Problem 1

a. #4 $S \to y^2+z^2=4 \quad x\in[0,5] \quad \iint_S (x+z)\,dS$

$\Phi(t,\theta)=(t, 2\cos\theta, 2\sin\theta) \quad 0\le t\le 5 \quad 0\le\theta\le 2\pi$

$T_t=(1,0,0) \quad T_\theta=(0,-2\sin\theta, 2\cos\theta)$

$$T_t\times T_\theta=\begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & -2\sin\theta & 2\cos\theta \end{vmatrix}=(0)i-(2\cos\theta)j+(-2\sin\theta)k$$

$$=(0,-2\cos\theta,-2\sin\theta)$$

$\|T_t\times T_\theta\|=\sqrt{4\cos^2\theta+4\sin^2\theta}=2$

$\int_0^{2\pi}\int_0^5 2(t+2\sin\theta)\,dt\,d\theta=\int_0^{2\pi}2\left(\frac{t^2}{2}+2t\sin\theta\right)\Big|_0^5 d\theta$

$=\int_0^{2\pi} 25+20\sin\theta\,d\theta=25\theta-20\cos\theta\Big|_0^{2\pi}=50\pi-20$

$-(0-20)=50\pi$

Problem 1

b. #6 $S \to z = 4+x+y \quad x^2+y^2=4 \quad \iint_S (x^2z+y^2z)\,dS$

$\Phi(x,y) = (x,y,4+x+y) \quad T_x = (1,0,1) \quad T_y = (0,1,1)$

$$T_x \times T_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = (0-1)i - (1-0)j + (1-0)k = (-1,-1,1)$$

$\|T_x \times T_y\| = \sqrt{1+1+1} = \sqrt{3} \quad x = r\cos\theta \quad y = r\sin\theta$

$0 \le r \le 2 \quad 0 \le \theta \le 2\pi \quad \iint_S (x^2+y^2)z\,dS$

$= \int_0^{2\pi}\int_0^2 r^2(4+r\cos\theta+r\sin\theta)\,dr\,d\theta$

$$\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r \longrightarrow \int_0^{2\pi}\int_0^2 \sqrt{3}\,r^3(4+r\cos\theta+r\sin\theta)\,dr\,d\theta$$

$= \int_0^{2\pi} \sqrt{3}\left(\frac{4r^4}{4} + \frac{r^5}{5}\cos\theta + \frac{r^5}{5}\sin\theta\right)\Big|_0^2 d\theta$

$= \int_0^{2\pi} \sqrt{3}\left(16 + \frac{32}{5}\cos\theta + \frac{32}{5}\sin\theta\right) d\theta$

$= \sqrt{3}\left(16\theta + \frac{32}{5}\sin\theta - \frac{32}{5}\cos\theta\right)\Big|_0^{2\pi}$

$= \sqrt{3}\left(32\pi + \frac{32}{5}(0-1-0+1)\right) = 32\sqrt{3}\,\pi$

Problem 1

c. #8 $S \to$ triangle w/ vertices $\overset{A}{(1,0,0)}, \overset{B}{(0,2,0)}, \overset{C}{(0,1,1)}$

$\iint_S xyz\, dS$ AB: $(y-0) = \frac{2-0}{0-1}(x-1)$ $y = -2x+2$

AC: $(y-0) = \frac{1-0}{0-1}(x-1)$ $y = -x+1 \longrightarrow 0 \le x \le 1$

$-x+1 \le y \le -2x+2$ $\vec{AB} = (-1,2,0)$ $\vec{AC} = (-1,1,1)$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{vmatrix} = (2-0)i - (-1-0)j + (-1+2)k = (2,1,1)$$

$(x-1, y-0, z-0) \cdot (2,1,1) = 2x-2+y+z \to z = 2-2x-y$

$\Phi(x,y) = (x, y, 2-2x-y)$ $0 \le x \le 1$ $-x+1 \le y \le -2x+2$

$T_x = (1,0,-2)$ $T_y = (0,1,-1)$

$$T_x \times T_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{vmatrix} = (0+2)i - (-1-0)j + (1-0)k = (2,1,1)$$

$\|T_x \times T_y\| = \sqrt{4+1+1} = \sqrt{6}$ $\int_0^1 \int_{-x+1}^{-2x+2} \sqrt{6}(2xy - 2x^2y - xy^2)\,dy\,dx$

