Extension to particular non-holonomic systems

$$f_\alpha(q_1 \dots q_n, \dot{q}_1 \dots \dot{q}_n, t) = 0$$

semi-holonomic constraints $\alpha = 1 \dots m$

$$\sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha = 0$$

Lagrange multipliers λ_α

$$\delta \int_{t_1}^{t_2} L\,dt = 0$$

Hamilton's principle still applies

$$\delta \int_{t_1}^{t_2} \left(L + \sum_{\alpha=1}^{m} \lambda_\alpha f_\alpha\right) dt = 0$$

'm+n' eqns to solve for the
'n' qj and the 'm' lambda_alphas

for simplicity $\lambda_\alpha = \lambda_\alpha(t)$

$$\frac{\partial f'}{\partial q_j} - \frac{d}{dt}\frac{\partial f'}{\partial \dot{q}_j} = 0 \qquad f' = L + \sum \lambda_\alpha f_\alpha$$

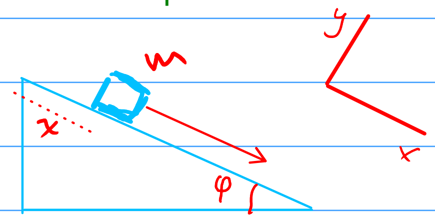
$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j}$$

$$\boxed{Q_j = \sum_{\alpha=1}^{m} \left\{ \lambda_\alpha \left[\frac{\partial f_\alpha}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial f_\alpha}{\partial \dot{q}_j}\right)\right] - \frac{d\lambda_\alpha}{dt}\frac{\partial f_\alpha}{\partial \dot{q}_j} \right\}}$$

this is also good to find the generalized forces in holonomic problems

$$\boxed{Q_j = \sum_{\alpha=1}^{m} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_j}}$$

Trivial example: mass down frictionless incline



Generalized coordinates x, y

Constraint $f(x,y) = y = 0$

GOALS:
1. eqns of motion
2. forces of constraint

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\dot{y}^2$$

$$V = -mgx\sin\varphi - mgy\cos\varphi$$

$$L = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\dot{y}^2 + mgx\sin\varphi + mgy\cos\varphi$$

Lagrange's eqns

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j$$

$$x: \quad \frac{d}{dt} m\dot{x} - mg\sin\varphi = Q_x = \lambda \frac{\partial f}{\partial x} = 0$$

$$\boxed{m\ddot{x} - mg\sin\varphi = 0} \quad (1)$$

$$y: \quad \frac{d}{dt} m\dot{y} - mg\cos\varphi = Q_y = \lambda \frac{\partial f}{\partial y} = \lambda$$

$$\boxed{m\ddot{y} - mg\cos\varphi = \lambda} \quad (2)$$

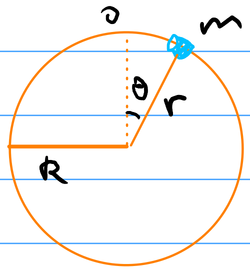
constraint $\quad y = 0 \Longrightarrow \boxed{\ddot{y} = 0}$ (3)

three eqns for three unknowns x, y, λ

(1) $\boxed{\ddot{x} = g\sin\varphi}$

(3) into (2) $\boxed{\lambda = mg\cos\varphi}$

2. Example: bead constrained to a circular wire - no friction



Initial conditions: start at the top from rest

generalized coordinates: r, θ

constraints: $r = R$, $f(r,\theta) = r - R = 0$

GOALS:
1. eqns of motion
2. normal force

$$T = \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2)$$

$$V = -mg(R - r\cos\theta)$$

$$L = \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2) + mg(R - r\cos\theta)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_j} - \frac{\partial L}{\partial q_j} = Q_j$$

r: $\quad m\ddot r - mr\dot\theta^2 + mg\cos\theta = Q_r = \lambda\frac{\partial f}{\partial r} = \lambda$

θ: $\quad mr^2\ddot\theta - mgr\sin\theta = Q_\theta = \lambda\frac{\partial f}{\partial \theta} = 0$

$$\boxed{\begin{aligned} m\ddot r - mr\dot\theta^2 + mg\cos\theta &= \lambda \\ mr^2\ddot\theta - mgr\sin\theta &= 0 \\ r - R &= 0 \end{aligned}}$$

(1)
(2)
(3)

using $r = R \Rightarrow \dot r = 0$

into (1) $\quad -r\dot\theta^2 + g\cos\theta = \frac{\lambda}{m} \Rightarrow \frac{\lambda}{m} = -R\dot\theta^2 + g\cos\theta$ (4)

into (2) $\quad \ddot\theta = \frac{g}{R}\sin\theta$ (5)

$$\ddot\theta = \frac{d\dot\theta}{d\theta}\frac{d\theta}{dt} = \dot\theta\frac{d\dot\theta}{d\theta} \qquad // \quad a = v\frac{dv}{dx}$$

$\dot\theta(\theta=0)=0$

$c = g/R$

into (5) $\quad \dot\theta \, d\dot\theta = \frac{g}{R} \sin\theta \, d\theta$

$$\frac{\dot\theta^2}{2} = -\frac{g}{R}\cos\theta + c$$

$$\dot\theta^2 = \frac{2g}{R}(1-\cos\theta)$$

into (4)

