2025/2026

Lecturer : H. MOUFEK

Algebra 1 - Tutorial 2

Basic Training Cycle

Sets and Relations

Exercise 1

1 Complete with, as appropriate, $\in, \notin, \subset, \not\subset, =$.
$\mathbb{N} \ldots \mathbb{C}$, $\frac{2}{3} \ldots \mathbb{Z}$, $[0,7[\ldots \mathbb{R}$, $[-1,8] \ldots \mathbb{Z}$, $\frac{\sqrt{2}}{2} \ldots \mathbb{Q}$, $]-1,1[\times\{0\} \ldots \mathbb{R}^2$,
$\{(0,1)\} \ldots \mathbb{R}^2$, $\{1\} \ldots \mathbb{N}$, $\mathbb{R} \times \mathbb{R}^* \ldots (\mathbb{R} \times \mathbb{R}) \backslash \{(0,0)\}$ $(\mathbb{R} \times \mathbb{R}^*) \cap (\mathbb{R}^* \times \mathbb{R}) \ldots (\mathbb{R}^*)^2$
$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \ldots \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$, $(i,1) \ldots \mathbb{N} \times \mathbb{C}$, $\{0;2\} \ldots$ $\mathcal{P}(\mathbb{N})$,
$\{(m, 2m, -m) \mid m \in \mathbb{R}\} \ldots \mathbb{R}^3$, $\{(a, a+b, c+d) \mid (a,b,c,d) \in \mathbb{R}^4\} \ldots \mathbb{R}^4$

2 Let $E = \{x, y, z\}$ be a set. Determine whether the following statements are true or false :
$x \in E$, $\{x\} \subset E$, $\varnothing \subset E$, $\varnothing \in E$, $\{\varnothing\} \subset E$, $\varnothing \in \mathcal{P}(E)$, $\{\varnothing\} \subset \mathcal{P}(E)$,
$\mathcal{P}(\{x; y\}) \subset \mathcal{P}(E)$.

Exercise 2

We define the following five sets : $A_1 = \{(x,y) \in \mathbb{R}^2, x+y<1\}$, $A_2 = \{(x,y) \in \mathbb{R}^2, x+y>-1\}$,
$A_3 = \{(x,y) \in \mathbb{R}^2, |x+y|<1\}$, $A_4 = \{(x,y) \in \mathbb{R}^2, |x-y|<1\}$,
$A_5 = \{(x,y) \in \mathbb{R}^2, |x|+|y|<1\}$.

1 Graphically, represent in an orthonormal system $(O, \vec{i}, \vec{j})$ the lines of equations : $x-y=1$, $x-y=-1$, $x+y=1$ and $x+y=-1$.

2 Hatch the regions representing the sets A_1 to A_5 on 5 different graphs.

3 Show geometrically, then by reasoning, that :

$$\forall (x,y) \in \mathbb{R}^2, (|x+y|<1 \text{ and } |x-y|<1) \Leftrightarrow |x|+|y|<1$$

Exercise 3

Show that :

1 $E = \{(m, 2m, -m) \mid m \in \mathbb{R}\} \subset F = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+3z=0\}$.

2 $E = \{t \longmapsto \lambda \sin(t) + \mu \cos(t) \mid \lambda, \mu \in \mathbb{R}\} \subset F = \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) \mid f'' = -f\}$.

3 $E = \{(x,y) \in \mathbb{R}^2 \mid y = 3x+1\} = F = \{(a+1, 3a+4), a \in \mathbb{R}\}$.

Exercise 4

1 Find $\{(t, t+1) \mid t \in \mathbb{R}\} \cap \{(x,y) \in \mathbb{R}^2 \mid x+y=0\}$.

2 Find $\{t \longmapsto \lambda\cos(t) + \mu e^{-t} \mid \lambda, \mu \in \mathbb{R}\} \cap \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) \mid f'' = -f\}$.

