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Algebra 1 - Tutorial 5

Basic Training Cycle
Polynomials and Rational Fractions

Exercise 1

Determine the degrees and leading coefficients of the following polynomials :

1. $P_3 = (1+X)^n + (1-X)^n$, $n \in \mathbb{N}^*$
2. $P_4 = (X+3)^n - (X-2)^n$
3. $P_5 = \prod_{k=0}^{n} (2X-k)$
4. $P_6 = \prod_{k=0}^{n} (X-2)^k$
5. $P_7 = (X+1)^{2n} - X^{2n-1}(X+2n)$

Exercise 2

State whether each of the following functions is a polynomial function and justify your answer :

1. $f_1 : x \mapsto \sin(x)$
2. $f_2 : x \mapsto \dfrac{1}{x}$
3. $f_3 : x \mapsto \dfrac{1}{1+x^2}$
4. $f_4 : x \mapsto e^x$
5. $f_5 : x \mapsto \dfrac{(x+1)^{42} - (1-x)^{42}}{x}$

Exercise 3

Let n be a natural number.
Prove that there does not exist a polynomial P such that : $P^2 = X(X^{2n}+1)$.

Exercise 4

Determine all real polynomials P such that :

1. $P(1) = 0$ and $P(2) = 0$
2. $P(1) = 1$ and $P(2) = 2$
3. $P^3 = X^2 P$
4. $P(X) = \frac{1}{2} X P'(X)$ where $P \in \mathbb{R}_2[X]$
5. $(X^2+1)P'' = 6P$
6. $P \circ P = P$
7. $P(0) = 0$, $P(1) = 1$, $P'(0) = 2$ et $P'(1) = 3$.

Exercise 5

Let E be the set of polynomials $P \in \mathbb{C}[X]$ satisfying $P(X^2+1) = P^2+1$ et $P(0) = 0$.

1. In this question, we consider a polynomial $P \in \mathbb{C}[X]$ and assume that $P \in E$.

   a. Compute $P(1)$, $P(2)$, and $P(5)$.

      We consider the sequence $(u_n)_{n\in\mathbb{N}}$ defined by $u_0 = 0$, and : $\forall n \in \mathbb{N}, u_{n+1} = u_n^2 + 1$.

   b. Prove that for every natural number n, $P(u_n) = u_n$.

   c. Prove that the sequence $(u_n)_{n\in\mathbb{N}}$is strictly increasing.

   d. Deduce that $P(X) = X$.

2. What is the set E ?

Exercise 6

What can be said about three elements P, Q and R in $\mathbb{R}[X]$, if they satisfy the equation $P^2 - XQ^2 + R^2 = 0$?

Exercise 7

Let P be the polynomial defined by $P(X) = (X-1)^4$. Compute $P^{(k)}(1)$ for any integer $k > 0$.
Generalize this result for $P(X) = (X-a)^n$, (with $a \in \mathbb{R}, n > 0$).

Exercise 8

Use the Euclidean division of A by B in the following cases :

1. $A = 4X^4 + X^3 - 2X^2 - 5$ et $B = 2X^2 + X + 1$.

2. $A = iX^3 - X^2 + 1 - i$ et $B = (1+i)X^2 - iX + 3$.

Exercise 9

The remainder of the division of a polynomial $A(X)$ by $X-1$ is 1, the remainder of $A(X)$ by $X+1$ is -1, and the remainder of $A(X)$ by $X-2$ is 2.
What is the remainder of the division of $A(X)$ by $(X-1)(X+1)(X-2)$?

Exercise 10

Let $n \in \mathbb{N}^*$. Determine the remainder of the Euclidean division of A by B :

1. $A = X^{2n} + X^n + X + 1$ et $B = (X-1)^2$.

2. $A = (X-1)^n + (X+1)^n - 1$ et $B = X^2 - 1$.

3. $A = X^{2n} + 2X^n + 1$ et $B = X^2 + 1$.

Exercise 11

Prove that : :

1. $(X+1)^{2n+1} + X^{n+2}$ is divisible by $X^2 + X + 1$ $(n \in \mathbb{N})$.
2. $\left(\sum_{k=0}^{n-1} X^k\right)^2 - n^2 X^{n-1}$ is divisible by $(X-1)^2$ $(n \geqslant 2)$. Is it also divisible $(X-1)^3$?

Exercise 12

Let $n > 2$ be a natural number. Determine all polynomials in $\mathbb{R}_n[X]$ that are divisible by $X+1$ and whose remainders in the Euclidean division by $X+2$, $X+3$, $\ldots$, $X+n+1$ are equal.

Exercise 13

Determine $n \in \mathbb{N}$ such that $(X+1)^n - X^n - 1$ is divisible by $P = X^2 + X + 1$.

