Midterm Exam Solutions - Algebra 2

Exercice 1:

1) The standard basis of $\mathbb{R}_6[X]$ is:

$$\mathcal{B} = \{1, X, X^2, X^3, X^4, X^5, X^6\}$$ 0,25

$\dim(\mathbb{R}_6[X]) = 7$. 0,25

2) Let $P \in G$. Prove that $X+1 \mid P$.

$P(X) = \lambda X^3 + \mu X^2 + \mu X + \lambda$, $(\lambda, \mu) \in \mathbb{R}^2$. 0,5

We have $P(-1) = -\lambda + \mu - \mu + \lambda = 0$ $\quad (X-a \mid P \Leftrightarrow P(a) = 0)$

Then all elements of G are divisible by $X+1$.

3. a) Let $P \in E$

- $P \in F \Leftrightarrow P = \delta X^6 + \alpha X^5 + \beta X^4 + \gamma X^3 + \beta X^2 + \alpha X + \delta$

$\Leftrightarrow P = \delta(X^6+1) + \alpha(X^5+X) + \beta(X^4+X^2) + \gamma X^3$

$\Leftrightarrow P \in \operatorname{Span}\{X^6+1, X^5+X, X^4+X^2, X^3\}$ 0,5

So $F = \operatorname{Span}\{X^6+1, X^5+X, X^4+X^2, X^3\}$

Then F is a subspace of $E = \mathbb{R}_6[X]$ spanned (generated) by $\{X^6+1, X^5+X, X^4+X^2, X^3\}$

- $P \in G \Leftrightarrow P = \lambda X^3 + \mu X^2 + \mu X + \lambda$

$\Leftrightarrow P = \lambda(X^3+1) + \mu(X^2+X)$

$\Leftrightarrow P \in \operatorname{Span}\{X^3+1, X^2+X\}$ 0,5

Then $G = \operatorname{Span}\{X^3+1, X^2+X\}$.

So G is a subspace of $\mathbb{R}_6[X]$ spanned by: $\{X^3+1, X^2+X\}$.

b) - $\mathcal{B}_F = \{X^6+1, X^5+X, X^4+X^2, X^3\}$ spans F and it is linearly independent (all polynomials in $\mathcal{B}_F$ have different degrees). 0,25

Then $\mathcal{B}_F$ is a basis for F

So: $\dim F = 4$ 0,25

* $\mathcal{B}_G = \{X^3+1, X^2+X\}$ spans G and the vectors X^3+1 and X^2+X are not colinear, then $\mathcal{B}_G$ is linearly independent. Then $\mathcal{B}_G$ is a basis for G. So $\dim G = 2$ 0,2 0,2

c) • Do we have $F \oplus G$? $F \oplus G \Leftrightarrow F \cap G = \{0\}$

Let $P \in F \cap G$

$$P \in F \cap G \Leftrightarrow P \in F \text{ and } P \in G$$

$$\Leftrightarrow P = \delta X^6 + \alpha X^5 + \beta X^4 + \gamma X^3 + \beta X^2 + \alpha X + \delta \text{ and } P = \lambda X^3 + \mu X^2 + \mu X + \lambda, \ (\delta, \alpha, \beta, \gamma, \lambda, \mu) \in \mathbb{R}^6$$

$$\Leftrightarrow \begin{cases} \delta = \alpha = \beta = 0 \\ \gamma = \lambda \\ \beta = \mu \\ \alpha = \mu \\ \delta = \lambda \end{cases} \Leftrightarrow \alpha = \beta = \gamma = \delta = \lambda = \mu = 0$$

0,1

Then : $F \cap G = \{0\} \Leftrightarrow F \oplus G$.

Is $E = F \oplus G$?

We have $\dim(F \oplus G) = \dim F + \dim G = 4 + 2 = 6 \neq \dim E$

Then $F \oplus G \neq E$. 0,1

4. a) $H = \{P \in \mathbb{R}_6[X], P'(-1) = 0\}$

* $H \subset \mathbb{R}_6[X]$ 0,2
* $P = 0_{\mathbb{R}_6[X]} \in H$ because $P'(-1) = 0$ 0,2
* Let $P, Q \in H$ and $\alpha, \beta \in \mathbb{R}$.

Prove that $\alpha P + \beta Q \in H$. Denote $R = \alpha P + \beta Q$.

$$\begin{aligned} R'(-1) &= \alpha P'(-1) + \beta Q'(-1) \\ &= 0 + 0 \quad (P \in H \text{ and } Q \in H) \\ &= 0 \end{aligned}$$

0,1

Then $R = \alpha P + \beta Q \in H$.

It follows that H is a subspace of $\mathbb{R}_6[X]$

b) H is a subspace of $\mathbb{R}_6[X] \Rightarrow \dim H \leq \dim E = 7$. 0,2

Suppose $\dim H = 7$.

