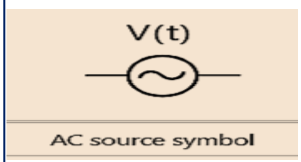
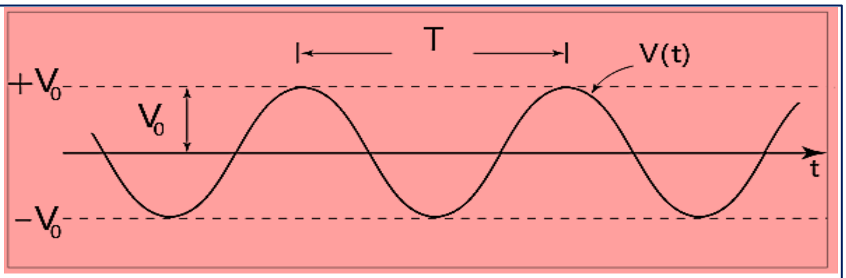
Chapter 2 ALTERNATING CURRENT (AC current)

I. Introduction to AC Circuits

- When a loop of wire rotates inside a magnetic field, Faraday's law predicts that the changing magnetic flux induces **an emf (electromotrice force)** that oscillates sinusoidally in time with a frequency determined by the angular speed of the coil.
- This is a source of *alternating current* (AC). More precisely, the emf in the coil is **a source of alternating *voltage***, which will create alternating current in whatever circuit is connected to the coil. The symbol used to represent an AC source is shown below.





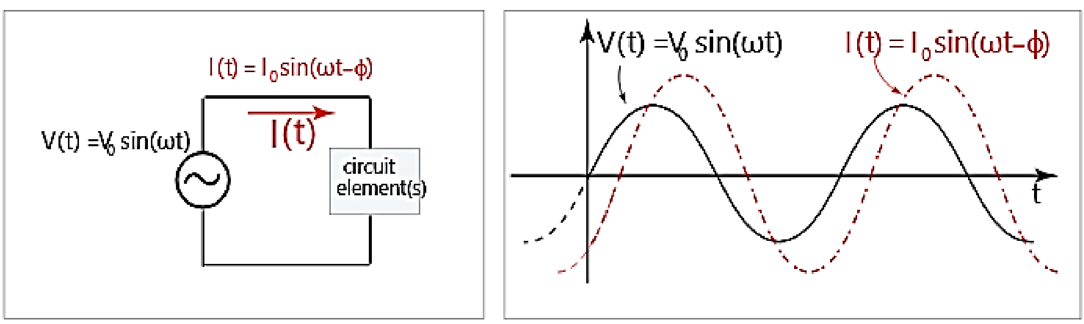
The AC voltage source is a sinusoidal function of time with a **frequency *f*** (or **pulsation: ω**) and **Amplitude *V*$_0$**, determined by the characteristics of the electric generator (the rotating coil in the B-field). **Unit of frequency**: **Hz** and **unit of pulsation**: **rd/s**

A plot of the voltage as a function of time, *V*(*t*)

The period $\boxed{T = \frac{1}{f} = \frac{2\pi}{\omega} \quad (s: second)}$

$$(\omega = 2\pi f)$$

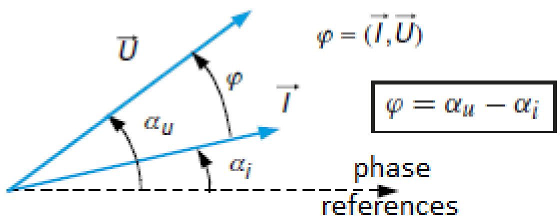
The AC source is connected to an electrical circuit where it can contain a resistor, a capacitor, an inductor, or a combination of these elements in series or in parallel. The current $I(t)$ flows through the circuit. If the source is oscillating with an angular frequency ω , we expect the current to oscillate with the same frequency, therefore the current is also a sinusoidal function of time given by: $\boxed{I(t) = I_0 sin(\omega t - \phi) \; or \; I(t) = I_0 sin(\omega t + \phi)}$.



where I_0 is the amplitude and ϕ the **phase shift** of the current with respect to the voltage. The choice of a minus sign will be explained below. The angular frequency is called the **driving frequency**, the voltage source is **driving** the circuit with frequency ω.

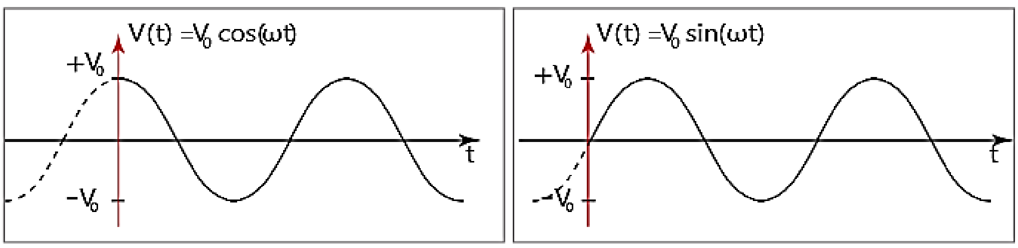
Specific parameters of AC voltage:

- ***Angular frequency (ω)***, also known as radial or circular frequency, measures angular displacement per unit time ($\omega = 2\pi f = \frac{2\pi}{T}$). It's units are therefore degrees (or radians) per second : $rads^{-1}$.
- ***Regular or linear frequency (f)***, sometimes also denoted by the Greek symbol "nu" (ν), counts the number of complete oscillations or rotations in a given period of time. Its units are therefore cycles per second (cps), also called ***hertz (Hz)***.
- The current is not in phase with the voltage. Since the current peaks *after* the voltage, we say that it "**lags**" the voltage (the opposite shift is called "**leading**" the voltage).
- The shift between the two curves is given by **the phase shift φ**.

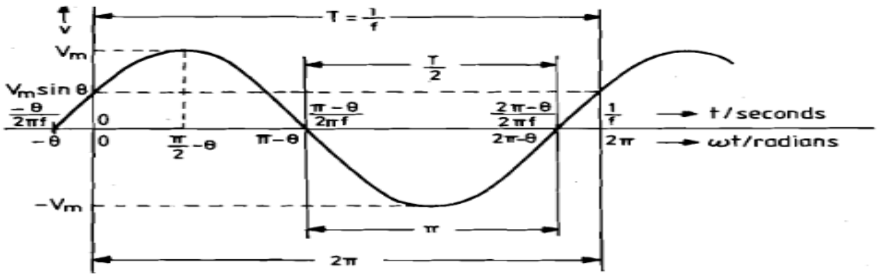


In summary:

- In an AC circuit, the voltage oscillates with an angular frequency ω and if we use it as a reference signal we can describe it as $V(t)=V_0\cos(\omega t)$ or $V(t)=V_0\sin(\omega t)$ depending on our choice of zero for the time.
- The resulting current is not necessarily in phase with the driving voltage, therefore we will assume that is expressed as $I(t)=I_0\cos(\omega t-\varphi)$ or $I(t)=I_0\sin(\omega t-\varphi)$, with the choice of triangular function being the same for voltage and current.



Example :



Waveform of a general sinusoidal voltage $v\ (t) = V_{,m}\, sin\ (\omega t\ +\ \theta)$

- ***Root Mean Square current*** (Effective current).

The Root Mean Square (RMS) value of an alternating current (or alternating voltage) is defined as the square root of the average of the square of the intensity (or voltage) calculated over one period. It is written as :

$$I_{eff} = \sqrt{\frac{1}{T}\int_0^T i(t)^2\, dt} \quad \text{and} \quad V_{eff} = \sqrt{\frac{1}{T}\int_0^T v(t)^2\, dt}$$

In the case of a sinusoidal alternating current, we obtain:

$$\boxed{I_{eff} = \frac{I_{max}}{\sqrt{2}} = 0.707.I_{max} \quad \text{and} \quad V_{eff} = \frac{V_{max}}{\sqrt{2}} = 0.707.V_{max}}$$

$V_{max}$and $I_{max}$are known as the peak values of the alternating voltage and current respectively.

The instantaneous value of such a current and voltage are then written :

$$\boxed{i(t) = I_{eff}\sqrt{2}\sin(\omega t + \varphi) \qquad and \qquad v(t) = V_{eff}\sqrt{2}\sin(\omega t + \varphi}$$

The mean value of a periodic signal $v(t)\ or\ i(t)$ of period T is given by:

$$\boxed{V_{moy} = \bar{V} =< V >= \frac{1}{T}\int_0^T v(t)\, dt \quad \text{and} \quad I_{moy} = \bar{I} =< I >= \frac{1}{T}\int_0^T i(t)\, dt}$$

II. Magnetic field and Electromagnetic induction

Throughout this part of chapter, we'll be looking at the case of **magnetic fields** created by fixed wire-shaped circuits of simple geometry, through which permanent currents flow (electric charges are in motion, but the intensity of the electric current does not depend on time). The fields created are not time-dependent, and this area of physics is generally referred to as **magnetostatics** (by analogy with electrostatics. However, to simplify the language, the term magnetostatic field is rarely used, **in favor of magnetic field**, which is also the case here.

Question: **The important question is: how do you generate a magnetic field from permanent currents?**

Experience 1: It all began with **Oersted's experiment in 1820**. He placed a conducting wire over a compass and passed a current through it. In the presence of a current, the compass needle was indeed deflected, unambiguously proving a link between electric current and magnetic field. He also observed:

- If you reverse the direction of the current, the deflection changes direction.
- The force deflecting the needle is non-radial

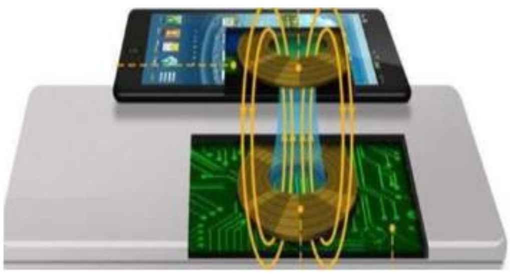
Remember that Oersted in 1820, discovered that a steady current produce a steady magnetic field and that connected electricity with magnetism.

A little time later, Faraday therefore suggest that maybe a steady magnetic field produce a steady current

II.1. Magnetic field applications

Electric and magnetic forces both act only on particles carrying charge. Moving electric charge create a magnetic field. A changing magnetic field creates a magnetic field, this elect is called **magnetic induction**. This links electricity and magnetism in a fundamental way. Magnetic induction is also the key to many practical applications

- ***Induction charging*** :
  - Inductive recharging requires no physical connection between the charger and the device.
  - is based on the principle of electrical induction, whereby the circulation of an electric current in a copper coil (wire winding) creates a current in a nearby coil. When the device to be recharged is placed on the charger, their proximity is such that the magnetic field created induces an electric current in the "receiver" coil, powering its battery.

- ***Electric motors*** : The electric motor consists of two main components:
  - The stator is a ferromagnetic frame containing electrical windings (coils). When an electric current passes through a coil, which becomes an electromagnet, it creates a magnetic field rotating inside the stator.
  - The rotor is the rotating element at the centre of the motor, and is subjected to the magnetic field created by the stator, transforming its power into mechanical power.

- ***Magnetic levitation trains*** : A magnetic levitation train is a train that uses magnetic forces to levitate and move forward.

II.2. ELECTROMAGNETIC INDUCTION

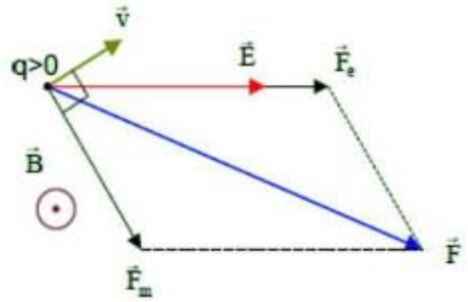
The term electromagnetic induction refers to the production of currents, and therefore of emf, from magnetic fields; we speak of induced currents and induced emf. Electromagnetic induction is responsible for the operation of generators and transformers, and for the production of electromagnetic waves such as light and radio waves. So, the Electromagnetic induction is the phenomenon in which electric current induced in a conductor by varying magnetic field.

1- Lorentz force

The total force, electrical and magnetic (known as electromagnetic) experienced by a particle of charge q and velocity measured in a Galilean frame of reference is:

$$\boxed{\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}}$$

The expression of the Lorentz force can be regarded the definition of electric $\vec{E}$ and magnetic $\vec{B}$ fields. The magnetic field $\vec{B}$, unlike the electric field $\vec{E}$ does not exert any force on a stationary charge.



