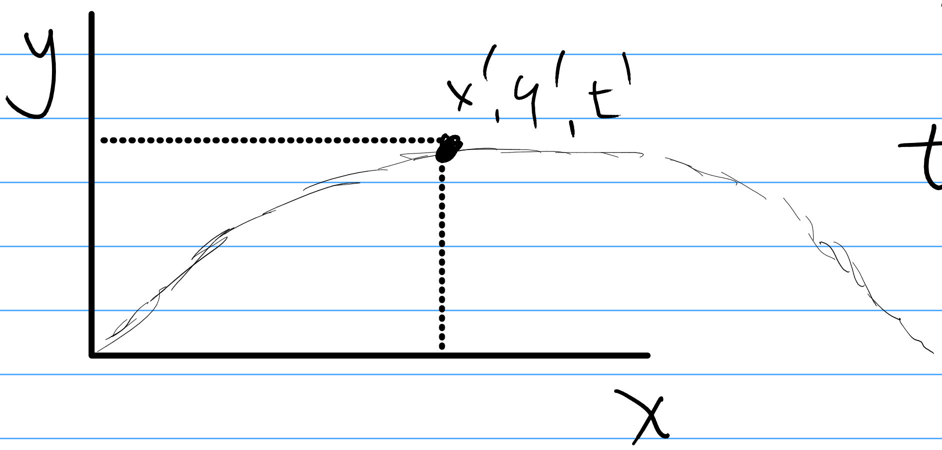
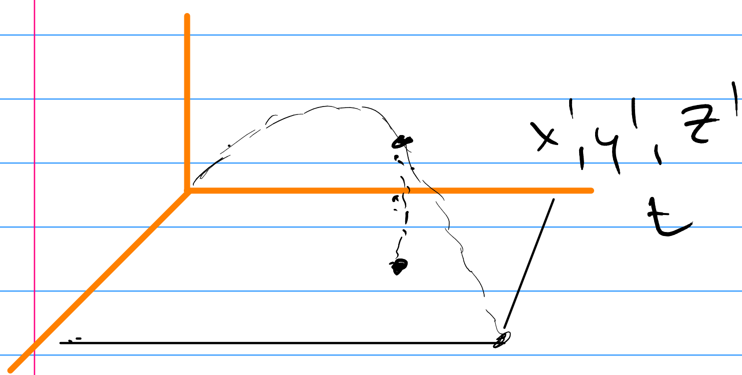
paths in configuration space

when a dynamical system evolves in time, its coordinates change according to the equations of motion

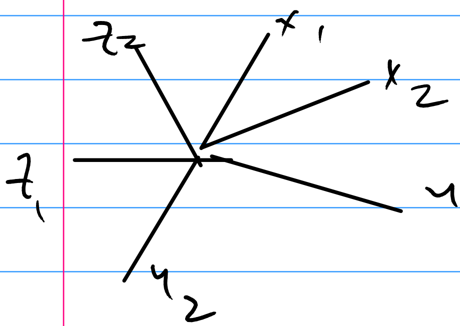


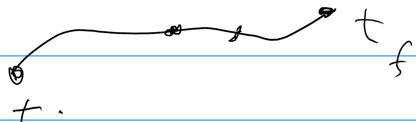
x, y

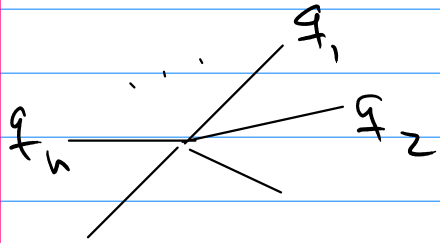
t parameterizes the motion



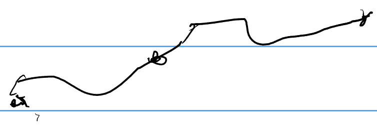
for two objects in 3D space - 6 coordinates, this is a 6-dimensional coordinate space







