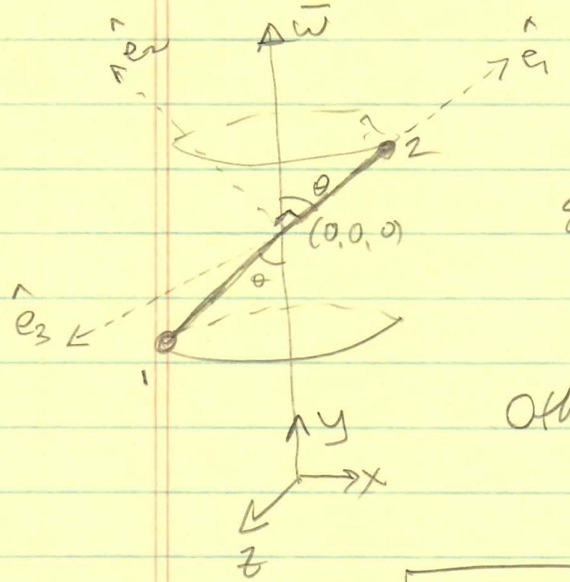
(5.18) ⊛ 1.(a)



Use the body set axes $\hat{e}_i$

$\hat{e}_1$ is along the bar

$\hat{e}_3$ only has components in x–z plane

$\omega_3 = 0$

Euler equations: $\begin{cases} I_1\dot{\omega}_1 = N_1 \\ I_2\dot{\omega}_2 = N_2 \\ I_3\dot{\omega}_3 + \omega_1\omega_2 I_2 = N_3 \end{cases}$

Other facts: $I_1 = 0$, $I_2 = I_3 = m\,l^2/2$

$\omega_1 = \omega\cos\theta$, $\omega_2 = \omega\sin\theta$

$\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$

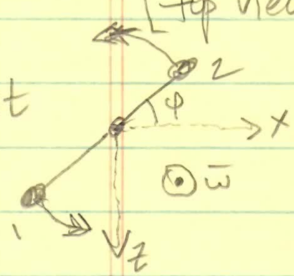
$\Rightarrow \boxed{N_1 = N_2 = 0}$, $N_3 = \underbrace{I_3\dot{\omega}_3}_{0} + \omega^2\sin\theta\cos\theta\, m\,l^2/2$

$\boxed{N_3 = m\frac{l^2}{2}\omega^2\sin\theta\cos\theta}$

⊛ (b) By construction $\bar{N}_3 \perp \bar{\omega}$
and $\bar{N}_3 \perp \hat{e}_1$ (bar)

⊛ 2.(a) First find position vectors for 1 and 2

[top view]



$\varphi = \omega t$

$\bar{r}_1 = \left(-\frac{l}{2}\sin\theta\cos\varphi,\ -\frac{l}{2}\cos\theta,\ \frac{l}{2}\sin\theta\sin\varphi\right)$

$\bar{r}_2 = \left(\frac{l}{2}\sin\theta\cos\varphi,\ \frac{l}{2}\cos\theta,\ -\frac{l}{2}\sin\theta\sin\varphi\right)$

Velocities:

$\bar{v}_1 = \frac{d\bar{r}_1}{dt} = \left(\frac{l}{2}\omega\sin\theta\sin\omega t,\ 0,\ \frac{l}{2}\omega\sin\theta\cos\omega t\right)$

$\bar{v}_2 = \frac{d\bar{r}_2}{dt} = \left(-\frac{l}{2}\omega\sin\theta\sin\omega t,\ 0,\ -\frac{l}{2}\omega\sin\theta\cos\omega t\right)$

Angular momenta:

$\bar{L}_1 = \bar{r}_1 \times m\bar{v}_1 = \Big(-\frac{l^2}{4}m\omega\sin\theta\cos\theta\cos\omega t,\ \frac{l^2}{4}m\omega\sin^2\theta\cos^2\omega t + \frac{l^2}{4}m\omega\sin^2\theta\sin^2\omega t,\ \frac{l^2}{4}m\omega\sin\theta\cos\theta\sin\omega t\Big)$

