Recap 1: First and Second solutions

- SOLDE has two independent solutions – general solution is linear combination
- $\sin x$, $\cos x$
- $P_l(x),\ Q_l(x)$
- $J_m(x),\ Y_m(x)$
- $H_m^{(1,2)}(x) = J_m(x)\ \pm\ iY_m(x)$
- $j_m(x),\ y_m(x), h_m^{(1,2)}(x)$
- behavior at $x = 0$ and $x = \infty$

Recap 2: Sturm-Liouville Theory

- Linear differential operator $\mathcal{L}$

$$\mathcal{L}y \equiv p_0(x)y'' + p_1(x)y' + p_2(x)y = 0$$

- Inner product of functions f and g

$$(f, g) = \int_a^b f^*(x)g(x)dx$$

- Adjoint operator $\bar{\mathcal{L}}$

$$(\bar{\mathcal{L}}v, u) = (v, \mathcal{L}u)$$

- If $\bar{\mathcal{L}} = \mathcal{L}$, then $\mathcal{L}$ is self-adjoint.
- Under what circumstances is that the case?

Recap 3: Sturm-Liouville Theory

- For real p_i, $(v, \mathcal{L}u) = (\mathcal{L}v, u)$ iff

$$\left[v^* p_0 u' - v^{*\prime} p_0 u - v^* p_0' u + v^* p_1 u\right]_a^b = 0$$

boundary conditions

and

$$(v^* p_0)'' - (v^* p_1)' + \cancel{v^* p_2} = p_0 v^{*\prime\prime} + p_1 v^{*\prime} + \cancel{p_2 v^*}$$

form of the ODE

$$\Longrightarrow \quad \cancel{v^{*\prime\prime} p_0} + 2v^{*\prime} p_0' + v^* p_0'' - v^{*\prime} p_1 - v^* p_1' = \cancel{p_0 v^{*\prime\prime}} + p_1 v^{*\prime}$$

$$\Longrightarrow \quad 2v^{*\prime}(p_0' - p_1) + v^*(p_0'' - p_1') = 0$$

independent of p_2

- Latter condition is satisfied if $p_1 = p_0'$
- Then the boundary conditions imply

$$\left[p_0 (v^* u' - v^{*\prime} u)\right]_a^b = 0$$

Sturm-Liouville Theory

- Suspend "why are we doing this?" for just a little longer…
- In the self-adjoint case, $p_1 = p_0'$, we can rewrite, again conventionally (sorry!)

$$\mathcal{L}y \equiv (\underbrace{p(x)}_{p_0}y')' + \underbrace{q(x)}_{p_2}y = 0$$

- How do our equations of interest stack up?
- SHO: $p_0 = 1,\ p_1 = 0,$ already self adjoint
- Legendre: $p_0 = 1 - x^2,\ p_1 = -2x,$ already self adjoint
- Bessel: $p_0 = x^2,\ p_1 = x,$ not self adjoint, but…
- Can <u>always</u> transform into self-adjoint form by multiplying by an <u>integrating factor</u>:

$$\mathcal{L} \to \frac{1}{p_0} e^{\int \frac{p_1}{p_0} dx} \mathcal{L}$$

works because $\left(e^{\int \frac{p_1}{p_0}}\right)' = \frac{p_1}{p_0} e^{\int \frac{p_1}{p_0}}$

Sturm-Liouville Theory

- Example: Bessel, $p_0 = x^2, p_1 = x$, integrating factor is

$$\frac{1}{p_0} e^{\int \frac{p_1}{p_0}} = \frac{1}{x^2} e^{\int \frac{1}{x} dx} = \frac{1}{x^2} e^{\log x} = \frac{1}{x}$$

- Self-adjoint form is

$$xy'' + y' + \left(\frac{x^2 - m^2}{x}\right) y = 0$$

again, q doesn't matter

- Bottom line: <u>all</u> SOLDEs can be put into self-adjoint form
- All the functions we have been studying play by the same rules!

$$\mathcal{L}y \equiv \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = 0$$

- <u>close</u> connection between self-adjoint operators here and hermitian matrices in linear algebra, and hermitian operators in QM.