In the case where a charge at rest is surrounded by point charges also at rest, the charge q undergoes an electric force: $\vec{F_e} = q\vec{E}$.
In the case where the various charges are in motion, we see the appearance of a force different from $\vec{F_e}$ and which depends on the charge speed $\vec{v}$ of the magnetic field created by all the moving charges other than q . The expression of this force is :

$$\boxed{\vec{F}_{magn} = q.\vec{v} \times \vec{B}}$$

This force is called the **magnetic force**, or the **magnetic part of the Lorentz force**. The magnetic field is defined by its action on a charged particle, of electric charge q , moving at speed in a reference frame. The magnitude is given by:

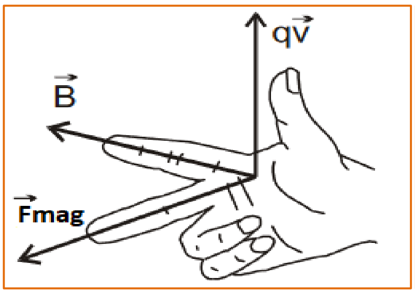
$$\boxed{F_{magn} = q.v\,.B\sin(\theta)}$$

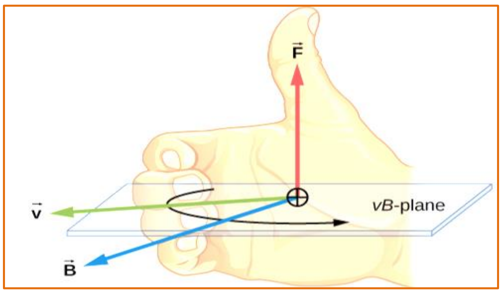
Where q is the electric charge (C), v is the speed charge (m/s), B is the magnitude of magnetic field vector (Tesla: T) and α formed by $\vec{v}$ and $\overrightarrow{B}$. The SI unit of magnetic field is called the **Tesla** (T): the Tesla equals a $Newton/(coulomb \times meter/sec)$. A smaller unit, called the **gauss** (G) is sometimes used, where $1G = 10^{-4}\,T$.

➢ *Direction of the Magnetic Field by the Right-Hand Rule (Fleming's Right Hand Rule)*

The direction of magnetic Lorentz force is determined by the three-finger rule of the right hand (see figure below (a)):

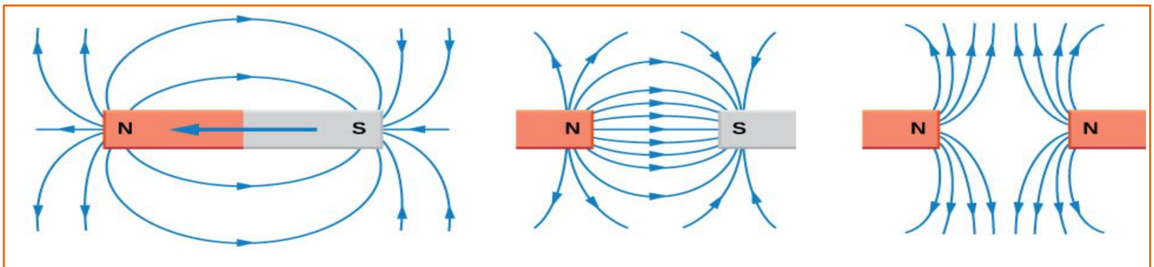
- Thumb: direction of v_q(= direction of $q\vec{v}$, if $q > 0$; = opposite direction to $q\vec{v}$, if $q < 0$),
- Index finger: direction of $\vec{B}$
- Middle finger: direction of $\vec{F}_{mag}$





(a) (b)

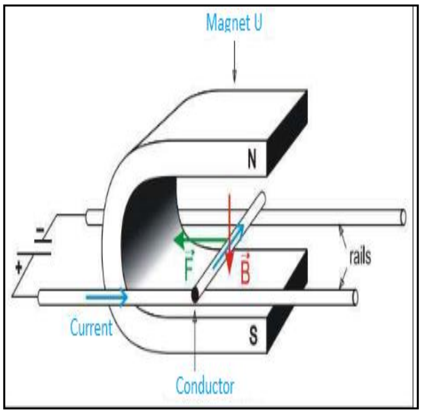
So, the magnetic fields exert forces on moving charges. The direction of the magnetic force on a moving charge is perpendicular to the plane formed by $\vec{v}$ and $\vec{B}$ (vB plane) and follows the right-hand rule (RHR-1) as shown (b). The magnitude of the force is proportional to q, v, B, and the sine of the angle between $\vec{v}$ and $\vec{B}$. As shown in Figure below, each of the **magnetic field lines** forms a closed loop. The field lines emerge from the north pole (N), loop around to the south pole (S), and continue through the bar magnet back to the north pole.



Magnetic field lines of a bar magnet (a) — Magnetic field lines between unlike poles (b) — Magnetic field lines between like poles (c)

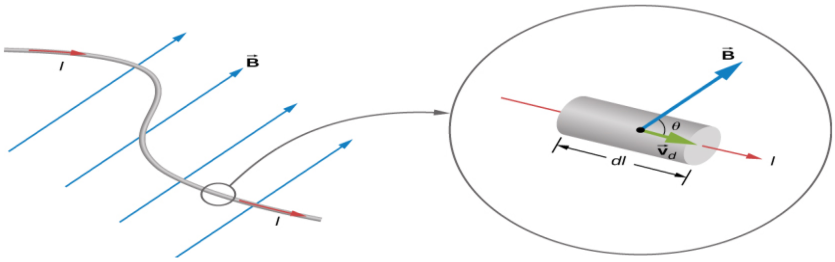
2- Force on a current-carrying wire

When a wire carrying an electric current is placed in a magnetic field, each of the moving charges, which comprise the current, experiences the Lorentz force, and together they can create a macroscopic force on the wire (sometimes called the **Laplace force**). By combining the Lorentz force law above with the definition of electric current, the following equation results, in the case of a straight stationary wire in a homogeneous field: $\boxed{\vec{F} = I.\,\vec{L}\mathrm{x}\overrightarrow{B}}$ **or** $\boxed{\overrightarrow{dF} = I.\,\overrightarrow{dl}\mathrm{x}\overrightarrow{B}}$



The magnitude of the force on a wire carrying current I with length L in a magnetic field is given by the equation:

$\boxed{F = I.L.B.\,sin\theta}$



When the magnetic force relationship is applied to a current-carrying wire, the right-hand rule is used to determine the direction of force on the wire. **This force is perpendicular to the plane formed by the field $\vec{B}$ and the element of current $\vec{L}$.**

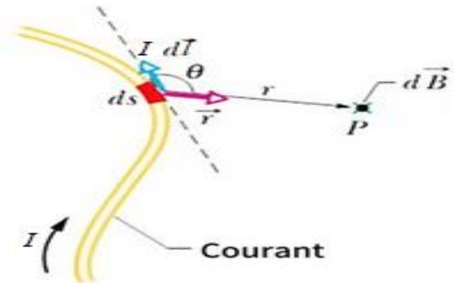
2-1. Magnetic field created by a straight current

- ***Definition of Magnetic field created by an electric current (H): BIOT AND SAVART'S LAW:***

An electric circuit (C), through which a current flows, causes, by “induction”, the appearance of an excitation magnetic field H at any point P in space such as M, located at a distance r from an element $\boldsymbol{dl}$ of the circuit. The magnetic induction vector B (expressed in tesla, T) is given by the BIOT and SAVART formula.

$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{l} \times \vec{r}}{r^2}$ and the magnitude is equal :

$$\boxed{dB = \frac{1}{4\pi}\frac{Idl.sin\theta}{r^2} \quad (Tesla : T)}$$



H : the magnetic excitation (SI unit A/m), r: distance between point P and the portion dl, $\mu_0 = 4\pi 10^{-7} T \cdot m/A$: Magnetic permeability of vacuum. The two quantities are linked by the scalar relationship:

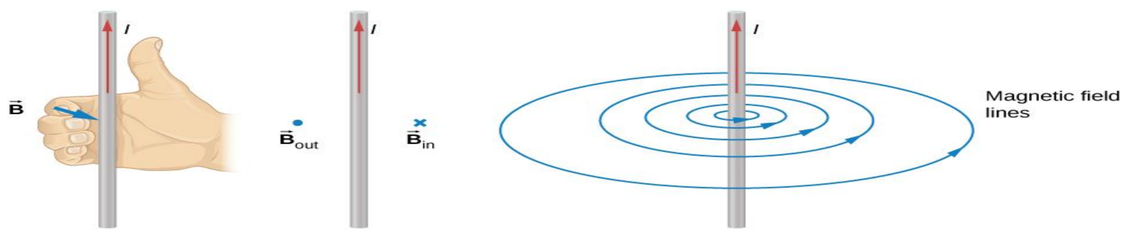
$B = \mu \times H \quad [T]$	$\mu_0 = 4\pi 10^{-7}$: Magnetic permeability of vacuum.
$\mu = \mu_0 \times \mu_r$	μ_r: Relative magnetic permeability of the material

A rectilinear conductor with a current flowing through it creates a magnetic field . in space, such that :

- The field lines are circles centered on the conductor.
- The vector is perpendicular to the conductor.
- Its direction is given by the right-hand rule.

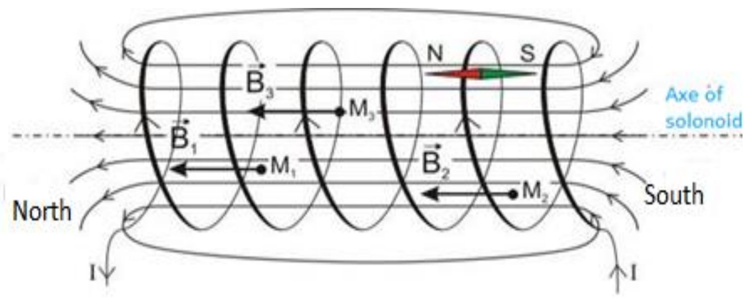
$$\boxed{B = \frac{\mu_0 I}{2\pi R}}$$

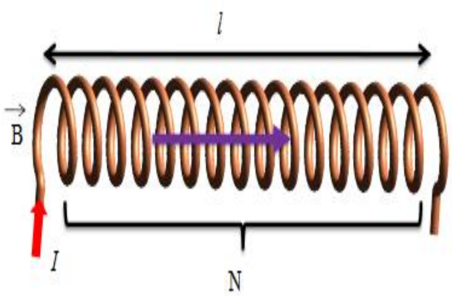
where I: current intensity in A, d: distance between point P and the conducting wire, $\mu_0 = 4\pi 10^{-7} T \cdot m/A$: Magnetic permeability of vacuum and B in tesla (T) and is measured with a teslameter.



- ***2.2 Magnetic field created by a solenoid*** **(Ampère Law)**

A solenoid is a long helical coil of wire through which a current is run in order to create a magnetic field. Inside a solenoid, the magnetic field is uniform.





The current enclosed in the loop will be the number of turns N in the length L that go thru the loop multiplied by the current I in each coil. It given by:

$$\boxed{B = \mu_0 I n} \quad \text{Or} \quad \boxed{B = \mu_0 \frac{N}{L} I}$$

Where n: number of turns per meter N: total number of turns of the solenoid L : length of solenoid. The absolute permeability for other materials can be expressed relative to the permeability of free space as: $\mu = \mu_r \mu_0$; Where μ_r is the relative permeability which is a dimensionless quantity.

Permeability and Relative Permeability of Materials

Material	Permeability (μ) (H.m-1)	Relative Permeability (μr)
Air	1.257×10^{-6}	1.000
Copper	1.257×10^{-6}	0.999
Vacuum	$4\pi \times 10^{-7}$	1
Water	1.256×10^{-6}	0.999
Wood	1.257×10^{-6}	1.00

3- Induction Electromagnetic

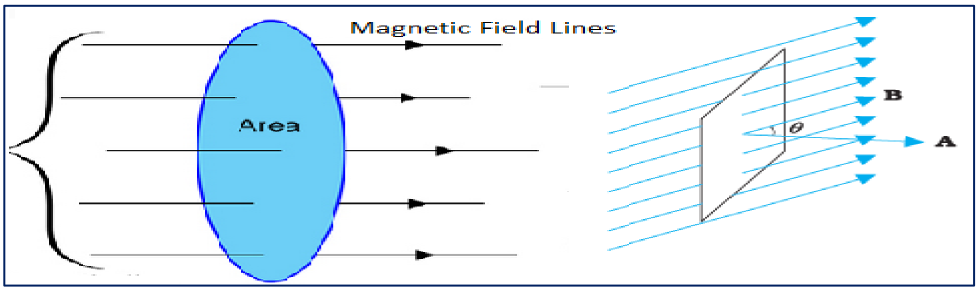
Up to now, we've been mainly interested in the creation of a magnetic field from a permanent current. This was motivated by **Oersted's experiment**. At the same time, **the English physicist Faraday** was preoccupied with the opposite question: since these two phenomena are linked, how can a current be produced from a magnetic field? He carried out a number of experiments, but **failed because he was trying to produce a permanent current**. In fact, he did notice some disturbing effects, but they were always **transient**.