$= \int_0^1 \sqrt{6}\left(2x\frac{y^2}{2} - 2x^2\frac{y^2}{2} - x\frac{y^3}{3}\right)\Big|_{-x+1}^{-2x+2} dx$

$= \int_0^1 \frac{2\sqrt{6}}{3}(-x^4 + 3x^3 - 3x^2 + x)\,dx = \frac{2\sqrt{6}}{3}\left(-\frac{x^5}{5} + \frac{3x^4}{4} - \frac{3x^3}{3} + \frac{x^2}{2}\right)\Big|_0^1$

$= \frac{2\sqrt{6}}{3}\left(-\frac{1}{5} + \frac{3}{4} - 1 + \frac{1}{2}\right) = \frac{2\sqrt{6}}{3}\left(\frac{1}{20}\right) = \frac{\sqrt{6}}{30}$

Problem 1

d. #10 $S \rightarrow x^2+y^2+z^2=1$ $\iint_S (x+y+z)\,dS = 3\iint_S z\,dS$

$\Phi(\theta,\phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$

$0 \leq \theta \leq 2\pi$ $0 \leq \phi \leq \pi$ $T_\theta = (-\sin\theta\sin\phi, \cos\theta\sin\phi, 0)$

$T_\phi = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)$

$$T_\theta \times T_\phi = \begin{vmatrix} i & j & k \\ -\sin\theta\sin\phi & \cos\theta\sin\phi & 0 \\ \cos\theta\cos\phi & \sin\theta\cos\phi & -\sin\phi \end{vmatrix} = (-\cos\theta\sin^2\phi)i - (\sin\theta\sin^2\phi)j$$

$+(-\sin^2\theta\sin\phi\cos\phi - \cos^2\theta\sin\phi\cos\phi)k$

$= (-\cos\theta\sin^2\phi, -\sin\theta\sin^2\phi, -\sin\phi\cos\phi)$

$\|T_\theta \times T_\phi\| = \sqrt{\cos^2\theta\sin^4\phi + \sin^2\theta\sin^4\phi + \sin^2\phi\cos^2\phi}$

$= \sin\phi$ $3\int_0^\pi \int_0^{2\pi} \sin\phi\cos\phi\,d\theta\,d\phi = 3\int_0^\pi 2\pi\cos\phi\sin\phi\,d\phi$

$= 3\pi\int_0^\pi \sin(2\phi)\,d\phi = -\frac{3\pi}{2}\cos(2\phi)\Big|_0^\pi = -\frac{3\pi}{2}(1-1) = 0$

$\iint_S (x+y+z)\,dS = 3\iint_S z\,dS = 0$

Problem 2

#16 $S \to z = \sqrt{R^2 - x^2 - y^2}$ $\quad 0 \leq x^2 + y^2 \leq R^2$

$m(x,y,z) = x^2 + y^2$ $\quad \Phi(\theta, \phi) = (R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi)$

$0 \leq \theta \leq 2\pi$ $\quad 0 \leq \phi \leq \frac{\pi}{2}$ $\quad T_\theta = (-R\sin\theta\sin\phi, R\cos\theta\sin\phi, 0)$

$T_\phi = (R\cos\theta\cos\phi, R\sin\theta\cos\phi, -\sin\phi)$

$T_\theta \times T_\phi = (-R^2\cos\theta\sin^2\phi, -R^2\sin\theta\sin^2\phi, -R^2\sin\phi\cos\phi)$

$\|T_\theta \times T_\phi\| = R^2\sin\phi$ $\quad m(\Phi(\theta,\phi)) = R^2\sin^2\phi$

$\int_0^{\pi/2}\int_0^{2\pi} R^2\sin^2\phi \cdot R^2\sin\phi \, d\theta \, d\phi = \int_0^{\pi/2} 2\pi R^4 \sin^3\phi \, d\phi$

$= \int_0^{\pi/2} 2\pi R^4 (1-\cos^2\phi)\sin\phi \, d\phi$ $\quad u = -\cos\phi$ $\quad \sin\phi \, d\phi = du$

$= \int_{-1}^{0} 2\pi R^4 (1-u^2) \, du = 2\pi R^4 \left(u - \frac{u^3}{3}\right)\Big|_{-1}^{0}$

