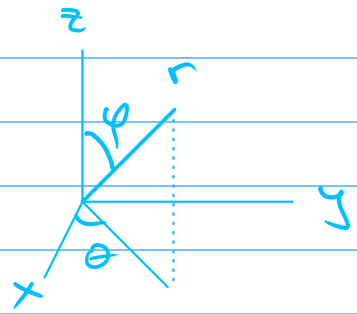
$$L = L_0 + \dot{q}^T a + \underbrace{\frac{1}{2}\dot{q}^T T \dot{q}}_{T_2}$$

$$p = \frac{\partial L}{\partial \dot{q}} = T\dot{q} + a$$

$$H = \frac{1}{2}(p^T - a^T)\,T^{-1}(p - a) - L_0 \qquad (4)$$

Example 1. Particle under a central force

1. spherical coordinates



$$\bar{v} = \dot{r}\hat{r} + r\dot{\theta}\sin\varphi\,\hat{\theta} + r\dot{\varphi}\hat{\varphi}$$

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2\sin^2\varphi + r^2\dot{\varphi}^2$$

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2\sin^2\varphi + r^2\dot{\varphi}^2)$$

$$\dot{q} = \begin{pmatrix}\dot{r}\\ \dot{\theta}\\ \dot{\varphi}\end{pmatrix}$$

$$T = \frac{m}{2}(\dot{r}, \dot{\theta}, \dot{\varphi})\underbrace{\begin{pmatrix}1 & 0 & 0\\ 0 & r^2\sin^2\varphi & 0\\ 0 & 0 & r^2\end{pmatrix}}_{[T]}\begin{pmatrix}\dot{r}\\ \dot{\theta}\\ \dot{\varphi}\end{pmatrix}$$

$$T^{-1} = \frac{1}{m}\begin{pmatrix}1 & & 0\\ & r^{-2}\sin^{-2}\varphi & \\ 0 & & r^{-2}\end{pmatrix}$$

Hamiltonian $H = \frac{1}{2}(p^T - a^T)\,T^{-1}(p - a) - L_0$

$$p = \begin{pmatrix}p_r\\ p_\theta\\ p_\varphi\end{pmatrix} \qquad a = 0 \qquad L_0 = -V$$

$$p = T\dot{q}$$

$$H = \frac{1}{2m}(P_r\ P_\theta\ P_\varphi)\left(\quad T^{-1} \quad \right)\begin{pmatrix} P_r \\ P_\theta \\ P_\varphi \end{pmatrix} + V$$

$$\boxed{H = \frac{1}{2m}\left(P_r^2 + \frac{P_\theta^2}{r^2 \sin^2\varphi} + \frac{P_\varphi^2}{r^2} \right) + V}$$

2. Cartesian coordinates

$$T = \frac{m}{2}(\dot{x}\ \dot{y}\ \dot{z}) \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$

$$H = \frac{1}{2m}(P_x\ P_y\ P_z) \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} + V$$

$$\boxed{H = \frac{1}{2m}(P_x^2 + P_y^2 + P_z^2) + V}$$

could have also used
H = T + V

Example 2. Charge in an EM field.

$$L = T - V = \underbrace{\frac{1}{2}mv^2}_{L_2} \underbrace{- q\varphi}_{L_0} + \underbrace{q\bar{A}\cdot\bar{v}}_{L_1}$$

now H=T+V does not work b/c 'L_o' does not have the entirety of 'V'.
'L_1' has part of the potential.
'H' is still the total energy b/c 'phi' constains *all* of the "potential" energy.

Using Cartesian coordinates

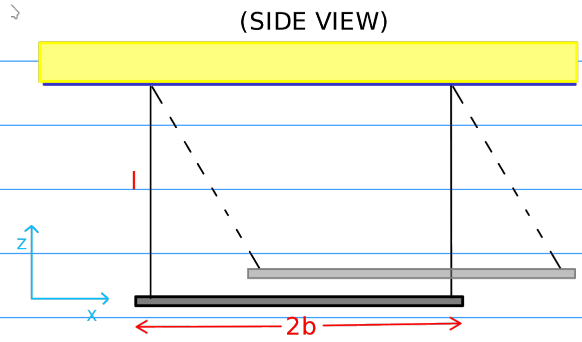
$$L = \frac{1}{2}m\dot{x}_i\dot{x}_i - q\varphi + qA_i\dot{x}_i$$

