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Mathematical Analysis 1

The Set of Real Numbers
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1 Real Numbers and some Subsets of Real Numbers

we will use **specific notations** to designate some famous or **well known sets of numbers**

- $\mathbb{N}$: The set of natural numbers.*i.e* $\mathbb{N} = \{1, 2, 3, 4, ...\}$
- $\mathbb{Z}$: The set of integers *i.e* $\mathbb{Z} = \{\cdot\cdot - 3, -2, -1, 0, 1, 2, 3, ...\}$

All these are **infinite** sets and it is clear that

$$\mathbb{N} \subset \mathbb{Z}.$$

Remark 1. *We can write*

$$\mathbb{Z} = \mathbb{Z}^+ \cup \mathbb{Z}^-,$$

where

$$\mathbb{Z}^- = \{\cdot\cdot - 3, -2, -1, 0\}$$

and

$$\mathbb{Z}^+ = \{0, 1, 2, 3, ...\}.$$

Definition 1. *(Rational Numbers) A **rational number** is a number that can be expressed as a ratio (fraction) of two integers (with the second integer **not equal to zero**). Hence, a rational number can be written as $\frac{p}{q}$ for some integers p and q , where $q \neq 0$. This set is designated by a capital Q and one has*

$$\mathbb{Q} := \left\{ \frac{p}{q} : p \in \mathbb{Z},\ q \in \mathbb{N}^* \right\}.$$

Example 1. $\frac{3}{7}, \frac{6}{4} = \frac{3}{2}, \frac{-3}{2}; \frac{6}{1} = 6.$

Remark 2. *it is easy to see that*

$$\mathbb{N} \subset \mathbb{Z} \subset Q.$$

Definition 2. *(**Irrational Numbers**) An **irrational number** is a real number that **can not be expressed** as a ratio (fraction) of two integers..*

Proposition 1. $\sqrt{2}$ *is an irrational number.*

Proof. Let us assume **on the contrary** that $\sqrt{2}$ is a rational number. Then, there exist positive integers a and b such that

$$\sqrt{2} = \frac{a}{b}$$

we assume also that $\frac{a}{b}$ is in its lowest term which means that a and b are coprime **(relatively prime** or **mutually prime**) *i.e.* This is equivalent to their **greatest common divisor** (GCD) being **1**. Then

$$\left(\sqrt{2}\right)^2 = \left(\frac{a}{b}\right)^2 \Rightarrow 2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2 \Rightarrow 2b^2 = a^2$$

this implies that a^2 is an **even integer** which also implies that a is an **even integer**. At this level, we deduce that

$$a = 2k \text{ for some integer } k \qquad (i)$$

and one has

$$2b^2 = 4k^2 \Rightarrow b^2 = 2k^2 \qquad (ii)$$

which implies b^2 is an **even integer** which also implies that b is an **even integer**. that From (i) and (ii), we obtain that 2 is a common factor of a and b. But, this **contradicts** the fact that a and b have no common factor other than 1. This means that our supposition is wrong. Hence, $\sqrt{2}$ is an irrational number. □

At this level, the set Q is probably the largest system of numbers with which you feel comfortable.

Real numbers The set of real numbers is denoted by $\mathbb{R}$ and **contains all of the previous number types**

$$\mathbb{N} \subset \mathbb{Z} \subset Q \subset \mathbb{R}.$$

We can represent the real numbers by the **set of points on a line**. The origin **corresponds to the number** 0. Numbers on the **right of** 0 are positive or > 0 and numbers on **the left of** 0 are negative or < 0.

2 $(\mathbb{R}, +, \times, \leq)$ as a complete totally ordered field

2.1 Basic notions

Definition 3. ***The field of real numbers*** $\mathbb{R}$ *is a set with two operations* ***addition and multiplication*** *and a* ***binary relation*** *termed "****less then or equal****" defined with following axioms*

R1_ $(\forall x, y, z \in \mathbb{R}) : (x + y) + z = x + (y + z)$ ***associative*** *law for addition*

R2_ $(\exists 0 \in \mathbb{R})\,(\forall x \in \mathbb{R}) : 0 + x = x + 0 = x$ *existence of the* ***additive identity element***

R3_ $(\forall x \in \mathbb{R})\,(\exists -x \in \mathbb{R}) : x + (-x) = (-x) + x = 0$ *existence of the* ***additive inverse***

R4_ $(\forall x, y, z \in \mathbb{R}) : x + y = y + x$ ***commutative*** *law for addition*

R5_ $(\forall x, y, z \in \mathbb{R}) : (x.y).z = x.(y.z)$ ***associative*** *law for multiplication*

R6_ $(\exists 1 \in \mathbb{R} - \{0\}) (\forall x \in \mathbb{R}) : 1.x = x.1 = x$ *existence of the* ***multiplicative identity element***

R7_ $(\forall x \in \mathbb{R} - \{0\}) (\exists x^{-1} \in \mathbb{R} - \{0\}) : x^{-1}.x = x.x^{-1} = 1$ *existence of* ***the multiplicative inverse***

R8_ $(\forall x, y, z \in \mathbb{R}) : x.(y + z) = x.y + x.z$ ***distributive law*** *of multiplication over addition*

R9_ $(\forall x, y, z \in \mathbb{R}) : x.y = y.x$ ***commutative law*** *for addition*

Example 2. *Prove that* $(\forall a, b, c \in \mathbb{R}) : a + c = b + c \Rightarrow a = b$
One has

$$a + c + (-c) = b + c + (-c)$$

then $a + 0 = b + 0$ *then* $a = b$

Example 3. *Prove that* $\forall a \in \mathbb{R} : a.0 = 0$
One has

$$a.0 = a.(0 + 0) = a.0 + a.0$$

then $a.0 = 0$.

Definition 4.

