3-MANIFOLDS WITH PRESCRIBED $\mathbb{Z}_2$-COHOMOLOGY RINGS

by

Kerem Kubilay

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3-MANIFOLDS WITH PRESCRIBED $\mathbb{Z}_2$-COHOMOLOGY RINGS

APPROVED BY:

Prof. Ferit Öztürk . . . .
(Thesis Supervisor)

Prof. Çağrı Karakurt . . . .

Assist. Prof. Yasemin Kara . . . .

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ABSTRACT

3-MANIFOLDS WITH PRESCRIBED $\mathbb{Z}_2$-COHOMOLOGY RINGS

The question of whether a graded $\mathbb{Z}_2$-algebra can be realized as a $\mathbb{Z}_2$-cohomology ring of a closed 3-manifold is settled by M. M. Postnikov in 1948. Recently in 2024 J. A. Hillman provided a new argument, giving explicit surgery descriptions for 3-manifolds that realize the given ring. We present Hillman's construction, provide illustrations and streamline the argument by appealing only to intersection theory to obtain the desired cup products. Central to both Postnikov's and Hillman's approaches is the Postnikov–Wu identity, which pertains to a distinguished element in the ring and its cup products. We also provide an alternative proof of this identity, again using intersection theory.

ÖZET

$\mathbb{Z}_2$-KOHOMOLOJİ HALKASI BELİRTİLMİŞ 3-MANİFOLDLAR

Kapalı bir 3-manifoldun $\mathbb{Z}_2$-kohomoloji halkası olarak dereceli bir $\mathbb{Z}_2$-cebirinin gerçekleştirilebilmesi sorusu, 1948 yılında M. M. Postnikov tarafından çözülmüştür. Yakın zamanda, 2024'te, J. A. Hillman bu duruma yeni bir yaklaşım sunmuş ve verilen halkayı gerçekleştiren 3-manifoldlar için açık surgery tanımları vermiştir. Bu çalışmada Hillman'ın inşasını sunuyor, çizimlerle destekliyor ve istenen cup çarpımlarını elde etmek için yalnızca kesişim kuramına başvurarak argümanı bir miktar sadeleştiriyoruz. Hem Postnikov'un hem de Hillman'ın yaklaşımlarında merkezi rol oynayan unsur, halkadaki ayırt edici bir elemana ve onun çarpımlarına ilişkin olan Postnikov–Wu özdeşliğidir. Bu özdeşlik için kesişim kuramına dayalı alternatif bir ispat da sunuyoruz.

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1. INTRODUCTION

Given a manifold M its homology groups can be thought of as a count of distinct lower-dimensional pieces of M, which also provides some idea of how these pieces are attached together. Consequently, homology is a loose invariant: wildly different spaces can have the same homology groups. Interestingly, passing to the dual notion—cohomology—offers deeper insight into the structure of M. Cohomology groups come equipped with the cup product, endowing them with the structure of a graded ring. The product structure of this ring captures additional information about the aforementioned lower-dimensional pieces of M, namely, how they intersect. Hence, different arrangements of similar pieces can result in different cup products. A standard example is the distinction between $S^2 \vee S^4$ and $\mathbb{C}P^2$.

In this sense, choosing different coefficients for cohomology corresponds to choosing different ways to count intersections. In this thesis, we are primarily interested in cohomology with $\mathbb{Z}_2$ coefficients, which come with the following quirks. First, counting intersections with $\mathbb{Z}_2$ amounts to simply determining whether an intersection exists at all, which is comparatively easier. Second, since $\mathbb{Z}_2$ is a field, we do not need to worry about torsion—a measure of twisting in the structure of M. Finally, every manifold is $\mathbb{Z}_2$-orientable, which allows the cup product–intersection correspondance to extend naturally to non-orientable manifolds. These advantages come at a cost: discriminative power is traded for computational ease, and we once again end up with a loose invariant. For example, $L(2,1)$ and $L(4,1)$ have the same $\mathbb{Z}_2$-cohomology rings.

It can then be said that, if one has a sense of what M looks like in terms of the arrangement of its lower-dimensional pieces and knows how they intersect, this information is sufficient to compute its $\mathbb{Z}_2$-cohomology ring $H^*(M; \mathbb{Z}_2)$. It is natural to wonder whether this process can be reversed. That is, given a ring $\mathcal{A}^*$ that "looks like" a $\mathbb{Z}_2$-cohomology ring, can we construct a manifold M such that $\mathcal{A}^* \cong H^*(M; \mathbb{Z}_2)$? This question—considered for closed 3-manifolds—was answered

affirmatively by M. M. Postnikov in 1948 in his first published paper [1], provided, of course, that one has the correct notion of "looking like a $\mathbb{Z}_2$-cohomology ring." The same problem, when considered with integral coefficients, was solved partially by D. Sullivan [2] and then completely by V. G. Turaev [3]. Recently, in 2024, J. A. Hillman provided a new argument for the $\mathbb{Z}_2$ case [4], using modern surgery terminology and giving explicit descriptions of the links used in these surgeries. This thesis presents his argument.

In Chapter 2, we study the product structure of $\mathbb{Z}_2$-cohomology rings of closed 3-manifolds. As a standard fact, the cup product, Poincaré duality, and the Universal Coefficient Theorem together induce a perfect symmetric pairing

$$\smile : H^1(M; \mathbb{Z}_2) \odot H^2(M; \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2. \tag{1.1}$$

Using this and the interplay between Poincaré duality maps D_i and the cup product, one can define a $\mathbb{Z}_2$-valued triple product on $H^1(M; \mathbb{Z}_2)$ by

$$xyz = D_3(x \smile (y \smile z)) = x(D_2(y \smile z)) = x(D_1(y) \cap D_1(z)). \tag{1.2}$$

This characterizes the entire product structure of the cohomology ring and reduces the problem of understanding the ring to understanding its triple products. The integral and $\mathbb{Z}_p$ versions of this also appear in Sullivan's and Turaev's work. The main property of the $\mathbb{Z}_2$ triple product is the Postnikov–Wu identity

$$wxy = x^2 y + xy^2, \tag{1.3}$$

which holds for a distinguished element w and any degree-1 cohomology classes x and y. As one of our contributions we provide a simple proof of this by taking w to be the first Stiefel–Whitney class $\omega_1(TM)$, and applying intersection theory (See [5] for a through introduction to Stiefel–Whitney class).

The properties of $H^*(M; \mathbb{Z}_2)$ we collect culminate in Postnikov's definition of an MS-algebra as a finite graded commutative $\mathbb{Z}_2$-algebra $\mathcal{A}^* = \bigoplus_0^3 \mathcal{A}^i$ with:

(i) $\dim \mathcal{A}^1 = \dim \mathcal{A}^2$ and $\dim \mathcal{A}^0 = \dim \mathcal{A}^3 = 1$.

(ii) Multiplication gives a perfect pairing $\mathcal{A}^1 \odot \mathcal{A}^2 \to \mathcal{A}^3$.

(iii) There exists an element $w \in \mathcal{A}^1$ satisfying $wxy = x^2 y + xy^2$ for every $x, y \in \mathcal{A}^1$.

This is the aforementioned correct notion of "looking like a $\mathbb{Z}_2$-cohomology ring." Hence Postnikov's result can be stated as:

Theorem 1.1. (See Theorem 2.4) Every MS-algebra can be realized as the $\mathbb{Z}_2$-cohomology ring of a closed 3-manifold.

In the subsequent chapters, we outline Hillman's alternative proof for Theorem 2.4. Chapter 3 will focus on realizing orientable MS-algebras, i.e., those for which the distinguished element w is zero, which reduces (1.2) to $x^2y = xy^2$. Chapter 4, on the other hand, will tackle the non-orientable case—with w nonzero. In both cases, Hillman showed, using (1.2) or its reduced form, that there exists a convenient choice of basis for $\mathcal{A}^1$, greatly simplifying the realization. We outline this at the end of Chapter 2.

Orientable manifolds will be constructed via surgery on links in S^3 (we suggest [6] for a comprehensive treatment of surgery on links). Given $\mathcal{A}^*$, Hillman's main idea is to first assemble a trivial unlink L inside S^3, having $\dim \mathcal{A}^1$ many components, and then to modify L using a series of 'local' tangle moves selected from a limited repertoire. After surgery, the Kronecker duals of the meridians of the components will represent the basis for $\mathcal{A}^1$. The triple products will be obtained using the last equality in (1.1)—as intersections of Poincaré duals [7], which will be represented as the union of a Seifert surface and a meridian disk that we attach during surgery. Thus, the values of the x^3 will be determined by the framings of the components, which control the self-intersection of the meridian disk, while the other triple products of $\mathcal{A}^1$ will be dictated by the relative arrangements of triples of link components, forcing their Seifert surfaces to intersect in a specific manner. Our contribution here is merely to provide illustrations that clearly depict these intersections.