Eigenfunction Problems

- Look at eigenvalue problems of the form

$$\mathcal{L}u \; + \; \lambda w(x)u = 0, \;\; \mathcal{L}u \; \equiv (p(x)u')' + q(x)u$$

- Eigenvalue is λ, $w(x) \geq 0$ is an optional weighting function; often $w(x) = 1$; solution is subject to boundary conditions

- Connection to problems of interest:
  - SHO: $u'' + \lambda u = 0$

    $\mathcal{L}u = u'', \; q(x) = 0, \; w(x) = 1, \; \lambda = k^2$ (say)
  - Legendre: $(1 - x^2)u'' - 2xu' + l(l+1)u = 0$

    $\mathcal{L}u = (1 - x^2)u'' - 2xu', \; q(x) = 0, \; w(x) = 1, \; \lambda = l(l+1)$
  - Bessel: $xu'' + u' + \left(x - \frac{m^2}{x}\right)u = 0$?? not in standard form

Eigenfunction Problems

- For Bessel, need to go back to the original context $(x = n\rho)$:

$$\rho(\rho \mathrm{P}')' + (n^2\rho^2 - m^2)\mathrm{P} = 0$$

$$(\rho \mathrm{P}')' + (n^2\rho - \frac{m^2}{\rho})\mathrm{P} = 0$$

$$\rho \mathrm{P}'' + \mathrm{P}' + \left(n^2\rho - \frac{m^2}{\rho}\right)\mathrm{P} = 0$$

now $\mathcal{L}^{(m)}\mathrm{P} = \rho \mathrm{P}'' + \mathrm{P}' - \frac{m^2}{\rho}\mathrm{P},\ w(x) = 1,\ \lambda = n^2$

- Solutions are $J_m(n\rho),\ Y_m(n\rho)$

Summary of Solutions

- SHO:

$$u'' + k^2 u = 0$$
$$\mathcal{L}u = u'',\ \ q(x) = 0,\ \ w(x) = 1,\ \ \lambda = k^2$$

eigenfunctions: $\sin kx,\ \ \cos kx$

- Legendre:

$$(1 - x^2)u'' - 2xu' + l(l+1)u = 0$$
$$\mathcal{L}u = (1 - x^2)u'' - 2xu',\ \ q(x) = 0,\ \ w(x) = 1,\ \ \lambda = l(l+1)$$

eigenfunctions: $P_l(x), Q_l(x)$

- Bessel:

$$\rho u'' + u' + \left(n^2\rho - \frac{m^2}{\rho}\right)u = 0$$
$$\mathcal{L}^{(m)}u = \rho u'' + u' - \frac{m^2}{\rho}u,\ \ w(\rho) = \rho,\ \ \lambda = n^2$$

eigenfunctions: $J_m(n\rho),\ \ Y_m(n\rho)$

Eigenfunction Problems

- Eigenfunctions u, v must satisfy boundary conditions:

  e.g. vibrations on a string, $u, v = 0$ at $x = 0, L$

  so $\quad \left[p_0(v^* u' - v^{*\prime} u)\right]_0^L = 0$ necessarily.
- Sometimes the form of p_0 helps:

  e.g. Legendre, on $[-1,1]$, $p_0 = 1 - x^2 = 0$ at each end of the range.
- Sometimes it's a combination:

  e.g. Bessel, on $[0, \infty]$, $p_0 = \rho = 0$ at $\rho = 0$,

  but $J \propto \rho^{-\frac{1}{2}}$, $J' \propto \rho^{-\frac{3}{2}}$, so $\rho J J' \to 0$ as $\rho \to \infty$
- Must check every time for the specific domain of interest.

Properties of Eigenfunctions

- Assume going forward that the eigenfunctions u_i satisfy the appropriate boundary conditions.
- Given the connection between the eigenfunctions of self-adjoint operators and eigenvectors of hermitian matrices, might expect:
- If $\mathcal{L}$ is self-adjoint and $\mathcal{L}u_i + \lambda_i w(x) u_i = 0$, then
  1. the eigenvalues λ_i are <u>real</u>,
  2. the eigenfunctions u_i are <u>orthogonal</u>,
  3. the eigenfunctions u_i are <u>complete</u>.

$$\int w(x)\, u_i^* u_j \, dx = \delta_{ij}$$

$$f(x) = \sum_i a_i u_i(x)$$

- Prove (1) and (2), come back to (3) in a moment.

