Solution - Midterm Test Algebra 2

Exercise 1:

1/ $A = 2I_3 + J$ where $J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

0,25 $J^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $J^3 = O_3$, $J^k = 0$ for $k \geq 3$. 0,25

$$A^k = (2I_3 + J)^k = \sum_{i=0}^{k} \binom{k}{i} 2^{k-i} J^i \quad 0{,}25$$

$$= \sum_{i=0}^{2} \binom{k}{i} 2^{k-i} J^i$$

$$= 2^k I_3 + k\,2^{k-1} J + k(k-1)2^{k-3} J^2 \quad 0{,}25$$

$$A^k = \begin{pmatrix} 2^k & k\,2^{k-1} & k(k-1)2^{k-3} \\ 0 & 2^k & k\,2^{k-1} \\ 0 & 0 & 2^k \end{pmatrix} \quad 0{,}5$$

2) $$P(X) = \sum_{k=0}^{n} \frac{P^{(k)}(2)}{k!} (X-2)^k$$

$$P(A) = \sum_{k=0}^{n} \frac{P^{(k)}(2)}{k!} (A - 2I)^k \quad 0{,}25$$

$$= \sum_{k=0}^{n} \frac{P^{(k)}(2)}{k!} J^k \quad 0{,}25$$

$$= P(2)\, I_2 + P'(2)\, J + \frac{1}{2} P''(2)\, J^2 \quad (J^k = 0 \text{ for } k \geq 3) \quad 0{,}5$$

$$= \begin{pmatrix} P(2) & P'(2) & \frac{1}{2} P''(2) \\ 0 & P(2) & P'(2) \\ 0 & 0 & P(2) \end{pmatrix}$$

Exercise 2 :

1/ $P(A) = A^2 - A = A - A = 0 \Rightarrow P$ is an annihilating polynomial for A. 0,5

* Suppose that A is invertible. We have :

$$A \cdot A^{-1} = I \Rightarrow A^2 A^{-1} = A$$ 0,5

$$A^2 A^{-1} - A A^{-1} = A - I \Rightarrow (A^2 - A) A^{-1} = A - I$$

$$\Rightarrow P(A) \cdot A^{-1} = 0 = A - I$$

$$\Rightarrow A = I \quad \text{contradiction}$$

* Suppose that $A - I$ is invertible.

$$P(A) = A^2 - A = 0 \Rightarrow A(A - I) = 0$$ 0,5

$$\Rightarrow A(A-I)(A-I)^{-1} = 0 \cdot (A-I)^{-1} = 0$$

$$\Rightarrow A = 0 \quad \text{Contradiction.}$$

2) $A = \begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix} \sim \begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix} \quad R_2 \leftarrow R_2 - 2R_1$

A is in row echelon form with one zero entry in his diagonal. 0,5
So A is not invertible.

3) We have $C_3 = -3C_2 \Rightarrow A$ is not invertible.

Or : $R_2 = 2R1 \Rightarrow A$ is not invertible 0,5

Or : $\det A = 0 \Rightarrow A$ is not invertible

Exercise 3 :

1/ $K^2 = -I_4$ 0,25

$K(-K) = I_4 \Rightarrow K$ is invertible with $K^{-1} = -K$.

0,25 0,25 0,25

2/ We have $aI \cdot bK = bK \cdot aI$

Then: $L^2 = (aI + bK)^2 = a^2 I + b^2 K^2 + 2abK$

$= a^2 I - b^2 I + 2abK$

1

$= a^2 I - b^2 I + 2a(bK + aI) - 2a^2 I$

$= -a^2 I - b^2 I + 2aL$

$= -(a^2+b^2) I + 2aL$

3/ We have a and b are not both zero $\Rightarrow a^2+b^2 \neq 0$.