$= 2\pi R^4 \left(1 - \frac{1}{3}\right) = \frac{4}{3}\pi R^4$

3) 7.5.26

Let S be a sphere of radius r and p be a point inside or outside the sphere (but not on it) Show that $\iint_S \frac{1}{\|x-p\|} dS = \begin{cases} 4\pi r & \text{if } p \text{ is inside } S \\ 4\pi r^2/d & \text{if } p \text{ is outside } S, \end{cases}$

where d is the distance from p to the center of the sphere and the integration is over the sphere (HINT: Assume p is on the z axis]

so p is $(0,0,d)$ d is distance b/c on z axis and center @ origin $(0,0,0)$

need to parameterize S. (use spherical coordinates)

$$\begin{cases} \Phi(\theta,\varphi) = (r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi) \\ 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \end{cases}$$

$$\|x-p\| = \sqrt{x^2+y^2+(z-d)^2}$$
$$= \sqrt{(r\cos\theta\sin\varphi)^2+(r\sin\theta\sin\varphi)^2+(r\cos\varphi-d)^2}$$
$$= \sqrt{r^2\cos^2\theta\sin^2\varphi + r^2\sin^2\theta\sin^2\varphi + r^2\cos^2\varphi - 2rd\cos\varphi + d^2}$$
$$= \sqrt{r^2\sin^2\varphi + r^2\cos^2\varphi - 2rd\cos\varphi + d^2}$$
$$= \sqrt{r^2 - 2rd\cos\varphi + d^2}$$

$T_\theta \times T_\varphi = -r^2\sin\varphi\,\hat{e}_p$ (from Lecture) ex 1

$$\iint_S \frac{1}{\|x-p\|} dS = \int_0^{2\pi}\int_0^{\pi} \frac{\|T_\theta \times T_\varphi\|}{\|x-p\|} dS$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{r^2\sin\varphi}{\sqrt{r^2-2rd\cos\varphi+d^2}} d\varphi\, d\theta$$

$u = r^2 - 2rd\cos\varphi + d^2$
$du = +2rd\sin\varphi\, d\varphi$

$$= \frac{r^2}{2rd}\int_0^{2\pi}\int_{r^2-2rd+d^2}^{r^2+2rd+d^2} \frac{1}{\sqrt{u}} du\, d\theta$$

$$= \frac{\cancel{2}\pi r}{\cancel{2}d}\int_{(r-d)^2}^{(r+d)^2} \frac{1}{\sqrt{u}} du = \frac{2\pi r}{d}\sqrt{u}\Big|_{(r-d)^2}^{(r+d)^2}$$

$$= \frac{2\pi r}{d}\left(|r+d| - |r-d|\right)$$ so $\begin{cases} 4\pi r & r > d \text{ (inside)} \\ \frac{4\pi r^2}{d} & r < d \text{ (outside)} \end{cases}$ ✓

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4a $\iint_S \vec{F}\cdot d\vec{S}$ $\quad \vec{F}(x,y,z) = x\hat{i} + y\hat{j} + z^2\hat{k}$

$$\phi(u,v) = (2\sin u, 3\cos u, v)$$

$$0 \le u \le 2\pi \qquad 0 \le v \le 1$$

$$\vec{T}_u = \frac{d((\phi(u,v)))}{du} = 2\cos u\,\hat{i} - 3\sin u\,\hat{j}$$

$$\vec{T}_v = \frac{d((\phi(u,v)))}{dv} = \hat{k}$$

$$\vec{T}_u \times \vec{T}_v = ((2\cos u\,\hat{i} - 3\sin u\,\hat{j}) \times (\hat{k})) = -3\sin u\,\hat{i} - 2\cos u\,\hat{j}$$

$$\vec{F}\cdot(\vec{T}_u \times \vec{T}_v) = (x\hat{i} + y\hat{j} + z^2\hat{k})\cdot(-3\sin u\,\hat{i} - 2\cos u\,\hat{j})$$
$$= -6\sin^2 u - 6\cos^2 u$$
$$= -6$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_\phi \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{T}_u \times \vec{T}_v)\,du\,dv = \int_0^1\int_0^{2\pi} -6\,du\,dv$$

$$\int_0^1 -6\,[u]_0^{2\pi}\,dv = [-12\pi v]_0^1 = \boxed{-12\pi}$$

4b $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$ $\quad \vec{F} = (x^2+y-4)\hat{i} + 3xy\hat{j} + (2xz+z^2)\hat{k}$