The set $\mathbb{R}$*.has* ***an order structure***

The set of real numbers $\mathbb{R}$ is equipped with an **order relation denoted by "$\leq$,"** which means for any $x, y \in \mathbb{R}$, we have $x \leq y$. Whether this is true or false depends on the values of x and y. This order relation satisfies the following properties:

R10. $(\forall x, y, z \in \mathbb{R}) : x \leq y$ and $y \leq z \Rightarrow x \leq z$ **transitivity property** of the binary relation $\leq$

Definition 5.

R11_ $(\forall x, y \in \mathbb{R}) : x \leq y$ *and* $y \leq x \Rightarrow x = y$ ***antisymmetric property*** *of the binary relation* $\leq$

R12_ $(\forall x, y \in \mathbb{R}) : x \leq y$ *or* $y \leq x$ ***total ordering*** *of of the binary relation* $\leq$

Exercise 1. *Show that* ***for every*** $x, y, x', y' \in \mathbb{R}$ *it holds*

1. $x \cdot 0 = 0$

2. $(-x).y = -xy$

3. $x \cdot (-y) = -(xy) = -(xy)$

4. $x \leq 0 \Leftrightarrow -x \geq 0$

5. $x \leq y \Leftrightarrow -x \geq -y$

1. One has
$$x \cdot 0 = x \cdot (0+0) = x.0 + x.0 \Rightarrow x.0 = 0.$$

2. since $(-x) + x = 0$, we have
$$xy + (-x)\,y = [x + (-x)]\,y = 0.y = 0$$
$$\Rightarrow -xy = (-x)\,y$$

Example 2.1. *What can we say about a real number x such that:*

1. $\forall \varepsilon > 0$, $x \leq 4000\varepsilon$*.we say that* $x = 0$

2. $\forall \varepsilon > 0$, $-4\varepsilon \leq x \leq \varepsilon$, *we say that* $x = 0$

3. $\forall \varepsilon > 0$, $-4 + \varepsilon \leq x < \varepsilon$, *we say that* $-4 \leq x \leq 0$

Exercise 2. *Let α, β and γ be three real numbers such that*
$$\alpha = 20 + 14\sqrt{2}, \beta = 20 - 14\sqrt{2}, \gamma = \sqrt[3]{\alpha} + \sqrt[3]{\beta}$$
Show that γ is solution of
$$\gamma^3 - 6\gamma - 40 = 0 \tag{2.1}$$
and that $\gamma \in \mathbb{Q}$.

2.1.1 The Axiom of Archimedes

Axiom 1. *.For an arbitrary real number b, there exists a* $\boxed{\textbf{natural number } n}$ ***larger than*** b

If we replace b by $\frac{b}{a}$ (where a and b are positive numbers) in the above axiom, we get the following consequence:
If a and b are arbitrary positive numbers, then there exists a natural number n such that
$$na > b.$$
And if instead of b we write $\frac{1}{\varepsilon}$, where $\varepsilon > 0$, then we get:

Axiom 2. *If ε is an arbitrary positive number, then there exists* $\boxed{\textbf{a natural number } n}$ ***such that*** $\frac{1}{n} < \varepsilon$

An important consequence of the axiom of Archimedes is that the rational numbers are "everywhere dense" within the real numbers.

Theorem 2. *There exists a* ***rational number between any two real numbers***

3 The absolute value of a real number

Definition 6. *The absolute value of a real a is defined by*

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

3.0.2 Properties

Let x, y in $\mathbb{R}$

1. $|x| = 0 \Leftrightarrow x = 0$
2. $-|x| \leq x \leq |x|$
3. $\forall a > 0 : |x| \leq a \Leftrightarrow -a \leq x \leq a$
4. $\forall a > 0 : |x| \geq a \Leftrightarrow x \leq -a$ or $x \geq a$
5. Triangular inequality: $|x + y| \leq |x| + |y|$
6. Second triangular inequality: $||x| - |y|| \leq |x - y|$

Proof. One have for all $x, y \in \mathbb{R}$:

5.
$$\begin{aligned} |x+y|^2 &= (x+y)^2 \\ &= x^2 + y^2 + 2xy \\ &= |x|^2 + |y|^2 + 2xy \end{aligned}$$

we know that

$$\forall a \in \mathbb{R}, a \leq |a|$$

then

$$\begin{aligned} |x+y|^2 &= |x|^2 + |y|^2 + 2xy \\ &\leq |x|^2 + |y|^2 + 2|xy| = (|x| + |y|)^2 \end{aligned}$$

6. We write

$$|x| = |(x - y) + y|$$

by the triangular inequality, we get

$$|x| \leq |x - y| + |y| \Rightarrow |x| - |y| \leq |x - y|$$

We do the same for $|y|$, one finds

$$|y| = |(y - x) + x| \leq |y - x| + |x| \Rightarrow |y| - |x| \leq |y - x| = |x - y|$$

then

$$-|x-y| \leq |x| - |y| \leq |x-y|$$

then

$$||x| - |y|| \leq |x-y|$$

$\square$

Exercise 3. *Show that for all x, y in $\mathbb{R}$*

$$\frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}$$

Let x, y in $\mathbb{R}$. One has

$$\frac{|x+y| \mp 1}{1+|x+y|} = 1 - \frac{1}{1+|x+y|}$$

Observe that

$$\begin{aligned}
1+|x+y| &\leq 1+|x|+|y| \Rightarrow \frac{1}{1+|x+y|} \geq \frac{1}{1+|x|+|y|} \\
&\Rightarrow -\frac{1}{1+|x+y|} \leq -\frac{1}{1+|x|+|y|} \\
&\Rightarrow 1 - \frac{1}{1+|x+y|} \leq 1 - \frac{1}{1+|x|+|y|} \\
&\Rightarrow \frac{|x+y|}{1+|x+y|} \leq \frac{|x|+|y|}{1+|x|+|y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}
\end{aligned}$$

