Recap 1: Applications of the Residue Theorem

- Consider

$$J = \int_{-\infty}^{\infty} d\omega \, \frac{1-e^{2i\omega a}}{\omega^2}$$

Near $\omega = 0$

$$1 - e^{2i\omega a} = -2i\omega a + O(\omega^2)$$

so integrand has a <u>simple</u> pole and $\mathrm{Res}(0) = -2ia$

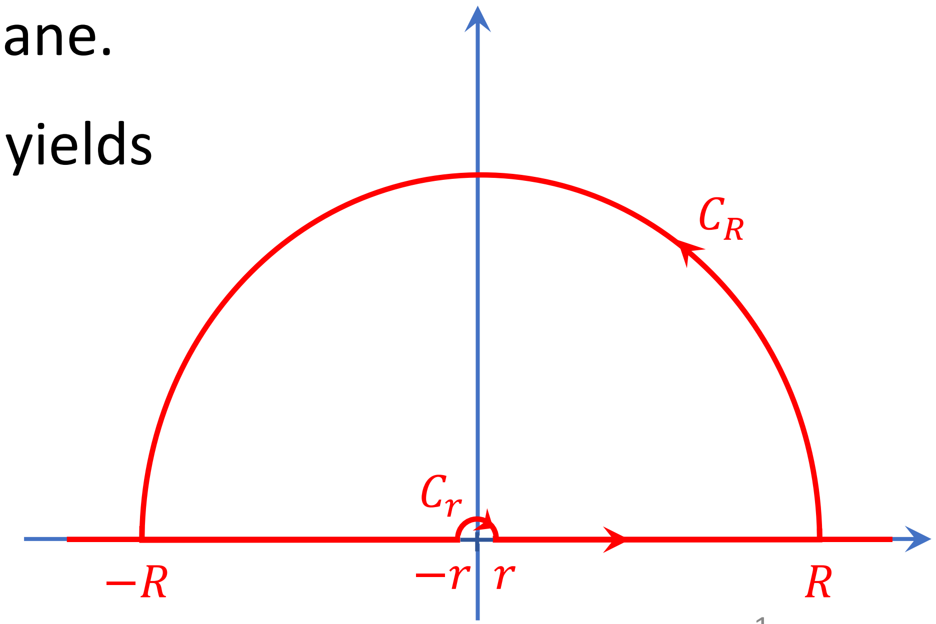
- Pole is on the axis, so avoid it with a small semicircle.
- If $a > 0$, OK to close with C_R in the upper half plane.
- <u>No</u> poles inside the contour, and the C_r integral yields

$$-\pi i \, (-2ia) = \; -2\pi a$$

clockwise, residue
half way

- Hence $J - 2\pi a = 0$

$$J = 2\pi a$$



Recap 2: Applications of the Residue Theorem

- Write

$$\mathrm{I} + \underbrace{\int_{R}^{R+i}}_{I_1} + \underbrace{\int_{R+i}^{-R+i}}_{I_2} + \underbrace{\int_{-R}^{-R+i}}_{I_3} = 2\pi i \operatorname{Res}\left(\frac{i}{2}\right) \qquad \mathrm{I} = \int_{-\infty}^{\infty} \frac{e^{az}\, dz}{\cosh \pi z}$$

- Strategy

i. Argue I_1 and I_3 away:

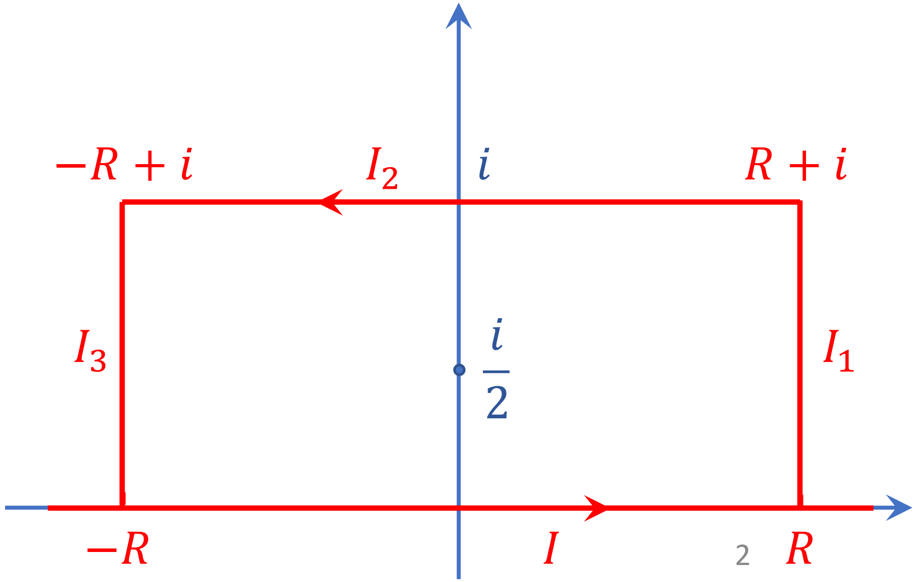
$$\text{e.g. } |I_1| \leq 2e^{(a-\pi)R} \to 0 \text{ as } R \to \infty$$

i. Relate I_2 to I: $\quad I_2 = e^{ai} I$

ii. Evaluate the residue at $z = \frac{i}{2}$:

$$\left(1 + e^{ai}\right) I = 2e^{ai/2}$$

$$I = \sec\frac{a}{2}$$



Recap 3: Applications of the Residue Theorem

$$\mathrm{I} = \int_0^\infty \frac{x^{\alpha-1}\,dx}{1+x}$$

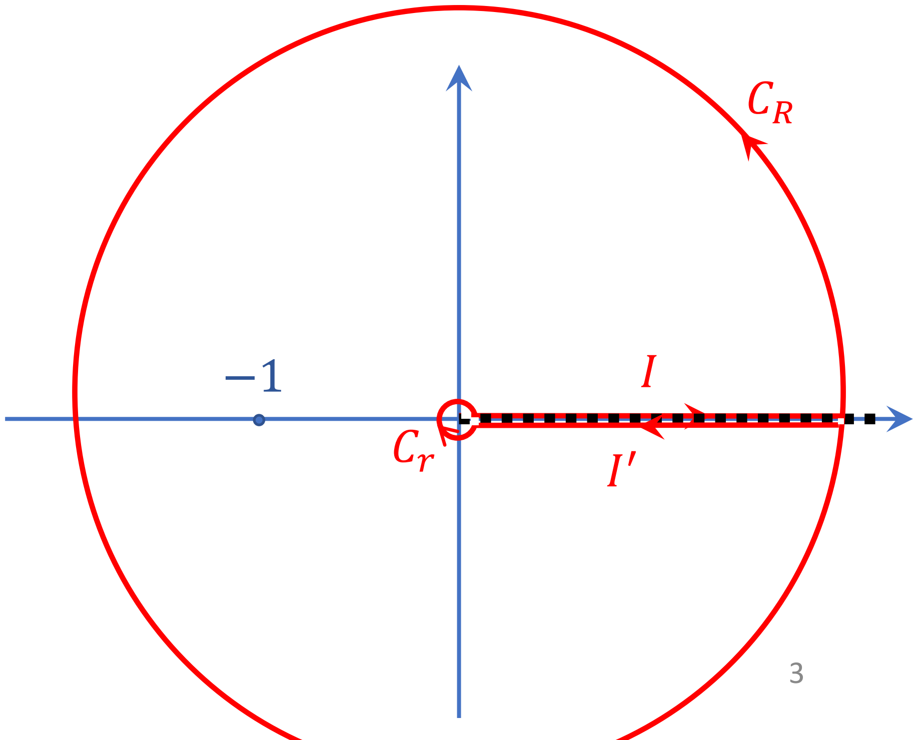
- Residue at $z = -1$ is $(-1)^{\alpha-1} = e^{\pi i(\alpha-1)}$
- $\left|\int_{C_R}\right| \leq \frac{R^{\alpha-1}}{R}\, 2\pi R = 2\pi R^{\alpha-1} \to 0$ as $R \to \infty$ if $\alpha < 1$.
- $\left|\int_{C_r}\right| \leq r^{\alpha-1} 2\pi r = 2\pi r^{\alpha} \to 0$ as $r \to 0$ if $\alpha > 0$.
- On the lower contour,

$$I' = \int_R^0 \frac{x^{\alpha-1} e^{2\pi i(\alpha-1)}\,dx}{1+x}$$
$$= e^{2\pi i(\alpha-1)}(-I)$$

- Hence

$$I + I' = \left[1 - e^{2\pi i(\alpha-1)}\right] I = 2\pi i\, e^{\pi i(\alpha-1)}$$

$$\Longrightarrow \quad \mathrm{I} = \frac{-2\pi i\, e^{\pi i \alpha}}{1-e^{2\pi i\alpha}} = \frac{-2\pi i}{e^{-\pi i\alpha}-e^{\pi i\alpha}} = \frac{\pi}{\sin \pi\alpha}$$



Integral Definitions of Functions

- Besides Fourier transform, many functions have integral definitions.
- Gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \quad [= (x-1)!]$$

- Legendre function

$$P_l(x) = \frac{2^{-l}}{2\pi i} \int_C \frac{\left(t^2-1\right)^l}{(t-x)^{l+1}}\, dt$$

- Bessel function

$$J_\nu(z) = \frac{2^{-l}}{2\pi i} \int_C e^{\frac{1}{2}x(t-1/t)} \frac{dt}{t^{\nu+1}}$$

and lots of others…

- Could start here and derive all other functional properties.

Convolution Theorem

- Define convolution of two functions f and g as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy$$

Interpretation: f is the incoming signal, and g is a "redistribution function" that dictates how much of the output at x is polluted by input data at $x - y$.

- Examples:
    1. Instrumental response: x is time, channel x picks up contribution from channel $x - y$; $g(y)$ is a property of the instrument.
    2. Telescope: x is spatial, $g(y)$ is the point-spread function of the instrument.
- Instrumental response smears the incoming ("true") data out over a range governed by g.
- Often the goal is to remove as much of this smearing as possible.

Convolution Theorem

- Fourier transforms provide a way of removing instrumental effects.
- Again, with the definition

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy$$

and supposing that $F(\omega)$ and $G(\omega)$ are the Fourier transforms of $f(x)$ and $g(x)$, then

$$\begin{aligned}(f * g)(x) &= \int_{-\infty}^{\infty} dy\, g(y) \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, F(\omega) e^{i\omega(x-y)} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, F(\omega)\, \left[\int_{-\infty}^{\infty} dy\, g(y) e^{-i\omega y}\right] e^{i\omega x} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, F(\omega)\, G(\omega)\, e^{i\omega x}\end{aligned}$$

- Note: works better with the $\frac{1}{2\pi}$ version of the FT/inverse formulation.
- Bottom line: transform of a convolution is a <u>product</u>, and vice versa.

Convolution Theorem

- Important application is deconvolution – removing instrumental effects.
- What we measure is $f * g$.
- We can determine the instrumental response g/G by calibrating our experiment

  e. g. if the input is a delta function, then the output is $g(x)$

$$\int_{-\infty}^{\infty} \delta(x-y)\, g(y)\, dy = g(x)$$

- Then in principle if we measure $f * g$ and know g, we can extract the true signal f by dividing out in Fourier space and transforming back.

$$f(x) = \mathcal{F}^{-1}(\mathcal{F}(f * g)/G)$$

- Major problem: instrumental response contains noise, and g/G is are typically Gaussians, so we amplify high-frequency noise and corrupt the recovered signal.
- Return to this later, in context of numerical applications.