Non-orientable manifolds, meanwhile, will be constructed via surgery on links in $S^2 \tilde{\times} S^1$, the twisted product of S^2 and S^1 (In [8] Lickorish shows any closed non-rientable 3-manifold can be constructed with this method). While the essence of Hill-

man's construction remains the same—namely, to represent the basis for $\mathcal{A}^1$ as the Kronecker duals of the meridians of the components of a link $\mathcal{L}$, use 'local' tangle moves and to realize the triple products via specific configurations of triples of link components—the presence of nontrivial products wxy slightly complicates the assembly of such a link. First, the identity (1.2) does not simplify, and the choice of basis discussed in Chapter 2 is not as simple. Second, since w represents the orientation character, such products can only be realized by incorporating 'global' sublinks in $\mathcal{L}$ that wrap around $S^2 \tilde{\times} S^1$. Essentially, we will have to slice $S^2 \tilde{\times} S^1$ along the orientation-reversing direction, weave these global sublinks into the resulting pieces, and then glue the manifold back together with $\mathcal{L}$ sitting inside it.

Our main contribution lies in this non-orientable case. First we provide clear pictures showing how these 'global' sublinks are arranged inside $S^2 \tilde{\times} S^1$. Second, in [4], Hillman calculates fundamental groups of the link exteriors and obtains triple products via group relations. We, however, provide new illustrations depicting dual surfaces. We rely on these diagrams to compute all triple products via intersection theory, thereby avoiding any calculations involving link groups or fundamental groups entirely.

2. RING STRUCTURE

We will examine the product structure of $\mathbb{Z}_2$-cohomology rings of closed 3-manifolds. Using the facts we prove, we will attempt to arrive at necessary conditions for a $\mathbb{Z}_2$-algebra to be realized as a cohomology ring. These conditions will turn out to be sufficient as well. Among the properties these rings possess, a certain identity will be of prime importance—the Postnikov–Wu identity. We will provide a geometric proof of this. In the second part of this chapter, we show the existence of a convenient choice of basis for the $\mathbb{Z}_2$-algebra we wish to realize.

2.1. Realizability

First, we define realizability.

Definition 2.1.1. *A finite, commutative, graded $\mathbb{Z}_2$-algebra $\mathcal{A}^*$ is said to be* realizable *if there exists a closed* 3-*manifold M such that*

$$\mathcal{A}^* \cong H^*(M; \mathbb{Z}_2). \tag{2.1}$$

To characterize realizable algebras, we must gather structural information about $\mathbb{Z}_2$-cohomology rings. For some algebraic facts, we need another definition.

Definition 2.1.2. *Let R be a commutative ring with unit. An R-linear map*

$$\Phi : \mathcal{M} \otimes_R \mathcal{N} \longrightarrow \mathcal{L} \tag{2.2}$$

of R-modules is called a pairing. *We say the pairing is* perfect *if the associated maps*

$$m \longmapsto \Phi(m \otimes \bullet) \quad and \quad n \longmapsto \Phi(\bullet \otimes n) \tag{2.3}$$

yield isomorphisms

$$\mathcal{M} \xrightarrow{\cong} \mathrm{Hom}_R(\mathcal{N}, \mathcal{L}) \quad and \quad \mathcal{N} \xrightarrow{\cong} \mathrm{Hom}_R(\mathcal{M}, \mathcal{L}). \tag{2.4}$$

With this in hand, we state some immediate facts.

Proposition 2.1. Let M be a closed 3-manifold. Then:

(i) Via evaluation we have Kronecker pairings

$$\langle \bullet, \bullet \rangle : H^i(M; \mathbb{Z}_2) \otimes H_i(M; \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2, \tag{2.5}$$

and since $\mathbb{Z}_2$ is a field, the Universal Coefficient Theorem yields

$$H^i(M; \mathbb{Z}_2) \cong \mathrm{Hom}\big(H_i(M; \mathbb{Z}_2), \mathbb{Z}_2\big) \quad \text{and} \quad H_i(M; \mathbb{Z}_2) \cong \mathrm{Hom}\big(H^i(M; \mathbb{Z}_2), \mathbb{Z}_2\big). \tag{2.6}$$

Hence the Kronecker pairings are perfect.

(ii) Cap product with the fundamental class $[M]$ yields the Poincaré duality isomorphisms

$$D_i : H^i(M; \mathbb{Z}_2) \longrightarrow H_{3-i}(M; \mathbb{Z}_2). \tag{2.7}$$

(iii) The standard cup product induces symmetric pairings

$$\smile : H^1(M; \mathbb{Z}_2) \odot H^1(M; \mathbb{Z}_2) \longrightarrow H^2(M; \mathbb{Z}_2), \tag{2.8}$$

and

$$\smile : H^1(M; \mathbb{Z}_2) \odot H^2(M; \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2, \tag{2.9}$$

the latter being a perfect pairing.

Proof. The only fact that warrants proof is that the pairing

$$\smile : H^1(M; \mathbb{Z}_2) \odot H^2(M; \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2 \tag{2.10}$$

is perfect. Let "$*$" denote the Kronecker dualities in (i). For $x \in H^1(M; \mathbb{Z}_2)$, the map $x \smile \bullet$ is given by $[D_2^{-1}(x^*)]^*(\bullet)$, while for $\Gamma \in H^2(M; \mathbb{Z}_2)$ the map $\bullet \smile \Gamma$ is $[D_1^{-1}(\Gamma^*)]^*(\bullet)$. Hence the associated maps

$$x \mapsto x \smile \bullet \quad \text{and} \quad \Gamma \mapsto \bullet \smile \Gamma \tag{2.11}$$

are just compositions of Poincaré and Kronecker dualities, and therefore are isomorphisms. $\square$

Proposition 2.1 allows us to define the symmetric trilinear form:

$$\mu : \odot^3 H^1(M; \mathbb{Z}_2) \to \mathbb{Z}_2, \tag{2.12}$$

given by

$$\mu(x \odot y \odot z) = D_3((x \smile (y \smile z)) = x(D_2(y \smile z)) = x(D_1(y) \cap D_1(z)). \tag{2.13}$$

The form μ contains a lot of information; in fact it captures the entire product structure of the ring. Since $H^1(M; \mathbb{Z}_2) = Hom(H_1(M; \mathbb{Z}_2), \mathbb{Z}_2) \cong H_1(M; \mathbb{Z}_2) \cong H^2(M; \mathbb{Z}_2)$ and $H^0(M; \mathbb{Z}_2) \cong H^3(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$, we have the following:

Proposition 2.2. The pair $(H^1(M; \mathbb{Z}_2), \mu)$ determines $H^*(M; \mathbb{Z}_2)$.

Proof. Let $\{\gamma_1, \gamma_2 \ldots \gamma_N\}$ be a basis for $H_1(M; \mathbb{Z}_2)$. Then we have bases $\{x_1, x_2 \ldots x_N\}$ and $\{\Gamma_1, \Gamma_2 \ldots \Gamma_N\}$ for $H^1(M; \mathbb{Z}_2)$ and $H^2(M; \mathbb{Z}_2)$ given by Kronecker and Poincaré duals of γ_i's respectively. Then we have $D(x_i \smile \Gamma_j) = x_i(\gamma_j) = 1$ if and only if $i = j$. The only degree of freedom we are left with is the values of the products $x_i \smile x_j$ which μ determines as follows. Fixing i and j we collect nonzero values of μ and define $K_{ij} := \{\, k \mid k \leq N, \ \mu(x_i \odot x_j \odot x_k) = 1\}$. Then by our choice of bases:

$$x_i \smile x_j = \sum_{K_{ij}} \Gamma_k \tag{2.14}$$

$\square$

For brevity, from this point on we write xyz instead of $\mu(x \odot y \odot z)$ and call it the triple product. A property of the triple product we will use extensively is due to Postnikov.

