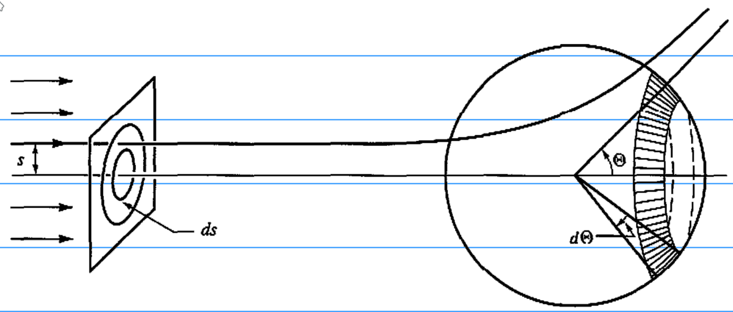
Scattering in a central force field

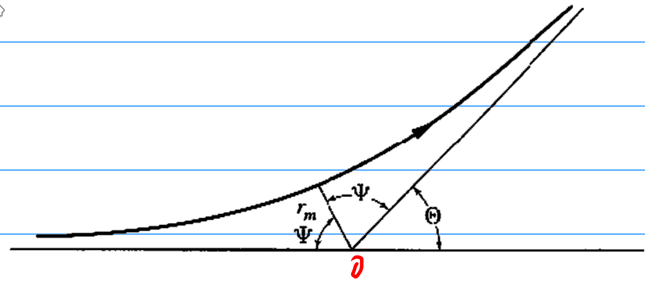


$$\sigma(\bar{\Omega})\,d\Omega$$

$$\underbrace{I\,\underbrace{2\pi s\,|ds|}_{\text{area ring}}}_{\text{Part/time}} = I\,\sigma(\bar{\Omega})\,d\Omega = I\,\sigma(\theta)\,2\pi\sin\theta\,|d\theta|$$

$$\sigma(\theta) = \frac{s}{\sin\theta}\left|\frac{ds}{d\theta}\right| \qquad s = s(\theta, E)$$

$$\theta = \pi - 2\psi$$



$$\psi = \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2m}{\ell^2}(E-V) - 1/r^2}} \qquad \ell = s\sqrt{2mE}$$

$$\theta = \pi - 2\int_{r_m}^{\infty} \frac{dr}{r^2\sqrt{\frac{1}{s^2E}(E-V) - 1/r^2}} \qquad r = 1/u, \quad dr = -\frac{du}{u^2}$$

$$\theta = \pi - 2\int_{0}^{u_m} \frac{s\,du}{\sqrt{1 - \frac{V}{E} - s^2u^2}}$$

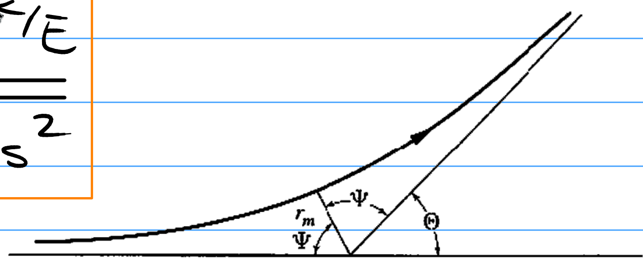
Lets look at the specific example of scattering by a Coulombic repulsive force.

scatt ctr Ze Part. $Z'e$

$$f = \frac{\overset{\equiv k}{ZZ'e^2}}{r^2} = ku^2 \qquad V = \frac{k}{r} = ku$$

$$\psi = \int_u^{u_m} \frac{s\,du}{\sqrt{1 - \frac{k}{E}u - s^2u^2}}$$

$$\boxed{\psi(u) = \sin^{-1} \frac{2s^2u + k/E}{\sqrt{\left(\frac{k}{E}\right)^2 + 4s^2}}}$$