$$f(q_1, q_2 \ldots q_n, \dot{q}_1 \ldots \dot{q}_n, t)$$

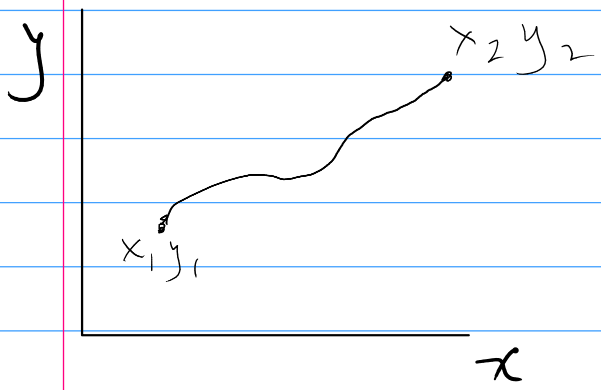


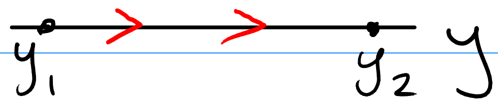
Calculus of Variations

$f(y, \dot{y}, x)$ $\quad y(x)$ path $\quad y(x_1) = y_1$

$\dot{y} = \frac{dy}{dx}$ $\quad y(x_2) = y_2$

$x_1 \to x_2$





not configuration space

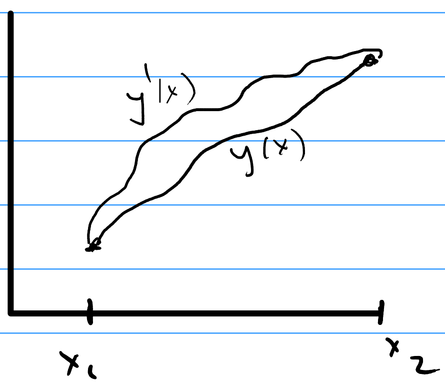
$y(x)$

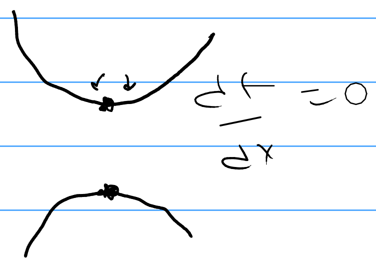
the issue here is on how 'y' varies wrt to 'x'

The answer is a curve, not a point. The variable is the functional form of 'y(x)'

$$J = \int_{x_1}^{x_2} f(y, \dot{y}, x)\, dx$$

question: what form and shape of 'y(x)' will render J at an extremum





$$y(x, \alpha) = y(x, 0) + \alpha\eta(x)$$

$\eta(x_1) = \eta(x_2) = 0$

$$J = J(\alpha) \qquad \left(\frac{dJ}{d\alpha}\right)_{\alpha=0} = 0$$

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \underbrace{\frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}}_{(1)} \right) dx$$

$$\frac{\partial x}{\partial \alpha} = 0$$

(1) $$\int_{x_1}^{x_2} \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}\, dx = \int \frac{\partial f}{\partial \dot{y}}\frac{\partial^2 y}{\partial x \partial \alpha}\, dx$$

integration by parts

$$\text{ibp} = \cancel{\frac{\partial f}{\partial \dot{y}}\frac{\partial y}{\partial \alpha}\Big|_{x_1}^{x_2}} - \int \frac{\partial y}{\partial \alpha}\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) dx$$

0

$$\frac{\partial y}{\partial \alpha} = \eta(x)\Big|_{x_1}^{x_2} = 0$$

$$\frac{dJ}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\right)\frac{\partial y}{\partial \alpha}\, dx$$

$$= \int \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\right)\eta(x)\, dx$$

$$= 0$$

use the "fundamental lemma" of the calculus of variations says

$$\int_{x_1}^{x_2} M(x)\,\eta(x)\,dx = 0 \qquad \text{for all } \eta(x)$$

$$\Rightarrow \quad M(x) = 0$$

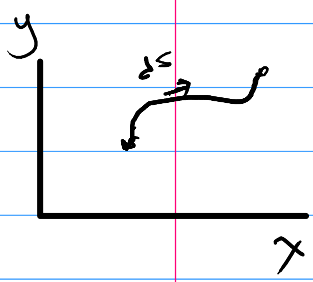
$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0}$$

Example 1 - extrema of the integral of a path integral

Find the path that gives an extrema of the integral 'I' where

$$I = \int_1^2 ds$$

b/c the integral is the length of a path,
then the extrema should give either max or min,
but we know that it will be a minimum.



