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Mathematical Analysis 1
Real Functions--Draft Version
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1 Real Functions: Definition, Properties and Examples

A real function is a type of function that $\boxed{\textbf{accept real values as input}}$ and $\boxed{\textbf{return real values as output}}$ in other words

Real function is a kind of **mapping between two sets of real numbers** namely from set A to set B that follows some important properties mentioned below:

1. $\boxed{\textbf{All elements}}$ of set A are associated with elements in the set B.
2. $\boxed{\textbf{Each element}}$ of set A is associated with $\boxed{\text{a } \textbf{unique element}}$ in the set B.

A function from set A to set B is written as

$$f : A \to B,$$

where

1. The set A is the **domain** denoted by D_f of the function, i.e. **set of input values**
$$\{x : f(x) \text{ exists}\}$$
2. The set B is called the **codomain** of the function which consists of the **output values of the function**.
$$\{f(x) : x \in \textbf{ domain} \text{ of the function}\}$$

Notation 1 *We write*

$$\begin{array}{lll} f: & D_f & \to J \subset \mathbb{R} \\ & x & \mapsto f(x) \end{array} \tag{1}$$

Remark 2 *The notation $f : D_f \to J \subset \mathbb{R}$ is read as "f* ***goes from*** *D_f* ***to*** *J" and $x \mapsto f(x)$ is read as "****x maps to f(x)***

Remark 3 *We can represent functions in four ways,*

1. ***visually*** *(a graph on the $xy-$plane)*

2. ***numerically*** *(using a table of values)*

3. ***algebraically*** *(by an explicit formula).*

Example 4 *Examples of different kinds of real functions are*

1. **Trigonometric** Functions: $\sin x, \cos x, \tan x,$, etc.

2. **Logarithmic** Functions: $\log(2x+3), 2\ln x$, etc.

3. **Exponential** Functions: a^x, e^x, etc.

4. **Modulus** Function: $|x|, |2x+3|$, etc.s

Definition 5 *The set of definition of f, denoted D_f , is the set of elements in I that have* ***exactly one image*** *in J under the function f. It is written as*

Graph of a function

Definition 6 *Let $f : D_f \to J \subset \mathbb{R}$. The graph of f or the representative curve of f, denoted G_f , is a subset of the set $D_f \times J$ given by*

$$G_f = \{(x, f(x)) : x \in D_f\}$$

Definition 7 *In mathematics,*

1. *the (1) is* $\boxed{\textbf{\textit{one-to-one or injective}}}$ *is a function f that maps* ***distinct elements*** *of its domain to* ***distinct*** *elements; that is,*

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

2. *the (1) is* $\boxed{\textbf{\textit{surjective}}}$ *(also known as surjection, or* ***onto function*** *) is a function f such that, for every element y of the* ***function's codomain****, there e****xists at least one element*** *x in the function's domain such that $f(x) = y$*

$$\forall y \in J : \exists x \in D_f, y = f(x)$$

3. *the (1) is a* $\boxed{\textbf{\textit{bijection}}}$ *if it is bothone toone and surjection*

1.1 Composition of Functions

Definition 8 *Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be two functions such that*

$$f(I) \subset J.$$

The composite function of f and g, denoted $g \circ f$ (read as g ***circle*** *f), is defined for all $x \in I$ by*

$$(g \circ f)(x) = g(f(x)).$$

It has the form

$$\begin{array}{ll} g \circ f & I \to J \to \mathbb{R} \\ & x \mapsto f(x) \mapsto g(f(x)) \end{array}$$

Exercise 9 *Let the function*

$$f :]0, +\infty[\to \mathbb{R}$$

be given with the formula

$$f\left(\frac{1}{x}\right) = x + \sqrt{1 + x^2}$$

find the formula of f

Solution 10

$$\begin{aligned} f\left(\frac{1}{x}\right) &= x + \sqrt{1 + x^2} \\ &= x\left(1 + \sqrt{\frac{1}{x^2} + 1}\right) \\ &= \frac{1}{\frac{1}{x}}\left(1 + \sqrt{\frac{1}{x^2} + 1}\right) \end{aligned}$$