$\psi(u_m)$: $$E = \underbrace{\tfrac{1}{2}m\dot{r}_m^2}_{0} + \tfrac{1}{2}m r_m^2\dot{\theta}^2 + \frac{k}{r_m}$$

$$mr^2\dot{\theta} = l = mv_0 s = s\sqrt{2Em} \qquad // \qquad E = \tfrac{1}{2}mv_0^2$$

$$\dot{\theta} = \frac{s}{r^2}\sqrt{\frac{2E}{m}}$$

$$E = \tfrac{1}{2}m r_m^2 \frac{s^2}{r_m^4}\frac{2E}{m} + \frac{k}{r_m} = \frac{s^2}{r_m^2}E + \frac{k}{r_m}$$

$$\Rightarrow s^2 = r_m^2\left(1 - \frac{k}{E r_m}\right) = \frac{1}{u_m^2} - \frac{k}{E}\cdot\frac{1}{u_m}$$

$$\psi(u_m) = \sin^{-1} \frac{\frac{2}{u_m}\left(1 - \frac{k}{E}u_m\right) + \frac{k}{E}}{\sqrt{\left(\frac{k}{E}\right)^2 + \frac{4}{u_m^2}\left(1 - \frac{k}{E}u_m\right)}} = \sin^{-1}(1) = \pi/2$$

$$\psi(u) = \sin^{-1} \frac{2s^2u + k/E}{\sqrt{\left(\frac{k}{E}\right)^2 + 4s^2}}$$

$\ell = s\sqrt{2mE}$

$s^2 = \frac{\ell^2}{2mE}$

$$= \sin^{-1} \frac{\frac{2}{r}\left(\frac{\ell^2}{2mE} + \frac{k}{E}\right)\frac{E}{k}}{\sqrt{\left[\left(\frac{k}{E}\right)^2 + \frac{4\ell^2}{2mE}\right]\frac{E^2}{k^2}}}$$

$$= \sin^{-1} \frac{\frac{\ell^2}{mkr} + 1}{\sqrt{1 + \frac{2\ell^2 E}{mk^2}}}$$

$\epsilon = \sqrt{1 + \frac{2E\ell^2}{mk^2}}$

$$\psi(u) = \sin^{-1} \frac{1}{\epsilon}\left(\frac{\ell^2}{mkr} + 1\right)$$

$$\psi(r) = \frac{\pi}{2} - \left[\frac{\pi}{2} - \cos^{-1} \frac{1}{\epsilon}\left(\frac{\ell^2}{mkr} + 1\right)\right]$$

$$\boxed{\frac{1}{r} = \frac{mzz'e^2}{\ell^2}\left(\epsilon\cos\psi - 1\right)}$$

$r \to \infty : \frac{1}{r} \to 0 \Longrightarrow \epsilon\cos\psi = 1$

$\theta = \pi - 2\psi$

$\cos\psi = \sin\frac{\theta}{2}$

$\sin\frac{\theta}{2} = \frac{1}{\epsilon}$

$\frac{1}{\sin^2\frac{\theta}{2}} = \epsilon^2$

$\frac{1}{\sin^2\frac{\theta}{2}} - 1 = \epsilon^2 - 1$

$\frac{\cos^2\frac{\theta}{2}}{\sin^2\theta/2} = \epsilon^2 - 1$

$\cot^2\frac{\theta}{2} = \epsilon^2 - 1$

$$\ell = s\sqrt{2mE}$$

$$\cot^2\frac{\theta}{2} = \frac{2E\,\ell^2}{m k^2} = \left(\frac{2Es}{zz'e^2}\right)^2$$

$$\boxed{s = \frac{zz'e^2}{2E}\cot\frac{\theta}{2}}$$

$$\frac{ds}{d\theta} = \frac{zz'e^2}{2E}\left(-\frac{1}{2}\right)\csc^2\frac{\theta}{2}$$

