Vector Spaces

# 1 Vector space structure

Let $V$ be a set and $(\mathbb{K}, +, \times)$ a field. We provide $V$ with two binary operations: the first one is internal, denoted " $\oplus$ " (called vector addition) defined by

$$\begin{array}{rcl} \mathbb{V} \times V & \rightarrow & V \\ (x, y) & \mapsto & x \oplus y. \end{array}$$

The second one is external (called scalar multiplication) denoted "$\triangle$"

$$\begin{array}{rcl} \mathbb{K} \times V & \rightarrow & V \\ (\lambda, x) & \mapsto & \lambda \triangle x. \end{array}$$

The set $V$ equipped with these binary operations is said to be a vector space over $\mathbb{K}$ (or $\mathbb{K}$-vector space and some times simply vector space) if the following conditions are satisfied:

1. $(V, \oplus)$ is an abelian group,
2. For any $\lambda \in \mathbb{K}$ and for any $u, v \in V$ : $\lambda \triangle (u \oplus v) = \lambda \triangle u \oplus \lambda \triangle v$,
3. For any $\lambda, \mu \in \mathbb{K}$ and for any $v \in V$ : $(\lambda + \mu) \triangle v = \lambda \triangle u \oplus \mu \triangle v$,
4. For any $\lambda, \mu \in \mathbb{K}$ and for any $v \in V$: $(\lambda\mu) \triangle v = \lambda \triangle (\mu \triangle v)$,
5. For any $v \in V$: $1 \triangle v = v$, where 1 is the neutral element of multiplication in $\mathbb{K}$.

Note that:

- When $\mathbb{K}$ is only a commutative ring instead of a field, we say that $V$ is a $\mathbb{K}$-module.
- The elements of $V$ are called vectors and those of $\mathbb{K}$ are called scalars.
- When the scalars are real numbers we shall often say that $V$ is a real vector space; and when the scalars are complex numbers we say that $V$ is a complex vector space.
- The neutral element of $V$ is noted $0_V$ (or simply 0) and is called the zero vector.

In the remainder of this course, we will use more familiar symbols ($+$ for $\oplus$ and $\times$ or $\cdot$ or nothing for $\triangle$) and that the context will prevent any potential confusion.

**Example 1**

1. *Let $V$ and $W$ be two $\mathbb{K}$-vector spaces. Then $V \times W$ is a vector space under the operations:*

$$\forall (x_1, y_1), (x_2, y_2) \in V \times W : (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

   *and* $\forall (x_1, y_1) \in V \times W, \forall \lambda \in \mathbb{K} : \lambda(x_1, y_1) = (\lambda x_1, \lambda y_1)$.

2. *Let $\mathbb{K}$ be a field and $n$ a positive integer. Then consider the set*

$$\mathbb{K}^n = \{(x_1, ..., x_n) : x_i \in \mathbb{K}\}$$

   *that is, the set of all $n$-tuples of elements from $\mathbb{K}$. We equip $\mathbb{K}^n$ with the internal operation:*

$$\forall (x_1, ..., x_n), (y_1, ..., y_n) \in \mathbb{K}^n : (x_1, ..., x_n) + (y_1, ..., y_n) = (x_1 + y_1, ..., x_n + y_n)$$

   *and the external operation:*

$$\forall \lambda \in \mathbb{K}, (x_1, ..., x_n) \in \mathbb{K}^n : \lambda(x_1, ..., x_n) = (\lambda x_1, ..., \lambda x_n).$$

   *We can verify easily that $\mathbb{K}^n$ with these two operations is a vector space over the field $\mathbb{K}$. In the particular cases $\mathbb{K} = \mathbb{R}$ and $n = 2$ or $n = 3$, we find the familiar vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$. Geometrically, $\mathbb{R}^2$ represents the cartesian plane, whereas $\mathbb{R}^3$ represents three-dimensional space.*

3. *Let $\mathbb{K}$ be a field. $\mathbb{K}[X]$ the set of polynomials with coefficients in $\mathbb{K}$ is a $\mathbb{K}$-vector space under the operations:*

$$(P + Q)(X) = P(X) + Q(X)$$

   *and*

$$(\lambda P)(X) = \lambda P(X)$$

   *. In the same way, the set $\mathbb{K}(X)$ of rational fractions with coefficients in $\mathbb{K}$ is a vector space over $\mathbb{K}$.*

4. *The set $C([a, b], \mathbb{R})$ of all continuous functions defined on the interval $[a, b]$ is a vector space over $\mathbb{R}$ under the operations:*

$$(f + g)(x) = f(x) + g(x)$$

   *and*

$$(\lambda f)(x) = \lambda f(x).$$

## 1.1 Rules of calculation in a vector space

**Proposition 2** *For any $\lambda \in \mathbb{K}$ and any vector $v$ in a vector space $V$ we have*

1. $\lambda 0_V = 0_V$.
2. $0_{\mathbb{K}} v = 0_V$.
3. $\lambda v = 0_V$, *then* $\lambda = 0_{\mathbb{K}}$ *or* $v = 0_V$.
4. $(-\lambda)v = \lambda(-v) = -(\lambda v)$.
5. $(-\lambda)(-v) = \lambda v$.

