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Lecturer : H. MOUFEK

Algebra 2 - Tutorial 3

Basic Training Cycle

Linear Transformations

Exercise 1

We define the matrix $A = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$ and the map $\varphi : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$ by $\varphi(M) = AMA$. We consider the following basis of $\mathcal{M}_2(\mathbb{R})$:

$$\mathcal{B} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

1. Show that φ is an endomorphism of $\mathcal{M}_2(\mathbb{R})$.
2. Determine the matrix B of φ in the basis $\mathcal{B}$.
3. Using the matrix B, determine the rank of φ.
4. Is φ an automorphism ?

Exercise 2

Let $f : \mathbb{R}^3 \to \mathbb{R}^2$, defined by $(x, y, z) \mapsto (2x - y, x + z)$.

1. Verify that f is linear.
2. Determine $\ker(f)$ and $\mathrm{Im}(f)$.
3. Discuss the injectivity, surjectivity, and bijectivity of f.
4. Determine the matrix of f relative to the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$.
5. Determine the matrix of f relative to the bases :

$$\mathcal{B}_1 = ((1, 0, 0), (1, 1, 0), (1, 1, 1)) \quad \text{and} \quad \mathcal{B}_2 = ((1, 0), (1, 1))$$

Exercise 3

Let $n \in \mathbb{N}$, $a \in \mathbb{R}$, and $\varphi : \mathbb{R}_n[X] \to \mathbb{R}_n[X]$ be the map defined by :

$$\forall P \in \mathbb{R}_n[X], \quad \varphi(P) = P(X - a)$$

1. Show that φ is an automorphism of $\mathbb{R}_n[X]$.
2. Deduce that $(1, X - a, (X - a)^2, \dots, (X - a)^n)$ is a basis of $\mathbb{R}_n[X]$.