NATIONAL SCHOOL OF CYBERSECURITY

2025/2026 Basic Training Cycle

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Anal 1- Tutorial 2

Real Sequences

Only a few exercises will be covered during the tutorial session.

Exercise 1

Determine the limit of each of the following sequences

$\boxed{1}$ $u_n = \dfrac{1+2+3+..+n}{n^2} \qquad v_n = \dfrac{1.2+2.3+..+n(n+1)}{n^3}$

$\boxed{2}$ $w_n = \dfrac{1+2^2+3^2+..+n^2}{n^3} \qquad v_n = \dfrac{1+a+...+a^n}{1+b+...+b^n}, 0 < |a|, |b| < 1.$

Exercise 2

Show using the $\varepsilon-$defintion of the limit that the following equences have a limit which is equal to zero

$\boxed{1}$ $u_n = \dfrac{1}{n} \qquad v_n = \dfrac{n-1}{n^2+2}$

$\boxed{2}$ $u_n = \dfrac{1}{n!} \qquad v_n = \dfrac{1}{\sqrt[3]{n+2}}$

Exercise 3

Let us consider the follwing real fraction

$$f(k) = \frac{1}{k^2+3k+2}$$

$\boxed{1}$ Determine a and b in $\mathbb{R}$ such that

$$f(k) = \frac{a}{k+2} + \frac{b}{k+1}$$

$\boxed{2}$ Deduce the limit of the sequence

Exercise 4

Calculate, if it exists, the limit of the following sequences:

$$a_n = \frac{\sin\left(\sum_{k=1}^{n} \exp\left(k^2+2^k\right)\right)}{n^2} \qquad b_n = \frac{n^2+(-1)^n\sqrt{n}}{n^2+n+1} \qquad c_n = \frac{(n+1)!+(n-1)!}{(n+2)!} \qquad d_n = \frac{\sum_{k=1}^{n} \frac{1}{2^k}}{\sum_{k=1}^{n} \frac{1}{3^k}}$$

$$e_n = \left(1+\frac{1}{n^2}\right)^n \qquad f_n = \sum_{k=2}^{n} \frac{1}{k^2-1} \qquad g_n = \frac{a^n+b^n}{a^n-b^n} \qquad h_n = \frac{\left[\left(5n-\frac{1}{2}\right)^2\right]}{\left[\left(4n+\frac{1}{2}\right)^2\right]}$$

$$i_n = \frac{\sum_{k=1}^{n} [kx]}{\left[\left(4n+\frac{1}{2}\right)^2\right]}, x \in \mathbb{R}. \qquad k_n = \frac{\sum_{k=1}^{n} [kx]}{\left[\left(4n+\frac{1}{2}\right)^2\right]}, \qquad l_n = \sum_{k=n}^{2n} \frac{k}{\sqrt{k^2+n^2}} \qquad m_n = \sqrt{n}\ln\left(\frac{\sqrt{n}+1}{\sqrt{n}-1}\right)$$

Exercise 5

Show that the following sequences are convergent and determine their limits

$$(1)\ u_n = \frac{\sum\limits_{k=1}^{n} E(kx)}{n^2};\ x \in \mathbb{R}; \qquad (2)\ u_n = \sum_{k=0}^{2n} \frac{k}{k+n^2}; \qquad (3)\ u_n = \sum_{k=0}^{n} \frac{1}{\mathscr{C}_n^k}.$$

Exercise 6

Show,using the criterion of monotone sequences, that the given sequences below are convergent.

$$u_n = \sum_{k=1}^{n} \tfrac{1}{n+k} \qquad v_n = \sum_{k=1}^{n} \tfrac{n+k}{n+k+1} \qquad v_n = \prod_{k=2}^{2n} \tfrac{n+k}{n+k+1} \qquad v_n = \sum_{k=1}^{n} \tfrac{1}{\sqrt{(n+k)(n+k+1)}}$$

Exercise 7

Consider the numerical sequences $(u_n), (v_n)$,and (w_n) defined for $n \in \mathbb{N}^*$ by

$$u_n = \frac{1!+2!+3!+..+n!}{(n-1)!} \qquad v_n = \frac{1!+2!+3!+..+n!}{n!} \qquad w_n = \frac{1!+2!+3!+..+n!}{(n+1)!}$$

$\boxed{1}$ Show that for all $n \in \mathbb{N}^*$

$$1 \leq v_n \leq 1 + \frac{1}{n} + \frac{n+2}{n(n+1)}$$

$\boxed{2}$ Deduce the limits of the sequences$(u_n), (v_n)$ and (w_n).