4 Intervals in $\mathbb{R}$

The naive definition of a real interval is the set of all real numbers lying between any two numbers. More precisely

Definition 7. *A non-empty subset I of $\mathbb{R}$ is called an interval if $\forall a, b \in I$ satisfying $a \leq b$, the relation $a \leq x \leq b$ implies $x \in I$.*

Example 4. *Let A be a non-empty subset of $\mathbb{R}$ defined as*

$$A = \left\{ \frac{1}{n}, n \in \mathbb{N}^* \right\}$$

A ***is not an interval.*** *For example, if we take $a = \frac{1}{2}$ and $b = \frac{1}{3}$ which are elements of A and $x = \frac{2}{5}$ which lies between a and b but it is not in A*

We distinguish several forms of intervals:

4.0.3 1-Bounded intervals

Let a and b in $\mathbb{R}$ such that $b > a$
• **Open interval** The set $\{x \in \mathbb{R} : a < x < b\}$ is called open interval and we give this usual notation

$$]a, b[= \{x \in \mathbb{R}, a < x < b\}$$

The points a and b are called the endpoints of the interval. Furthermore, one has

$$]a, a[= \emptyset.$$

We list below other type of intervals
• **Closed interval**

$$[a, b] = \{x \in \mathbb{R}, a \leq x \leq b\}$$

• **Half-open interval (left-open)**

$$]a, b] = \{x \in \mathbb{R}, a < x \leq b\}$$

• **Half-open interval (right-open)**

$$[a, b[= \{x \in \mathbb{R}, a \leq x < b\}$$

5 The Integer part of a real number

For any real number $x \in \mathbb{R}$ we denote by $[x]$ the **integer part** of x defined as follows

$$E(x) = [x] = \max\left(n \in \mathbb{Z} : n \leq x\right)$$

Example 5. $E(e) = 2, E\left(\sqrt{2}\right) = 1,\ E(-\pi) = -4,\ \textit{and}\ E(-5,2) = -6.$

Consequently, $E(x)$ is **the unique** integer that satisfies

$$E(x) \leq x < E(x) + 1$$

Properties Let $x, y \in \mathbb{R}$, we have:

1. $\forall k \in \mathbb{Z}, E(x + k) = E(x) + k$
2. $E(x) + E(-x) = \begin{cases} 0 & x \in \mathbb{Z}, \\ -1 & x \in \mathbb{R}/\mathbb{Z}. \end{cases}$
3. $x - 1 < E(x) \leq x$
4. $x \leq y \Rightarrow E(x) \leq E(y)$
5. $E(x) + E(y) \leq E(x + y) \leq E(x) + E(y) + 1.$

Proof. we just give the demonstration of some properties

1. Let $k \in \mathbb{Z}$ and $x \in \mathbb{R}$. One has

$$\begin{aligned} E(x) &\leq x < E(x)+1 \\ &\Rightarrow E(x)+\mathbf{k} \leq x+\mathbf{k} < E(x)+1+\mathbf{k} \\ &\Rightarrow (E(x)+\mathbf{k}) \leq x+\mathbf{k} < (E(x)+\mathbf{k})+1 \end{aligned}$$

   the uniqueness of the integer part implies that

$$E(x+\mathbf{k}) = E(x)+\mathbf{k}$$

2. Let $x \in \mathbb{R}$. One has

$$E(x) \leq x < E(x)+1$$

   multiplying by $-\mathbf{1}$, we get

$$-E(x)-1 < -x \leq -E(x)$$

   keeping in mind that $x \in \mathbb{R}/\mathbb{Z}$ then

$$-E(x) \neq -x$$

   then

$$\begin{aligned} -E(x)-1 &< -x<-E(x) \\ &\Rightarrow E(x)-1 \leq -x<-E(x) \\ &\Rightarrow E(x)-1 \leq -x<\{-E(x)-1\}+1 \end{aligned}$$

   the uniqueness of the integer part implies that

$$E(-x) = -E(x)-1.$$

$\square$

Exercise 4. *Show that for all $(m,n) \in \mathbb{Z}^2$, prove that*

$$E(\frac{n+m}{2}) + E(\frac{n-m+1}{2}) \in \mathbb{Z}$$

We will consider two cases:

1. **First case:** If we assume that $n+m$ is **even**, then

$$n+m = 2k,$$

   which gives

$$E\left(\frac{n+m}{2}\right) = k$$

Furthermore, we have

$$\frac{n-m+1}{2} = \frac{2k-m-m+1}{2} = k-m+\frac{1}{2}$$

Since $\dfrac{n+m}{2} - m \in \mathbb{Z}$ and based on the previous proposition, we have

$$E\left(\frac{n-m+1}{2}\right) = k-m+E\left(\frac{1}{2}\right) = k-m$$

Therefore

$$E(\frac{n+m}{2}) + E(\frac{n-m+1}{2}) = 2k-m = n$$

2. **Second case:** If $n+m$ is **odd**, then

$$n+m = 2k+1,$$

and

$$E\left(\frac{n+m}{2}\right) = E\left(\frac{2k+1}{2}\right) = k+E\left(\frac{+1}{2}\right) = k$$

we observe that

$$n-m+1 = 2k+1-m-m+1$$

which **is even since it is the sum of two even integer**. Thus

$$E\left(\frac{n-m+1}{2}\right) = E\left(\frac{2k+1-m-m+1}{2}\right) = k+1-2m$$

. We also

$$E(\frac{n+m}{2}) = E(\frac{n-m+1+2m-1}{2}) = k+k+1-2m$$

Hence

$$E(\frac{n+m}{2}) + E(\frac{n-m+1}{2}) = n.$$

Exercise 5. *Show that for all $n \in \mathbb{Z}$, prove that*

$$E(\frac{n-1}{2}) + E(\frac{n+2}{4}) + E(\frac{n+4}{4}) = n$$

We will consider 4 cases:

$$\begin{cases} n = 4k \\ n = 4k+1 \\ n = 4k+3 \\ n = 4k+3 \end{cases}$$

For the fist situation one has

$$\begin{aligned}
& E(\frac{n-1}{2})+E(\frac{n+2}{4})+E(\frac{n+4}{4}) \\
= \;& E(\frac{4k-1}{2})+E(\frac{4k+2}{4})+E(\frac{4k+4}{4}) \\
= \;& E(2k-\frac{1}{2})+E(k+\frac{1}{2})+E(k+1) \\
= \;& 2k+E(-\frac{1}{2})+k+E(+\frac{1}{2})+k+1 \\
= \;& 2k+(-1)+k+0+k+1 \\
= \;& 4k=n
\end{aligned}$$

we obtain a similar results for the other situations

Exercise 6. *Show that for all $(m,n)\in\mathbb{Z}^2$, prove that*

$$E(\frac{n-1}{2})+E(\frac{n+2}{4})+E(\frac{n+4}{4})=n$$

We will consider 4 cases:

$$\begin{cases} n=4k \\ n=4k+1 \\ n=4k+3 \\ n=4k+3 \end{cases}$$

Exercise 7. *Show that for all $n\in\mathbb{Z}$,*

$$E\left((\sqrt{n}+\sqrt{n+1})^2\right)=4n+1$$

One has

$$E\left((\sqrt{n}+\sqrt{n+1})^2\right)=E\left((n+n+1+2\sqrt{n^2+n}\right)=2n+1+E\left(2\sqrt{n^2+n}\right)$$

recall that

$$n^2+n=\left(n+\frac{1}{2}\right)^2-\frac{1}{4}$$

then,

$$\begin{aligned}
n^2 \;&\leq\; n^2+n<\left(n+\frac{1}{2}\right)^2 \\
&\Rightarrow\; \sqrt{n^2}\leq\sqrt{n^2+n}<\sqrt{\left(n+\frac{1}{2}\right)^2} \\
&\Rightarrow\; 2\sqrt{n^2}\leq 2\sqrt{n^2+n}<2\sqrt{\left(n+\frac{1}{2}\right)^2} \\
&\Rightarrow\; 2n\leq 2\sqrt{n^2+n}<2n+1
\end{aligned}$$

the uniqueness of the integer part implies that

$$E(2\sqrt{n^2+n}) = 2n.$$

then,

$$E\left((\sqrt{n}+\sqrt{n+1})^2\right) = 4n+1$$

Exercise 8. *Let $n \in \mathbb{N}$. Determine*

$$E\left(\sqrt{n^2+3n+4}\right))$$

Let $n \in \mathbb{N}$. One has

$$(n+1)^2 = n^2+2n+1 < n^2+3n+4 < n^2+4n+4 = (n+2)^2$$

then

$$(n+1) < \sqrt{n^2+3n+4} < n+2 = (n+1)+1$$

the uniqueness of the integer part implies that

$$E\left(\sqrt{n^2+3n+4}\right)) = n+1.$$

6 Upper Bounds

Let $S \subset \mathbb{R}$ be a no empty set.

Definition 8. *We say "a set S is* ***bounded above****" if there exists a number M such that*

$$\forall x \in S : x \leq M.$$

In this situation, we call M $\boxed{\textbf{\textit{an upper bound}}}$ *of S or just* ***u.b***

Example 6. 7 *is a upper bound of the set $\{5,6,7\}$. So are $7, 8,$ and 9.*

Remark 3. *If A has an upper bound M, one can* ***easily find other upper bounds****, such as $M+1, M+2, \cdot$*

Remark 4. *$\cdot$An* ***upper bound*** *of a set A,* $\boxed{\textbf{may or may not belongs}}$ *to the set A.*

Example 7. *Let $A = \{1,2\}$ be a subset of the set of natural numbers $\mathbb{N}$, then $2, 3, 4, 5, \cdots$ will all be upper bounds of A,* ***but only*** 2 ***belongs to*** *A.*

Example 8. *Let $A = \{x \in \mathbb{Q} : 0 < x < 1\}$ be subset of rational numbers $\mathbb{R}$, then $1, 1+\frac{1}{2}, 1+\frac{3}{4}, \cdots$ are all upper bounds of A. However,* ***none of them belong to*** *A*

7 Lower Bounds

Definition 9. *We say "a set S is **bounded below"** if there is a number m such that*

$$\forall x \in S : x \geq m.$$

*In this situation, we call m $\boxed{\textbf{\textit{a lower bound}}}$ of S or just **l.b***

Definition 10. *If a set A is both **bounded above and bounded below**, then we say A is **bounded.***

Example 9. *Let $S = \{(x, y) \in \mathbb{R}^2 : y = x^2$ and $1 \leq x \leq 2\}$. there are many **u.b** such $M = 5, M = 4.1...$*

Example 10. *Let $S = \{(x, y) \in \mathbb{R}^2 : y = x^2$ and $-1 \leq x \leq 2\}$. there are many **l.b** such $m = -2, m = -1...$*

7.1 Supremum and Infimum

Definition 11. *A real number M is called the **supremum** of A, denoted by*

$$M = \sup(A),$$

if and only if

$$\begin{cases} M \text{ is an } \textbf{upper bound} \text{ of } A, i.e & \forall x \in A, x \leq M \\ \\ If\ M' \text{ is another upper bound of } A, \text{ then} & M \leq M' \end{cases}$$

In the other words, M is the $\boxed{\textbf{\textit{smallest among all the upper bounds}}}$ of A.