Theorem 2.3 (Postnikov–Wu Identity [1, 4])**.** Let M be a closed 3-manifold. Then there exists a distinguished element $w \in H^1(M; \mathbb{Z}_2)$ such that for every $x, y \in H^1(M; \mathbb{Z}_2)$,

$$wxy = x^2 y + xy^2, \tag{2.15}$$

where $w = 0$ if and only if M is orientable.

Proof. We assume M is smooth and let $w = \omega_1(TM) \in H^1(M; \mathbb{Z}_2)$ be the first Steifel-Whitney class so that $w = 0$ if and only if M is orientable [5]. For $x, y \in H^1(M; \mathbb{Z}_2)$ with $x \smile y \neq 0$ Poincaré duals $D(x) = X$ and $D(y) = Y$ can be chosen as smoothly immersed surfaces in M that intersect transversely at a closed curve $X \cap Y = C$. Restricting TM to this curve we get a splitting $TM|_C = TC \oplus N_{C/M}$, tangent and normal bundle of C in M. Hence,

$$wxy = w(X \cap Y) = \omega_1(TM|_C) = \omega_1(TC) + \omega_1(N_{C/M}). \tag{2.16}$$

Now $\omega_1(TC) = 0$ since TC is trivial and $N_{C/M}$ can be further split down as $N_{C/Y} \oplus N_{C/X}$ as the parts of the normal bundle tangent to Y and X respectively. So (2.16) becomes

$$wxy = \omega_1(N_{C/Y}) + \omega_1(N_{C/X}). \tag{2.17}$$

If $\omega_1(N_{C/Y}) = 0$ then in a small enough neighborhood of C, X can be pushed away from itself using $N_{C/Y}$, which implies $X \cap X' \cap Y = \emptyset$. If $\omega_1(N_{C/Y}) = 1$ then going along C reverses the orientation of $N_{C/Y}$ and pushing X with $N_{C/X}$ slightly creates a self-intersection $X \cap X$, which is a curve in X intersecting C transversely once. Thus,

$$\omega_1(N_{C/Y}) = \mathrm{mod}_2 \, \#(X \cap X' \cap Y) = x^2 y \tag{2.18}$$

Putting it all together, we obtain

$$wxy = \omega_1(N_{C/Y}) + \omega_1(N_{C/X}) = x^2 y + xy^2. \tag{2.19}$$

$\square$

These observations motivates the following definition due to Postnikov.

Definition 2.1.3 ([1,4])**.** *An MS-algebra is a finite graded commutative $\mathbb{Z}_2$-algebra $\mathcal{A}^* = \bigoplus_0^3 \mathcal{A}^i$ such that;*

(i) *$dim\mathcal{A}^1 = dim\mathcal{A}^2$ and $dim\mathcal{A}^0 = dim\mathcal{A}^3 = 1$*

(ii) *Multiplication gives a perfect pairing $\mathcal{A}^1 \odot \mathcal{A}^2 \to \mathcal{A}^3$.*

(iii) *There exist a distinguished element $w \in \mathcal{A}^1$ satisfying $wxy = x^2 y + xy^2$ for every $x, y \in \mathcal{A}^1$.*

We call $\mathcal{A}^*$ orientable if $w = 0$ and nonorientable if $w \neq 0$.

From the previous discussion it is clear that a realizable algebra is an MS-algebra. The rest of this thesis will try to present the proof of the reverse assertion. That is;

Theorem 2.4. (Postnikov [1, 4]) Every MS-algebra is realizable.

2.2. Choosing a Basis

When $\mathcal{A}^*$ is an orientable MS-algebra, Postnikov-Wu identity reduces to $x^2y = xy^2$ which can be used to construct a basis for $\mathcal{A}^1$ where nonzero triple products x^2y comes in consecutive pairs and in a sense normal to each other. This can be formalized with the following:

Theorem 2.5 (Hillman [4]). If $\mathcal{A}^*$ is orientable and $dim\mathcal{A}^1 = N$ then there exist an even $K \leq N$ and a basis $\{x_1, x_2, \ldots x_K, \ldots, x_N\}$ for $\mathcal{A}^1$ such that:

(a) $x_i^2 x_j \neq 0$ iff there exist a $k \leq K/2$ with $i = 2k - 1$ and $j = 2k$.

(b) $x_i^3 = 0$ for all $i \leq K$.

Proof. Let $\mathcal{A}^1 = \langle x_1, x_2, \ldots, x_N \rangle$. If $x_i^2 = 0$ for all i, then $K = 0$ and we are done. If not, let $x_i^2 \neq 0$ for $i \leq M \leq N$. Define $\mathcal{S}^1 := \langle x_1, x_2, \ldots, x_M \rangle$ and $\mathcal{S}^2 := \langle x_1^2, x_2^2, \ldots, x_M^2 \rangle$. If $x^2 \in \mathcal{S}^2$, then by non-singularity and the Postnikov-Wu identity, there exists a $y \in \mathcal{S}^1$ such that $x^2y = xy^2 \neq 0$. This makes the restricted pairing $\mathcal{S}^1 \odot \mathcal{S}^2 \to \mathcal{A}^3$ perfect. Hence, there is a modification of our basis where the pairing is block diagonal with blocks either (1) or $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$. The former corresponds to $x^3 \neq 0$, while the latter corresponds to nonzero products x^2y. A final re-enumeration gives the desired basis $\{x_1, x_2, \ldots, x_K, \ldots, x_N\}$ where $K/2$ is the number of blocks $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$. $\qquad\square$

If $\mathcal{A}^*$ is not orientable we have the full identity $wxy = x^2y + xy^2$. A similar argument can be used to normalize products wxy.

Theorem 2.6 (Hillman [4]). If $\mathcal{A}^*$ is nonorientable with $dim\mathcal{A}^1 = N$ then there exist an even $K \leq N$ and a basis $\{w = x_1, x_2, \ldots x_K, \ldots, x_N\}$ for $\mathcal{A}^1$ such that:

(a) When $w^2 = 0$, $wx_ix_j \neq 0$ iff there exists a $k \leq K/2$ with $i = 2k$ and $j = 2k + 1$.

(b) When $w^2 \neq 0$, $wx_ix_j \neq 0$ iff there exists a $k \leq K/2$ with $i = 2k - 1$ and $j = 2k$.

Proof. Start with $\mathcal{A}^1 = \langle w = x_1, x_2, \ldots, x_N \rangle$. First, assume $w^2 = 0$. Now, since $w \neq 0$, we can let $wx_i \neq 0$ for $1 < i \leq K + 1$ and define $\mathcal{W}^1 := \langle x_2, x_3, \ldots, x_{K+1} \rangle$ and $\mathcal{W}^2 := \langle wx_2, wx_3, \ldots, wx_{K+1} \rangle$. Note that $wxy \neq 0$ implies $wx \neq 0$ and $wy \neq 0$. Hence, once again, the restricted pairing $\mathcal{W}^1 \odot \mathcal{W}^2 \to \mathcal{A}^3$ is perfect and can be made block diagonal. But since $wx^2 = x^3 + x^3 = 0$ for all x, every block is of the form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$ and there are $K/2$ many of them. If $w^2 \neq 0$, then we have $\mathcal{W}^1 := \langle w = x_1, x_2, \ldots, x_K \rangle$ and $\mathcal{W}^2 := \langle w^2 = wx_1, wx_2, \ldots, wx_K \rangle$, and the argument is the same. $\square$

3. ORIENTABLE CASE

We will realize orientable MS-algebras via surgery on links in S^3 [4]. Given $\dim \mathcal{A}^1 = N$, the general idea is to first assemble an N-component trivial unlink NU inside S^3, and then use a series of local tangle moves (Figure 3.2) to introduce the desired triple products as intersections of Seifert surfaces for the components. Our normalized basis from the previous section allows us to focus only on the relative configuration of pairs and triples of link components. The relevant configurations are the two component link $L_{2,4}$, the Borromean link Bo (Figure 3.1) and the three component link U_{002} (Figure 3.3).

Figure 3.1. (a) picture of $L_{2,4}$, (b) picture of Bo.