$S = x^2+y^2+z^2 = 16,\ z \geq 0$

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_D (\nabla \times \vec{F}) \cdot (\vec{T_x} \times \vec{T_y})\, dy\, dx$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ d/dx & d/dy & d/dz \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix}$$

$= \hat{i}\,(d/dy(2xz+z^2) - d/dz(3xy))$
$- \hat{j}\,(d/dx(2xz+z^2) - d/dz(x^2+y-4))$
$+ \hat{k}\,(d/dx(3xy) - d/dy(x^2+y-4))$

$= \hat{i}(0-0) - \hat{j}(2z) + \hat{k}(3y-1)$

$0\hat{i} - 2z\hat{j} + (3y-1)\hat{k} = (0, -2z, 3y-1)$

$x^2+y^2+z^2 = 16 \rightarrow z = \pm\sqrt{16-x^2-y^2}$, $z \geq 0$

$z = \sqrt{16-x^2-y^2}$

$T(x,y,z) = (x, y, \sqrt{16-x^2-y^2})$, $-4 \leq x \leq 4$, $-\sqrt{16-x^2} \leq y \leq \sqrt{16-x^2}$

$T_x(x,y) = d/dx\,(x, y, \sqrt{16-x^2-y^2}) = \left(1,\ 0,\ \frac{-x}{\sqrt{16-x^2-y^2}}\right)$

$T_y(x,y) = d/dy\,(x, y, \sqrt{16-x^2-y^2}) = \left(0,\ 1,\ \frac{-y}{\sqrt{16-x^2-y^2}}\right)$

$$\vec{T_x} \times \vec{T_y} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \dfrac{-x}{\sqrt{16-x^2-y^2}} \\ 0 & 1 & \dfrac{-y}{\sqrt{16-x^2-y^2}} \end{vmatrix}$$

$$= \left(\frac{x}{\sqrt{16-x^2-y^2}}, \frac{y}{\sqrt{16-x^2-y^2}}, 1 \right)$$

$$(\nabla \times F) \cdot (\vec{T_x} \times \vec{T_y}) = (0, -2z, 3y-1) \cdot \left(\frac{x}{\sqrt{16-x^2-y^2}}, \frac{y}{\sqrt{16-x^2-y^2}} \right)$$

* sub $z = \sqrt{16-x^2-y^2}$

$$= \left(0 + \frac{-2zy}{\sqrt{16-x^2-y^2}} + 3y-1 \right) \quad \frac{-2zy}{\sqrt{16-x^2-y^2}} + 3y-1$$

$$= \frac{-2(\sqrt{16-x^2-y^2})y}{\sqrt{16-x^2-y^2}} + 3y-1 = -2y+3y-1 = y-1$$

$$\iint_S (\nabla \times \vec{T}) \, d\vec{S} = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (y-1) \, dy \, dx \quad \int_{-4}^{4} \left[\frac{y^2}{2} - y \right]_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} dx$$

$$= \int_{-4}^{4} -2\sqrt{16-x^2} \, dx = -2 \int_{-4}^{4} \sqrt{4^2-x^2} \, dx = -2 \left[\frac{x\sqrt{16-x^2}}{2} + \frac{16}{2} \sin^{-1}\left(\frac{x}{4}\right) \right]_{-4}^{4}$$

$$-2 \left[8\left[\frac{\pi}{2}\right] + 8\left[\frac{\pi}{2}\right] \right) \quad -2(8\pi) = \boxed{-16\pi}$$

5)a) 7.6.6

Compute heat flux across the unit sphere S if $T(x,y,z) = x$

Can you interpret this answer physically?

$$\iint_D F(\Phi(u,v)) \cdot (T_u \times T_v)\, du\, dv$$

$$\begin{cases} \Phi(\theta,\varphi) = (\cos\theta\sin\varphi,\ \sin\theta\sin\varphi,\ \cos\varphi) \\ 0 \le \theta \le 2\pi \\ 0 \le \varphi < \pi \end{cases}$$

