Problem 1

a. #4 $S \to y^2+z^2=4 \quad x \in [0,5] \quad \iint_S (x+z)\,dS$

$\Phi(t,\theta) = (t, 2\cos\theta, 2\sin\theta) \quad 0 \le t \le 5 \quad 0 \le \theta \le 2\pi$

$T_t = (1,0,0) \quad T_\theta = (0, -2\sin\theta, 2\cos\theta)$

$$T_t \times T_\theta = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & -2\sin\theta & 2\cos\theta \end{vmatrix} = (0)i - (2\cos\theta)j + (-2\sin\theta)k$$

$$= (0, -2\cos\theta, -2\sin\theta)$$

$\|T_t \times T_\theta\| = \sqrt{4\cos^2\theta + 4\sin^2\theta} = 2$

$\int_0^{2\pi}\int_0^5 2(t+2\sin\theta)\,dt\,d\theta = \int_0^{2\pi} 2\left(\frac{t^2}{2} + 2t\sin\theta\right)\Big|_0^5 d\theta$

$= \int_0^{2\pi} 25 + 20\sin\theta\,d\theta = 25\theta - 20\cos\theta\Big|_0^{2\pi} = 50\pi - 20$

$-(0-20) = 50\pi$

Problem 1

b. #6 $S \to z = 4 + x + y$ $x^2 + y^2 = 4$ $\iint_S (x^2 z + y^2 z)\, dS$

$\Phi(x,y) = (x, y, 4+x+y)$ $T_x = (1,0,1)$ $T_y = (0,1,1)$

$$T_x \times T_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = (0-1)i - (1-0)j + (1-0)k = (-1,-1,1)$$

$\|T_x \times T_y\| = \sqrt{1+1+1} = \sqrt{3}$ $x = r\cos\theta$ $y = r\sin\theta$

$0 \le r \le 2$ $0 \le \theta \le 2\pi$ $\iint_S (x^2+y^2) z\, dS$

$= \int_0^{2\pi}\int_0^2 r^2(4 + r\cos\theta + r\sin\theta)\, dr\, d\theta$

$$\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$\longrightarrow \int_0^{2\pi}\int_0^2 \sqrt{3}\, r^3 (4 + r\cos\theta + r\sin\theta)\, dr\, d\theta$

$= \int_0^{2\pi} \sqrt{3}\left(\frac{4r^4}{4} + \frac{r^5}{5}\cos\theta + \frac{r^5}{5}\sin\theta\right)\Big|_0^2 d\theta$

$= \int_0^{2\pi} \sqrt{3}\left(16 + \frac{32}{5}\cos\theta + \frac{32}{5}\sin\theta\right) d\theta$

$= \sqrt{3}\left(16\theta + \frac{32}{5}\sin\theta - \frac{32}{5}\cos\theta\right)\Big|_0^{2\pi}$

$= \sqrt{3}\left(32\pi + \frac{32}{5}(0 - 1 - 0 + 1)\right) = 32\sqrt{3}\,\pi$

Problem 1

c. #8 $S \to$ triangle w/ vertices $\overset{A}{(1,0,0)}, \overset{B}{(0,2,0)}, \overset{C}{(0,1,1)}$

$\iint_S xyz \, dS$ AB: $(y-0) = \frac{2-0}{0-1}(x-1)$ $y = -2x+2$

AC: $(y-0) = \frac{1-0}{0-1}(x-1)$ $y = -x+1 \longrightarrow 0 \leq x \leq 1$

$-x+1 \leq y \leq -2x+2$ $\vec{AB} = (-1,2,0)$ $\vec{AC} = (-1,1,1)$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{vmatrix} = (2-0)i - (-1-0)j + (-1+2)k = (2,1,1)$$

$(x-1, y-0, z-0) \cdot (2,1,1) = 2x-2+y+z \to z = 2-2x-y$

$\Phi(x,y) = (x, y, 2-2x-y)$ $0 \leq x \leq 1$ $-x+1 \leq y \leq -2x+2$

$T_x = (1,0,-2)$ $T_y = (0,1,-1)$

$$T_x \times T_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{vmatrix} = (0+2)i - (-1-0)j + (1-0)k = (2,1,1)$$

$\|T_x \times T_y\| = \sqrt{4+1+1} = \sqrt{6}$ $\int_0^1 \int_{-x+1}^{-2x+2} \sqrt{6}\left(2xy - 2x^2y - xy^2\right) dy\,dx$

