Boolean Algebra

Learning Objectives

In this course you will learn about:

- Boolean algebra
- Fundamental concepts and basic laws of Boolean algebra
- Boolean function and minimization
- Logic gates
- Logic circuits and Boolean expressions
- Combinational circuits and design

Boolean Algebra

- An algebra that deals with binary number system
- George Boole (1815-1864), an English mathematician, developed it for:
  - Simplifying representation
  - Manipulation of propositional logic
- In 1938, Claude E. Shannon proposed using Boolean algebra in design of relay switching circuits
- Provides economical and straightforward approach
- Used extensively in designing electronic circuits used in computers

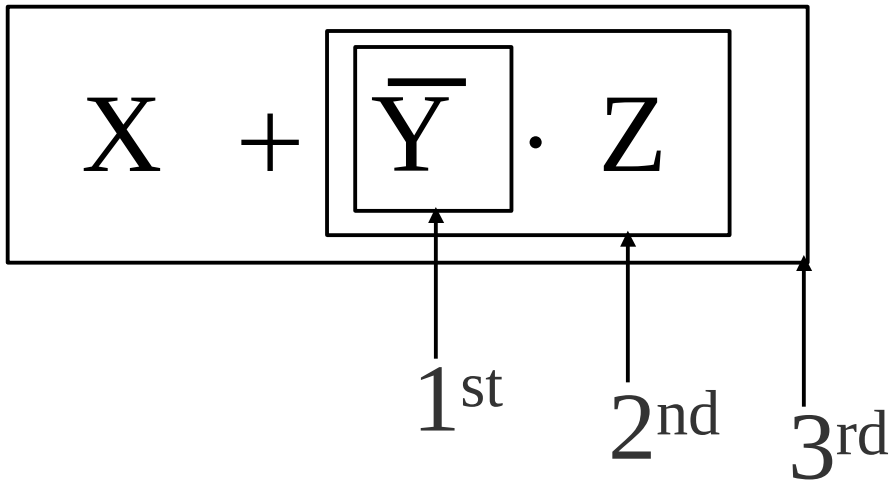
Fundamental Concepts of Boolean Algebra

- Use of Binary Digit
  - Boolean equations can have either of two possible values, 0 and 1
- Logical Addition
  - Symbol '**+**', also known as '**OR**' operator, used for logical addition. Follows law of binary addition
- Logical Multiplication
  - Symbol '**.**', also known as '**AND**' operator, used for logical multiplication. Follows law of binary multiplication
- Complementation
  - Symbol '**-**', also known as '**NOT**' operator, used for complementation. Follows law of binary compliment

Operator Precedence

- Each operator has a precedence level
- Higher the operator's precedence level, earlier it is evaluated
- Expression is scanned from left to right
- First, expressions enclosed within parentheses are evaluated
- Then, all complement (NOT) operations are performed
- Then, all '$\cdot$' (AND) operations are performed
- Finally, all '+' (OR) operations are performed

Operator Precedence



Postulates of Boolean Algebra

Postulate 1:

(a) A = 0, if and only if, A is not equal to 1

(b) A = 1, if and only if, A is not equal to 0

Postulate 2:

(a) $x + 0 = x$

(b) $x \cdot 1 = x$

Postulate 3: Commutative Law

(a) $x + y = y + x$

(b) $x \cdot y = y \cdot x$

Postulates of Boolean Algebra

Postulate 4: Associative Law

(a) $x + (y + z) = (x + y) + z$

(b) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

Postulate 5: Distributive Law

(a) $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

(b) $x + (y \cdot z) = (x + y) \cdot (x + z)$

Postulate 6:

(a) $x + \overline{x} = 1$

(b) $x \cdot \overline{x} = 0$

The Principle of Duality

There is a precise duality between the operators . (AND) and + (OR), and the digits 0 and 1.

For example, in the table below, the second row is obtained from the first row and vice versa simply by interchanging '+' with '.' and '0' with '1'

	Column 1	Column 2	Column 3
Row 1	$1 + 1 = 1$	$1 + 0 = 0 + 1 = 1$	$0 + 0 = 0$
Row 2	$0 \cdot 0 = 0$	$0 \cdot 1 = 1 \cdot 0 = 0$	$1 \cdot 1 = 1$

Therefore, if a particular theorem is proved, its dual theorem automatically holds and need not be proved separately

Some Important Theorems of Boolean Algebra

Sr. No.	Theorems/ Identities	Dual Theorems/ Identities	Name (if any)
1	$x + x = x$	$x \cdot x = x$	Idempotent Law
2	$x + 1 = 1$	$x \cdot 0 = 0$	
3	$x + x \cdot y = x$	$x \cdot (x + y) = x$	Absorption Law
4	$\overline{\overline{x}} = x$		Involution Law
5	$x \cdot (\overline{x} + y) = x \cdot y$	$x + \overline{x} \cdot y = x + y$	
6	$\overline{x+y} = \overline{x} \cdot \overline{y}$	$\overline{x \cdot y} = \overline{x} + \overline{y}$	De Morgan's Law

Methods of Proving Theorems

The theorems of Boolean algebra may be proved by using one of the following methods:

1. By using postulates to show that L.H.S. = R.H.S
2. By *Perfect Induction* or *Exhaustive Enumeration* method where all possible combinations of variables involved in L.H.S. and R.H.S. are checked to yield identical results
3. By the *Principle of Duality* where the dual of an already proved theorem is derived from the proof of its corresponding pair

Proving a Theorem by Using Postulates (Example)

Theorem:

$$x + x \cdot y = x$$

Proof:

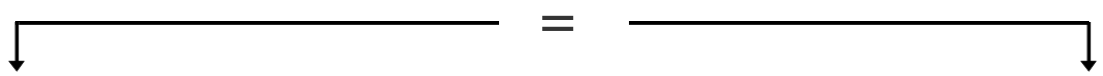
L.H.S.

$= x + x \cdot y$	
$= x \cdot 1 + x \cdot y$	by postulate 2(b)
$= x \cdot (1 + y)$	by postulate 5(a)
$= x \cdot (y + 1)$	by postulate 3(a)
$= x \cdot 1$	by theorem 2(a)
$= x$	by postulate 2(b)
= R.H.S.	

Proving a Theorem by Perfect Induction (Example)

Theorem:

$$x + x \cdot y = x$$



x	y	x . y	x + x y
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

Proving a Theorem by the Principle of Duality (Example)

Theorem:

$x + x = x$

Proof:

L.H.S.	
$= x + x$	
$= (x + x) \cdot 1$	by postulate 2(b)
$= (x + x) \cdot (x + \overline{x})$	by postulate 6(a)
$= x + x \cdot \overline{x}$	by postulate 5(b)
$= x + 0$	by postulate 6(b)
$= x$	by postulate 2(a)
$=$ R.H.S.	

