HW 4 - 8.4 a) and b) (just TM), 8.6

8.4 transverse E and B waves propogate along hollow, right cylinder w/ inner radius R and σ

a. cutoff frequencies of TE + TM modes?

modes must satisfy Helmholtz eqn: $(\nabla_t^2 + \gamma^2)\,\psi(\rho,\phi) = 0$

for TM: $\psi = E_z = E_0 J_m(\gamma_{mn}\rho)e^{\pm im\phi}$

TE: $\psi = H_z = H_0 J_m(\gamma_{mn}\rho)e^{\pm im\phi}$

for TM, need $E_z(\rho=R)=0$ so $J_m(\gamma R) = 0$

X_{mn} are zeros of J_m so $\gamma_{mn} = \frac{X_{mn}}{R}$

$$\boxed{\text{cutoff } \omega_{mn} = \frac{cX_{mn}}{R} = \frac{X_{mn}}{\sqrt{\mu\varepsilon}\,R}}$$

for TE, need $\left[\frac{\partial B_z}{\partial\rho}\right]_{\rho=R} = 0 = \left[\frac{\partial}{\partial\rho}\left(J_m(\gamma\rho)\right)\right]_{\rho=R}$

$\gamma_{mn} = \frac{X'_{mn}}{R}$ so $\boxed{\text{cutoff } \omega_{mn} = \frac{cX'_{mn}}{R} = \frac{X'_{mn}}{\sqrt{\mu\varepsilon}\,R}}$

need dominant mode + next 4 frequencies

(TM)

X_{mn}	m=0	m=1	m=2
n=1	2.4	3.83	5.14
n=2	5.52	7.02	8.42
n=3	8.65	10.17	11.62

(TE)

X'_{mn}	m=0	m=1	m=2
n=1	$\boxed{0}$	1.84	3.05
n=2	3.83	5.33	6.71
n=3	7.02	8.54	9.97

not a valid mode ($\gamma=0$)

dominant mode is TE_{11} with $\omega_c = 1.84\,c/R$

next 4 modes: TM_{01} with $\omega_c = 2.40\,c/R$

TE_{21} with $\omega_c = 3.05\,c/R$

(degenerate) $\rightarrow$ TE_{02} and TM_{11} with $\omega_c = 3.83\,c/R$

so dominant frequency is $\omega_D = 1.84\, c/R$

other 4 frequencies as ratios to dominant mode:

$TM_{01} \rightarrow \omega_c/\omega_D = 1.30$

$TE_{21} \rightarrow \omega_c/\omega_D = 1.66$

TE_{02} and $TM_{11} \rightarrow \omega_c/\omega_D = 2.08$

b) calculate attenuation constants of waveguide as function of frequency for lowest 2 distinct modes + plot

do just TM so TM_{01} and TM_{11}

eq 8.51: $$P = \frac{1}{2\sqrt{\mu\epsilon}}\left(\frac{\omega}{\omega_\lambda}\right)^2\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{1/2}\begin{Bmatrix}\overset{TM}{\varepsilon} \\ \underset{TE}{\mu}\end{Bmatrix}\int_A \psi^*\psi\, da$$

attenuation constant $$B_\lambda = -\frac{1}{2P}\frac{dP}{dz} = \frac{1}{2P}\left|\frac{dP}{dz}\right|$$

so also need power loss

$$-\frac{dP}{dz} = \frac{1}{2\sigma\delta}\oint_C |n\times H|^2 dl = \frac{1}{2\sigma\delta}\, 2\pi R|H|^2$$

$$E_z = \psi = E_0 J_m\left(\frac{x_{mn}\rho}{R}\right)$$

$$H_t = \frac{1}{Z}\hat{z}\times E_t$$

we know $E_t = \dfrac{ik\nabla_t\psi}{\gamma^2}$ and for TM, $\gamma = \dfrac{x_{mn}}{R}$

so $$E_t = \frac{ikE_0}{\gamma^2}\left[\hat{\rho}\frac{x_{mn}}{R}J_m'\left(\frac{x'_{mn}\rho}{R}\right)\right]$$

$$= \frac{ikE_0R}{x_{mn}}J_m'\left(\frac{x_{mn}\rho}{R}\right)\hat{\rho}$$

then $$H_t = \frac{ikE_0R}{Z\,x_{mn}}J_m'\left(\frac{x_{mn}\rho}{R}\right)\hat{\phi}$$

