# Separation of Variables in Cylindrical Polar Coordinates

- Seeking separable solution

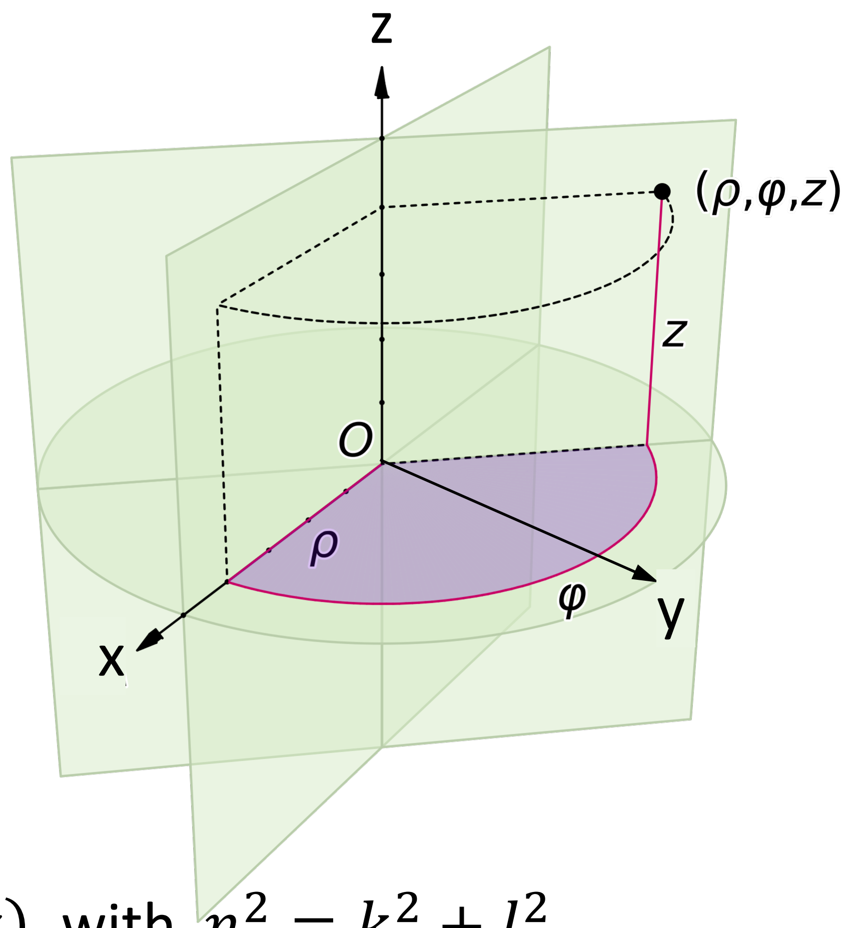
$$\chi(\rho,\varphi,z) = \mathrm{P}(\rho)\Phi(\varphi)Z(z)$$

$Z'' - l^2 Z = 0$ — solution $Z_l(z)$

$\Phi'' + m^2\Phi = 0$ — solution $\Phi_m(\varphi)$

and $\rho(\rho\mathrm{P}')' + \underbrace{(k^2 + l^2)}_{\text{call this } n^2}\rho^2\mathrm{P} - m^2\mathrm{P} = 0$

solution $\mathrm{P}_{nm}(\rho)$




- General solution is

$$\chi(\rho,\varphi,z) = \sum_{lmn} a_{lmn}\,\mathrm{P}_{nm}(\rho)\Phi_m(\varphi)Z_l(z), \text{ with } n^2 = k^2 + l^2$$

(generalized sum — not clear yet what $l, m, n$ really are)

# Separation of Variables in Cylindrical Polar Coordinates

- $z$ and φ equations are easy to solve:

$$Z'' - l^2 Z = 0 \implies Z_l(z) = e^{\pm lz} \quad (l \text{ may be real or complex})$$

$$\Phi'' + m^2\Phi = 0 \implies \Phi_m(\varphi) = e^{\pm im\varphi}$$

- φ is an angular coordinate, so the solution must be periodic

$\implies$ $m$ must be an <u>integer</u>

- Left with the radial equation

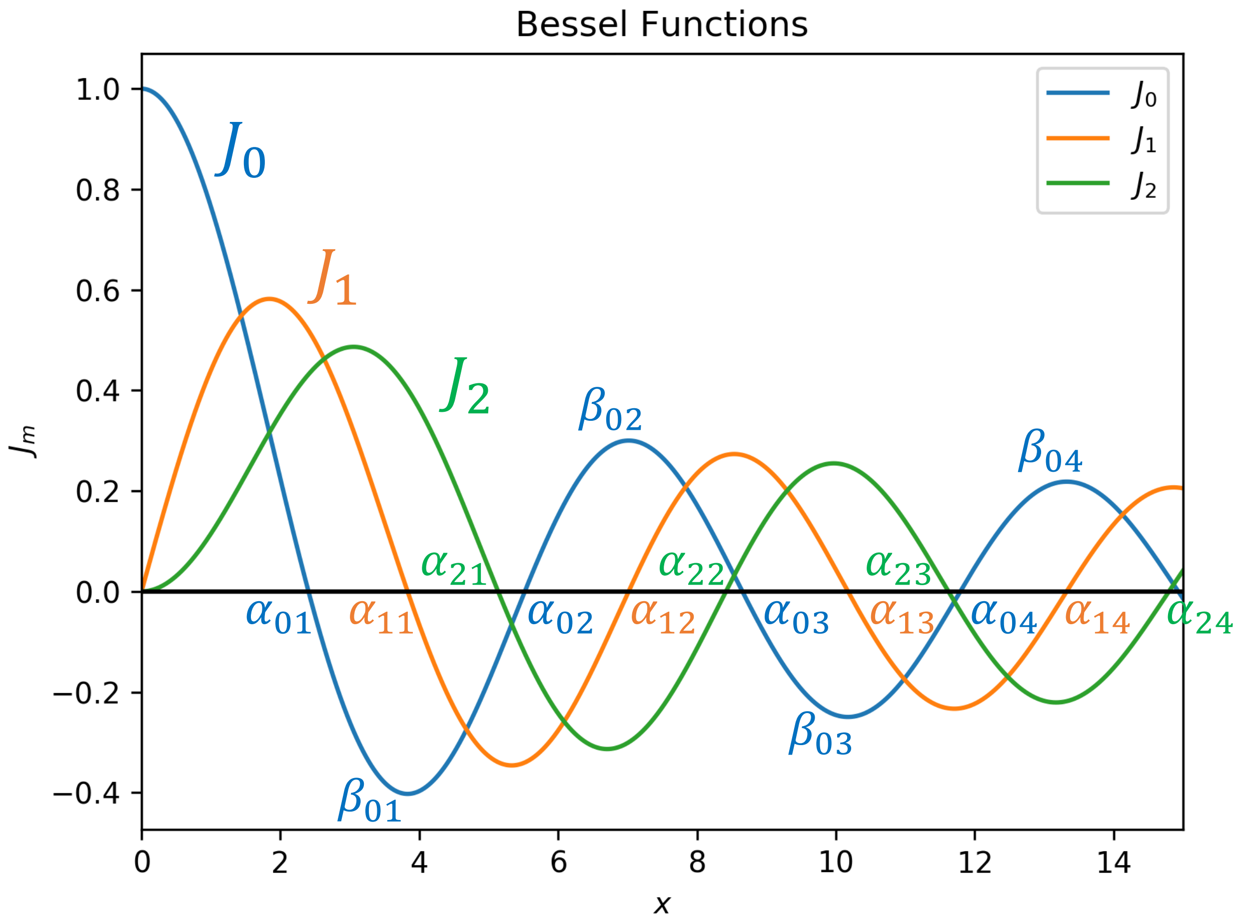
$$\rho(\rho \mathrm{P}')' + (n^2\rho^2 - m^2)\mathrm{P} = 0$$