$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \dot{y}^2}\,dx$$

$$I = \int_1^2 \sqrt{1 + \dot{y}^2}\,dx$$

$$f = \sqrt{1 + \dot{y}^2}$$

$$\frac{\partial f}{\partial y} = 0$$

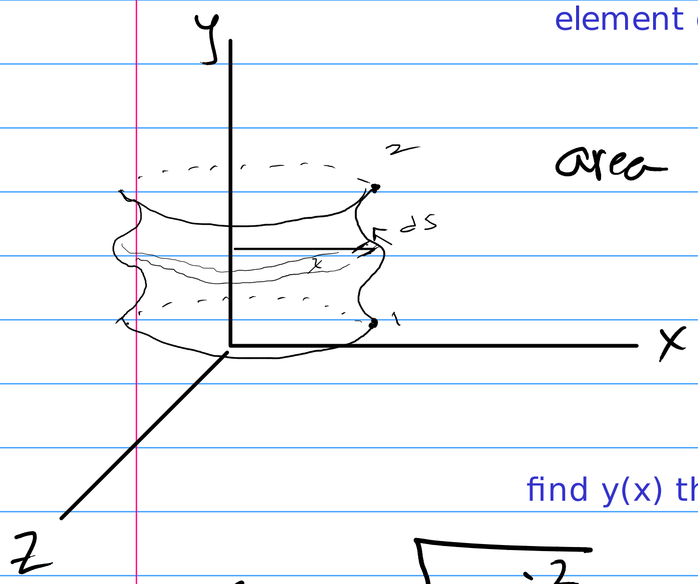
$$\frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{1 + \dot{y}^2}}$$

$$\frac{d}{dx}\frac{\partial f}{\partial \dot{y}} = 0 = \frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}\right)$$

$$\frac{\dot{y}}{\sqrt{1+\dot{y}^2}} = c \Rightarrow \dot{y} = a$$

$$\boxed{y = ax + b}$$

Example 2 - minimum surface of revolution



element of area $2\pi x\, ds$

$$\text{area} \quad I = \int_1^2 2\pi x\, ds$$

$$I = 2\pi \int_1^2 x\sqrt{1+\dot{y}^2}\, dx$$

find y(x) that extremizes the integral 'I'

$$f = x\sqrt{1+\dot{y}^2} \qquad \frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial \dot{y}} = \frac{x\dot{y}}{\sqrt{1+\dot{y}^2}}$$

$$\frac{d}{dx}\frac{\partial f}{\partial \dot{y}} = 0 \Longrightarrow \frac{x\dot{y}}{\sqrt{1+\dot{y}^2}} = c$$

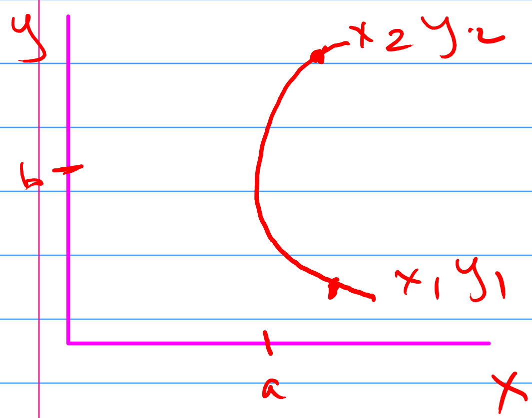
$$x^2\dot{y}^2 = c'(1+\dot{y}^2)$$

$$\dot{y}^2(x^2-a^2)=a^2$$

$$\frac{dy}{dx}=\frac{a}{\sqrt{x^2-a^2}} \Rightarrow y=a\cosh^{-1}\frac{x}{a}+b$$