**Proof.** Let $\lambda \in \mathbb{K}$ and $v \in E$:

1. $\lambda 0_V = \lambda(0_V + 0_V) = \lambda 0_V + \lambda 0_V$, then $\lambda 0_V = 0_V$.
2. $0_{\mathbb{K}} v = (0_{\mathbb{K}} + 0_{\mathbb{K}})v = 0_{\mathbb{K}} v + 0_{\mathbb{K}} v$, then $0_{\mathbb{K}} v = 0_V$.
3. Suppose $\lambda v = 0_V$ with $\lambda \neq 0$. Then $\lambda$ admits an inverse $\lambda^{-1}$. This gives $\lambda^{-1}\lambda v = 0_V$, that is, $1.v = 0$, then $v = 0_V$.
4. On one hand $(\lambda - \lambda)v = 0_{\mathbb{K}} v = 0_V$. On the other hand $(\lambda - \lambda)v = \lambda v + (-\lambda v)$. Then $(-\lambda)v = -(\lambda v)$. We have also $\lambda(v - v) = 0_V$, then $\lambda v + \lambda(-v) = 0_V$ which gives $\lambda(-v) = -(\lambda v)$ an so the result.
5. A direct consequence of 4).

■

## 1.2 Subspaces

**Definition 3** *A subspace $W$ of a vector space $V$ over a field $\mathbb{K}$ is a subset of $V$, which, under the same addition and scalar multiplication operations as $V$ is itself a vector space.*

**Example 4**

1. $\mathbb{R}$ *is a subspace of the real vector space* $\mathbb{C}$.
2. $\mathbb{Q}$ *is a subspace of the real vector space* $\mathbb{R}$.
3. *In any $\mathbb{K}$-space $V$, the singleton $\{0_V\}$ and the whole space $V$ are subspaces of $V$; called trivial subspaces. $\{0_V\}$ is then the smallest (for the inclusion) subspace of $V$, since, we have $0_V \in W$ for every subspace $W$ of $V$, and $V$ is therefore the biggest (for the inclusion) subspace of $V$.*
4. *Consider the vector space $\mathbb{K}^n$ over $\mathbb{K}$. For each $1 \leq i \leq n$, the subset* $W_i = \left\{ (x_1, x_2, ..., \underbrace{0}_{ith\ position}, ..., x_n) \in \mathbb{K}^n \right\} \subseteq \mathbb{K}^n$ *is a subspace of* $\mathbb{K}^n$.

5. *Let $V$ be a $\mathbb{K}$-vector space and $v$ a vector of $V$, then the subset of $V$ defined by $W = \{\lambda v : \lambda \in \mathbb{K}\}$ is a subspace of $V$ called a vector line generated by the vector $v$.*

6. *In $\mathbb{R}^3$, the lines passing by the origin, the Planes passing by the origin are subspaces.*

7. *For any positive integer $n$, the subset $\mathbb{K}_n[X]$ of $\mathbb{K}[X]$ of polynomials of degree at most $n$ is a subspace of $\mathbb{K}[X]$.*

The following caracterisation of a subspace is easy to establish.

**Proposition 5** *Let $V$ be a $\mathbb{K}$-vector space. A nonempty subset $W$ of $V$ is said to be a subspace of $V$ if:*

1. $W$ is a subgroup of the group $(V, +)$,

2. For any $\lambda \in \mathbb{K}, v \in W$: $\lambda v \in W$ ($W$ is closed under the external operation).

In other words, $W$ a nonempty subset of $\mathbb{K}$-vector space $V$ is subspace if and only if $\forall \lambda, \mu \in \mathbb{K}, \forall u, v \in W : \lambda u + \mu v \in W$.

### 1.3 Intersection and union of vector subspaces

**Theorem 6** *The intersection of any set of subspaces of a vector space $V$ is a subspace of $V$.*

**Proof.** We have shown that the intersection of subgoups is a subgroup. It is easy to show the stability by multiplication by a scalar. ■

But in general, the union of two subspaces $W_1$ and $W_2$ of $V$ is not a subspace of $V$, since it has been shown that $W_1 \cup W_2$ is not a subgroup in general.

### 1.4 Linear combinations

**Definition 7** *Let $V$ be a vector space and $G$ a nonempty subset of $V$. A vector $v$ of $V$ is said to be a linear combination of vectors of $G$ if there exist scalars $\lambda_1, \lambda_2, ..., \lambda_n$ and vectors $v_1, v_2, ..., v_n \in G$ such that*

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n.$$

**Example 8**

1. *In any $\mathbb{K}$-vector space, the null vector is a linear combination of any collection of vectors, the coefficients are all zero.*

2. *In the $\mathbb{K}$-vector space $\mathbb{K}^n$, any vector $v = (x_1, x_2, ..., x_n)$ is a linear combination of the set of vectors of*

$$G = \{e_1 = (1, 0, ..., 0), e_2 = (0, 1, 0, ..., 0), ..., e_n = (0, 0, ..., 1)\}.$$

*In fact, according to the definition of operations in $\mathbb{K}^n$, we have*

$$v = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n.$$

**Proposition 9** *Let $V$ be a vector space and $G$ be a set of vectors of $V$. The set of all linear combinations of vectors of $G$ denoted by $\langle G \rangle$ or $\operatorname{Span} G$ is a subspace of $V$. Moreover, $\operatorname{Span} G$ is the smallest subspace of $V$ containing $G$, that is, any subspace $W$ of $V$ that contains $G$ also contains $\operatorname{Span} G$.*

