PHYS 501: Mathematical Physics I

Fall 2020, Homework #2

(Due October 7, 2020)

1. (a) On performing separation of variables of Laplace's equation

$$\nabla^2 u = 0$$

in plane polar coordinates, with

$$u(r, \theta) = R(r)\Theta(\theta),$$

show that the radial function $R(r)$ corresponding to angular dependence $\Theta(\theta) = e^{im\theta}$ satisfies the ODE

$$r^2 R'' + rR' - m^2 R = 0,$$

and that this equation has solutions $R = r^{\pm m}$.

(b) Hence write down the general solution to Laplace's equation in polar coordinates.

(c) Find the solution $u(r, \theta)$ of Laplace's equation inside a circle of radius a, where u is regular inside the circle and satisfies the boundary conditions

$$u(a, \theta) = U \cos^2 \theta \, .$$

2. Find the three lowest-frequency modes of oscillation of acoustic waves in a hollow sphere of radius R. Assume a boundary condition $\partial u/\partial r = 0$ at $r = R$, where u obeys the differential equation

$$\nabla^2 u = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \, .$$

3. (a) A particle of mass m is contained in a cylinder of radius R and height H. The particle is described by a wavefunction $\psi(\rho, \phi, z)$ satisfying

$$-\frac{\hbar^2}{2m} \nabla^2 \psi = E\psi,$$

where ρ, ϕ, z are cylindrical polar coordinates and $\psi = 0$ on the surface of the cylinder $(\rho = R, z = 0, H)$. Find the ground-state energy of the system, and write down an explicit expression for the (unnormalized) lowest-energy wavefunction.

(b) Repeat part (a), but now for a particle moving in *two* dimensions, within a semicircular region of radius R, again with $\psi = 0$ on the boundary of the region.

4. The neutron density n inside a spherical sample of fissionable material obeys the equation

$$\nabla^2 n + \lambda n = \frac{1}{\kappa} \frac{\partial n}{\partial t} \, ,$$

where $\lambda > 0$, $\kappa > 0$, and $n = 0$ on the surface of the sample.

(a) Suppose the sample is spherical, of radius R. By seeking spherically symmetric modes with time dependence $e^{\alpha t}$, find the critical radius R_0 such that n is unstable and *increases* exponentially with time for $R > R_0$.

(b) Now suppose the sample is a hemisphere, again of radius R. Repeat part (a), for axially symmetric modes.

(c) Two hemispheres of the material, each just barely stable as in part (b), are brought together to form a sphere. This sphere is unstable, with

$$n \sim e^{t/\tau} .$$

Find the time constant τ of the resulting explosion.