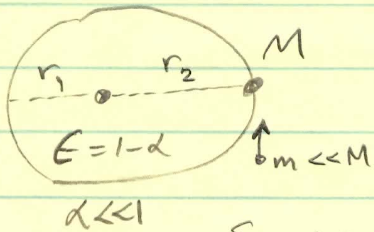
(3.10)



Minimum KE of m for (M+m) to go from ellipse to parabolic orbit.

Suppose v - velocity of planet before collision

v_c - " comet "

v' - " of planet + comet after collision

E - energy of the planet before collision that b/c it is an ellipse $E<0$.

the condition of a parabolic orbit is that

E' - energy after collision $E' = 0 = \frac{1}{2}(M+m)v'^2 - \frac{k'}{r_2}$

We used $\dot{r}$ at the aphelion and

$\boxed{\dot{r}=0}$

$K = GMM_s$ and $k' = G(M+m)M_s$

$$E = \frac{l^2}{2Mr_2^2} - \frac{k}{r_2} \Rightarrow \frac{k}{r_2} = \frac{l^2}{2Mr_2^2} - E$$

Substituting $\frac{1}{2}(M+m)v'^2 \simeq \boxed{\frac{1}{2}Mv'^2 = \frac{l^2}{2Mr_2^2} - E}$

Linear momentum is conserved $Mv + mv_c = (M+m)v' \simeq Mv'$

$$\boxed{l = Mr_2 v} \longrightarrow v' = \frac{m}{M}v_c + v = \frac{m}{M}v_c + \frac{l}{Mr_2}$$

Substitute v' above $\frac{1}{2}M\Big(\underbrace{\left(\frac{m}{M}v_c\right)^2}_{\approx 0} + \cancel{\frac{l^2}{M^2r_2^2}} + \frac{2m}{M^2}\cdot\frac{l}{r_2}v_c\Big) = \cancel{\frac{l^2}{2Mr_2^2}} - E$

$$\frac{m}{M}\frac{l}{r_2}v_c = -E$$

Lets get an expression for E:

$$\boxed{\epsilon = \sqrt{1 + \frac{2El^2}{Mk^2}}} \Rightarrow \epsilon^2 = 1 + \frac{2El^2}{Mk^2}$$

$$(1-\alpha)^2 = 1 - 2\alpha \simeq 1 + \frac{2El^2}{Mk^2} \Rightarrow E = -\frac{\alpha Mk^2}{l^2}$$

Substitute E to get

$$\frac{m}{M}\frac{l}{r_2}v_c = \frac{\alpha Mk^2}{l^2} \Rightarrow \boxed{v_c = \alpha\frac{M^2 r_2 K^2}{ml^3}}$$

Now lets get an expression for r_2 and l:

We know $\boxed{r_2 = a(1+\epsilon)} = a(1+1-\alpha) \cong 2a$

Semi major axis is

And we also know $\boxed{a = -\frac{k}{2E}} = \frac{k}{2}\cdot\frac{l^2}{\alpha M k^2} = \frac{l^2}{2\alpha M k} \Rightarrow l^2 = 2a\alpha M k$

Substituting r_2 and l^2 in above:

$$v_c^2 = \alpha^2 \frac{M^4 r_2^2 k^4}{m^2 l^6} = \alpha^2 \frac{M^4 k^4}{m^2} 4a^2 \frac{1}{8a^3\alpha^3 M^3 k^3}$$

$$= \frac{1}{\alpha}\cdot\frac{1}{2a}\frac{Mk}{m^2}$$

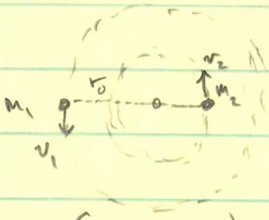
So that the energy of the comet is at least

$$E_c = \frac{1}{2} m v_c^2 = \frac{1}{2} m \frac{1}{2a\alpha}\frac{Mk}{m^2}$$

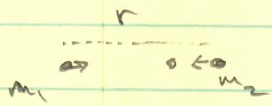
$$\boxed{E_c = \frac{1}{2\alpha}\cdot\frac{M}{m}\cdot\frac{k}{2a}}$$

this is a big number due to that are big.