Application: Solving an ODE

- Driven harmonic oscillator

$$\ddot{x} + \omega_0^2 x = f(t)$$

- Solve using Fourier transform $[X(\omega), F(\omega)]$

$$\Longrightarrow \quad (\omega_0^2 - \omega^2)X = F$$

$$\Longrightarrow \quad X(\omega) = \frac{F(\omega)}{\omega_0^2 - \omega^2}$$

- Product of two Fourier transforms, so the solution $x(t)$ is the <u>convolution</u> of $f(t)$ and $g(t)$, where

$$g(t) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega^2 - \omega_0^2} = \frac{-1}{2\pi} I$$

- This is a "big semicircle" integral, with poles on the path of integration.

Application: Solving an ODE

- This is a "big semicircle" integral, with poles on the path of integration.
- For $t > 0$, close in the upper half plane.
- Avoid poles with small semicircles.
- Jordan's lemma $\Longrightarrow \int_{C_R} \to 0$
- Residue at $\omega = -\omega_0$ is $-\dfrac{e^{-i\omega_0 t}}{2\omega_0}$
- Residue at $\omega = \omega_0$ is $\dfrac{e^{i\omega_0 t}}{2\omega_0}$

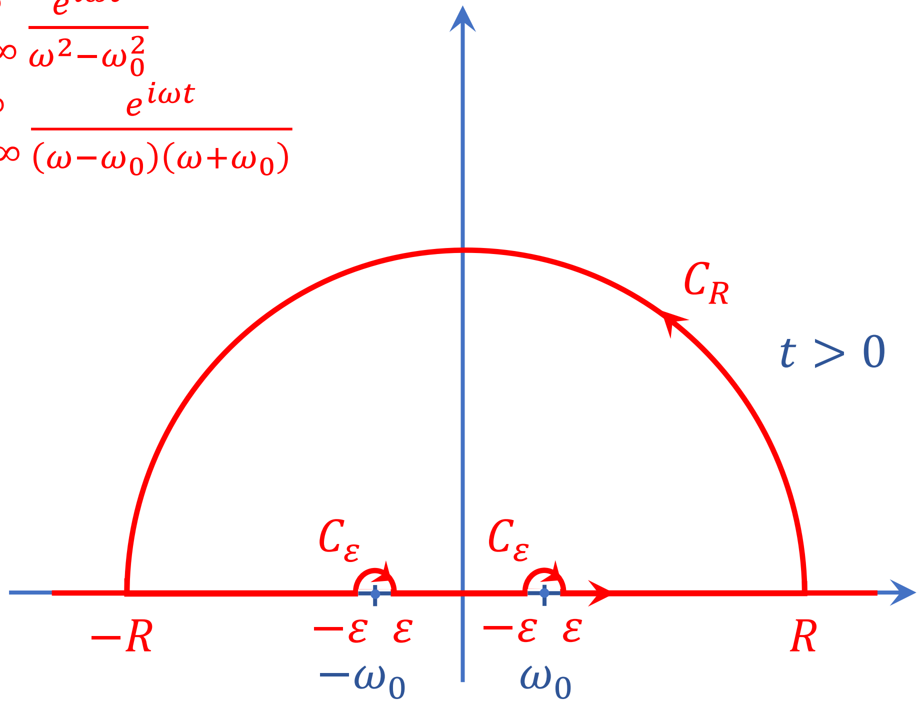
$$\Longrightarrow \quad I - \pi i\left(\frac{e^{i\omega_0 t}}{2\omega_0} - \frac{e^{-i\omega_0 t}}{2\omega_0}\right) = 0$$

$$\Longrightarrow \quad I = \pi i\left(\frac{e^{i\omega_0 t}}{2\omega_0} - \frac{e^{-i\omega_0 t}}{2\omega_0}\right) = -\frac{\pi}{\omega_0}\sin\omega_0 t$$

$$\Longrightarrow \quad g(t) = \frac{\sin\omega_0 t}{2\omega_0}$$

$$I = \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega^2 - \omega_0^2}$$

$$= \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega-\omega_0)(\omega+\omega_0)}$$



