Chapter 1 (Part 3):Functions

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Algebra 1

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Outlines of this talk

- Functions
- Exercise

Definition

Let X and Y be non empty sets and let f be a relation from X to Y

- f is called a partial function from X to Y, denoted by $f : X \to Y$, if for each $x \in X, f(\{x\})$ is either a singleton or $\emptyset$.
- For an element $x \in X$, if $f(\{x\}) = \{y\}$, a singleton, we write $f(x) = y$. Hence, y is referred to as the image of x under f; and x is referred to as the pre-image of y under f.
- Let $f : E \longrightarrow F$ be a function. Then E is called the domain of f and $F$is called the codomain of f.
- The image of a function is sometimes written $Im(f)$.
- If f is a partial function from X to Y such that for each $x \in X, f(x)$ is a singleton then f is called a function and is denoted by $f : X \longrightarrow Y$.

Correspondence

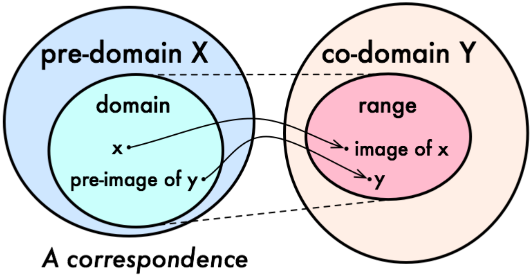


Figure: Correspondence

- f is a function because

$$\forall x \in A, f(\{x\})\textbf{is a singleton}$$

- We have $f(a) = 3, f(b) = f(c) = 5, f(d) = 4$ then a is the **image** of 3 and 3 is the **pre-image** of a.

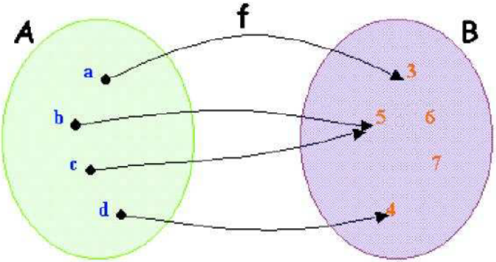


Figure: Example 1

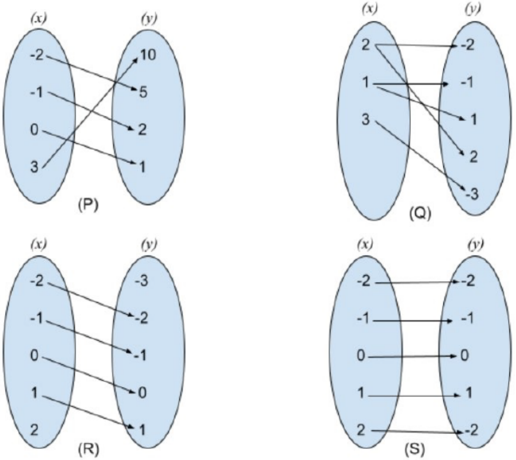


Figure: Example 2

Example 2

- P, S are a functions
- R is a partial function from (x) to (y) because

$$\forall x \in (x), f(\{x\})\textbf{is a singleton or} \quad \emptyset$$

- In Q we have $f(\{1\}) = \{1, -1\}$ then f is not a function.

The function

$$Id_E : E \to E$$

$$x \mapsto y = x$$

is called identity function.

the relation

$$f : \mathbb{R} \to \mathbb{R}$$

$$x \mapsto \frac{1}{x}$$

is not an function because element 0 does not have an image in $\mathbb{R}$.it is a partial function

Example 3

- Suppose $A \subseteq E$. The characteristic function of A , $\chi_A : E \to \{0, 1\}$, is defined by

$$\chi_A = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}$$

- Among the properties of this function ,Let $A, B \in \mathcal{P}(E)$

  1- $\chi_A = \chi_B \Leftrightarrow A = B$

  2- $\chi_{A \cap B} = \chi_A \chi_B$

  3- $\chi_{A \cup B} = \chi_A + \chi_B - \chi_A \chi_B$

  4- $\chi_{\overline{A}} = 1 - \chi_A$

Domaine of definition

Definition

For a function $f : X \longrightarrow Y$ the domain of definition of f is the set X.
the domain is taken to be the set of all real x for which the function is defined.We noted it by $\mathcal{D}_f$

Example

- $$f : \mathbb{R} \to \mathbb{R}$$
  $$x \mapsto \frac{1}{x}$$

  f is a partial function but if we take

  $$f : E \longrightarrow \mathbb{R}, x \longmapsto \frac{1}{x}$$

  with

  $$E \subseteq \mathbb{R} - \{0\}$$

  Then f is a function.