3.1. Magnetic Flux

The current induced by a variation in flux in a conductive loop generates a magnetic field whose effects are such as to oppose the movement that induced the current. Faraday's great insight lay in discovering a simple mathematical relationship to explain the series of experiments he carried out on electromagnetic induction. First, we need to explain the concept of magnetic flux, ϕ_B.

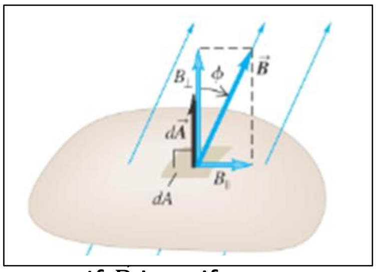
Definition of magnetic Flux

Magnets and electromagnets produce a magnetic field represented by imaginary lines known as magnetic field lines or magnetic lines of force. ***Magnetic flux is the number of lines passing through a given area, whether in air or vacuum or inside a magnetic material***. The magnetic flux is analogous to the electric flux.



The magnetic flux is a scalar quantity and can be quantified by establishing an imaginary surface represented by an area vector in the vicinity of a magnetic field. The magnetic flux ϕ is given by the dot product of the magnetic field and the area vector:

$$\boxed{\phi_B = \int_A \vec{B}.\overrightarrow{dA} = \int B\cos\varphi . dA}$$



Where φ: the angle between the magnetic field vector and the area vector. If B is uniform over the flat of area, then the magnetic flux is given by:

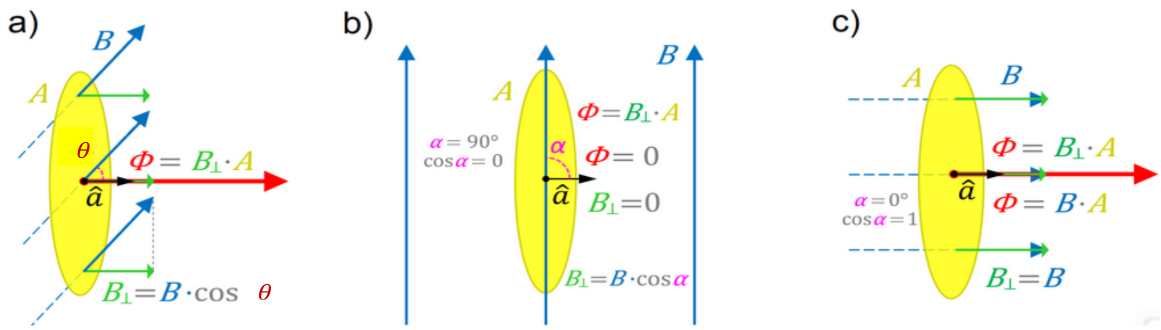
$$\boxed{\phi_B = B.A.cos\varphi}$$

The SI unit of magnetic flux is Weber (Wb), One weber is the amount of magnetic flux over an area of 1 meter held normal to a uniform magnetic field of one tesla. Thus:

$$\boxed{1\,weber = 1\,tesla \text{ x } 1\,m^2 = 10^8\text{maxwell}}$$

If the coil has **N turns**, total amount of magnetic flux linked with the coil is:

$$\boxed{\phi_B = \mathbf{N.B.A\cos\varphi}}$$

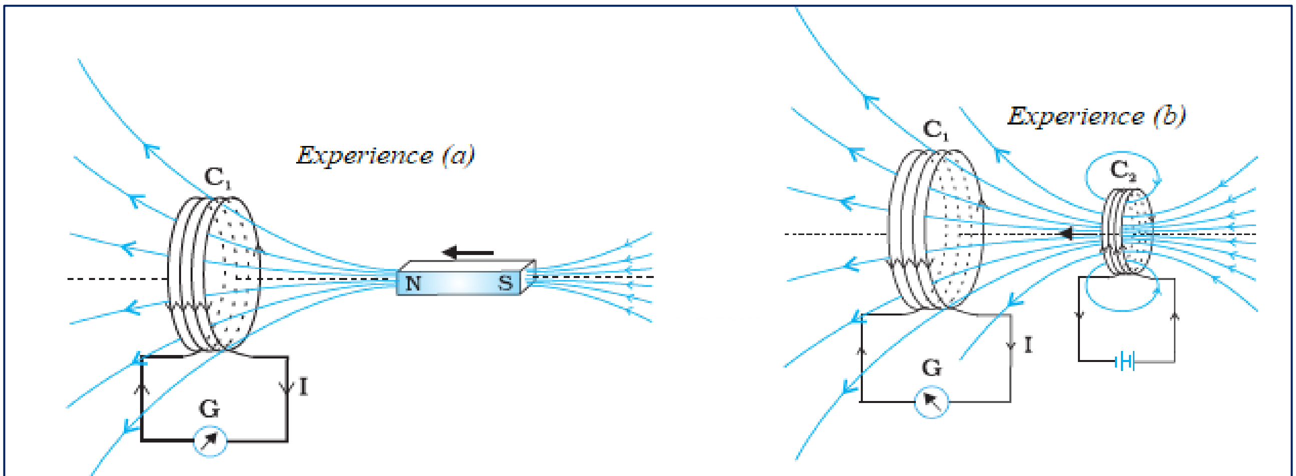


3.2. Induction laws (Faraday law and Lenz law)

These are two laws of induction: (1) **Faraday's** law defining the induced emf, and (2**) Lenz's law** concerning the direction of the induced current. These two closely related laws are based on magnetic flux.

A- FARADAY'S LAW OF INDUCTION

From the experimental observations, Faraday arrived at a conclusion that an emf is induced in a coil when magnetic flux through the coil changes with time. The discovery and understanding of electromagnetic induction are based on a long series of experiments carried out by Faraday and Henry (see figure below).



When the bar magnet is pushed towards the coil the pointer in the galvanometer G deflects.

Current is induced in coil C_1 due to motion of the current carrying coil C_2.

Interpretation of Faraday:

1- The change in magnetic flux $\frac{d\phi}{dt}$ induces emf in coil C_1. It was this induced emf which caused electric current to flow in coil C_1 and through the galvanometer.
2- The common point in all these observations is that the time rate of change of magnetic flux through a circuit induces emf in it.

Faraday stated experimental observations in the form of a law called Faraday's law of electromagnetic induction. The law is stated below:

The magnitude of the induced emf (Motional ElectroMotive Force) *in a circuit is equal to the time rate of change of magnetic flux through the circuit.*

Mathematically, the induced emf $\varepsilon(t)$ is given by:

$$\boxed{\varepsilon = -\frac{d\Phi_B}{dt}}$$

However, **that direction is most easily determined** with a rule known as **Lenz's law**, which we will discuss shortly

Faraday's Law indicates how to calculate the potential difference that produces the induced current. The magnitude of the induced emf equals the rate of change of the magnetic flux

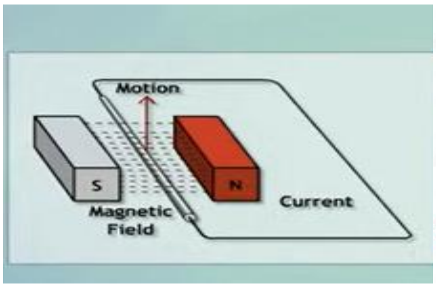
The induced electromotive force is expressed in **volts** and the magnetic flux in **Weber**.

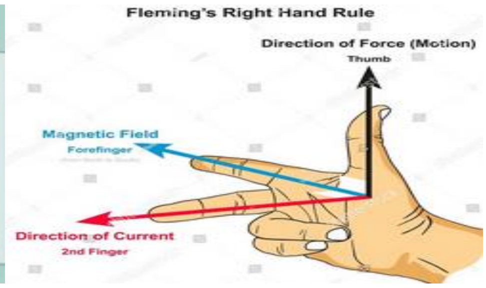
In the case of a closely wound coil of *N* turns, change of flux associated with each turn, is the same. Therefore, the expression for the total induced emf is given by :

$$\boxed{\varepsilon(t) = -N\frac{d\phi_B}{dt}} \qquad \boxed{\phi_{solonoid} = N.\,\phi_{spire}}$$

B- Lenz's Law

The current induced by a flux variation in a conductive loop generates a magnetic field whose effects are such as to oppose the movement that induced the current. ***The polarity of induced emf is such that it tends to produce a current which opposes the change in magnetic flux that produced it.***





Faraday's law can be used to determine the intensity of the induced emf and to deduce the intensity of the induced current using Ohm's law:

$$\boxed{I_{\ induit} = \frac{|\varepsilon|}{R}}$$

where R is the resistance of the conductor loop

Changing the current in the right-hand coil induces a current in the left-hand coil. The induced current does not depend on the size of the current in the right-hand coil.

III. INDUCTANCE

The **inductance** of an electrical circuit is a coefficient that reflects the fact that a current flowing through it creates a magnetic field across the section surrounded by the circuit. As a result, the magnetic field flows through the section bounded by the circuit. The **inductance** is equal to the quotient of the flux of this magnetic field and the intensity of the current flowing through the circuit or electric dipole. **The unit of inductance is the Henry (H**). These dipoles are generally coils, often called **inductors** or **chokes**.

$$\frac{d\Phi_B}{dt} \propto \frac{dI}{dt}$$

III.1. Self-inductance

The emf is induced in a single isolated coil due to change of flux ϕ_B through the coil by means of varying the current I through the same coil. This phenomenon is called ***self-induction***. In this case, flux linkage through a coil of N turns is proportional to the current through the coil. The self-inductance L of the electric circuit is then defined as the ratio between the flux embraced by the circuit and the current and is expressed as:

$$N\Phi_B \propto I$$

$$N\Phi_B = L\,I \qquad \text{So : } \boxed{L = \frac{N\phi_B}{I} \ \ if\ N = 1 \ \rightarrow \ L = \frac{\phi_B}{I}}$$

When the current is varied, the flux linked with the coil also changes and an emf is induced in the coil. The induced emf is given by:

$$\boxed{\varepsilon(t) = -\frac{d\phi_B}{dt}\ so\ \varepsilon(t) = -L\frac{dI}{dt} \rightarrow V_L(t) = L\frac{dI}{dt}}$$

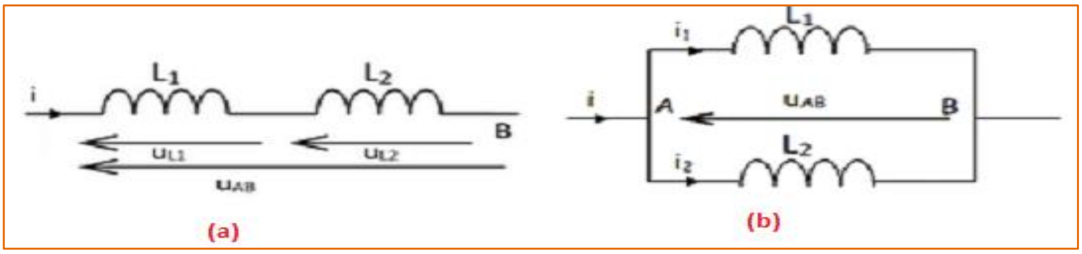
➢ **SELF INDUCTANCE OF A LONG SOLENOID**

We have $B = \mu_0 \frac{N}{lenght} I$, The Total magnetic flux linked with the solenoid $\phi_B = B.A = LI$ so the self-inductance of solenoid is given by: $\boxed{L = \mu_0 \frac{N^2}{lenght}.A}$. If core is of any other magnetic material, μ_0 is replaced by μ, where $\boxed{L = \mu_0.\mu_r \frac{N^2}{lenght}.A}$

The symbol for an *inductor :*  *If the coil is wrapped around an iron core so as to enhance its magnetic effect, it is symbolised by putting two lines above it, as shown here*

III.2. Grouping of coils

The grouping of inductances (coils) can be in series or in parallel (see following figure).