equation of a catenary

$$\Rightarrow \boxed{x=a\cosh\frac{y-b}{a}}$$

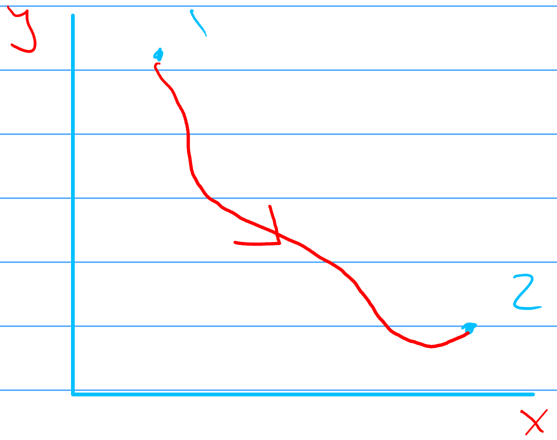


e.g. soap bubble between two rings

Example 3 - the brachistochrone

Basic question: find the curve along which a particle falls under gravity from rest in the least time.

$$t_{12}=\int_1^2 dt=\int_1^2\frac{ds}{v}$$



Use the energy theorem for find 'v'

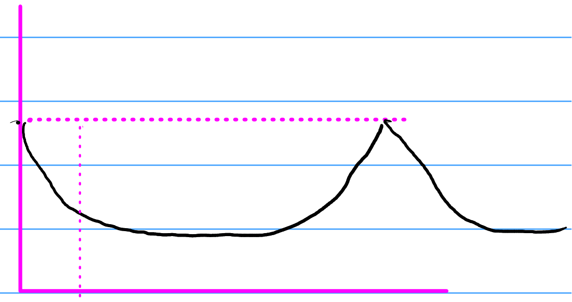
$$0=\frac{1}{2}mv^2-mgy$$

$$v=\sqrt{2gy}$$

$$t_{12}=\int\frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}}\,dx \Rightarrow f=\frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}}$$

$$\boxed{\begin{aligned} x &= a(\theta - \sin\theta) \\ y &= a(1 - \cos\theta) \end{aligned}}$$

these equations describe
a cycloid



We extend now to many coordinates

$$\delta J = \delta \int_1^2 f(y_1(x), y_2(x) \ldots \dot{y}_1(x), \dot{y}_2(x) , \ldots x) \, dx$$

$$y_1(x, \alpha) = y_1(x, 0) + \alpha \eta_1(x)$$

$$y_2(x, \alpha) = y_2(x, 0) + \alpha \eta_2(x)$$

$$\vdots$$

$$\boxed{\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}_i}\right) = 0} \qquad i = 1, \ldots n$$

Euler-Lagrange differential eqns

$$I = \int_{t_1}^{t_2} L \, dt \qquad L = T - V$$

Hamilton's Principle: The motion of a system from time 't1' to 't2' is such that the "action integral" I has a stationary value for the path of motion (in configuration space)

$x \to t \qquad y_i \to q_i$

$$f(y, \dot{y}, x) \longrightarrow L(q, \dot{q}, t)$$

then we obtain back Lagrange's equations of motion

This derivation of equations of motion is totally independent of Newton's laws: this is to say that if you assume Hamilton's principle true, then you can obtain the eqns of motion.

forces are derivable from a potential (may be a function of coordinates, velocities, and time) - system is called "monogenic"

if the potential is only a function of coordinates, then the monogenic system is also conservative.

Extension to specific non-holonomic systems

What we know up to now:

1. Lagrange's eqns require independent qj.
2. Virtual displacements have to be consistent with the constraints.

In general, non-holonomic constraints cannot solve for the dependent coordinates (qj's), so these might not comply with either 1 or 2 above.

There is a special case of non-holonomic constraints that can be put in a Lagrangian formalism.

$$f_\alpha(q_1 \dots q_n, \dot{q}_1 \dots \dot{q}_n, t) = 0$$

α index of constraint

m $<$ 3 s of "

if this is possible, then these are called semi-holonomic constraints.

NOTE: if f_alpha do not depend on the q-dots, all of them reduce to holonomic constraints.

$$\sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha = 0$$ also true

lambdas are called the Lagrange undetermined multipliers

$$\delta \int_{t_1}^{t_2} L\, dt = 0$$ Hamilton's principle

also true that

$$\delta \int_{t_1}^{t_2} \left(L + \sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha \right) dt = 0$$

this will result in 'm+n' equations that will solve for the 'n' qjs and the 'm' lambdas.