If $\dim H = 7$, then $H = \mathbb{R}_6[X]$ ($H \subset \mathbb{R}_6[X]$) .

But : $X^2 \in \mathbb{R}_6[X]$ and $X^2 \notin H$ ($P'(-1) = -2 \neq 0$)

Contradiction. 0,2

So $\dim H \leq 6$.

c) $1 \in H$ ($(1)' = 0$)

$\forall k \in [\![2, 6]\!]$, $P_k = (X+1)^k \in H$ because :

$(X+1)^k \in E$ 0,1

$P'_k(X) = k(X+1)^{k-1}$ where $k-1 \geq 1$.

$P'_k(-1) = k \times 0^{k-1} = 0$ where $k-1 \geq 1$.

Moreover, $\mathcal{F}$ is a collection of polynomials with different degrees ($\forall i \neq j$, $\deg(P_i) \neq \deg(P_j)$). 0,2

Then $\mathcal{F}$ is linearly independent.

d) $\mathcal{F}$ is linearly independent, the $\mathcal{F}$ is a basis for Span$\{\mathcal{F}\}$

So $\dim(\text{Span}\{\mathcal{F}\}) = 6$.

$\mathcal{F}$ is a collection of elements of H, then Span$\{\mathcal{F}\} \subset H$. 0,7

It follows that : $\dim(\text{Span}\{\mathcal{F}\}) \leq \dim H$

Then : $6 \leq \dim H \leq 6$ (from question 4.b)

5. a) We have $G \cap H \subset G \Rightarrow \dim(G \cap H) \leq \dim G = 2$ 0,2

$\dim(G+H) = \dim G + \dim H - \dim(G \cap H) \leq \dim E$

$\Leftrightarrow \dim(G \cap H) \geq \dim G + \dim H - \dim H$

$\Leftrightarrow \dim(G \cap H) \geq 2 + 6 - 7 = 1$. 0,2

Then : $1 \leq \dim(G \cap H) \leq 2$

5\. b) Suppose $\dim(G \cap H) = 2 = \dim G$.

Then $G \cap H = G$ (because $G \cap H \subset G$)

So $G \subset H$ $(G \cap H \subset H)$

But: $P = X^3 + 1 \in G$ and $P'(-1) = 3 \neq 0$ 0,1

$P \notin H \Rightarrow G \not\subset H$ Contradiction.

Then $\dim(G \cap H) = 1$.

c) Let $P \in G \cap H$.

$P \in G \cap H \iff P \in G$ and $P \in H$

$\iff$ P is divisible by $X+1$ (from 2)) and $P'(-1) = 0$

0,1 $\iff P(-1) = 0$ and $P'(-1) = 0$

$\iff$ -1 is a root of multiplicity at least 2 of P.

d) $\dim(G \cap H) = 1$, So $G \cap H$ is spanned by one nonzero vector.

Let $P = (X+1)^3 \in H$ because -1 is a root of multiplicity 3 of P.

and $P \in G$ ($P(-1) = 0 \iff$ P is divisible by $X+1$) $P'(-1) = 0$

Finally $\{(X+1)^3\}$ is a basis for $G \cap H$. 0,1

Exercise 2:

1)

n	0	1	2	3	4	5	6	7	8	9	10	11	12
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144

0,1

2) $A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A + I_2$ 0,1

3) $A^{2n} = (A^2)^n = (A + I_2)^n = \sum_{k=0}^{n} \binom{n}{k} A^k$ 0,1

4) $A = \begin{pmatrix} F_0 & F_1 \\ F_1 & F_2 \end{pmatrix}$, $A^2 = \begin{pmatrix} F_1 & F_2 \\ F_2 & F_3 \end{pmatrix}$, by induction: $A^k = \begin{pmatrix} F_{k-1} & F_k \\ F_k & F_{k+1} \end{pmatrix}$

By identification: $A^{2n} = \begin{pmatrix} F_{2n-1} & F_{2n} \\ F_{2n} & F_{2n+1} \end{pmatrix} = \sum_{k=0}^{n} \binom{n}{k} \begin{pmatrix} F_{k-1} & F_k \\ F_k & F_{k+1} \end{pmatrix}$

0,1

$$A^{2n} = \begin{pmatrix} F_{2n-1} & F_{2n} \\ F_{2n} & F_{2n+1} \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{n} \binom{n}{k} F_{k-1} & \sum_{k=0}^{n} \binom{n}{k} F_k \\ \sum_{k=0}^{n} \binom{n}{k} F_k & \sum_{k=0}^{n} \binom{n}{k} F_{k+1} \end{pmatrix}$$ 0,5

Then: $F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_k$.