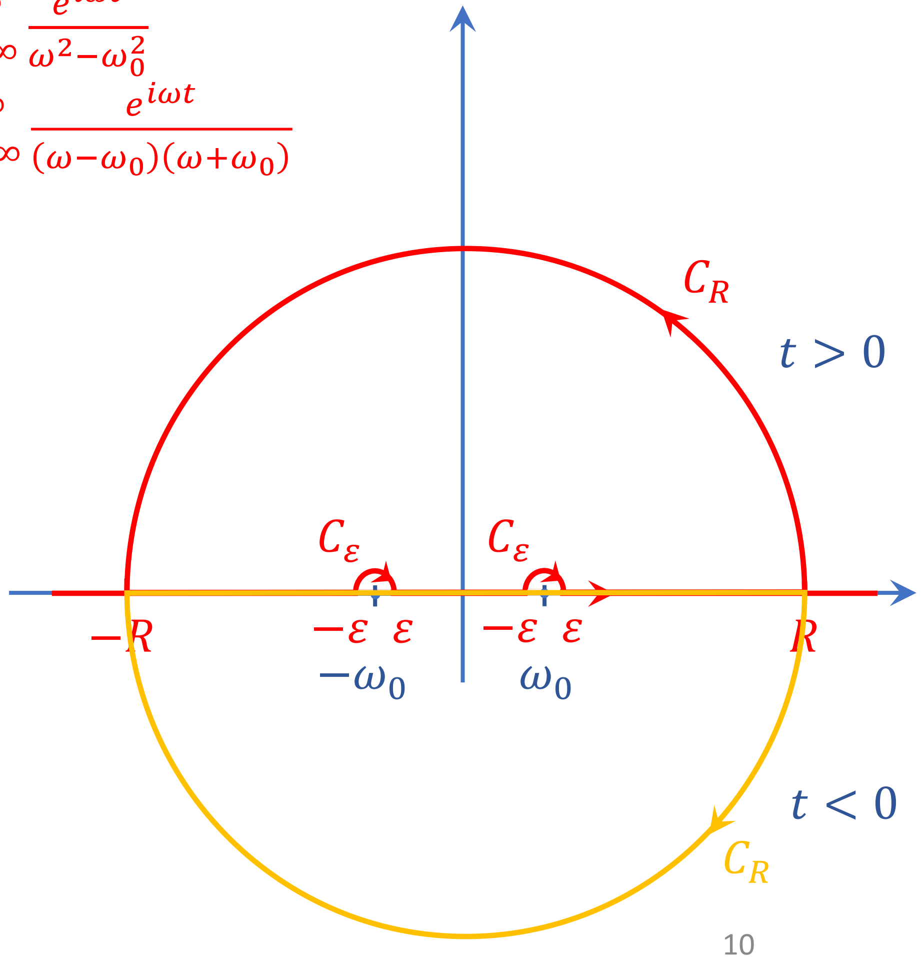
Application: Solving an ODE

- This is a "big semicircle" integral, with poles on the path of integration.
- For $t < 0$, close in the lower half plane.
- Avoid poles with small semicircles.
- Jordan's lemma $\Longrightarrow \int_{C_R} \to 0$
- Residue at $\omega = -\omega_0$ is $-\frac{e^{-i\omega_0 t}}{2\omega_0}$
- Residue at $\omega = \omega_0$ is $\frac{e^{i\omega_0 t}}{2\omega_0}$

$$\Longrightarrow \quad I - \pi i\left(\frac{e^{i\omega_0 t}}{2\omega_0} - \frac{e^{-i\omega_0 t}}{2\omega_0}\right) = -2\pi i\left(\frac{e^{i\omega_0 t}}{2\omega_0} - \frac{e^{-i\omega_0 t}}{2\omega_0}\right)$$

$$\Longrightarrow \quad I = -\pi i\left(\frac{e^{i\omega_0 t}}{2\omega_0} - \frac{e^{-i\omega_0 t}}{2\omega_0}\right) = \frac{\pi}{\omega_0}\sin\omega_0 t$$

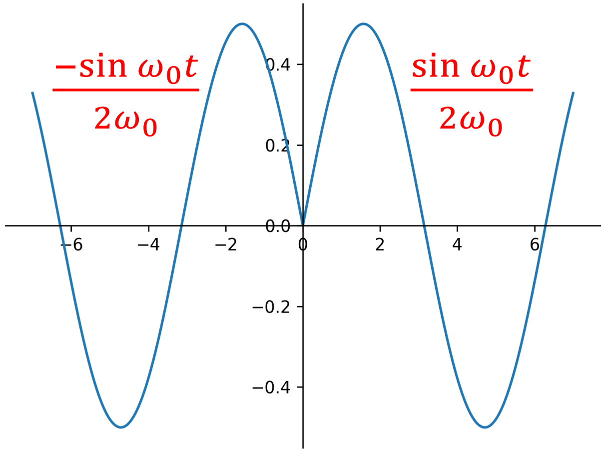
$$I = \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega^2 - \omega_0^2} = \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega-\omega_0)(\omega+\omega_0)}$$



Application: Solving an ODE

- In either case,

$$x(t) = g(t) = \frac{\sin \omega_0 |t|}{2\omega_0}$$



- But in a physical problem, might expect <u>initial conditions</u> to come into play.

e.g. if $g(t)$ is the response to driving by a delta function

– impulsive input at time $t = 0$

– what if we want a <u>causal</u> solution: $x(t) = 0$ for $t < 0$?

– no provision for this in the Fourier formulation

– <u>does</u> have a discontinuity in $\dot{x}$ at $t = 0$, but no way to force $x(t) = 0$ for $t < 0$.

Application: Solving an ODE

- Can solve the causal problem for $f(t) = \delta(t)$:

$$\ddot{x} + \omega_0^2 x = \delta(t)$$

- Let $x(t) = 0,\ \dot{x}(t) = 0$ for $t < 0$.
- Assume $x(t)$ is continuous, $\dot{x}(t)$ has a jump, and integrate the equation from $t = 0_-$ to $t = 0_+$:

$$[\dot{x}]_-^+ = 1$$

$$\Longrightarrow \qquad \dot{x}(0_+) = 1$$

- Then for $t > 0,\ \ddot{x} + \omega_0^2 x = 0$, so

$$x = A \sin \omega_0 t + \mathrm{B} \cos \omega_0 t$$

$$x(0) = 0,\ \dot{x}(0) = 1 \implies B = 0,\ A = 1/\omega_0$$

$$\Longrightarrow \qquad x(t) = \frac{\sin \omega_0 t}{\omega_0}$$

Note: <u>same</u> $\Delta\dot{x}$ at $x = 0$ as in previous solution.

Application: Solving an ODE

- Consider a damped, driven harmonic oscillator with delta-function driving:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \delta(t)$$

- Solve using Fourier transform

$$\Longrightarrow \qquad (-\omega^2 + 2i\gamma\omega + \omega_0^2)X = 1$$

$$\Longrightarrow \qquad X(\omega) = \frac{1}{\omega_0^2 + 2i\gamma\omega - \omega^2} = \frac{-1}{f(\omega)}$$

- Poles of $X(\omega)$ where $f(\omega) = 0$.