Definition 12. *A real number m is called the **infimum** of A, denoted by*

$$m = \inf(A),$$

if and only if

$$\begin{cases} m \text{ is a } \textbf{lower bound} \text{ of } A, i.e & \forall x \in A, x \geq m \\ \\ If\ m' \text{ is } \textbf{another} \text{ lower bound of } A, \text{ then} & m \geq m' \end{cases}$$

In the other words, m is the $\boxed{\textbf{\textit{largest among all the lower bounds}}}$ of A

Example 11. *If $a < b$, then $b = \sup[a, b] = \sup[a, b)$ and $a = \inf[a, b] = \inf(a, b]$.*

Example 12. *If $S = \{q \in \mathbb{Q} : e < q < \pi\}$, then $\inf\ S = e$, $\sup S = \pi$*

Example 13. *If $S = \{x \in \mathbb{R} : x^2 < 3\}$, then $\inf S = -\sqrt{3}$, $\sup S = \sqrt{3}$*

Let $A \subset \mathbb{R}$ be a non empty subset

Proposition 3. *The **supremum** or **infimum** of a set A is* $\boxed{\textbf{\textit{unique if it exists}}}$. *Moreover, if **both exist**, then*

$$\inf A \leq \sup A.$$

Proposition 4. *Every non empty set of real numbers that is* $\boxed{\textbf{\textit{bounded from above has a supremum}}}$, *and every non empty set of real numbers that is* $\boxed{\textbf{\textit{bounded from below has an infimum}}}$

Proposition 5. *if A is bounded above, the supremum of A i.e $M = supA$ is the unique real number satisfying*

$$\begin{cases} M \text{ is an } \textbf{upper bound} \text{ of } A, \text{i.e} & \forall x \in A, x \leq M \\ \\ \forall \varepsilon > 0, \exists x \in A\ ; & M - \varepsilon < x \leq M \end{cases}$$

Proposition 6. *if A is bounded below, the infimum of A i.e $m = \inf A$ is the unique real number satisfying*

$$\begin{cases} m \text{ is an } \textbf{lower bound} \text{ of } A, \text{i.e} & \forall x \in A, x \geq m \\ \\ \forall \varepsilon > 0, \exists x \in A\ ; & m \leq x < m + \varepsilon \end{cases}$$

Remark 5. *There **are four possibilities** for a non empty subset S of $\mathbb{R}$:it can*

1. *have **both** a supremum and an infimum,*
2. *have a supremum but **no** infimum,*
3. *have a infimum but **no** supremum,*
4. *have **neither** a supremum nor an infimum.*

Example 14. *Find $\sup A$ and $\inf A$ (if they exist) where*

$$A = \left\{ x_n = \frac{n+3}{\frac{n}{4}+1}, n \in \mathbb{N} \right\}$$

*Let us us prove that A is **bounded**. One has*

$$\frac{n+3}{\frac{n}{4}+1} = 4\frac{n+3}{n+4} = 4\frac{n+3\pm 1}{n+4} = 4\left(1 - \frac{1}{n+4}\right)$$

but

$$n+4 \geq 4 \Rightarrow \frac{1}{n+4} \leq \frac{1}{4} \Rightarrow -\frac{1}{n+4} \geq -\frac{1}{4} \Rightarrow -\frac{4}{n+4} \geq -1 \Rightarrow 4 - \frac{4}{n+4} \geq 4 - 1 = 3$$

then,

$$x_n \geq 3$$

on the other hand, one has

$$n \geq 0 \Rightarrow n+4 > 4 > 0 \Rightarrow \frac{1}{n+4} > 0 \Rightarrow -\frac{4}{n+4} < 0 \Rightarrow 4 - \frac{4}{n+4} < 4$$

then,

$$x_n < 4.$$

Summing up

$$x_n \in [3, 4[$$

which implies

$$\min A = \inf A = 3, \sup A = 4$$

Let us that $\sup A = 4$. *In fact,*

$$\sup A = 4 \Leftrightarrow \begin{cases} 4 \textit{ is an } \boldsymbol{upper\ bound} \textit{ of } A, i.e & \forall a_n \in A, a_n \leq 4 \\ \\ \forall \varepsilon > 0, \exists a_n \in A \ ; & 4 - \varepsilon < a_n \leq 4 \end{cases}$$

let $\varepsilon > 0$ *and we look for* a_n *satisfying*

$$4 - \varepsilon < a_n$$

which implies that

$$\begin{aligned} 4 - \varepsilon &< 4 - \frac{4}{n+4} \\ &\Rightarrow \frac{4}{n+4} < \varepsilon \\ &\Rightarrow n + 4 > \frac{\varepsilon}{4}, \end{aligned}$$

it suffices to take $n > n_0 = \left[\frac{\varepsilon}{4} - 4\right] + 1$

Example 15. *Find* $\sup B$ *and* $\inf B$ *(if they exist) where*

$$B = \left\{ y_n = \frac{1}{n^2} + \frac{2}{n} + 4, n \in \mathbb{N}^* \right\}$$