We can achieve all three of these by using the two tangle moves shown in Figure 3.2. Specifically, if $2U$ and $3U$ are trivial unlinks of two and three components respectively, then using move (a) on $2U$ results in $L_{2,4}$, while using move (b) on $3U$ gives us Bo. Using both moves on $3U$, we obtain the link U_{002}.
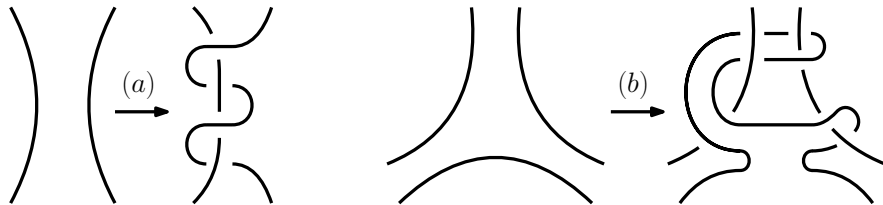

Figure 3.2. (a) increases linking number by 2, (b) does not change it.

The fact that these moves can be performed locally i.e in a small enough ball of our choosing (Fig 3.7), will be crucial to the construction. The triple products we have already realized will remain unaffected by the addition of new basis elements and new products.
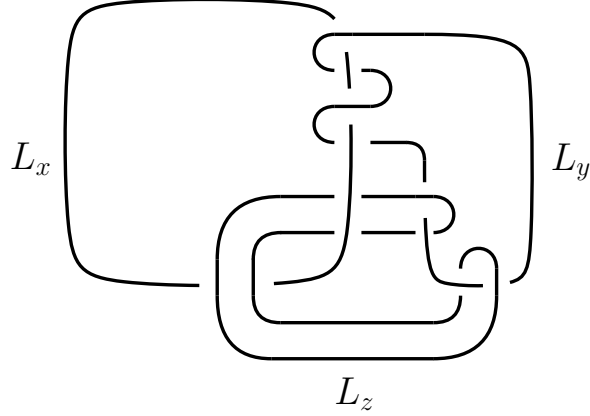

Figure 3.3. The link U_{002} with linking numbers 0, 0 and 2.

3.1. Surgery in S^3

We choose the following convention for surgery [4]. Let K be a knot in S^3 with longitude λ_K and meridian μ_K. Let $X(K)$ denote its exterior. A p-framed surgery on K is given by a homoemorphism $\phi : \partial X(K) \to S^1 \times \partial D^2$ where $\phi(\lambda_K + p\mu_K) = * \times \partial D^2$ and $\phi(\mu_K) = S^1 \times *$. That is, if $\{[\lambda_K], [\mu_K]\}$ and $\{[S^1], [\partial D^2]\}$ are taken as bases for $H_1(\partial X(K))$ and $H_1(S^1 \times \partial D^2)$ respectively , then ϕ induces the following map on homology:

$$\begin{pmatrix} -p & 1 \\ 1 & 0 \end{pmatrix} : H_1(\partial X(K)) \to H_1(S^1 \times \partial D^2) \tag{3.1}$$

Hence $\phi(\lambda_K)$ is a simple closed curve representing the homology class $-p[S^1] + [\partial D^2]$ in $H_1(S^1 \times \partial D^2)$, so if $p = 0$, $\phi(\lambda_K)$ bounds a copy of D^2 in $S^1 \times D^2$ and if $p = 2$ it bounds a Möbius band.

Now take an N-component framed link L in S^3 where each component is an unknot with even framing and each pair has even linking number. Perform surgery on L and let $M(L)$ denote the resulting manifold. The meridians μ_{L_i} of the components

L_i form a canonical basis for $H_1(M(L); \mathbb{Z}_2)$, and their Kronecker duals yield a basis $\{x_1, x_2, \ldots, x_N\}$ for $H^1(M(L); \mathbb{Z}_2)$. The Poincaré duals of the x_i's will then be represented by surfaces $F_i = \Sigma_i \cup \Lambda_i$, where Σ_i is a Seifert surface for L_i lying entirely in $X(L_i)$, and Λ_i is a surface bounded by $\phi(\lambda_{L_i})$ inside $S^1 \times D^2$. Since all linking numbers are even, we may attach handles to Σ_i so that $F_i \cap L_j = \emptyset$ for $i \neq j$. In particular, if $l(L_i, L_j) = 0$, then both feet of the handle can be attached on the same side of Σ_i, and if, further, L_i has zero framing, then F_i will be orientable. Whereas if $l(L_i, L_j) = 2$, then the feet of the handle attach on opposite sides, and F_i will be non-orientable (Figure 3.4).
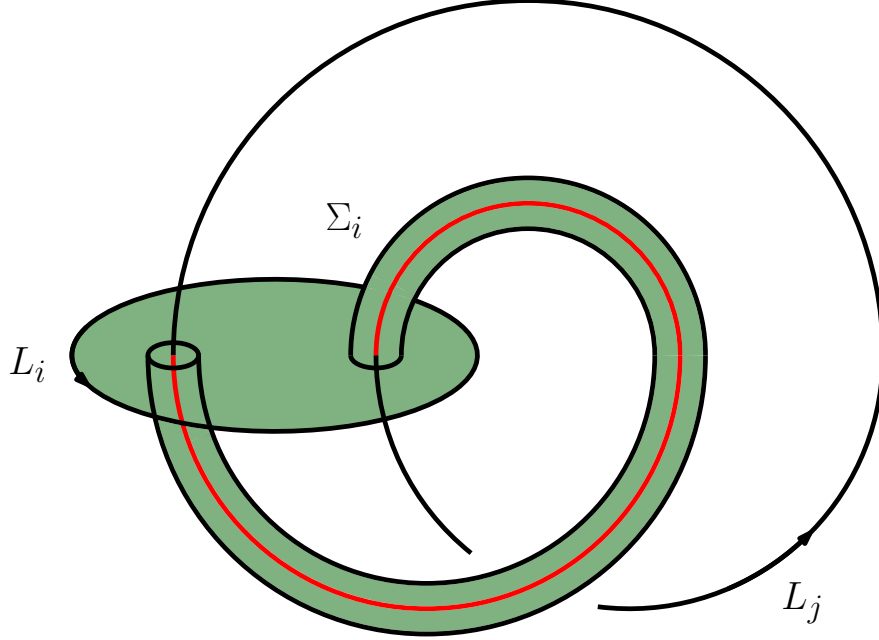

Figure 3.4. When $l(L_i, L_j) = 2$, the feet of the handle attach to opposite sides of the disc bounded by L_i. The resulting Seifert surface Σ_i-colored green-is a punctured Klein bottle which will be capped off by Λ_i after surgery.

3.2. Small Examples

We will go through the construction of the cases with $N = \dim \mathcal{A}^1 \leq 3$.

(i) $N = 1$: Let $\mathcal{A}^1 = \langle x \rangle$. Take an unknot L in S^3; we will represent x as the Kronecker dual of the meridian μ_L in $M(L)$. If $x^2 = 0$, then give L zero framing so that $\phi(\lambda_L)$ bounds a disc in both $S^1 \times D^2$ and $X(L)$. Hence, $F_x = \Sigma_x \cup \Lambda_x \cong D^2 \cup D^2 \cong S^2$. We get $x^2 = D^{-1}(F_x \cap F'_x) = 0$ as desired. It is easily seen that $M(L) \cong S^2 \times S^1$. If, on the other hand, $x^2 \neq 0$, then $x^3 \neq 0$ by non-singularity, and we give L framing 2. We then obtain the familiar manifold $L(2,1) = \mathbb{R}P^3$. In our setup, its triple product is given as follows: $\phi(\lambda_L)$ bounds a disc outside and a Möbius band inside—F_x is a copy of $\mathbb{R}P^2$. The disc part of F_x can be pushed away from itself, while the Möbius band part creates a self-intersection, which is a simple closed curve γ homotopic to the core of $S^1 \times D^2$. Since $\phi(\mu_L) = S^1 \times *$, pushing γ outside of $S^1 \times D^2$ gives a curve homotopic to the meridian of L. Thus, $x^3 = x(F_x \cap F'_x) = 1$.

