الجمهورية الجزائرية الديمقراطية الشعبية

People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research National School of Cyber Security	NSCS	وزارة التعليم العالي والبحث العلمي المدرسة الوطنية العليا في الأمن السيبراني
Level : 1st Year Basic Training **Section / Group :** A & B **Module :** Algebra 1 **Full Name :**................................	**Midterm Test 1**	**Module's Teacher :** Dr. Hamza Moufek **Date :** 26 / 11/ 2025 **Duration :** 1h30 *Note : No documents are allowed.*

Exercise 1 : (5 points)

We consider on $\mathbb{R}^*$ the binary relation R defined by :

$$x, y \in \mathbb{R}^*, \quad x\,R\,y \iff x \cdot y > 0.$$

1. Show that R is an equivalence relation.
2. Determine the equivalence class $\overline{x}$ associated with x.
3. Indicate the number of elements of the quotient set $\mathbb{R}^*/R$.

Exercise 2 : (12 points)

A set E is said to be *countable* if and only if there exists a bijection from the set $\mathbb{N}$ of natural numbers onto E.

1. (a) For every $n \in \mathbb{N}$, define $f(n) = n + 1$. Prove that $f : \mathbb{N} \to \mathbb{N}^*$ is a bijection.
   (b) Determine a bijection g from $\mathbb{N}$ onto the set of even natural numbers $2\mathbb{N}$.
2. The goal of this question is to show that $\mathbb{Z}$ is countable. For every $n \in \mathbb{N}$, define :

$$\varphi(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even,} \\ -\dfrac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

   (a) Show that this defines a function $\varphi : \mathbb{N} \to \mathbb{Z}$.
   (b) Show that φ is injective.
   (c) Show that φ is surjective.
   (d) Explicitly give the inverse bijection φ^{-1}.
3. The goal of this question is to show that $\mathbb{N}^2$ is countable.
   (a) Prove by strong induction on $n \in \mathbb{N}^*$ that :

$$\forall n \in \mathbb{N}^*, \ \exists (p, q) \in \mathbb{N}^2, \quad n = 2^p(2q + 1).$$

   (b) Show that the function

$$\psi : \begin{cases} \mathbb{N}^2 \longrightarrow \mathbb{N}^*, \\ (p, q) \longmapsto 2^p(2q + 1) \end{cases}$$

   is bijective.
   (c) Deduce from the previous question and from 1.(a) that $\mathbb{N}^2$ is countable.
4. Deduce from the previous question that $\mathbb{Z}^2$ is countable.

5. A set E is said to be *at most countable* if and only if there exists a surjection from $\mathbb{N}$ onto E.

   (a) Consider the function

$$h : \begin{cases} \mathbb{Z} \times \mathbb{N}^* \longrightarrow \mathbb{Q}, \\ (p, q) \longmapsto \dfrac{p}{q} \end{cases}$$

   Determine whether h is injective, surjective, or bijective.

   (b) Conclude that $\mathbb{Q}$ is at most countable.

Exercise 3 : (3 points)

In this statement, the negation of a propositional formula P will be written indifferently as $\neg P$ or $\overline{P}$.

1. The *Sheffer stroke*, denoted "$|$", is the logical connective defined as follows : For any propositional formulas P and Q, the formula $(P \mid Q)$ is logically equivalent to $(\overline{P} \vee \overline{Q})$. Show that $(P \mid Q) \equiv \overline{P \wedge Q}$.
2. Express the connectives $\neg$, $\wedge$, $\vee$, and $\Rightarrow$ using only the Sheffer stroke.