Recap 1: Special Functions in Physics

- Harmonic Oscillator

  $$y'' + k^2 y = 0$$ all

- Bessel

  $$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$ polar, cyl. polar, sph. polar

- Legendre ($m = 0$)

  $$(1 - x^2) y'' - 2x y' + l(l+1) y = 0$$ spherical polar

- Legendre (general)

  $$(1 - x^2) y'' - 2x y' + \left[l(l+1) - \frac{m^2}{1-x^2} \right] y = 0$$ spherical polar

- Hermite

  $$y'' - 2x y' + 2n y = 0$$ QM harmonic oscillator

- Laguerre

  $$x y'' - (1 - x) y' + n y = 0$$ QM hydrogen atom

Recap 2: Series Solutions to SOLDEs

- In the vicinity of some point (here $x = 0$), seek a power-series solution of the form (power series with an x^k multiplier)

$$y(x) = x^k \sum_{i=0}^{\infty} a_i x^i$$

- Notes: k and a_i are formally undetermined, no constraint on k, $a_0 \neq 0$.
- Basic approach: assume convergence and substitute the series into the ODE, then compare powers of x.
- Fuchs: this <u>will</u> create at least one solution to the ODE for problems of interest.
- Form of the series allows study of the properties of the first solution.
- Second solution: may come from the series or from the Wronskian

$$y_2(x) = y_1(x) \int_{x_0}^{x} \frac{e^{-\int_{x_0}^{x_2} P(x_1)dx_1}}{y_1^2(x_2)} dx_2$$

Example 1: Harmonic Oscillator

- Differential equation:

$$y'' + y = 0$$

- First series solution (larger k: $k = 1$)

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) = \sin x$$

- Wronskian second solution ($P = 0$):

$$\int_{x_0}^{x_2} P x_1 dx_1 = 0$$

$$y_2(x) = y_1(x) \int_{x_0}^{x} \frac{1}{\sin^2 x_2} dx_2$$

$$= \sin x \int_{x_0}^{x} \operatorname{cosec}^2 x_2 dx_2$$

$$= \sin x \; (-\cot x)$$

$$= -\cos x$$

Example 2: Legendre Equation

- Wronskian integrals not always so easy to do
- Differential equation:

$$(1 - x^2)y'' - 2xy' + l(l + 1)y = 0$$

- First series solution (larger k: $k = 1$)

$$y_1(x) = P_l(x)$$

- Wronskian second solution $\left(P = \frac{-2x}{1-x^2}\right)$:

$$\int_{x_0}^{x_2} P(x_1)dx_1 = \log(1 - x_2^2)$$

$$y_2(x) = y_1(x) \int_{x_0}^{x} \frac{1}{(1-x_2^2)P_l^2(x_2)} dx_2$$

$$= ?$$

Behavior of Second Solutions

- Can use the Wronskian approach to study the properties of first and second solutions.
- Suppose we can write, near an ordinary or regular singular point,

$$P(x) = \sum_{m=-1}^{\infty} p_m x^m, \qquad Q(x) = \sum_{n=-2}^{\infty} q_n x^n$$

- Then if $\quad y(x) = \sum_{i=0}^{\infty} a_i x^{k+i},$

  substitute to find

$$\sum_{i=0}^{\infty} a_i (k+i)(k+i-1) x^{k+i-2} + \left(\sum_{m=-1}^{\infty} p_m x^m\right)\left(\sum_{i=0}^{\infty} a_i (k+i) x^{k+i-1}\right) + \left(\sum_{n=-2}^{\infty} q_n x^n\right)\left(\sum_{i=0}^{\infty} a_i x^{k+i}\right)$$

- Leading term is $x^{k-2} \quad \Rightarrow \quad i = 0,\ m = -1,\ n = -2$

$$\Rightarrow \quad k(k-1) + p_{-1} k + q_{-2} = 0$$

Behavior of Second Solutions

- Indicial equation
$$k(k-1)+p_{-1}k+q_{-2}=0$$
- Note that, at an <u>ordinary</u> point, $p_{-1}=q_{-2}=0$, so $k=0,1$ <u>always</u>, but for (say) Bessel's equation, with $p_{-1}=1$, $q_{-2}=-\nu^2$, we recover
$$k(k-1)+k-\nu^2=0$$
$$k^2=\nu^2.$$
- In general,
$$k^2+(p_{-1}-1)k+q_{-2}=0$$
[and note that the solution(s) k may be complex].
- Call the root with the larger real part α, and the other root $\alpha-n$, where $Re(n)>0$. Fuchs says that the only issue is when n is an <u>integer</u>.