**Proof.** The null vector in $V$ is a linear combination of vectors from $G$, it is sufficient to take the scalars as null. Since the sum of two linear combinations of vectors from $G$ is itself a linear combination of vectors from $G$, and multiplying a linear combination of vectors from $G$ by a scalar results in another linear combination of vectors from $G$, we can conclude that $\operatorname{Span}(G)$ is a subspace of $V$. Any subspace $W$ containing $G$ must contain also all the linear combinations of the vectors of $G$. Then it contains $\operatorname{Span} G$. ■

- $\operatorname{Span} G$ is called the subspace spanned (or generated) by the set $G$.
- The elements of $G$ are called generators.
- We define the Span of the empty set to be the zero vector ($\operatorname{Span} \varnothing = \langle \varnothing \rangle = \{0_V\}$).

**Example 10**

1. *In* $\mathbb{R}^3, \operatorname{Span}\{(0, 1, 0), (0, 0, 1)\} = \{(0, y, z), y, z \in \mathbb{R}\}.$
2. *In* $\mathbb{R}_2[X]$, $\operatorname{Span}\{1, 1 + X\} = \mathbb{R}_1[X].$

# 2 Bases and dimension

## 2.1 Linear independence-Linear dependence

**Definition 11** *Let $V$ be a $\mathbb{K}$-vector space and $G = \{v_1, v_2, ..., v_n\} \subset V$.*

1. *We say that $G$ is a linearly independent if:*

$$\forall \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K} : \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0$$

*gives necessarily*

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0.$$

2. *We say that $G$ is linearly dependent if it is not linearly independent; that is, if there exist scalars $\lambda_1, \lambda_2, ..., \lambda_n$ not all zero such that $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0$.*

When $G$ is an infinite set of vectors, we say that $G$ is linearly independent if every finite subset $G^{'}$ of $G$ is linearly independent. Otherwise, if some finite subset $G^{'}$of $G$ is linearly dependent, we say that $G$ is linearly dependent.

**Example 12**

1. *In $\mathbb{K}^n$, the set of vectors*

$$B = \{e_1 = (1, 0, 0, ..., 0), e_2 = (0, 1, 0, 0, ..., 0), ..., e_n = (0, 0, ..., 0, 1)\}$$

*is linearly independent, since*

$$\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n = 0$$

*gives necessarily*

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0.$$

2. *In $\mathbb{C}[X]$ as $\mathbb{C}$-vector space the set $\left\{1, 1+X, 1+X+X^2\right\}$ is linearly independent since the equation $\lambda_1 + \lambda_2(1+X) + \lambda_3(1+X+X^2) = 0$ gives necessarily $\lambda_1 = \lambda_2 = \lambda_3 = 0$.*

3. *Let $\mathbb{K}$ be a field. The set $\{1, X, ..., X^n, ...\}$ is an infinite linearly independent set of vectors of the vector space $\mathbb{K}[X]$.*

The following properties are easy to establish.

**Proposition 13** *Let $V$ be a $\mathbb{K}$-vector space.*

1. *Any subset of $V$ containing the null vector is linearly dependent.*

2. *Any subset of $V$ containing linearly dependent subset is itself linearly dependent.*

3. *A subset $G$ of $V$ is linearly dependent if and only if one of its vectors is a linear combination of others.*

4. *Two vectors $v_1$ and $v_2$ of $V$ are linearly dependent if one is a scalar multiple of the other, we say they are colinear.*

**Definition 14** *A $\mathbb{K}$-vector space $V$ is said to be finite dimensional if it is spanned (generated) by a finite number of its vectors. In other words, if there exists a finite set $G$ of vectors of $V$ such that $V = \operatorname{Span} G$.*

**Theorem 15** *Let $V$ be a $\mathbb{K}$-vector space spanned by $n$ vectors. Then any set $G$ of vectors in $V$ of cardinality greater than or equal to $n+1$ is linearly dependent.*

**Definition 16** *Let $V$ be a vector space over $\mathbb{K}$. A basis for $V$ is any linearly independent subset $B$ of $V$ that generates $V$. We also say that the vectors of $B$ form a basis for $V$.*

**Theorem 17** *Any $\mathbb{K}$-finite vector space admits a basis.*

**Example 18**

1. *In $\mathbb{K}^n$, the set of vectors*

$$B = \{e_1 = (1, 0, 0, ..., 0), e_2 = (0, 1, 0, , ..., 0), ..., e_n = (0, 0, ..., 0, 1)\}$$

*is a basis for $\mathbb{K}^n$.*

2. *In $\mathbb{K}_n[X]$, the set $\{1, X, ..., X^n\}$ is a basis for $\mathbb{K}_n[X]$.*

3. *In $\mathbb{K}[X]$, the set $\{1, X, ..., X^n, ...\}$ is a basis for $\mathbb{K}[X]$.*

The above bases are called the natural (or canonical) bases.

**Proposition 19** *Let $V$ be a $\mathbb{K}$-finite dimensional space. Any two bases of $V$ have the same number of vectors. This common number to all bases of $V$ is called the dimension of $V$ over $\mathbb{K}$ and is denoted $\dim_{\mathbb{K}} V$ (or simply $\dim V$ if no confusion arises).*

**Proof.** Let $B_1 = \{v_1, v_2, ..., v_n\}$ and $B_1 = \{u_1, u_2, ..., u_m\}$ be two basese of $V$. Since $\{v_1, v_2, ..., v_n\}$ is spanning set and $\{u_1, u_2, ..., u_m\}$ linearly independent then $m \leq n$. Using the same reasonning we get $n \leq m$. Then $n = m$. ■

Convention: the dimension of the null subspace is 0.