(ii) $N = 2$: Set $\mathcal{A}^1 = \langle x, y \rangle$. We start with two unknots L_x and L_y. If $x^2 y = yx^2 = 0$, we proceed as in (i) and take a connected sum. If not, we apply tangle move (a) (Figure 3.2) so that $L_x \sqcup L_y \cong L_{2,4}$. In particular, $l(L_x, L_y) = 2$. By Theorem 2.5, we may further assume $x^3 = y^3 = 0$, so we give each component zero framing. After attaching a handle to Σ_y around a segment of L_x (colored red in Figure 3.5) to make F_y and L_x disjoint, we see that F_y is a Klein bottle intersecting itself at a curve homotopic to the meridian of L_x. Hence, $xy^2 = x(F_y \cap F'_y) = 1$ but $y^3 = y(F_y \cap F'_y) = 0$. This is the quaternionic manifold S^3/Q_8 [4].
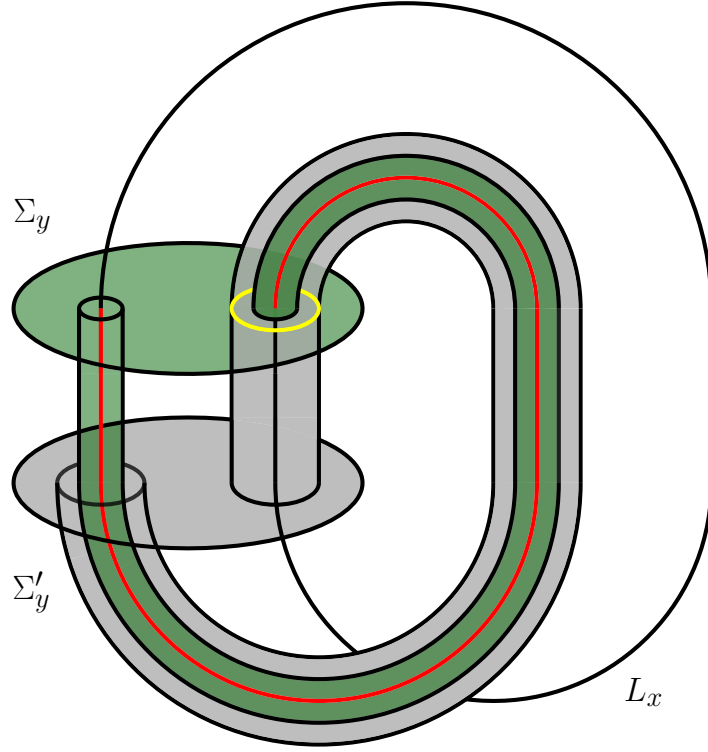
Figure 3.5. A picture showing how two copies of a Seifert surface for L_y-gray and green surfaces-intersect each other at a meridian for L_x-yellow curve.

(iii) $N = 3$: Let $\mathcal{A}^1 = \langle x, y, z \rangle$. If $xyz = 0$, then we proceed as in (i) and (ii), and then take connected sums. Hence, we assume $xyz \neq 0$. If $x^2 = y^2 = z^2 = 0$, then we use tangle move (b) (Figure 3.2) on L_x to get $L_x \sqcup L_y \sqcup L_z \cong Bo$. We give all components zero framing. Both Σ_y and Σ_z receive a handle going around a segment of L_x. In fact, these segments overlap and the handles intersect along a meridian of L_x (Figure 3.6). Thus, we have $xyz = x(F_y \cap F_z) = 1$, but since F_x, F_y, and F_z are orientable and homeomorphic to tori, there is no self-intersection, and $x^2 = y^2 = z^2 = 0$. In fact the resulting manifold is the 3-torus T^3 and the dual surfaces we found are homotopic to $S^1 \times S^1 \times *$, $S^1 \times * \times S^1$ and $* \times S^1 \times S^1$ respectively.
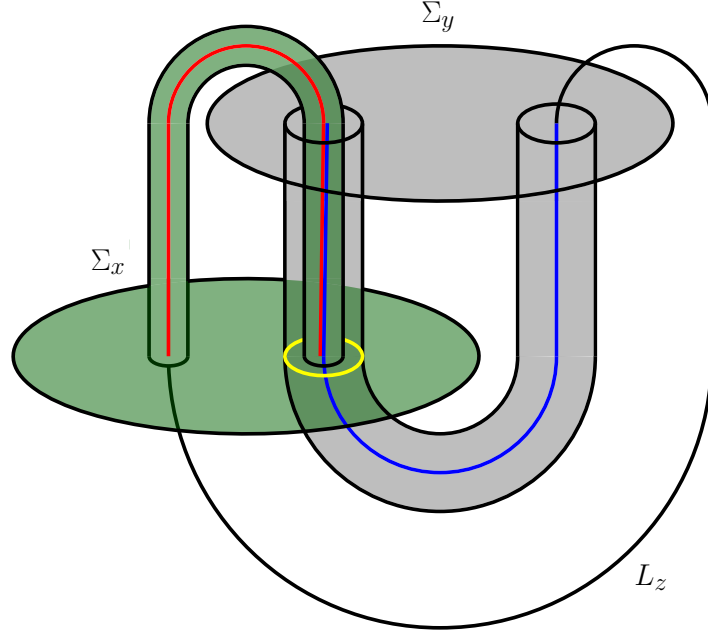
Figure 3.6. The core for the handle of Σ_x is shown in red while that of Σ_y is shown in blue. The overlap of these segments forces an intersection along the yellow curve.

If $x^2 \neq 0$ but $x^3 = 0$, then by Theorem 2.5, $y^3 = 0$ and $x^2y = xy^2 \neq 0$. Using tangle move (a) (Figure 3.2) on L_y and move (b) (Figure 3.2) on L_z, we arrange $L_x \sqcup L_y \sqcup L_z$ as a copy of the link U_{002} in Figure 3.3, with $L_x \sqcup L_y \cong L_{2,4}$. Give L_x and L_y zero framing. Now $F_x \cong F_y \cong T^2 \# Kb$, a combination of the previous cases. The torus parts intersect each other along a meridian of L_z, as before, while the Klein bottle parts provide the desired self-intersections, as in case (ii). Thus, $xyz = z(F_x \cap F_y) = 1$ and $xy^2 = x(F_y \cap F_y') = 1$. Finally, give L_z framing 2 if and only if $z^3 \neq 0$, to obtain $z^3 = z(F_z \cap F_z') = 1$. This does not affect the value of either xyz or xy^2, since the self-intersection of F_z happens inside $S^1 \times D^2$, while the interaction with other surfaces occurs outside.
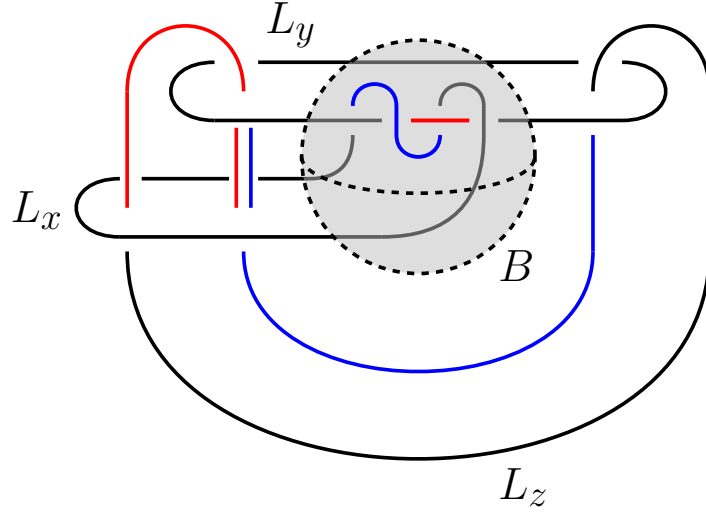
Figure 3.7. Handle placement for the link U_{002}. Only the cores of the handles are shown, red for Σ_x and blue for Σ_y. Note that, product x^2y can be realized inside a ball-denoted B here-while xyz is realized outside.

3.3. Realization

Let $\mathcal{A}^*$ be an orientable MS-algebra, and let $\{x_1, x_2, \ldots, x_K, \ldots, x_N\}$ be the basis for $\mathcal{A}^1$ as given by Theorem 2.5. Take L, the N-component trivial unlink inside S^3, so that the Kronecker duals of the meridians of L_i represent $\{x_1, x_2, \ldots, x_K, \ldots, x_N\}$. Use move (a) (Figure 3.2) $K/2$ times to ensure that $L_{2k-1} \sqcup L_{2k} \cong L_{2,4}$ for every $k \leq K/2$. For each triple product $x_i x_j x_k$ with $i < j < k$, use move (b) (Figure 3.2) to arrange $L_i \sqcup L_j \sqcup L_k$ as a copy of either Bo (when $K < i < j < k$) or the link U_{002} (Figure 3.3) (when $i < j \leq K < k$). Finally, choose appropriate framings to realize the values of the cubes. Hence, $\mathcal{A}^* \cong H^*(M(L); \mathbb{Z}_2)$.