$$\implies \rho^2 \mathrm{P}'' + \rho \mathrm{P}' + (n^2\rho^2 - m^2)\mathrm{P} = 0$$

$$\implies x^2 \mathrm{P}'' + x\,\mathrm{P}' + (x^2 - m^2)\mathrm{P} = 0$$

<u>Bessel's Equation</u> $\implies \mathrm{P}_{nm}(\rho) = J_m(n\rho)$

# Zeros of Bessel Functions

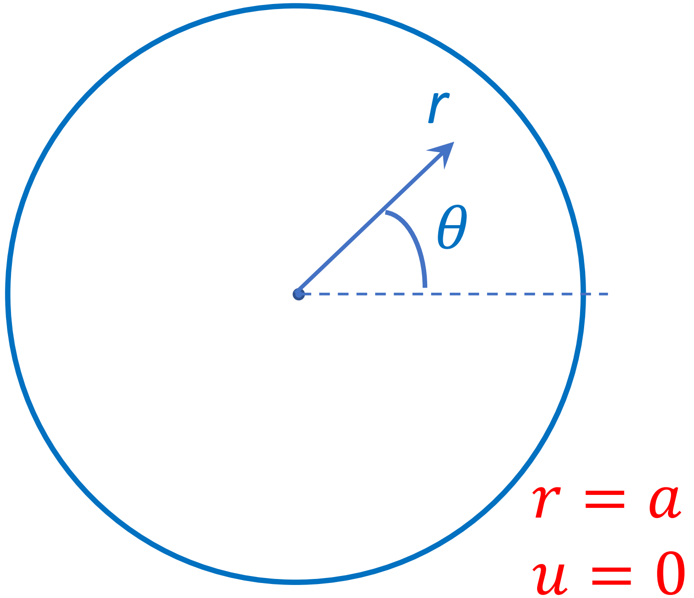



| | $n$=1 | $n$=2 | $n$=3 | $n$=4 |
|---|---|---|---|---|
| $m$=0 | 2.40 | 5.52 | 8.65 | 11.79 |
| $m$=1 | 3.83 | 7.02 | 10.17 | 13.32 |
| $m$=2 | 5.14 | 8.42 | 11.62 | 14.80 |
| $m$=3 | 6.38 | 9.76 | 13.02 | 16.22 |

- zeros of $J_m$ interleave those of $J_m$
- ordering is irregular
- turning points $\beta_{mn}$ also tabulated

| $m$ | $n$ |
|---|---|
| 0 | 1 |
| 1 | 1 |
| 2 | 1 |
| 0 | 2 |
| 3 | 1 |
| 1 | 2 |
| 4 | 1 |
| 2 | 2 |
| 0 | 3 |
| 5 | 1 |
| 3 | 2 |
| 6 | 1 |
| 1 | 3 |
| 4 | 2 |
| 7 | 1 |
| 2 | 3 |
| 0 | 4 |
| 8 | 1 |
| 5 | 2 |
| 3 | 3 |

# Vibration of a Circular Membrane




- Same problem as before, but in a different geometry.
- Seek separable solution: $u(r,\theta) = R(r)\Theta(\theta)$
- where $\Theta'' + m^2\Theta = 0$

$$r^2R'' + rR' + (k^2r^2 - m^2)R = 0$$

- Solutions have the form

$$u(r,\theta) = \begin{cases} J_m(kr)\cos m\theta \\ J_m(kr)\sin m\theta \end{cases}$$

- Boundary condition at $r = a$

$\implies$ $J_m(ka) = 0$

$\implies$ $ka$ is a zero of $J_m$

$\implies$ $ka = \alpha_{mn}$ for some integer $n$.

# Vibration of a Circular Membrane

- Recall that $k = \omega c$, where $\omega$ is frequency, so the fundamental mode is

$$u_0(r,\theta) = J_0(\alpha_{01} r/a), \text{ with } \omega_0 = \alpha_{01} c/a$$

- The next few, in order, are

$$u_1(r,\theta) = J_1(\alpha_{11} r/a)\begin{cases}\cos\theta \\ \sin\theta\end{cases}, \text{ with } \omega_1 = \alpha_{11} c/a$$

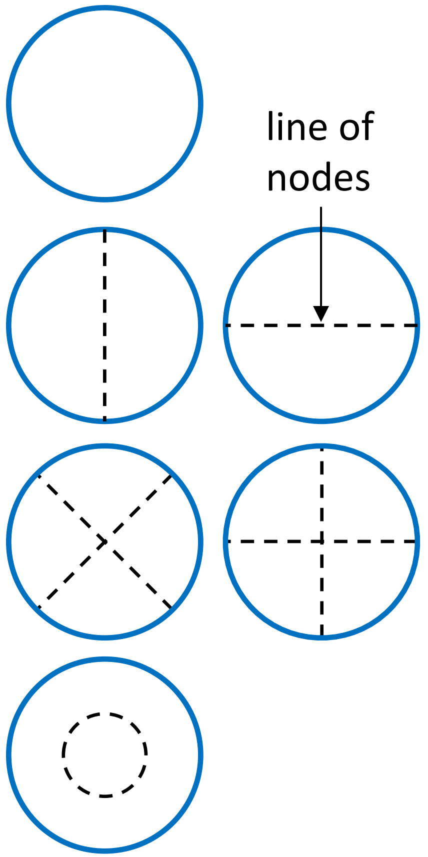
$$u_2(r,\theta) = J_2(\alpha_{21} r/a)\begin{cases}\cos 2\theta \\ \sin 2\theta\end{cases}, \text{ with } \omega_2 = \alpha_{21} c/a$$

$$u_3(r,\theta) = J_0(\alpha_{02} r/a), \text{ with } \omega_3 = \alpha_{02} c/a$$