Let us us prove that B *is bounded. One has*

$$n \geq 1 \Rightarrow \frac{2}{n} \leq 2 \textit{ and } \frac{1}{n^2} \leq 1 \Rightarrow \frac{1}{n^2} + \frac{2}{n} + 4 \leq 2 + 1 + 4 = 7$$

then,

$$y_n < 4.$$

On the other hand, one has

$$n > 0 \Rightarrow \frac{2}{n} > 0 \textit{ and } \frac{1}{n^2} > 0 \Rightarrow \frac{1}{n^2} + \frac{2}{n} + 4 > 4.$$

then,

$$y_n > 4.$$

summing up

$$y_n \in]4, 7]$$

which implies

$$\inf A = 4, \sup A = 7$$

Let us show that $\inf A = 4$. *In fact, one has*

$$\inf A = 4 \Rightarrow \begin{cases} 4 \textit{ is an } \textbf{\textit{lower bound}} \textit{ of } A, i.e & \forall y_n \in A, y_n \geq 4 \\ \\ \forall \varepsilon > 0, \exists y_n \in A \ ; & 4 \leq y_n < 4 + \varepsilon \end{cases}$$

then,

$$y_n < 4 + \varepsilon \Rightarrow \frac{1}{n^2} + \frac{2}{n} + 4 < 4 + \varepsilon \Rightarrow \frac{1}{n^2} + \frac{2}{n} < \varepsilon$$

then

$$\frac{2}{n} < \varepsilon \Rightarrow n > \frac{2}{\varepsilon}$$

it suffices to take

$$n_0 = \left[\frac{2}{\varepsilon}\right] + 1.$$

Exercise 9. *Find the supremum and infimum of each sets (if they exist) where*

$$A = [0, 1], \ B =]0, 1[, \ C = \{1 - \frac{1}{n}, n \in \mathbb{N}^*\}, \ D = \{\sqrt{2n} - \sqrt{n+2}, n \in \mathbb{N}^*\}$$

1. A is bounded above by 1 and bounded below by 0, so the lower and upper bounds exist and are respectively 0 and 1

2. B is bounded above by 1 and bounded below by 0, so the lower and upper bounds exist and are respectively 0 and 1

3. C is bounded above by 1 and bounded below by 0. In fact

$$n \geq 1 : 0 < \frac{1}{n} \leq 1$$

and

$$-1 \leq -\frac{1}{n} < 0$$

then

$$0 - 1 + 1 \leq +1 - \frac{1}{n} < 0 + 1 = 1$$

4. Let $n \geq 1$:
$$\sqrt{2n} - \sqrt{n+2} = \sqrt{n}\left(\sqrt{2} - \sqrt{1+\frac{2}{n}}\right) \to +\infty$$
D is not bounded above. On the other hand, one has
$$1 + \frac{2}{n} \leq 3 \text{ for } n \geq 1$$
so
$$\sqrt{1+\frac{2}{n}} \leq \sqrt{3} \text{ for } n \geq 1$$
which implies that
$$\sqrt{2} - \sqrt{1+\frac{2}{n}} \geq \sqrt{2} - \sqrt{3} \text{ for } n \geq 1$$
then, D is bounded below

Exercise 10. *Find the supremum and infimum of each sets (if they exist) where*
$$A = \{\frac{k+1}{k+m}, k, m \in \mathbb{N}^*\}$$

Exercise 11. *Let $A \subset \mathbb{R}$ the set defined by*
$$A = \{\frac{[x]+1}{x}, x > \frac{1}{2}\}$$

1. *Show that A is bounded*
2. *Find* $\inf A$ *and* $\sup A$

We can distinguish two cases

1. $\frac{1}{2} < x < 1 \Rightarrow [x] = 0$ and
$$\frac{[x]+1}{x} = \frac{1}{x} \in \,]1, 2[\,.$$
2. $x \geq 1$: We know that
$$[x] \leq x < [x] + 1$$
Putting $k_x := [x]\,(\in \mathbb{N}^*)$. Then,
$$\frac{[x]+1}{x} = \frac{k_x+1}{x}$$
then
$$\begin{aligned} k_x &\leq x < k_x + 1 \Rightarrow \frac{1}{k_x+1} < \frac{1}{x} \leq \frac{1}{k_x} \\ &\Rightarrow 1 < \frac{[x]+1}{x} \leq \frac{k_x+1}{k_x} = 1 + \frac{1}{k_x} \leq 2 \end{aligned}$$
Summing up
$$\forall x > \frac{1}{2} : 1 < \frac{[x]+1}{x} \leq 2$$
then A is bounded

Note that for $x=1$, we get

$$\frac{[x]+1}{x} = 2 \in A$$

Then

$$\sup A = \max A = 2.$$

On the other hand, one has

$$\inf A = 1.$$

Exercise 12. *Let A and B two non empty subsets of $\mathbb{R}$ and $f\colon A\times B \to \mathbb{R}$ bounded. Compare between*

$$\inf(\sup(f(x,y); x \in A); y \in B)$$

and

$$\sup(\inf(f(x,y); y \in B); x \in A).$$

Let's start by fixing $\boxed{x_0 \in A}$ and let $y \in B$. Then,

$$f(\mathbf{x}_0, y) \leq \sup(f(\mathbf{x}, y); \mathbf{x} \in \mathbf{A}). \, (x)$$

Let us move to the **lower bound in this inequality for y∈B**. we obtain

$$\inf(f(\mathbf{x}_0, y); \mathbf{y} \in \mathbf{B}) \leq \inf(\sup(f(x,y); x \in A); \mathbf{y} \in \mathbf{B}). \qquad (y)$$

We then **take the upper bound on** $x_0 \in A$ and we find the inequality

$$\sup\left([\inf(f(\mathbf{x}_0, y); \mathbf{y} \in \mathbf{B})], \boxed{\mathbf{x}_0 \in \mathbf{A}}\right) \leq \inf(\sup(f(x,y); x \in A); y \in B).$$

Exercise 13. *Let A and B be two non empty, bounded subsets of $\mathbb{R}$. Show that;*

1. *$A \subset B$ then*
$$\inf B \leq \inf A \leq \sup A \leq \sup B,$$
2. $\min(\inf A, \inf B) = \inf(A \cup B) \leq \sup(A \cup B) = \max(\sup A, \sup B)$,
3. *If $A \cap B = \emptyset$, one have*
$$\max(\inf A, \inf B) \leq \inf(A \cap B) \leq \sup(A \cap B) \leq \min(\sup A, \sup B)$$

8 Annex Proof Techniques

8.1 Process of Proof by Induction

A proof by induction of $\mathcal{P}(n)$, a mathematical statement involving a value n, involves these main steps:
Base Case: • **Prove directly** that $\mathcal{P}$ is correct for the initial value of n (for most examples you will see this is **zero or one**). This is called the **base case**.
Induction Hypothesis: Assume that the statement $\mathcal{P}(n)$ is true for any positive integer $n = k$, for some $k \geq n_0$.
Inductive Step: Show that the statement $\mathcal{P}(n)$ is true for $n = k + 1$..