Exercise 8

Study the convergence of the complex sequence defined by $u_0 \in \mathbb{C}$ and

$$\forall n \in \mathbb{N}; u_{n+1} = \frac{2u_n - \overline{u_n}}{3}.$$

Exercise 9

Let $(a;b) \in \mathbb{C}^2$ and $(z_n)_{n\in\mathbb{N}}$ a complex sequence such that

$$z_{2n} \to a \text{ and } z_{2n+1} \to b$$

Show that the sequence $(z_n z_{n+1})_{n\in\mathbb{N}}$ is convergent and determine its limit.

Exercise 10

Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ a real sequences in $\mathbb{R}_+^*$. Set

$$w_n = \frac{(u_n)^3 + (v_n)^3}{(u_n)^2 + (v_n)^2}$$

Show that

$$(u_n \to 0 \text{ and } u_n \to 0) \Rightarrow w_n \to 0.$$

Exercise 11

Let $a \in \mathbb{R}$ and $(u_n)_{n\in\mathbb{N}}$; $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ three real sequences such that

$$u_n + v_n + w_n \to 3a$$

and

$$(u_n)^2+(v_n)^2+(w_n)^2 \to 3a^2$$

Show that

$$u_n \to a \; ; \; v_n \to a ; w_n \to a.$$

Exercise 12

Show in each case that the sequences (u_n) and (v_n) are adjacent

$\boxed{1}$ $u_n = \sum\limits_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\sqrt{n} \qquad v_n = \sum\limits_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\sqrt{n+1}$

$\boxed{2}$ $u_n = \prod\limits_{k=1}^{n} \left(1+\frac{1}{k.k!}\right) \qquad v_n = \left(1+\frac{1}{n.n!}\right) u_n, \; n \geq 1.$

Exercise 13

Examine whether the following are Cauchy sequences

$\boxed{1}$ $u_n = 1+\frac{1}{4}+....+\frac{1}{n^2} \qquad v_n = \frac{\sin 1}{2}+\frac{\sin 2}{4}+....+\frac{\sin n}{2^n}$

$\boxed{2}$ $w_n = \frac{\cos 1!}{1.2}+\frac{\cos 2!}{2.3}+....+\frac{\cos n!}{n(n+1)} \qquad v_n = 1+\frac{1}{\ln 2}+\frac{1}{\ln 3}+....+\frac{1}{\ln n}.$

Exercise 14

Let $(u_n)_{n\in\mathbb{N}}$ the sequence defined by

$$\begin{cases} u_0 = 0 & u_1 = 1 \\ \forall n \in \mathbb{N} & u_{n+2} = u_{n+1}+u_n \end{cases}$$

$\boxed{1}$ Give the explicit expression of this sequence

$\boxed{2}$ Show that for all $n \in \mathbb{N}$

$$(u_{n+1})^2 - u_{n+2}u_n = (-1)^n$$

$\boxed{3}$ Deduce that $\left(\frac{u_{n+1}}{u_n}\right)_{n\in\mathbb{N}^*}$ converges

$\boxed{4}$ Show that for all $n \in \mathbb{N}$

$$\sum_{k=0}^{n} C_n^k u_k = u_{2n} \text{ and } \sum_{k=0}^{n} (-1)^k C_n^k u_k = -u_n.$$

Exercise 15

Let $(u_n)_{n\in\mathbb{N}}$ the sequence defined by

$$\begin{cases} u_0 = 0 & u_1 = 1 \\ \forall n \in \mathbb{N} & u_{n+2} = 10u_{n+1} - 21u_n + 12n \end{cases}$$

Calculate u_n.