Exercise 14

Let $n \in \mathbb{N}$, and define $P = (X^2 - 1)^n$.

1. Calculate $P^{(n)}$ using the Leibniz's formula.
2. Compute the leading coefficient of $P^{(n)}$ using two different methods and deduce the value of $\sum_{k=0}^{n} \binom{n}{k}^2$.

Exercise 15

Determine the gcd and lcm of P and Q, as well as a Bézout identity, for the following cases :

1. $P = X^4 + X^3 - 3X^2 - 4X - 1$ et $Q = X^3 + X^2 - X - 1$.
2. $P = X^4 + X^3 - 2X + 1$ et $Q = X^2 + X + 1$.

Exercise 16

Prove that $(X-1)^2$ and $(X+1)^2$ are coprime.

Exercise 17

We Define a sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ by setting $P_0 = X$ and $P_{n+1} = (P_n - 2)^2$ for all $n \in \mathbb{N}$. Determine the remainder of the Euclidean division of P_n by X^3.

Exercise 18

Without expanding, show that the polynomial $P(X) = (X-3)^2 - 2(X-2)^2 + (X-1)^2 - 2$ is the zero polynomial.

Exercise 19

Let $p, q, r \in \mathbb{N}$. Prove that $P = X^{3p+2} + X^{3q+1} + X^{3r}$ is divisible by $Q = X^2 + X + 1$ in $\mathbb{R}[X]$, using two different methods : :

1. By using the roots of the polynomial Q.
2. By computing $P - Q$ and factoring $X^3 - 1$.

Exercise 20

1. Show that -1 is a triple root of $P(X) = X^5+2X^4+2X^3+4X^2+5X+2$. Deduce its factorization.
2. Determine the multiplicity order of the root 1 for the polynomials : $P(X) = X^{2n} - nX^{n+1} + nX^{n-1} - 1$ et $Q(X) = X^{2n+1} - (2n+1)X^{n+1} + (2n+1)X^n - 1$.

Exercise 21

Factor the following polynomials in $\mathbb{C}[X]$ and then in $\mathbb{R}[X]$:

1. $X^3 - 5X^2 + 3X + 9$.
2. $X^4 + 3X^3 - 14X^2 + 22X - 12$, knowing that $i + 1$ is a complex root.
3. $2X^4 - 3X^2 - 2$.
4. $X^4 + X^3 - X - 1$.
5. $(X^2 - X + 1)^2 + 1$.
6. $(X^2 - X + 2)^2 + (X - 2)^2$.
7. $X^6 - 7X^3 - 8$.
8. $X^6 + 1$.
9. $6X^5 + 15X^4 + 20X^3 + 15X^2 + 6X + 1$.
10. $X^6 - X^5 + X^4 - X^3 + X^2 - X + 1$

Exercise 22

1 Factor the polynomial $P(X) = (X+1)^n - (X-1)^n$ in $\mathbb{C}[X]$ ($n \geqslant 2$).

2 Deduce that for all $p \in \mathbb{N}^*$, $\displaystyle\prod_{k=1}^{p} \operatorname{cotan} \frac{k\pi}{2p+1} = \frac{1}{\sqrt{2p+1}}$.

Exercise 23

Solve the following system in $\mathbb{C}^3$

$$\left\{\begin{array}{ccccccc} x & + & y & + & z & = & 1 \\ x^2 & + & y^2 & + & z^2 & = & 1 \\ x^3 & + & y^3 & + & z^3 & = & -5 \end{array}\right.$$

To do this, we will look for a monic polynomial of degree 3 with roots x, y, and z, and we will calculate each of its coefficients using the given conditions.

Exercise 24

Consider the polynomials :

$$P(X) = X^8 - 2X^4 + 8X^3 + 1 \quad \text{and} \quad Q(X) = X^6 - X^4 - X^2 + 1.$$

1 Find the obvious roots of $Q(X)$. What is their multiplicity ?

2 Decompose $Q(X)$ into a product of irreducible factors in $\mathbb{R}[X]$.

3 Provide the theoretical form of the partial fraction decomposition of the rational fraction $F(X) = \frac{P(X)}{Q(X)}$ over $\mathbb{R}$.

4 Calculate the coefficients of this decomposition.

Exercise 25

Decompose the following fractions into sums of partial fractions in $\mathbb{R}$:

1 $A(X) = \frac{X^2+2X+5}{X^2-3X+2}$

2 $B(X) = \frac{X^3-X+2}{X(X^2+X+1)(X^2+1)^4}$

3 $E(X) = \frac{n!}{X(X+1)\cdots(X+n)} \quad (n \geq 0)$

4 $F(X) = \frac{X^5+4}{X^4+4X^2}$

Additional Exercises

Exercise 26

Let $P(X) = X^3 - 2X^2 - 5X + 6$.

