

Exercise 1

Using the syntax of the predicates write the following sentences (f and g denote real functions) :

- 1 g is not a constant function.
- 2 f is even.
- 3 g is periodic of period 2π .
- 4 g is periodic.
- 5 f is an increasing function on $[0, +\infty[$.
- 6 The curves representing f and g intersect at least once.
- 7 f is strictly less than g .
- 8 f is not strictly less than g .

Exercise 2

Let P, Q, R three logical propositions.

- 1 Show that $P \Rightarrow Q$ is equivalent to $\neg P \vee Q$. Write the negation of $P \Rightarrow Q$.
- 2 Show that the following assertions are equivalent :
 - a $(P \vee Q) \Rightarrow R$ and $(P \Rightarrow R) \wedge (Q \Rightarrow R)$.
 - b $(P \Rightarrow (Q \Rightarrow R))$ and $((P \wedge Q) \Rightarrow R)$.

Exercise 3

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. Write the negation of the following statements :

- 1 $\exists M \in \mathbb{R}, \forall x \in \mathbb{R}, f(x) \leq M$.
- 2 $\forall x \in \mathbb{R}, (f(x) \geq 0 \implies x \geq 0)$.
- 3 $\forall x \in \mathbb{R}, f(x) \geq 1$ or $f(x) \leq -1$.
- 4 $\exists \ell \in \mathbb{R}, \forall \varepsilon > 0, \exists A > 0, \forall x \geq A, |f(x) - \ell| \leq \varepsilon$
- 5 $\forall x \in \mathbb{R}, (f(x) < 0 \iff x \in [0, 1])$.

Exercise 4

Say if the following statements are true or false. Carefully justify each answer (demonstrate true results, give counterexamples when false, optionally cite a hypothesis to add for making the statement true) :

- 1 $\exists n \in \mathbb{Z}, \forall p \in \mathbb{Z}, n \leq p$.
- 2 $\forall x \in \mathbb{R}, \exists y > 0, x = e^y$.
- 3 $\forall a \in \mathbb{R}, \exists x \in \mathbb{R}, \ln x < a$.
- 4 $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \exists z \in \mathbb{R}, x < y < z$.

Exercise 5

For each of the statements $P(\cdot)$ below, determine if the proposition $Q(\cdot)$ is necessary, sufficient, both at the same time or nothing at all (justify your answer).

- 1 Parameter : $x \in \mathbb{R}$.
Propositions : $P(x) : (x \geq 0)$ and $Q(x) : (x \geq 1)$.
- 2 Parameter : $(a, b) \in \mathbb{R}^2$.
Propositions : $P(a, b) : ((a + b)^2 = a^2 + b^2)$ and $Q(a, b) : (a = b = 0)$.
- 3 Parameter : $(a, b, c) \in \mathbb{R}^3$.
Propositions : $P(a, b, c) : (|a + b + c| = 0)$ and $Q(a, b, c) : (|a + b| + |c| = 0)$.
- 4 Parameter : $(u_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$.
Propositions : $P((u_n)_{n \in \mathbb{N}}) : (\forall N \in \mathbb{N}^*, (u_1 - u_0)(u_2 - u_1) \cdots (u_N - u_{N-1}) > 0)$ and $Q((u_n)_{n \in \mathbb{N}}) : (\text{the sequence } (u_n)_{n \in \mathbb{N}} \text{ is strictly increasing})$.

Exercise 6

For any $n \geq 1$, we set :

$$A_n = \sum_{k=1}^n (-1)^{k-1} k$$

- 1 Compute A_n for $1 \leq n \leq 6$ and report the results in a table.
- 2 Prove using induction the following property :

$$\forall n \in \mathbb{N}^*, A_n = \frac{(-1)^{n-1}(2n+1)+1}{4}$$

- 3 Check that :

$$\forall n \in \mathbb{N}^*, A_n = (-1)^{n-1} \left\lfloor \frac{n+1}{2} \right\rfloor$$

Exercise 7

Prove by induction that for all $n \in \mathbb{N}^*$, there exist natural integers p and q such that $n = 2^p(2q + 1)$.

Exercise 8

Prove the following statements :

- 1 $\forall n \in \mathbb{N}^*, 6 \text{ divides } 5n^3 + n.$
- 2 Two positive integers are multiples of each other if and only if they are equal.

Exercise 9

- 1 Show that : $\sqrt{3} \notin \mathbb{Q}.$
- 2 Let $n \in \mathbb{N}$. We admit : $(2 + \sqrt{3})^n \in \mathbb{Q} \Leftrightarrow n = 0$
Deduce a necessary and sufficient condition on n so that $(2 - \sqrt{3})^n \in \mathbb{Q}.$

Exercise 10

Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Show that : $\forall n \in \mathbb{Z}^*, nx \in \mathbb{R} \setminus \mathbb{Q}.$

Exercise 11

Show that any function from \mathbb{R} to \mathbb{R} can be written as the sum of a constant function and a function whose image of 0 is 0.