

## Exercise 1

1 Complete with, as appropriate,  $\in, \notin, \subset, \not\subset, =$ .

$$\begin{aligned} \mathbb{N} &\dots \mathbb{C}, \quad \frac{2}{3} \dots \mathbb{Z}, \quad [0, 7] \dots \mathbb{R}, [-1, 8] \dots \mathbb{Z}, \quad \frac{\sqrt{2}}{2} \dots \mathbb{Q}, \quad ]-1, 1[ \times \{0\} \dots \mathbb{R}^2, \\ \{(0, 1)\} &\dots \mathbb{R}^2, \quad \{1\} \dots \mathbb{N}, \quad \mathbb{R} \times \mathbb{R}^* \dots (\mathbb{R} \times \mathbb{R}) \setminus \{(0, 0)\}, \quad (\mathbb{R} \times \mathbb{R}^*) \cap (\mathbb{R}^* \times \mathbb{R}) \dots (\mathbb{R}^*)^2 \\ \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) &\dots \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}, \quad (i, 1) \dots \mathbb{N} \times \mathbb{C}, \quad \{0; 2\} \dots \mathcal{P}(\mathbb{N}), \\ \{(m, 2m, -m) \mid m \in \mathbb{R}\} &\dots \mathbb{R}^3, \quad \{(a, a+b, c+d) \mid (a, b, c, d) \in \mathbb{R}^4\} \dots \mathbb{R}^4 \end{aligned}$$

2 Let  $E = \{x, y, z\}$  be a set. Determine whether the following statements are true or false:

$$\begin{aligned} x \in E, \quad \{x\} &\subset E, \quad \emptyset \subset E, \quad \emptyset \in E, \quad \{\emptyset\} \subset E, \quad \emptyset \in \mathcal{P}(E), \quad \{\emptyset\} \subset \mathcal{P}(E), \\ \mathcal{P}(\{x; y\}) &\subset \mathcal{P}(E). \end{aligned}$$

## Exercise 2

We define the following five sets:

$$\begin{aligned} A_1 &= \{(x, y) \in \mathbb{R}^2, x + y < 1\} \\ A_2 &= \{(x, y) \in \mathbb{R}^2, x + y > -1\} \\ A_3 &= \{(x, y) \in \mathbb{R}^2, |x + y| < 1\} \\ A_4 &= \{(x, y) \in \mathbb{R}^2, |x - y| < 1\} \\ A_5 &= \{(x, y) \in \mathbb{R}^2, |x| + |y| < 1\} \end{aligned}$$

- 1 Graphically, represent in an orthonormal system  $(O, \vec{i}, \vec{j})$  the lines of equations:  $x - y = 1, x - y = -1, x + y = 1$  and  $x + y = -1$ .
- 2 Hatch the regions representing the sets  $A_1$  to  $A_5$  on 5 different graphs.
- 3 Show geometrically, then by reasoning, that:

$$\forall (x, y) \in \mathbb{R}^2, (|x + y| < 1 \text{ and } |x - y| < 1) \Leftrightarrow |x| + |y| < 1$$

## Exercise 3

Show that:

- 1  $E = \{(m, 2m, -m) \mid m \in \mathbb{R}\} \subset F = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + 3z = 0\}$ .

- 2  $E = \{t \rightarrow \lambda \sin(t) + \mu \cos(t) \mid \lambda, \mu \in \mathbb{R}\} \subset F = \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) \mid f'' = -f\}$ .

- 3  $E = \{(x, y) \in \mathbb{R}^2 \mid y = 3x + 1\} = F = \{(a + 1, 3a + 4), a \in \mathbb{R}\}$ .

- 4  $E = \{z \in \mathbb{C}, |z + i| = |z - i|\} = F = \mathbb{R}$ .

### Exercise 9

Let  $I$  be a set and  $\{A_i\}_{i \in I}$  a subset of  $\mathcal{P}(E)$ . Show that:

1  $\overline{\left(\bigcup_{i \in I} A_i\right)} = \bigcap_{i \in I} \overline{A_i}.$

2  $\overline{\left(\bigcap_{i \in I} A_i\right)} = \bigcup_{i \in I} \overline{A_i}.$

### Exercise 10

Let  $E$  be a set.

- 1 Let  $A, B \in \mathcal{P}(E)$ . Solve the equation for the unknown  $X \in \mathcal{P}(E)$ :

$$X \cup B = A \cup B.$$

- 2 Let  $B \in \mathcal{P}(E)$ . Consider the following relation:

$$\forall X, Y \in \mathcal{P}(E), \quad X \mathcal{R} Y \iff X \cup B = Y \cup B.$$

- (a) Show that  $\mathcal{R}$  is an equivalence relation on  $\mathcal{P}(E)$ .  
(b) Let  $X \in \mathcal{P}(E)$ . Find:  $\text{cl}_{\mathcal{R}}(X)$ .

### Exercise 11

We define a relation  $\mathcal{R}$  on  $\mathbb{R}$  by  $x \mathcal{R} y$  if and only if

$$\cos^2 x + \sin^2 y = 1.$$

Show that  $\mathcal{R}$  is an equivalence relation and determine its equivalence classes.