Proving a Theorem by the Principle of Duality (Example)

Dual Theorem:

$$x \cdot x = x$$

Proof:

L.H.S.	
$= x \cdot x$	
$= x \cdot x + 0$	by postulate 2(a)
$= x \cdot x + x \cdot \overline{x}$	by postulate 6(b)
$= x \cdot (x + \overline{x})$	by postulate 5(a)
$= x \cdot 1$	by postulate 6(a)
$= x$	by postulate 2(b)
= R.H.S.	

Notice that each step of the proof of the dual theorem is derived from the proof of its corresponding pair in the original theorem

Boolean Functions

- A Boolean function is an expression formed with:
  - Binary variables
  - Operators (OR, AND, and NOT)
  - Parentheses, and equal sign
- The value of a Boolean function can be either 0 or 1
- A Boolean function may be represented as:
  - An algebraic expression, or
  - A truth table

Representation as an Algebraic Expression

$$W = X + \bar{Y} \cdot Z$$

- Variable W is a function of X, Y, and Z, can also be written as W = f (X, Y, Z)
- The RHS of the equation is called an ***expression***
- The symbols X, Y, Z are the ***literals*** of the function
- For a given Boolean function, there may be more than one algebraic expressions

Representation as a Truth Table

X	Y	Z	W
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Representation as a Truth Table

- The number of rows in the table is equal to 2^n, where *n* is the number of literals in the function
- The combinations of 0s and 1s for rows of this table are obtained from the binary numbers by counting from 0 to $2^n - 1$

Minimization of Boolean Functions

- Minimization of Boolean functions deals with
  - Reduction in number of literals
  - Reduction in number of terms

- Minimization is achieved through manipulating expression to obtain equal and simpler expression(s) (having fewer literals and/or terms)

Minimization of Boolean Functions

$$F_1 = \overline{x} \cdot \overline{y} \cdot z + \overline{x} \cdot y \cdot z + x \cdot \overline{y}$$

F_1 has 3 literals (x, y, z) and 3 terms

$$F_2 = x \cdot \overline{y} + \overline{x} \cdot z$$

F_2 has 3 literals (x, y, z) and 2 terms

F_2 can be realized with fewer electronic components, resulting in a cheaper circuit

Minimization of Boolean Functions

x	y	z	F_1	F_2
0	0	0	0	0
0	0	1	1	1
0	1	0	0	0
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	0	0
1	1	1	0	0

Both F_1 and F_2 produce the same result

Complement of a Boolean Function

- The complement of a Boolean function is obtained by interchanging:
  - Operators OR and AND
  - Complementing each literal
- This is based on ***De Morgan's theorems***, whose general form is:

$$\overline{A_1 + A_2 + A_3 + \ldots + A_n} = \bar{A}_1 \cdot \bar{A}_2 \cdot \bar{A}_3 \cdot \ldots \cdot \bar{A}_n$$

$$\overline{A_1 \cdot A_2 \cdot A_3 \cdot \ldots \cdot A_n} = \bar{A}_1 + \bar{A}_2 + \bar{A}_3 + \ldots + \bar{A}_n$$

Complementing a Boolean Function (Example)

$$F_1 = \bar{x} \cdot y \cdot \bar{z} + x \cdot \bar{y} \cdot z$$

To obtain $\overline{F_1}$, we first interchange the OR and the AND operators giving

$$(\bar{x} + y + \bar{z}) \cdot (x + \bar{y} + z)$$

Now we complement each literal giving

$$\overline{F_1} = (x + \bar{y} + z) \cdot (\bar{x} + y + \bar{z})$$

Canonical Forms of Boolean Functions

Minterms : *Minterm is the product of N distinct literals where each literal occurs exactly once. The output of the minterm functions is 1, provide 2^n possible combinations called minterms or standard products. Minterm is represented by m.*

Maxterms : *Maxterm is the sum of N distinct literals where each literals occurs exactly once. The output of the maxterm functions is 0, provide 2^n possible combinations called maxterms or standard sums. Maxterm is represented by M.*

Minterms and Maxterms for three Variables

Variables			Minterms (standard products)		Maxterms (standard sums)	
x	y	z	Term	Designation	Term	Designation
0	0	0	$\bar{x} \cdot \bar{y} \cdot \bar{z}$	m_0	$x + y + z$	M_0
0	0	1	$\bar{x} \cdot \bar{y} \cdot z$	m_1	$x + y + \bar{z}$	M_1
0	1	0	$\bar{x} \cdot y \cdot \bar{z}$	m_2	$x + \bar{y} + z$	M_2
0	1	1	$\bar{x} \cdot y \cdot z$	m_3	$x + \bar{y} + \bar{z}$	M_3
1	0	0	$x \cdot \bar{y} \cdot \bar{z}$	m_4	$\bar{x} + y + z$	M_4
1	0	1	$x \cdot \bar{y} \cdot z$	m_5	$\bar{x} + y + \bar{z}$	M_5
1	1	0	$x \cdot y \cdot \bar{z}$	m_6	$\bar{x} + \bar{y} + z$	M_6
1	1	1	$x \cdot y \cdot z$	m_7	$\bar{x} + \bar{y} + \bar{z}$	M_7

Note that each minterm is the complement of its corresponding maxterm and vice-versa

Sum-of-Products (SOP) Expression

A sum-of-products (SOP) expression is a product term (minterm) or several product terms (minterms) logically added (OR) together. Examples are:

x $\qquad$ $x+y$

$x+y\cdot z$ $\qquad$ $x\cdot y+z$

$x\cdot y+\overline{x}\cdot\overline{y}$ $\qquad$ $\overline{x}\cdot\overline{y}+x\cdot\overline{y}\cdot z$

Steps to Express a Boolean Function in its Sum-of-Products Form

1. Construct a truth table for the given Boolean function
2. Form a minterm for each combination of the variables, which produces a 1 in the function
3. The desired expression is the sum (OR) of all the minterms obtained in Step 2

Expressing a Function in its Sum-of-Products Form (Example)

x	y	z	F_1
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

The following 3 combinations of the variables produce a 1:

001, 100, and 111

Expressing a Function in its Sum-of-Products Form (Example)

- Their corresponding minterms are:

$$\bar{x} \cdot \bar{y} \cdot z, \quad x \cdot \bar{y} \cdot \bar{z}, \quad \text{and} \quad x \cdot y \cdot z$$