(3.11)



(orbiting)



(stopped)

r_0 — initial distance

$\mu = \frac{m_1 m_2}{m_1 + m_2}$

When it stops, $E = \frac{1}{2}\mu\dot{r}^2 - \frac{k}{r}$ (note $l = 0$)

reduced mass

distance between masses

$k = G m_1 m_2$

Just after stopping (initial condition) $E = -\frac{k}{r_0}$

Energy conservation $\frac{1}{2}\mu\dot{r}^2 - \frac{k}{r} = -\frac{k}{r_0}$

$$\dot{r}^2 = \frac{2k}{\mu}\left(\frac{r_0 - r}{r_0 r}\right) \Rightarrow \dot{r} = -\sqrt{\frac{2k}{\mu}}\sqrt{\frac{r_0 - r}{r_0 r}}$$

choose negative root b/c $\dot{r} < 0$, r is decreasing

$$\int_0^t dt' = \int_{r_0}^0 -\sqrt{\frac{\mu}{2k}}\sqrt{\frac{r_0 r}{r_0 - r}}\, dr$$

change variables to
$r = r_0 \sin^2\theta$
$dr = 2r_0 \sin\theta\cos\theta\, d\theta$

$$t = -\sqrt{\frac{\mu r_0}{2k}}\, 2r_0 \int_{\pi/2}^0 \sin^2\theta\, d\theta$$

$$t = -r_0\sqrt{\frac{\mu r_0}{2k}}\,(\theta - \sin\theta\cos\theta)\Big|_{\pi/2}^0 = r_0\sqrt{\frac{\mu r_0}{2k}}\left(\frac{\pi}{2}\right) = \sqrt{\frac{\mu}{2k}}\, r_0^{3/2}\,\frac{\pi}{2}$$

But we know that $\tau = 2\pi r_0^{3/2}\sqrt{\frac{\mu}{k}}$

So that $\boxed{t = \frac{\tau}{4\sqrt{2}}}$

(3.14) A) Compare perihelion distance of a parabolic orbit with radius of circular orbit. Both have same l.

Use $\frac{1}{r} = \frac{mk}{l^2}(1+\epsilon\cos\theta)$

Parabolic: perihelion occurs at $\theta = 0$ with $\epsilon = 1$

$$\frac{1}{r_p^p} = \frac{mk}{l^2}(1+1) = 2\frac{mk}{l^2}$$

Circular: $\epsilon = 0$ $\quad \frac{1}{r_c} = \frac{mk}{l^2}$

$$\Rightarrow r_p^p = \frac{1}{2}\cdot\frac{l^2}{mk} \Rightarrow \boxed{r_p^p = \frac{1}{2} r_c}$$

B) Starting from $r = \frac{l^2}{mk}\cdot\frac{1}{(1+\epsilon\cos\theta)}$

$$\dot{r} = +\frac{l^2}{mk}\frac{\epsilon\sin\theta}{(1+\epsilon\cos\theta)^2}\dot\theta = r\frac{\epsilon\sin\theta}{1+\epsilon\cos\theta}\dot\theta$$

Because $\bar{v} = \dot r\hat r + r\dot\theta\hat\theta \Rightarrow v^2 = \dot r^2 + r^2\dot\theta^2$ $\quad \dot\theta = l/mr^2$

so that $v^2 = \left[\left(\frac{\epsilon\sin\theta}{1+\epsilon\cos\theta}\right)^2 + 1\right] r^2\dot\theta^2 = \frac{(1+\epsilon^2+2\epsilon\cos\theta)}{(1+\epsilon\cos\theta)^2} r^2\left(\frac{l}{mr^2}\right)^2$

Circle: $\epsilon = 0$ $\quad v_c^2 = \frac{l^2}{m^2r^2} = \frac{l^2}{m^2r}\cdot\frac{mk}{l^2} = \frac{k}{mr}$ $\quad$ (*)

parabola: $\epsilon = 1$: $v_p^2 = \frac{2+2\cos\theta}{(1+\cos\theta)^2}\cdot\frac{l^2}{m^2r^2} = \frac{2l^2}{m^2r^2}\left(\frac{1}{1+\cos\theta}\right)$

but $r = \frac{l^2}{mk}\left(\frac{1}{1+\cos\theta}\right)$

$$v_p^2 = \frac{2l^2}{m^2r^2}\cdot\frac{mkr}{l^2} = \frac{2k}{mr} \quad (**)$$

So that $\boxed{v_p = \sqrt{2}\, v_c}$