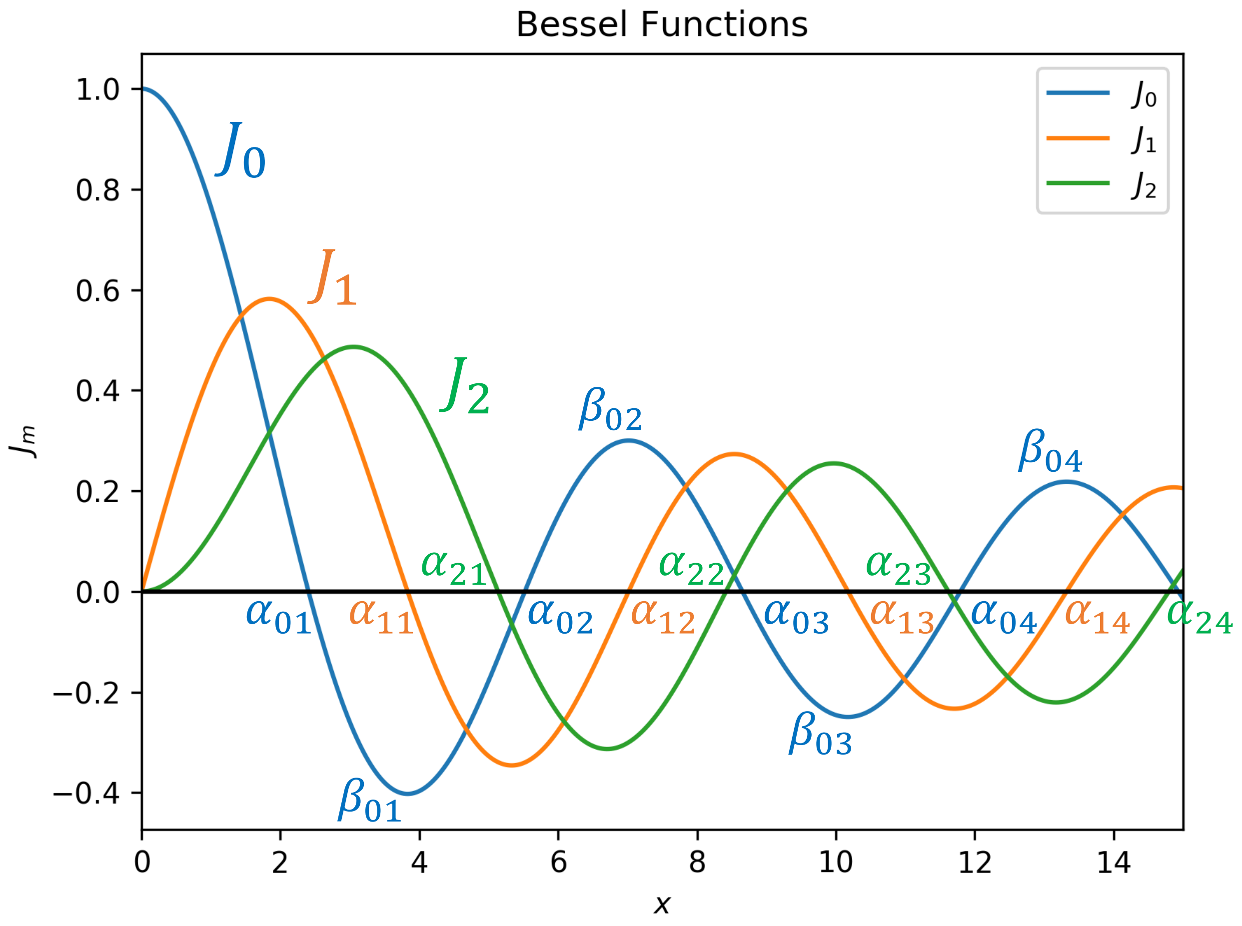
"etc."

- QUESTION: What are the next 2 modes, ordered by frequency?

Just ordering the solutions in $\omega$.



# Zeros of Bessel Functions




| | $n$=1 | $n$=2 | $n$=3 | $n$=4 |
|---|---|---|---|---|
| $m$=0 | 2.40 | 5.52 | 8.65 | 11.79 |
| $m$=1 | 3.83 | 7.02 | 10.17 | 13.32 |
| $m$=2 | 5.14 | 8.42 | 11.62 | 14.80 |
| $m$=3 | 6.38 | 9.76 | 13.02 | 16.22 |

- zeros of $J_m$ interleave those of $J_m$
- ordering is irregular
- turning points $\beta_{mn}$ also tabulated

| $m$ | $n$ |
|---|---|
| 0 | 1 |
| 1 | 1 |
| 2 | 1 |
| 0 | 2 |
| 3 | 1 |
| 1 | 2 |
| 4 | 1 |
| 2 | 2 |
| 0 | 3 |
| 5 | 1 |
| 3 | 2 |
| 6 | 1 |
| 1 | 3 |
| 4 | 2 |
| 7 | 1 |
| 2 | 3 |
| 0 | 4 |
| 8 | 1 |
| 5 | 2 |
| 3 | 3 |

# Vibration of a Circular Membrane

$$u_0(r,\theta) = J_0(\alpha_{01} r/a),\ \text{with } \omega_0 = \alpha_{01} c/a$$

$$u_1(r,\theta) = J_1(\alpha_{11} r/a)\begin{cases}\cos\theta \\ \sin\theta\end{cases},\ \text{with } \omega_1 = \alpha_{11} c/a$$

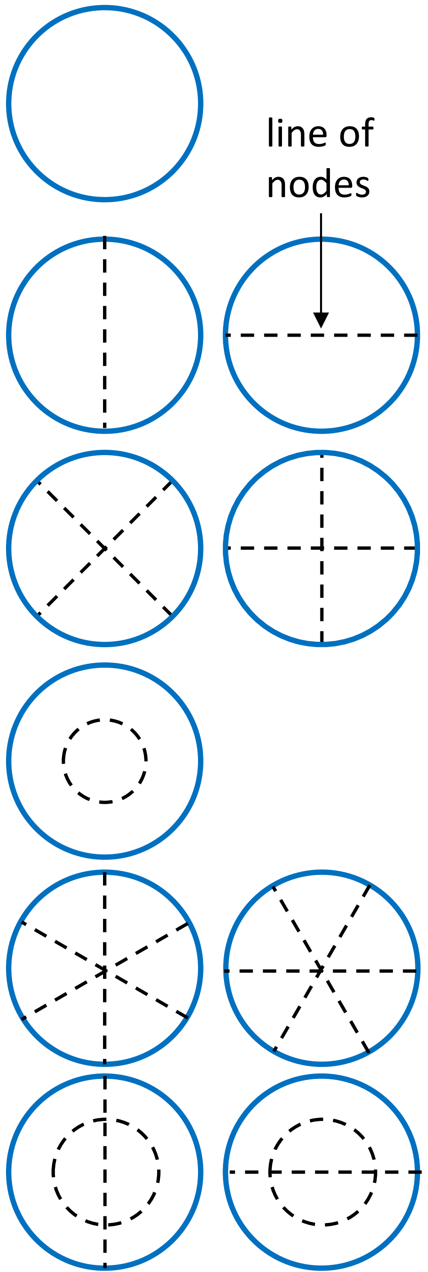
$$u_2(r,\theta) = J_2(\alpha_{21} r/a)\begin{cases}\cos 2\theta \\ \sin 2\theta\end{cases},\ \text{with } \omega_2 = \alpha_{21} c/a$$

$$u_3(r,\theta) = J_0(\alpha_{02} r/a),\ \text{with } \omega_3 = \alpha_{02} c/a$$