Coils in series: Coils equivalent to series coils are added together: in effect, the magnetic fields are added together (figure (a)).So the equivalence inductance is given by: $= L1 + L2$. For N coils we obtain :

$$\boxed{L_{eq} = \sum_{i=1}^{N} L_i}$$

➢ *Coils in parallel:* The inverse of the equivalent inductance of parallel coils is the sum of the inverses of each of the inductances (figure (b)). So, $\frac{1}{Leq} = \frac{1}{L1} + \frac{1}{L2}$. For N coils we obtain:

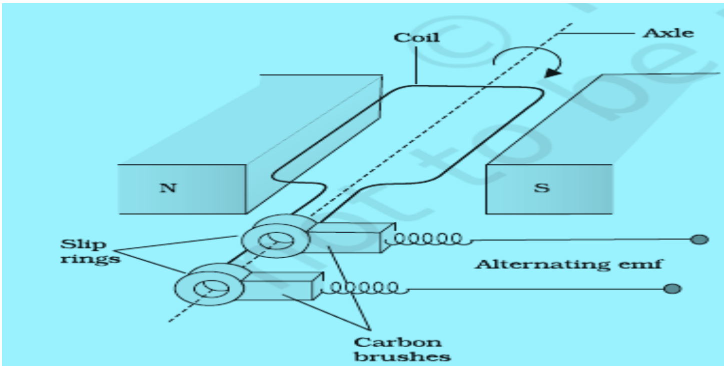
$$\boxed{\frac{1}{L_{eq}} = \sum_{i=1}^{N} \frac{1}{L_i}}$$

IV. AC GENERATOR

The phenomenon of electromagnetic induction has been technologically exploited in many ways. An exceptionally important application is the generation of alternating currents (ac). The main application of Faraday's law is undoubtedly the electric generator or **Alternator** (dynamo). transforms mechanical energy into electrical energy.

The mechanical energy supplied to the generator rotates its axis⇒ turn drives a conductor (several, in fact) between the poles of a magnet.

The result is a magnetic flux through the coil, and an e.m.f. and current are induced in the conductor.



When the coil is rotated with a constant angular speed ω, the angle θ between the magnetic field vector B and the area vector A of the coil at any instant t is $\theta = \omega t$ (assuming θ = 0° at t = 0). If the generator pivot is rotated with a constant angular velocity, ω, we have : $\omega = \frac{d\theta}{dt}$ $\boxed{\theta = \omega t}$

The flux at any time t is given by :

$$\boxed{\phi_B = BA\cos\theta = BA\cos\omega t}$$

The emf induced in such a generator can be calculated using Faraday's law:

$$\boxed{\varepsilon = -N\frac{d\phi_B}{dt} = -N\frac{d}{dt}(BA\cos\theta) = -N\frac{d}{dt}(BA\cos\omega t)}$$

Thus, the instantaneous value of the emf is:

$$\boxed{\varepsilon(t) = NBA\omega \sin\omega t = \varepsilon_0 \sin(\omega t)}$$

where $\boldsymbol{\varepsilon_0}$=$\boldsymbol{NBA\omega}$is the maximum value of the emf.

The emf is an alternating emf that varies sinusoidally with time. The direction of the current changes periodically and therefore the current is called ***alternating current*** **(ac).** Since $\omega = 2\pi f$, So, we can be written as:

$$\boldsymbol{\varepsilon(t) = \varepsilon_0 sin(2\pi f t)}$$

where $\boldsymbol{f}$ is the frequency of revolution of the generator's coil. The alternative voltage is $\boxed{V(\boldsymbol{t}) = \boldsymbol{V_M sin(\omega t)}}$

V. AC circuits (hase shift, Fresnel representation, phasors and reactance).

V.1. Complex Numbers in AC circuits

The complex numbers may be used to analyze and compute ***currents and voltages in AC*** circuits. The **resistance**, the **impedance of a capacitor** and the **impedance of an inductor** are represented by complex numbers. It is also shown how the use of complex impedances allows the use of a law similar to Ohm's law in order to mathematically model AC circuits. Two main reasons that make the use of complex numbers suitable to model AC circuits, and many other sine wave phenomena in several branches of engineering, are:

1) The AC signals (and many other sine wave phenomena) are characterized by a magnitude and a phase that are, respectively, very similar to the modulus and argument of complex numbers.
2) The basic operations such as addition, subtraction, multiplication and division of complex numbers are easier to carry out and to program on a computer.

Note:

1) Because the symbol i is used for currents in AC circuits, here we use j as the imaginary unit defined by $j^2 = -1$ or $= \sqrt{-1}$.

2) The symbol **Re** represents the real part of a complex number and **Im** represents the imaginary part.

A complex number in standard form $Z = a + jb$ may be written

- In exponential form as follows $Z = re^{j\theta}$ with $j^2 = -1$
- In polar form as follows : $Z = r\angle\theta$

where $r = \sqrt{a^2 + b^2}$ is the **modulus of** Z or **Magnitude** of Z and $\boxed{tan\theta = \frac{b}{a} \ \ so \ \ \theta = \arg(Z) = tan^{-1}\frac{b}{a}}$ its **argument**.

Take the real part, written as Re , of each side of a complex number in exponential form

$$Re(re^{j\theta}) = Re(r(cos\theta + jsin\theta)) = Re(rcos\theta + jrsin\theta) = rcos\ \theta$$

- **Phasor Diagramm :** use complex numbers to represent the important information from the time functions (magnitude and phase angle) in vector form.

Phasor Notation

$\mathbf{V} = V_{RMS}\angle\theta°$ or $\overline{V} = V_{RMS}\angle\theta°$

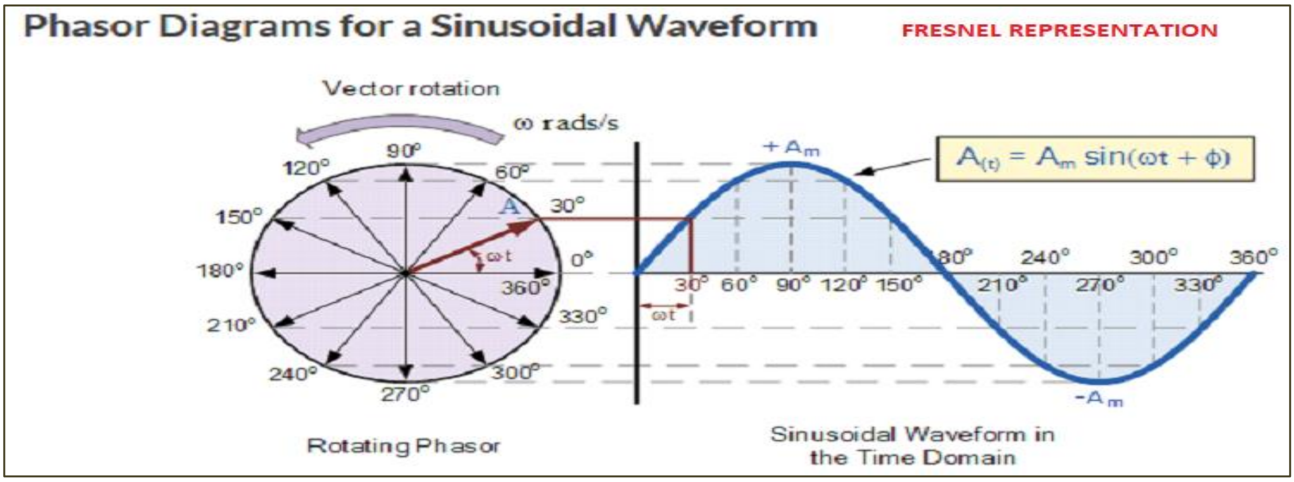
$\mathbf{I} = I_{RMS}\angle\theta°$ or $\overline{I} = I_{RMS}\angle\theta°$

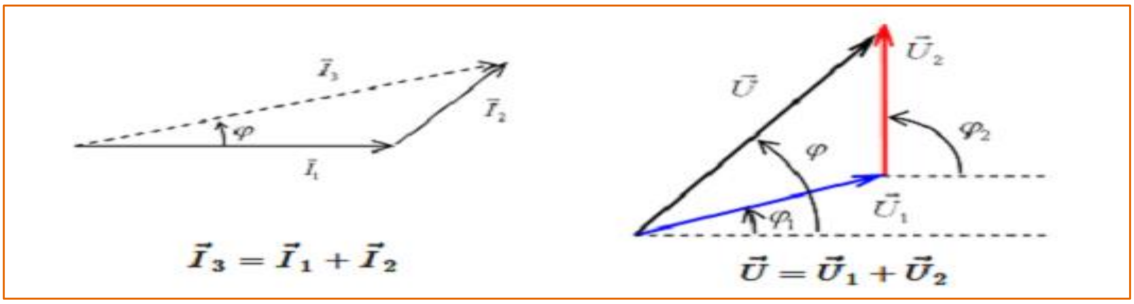
Where: V_{RMS}, I_{RMS} = RMS magnitude of voltages and currents
θ= phase shift in degrees for voltages and currents

- **Phasor Representation (Representation Fresnel)**

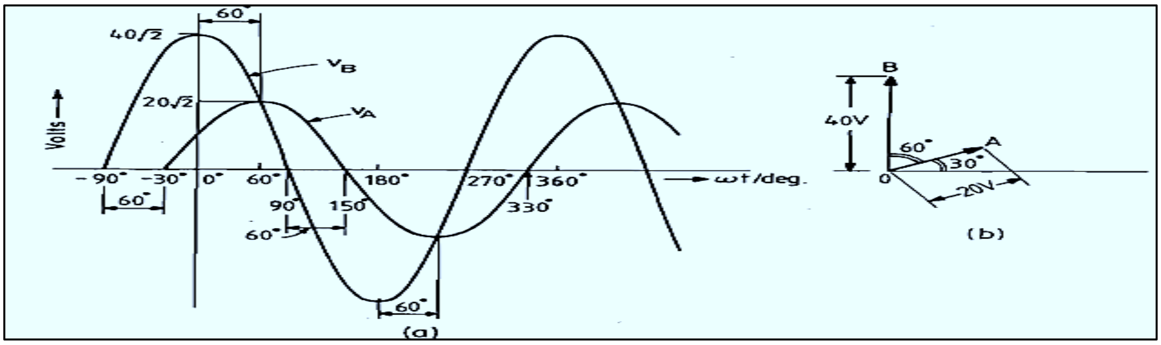
The advantage of the Fresnel representation is that it makes it easy to sum two sinusoidal quantities of the same pulsation. In electricity, this representation makes it easy to find:

- a voltage by means of a loop law
- or a current by means of a junction law.

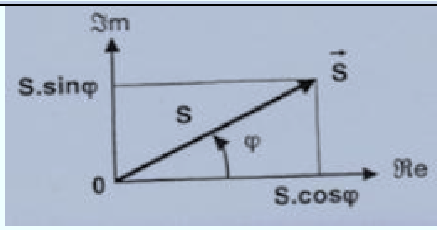




Example, consider two voltages $v_A(t) = 20\sqrt{2}\,sin\,(wt + 30°)$ and $v_B(t) = 40\sqrt{2}\,sin\,(wt + 90°)$ whose waveforms are shown in Figure. Two sinusoids with a phase difference of 60' (a) Waveforms (b) Phasors

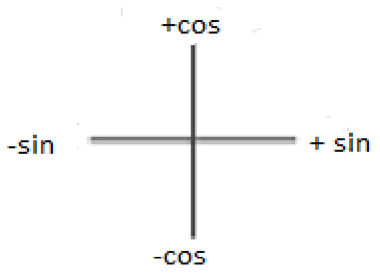


- **Representation of a sinusoidal quantity**

Temporel	Complexe	Vectoriel (Phasor: Fresnel)
$s(t) = \sqrt{2}\, S_{eff}\ \sin(\omega t+\varphi)$ S_{eff} : Effective Value: rms ω: Angle frequency (rd/s) $\omega=2\pi f$ φ: Initial angle shift	$S = S_{eff}\ \varphi^{\circ}$ S: **Complexe Value** S_{eff} : Magnitdue efefctive Value φ°: argument	

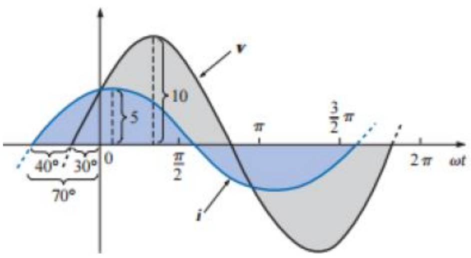
- ***Definition of Phasor***

A phasor is a rotating line whose projection on a vertical axis can be used to represent sinusoidally varying quantities. To get at the idea, consider the line of length V_m shown in Figure (It is the phasor.). The vertical projection of this line (indicated in dotted line) is $V_m \sin(\omega t)$, We assume that the phasor rotates at angular velocity of ωrad/s in the counterclockwise direction. The geometric relationship between various forms of the sine and cosine functions can be derived from Figure below.