Domaine of definition

Remark

Let $f : E \longrightarrow \mathbb{R}$

$$f \textbf{ is a function } \Leftrightarrow E \subseteq \mathcal{D}_f$$

Example

- $E \longrightarrow \mathbb{R}, x \longmapsto \frac{x+2}{x^2-3x}$

$$f \textbf{ is a function } \Leftrightarrow E \subseteq \mathbb{R} - \{0, 3\}$$

- $E \longrightarrow \mathbb{R}, x \longmapsto \sqrt{x+1}$

$$f \textbf{ is a function } \Leftrightarrow E \subseteq [-1, +\infty[$$

- $E \longrightarrow \mathbb{R}, x \longmapsto \ln\left(\frac{x+1}{x-1}\right)$

$$f \textbf{ is a function } \Leftrightarrow E \subseteq]-\infty, -1[\cup]1, +\infty[$$

Equality of Functions

Definition

We say that two functions f and g are equal if :

1- They have the same domain set E and the same codomain set F

2- $\forall x \in E,\ f(x) = g(x)$.

Remark f, g and h are differente because they havent the same starting or arrival set.

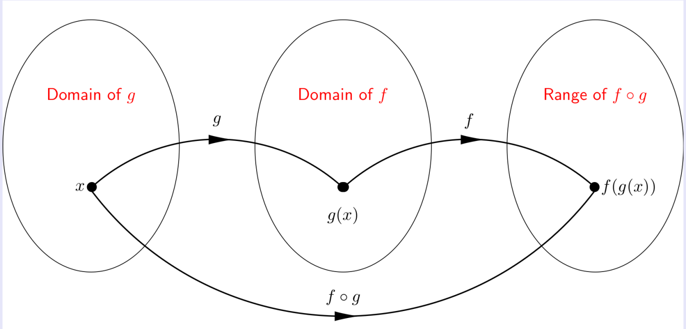
$$\begin{aligned} f: \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^2, \end{aligned} \quad , \quad \begin{aligned} g: \mathbb{R}_+ &\to \mathbb{R} \\ x &\mapsto x^2, \end{aligned} \quad \text{and} \quad \begin{aligned} h: \mathbb{R} &\to \mathbb{R}_+ \\ x &\mapsto x^2, \end{aligned}$$

Composition of Functions

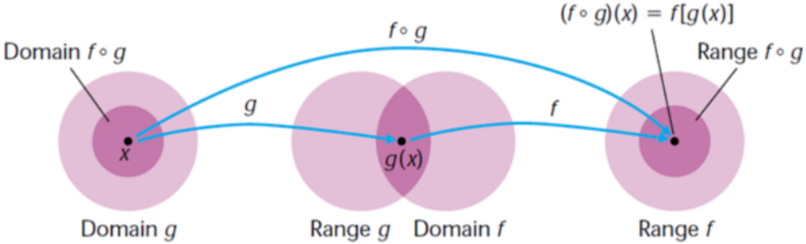
Definition

Let $f : G \to E$ and $g : F \to G$ two functions. We call the composite of the functions g and f, the function denoted $f \circ g$ defined from F in E by

$$\forall x \in F, \; f \circ g(x) = f(g(x))$$



Composition of Functions



Composition of Functions

Example

Let $f, g : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2$ and $g(x) = x + 1$. Then

$$(g \circ f)(x) = g(x^2) = x^2 + 1,$$

while

$$(f \circ g)(x) = f(x + 1) = (x + 1)^2 = x^2 + 2x + 1,$$

Therefore, in general,

$$g \circ f \neq f \circ g$$

Composite Functions

Proposition

LetE, F, G and H be four sets. For all functions $f : E \to F$, $g : F \to G$ and $h : G \to H$, we have:

$$(h \circ g) \circ f = h \circ (g \circ f)$$

Remark We say $\circ$ is an **associative operation** on set of functions

Direct Image

Let E, F be two sets.