- Taking the OR of these minterms, we get

$$F_1 = \bar{x} \cdot \bar{y} \cdot z + x \cdot \bar{y} \cdot \bar{z} + x \cdot y \cdot z = m_1 + m_4 + m_7$$

$$F_1(x \cdot y \cdot z) = \Sigma(1,4,7)$$

Product-of Sums (POS) Expression

A product-of-sums (POS) expression is a sum term (maxterm) or several sum terms (maxterms) logically multiplied (ANDed) together. Examples are:

x $\qquad$ $(x+\bar{y})\cdot(\bar{x}+y)\cdot(\bar{x}+\bar{y})$

$\bar{x}+y$ $\qquad$ $(x+y)\cdot(\bar{x}+y+z)$

$(\bar{x}+\bar{y})\cdot z$ $\qquad$ $(\bar{x}+y)\cdot(x+\bar{y})$

Steps to Express a Boolean Function in its Product-of-Sums Form

1. Construct a truth table for the given Boolean function
2. Form a maxterm for each combination of the variables, which produces a 0 in the function
3. The desired expression is the product (AND) of all the maxterms obtained in Step 2

Expressing a Function in its Product-of-Sums Form

x	y	z	F_1
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

- The following 5 combinations of variables produce a 0:
  000, 010, 011, 101, and 110

Expressing a Function in its Product-of-Sums Form

- Their corresponding maxterms are:

$$(x+y+z),(x+\bar{y}+z),(x+\bar{y}+\bar{z}),$$

$$(\bar{x}+y+\bar{z}) \text{ and } (\bar{x}+\bar{y}+z)$$

- Taking the AND of these maxterms, we get:

$$F_1=(x+y+z)\cdot(x+\bar{y}+z)\cdot(x+\bar{y}+\bar{z})\cdot(\bar{x}+y+\bar{z})$$

$$(\bar{x}+\bar{y}+z)=M_0\cdot M_2\cdot M_3\cdot M_5\cdot M_6$$

$$F_1(x,y,z)=\Pi(0,2,3,5,6)$$

Conversion Between Canonical Forms (Sum-of-Products and Product-of-Sums)

To convert from one canonical form to another, interchange the symbol and list those numbers missing from the original form.

Example:

$$F(x,y,z) = \Pi(0,2,4,5) = \Sigma(1,3,6,7)$$

$$F(x,y,z) = \Pi(1,4,7) = \Sigma(0,2,3,5,6)$$

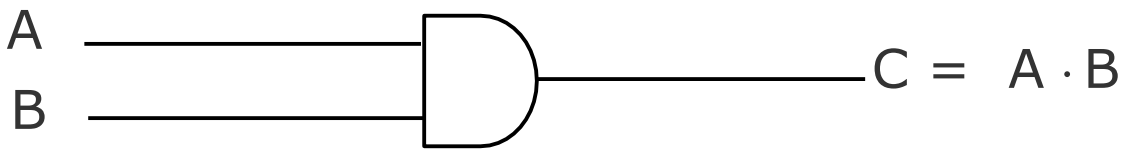
Logic Gates

- Logic gates are electronic circuits that operate on one or more input signals to produce standard output signal
- Are the building blocks of all the circuits in a computer
- Some of the most basic and useful logic gates are AND, OR, NOT, NAND and NOR gates

AND Gate

- Physical realization of logical multiplication (AND) operation
- Generates an output signal of 1 only if all input signals are also 1

AND Gate (Block Diagram Symbol and Truth Table)

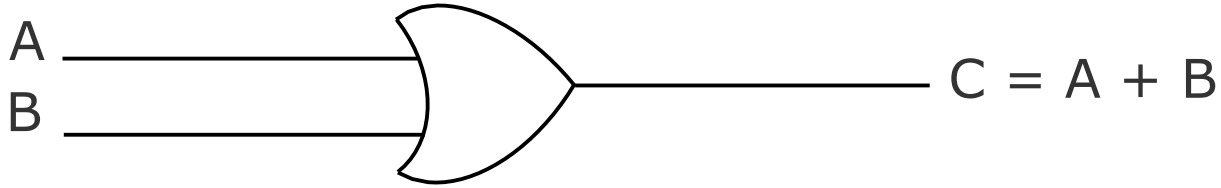


Inputs		Output
A	B	C = A · B
0	0	0
0	1	0
1	0	0
1	1	1

OR Gate

- Physical realization of logical addition (OR) operation
- Generates an output signal of 1 if at least one of the input signals is also 1

OR Gate (Block Diagram Symbol and Truth Table)

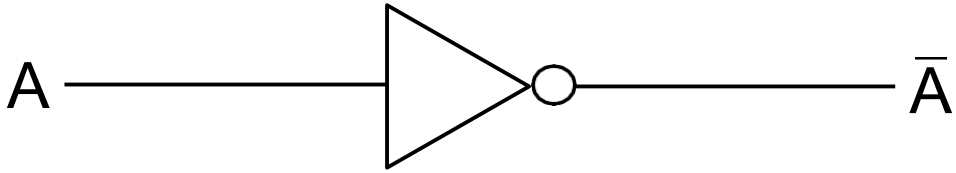


Inputs		Output
A	B	C = A + B
0	0	0
0	1	1
1	0	1
1	1	1

NOT Gate

- Physical realization of complementation operation
- Generates an output signal, which is the reverse of the input signal

NOT Gate (Block Diagram Symbol and Truth Table)

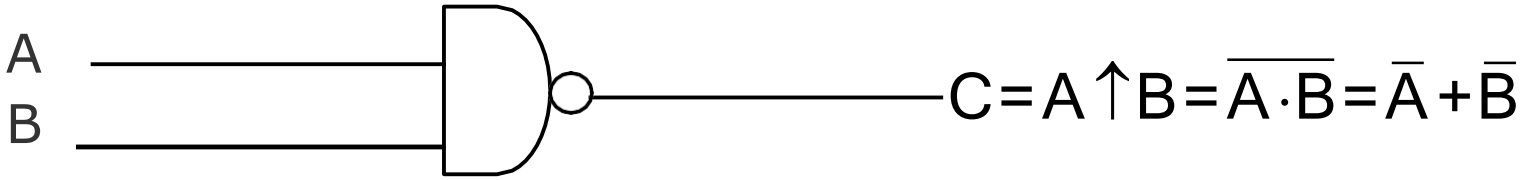


Input	Output
A	$\bar{A}$
0	1
1	0

NAND Gate

- Complemented AND gate
- Generates an output signal of:
  - 1 if any one of the inputs is a 0
  - 0 when all the inputs are 1

NAND Gate (Block Diagram Symbol and Truth Table)