$T_\theta \times T_\varphi = -\sin\varphi\, \hat{e}_\rho$ (From Lecture Ex 1) $= -\sin\varphi(\sin\varphi\cos\theta\, i + \sin\varphi\sin\theta\, j + \cos\varphi\, k)$

$= -\sin^2\varphi\cos\theta\, i + \sin^2\varphi\sin\theta\, j + \sin\varphi\cos\varphi\, k$

$F = -k\nabla T = (-k, 0, 0)$

$F(\Phi(\theta,\varphi)) = (-k, 0, 0)$

$$\int_0^{\pi}\int_0^{2\pi} (-k,0,0)\cdot(\sin^2\varphi\cos\theta,\ \sin^2\varphi\sin\theta,\ \sin\varphi\cos\varphi)\, d\theta\, d\varphi$$

$$= \int_0^{\pi}\int_0^{2\pi} -k\sin^2\varphi\cos\theta\, d\theta\, d\varphi$$

$$= \int_0^{\pi} -k\sin^2\varphi\sin\theta \Big|_0^{2\pi} d\varphi = \int_0^{\pi} 0\, d\varphi$$

So the heat flux is $\boxed{0}$

The total rate of heat flux is 0, so the amount leaving is the same as the amount entering.

S) b) 7.6.8

Let the velocity field of a fluid be described by $F = \sqrt{y}\, i$ (m/s) compute how many cubic meters of fluid per second are crossing the surface $x^2 + z^2 = 1$, $0 \le y \le 1$, $0 \le x \le 1$

So flux!

$$\iint_D F\, ds = \iint_D F(\Phi(u,v)) \cdot (T_u \times T_v)\, du\, dv$$

$\Phi(u,v) = (\sin u, v, \cos u) = (x, y, z)$ (circle/cylinder unit along y axis)

$\begin{cases} 0 \le u \le \pi \\ 0 \le v \le 1 \end{cases}$

(sinθ, cosθ)

$T_u = \begin{bmatrix} \cos u \\ 0 \\ -\sin u \end{bmatrix}$ $T_v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$T_u \times T_v = \begin{vmatrix} i & j & k \\ \cos u & 0 & -\sin u \\ 0 & 1 & 0 \end{vmatrix} \begin{matrix} i & j \\ \cos u & 0 \\ 0 & 1 \end{matrix}$

$i(0 + \sin u) - j(0 - 0) + k(\cos u - 0)$

$= \sin u\, i + \cos u\, k$

$$\int_0^1 \int_0^\pi (\sqrt{v}, 0, 0) \cdot (\sin u, 0, \cos u)\, du\, dv$$

$$= \int_0^1 \int_0^\pi \sqrt{v} \sin u\, du\, dv = \int_0^1 \sqrt{v}(-\cos u)\Big|_0^\pi dv = \int_0^1 2\sqrt{v}\, dv$$

$$= \frac{4}{3} v^{3/2} \Big|_0^1 = \frac{4}{3} - 0 = \boxed{\frac{4}{3}\ m^3/s}$$

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6) 7.6.20

a) a uniform fluid that flows vertically downward (heavy rain) is described by the vector field $F(x,y,z) = (0,0,-1)$
Find the total flux through cone $z = (x^2+y^2)^{1/2}$, $x^2+y^2 \le 1$

cone is $\begin{cases} \Phi(\theta, r) = (r\cos\theta, r\sin\theta, r) \\ 0 \le \theta \le 2\pi \\ 0 \le r \le 1 \end{cases}$

$$\text{Flux} = \iint_S F \cdot ds = \iint_D F \cdot (T_\theta \times T_r)\, ds$$

$T_\theta = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix}$ $T_r = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 1 \end{bmatrix}$ $T_\theta \times T_r = \begin{vmatrix} i & j & k \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix} \begin{matrix} i & j \\ -r\sin\theta & r\cos\theta \\ \cos\theta & \sin\theta \end{matrix}$

$i(r\cos\theta - 0) - j(-r\sin\theta - 0) + k(-r\sin^2\theta - r\cos^2\theta)$

$(r\cos\theta, r\sin\theta, -r)$

$$\int_0^{2\pi}\int_0^1 (0,0,-1)\cdot(r\cos\theta, r\sin\theta, -r)\, dr\, d\theta = \int_0^{2\pi}\int_0^1 r\, dr\, d\theta = \boxed{\pi}$$

b) Rain is driven sideways so it falls @ 45° angle $F(x,y,z) = \left(-\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right)$
Now what is the flux through the cone?