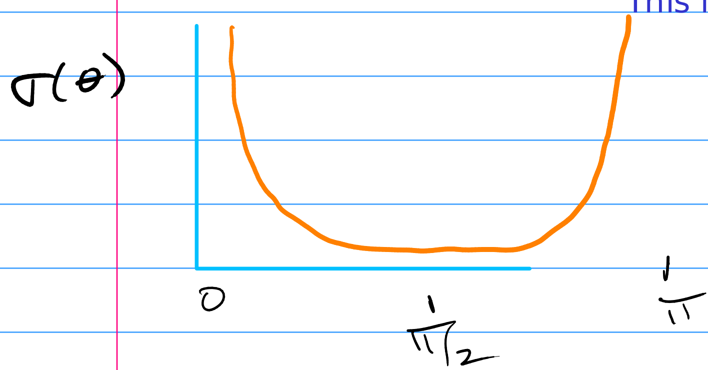
$$\sigma(\theta) = \frac{s}{\sin\theta}\left|\frac{ds}{d\theta}\right|$$

$$= \frac{s}{\sin\theta}\,\frac{zz'e^2}{2E}\left(\frac{1}{2}\right)\csc^2\frac{\theta}{2}$$

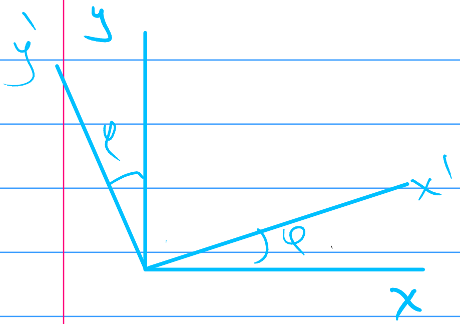
$$= \underbrace{\left(\frac{zz'e^2}{2E}\cot\frac{\theta}{2}\right)}_{s}\;\underbrace{\frac{1}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}}_{1/\sin\theta}\left(\frac{zz'e^2}{2E}\right)\frac{1}{2}\csc^2\frac{\theta}{2}$$

$$\boxed{\sigma(\theta) = \frac{1}{4}\left(\frac{zz'e^2}{2E}\right)^2\csc^4\frac{\theta}{2}}$$

This is the Rutherford scattering cross section



Rotation of rigid bodies



$$x' = x\cos\varphi + y\sin\varphi$$
$$y' = -x\sin\varphi + y\cos\varphi$$

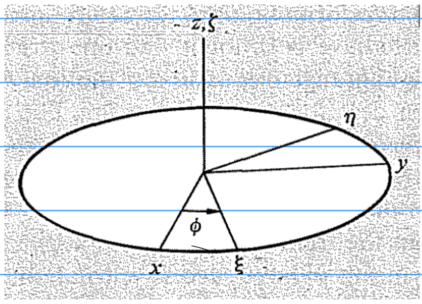
$$A = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow x' = A x$$

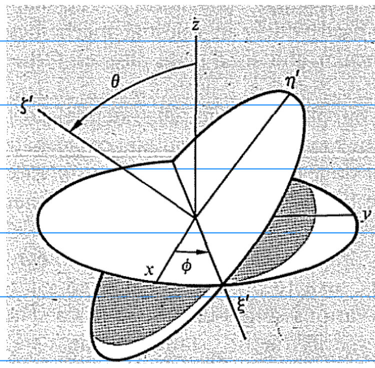
The Euler Angles

φ, θ, ψ

(1)



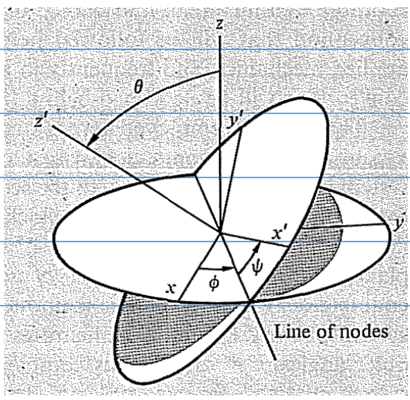
rotate by φ along z



(2)

rotate by θ along ξ

(3)



rotate by ψ along ζ'