$$P = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \qquad a = \begin{pmatrix} qA_x \\ qA_y \\ qA_z \end{pmatrix} \qquad L_0 = -q\varphi$$

$$H = \frac{1}{2m}(p_i - qA_i)(p_i - qA_i) + q\varphi$$

in vector form

$$\boxed{H = \frac{1}{2m}\left(\bar{p} - q\bar{A}\right)^2 + q\varphi}$$



$$L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \frac{1}{3}b^2\dot{\alpha}^2\right) \underbrace{- \frac{mg}{2l}\left(x^2 + y^2 + b^2\alpha^2\right)}_{L_0}$$

KE matrix

$$T = m\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & b^2/3 \end{pmatrix} \qquad T^{-1} = \frac{1}{m}\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 3/b^2 \end{pmatrix}$$

$$H = \frac{1}{2} P^T T^{-1} P - L_0$$

$$P = \begin{pmatrix} P_x \\ P_y \\ P_\alpha \end{pmatrix}$$

$$= \frac{1}{2} (P_x \, P_y \, P_\alpha) \frac{1}{m} \begin{pmatrix} & & \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_\alpha \end{pmatrix} - L_0$$

$$\boxed{H = \frac{1}{2m}\left(P_x^2 + P_y^2 + P_\alpha^2 \frac{3}{b^2}\right) + \frac{mg}{2\ell}\left(x^2 + y^2 + b^2\alpha^2\right)}$$

b/c $q_i \neq q_i(t)$ & $V \neq V(\dot{q}_i)$

H = tot energy

also $V \neq V(t) \Rightarrow E$ is conserved

Eqns of motion

$$\dot{x} = \frac{\partial H}{\partial P_x} = \frac{P_x}{m} \qquad \dot{y} = \frac{\partial H}{\partial P_y} = \frac{P_y}{m} \qquad \dot{\alpha} = \frac{\partial H}{\partial P_\alpha} = \frac{P_\alpha}{m} \cdot \frac{3}{b^2}$$

$$\dot{P}_x = -\frac{\partial H}{\partial x} = -\frac{mg}{\ell} x \qquad \dot{P}_y = -\frac{\partial H}{\partial y} = -\frac{mg}{\ell} y \qquad \dot{P}_\alpha = -\frac{\partial H}{\partial \alpha} = -\frac{mg}{\ell} b^2 \alpha$$

$$m\ddot{x} = -\frac{mg}{\ell} x \qquad m\ddot{y} = -\frac{mg}{\ell} y \qquad \ddot{\alpha} = -\frac{3g}{\ell}\alpha$$

$$\omega_0 = \sqrt{g/\ell} \qquad \omega_0 = \sqrt{g/\ell} \qquad \omega_1 = \sqrt{3}\,\omega_0$$

The generalized momenta

$$P_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad P_y = m\dot{y} \qquad P_\alpha = m\frac{b^2}{3}\dot{\alpha} = I\dot{\alpha}$$

Conservation theorems

The conservation theorems and considerations that we saw regarding the Lagrangian are also applicable to the Hamiltonian.

- cyclic coordinates in 'L' will also be absent in 'H'
- conjugate momentum will be conserved

Symmetry properties and conservation are also preserved.

- e.g. translational or rotational symmetry will lead to conservation of the corresponding conjugate momenta.

Possibility of 'H' as a constant of motion also comes from the same reasons:

$$\frac{dH}{dt} = \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t}$$

we already know from Hamilton's equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

cancelling those two, resulting in

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

from (3)

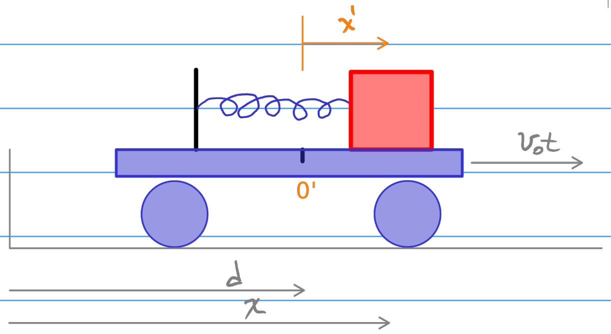
reasserts conservation *if* 'H' and/or 'L' are not explicitly dependent on 't'

Same as for 'h' above, identification of 'H' as a constant of motion and as the total energy are two separate issues.
=> the conditions for one do not determine the other.