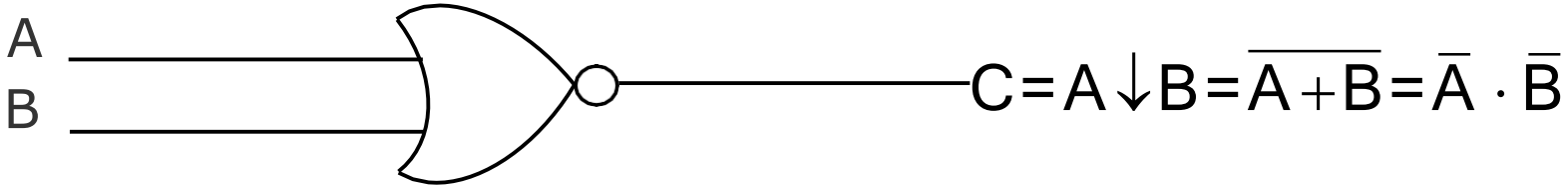


Inputs		Output
A	B	$C = \bar{A} + \bar{B}$
0	0	1
0	1	1
1	0	1
1	1	0

NOR Gate

- Complemented OR gate
- Generates an output signal of:
  - 1 only when all inputs are 0
  - 0 if any one of inputs is a 1

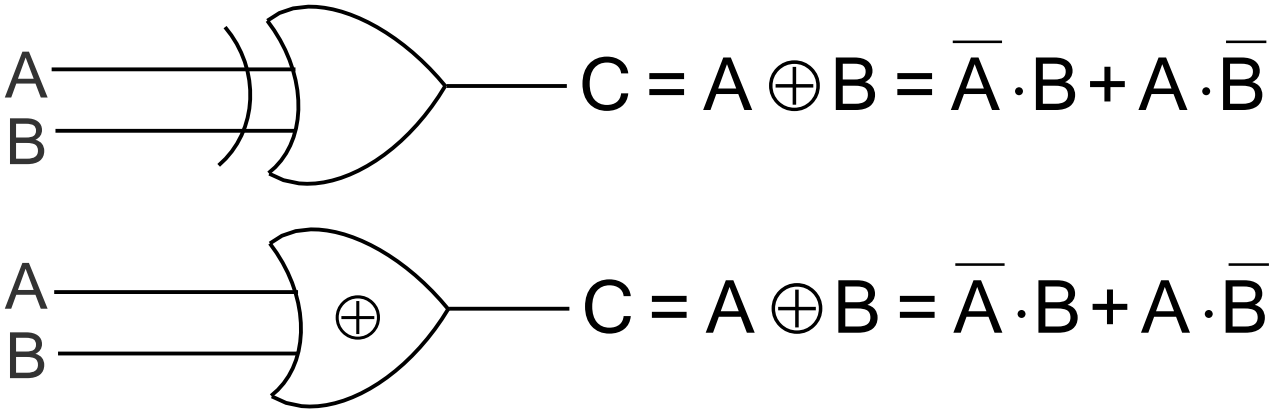
NOR Gate (Block Diagram Symbol and Truth Table)



Inputs		Output
A	B	$C = \bar{A}\,\bar{B}$
0	0	1
0	1	0
1	0	0
1	1	0

Exclusive-OR Function (XOR)

$$A \oplus B = A \cdot \overline{B} + \overline{A} \cdot B$$



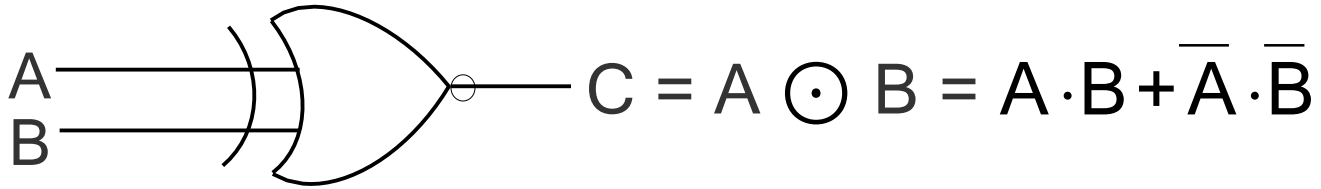
Also, $(A \oplus B) \oplus C = A \oplus (B \oplus C) = A \oplus B \oplus C$

Exclusive-OR Function (Truth Table)

Inputs		Output
A	B	$C = A \oplus B$
0	0	0
0	1	1
1	0	1
1	1	0

Not Exclusive-OR Function (XNOR)

$$A \odot B = A \cdot B + \overline{A} \cdot \overline{B}$$



Also, $(A \odot B) \odot = A \odot (B \odot C) = A \odot B \odot C$

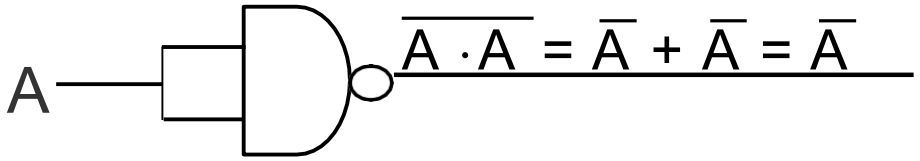
Equivalence Function (Truth Table)

Inputs		Output
A	B	$C = A \odot B$
0	0	1
0	1	0
1	0	0
1	1	1

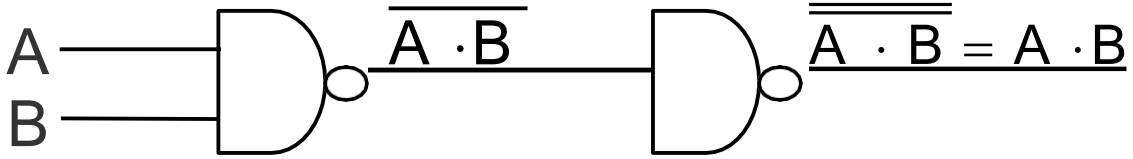
Universal NAND Gate

- NAND gate is an universal gate, it is alone sufficient to implement any Boolean expression
- **To understand this, consider:**
  - Basic logic gates (AND, OR, and NOT) are logically complete
  - Sufficient to show that AND, OR, and NOT gates can be implemented with NAND gates

