Differentiability of real-valued functions

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$$f'_-(x_0) = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}$$

(provided it exists and it is finite)

Definition

Let f be a real function defined on an open interval $]a, b[$ and let $x_0 \in]a, b[$. f is differentiable at x_0 if the following limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists and it is finite

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Example

Consider the function

$$f(x) = \sqrt{x}$$

and study the differentiability of f in its domain of definition

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Let $x_0 \in [0, +\infty[$.

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Example

Let $x_0 \in [0, +\infty[$.

1. If $x_0 > 0$, we have

$$\lim_{h \longrightarrow 0} \frac{f(x_0 + \mathbf{h}) - f(x)}{h} = \lim_{h \longrightarrow 0} \frac{\sqrt{x_0 + \mathbf{h}} - \sqrt{x_0}}{h}$$

$$=$$

Example

Let $x_0 \in [0, +\infty[$.

1. If $x_0 > 0$, we have

$$\begin{aligned}
\lim_{h \to 0} \frac{f(x_0+\mathbf{h}) - f(x)}{h} &= \lim_{h \to 0} \frac{\sqrt{x_0+\mathbf{h}} - \sqrt{x_0}}{h} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{x_0+\mathbf{h}} + \sqrt{x_0}} \\
&=
\end{aligned}$$

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&= \lim_{h \to 0} \frac{1}{\sqrt{x_0 + \mathbf{h}} + \sqrt{x_0}} \\
&= \frac{1}{2\sqrt{x_0}}
\end{aligned}$$

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Example

2. If $x_0 = 0$, we have

$$\begin{aligned}\lim_{h\to 0^+} \frac{f(0+\mathbf{h}) - f(0)}{h} &= \lim_{h\to 0^+} \frac{\sqrt{0+\mathbf{h}} - \sqrt{0}}{h} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{h}} = +\infty.\end{aligned}$$

Example

Consider the function

$$f(x) = x^2$$

and study the differentiability of f in its domain.

Example

Let $x_0 \in \mathbb{R}$

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$$\lim_{h \longrightarrow 0} \frac{f(x_0+\mathbf{h}) - f(x)}{h}$$

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$$\lim_{h \to 0} \frac{f(x_0 + \mathbf{h}) - f(x)}{h} = \lim_{h \to 0} \frac{(x_0 + \mathbf{h})^2 - (x_0)^2}{h}$$

Example

Let $x_0 \in \mathbb{R}$

$$\begin{aligned}\lim_{h\to 0}\frac{f\left(x_0+\mathbf{h}\right)-f\left(x\right)}{h} &= \lim_{h\to 0}\frac{\left(x_0+\mathbf{h}\right)^2-\left(x_0\right)^2}{h}\\ &= \lim_{h\to 0}\frac{\left(\left(x_0\right)^2+2\left(x_0\right)\mathbf{h}+\mathbf{h}^2\right)-\left(x_0\right)^2}{h}\end{aligned}$$

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\begin{aligned}
\lim_{h \to 0} \frac{f(x_0 + \mathbf{h}) - f(x)}{h} &= \lim_{h \to 0} \frac{(x_0 + \mathbf{h})^2 - (x_0)^2}{h} \\
&= \lim_{h \to 0} \frac{\left((x_0)^2 + 2(x_0)\mathbf{h} + \mathbf{h}^2\right) - (x_0)^2}{h} \\
&= \lim_{h \to 0} 2(x_0)\mathbf{h} + \mathbf{h}
\end{aligned}
$$

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&= \lim_{h\to 0} \frac{\left((x_0)^2 + 2(x_0)\mathbf{h} + \mathbf{h}^2\right) - (x_0)^2}{h} \\
&= \lim_{h\to 0} 2(x_0)\mathbf{h} + \mathbf{h} \\
&= 2x_0
\end{aligned}$$

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$$f(x) = \cos x$$

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$$
\begin{aligned}
& \lim_{h \to 0} \frac{f(x_0 + \mathbf{h}) - f(x)}{h} \\
= \quad & \lim_{h \to 0} \frac{\cos(x_0 + \mathbf{h}) - \cos(x_0)}{h}
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= \quad & -2 \lim_{h\to 0} \left(\frac{\sin\left(\frac{x_0+\mathbf{h}+x_0}{2}\right) \cdot \sin\left(\frac{x_0+\mathbf{h}-x_0}{2}\right)}{h} \right)
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= \quad & -\lim_{h\to 0} \sin\left(x_0 + \frac{\mathbf{h}}{2}\right) \lim_{h\to 0} \frac{\sin\left(\frac{\mathbf{h}}{2}\right)}{\frac{\mathbf{h}}{2}}
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Corollary

If f is differentiable at x_0, then

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Rules of differentiation

Corollary

Let f and g be two functions defined on I and *differentiable* *at a point $x_0 \in I$. Then*

1. *For any $\alpha, \beta \in \mathbb{R}$, the function $\alpha f + \beta g$ is*

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$$(fg)'(x_0) = f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0)$$

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1. *For any $\alpha, \beta \in \mathbb{R}$, the function $\alpha f + \beta g$ is differentiable at x_0, and*
$$(\alpha f + \beta g)' (x_0) = \alpha f' (x_0) + \beta g' (x_0)$$
2. *The function fg is differentiable at x_0, and*
$$(fg)' (x_0) = f' (x_0) g (x_0) + f (x_0) g' (x_0)$$
3. *If $g(x_0) \neq 0$, then the function $\dfrac{f}{g}$ is differentiable at x_0, and*
$$\left(\frac{f}{g}\right)' (x_0) = \frac{f' (x_0) g (x_0) - f (x_0) g' (x_0)}{g^2 (x_0)}$$