Exercise 16

Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ a real sequences such that

$$u_0 = v_0 = 0$$

and

$$\forall n \in \mathbb{N} : \begin{cases} u_{n+1} = -u_n + 2v_n + 1 \\ \\ v_{n+1} = -4u_n + 5v_n + 2^n \end{cases}$$

Give the explicit expression of these sequences.

Exercise 17

Give a complete study of the following sequences

$$\begin{cases} u_0 = 1 \\ \\ u_{n+1} = \dfrac{(u_n)^2}{u_n + 1} \end{cases} ; \begin{cases} u_0 = 2 \\ \\ u_{n+1} = \sqrt{u_n + 1} \end{cases} ; \begin{cases} u_0 \in \left[\frac{1}{3}; +\infty\right[\\ \\ u_{n+1} = \sqrt{u_n - \dfrac{2}{9}} \end{cases}$$

Exercise 18

Give a complete study of the following real sequence

$$\begin{cases} u_1 > 0 \\ \\ u_{n+1} = \dfrac{\sqrt{nu_n}}{u_n + 1}; n \in \mathbb{N}^* \end{cases}$$

Exercise 19

Let $(u_n)_{n\in\mathbb{N}}$ complex sequence such that the follwing . Assume that subsequences

$$(u_{2p})_{p\in\mathbb{N}};\ (u_{2p+1})_{p\in\mathbb{N}};\ (u_{3p})_{p\in\mathbb{N}}$$

are convergent. Show that $(u_n)_{n\in\mathbb{N}}$ is a convergent sequence.

Exercise 20

Let $(u_n)_{n\in\mathbb{N}}$ a $\mathbb{Z}-$valued sequence such that the follwing . Assume that subsequences Show that

$(u_n)_{n\in\mathbb{N}}$ converges iff $(u_n)_{n\in\mathbb{N}}$ is constant.

Exercise 21

Let $(u_n)_{n\in\mathbb{N}}$ a complex bounded sequence such that

$$u_0 = v_0 = 0$$

and

$$\forall n \in \mathbb{N} : u_{2n} = 2u_n - 1$$

Show that $(u_n)_{n\in\mathbb{N}}$ is constant.

Exercise 22

Let $(u_n)_{n\in\mathbb{N}}$ a real sequence such that

$$u_0 = u_1 = u_2 = 1$$

and

$$\forall n \in \mathbb{N} : u_{n+3} = \frac{u_{n+2}u_{n+1} + 1}{u_n}$$

$\boxed{1}$ Prove that

$$\forall n \in \mathbb{N} : u_{n+4} = 4u_{n+2} - u_n$$

$\boxed{2}$ Deduce that

$$\forall n \in \mathbb{N} : (u_n)_{n\in\mathbb{N}} \in \mathbb{N}^*$$

Exercise 23

Let $a < b \in]0;1[$ and Let $(u_n)_{n\in\mathbb{N}}$; $(v_n)_{n\in\mathbb{N}}$ real sequences such that

$$u_0 = a;\ v_0 = b$$

such that

$$u_0 = v_0 = 0$$

and

$$\forall n \in \mathbb{N} : u_{n+1} = (u_n)^{v_n} \text{ and } v_{n+1} = (v_n)^{u_n}$$

Show that $(u_n)_{n\in\mathbb{N}}$ converges and $(v_n)_{n\in\mathbb{N}}$ converges to 1

Exercise 24

Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be the sequences defined for $n \in \mathbb{N}$ by

$$\begin{cases} u_0 > 0 \\ u_{n+1} = \dfrac{u_n + v_n}{2} \end{cases} ; \quad \begin{cases} v_0 > 0 \\ v_{n+1} = \dfrac{u_n + \sqrt{u_n.v_n} + v_n}{2} \end{cases} ;$$

Show that $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ converge to the same limit ℓ and

$$v_1 < l < u_1.$$

Exercise 25

(NHSM 25) Consider the sequence defined recursively by

$$x_1 = \sqrt{2}, \quad x_n = \sqrt{2 + x_{n-1}}.$$

$\boxed{1}$ Show, by induction, that $x_n < 2$ for all n.

$\boxed{2}$ Show, by induction, that $x_n < x_{n+1}$ for all n.

$\boxed{3}$ Find the limit of $\{x_n\}$.