Implementation of NOT, AND and OR Gates by NAND Gates

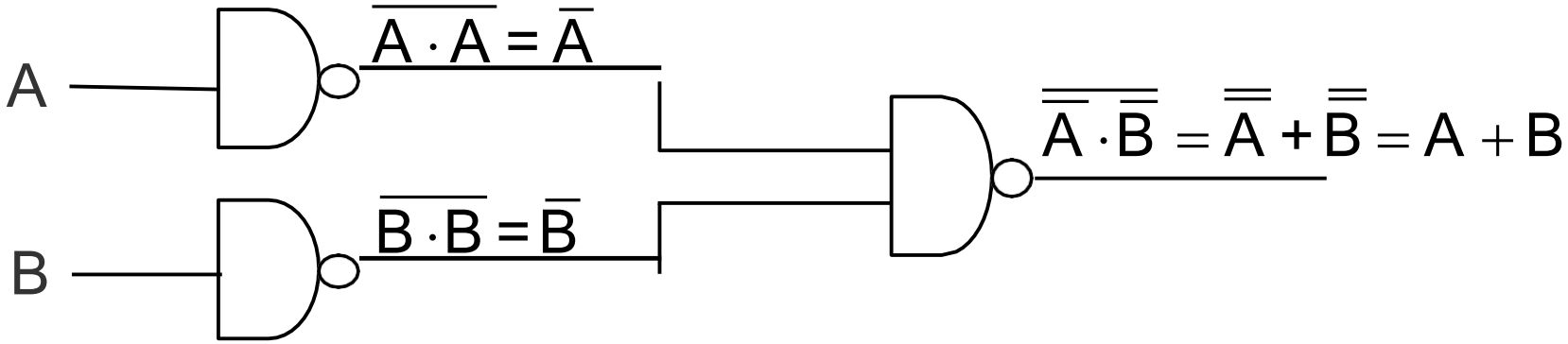


(a) NOT gate implementation.



(b) AND gate implementation.

Implementation of NOT, AND and OR Gates by NAND Gates



(c) OR gate implementation.

Method of Implementing a Boolean Expression with Only NAND Gates

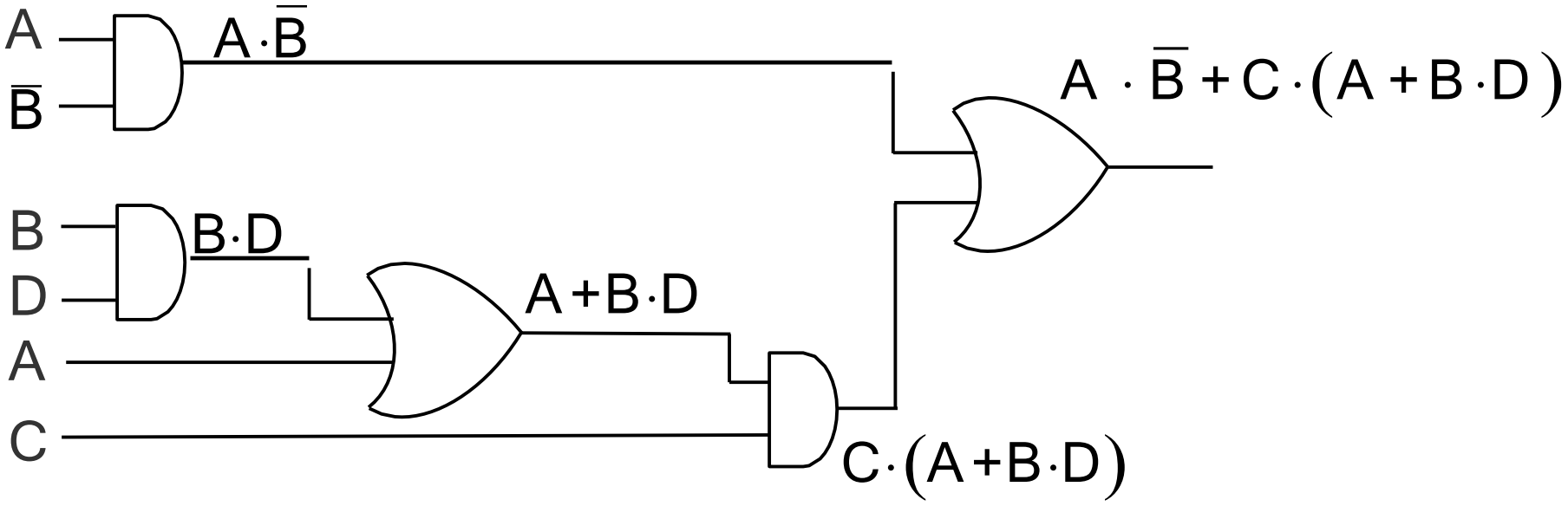
Step 1: From the given algebraic expression, draw the logic diagram with AND, OR, and NOT gates. Assume that both the normal (A) and complement ($\overline{A}$) inputs are available

Step 2: Draw a second logic diagram with the equivalent NAND logic substituted for each AND, OR, and NOT gate

Step 3: Remove all pairs of cascaded inverters from the diagram as double inversion does not perform any logical function. Also remove inverters connected to single external inputs and complement the corresponding input variable