Is is also possible that under a set of generalized coordinates 'H' is conserved, but under another it is not.

Example. Massless cart moves with constant 'v_o' with a block 'm' attached on top, and constrained to move under a spring 'k'.

Notice that as 'm' oscillates, the cart's CM does *not*. The cart only moves to the front uniformly.



1. Relative to ground:

$d = v_o t$

$$L(x, \dot{x}, t) = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k (x - d)^2$$

$$= \underbrace{\frac{1}{2} m \dot{x}^2}_{L_2} \underbrace{- \frac{1}{2} k (x - v_o t)^2}_{L_0}$$

$L_1 = 0$

$a = 0$

$T = m$

$T^{-1} = 1/m$

$$H(x, p, t) = \frac{1}{2} p^T T^{-1} p - L_0$$

$$\boxed{H = \frac{p^2}{2m} + \frac{1}{2} k (x - v_o t)^2}$$

b/c $x \neq x(t)$ & $V \neq V(\dot{x})$ $\Rightarrow$ 'H' = total energy

But b/c $H = H(t)$ energy is not conserved!

Energy fluctuates to make the cart go at constant 'v_o'.

2. Now change coordinates $x' = x - V_o t$ $x = x' + V_o t$

$$L(x', \dot{x}', t) = \frac{1}{2} m \left[\frac{d}{dt}(x' + v_0 t)\right]^2 - \frac{1}{2} k x'^2$$

$$= \frac{1}{2} m (\dot{x}'^2 + V_0^2 + 2\dot{x}' V_0) - \frac{1}{2} k x'^2$$

$$= \underbrace{\frac{1}{2} m \dot{x}'^2}_{L_2} + \underbrace{m V_0 \dot{x}'}_{L_1} \underbrace{- \frac{1}{2} k x'^2 + \frac{1}{2} m V_0^2}_{L_0}$$

$a = m v_0$

$$H(x', p', t) = \frac{1}{2} (p' - a) T^{-1} (p' - a) - L_0$$

$$\boxed{H(x' p' t) = \frac{1}{2m} (p' - m v_0)^2 + \frac{1}{2} k x'^2 - \frac{1}{2} m v_0^2}$$

b/c $x' = x'(t) \Rightarrow$ 'H' is *not* the total energy

but b/c $H \neq H(t)$ it is conserved

$$\frac{dH}{dt} = \frac{\partial H}{\partial p'} \dot{p}' + \frac{\partial H}{\partial x'} \dot{x}' + \cancel{\frac{\partial H}{\partial t}}$$

$* \ \frac{\partial H}{\partial p'} = \frac{1}{m} (p' - m v_0)$ $\qquad * \ \frac{\partial H}{\partial x'} = k x'$

$* \ p' = m \dot{x}' + m v_0 \longrightarrow \dot{x}' = \frac{1}{m} (p' - m V_0)$

$$\frac{dH}{dt} = \frac{1}{m} (p' - m v_0) \dot{p}' + k x' \dot{x}'$$

$$= \frac{1}{m}(p' - mv_0)\,\dot{p}' + kx'\left(\frac{1}{m}\right)(p' - mv_0)$$

$$* \quad \dot{p}' = \frac{-\partial H}{\partial x'} = -kx'$$

$$\frac{dH}{dt} = \frac{1}{m}(p' - mv_0)(-kx') + \frac{kx'}{m}(p' - mv_0)$$

$$= 0$$

3. Equations of motion

$\underline{x}$ $\qquad\qquad\qquad\qquad$ $\underline{x'}$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad\qquad \dot{x}' = \frac{\partial H}{\partial p'} = \frac{1}{m}(p' - mv_0)$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -k(x - v_0 t) \qquad\qquad \dot{p}' = \frac{-\partial H}{\partial x'} = -kx'$$

$$m\ddot{x}' = -kx'$$

$$m\ddot{x} = -k(x - v_0 t)$$

$$\left\{\begin{aligned} x' &= x - v_0 t \\ \dot{x}' &= \dot{x} \\ \ddot{x}' &= \ddot{x} \end{aligned}\right.$$

$$m\ddot{x} = -k(x - v_0 t)$$