PHYS 501: Mathematical Physics I

Make-up Final Examination, December 30, 2020

Answer four parts of question 1 and two other questions.

Time allowed: 2 hours

1. (a) [*8 points*] Write down (do *not* derive!) the general solution of the three-dimensional Helmholtz equation

$$\nabla^2 u + k^2 u = 0$$

for the infinite domain (in spherical polar coordinates) $0 \le r < \infty$, with the boundary condition that u represents the spatial part of an outgoing spherical wave ($e^{-i\omega t}$ behavior) as $r \to \infty$.

(b) [*8 points*] What is a generating function? Give an example of a generating function, along with one example of its use in developing the properties of Bessel or Legendre functions.

(c) [*8 points*] By writing down the general solution of the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(\mathbf{x})\psi = E\psi,$$

find the ground-state energy of a particle confined to a half-cylinder (i.e. semi-circular cross section) of radius R and length L, with $V = 0$ inside the half-cylinder and $\psi = 0$ on the surface.

(d) [*8 points*] What is a Legendre series? Give an example of how one might be used to solve a three-dimensional electrostatics problem with a spherical boundary condition.

(e) [*8 points*] Explain, with an example, how a Fourier transform can be used to solve a linear ordinary differential equation. List two practical limitations of this approach.

(f) [*8 points*] Give an *operational* definition of a Green's function, in terms of the solution to the inhomogeneous partial differential equation $\mathcal{L}u = f(\mathbf{x})$ (where $\mathcal{L}$ is a linear differential operator). What differential equation does G satisfy?

(g) [*8 points*] Explain how to separate the Green's function G for the linear differential equation $\mathcal{L}u = f(\mathbf{x})$ into a fundamental solution u and another function v satisfying $\mathcal{L}v = 0$, and describe the roles of both u and v in determining G. Give an example of a fundamental solution in *three* dimensions.

2. (a) [*17 points*] Find the solution $\Phi(x, y)$ of Laplace's equation

$$\nabla^2\Phi = 0$$

in the square $0 < x < 1$, $0 < y < 1$, with boundary conditions

$$\begin{aligned} \Phi(x,0) = \Phi(x,1) = \Phi(0,y) &= 0\,, \\ \Phi(1,y) &= \sin 2\pi y\,. \end{aligned}$$

(b) [*17 points*] Repeat part (a) for the potential $\Phi(r, \theta)$, which is regular within a circle of radius 1 and satisfies

$$\Phi(1,\theta) = 2\cos^2\theta\,.$$

3. (a) [*5 points*] Write down the Taylor series for $f(z) = e^z$ about $z = 0$.

   (b) [*9 points*] Hence compute the residue at $z = 0$ of the function $g(z) = e^{tz}/z^n$, for any integer $n > 0$.

   (c) [*20 points*] Evaluate the integral

$$\lim_{h\to\infty} \frac{1}{2\pi i} \int_{\gamma - ih}^{\gamma + ih} \frac{e^{tz}}{z^4}\, dz,$$

   where $\gamma > 0$ and the path of integration runs parallel to the imaginary axis, for (i) $t < 0$ and (ii) $t > 0$. Sketch and justify the contours you use.

4. A field $u(x, t)$ satisfies the diffusion equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\kappa} \frac{\partial u}{\partial t},$$

   for $-\infty < x < \infty$ and $t \geq 0$, with $u(x, 0) = f(x)$. Let $U(k, t)$ be the Fourier transform of u.

   (a) [*4 points*] Write down the differential equation satisfied by $U(k, t)$.

   (b) [*10 points*] Solve this differential equation for the special case $f(x) = \delta(x)$, and hence determine the solution $u(x, t)$ in that case.

   (c) [*10 points*] Write down an integral expression for the general solution $u(x, t)$ for arbitrary $f(x)$.

   (d) [*10 points*] Evaluate this integral for $f(x) = e^{-ax^2}$.

5. A function $u(\mathbf{x})$ satisfies Laplace's equation in the half space $z > 0$, with the boundary condition $u(x, y, 0) = f(x, y)$, where $\mathbf{x} = (x, y, z)$.

   (a) [*10 points*] Using the method of images, show that the Green's function for the problem is

$$G(\mathbf{x}, \mathbf{x}') = \frac{-1}{4\pi|\mathbf{x} - \mathbf{x}'|} + \frac{1}{4\pi|\mathbf{x} - \mathbf{x}_1'|},$$

   where $\mathbf{x}_1' = (x', y', -z')$.

   (b) [*14 points*] Write down an integral expression for $u(\mathbf{x})$ in terms of G, and hence show that

$$u(x, y, z) = -\frac{z}{2\pi} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy'\, \frac{f(x', y')}{s^3},$$

   where $s^2 = (x - x')^2 + (y - y')^2 + z^2$.

   (b) [*10 points*] Verify the above expression by solving the problem for the case $f(x, y) = A$, where A is constant.