- According to the table, next terms are (3,1) and (1,2):

$$u_4(r,\theta) = J_3(\alpha_{31} r/a)\begin{cases}\cos 3\theta \\ \sin 3\theta\end{cases},\ \text{with } \omega_4 = \alpha_{31} c/a$$

$$u_5(r,\theta) = J_1(\alpha_{12} r/a)\begin{cases}\cos\theta \\ \sin\theta\end{cases},\ \text{with } \omega_5 = \alpha_{12} c/a$$



# Vibration of a Circular Membrane

- Solving the IV problem: as with the square membrane, the full solution is a weighted sum of normal-mode solutions:

  $$u(r,\theta,t) = \sum_{m,n} (A_{mn} \cos m\theta + B_{mn} \sin m\theta)\, J_m\left(\frac{\alpha_{mn} r}{a}\right) e^{i\omega_{mn} t}$$

  where $\omega_{mn} = \alpha_{mn} c/a$

- Complete the solution by fitting the initial conditions:

  $$u(r,\theta,0) = \sum_{m,n} (A_{mn} \cos m\theta + B_{mn} \sin m\theta)\, J_m\left(\frac{\alpha_{mn} r}{a}\right)$$

- Fourier-Bessel series for $A_{mn}$ and $B_{mn}$

  — know how to invert the Fourier series

  — will return to this problem later to invert the Bessel series

# Aside: Fourier and Bessel Series

- Fourier series solution to the vibrating string/membrane is of the form

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right)$$

  (more generally, sines and cosines).
- Radial Bessel series solution to the vibrating membrane is of the form

$$f(r) = \sum_{n=1}^{\infty} A_n \, J_m\left(\frac{\alpha_{mn} r}{a}\right)$$

- Note that $J_m$ here plays the <u>same</u> role as sine or cosine as a basis for the expansion, and the sum is over the zeros of sine $(n\pi)$ or $J_m$ $(\alpha_{mn})$ — <u>same</u> basic structure. The $m$ in $J_m$ relates to the second sum over the angular variables.

# Laplace's Equation in (Cylindrical) Polars

- Separate the potential $\phi$ as $\phi(\rho, \phi, z) = \mathrm{P}(\rho)\Phi(\varphi)Z(z)$
- Same derivation as before leads to

$$Z'' - l^2 Z = 0 \implies Z_l(z) = e^{\pm lz} \qquad \text{($l$ may be real or complex)}$$

$$\Phi'' + m^2\Phi = 0 \implies \Phi_m(\varphi) = e^{\pm im\varphi} \qquad \text{($m$ must be an integer)}$$

$$\rho(\rho\mathrm{P}')' + (l^2\rho^2 - m^2)\mathrm{P} = 0 \qquad (k = 0\text{, so now } n = l)$$

$$\implies \quad \mathrm{P}_{mn}(\rho) = J_m(l\rho)$$

- In 2D (i.e. Laplace in a circle), $l = 0$, so ($\rho \longrightarrow r, \varphi \longrightarrow \theta$)

$$r(r\mathrm{P}')' - m^2\mathrm{P} = 0$$

$$\implies r^2\mathrm{P}'' + r\mathrm{P}' - m^2\mathrm{P} = 0$$

$$\implies \mathrm{P}(r) = r^{\pm m}$$

# Example (2D): Laplace's Equation in a Circle

- Problem: $\phi(r,\theta)$, $\nabla^2\phi = 0$, $\phi(a,\theta) = f(\theta)$
- Interior solution is (assuming regular, so omitting negative powers of $r$)

$$\phi(r,\theta) = \sum_{m=0}^{\infty} r^m \left(A_m \cos m\theta + B_m \sin m\theta\right)$$

- Again, coefficients come from the boundary conditions and inversion of the Fourier series:

$$A_m = \frac{1}{\pi}\int_0^{2\pi} d\theta\, f(\theta)\cos m\theta, \qquad B_m = \frac{1}{\pi}\int_0^{2\pi} d\theta\, f(\theta)\sin m\theta$$

- e.g. suppose $f(\theta) = \phi_0 \cos 2\theta$

then $B_m = 0$ for all $m$, $A_m = 0$ unless $m = 2$, so

$$A_2 a^2 = \frac{\phi_0}{\pi}\int_0^{2\pi} d\theta\ \cos^2 2\theta = \phi_0$$

$$\Longrightarrow \quad \phi(r,\theta) = \phi_0 \left(\frac{r}{a}\right)^2 \cos 2\theta$$

Exterior solution?

# Example (3D): Diffusion Equation in a Long Cylinder

- Long uniform cylinder (i.e. infinitely long), radius $a$, initially at internal temperature $T_0$, immersed in liquid of temperature $0$.
- IC and BC are independent of $z$, so this is really a 2D problem.
- Seek separated solution with $\chi(\rho,\varphi)e^{-\kappa k^2 t}$ time dependence, as before, to find

$$\nabla^2\chi + k^2\chi = 0 \quad \Longrightarrow \quad \chi(\rho,\varphi) = \sum_{k,m} A_{km} J_m(k\rho) e^{im\varphi}$$

- but axisymmetric, so only $m = 0$ survives

$$\Longrightarrow \quad \chi(\rho) = \sum_k A_k J_0(k\rho)$$

- BC at $\rho = a \Longrightarrow J_0(ka) = 0$, so $k_n a = \alpha_{0n}$

$$\Longrightarrow \quad T(\rho,\theta) = \sum_n A_n J_0(\alpha_{0n}\rho/a)\; e^{-\kappa k_n^2 t}$$

- Coefficients $A_n$ entail another Bessel series for the ICs.