Behavior of Second Solutions

- Roots of
$$k^2 + (p_{-1} - 1)k + q_{-2} = 0$$
are α and $\alpha - n$.
- In terms of these roots, the equation is
$$(k - \alpha)(k - \alpha + n) = 0$$
$$\Longrightarrow \qquad k^2 + (n - 2\alpha)k + \alpha(\alpha - n) = 0,$$
so $\qquad p_{-1} - 1 = n - 2\alpha, \qquad q_{-2} = \alpha(\alpha - n).$
- For the first solution y_1, with $k = \alpha$, Fuchs tells us
$$y_1 = x^{\alpha} \sum_{i=0}^{\infty} a_i x^i$$
- Second solution y_2 comes from the Wronskian.

Behavior of Second Solutions

- Second solution is

$$y_2(x) = y_1(x) \int_{x_0}^{x} \frac{e^{-\int_{x_0}^{x_2} P(x_1)dx_1}}{y_1^2(x_2)} dx_2$$

$$= y_1(x) \int_{x_0}^{x} \frac{e^{-\int_{x_0}^{x_2} \left(\sum_{m=-1}^{\infty} p_m x_1^m\right) dx_1}}{x^{2\alpha}\left(\sum_i a_i x_2^i\right)^2} dx_2$$

- Only interested in the leading terms, so write

$$P(x_1) = p_{-1} x_1^{-1} + TS,$$

where "TS" just means a Taylor series—a sum over non-negative powers of x.

- Assert that, for x sufficiently close to 0,

$$\int TS = TS,\ e^{TS} = TS,\ (TS)^2 = TS,\ TS \times TS = TS,\ \frac{1}{TS} = TS,\ \frac{TS}{TS} = TS$$

where the details of each TS are unimportant.

Behavior of Second Solutions

- With that convention, the numerator in the x_2 integral is

$$e^{-\int_{x_0}^{x_2}\int[p_{-1}x_1^{-1}dx_1+TS(x_1)]\,dx_1} = e^{-p_{-1}\log x_2+TS(x_2)} = x_2^{-p_{-1}}TS(x_2)$$

and the denominator is

$$x_2^{2\alpha}TS(x_2)^2 \;= x_2^{2\alpha}TS(x_2),$$

so 1/denominator is

$$x_2^{-2\alpha}TS(x_2)$$

- Combining, the Wronskian solution is

$$y_2(x) = y_1(x)\int_{x_0}^{x} x_2^{-p_{-1}-2\alpha}TS(x_2)\,dx_2$$

$$= y_1(x)\int_{x_0}^{x} x_2^{-n-1}\sum_{i=0}^{\infty} b_i x_2^i \;dx_2$$

$$p_{-1} - 1 = (n - 2\alpha)$$
$$\Longrightarrow -p_{-1} - 2\alpha = -n - 1$$

Behavior of Second Solutions

- Second solution is

$$y_2(x) = x^\alpha TS \int_{x_0}^{x} x_2^{-n-1} \sum_{i=0}^{\infty} b_i x_2^i \ dx_2$$

- If n is an integer, then the $i = n$ term in the sum yields x_2^{-1} and the integral of that term is $b_n \log x$ — an <u>additional</u> term not included in the original series solution.
- For all other terms, or if n is not an integer, the integral yields (neglecting the x_0 part)

$$\sum_{i=0}^{\infty} \int_{x_0}^{x} x_2^{-n-1} b_i x_2^i \, dx_2 = \sum_{i=0}^{\infty} \frac{b_i x^{i-n}}{i-n} = x^{-n} TS$$

$$\Longrightarrow \quad y_2(x) = x^\alpha TS \times x^{-n} TS = x^{\alpha-n} \, TS$$

