NATIONAL SCHOOL OF CYBERSECURITY

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Algebra 1 - Tutorial 1

Basic Training Cycle

Mathematical Logic

Exercise 1

Using the syntax of the predicates write the following sentences (f and g denote real functions):

1. g is not a constant function.
2. f is even.
3. g is periodic of period 2π.
4. g is periodic.
5. f is an increasing function on $[0,+\infty[$.
6. The curves representing f and g intersect at least once.
7. f is strictly less than g.
8. f is not strictly less than g.
9. g is not bounded from above by any real number (we say that g is bounded from above by a real number M if all the values taken by the function g are less than M).

Exercise 2

Let P, Q, R three logical propositions.

1. Show that $P \Rightarrow Q$ is equivalent to $\neg P \vee Q$. Write the negation of $P \Rightarrow Q$.
2. Show that the following assertions are equivalent:
   (a) $(P \vee Q) \Rightarrow R$ and $(P \Rightarrow R) \wedge (Q \Rightarrow R)$.
   (b) $(P \Rightarrow (Q \Rightarrow R))$ and $((P \wedge Q) \Rightarrow R)$.

Exercise 3

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Write the negation of the following statements:

1. $\exists M \in \mathbb{R}, \forall x \in \mathbb{R}, f(x) \leq M$.
2. $\forall x \in \mathbb{R}, (f(x) \geq 0 \Longrightarrow x \geq 0)$.
3. $\forall x \in \mathbb{R}, f(x) \geq 1$ or $f(x) \leq -1$.
4. $\exists \ell \in \mathbb{R}, \forall \varepsilon > 0, \exists A > 0, \forall x \geq A, |f(x) - \ell| \leq \varepsilon$
5. $\forall x \in \mathbb{R}, (f(x) < 0 \Longleftrightarrow x \in [0,1])$.

Exercise 4

Say if the following statements are true or false. Carefully justify each answer (demonstrate true results, give counterexamples when false, optionally cite a hypothesis to add for making the statement true):

1. $\exists n \in \mathbb{Z}, \forall p \in \mathbb{Z}, n \leq p.$
2. $\forall x \in \mathbb{R}, \exists y > 0, x = e^y.$
3. $\forall a \in \mathbb{R}, \exists x \in \mathbb{R}, \ln x < a.$
4. $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \exists z \in \mathbb{R}, x < y < z.$
5. $\forall a > 0, \exists q \in \mathbb{Q}, \sqrt{2} \leq q \leq \sqrt{2} + a.$

Exercise 5

For each of the statements $P(\cdot)$ below, determine if the proposition $Q(\cdot)$ is necessary, sufficient, both at the same time or nothing at all (justify your answer).

1. Parameter: $x \in \mathbb{R}$.

   Propositions: $P(x) : (x \geq 0)$ and $Q(x) : (x \geq 1)$.

2. Parameter: $(a,b) \in \mathbb{R}^2$.

   Propositions: $P(a,b) : \left((a+b)^2 = a^2 + b^2\right)$ and $Q(a,b) : (a = b = 0)$.

3. Parameter: $(a,b,c) \in \mathbb{R}^3$.

   Propositions: $P(a,b,c) : (|a+b+c| = 0)$ and $Q(a,b,c) : (|a+b| + |c| = 0)$.

4. Parameter: $(u_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$.

   Propositions : $P\left((u_n)_{n\in\mathbb{N}}\right) : (\forall N \in \mathbb{N}^*, (u_1 - u_0)(u_2 - u_1)\cdots(u_N - u_{N-1}) > 0)$ and $Q\left((u_n)_{n\in\mathbb{N}}\right)$: (the sequence $(u_n)_{n\in\mathbb{N}}$ is strictly increasing).

Exercise 6

For any $n \geqslant 1$, we set:

$$A_n = \sum_{k=1}^{n} (-1)^{k-1} k$$

1. Compute A_n for $1 \leqslant n \leqslant 6$ and report the results in a table.
2. Prove using induction the following property:

$$\forall n \in \mathbb{N}^*, A_n = \frac{(-1)^{n-1}(2n+1)+1}{4}$$

3. Check that:

$$\forall n \in \mathbb{N}^*, A_n = (-1)^{n-1} \left\lfloor \frac{n+1}{2} \right\rfloor$$

Exercise 7

Prove by induction that for all $n \in \mathbb{N}^*$, there exist natural integers p and q such that $n = 2^p(2q+1)$.

Exercise 8

Consider the Fibonacci sequence $\{F_n\}_{n\in\mathbb{N}}$, defined by the relations $F_1 = 1, F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

$\boxed{1}$ Compute F_{20}.

$\boxed{2}$ Use an extended Principle of Mathematical Induction in order to show that for $n \geq 1$,

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

$\boxed{3}$ Use the result of part (2) to compute F_{20}.

Exercise 9

Prove the following statements:

$\boxed{1}$ $\forall n \in \mathbb{N}^*, 6$ divides $5n^3 + n$.

$\boxed{2}$ Two positive integers are multiples of each other if and only if they are equal.

Exercise 10

$\boxed{1}$ Show that : $\sqrt{3} \notin \mathbb{Q}$.

$\boxed{2}$ Let $n \in \mathbb{N}$. We admit:

$$(2+\sqrt{3})^n \in Q \quad \Leftrightarrow \quad n = 0$$

Deduce a necessary and sufficient condition on n so that $(2-\sqrt{3})^n \in \mathbb{Q}$.

Exercise 11

Let $x \in \mathbb{R}\backslash\mathbb{Q}$. Show that: $\forall n \in \mathbb{Z}^*, nx \in \mathbb{R}\backslash\mathbb{Q}$.

Exercise 12

Show that any function from $\mathbb{R}$ to $\mathbb{R}$ can be written as the sum of a constant function and a function whose image of 0 is 0.