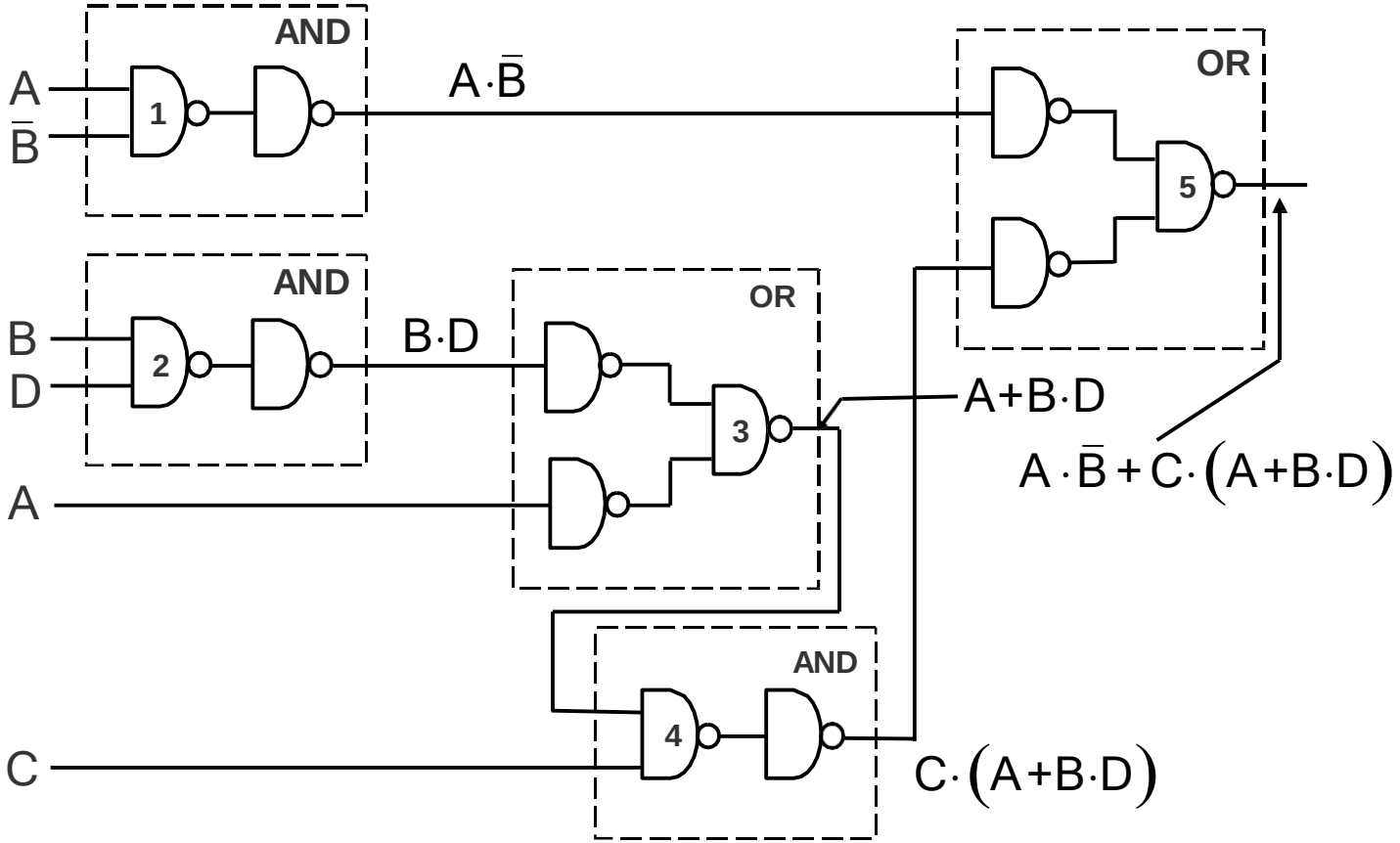
Implementing a Boolean Expression with Only NAND Gates (Example)

Boolean Expression = $A \cdot \overline{B} + C \cdot (A + B \cdot D)$



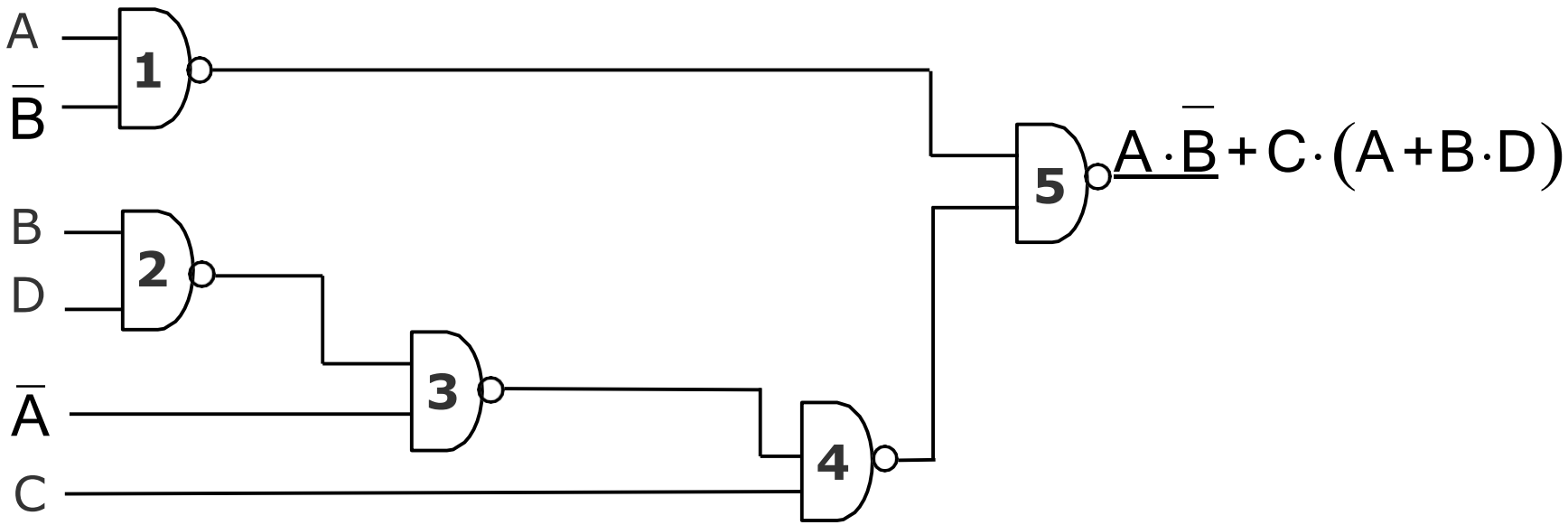
(a) **Step 1:** AND/OR implementation

Implementing a Boolean Expression with Only NAND Gates (Example)



(b) **Step 2:** Substituting equivalent NAND functions

Implementing a Boolean Expression with Only NAND Gates (Example)

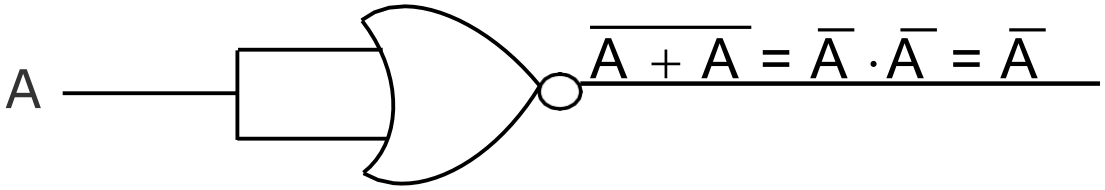


(c) **Step 3:** NAND implementation.

