2024/2025

Lecturer: B. CHAOUCHI

Analysis 1 - W.S 5

Basic Training Cycle

Taylor Series

Exercise 1

Find the the Taylor series for the following functions:

$(*)\, f_1(x) = \ln x,\ x_0 = 1, n = 4 \qquad f_2(x) = \frac{1}{x},\ x_0 = 1, n = 4$

$(*)\, f_3(x) = \sin x,\ x_0 = \frac{\pi}{4}, n = 4 \qquad f_4(x) = x^4 + x - 2,\ x_0 = 1, n = 3$

$f_5(x) = (x-1)e^x, x_0 = 1, n = 3$

Exercise 2

Find the first the 3 terms in the Maclaurin series for the following functions:

$(*)\, f_1(x) = \sin^2 x, n = 3 \qquad (*)\, f_2(x) = \frac{x}{\sqrt{x^2-1}} \qquad (*)\, f_3(x) = \frac{x}{x^2+1}$

Exercise 3

Find the Maclaurin series for

$$(*)\, f_1(x) = \sqrt{1+x+x^2}, n = 3 \qquad (*)\, f_2(x) = \sqrt{\cos x}, n = 4$$

$$(*)\, f_3(x) = \ln(\cos(3x)), n = 3$$

Exercise 4

Find the Maclaurin series for

$(*)\, f_1(x) = \frac{1}{\sin^2 x} - \frac{1}{\sinh^2 x}, n = 2$

$(*)\, f_2(x) = \left((\cos x)^{x^2} - 1\right)\tan^3 x, n = 8$

$f_3(x) = \sqrt{\cos(x^3) + \cosh(x^3) - 2}, n = 17$

Exercise 5

$(*)$Find the Maclaurin series for

$(*)\, f_1(x) = \ln\left(e^{2x} + 2e^x + 3\right), n = 2 \qquad f_2(x) = \sqrt{8 + \sqrt{1+6x}}, n = 2$

$f_3(x) = \cosh(\ln(\cosh x)), n = 6$

Exercise 6

$\boxed{1}$ $(*)$Find the Maclaurin series for $\sin^2 x$ using the series for $\cos 2x$. Hence find

$$\lim_{x\to 0} \frac{\sin^2 x - x^2}{x^4}.$$

2 Find the first 3 terms in the Maclaurin series for $\cos(\sin x)$. Hence or otherwise find

$$\lim_{x\to 0}\frac{1-\cos(\sin x)}{x^2}.$$

3 Find the first 3 terms in the Maclaurin series for $\sin(\sin x)$. Hence or otherwise find

$$\lim_{x\to 0}\frac{x-\sin(\sin x)}{x^3}.$$

Exercise 7

Find

$$(*)\,1.\lim_{x\to 0}\left(\frac{1}{(\tanh x)^2}-\frac{1}{(\tan x)^2}\right)\qquad 2.\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}\qquad 3.\lim_{x\to 0}\left(2^x+3^x-4^x\right)^{\frac{1}{x}}$$

Exercise 8

1 Evaluate the integral

$$\int_0^{\frac{\pi}{6}}\sin^2 x dx$$

by first finding the Maclaurin approximation tothe integrand with 3 terms.

2 Evaluate the integral exactly and compare.

Exercise 9

Find the Maclaurin series for $f_1(x)=\exp\left(\sum_{k=1}^{20}\frac{(-1)^{k+1}}{k}x^k\right),\ n=22\qquad f_2(x)=\int_x^{2x}\ln(1+t)\ln(1-t)\,dt,\ n=3$

Exercise 10

1 Find the Maclaurin series for the functions e^x and $\sin x$ and hence expand $f(x) = e^{\sin x}$ up to the term in x^4 .

2 Integrate $\int_0^1 e^{\sin x}dx$

Exercise 11

1 Find the Maclaurin series for the function $f(x)=\ln(1+x)$ and hence that for $\dfrac{\ln(1+x)}{x}$

2 By keeping the first four terms in the Maclaurin series for $\dfrac{\ln(1+x)}{x}$ integrate the function $\dfrac{\ln(1+x)}{x}$ from $x=0$ to $x=1$

3 Estimate the magnitude and sign of the error made in evaluating the integral in this way

Exercise 12

1 Find the Maclaurin series for the function $f(x)=\dfrac{1}{1+x^2}$.

$\boxed{2}$ Hence by integrating each term, show that

$$\arctan x = \sum_{1}^{+\infty} (-1)^{n+1} \frac{x^{2n-1}}{2n-1}.$$

$\boxed{3}$ Show that this series converges for $-1 \leq x \leq 1$ and hence show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4}.$$

Exercise 13

(∗)Let

$$f(x) = \ln\left(\frac{1+\tan x}{1-\tan x}\right)$$

$\boxed{1}$ Show that f is well defined on $\left]-\frac{\pi}{4}, \frac{\pi}{4}\right[$

$\boxed{2}$ Determine the tangent's equation at $x = 0$

$\boxed{3}$ Study the relative position of this function at the indicated location

Exercise 14

$\boxed{1}$ Find the Maclaurin series for the functions e^x and $\sin x$ and hence expand $f(x) = e^{\sin x}$ up to the term in x^4 .

$\boxed{2}$ Integrate $\int_0^1 e^{\sin x} dx$

Exercise 15

(∗)Let

$$f(x) = x\sqrt{1 - \frac{2}{3}x^2} \text{ and } g(x) = \arctan x$$

Study the relative position of these functions near $x = 0$