$$\left|\frac{dP}{dz}\right| = \frac{1}{2\sigma\delta} 2\pi R |H|^2$$

$$= \frac{\pi R}{\sigma\delta}\left|\frac{ikE_0 R}{Z X_{mn}} J_m'\left(\frac{X_{mn}\rho}{R}\right)\right|^2$$

$$= \frac{\pi R}{\sigma\delta}\left[\frac{k^2 E_0^2 R^2}{Z^2 X_{mn}^2} J_m'^2\left(\frac{X_{mn}\rho}{R}\right)\right]$$

$$P = \frac{1}{2\sqrt{\mu\varepsilon}}\left(\frac{\omega}{\omega_\lambda}\right)^2\left(1-\frac{\omega_\lambda^2}{\omega^2}\right)^{1/2} \varepsilon \int E_0^2 J_m^2\left(\frac{X_{mn}\rho}{R}\right) da$$

so $P(TM_{01}) = \frac{\varepsilon}{2\sqrt{\mu\varepsilon}}\left(\frac{\omega}{\omega_{01}}\right)^2\left(1-\frac{\omega_{01}^2}{\omega^2}\right)^{1/2}\int E_0^2 J_0^2\left(\frac{X_{01}\rho}{R}\right) da$

and $\left|\frac{dP}{dz}\right| = \frac{\pi R}{\delta\sigma}\left[\frac{k^2 E_0^2 R^2}{Z^2 X_{01}^2} J_0'^2\left(\frac{X_{01}\rho}{R}\right)\right]$

since waveguide is hollow, $\mu = \mu_c = \mu_0$ and $\varepsilon = \varepsilon_0$,

$$\boxed{B_{TM01}(\omega) = \frac{1}{R}\left[\frac{\varepsilon_0 \omega^3}{2\sigma(\omega^2-\omega_{01}^2)}\right]^{1/2}}$$

8.6 Resonant cavity of copper - hollow, circular cylinder w/ R, L

a) resonant frequencies for all types of waves?

use $\frac{1}{\sqrt{\mu\varepsilon}\, R}$ as unit of frequency, plot lowest 4 resonant frequencies as function of R/L for 0 < R/L < 2 - does same mode have lowest frequency for all R/L?

TM modes: $(\nabla_t^2 + \gamma^2) E_z = 0$ w/ $E_z|_S = 0$

$$\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial \phi^2} + \gamma^2 E_z = 0$$

w/ $E_z(\rho, \phi) = R(\rho)\, P(\phi)$:
$$\frac{\partial^2 P}{\partial \phi^2} + mP = 0$$
$$\frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial R}{\partial \rho} + \left(\gamma^2 - \frac{m^2}{\rho^2}\right) = 0$$

so $P(\phi) = e^{\pm i m \phi}$ and $R(\rho) = J_m(\gamma \rho)$

Since $\gamma = x_{mn}/R$, $E_z = A J_m\left(x_{mn}\frac{\rho}{R}\right) e^{\pm i m\phi} e^{-i\omega t}$

$$E_z = A J_m\left(x_{mn}\frac{\rho}{R}\right) e^{\pm i m \phi} e^{-i\omega t} \begin{cases} \sin kz & k \text{ even} \\ \cos kz & k \text{ odd} \end{cases}$$

so $k = n\pi/L$ and $\omega_{mn} = c\sqrt{\gamma_{mn}^2 + k_n^2}$

for $\rho \geq 0$, $\boxed{\omega_{mn} = \frac{1}{\sqrt{\mu\varepsilon}\, R}\sqrt{x_{mn}^2 + \left(\frac{n\pi R}{L}\right)^2}}$

