# PHYS 501: Mathematical Physics I

*Fall 2020*

Homework #1
(Due: September 30, 2020)

1. The Kortweg-de Vries (KdV) equation

$$\frac{\partial^3 \psi}{\partial x^3} + 6\psi \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial t} = 0$$

is a well-known two-dimensional nonlinear partial differential equation. In general, the solutions are *dispersive*, meaning that, in a solution of the form $e^{ikx-i\omega t}$, the frequency $\omega$ depends on the wavenumber $k$. However, non-dispersive *soliton* solutions also exist.

(a) Let $\xi = x - ct$ and seek a traveling wave solution $\psi(\xi)$. Rewrite the above equation in terms of $\xi$ and show that it implies

$$(6\psi - c)\frac{d\psi}{d\xi} + \frac{d^3\psi}{d\xi^3} = 0,$$

and integrate to find

$$\frac{d^2\psi}{d\xi^2} = c\psi - 3\psi^2.$$

(b) Hence show that

$$\left(\frac{d\psi}{d\xi}\right)^2 = c\psi^2 - 2\psi^3,$$

and solve this equation to find the solution $\psi(\xi)$. Show your work—that is, don't just dump the equation into Wolfram Alpha for solution!

2. A second-order linear partial differential equation in two dimensions $(x, y)$ has the form

$$A(x,y)\frac{\partial^2\psi}{\partial x^2} + 2B(x,y)\frac{\partial^2\psi}{\partial x \partial y} + C(x,y)\frac{\partial^2\psi}{\partial y^2} = 0.$$

As discussed in class, the characteristic equation for this system is

$$A\left(\frac{dy}{dx}\right)^2 - 2B\frac{dy}{dx} + C = 0.$$

Assume that the system is hyperbolic, and denote the two solution families (corresponding to the two roots of the above quadratic equation) of this ODE by

$$\begin{aligned} \xi(x,y) &= \text{constant}, \\ \eta(x,y) &= \text{constant}. \end{aligned}$$

Transform the PDE from the $(x, y)$ to the $(\xi, \eta)$ coordinate system, and show that it takes the form

$$\frac{\partial^2 \psi}{\partial \xi \partial \eta} = \cdots$$

where the (very ugly) right-hand side depends only on known functions and first derivatives of $\psi$.

3. (a) Write down and solve the characteristic equation for the wave equation

$$\frac{\partial^2 \psi}{\partial t^2} \; - \; c(x)^2 \, \frac{\partial^2 \psi}{\partial x^2} = 0.$$

where the signal speed is $c(x) = c_0 \, (1 + |x|/a)^{-1}$. Sketch some representative characteristic curves.

(b) For the case $c =$ constant $(a \to \infty)$ find, using the method of characteristics, the solution to the equation satisfying the initial conditions

$$\psi(x, 0) = 0, \qquad \left.\frac{\partial \psi}{\partial t}\right|_{t=0} = e^{-|x|},$$

for $x > 0, t > 0$.

4. A uniform cube of side $L$ initially is at temperature $T = 0$. At time $t = 0$ the cube is immersed in a heat bath of temperature $T_0 > 0$. The temperature within the cube obeys the diffusion equation

$$\nabla^2 T = \frac{1}{\kappa} \frac{\partial T}{\partial t} \, .$$

Write down a general expression for the temperature in the cube, fit it to the boundary and initial conditions, and hence derive a formula (in the form of an infinite sum) for the temperature at any point within the cube at any subsequent time.

(Hint: It is easiest to work with the variable $T' = T - T_0$, which satisfies the same differential equation with the boundary condition $T' = 0$ on the surface of the cube.)