Eigenfunctions and Eigenvalues

- Suppose $\mathcal{L}u_i + \lambda_i w(x) u_i = 0$ (†)

  and $\mathcal{L}u_j + \lambda_j w(x) u_j = 0$

  so $\mathcal{L}u_j^* + \lambda_j^* w(x) u_j^* = 0$ (††) $\mathcal{L}$ and w real

- Then u_j^* (†) $- u_i$ (††)

  $\Longrightarrow$ $u_j^* \mathcal{L}u_i + \lambda_i\, w(x) u_i u_j^* - u_i \mathcal{L}u_j^* - \lambda_j^* w(x) u_i u_j^* = 0$

  $\Longrightarrow$ $u_j^* \mathcal{L}u_i - u_i \mathcal{L}u_j^* = (\lambda_j^* - \lambda_i)\, w(x)\, u_i u_j^*$

  Integrate from a to b

  $\Longrightarrow$ $\int_a^b u_j^* \mathcal{L}u_i\, dx - \int_a^b \mathcal{L}u_j^* u_i dx = (\lambda_j^* - \lambda_i) \int_a^b w(x)\, u_i u_j^*\, dx$

  $(u_j, \mathcal{L}u_i) - (\mathcal{L}u_j, u_i) = 0$ $\mathcal{L}$ self-adjoint

Eigenfunctions and Eigenvalues

- We have shown, for eigenfunctions u_i and u_j, that

$$(\lambda_j^* - \lambda_i) \int_a^b w(x)\, u_i u_j^*\, dx = 0$$

- If $i = j$, then

$$\int_a^b w(x)\, u_i u_j^*\, dx = \int_a^b w(x)\, |u_i|^2\, dx > 0,$$

so $\lambda_i^* = \lambda_i$ $\qquad$ λ_i is <u>real</u>

- If $i \neq j$, then, if $\lambda_j^* \neq \lambda_i$,

$$\int_a^b w(x)\, u_i u_j^*\, dx = 0$$

u_i, u_j <u>orthogonal</u>

(with caveats)

Eigenfunctions and Eigenvalues

- Definition of orthogonality for eigenfunctions <u>includes</u> the weighting function w — part of the eigenvalue problem

$$\int_a^b w(x)\, u_i u_j^*\, dx = 0\,, \quad i \neq j$$

- Let's <u>redefine</u> the inner product from here on to include w

$$(u_j, u_i) \equiv \int_a^b w(x)\, u_i u_j^*\, dx$$

- Can always <u>normalize</u> the eigenfunctions (linear system) so that

$$\|u_i\|^2 \equiv (u_i, u_i) = \int_a^b w(x)\, |u_i|^2\, dx = 1$$

- Eigenfunctions form an <u>orthonormal</u> set, so

$$(u_j, u_i) = \int_a^b w(x)\, u_i u_j^*\, dx = \delta_{ij}$$

- Small problem: proof doesn't work if $\lambda_j^* = \lambda_j = \lambda_i$

Eigenfunctions and Eigenvalues

- Proof fails if $\lambda_j^* = \lambda_j = \lambda_i$ — degeneracy: we have a subspace of eigenfunctions all with the same eigenvalue.
- Use Gram-Schmidt orthogonalization to create an orthonormal set.