$F(x,y,z) = \left(-\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right)$

$$\int_0^{2\pi}\int_0^1 \left(-\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right)\cdot(r\cos\theta, r\sin\theta, -r)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 -\frac{\sqrt{2}}{2} r\cos\theta + \frac{\sqrt{2}}{2} r\, dr\, d\theta = \int_0^{2\pi} -\frac{\sqrt{2}}{4} r^2\cos\theta + \frac{\sqrt{2}}{4} r^2 \Big|_0^1 d\theta$$

$$= \int_0^{2\pi} -\frac{\sqrt{2}}{4}\cos\theta + \frac{\sqrt{2}}{4}\, d\theta = -\frac{\sqrt{2}}{4}\sin\theta + \frac{\sqrt{2}}{4}\theta \Big|_0^{2\pi}$$

$$= 0 + \frac{\sqrt{2}\pi}{2} + 0 + 0 = \boxed{\frac{\pi\sqrt{2}}{2}}$$

7a $D = [-1,1] \times [-1,1]$, $P(x,y) = x$, $Q(x,y) = y$

$$\iint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy \quad (1)$$

$\langle x,y \rangle = \langle -1, 1 \rangle + t\langle 0,-2 \rangle$

$\langle x,y \rangle = \langle -1, -2t+1 \rangle$

$\left.\begin{array}{l} x = -1 \\ y = -2t+1 \end{array}\right\} t = [0,1]$

path btwn $(-1,1)$ + $(-1,-1)$

C_1 is $\left.\begin{array}{l} x = -1 \\ y = -2+t \end{array}\right\} t = [0,1]$

path btwn $(-1,-1)$ + $(1,-1)$ is C_2

$\langle x,y \rangle = \langle -1,-1 \rangle + (t-1)\langle 2,0 \rangle$

$\langle x,y \rangle = \langle 2t-3, -1 \rangle$

$\left.\begin{array}{l} x = 2t-3 \\ y = -1 \end{array}\right\} t = [1,2]$

path btwn $(1,-1)$ + $(1,1)$ is C_3

$\langle x,y \rangle = \langle 1,-1 \rangle + (t-2)\langle 0,2 \rangle$

$\langle x,y \rangle = \langle 1, 2t-5 \rangle$

$\left.\begin{array}{l} x = 1 \\ y = 2t-5 \end{array}\right\} t = [2,3]$

path btwn $(1,1)$ + $(-1,1)$ is C_4

$\langle x,y \rangle = \langle 1,1 \rangle + (t-3)\langle -2,0 \rangle$

$\langle x,y \rangle = \langle -2t+7, 1 \rangle$

$\left.\begin{array}{l} x = -2t+7 \\ y = 1 \end{array}\right\} t = [3,4]$

path	$x(t)$	$y(t)$	t	$x'(t)$	$y'(t)$
C_1	-1	$-2t+1$	$0 \le t \le 1$	0	-2
C_2	$2t-3$	-1	$1 \le t \le 2$	2	0
C_3	1	$2t-5$	$2 \le t \le 3$	0	2
C_4	$-2t+7$	-1	$3 \le t \le 4$	-2	0

(2)

$$\int_{\partial D} P\,dx + Q\,dy = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy + \int_{C_3} P\,dx + Q\,dy + \int_{C_4} P\,dx + Q\,dy$$

$$\int_{C_1} P\,dx + Q\,dy = \int_{C_1} x\,dx + y\,dy \int_0^1 -1(0) + (-2t+1)(-2)\,dt$$

$$= -2\left[\frac{-2t^2}{2} + t\right]_0^1 = 0$$

$$\int_{C_2} P\,dx + Q\,dy \int_{C_2} x\,dx + y\,dy = \int_1^2 2(2t-3) + 0(-1)\,dt$$

$$= 2\left[\frac{2t^2}{2} - 3t\right]_1^2 = 0$$

$$\int_{C_3} P\,dx + Q\,dy = \int_2^3 1(0) + (2t-5)(2)\,dt = 2\left[\frac{2t^2}{2} - 5t\right]_2^3 = 0$$