4. NONORIENTABLE CASE

We shall construct non-orientable manifolds via surgery in $S^2 \tilde{\times} S^1$ [4], twisted S^2 bundle over S^1, which can be obtain by identiflying the ends of $S^2 \times [0, 1]$ by an an orientatiton reversing homeomorphism of S^2. As opposed to S^3, there is a preferred direction in $S^2 \tilde{\times} S^1$: the one reversing the orientation. Therefore assembling our link will require more care. We must ensure that some components run parallel to this direction. Another challenge is that the identity $wxy = x^2 y + xy^2$ has no further simplification , which not only presents a more complicated relation but also prevents us from normalizing the products $x^2 y$ as we did in the orientable case.

4.1. Surgery in $S^2 \tilde{\times} S^1$

To tackle the aforementioned complications, we adopt the following convention [4]. Take a tangle in $D^2 \times [0, 1]$ whose ends lie on a fixed diameter d of $D^2 \times \{0\}$ and $D^2 \times \{1\}$. Identifying the ends of $D^2 \times [0, 1]$ via reflection across d—which also pairs up the ends of our tangle—we obtain a link in $D^2 \tilde{\times} S^1$. If we further assume that each component of L is orientation-preserving in $D^2 \tilde{\times} S^1$, then tubular neighborhoods of all components would be homeomorphic to tori, and after choosing framings, we can perform surgery. Let $Y(L)$ denote the result of this surgery. Gluing another copy of $D^2 \tilde{\times} S^1$ along the boundary, we define $Z(L) = Y(L) \cup_\partial D^2 \tilde{\times} S^1$, the result of surgery on L in $S^2 \tilde{\times} S^1$. We set the core of this second copy to be the standard orientation-reversing loop γ_w.
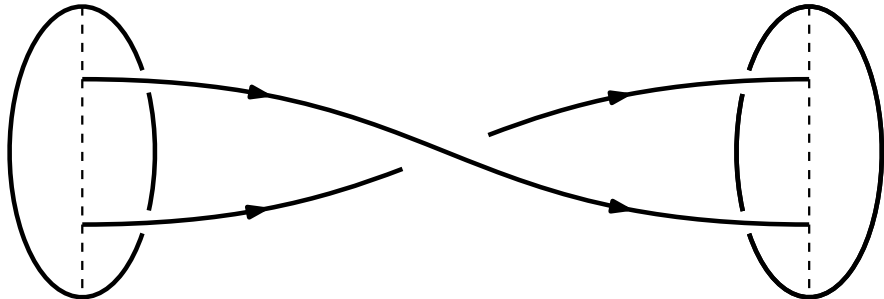

Figure 4.1. A tangle in $D^2 \times [0, 1]$ giving the one component link L_T in $S^2 \tilde{\times} S^1$.

Now let L be an $(N-1)$-component framed link in $D^2 \tilde{\times} S^1$, where each component is orientation-preserving and has even framing. Assume, in addition, that all pairwise linking numbers are also even. As in the orientable case, the meridians of L, together with γ_w, represent a basis for $H_1(Z(L); \mathbb{Z}_2)$. Let $\{w = x_1, x_2, \ldots, x_N\}$ be the Kronecker dual basis for $H^1(Z(L); \mathbb{Z}_2)$, where w is dual to γ_w. The Poincaré duals of $x_{i \geq 2}$ are again given by $F_i = \Sigma_i \cup \Lambda_i$, while the dual F_w of w is a fiber S^2. Now, if L_i lies entirely in $S^2 \times (0, 1)$, then Σ_i is a disc. If not, then it wraps around an even number of times, and Σ_i is a twisted ribbon. In particular, if L_i represents the class $2[S^1]$ in $H_1(D^2 \tilde{\times} S^1)$, then its Seifert surface is a Möbius band.

4.2. Building Blocks

We follow a similar strategy to Section 3.2 and collect examples with $N = dim \mathcal{A}^1 \leq 3$. This list will not be exhaustive, however it will provide the building blocks for the general case.

(i) $N = 1$: We let $\mathcal{A}^1 = \langle w \rangle$. Since $w^3 = w^3 + w^3 = 0$ by the Postnikov–Wu identity, we must have $w^2 = 0$ by non-singularity. The space $S^2 \tilde{\times} S^1$ itself realizes this case, where γ_w is homotopic to $* \times S^1$, and the Poincaré dual of w is a sphere that does not intersect itself.

(ii) $N = 2$ and $w^2 \neq 0$: Setting $\mathcal{A}^1 = \langle w, x \rangle$, by Theorem 2.6 we may assume $w^2 x \neq 0$, and after replacing x with $x + w$ if necessary, we may further assume $x^3 = 0$. We represent x with the meridian of L_T, the link in Figure 4.1. After performing zero-framed surgery, we observe that Σ_x is a Möbius band and Λ_x is a disc. But since this Möbius band is co-oriented with $Z(L_T)$, we have $F_x \cap F_x' = \emptyset$. To have F_w and L_T disjoint, we add a handle to F_w, a handle enclosing exactly half of L_T (Figure 4.2). As Σ_x intersects F_w at an arc connecting the feet of this handle, the handle itself intersects Σ_x along an arc homotopic to the segment of L_T it encloses. The union of these arcs is the curve $F_w \cap F_x$, and it is homotopic to γ_w. Hence, $wwx = w(F_w \cap F_x) = 1$.
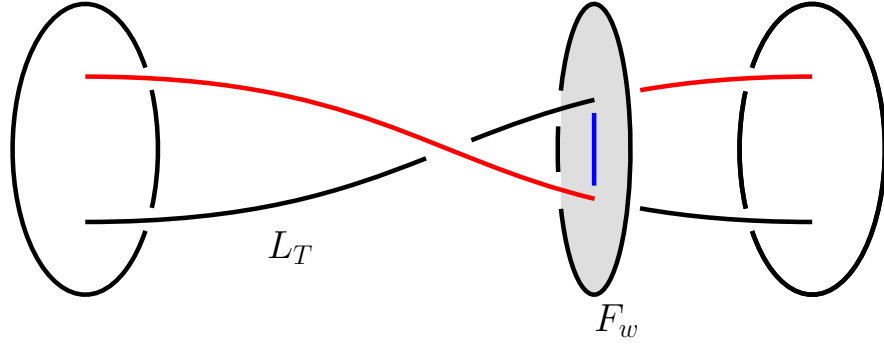
Figure 4.2. The core of the handle is shown in red while blue is the arc connecting the feet. Red and blue combined is the curve $F_w \cap F_x$

The resulting manifold is $S^1 \times \mathbb{R}P^2$. To see this, note that the exterior $X(L_T)$ of L_T in $S^2 \tilde{\times} S^1$ is homeomorphic to $S^1 \times Mb$, while the tubular neighborhood of L_T is just a torus. Hence, we may write $S^2 \tilde{\times} S^1$ as the union $S^1 \times Mb \cup D^2 \times S^1$. Performing zero-framed surgery kills the longitude of L_T, and we obtain $S^1 \times Mb \cup S^1 \times D^2 \cong S^1 \times \mathbb{R}P^2$.

(iii) $N = 3$ and $w^2 = 0$: For $\mathcal{A}^1 = \langle w, x, y \rangle$, we, again by Theorem 2.6, may let $wxy \neq 0$. If $x^3 \neq 0$ and $y^3 = 0$, we can replace y with $y + x$, ensuring that either both x^3 and y^3 are zero or neither is. We realize x and y with the meridians of the link $L_C = L_x \sqcup L_y$, pictured below:
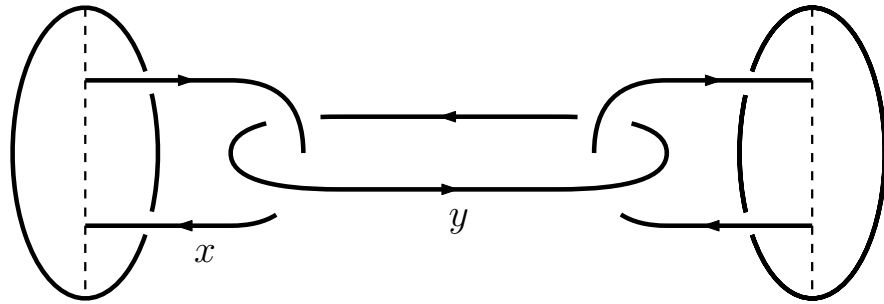

Figure 4.3. Another tangle in $D^2 \times [0, 1]$ giving the two component link L_C in $S^2 \tilde{\times} S^1$.

