Quantum Midterm Keri Hener

#1 $(x+iap)(x-iap) = x^2 \cancel{-iapx + iapx} - i^2a^2p^2$

$= x^2 + a^2p^2$

$\langle\alpha| x^2 + a^2p^2 |\alpha\rangle$ $\qquad (x+y)|\gamma\rangle = x|\gamma\rangle + y|\gamma\rangle$

$= \langle\alpha|x^2|\alpha\rangle + \langle\alpha|a^2p^2|\alpha\rangle$

$= \langle\alpha|x^2|\alpha\rangle + a^2\langle\alpha|p^2|\alpha\rangle = \boxed{f(x') + a^2 g(p')}$

#2 $|\vec{S}\cdot\hat{n};+\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle$

a) for $\theta = 0$, $|\vec{S}\cdot\hat{n};+\rangle = |+\rangle$

for $\theta = \frac{\pi}{2}$, $|\vec{S}\cdot\hat{n};+\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$ $\qquad$ since $\cos(\frac{\pi}{4}) = \sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

b) in S_z basis, $S_y = \frac{\hbar}{2}[-i|+\rangle\langle-| + i|-\rangle\langle+|]$

$\langle\vec{S}\cdot\hat{n};+| S_y |\vec{S}\cdot\hat{n};+\rangle = \langle\vec{S}\cdot\hat{n};+|\frac{\hbar}{2}[-i|+\rangle\langle-| + i|-\rangle\langle+|\,|\vec{S}\cdot\hat{n};+\rangle$

$\langle\vec{S}\cdot\hat{n};+|\frac{\hbar}{2}[-i|+\rangle\langle-|\vec{S}\cdot\hat{n};+\rangle + i|-\rangle\langle+|\vec{S}\cdot\hat{n};+\rangle]$

$= i\frac{\hbar}{2}[\langle\vec{S}\cdot\hat{n};+|-\rangle\langle+|\vec{S}\cdot\hat{n};+\rangle - \langle\vec{S}\cdot\hat{n};+|+\rangle\langle-|\vec{S}\cdot\hat{n};+\rangle]$

$= i\frac{\hbar}{2}[$

c) $\langle(\Delta S_z)^2\rangle\langle(\Delta S_x)^2\rangle \geq \frac{1}{4}|\langle[S_z, S_x]\rangle|^2$ $\qquad S_z = \frac{\hbar}{2}[|+\rangle\langle+| - |-\rangle\langle-|]$

$\geq \frac{1}{4}|\langle S_zS_x - S_xS_z\rangle|^2$ $\qquad S_x = \frac{\hbar}{2}[|+\rangle\langle-| - |-\rangle\langle+|]$

$\langle[S_z, S_x]\rangle$ in state $|\vec{S}\cdot\hat{n};+\rangle = \langle\vec{S}\cdot\hat{n};+|S_zS_x - S_xS_z|\vec{S}\cdot\hat{n};+\rangle$

$\langle\vec{S}\cdot\hat{n};+| = \cos\frac{\theta}{2}\langle+| + \sin\frac{\theta}{2}\langle-|$

$\cos\frac{\theta}{2}\langle+| + \sin\frac{\theta}{2}\langle-| \,|S_zS_x - S_xS_z|\, \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|+\rangle$

$$S_z S_x - S_x S_z = \frac{\hbar^2}{4}[|+\rangle\langle+| - |-\rangle\langle-|][|+\rangle\langle-| - |-\rangle\langle+|] - \frac{\hbar^2}{4}[|+\rangle\langle-| - |-\rangle\langle+|][|+\rangle\langle+| - |-\rangle\langle-|]$$

$$= \frac{\hbar^2}{4}[|+\rangle\langle+|+\rangle\langle-| - \cancel{|+\rangle\langle+|-\rangle}\langle+| - \cancel{|-\rangle\langle-|+\rangle}\langle-| + |-\rangle\langle-|-\rangle\langle+|]$$

$$-\frac{\hbar^2}{4}[\cancel{|+\rangle\langle-|+\rangle}\langle+| - |+\rangle\langle-|-\rangle\langle-| - |-\rangle\langle+|+\rangle\langle+| + \cancel{|-\rangle\langle+|-\rangle}\langle-|]$$

$$= \frac{\hbar^2}{4}[$$

$* \langle+|-\rangle = 0$

d) $U = \sum_k |b^k\rangle\langle a^k|$ need $U|S_z;\pm\rangle = |S_x;\pm\rangle$

U diagonal matrix

$U = |S_z;+\rangle\langle S_x;+| + |S_z;-\rangle\langle S_x;-|$

$|S_x;\pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{1}{\sqrt{2}}|-\rangle$

want $|S_z;\pm\rangle = \frac{1}{\sqrt{2}}|S_x;+\rangle \pm \frac{1}{\sqrt{2}}|S_x;-\rangle$

$$U = \begin{pmatrix} \langle S_z;+|S_x;+\rangle & \langle S_z;+|S_x;-\rangle \\ \langle S_z;-|S_x;+\rangle & \langle S_z;-|S_x;-\rangle \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \langle+|(|+\rangle+|-\rangle) & \langle+|(|+\rangle- \\ \langle-|(|+\rangle+|-\rangle) & \langle-|(|+\rangle-|-\rangle) \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} \langle+|+\rangle + \cancel{\langle+|-\rangle} & \langle+|+\rangle - \cancel{\langle+|-\rangle} \\ \cancel{\langle-|+\rangle} + \langle-|-\rangle & \cancel{\langle-|+\rangle} - \langle-|-\rangle \end{pmatrix} \boxed{= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}$$

e) $|\vec{S}\cdot\hat{n};+\rangle = \cos\frac{\theta}{2}|S_z;+\rangle + \sin\frac{\theta}{2}|S_z;-\rangle$

$U|\vec{S}\cdot\hat{n};+\rangle = \cancel{\cos\frac{\theta}{2}}$

$$= \cos\frac{\theta}{2}\left[\frac{1}{\sqrt{2}}|S_x;+\rangle + \frac{1}{\sqrt{2}}|S_x;-\rangle\right] + \sin\frac{\theta}{2}\left[\frac{1}{\sqrt{2}}|S_x;+\rangle - \frac{1}{\sqrt{2}}|S_x;-\rangle\right]$$

$$= \left(\cos\left(\frac{\theta}{2}\right)\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}\right)|S_x;+\rangle + \left(\frac{1}{\sqrt{2}}\cos\left(\frac{\theta}{2}\right) - \frac{1}{\sqrt{2}}\sin\left(\frac{\theta}{2}\right)\right)|S_x;-\rangle$$

f) S_y in S_x?

$$\boxed{S_y = \frac{\hbar}{2}\left[|+\rangle\langle -| - |-\rangle\langle +|\right]}$$

$[S_i, S_j] = i\hbar\,\varepsilon_{ijk} S_k$

x y z

to preserve, let S_y be S_x $= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

S_z be S_y

$$S_y \doteq \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathcal{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\mathcal{U}(S_y) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

g) $tr(S_y)$ in S_x and S_z basis

$$tr(X) = \sum_{a'} \langle a'|X|a'\rangle$$

in S_x: $tr(S_y) = \Sigma$