **Incidence matrix at the arcs:**
- **Directed graph**

Let's take the case of the graph opposite: It has 5 vertices and 6 edges, the incidence matrix will therefore have 5 rows and 6 columns

	e1	e2	e3	e4	e5	e6
v1	- 1	- 1	0	0	0	0
v2	+1	0	+1	0	+1	0
v3	0	0	0	0	- 1	- 1
v4	0	+1	0	- 1	0	+1
v5	0	0	- 1	+1	0	0

V1 has two incoming arcs e1 and e2 V2 has three outgoing arcs e1 e3 and e5



- 1 for incoming arc and +1 for outgoing arc and 0 if no arc

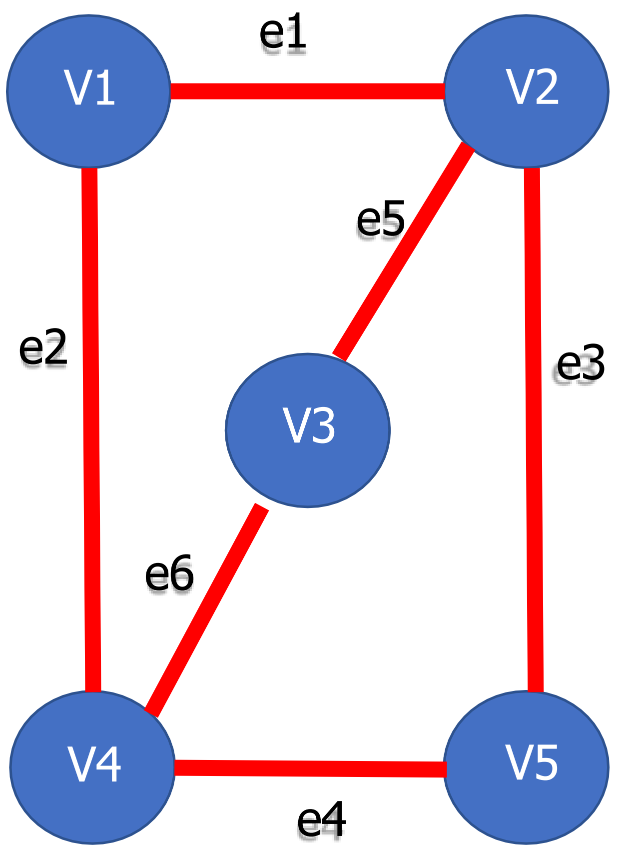
Representation of a graph

- **<u>Adjacency matrix</u>**
- **<u>Undirected graph= number of edges joining two vertices</u>**

Let's take the case of the graph opposite: It has 5 vertices, the incidence matrix will therefore have 5 rows and 5 columns

Symmetric matrix

	v1	v2	v3	v4	v5
v1	0	1	0	1	0
v2	1	0	1	0	1
v3	0	1	0	1	0
v4	1	0	1	0	1
v5	0	1	0	1	0



If the graph is not directed, we put 1 if the vertices are adjacent and 0 otherwise.

Representation of a graph

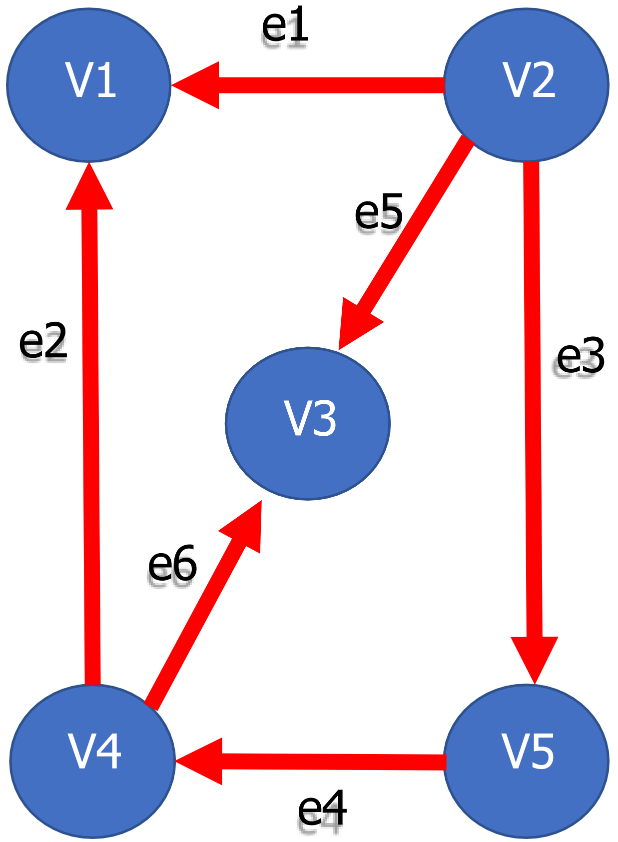
- **Adjacency matrix**
- **Directed graph: arcs outgoing**

Let's take the case of the graph opposite: It has 5 vertices,
the incidence matrix will therefore have 5 rows and 5 columns

Non-symmetric matrix

	v1	v2	v3	v4	v5
v1	0	0	0	0	0
v2	1	0	1	0	1
v3	0	0	0	0	0
v4	1	0	1	0	0
v5	0	0	0	1	0

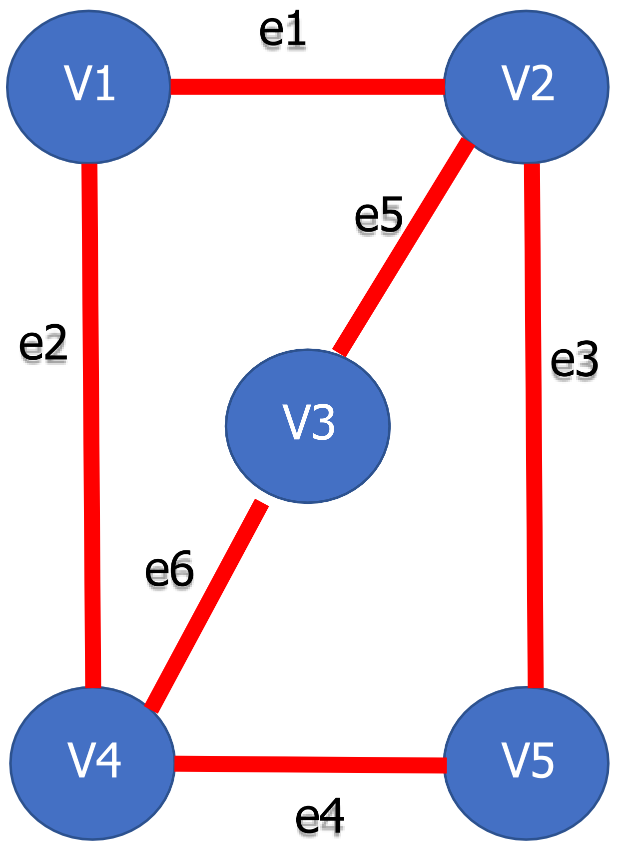
V1 has two incoming arcs e1 and e2 V2 has three outgoing arcs e1 e3 and e5



If the graph is oriented, we put 1 if there is an outgoing arc (from row to column) and 0 otherwise

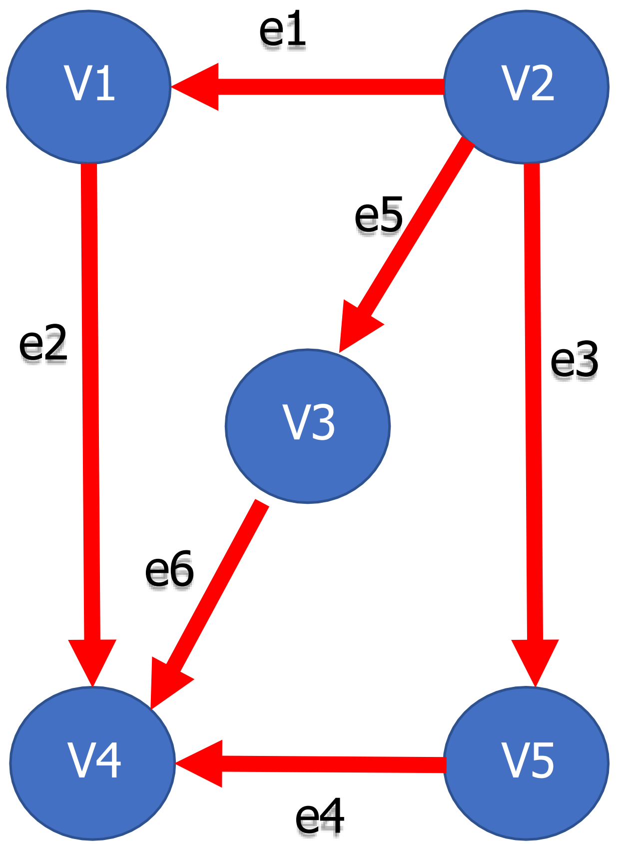
- *Connected graph*

A finite graph G = (V,E) with
V ={v1, v2, . . . , vn} and
E={e1, e2, . . . ,em}
It is said to be connected if whatever the vertices Vx and Vy of V there is a chain connecting Vx to Vy.



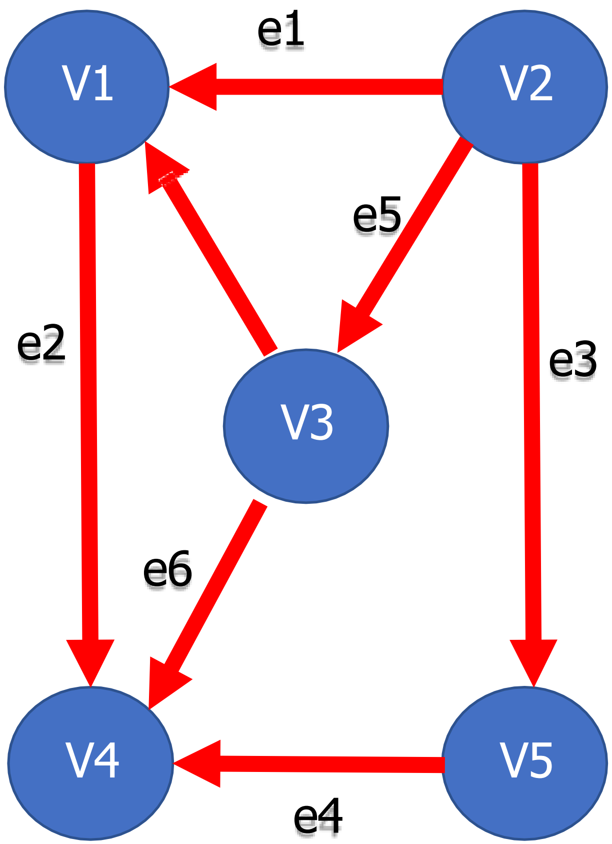
- *Cconnected graph*

For a digraph we speak of connectivity if, forgetting the orientation of the edges, the graph is connected.



- Strongly connected graph (*connected graph*)

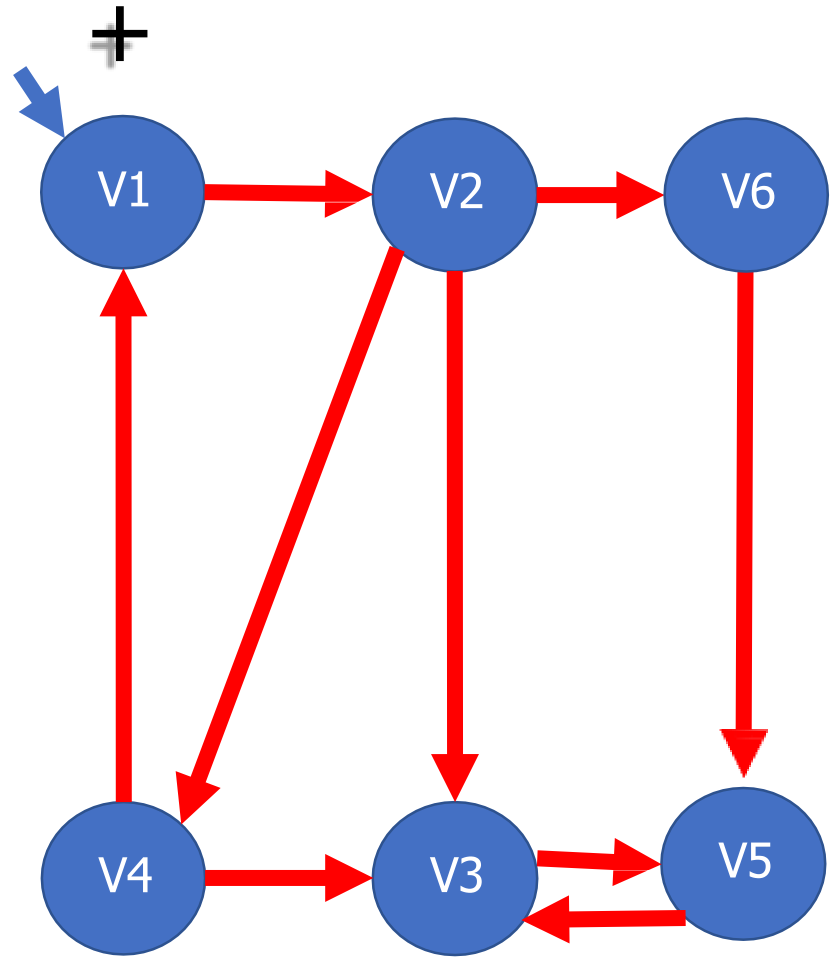
We are talking about strong connectivity if there exists a directed path from any vertex Vx to any vertex Vy..