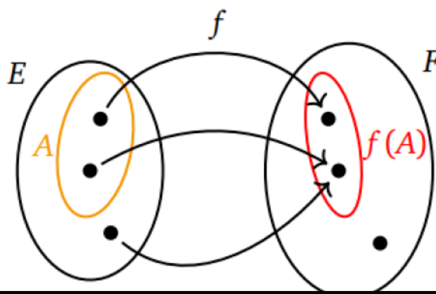
Definition

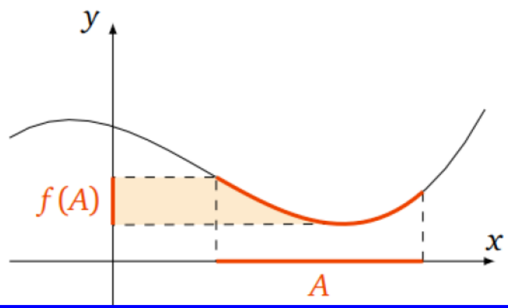
Let $A \subset E$ and $f : E \to F$, the direct image of A by f is the set

$$f(A) = \{f(x) \in F \mid x \in A\}$$

Formally we have,

$$\forall y \in F, (y = f(x) \Longleftrightarrow \exists x \in A \mid y = f(x))$$





Direct Image

Example

We consider the function

$$f : \mathbb{R} \to \mathbb{R}$$

$$x \mapsto 2 - x$$

$f([0, \frac{1}{2}]) = \{f(x) \in \mathbb{R}, x \in [0, \frac{1}{2}]\}$. We have

$$\begin{aligned} 0 \leq x \leq \tfrac{1}{2} &\Longrightarrow -\tfrac{1}{2} \leq -x \leq 0 \\ &\Longrightarrow \tfrac{3}{2} \leq 2 - x \leq 2. \end{aligned}$$

Then

$$f([0, \frac{1}{2}]) = [\frac{3}{2}, 2]$$

Preimage

Let E, F be two sets.

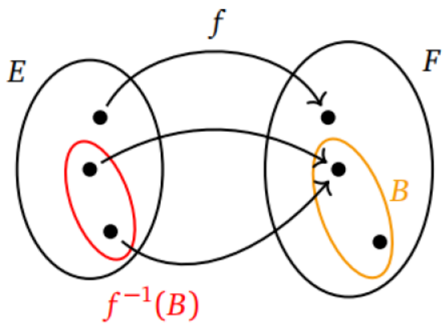
Definition

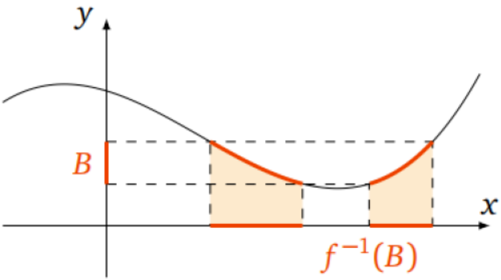
Let $B \subset F$ and $f : E \to F$, the preimage of B by f is the set

$$f^{-1}(B) = \{x \in E \mid f(x) \in B\}$$

Formally we have,

$$\forall x \in E, x \in f^{-1}(B) \iff f(x) \in B$$





Preimage

Example

We consider the function

$$f : \mathbb{R} \to \mathbb{R}$$

$$x \mapsto (x-1)^2$$

$$f^{-1}(0) = \{x \in \mathbb{R} \mid f(x) = 0\} = \{1\}$$

$$f^{-1}(]0, \frac{1}{2}[) = \{x \in \mathbb{R} \mid f(x) \in]0, \frac{1}{2}[\}$$

Solving the inequality $0 < (x-1)^2 < \frac{1}{2}$ gives:

$$f^{-1}(]0, \frac{1}{2}[) =]\frac{\sqrt{2}-1}{\sqrt{2}}, 1[\cup]1, \frac{\sqrt{2}+1}{\sqrt{2}}[$$

Direct Image and Preimage

Proposition

Let $f : E \to F$ a function, $A, B \subset E$ and $M, N \subset F$. We have

1- $A \subset B \Longrightarrow f(A) \subset f(B)$

2- $f(A \cup B) = f(A) \cup f(B)$

3- $f(A \cap B) \subset f(A) \cap f(B)$

4- $M \subset N \Longrightarrow f^{-1}(M) \subset f^{-1}(N)$

5- $f^{-1}(M \cup N) = f^{-1}(M) \cup f^{-1}(N)$

6- $f^{-1}(M \cap N) = f^{-1}(M) \cap f^{-1}(N)$

One-to-One Function or Injection function

Definition

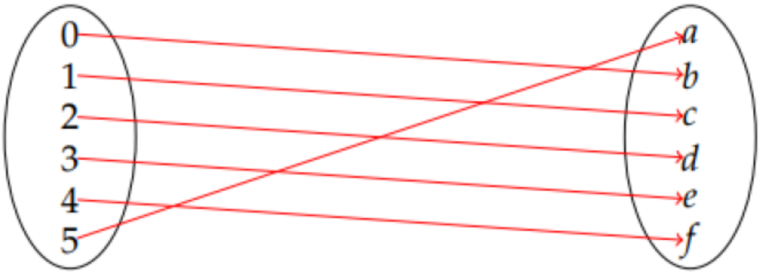
A function $f : E \to F$ is one-to-one (or injective) if we have