  e.g. suppose u_1, u_2, u_3 all have the same eigenvalue λ.
- Let $e_1 = u_1 / \|u_1\|$, so $\|e_1\| = 1$.
- Let $v_2 = u_2 - (u_2, e_1)e_1$, so $(v_2, e_1) = 0$.
- Let $e_2 = v_2 / \|v_2\|$, so $\|e_2\| = 1, (e_2, e_1) = 0$.
- Repeat: let $v_3 = u_3 - (u_3, e_1)e_1 - (u_3, e_2)e_2$, so $(v_3, e_1) = (v_3, e_2) = 0$.
- Let $e_3 = v_3 / \|v_3\|$, and (e_1, e_2, e_3) are orthonormal, all with eigenvalue λ.
- Can always do this, so can always construct a set of functions such that

$$(u_j, u_i) = \int_a^b w(x)\, u_i u_j^*\, dx = \delta_{ij}$$

Completeness of Eigenfunctions

- Completeness $\Longrightarrow$ in the vector space of functions on $[a, b]$, the eigenfunctions u_i of <u>any</u> self-adjoint differential operator form a basis set.
- More formally, <u>any</u> function $f(x)$ on $[a, b]$ can be expanded as a generalized Fourier series of the form

$$f(x) = \sum_{i=0}^{\infty} a_i u_i(x)$$

- Heuristically,

$$\begin{aligned}
(u_j, f) &= \int_a^b w(x)\, u_j^*(x)\, f(x)\, dx \\
&= \int_a^b w(x)\, u_j^*(x) \sum_{i=0}^{\infty} a_i u_i(x)\, dx \\
&= \sum_{i=0}^{\infty} a_i \int_a^b w(x)\, u_j^*(x) u_i(x)\, dx \\
&= \sum_{i=0}^{\infty} a_i \delta_{ij} \\
&= a_j
\end{aligned}$$

Need to justify the steps

Need to consider convergence

Convergence of the Expansion

- In general, the series converges in the "mean square" sense:

$$\lim_{n\to\infty} \int_a^b w(x)\, [f(x) - \textstyle\sum_{i=0}^{n} a_i u_i(x)]^2\, dx = 0,$$

where $a_i = (u_i, f)$.

- Weak kind of convergence
  - not pointwise
  - not uniform (necessary to justify integration, differentiation term by term, swapping integral and sum, etc.)
  - runs into problems near a discontinuity in f.
- Return to this later, but first focus on the big picture — we now have a way to <u>invert</u> the series solutions we saw earlier in the study of PDEs.

Fourier Series

- For the simple harmonic oscillator, $\mathcal{L}u = u''$, $w(x) = 1$, $\lambda = k^2$

  boundary conditions: <u>periodic</u> on $[0, L]$

  eigenfunctions: $u_k = \sin kx,\ \cos kx$, BC $\Longrightarrow k = \frac{2\pi n}{L}$, n integer

  normalization: $u_n = \sqrt{\frac{2}{L}}\sin\frac{2\pi nx}{L},\ \sqrt{\frac{2}{L}}\cos\frac{2\pi nx}{L},\ u_0 = \sqrt{\frac{1}{L}}$

  because $\int_0^L \sin^2\frac{2\pi nx}{L}\,dx = \frac{1}{2}L,$

  $\int_0^L \cos^2\frac{2\pi nx}{L}\,dx = \frac{1}{2}L$

  $\int_0^L dx = L.$

- Can check $(u_m, u_n) = \delta_{mn}$.

Fourier Series

- Then Fourier series for f is

$$f(x) \ = \sum_{n=1}^{\infty}\left(\alpha_n \sqrt{\frac{2}{L}}\cos\frac{2\pi nx}{L} + \beta_n \sqrt{\frac{2}{L}}\sin\frac{2\pi nx}{L}\right) + \alpha_0$$

where

$$\alpha_n = \int_0^L w(x)\sqrt{\frac{2}{L}}\cos\frac{2\pi nx}{L}\ f(x)\,dx\,,\ \ \beta_n = \int_0^L w(x)\sqrt{\frac{2}{L}}\sin\frac{2\pi nx}{L}\ f(x)\,dx$$

$$\alpha_0 = \int_0^L w(x)\sqrt{\frac{1}{L}}f(x)\ dx$$

- More conventionally,

$$f(x) \ = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{2\pi nx}{L} + b_n\sin\frac{2\pi nx}{L}\right),$$

where $\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \frac{2}{L}\int_0^L \begin{pmatrix} \cos\frac{2\pi nx}{L} \\ \sin\frac{2\pi nx}{L} \end{pmatrix} f(x)\,dx$