**Example 20**

1. *For any field $\mathbb{K}$ we have $\dim_{\mathbb{K}} \mathbb{K}^n = n$.*

2. $\dim \mathbb{K}_n[X] = n + 1$,

3. *If $V$ and $W$ are two finite dimensional $\mathbb{K}$-vector spaces then*

$$\dim_{\mathbb{K}}(V \times W) = \dim_{\mathbb{K}} V + \dim_{\mathbb{K}} W.$$

▶ Note that the dimension depends on the field considered. For example, a $\mathbb{C}$-space $V$ can be viewed as an $\mathbb{R}$-space or as a $\mathbb{Q}$-space. Look at it as an $\mathbb{R}$-space or as a $\mathbb{Q}$-space does not give the same dimension and we have $\dim_{\mathbb{R}} V = 2 \dim_{\mathbb{C}} V$.

**Proposition 21** *Let $V$ be a $\mathbb{K}$-vector space of dimension $n$. Then*

1. *Any linearly independent subset of $V$ is of cardinality at most $n$.*

2. *Any spanning set of $V$ is of cardinality at least $n$.*

3. *Any linearly independent set of* $n$ *vectors is a basis.*

4. *Any spanning set of* $V$ *of* $n$ *vectors is a basis.*

**Theorem 22** *Let* $V$ *be a finite dimensional* $\mathbb{K}$*-space. Then*

1. *Any linearly independent subset* $L$ *of* $V$ *can be extended to a basis, in other words, any linearly independent set is contained in a basis.*

2. *From any spanning set* $G$ *of* $V$, *we can extract a basis, in other words, any spanning set contains a basis.*

**Proof.** Let $G$ a be a spanning set of $V$ and put

$$X = \{M \subset V, M \text{ linearly independent and } L \subset M \subset L \cup G\}.$$

The set $X$ is not empty (since $L$ is an element of $X$), then it contains an element $B$ with the largest number of elements. This subset $B$ is a basis of $V$. Indeed, it is linearly independent by construction, Let us show that it is a spanning set of $V$. For any $v \in (L \cup G) \diagdown B$, since $B$ is maximal we have $B \cup \{v\}$ is linearly dependent and then $v \in \operatorname{Span} B$ and consequently $V = \operatorname{Span} B$. ■

**Example 23**

1. *Let* $T = \{(1,1,1),(1,2,2)\}$. $T$ *is linearly independent in* $\mathbb{R}^3$, *we can extend it to a basis by adding* $e_3 = (0,0,1)$.

2. *Let* $T = \{4, 1+X\}$. $T$ *is linearly independent in* $\mathbb{R}_2[X]$, *we can extend it to a basis by adding any polynomial of degree* 2.

3. *Let* $T = \{(1,1,1),(1,2,2),(1,1,3),(1,0,0)\}$. $T$ *is spanning set of* $\mathbb{R}^3$, *we can extract a basis from* $T$ *by taking* $T' = \{(1,1,3),(1,2,2),(1,0,0)\}$.

**Proposition 24** *Let* $V$ *be a vector space over* $\mathbb{K}$ *and* $B = \{v_1, v_2, ..., v_n\}$ *a subset of* $V$. *Then the following are equivalent:*

1. $B$ *is a basis for* $V$.

2. *Any* $v \in V$, *is uniquely written in the form* $v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$ *for* $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K}$.

   *The scalars* $\lambda_1, \lambda_2, ..., \lambda_n$ *are called the coordinates of* $v$ *in the basis* $B$.

**Proof.** Suppose that $B$ is a basis for $V$. Since $B$ is a spanning set of $V$. Then any $v \in V$ can be written as a linear combination of $v_1, v_2, ..., v_n$, say that, $v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$ for some $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K}$. Assume $v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$ and $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ are two manners to express $v$ as a combination of $v_1, v_2, ..., v_n$, we get $(\lambda_1 - \alpha_1)v_1 + (\lambda_2 - \alpha_2)v_2 + \cdots + (\lambda_n - \alpha_n)v_n = 0$. The uniqueness follows directly from the linear independence of $v_1, v_2, ..., v_n$. Conversely, suppose that any $v \in V$, is uniquely written in the form $v = \lambda_1 v_1 +$

$\lambda_2 v_2 + \cdots + \lambda_n v_n$ for $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K}$, then $B = (v_1, v_2, ..., v_n)$ is a spanning set of $V$. We also have $\forall \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K} : \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0$ gives necessarily $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$ since there is only one way to express zero as a linear combination of $v_1, v_2, ..., v_n$, hence $v_1, v_2, ..., v_n$ are linearly independent.
■

**Example 25** *1. In $\mathbb{R}^3$, the coordinates of the vector $v = (1, 2, 4)$ in the canonical basis $\{e_1, e_2, e_3\}$ are $1, 2, 4$ but in the basis $\{v_1 = (1, 1, 1), v_2 = (0, 1, 1), v_3 = (0, 0, 2)\}$ are $2, 2, 1$ since $v = v_1 + v_2 + v_3$.*

*2. In $\mathbb{R}_3[X]$, the coordinates of the polynomial $P(X) = 1 + 3X + 4X^2 + 2X^3$ in the canonical basis of $\mathbb{R}_3[X]$ are $1, 3, 4, 2$.*

**Proposition 26** *Let $V$ be a $\mathbb{K}$ finite dimensional vector space and $W$ a vector subspace of $V$. We have*

*1.* $\dim W \leq \dim V$.