Exercise 14. *Show that for all* $n \in \mathbb{N}$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + ... + \frac{1}{n \cdot (n+1)} = \frac{n}{(n+1)}$$

Let $\mathcal{P}(n)$ be the statement that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + ... + \frac{1}{n \cdot (n+1)} = \frac{n}{(n+1)}$$

1. **Base Case:** When $n = 1$ the left hand side of the equation is $\frac{1}{1 \cdot 2}$ and the right hand side is $\frac{1}{1+1}$ So $\mathcal{P}(1)$ is correct.

2. **Induction Hypothesis:** Assume that the statement $\mathcal{P}(n)$ is true for any positive integer n for some $n \geq 1$.

$$\mathcal{P}(n) : \forall n \in \mathbb{N}, \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + ... + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1} \text{ is true}$$

3. **Inductive Step:** Show that the statement $\mathcal{P}(n)$ is true for $n + 1.i.e$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + ... + \frac{1}{n \cdot (n+1)} + \frac{1}{(n+1) \cdot (n+2)} = \frac{n+1}{n+2}$$

One has

$$\begin{aligned}
& \underbrace{\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + ... + \frac{1}{n \cdot (n+1)}}_{\frac{n}{n+1}} + \frac{1}{(n+1) \cdot (n+2)} \\
= \quad & \frac{n}{n+1} + \frac{1}{(n+1) \cdot (n+2)} \\
= \quad & \frac{n^2 + 2n + 1}{(n+1) \cdot (n+2)} \\
= \quad & \frac{n+1}{n+2}
\end{aligned}$$

So $\mathcal{P}(n + 1)$ is correct. Hence by mathematical induction $\mathcal{P}(n)$ is correct.

Exercise 15. *Show that for all $n \in \mathbb{N}$*

$$1+2+3+...+n=\frac{n\left(n+1\right)}{2}$$

Let $\mathcal{P}(n)$ be the statement that

$$1+2+3+...+n=\frac{n\left(n+1\right)}{2}$$

1. **Base Case:** When $n = 1$ the left hand side of the equation is 1 and the right hand side is $\dfrac{1\left(1+1\right)}{2}$ So $\mathcal{P}(1)$ is correct.
2. **Induction Hypothesis:** Assume that the statement $\mathcal{P}(n)$ is true for any positive integer n for some $n \geq 1$.

$$\mathcal{P}(n) : \forall n \in \mathbb{N}, 1+2+3+...+n=\frac{n\left(n+1\right)}{2} \text{ is true}$$

3. **Inductive Step:** Show that the statement $\mathcal{P}(n)$ is true for $n+1.i.e$

$$1+2+3+...+n+1=\frac{\left(n+1\right)\left(n+2\right)}{2}$$

One has

$$\begin{aligned}
&\underbrace{1+2+3+...+n}_{\frac{n\left(n+1\right)}{2}}+\left(n+1\right)\\
&= \frac{n\left(n+1\right)}{2}+\left(n+1\right)\\
&= \frac{n\left(n+1\right)}{2}+\frac{2}{2}\left(n+1\right)\\
&= \frac{\left(n+1\right)\left(n+2\right)}{2}
\end{aligned}$$

So $\mathcal{P}(n+1)$ is correct. Hence by mathematical induction $\mathcal{P}(n)$ is correct.

Exercise 16. *Show that for all $n \in \mathbb{N}$*

$$1+4+7+\cdots+\left(3n-2\right)=\frac{n\left(3n-1\right)}{2}$$

Let $\mathcal{P}(n)$ be the statement that $\forall n \geq 1:$

$$1+4+7+\cdots+\left(3n-2\right)=\frac{n\left(3n-1\right)}{2}$$

1. **Base Case:** When $n = 1$ the left hand side of the equation is 1 and the right hand side is $\dfrac{1\left(3-1\right)}{2}$ So $\mathcal{P}(1)$ is correct.

2. **Induction Hypothesis:** Assume that the statement $\mathcal{P}(n)$ is true for any positive integer n for some $n \geq 1$.

$$\mathcal{P}(n) : \forall n \geq 1 : 1 + 4 + 7 + \cdots + (3n-2) = \frac{n(3n-1)}{2} \text{ is true}$$

3. **Inductive Step:** Show that the statement $\mathcal{P}(n)$ is true for $n+1.i.e$

$$1 + 4 + 7 + \cdots + (3(n+1) - 2) = \frac{(n+1)(3(n+1)-1)}{2}$$

it means

$$1 + 4 + 7 + \cdots + (3n+1) = \frac{(n+1)(3n+2)}{2}$$

One has

$$\begin{aligned}
&\underset{\underleftrightarrow{\frac{n(3n-1)}{2}}}{\overleftrightarrow{1 + 4 + 7 + \cdots + + (3n-2)}} + (3n+1) \\
&= \frac{n(3n-1)}{2} + (3n+1) \\
&= \frac{n(3n-1)}{2} + \frac{2}{2}(3n+1) \\
&= \frac{3n^2 + 5n + 2}{2} \\
&= \frac{(n+1)(3n+2)}{2}
\end{aligned}$$

So $\mathcal{P}(n+1)$ is correct. Hence by mathematical induction $\mathcal{P}(n)$ is correct