$$\forall x, x' \in E;\ f(x) = f(x') \Longrightarrow x = x'$$

or by taking the contrapositive of the implication,

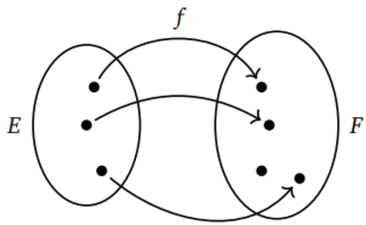
$$\forall x, x' \in E;\ x \neq x' \Longrightarrow f(x) \neq f(x')$$

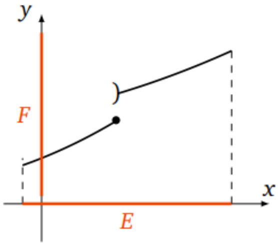
Here is a example showing an one-to-one function



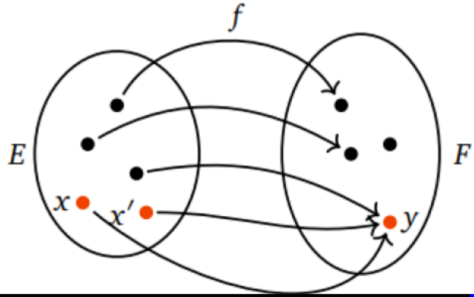
One-to-One

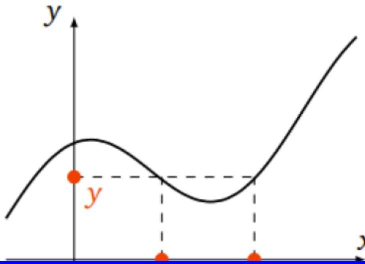
The functions f represented are one-to-one or injective:





Here are two non-injective functions:





Onto function or Surjection function

Definition

A function $f : E \to F$ is onto or surjective if we have

$$\forall y \in F, \ \exists x \in E, \ y = f(x)$$

A surjection is also known as an onto function. From the definition, f is surjective if, and only, if $f(E) = F$.

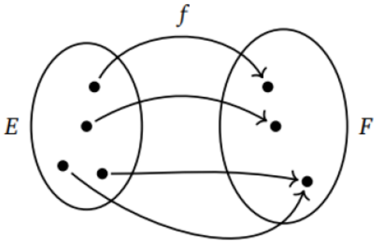
Example

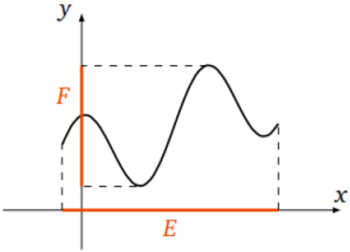
- Show that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = 2x + 1$ is one-to-one and onto.
- Show that the function $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ given by $f(n) = 2n + 1$ is one-to-one but not onto.

The function f is surjective if and only if the equation $y = f(x)$ admits at least one solution x of E for any element y of F.

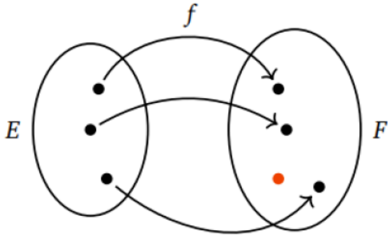
Onto function or Surjection function

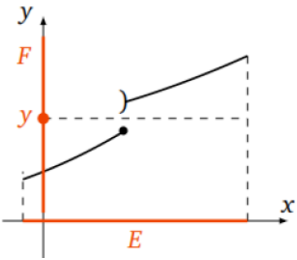
The function f represented are surjective:





Here are two non-surjective functions:





Bijection

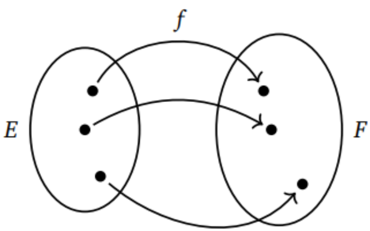
Definition

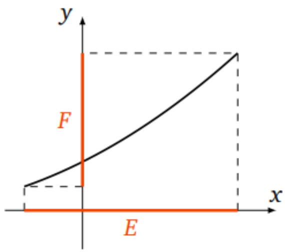
A function that is both injective and surjective is said to be bijective.