# Example (3D): Laplace's Equation in a Cylinder

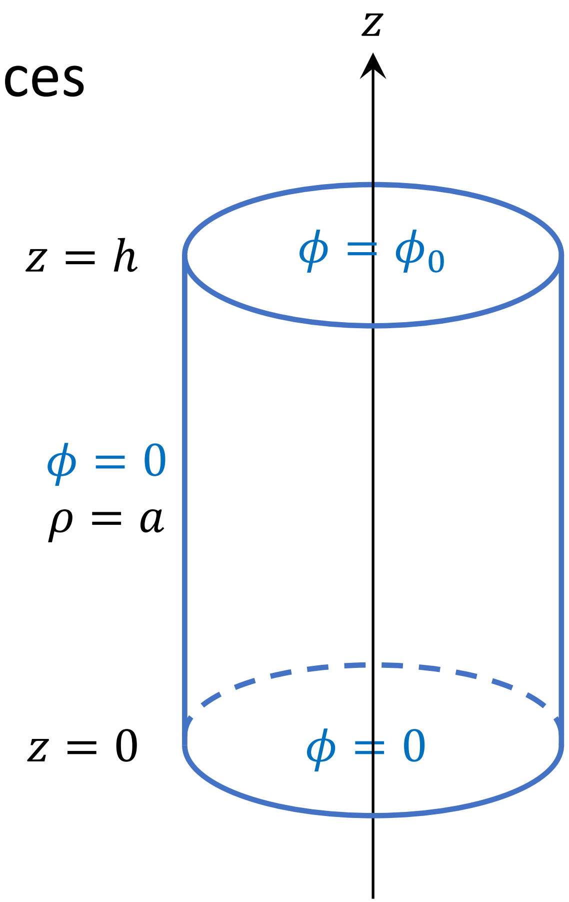
- Finite uniform cylinder, radius $a$, potential $\phi$ on all surfaces except the top is 0, potential on top ($z = h$) is $\phi_0$.
- Expect solution to be a sum of terms of the form

$$\phi_{ml}(\rho, \varphi, z) = J_m(\lambda\rho)\underbrace{e^{\pm im\varphi}e^{\pm\lambda z}}_{\text{shorthand}}$$

- Problem is axisymmetric, so expect $m = 0$.
- Boundary condition at $\rho = a \implies \lambda a = \alpha_{0l},\ l$ integer.
- Boundary condition at $z = 0 \implies \sinh \lambda z$ solution.

$$\implies \phi_{ml}(\rho, \varphi, z) = \textstyle\sum_l A_l\, J_0\left(\frac{\alpha_{0l}\,\rho}{a}\right)\sinh\left(\frac{\alpha_{0l}\,z}{a}\right)$$

- Coefficients $A_n$ again entail a Bessel series for the remaining BC at $z = h$.



# Separation of Variables in Spherical Polar Coordinates

- Spherical polars: $r, \theta, \varphi$
- Helmholtz: $\nabla^2\chi + k^2\chi = 0$

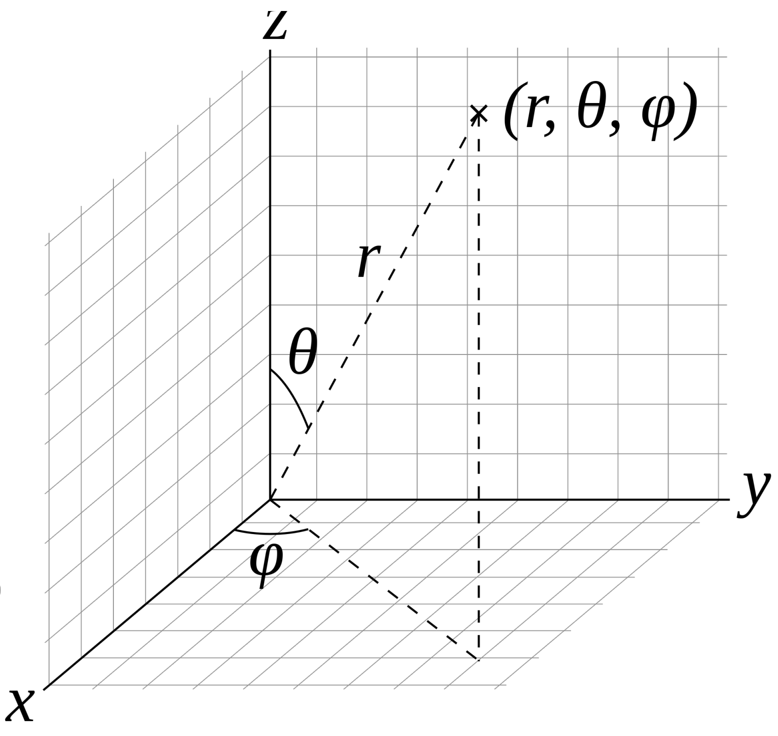
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\chi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\chi}{\partial\theta}\right)$$
$$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\chi}{\partial\varphi^2} + k^2\chi = 0$$

- Seek separable solution $\chi(r,\theta,\varphi) = \mathrm{R}(r)\Theta(\theta)\Phi(\varphi)$

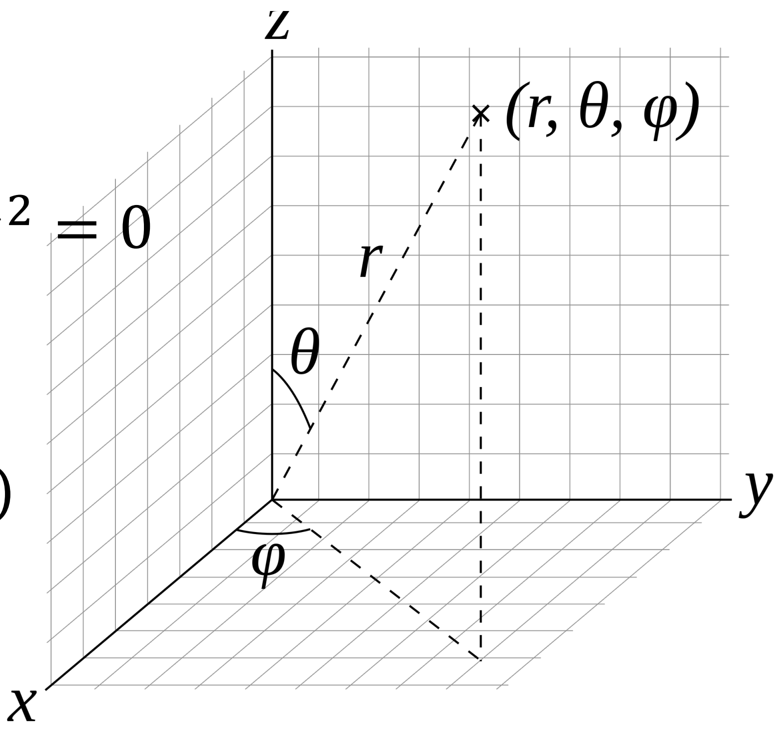
$$\Longrightarrow \frac{1}{r^2R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{r^2\Theta\sin\theta}\frac{d}{dr}\left(\sin\theta\frac{d\Theta}{d\theta}\right)$$
$$+\frac{1}{r^2\Phi\sin^2\theta}\frac{d^2\Phi}{d\varphi^2} + k^2 = 0$$

