Differentiability of real-valued functions

Impressive slide presentations

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Leibniz formula

Proposition

Assume that the functions $u = u(x)$ and $v = v(x)$ are n **times differentiable** *on an interval $]a, b[$ where $n \in \mathbb{N}$. Then,using the usual convention*

$$u^{(0)}(x) = u(x)$$

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the following so called **Leibniz formula** *holds*

$$(u.v)^{(n)}(x) = \sum_{j=0}^{n} \mathcal{C}_n^j u^{(j)}(x).v^{(n-j)}(x), \qquad x \in \,]a, b[\,.$$

Definition

A **critical point** x_0 of a continuous function f

Definition

A **critical point** x_0 of a continuous function f is a point at which the derivative at x_0 is zero

$$f'(x_0) = 0$$

Definition

A **local extremum** x_0

Definition

A **local extremum** x_0 is a maximum or minimum of the function x_0 on some interval of x-values

Definition

We say that f has a **local maximum** at x_0

Definition

We say that f has a **local maximum** at x_0 if there exists an interval J such that

$$\forall x \in J \cap D_f : f(x) \leq f(x_0)$$

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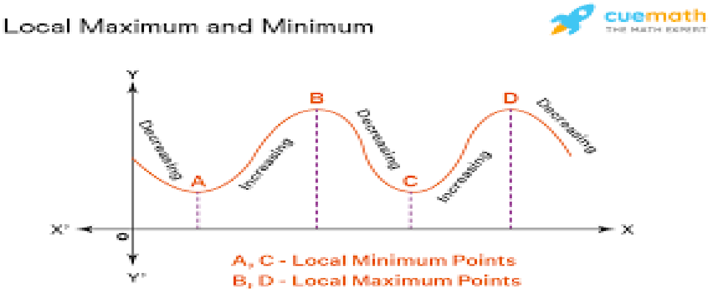
Local Maximum and Minimum
cuemath
Y
B
D
Decreasing
Decreasing
Decreasing
Increasing
Increasing
A
C
X'
X
O
Y'
A, C - Local Minimum Points
B, D - Local Maximum Points

Theorem

Let $I \subset \mathbb{R}$ an open interval and $f : I \to \mathbb{R}$ a differentiable function. if f has

(or a local minimum)

(or a local minimum) **at** x_0 **then**

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$$f'(x_0)$$

Remark the reciprocal of this result is false. Let

$$f(x) = x^3$$

and $x_0 = 0$

Rolle's Theorem

Theorem

$f : [a, b] \longrightarrow \mathbb{R}$ be a function such that

1. *f is* **continuous** *on the interval $[a, b]$.*
2. *f is* **differentiable** *on the open interval $]a, b[$.*
3. *$f(a) = f(b)$.*
   Then, $\exists c \in \,]a, b[$ such that

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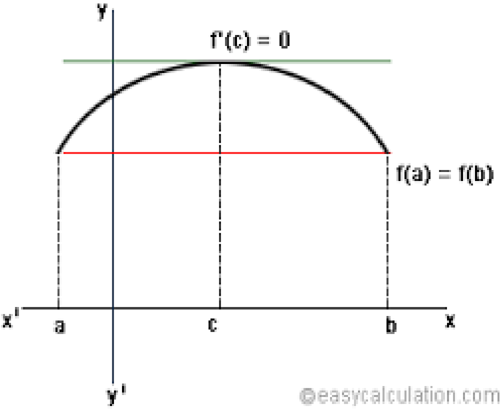
y
f'(c) = 0
f(a) = f(b)
x'
a
c
b
x
y'
©easycalculation.com

Mean Value Theorem (M.V.T) or Lagrange's mean value theorem

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2. *f is differentiable on the open interval $]a, b[$*
   Then, $\exists c \in \,]a, b[$ such that

$$f(b) - f(a) = (b - a)f'(c).$$

Example

Let P the function defined by

$$P(x) = 3x^4 - 11x^3 + 12x^2 - 4x + 2.$$

Show that there exists at least c$\in]0,1[$ such that $P'(c) = 0$. It is clear that P is continuous on $[0,1]$ and differntiable on $]0,1[$. Further

$$P(0) = P(1) = 2$$

Using MVT theorem, we deduce that there exists $c \in]0,1[$

$$\frac{P(1) - P(0)}{1 - 0} = P'(c) = 0.$$

Function Variation

Corollary

Let $f : I \longrightarrow \mathbb{R}$ be a differentiable function on I. We have

1. $\forall x \in I, f(x) \geq 0 \Leftrightarrow f$ is increasing
2. $\forall x \in I, f(x) \leq 0 \Leftrightarrow f$ is decreasing
3. $\forall x \in I, f(x) = 0 \Leftrightarrow f$ is constant.

Corollary

L'Hospital's Rule *Let $f, g : I \longrightarrow \mathbb{R}$ be two differentiable functions and let $x_0 \in I$. Additionally, suppose that*

1. $\lim\limits_{x \longrightarrow x_0} f(x_0) = \lim\limits_{x \longrightarrow x_0} g(x_0) = 0$
2. $\forall x \in I - \{x_0\} : g'(x_0) \neq 0.$

$$\textit{if } \lim_{x \longrightarrow x_0} \frac{f'(x)}{g'(x)} = l \,(l \in \mathbb{R}) \;\; \textit{Then } \lim_{x \longrightarrow x_0} \frac{f(x)}{g(x)} = l.$$

Note This rule remains valid if $\lim\limits_{x \longrightarrow x_0} f(x_0) = \lim\limits_{x \longrightarrow x_0} g(x_0) = \pm\infty$