$L^2 = -(a^2+b^2)I + 2aL \Rightarrow L^2 - 2aL = -(a^2+b^2)I$

$\Rightarrow L\left(-\frac{1}{a^2+b^2}(L - 2aI)\right) = I$ 0,25

$\Rightarrow L$ is invertible with: 0,25

$L^{-1} = -\frac{1}{a^2+b^2} L + \frac{2a}{a^2+b^2} I$ 0,25

$\Rightarrow L^{-1} = \underbrace{\frac{a}{a^2+b^2}}_{c} I - \underbrace{\frac{b}{a^2+b^2}}_{d} K$ 0,25

$= cI + dK$

4) $A = \sqrt{2}\, I + K$ 0,25 $\Rightarrow A^{-1} = \frac{\sqrt{2}}{(\sqrt{2})^2 + (1)^2} I - \frac{1}{3} K$

$= \frac{\sqrt{2}}{3} I - \frac{1}{3} K$ 0,25

$$A^{-1} = \begin{pmatrix} \frac{\sqrt{2}-1}{3} & -\frac{1}{3} & \frac{1}{3} & 1 \\ -\frac{1}{3} & \frac{\sqrt{2}-1}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & \frac{1}{3} & \frac{\sqrt{2}}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 0 & \frac{\sqrt{2}+2}{3} \end{pmatrix}$$ 0,1

Exercise 4 :

1. a) Let $\alpha, \beta \in \mathbb{R}$

$$\alpha . 1 + \beta . i = 0 \Rightarrow \alpha = \beta = 0 .$$ 0,75

The $\mathcal{F}$ is linearly independent.

b) We have : $\alpha . 1 + \beta i = 0$ for $\alpha = i$ and $\beta = -1 \neq 0$.

So $\mathcal{F}$ is linearly dependent. 0,75

2. a) * $\mathbb{R}.w \subset \mathbb{C}$ 0,25

* $0_{\mathbb{C}} = 0_{\mathbb{R}} . w \in \mathbb{R}.w$ 0,25

* Let $u, v \in \mathbb{R}.w$ and $\alpha, \beta \in \mathbb{R}$

$u \in \mathbb{R}.w \Rightarrow u = x.w \ , x \in \mathbb{R}$

$v \in \mathbb{R}.w \Rightarrow v = x'.w \ , x' \in \mathbb{R}$

$$\alpha u + \beta v = \alpha x.w + \beta x'.w$$ 1

$$= \underbrace{(\alpha x + \beta x')}_{\in \mathbb{R}} . w \in \mathbb{R}.w$$

Then $\mathbb{R}.w$ is a subspace of the $\mathbb{R}$-vector space $\mathbb{C}$.

b) When $\mathbb{C}$ is considered as a $\mathbb{C}$-vector space,

$\mathbb{R}.w$ is a subspace of $\mathbb{C}$ if and only if $w = 0$. 1

In this case, $\mathbb{R}.w = \{0\}$ is the only subspace of $\mathbb{C}$.

Exercise 5 :

1) * $F \subset \mathcal{C}^2(\mathbb{R}, \mathbb{R})$ 0,25

* $0_{\mathcal{C}^2(\mathbb{R},\mathbb{R})} \in F$ because : $\left(0_{\mathcal{C}^2(\mathbb{R},\mathbb{R})}\right)'' = 0 = (1+x^2) . 0_{\mathcal{C}^2(\mathbb{R},\mathbb{R})}$. 0,25

* Let $\varphi_1, \varphi_2 \in F$ and $\alpha, \beta \in \mathbb{R}$

We denote : $\psi = \alpha \varphi_1 + \beta \varphi_2$

$$\begin{aligned}\psi''(x) &= \alpha\,\varphi_1''(x) + \beta\,\varphi_2''(x)\\ &= \alpha\,(1+x^2)\,\varphi_1(x) + \beta\,(1+x^2)\,\varphi_2(x)\\ &= (1+x^2)\,(\alpha\,\varphi_1(x)+\beta\,\varphi_2(x))\\ &= (1+x^2)\,\psi(x)\end{aligned}$$ 1

Then F is a subspace of $\mathcal{C}^2(\mathbb{R},\mathbb{R})$.

