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Algebra 2 - Tutorial 2

Basic Training Cycle

Vector Spaces

Exercise 1

Prove that the only vector subspaces of the vector space $(\mathbb{K}, +, \cdot)$ are $\{0\}$ and $\mathbb{K}$ itself.

Exercise 2

Are the sets described below vector spaces ? Justify your answer without finding a spanning set.

1. $E_1 = \{f \in \mathcal{F}(\mathbb{R}), f(0) = -1\}$.
2. $E_2 = \{(x, y) \in \mathbb{R}^2, xy \in \mathbb{Q}\}$.
3. $E_3 = \{M \in \mathcal{M}_n(\mathbb{R}) \mid \text{all coefficients of } M \text{ are equal }\}$.
4. $E_4 = \{(x, y, z, t) \in \mathbb{R}^4, x + z = y + t \text{ and } y = t^2\}$.
5. $E_5 = \{P \in \mathbb{R}_n[X] \mid P(X + 1) = P(X)\}$ where $n \in \mathbb{N}^*$.
6. $E_6 = \{f \in \mathcal{F}(\mathbb{R}), f(x) = 0 \text{ for infinitely many values of } x\}$.
7. $E_7 = \{f \in \mathcal{F}(\mathbb{R}), f \text{ is increasing on } \mathbb{R}\}$.
8. E_8 : The set of nilpotent matrices.
9. $E_9 = \{A \in \mathcal{M}_3(\mathbb{R}) \mid A^t = -A\}$.
10. $E_{10} = \{(u_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}, \forall n \in \mathbb{N}, u_{n+2} = u_{n+1} + 4u_n\}$.
11. E_{11} : The set of real arithmetic sequences.
12. $E_{12} = \{(u_n) \in \mathbb{R}^{\mathbb{N}}, u_n u_{n+2} \leq 0\}$.

Exercise 3

Using the notion of *Span*, show that E_3, E_9, and E_{11} are vector spaces and determine a basis and the dimension.

Exercise 4

Let F, G, H be vector subspaces of a vector space E.
Show that $F + (G \cap H) \subset (F + G) \cap (F + H)$, and that if $F \subset G$, then the previous inclusion becomes an equality.

Exercise 5

Let E_1, E_2, E_3 be three vector subspaces satisfying :

$$\begin{cases} E_1 + E_3 = E_2 + E_3, \\ E_1 \cap E_3 = E_2 \cap E_3, \\ E_1 \subset E_2. \end{cases}$$

Prove that $E_1 = E_2$.

Exercise 6

In each of the following cases, determine whether the collection (u_i) generates E :

1. $E = \mathbb{R}^3$ and $u_1 = (1, -1, -2)$, $u_2 = (7, 10, 3)$, $u_3 = (3, -4, -7)$.
2. $E = \mathbb{R}^4$ and $u_1 = (0, 0, 0, 1)$, $u_2 = (0, 0, 1, 1)$, $u_3 = (0, 1, 1, 1)$, $u_4 = (1, 1, 1, 1)$, $u_5 = (1, 1, 1, 0)$.

Exercise 7

Let $u_1 = (1, -2, 1)$ and $u_2 = (0, 1, 2)$.

1. Let $v = (1, 2, 3)$. Is $v \in \text{Vect}(u_1, u_2)$?
2. Determine $k \in \mathbb{R}$ such that $(2k + 1, k, -k) \in \text{Span}\{u_1, u_2\}$.
3. Let $a = (-1, 4, 2)$ and $b = (1, 0, 5)$. Show that $\text{Span}\{u_1, u_2\} = \text{Span}\{a, b\}$.

Exercise 8

For which values of the real number m is the collection $\{u_1, u_2, u_3, u_4\}$ linearly independent in $\mathbb{R}^4$?

$$u_1 = (1, 1, 0, 0), \quad u_2 = (1, m, 1, 0), \quad u_3 = (1, 0, m, 1), \quad u_4 = (1, 0, 0, m)$$

In the case where the sequence of vectors is linearly dependent, provide a dependency relation.