Marking algorithm

- Strongly connected components

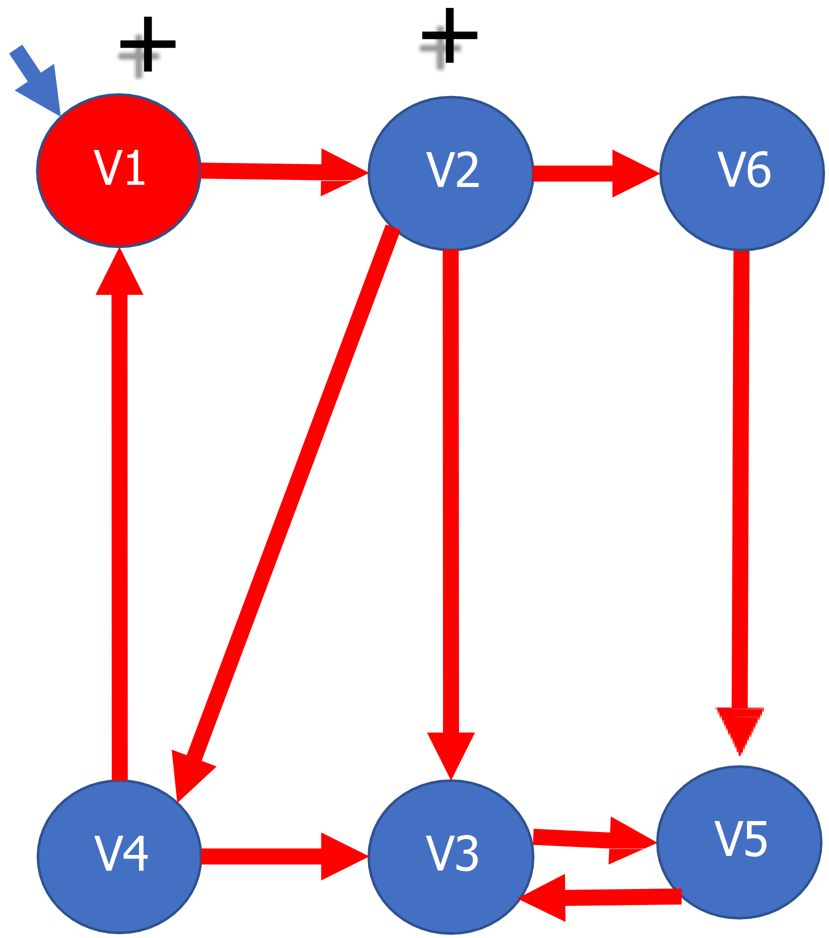
We select a vertex
We mark it with +



Marking algorithm

- Strongly connected components

We mark by + all the successors of the selected vertex as well as the successors of its successors



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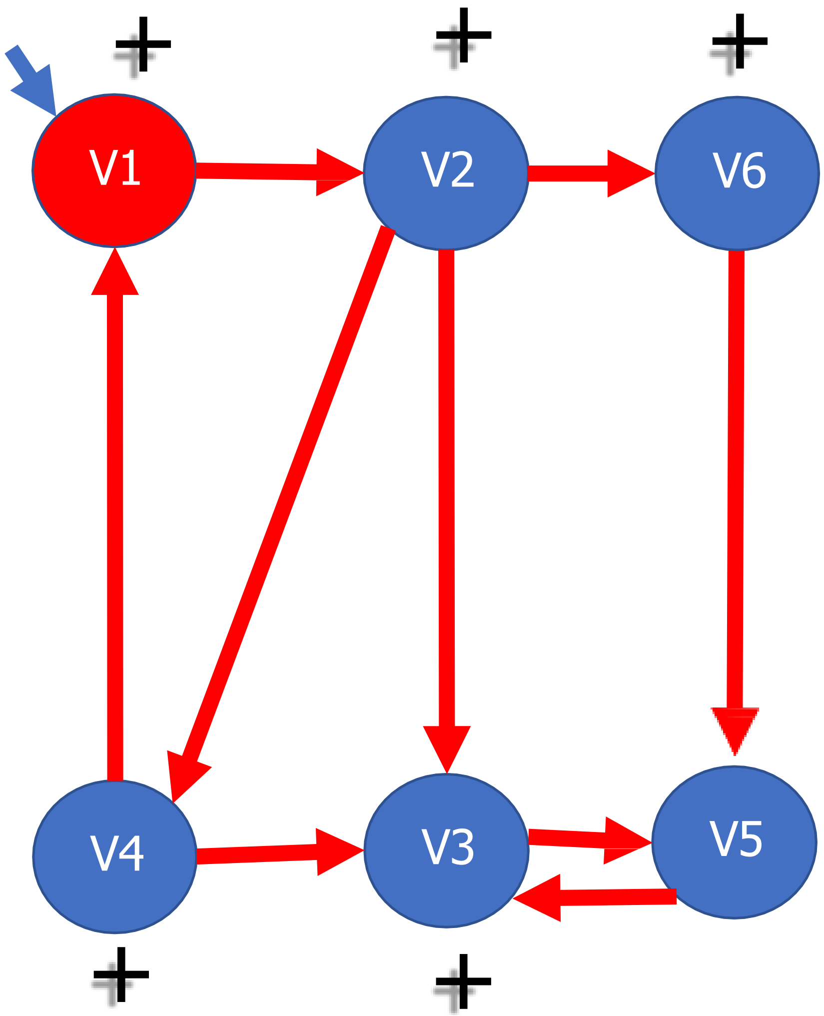

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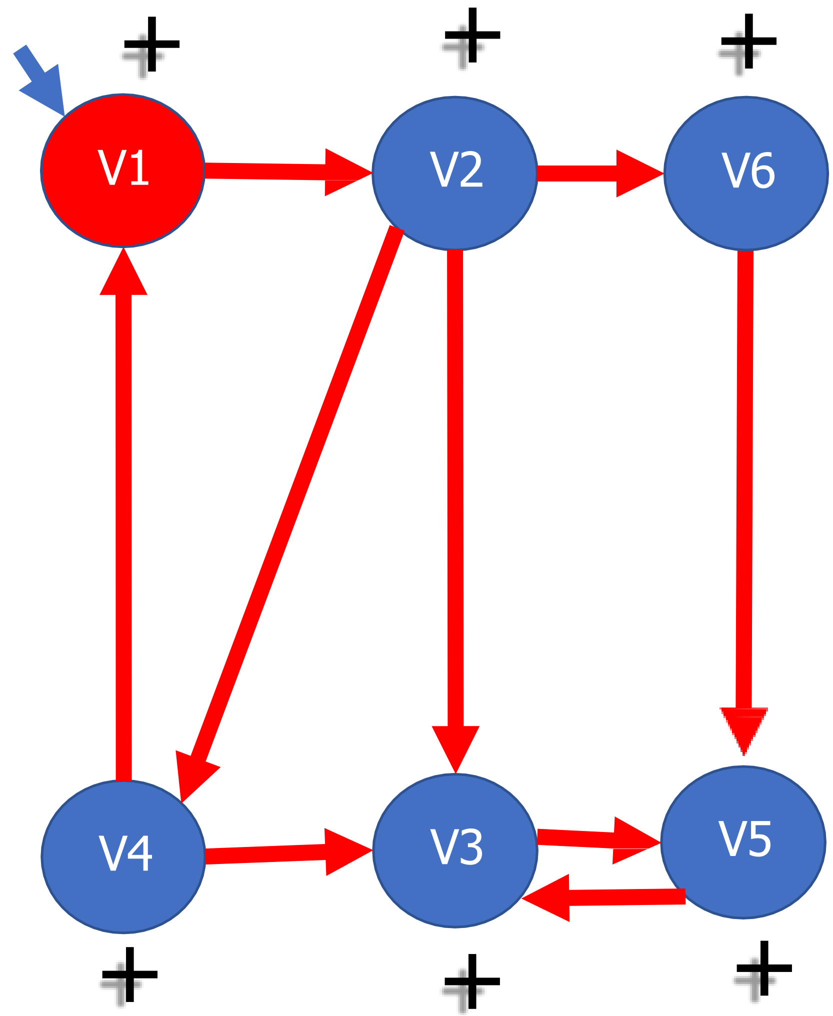
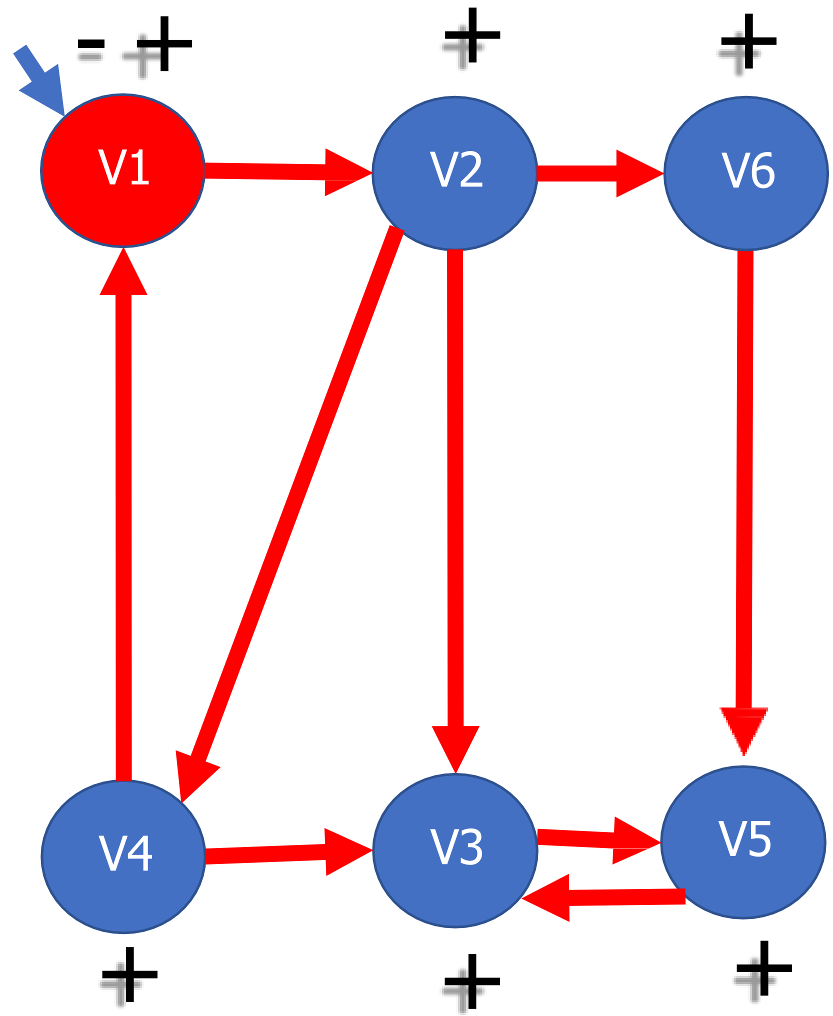