2) $v, w \in F \Rightarrow v''(x) = (1+x^2)\,v(x)$

$w''(x) = (1+x^2)\,w(x)$

$$\begin{aligned}(v w' - w v')' &= \cancel{v'w'} + v w'' - \cancel{w'v'} - w v''\\ &= v(1+x^2)w - w(1+x^2)v\\ &= (1+x^2)(vw - wv)\\ &= 0\end{aligned}$$ 0,5

Then $vw' - wv'$ is constant on $\mathbb{R}$.

3) Show that $f, g \in F$.

- We have, $f'(x) = x\,e^{\frac{x^2}{2}}$ 0,5

$f''(x) = e^{\frac{x^2}{2}} + x^2 e^{\frac{x^2}{2}} = (1+x^2)\,e^{\frac{x^2}{2}} = (1+x^2)\,f(x)$

So $f \in F$

- $$\begin{aligned}g'(x) &= x\,e^{\frac{x^2}{2}}\int_0^x e^{-t^2}\,dt + e^{\frac{x^2}{2}}\left(\int_0^x e^{-t^2}\,dt\right)'\\ &= x\,e^{\frac{x^2}{2}}\int_0^x e^{-t^2}\,dt + e^{\frac{x^2}{2}}\,e^{-x^2}\\ &= x\,e^{\frac{x^2}{2}}\int_0^x e^{-t^2}\,dt + e^{-\frac{x^2}{2}}\end{aligned}$$

$$g''(x) = x\left[x\, e^{\frac{x^2}{2}}\int_0^x e^{-t^2}dt + e^{-\frac{x^2}{2}}\right] + e^{\frac{x^2}{2}}\int_0^x e^{-t^2}dt - x\, e^{\frac{x^2}{2}}$$

$$= x^2 e^{\frac{x^2}{2}}\int_0^x e^{-t^2}dt + \cancel{x e^{-\frac{x^2}{2}}} + e^{\frac{x^2}{2}}\int_0^x e^{-t^2}dt - \cancel{x e^{-\frac{x^2}{2}}}$$

0,5

$$= (x^2+1)\left[e^{\frac{x^2}{2}}\int_0^x e^{-t^2}dt\right]$$

$$= (x^2+1)\, g(x)$$

Then $g \in F$.

* $\{f, g\}$ is linearly independent $\Leftrightarrow \forall \alpha, \beta \in \mathbb{R}$:

$$\alpha f + \beta g = 0 \Rightarrow \alpha = \beta = 0$$

We evaluate $\alpha f + \beta g$ at 0:

$$\alpha f(0) + \beta g(0) = 0 \Rightarrow \alpha + \beta \cdot 0 = 0$$

0,5

$$\Rightarrow \alpha = 0.$$

Then: $\beta\, e^{\frac{x^2}{2}}\int_0^x e^{-t^2}dt = 0 \Rightarrow \beta = 0.$

Because $e^{\frac{x^2}{2}} > 0$ and $\int_0^x e^{-t^2}$ is strictly positive or negative.

0,5

So $\{f, g\}$ is linearly independent.

4) We have $f(x) = e^{\frac{x^2}{2}} \neq 0$.

$$\left(\frac{h}{f}\right)' = \frac{f h' - h f'}{f^2} = \frac{b}{f^2}.$$

1,5

$$\frac{h}{f} = a + b\frac{g}{f} \Rightarrow \left(\frac{h}{f}\right)' = b\left(\frac{g}{f}\right)'$$

We have: $\left(\frac{g}{f}\right)' = \frac{1}{f^2}$, then there exist a and $b \in \mathbb{R}$ such that:

$$\frac{h}{f} = a + b\,\frac{g}{f}.$$

Then: $h = af + bg$.

5) From 4, we have: $h \in F \Rightarrow h \in \text{Span}\{f,g\}$.

So: $F \subset \text{Span}\{f,g\}$ 0,1

It is obvious that $\text{Span}\{f,g\} \subset F$

Then: $F = \text{Span}\{f,g\}$. 0,2

Moreover, from 3) $\{f,g\}$ is linearly independent. 0,2

It follows that $\{f,g\}$ is a basis for F.

Finally $\dim F = 2$. 0,1