– just the usual second series solution

Behavior of Second Solutions

- Fuller statement of Fuchs theorem:
  - if the two roots of the indicial equation, α and $\alpha - n$ do not differ by an integer, then each gives a valid series solution
  - if they do differ by an integer, then the second solution takes the form

    $$y_2(x) = b_n \log x \, y_1(x) + x^{\alpha - n} \, TS$$
- Possible that $b_n = 0$, so the divergent term may not appear.
- e.g. harmonic oscillator: $\alpha = 1, n = 1$, but the corresponding term in the sine series has $b_n = 0$, so no divergent term.

numerator: $e^{-\int_{x_0}^{x_2} P(x_1) dx_1} = 1$

1/denominator: $x_2^{-2}\left(1 - \frac{x_2^2}{3!} + \frac{x_2^4}{5!} - \cdots\right)^{-2} = x_2^{-2} \cdot [\underline{\text{even}}\ TS]$

$\Longrightarrow$ no b_1 term

Second Solutions of Legendre's Equation

- Legendre polynomials: $n = 1$, but now we have both even and odd solutions, so expect logarithmic divergence at $x = 0$.
- Example: $P_0(x) = 1$, so corresponding second solution is

$$Q_0(x) = \int_{x_0}^{x} \frac{1}{1-x_2^2}\, dx_2 = \frac{1}{2} \log \frac{1+x}{1-x}$$

- In the Legendre case, all second solutions are at least logarithmically divergent at $x = 0$ $(\theta = \pm\pi/2)$ — usually not what we want in a physical solution, so these solutions are rarely used.
- For Bessel functions, however, the second solutions are very important.

Second Solutions of Bessel's Equation

- First solution, $k = +\nu$, is $J_\nu(x)$, already seen
  - regular for all $x \geq 0$, $\sim x^\nu$ as $x \to 0$, $\to 0$ as $x \to \infty$
- If ν is <u>not</u> an integer, then can show (R&H Sec. 9.5.1) that $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent, so $J_{-\nu}(x)$ is a valid second solution

  $\Longrightarrow$ $c_1 J_\nu(x) + c_2 J_{-\nu}(x)$ is the general solution.
- Note no logarithmic behavior in $J_{-\nu}(x)$ when ν is half an odd integer (as in the spherical Bessel functions) even though the roots of the indicial equation differ by an integer — turns out $b_{2\nu} = 0$.
- For example, with $\nu = \frac{1}{2}$, we saw

$$y_1 = J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

- Substitute into the Wronskian expression and find $b_1 = 0$ again.

Second Solutions of Bessel's Equation

- If ν is an integer (m, say), straightforward to show (R&H Sec. 9.5.2) that the $k = \pm m$ solutions are <u>not</u> linearly independent:

$$J_{-m}(x) = (-1)^m J_m(x)$$

- This time, in the Wronskian expression, $P(x) = \frac{1}{x}$ and $y_1 = J_m(x)$ is a sum of <u>even</u> powers of x, so

numerator: $e^{-\int_{x_0}^{x_2} P(x_1)dx_1} = \frac{1}{x_2}$

1/denominator: $x_2^{-2m}[\text{even } TS]$

so

$$y_2(x) = y_1(x) \int_{x_0}^{x} x_2^{-2m-1} \sum_{i=0}^{\infty} b_{2i} x_2^{2i} \ dx_2$$

$\Longrightarrow b_{2m} \log x$ term appears.