Marking algorithm

- Strongly connected components

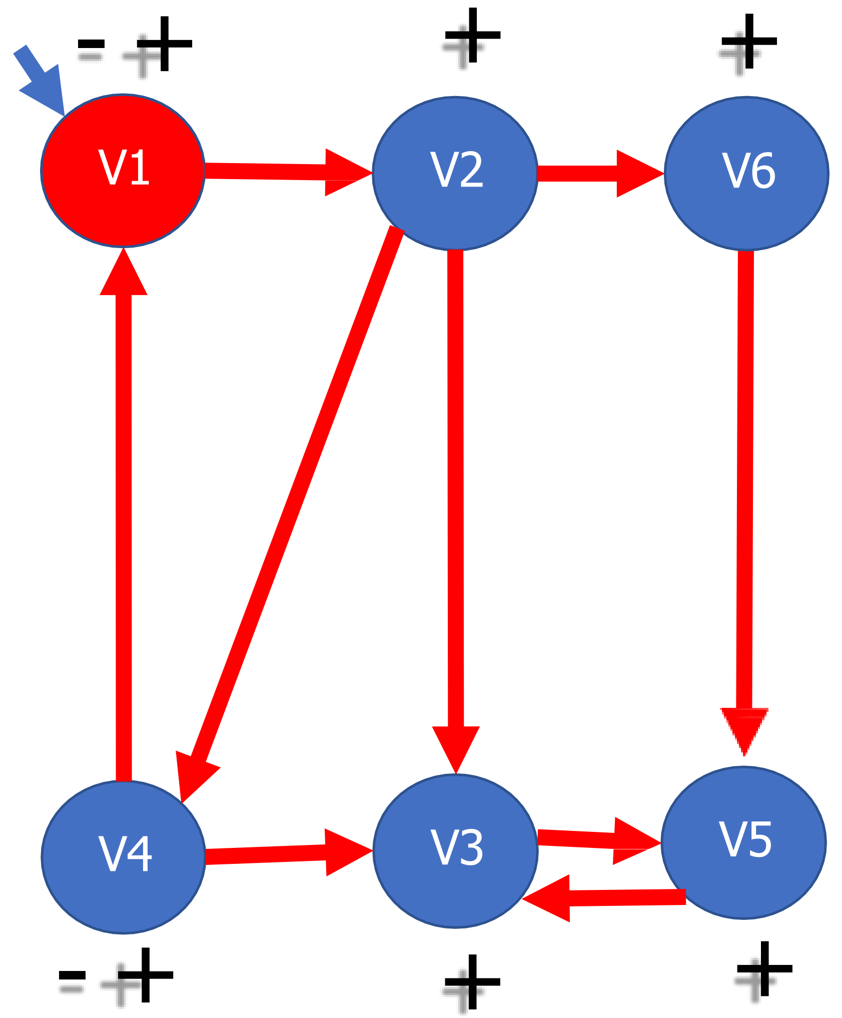
We mark by - the selected vertex



Marking algorithm

- Strongly connected components

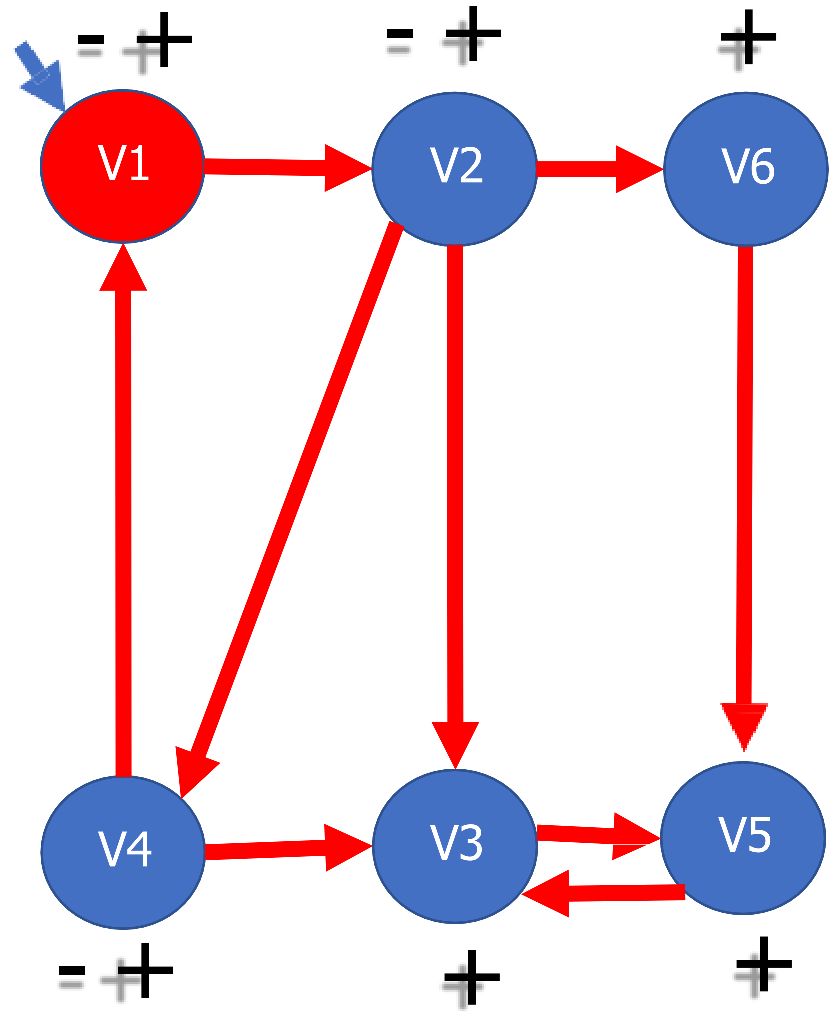
We mark by - all the predecessors of the selected vertex as well as the predecessors of its predecessors



Marking algorithm

- Strongly connected components

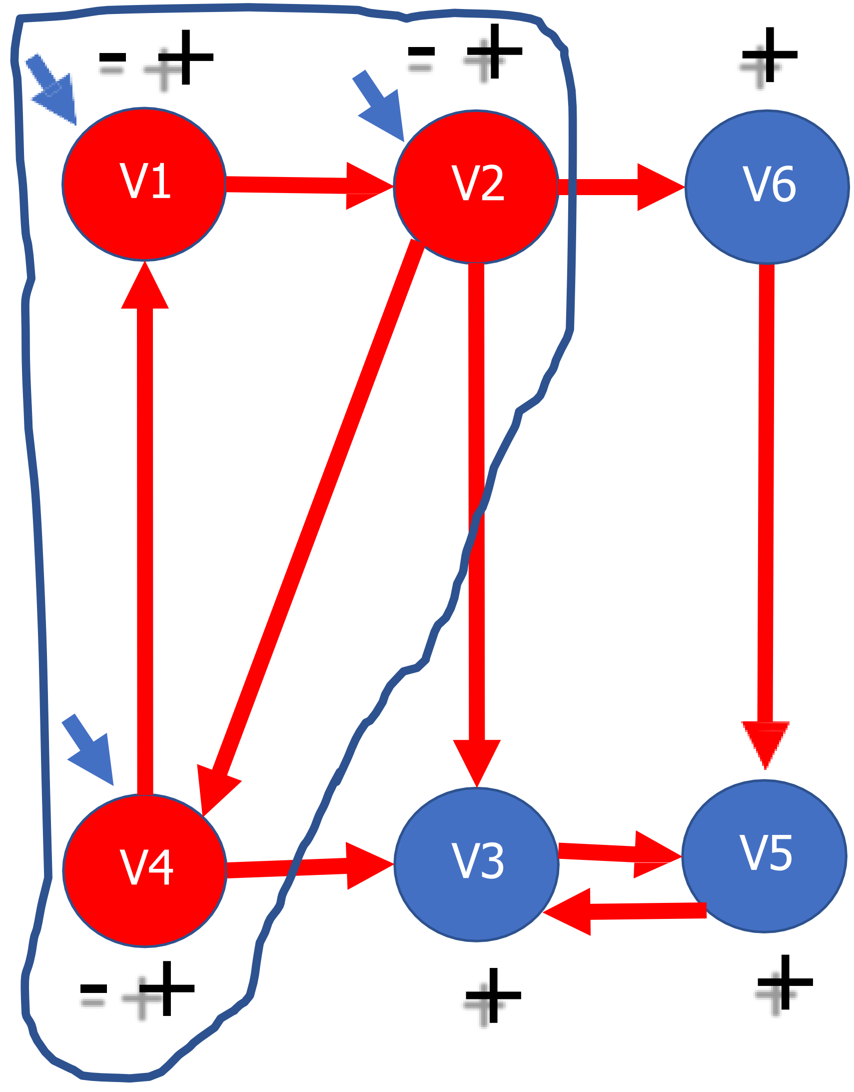
We mark by - all the predecessors of the selected vertex as well as the predecessors of its predecessors



Marking algorithm

- Strongly connected components

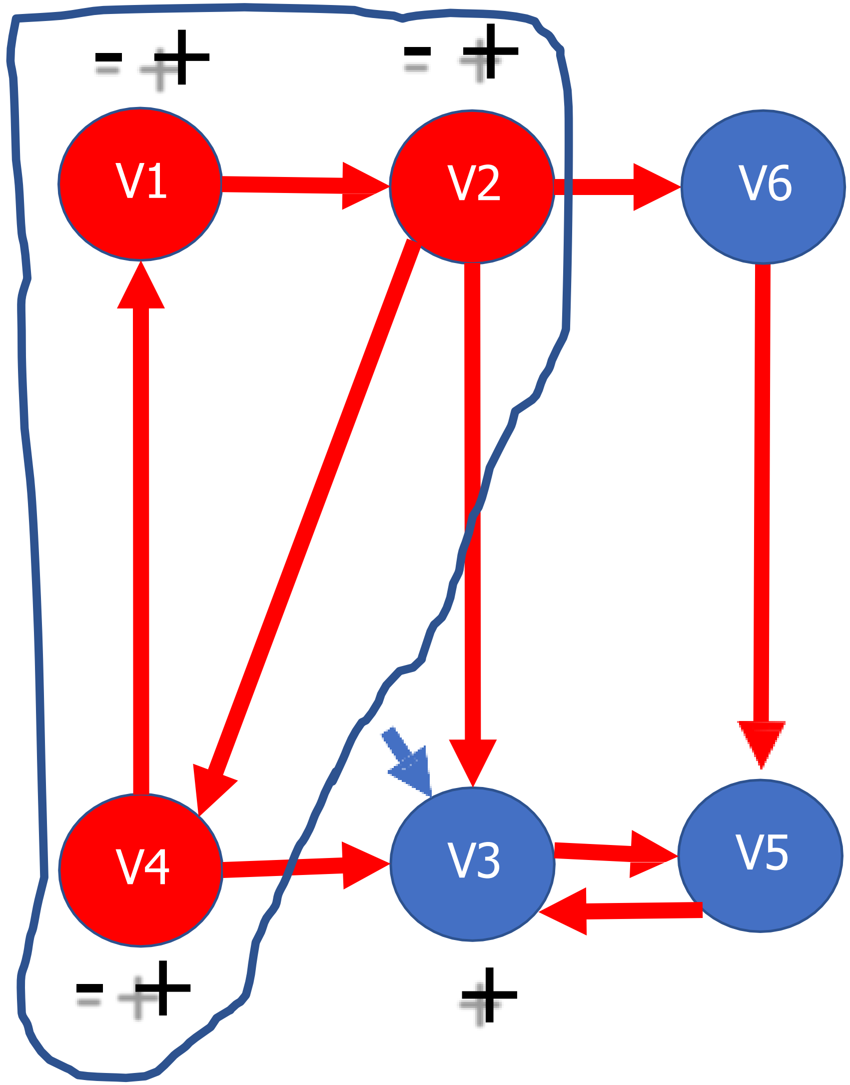
After the first iteration, we notice that only the vertices v1 v2 and v4 are marked by + and –
So v1 v2 v4 is a strongly connected component



Marking algorithm

- Strongly connected components

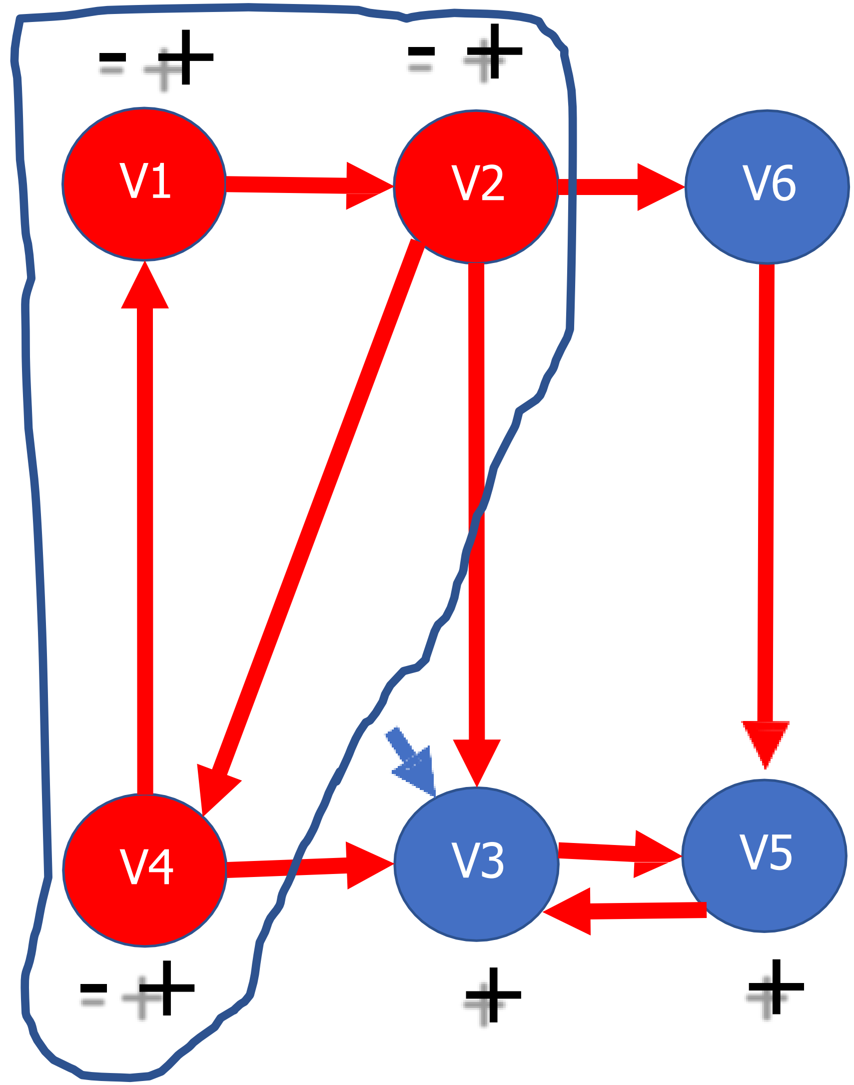
Second iteration:
We mark a vertex with +