$\bar{L}_1 = \frac{l^2}{4}m\omega\sin\theta\left(-\cos\theta\cos\omega t,\ \sin\theta,\ \cos\theta\sin\omega t\right)$

$$\bar{L}_2 = \bar{r}_2 \times m\bar{v}_2 = \Big(-\frac{\ell^2}{2} m\omega \sin\theta\cos\theta\cos\omega t,\ \frac{\ell^2}{4} m\omega \sin^2\theta\cos^2\omega t$$
$$+ \frac{\ell^2}{4} m\omega \sin^2\theta \sin^2\omega t,\ \frac{\ell^2}{4} m\omega \sin\theta\cos\theta\sin\omega t\Big)$$

$$\bar{L}_2 = \frac{\ell^2}{4} m\omega \sin\theta \left(-\cos\theta\cos\omega t,\ \sin\theta,\ \cos\theta\sin\omega t\right)$$

total angular momentum:

$$\bar{L} = \bar{L}_1 + \bar{L}_2 = \frac{\ell^2}{2} m\omega \sin\theta \left(-\cos\theta\cos\omega t,\ \sin\theta,\ \cos\theta\sin\omega t\right)$$

torque

$$\bar{N} = \frac{d\bar{L}}{dt} = m\frac{\ell^2}{2}\omega^2 \sin\theta\left(\cos\theta\sin\omega t,\ 0,\ \cos\theta\cos\omega t\right)$$

$$\boxed{\bar{N} = \frac{\ell^2}{2} m\omega^2 \sin\theta\cos\theta\left(\sin\omega t,\ 0,\ \cos\omega t\right)}$$

* (b) this last torque oscillates in the x-z plane and has the same magnitude as the one found above. In the body set $\bar{N}$ would not be oscillating, but fixed on $\hat{e}_3$.

To show that it is $\perp$ to $\bar{\omega}$:

$$\bar{N}\cdot\bar{\omega} \sim (\sin\omega t,\ 0,\ \cos\omega t)\cdot(0,1,0) = 0$$

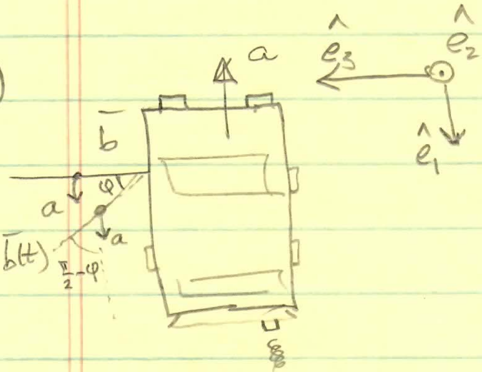
$$\boxed{\bar{N} \perp \bar{\omega}}$$

For the other one along the bar

$$\bar{N}\cdot\bar{r}_1 = \frac{\ell^2}{2} m\omega^2 \sin\theta\cos\theta\left(-\frac{\ell}{2}\right)\underbrace{(\sin\omega t, 0, \cos\omega t)\cdot(\sin\theta\cos\varphi, \cos\theta, -\sin\theta\sin\varphi)}_{\sin\theta\sin\varphi\cos\varphi - \sin\theta\sin\varphi\cos\varphi = 0}$$

$$\boxed{\bar{N} \perp \bar{r}_1}$$

(5.23)



We know $\omega_1 = \omega_3 = 0$

$N_1 = N_3 = 0$

From Euler equations:

$$I_2 \dot{\omega}_2 = N_2$$

$$I\ddot{\varphi} = bma\sin\left(\tfrac{\pi}{2}-\varphi\right) = bma\cos\varphi$$

$$\ddot{\varphi} = \frac{bma}{I}\cos\varphi$$

Use $\ddot{\varphi} = \frac{d\dot{\varphi}}{d\varphi}\dot{\varphi}$ $\longrightarrow$ $\dot{\varphi}\,d\dot{\varphi} = \frac{bma}{I}\cos\varphi\,d\varphi$

$$\frac{\dot{\varphi}^2}{2} = \frac{bma}{I}\sin\varphi$$

$$\dot{\varphi} = \sqrt{\frac{2bma}{I}\sin\varphi} \implies dt = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{\frac{2bma}{I}\sin\varphi}} = \underbrace{\sqrt{\frac{I}{2bma}}\int_0^{\pi/2}\frac{d\varphi}{\sqrt{\sin\varphi}}}_{T = ?}$$