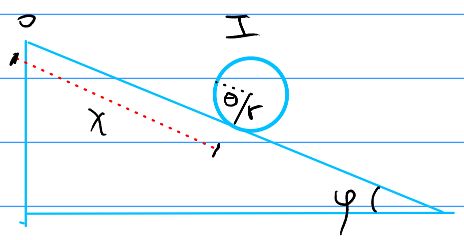
$$\frac{\lambda}{m} = -2g(1-\cos\theta) + g\cos\theta$$

$$\boxed{\lambda = mg\,(3\cos\theta - 2)}$$ normal force

note: $\quad \theta = 0 \qquad \lambda = mg$

$\qquad \theta = \pi \qquad \lambda = -5mg$

Another simple example: body rolling down an incline plane - w/ no slipping



generalized coordinates x, y, θ

constraints $x = r\theta \qquad y = 0$

$$f_1(x,y,\theta) = x - r\theta = 0$$

$$f_2(x,y,\theta) = y = 0$$

GOALS:
1. eqns of motion
2. frictional force
3. max phi such that there is no slipping

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2$$

$$V = -mgx\sin\varphi - mgy\cos\varphi$$

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2 + mgx\sin\varphi + mgy\cos\varphi$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j$$

x: $$m\ddot{x} - mg\sin\varphi = Q_x = \lambda_1\frac{\partial f_1}{\partial x} + \lambda_2\frac{\partial f_2}{\partial x} = \lambda_1$$

$$\boxed{m\ddot{x} - mg\sin\varphi - \lambda_1 = 0} \quad (1)$$

y: $$m\ddot{y} - mg\cos\varphi = Q_y = \lambda_1\frac{\partial f_1}{\partial y} + \lambda_2\frac{\partial f_2}{\partial y} = \lambda_2$$

$$\boxed{m\ddot{y} - mg\cos\varphi - \lambda_2 = 0} \quad (2)$$

θ: $$I\ddot{\theta} = \lambda_1\frac{\partial f_1}{\partial \theta} + \lambda_2\frac{\partial f_2}{\partial \theta} = -r\lambda_1 = Q_\theta$$

$$\boxed{I\ddot{\theta} + r\lambda_1 = 0}$$

$$\boxed{\begin{aligned} x - r\theta &= 0 \\ y &= 0 \end{aligned}}$$ (4) (5)

five unknowns

$x, y, \theta, \lambda_1, \lambda_2$

from (4) $\ddot{x} = r\ddot{\theta}$

into (3) $I\dfrac{\ddot{x}}{r} = -\lambda_1 r \Rightarrow \boxed{\lambda_1 = -\dfrac{I}{r^2}\ddot{x}}$

into (1) $m\ddot{x} - mg\sin\varphi + \dfrac{I}{r^2}\ddot{x} = 0$

$$\Rightarrow \boxed{\ddot{x} = \frac{g\sin\varphi}{1 + I/mr^2}}$$

into (3)

$$I\ddot{\theta} - \frac{I}{r^2}\cdot\frac{g\sin\varphi}{1 + I/mr^2} = 0$$

$$\boxed{\ddot{\theta} = \frac{1}{r}\cdot\frac{g\sin\varphi}{1 + I/mr^2}}$$

from (5) into (2) $y = 0 \Rightarrow \ddot{y} = 0$

$$\boxed{\lambda_2 = -mg\cos\varphi}$$

force that makes the body roll

$$Q_\theta = -\lambda_1 r = r\cdot\frac{I}{r^2}\cdot\frac{g\sin\varphi}{1 + I/mr^2}$$

$$= r F_f$$

$$\boxed{F_f = \frac{I}{r^2}\cdot\frac{g\sin\varphi}{1 + I/mr^2}}$$

$$F_f = \mu N = \mu m g \cos\varphi$$

critial angle comes from equating both frictional forces

$$\mu m g \cos\varphi = \frac{I}{r^2} \cdot \frac{g \sin\varphi}{1 + I/mr^2}$$

$$\boxed{\tan\varphi = \mu\left(\frac{mr^2}{I} + 1\right)}$$

Conservation laws and symmetry

Most systems for which eqns of motion are obtained are not really integrable (analytically).

Nonetheless, a great deal can be learned from looking at the symmetry of the problem.

Key points: look at the first integrals (first order derivatives)

Definition 1: suppose 'V' only depends on 'x' in 1D motion

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} - \frac{\partial V}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m\dot{x} = p_x$$

in general $$p_j = \frac{\partial L}{\partial \dot{q}_j}$$ called "canonical" or "conjugate" momentum

Definiton 2: suppose 'L' does not contain a given generalized coordinate 'qj'.
Then this coordinate is called a "cyclic" or "ignorable" coordinate.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \underbrace{\frac{\partial L}{\partial q_j}}_{=0} = 0 \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0$$

$$\dot{p}_j = \frac{d}{dt} p_j = 0$$

$$\boxed{p_j = const}$$

Generalized momentum conjugate to a cyclic coordinate is conserved