- Separate as usual, e.g. all the $\varphi$ terms to one side of the equation, so

$$\frac{\Phi''}{\Phi} = -m^2, \text{ so } \Phi'' + m^2\Phi = 0, \text{ so } \Phi = e^{\pm im\varphi},$$ where $m$ is an integer



# Separation of Variables in Spherical Polar Coordinates




- The $r, \theta$ equations are

$$\frac{1}{r^2 R}(r^2 R')' + \frac{1}{r^2 \Theta \sin\theta}(\sin\theta\, \Theta')' - \frac{m^2}{r^2 \sin^2\theta} + k^2 = 0$$

- Once again, separate into $f(r) = g(\theta)$, to find

$$\frac{1}{\Theta \sin\theta}(\sin\theta\, \Theta')' - \frac{m^2}{\sin^2\theta} = \text{constant } = l(l+1)$$

$$\Longrightarrow \frac{1}{\sin\theta}(\sin\theta\, \Theta')' - \frac{m^2 \Theta}{\sin^2\theta} = l(l+1)\Theta\,.$$

- With this separation constant, the radial equation becomes

$$\frac{1}{r^2}(r^2 R')' + \left(k^2 - \frac{l(l+1)}{r^2}\right) R = 0\,.$$

- Look first at the angular equation.

# The Associated Legendre Equation

- In the $\theta$ equation, set $\mu = \cos\theta$, so $d\mu = -\sin\theta d\theta$ and

$$(1-\mu^2)\frac{d^2\Theta}{d\mu^2} - 2\mu\frac{d\Theta}{d\mu} + \left[l(l+1) - \frac{m^2}{1-\mu^2}\right]\Theta = 0.$$

- This is the <u>Associated Legendre Equation</u>

  - solutions are $P_l^m(\mu)$

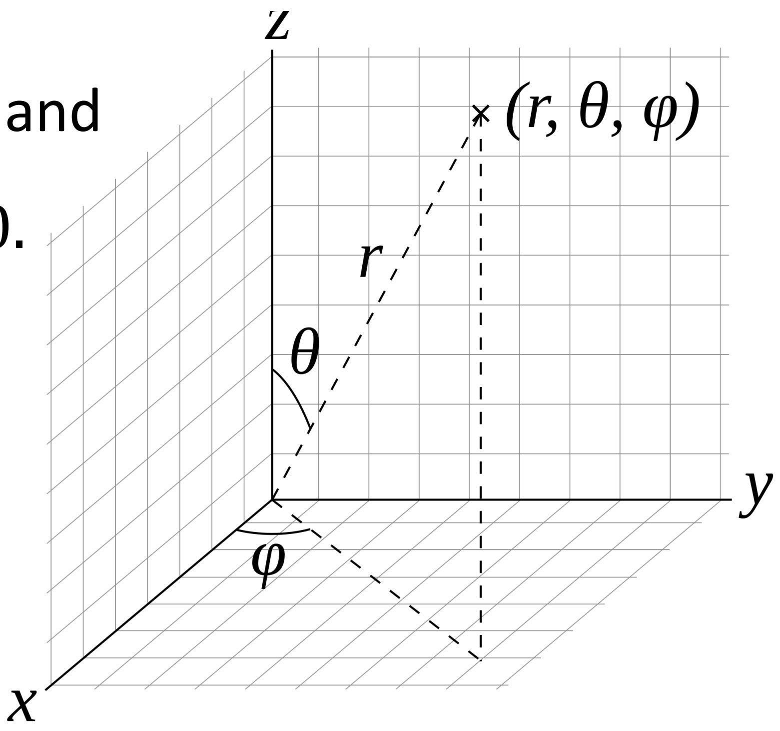
- Very common to combine the angular solutions:

$$Y_l^m(\theta,\varphi) = P_l^m(\cos\theta)e^{\pm im\varphi}$$

- Easy to show that

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_m^m}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_l^m}{\partial\varphi^2} + l(l+1)Y_l^m = 0.$$

- “Angular part” of the Helmholtz solution (note no $k$ dependence).
- The $Y_l^m$ are called <u>spherical harmonics</u>



# Legendre Functions

- Will study general properties in more detail later, and we'll see that $l$ must also be integral, with $|m| \leq l$
- The first few (with conventional normalization $P_l^0(1) = 1$) are

$$P_0^0(\mu) = 1$$

$$P_1^0(\mu) = \mu = \cos\theta$$

$$P_1^1(\mu) = \sqrt{1-\mu^2} = \sin\theta$$

pair with $\cos\varphi$

$\sin\varphi$

$$P_2^0(\mu) = \frac{1}{2}(3\mu^2 - 1) = \frac{1}{4}(1 + 3\cos 2\theta)$$

$$P_2^1(\mu) = 3\mu\sqrt{1-\mu^2} = \frac{3}{2}\sin 2\theta$$

$$P_2^2(\mu) = 3(1-\mu^2) = \frac{3}{2}(1-\cos 2\theta)$$

pair with $\cos 2\varphi$

$\sin 2\varphi$

"etc."

# Spherical Harmonics

- Spherical harmonics $Y_l^m(\theta,\varphi) = P_l^m(\cos\theta)e^{\pm im\varphi}$
- First few (with standard normalization, explained later) are

$$Y_0^0(\theta,\varphi) = \sqrt{\frac{1}{4\pi}}$$

$$Y_1^0(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^{\pm1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi}$$

$$Y_2^0(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$$

$$Y_2^{\pm1}(\theta,\varphi) = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\varphi}$$

$$Y_2^{\pm2}(\theta,\varphi) = \mp\sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\varphi}$$

Basis functions for expansion of <u>any</u> field on the surface of a sphere:

$$f(\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm} Y_l^m(\theta,\varphi)$$

# Radial Equation

- The radial equation is

$$\frac{1}{r^2}(r^2R')' + \left(k^2 - \frac{l(l+1)}{r^2}\right)R = 0$$

$$\Longrightarrow \quad R'' + \frac{2}{r}R' + \left(k^2 - \frac{l(l+1)}{r^2}\right)R = 0.$$

