NATIONAL SCHOOL OF CYBERSECURITY

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Lecturer : H. MOUFEK

Algebra 1 - Tutorial 3

Basic Training Cycle
Functions

Exercise 1

Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by $x \mapsto x^2$.
Find the following sets :

$$f^{-1}(\{1\}), f([1,4]), \quad f^{-1}([-1,4]), \quad f(f^{-1}([-1,4])), \quad f^{-1}(f([1,4])),$$
$$f([-3,-1] \cap [-2,1]), \quad f^{-1}(]-\infty, 2] \cap [1,+\infty[)$$

Exercise 2

Let E be a set. Recall that, for every $A \in \mathcal{P}(E)$, the indicator (or characteristic) function of A is the mapping

$$\mathbb{1}_A : E \longrightarrow \{0,1\}, \quad x \longmapsto \begin{cases} 0 & \text{if } x \notin A, \\ 1 & \text{if } x \in A. \end{cases}$$

We denote by 1 the constant function from $\mathcal{P}(E)$ to $\{0,1\}$ equal to 1.

1. Show that, for all $A, B \in \mathcal{P}(E)$:

$$A = B \iff \mathbb{1}_A = \mathbb{1}_B, \qquad \mathbb{1}_{\overline{A}} = 1 - \mathbb{1}_A,$$
$$\mathbb{1}_{A\cap B} = \mathbb{1}_A \mathbb{1}_B, \qquad \mathbb{1}_{A\cup B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_A\mathbb{1}_B, \qquad \mathbb{1}_{A\setminus B} = \mathbb{1}_A - \mathbb{1}_A\mathbb{1}_B.$$

2. Deduce that, for all $A, B \in \mathcal{P}(E)$:

$$A \cap (A \cup B) = A \quad \text{and} \quad A \cup (A \cap B) = A.$$

Exercise 3

Are the following functions injective ?

1. $f : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^2 \end{cases}$
2. $g : \begin{cases} \mathbb{R}_+ \to \mathbb{R} \\ x \mapsto x^2 \end{cases}$
3. $h : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto 2x - 3 \end{cases}$
4. $j : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto |x| \end{cases}$
5. $k : \begin{cases} \mathbb{R}_+^* \to \mathbb{R} \\ x \mapsto \ln(x^2) - \ln(3x) \end{cases}$
6. $l : \begin{cases} [0, \frac{\pi}{2}] \to [0,1] \\ x \mapsto \sqrt{\sin(x)} \end{cases}$
7. $m : \begin{cases} [0, \pi] \to [0,1] \\ x \mapsto |\cos(x)| \end{cases}$
8. $n : \begin{cases} \mathbb{Z} \to \mathbb{Z} \\ x \mapsto 2x \end{cases}$
9. $p : \begin{cases} \mathbb{R} \to \mathbb{Z} \\ x \mapsto \lfloor x \rfloor \end{cases}$

Exercise 4

Are the following functions surjective ?

1. $f : \left\{ \begin{array}{l} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^2 \end{array} \right.$
2. $g : \left\{ \begin{array}{l} \mathbb{R}_+ \to [0,1] \\ x \mapsto x^2 \end{array} \right.$
3. $h : \left\{ \begin{array}{l} \mathbb{R} \to \mathbb{R} \\ x \mapsto 2x - 3 \end{array} \right.$
4. $j : \left\{ \begin{array}{l} \mathbb{R} \to \mathbb{R}^+ \\ x \mapsto |x| \end{array} \right.$
5. $k : \left\{ \begin{array}{l} \mathbb{R} \to \mathbb{R}^+ \\ x \mapsto \sqrt{e^x} \end{array} \right.$
6. $l : \left\{ \begin{array}{l} \mathbb{Z} \to \mathbb{Z} \\ x \mapsto 2x \end{array} \right.$
7. $m : \left\{ \begin{array}{l} \mathbb{R} \to \mathbb{Z} \\ x \mapsto \lfloor x \rfloor \end{array} \right.$

Exercise 5

Prove that the following functions are bijective. For f, h and k, find their inverses.

1. $f : \left\{ \begin{array}{l} [0; +\infty[\to [-5, +\infty[\\ x \mapsto x^2 - 5 \end{array} \right.$
2. $g : \left\{ \begin{array}{l} \mathbb{R} \to \mathbb{R} \\ x \mapsto \sin(x) + 2x \end{array} \right.$
3. $h : \left\{ \begin{array}{l}]6, +\infty[\to \mathbb{R}_+^* \\ x \mapsto \frac{1}{x-6} \end{array} \right.$
4. $k : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+y, x-y). \end{array}$

Exercise 6

Consider the following function :

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^2 + 3x - 4.$$

1. Is it injective ? Surjective ?
2. Determine I and J, two intervals not reduced to a point such that $g = f|_I^J$ is bijective.
3. Give an expression for g^{-1}.

Exercise 7

Consider the function :

$$f : \mathbb{C} \setminus \{2i\} \to \mathbb{C}, \quad z \mapsto \frac{z^2}{z - 2i}.$$

1. Find the preimages of $1 + i$ under f.
2. Is f injective ? Surjective ? Bijective ? Justify your answer.

Exercise 8

Consider the functions :

$$f : \mathbb{N} \to \mathbb{N}, \quad n \mapsto 2n,$$

$$g : \mathbb{N} \to \mathbb{N}, \quad n \mapsto \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

1. Are f and g injective ? Surjective ?
2. Determine $g \circ f$ and $f \circ g$.

Exercise 9

Let f be defined as : $f : \mathbb{Z} \to \mathbb{Z}, \quad n \mapsto n + (-1)^n$.

1 Calculate $f \circ f$. What can be deduced about f ?

2 Solve the equation : $347 = n + (-1)^n$ where $n \in \mathbb{Z}$.

Exercise 10

Let $f : \begin{array}{rcl} \mathbb{N} \times \mathbb{N} & \to & \mathbb{N} \\ (p, q) & \mapsto & pq \end{array}$ and $g : \begin{array}{rcl} \mathbb{N} & \to & \mathbb{N} \times \{1, 2\} \\ n & \mapsto & \begin{cases} (n, 2) & \text{if } n \text{ even} \\ (n, 1) & \text{if } n \text{ odd} \end{cases} \end{array}$.

Let $2\mathbb{N}$ denote the set of even natural numbers and $4\mathbb{N} = \{4n \in \mathbb{N} \mid n \in \mathbb{N}\}$.

1 Is the function f :

- **a** injective from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$?
- **b** surjective from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$?

2 Is the function g :

- **a** injective from $\mathbb{N}$ to $\mathbb{N} \times \{1, 2\}$?
- **b** surjective from $\mathbb{N}$ to $\mathbb{N} \times \{1, 2\}$?

3 Consider the function $f \circ g$ defined from $\mathbb{N}$ to $\mathbb{N}$:

- **a** Determine $\{n \in \mathbb{N} \mid (f \circ g)(n) = n\}$.
- **b** Is the function $f \circ g$ injective from $\mathbb{N}$ to $\mathbb{N}$?
- **c** Determine $(f \circ g)^{-1}(4\mathbb{N})$.
- **d** Determine the image of $\mathbb{N}$ under $f \circ g$.