الجمهـــوريـــــــة الجزائريــــــــة الديمـــقـــراطيـــــــة الشعبيـــــــة

People`s Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research National School of Cyber Security	NSCS المدرسة الوطنية العليا في الأمن السيبراني NATIONAL SCHOOL OF CYBERSECURITY	وزارة التعـليـم العـــالـي والـبحــث الـعــلـمي المدرســـة الوطنيـــة العليـــا في الأمـن السيبراني
Foundation Training Department		قســم التكويـن القــاعدي
LEVEL : 1st Year Basic Training		MODULE'S TEACHER : **Dr. Hamza Moufek**
SECTION / GROUP : A & B		DATE : 12 / 05/ 2025
MODULE : Algebra 2	**Midterm Exam**	DURATION : 2h
FULL NAME :		*NOTE : No documents are allowed.*

Exercise 1 : (9 points)

Let $E = \mathbb{R}_6[X]$, and consider the following sets :

$$F = \left\{\delta X^6 + \alpha X^5 + \beta X^4 + \gamma X^3 + \beta X^2 + \alpha X + \delta,\ (\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4\right\}$$
$$G = \left\{\lambda X^3 + \mu X^2 + \mu X + \lambda,\ (\lambda, \mu) \in \mathbb{R}^2\right\}$$
$$H = \{P \in E,\ P'(-1) = 0\}$$

1. Recall the standard basis of E and its dimension.
2. Justify that all elements of G are divisible by the polynomial $X + 1$.
3. **Study of F and G**
   (a) Show that F and G are subspaces of E.
   (b) Give a basis of F and a basis of G. Deduce $\dim(F)$ and $\dim(G)$.
   (c) Do we have $F \oplus G$? Is $E = F \oplus G$?
4. **Study of H**
   (a) Without finding a generating set, show that H is a subspace of E.
   (b) Without finding a generating set, show that $\dim(H) \leq 6$.
   (c) Show that $\mathcal{F} = \{1, (X+1)^2, (X+1)^3, (X+1)^4, (X+1)^5, (X+1)^6\}$ is a linearly independent set of elements of H.
   (d) Justify (carefully) that $\dim(H) = 6$.
5. **Study of $G \cap H$**
   (a) Justify that $1 \leq \dim(G \cap H) \leq 2$.
   (b) Explain why one of the two choices leads to a contradiction.
   (c) Show that any polynomial in $G \cap H$ has -1 as a multiple root.
   (d) Deduce a basis for $G \cap H$.

Exercise 2 : (4 points)

Recall that the Fibonacci sequence is defined by :

$$F_0 = 0, F_1 = 1, \text{ and } \forall n \in \mathbb{N}^*, F_{n+1} = F_n + F_{n-1}$$

Consider the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

1. Compute F_n for $0 \leq n \leq 12$ (present the results in a table).
2. Express A^2 as a linear combination of A and the identity matrix I_2.
3. For all $n \in \mathbb{N}$, express A^{2n} as a linear combination of A^k for $0 \leq k \leq n$.

4. Deduce from the above that :

$$\forall n \in \mathbb{N}, F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_k \tag{1}$$

then verify this formula "by hand" for $n = 6$.

5. Using the formula (1), prove that :

$$\exists a > 0; \forall n \in \mathbb{N}, F_{2n} \leq a^n F_n.$$

Exercise 3 : (7 points)

Let $E = \mathbb{R}_3[X]$ and $\mathcal{B} = (1, X, X^2, X^3)$ be the standard basis of E.

1. Consider $L\colon E \longrightarrow \mathbb{R}_1[X]$ the map defined by :

$$\forall P \in E, \quad L(P) = \left(\int_{-1}^{1} P(t)\,\mathrm{d}t\right) X$$

In particular $L(1) = \left(\int_{-1}^{1} 1\,\mathrm{d}t\right) X = 2X$.

   (a) Compute $L(X)$, $L(X^2)$, and $L(X^3)$.

   (b) Show that L is a linear transformation.

   (c) Conclude that the set $F = \left\{P \in E \mid \int_{-1}^{1} P(t)\,\mathrm{d}t = 0\right\}$ is a subspace of E.

2. We now consider the linear map

$$f\colon \begin{array}{ccc} E & \longrightarrow & \mathbb{R}_2[X] \\ P & \longmapsto & P' + L(P) \end{array}$$

where P' denotes the derivative of P. Let $\mathcal{C} = (1, X, X^2)$ be the standard basis of $\mathbb{R}_2[X]$.

   (a) Determine the matrix A of f with respect to the standard bases $\mathcal{B}$ and $\mathcal{C}$.

   (b) Deduce the rank of f. Is f surjective ?

   (c) Deduce the dimension of $\ker(f)$.

   (d) Find a basis for $\ker(f)$.

3. Consider the restriction φ of f to F :

$$\varphi\colon \begin{array}{ccc} F & \longrightarrow & \mathbb{R}_2[X] \\ P & \longmapsto & P' + \left(\int_{-1}^{1} P(t)\,\mathrm{d}t\right) X \end{array}$$

   (a) Simplify the expression of $\varphi(P)$ for any polynomial $P \in F$.

   (b) Prove that φ is injective (using the kernel).