it follows that

$$f(x) = \frac{1}{x} + \sqrt{x^2 + 1}$$

Exercise 11 *Let the function*

$$f :]0, +\infty[\to \mathbb{R}$$

be given with the formula

$$f(x - 2) = \frac{1}{x + 3}, x \neq -3$$

find the formula of f

Solution 12 *Putting*

$$t = x - 2$$

we get

$$x = t + 2$$

clearly if $x \neq -3$ *then* $t \neq -5$. *Hence*

$$f(t) = \frac{1}{t + 5}, t \neq -5$$

Exercise 13 *Let the function*

$$f : \mathbb{R} - \{-1\} \to \mathbb{R};$$

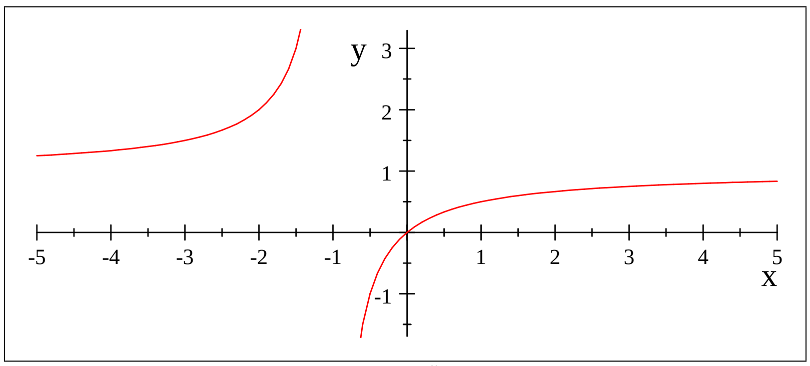
be given with the formula

$$f(x) = \frac{x}{x+1}, x \neq -1$$

find the function $f_n, n \in \mathbb{N}$ *where*

$$f_1 = f; \;\; f_2 = f \circ f, \;\; ......$$

Solution 14 *The range of the function* f *is* $\mathbb{R} - \{-1\}$



$x \mapsto \frac{x}{x+1}$

then it holds

$$\begin{aligned} f_2(x) &= (f \circ f)(x) = f[f(x)] \\ &= f\left[\frac{x}{x+1}\right] = \frac{\frac{x}{x+1}}{1 + \frac{x}{x+1}} = \frac{1}{1+2x}. \end{aligned}$$

thought the definition of f_2 *reduces the domain to the set*

$$\mathbb{R} - \left\{-1, -\frac{1}{2}\right\}.$$

by induction we will prove the following assertion

$$P(n) : f_n(x) = \frac{1}{1+nx}, \mathbb{R} - \left\{-1, -\frac{1}{2}, ..., -\frac{1}{n}\right\}.n > 1$$

We just proved that $P(2)$ *(base case).Suppose it holds true for* $P(n)$ *(induction hypothesis) Then*

$$P(n+1) : f_{n+1}(x) = f_n\left[\frac{1}{1+nx}\right] = \frac{\frac{1}{1+nx}}{1 + \frac{1}{1+nx}} = \frac{1}{1+(n+1)x}, \mathbb{R} - \left\{-1, -\frac{1}{2}, ..., -\frac{1}{n}, -\frac{1}{n+1}\right\}.n > 1$$

1.2 Even and Odd Functions

Definition 15 *Let $f : I \to J$ be a real function such that the domain I is* ***symmetric*** *with respect to the origin, i.e., for all $x \in I, -x \in I$. We say that:*

1. *f is* ***even*** *if for all $x \in I, f(-x) = f(x)$.*
2. *f is* ***odd*** *if for all $x \in I, f(-x) = -f(x)$.*

Remark 16 *Geometrically*

1. *Any constant function on I is even*
2. *If f is* ***even****, its graph is symmetric with respect to* ***the*** *$y-$****axis****.*
3. *If f is* ***odd****, its graph is symmetric with respect to* ***the origin***

Example 17 *Show that function*

$$f : \mathbb{R} \to \mathbb{R};$$

given by

$$f(x) = g(|x|)$$

is an even function for any function

$$g : \mathbb{R} \to \mathbb{R}.$$

In fact, for every $x \in \mathbb{R}$:

$$f(-x) = g(|-x|) = g(|x|) = f(x)$$

Example 18 *The function*

$$f : \mathbb{R} \to \mathbb{R};$$

given by

$$f(x) = \frac{e^{2x} - 1}{e^{2x} + 1}$$

is an odd function. In fact

$$f(-x) = \frac{e^{-2x} - 1}{e^{-2x} + 1} = \frac{e^{-2x}\left(1 - e^{2x}\right)}{e^{-2x}\left(1 + e^{2x}\right)} = \frac{\left(1 - e^{2x}\right)}{\left(1 + e^{2x}\right)} = -\frac{\left(e^{2x} - 1\right)}{\left(1 + e^{2x}\right)} = -f(x)$$

Example 19 *Let$f, g : \mathbb{R} \to \mathbb{R}$ two real odd functions. What about $f \circ g$?*

$$(f \circ g)(-x) = f[g(-x)] = f[-g(x)] = -f[g(x)]$$

1.3 Periodic Functions

Definition 20 *Let $f : D_f \to \mathbb{R}$ be a real function. Let $T \in \mathbb{R}_+^*$ such that for all*