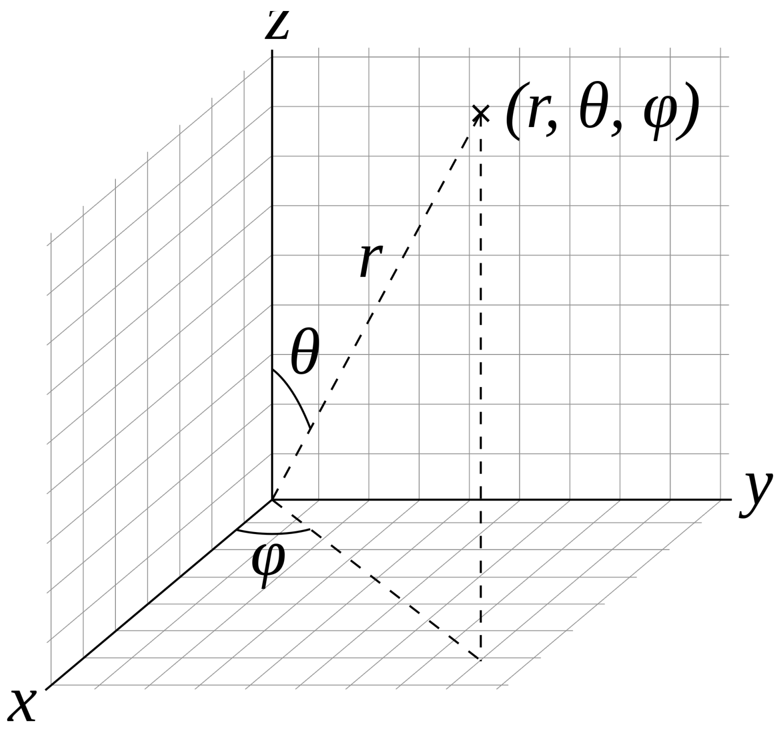
- Let $u = r^{\frac{1}{2}}R$, so

$$R = r^{-\frac{1}{2}}u$$

$$R' = r^{-\frac{1}{2}}u' - \frac{1}{2}r^{-\frac{3}{2}}u$$

$$R'' = r^{-\frac{1}{2}}u'' - r^{-\frac{3}{2}}u' + \frac{3}{4}r^{-\frac{5}{2}}u$$

- multiply equation by $r^{\frac{1}{2}}$ and substitute



# Radial Equation

$$\left(u'' - r^{-1}u' + \frac{3}{4}r^{-2}\,u\right) + (2r^{-1}u' - r^{-2}\,u) + \left(k^2 - \frac{l(l+1)}{r^2}\right)u = 0$$

$$\Longrightarrow \quad u'' + \frac{u'}{r} + \left[k^2 - \frac{l(l+1)+\frac{1}{4}}{r^2}\right] = 0$$

$$\Longrightarrow \quad u'' + \frac{u'}{r} + \left[k^2 - \frac{\left(l+\frac{1}{2}\right)^2}{r^2}\right] = 0.$$

- Recall that $J_m(x)$ satisfies

$$x^2y'' + xy' + (x^2 - m^2)y = 0$$

so $J_m(kr)$ satisfies

$$r^2y'' + ry' + (k^2r^2 - m^2)y = 0.$$

- Hence

$$u_l(r) = J_{l+\frac{1}{2}}(kr) \quad \text{so the solution is } R_l(r) = r^{-\frac{1}{2}}u_l(r) = j_l(kr)$$

Spherical Bessel function

# Spherical Bessel Functions

- Conventional definition of spherical Bessel functions

$$j_l(x) = \left(\frac{\pi}{2x}\right)^{1/2} J_{l+\frac{1}{2}}(x), \text{ so}$$

- First few:

$$J_{\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x$$

$$J_{-\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x$$

$$J_{\frac{3}{2}}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \left(\frac{\sin x}{x} - \cos x\right)$$

$$J_{-\frac{3}{2}}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \left(-\frac{\cos x}{x} - \sin x\right)$$

$$j_0(x) = \frac{\sin x}{x}$$

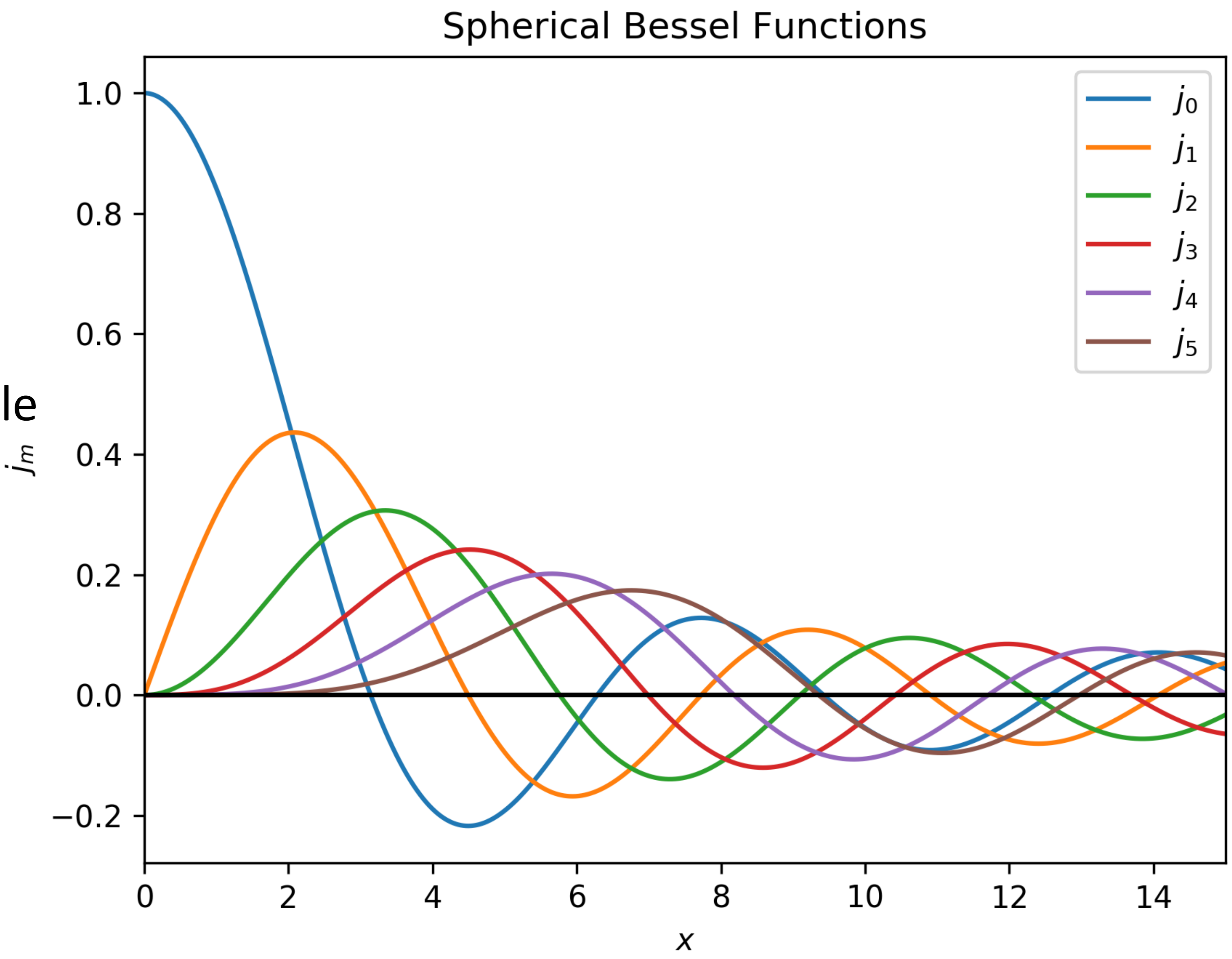
$$j_{-1}(x) = \frac{\cos x}{x}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

$$j_{-2}(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

# Spherical Bessel Functions

- Negative indices are singular at $x = 0$.
- Regular functions:
- Again, ordering of zeros starts off simple but soon becomes complicated.
- Standard functions, zeros tabulated in `Python`.