Exercise 9

Determine the rank of the following sets of vectors :

1. $\mathscr{F} = \big((1, 1, 1), (0, 1, 2), (-1, 0, 1), (0, 1, 2)\big)$.
2. $\mathscr{F} = (I_n, J, J^2, J^3)$ where $J \in \mathscr{M}_n(\mathbb{R})$ is the matrix with all entries equal to 1.
3. $\mathscr{F} = (\cos, \sin, x \mapsto \sin(2x))$.
4. $\mathscr{F} = \big((1, j, j^2), (j, j^2, 1), (j^2, 1, j)\big)$ in $\mathbb{C}^3$ as a $\mathbb{C}$-vector space, then in $\mathbb{C}^3$ as an $\mathbb{R}$-vector space, where $j = e^{2\pi i/3}$.

Exercise 10

We denote by $\mathcal{B}_c = (e_1, e_2, e_3, e_4)$ the standard basis of $\mathbb{R}^4$. Consider the following sets :

$$V = \left\{ \begin{pmatrix} \lambda+\mu \\ 2\lambda+2\mu \\ -2\lambda+\nu \\ \lambda+3\mu+\nu \end{pmatrix} \middle| (\lambda,\mu,\nu) \in \mathbb{R}^3 \right\}, W = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \middle| \left\{ \begin{array}{ccccccccc} x & - & y & - & z & + & t & = & 0 \\ & & 2y & + & z & - & t & = & 0 \\ 3x & + & y & - & z & + & t & = & 0 \end{array} \right. \right\}$$

1 Justify that V and W are vector subspaces of $\mathbb{R}^4$.

2
- **a** Show that the vector subspace V has dimension 2 and provide a basis for V. We will denote this basis as (v_1, v_2).
- **b** Show that the vector subspace W has dimension 2 and provide a basis for W. We will denote this basis as (w_3, w_4).

3
- **a** Compute the rank of the collection (v_1, v_2, w_3, w_4).
- **b** Is the collection $\{v_1, v_2, w_3, w_4\}$ linearly independent? Does it generate $\mathbb{R}^4$?
- **c** Extract a basis of the vector space $F = \text{Span}\{v_1, v_2, w_3, w_4\}$ from the collection $\{v_1, v_2, w_3, w_4\}$.

4 Let f_1, f_2, f_3 be the following vectors :

$$f_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 3 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

- **a** Show that the collection of vectors$\{f_1, f_2, f_3\}$ is a basis of F.
- **b** Show that the collection $\mathcal{B} = \{f_1, f_2, f_3, e_4\}$ is a basis of $\mathbb{R}^4$.

5 Determine the coordinate matrix of the vectors of the standard basis $\mathcal{B}_c$ in the basis $\mathcal{B}$.

Exercise 11

For each $k \in \mathbb{N}$, define the function $f_k : t \mapsto \cos(kt)$.

1 Compute the integral $\displaystyle\int_0^{2\pi} f_k(t)\, f_p(t)\, dt$ for two natural numbers k and p.

2 Deduce that the collection $(f_k)_{k \in [\![0,n]\!]}$ is linearly independent for any $n \in \mathbb{N}$. What can be said about the dimension of the vector space $\mathcal{F}(\mathbb{R}, \mathbb{R})$?

3 For each $k \in \mathbb{N}$, define $g_k(t) = \cos^k(t)$.
Show that the $\text{Span}\{f_0, \ldots, f_n\} = \text{Span}\{g_0, \ldots, g_n\}$, for every $n \in \mathbb{N}$.

4 Determine the rank of the colection $(g_k)_{k \in [\![0,n]\!]}$, for any $n \in \mathbb{N}$.

5 Deduce that the collection $(g_k)_{k \in [\![0,n]\!]}$ is linearly independent, for all $n \in \mathbb{N}$.

Exercise 12

Let $\mathbb{E}$ be a vector space over $\mathbb{K}$, and let $(e_1, \dots, e_n)$ be a linearly independent set of vectors in $\mathbb{E}$.