Second Solutions of Bessel's Equation

- Second solution, $k = -m$, is

$$y_2(x) = J_m(x)\,(\log x + x^{-m}TS).$$

  - all second solutions are singular at $x = 0$
  - logarithmic for $m = 0$, stronger singularity for $m > 0$
  - note that $J_m(x) \sim x^m$ as $x \to 0$, so $J_m(x)\log x \to 0$ if $m > 0$

- Example:

$$J_0(x) = 1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 \ldots$$

$$y_2(x) = J_0(x)\left[\log x + \frac{1}{4}x^2 + \frac{5}{128}x^4 \ldots\right]$$

Bessel Functions of the Second Kind

- Any linear combination of $y_1(x)$ and $y_2(x)$ solves the equation.
- Conventional to (re)define $J_\nu(x)$ as a Bessel function of the first kind.
- Conventional to define a Bessel function of the second kind by

$$Y_\nu(x) = \frac{\cos \nu\pi \, J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}$$

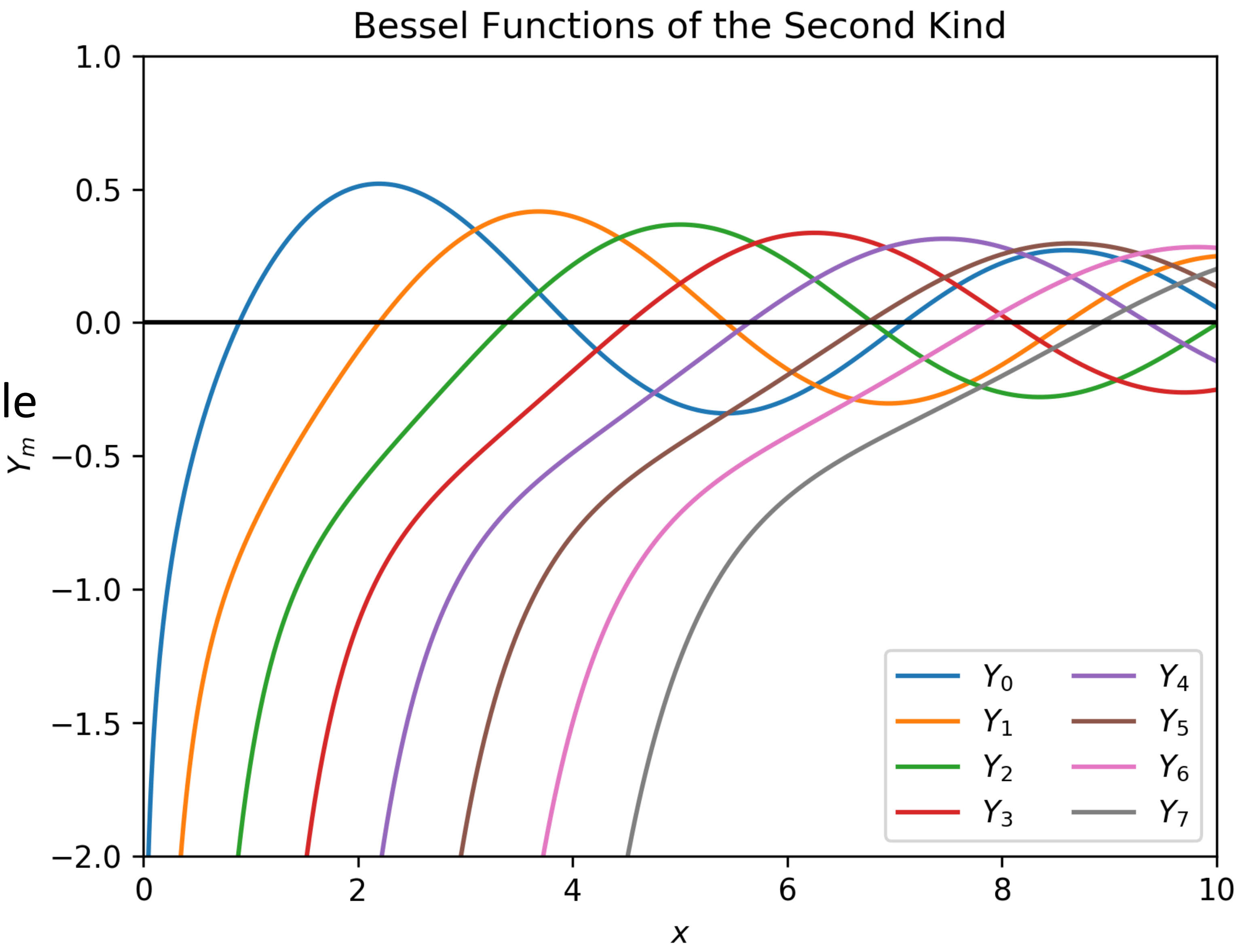
May see N_m instead of Y_m — Neumann functions.