For $n = 6$, we have: $F_{12} = \sum_{k=0}^{6} \binom{6}{k} F_k$ 0,5

$$= \binom{6}{0} F_0 + \binom{6}{1} F_1 + \binom{6}{2} F_2 + \binom{6}{3} F_3 + \binom{6}{4} F_4 + \binom{6}{5} F_5 + \binom{6}{6} F_6$$
$$= 6F_1 + 15F_2 + 20F_3 + 15F_4 + 6F_5 + F_6$$
$$= 144$$

5) From 4), we have:

$$F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_k \leq \underbrace{\sum_{k=0}^{n} \binom{n}{k}}_{2^n} F_n \quad (F_k \leq F_n, \forall k \leq n)$$

$$\leq 2^n F_n$$ 1

Then, $\exists a = 2 > 0, \forall n \in \mathbb{N}: F_{2n} \leq a^n F_n$.

Exercise 3:

1.a) $L(X) = \left(\int_{-1}^{1} t\, dt\right) X = \left[\frac{t^2}{2}\right]_{-1}^{1} X = 0$. 0,25

$L(X^2) = \frac{2}{3} X$ 0,25

$L(X^3) = 0$ 0,25

b) Let $\alpha, \beta \in \mathbb{R}$ and $P, Q \in \mathbb{R}_3[X]$.

$$L(\alpha P + \beta Q) = \left(\int_{-1}^{1} (\alpha P + \beta Q)(t)\, dt\right) X$$
$$= \left(\int_{-1}^{1} (\alpha P(t) + \beta Q(t))\, dt\right) X$$ 1
$$= \left(\int_{-1}^{1} \alpha P(t)\, dt + \int_{-1}^{1} \beta Q(t)\, dt\right) X$$
$$= \alpha \left(\int_{-1}^{1} P(t)\, dt\right) X + \beta \left(\int_{-1}^{1} Q(t)\, dt\right) X$$
$$= \alpha L(P) + \beta L(Q).$$ Then L is a linear map.

c) We notice that, for $P \in E$:

$$P \in Ker(L) \iff L(P) = 0 \iff \int_{-1}^{1} P(t)\,dt = 0$$ 0,75

Then: $F = Ker(L)$ which is a subspace of $E = \mathbb{R}_3[X]$.

2. a)

$$A = Mat\, f = \begin{array}{cc} \begin{array}{cccc} f(1) & f(X) & f(X^2) & f(X^3) \end{array} & \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & \frac{8}{3} & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} & \begin{array}{c} 1 \\ X \\ X^2 \end{array} \end{array}$$ 0,5

b) $$A \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{8}{3} & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \qquad C_1 \leftarrow C_1 - \frac{3}{4} C_3$$ 0,75

$rank(f) = rank(A) = 3$ 0,25

$$\left.\begin{array}{l} \dim(Im(f)) = rank(f) = 3 = \dim(\mathbb{R}_2[X]) \\ Im(f) \subset \mathbb{R}_2[X] \end{array}\right\} \Rightarrow Im(f) = \mathbb{R}_2[X]$$ 0,5

Then f is surjective.

c) From the Rank-nullity theorem, we have:

$$\dim E = rank(f) + \dim(Ker(f))$$

Thus: $\dim(Ker(f)) = 1$. 0,5

d) We observe that: $C_1 = \frac{3}{4} C_3$ (from the matrix A). So:

$$f(1) = \frac{3}{4} f(X^2).$$

$$f \in \mathcal{L}(E, \mathbb{R}_2[X]) \Rightarrow f\left(1 - \frac{3}{4}X^2\right) = 0$$ 0,5

$$\Rightarrow 1 - \frac{3}{4}X^2 \in Ker(f).$$

Then $\mathcal{B} = \left\{1 - \frac{3}{4}X^2\right\}$ is a basis for $Ker(f)$ (because $\dim(Ker f) = 1$)

3. a) $\varphi(P) = P' + \left(\int_{-1}^{1} P(t)\,dt\right) X = P'$

Because : $\int_{-1}^{1} P(t)\,dt = 0$ $\quad (P \in F)$ 0,1

b) $\text{Ker}(\varphi) = \{ P \in F \mid \varphi(P) = 0 \}$

let $P \in F$.

$\varphi(P) = 0 \Leftrightarrow P' = 0 \Leftrightarrow P = c,\ c \in \mathbb{R}$. 0,2

$P \in F \Leftrightarrow \int_{-1}^{1} P(t)\,dt = 0$

$\Leftrightarrow \int_{-1}^{1} c\,dt = 0$

$\Leftrightarrow [ct]_{-1}^{1} = 0$

$\Leftrightarrow 2c = 0$ 0,2

$\Leftrightarrow c = 0$ So $\varphi(P) = 0 \Leftrightarrow P = 0$.

Then $\text{Ker}(\varphi) = \{0\}$ 0,2

So φ is injective. 0,2