Theorem

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$$(g \circ f)'(x_0)$$

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Then the function $g \circ f$ is differentiable at the point x_0, and

$$(g \circ f)'(x_0) = f'(x_0)\, g'(f(x_0))$$

$$\lim_{x \longrightarrow x_0} \frac{\left(g \circ f\right)\left(x\right) - \left(g \circ f\right)\left(x_0\right)}{x - x_0}$$

$$
\begin{aligned}
& \lim_{x \longrightarrow x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} \\
= \; & \lim_{x \longrightarrow x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0}
\end{aligned}
$$

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= & \lim_{x \longrightarrow x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} \frac{f(x) - f(x_0)}{f(x) - f(x_0)}
\end{aligned}$$

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= \; & \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}
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= \; & g'(f(x_0)) f'(x_0)
\end{aligned}
$$

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Let $f : I \longrightarrow J$ be a function that is

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Theorem

Let $f : I \longrightarrow J$ be a function that is continuous **,strictly monotonic** *, and* **differentiable** *at the point $x_0 \in I$ such that $f(I) = J$ and $f'(x_0) \neq 0$. In that case, we have*

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'\left(f^{-1}(y_0)\right)}.$$

Proof.

Set $g = f^{-1}$

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$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0}$$

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\begin{aligned}
& \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} \\
= \quad & \lim_{y \to y_0} \frac{g(y) - x_0}{f(x) - f(x_0)}
\end{aligned}
$$

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$$\begin{aligned}
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= &\lim_{y \to y_0} \frac{g(y) - x_0}{f(x) - f(x_0)} \\
= &\lim_{y \to y_0} \frac{1}{\dfrac{f(x) - f(x_0)}{g(y) - x_0}}
\end{aligned}$$

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Set $g = f^{-1}$

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= \quad & \lim_{y \to y_0} \frac{1}{\dfrac{f(x) - f(x_0)}{g(y) - x_0}} \\
= \quad & \lim_{x \to x_0} \frac{1}{\dfrac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{f'(x_0)}
\end{aligned}
$$

$\square$

Example

$f(x) = 2x + 1, x \in \mathbb{R},$

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$y = 2x + 1$; it follows that$x = \frac{y-1}{2}$ and

$$f^{-1}(y) = \frac{y-1}{2} \Rightarrow \left(f^{-1}\right)'(y) = \frac{1}{2}$$

since $f'(x) = 2$

since $f'(x) = 2$ then $f'(x) = f'\left(f^{-1}(y)\right) = 2$

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$$\left(f^{-1}\right)'(y) = \frac{1}{f'\left(f^{-1}(y)\right)} = \frac{1}{2}$$

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$f(x) = \sqrt{x} + 2, x > 0$, we have

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$$f^{-1}(y) = (y-2)^2$$

then

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$$\left(f^{-1}\right)'(y)$$

Example

$f(x) = \sqrt{x} + 2, x > 0$, we have $f'(x) = \dfrac{1}{2\sqrt{x}}$ and

$$f^{-1}(y) = (y-2)^2$$

then

$$\begin{aligned} & \left(f^{-1}\right)'(y) \\ = \quad & \frac{1}{f'\left(f^{-1}(y)\right)} \end{aligned}$$

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= & 2\sqrt{x} = 2(y-2), x > 0
\end{aligned}$$

Example

$f(x) = \ln x$, $x > 0$, we have $f'(x) = \dfrac{1}{x}$ and

$$\left(f^{-1}\right)'(y) = \frac{1}{f'\left(f^{-1}(y)\right)} = \frac{1}{f'(x)} = \frac{1}{\frac{1}{x}} = x = e^{y}$$

Example

$f(x) = e^x, x > 0$, we have $f'(x) = e^x$ and

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(y)\right)} = \frac{1}{f'(x)} = \frac{1}{e^x} = \frac{1}{x}$$

Example

$f(x) = x + \ln x, x > 0$, we have $f'(x) = \dfrac{x+1}{x}$ and

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(y)\right)} = \frac{1}{f'(x)} = \frac{1}{\dfrac{x+1}{x}} = \frac{x}{x+1} = \frac{f^{-1}(y)}{f^{-1}(y)+1}$$

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$f(x) = \dfrac{x^2}{1+x^2}, x < 0$

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= & \frac{1}{f'\left(f^{-1}(y)\right)} \\
= & \frac{1}{f'(x)} = \\
= & \frac{1}{\dfrac{2x}{(1+x^2)^2}} \\
= & \frac{(1+x^2)^2}{2x} = \frac{\left(1+\left(f^{-1}(y)\right)^2\right)^2}{2f^{-1}(y)}
\end{aligned}
$$

Example

Soit $f : \mathbb{R} \to \mathbb{R}$ la fonction définie par

$$f(x) = x + e^x.$$

1. f is **differentiable** since it is

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3. f is **bijective** and $\lim_{x \to +\infty} f(x) \to +\infty$, $\lim_{x \to -\infty} f(x) \to -\infty$.

Example

We know that $f'(x)$

Example

We know that $f'(x)= 1 + e^x$.then

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)} = \frac{1}{1+e^x} = \frac{1}{1+e^{f^{-1}(y)}}$$

but $f^{-1}(f(x)) = x$

Definition

($\mathcal{C}^n$ Functions) We say that a function $f : I \longrightarrow \mathbb{R}$ is of class $\mathcal{C}^n$ if and only if

1. f is n times differentiable on I.
2. The function $f^{(n)}$is continuous on I.