6) 7.6.20

a) a uniform fluid that flows vertically downward (heavy rain) is described by the vector field $F(x,y,z) = (0,0,-1)$ Find the total flux through cone $z=(x^2+y^2)^{1/2}$, $x^2+y^2 \leq 1$

cone is $\begin{cases} \Phi(\theta,r) = (r\cos\theta, r\sin\theta, r) \\ 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 1 \end{cases}$

$$\text{Flux} = \iint_S F \cdot ds = \iint_D F \cdot (T_\theta \times T_r)\, ds$$

$T_\theta = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix}$ $T_r = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 1 \end{bmatrix}$ $T_\theta \times T_r = \begin{vmatrix} i & j & k \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix} \begin{matrix} i & j \\ -r\sin\theta & r\cos\theta \\ \cos\theta & \sin\theta \end{matrix}$

$$i(r\cos\theta - 0) - j(-r\sin\theta - 0) + k(-r\sin^2\theta - r\cos^2\theta)$$

$$(r\cos\theta, r\sin\theta, -r)$$

$$\int_0^{2\pi}\int_0^1 (0,0,-1)\cdot(r\cos\theta, r\sin\theta, -r)\, dr\, d\theta = \int_0^{2\pi}\int_0^1 r\, dr\, d\theta = \boxed{\pi}$$

b) Rain is driven sideways so it falls @ 45° angle $F(x,y,z) = \left(-\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right)$ Now what is the flux through the cone?

$$F(x,y,z) = \left(-\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right)$$

$$\int_0^{2\pi}\int_0^1 \left(\frac{-\sqrt{2}}{2}, 0, \frac{-\sqrt{2}}{2}\right)\cdot(r\cos\theta, r\sin\theta, -r)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{-\sqrt{2}}{2} r\cos\theta + \frac{\sqrt{2}}{2} r\, dr\, d\theta : \int_0^{2\pi} \frac{-\sqrt{2}}{4} r^2\cos\theta + \frac{\sqrt{2}}{4} r^2 \Big|_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{-\sqrt{2}}{4}\cos\theta + \frac{\sqrt{2}}{4}\, d\theta = -\frac{\sqrt{2}}{4}\sin\theta + \frac{\sqrt{2}}{4}\theta \Big|_0^{2\pi}$$

$$= 0 + \frac{\sqrt{2}\pi}{2} + 0 + 0 \quad = \boxed{\frac{\pi\sqrt{2}}{2}}$$

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