Kepler problem, $f = -\frac{k}{r^2}$: $V = \frac{-k}{r}$

substitute this in the equation for theta $u = \frac{1}{r}$

$$\theta = \theta' - \int \frac{du}{\sqrt{\frac{2m}{l^2}(E + ku) - u^2}}$$

$$\int \frac{dx}{\sqrt{\alpha x^2 + \beta x + \gamma}} = -\frac{1}{\sqrt{-\alpha}} \sin^{-1}\left(\frac{2\alpha x + \beta}{\sqrt{\beta^2 - 4\alpha\gamma}}\right)$$

$$\theta = \theta' - \cos^{-1} \frac{\frac{l^2 u}{mk} - 1}{\sqrt{1 + \frac{2El^2}{mk^2}}}$$

$$\sqrt{1 + \frac{2El^2}{mk^2}} \cos(\theta - \theta') = \frac{l^2}{mk} \cdot \frac{1}{r} - 1$$

$$\boxed{\frac{1}{r} = \frac{mk}{l^2}\left(1 + \sqrt{1 + \frac{2El^2}{mk^2}} \cos(\theta - \theta')\right)}$$

the general equation of a conic with one focus at the origin is given by

$$\frac{1}{r} = C\left[1 + \epsilon \cos(\theta - \theta')\right]$$

eccentricity

$$\boxed{\epsilon = \sqrt{1 + \frac{2El^2}{mk^2}}}$$

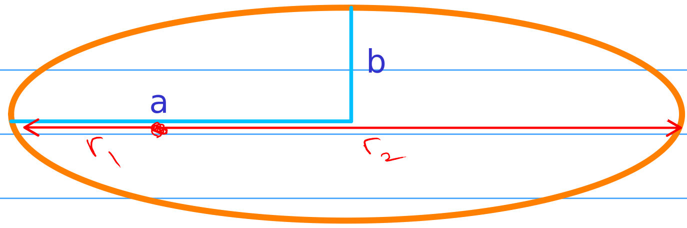
The following are true about conics

$\epsilon > 1$, $E > 0$ hyperbola

$\epsilon = 1$, $E = 0$ parabola

$\epsilon < 1$, $E < 0$ ellipse

$\epsilon = 0$, $E = -\frac{mk^2}{2l^2}$ circle elliptical orbits



$$a = \frac{1}{2}(r_1 + r_2)$$

At the apsidal distances, which are turning points, the r-dot = 0

$$E = \underbrace{\frac{1}{2} m \dot{r}^2}_{0} + \frac{1}{2} \cdot \frac{l^2}{m r^2} - \frac{k}{r}$$

$$E - \frac{1}{2} \cdot \frac{l^2}{m r^2} + \frac{k}{r} = 0$$

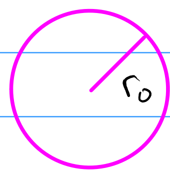
$$r^2 + \frac{k}{E} r - \frac{l^2}{2 m E} = 0$$

$$\frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} + \frac{-\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} = -\frac{\beta}{\alpha}$$

$$a = \frac{1}{2}(r_1 + r_2) = -\frac{k}{2E}$$

Note that for a circular orbit this means that the energy is

$$E = -\frac{k}{2 r_0}$$



$$\boxed{\epsilon = \sqrt{1 - \frac{l^2}{m k a}}}$$

$$\frac{l^2}{m k} = a(1 - \epsilon^2)$$

substitute into 1/r

$$\boxed{r = \frac{a(1 - \epsilon^2)}{1 + \epsilon \cos(\theta - \theta')}}$$

Note that the apsidal distances occur for $\theta - \theta' = (0, \pi)$

$$r_1, r_2 = a(1 \pm \epsilon)$$

Velocity vector along the path of an orbit:

$$\bar{v} = v_r \hat{r} + v_\theta \hat{\theta} \qquad v_r = \dot{r} \;:\; v_\theta = r\dot{\theta}$$

in a circle $\dot{r} = v_r = 0$

$$v_\theta = a\dot{\theta}_0 = \frac{\ell}{ma} \equiv v_0 \qquad \dot{\theta} = \frac{\ell}{mr^2}$$

in an ellipse

$$\dot{r} = \frac{a\epsilon(1-\epsilon^2)\sin(\theta-\theta')\dot{\theta}}{[1+\epsilon\cos(\theta-\theta')]^2}$$

$$v_r = \frac{r^2}{a(1-\epsilon^2)} \cdot \epsilon\sin(\theta-\theta')\dot{\theta}$$

$$= \frac{\ell}{ma} \cdot \frac{\epsilon\sin(\theta-\theta')}{1-\epsilon^2}$$

$$v_r = v_0 \frac{\epsilon\sin(\theta-\theta')}{1-\epsilon^2}$$

notice that at apsidal distances, v_r vanishes, as it should

$$v_\theta = r\dot{\theta} = \frac{\ell}{mr} = \frac{\ell}{m} \frac{1+\epsilon\cos(\theta-\theta')}{a(1-\epsilon^2)}$$

$$= v_0 \frac{1+\epsilon\cos(\theta-\theta')}{1-\epsilon^2}$$

is maximal at perihelion

$$\theta-\theta' = 0 \qquad v_\theta = \frac{v_0}{1-\epsilon}$$

Motion in time in the Kepler problem: the period of the orbit

$$t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \ell^2/2mr^2\right)}}$$

for the inverse-square force law

$$t = \sqrt{\frac{m}{2}} \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{k}{r} - \frac{\ell^2}{2mr^2} + E}}$$

$$\frac{\ell^2}{mk} = a(1-\epsilon^2) \quad : \quad E = -\frac{k}{2a}$$

$$t = \sqrt{\frac{m}{2k}} \int_{r_0}^{r} \frac{r\,dr}{\sqrt{r - \frac{r^2}{2a} - \frac{a(1-\epsilon^2)}{2}}}$$