Exercise 5

Let E be a set and A, B, C be subsets of E. Prove the following propositions :

1 $(A \cup C) \cap \overline{(A \cup B)} = \bar{A} \cap C \cap \bar{B}$

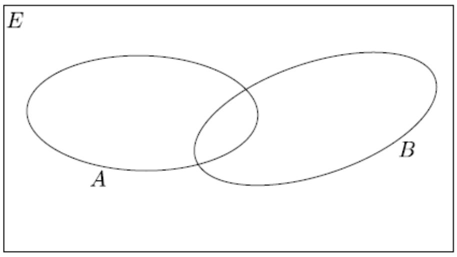
2 $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Exercise 6

Let A and B be two parts of a set E.
The set $A \Delta B = (A \backslash B) \cup (B \backslash A)$ is called the symmetric difference of $\boldsymbol{A}$ and $\boldsymbol{B}$.

1 Describe the set $A\Delta B$ in a simple sentence.

2 After reproducing this drawing on your paper, hatch the part corresponding to $A\Delta B$.



3 Show that $A\Delta B = (A \cup B) \backslash (A \cap B)$.

4 Let C be a subset of E. Prove that $A\Delta C = B\Delta C \Leftrightarrow A = B$.

Exercise 7

1 Let $x \in \mathbb{R}$. What is the mathematical meaning of $x \in \bigcup_{n \in \mathbb{N}^+} \left[1 - \frac{1}{n}, 1 + \frac{1}{n}\right]$ and $x \in \bigcap_{n \in \mathbb{N}^*} \left[1 - \frac{1}{n}, 1 + \frac{1}{n}\right]$?

2 Without proof, give the value of $\bigcup_{n \in \mathbb{N}^*} \left[1 - \frac{1}{n}, 1 + \frac{1}{n}\right]$.

3 Show by double inclusion that $\bigcap_{n \in \mathbb{N}^*} \left[1 - \frac{1}{n}, 1 + \frac{1}{n}\right[= \{1\}$.

Exercise 8

Let I be a set and $\{A_i\}_{i\in I}$ a subset of $\mathcal{P}(E)$. Show that :

1 $\overline{\left(\bigcup_{i\in I} A_i\right)} = \bigcap_{i\in I} \overline{A_i}$.

2 $\overline{\left(\bigcap_{i\in I} A_i\right)} = \bigcup_{i\in I} \overline{A_i}$.

Exercise 9

Let E be a set.

1 Let $A, B \in \mathcal{P}(E)$. Solve the equation for the unknown $X \in \mathcal{P}(E)$:

$$X \cup B = A \cup B.$$

2 Let $B \in \mathcal{P}(E)$. Consider the following relation :

$$\forall X, Y \in \mathcal{P}(E), \quad X\mathcal{R}Y \iff X \cup B = Y \cup B.$$

a Show that $\mathcal{R}$ is an equivalence relation on $\mathcal{P}(E)$.

b Let $X \in \mathcal{P}(E)$. Find : $\mathrm{cl}_{\mathcal{R}}(X)$.

Exercise 10

We define a relation $\mathcal{R}$ on $\mathbb{R}$ by $x\mathcal{R}y$ if and only if $\cos^2 x + \sin^2 y = 1$.
Show that $\mathcal{R}$ is an equivalence relation and determine its equivalence classes.

Exercise 11

We define a binary relation $\mathcal{R}$ on $\mathbb{Z}$ by $x\mathcal{R}y$ if and only if $x + y$ is even. Show that $\mathcal{R}$ is an equivalence relation and specify its equivalence classes.

Exercise 12

Let $\mathcal{R}$ be a relation defined on the set Z_+^* : for $a, b \in Z_+^*$, $a\mathcal{R}b \iff a \mid b$.

1 Show that $\mathcal{R}$ is an ordering relation on $\mathbb{Z}_+^*$.

2 In case $\mathcal{R}$ is defined on $\mathbb{Z}$, is $\mathcal{R}$ still an ordering relation on $\mathbb{Z}$? Why ?

3 Draw the Hasse diagram representing the ordering relation $\mathcal{R}$ on the set

$$A = \{1, 2, 3, 4, 6, 8, 12, 24\}.$$

4 Decide whether $\mathcal{R}$ is a total ordering relation on Z_+^*, and justify.

5 In case $\mathcal{R}$ is defined on $B = \{1, 3, 9, 27, 81\}$, is $\mathcal{R}$ a total ordering relation on B ? Why ?

6 Draw the Hasse diagram representing the ordering relation $\mathcal{R}$ on the set

$$B = \{1, 3, 9, 27, 81\}.$$