TE modes: $\gamma = x'_{mn}/R$

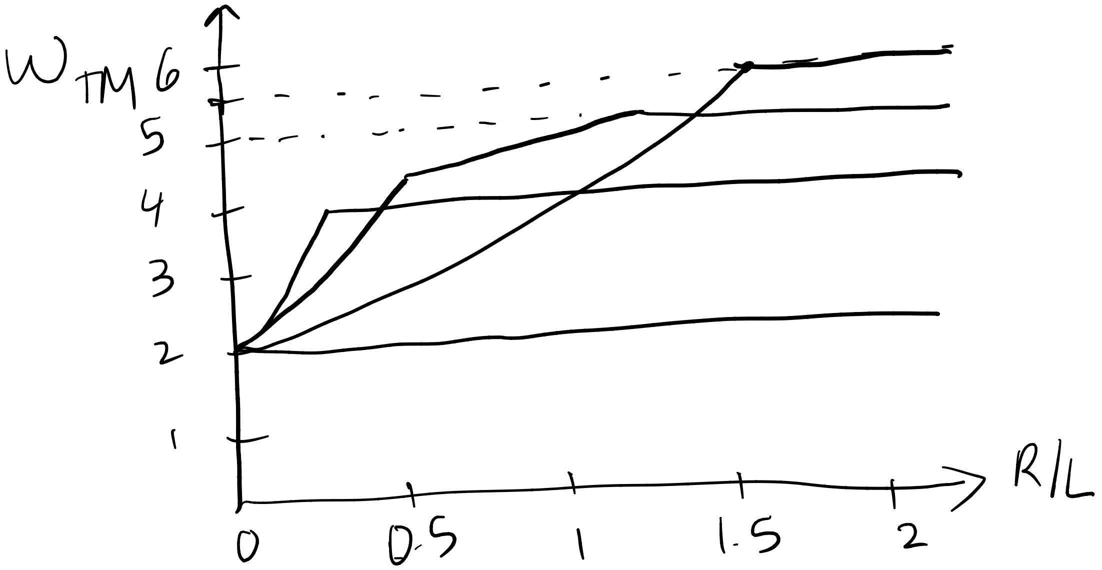
and $B_z = A J_m(x'_{mn}\, \rho/R)\, e^{\pm i m\phi} e^{-i\omega t} \begin{cases} \sin kz & k \text{ even} \\ \cos kz & k \text{ odd} \end{cases}$

so for $\rho > 0$, $\boxed{\omega_{mn} = \frac{1}{\sqrt{\mu\varepsilon}\, R}\sqrt{x'^2_{mn} + \left(\frac{p\pi R}{L}\right)^2}}$

(TM)

X_{mn}	m=0	m=1	m=2
n=1	2.4	3.83	5.14
n=2	5.52	7.02	8.42
n=3	8.65	10.17	11.62

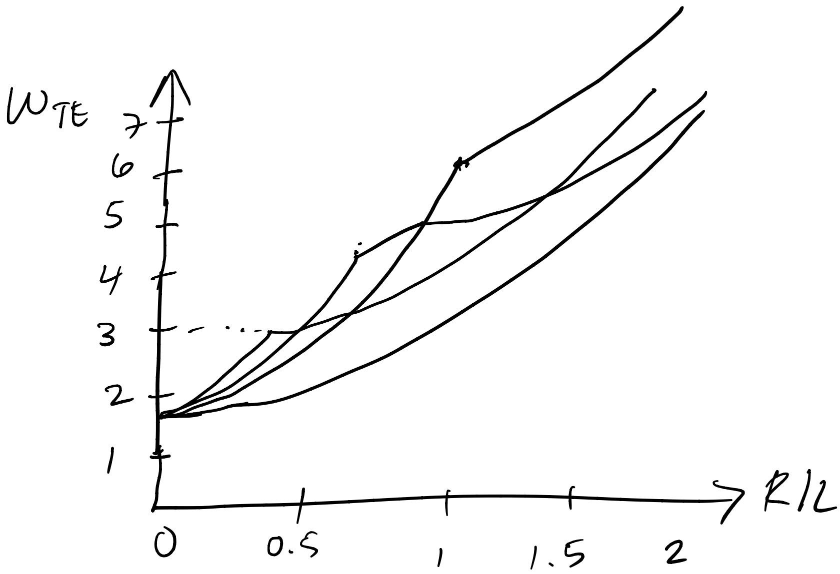
4 lowest TM resonant frequencies have m,n = 0 1, 1 1, 2 1, 0 2



(TE)

X'_{mn}	m=0	m=1	m=2	m=3
n=1	0	1.84	3.05	4.2
n=2	3.83	5.33	6.71	8.02
n=3	7.02	8.54	9.97	11.35

4 lowest resonant frequencies have m,n = 1,1
2,1
0,2
3,1



for TM, yes the same mode has lowest frequency for all values of R/L

for TE, yes as well