*2. If* $\dim W = \dim V$. *Then* $V = W$.

**Proof.** Let $V$ be a $\mathbb{K}$ finite dimensional vector space and $W$ a vector subspace of $V$ and let $n$ be then dimension of $V$.

1. Let $G = \{v_1, v_2, ..., v_p\}$ be a basis of $W$. Then $G$ is a linearly independent subset of $V$, hence $p \leq n$, this gives $\dim W \leq \dim V$.

2. If $\dim W = \dim V$, then a basis of $W$ is maximal linearly independent subset of $V$, hence it is also a basis of $V$ and so $V = W$.

■

## 2.2 Quotient subspaces

Let $V$ be a $\mathbb{K}$-vector space and $W$ a subspace. We define an equivalence relation on $V$ by $x \Re y \Leftrightarrow x - y \in W$. Then the equivalence class of a vector $x$ under the relation $\Re$ above is the set $\overline{x} = \{x + y : y \in W\}$.

The quotient set $V/W$ is a vector space over $\mathbb{K}$, with addition and scalar multiplication defined by:

$$\overline{x} + \overline{y} = \overline{(x + y)} \text{ and } \lambda \overline{x} = \overline{(\lambda x)}.$$

Indeed, it has been shown when we have defined the quotient group that the operation $(+)$ is well defined. Similarly for any $\lambda \in \mathbb{K}$, $x - y \in W$, we have also $\lambda(x - y) = \lambda x - \lambda y \in W$ and thus $\overline{\lambda x} = \overline{\lambda y}$, so scalar multiplication is well-defined. and it is easy to verify the axioms of vecor space since these are all essentially immediate from the appropriate axioms in $V$.

**Example 27** *In $V = \mathbb{R}^2$, consider the vector subspace $W = \{(x, y) \in \mathbb{R}^2 : y = x\} = \operatorname{Span}\{(1, 1)\}$. $V$ can be interpreted as the plane. Then, $W$ represents the first bisector (the line with the equation $y = x$). For any $(x_0, y_0) \in V$, we have $\overline{(x_0, y_0)} = \{(x_0, y_0) + f : f \in W\} = \{(x_0 + \alpha, y_0 + \alpha) : \alpha \in \mathbb{R}\}$. Thus, $\overline{(x_0, y_0)}$ is the line with the equation $y = x + (y_0 - x_0)$; it is the line parallel to $W$ passing through $(x_0, y_0)$. In this example, the quotient space $V/W$ is the set of lines in the plane parallel to $W$.*

Let $V$ be a vector space over $\mathbb{K}$ and $G = \{v_1, v_2, ..., v_n\} \subset V$. We define the rank of $G$ by $\operatorname{rank} G = \dim \operatorname{Span} G$.

**Proposition 28** *Let $V$ be a vector space over $\mathbb{K}$ and $G = \{v_1, v_2, ..., v_n\}$ a subset of $V$. Then the rank of $G$ is the maximum of linearly independent vectors extracted from $G$.*

**Proof.** Since $G$ is a spanning set of $\operatorname{Span} G$ then a basis of $\operatorname{Span} G$ is no thing else than a linearly independent subset extracted from $G$ and so $\operatorname{rank}(G)$ is the maximum of linearly independent vectors extracted from $G$. ■

**Example 29** *In $\mathbb{R}_3[X]$, let $G = \left\{P_1 = 1 + X, P_2 = 1 + X^2, P_3 = 1 + X^2 + X^3, \text{and } P_4 = 3 + 2X + X^2 + X^3\right\}$ we have $rank G = 3$ since $P_4 = P_1 + P_2 + P_3$ and $\{P_1, P_2, P_3\}$ linearly independent.*

**Theorem 30** *Let $V_1$ and $V_2$ be two finite dimensional $\mathbb{K}$ vector spaces. Then*

$$\dim V_1 + \dim V_2 = \dim(V_1 + V_2) + \dim(V_1 \cap V_2).$$

**Proof.** Let $B_0 = \{e_1, e_2, ..., e_n\}$ be a basis of $V_1 \cap V_2$. We complete $B_0$ to a basis of $V_1$, say $B_1 = \{e_1, e_2, ...e_n, f_1, f_2, ..f_p\}$, and to a basis of $V_2$, say $B_2 = \{e_1, e_2, ...e_n, g_1, g_2, ...g_q\}$, respectively. We will show that $B = \{e_1, e_2, ...e_n, f_1, f_2, ..., f_p, g_1, g_2, ..., g_q\}$ is a basis of $V_1 + V_2$. Any $z \in (V_1 + V_2)$ can be written $z = x + y$ with $x \in V_1$ and $y \in V_2$ hence

