2025/2026

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Algebra 1 - Tutorial 4

Basic Training Cycle

Algebraic Structures

Exercise 1

Let $(G, *)$ be a group with three elements. Construct its multiplication table.

Exercise 2

Let $(E, *)$ be a set equipped with a binary operation and let $e \in E$ such that :

(a) $\forall x, y, z \in E : \ (x * y) * z = (y * z) * x$.

(b) $\forall x \in E, \ x * e = x$.

(c) $\forall x \in E, \ \exists x' \in E : x * x' = e$.

1. Prove that the operation $*$ is commutative.
2. Prove that $(E, *)$ is a commutative group.

Exercise 3

Let $G =]-1, 1[$. We define for all elements x and y of G :

$$x * y = \frac{x + y}{1 + xy}.$$

1. Verify that $*$ is an associative binary operation on G.
2. Show that $(G, *)$ is a group. Is it commutative ?
3. Provide an expression for x^{*n}.

Exercise 4

Let $(E, *)$ be a set equipped with a binary operation, associative and possessing a neutral element e.
Assume moreover that $\forall x \in E, \ x * x = e$.
Show that $(E, *)$ is an abelian group.

Exercise 5

Let $G = \mathbb{R}^* \times \mathbb{R}$. For $(x, y), (x', y') \in G$, define the binary operation :

$$(x, y) * (x', y') = (xx',\ xy' + y).$$

1 Verify that $*$ is an associative binary operation on G.

2 Verify that $(G, *)$ is a group. Is it commutative ?

3 Give a closed form for $(x, y)^{*n}$.

Exercise 6

We define on $\mathbb{R}$ the internal composition law $\star$ as follows :

$$\forall x, y \in \mathbb{R} : x \star y = x + y - 2$$

1 Show that $(\mathbb{R}, \star)$ is an abelian group.

2 Let $n \in \mathbb{N}^*$. We define $x^{(1)} = x$ and $x^{(n+1)} = x^{(n)} \star x$.

- **a** Compute $x^{(2)}$, $x^{(3)}$, and $x^{(4)}$.
- **b** Show that $\forall n \in \mathbb{N}^* : x^{(n)} = nx - 2(n-1)$.

3 Let $H = \{x \in \mathbb{R} : x \text{ is even}\}$. Show that $(H, \star)$ is a subgroup of $(\mathbb{R}, \star)$.

Exercise 7

Let G be a group and let H and K be two subgroups of G.

1 Prove that $H \cap K$ is a subgroup of G.

2 Prove that $H \cup K$ is a subgroup of G if and only if $H \subseteq K$ or $K \subseteq H$.