$= \int_0^1 \sqrt{6}\left(2x\frac{y^2}{2} - 2x^2\frac{y^2}{2} - x\frac{y^3}{3}\right)\Big|_{-x+1}^{-2x+2} dx$

$= \int_0^1 \frac{2\sqrt{6}}{3}\left(-x^4 + 3x^3 - 3x^2 + x\right) dx = \frac{2\sqrt{6}}{3}\left(-\frac{x^5}{5} + \frac{3x^4}{4} - \frac{3x^3}{3} + \frac{x^2}{2}\right)\Big|_0^1$

$= \frac{2\sqrt{6}}{3}\left(-\frac{1}{5} + \frac{3}{4} - 1 + \frac{1}{2}\right) = \frac{2\sqrt{6}}{3}\left(\frac{1}{20}\right) = \frac{\sqrt{6}}{30}$

Problem 1

d. #10 $S \to x^2+y^2+z^2=1$ $\iint_S (x+y+z)\,dS = 3\iint_S z\,dS$

$\Phi(\theta,\phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$

$0 \le \theta \le 2\pi$ $0 \le \phi \le \pi$ $T_\theta = (-\sin\theta\sin\phi, \cos\theta\sin\phi, 0)$

$T_\phi = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)$

$$T_\theta \times T_\phi = \begin{vmatrix} i & j & k \\ -\sin\theta\sin\phi & \cos\theta\sin\phi & 0 \\ \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \end{vmatrix} = (-\cos\theta\sin^2\phi)i - (\sin\theta\sin^2\phi)j$$

$+(-\sin^2\theta\sin\phi\cos\phi - \cos^2\theta\sin\phi\cos\phi)k$

$= (-\cos\theta\sin^2\phi, -\sin\theta\sin^2\phi, -\sin\phi\cos\phi)$

$\|T_\theta \times T_\phi\| = \sqrt{\cos^2\theta\sin^4\phi + \sin^2\theta\sin^4\phi + \sin^2\phi\cos^2\phi}$

$= \sin\phi$ $3\int_0^\pi\int_0^{2\pi} \sin\phi\cos\phi\,d\theta\,d\phi = 3\int_0^\pi 2\pi\cos\phi\sin\phi\,d\phi$

$= 3\pi\int_0^\pi \sin(2\phi)\,d\phi = -\frac{3\pi}{2}\cos(2\phi)\Big|_0^\pi = -\frac{3\pi}{2}(1-1) = 0$

$\iint_S (x+y+z)\,dS = 3\iint_S z\,dS = 0$

Problem 2

#16 $S \rightarrow z = \sqrt{R^2 - x^2 - y^2}$ $\quad 0 \le x^2 + y^2 \le R^2$

$m(x,y,z) = x^2 + y^2$ $\quad \Phi(\theta, \phi) = (R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi)$

$0 \le \theta \le 2\pi \quad 0 \le \phi \le \frac{\pi}{2}$ $\quad T_\theta = (-R\sin\theta\sin\phi, R\cos\theta\sin\phi, 0)$

$T_\phi = (R\cos\theta\cos\phi, R\sin\theta\cos\phi, -\sin\phi)$

$T_\theta \times T_\phi = (-R^2\cos\theta\sin^2\phi, -R^2\sin\theta\sin^2\phi, -R^2\sin\phi\cos\phi)$

$\|T_\theta \times T_\phi\| = R^2\sin\phi$ $\quad m(\Phi(\theta,\phi)) = R^2\sin^2\phi$

$\int_0^{\pi/2}\int_0^{2\pi} R^2\sin^2\phi \cdot R^2\sin\phi \, d\theta \, d\phi = \int_0^{\pi/2} 2\pi R^4 \sin^3\phi \, d\phi$

$= \int_0^{\pi/2} 2\pi R^4 (1-\cos^2\phi)\sin\phi \, d\phi$ $\quad u = -\cos\phi \quad \sin\phi \, d\phi = du$

$= \int_{-1}^{0} 2\pi R^4 (1-u^2) \, du = 2\pi R^4 \left(u - \frac{u^3}{3}\right)\Big|_{-1}^{0}$

$= 2\pi R^4 \left(1 - \frac{1}{3}\right) = \frac{4}{3}\pi R^4$