Ministry of Higher Education and Scientific Research
National Higher School of Cybersecurity
1st Year Integrated Preparatory Cycle **Academic year : 2024 -2025**
Mathematical Analysis 1

Practice worksheet 1: Real numbers

Exercice 1. ** *Show that* ***for every*** $x, y, x', y' \in \mathbb{R}$ *it holds*

1. $x \cdot 0 = 0$
2. $-x = (-1) \cdot x$
3. $-(-x) = x$
4. $x \cdot (-y) = -(xy) = -(xy)$
5. $x \leq 0 \Leftrightarrow -x \geq 0$
6. $x \leq y \Leftrightarrow -x \geq -y$

Exercise 1. ***Show*** *that*

$$\forall n \geq 1 : 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Exercise 2. ***Show*** *that*

$$\forall n \geq 1 : 1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}.$$

Exercise 3. ***Show*** *that*

$$\forall n \geq 7 : n! \geq 3^n.$$

Exercise 4. ***Show*** *that*

$$\forall n \in \mathbb{N} - \{0\} : 2^{n-1} \leq n! \leq n^n.$$

Exercise 5. ***Prove*** *the following inequalities for* $n \in \mathbb{N}$

1. $(1+x)^n \geq 1 + nx,\ x > -1$
2. $(1+x_1)(1+x_2)(1+x_3)....(1+x_n) \geq 1 + x_1 + x_2 + x_3 + ... + x_n$, *where* $x_k > -1$
3. $n! < \left(\frac{1+n}{2}\right)^n,\ n > 1.$

Exercise 6. *Let* $n \in \mathbb{N}$. ***Show*** *that*

$$n^2 \text{ is odd } \Rightarrow\ n \text{ is odd}$$

Exercise 7. *Let* $a, b \in \mathbb{R}^+$. ***Prove*** *that*

$$\sqrt{a} + \sqrt{b} \leq \sqrt{2}\sqrt{a+b}$$

Exercise 8. *Let* $a, b \in \mathbb{R}^+ - \{0\}$. ***Prove*** *that*

1. $\dfrac{2}{\dfrac{1}{a} + \dfrac{1}{b}} \leq \sqrt{ab}$,

2. $\sqrt{a} + \sqrt{b} \leq \sqrt{2}\sqrt{a+b}$

Exercise 9. *Prove that for every* $x, y \in \mathbb{R}$ *the following properties hold*

1. $|x+y| \leq |x| + |y|$,

2. $|x-y| \geq ||x| - |y||$.

Exercise 10. *Solve the following equations*

1. $|x-1| + |x-2| = 2$,

2. $\sqrt{41-x} + \sqrt{41+x} = 10$.

Exercise 11.
1. *Solve the following equation* $|u-1| + |u+1| = 4$,

2. ***Deduce*** *the solutions of* $\left|\sqrt{1+x} - 1\right| + \left|\sqrt{1+x} + 1\right| = 4$,

3. ***Deduce*** *the solutions of* $\left|\sqrt{x+2-2\sqrt{1+x}}\right| + \left|\sqrt{x+2+2\sqrt{1+x}}\right| = 4$.

Exercise 12. ***Prove*** *that* $\sqrt[3]{3+2\sqrt{6}}$ *is* ***irrational***.

Exercise 13. ***Prove*** *that* $\sqrt{3}$ *is irrational.*

Exercise 14. ***Assume*** *that* $\sqrt{2}$, $\sqrt{3}$,*and* $\sqrt{6}$ *are* ***irrationals***. *Prove that*

1. $\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$,

2. $\left(\sqrt{2} + \sqrt{3}\right)^2 \notin \mathbb{Q}$,

3. $\sqrt{2} + \sqrt{3} + \sqrt{6} \notin \mathbb{Q}$.

Exercise 15. ***Determine*** *the following sets, put these sets in the form of an* ***interval*** *of* $\mathbb{R}$ *or a* ***union*** *of intervals*

1. $A_1 = \{x \in \mathbb{R} : x^2 < 1\}$,

2. $A_2 = \{x \in \mathbb{R} : x^3 < 1\}$,

3. $A_3 = \left\{x \in \mathbb{R} : -1 < \dfrac{2x}{1+x^2} < 1\right\}$,

4. $A_4 = \left\{ x \in \mathbb{R}^* : \dfrac{1}{|x|} > 1 \right\}$.

Exercise 16. ***Find*** *if any, the* ***infimum*** *and the* ***supremum*** *of the following set*

$$A = \left\{ \frac{mn}{(m+n)^2} : m, n \in \mathbb{N}^* \right\}.$$

Exercise 17. ***Find*** *if any, the* ***infimum*** *and the* ***supremum*** *of the following sets*

1. $A = \left\{ \dfrac{2^n}{2^n - 1} : m, n \in \mathbb{N}^* \right\}$,

2. $B = \left\{ \dfrac{1}{1 - 2^{-n}} : m, n \in \mathbb{N}^* \right\}$,

3. $C = \left\{ \left| \dfrac{x^3}{x^3 - 1} \right| : x \in]0, 1[\cup]1, +\infty[\right\}$.

Exercise 18. *Find if any, the infimum and the supremum of the following set*

$$S = \left\{ \frac{1}{p} + \frac{1}{p} : p, q \in \mathbb{N}^* \right\}.$$

Exercise 19. *Find if any, the infimum and the supremum of the following set*

$$M = \left\{ \frac{(-1)^n}{n} + \frac{2}{n} : n \in \mathbb{N}^* \right\}.$$

Exercise 20. *Find if any, the infimum and the supremum of the following set*

$$C = \left\{ \frac{x+1}{x+2} : x \in \mathbb{R}, x \leq 3 \right\}.$$

Exercise 21. *Find if any, the infimum and the supremum of the following set*

$$X = \left\{ (-1)^n + \frac{1}{n} : n \in \mathbb{N}^* \right\}.$$

Exercice 2. **
a. For any real number x, we denote by $[x]$ for the integer part of x defined by $[x] \in \mathbb{Z}$ and $[x] \leq x < [x] + 1$. Show that

1 . $x \leq y \Rightarrow [x] \leq [y]$

2. $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z} : [x + n] = [x] + n$

3. $\forall x, y \in \mathbb{R} : [x] + [y] \leq [x + y] \leq [x] + [y] + 1$

b. Show that for any $x \in \mathbb{R}$ and for each $n \in \mathbb{N}$: $n[x] \leq [nx]$, then deduce that $[x] \leq \left[\frac{[nx]}{n}\right]$.

c. Calculate $[x] + [-x]$ for all real x.

Exercice 3. ***Let a, b, c be positive real numbers.*

1. *Show that* $a^2 + b^2 + c^2 \geq ab + ac + bc$
2. *Deduce that* $(a + b + c)^2 \geq 3(ab + ac + bc)$
3. *Show that* $8(a^4 + b^4) \geq (a + b)^4$