$$\begin{aligned} z &= (\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n + \beta_1 f_1 + \beta_2 f_2 + \cdots + \beta_p f_p) \\ &\quad +(\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n + \delta_1 g_1 + \delta_2 g_2 + \cdots + \delta_q g_q) \\ &= ((\alpha_1 + \lambda_1) e_1 + (\alpha_2 + \lambda_2) e_2 + \cdots + (\alpha_n + \lambda_n) e_n \\ &\quad +\beta_1 f_1 + \beta_2 f_2 + \cdots + \beta_p f_p + \delta_1 g_1 + \delta_2 g_2 + \cdots + \delta_q g_q \end{aligned}$$

and so $\{e_1, e_2, ..., e_n, f_1, f_2, ..., f_p, g_1, g_2, ..., g_q\}$ is a spanning set of $V_1 + V_2$. Let us show that it is a linearly independent. Assume

$$\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n + \beta_1 f_1 + \beta_2 f_2 + \cdots + \beta_p f_p + \delta_1 g_1 + \delta_2 g_2 + \cdots + \delta_q g_q = 0.$$

Then

$$\underbrace{\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n + \beta_1 f_1 + \beta_2 f_2 + \cdots + \beta_p f_p}_{A} = \underbrace{-\delta_1 g_1 - \delta_2 g_2 - \cdots - \delta_q g_q}_{B}$$

But $A \in V_1$ and $B \in V_2$, then $A \in V_2$ so $A \in V_1 \cap V_2$, this gives

$$A = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n + \beta_1 f_1 + \beta_2 f_2 + \cdots + \beta_p f_p = t_1 e_1 + t_2 e_2 + \cdots + t_n e_n + 0 f_1 + 0 f_2 + \cdots + 0 f_p$$

The uniqueness gives $\alpha_1 = t_1$, $\alpha_2 = t_2$, ..., $\alpha_n = t_n$ and $\beta_1 = \beta_2 = \cdots = \beta_p = 0$.

Replacing this in

$$\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n + \beta_1 f_1 + \beta_2 f_2 + \cdots + \beta_p f_p + \delta_1 g_1 + \delta_2 g_2 + \cdots + \delta_q g_q = 0$$

and knowing that $\{e_1, e_2, ...e_n, g_1, g_2, ...g_q\}$ is a basis, we get

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = \delta_1 = \delta_2 = \cdots = \delta_q = 0.$$

Then, we conclude that $B$ is a basis of $V_1 + V_2$ and so the formula holds. ■

**Definition 31** *Let $V_1$ and $V_2$ be two $\mathbb{K}$-vector subspaces of a vector space $V$.*

1. *Then the sum $V_1 + V_2$ is said to be direct if $V_1 \cap V_2 = \{0_E\}$ and we write $V_1 + V_2 = V_1 \oplus V_2$.*

2. *$V_1$ and $V_2$ are said to be in complementary if $V = V_1 \oplus V_2$. We say also that $V$ is a direct sum of $V_1$ and $V_2$.*

**Proposition 32** *Let $V$ be a vector space, and $V_1$ and $V_2$ subspaces of $V$. Then $V = V_1 \oplus V_2$ if and only if every vector $v \in V$ is written in a unique way as $z = u + w, u \in V_1, w \in V_2$.*

**Proof.** It is clear since every $v \in V$ can be written (uniquely) as $v = u + w$ with $u \in V_1, w \in V_2$, that is, $V = V_1 + V_2$. Now, let $v \in V_1 \cap V_2$. Then since $v \in V_1$ and $v \in V_2$, we can write: $v = v + 0$ (where $v \in V_1, 0 \in V_2$) and $v = 0 + v$ (where $0 \in V_1, v \in V_2$). But the expression $v = u + w$ is unique, hence $v = 0$. Then $V_1 \cap V_2 = \{0\}$ and so $V = V_1 \oplus V_2$. Conversely, Since $V = V_1 + V_2$, we must only check the uniqueness. Suppose $v = u_1 + w_1$ and $v = u_2 + w_2$, where $u_1, u_2 \in V_1$ and $w_1, w_2 \in V_2$. Then $u_1 + w_1 = u_2 + w_2$ and thus $u_1 - u_2 = w_2 - w_1$. Put $t = u_1 - u_2 = w_2 - w_1$. Then $t \in V_1$ and $t \in V_2$, so $t \in V_1 \cap V_2 = \{0\}$ and hence $t = 0$. Thus, $u_1 = u_2$ and $w_1 = w_2$ so we have the uniqueness. ■

**Theorem 33** *Every vector subspace $W$ of a finite-dimensional vector space $V$ has at least one complement $S$ in $V$.*