- Look at $f(\omega) = \omega^2 - 2i\gamma\omega - \omega_0^2$

$$f(\omega) = 0 \text{ when } \omega = \frac{1}{2}\left(2i\gamma \pm \sqrt{-4\gamma^2 + 4\omega_0^2}\right)$$

Application: Solving an ODE

$$f(\omega) = 0 \text{ when } \omega = \frac{1}{2}\left(2i\gamma \pm \sqrt{-4\gamma^2 + 4\omega_0^2}\right)$$

- for <u>small</u> γ, expand to find

$$\omega = i\gamma \pm (-\gamma^2 + \omega_0^2)^{1/2} = i\gamma \pm \omega_0\left(1 - \frac{\gamma^2}{\omega_0^2}\right)^{1/2}$$

$$= i\gamma \pm \omega_0\left(1 - \frac{\gamma^2}{2\omega_0^2} + \cdots\right)$$

$$= \pm\omega_0 + i\gamma \quad \text{to first order}$$

$\Longrightarrow$ poles have moved <u>above</u> the real axis for $\gamma > 0$.

<u>No</u> poles on the integration path.

Application: Solving an ODE

- Write

$$f(\omega) = (\omega - \omega_1)(\omega - \omega_2),$$

where $\omega_1 = -\omega_0 + i\gamma,\ \omega_2 = \omega_0 + i\gamma$

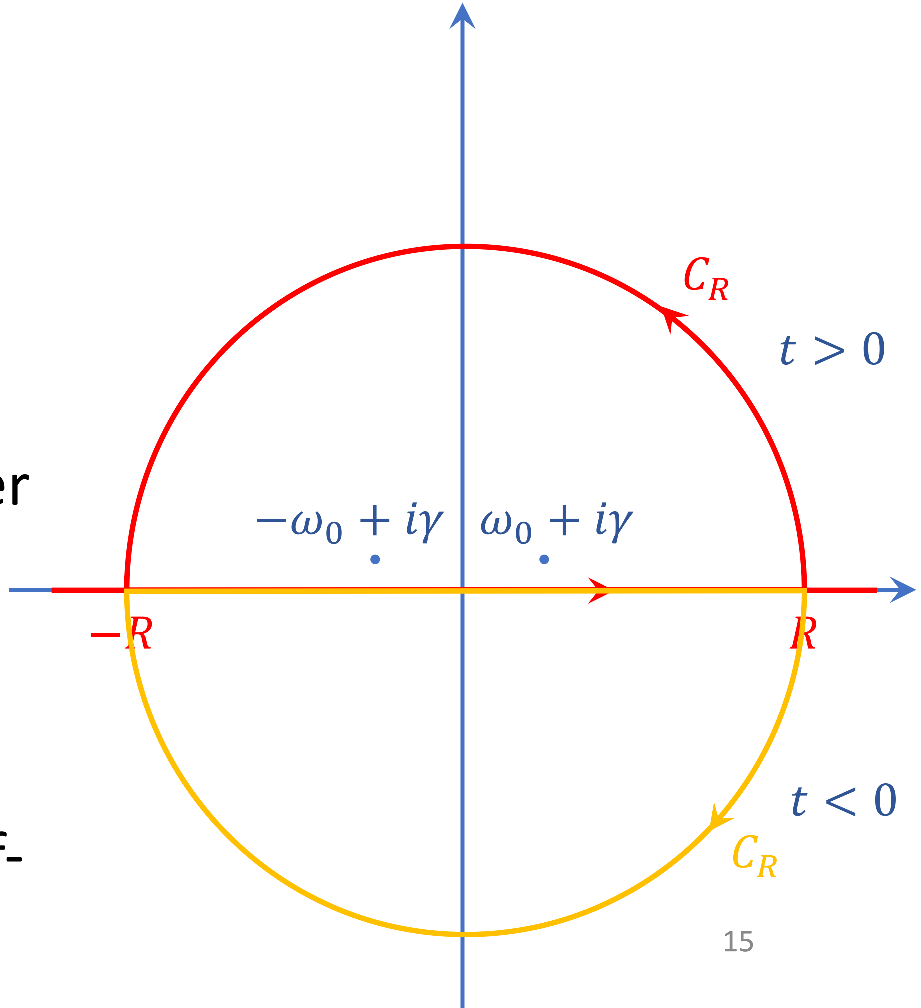
- Now the inversion integral is

$$x(t) = \frac{-1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega-\omega_1)(\omega-\omega_2)}$$

- For $t < 0$, must close the contour in the lower half-plane, no poles on the axis or inside the contour, so $x(t) = 0$.

Restored the causal solution!

- For $t > 0$, close the contour in the upper half-plane, pick up 2 poles.



Application: Solving an ODE

- For $t > 0$, close the contour in the upper half-plane, pick up 2 poles at $\omega = \omega_1, \omega_2$

$$x(t) = \frac{-1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega-\omega_1)(\omega-\omega_2)}$$