### Exercise 12

We define a binary relation  $\mathcal{R}$  on  $\mathbb{Z}$  by  $x \mathcal{R} y$  if and only if  $x + y$  is even. Show that  $\mathcal{R}$  is an equivalence relation and specify its equivalence classes.

### Exercise 13

Let  $\mathcal{R}$  be a relation defined on the set  $\mathbb{Z}^+$ : for  $a, b \in \mathbb{Z}^+$ ,  $a \mathcal{R} b \iff a | b$ .

- 1 Show that  $\mathcal{R}$  is an ordering relation on  $\mathbb{Z}^+$ .  
2 In case  $\mathcal{R}$  is defined on  $\mathbb{Z}$ , is  $\mathcal{R}$  still an ordering relation on  $\mathbb{Z}$ ? Why?  
3 Draw the Hasse diagram representing the ordering relation  $\mathcal{R}$  on the set

$$A = \{1, 2, 3, 4, 6, 8, 12, 24\}.$$

- 4 Decide whether  $\mathcal{R}$  is a total ordering relation on  $\mathbb{Z}^+$ , and justify.  
5 In case  $\mathcal{R}$  is defined on  $B = \{1, 3, 9, 27, 81\}$ , is  $\mathcal{R}$  a total ordering relation on  $B$ ? Why?  
6 Draw the Hasse diagram representing the ordering relation  $\mathcal{R}$  on the set

$$B = \{1, 3, 9, 27, 81\}.$$

### Exercise 4

- 1 Find  $\{(t, t+1) \mid t \in \mathbb{R}\} \cap \{(x, y) \in \mathbb{R}^2 \mid x+y=0\}$ .
- 2 Find  $\{t \mapsto \lambda \cos(t) + \mu e^{-t} \mid \lambda, \mu \in \mathbb{R}\} \cap \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) \mid f'' = -f\}$ .

### Exercise 5

Let  $E$  be a set and  $A, B, C$  be subsets of  $E$ . Prove the following propositions:

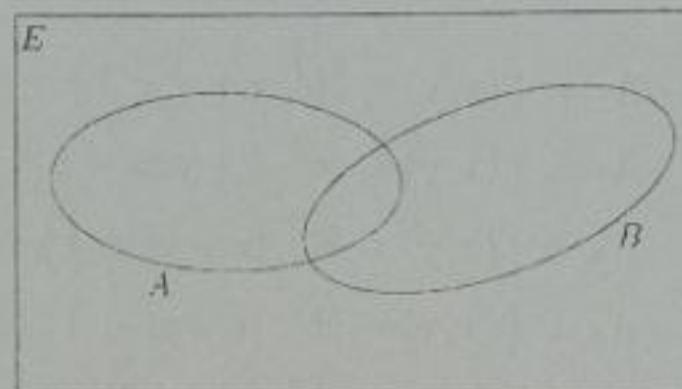
- 1  $(A \cup C) \cap \overline{(A \cup B)} = \bar{A} \cap C \cap \bar{B}$
- 2  $(A \cap C) \cap \overline{(A \cap B)} = A \cap C \cap \bar{B}$
- 3  $A \times (B \cap C) = (A \times B) \cap (A \times C)$
- 4  $A \times (B \cup C) = (A \times B) \cup (A \times C)$

### Exercise 6

Let  $A$  and  $B$  be two parts of a set  $E$ .

The set  $A \Delta B = (A \setminus B) \cup (B \setminus A)$  is called the symmetric difference of  $A$  and  $B$ .

- 1 Describe the set  $A \Delta B$  in a simple sentence.
- 2 After reproducing this drawing on your paper, hatch the part corresponding to  $A \Delta B$ .



- 3 Show that  $A \Delta B = (A \cup B) \setminus (A \cap B)$ .
- 4 Let  $C$  be a subset of  $E$ . Prove that  $A \Delta C = B \Delta C \Leftrightarrow A = B$ .

### Exercise 7

Let  $(A, B) \in \mathcal{P}(E)^2$ .

- 1 Show that  $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$ .
- 2 Compare  $\mathcal{P}(A \cup B)$  and  $\mathcal{P}(A) \cup \mathcal{P}(B)$ .
- 3 Compare  $\mathcal{P}(A \times B)$  and  $\mathcal{P}(A) \times \mathcal{P}(B)$ .

### Exercise 8

- 1 Let  $x \in \mathbb{R}$ . What is the mathematical meaning of  $x \in \bigcup_{n \in \mathbb{N}^+} \left[1 - \frac{1}{n}, 1 + \frac{1}{n}\right]$  and  $x \in \bigcap_{n \in \mathbb{N}^+} \left[1 - \frac{1}{n}, 1 + \frac{1}{n}\right]$ ?
- 2 Without proof, give the value of  $\bigcup_{n \in \mathbb{N}^+} \left[1 - \frac{1}{n}, 1 + \frac{1}{n}\right]$ .
- 3 Show by double inclusion that  $\bigcap_{n \in \mathbb{N}^+} \left[1 - \frac{1}{n}, 1 + \frac{1}{n}\right] = \{1\}$ .