- Obviously a solution for non-integer ν.
- But indeterminate $(0/0)$ for integer ν.
- Can define the integer functions by

$$Y_m(x) = \lim_{\nu \to m} \frac{\cos \nu\pi \, J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}$$

evaluate by l'Hopital's rule $\Longrightarrow Y_m(x) = \frac{1}{\pi}\left[\frac{\partial J_\nu(x)}{\partial \nu} - (-1)^m \frac{\partial J_{-\nu}(x)}{\partial \nu}\right]_{\nu=m}$

- Singular at $x = 0$.
- Oscillatory, damped as $x \to \infty$.
- Again, ordering of zeros starts off simple but soon becomes complicated.
- Standard functions, zeros tabulated in (e.g.) `Python`.



Bessel Functions

- Why make such a spectacularly opaque definition of a standard function?
- Answer comes when we look at the <u>asymptotic</u> behavior of $J_\nu(x)$ and $Y_\nu(x)$ as $x \to \infty$
- With this definition (and integer limit), can show

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

$$Y_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

- Aside from phase, $J_\nu(x)$ plays the role of $\cos x$, $Y_\nu(x)$ the role of $\sin x$.
- Already seen a hint of this behavior in the spherical Bessel functions.
- Asymptotic behavior is true for <u>all</u> Bessel functions.

Bessel Functions

- Example: in the 2D wave equation, with assumed $u(r,\theta,t) = \chi(r,\theta)e^{-i\omega t}$ time dependence, the spatial part reduces to the Helmholtz equation with $k = \omega c$:

$$\nabla^2\chi + k^2\chi = 0$$

- Assume looking at BCs on a circle of radius a, with $\chi(a,\theta)$ specified (so really talking about a wave driven by oscillations $\chi(a,\theta)e^{-i\omega t}$ on the boundary).
- We know the general solution:

$$\chi(r,\theta) = \sum_{m=0}^{\infty} [J_m(kr) + B_m Y_m(kr)]\,[C_m \cos m\theta + D_m \sin m\theta]$$

- Expect the interior solution $(r < a)$ to be regular at $r = 0$, so $B_m = 0$.
- BC at $r = a$ gives a Fourier series:

$$\sum_{m=0}^{\infty} J_m(ka)\,[C_m \cos m\theta + D_m \sin m\theta] = \chi(a,\theta)$$

Bessel Functions

- What about the exterior solution $(r > a)$?
- $Y_m(kr)$ solution no longer excluded (goes to zero as $r \to \infty$)
- Free to choose the linear combination of $J_m(kr)$ and $Y_m(kr)$ to describe the expected behavior at infinity.

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \quad Y_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

$$\Rightarrow \; J_m(kr) + i\, Y_m(kr) \; \sim \sqrt{\frac{2}{\pi kr}}\, e^{i\left(kr - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)}$$

- This overall solution has r, t-dependence:

$$u(r,\theta,t) \sim \sqrt{\frac{2}{\pi kr}}\, e^{i(kr - \omega t)} \quad \text{as} \quad r \to \infty$$

Outgoing wave

Hankel Functions

- The specific combinations of J_ν and Y_ν

$$H_\nu^{(1)} = J_\nu + iY_\nu$$

$$H_\nu^{(2)} = J_\nu - iY_\nu$$

are called <u>Hankel functions</u>.

- Useful because, coupled with an $e^{-i\omega t}$ time dependence, they represent outgoing and incoming wave solutions in the 2D and 3D wave problems (very common BCs).
- For outgoing wave BC, exterior solution is

$$\chi(r,\theta) = \sum_{m=0}^{\infty} H_m^{(1)}(kr)\,[E_m \cos m\theta + F_m \sin m\theta]$$

$$\Longrightarrow \sum_{m=0}^{\infty} H_m^{(1)}(ka)\,[E_m \cos m\theta + F_m \sin m\theta] = \chi(a,\theta)$$

another (well, essentially the same) Fourier series.