Marking algorithm

- Strongly connected components

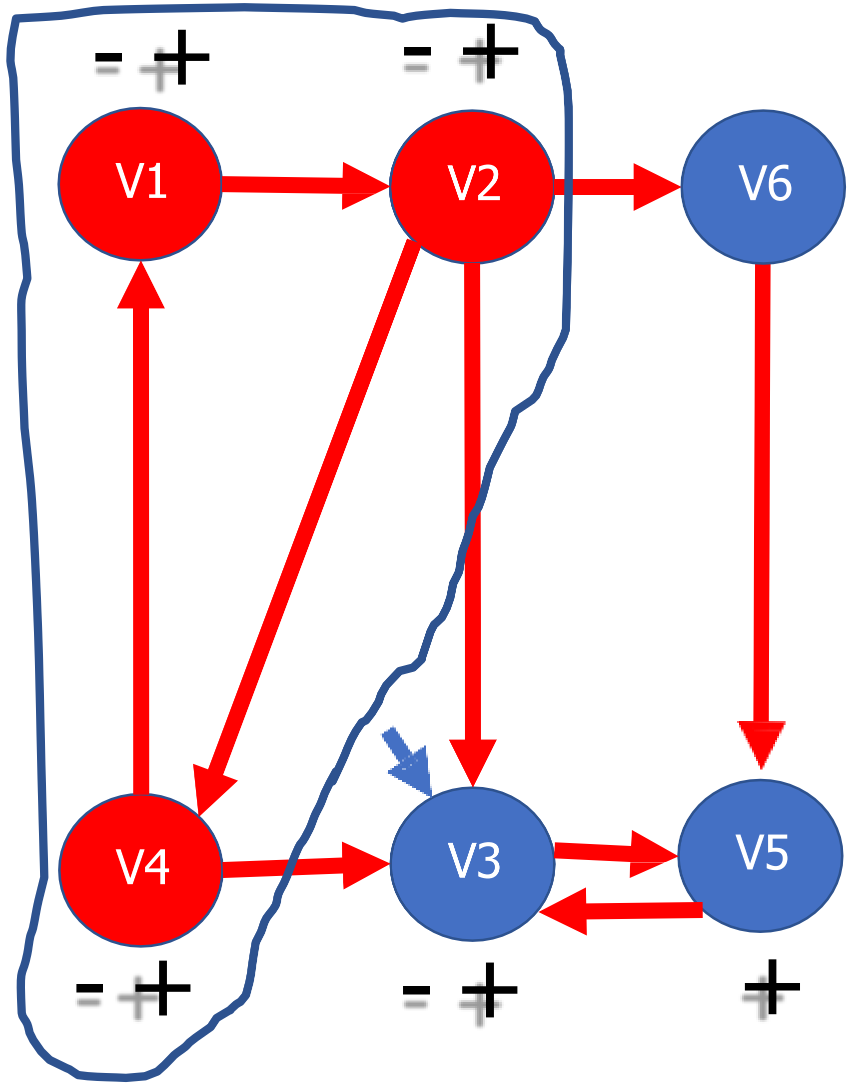
We mark by + all the successors of the selected vertex as well as the successors of its successors



Marking algorithm

- Strongly connected components

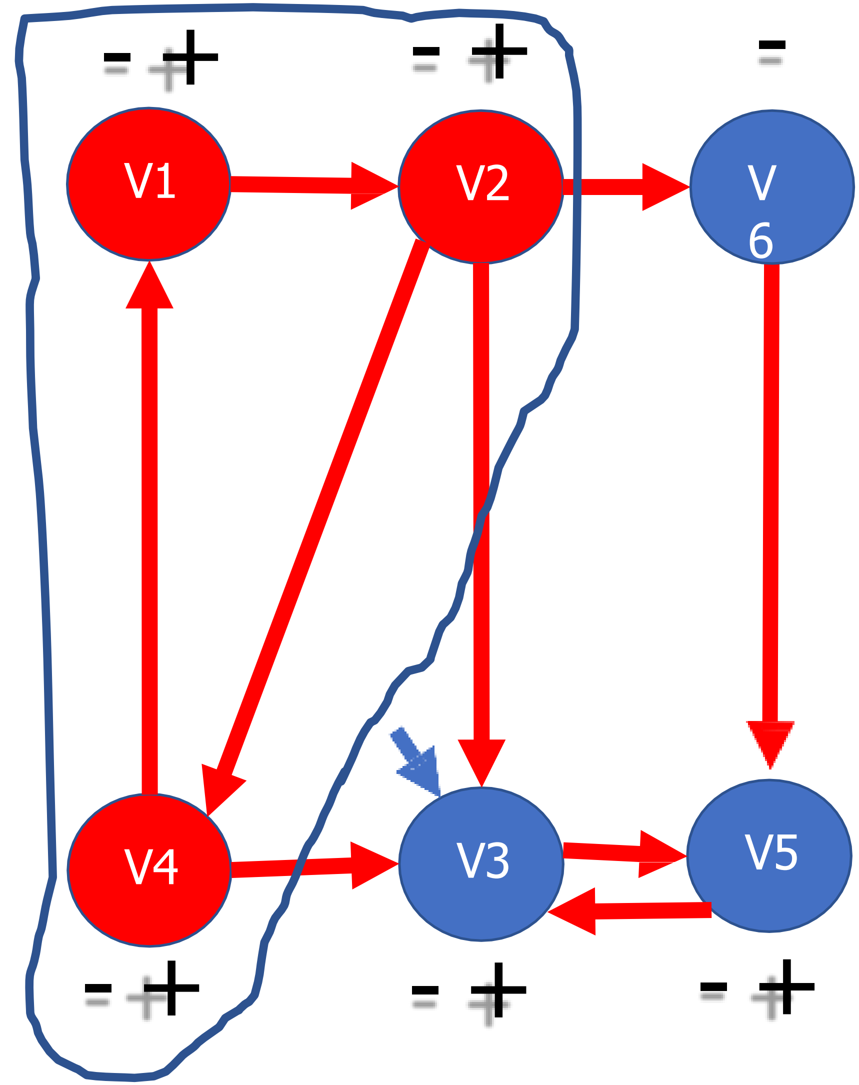
We mark by - the selected vertex



Marking algorithm

- Strongly connected components

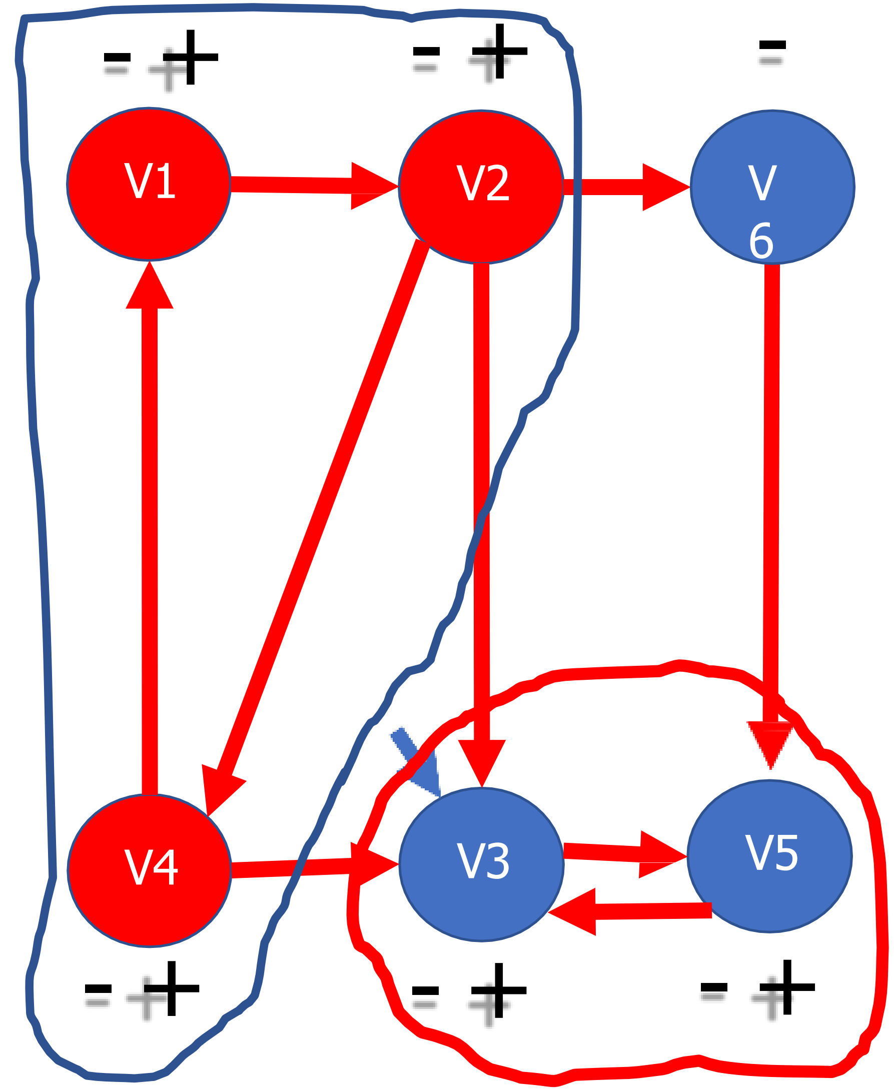
We mark by - all the predecessors of the selected vertex as well as the predecessors of its predecessors



Marking algorithm

- Strongly connected components

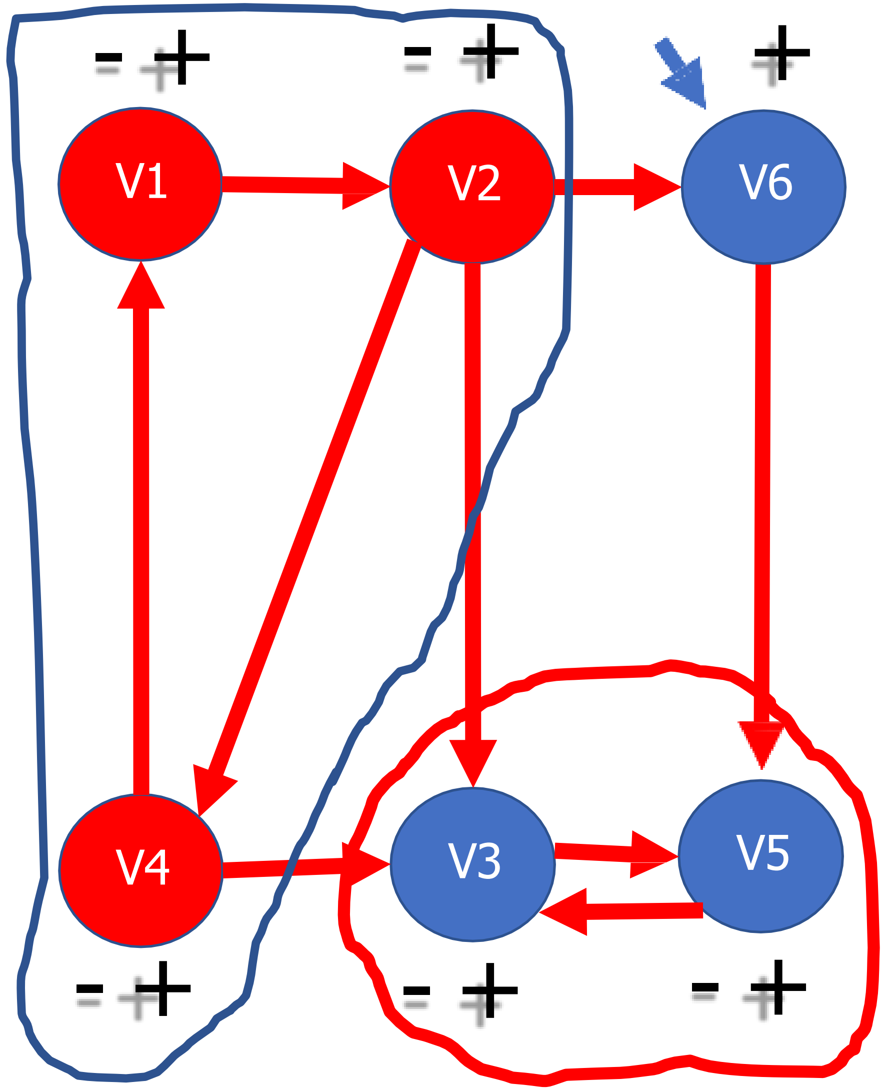
After the second iteration, we notice that only vertices v3 and v5 are marked by + and –
So v3 v5 is a strongly connected component