# Particle in a Sphere

- Reminder: for a quantum-mechanical particle in a box, with $V = 0$ inside the box and $\psi = 0$ on the boundary, the wave function satisfies

$$\nabla^2\psi + k^2\psi = 0, \text{ where } k^2 = \frac{2mE}{\hbar^2}$$

- In a sphere, radius $a$, the general solution is a linear combination of modes of the form

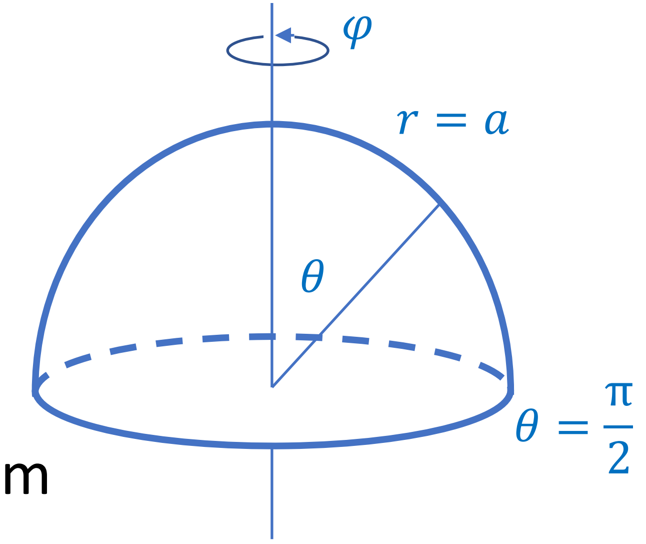
$$\psi_{lmn} = j_l(kr)\, Y_l^m(\theta, \varphi) \sim j_l(kr)\, P_l^m(\cos\theta)e^{\pm im\varphi}$$

where $l, m$ are integers, with $l \geq 0$, $-l \leq m \leq l$

- $\psi = 0$ on the boundary $\Longrightarrow$ $j_l\,(ka) = 0$
- Function with the lowest first zero is $j_0$, with $k_0 a = \pi$, so the ground-state energy is ($l = m = 0$)

$$E_0 = \frac{\hbar^2 k_0^2}{2m} = \frac{\hbar^2\pi^2}{2ma^2}, \text{ with } \psi_0 = j_0\left(\frac{\pi r}{a}\right) = \frac{\sin \pi r/a}{\pi r/a}$$

# Particle in a Hemisphere




- Hemisphere, radius $a$, flat face at $\theta = \frac{\pi}{2}$, axisymmetric.
- Wave function again satisfies

$$\nabla^2\psi + k^2\psi = 0, \text{ where } k^2 = \frac{2mE}{\hbar^2}$$

- General solution is a linear combination of modes of the form

$$\psi_{lmn} = j_l(kr)\, Y_l^m(\theta,\varphi) \;\sim\; j_l(kr)\, P_l^m(\cos\theta)e^{\pm im\varphi}$$

- Axisymmetry $\Longrightarrow$ expect $m = 0$, but now $l = 0$ won't do because of the boundary condition on the flat face — need $\sim \cos\theta$ behavior: need $l$ odd
- $\psi = 0$ on the spherical surface $\Longrightarrow$ $j_l\,(ka) = 0$
- Suitable function with the lowest first zero is $j_1(kr)$, with $\tan k_0a = k_0a$, so $k_0a = 4.49 = \beta$ (say), and the ground-state energy is

$$E_0 = \frac{\hbar^2 k_0^2}{2m} = \frac{\hbar^2\beta^2}{2ma^2}, \quad \text{with } \psi_0 = j_1\left(\frac{\beta r}{a}\right) = \frac{\sin \beta r/a}{\beta^2 r^2/a^2} - \frac{\cos\beta r/a}{\beta r/a}$$

# Laplace's Equation in a Sphere

- Previous derivation of the radial equation fails when $k = 0$.
- Radial equation becomes

$$\frac{1}{r^2}(r^2 R')' - \frac{l(l+1)}{r^2} R = 0$$

- Solution is easily found:

$$R(r) = r^l \text{ or } r^{-l-1}$$

- General solution is a linear combination of modes of the form

$$\phi_{lmn} = \left(A_l r^l + B_l r^{-l-1}\right) Y_l^m(\theta, \varphi)$$

- e.g. look for an interior solution $(r < a)$, expect $\phi$ to be non-singular, so $B_l = 0$

$$\phi(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_l r^l Y_l^m(\theta, \varphi)$$

- Boundary condition

$$\phi(a, \theta, \varphi) = f(\theta, \varphi)$$

# Laplace's Equation in a Sphere

- Evaluating the general solution at the boundary

$$\Longrightarrow \; f(\theta, \varphi) \; = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_l a^l \, Y_l^m(\theta, \varphi)$$

- Laplace series for $f$ — analogous to Fourier expansion of BCs in 2D.
- For simple cases, can just read off the answer

$f \; = \;$ constant $\; \Longrightarrow \;$ only $\; Y_0^0 \;$ contributes

$f \; = \; \cos\theta \; \Longrightarrow \; Y_1^0$

$f \; = \; \sin\theta \cos\varphi \; \Longrightarrow \; Y_1^{\pm 1}$

etc.

- Need a robust way to invert the series – to come!

# PDEs and Coordinate Systems

| | Cartesian | (Cylindrical) Polar | Spherical Polar |
|---|---|---|---|
| **Hyperbolic** | 1D string ✓<br>2D membrane ✓<br>3D volume | Circular membrane ✓<br>Wave in cylinder | Wave in sphere ✓ |
| **Elliptic** | Laplace in square ✓<br>Laplace in cube | Laplace in circle ✓<br>Laplace in cylinder ✓ | Laplace in sphere ✓<br>Sphere in E-field ✓ |
| **Parabolic** | Diffusion in square<br>Diffusion in cube ✓ | Diffusion in cylinder ✓ | Diffusion in sphere ✓ |
| **Schrödinger*** | Particle in line<br>Particle in rectangle<br>Particle in cuboid | Particle in a circle<br>Particle in a cylinder ✓ | Particle in sphere ✓<br>Particle in hemisphere ✓ |