(1) $\xi = DX$ (2) $\xi' = C\xi$ (3) $x' = B\xi'$

$x' = Ax \implies A = BCD$

$$D = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{about } z$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \quad \text{about } \xi$$

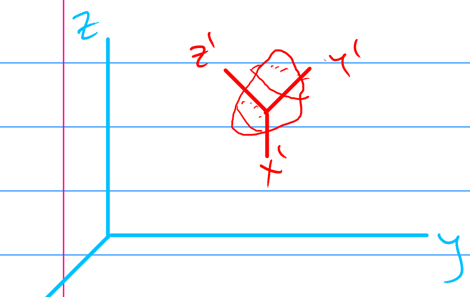
$$B = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{about } \xi'$$

$A = BCD$

$$\mathbf{A} = \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}$$

$x' = Ax$ $\qquad x = A^{-1}x' = A^{T}x'$

Rate of change of a vector



x, y, z space set of axes

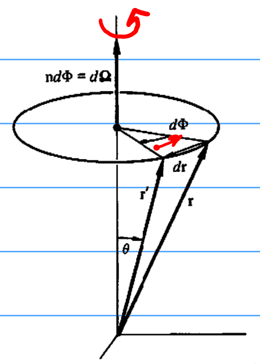
x', y', z' body set of axes

e.g. a position vector relative to the body set will change relative to the space set but not to the body set.

The time-rate of change of the components will be different depending on the axes of reference.

$\bar{G}$

$$(d\bar{G})_{space} = (d\bar{G})_{body} + (d\bar{G})_{rot}$$



$d\bar{\Omega}$ $\bar{r}$

$d\bar{\Omega} \times \bar{r}$

$$d\bar{G}_{rot} = d\bar{\Omega} \times \bar{G}$$

$$(d\bar{G})_{space} = (d\bar{G})_{body} + d\bar{\Omega} \times \bar{G}$$

$$\left(\frac{d\bar{G}}{dt}\right)_s = \left(\frac{d\bar{G}}{dt}\right)_r + \bar{\omega} \times \bar{G}$$

$\bar{\omega}\, dt = d\bar{\Omega}$

$$\boxed{\left(\frac{d}{dt}\right)_s = \left(\frac{d}{dt}\right)_r + \bar{\omega} \times}$$

The Coriolis Effect

$\bar{G} = \bar{r}$

$$\left(\frac{d}{dt}\bar{r}\right)_s = \left(\frac{d}{dt}\bar{r}\right)_r + \bar{\omega} \times \bar{r}$$

$$\bar{v}_s = \bar{v}_r + \bar{\omega} \times \bar{r}$$

velocity rel. to space set — vel. rel. body set — angular velocity

$$\left(\frac{d\bar{v}_s}{dt}\right)_s = \left(\frac{d\bar{v}_s}{dt}\right)_r + \bar{\omega} \times \bar{v}_s$$

$$= \left(\frac{d}{dt}(\bar{v}_r + \bar{\omega}\times\bar{r})\right)_r + \bar{\omega} \times (\qquad)$$

$$= \bar{a}_r + 2\bar{\omega}\times\bar{v}_r + \bar{\omega}\times(\bar{\omega}\times\bar{r})$$

$\bar{F} = m\bar{a}_s$

$$\boxed{\begin{aligned} m\bar{a}_r &= \bar{F} \underbrace{- 2m\bar{\omega}\times\bar{v}_r}_{\text{Coriolis eff}} \underbrace{- m\bar{\omega}\times(\bar{\omega}\times\bar{r})}_{\text{centrifugal force}} \\ \bar{F}_{eff} &= \end{aligned}}$$

$\omega \sim 7\times 10^{-5}$