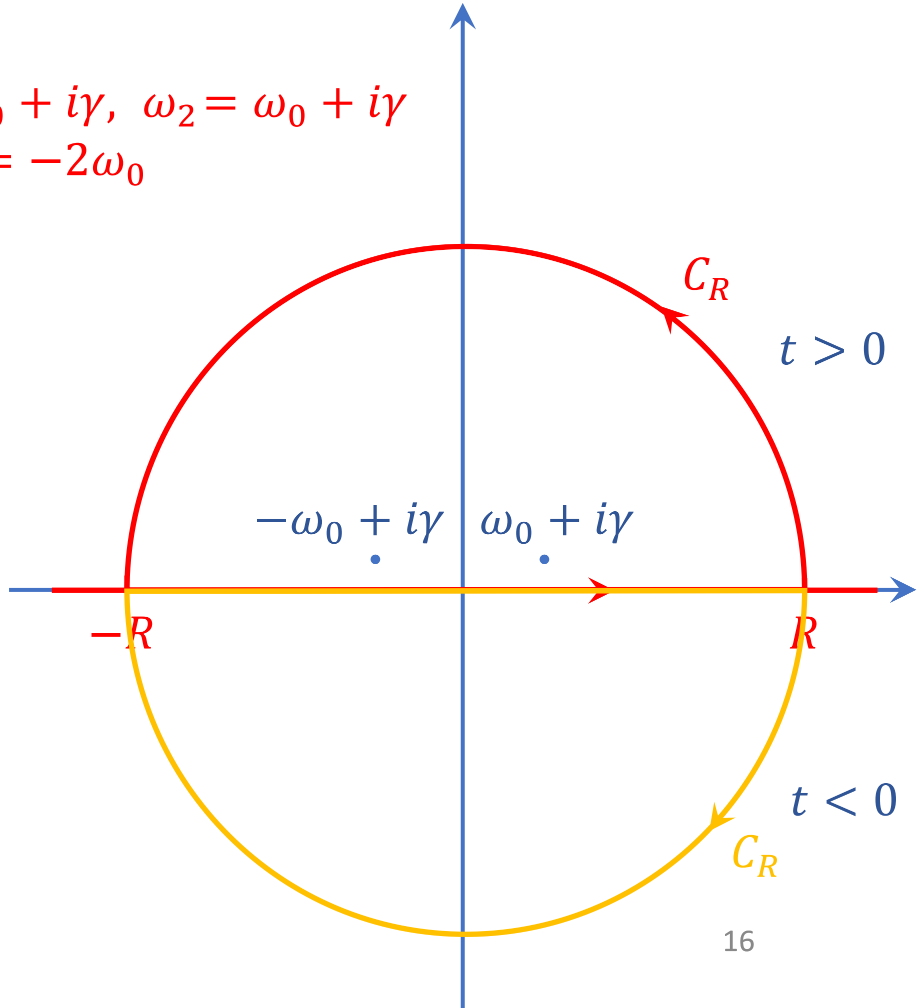
$\omega_1 = -\omega_0 + i\gamma,\ \ \omega_2 = \omega_0 + i\gamma$
$\omega_1 - \omega_2 = -2\omega_0$

- Residues are

$$r_1 = \frac{-1}{2\pi}\frac{e^{i\omega_1 t}}{\omega_1-\omega_2}, \qquad r_2 = \frac{-1}{2\pi}\frac{e^{i\omega_2 t}}{\omega_2-\omega_1}$$

so

$$\begin{aligned} x(t) &= 2\pi i(r_1 + r_2) \\ &= -i\left(\frac{e^{i\omega_1 t}}{\omega_1-\omega_2} + \frac{e^{i\omega_2 t}}{\omega_2-\omega_1}\right) \\ &= \frac{i}{2\omega_0} e^{-\gamma t}\left(e^{-i\omega_0 t} - e^{i\omega_0 t}\right) \\ &= \frac{1}{\omega_0} e^{-\gamma t} \sin \omega_0 t \end{aligned}$$



Application: Solving an ODE

- Original FT result with poles on axis:

$$x(t) = \frac{\sin \omega_0 |t|}{2\omega_0}$$

  – non-causal result.

- Introduction of damping restores causality:

$$x(t) = \begin{cases} 0, & t < 0 \\ e^{-\gamma t} \dfrac{\sin \omega_0 t}{\omega_0}, & t > 0 \end{cases}$$

- Now let $\gamma \rightarrow 0$; we still retain causal result.

  – alternate means of doing the FT integral and preserving causality.

- Difference between the two solutions is a solution to the homogeneous equation, $\dfrac{\sin \omega_0 t}{\omega_0}$.
- Can <u>always</u> add any multiple of this solution.

Application: Solving an ODE

- Bottom line:

  Even though we can easily handle poles on the axis mathematically, often they are there because of some unphysical assumption (e.g. no friction)

  Offsetting the poles from the axis and letting the offset go to zero is a common technique to force a causal result.

Application: Poisson's Equation

- Another inhomogeneous example: Poisson's equation

$$\nabla^2\phi = 4\pi G\rho$$

- Solve on infinite domain using Fourier transforms

$$\tilde{\phi}(\boldsymbol{k}) = (2\pi)^{-\frac{3}{2}} \int d^3x\, \phi(\boldsymbol{x}) e^{-i\boldsymbol{k}.\boldsymbol{x}}$$

$$\tilde{\rho}(\boldsymbol{k}) = (2\pi)^{-\frac{3}{2}} \int d^3x\, \rho(\boldsymbol{x}) e^{-i\boldsymbol{k}.\boldsymbol{x}}$$

- → this is Homework 6, problem 1(a)

x, k are 3-D vectors

Turns out to be a very efficient way of computing ϕ on a grid.

Application: Poisson's Equation

- Suppose ρ represents a point mass m at the origin

$$\rho(\boldsymbol{x}) = m\delta(\boldsymbol{x})$$

$\rightarrow$ Homework 6, problem 1(b)

.

.

.

$$\Longrightarrow \qquad \phi(\boldsymbol{x}) = \int d^3k\, f(k)\, e^{i\boldsymbol{k}.\boldsymbol{x}}$$

- Very common integral in 3-D — there's a "trick" for doing it.