$$\int_{C_4} = \int_3^4 (-2t+7)(-2) + (-1)(0)\,dt = -2\left[\frac{-2t^2}{2} + 7t\right]_3^4 = 0$$

$$\int_{\partial D} P\,dx + Q\,dy = 0$$

$$\iint_D \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dx\,dy = \int_{-1}^{1}\int_{-1}^{1}\left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dx\,dy$$

$$\int_{-1}^{1}\int_{-1}^{1}\left(\frac{d(y)}{dx} - \frac{d(x)}{dy}\right) dx\,dy = 0$$

both sides equal 0 so Green's Theorem is valid for the above integral

7b $D = [0, \pi/2] \times [0, \pi/2]$, $P(x,y) = \sin x$, $Q(x,y) = \cos y$

$$\iint_C P\,dx + Q\,dy = \iint_D \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dx\,dy \quad (1)$$

$C_1 : (0,0) \rightarrow (\pi/2, 0)$

$\langle x,y \rangle = \langle 0,0 \rangle + t\langle \pi/2, 0 \rangle$

$\langle x,y \rangle = \langle \pi t/2, 0 \rangle$

$\left.\begin{array}{l} x = \pi t/2 \\ y = 0 \end{array}\right\} t = [0,1]$

$C_2 : (\pi/2, 0) \rightarrow (\pi/2, \pi/2)$

$\langle x,y \rangle = \langle \pi/2, 0 \rangle + (t-1)\langle \pi/2, \pi/2 \rangle$

$\langle x,y \rangle = \langle \pi/2, \pi(t-1)/2 \rangle$

$\left.\begin{array}{l} x = \pi/2 \\ y = \pi(t-1)/2 \end{array}\right\} t = [1,2]$

$C_3 : (\pi/2, \pi/2) \longrightarrow (0, \pi/2)$

$\langle x,y \rangle = \langle \pi/2, \pi/2 \rangle + (t-2)\langle 0, \pi/2 \rangle$

$\langle x,y \rangle = \langle \pi(-t+3)/2, \pi/2 \rangle$

$\left.\begin{array}{l} x = \pi(-t+3)/2 \\ y = \pi/2 \end{array}\right\} t = [2,3]$

$C_4: (0, \pi/2) \to (0,0)$

$\langle x,y\rangle = \langle 0, \pi/2\rangle + (t-3)\langle 0,0\rangle$
$\langle x,y\rangle = \langle 0, \pi(-t+4)/2\rangle$

$\left.\begin{array}{l} x=0 \\ y=\pi(-t+4)/2 \end{array}\right\} t\in[3,4]$

$x(t)$	$y(t)$	t	$x'(t)$	$y'(t)$
$\pi t/2$	0	$0\le t\le 1$	$\pi/2$	0
$\pi/2$	$\pi(t-1)/2$	$1\le t\le 2$	0	$\pi/2$
$\pi(-t+3)/2$	$\pi/2$	$2\le t\le 3$	$-\pi/2$	0
0	$\pi(-t+4)/2$	$3\le t\le 4$	0	$-\pi/2$

$$\int_{\partial D} Pdx+Qdy = \int_{C_1} Pdx+Qdy + \int_{C_2} Pdx+Qdy + \int_{C_3} Pdx+Qdy + \int_{C_4} Pdx+Qdy$$

$$\int_{C_1} Pdx+Qdy = \int_0^1 \sin(\pi t/2)(\pi/2) + \cos(0)(0)\,dt$$
$$= \frac{\pi}{2}\left[\frac{-\cos(\pi t/2)}{\pi/2}\right]_0^1 = 1$$

$$\int_{C_2} Pdx+Qdy = \int_1^2 \sin(\pi/2)(0) + \cos(\pi(t-1)/2)(\pi/2)\,dt$$

$$= \left[\sin\frac{\pi}{2}(t-1)\right]_1^2 = 1$$

$$\int_{C_3} Pdx+Qdy = \int_2^3 \sin(\pi(-t+3)/2)(-\pi/2) + \cos(\pi/2)(0)\,dt$$
$$= -\left[\cos\frac{\pi}{2}(-t+3)\right]_2^3 = 1$$