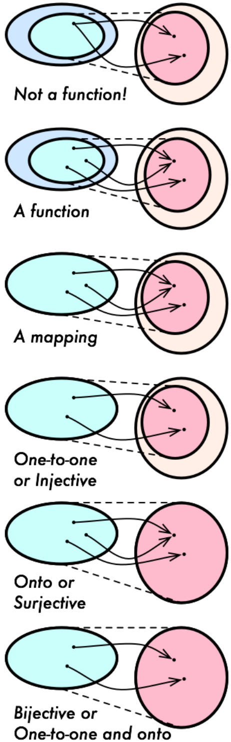
Proposition

The function f is bijective if and only if

$$\forall y \in F,\ \exists!\ x \in E,\ y = f(x)$$






Not a function!
A function
A mapping
One-to-one
or Injective
Onto or
Surjective
Bijective or
One-to-one and onto

Inverse function

When a function is bijective, it is possible to introduce the notion of inverse function.

Definition

Let $f : E \to F$ a bijective map from E to F. We then define a function from F to E by

$$\forall x \in E,\ y \in F, y = f(x) \Longleftrightarrow x = f^{-1}(y)$$

$$\begin{array}{rcccl} f & : & E & \longrightarrow & F \\ & & x & \longmapsto & y = f(x) \end{array} \Longleftrightarrow \begin{array}{rcccl} f^{-1} & : & F & \longrightarrow & E \\ & & y & \longmapsto & x = f^{-1}(y) \end{array}$$

Proposition

Let $f : E \to F$ be a bijective map, then $f \circ f^{-1} = Id_F$ and $f^{-1} \circ f = Id_E$

Inverse function

Remark :

1- $f \circ f^{-1} = Id_F$ is reformulated as follows: $\forall y \in F,\ f(f^{-1}(y)) = y$.

While $f^{-1} \circ f = Id_E$ is written $\forall x \in E,\ f^{-1}(f(x)) = x$.

2- If f is bijective, we have $(f^{-1})^{-1} = f$.

Examples

1- If E is a set, Id_E is bijective and $Id_E^{-1} = Id_E$

2- Let $f : \mathbb{R} \to \mathbb{R}_+^*$ defined by: $f(x) = exp(x)$ is bijective and its inverse bijection is $f^{-1} : \mathbb{R}_+^* \to \mathbb{R}$ defined by $f^{-1}(y) = ln(y)$. We have $exp(ln(y)) = y$ for all $y \in \mathbb{R}_+^*$ and $ln(exp(x)) = x$ for all $x \in \mathbb{R}$.

3- we take the previous example. We had previously shown that it is injective and surjective so it is a bijection. its inverse map is:

$$f^{-1} : \mathbb{R} \to \mathbb{R}$$

$$x \mapsto f^{-1}(x) = \frac{x-1}{2}$$

Composition of functions

Proposition 1

Let E, F, G be three sets, $f : E \to F$ and $g : F \to G$ two functions.

1- If f and g are both one-to-one then $g \circ f$ is one-to-one.

2- If f and g are both onto then $g \circ f$ is onto.

3- If $g \circ f$ is one-to-one then f is one-to-one.

4- If $g \circ f$ is onto then g is onto.

Proposition 2

Let $f : E \to F$ and $g : F \to G$ be a bijective functions, then $g \circ f$ is bijective and its inverse bijection is $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Let E be a set.
Recall that, for any $A \in \mathcal{P}(E)$, the indicator function of A is the map

$$\mathbf{1}_A : E \longmapsto \{0,1\}, \quad x \longmapsto \begin{cases} 0 & \text{si} \quad x \notin A \\ 1 & \text{si} \quad x \in A. \end{cases}$$

We denote by 1 the constant map from $\mathcal{P}(E)$ to $\{0,1\}$ equal to 1.

- Show that, for all $A, B \in \mathcal{P}(E)$:

$$\begin{array}{ll} (1) & A = B \iff 1_A = 1_B, \\ (2) & 1_{\overline{A}} = 1 - 1_A, \\ (3) & 1_{A\cap B} = 1_A \cdot 1_B, \\ (4) & 1_{A\cup B} = 1_A + 1_B - 1_{A\cap B}, \\ (5) & 1_{A\setminus B} = 1_A - 1_{A\cap B}. \end{array}$$

- Deduce that, for all $A, B \in \mathcal{P}(E)$:

$$A \cap (A \cup B) = A \quad \text{and} \quad A \cup (A \cap B) = A.$$