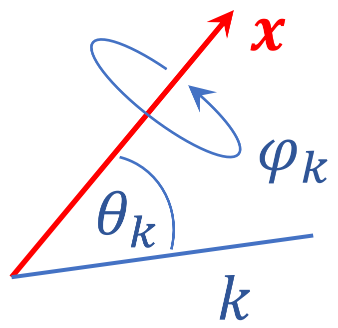
Application: Poisson's Equation

- Have

$$\phi(\boldsymbol{x}) = \int d^3k\, f(k)\, e^{i\boldsymbol{k}.\boldsymbol{x}}$$

- Note that $\boldsymbol{x}$ is just a parameter in the $\boldsymbol{k}$ integral, and we are free to choose the $\boldsymbol{k}$ coordinate system any way we like.
  - choose the z-axis in $\boldsymbol{k}$ space to be parallel to $\boldsymbol{x}$
  - define (k, θ_k, φ_k) spherical polar coordinates in $\boldsymbol{k}$ space
  - can write $d^3k = k^2 \sin\theta_k \; dk\, d\theta_k\, d\varphi_k$
  - useful because, by choice of coordinates,

$$\boldsymbol{k}.\boldsymbol{x} = kr\cos\theta_k \qquad\qquad (r = |\boldsymbol{x}|)$$



- Now

$$\phi(\boldsymbol{x}) = \iiint k^2 f(k)\, dk\, d\theta_k\, d\varphi_k \sin\theta_k\, e^{ikr\cos\theta_k}$$

Application: Poisson's Equation

- Solution for the point mass is

$$\phi(\boldsymbol{x}) = \frac{-Gm}{r}$$

- Note also, for general $\rho(\boldsymbol{x})$, the solution is the convolution of ρ and

$$g(\boldsymbol{x}) = \frac{-G}{|\boldsymbol{x}|}$$

- In 1-D, would write the convolution as

$$y(x) = \int_{-\infty}^{\infty} dx' \, \rho(x') \, g(x - x')$$

$g(x - x')$ dictates how the solution at x depends on the source at x'

- In 3-D, this becomes

$$\phi(\boldsymbol{x}) = -G \iiint d^3x' \, \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}$$

principle of superposition

Green's Functions

- Have seen that the solution to the linear differential equation

$$\mathcal{L}y = f$$

can be written as

$$y(x) = (f * g)(x) = \int_{-\infty}^{\infty} dx'\, f(x')\, g(x - x')$$

- Have only seen a proof using Fourier transforms on an infinite domain.
- Principle of superposition suggests it may be more general.
- Heuristically, if we can write

$$f(x) = \int_{-\infty}^{\infty} dx'\, f(x')\, \delta(x - x')$$

we might seek a solution of the form

$$y(x) = \int_{-\infty}^{\infty} dx'\, f(x')\, G(x, x')$$

field source redistribution

$G(x, x')$ dictates how the solution at x depends on the source at x'

Green's function

Green's Functions

- Suppose

$$y(x) = \int dx' \, f(x') \, G(x, x')$$

- What is the condition for $y(x)$ to satisfy the differential equation

$$\mathcal{L}y = f$$

- Substitute in:

$$\mathcal{L}_x \int dx' \, f(x') \, G(x, x') = \int dx' \, f(x') \, \delta(x - x')$$

$$\Longrightarrow \qquad \int dx' \, f(x') \, \mathcal{L}_x G(x, x') = \int dx' \, f(x') \, \delta(x - x')$$

- Want this to be true for <u>any</u> source f, so require

$$\mathcal{L}_x G(x, x') = \delta(x - x')$$

- ODE for the Green's function G, subject to boundary conditions on the original problem.

Green's Functions

- G is the solution to the "point-source" problem
  - property of $\mathcal{L}$ and the boundary conditions, <u>not</u> of f
- <u>Generalization</u> of the principle of superposition to any linear operator.
- How to determine in a specific case?
  1. Solve the ODE directly, with boundary conditions
  2. Eigenfunction expansion gives formal expression for G
  3. Fundamental solutions
     - applications in potential theory, Helmholtz
     - method of images
- Only going to scratch the surface — E&M!

Green’s Functions: Example 1

- Vibrating string

  ODE: $y'' + k^2 y = 0, \qquad k = \omega / c$

  BCS: $y(0) = y(L) = 0$

- Green’s function $G(x, x')$ satisfies

$$G'' + k^2 G = \delta(x - x')$$

where $'$ means differentiation with respect to x and

$$G(0, x') = G(L, x') = 0$$

- For $x \neq x'$,

$$G'' + k^2 G = 0$$

$$\Longrightarrow \quad G(x, x') = \begin{cases} A \sin kx\,, & x < x' \\ B \sin k(x - L)\,, & x > x' \end{cases}$$

Form of the solutions is a convenient way of enforcing the boundary conditions.

Green's Functions: Example 1

- What is the connection between solutions in $x < x'$ and $x > x'$?

  G'' is a δ-function, but only a δ-function

  G' has a discontinuity ("jump")

  G is continuous

- Determine the jump in G' by integrating the ODE across $x = x'$:

$$\int_{x'_-}^{x'_+} G'' + k^2 \int_{x'_-}^{x'_+} G = \int_{x'_-}^{x'_+} \delta(x - x')$$

$$\Longrightarrow \quad [G']_{x'_-}^{x'_+} + k^2 . 0 = 1$$

$$\Longrightarrow \quad \left[\frac{\partial G}{\partial x}\right]_{x'_-}^{x'_+} = 1$$

- Allows us to make the connection between A and B.

Green's Functions: Example 1

$$G(x,x') = \begin{cases} A\sin kx\,, & x < x' \\ B\sin k(x-L)\,, & x > x' \end{cases}$$

- Continuity in G at $x = x'$

  $\Longrightarrow \quad A\sin kx' = B\sin k(x'-L)$

- Jump in G'

  $\Longrightarrow \quad Ak\cos kx' + 1 = Bk\cos k(x'-L)$

- Can easily solve to find

$$A = \frac{\sin k(x'-L)}{k\sin kL}$$

$$B = \frac{\sin kx'}{k\sin kL}$$

$$\Longrightarrow \quad G(x,x') = \frac{1}{k\sin kL}\begin{cases} \sin kx\sin k(x'-L), & 0 < x < x' \\ \sin kx'\sin k(x-L)\,, & x' < x < L \end{cases}$$

- Note that $G(x,x') = G(x',x)^*$ Generic property of Green's functions

Green's Functions: Example 1

- Now, for any given source function f, have a complete (integral) solution to the problem

$$y(x) = \int_0^L dx'\, f(x')\, G(x,x')$$

$$= \frac{\sin k(x-L)}{k \sin kL} \int_0^x dx'\, f(x') \sin kx'$$

$$+ \frac{\sin kx}{k \sin kL} \int_x^L dx'\, f(x') \sin k(x'-L)$$

- Not particularly transparent, but the fact the we can write down a solution then allows us to go on to explore its properties.