To integrate this, we change variables using the eccentricity anomaly ψ

$$r = a(1-\epsilon\cos\psi)$$

where at perihelion $\theta = 0 = \psi$ $\qquad \theta' = 0$

and at aphelion $\theta = \pi = \psi$

$$t = \sqrt{\frac{ma^3}{k}} \int_0^{\psi} (1-\epsilon\cos\psi)\,d\psi$$

Using this expression to calculate the period, we take psi to be 2pi

$$\boxed{\tau = 2\pi a^{3/2} \sqrt{\frac{m}{k}}}$$

Note that this is not yet Kepler's 3rd law ...

putting back the reduced mass, and k

$$\frac{m}{k} = \frac{m_1 m_2}{m_1 + m_2} \cdot \frac{1}{G m_1 m_2} = \frac{1}{G(m_1+m_2)}$$

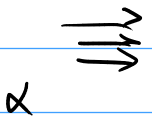
$$\tau = \frac{2\pi a^{3/2}}{\sqrt{G(m_1+m_2)}} \simeq \frac{2\pi a^{3/2}}{\sqrt{Gm_2}}$$

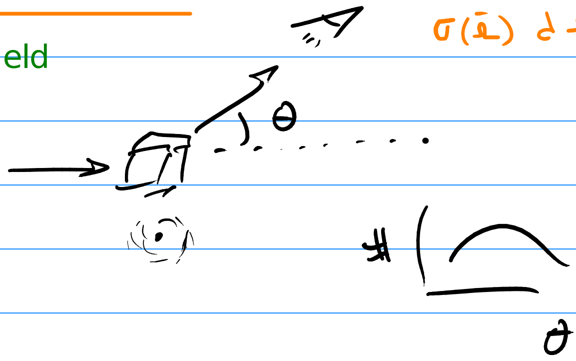
Kepler's 3rd: "square of the periods of the *various* planets are proportional to the cube of their major axes"

In reality, the constant of proportionality depends on the mass of each planet!

$\sigma(\bar{\Omega})\, d\Omega$

Scattering in a central force field

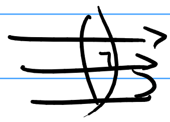




Several assumptions:

1. scattering is provoked by a central force,
2. there is a uniform stream of identical particles characterized by its intensity I = #particles crossing unit area normal to the beam in unit time,
3. forces fall off to zero for large distances.

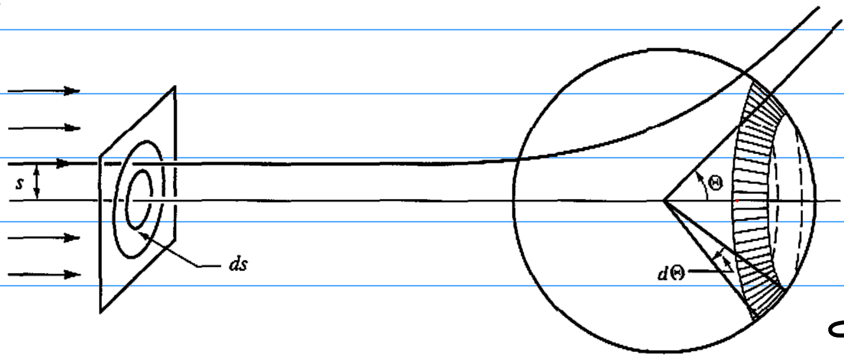
$$I = \frac{\#\text{part}}{\text{area} \cdot \text{time}}$$



$\sigma(\bar{\Omega})\, d\Omega =$ # particles scattered into a solid angle dOmega per unit time divided by the intensity

$$\frac{\#\text{part}}{\text{time}} \cdot \frac{\text{area} \cdot \text{time}}{\#\text{part}} \rightarrow \text{area}$$

$d\Omega$ element of solid angle in the direction of $\bar{\Omega}$



$$d\Omega = 2\pi \sin\theta\, d\theta$$

θ — scattering angle - angle between scattered and incident directions

s — impact parameter - perp. distance from center of force to an incident particle

v_0 — incident speed of the particle

E — energy of the particle

$E = \frac{1}{2} m v_0^2$

Angular momentum $\ell = m v_0 s = s\sqrt{2mE}$

Note that # particles scattered into solid angle $d\Omega$ between $\theta + d\theta$

•must equal # incident particles between s and $s+ds$

intensity

$$\underbrace{I \underbrace{2\pi s|ds|}_{\text{area of ring}}}_{\text{\# particles}} = I\,\sigma(\Omega)\,d\Omega$$

$$= I\,\sigma(\theta)\,2\pi \sin\theta\,|d\theta|$$