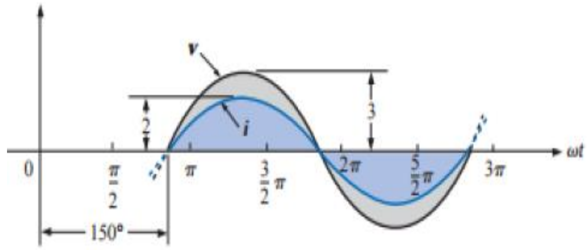


$$\cos\alpha = \sin(\alpha + 90^{\circ})$$
$$\sin\alpha = \cos(\alpha - 90^{\circ})$$
$$-\sin\alpha = \sin(\alpha \pm 180^{\circ})$$
$$-\cos\alpha = \sin(\alpha + 270^{\circ}) = \sin(\alpha - 90^{\circ})$$

The phase shift between two waveforms indicates which one **leads** or **lags**, and by how many *degrees or radians*. Phase difference refers to the angular displacement between different waveforms of the *same frequency*. If the angular displacement is 0°, the waveforms are said to be ***in phase***, otherwise, they are ***out of phase***.



i(t) leads v(t) by 40°

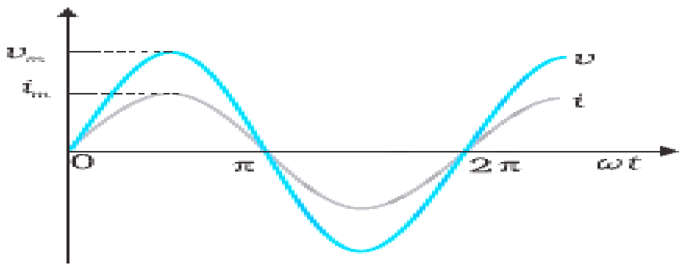


v(t) and i(t) in phase

V.2. VOLTAGE APPLIED TO A RESISTOR

The Figure shows a resistor connected to a source ε of AC voltage (~). We consider a source which produces sinusoidally varying potential difference across its terminals. Let this potential difference, also called ac voltage, be given by: $v(t) = v_M \, sin\omega \, t$ where v_M is the amplitude of the oscillating potential difference and ω is its angular frequency.

To find the value of current through the resistor, we apply Kirchhoff's loop rule $\sum \varepsilon(t) = 0$. So $v_m \sin(\omega t) = R.i(t)$ or $i(t) = \frac{v_m}{R}\sin(\omega t) = i_m \sin(\omega t)$. Since R is a constant, $i_m = \frac{v_m}{R}$ the peak amplitude of current. This equation is Ohm's law, which for resistors, works equally well for both AC and DC voltages. The voltage across a pure resistor and the current through it, are plotted as a function of time as below:



We note, that both *v(t)* and $i(t)$ reach zero, minimum and maximum values at the same time. So, ***the voltage* and *current are in phase*** *with each other*.

- ***Power dissipated in AC circuit***

There is Joule heating and dissipation of electrical energy when an AC current passes through a resistor. The instantaneous power dissipated in the resistor is:

$$\boxed{P = R.i(t)^2 = R.i_m^2 \, sin^2(\omega \, t)} \; (Watt)$$

The average value of *P* over a cycle is:

$$\bar{P} = < R.i(t)^2 > \; = < R.i_m^2 \, sin^2(\omega \, t) > = \frac{1}{T}\int_0^T p(t)dt$$

where the bar over a letter (here, p) denotes its average value and <......> denotes taking average of the quantity inside the bracket. Since, R and i_m^2 are constants, so:

$$\bar{p} = R.i_m^2 < sin^2(\omega\ t) > = \frac{R.i_m^2}{T}\int_0^T sin^2(\omega\ t)dt = \frac{R.i_m^2}{T}\int_0^T sin^2(\omega\ t)dt$$

$$= \frac{R.i_m^2}{T}\int_0^T \left(\frac{1}{2}(1-\cos(2\omega t)\right) dt$$

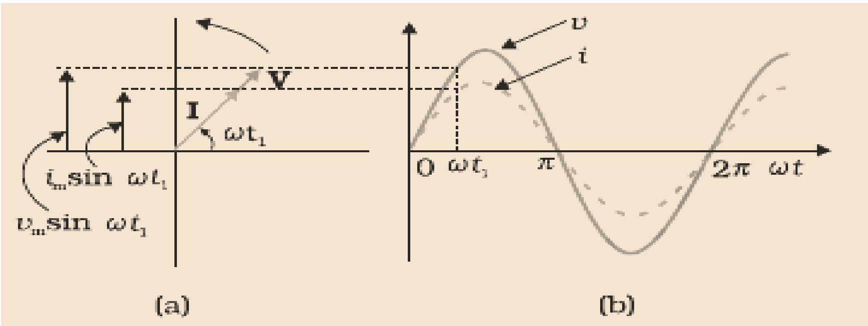
We have $< \cos 2\omega t > = 0$, finally we obtain:

$$\boxed{\bar{P} = \frac{1}{2}R.i_m^2 = R.I_{rms}^2}$$

With : $I_{rms} = \frac{I_{max}}{\sqrt{2}}$

➢ ***REPRESENTATION OF AC CURRENT AND VOLTAGE BY ROTATING VECTORS — PHASORS***

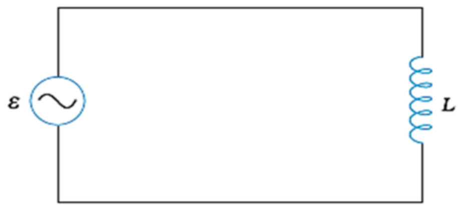
In order to show phase relationship between voltage and current in an AC circuit, we use the notion of ***phasors***. The analysis of an AC circuit is facilitated by the use of



V.3. VOLTAGE APPLIED TO AN INDUCTOR

The figure below shows an AC source connected to an inductor. **Usually, inductors have appreciable resistance in their windings, but we shall assume that this inductor has negligible resistance**. Thus, the circuit is a purely inductive ac circuit. Let the AC source voltage, be given by: $v(t) = v_M\ sin\omega\ t$. Using the Kirchhoff's loop rule, $\sum \varepsilon\ (t\ ) = 0$, and since there is no resistor in the circuit.

$$v(t) - L\frac{di(t)}{dt} = 0$$



where the second term is the self-induced Faraday emf in the inductor; and L is the self-inductance of the inductor. The negative sign follows from Lenz's law. So we obtain :

$$\frac{di(t)}{dt} = \frac{v(t)}{L} = \frac{v_M}{L}\ sin(\omega t)$$

To obtain the current, we integrate di/dt with respect to time:

$$\int \frac{di(t)}{dt} = \frac{v_M}{L}\int sin(\omega t) \Longrightarrow i(t) = -\frac{v_M}{\omega L}.\cos(\omega t)$$

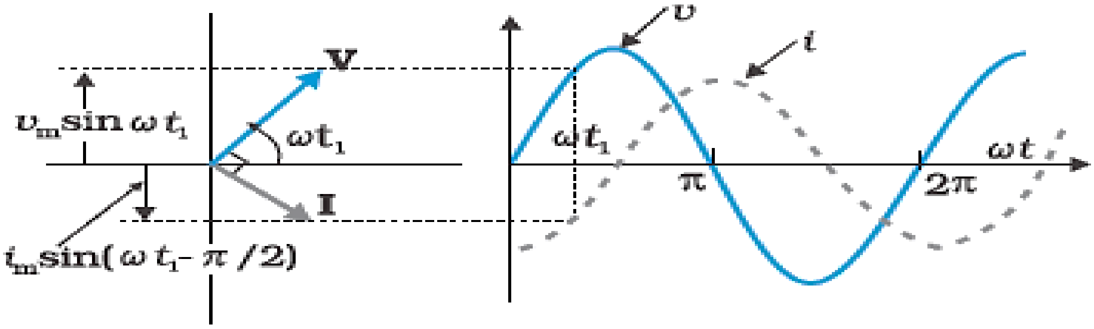
Using $-\cos(\omega t) = \sin\left(\omega t \frac{v_M}{L}.\cos(\omega t)\right)$, we obtain finaly

$$\boxed{\boldsymbol{i(t) = \frac{v_M}{\omega L}.\sin\left(\omega t - \frac{\pi}{2}\right) = i_m \sin\left(\omega t - \frac{\pi}{2}\right)}}$$

where $\boldsymbol{i_m}$ is the amplitude of the current (Peak value of amplitude). The quantity ωL is analogous to the resistance and is called ***inductive reactance***, denoted by X_L:

$$\boxed{\boldsymbol{X_L = \omega L}}\, it's\ unit\ is\ Ohm\ (\Omega)$$

The unit of inductive reactance is the same as that of resistance and its SI unit is ohm (Ω). The inductive reactance limits the current in a purely inductive circuit. **The inductive reactance is directly proportional to the inductance and to the frequency of the current**. The source voltage and the current in an inductor show that **the current lags the voltage by $\pi/2$** or one-quarter (1/4) cycle. The figure shows the voltage and the current phasors in the present case at instant t_1. The current phasor I is $\pi/2$ behind the voltage phasor V. When rotated with frequency ω counterclockwise.



We see that the current reaches its maximum value later than the voltage by one-fourth of periode $\left[\frac{T}{4} = \frac{\pi/2}{\omega}\right]$.

The instantaneous power supplied to the inductor is

$$p_L = v(t).i(t) = v_m sin(\omega\, t).i_m \sin\left(\omega t - \frac{\pi}{2}\right) = \frac{-v_m.i_m}{2} sin(\omega\, t) cos(\omega\, t)$$

So:

$$\boxed{p_L = -\frac{v_m.i_m}{2}\sin(2\omega\, t)}$$

Since the average of $sin\,(2\omega t)$ over a complete cycle is zero.

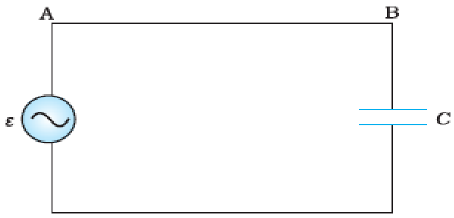
The average value of power over one cycle is given by :

$$\bar{p} = -\frac{v_m.i_m}{2} < sin(2\omega\, t) >$$

Thus, the *average power supplied to an inductor over one complete cycle is zero.*

V.4. AC VOLTAGE APPLIED TO A CAPACITOR

The figure below shows an AC source e generating AC voltage $v(t) = v_M \, sin\omega \, t$ connected to a capacitor only, a purely capacitive ac circuit. Using the Kirchhoff's loop rule, $\sum \varepsilon\,(t) = 0$



When the capacitor is connected to an AC source, ***it limits or regulates the current, but does not completely prevent the flow of charge***. The capacitor is alternately charged and discharged as the current reverses each half cycle. Let q be the charge on the capacitor at any time t. The instantaneous voltage v across the capacitor is:

$$v(t) = \frac{q}{C}$$

From the Kirchhoff's loop rule, the voltage across the source and the capacitor are equal:

$$v_M \sin(\omega\, t) = \frac{q}{C}$$

To find the current, we use the relation $i(t) = \frac{dq}{dt}$

$$i(t) = \frac{d}{dt}(v_M\, C.\sin(\omega\, t)) = \omega C v_M \cos(\omega\, t)$$

Using relation, $\cos(\omega\, t) = \sin(\omega t + \frac{\pi}{2})$, we have :

$$\boxed{i(t) = i_m sin\left(\omega t + \frac{\pi}{2}\right)}$$

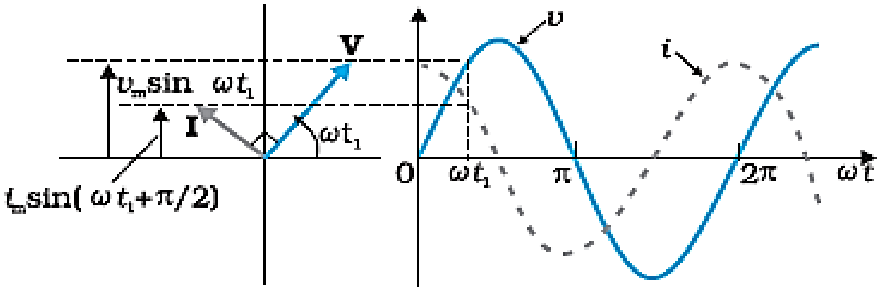
Where the amplitude of oscillating current is $i_m = \omega C v_m$. So, we can rewrite it as: $\boxed{\boldsymbol{i_m = \frac{v_m}{(1/\omega C)}}}$.

