NSCS

المدرسة الوطنية العليا في الأمن السيبراني

NATIONAL SCHOOL OF CYBERSECURITY

Date: January 19, 2026
Lecturer: Pr B.CHAOUCHI

Final Exam - Analysis I

Basic Training Cycle
Duration: 2 hours

1. Choose either exercise 2 or 2* .
2. Choose either exercise 5 or 5* .
3. Only **non-programmable calculators** are allowed.
4. The graphs must be drawn on graph paper and must include the students first and last name.
5. All unclear responses or sheets without your name will not be graded.
6. The mark scheme : **(4)+(3)+(2)+(5)+(6)**

Exercise 1 (4 Pts)

1

Let

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

be a real-valued function defined by

$$f(x) = \lfloor x \rfloor + (x - \lfloor x \rfloor)^2,$$

where $\lfloor x \rfloor$ denotes the integer part (floor) of x.

1. Evaluate $f(x+1)$
2. Let $x \in [0, 1)$. Sketch the graph of the function f in an orthonormal coordinate system.
3. Let $x \in \mathbb{R}$. Sketch the graph of the function f in an orthonormal coordinate system.
4. Using the previous question, study the **continuity** of the function f.

Exercise 2 (3 Pts)

2

Let $A = \{x \in \mathbb{Q} : 1 < x \text{ and } x^2 < 2\}$.

1. Show that $A \neq \emptyset$ and that A is upper bounded in $\mathbb{Q}$.
2. Let $r \in A$. Show that there exists $n \in \mathbb{N}$ for which $n(2 - r^2) > 2r + 1$.

3. Deduce that $r' = r + \frac{1}{n} \in A$.

4. Let $M \in \mathbb{Q}$ be an upper bound of A. Show that $M > \sqrt{2}$.

5. Deduce that $\sup A \notin \mathbb{Q}$.

$\hookrightarrow$ Use Archimedes' result : For real numbers a, b such that $a > 0$, there exists $n \in \mathbb{N}$ for which $na > b$.

Exercise 2* (3 Pts)

3

Let us consider the set

$$\mathcal{A} = \left\{ \left(\frac{n+m+1}{n+m} \right)^{n+m} : n, m \in \mathbb{N}^* \right\}$$

1. Give the table of variations of the function $f(x) = \ln(x+1) - x$ and draw its graph in an orthonormal coordinate system.
2. Using question 1, show that $\mathcal{A}$ is bounded above
3. Determine $\sup(\mathcal{A})$

Exercise 3 (2 Pts)

4

Compute the following sums, $n \geq 1$

1. $S = \sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} \min(i, j) \right)$

Exercise 4 (5 Pts)

5

Let $a > 0$ and let (u_n) be a real sequence satisfying $u_0 > 0$ and

$$u_{n+1} = \frac{1}{2} \left(u_n + \frac{a}{u_n} \right)$$

1. Show first that (u_n) is well-defined, that is, $u_n \neq 0$, $n \in \mathbb{N}$.

2. Show that $u_{n+1}^2 - a = \dfrac{(u_n^2 - a)^2}{4u_n^2}$, $n \geq 1$.

3. Deduce that $u_n \geq \sqrt{a}$, $n \geq 1$ and that (u_n) is decreasing.

4. Deduce that (u_n) is convergent and find its limit.

5. Using the relation

$$u_{n+1}^2 - a = (u_{n+1} + \sqrt{a})(u_{n+1} - \sqrt{a})$$

, give an upper bound of $u_{n+1} - \sqrt{a}$ in terms of $u_n - \sqrt{a}$.

6. If $u_1 - \sqrt{a} \leq k$, show that

$$u_n - \sqrt{a} \leq 2\sqrt{a}\left(\frac{k}{2\sqrt{a}}\right)^{2^{n-1}} \quad , \ n \geq 1$$

Exercise 5 (6 Pts)

6

For every integer $n \in \mathbb{N}^*$, define

$$f_n(x) = x(\ln x)^n$$

Let C_n denote the graph of the function f_n.

1. What are the limits of f_n at the boundaries of its domain?
2. Study the variations of the functions f_1 and f_2 (a complete table of variations should be provided in each case).
3. Solve the equation $f_1(x) = f_2(x)$ and deduce the relative positions of the curves C_1 and C_2. More generally, verify that there exist two points in the plane that are common to all the curves C_n.
4. More generally, study the relative positions of C_n and C_{n+1} on the interval $[1, +\infty)$.
5. *What can be said about the positions of all the curves C_n on the interval $(0, 1)$ (be as precise as possible)?*
6. Draw, in the same coordinate system, a neat sketch of the curves C_1 and C_2.
7. Generalize the results of question 2 by studying the variations of f_n for every integer $n > 1$ (one may distinguish two cases depending on whether n is even or odd).

Exercise 5* (6 Pts)

7

Definitions (included in this exercise):

- **Virus:** Infects files and spreads through user actions (opening an infected file, using a USB drive, etc.).
- **Worm:** Self-replicates and spreads automatically through networks without user interaction.
- **Hybrid Program:** A malicious program that is neither a virus nor a worm.
- **IoT Threats:** Infect Internet-of-Things devices; spread depends on network connectivity and weak security; growth is intermediate.
- **Severity:** Measured by the absolute number of devices infected after a certain period of time.

The number of infected devices on day n is modeled as follows:

$$\begin{cases} V_0 = 10, \quad V_{n+1} = V_n + 5\sqrt{1 - \dfrac{V_n}{500}} \\ W_0 = 10, \quad W_{n+1} = W_n\Big(1 + 0.8(1 - \dfrac{W_n}{1000})\Big) \\ H_0 = 10, \quad H_{n+1} = 1.4H_n - 0.001{H_n}^2 + 2 \\ I_0 = 5, \quad I_{n+1} = I_n + 0.5I_n\Big(1 - \dfrac{I_n}{200}\Big) + 3 \end{cases}$$

1. Compute approximately the first five terms of each sequence.
2. Which type of malicious program spreads fastest at the beginning? Which one slows down first?
3. Find the limit of each sequence and compare the severity of the different threats after 5 days.