Marking algorithm

- Strongly connected components

Third iteration:
We mark a vertex by + Here only v1 remains



Marking algorithm

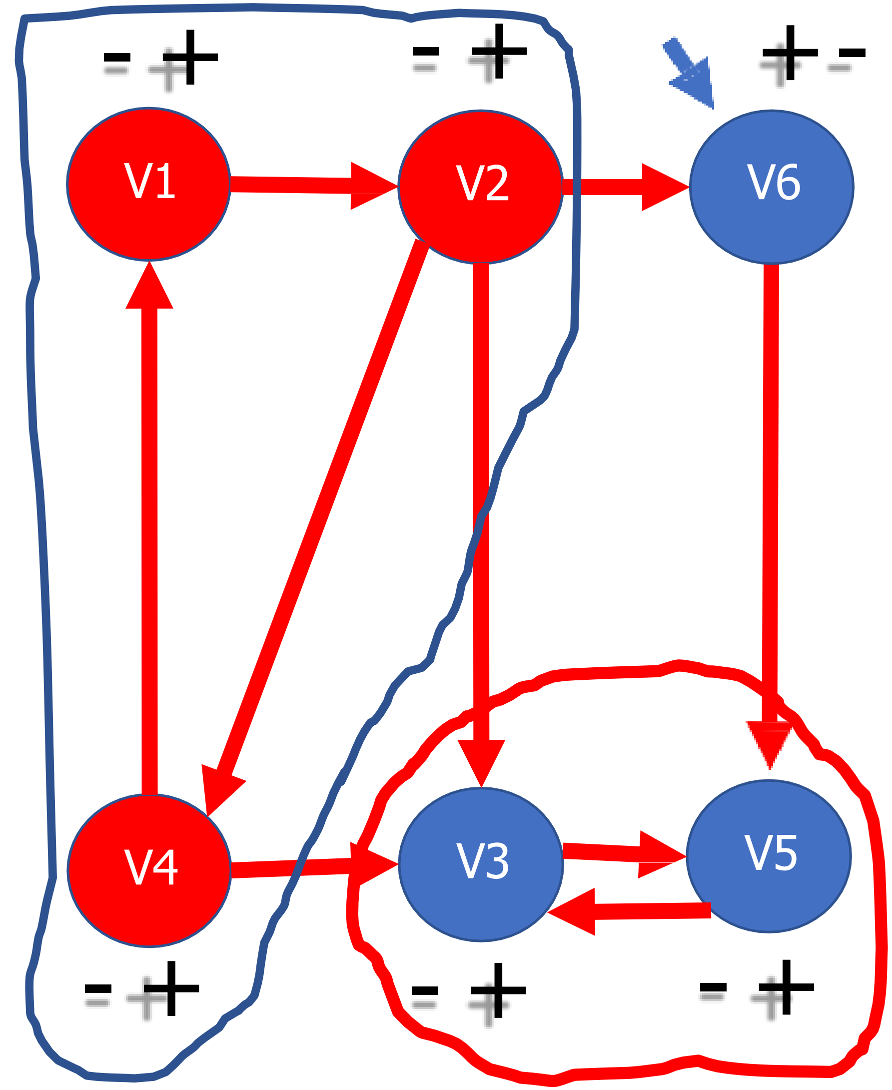
- Strongly connected components

Third iteration:

We mark v1 by -

Here only v1

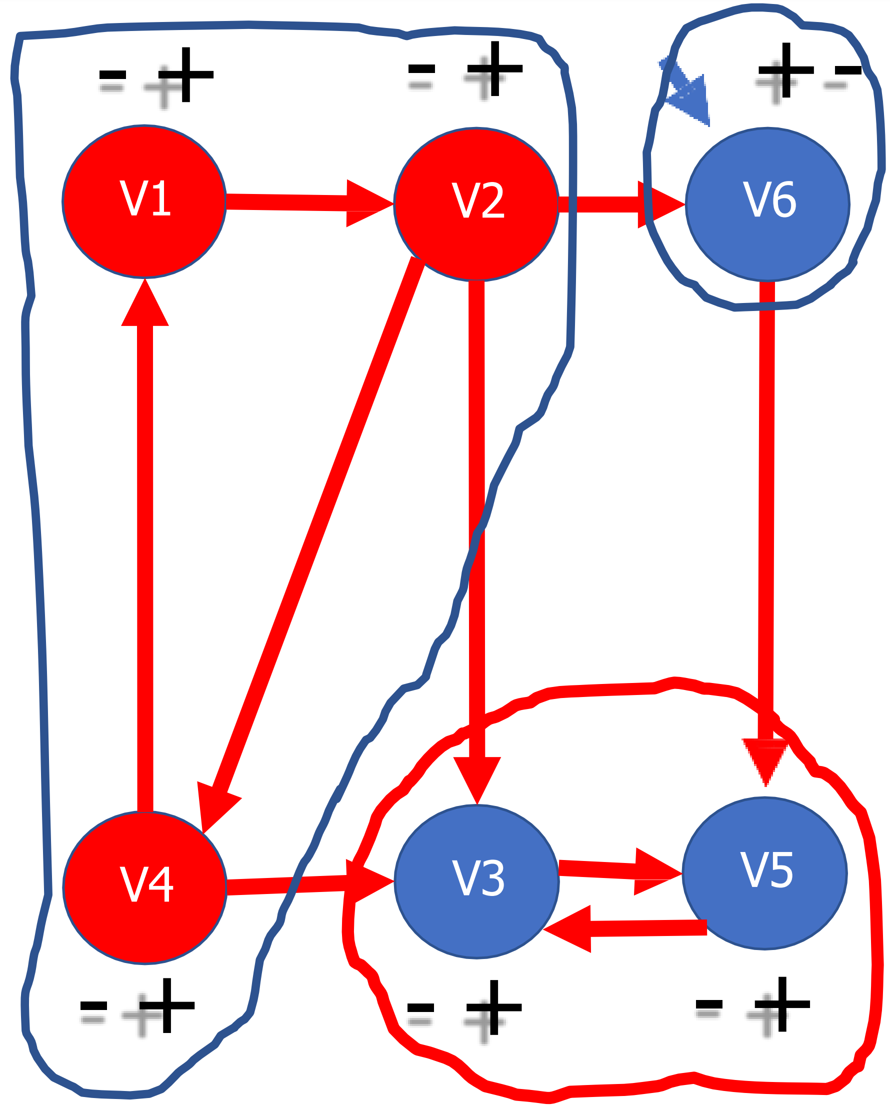
remains



Marking algorithm

- Strongly connected components

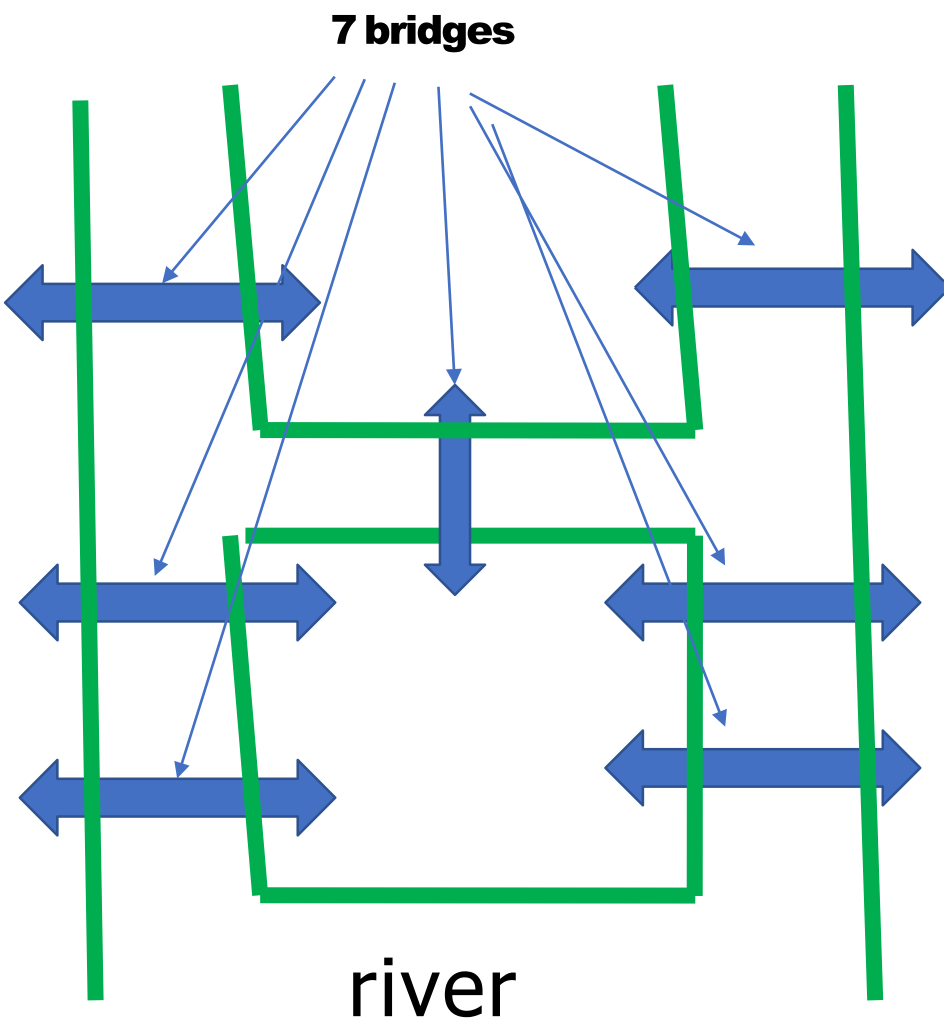
V1 is marked by + and – so it is a strongly connected component



Eulerian problem

- **Is it possible to walk across each bridge once and only once?**

It comes from the famous problem posed and solved by EULER in 1736: the city of KOENIGSBERG (KALININGRAD) watered by the PREGEL river and having seven bridges, can be modeled by the following figure



Eulerian problem

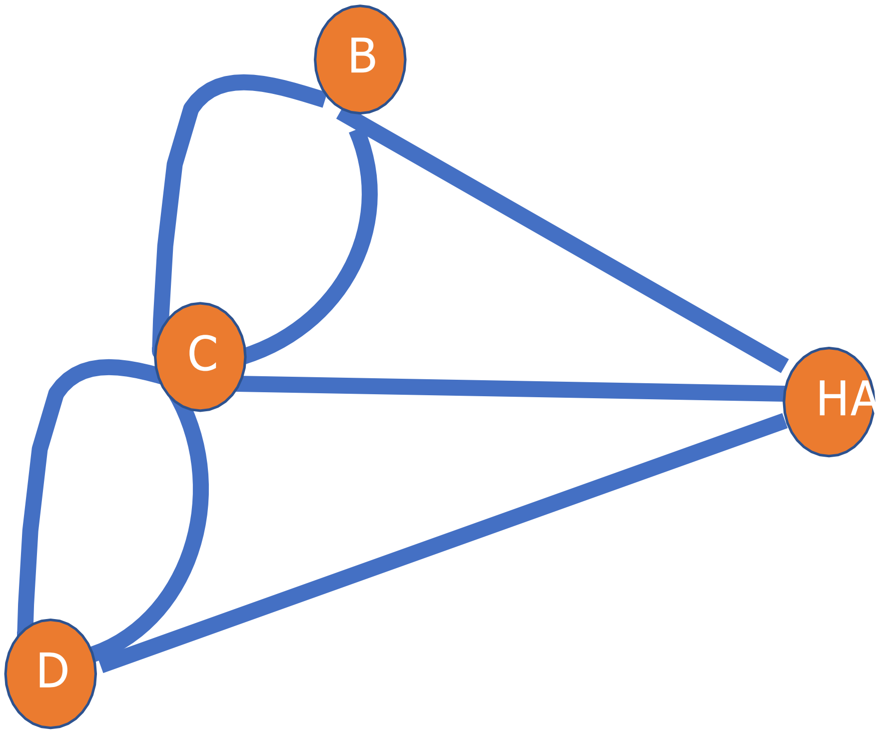
- **solution**

If a vertex is of EVEN degree, it does not pose a problem because it has as many edges to get there as edges to leave it.

- - If a vertex is of ODD degree, it necessarily constitutes a beginning and an end of the itinerary, otherwise there will be a time when we will not be able to either return or leave.

