Ministry of Higher Education and Scientific Research
National School of Cyber Security
Foundation Training Department

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# Midterm Test — Analysis I

Duration: 1h30

## Part 1: Questions on the course material (8 pts)

1. Dertemine the basic period of the following functions , if they exist

$$f(x) = (-1)^{[x]}[x-[x]] \quad (1)$$

$$f(x) = x - 4\left[\frac{x}{4}\right] \quad (1)$$

2. Dertemine **without details**, the domain of defintion of the following functions

   1. $f(x) = \sqrt[4]{x - x^3}$ (0.5)
   2. $g(x) = x + 2\sqrt{x+1}$ (0.5 )
   3. $h(x) = \dfrac{1}{\sqrt{-\left(|x|^3 + 5x^2 - 2|x| - 24\right)}}$ (2)
   4. $k(x) = \ln(1 + [x])$ (0.5 )

3. Using the $\varepsilon - n_0$ definition of the limit of a sequence, prove that

$$\lim_{n \to +\infty} \frac{5n^2 + 3n - 2}{2n^2 + 7n + 4} = \frac{5}{2} \quad (1.5\ )$$

4. Compute

$$\lim_{n \to +\infty} \frac{[\sqrt{n}]}{n} \quad (1\ )$$

and justify your result using the $\varepsilon - n_0$ definition of the limit of a sequence as in the previous question

## Part 2: Exercises (12 pts)

### Exercise 1 (4 pts)

Choose one of the two following questions

1. Show by induction that for strictly positive real numbers $a_k$, with $0 \leq k \leq n$, we have

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) \geq n^2$$

2. Let

$$B = \left\{2^x + 2^{\frac{1}{x}},\ x > 0\right\},$$

Find $\inf(B)$ without using the notion of the graph of a real function

### Exercise 2 (4 pts))

We consider the following well defined real sequence $(u_n)_{n\in\mathbb{N}}$ given by

$$\begin{cases} u_0 \geq 1 \\ u_{n+1} = \sqrt{(u_n)^2 + \dfrac{1}{2^n}} \end{cases}$$

1. Prove that $\forall n \in \mathbb{N} : u_n \geq 1$.
2. Prove that $\forall n \in \mathbb{N} : u_{n+1} - u_n \leq \dfrac{1}{2^{n+1}}$ and deduce that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy's sequence
3. Set for all $n \in \mathbb{N}$ :

$$v_n = (u_{n+1})^2 - (u_n)^2$$

give the explicit formula of $(v_n)_{n\in\mathbb{N}}$

4. Using two methods compute $\sum\limits_{k=0}^{n} v_k$ and deduce $\lim\limits_{n\to+\infty} u_n$.

### Exercise 3 (4 pts)

We consider the real sequence $(u_n)_{n\in\mathbb{N}^*}$ defined by

$$\forall n \in \mathbb{N}^* : u_n = \frac{\ln(n!)}{n}$$

1. Prove that

$$\forall n \in \mathbb{N}^* : u_{2n} \geq \frac{\ln(n)}{2},$$

and deduce that the subsequence $u_{2n+1}$ diverges.

2. We consider the sequence $(z_n)_{n\in\mathbb{N}^*}$ defined by

$$\forall n \geq 2 : z_n = \frac{1}{\ln(n!)} \sum_{k=1}^{n} [\ln(k)]$$

Show that $(z_n)_n$ is convergent and determine its limit