Legendre Series

- For Legendre's equation, $\mathcal{L}u = (1 - x^2)u'' - 2xu', \ w(x) = 1, \ \lambda = l(l+1)$

  boundary conditions: any, on $[-1, 1]$

  eigenfunctions: $u_l = P_l(x), \ \ l$ integer

  orthogonality: $(P_l, P_m) = \ A_l \delta_{lm}$

  will see $A_l = \frac{1}{l + \frac{1}{2}}$

- Hence

$$f(x) = \sum_{i=0}^{\infty} (u_i, f)\, u_i(x)$$

$$= \sum_{l=0}^{\infty} a_l P_l(x)$$

where

$$a_l = \left(l + \frac{1}{2}\right) \int_{-1}^{1} P_l(x) f(x)\, dx$$

Bessel Series

- For Bessel's equation, $\mathcal{L}^{(m)}u = \rho u'' + u' - \frac{m^2}{\rho}u,\ w(\rho) = \rho,\ \lambda = n^2$

  boundary conditions: u regular at $\rho = 0,\ u(a) = 0$

  eigenfunctions: $u_i = J_m\left(\frac{\alpha_{mi}\rho}{a}\right),\quad i =$ integer

  orthogonality: $(u_i, u_j) = \int_0^a \rho\, J_m\left(\frac{\alpha_{mi}\rho}{a}\right) J_m\left(\frac{\alpha_{mj}\rho}{a}\right)\, d\rho = B_{mi}^2 \delta_{ij}$

- Hence, can expand for $0 \leq \rho \leq a$

$$f(\rho) = \sum_{i=0}^{\infty} a_i \frac{1}{B_{mi}} J_m\left(\frac{\alpha_{mi}\rho}{a}\right) \qquad f(\rho) = \sum_{i=0}^{\infty} a_i J_m\left(\frac{\alpha_{mi}\rho}{a}\right)$$

where

$$a_i = \int_0^a \frac{1}{B_{mi}} J_m\left(\frac{\alpha_{mi}\rho}{a}\right) f(\rho)\, d\rho \qquad a_i = \int_0^a \frac{1}{B_{mi}^2} J_m\left(\frac{\alpha_{mi}\rho}{a}\right) f(\rho)\, d\rho$$

Vibration of a Circular Membrane

- Lecture 4: full solution is a weighted sum of normal-mode solutions:

$$u(r,\theta,t) = \sum_{m,i} J_m\left(\frac{\alpha_{mi} r}{a}\right)\left(C_{mi}\cos m\theta + D_{mi}\sin m\theta\right) e^{i\omega_{mi} t}$$

where $\omega_{mi} = \alpha_{mi} c/a$

- Complete the solution by fitting the initial conditions:

$$u(r,\theta) = \sum_{m,i} J_m\left(\frac{\alpha_{mi} r}{a}\right)\left(C_{mi}\cos m\theta + D_{mi}\sin m\theta\right)$$

- Fourier-Bessel series for C_{mi} and D_{mi}
- Now we know how to invert:

$$C_{mi} = \frac{1}{\pi B_{mi}^2}\int_0^a r dr \int_0^{2\pi} d\theta \; J_m\left(\frac{\alpha_{mi} r}{a}\right)\cos m\theta \, u(r,\theta,0)$$

and similarly for D_{mi}.

Application of Fourier Series to PDEs

- String, fixed at $x = 0,\ L$, displacement $u(x,t)$, satisfies wave equation

  $$u_{tt} = c^2 u_{xx}$$

- Expand u as a Fourier series in x satisfying the boundary conditions

  $$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\frac{n\pi x}{L}$$

- Substitute:

  $$\sum_{n=1}^{\infty} \ddot{a}_n(t) \sin\frac{n\pi x}{L} = c^2 \sum_{n=1}^{\infty} a_n(t) \left(-\frac{n^2\pi^2}{L^2}\right) \sin\frac{n\pi x}{L}$$

  $$\Longrightarrow \quad \ddot{a}_n + \left(\frac{n\pi c}{L}\right)^2 a_n = 0$$