1. For every $k \in [\![1, n-1]\!]$, define $\varepsilon_k = e_k + e_{k+1}$. Show that the set $(\varepsilon_1, \dots, \varepsilon_{n-1})$ is linearly independent.
2. Define $\varepsilon_n = e_1 + e_n$. Is the set $(\varepsilon_1, \dots, \varepsilon_{n-1}, \varepsilon_n)$ linearly independent?

Exercise 13

Let $u = (1, -1, 1)$, $v = (0, -1, 2)$, and $w = (1, -2, 3)$ be three vectors in $\mathbb{R}^3$.

1. Determine the rank of the family (u, v, w).
2. Let $G = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2y + z = 0\}$.
   - a. Show that G is a vector subspace of $\mathbb{R}^3$ and determine its dimension.
   - b. Deduce that $G = \text{Span}(u, v, w)$.
3. Let u be a vector that does not belong to G.
   - a. What is the dimension of $\text{Span}(u)$?
   - b. Determine $G \cap \text{Span}(u)$.
   - c. Deduce that G and $\text{Span}(u)$ are complementary subspaces in $\mathbb{R}^3$.

Exercise 14

Let F and G be two vector subspaces of $\mathbb{R}^{12}$. Assume that $\dim(F) = 5$, $\dim(G) = 8$, and F is not included in G.

1. Show that $1 \leq \dim(F \cap G)$.
2. Show that $\dim(F \cap G) \leq 4$.

Exercise 15

Let F and G be two subspaces of a vector space of dimension 5 such that $\dim(F) = \dim(G) = 3$. Show that $F \cap G \neq \{0\}$.

Exercise 16

Let E be a finite-dimensional vector space and F_1, F_2, F_3 be vector subspaces of E. Show that $E = F_1 \oplus F_2 \oplus F_3$ if and only if the following conditions hold :

$$\begin{cases} \dim E = \dim F_1 + \dim F_2 + \dim F_3, \\ F_1 \cap F_2 = \{0\}, \\ (F_1 + F_2) \cap F_3 = \{0\}. \end{cases}$$

Exercise 17

Let $E = \mathbb{R}_3[X]$ be the vector space of real-valued polynomials with degree less than or equal to 3. Define :

$$F = \{P \in E \mid P(0) = P(1) = P(2) = 0\}, \quad G = \{P \in E \mid P(1) = P(2) = P(3) = 0\},$$
$$\text{and } H = \{P \in E \mid P(X) = P(-X)\}.$$

Show that $\mathbb{R}_3[X] = F \oplus G \oplus H$.

Exercise 18

Define $F = \{P \in \mathbb{R}_3[X] \mid P(1) = P(2)\}$..

1. Without computation, show that $\dim(F) \leq 3$.
2. Determine the dimension of F.

Exercise 19

Let $(p_n)_{n\in\mathbb{N}}$ be the sequence of prime numbers. Show that the set $(\ln p_n)_{n\in\mathbb{N}}$ is linearly independent in the $\mathbb{Q}$-vector space $\mathbb{R}$. What can we say in the case of the $\mathbb{Q}$-vector space $\mathbb{R}$?

Exercise 20

Prove that in each of the following cases, F and G are complementary subspaces of E :

1. $E = \{(u_n) \in \mathbb{R}^{\mathbb{N}} \mid \text{convergent sequences}\}$, $F = \{(u_n) \in \mathbb{R}^{\mathbb{N}} \mid \text{sequences converging to } 0\}$ and $G = \{(u_n) \in \mathbb{R}^{\mathbb{N}} \mid \text{constant sequences}\}$.
2. $E = \{f \in \mathbb{R}^{\mathbb{R}} \mid f \text{ is differentiable}\}$, $F = \mathrm{Span}(\cosh, \sinh)$ and $G = \{f \in E \mid f(0) = f'(0) = 0\}$.