Green's Functions: Example 2

- Now reconsider the same ODE, but now on an infinite time domain (same study as before, using Fourier transforms, $t' = 0$)

$$\ddot{G} + \omega_0^2 G = \delta(t)$$

– for $t < 0$, solution is

$$G = Ae^{i\omega_0 t} + Be^{-i\omega_0 t} = G_-(t)$$

– for $t > 0$, solution is

$$G = Ce^{i\omega_0 t} + De^{-i\omega_0 t} = G_+(t)$$

- G is continuous at $t = 0$

$$\Longrightarrow \qquad A + B = C + D$$

- $\dot{G}$ is discontinuous at $t = 0$, $\left[\dot{G}\right]_{0-}^{0+} = 1$

$$\Longrightarrow \qquad i\omega_0 A - i\omega_0 B + 1 = i\omega_0 C - i\omega_0 D \qquad \Longrightarrow A - B = C - D + \frac{1}{i\omega_0}$$

Green's Functions: Example 2

- Clearly can't determine all of A, B, C, D, but can write

$$C = A - \frac{i}{2\omega_0}, \quad D = B + \frac{i}{2\omega_0}$$

- So solution is indeterminate, but can say that

$$G_+(t) = G_-(t) - \frac{i}{2\omega_0} e^{i\omega_0 t} + \frac{i}{2\omega_0} e^{-i\omega_0 t}$$

$$= G_-(t) + \frac{\sin \omega_0 t}{\omega_0}$$

- <u>Same</u> jump as we found before using Fourier techniques.

$$A + B = C + D$$

$$A - B = C - D + \frac{1}{i\omega_0}$$

Green's Functions: Example 3

- Can do something similar in 2-D

  e.g. circular drum problem

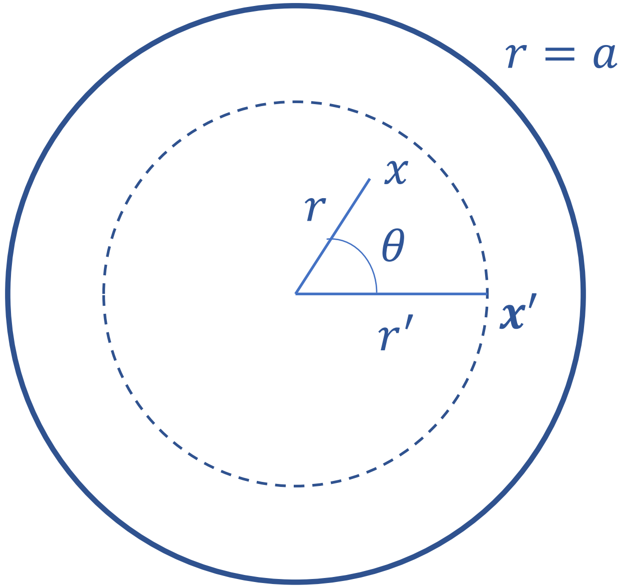
$$\nabla^2 u + k^2 u = 0, \qquad u(a,\theta) = 0$$

- Choose $\boldsymbol{x}'$ as the axis of polar coordinates for $\boldsymbol{x}$, and assume G is even in θ
- Then

$$\nabla^2 G + k^2 G = \delta(\boldsymbol{x} - \boldsymbol{x}')$$

- For $\boldsymbol{x} \neq \boldsymbol{x}'$,

$$\nabla^2 G + k^2 G = 0$$



$$\Longrightarrow \quad G = \begin{cases} \sum_m A_m J_m(kr) \cos m\theta\,, & r < r' \\ \sum_m B_m[(J_m(kr)Y_m(ka) - Y_m(kr)J_m(ka)] \cos m\theta\,, & r > r' \end{cases}$$

Combination chosen to set $G = 0$ on the outer boundary

Green's Functions: Example 3

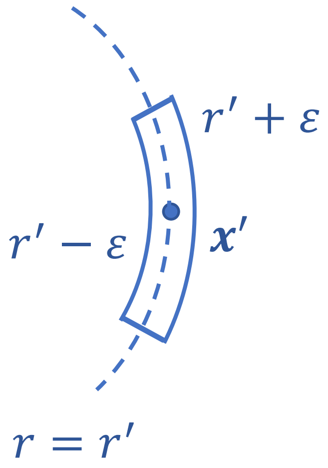
- Continuity at $r = r'$ applies term by term, so

$$A_m J_m(kr') = B_m[(J_m(kr')Y_m(ka) - Y_m(kr')J_M(ka)]$$

- Discontinuity in G' at $\boldsymbol{x} = \boldsymbol{x}'$

$$\int \nabla^2 G = 1$$

$$= \left[\frac{\partial G}{\partial r}\right]_{r'_-}^{r'_+}$$



Can show

$$\left[\frac{\partial G}{\partial r}\right]_{r'_-}^{r'_+} = \frac{1}{r'}\delta(\theta) = \frac{1}{\pi r'}\sum_m \frac{\cos m\theta}{\beta_m}$$

where $\beta_0 = 2, \beta_m = 1, m > 0$

$$\Longrightarrow \qquad A_m J'_m(kr') + \frac{1}{\pi \beta_m k r'} = B_m[(J'_m(kr')Y_m(ka) - Y'_m(kr')J_M(ka)]$$

$\Longrightarrow$ can solve for A_m, B_m