Sturm-Liouville Theory

- Still have that pesky problem of inverting the Bessel, Legendre, and Laplace series we encountered.
- Need a more general theory of the properties of SOLDEs.
- Convention: modify the standard form of a SOLDE and define a linear differential operator $\mathcal{L}$ by

$$\mathcal{L}y \equiv p_0(x)y'' + p_1(x)y' + p_2(x)y = 0$$

(Not so different: $P = \frac{p_1}{p_0}$, $\quad Q = \frac{p_2}{p_0}$ previously.) Assume the p_i are <u>real</u>.

- Functions on some interval $[a, b]$ on the real line form a vector space. Define an <u>inner product</u> of f and g by

aka $f \cdot g$, $\langle f|g\rangle$

$$(f, g) = \int_a^b f^*(x)g(x)dx \qquad (f^* \text{ is complex conjugate})$$

Sturm-Liouville Theory

- For two functions u and v, consider

$$(v, \mathcal{L}u) = \int_a^b v^* \mathcal{L}u \, dx = \int_a^b v^* (p_0 u'' + p_1 u' + p_2 u) \, dx$$

- Some terminology: define the adjoint operator $\bar{\mathcal{L}}$ by

$$(\bar{\mathcal{L}}v, u) \; = \; (v, \mathcal{L}u)$$

- If $\bar{\mathcal{L}} = \mathcal{L}$, then $\mathcal{L}$ is a self-adjoint operator.

Think hermitian operators in QM…

- Under what circumstances is that the case?
- Study of such operators and their eigenfunctions is Sturm-Liouville Theory.

Sturm-Liouville Theory

- Jump in…

$$(v, \mathcal{L}u) = \int_a^b v^*(p_0 u'' + p_1 u' + p_2 u)\, dx$$

1 $$\int_a^b v^* p_0 u'' dx \;=\; [v^* p_0 u']_a^b - \int_a^b (v^* p_0)' u'\, dx$$

$$= \; [v^* p_0 u' - (v^* p_0)' u]_a^b + \int_a^b (v^* p_0)'' u\, dx$$

$$= \; \left[v^* p_0 u' - v^{*\prime} p_0 u - v^* p_0' u\right]_a^b + \int_a^b (v^* p_0)'' u\, dx$$

2 $$\int_a^b v^* p_1 u' dx \;=\; [v^* p_1 u]_a^b - \int_a^b (v^* p_1)' u\, dx$$

$$\Longrightarrow (v, \mathcal{L}u) = \left[v^* p_0 u' - v^{*\prime} p_0 u - v^* p_0' u + v^* p_1 u\right]_a^b$$

$$+ \int_a^b [(v^* p_0)'' - (v^* p_1)' + v^* p_2]\, u\, dx$$

Sturm-Liouville Theory

- For <u>real</u> p_i, $(v, \mathcal{L}u) = (\mathcal{L}v, u)$ iff

$$\left[v^* p_0 u' - v^{*\prime} p_0 u - v^* p_0' u + v^* p_1 u\right]_a^b = 0$$

boundary conditions

and

$$(v^* p_0)'' - (v^* p_1)' + v^* \cancel{p_2} = p_0 v^{*\prime\prime} + p_1 v^{*\prime} + \cancel{p_2} v^*$$

form of the ODE

$$\implies \quad \cancel{v^{*\prime\prime} p_0} + 2v^{*\prime} p_0' + v^* p_0'' - v^{*\prime} p_1 - v^* p_1' = \cancel{p_0 v^{*\prime\prime}} + p_1 v^{*\prime}$$

$$\implies \quad 2v^{*\prime}(p_0' - p_1) + v^*(p_0'' - p_1') = 0$$

independent of p_2

- Latter condition is satisfied if $p_1 = p_0'$
- Then the boundary conditions imply

$$\left[p_0 (v^* u' - v^{*\prime} u)\right]_a^b = 0$$