Similar to before, we attach a handle to F_w, enclosing a segment of L_y (Figure 4.4), which intersects Σ_y along an arc homotopic to that exact segment. Meanwhile, Σ_y intersects F_w along an arc connecting the feet of the handle. The union of these yields the curve $F_w \cap F_y$, which is homotopic to a meridian of L_x. Thus, $wxy = x(F_w \cap F_y) = 1$.

Note, however, that we may push away F_w from itself by translating the sphere part while enlarging the handle. Thus, $F_w \cap F'_w = \emptyset$, and we obtain $w^2 = 0$. Finally, we choose appropriate framings to realize the values of cubes as we did in the orientable case.
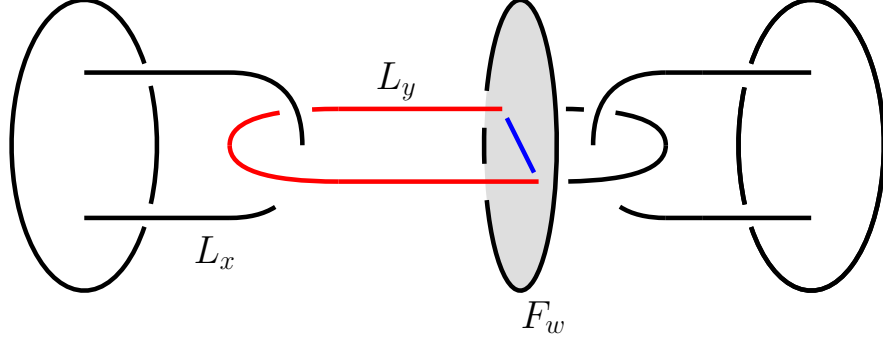

Figure 4.4. The core of the handle is shown in red while blue is the arc connecting the feet.

This time, if framings are zero, then $Z(L_C) \cong S^1 \times Kb$, while 2-framed surgery results in $MT\left(\left[\begin{smallmatrix} -1 & -2 \\ 2 & 5 \end{smallmatrix}\right]\right)$, the mapping torus of the linear map $\left[\begin{smallmatrix} -1 & -2 \\ 2 & 5 \end{smallmatrix}\right]$. These equalities, however, are not as easily seen—one has to compute link groups and use the fact that aspherical manifolds are determined by their fundamental groups [4].

4.3. Assembly

We will construct manifolds for $dim\mathcal{A}^1 > 3$ by assembling smaller examples constructed before [4]. Let $\{L_1, L_2, \ldots, L_n\}$ be a finite collection of links where $L_1 = L_C$ or L_T while $L_{i \geq 2} = L_C$. Inside D^2, we identify small disjoint discs D_i, all of which are centered along a fixed diameter d. We imagine within each $D_i \times [0, 1]$ sits the tangle for the link L_i as in Figure 4.1 and 4.3. Identifying the ends of $D^2 \times [0, 1]$ via reflection across d, we get a link $\mathcal{L} = \sqcup^n L_i$ inside $D^2 \tilde{\times} S^1$. Performing surgery and then gluing another copy of $D^2 \tilde{\times} S^1$, we obtain the manifold $W(\mathcal{L})$, the result of surgery on $\mathcal{L}$ in $S^2 \tilde{\times} S^1$. As before, we choose the core of the second $D^2 \tilde{\times} S^1$ as our standard orientation-reversing loop γ_w.
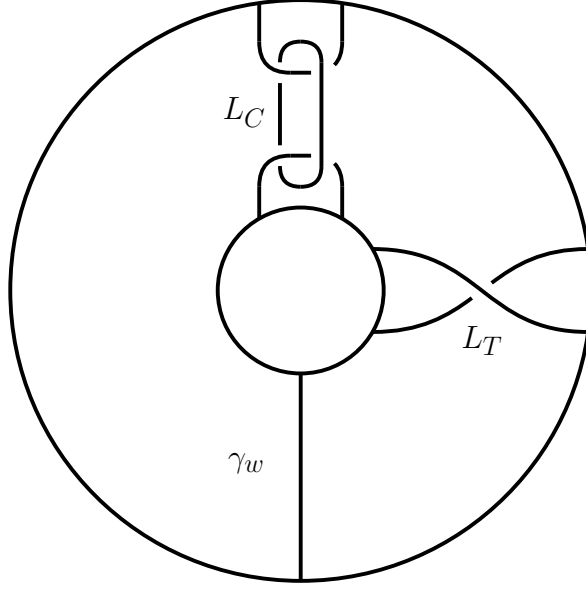
Figure 4.5. Diagram for $W(\mathcal{L})$ of Example 4.3.1, showing $\mathcal{L}$ inside $S^2 \times [0,1]$.

By construction, $W(\mathcal{L})$ contains each $Y(L_i)$, the result of surgery on L_i in $D_i \tilde{\times} S^1$ (Section 4.1). Since the components of L_i have image zero in $H_1(D_i \tilde{\times} S^1; \mathbb{Z}_2)$, the surfaces they bound are entirely contained in $\iota_i Y(L_i)$—the inclusion of $Y(L_i)$ in $W(\mathcal{L})$. Thus, no new intersections between Seifert surfaces coming from different links are introduced. On the other hand, a dual sphere $S^2 \times *$ of w in $W(\mathcal{L})$ intersects $\iota_i Y(L_i)$ at a disc $D_i \times *$ and any intersection that occurs between this disc and Seifert surfaces of L_i does so exactly as in Figure 4.2 and 4.4. As a result, $W(\mathcal{L})$ inherits precisely the triple products we intend.

Example 4.3.1. Let $\mathcal{A}^1 = \langle w, x, y, z \rangle$ with $w^2 x$ and wyz nonzero. Construct $W(\mathcal{L})$ where $\mathcal{L} = L_T \sqcup L_C$. We represent x with the meridian of L_T while the meridians of L_C correspond to y and z. Hence, $H^1(W(\mathcal{L}); \mathbb{Z}_2) = \langle w, x, y, z \rangle$ and since $\Sigma_x \cap \Sigma_y = \Sigma_x \cap \Sigma_z = \emptyset$, we have $xy = xz = 0$. Whereas from the previous section we know that $H^1(Z(L_T); \mathbb{Z}_2) = \langle w, x \mid wwx \neq 0 \rangle$ and $H^1(Z(L_C); \mathbb{Z}_2) = \langle w, y, z \mid wyz \neq 0 \rangle$. Hence, $w^2 x = wyz = 1$ in $H^*(W(\mathcal{L}); \mathbb{Z}_2)$ and we obtain $\mathcal{A}^* \cong H^*(W(\mathcal{L}); \mathbb{Z}_2)$ (Figure 4.5).

4.4. Other Products

We have laid out a procedure to obtain the products wxy. Now we turn our attention to the other triple products xyz and x^2y. The idea is to first add additional components to $\mathcal{L}$, one for each basis element x with $wx = 0$, and then modify $\mathcal{L}$ via moves (a) and (b) to introduce the desired new products. Since both moves can be performed sufficiently far from a chosen dual sphere of w, this will not upset the previously established multiplicative structure.
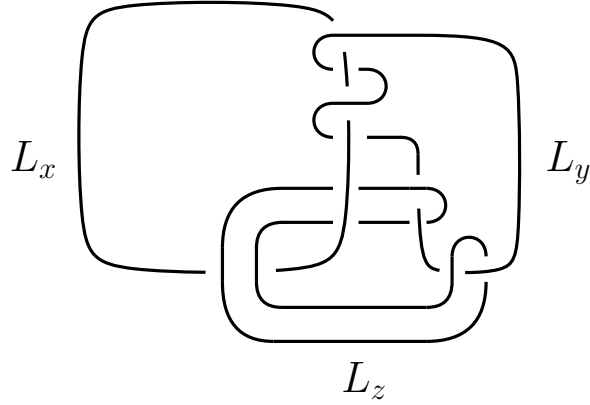

Figure 4.6. The link U_{022}, with linking numbers 0, 2, and 2. The difference from Figure 3.3 is at the lower right crossing.

