5) a) 7.6.6

Compute heat flux across the unit sphere S if $T(x,y,z) = x$

Can you interpret this answer physically?

$$\iint_D F(\Phi(u,v)) \cdot (T_u \times T_v)\, du\, dv$$

$$\begin{cases} \Phi(\theta,\varphi) = (\cos\theta\sin\varphi,\ \sin\theta\sin\varphi,\ \cos\varphi) \\ 0 \le \theta \le 2\pi \\ 0 \le \varphi < \pi \end{cases}$$

$T_\theta \times T_\varphi = -\sin\varphi\, \hat{e}_\rho$ (From Lecture Ex 1) $= -\sin\varphi(\sin\varphi\cos\theta\, i + \sin\varphi\sin\theta\, j + \cos\varphi\, k)$

$= -\sin^2\varphi\cos\theta\, i + \sin^2\varphi\sin\theta\, j + \sin\varphi\cos\varphi\, k$

$F = -k\nabla T = (-k, 0, 0)$

$F(\Phi(\theta,\varphi)) = (-k, 0, 0)$

$$\int_0^{\pi}\int_0^{2\pi} (-k,0,0)\cdot(\sin^2\varphi\cos\theta,\ \sin^2\varphi\sin\theta,\ \sin\varphi\cos\varphi)\, d\theta\, d\varphi$$

$$= \int_0^{\pi}\int_0^{2\pi} -k\sin^2\varphi\cos\theta\, d\theta\, d\varphi$$

$$= \int_0^{\pi} -k\sin^2\varphi\sin\theta \Big|_0^{2\pi} d\varphi = \int_0^{\pi} 0\, d\varphi$$

So the heat flux is $\boxed{0}$

The total rate of heat flux is 0, so the amount leaving is the same as the amount entering.

S) b) 7.6.8

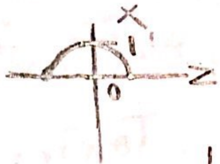
Let the velocity field of a fluid be described by $F = \sqrt{y}\, i$ (m/s) compute how many cubic meters of fluid per second are crossing the surface $x^2 + z^2 = 1$, $0 \le y \le 1$, $0 \le x \le 1$

So flux!

$$\iint_D F\, ds = \iint_D F(\Phi(u,v)) \cdot (T_u \times T_v)\, du\, dv$$

$\Phi(u,v) = (\sin u, v, \cos u) = (x, y, z)$ (unit circle/cylinder along y axis)

$\begin{cases} 0 \le u \le \pi \\ 0 \le v \le 1 \end{cases}$



$(\overset{x}{\sin\theta}, \overset{z}{\cos\theta})$

$$T_u = \begin{bmatrix} \cos u \\ 0 \\ -\sin u \end{bmatrix} \quad T_v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad T_u \times T_v = \left| \begin{matrix} i & j & k \\ \cos u & 0 & -\sin u \\ 0 & 1 & 0 \end{matrix} \right| \begin{matrix} i & j \\ \cos u & 0 \\ 0 & 1 \end{matrix}$$

$i(0 + \sin u) - j(0 - 0) + k(\cos u - 0)$

$= \sin u\, i + \cos u\, k$

$$\int_0^1 \int_0^\pi (\sqrt{v}, 0, 0) \cdot (\sin u, 0, \cos u)\, du\, dv$$

$$= \int_0^1 \int_0^\pi \sqrt{v} \sin u\, du\, dv = \int_0^1 \sqrt{v}(-\cos u)\Big|_0^\pi dv = \int_0^1 2\sqrt{v}\, dv$$

$$= \frac{4}{3} v^{3/2} \Big|_0^1 = \frac{4}{3} - 0 = \boxed{\frac{4}{3}\ m^3/s}$$

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