Exercise 17. *Verify that for all $n \geq 1$, the sum of the squares of the first $2n$ positive integers is given by the formula*

$$1^2 + 2^2 + 4^2 + ... + (2n)^2 = \frac{n(2n+1)(4n+1)}{3}$$

1. ***Base Case.*** *The statement $\mathcal{P}(1)$ says that*

$$1^2 + 2^2 = 5 = \frac{1(2(1)+1)(4(1)+1)}{3} = \frac{15}{3} = 5$$

which is true

2. ***Induction Hypothesis:*** *Assume that the statement $\mathcal{P}(n)$ is true for any positive integer n for some $n \geq 1$.*

$$\mathcal{P}(n) : \forall n \geq 1 : 1^2 + 2^2 + 4^2 + ... + (2n)^2 = \frac{n(2n+1)(4n+1)}{3} \textit{ is true}$$

1. **Inductive Step:** Show that the statement $\mathcal{P}(n)$ is true for $n+1.i.e$

$$1^2+2^2+4^2+...+(2n)^2+(n+1)^2=\frac{(n+1)(2(n+1)+1)(4(n+1)+1)}{3}$$

One has

$$\begin{aligned} & 1^2+2^2+4^2+...+(2n)^2+(2(n+1))^2 \\ = & \frac{n(2n+1)(4n+1)}{3}+(2n+1)^2+(2n+2)^2 \\ = & \frac{n(2n+1)(4n+1)}{3}+\frac{3}{3}(2n+1)^2+\frac{3}{3}(2n+2)^2 \\ = & \frac{8n^3+30n^2+37n+15}{3} \end{aligned}$$

On the other side of $\mathcal{P}(n+1)$ is

$$\frac{(n+1)(2n+3)(4n+5)}{3}=\frac{8n^3+30n^2+37n+15}{3}$$

Therefore $\mathcal{P}(n+1)$ holds. Thus, by the principle of mathematical induction, for all $n\geq 1$, $\mathcal{P}(n)$ holds

Exercise 18. *Show that*

$$n!>3^n,$$

for $n\geq 7$

For any $n\geq 7$, let $\mathcal{P}(n)$ be the statement that

$$n!>3^n$$

1. **Base Case:** When $n=7$ the left hand side of the equation is

$$7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1=5040,$$

and the right hand side is

$$3^7=2187$$

So $\mathcal{P}(1)$ is correct.

Exercise 19. *1.* ***Induction Hypothesis:*** *Assume that the statement* $\mathcal{P}(n)$ *is true for any positive integer* n *for some* $n\geq 1$.

$$\mathcal{P}(n):\forall n\geq 7:n!>3^n \text{ is true}$$

2. ***Inductive Step:*** *Show that the statement* $\mathcal{P}(n)$ *is true for* $n+1.i.e$

$$(n+1)!>3^{n+1}$$

One has

$$\begin{aligned}(n+1)! &= (n+1)\cdot n! \\ &> (n+1)\cdot 3^n\geq(7+1)\cdot 3^n\geq 8\cdot 3^n>3\cdot 3^n=3^{n+1}\end{aligned}$$

Therefore $\mathcal{P}(n+1)$ *holds*

8.2 Proof by contradiction

In logic, proof by contradiction is a form of proof that establishes the truth or the validity of a proposition by showing that assuming the proposition to be false leads to a contradiction. Although it is quite freely used in mathematical proofs, not every school of mathematical thought accepts this kind of nonconstructive proof as universally valid

Exercise 20. *For every real number* $x \in [0, \frac{\pi}{2}]$*, we have* $\sin x + \cos x \geq 1$

Suppose for the sake of contradiction that this is not true. Then there exists an $x \in [0, \frac{\pi}{2}]$, we have

$$\sin x + \cos x < 1.$$

Since $x \in [0, \frac{\pi}{2}]$, neither $\sin x$ nor $\cos x$ is negative, so

$$0 \leq \sin x + \cos x < 1.$$

Thus

$$0^2 \leq (\sin x + \cos x)^2 < 1^2,$$

which gives

$$0^2 \leq \sin^2 x + 2 \sin x \cos x + \cos^2 x < 1^2.$$

As

$$\sin^2 x + \cos^2 x = 1,$$

this becomes

$$0 \leq 1 + 2 \sin x \cos x < 1,$$

so

$$1 + 2 \sin x \cos x < 1.$$

Subtracting 1 from both sides gives

$$2 \sin x \cos x < 0.$$

But this contradicts the fact that neither $\sin x$ nor $\cos x$ is negative

Exercise 21. *If* $a, b \in \mathbb{Z}$*, then*

$$a^2 - 4b \neq 2.$$

Suppose this proposition is false. This conditional statement being false means there exist numbers a and b for which $a, b \in \mathbb{Z}$ is true but

$$a^2 - 4b \neq 2.$$

is false.Thus there exist integers $a, b \in \mathbb{Z}$ for which

$$a^2 - 4b = 2$$

From this equation we get

$$a^2 = 4b + 2 = 2(2b + 1),$$

so a^2 is even. Since a^2 is even, it follows that a is even, so

$$a = 2c$$

for some integer c. Now plug

$$a = 2c$$

back into the boxed equation

$$a^2 - 4b = 2$$

We get

$$(2c)^2 - 4b = 2,$$

so

$$4c^2 - 4b = 2.$$

Dividing by 2, we get

$$2c^2 - 2b = 1.$$

Therefore

$$1 = 2(c^2 - b),$$

and since

$$c^2 - b \in \mathbb{Z},$$

it follows that 1 is even. Since we know 1 is not even, something went wrong. But all the logic after the first line of the proof is correct, so it must be that the first line was incorrect. In other words, we were wrong to assume the proposition was false. Thus the proposition is true.