Note, however, that having already normalized our basis for products wxy, we may no longer do so for the x^2y's. That means we will need more links, namely U_{022} from Figure 4.4 and U_{222} from Figure 4.7. Let $xyz = 1$. There are two main cases to consider, each with its own subcases:
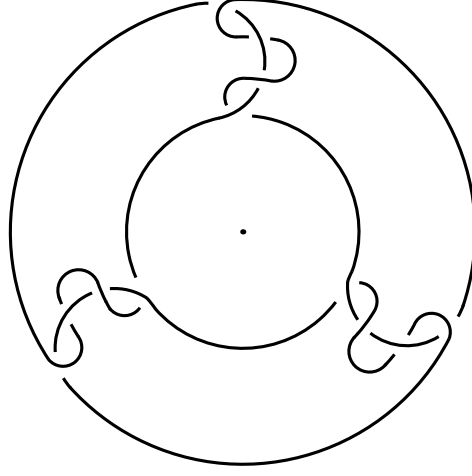
Figure 4.7. U_{222} with linking numbers 2. Notice the rotational symmetry about the center, interchanging components.

(i) First is when $wxy = wxz = wyz = 0$, which implies $x^2y = xy^2$, $x^2z = xz^2$, and $y^2z = yz^2$. If $x^2y = x^2z = y^2z = 0$, then we use move (b) and arrange the respective components as a copy of Bo. If $x^2y = 1$ but $x^2z = y^2z = 0$, we use both moves once and have $L_x \sqcup L_y \sqcup L_z \cong U_{002}$ (Figure 3.3) with L_x and L_y linked. So far everything is as was in Section 3.2. Now if both x^2y and y^2z are nonzero but $x^2z = 0$, then we use (a) twice and (b) once to obtain $L_x \sqcup L_y \sqcup L_z \cong U_{022}$ (Figure 4.4), with L_x and L_z unlinked. Using the handle placement shown in Figure 4.8, we have $\Sigma_y \cap \Sigma_y' = \mu_x + \mu_z$ and $\Sigma_x \cap \Sigma_y = \mu_z$. Hence, $xyz = xy^2 = y^2z = 1$, but since $\Sigma_z \cap \Sigma_z' = \mu_y$, we get $xz^2 = 0$.
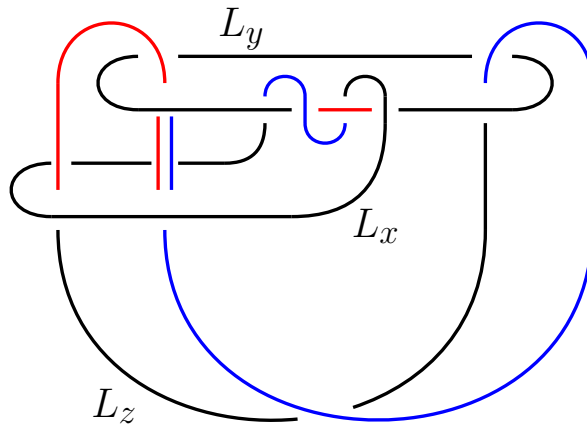

Figure 4.8. The cores of the handles of Σ_x are shown in red, and those of Σ_y are shown in blue.

Finally, if none of these products is zero, we apply move (a) three times. The resulting link is U_{222} (Figure 4.7). Attaching handles as prescribed by Figure 4.9, we observe that $\Sigma_x \cap \Sigma'_x = \mu_y + \mu_z$ and $\Sigma_x \cap \Sigma_y = \mu_z$. Thus, $xyz = x^2 y = 1$, and by the symmetry exhibited by U_{222}, we also obtain $x^2 z = y^2 z = 1$.
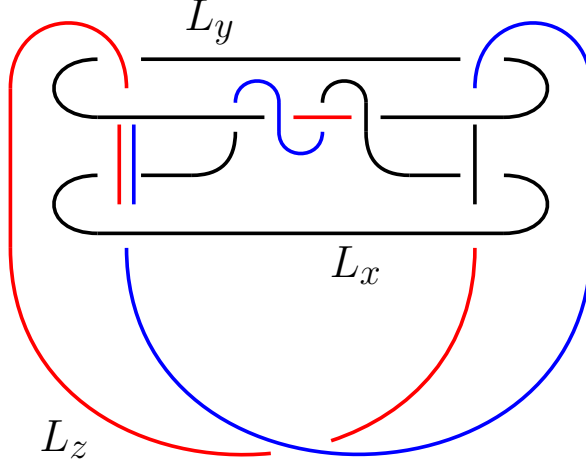

Figure 4.9. The cores of the handles of Σ_x are shown in red, and those of Σ_y are shown in blue.

(ii) The other possibility is $wxy = x^2 y + xy^2 \neq 0$. Then, by our choice of basis, $wxz = wyz = 0$, and we have $x^2 z = xz^2$ and $y^2 z = yz^2$. We proceed in a similar fashion. If $x^2 z = xz^2 = 0$, we use move (b). If $x^2 z = 1$ but $y^2 z = 0$, then we use moves (a) and (b) once, and if neither product is zero, then we use move (a) twice. The resulting links are again based on Bo, U_{002}, and U_{022} respectively, with the added technicality that $L_x \sqcup L_y \cong L_C$, realizing wxy.

4.5. Realization II

Let $\mathcal{A}^*$ be a non-orientable MS-algebra, and let $\{w = x_1, x_2, \ldots, x_K, \ldots, x_N\}$ be the basis for $\mathcal{A}^1$ given by Theorem 2.6. Take the $N-1$-component link $\mathcal{L} = L_2 \sqcup L_3 \ldots \sqcup L_N$ inside $S^2 \tilde{\times} S^1$. The Kronecker duals of the meridians of L_i represent x_i, while $w = x_1$ is represented by the dual of γ_w, the standard orientation-reversing loop. If $w^2 = 0$, then $L_{2k} \sqcup L_{2k+1} \cong L_C$ for $k \leq K/2$, situated in $S^2 \tilde{\times} S^1$ as described in Section 4.3, and $L_{K+2} \sqcup L_{K+3} \ldots \sqcup L_N$ is a trivial unlink lying entirely in $S^2 \times (0, 1)$.

Table 4.1. Given $xyz = 1$, use the moves as prescribed. (*) wxy determines x^2y. (**) The links are based on Bo, U_{002}, and U_{022}, respectively, with the added technicality that $L_x \sqcup L_y \cong L_C$.

wxy	x^2y	x^2z	y^2z	Moves	Link
0	0	0	0	(b)	Bo
0	0	0	1	(a) (b)	U_{002}
0	0	1	1	(a) (a) (b)	U_{022}
0	1	1	1	(a) (a) (a)	U_{222}
1	(*)	0	0	(b)	Bo (**)
1	(*)	0	1	(a) (b)	U_{002} (**)
1	(*)	1	1	(a) (a)	U_{022} (**)

Whereas if $w^2 \neq 0$, then $L_2 \cong L_T$, $L_{2k-1} \sqcup L_{2k} \cong L_C$ for $1 < k \leq K/2$, and $L_{K+1} \sqcup L_{K+2} \ldots \sqcup L_N$ is a trivial unlink (Figure 4.10). For each non-zero product $x_i x_j x_k$ with $1 < i < j < k$, use tangle moves as prescribed in Table 4.1. Finally, choose appropriate framings for components lying entirely in $S^2 \times (0,1)$, realizing the values of cubes. Hence, $\mathcal{A}^* \cong H^*(W(\mathcal{L}); \mathbb{Z}_2)$.
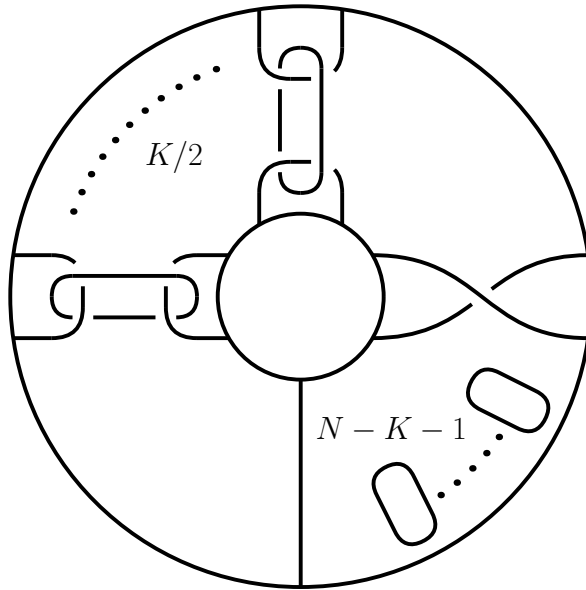

Figure 4.10. Generalized diagram for $W(\mathcal{L})$, showing $\mathcal{L}$ just before the application of tangle moves.

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