EEE 485-585 FALL 2018 PROBLEM SET 1

Due Date: Tuesday, 23 October 2018, 17:00 (room EE-212)

Question 1 [25 pts]

Consider independent and identically distributed random variables $X_1, \ldots, X_n$, where $X_i \sim$ Poisson(λ).

(a) [5 pts] Given a realization (observation) $x_1, \ldots, x_n$ of these random variables find the maximum likelihood estimate of λ.

(b) [12 pts] Assume that we only observe events $X_i = 0$ or $X_i > 0$ for each random variable. Given a realization $y_1, \ldots, y_n$ of these events ($y_i = 0$ if $x_i = 0$ and $y_i = 1$ if $x_i > 0$), find the maximum likelihood estimate of λ.

(c) [8 pts] Assume that $X_i \sim$ Poisson$(e^{-\lambda})$ for $i = 1, \ldots, n$. Given a realization (observation) $x_1, \ldots, x_n$ of these random variables find the maximum likelihood estimate of λ.

Question 2 [25 pts]

Consider n independent and identically distributed trials $X_1, \ldots, X_n$, where each X_i represents the outcome of six-sided die. Let θ_j be the parameter that represents the probability that die takes value j, and let $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_6)$. Given a realization (observation) $x_1, \ldots, x_n$ of $X_1, \ldots, X_n$

(a)[13 pts] Compute the maximum likelihood estimate of $\boldsymbol{\theta}$.

(b)[12 pts] Assume that the prior on $\boldsymbol{\theta}$ is given as

$$p(\theta_1, \ldots, \theta_6) = \frac{\theta_1^{(\alpha_1 - 1)} \theta_2^{(\alpha_2 - 1)} \ldots \theta_6^{(\alpha_6 - 1)}}{A(\alpha_1, \ldots, \alpha_6)}$$

where $A(\cdot)$ is independent of $\boldsymbol{\theta}$. Based on this, compute the MAP estimate of $\boldsymbol{\theta}$.

Question 3 [25pts]

(a)[10 pts] Let $X \sim \mathcal{N}(\mu, 1)$. Generate n (take n as $100, 1000, 1000$) realizations from this random variable, then using these observations estimate μ (maximum likelihood estimation). Repeat this experiment 100 times and for each of these 100 runs, calculate the confidence intervals (95%) using this estimation. Report the proportion of these confidence intervals that contain the true value of μ, for $\mu = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$. Are your results consistent with the theory?

(b)[5 pts] What do you observe for different values of n? Does the estimated mean converge towards the true mean as n gets larger? Explain your reasoning.

(c)[10 pts] Repeat experiment in part a with $X \sim \mathcal{N}(\mu, 0.1)$ and $X \sim \mathcal{N}(\mu, 10)$. Compare the results with part a. Comment on the effect of variance. Is the result consistent with the theory?

Question 4 [25 pts]

Consider dataset $\mathcal{D} = \{(x_i, y_i)\}_{i=1}^n$. Assume that the following linear relationship holds between y_i and x_i: $y_i = ax_i + \epsilon_i$, for all $i = 1, \ldots, n$, where $\epsilon_1, \ldots, \epsilon_n$ are independent and identically distributed zero-mean normal variables with variance σ^2.

(a)[10 pts] Find the estimate $\hat{a}$ of a that minimizes the residual sum of squares on $\mathcal{D}$.

(b)[15 pts] With x_i fixed, y_i is a random variable. Using this, compute the distribution of $\hat{a}$. Comment on the dependence of this distribution on σ^2.