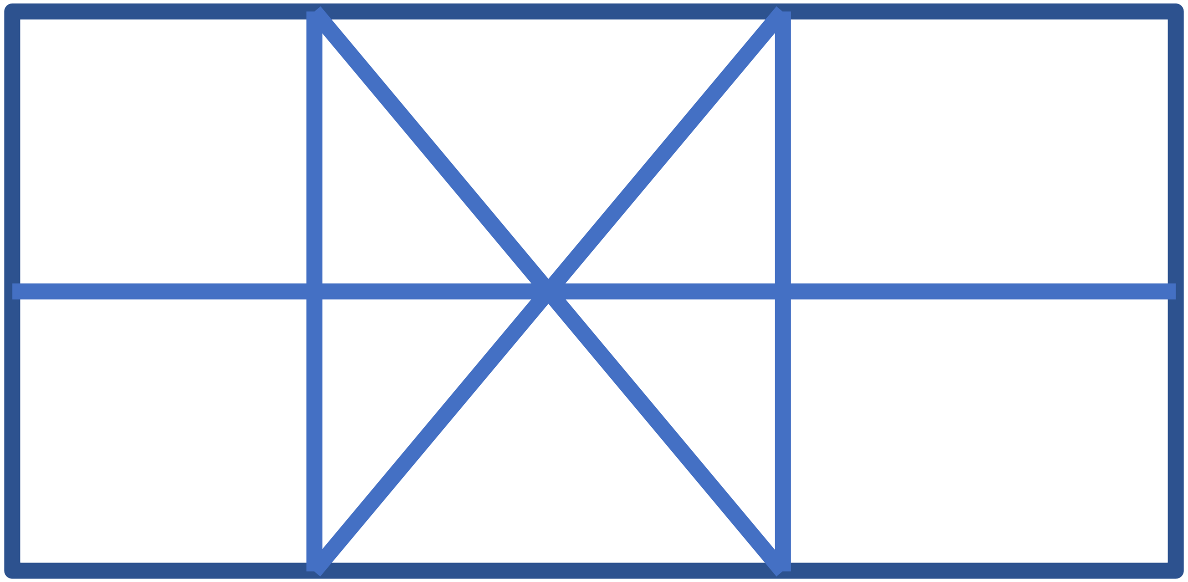
- - Only the vertices of departure and arrival can be of ODD degree. But since the graph has more than 2 vertices of ODD degree, so it is IMPOSSIBLE to find a route that verifies the imposed conditions.

7 bridges



- Example: is this graph Eulerian?

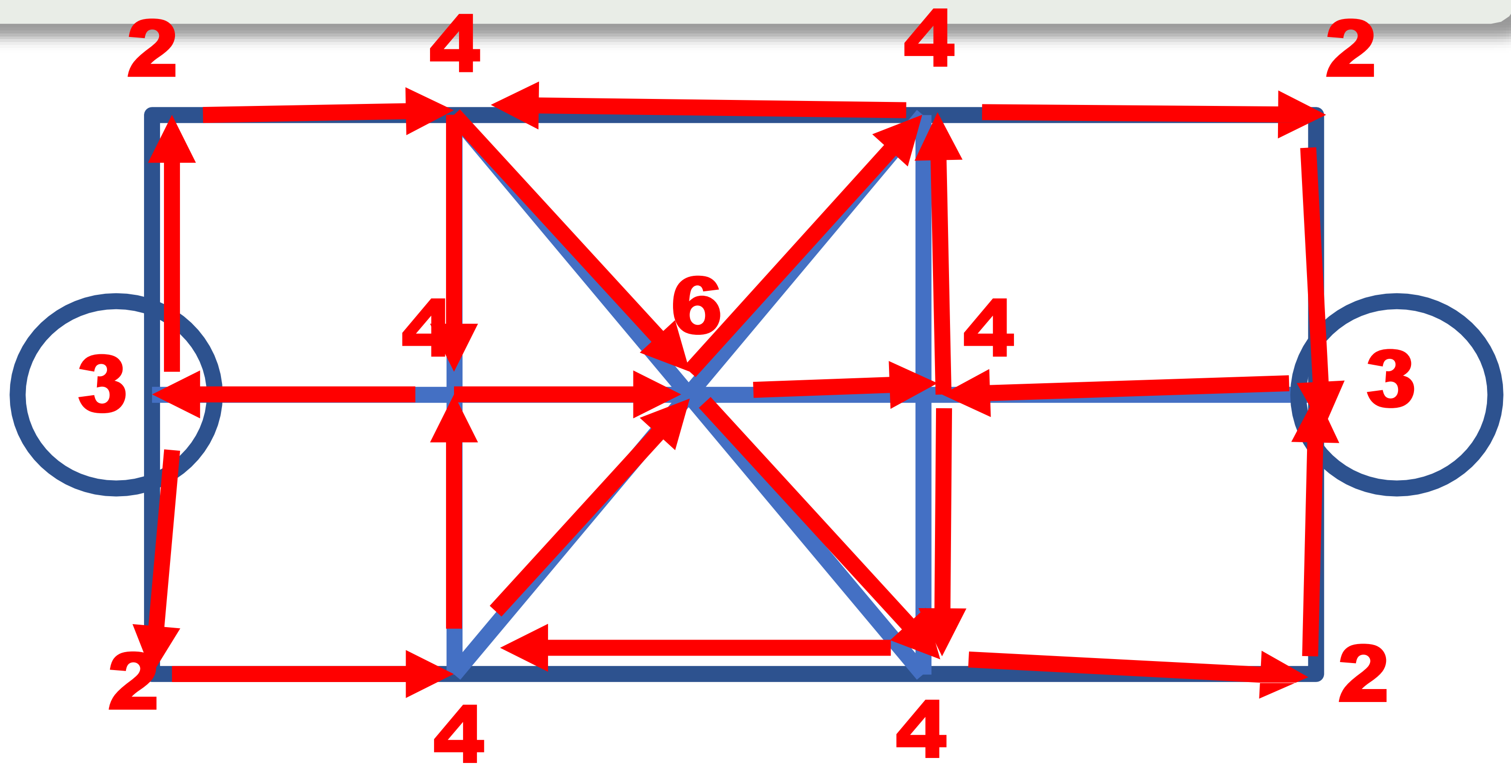
Can we draw this graph without passing twice through the same edge and without raising our hand?



Eulerian problem

- Example: is this graph Eulerian?

There are exactly two vertices of odd degree so the graph is Eulerian provided we take these vertices as starting and ending points



Eulerian problem

- summary

To draw the graph of without passing twice through the same edge and without raising your hand

If
There are more than two vertices of odd degree No Eulerian graph If

There are exactly 2 vertices of odd degree
The graph is Eulerian provided that we take these vertices as starting and ending points.

All points have even degree
The path is possible, we can start from any point and we will end the path at this same point.

Hamiltonian problem

Hamilton 1859

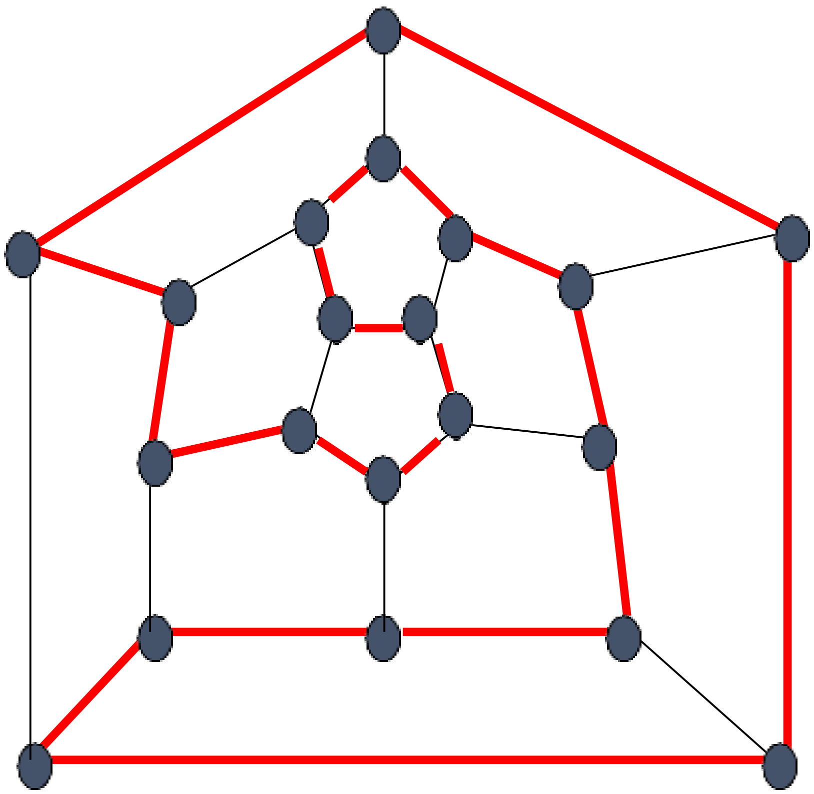
This is a game invented by W. HAMILTON in 1859.

Problematic:

We are given 20 cities and we propose to go through each of these cities once and only once and to return to the starting city. (Ex: Traveling salesman problem).

Solution:

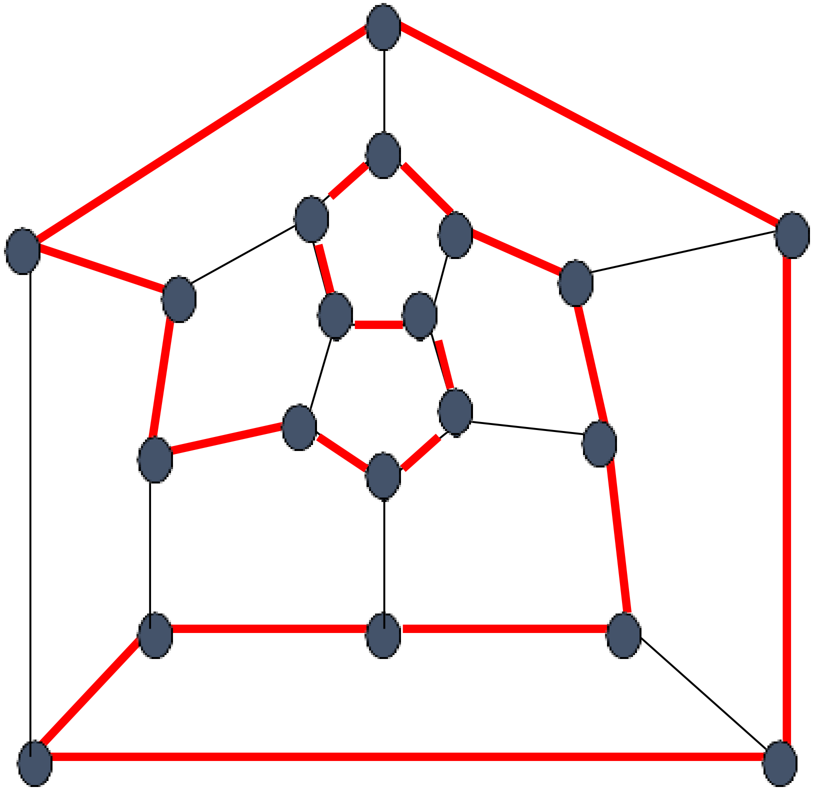
Solving this problem involves finding a Hamiltonian circuit in the graph G that models the different cities connected by different routes.



Hamiltonian problem

Hamiltonian chain

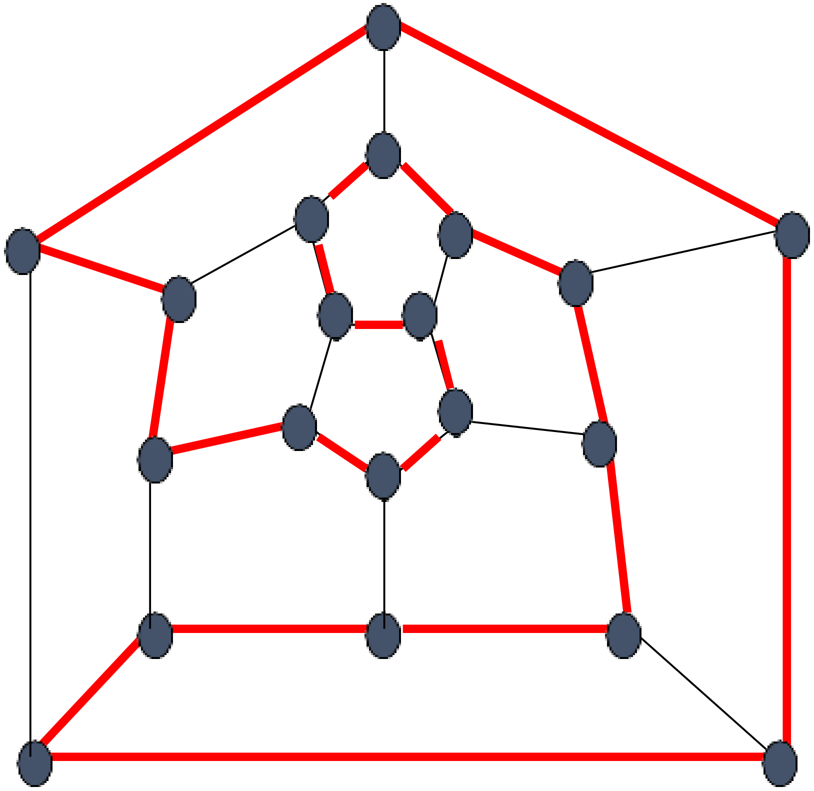
A chain joining the vertices of a graph G = (V,E) is Hamiltonian if it is elementary and includes |V| - 1 edges, that is to say if it passes through all the vertices of G