**Proof.** Let $B_0 = \{w_1, w_2, ..., w_m\}$ be a basis of $W$ and $B = \{v_1, v_2, ..., v_n\}$ a basis of $V$. $B_0$ is linearly independent in $V$ and $B$ is a spanning set of $V$ we know that there exists $C = \{v_{i_1}, v_{i_2}, ..., v_{i_k}\} \subset B$ such that $B_0 \cup C$ is a basis of $V$. Put $S = \mathrm{Span}\{v_{i_1}, v_{i_2}, ..., v_{i_k}\}$. Let us show that $V = W \oplus S$. Let $v \in V$, since $B_0 \cup C$ is a basis of $V$ we have $v = \lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_m w_m + \mu_1 v_{i_1} + \mu_2 v_{i_2} + \cdots + \mu_k v_{i_k}$ for some $\lambda_1, \lambda_2, ..., \lambda_m, \mu_1, \mu_2, ..., \mu_k \in \mathbb{K}$ but $(\lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_m w_m) \in V$ and $(\mu_1 v_{i_1} + \mu_2 v_{i_2} + \cdots + \mu_k v_{i_k}) \in S$. Then $V = W + S$. Let $v \in W \cap S$. Then $v = \lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_m w_m = \mu_1 v_{i_1} + \mu_2 v_{i_2} + \cdots + \mu_k v_{i_k}$ for some $\lambda_1, \lambda_2, ..., \lambda_m, \mu_1, \mu_2, ..., \mu_k \in \mathbb{K}$. This gives $\lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_m w_m - \mu_1 v_{i_1} - \mu_2 v_{i_2} - \cdots - \mu_k v_{i_k} = 0$ then since $B_0 \cup C$ is a basis of $V$ we have $\lambda_1 = \lambda_2 = \cdots = \lambda_m = \mu_1 = \mu_2 = \cdots = \mu_k = 0$. ■

**Proposition 34** *Let $V_1$ and $V_2$ be two subspaces of a finite dimensional $\mathbb{K}$-vector space $V$, and let $B_1 = \{f_1, f_2, .., f_p\}$ a basis of $V_1$ and $B_2 = \{g_1, g_2, ..., g_q\}$ a basis of $V_2$. Then $V = V_1 \oplus V_2$ if and only if $B_1 \cup B_2$ is a basis of $V$.*

**Proof.** First, let us show that $V = V_1 + V_2$ if and only if $B_1 \cup B_2$ is a spanning set of $V$. $V = V_1 + V_2$ if and only if any $v \in V$, $v = x + y$ with $x \in V_1$ and $y \in V_2$, this is equivalent to $v = x + y = \lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_p f_p + \mu_1 g_1 + \mu_2 g_2 + \cdots + \mu_q g_q$ for some $\lambda_1, \lambda_2, ..., \lambda_m, \mu_1, \mu_2, ..., \mu_k \in \mathbb{K}$, that is $B_1 \cup B_2$ is a spanning set of $V$. Now, let us show that $V_1 \cap V_2 = \{0\}$ if and only if $B_1 \cup B_2$ is linearly independent set in $V$. suppose $V_1 \cap V_2 = \{0\}$ Assume $\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_p f_p + \mu_1 g_1 + \mu_2 g_2 + \cdots + \mu_q g_q = 0$, this gives $\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_p f_p = -\mu_1 g_1 - \mu_2 g_2 - \cdots - \mu_q g_q = v \in V_1 \cap V_2 = \{0\}$ this is equivalent to $v = 0$. Since $B_1$ and $B_2$ are linearly independent this gives $\lambda_1 = \lambda_2 = \cdots = \lambda_p = \mu_1 = \mu_2 = \cdots = \mu_q = 0$. Conversely, suppose $B_1 \cup B_2$ is linearly independent and let $v \in V_1 \cap V_2$ this gives $v = \lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_p f_p = \mu_1 g_1 + \mu_2 g_2 + \cdots + \mu_q g_q$ for some $\lambda_1, \lambda_2, ..., \lambda_m, \mu_1, \mu_2, ..., \mu_k \in \mathbb{K}$ then $\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_p f_p - \mu_1 g_1 - \mu_2 g_2 - \cdots - \mu_q g_q = 0$ since $B_1 \cup B_2$ is linearly independent we have $\lambda_1 = \lambda_2 = \cdots = \lambda_p = \mu_1 = \mu_2 = \cdots = \mu_q = 0$. ■

**Proposition 35** *Let $V_1$ and $V_2$ be two subspaces of a finite dimensional $\mathbb{K}$-vector space $V$. Then $V = V_1 \oplus V_2$ if and only if $V_1 \cap V_2 = \{0\}$ and $\dim V_1 + \dim V_2 = \dim V$.*

**Proof.** We have $V = V_1 \oplus V_2$ if and only if $B_1 \cup B_2$ is a basis is of $V$ which is equivalent to $\dim V_1 + \dim V_2 = \dim V$ and $B_1 \cup B_2$ linearly independent or spanning set of $V$, that is, $V_1 \cap V_2 = \{0\}$ and $\dim V_1 + \dim V_2 = \dim V$. ■

**Example 36**

1. *In $\mathbb{R}^3$, Let $L$ be a line and $P$ a plan. Then $\mathbb{R}^3 = L \oplus P$ if and only if $L \cap P = \{0\}$.*
2. *In $\mathbb{R}^3$, Let $L_1, L_2$ be two lines. Then $L_1 + L_2 = L_1 \oplus L_2$ if and only if $L_1 \cap L_2 = \{0\}$.*
3. *Let $F(\mathbb{R}, \mathbb{R})$ be the set of functions from $\mathbb{R}$ into $\mathbb{R}$. We have $F(\mathbb{R}, \mathbb{R}) = F^+(\mathbb{R}, \mathbb{R}) \oplus F^-(\mathbb{R}, \mathbb{R})$, where $F^+(\mathbb{R}, \mathbb{R})$ is the set of the even functions from $\mathbb{R}$ into $\mathbb{R}$ and $F^-(\mathbb{R}, \mathbb{R})$ is the set of the odd functions from $\mathbb{R}$ into $\mathbb{R}$.*