The quantity $\boldsymbol{\frac{1}{\omega C}}$ is analogous to the resistance and is called ***capacitive reactance***, denoted by X_C:

$$\boxed{\boldsymbol{X_C = \frac{1}{\omega C}}}$$

The unit of capacitive reactance is the same as that of resistance and its SI unit is ohm (Ω).The capacitive reactance limits the amplitude of the current in a purely capacitive circuit in the same way as the resistance limits the current in a purely resistive circuit. But it is inversely proportional to the frequency and the capacitance.

The Figure shows the phasor diagram at an instant t_1 . Here the current phasor $\boldsymbol{I}$ is π/2 **ahead of the voltag**e phasor $\boldsymbol{V}$ as they rotate counterclockwise. We see that the current reaches its maximum value earlier than the voltage by one-fourth (1/4) of a period.



Thus, in the case of a capacitor, the current leads the voltage by $\pi/2$.

The instantaneous power supplied to the capacitor is:

$$p_C = v(t).i(t) = v_m sin(\omega\, t).i_m \cos(\omega t) = \frac{v_m.i_m}{2} sin(\omega\, t) cos(\omega\, t)$$

So, as in the case of an inductor, the average power:

$$\boxed{p_C = \frac{v_m.i_m}{2}\sin(2\omega\, t) = 0}$$

Since the average of $sin\,(2\omega t)$ over a complete cycle is zero.

Unlike a resistor, a capacitor does not dissipate energy over a half-integer number of periods. This is due to the phase shift between current and voltage. During a quarter cycle, the capacitor stores energy by accumulating charges on its armatures, the product $v.i$ is positive, energy which it returns entirely to the source during the next quarter of the cycle, the product product $v\,.\,i$ is negative

VI. IMPEDANCE CONCEPT

In practice, we represent circuit elements by their impedance, and determine magnitude and phasor relationships in one step. Before we do it, however, we need to learn how to represent circuit elements as **impedance**. These components have a reactance that opposes current fluctuations. In particular, this reactance is responsible for the phase shift between current and voltage. As with resistors, **impedance is expressed in Ohm**. It is represented by the letter $\boldsymbol{Z}$. For example, **Ohm's law can be written as:**

$$\boxed{\boldsymbol{V = Z.I \quad or\; I = \frac{V}{Z}\; or\; Z = \frac{V}{I}\; [\Omega]}}$$

But don't confuse impedance Z with resistance R:

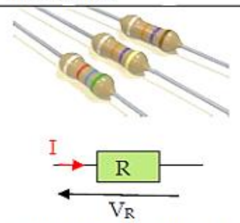
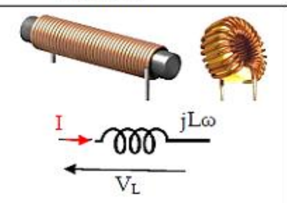
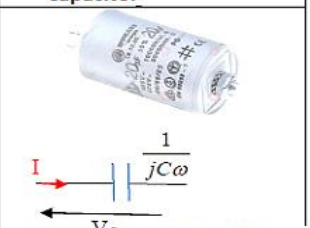
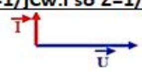
- Resistance R does not depend on the nature of the current (AC or DC).
- Impedance Z is only to be considered for alternating current, and is frequency dependent.

Certain circuit elements oppose current (coils) or voltage (capacitors) fluctuations without consuming energy as a simple resistor.

Complex Impedance	**Z= U/I** $=\lvert Z\rvert, \varphi°$ $= R + iX$	**R= Re(Z) : Resitance** (Ω) **X=Im (Z): Reactance** (Ω) - sign (+) indicates that reanctance is inductive $(X_L = \omega L)$ sign (-) indicates that reanctance is capacitive $(X_C = 1/\omega C)$
▪ if ϕ > 0 the voltage leads the current. ▪ If ϕ < 0 the voltage lags the current. ▪ If ϕ =0 voltage and current are in phase.		
Complex Admittance	**Y=1/Z:** $=\lvert Y\rvert, -\varphi°$ $= G + iB$	**G=Re(Y) : Conductance (Siemens)** **B=Im(Y): Susceptance (siemens)** - Sign (-), the suscepance is inductive - Sign(+), the susceptance is capacitive

Complex impedances

Symbol	Admittance, Y	Impedance, $Z = 1/Y$
R	$\frac{1}{R}$	R
L	$\frac{1}{j\omega L}$	$j\omega L$
C	$j\omega C$	$\frac{1}{j\omega C}$
	$I = YV$	$V = ZI$

	Resistance	Inductor	Capacitor
Electrical Dipole	 VR=R.I so Z=R=[R; 0°]	 VL=jLw.I so Z=jLw = [Lw, π /2]	 Vc=1/jCw.I so Z=1/jCw =[1/Cw ;- π /2]
Fresnel Diagram			

➢ Phasors : Complex impedances

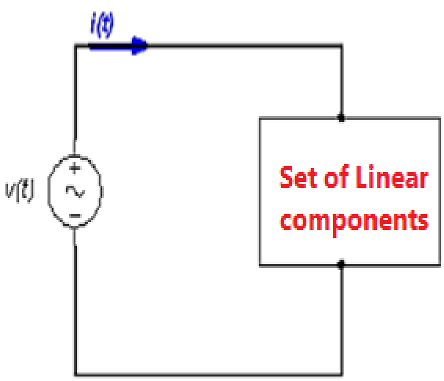
Impedance is the modulus of the complex impedance. The angle of the complex impedance is the phase shift of current $\boldsymbol{i}$ with respect to voltage v.

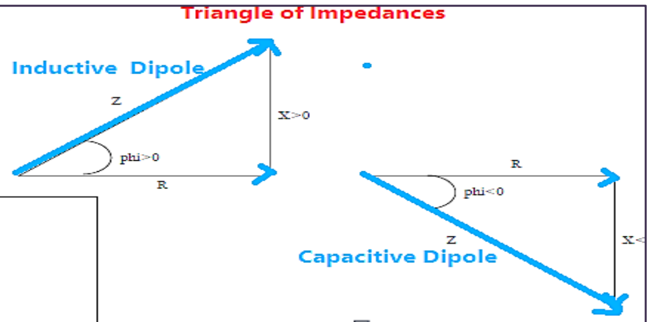
❑ **Complex Impedance** :

$$\overline{Z} = \frac{\overline{V}}{\overline{I}} \quad \begin{cases} \overline{V} & \text{volts } (V) \\ \overline{I} & \text{ampères } (A) \\ \overline{Z} & \text{ohms } (\Omega) \end{cases}$$

$$\begin{cases} \overline{V} = V\angle\theta_v \\ \overline{I} = I\angle\theta_i \end{cases} \Rightarrow \overline{Z} = \frac{V\angle\theta_v}{I\angle\theta_i} = \frac{V}{I}\angle\theta_v - \theta_i$$

$$\overline{Z} = \underbrace{\frac{V}{I}}_{Z}\angle\underbrace{\theta_v - \theta_i}_{\varphi} = Z\angle\varphi$$





- **Impedance complexe**

$$\overline{Z} = R + jX$$

- ✓ **Résistance** :

$$R = \mathcal{R}_e(\overline{Z}) = \frac{V}{I}\cos\varphi = \frac{V}{I}\,FP$$

- ✓ **Reactance**:

$$X = \mathcal{I}_m(\overline{Z}) = \frac{V}{I}\sin\varphi$$

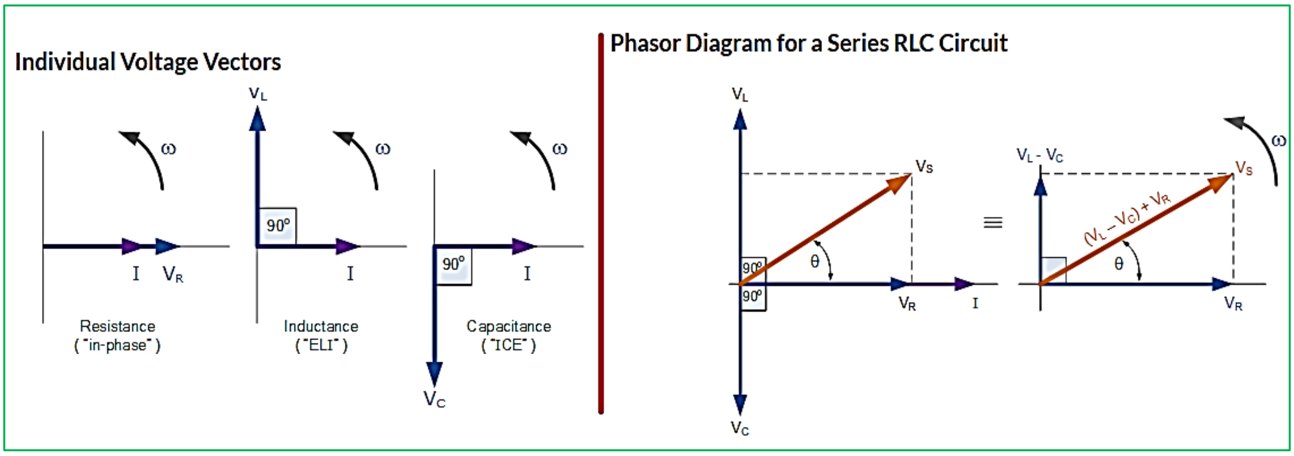
$$\begin{cases} R = Z\cos\varphi \\ X = Z\sin\varphi \end{cases} \; ; \; \begin{cases} Z = \sqrt{R^2 + X^2} \\ \varphi = \arctan\left(\frac{X}{R}\right) \end{cases}$$

$U = Z \times I$; $\omega = 2\pi f$.

IMPEDANCE CONCEPT: SUMMARY

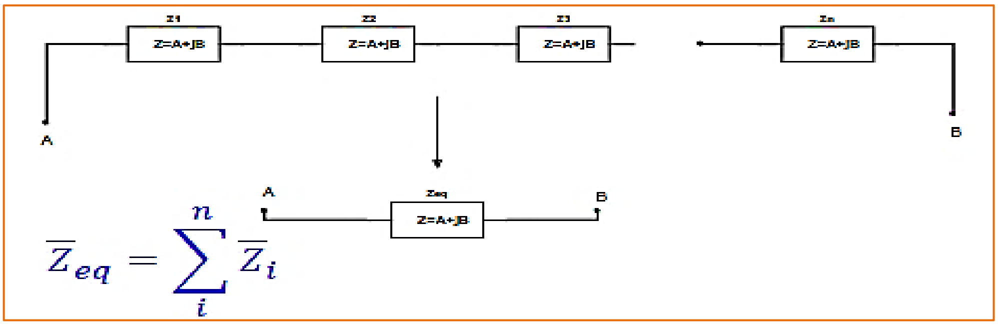
	Impedance Z	Power factor : $\cos\varphi$	phase shift: φ	Scheme
Ohmic conductor	R	1	0	U_R I
Inductance	$L\omega$	0	$\frac{\pi}{2}$	U_L I
Capacitor	$\frac{1}{C\omega}$	0	$-\frac{\pi}{2}$	I U_C
RL Circuit	$\sqrt{R^2+(L\omega)^2}$	$\frac{R}{Z}$	Value to calculate	U U_L φ I U_R
RC Circuit	$\sqrt{R^2+\left(\frac{1}{C\omega}\right)^2}$	$\frac{R}{Z}$	Value to calculate	U_R φ I U_C U
RLC Circuit	$\sqrt{R^2+\left(L\omega-\frac{1}{C\omega}\right)^2}$	$\frac{R}{Z}$	Value to calculate	U_C U U_L φ I U_R

Electrical impedance, or simply impedance, describes a measure of opposition to alternating current (AC). Electrical impedance extends the concept of resistance to AC circuits, describing not only the relative amplitudes of the voltage and current, but also the relative phases.