Hamiltonian problem

Hamiltonian cycle

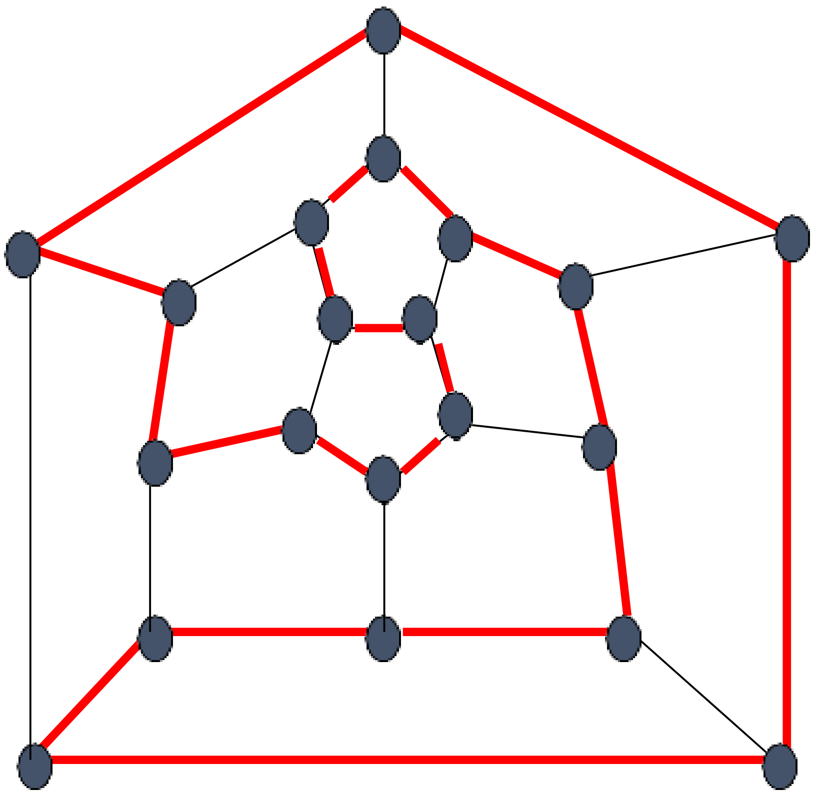
A cycle of a graph G = (V,E) is Hamiltonian if it is elementary and includes |V| edges, that is to say if it passes through all the vertices of G.



Hamiltonian problem

Hamiltonian graph

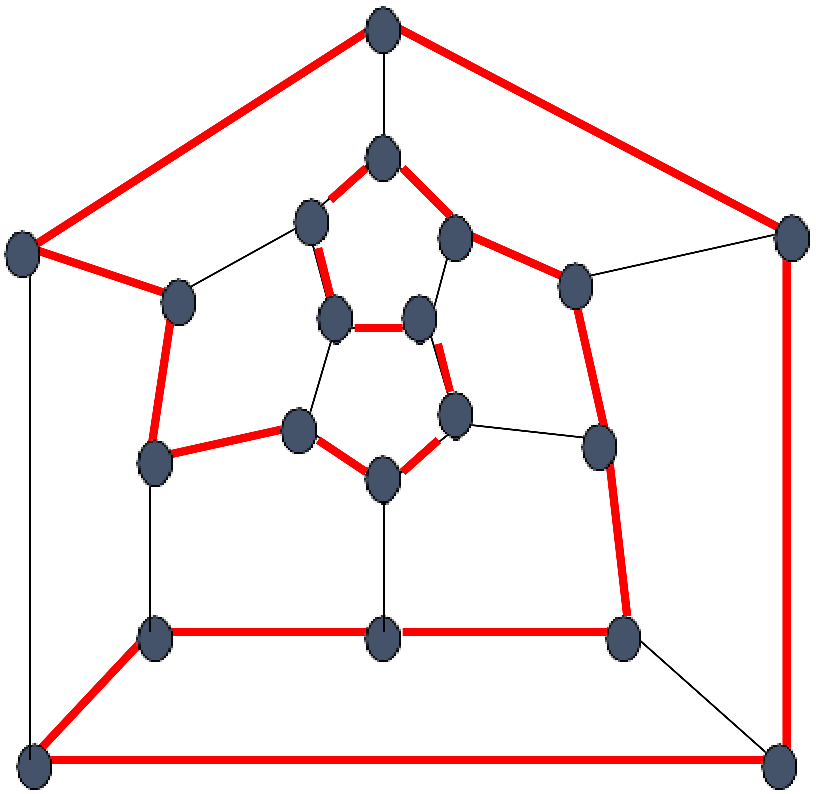
A graph G is said to be Hamiltonian if it admits a Hamiltonian cycle.



Hamiltonian problem

Hamiltonian path

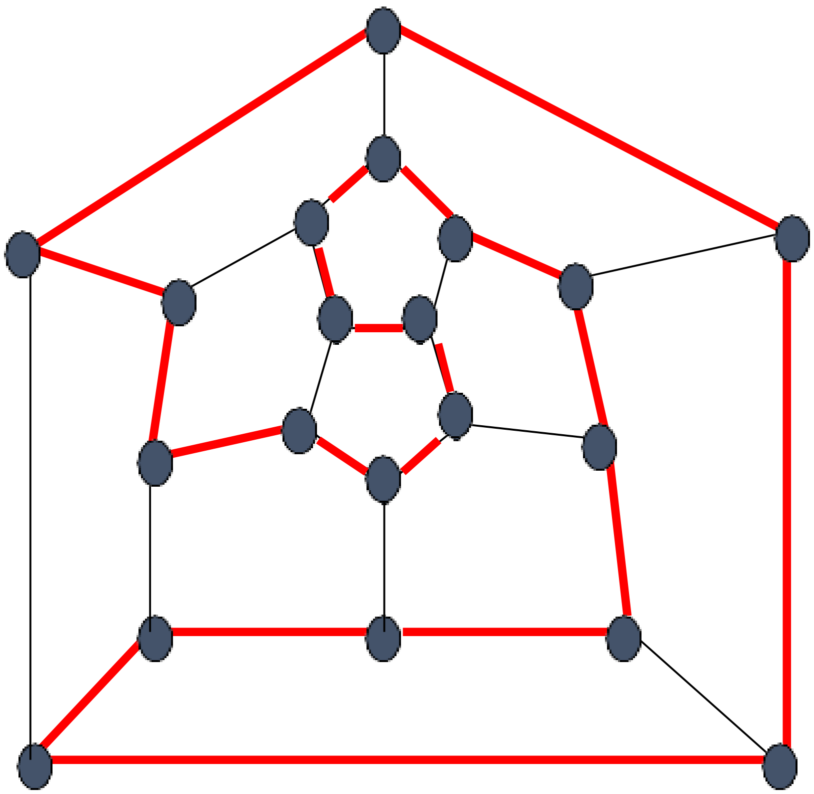
It is a path that passes once and only once through all the vertices of the graph.



Hamiltonian problem

Amiltonian Chircuit

This is a closed Hamiltonian path.



Bipartite graph

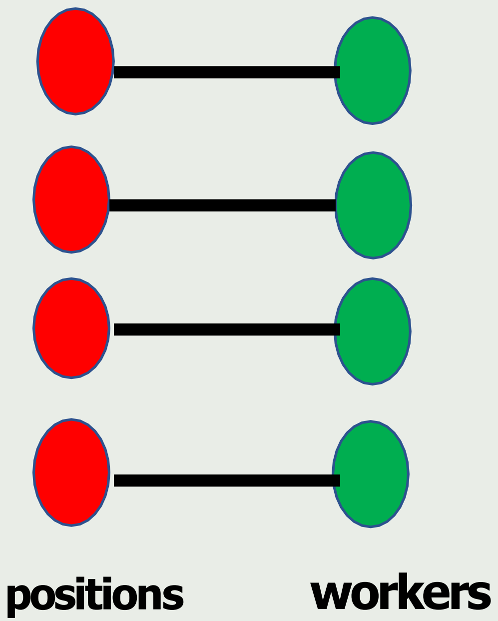
example

Let p workers and q positions be. Each worker is qualified to work at the position to which he will be assigned.

We can model this kind of problem by a graph G=(X1,X2,U) with X1 U X2 = X.

Bipartite graph

definition



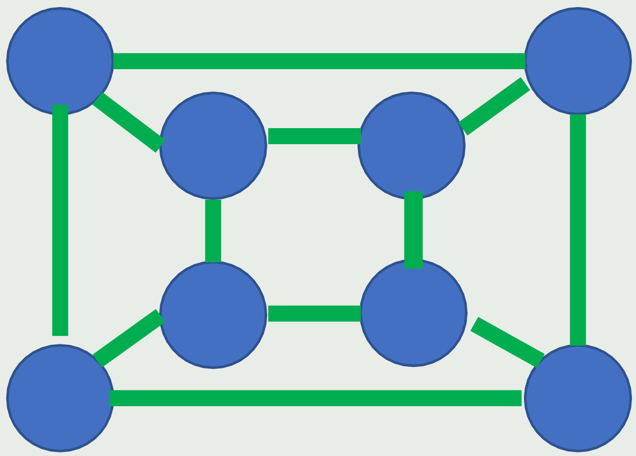
It is a graph G = (X,U) admitting a partition of X into two classes X1 and X2 such that all u ϵ U has one of its ends in X1 and the other end in X2. Such a graph is called a bipartite graph and is denoted: G = (X1,X2,U).

A graph is bipartite if

- Its chromatic number = 2

Or

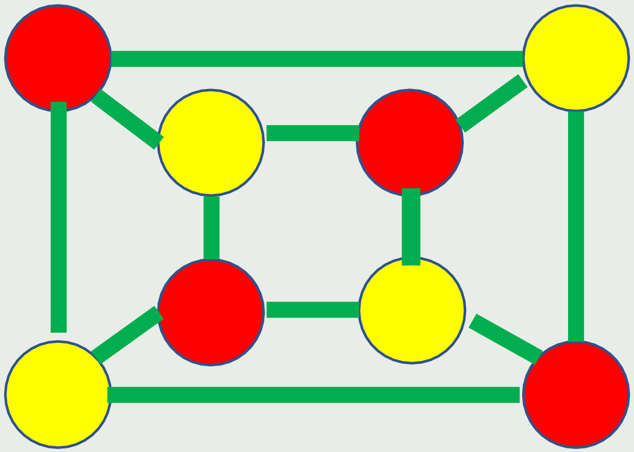
- No odd cycle



Is this graph bipartite?

This graph is bipartite because

- The chromatic number = 2
- No odd cycle



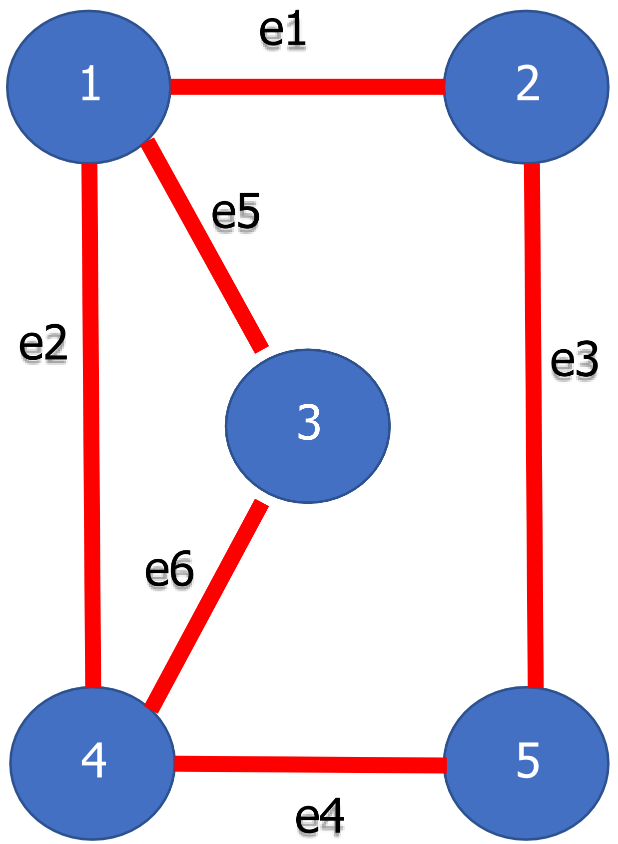
Adjoint graph(*line graph)*

- Adjoint graph(*Line graph*)

A finite graph G = (V,E)
Its adjoint graph L(G) is defined as follows:

* Each vertex of L(G) represents an edge of G

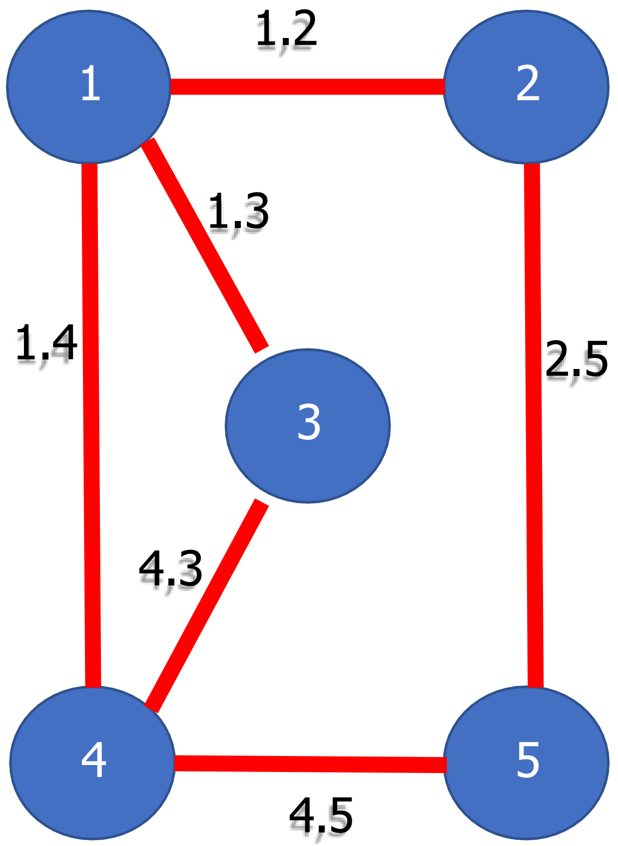
* Two vertices of L(G) are adjacent if and only if the corresponding edges share a common endpoint in G



Adjoint graph(*line graph)*

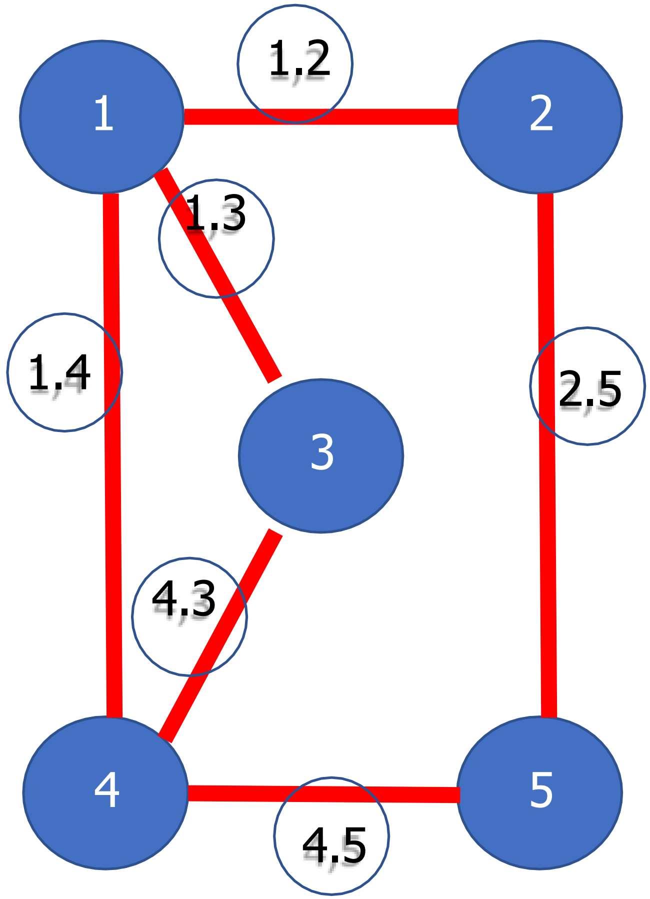
- Adjoint graph(*Line graph*)

The vertices of the adjoint graph L(G) are constructed from the edges of G



- Adjoint graph(*Line graph*)

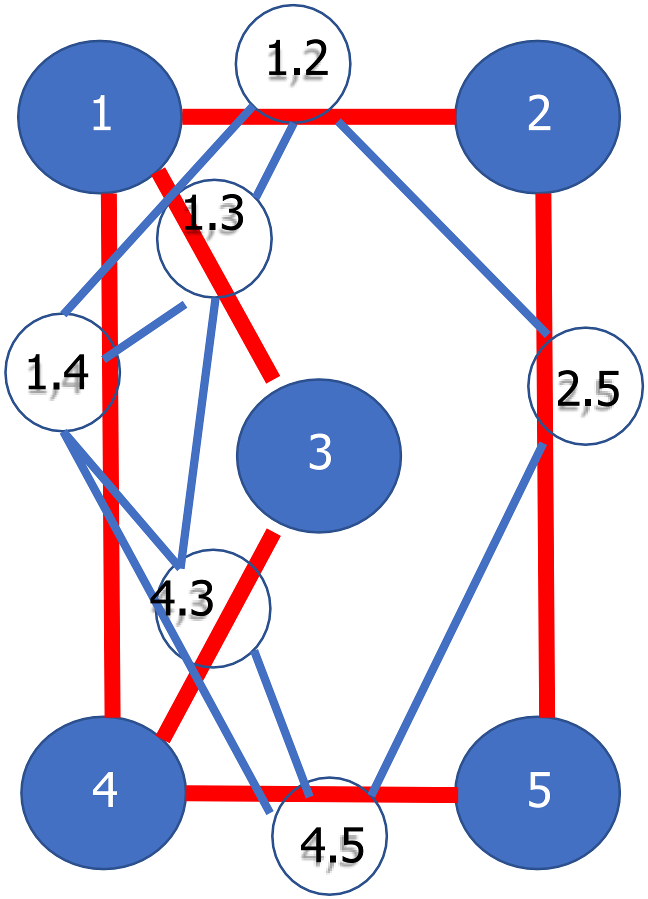
The vertices of the adjoint graph L(G) are constructed from the edges of G



Adjoint graph*(line graph)*

- Adjoint graph(*Line graph*)

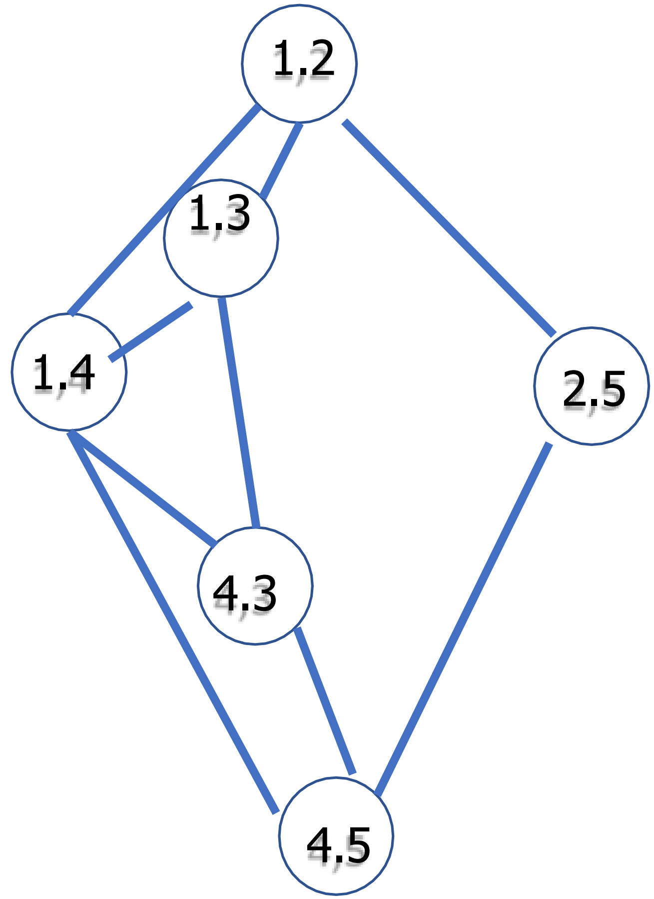
Connect new vertices if common endpoint



Adjoint graph(*line graph*)

- Adjoint graph(*Line graph*)

Here is L(G)



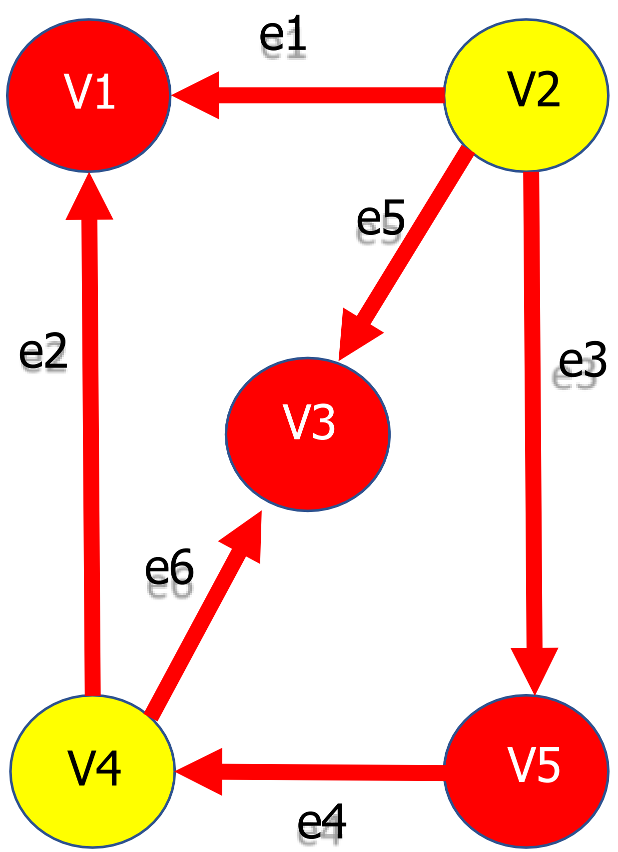
k-coloring of a graph

We say that G=(X,U) admits K-Coloration or G is Kchromatic if there exists a partition of its vertices into K stable sets or K classes

($X_1, X_2, \ldots, X_k$) so that two vertices of the same class are not adjacent (Vertices of class Xi are colored with the same color).

A graph admitting a K-Coloring is said to be K-Colorable.

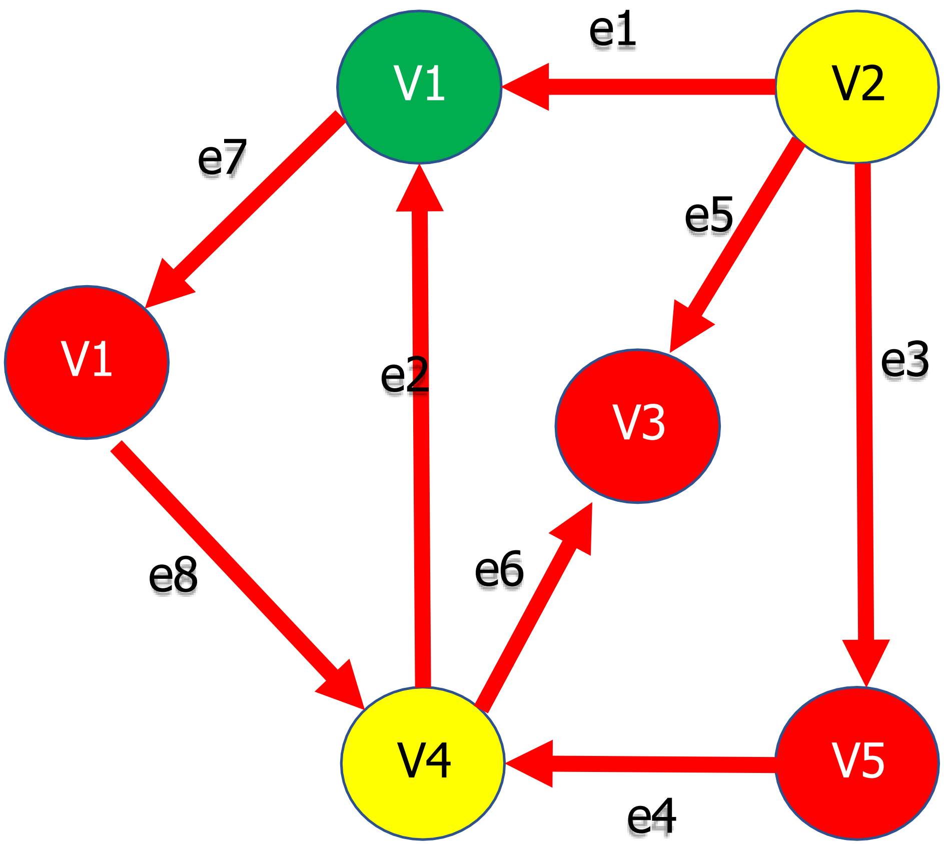
The chromatic number Ƴ(G) is the number K minimum for which G is K-Colorable.



2-colorable

γ(G) = 2 is called the chromatic number of G.

k-coloring of a graph



3-colorable

γ(G) = 3 is called the chromatic number of G.