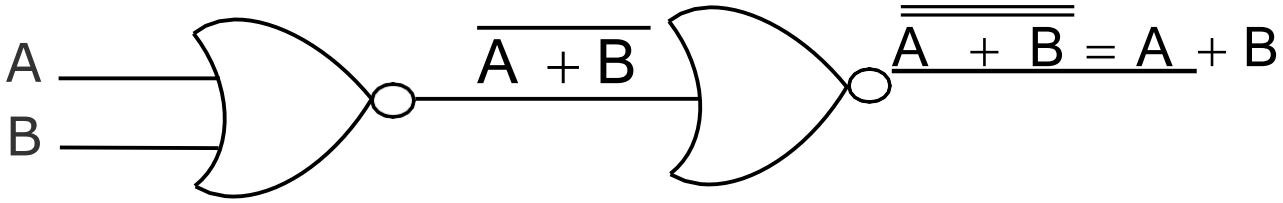
Universal NOR Gate

- NOR gate is an universal gate, it is alone sufficient to implement any Boolean expression
- To understand this, consider:
  - Basic logic gates (AND, OR, and NOT) are logically complete
  - Sufficient to show that AND, OR, and NOT gates can be implemented with NOR gates

Implementation of NOT, OR and AND Gates by NOR Gates

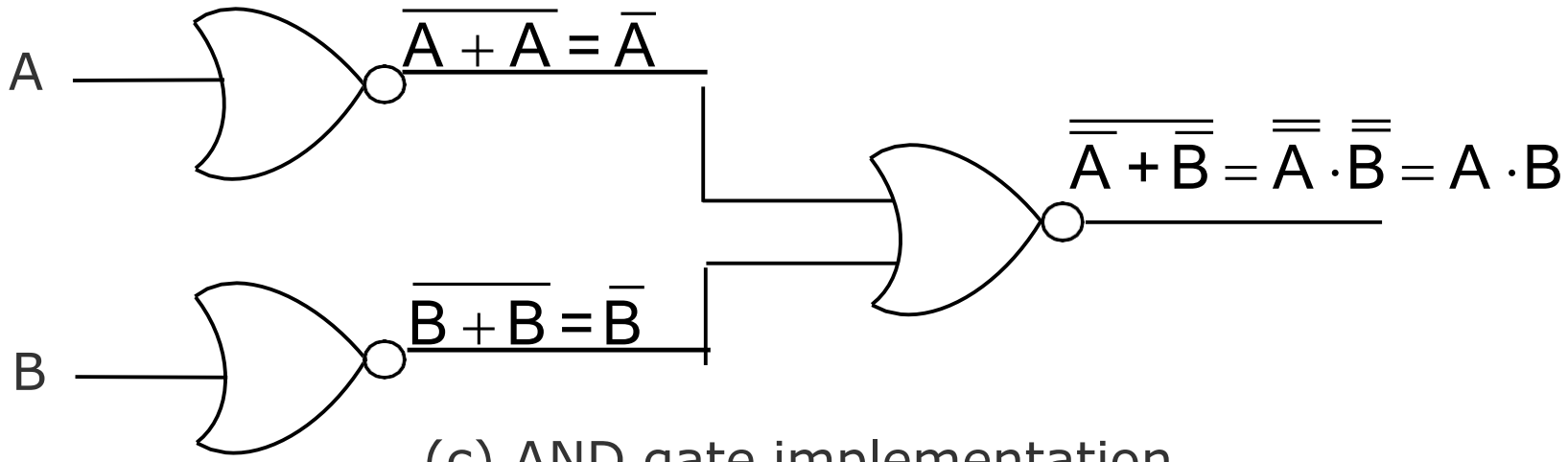


(a) NOT gate implementation.



(b) OR gate implementation.

Implementation of NOT, OR and AND Gates by NOR Gates



(c) AND gate implementation.

Method of Implementing a Boolean Expression with Only NOR Gates

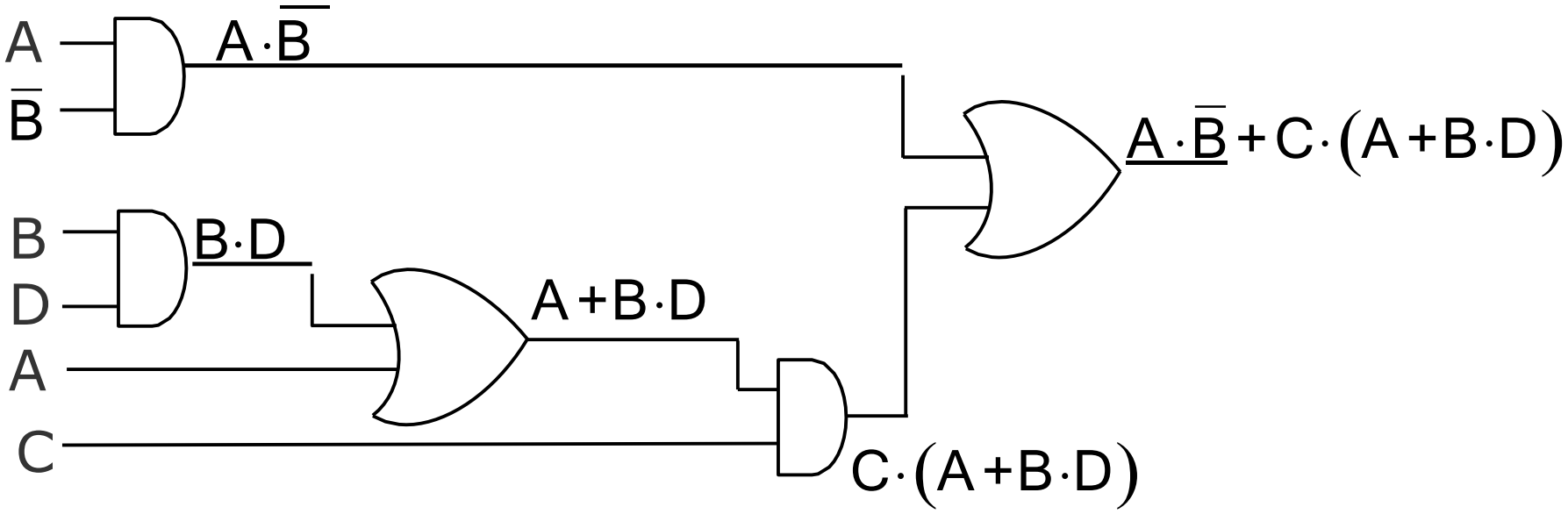
Step 1: For the given algebraic expression, draw the logic diagram with AND, OR, and NOT gates. Assume that both the normal (A) and complement $\overline{(A)}$ inputs are available

Step 2: Draw a second logic diagram with equivalent NOR logic substituted for each AND, OR, and NOT gate

Step 3: Remove all parts of cascaded inverters from the diagram as double inversion does not perform any logical function. Also remove inverters connected to single external inputs and complement the corresponding input variable

Implementing a Boolean Expression with Only NOR Gates (Examples)

