الجمهورية الجزائرية الديمقراطية الشعبية
People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research
National School of Cyber Security
Foundation Training Department

وزارة التعليم العالي والبحث العلمي
المدرسة الوطنية العليا في الأمن السيبراني
قسم التكوين القاعدي

NSCS
المدرسة الوطنية العليا في الأمن السيبراني
NATIONAL SCHOOL OF CYBERSECURITY

LEVEL: 1st Year Basic Training
SECTION / GROUP: A & B
MODULE: Algebra 1
FULL NAME:.................................

MODULE'S TEACHER: Dr. Hamza Moufek
DATE: 18 / 01/ 2026
DURATION: 2h
NOTE: No documents are allowed.

Midterm Exam

Exercise 1 : (10 points)

Throughout this exercise, $\mathbb{R}$ and $\mathbb{C}$ denote respectively the field of real numbers and the field of complex numbers, equipped with their usual additions and multiplications.
Let $E = \{(a, b) \mid a, b \in \mathbb{R}\}$.
We equip E with two binary operations $\oplus$ and $\otimes$ defined by

$$\forall (a, b), (c, d) \in E, \quad (a, b) \oplus (c, d) = (a + c, b + d),$$
$$\forall (a, b), (c, d) \in E, \quad (a, b) \otimes (c, d) = (ac - bd, ad + bc).$$

1. Prove that $(E, \oplus, \otimes)$ is a commutative ring with identity.
2. Let $\varphi : E \longrightarrow \mathbb{C}, \quad (a, b) \longmapsto a + ib$.
   (a) Show that φ is a ring homomorphism.
   (b) Determine $\ker(\varphi)$ and $\mathrm{Im}(\varphi)$. φ is it a ring isomorphism ?
   (c) Deduce the multiplicative inverse of any $(a, b) \neq (0, 0)$ with respect to the operation $\otimes$.
3. For $n \in \mathbb{N}^*$, denote by $(a, b)^n$ the product $(a, b) \otimes \cdots \otimes (a, b) \quad (n \text{ times})$.
   Compute $(1, -1)^{2026}$, and express the result in the form (α, β), where α and β are real numbers to be determined.

Exercise 2 : (8 points)

Let $E = \mathbb{R}^{\mathbb{R}}$ be the set of all functions from $\mathbb{R}$ into $\mathbb{R}$. We denote by $S(\mathbb{R})$ the set of all bijections from $\mathbb{R}$ onto itself.
For $(f, g) \in E^2$, we define the relation $\mathcal{R}$ by

$$f \mathcal{R} g \Leftrightarrow \exists b \in S(\mathbb{R}) \text{ such that } b^{-1} \circ f \circ b = g.$$

If $f \in E$, we denote by $\overline{f}$ the equivalence class of f.

1. Prove that the binary relation $\mathcal{R}$ is an equivalence relation on E.
2. (a) Recall the definition of the equivalence class $\overline{f}$ of a function $f \in E$.
   (b) Determine the equivalence class $\overline{\mathrm{Id}_{\mathbb{R}}}$ of the identity function $\mathrm{Id}_{\mathbb{R}}$.
   (c) Show that the equivalence class $\overline{1_{\mathbb{R}}}$ of the constant function equal to 1 is the set of all constant functions.
3. Is the relation $\mathcal{R}$ an order relation on E ?
4. Let $f \in E$.
   (a) Show that if f is injective, then every element of $\overline{f}$ is injective.
   (b) Show that if f is surjective, then every element of $\overline{f}$ is surjective.
5. Show that the collection $(\overline{f})_{f \in S(\mathbb{R})}$ is a partition of $S(\mathbb{R})$.

Exercise 3 : (2 points)

Let F be a subfield of $(\mathbb{Q}, +, \times)$. Show that $F = \mathbb{Q}$.