Ministry of Higher Education and Scientific Research National School of Cyber Security		وزارة التعليم العالي والبحث العلمي المدرسة الوطنية العليا في الأمن السيبراني
Foundation Training Department		قسم التكوين القاعدي
LEVEL : 1st Year Basic Training		MODULE'S TEACHER : **Dr. Hamza Moufek**
SECTION / GROUP : A & B		DATE : 05 / 04/ 2025
MODULE : Algebra 2	**Midterm Test**	DURATION : 2h
FULL NAME :		*NOTE : No documents are allowed.*

Exercise 1 : (2,5 points)

Consider the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

1. Compute A^k for $k \in \mathbb{N}$.
2. Let $P \in \mathbb{R}_n[X]$, with $n \geq 2$. Prove that :

$$P(A) = \begin{pmatrix} P(2) & P'(2) & \frac{1}{2}P''(2) \\ 0 & P(2) & P'(2) \\ 0 & 0 & P(2) \end{pmatrix}.$$

Exercise 2 : (2,5 points)

Let $A = \begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix}$.

1. Show that $P(X) = X^2 - X$ is an annihilating polynomial for A. Deduce, by a proof by contradiction, that A is not invertible. Justify, using the same reasoning, that $A - I$ is also not invertible.
2. Verify using the pivot method that A is indeed not invertible.
3. Use a third method to prove differently that A is not invertible.

Exercise 3 : (4 points)

We consider the matrix K given by :

$$K = \begin{pmatrix} 1 & 1 & -1 & -3 \\ 1 & 1 & 1 & -2 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & 0 & -2 \end{pmatrix}$$

1. Compute K^2. Deduce that K is invertible and compute K^{-1}.
2. Let a and b be two real numbers ; we define $L = aI + bK$. Show that $L^2 = -(a^2+b^2)I + 2aL$.
3. Deduce that if a and b are not both zero, L is invertible, and express L^{-1} in the form $cI + dK$.
4. Deduce the inverse of the matrix $A = \begin{pmatrix} 1+\sqrt{2} & 1 & -1 & -3 \\ 1 & 1+\sqrt{2} & 1 & -2 \\ 0 & -1 & \sqrt{2} & 1 \\ 1 & 1 & 0 & -2+\sqrt{2} \end{pmatrix}$.

Exercise 4 : (4 points)

1. We consider the set of complex numbers $\mathbb{C}$. We define the collection $\mathcal{F} = \{1, i\}$.
   (a) In this question, we consider $\mathbb{C}$ as an $\mathbb{R}$-vector space. Prove that the collecion $\mathcal{F}$ is linearly independent.
   (b) In this question, we consider $\mathbb{C}$ as a $\mathbb{C}$-vector space. Prove that the collection $\mathcal{F}$ is linearly dependent.
2. Let $\omega \in \mathbb{C}$. We define $\mathbb{R} \cdot \omega = \{x \cdot \omega \mid x \in \mathbb{R}\}$.
   (a) Show that $\mathbb{R} \cdot \omega$ is a subspace of $\mathbb{C}$ when $\mathbb{C}$ is considered as an $\mathbb{R}$-vector space.
   (b) Under what condition is $\mathbb{R} \cdot \omega$ a subspace of $\mathbb{C}$ when $\mathbb{C}$ is considered as a $\mathbb{C}$-vector space ?

Exercise 5 : (7 points)

Let $E = \mathcal{C}^2(\mathbb{R}, \mathbb{R})$ be the vector space of twice continuously differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. We consider the set F of elements φ of E such that :

$$\forall x \in \mathbb{R}, \quad \varphi''(x) = (1 + x^2)\varphi(x).$$

Moreover, we denote by f and g the functions defined for all $x \in \mathbb{R}$ by :

$$f(x) = e^{\frac{x^2}{2}}, \quad g(x) = e^{\frac{x^2}{2}} \int_0^x e^{-t^2} dt.$$

1. Show that F is a subspace of the vector space $\mathcal{C}^2(\mathbb{R}, \mathbb{R})$.
2. Show that if $v, w \in F$, then the function $vw' - wv'$ is constant on $\mathbb{R}$.
3. Show that the functions f and g belong to F, and that the collection (f, g) is linearly independent.
4. Let $h \in F$. Show that there exist $(a, b) \in \mathbb{R}^2$ such that $h = af + bg$.
5. Deduce the dimension of the vector space F.