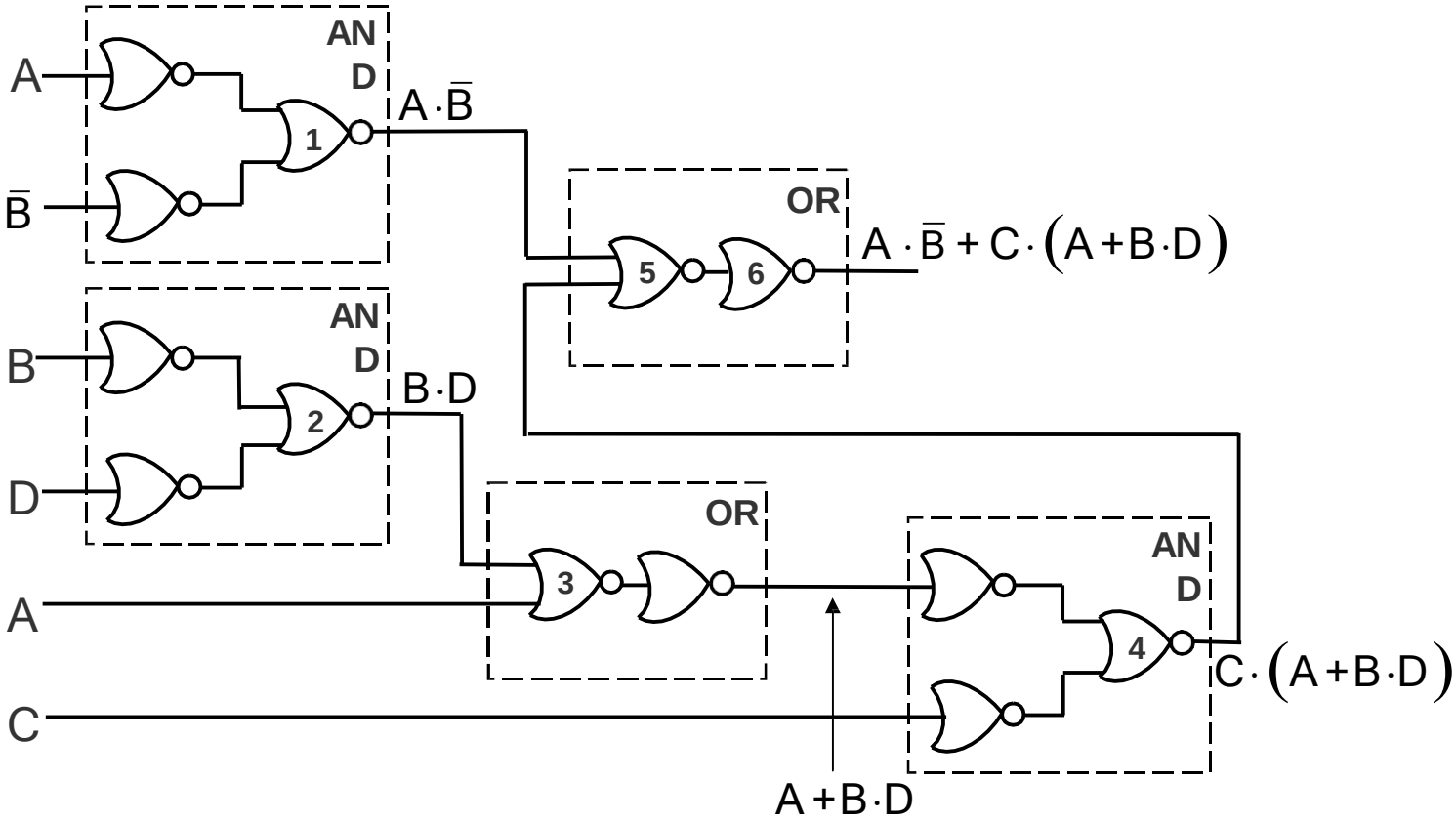
Boolean Expression $= \overline{A} \cdot B + C \cdot (A + B \cdot D)$



(a) **Step 1:** AND/OR implementation.

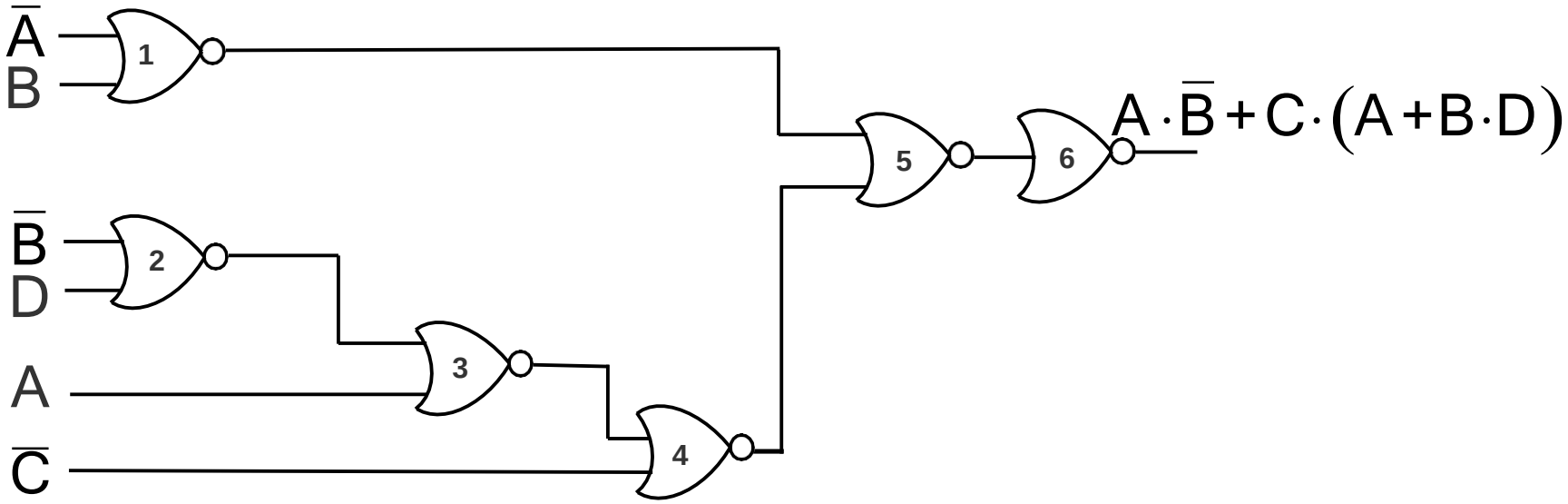
Implementing a Boolean Expression with Only NOR Gates (Examples)

(Continued from previous slide..)



(b) **Step 2:** Substituting equivalent NOR functions.

Implementing a Boolean Expression with Only NOR Gates (Examples)



(c) **Step 3:** NOR implementation.