$\circledast$ change variables $\theta = \pi/2 - \varphi$: $\sin\varphi = \cos\theta$: $d\varphi = -d\theta$

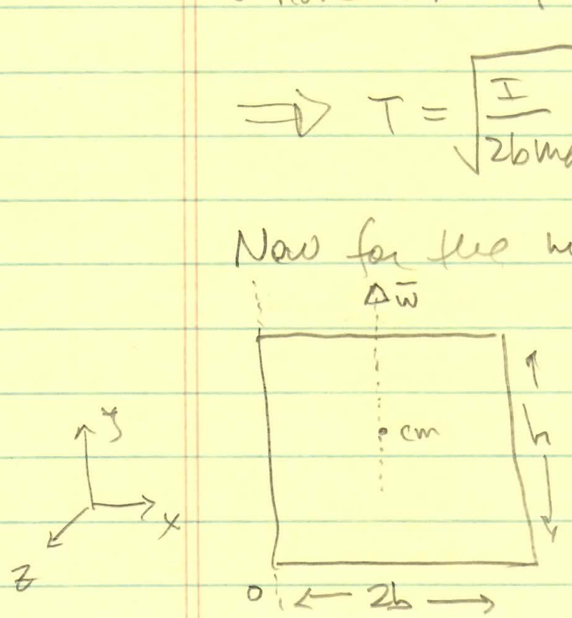
$$dt = \sqrt{\frac{I}{2bma}}\int_{\pi/2}^{0}\frac{-d\theta}{\sqrt{\cos\theta}} = \sqrt{\frac{I}{2bma}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{\cos\theta - \underbrace{\cos\tfrac{\pi}{2}}_{\text{added zero}}}}$$

(elliptical integral) $\sqrt{2}\,K(\sin\frac{\pi}{4}) = \sqrt{2}\,K(\frac{\sqrt{2}}{2})$

The $\frac{\sqrt{2}}{2}$ is a singular value of the elliptical integral [see links in hmwk] where $K_r = K_1$ so $K(\frac{\sqrt{2}}{2}) = \frac{1}{4\sqrt{\pi}}\Gamma^2(\frac{1}{4})$ gamma function

$$\implies T = \sqrt{\frac{I}{2bma}}\cdot\frac{1}{4}\cdot\sqrt{\frac{2}{\pi}}\,\Gamma^2\left(\tfrac{1}{4}\right) \quad \text{where } \Gamma\left(\tfrac{1}{4}\right) \simeq 3.63$$

Now for the moment of inertia of the door, 1st rotate around center of mass



$$I_s^y = I_{yy} = \iint \sigma(x^2 + \underbrace{z^2}_{0})\,dx\,dy$$

(diagonal matrix)

$$I_s^y = \sigma \int_0^h \int_{-b}^{b} x^2 \, dx \, dy = \sigma \left. \frac{x^3}{3} \right|_{-b}^{b} y \Big|_0^h = \sigma \frac{2b^3}{3} h$$

density $\sigma = \frac{M}{2bh}$: $I_s^y = \frac{M}{2bh} \cdot \frac{2b^3}{3} h = \frac{1}{3} M b^2$

By the parallel axis theorem, for I around door hinges

$$I = \frac{1}{3} M b^2 + M b^2 = \frac{4}{3} M b^2$$

Putting back

$$T = \sqrt{\frac{4}{3} M b^2 \cdot \frac{1}{2b M a} \cdot \frac{2}{\pi}} \cdot \frac{1}{4} \Gamma^2\left(\frac{1}{4}\right) = \frac{1}{2} \sqrt{\frac{b}{3a\pi}} \, \Gamma^2\left(\frac{1}{4}\right)$$

Using $b = 0.6\,m$ and $a = 0.3\,m/s^2$

$$T = \frac{1}{2} \sqrt{\frac{0.6}{3\pi} \cdot \frac{1}{0.3}} \, (3.63)^2 \simeq 3.035 \text{ sec}$$