# 3 Exercices

**Exercise 37** *In $\mathbb{R}^4$, let*

$$V = \operatorname{Span}((0, -1, 6, -4), (1, 0, 1, -1), (4, 1, -2, 0), (-3, -2, 9, -5))$$

*and*

$$W = \{(x, y, z, t) \in \mathbb{R}^4 : x - 2z = y + z + t = y + t = 0\}.$$

1. Show that $W$ is a subspace of $\mathbb{R}^4$.

2. Give a basis and the dimension of $V$ and $W$.

3. Determine $V+W$ and show that $V+W$ is a direct sum. Do we have $\mathbb{R}^4 = V \oplus W$?

4. Find a subspace $Z$ of $\mathbb{R}^4$ such that $V \oplus W \oplus Z = \mathbb{R}^4$.

**Solution 38** *1. We have*

$$W = \{(x, y, z, t) \in \mathbb{R}^4 : x - 2z = y + z + t = y + t = 0\}.$$

*It is clear that* $(0, , 0, 0, 0) \in W$. *Let* $(x_1, y_1, z_1, t_1)$, $(x_2, y_2, z_2, t_2) \in W, \alpha \in \mathbb{R}$. *We have*

$$(x_1, y_1, z_1, t_1) \in \mathbb{W} \Leftrightarrow x_1 - 2z_1 = y_1 + z_1 + t_1 = y_1 + t_1 = 0 \text{ (1)}$$

$$(x_2, y_2, z_2, t_2) \in \mathbb{W} \Leftrightarrow x_2 - 2z_2 = y_2 + z_2 + t_2 = y_2 + t_2 = 0 \text{ (2)}$$

*Multplying (1) by* $\alpha$ *and adding with (2) we get*

$$\alpha x_1 + x_2 - 2\alpha z_1 - 2z_2 = \alpha y_1 + y_2 + \alpha z_1 + z_2 + \alpha t_1 + t_2 = \alpha y_1 + y_2 + \alpha t_1 + t_2 = 0$$

*This gives* $(\alpha x_1 + x_2, \alpha y_1 + y_2, \alpha z_1 + z_2, \alpha t_1 + t_2) \in W$ *that is,* $\alpha(x_1, y_1, z_1, t_1) + (x_2, y_2, z_2, t_2) \in W$. *Then* $W$ *is a subspace of* $\mathbb{R}^4$.

2. *We have* $(4, 1, -2, 0) = 4(1, 0, 1, -1) - (0, -1, 6, -4)$ *and* $(-3, -2, 9, -5) = -3(1, 0, 1, -1) + 2(0, -1, 6, -4)$, *so* $V = \mathrm{Span}\{(0, -1, 6, -4), (1, 0, 1, -1)\}$. *Since* $(0, -1, 6, -4)$ *and* $(1, 0, 1, -1)$ *are not collinear, then* $\{(0, -1, 6, -4), (1, 0, 1, -1)\}$ *is linearly independent. The set* $B_1 = \{(0, -1, 6, -4), (1, 0, 1, -1)\}$ *is a spanning set of* $V$ *and it is linearly independent so it is a basis of* $V$ *and we have* $\dim V = 2$.

$$\begin{aligned} W &= \{(x, y, z, t) \in \mathbb{R}^4 : t = -y \wedge z = 0 \wedge x = 2z = 0\} \\ &= \{(0, y, 0, -y) : y \in \mathbb{R}\} \\ &= \{y(0, 1, 0, -1) : y \in \mathbb{R}\} \\ &= \mathrm{Span}\{(0, 1, 0, -1)\}. \end{aligned}$$

*Since* $(0, 1, 0, -1) \neq 0_{\mathbb{R}^4}$, *then* $\{(0, 1, 0, -1)\}$ *is linearly independent. The set* $B_2 = \{(0, 1, 0, -1)\}$ *is a spanning set of* $W$ *and it is linearly independent so it is a basis of* $W$ *and we have* $\dim W = 1$.

3. *We have* $V + W = \mathrm{Span}\{(0, -1, 6, -4), (1, 0, 1, -1), (0, 1, 0, -1)\}$. *Let* $v \in V \cap W$, *then* $v = \lambda(0, 1, 0, -1) = X = \alpha(0, -1, 6, -4) + \beta(1, 0, 1, -1)$ *for some* $\alpha, \beta, \lambda \in \mathbb{R}$. *Hence we obtain the system of equations*

$$\begin{cases} 0 &= \beta, \\ \lambda &= -\alpha \\ 0 &= 6\alpha + \beta \\ -\lambda &= -4\alpha - \beta \end{cases}$$

*which gives us* $\alpha = \beta = \lambda = 0$*, thus* $v = 0_{\mathbb{R}^4}$*. Therefore* $V \cap W = \{0_{\mathbb{R}^4}\}$ *which means that* $V + W$ *is a direct sum.*
*The fact that* $V + W$ *is a direct sum implies that* $\dim(V + W) = \dim V + \dim W = 3 \neq \dim \mathbb{R}^4$*, hence* $V + W \neq \mathbb{R}^4$*, and so* $V \oplus W \neq \mathbb{R}^4$*.*

4. *Using Gauss elemination we can verify thatv* $B_1 \cup B_2 = \{(0, -1, 6, -4), (1, 0, 1, -1), (0, 1, 0, -1)\}$ *is of rank* 3 *and then we can take* $Z = \operatorname{Span}\{(0, 0, 0, 1)\}$*.*