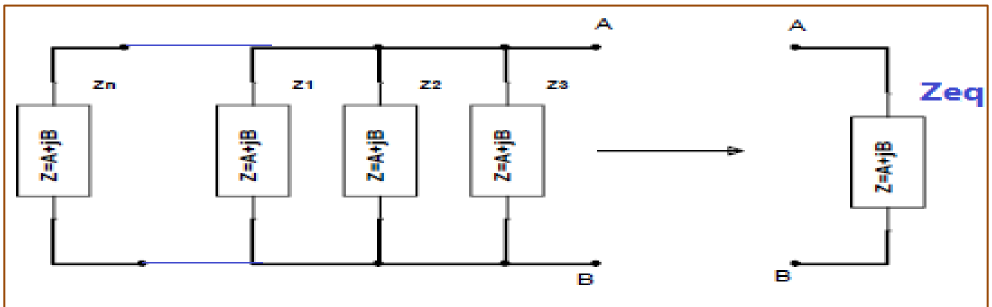
➢ Equivalent complex impedances

- **Impedances connected in series**



$$\overline{Z}_{eq} = \sum_{i}^{n} \overline{Z}_i$$

- **Impedances connected in parallel.**



$$\frac{1}{\overline{Z}_{eq}} = \left(\sum_{i}^{n} \frac{1}{\overline{Z}_i}\right)$$

Resistance

❑ Impedance

$$Z_R = \frac{V}{I} \Rightarrow \boxed{Z_R = R}$$

❑ Complex Impedance

$$\overline{Z}_R = \frac{\overline{V}}{\overline{I}} = \frac{V\angle 0^\circ}{I\angle 0^\circ} = \boxed{R\angle 0^\circ = R}$$

Inductance

❑ Impedance

$$Z_L = \frac{V}{I} = \frac{V}{\frac{V}{L\omega}} = L\omega \Rightarrow \boxed{Z_L = L\omega}$$

❑ Complex Impedance

$$\overline{Z}_L = \frac{\overline{V}}{\overline{I}} = \frac{V\angle 0^\circ}{I\angle \theta_i} = \frac{V\angle 0^\circ}{\frac{V}{L\omega}\angle -\frac{\pi}{2}} = \boxed{L\omega\angle\frac{\pi}{2} = jX_L}$$

$$X_L = L\omega$$

Capacitor

❑ Impedance

$$Z_C = \frac{V}{I} = \frac{V}{C\omega V} = \frac{1}{C\omega} \Rightarrow \boxed{Z_C = \frac{1}{C\omega}}$$

❑ Complex Impedance

$$\overline{Z}_C = \frac{\overline{V}}{\overline{I}} = \frac{V\angle 0^\circ}{I\angle \theta_i} = \frac{V\angle 0^\circ}{C\omega V\angle +\frac{\pi}{2}} = \boxed{\frac{1}{C\omega}\angle -\frac{\pi}{2} = jX_C}$$

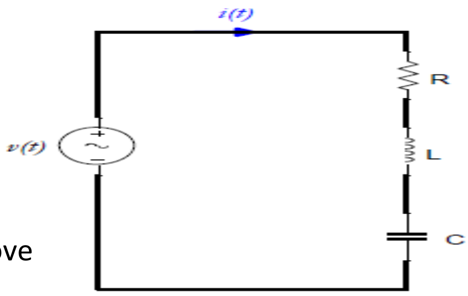
$$X_C = -1/C\omega$$

VI . Analysis of a series RLC circuit

Figure shows a series RLC circuit connected to an AC source. As usual, we take the voltage of the source to be $v = v_m \sin \omega t$. If q is the charge on the capacitor and i the current, at time t, we have, from Kirchhoff's loop rule:

$$L\frac{di}{dt} + iR + \frac{q}{C} = v \qquad v_L + v_R + v_C =$$

$$v_{Rm} = i_m R,\ v_{Cm} = i_m X_C,\ v_{Lm} = i_m X_L$$



The phasor relation whose vertical component gives the above equation is :

The three impedances: Z_R , $Z_L = jL\omega$ and $Z_C = -\frac{j}{C\omega}$ are connected in series, so the equivalent impedance:

$$\bar{Z}_{eq} = R + jL\omega - \frac{j}{C\omega} \qquad so: \quad \bar{Z}_{eq} = R + j\left[L\omega - \frac{1}{C\omega}\right]$$

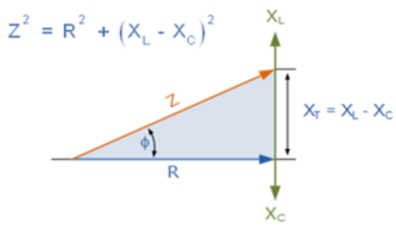
or in polar form

$$\bar{Z} = Z e^{-j\varphi} \quad \text{with}, \quad Z = \sqrt{R^2 + \left[L\omega - \frac{1}{C\omega}\right]^2} \quad \text{and } tg\,\varphi = \frac{\frac{1}{C\omega} - L\omega}{R}$$

Impedance equivalente

$$\bar{Z}_{tot} = \bar{Z}_{eq} = \bar{Z}_R + \bar{Z}_L + \bar{Z}_C = R + j(X_L + X_C)$$
$$= \sqrt{R^2 + (X_L + X_C)^2}\angle \arctan\left(\frac{X_L + X_C}{R}\right)$$

The Impedance Triangle for a Series RLC Circuit



Current Phasor

$$\bar{I} = \frac{\bar{V}}{\bar{Z}_{eq}} = \frac{V\angle 0°}{\sqrt{R^2 + (X_L + X_C)^2}\angle \arctan\left(\frac{X_L + X_C}{R}\right)}$$

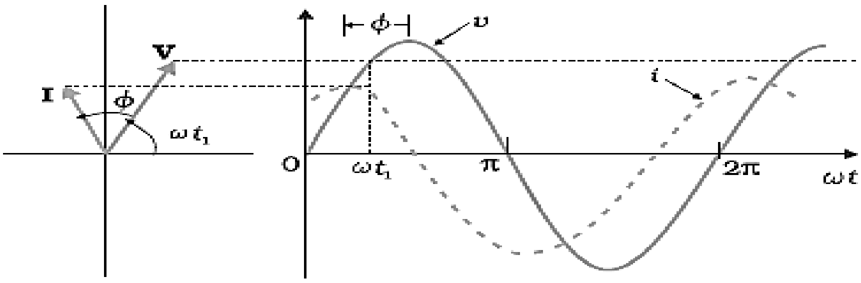
$$\Rightarrow \bar{I} = \frac{V}{\sqrt{R^2 + (X_L + X_C)^2}}\angle 0 - \arctan\left(\frac{X_L + X_C}{R}\right)$$

$$\Rightarrow \boxed{\bar{I} = \frac{V}{\sqrt{R^2 + (X_L + X_C)^2}}\angle - \arctan\left(\frac{X_L + X_C}{R}\right)}$$

Temporel Current Expression

$$i(t) = \frac{V}{\sqrt{R^2 + (X_L + X_C)^2}}\sqrt{2}\sin\left(\omega t - \arctan\left(\frac{X_L + X_C}{R}\right)\right)$$

$$i_m = \frac{v_m}{\sqrt{R^2 + (X_C + X_L)^2}}$$



(a) (b)

VII. Resonance

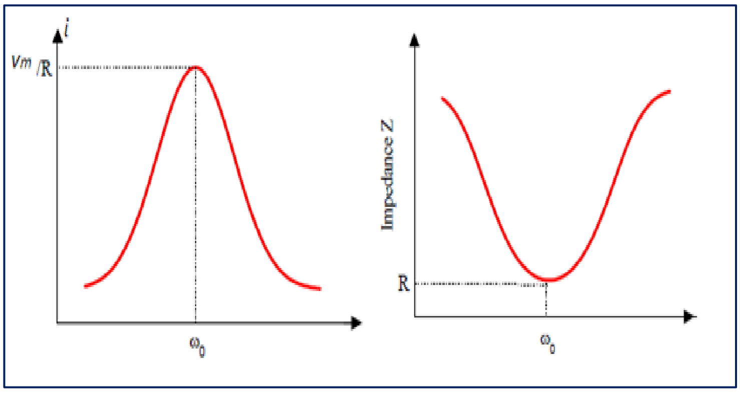
An interesting characteristic of the series *RLC* circuit is the phenomenon of resonance. The phenomenon of resonance is common among systems that have a tendency to oscillate at a particular frequency. This frequency is called the system's ***natural frequency***.

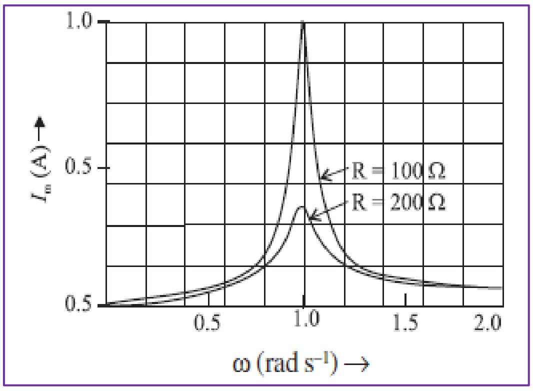
For an RLC circuit driven with voltage of amplitude v_m and frequency f or pulsation ω. The current amplitude is given using ohm's law by:

$$\boxed{i_m = \frac{v_m}{Z} = \frac{v_m}{\sqrt{R^2 + (X_c - X_L)^2}}}$$

With $X_c = 1/\omega C$ and $X_L = \omega L$, so if ω is varied, then a particular frequency ω_0, $X_c = X_L$ and the impedance is minimum $\left(Z = \sqrt{R^2 + (X_c - X_L)^2} = \sqrt{R^2 + 0}\right) = R$. This frequency is called the resonant frequency. $X_c = X_L \; or \; \frac{1}{\omega_0 C} = \omega_0 L$

At $\omega_0$impedance Z is minimum and equals R; current i is maximum





VIII. POWER IN AC CIRCUIT: THE POWER FACTOR

Electrical power is the quantity of energy supplied or received during one second. There are three types of AC power:

- Active power;
- Reactive power;
- Apparent power

1- Active power

Active power is referred to as (P). Its unit is W (Watt). The expression of active power (P) depends on the rms voltage (V_{rms}), the rms electric current (I_{rms}) and the phase shift ($\varphi = \varphi_v - \varphi_i$). The symbol (ϕ) is the phase shift of the current with respect to the voltage. Its expression is :

$$\boxed{P = V_{rms}.I_{rms}.\cos(\varphi)} \quad with \begin{cases} P: Active\ Power\ (W) \\ V: Electric\ Voltage\ (V) \\ I_{rms}: Current\ (A) \\ \varphi: Shift\ phase\ (rad) \end{cases}$$

La puissance active (P) est mesurée à l'aide d'un wattmètre.

The average power delivered by the source to the circuit is given by:

$$\text{Average Power} = \frac{V_m^2}{2Z}\cos\phi$$

$$= \frac{V_m}{\sqrt{2}} \cdot \frac{V_m}{\sqrt{2}Z}\cos\phi = V_{rms} I_{rms} \cos\phi$$

The $cos\varphi$ is called ***power factor*** and is given by:

$$\boxed{\cos\phi = \frac{R}{Z} = \frac{R}{\sqrt{R^2 + (X_L - X_C)^2}}}$$

2- Apparent power

Apparent power is denoted by (S). Its unit is VA (VoltAmpere). It depends on the rms voltage (V_{rms}) and the rms current (I_{rms}). Its expression is:

$$\boxed{S = V_{rms}.I_{rms}}$$

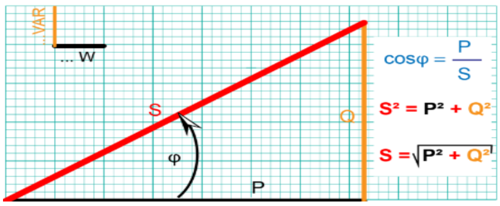
3- Reactive power

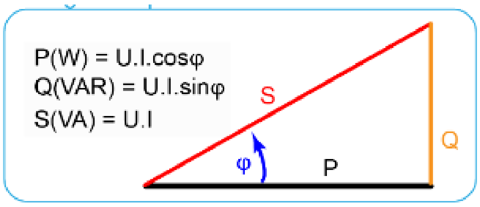
Reactive power is designated (Q). Its unit is (**VAR**: stands for Voltampere Reactive). It is absorbed by inductive receivers such as motors, electric welders, etc., or supplied by capacitive receivers such as batteries; electric welders, etc., or supplied by capacitive receivers such as capacitor banks; capacitor banks. Its expression is:

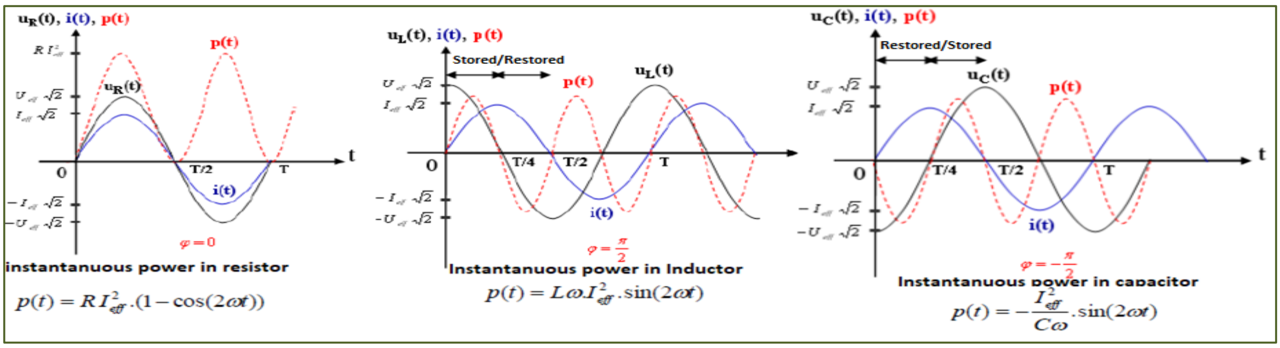
$$\boxed{Q = V_{rms}.I_{rms}.\sin(\varphi)}$$

The power factor ($cos\varphi$) is also defined as the ratio of active power to apparent power. Its expression is :

$$\boxed{cos\varphi = \frac{P}{S} \qquad with\ \boldsymbol{co\varphi \le 1}}$$

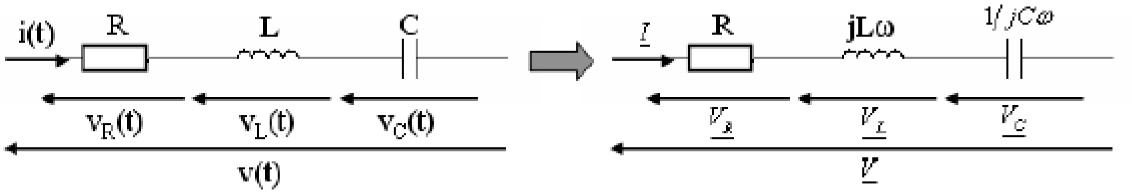






Summary

- **Case of an RLC series circuit :**



We can associate i(t) and v(t) with their complex notations, giving :

$$\underline{V} = \underline{V}_R + \underline{V}_L + \underline{V}_C$$

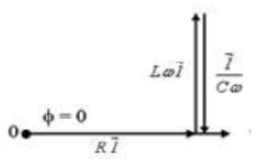
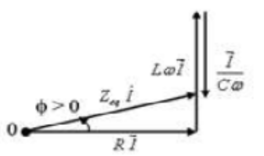
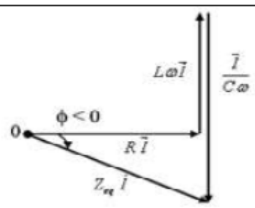
$$\underline{V} = R.\underline{I} + jL\omega.\underline{I} + \frac{1}{jC\omega}.\underline{I} = \left(R + j\left(L\omega - \frac{1}{C\omega}\right)\right).\underline{I} = \underline{Z}_{eq}.\underline{I}$$

We find the impedance of the series RLC dipole :

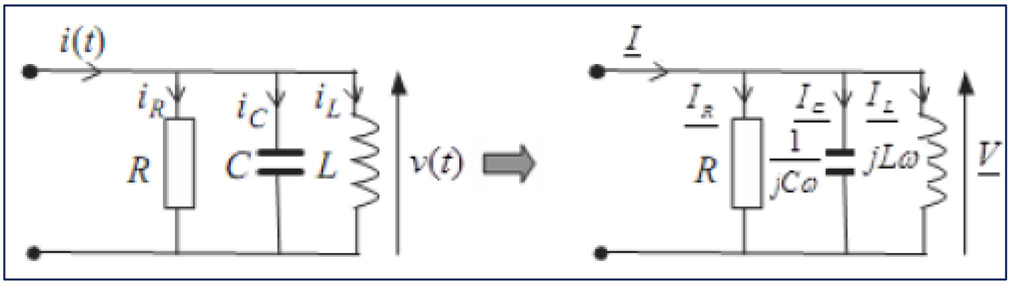
$$\underline{Z}_{eq} = R + j\left(L\omega - \frac{1}{C\omega}\right)$$

The modulus and phase of $\underline{Z}_{eq}$ are :

$$\underline{Z}_{eq} = \left[\underbrace{\sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}}_{Magnitude}, \underbrace{\arctan\frac{\left(L\omega - \frac{1}{C\omega}\right)}{R}}_{Phase(\varphi)}\right]$$

Three Cases are possible:		
$L\omega - \frac{1}{C\omega} = 0 \Rightarrow L\omega = \frac{1}{C\omega}$ ***Resistive circuit***	$L\omega - \frac{1}{C\omega} > 0 \Rightarrow L\omega > \frac{1}{C\omega}$ ***Inductive circuit***	$L\omega - \frac{1}{C\omega} < 0 \Rightarrow L\omega < \frac{1}{C\omega}$ ***Capacitive Circuit***
 $\varphi - 0$ $v(t)$and $v_R(t)$ are in phase (v(t) and i(t) are in phase).	 $\varphi > 0$ $v(t)$ voltage is ahead of current *i(t)* in ahead of $v_R(t)$	 $\varphi < 0$ $v(t)$ is lags of $v_R(t)$ (i.e $i(t)$).

- **Parallel RLC circuit :**



$$i(t) = i_R(t) + i_L(t) + i_C(t) = \frac{v(t)}{R} + \frac{1}{L}.\int v(t).dt + C.\frac{dv(t)}{dt}$$

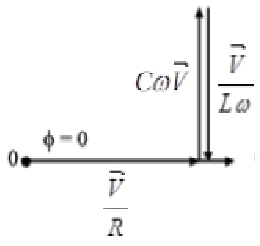
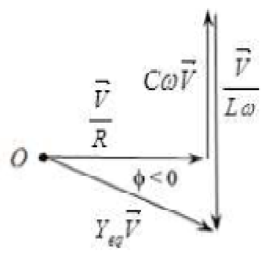
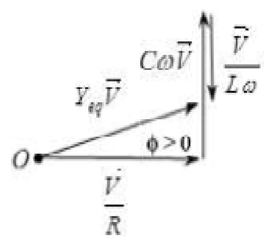
We can associate i(t) and v(t) with their complex notations, giving :

$$\underline{I} = \left(\frac{1}{R}.\underline{V}\right) + \left(\frac{1}{jL\omega}.\underline{V}\right) + (jC\omega.\underline{V}) = \left(\frac{1}{R} + \frac{1}{jL\omega} + jC\omega\right).\underline{V} = \underline{Y_{eq}}.\underline{V}$$

$$\underline{I} = \left[G + j\left(C\omega - \frac{1}{L\omega}\right)\right].\underline{V} = \underline{Y_{eq}}.\underline{V}$$

$$\underline{Y_{eq}} = \left[\underbrace{\sqrt{G^2 + \left(C\omega - \frac{1}{L\omega}\right)^2}}_{Magnitude}, \underbrace{\arctan\frac{C\omega - \frac{1}{L\omega}}{G}}_{Phase(\varphi)}\right]$$

$$\frac{1}{\underline{Z_{eq}}} = \frac{1}{\left(\frac{1}{R} + \frac{1}{jL\omega} + jC\omega\right)}$$

Three Cases are possible:		
$C\omega \quad \frac{1}{L\omega} = 0 \Rightarrow C\omega = \frac{1}{L\omega}$ ***Resistive Circuit***	$C\omega \quad \frac{1}{L\omega} > 0 \Rightarrow C\omega > \frac{1}{L\omega}$ ***Inductive Circuit***	$C\omega \quad \frac{1}{L\omega} < 0 \Rightarrow C\omega < \frac{1}{L\omega}$ ***Capacitive Circuit***
 $\varphi = 0$ $v(t)$ and $v_R(t)$ are in phase ie: $v(t)$ and $i(t)$).	 $\varphi < 0$ $v(t)$ is lags of $v_R(t)$ (i.e. $i(t)$).	 $\varphi > 0$ $v(t)$ is ahead of $v_R(t)$ (i.e. $i(t)$).

- **Active Power**

1- For resistance : $P_R = U_{eff}.I_{eff}.\cos(0) \Rightarrow P_R = U_{eff}.I_{eff} = R.I_{eff}^2 = \frac{U_{eff}^2}{R}$

Active power $P_R > 0$: The resistor **consumes** active power.

2- For Inductor :Voltage leads current by $\varphi = \frac{\pi}{2}$, so : $P_L = U_{eff}.I_{eff}.\cos\left(\frac{\pi}{2}\right) = 0$

3- For Capacitor: Voltage lags current by $\varphi = -\frac{\pi}{2}$, so : $P_C = U_{eff}.I_{eff}.\cos\left(-\frac{\pi}{2}\right) = 0$

- **Reactive Power**

$$Q = U_{eff} . I_{eff} . \sin(\varphi)$$ $$Q = V_{rms} . I_{rms} . \sin(\varphi)$$

For an inductive dipole : $0 < \varphi \leq \frac{\pi}{2}$ *et* $Q_L > 0.$

For a capacitive dipole : $-\frac{\pi}{2} \leq \varphi < 0$ *et* $Q_C < 0.$

1- For resistance: $Q_R = U_{eff} . I_{eff} . \sin(0) = 0$

Reactive power in a resistor is zero. The resistor does not consume reactive power.

2- For an inductor (coil) :The reactive power in the case of an inductance is then written : $Q_L = U_{eff} . I_{eff} . \sin\left(\frac{\pi}{2}\right) \Rightarrow Q_L = U_{eff} . I_{eff} = L\omega . I_{eff}^2 = \frac{U_{eff}^2}{L\omega}$

Reactive power $Q_L > 0$: The coil **consumes** reactive power.

3- For a capacitor : The reactive power in the case of a capacitor is then written : $Q_C = U_{eff} . I_{eff} . \sin\left(-\frac{\pi}{2}\right) \Rightarrow Q_C = -U_{eff} . I_{eff} = -C\omega . U_{eff}^2 = -\frac{I_{eff}^2}{C\omega}$

Reactive power $Q_C < 0$: The capacitor consumes negative power, i.e. it **produces** reactive power.

- **Apparent Power**

$$S = \sqrt{P^2 + Q^2} = U_{rms} . I_{rms}$$

The power factor ($cos\varphi$) is also defined as the ratio of active power to apparent power. Its expression is :

$$cos\varphi = \frac{P}{S} \qquad with\ \mathbf{co\varphi} \leq \mathbf{1}$$

- **For Resistance:** : $S_R = \sqrt{P_R^2 + Q_R^2} = R . I_{eff}^2 = \frac{U_{eff}^2}{R}.$
- **For Inductor** : $S_L = \sqrt{P_L^2 + Q_L^2} = L\omega . I_{eff}^2 = \frac{U_{eff}^2}{L\omega}.$
- **For Capacitor** : $S_C = \sqrt{P_C^2 + Q_C^2} = C\omega . U_{eff}^2 = \frac{I_{eff}^2}{C\omega}.$

$$\cos(\varphi) = \frac{P}{S} \quad \sin(\varphi) = \frac{Q}{S} \quad \tan(\varphi) = \frac{\sin(\varphi)}{\cos(\varphi)} = \frac{Q}{P} \quad S = \sqrt{P^2 + Q^2}$$

Symbol	Name	Unit	Formula	Mesure
P	**Active power or Average Power**	Watt [W]	$P = U_{eff}.I_{eff}\cos(\varphi)$ $P = S\cos(\varphi)$	**Wattmeter**
Q	**Reactive Power**	**Volt Ampere Reactive** [var]	$Q = U_{eff}.I_{eff}\sin(\varphi)$ $Q = S\sin(\varphi)$	**By calculation**
S	**Apparent Power**	Volt-Ampere [VA]	$S = U_{eff}.I_{eff}$ $S = \sqrt{P^2 + Q^2}$	**Voltmeter and Ampermeter**

The power consumption of each basic element is shown below

Dipole	$\underline{Z}$	$\varphi_u - \varphi_i$	P	Q	S
Resistance	R	0	$P_R = R.I_{eff}^2 = \frac{U_{eff}^2}{R}$	0	$R.I_{eff}^2 = \frac{U_{eff}^2}{R}$
Inductor	$jL\omega$	$\frac{\pi}{2}$	0	$Q_L = L\omega.I_{eff}^2 = \frac{U_{eff}^2}{L\omega}$	$L\omega.I_{eff}^2 = \frac{U_{eff}^2}{L\omega}$
Capacitor	$-j\frac{1}{C\omega}$	$-\frac{\pi}{2}$	0	$Q_C = -C\omega.U_{eff}^2 = -\frac{I_{eff}^2}{C\omega}$	$C\omega.U_{eff}^2 = \frac{I_{eff}^2}{C\omega}$