3) 7.5.26

Let S be a sphere of radius r and p be a point inside or outside the sphere (but not on it) Show that

$$\iint_S \frac{1}{\|x-p\|}\,dS = \begin{cases} 4\pi r & \text{if } p \text{ is inside } S \\ 4\pi r^2/d & \text{if } p \text{ is outside } S, \end{cases}$$

where d is the distance from p to the center of the sphere and the integration is over the sphere (HINT: Assume p is on the z axis]

so p is $(0,0,d)$ d is distance b/c on z axis and center @ origin $(0,0,0)$

need to parameterize S. (use spherical coordinates)

$$\begin{cases} \Phi(\theta,\varphi) = (r\cos\theta\sin\varphi,\ r\sin\theta\sin\varphi,\ r\cos\varphi) \\ 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \end{cases}$$

$$\|x-p\| = \sqrt{x^2+y^2+(z-d)^2}$$
$$= \sqrt{(r\cos\theta\sin\varphi)^2+(r\sin\theta\sin\varphi)^2+(r\cos\varphi-d)^2}$$
$$= \sqrt{r^2\cos^2\theta\sin^2\varphi + r^2\sin^2\theta\sin^2\varphi + r^2\cos^2\varphi - 2rd\cos\varphi + d^2}$$
$$= \sqrt{r^2\sin^2\varphi + r^2\cos^2\varphi - 2rd\cos\varphi + d^2}$$
$$= \sqrt{r^2 - 2rd\cos\varphi + d^2}$$

$$\iint_S \frac{1}{\|x-p\|}\,dS = \int_0^{2\pi}\int_0^{\pi} \frac{\|T_\theta \times T_\varphi\|}{\|x-p\|}\,dS$$

$T_\theta \times T_\varphi = -r^2\sin\varphi\,\hat{e}_\rho$ (from Lecture) ex 1

$$= \int_0^{2\pi}\int_0^{\pi} \frac{r^2\sin\varphi}{\sqrt{r^2-2rd\cos\varphi+d^2}}\,d\varphi\,d\theta$$

$u = r^2 - 2rd\cos\varphi + d^2$
$du = +2rd\sin\varphi\,d\varphi$

$$= \frac{r^2}{2rd}\int_0^{2\pi}\int_{r^2-2rd+d^2}^{r^2+2rd+d^2} \frac{1}{\sqrt{u}}\,du\,d\theta$$

$$= \frac{\cancel{2}\pi r}{\cancel{2}d}\int_{(r-d)^2}^{(r+d)^2} \frac{1}{\sqrt{u}}\,du = \frac{2\pi r}{d}\sqrt{u}\Big|_{(r-d)^2}^{(r+d)^2}$$

$$= \frac{2\pi r}{d}\left(|r+d| - |r-d|\right) \text{ so } \begin{cases} 4\pi r & r>d \text{ (inside)} \\ \dfrac{4\pi r^2}{d} & r<d \text{ (outside)} \end{cases}$$

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