1. Determine an obvious root of the polynomial P.
2. Factor P in the form $(X+2)Q(X)$, where Q is a polynomial of degree 2 .
3. Deduce the sign table of P on $\mathbb{R}$.
4. Solve the inequalities $(\ln x)^3 - 2(\ln x)^2 - 5\ln x + 6 > 0$ and $e^{2x} - 2e^x \leqslant 5 - 6e^{-x}$.

Exercise 27

Let $(a, n) \in \mathbb{R}^* \times \mathbb{N}^*$.
We Define $A(X) = X^2 - 2X \operatorname{ch}(a) + 1$ and $P_n(X) = X^{n+1}\operatorname{sh}(na) - X^n \operatorname{sh}((n+1)a) + \operatorname{sh}(a)$.
Show that A divides P_n in $\mathbb{R}[X]$ and determine the quotient by seeking to factor A in the explicit expression of P_n.

Exercise 28

Let $P = aX^3 + bX^2 + cX + d \in \mathbb{C}[X]$ with $a \neq 0$. We denote α_1, α_2 and α_3 as the three roots of P.

1. Express $\alpha_1 + \alpha_2 + \alpha_3$, $\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1$ and $\alpha_1\alpha_2\alpha_3$ in terms of a, b, c and d.
2. Expanding $(\alpha_1 + \alpha_2 + \alpha_3)^2$, express $\alpha_1^2 + \alpha_2^2 + \alpha_3^2$ in terms of a, b, c and d.
3. Provide a necessary and sufficient condition on d such that 0 is not a root of P, and in this case, compute the expression $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3}$ in terms of a, b, c and d.

Exercise 29

Given $(n+1)$ distinct complex numbers $(a_0, a_1, \cdots, a_n)$ and $(n+1)$ complex numbers $(b_0, b_1, \cdots, b_n)$, we seek a polynomial P of minimal degree such that

$$\forall i \in [\![0, n]\!], \quad P(a_i) = b_i.$$

Proposition. Such a polynomial exists. It is moreover unique if we assume that $\deg(P) \leq n$.

1. We define $L_i = \displaystyle\prod_{k \neq i} \frac{X - a_k}{a_i - a_k}$. Show that $L_i \in \mathbb{C}_n[X]$ for all i, and calculate $L_i(a_j)$.
   Deduce the existence of the polynomial $P \in \mathbb{C}_n[X]$ such that $\forall i \in [\![0, n]\!], P(a_i) = b_i$.
2. Prove the uniqueness of such a polynomial. Is there always uniqueness if we do not assume $\deg(P) \leq n$?

Exercise 30

We consider the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ defined by :

$$\begin{cases} P_0 = 1 \text{ and } P_1 = X \\ \forall n \in \mathbb{N}, \quad P_{n+2} = 2XP_{n+1} - P_n \end{cases}$$

1 For all $n \in \mathbb{N}$, determine the parity, the degree, and the leading coefficient of P_n.

2 (a) Show that for all $a, b \in \mathbb{R}$, we have $2\cos(a)\cos(b) = \cos(a+b) + \cos(a-b)$.

(b) Establish that for all $n \in \mathbb{N}$ and for all $x \in \mathbb{R}$, $P_n(\cos(x)) = \cos(nx)$.

3 (a) Let $n \in \mathbb{N}^*$. Solve the equation $\cos(n\theta) = 0$ on $[0, \pi]$.

(b) Deduce that P_n splits in $\mathbb{R}$ and determine its roots.

(c) Give a factored expression of $P_n(X)$.

(d) By calculating $P_n(0)$ using two different methods, show that :

$$\prod_{k=0}^{n-1} \cos\left(\frac{(2k+1)\pi}{2n}\right) = \begin{cases} \frac{(-1)^{n/2}}{2^{n-1}} & \text{if } n \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

4 Using De Moivre's formula and Newton's binomial theorem, give another expression for $P_n(X)$.

Exercise 31

Determine $m > 0$ such that the polynomial $P(X) = X^4 - (3m+2)X^2 + m^2$ has four roots in arithmetic progression.

Exercise 32

Let P be a polynomial of degree n such that for all $k \in \{0, \ldots, n\}$,

$$P(k) = \frac{k}{k+1}.$$

Determine $P(n+1)$.

Exercise 33

Find $a \in \mathbb{R}$ such that the polynomial $P = (1-a)x^3 + (2+a)x^2 + (a-1)x + 7 - a$ has a complex root with modulus 1. Find the other roots.