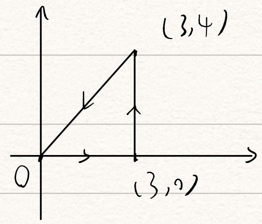
$$\int_{C_4} P\,dx + Q\,dy = \int_3^4 \sin(0)(0) + \cos(\pi(-t+4)/2)(-\pi/2)\,dt$$

$$= -\frac{\pi}{2}\left[\frac{\sin \pi/2(-t+4)}{-\pi/2}\right]_3^4 = -1$$

$$\int_{\partial D} P\,dx + Q\,dy = 1+1-1-1 = 0$$

Green Theorem is VALID for the integral

8:



$$D = \{ 0 \le x \le 3, \quad 0 \le y \le \tfrac{4}{3}x \}$$

$$\therefore \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dy\,dx$$

$$= \int_0^3 \int_0^{\frac{4}{3}x} (10y - 8y)\,dy\,dx$$

$$= \int_0^3 \frac{16}{9} x^2\,dx$$

$$= \frac{16}{27} x^3 \Big|_0^3$$

$$= 16$$

9. Using divergence theorem:

$$\int_{\partial D} \vec{F} \cdot \vec{n} \cdot ds = \iint_D \operatorname{div} \vec{F} \cdot dA$$

Note that $\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0$

$$\therefore \iint_D \operatorname{div} \vec{F} \cdot dA = 0$$

10. a): The area of Region D is

$$A(D) = \frac{1}{2} \int_C x\,dy - y\,dx$$

since $\begin{cases} x = R\cos t \\ y = R\sin t \end{cases} \quad (0 \le t \le 2\pi)$

Hence, $A(D) = \frac{1}{2} \int_C [R\cos t \cdot R\cos t - R\sin t \cdot (-R\sin t)]\,dt$

$$= \frac{1}{2} \int_0^{2\pi} R^2 \cdot dt$$

$$= \frac{1}{2} \cdot 2\pi R^2$$
$$= \pi R^2 .$$

b): since $A = \frac{1}{2} \int_C x\,dy - y\,dx$

We let $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$\therefore \quad A = \frac{1}{2} \int_C (r^2\cos^2\theta + r^2\sin^2\theta)\, d\theta$$
$$= \frac{1}{2} \int_C r^2 d\theta$$

11) 8.1.20 Remark: This problem is asking you to compute $\int_{\partial D} F \cdot ds$ and $(\nabla \times F) \cdot k$ and explain why $\int_{\partial D} F \cdot ds \neq \iint_D (\nabla \times F) \cdot k \, dA$ does not violate Green's Theorem.

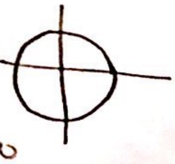
Let $P(x,y) = -y/(x^2+y^2)$ and $Q(x,y) = x/(x^2+y^2)$

Assuming D is the unit disc, investigate why greens theorem fails for this $P + Q$.

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} \begin{matrix} i & j \\ \partial x & \partial y \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{matrix} \quad i(0-0) - j(0-0) + k\left(\frac{\partial}{\partial x}\frac{x}{x^2+y^2} + \frac{\partial}{\partial y}\frac{y}{x^2+y^2}\right)$$

$$= \frac{\cancel{x^2}+\cancel{y^2}-2x^2+x^2+\cancel{y^2}-2y^2}{(x^2+y^2)^2} k = \vec{0} k$$

So $\nabla \times \vec{F} = 0$ so $\iint_D (\nabla \times F) \cdot k \, dA = 0$

$\int_{\partial D} F \cdot ds$  (from Lecture "More on Green's Theorem EX1")

can deform to any e so unit circle

$$\oint_{C^+} F \cdot \vec{dr} = 2\pi N \leftarrow \text{\# of times curve } C \text{ wraps around origin}$$

for $F = \frac{-y}{x^2+y^2} i + \frac{x}{x^2+y^2} j$

so $\int_{\partial D} \vec{F} \cdot dr = 2\pi$

$2\pi \neq 0$ BUT!

Unit disk D contains the origin where $P(x,y)$ and $Q(x,y)$ aren't defined, and for Green's Theorem to be applicable F must be "nicely defined on D" so the conditions aren't satisfied so Green's theorem isn't violated