$$x \in I, x \pm T \in I.$$

f is called $T-$periodic if for all $x \in D_f$,

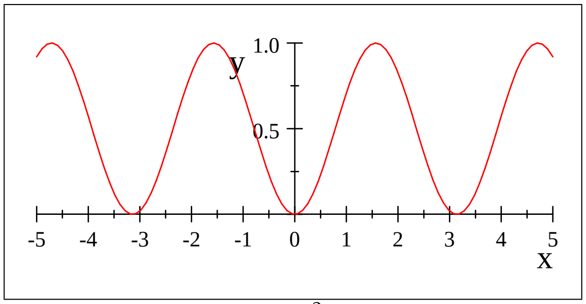
$$f(x \pm T) = f(x).$$

Exercise 21 *Study the periodicity of the functions*

$$f(x) = \sin^2(x) \qquad g(x) = \sin\left(x^2\right) \qquad h(x) = \sin(|x|)$$

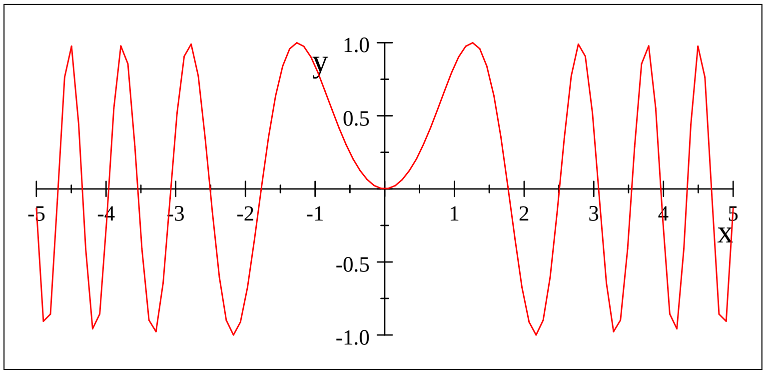
Solution 22 *It holds that*

1.
$$f(x) = \sin^2(x) = \frac{1 - \cos(2x)}{2}$$
since the function $x \mapsto \cos(2x)$ is periodic with a period π. Then, f is also periodic with basic period π. $\sin^2(x)$



$x \mapsto \sin^2(x)$

2. *The zeros of $g(x) = \sin\left(x^2\right)$ are fo the form $\sqrt{k\pi}, k \in \mathbb{N},$. We show that the distance between zeros tens to* 0.
$$\lim_{k \to +\infty} \sqrt{(k+1)\pi} - \sqrt{k\pi} = \lim_{k \to +\infty} \frac{1}{\sqrt{(k+1)\pi} + \sqrt{k\pi}} = 0.$$
This implies that this function is not periodic



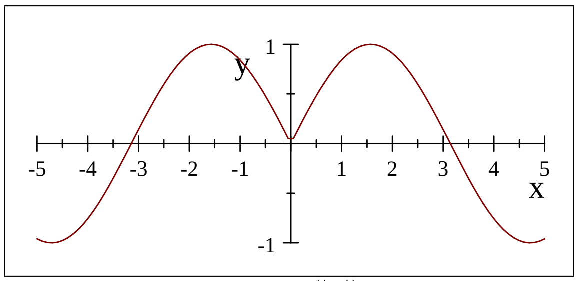
$$x \mapsto \sin\left(x^2\right)$$

3. *Only and are candidates as periods of this function. However*

$$h\left(-\frac{\pi}{2}\right) = 1, h\left(-\frac{\pi}{2} + 2\pi\right) = h\left(-\frac{3\pi}{2}\right) = -1$$

$$h\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, h\left(\frac{\pi}{4} + \pi\right) = h\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

Hence, it is not pêriodic function.$\sin(|x|)$



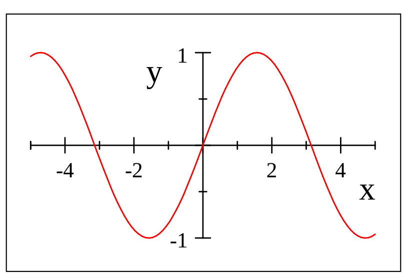
$x \mapsto \sin(|x|)$

1.4 Monotony of functions

Definition 23 *Consider $f : D_f \subset \mathbb{R} \to \mathbb{R}$. Let $x_1, x_2 \in M$. Then.:*

1. f is called increasing function if and only for all $x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$.
2. f is called decreasing function if and only if all $x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$.
3. f is **monotonic** if and only if f is increasing $\boxed{\textbf{or}}$ decreasing

Remark 24 *Every monotonic function is injective but the converse is not true. In fact, consider the function $x \mapsto \sin(x)$ in the interval $]0, 2\pi[$. The sine function is **one-to-one** over this interval (since each unique $x-$value maps to a unique $y-$value), but it is not strictly monotonic, because it increases from 0 to $\frac{\pi}{2}$, decreases from$\frac{\pi}{2}$ to $\frac{3\pi}{2}$, and then increases again from $\frac{3\pi}{2}$ to 2π.*



$x \mapsto \sin x$

1.5 Bounded Function, Upper Bounded, Lower Bounded, Bounded

Definition 25 *Let $f : D_f \to \mathbb{R}$ be a real function. f is called:*

1. *upper-bounded if and only if* ***there exists*** *$M \in \mathbb{R}$ such that* ***for every*** *$x \in D_f$, $f(x) \leq M$.*

2. *lower-bounded if and only if there exists $m \in \mathbb{R}$ such that for every $x \in D_f$, $f(x) \geq m$.*

3. *bounded if and only if there exist M, $m \in \mathbb{R}$ such that for every $x \in D_f$, $m \leq f(x) \leq M$.*

Remark 26 *f is called bounded if and only if $|f|$is upper-bounded.*