PHYS 501: Mathematical Physics I

Fall 2020

Solutions to Homework #4

1. (a) (i) Setting $z = 1$, we find

$$\sum_{n=0}^{\infty} P_n(1) h^n \;=\; \left(1 - 2h + h^2\right)^{-1/2} \;=\; (1-h)^{-1}.$$

Application of the binomial theorem gives

$$(1-h)^{-1} = \sum_{n=0}^{\infty} h^n,$$

so $P_n(1) = 1$ for all n.

(ii) Setting $z = 0$, we have

$$\sum_{n=0}^{\infty} P_n(1) h^n \;=\; \left(1 + h^2\right)^{-1/2}.$$

The binomial expansion of the expression on the right is

$$\left(1 + h^2\right)^{-1/2} \;=\; 1 - \tfrac{1}{2}h^2 + \tfrac{3}{8}h^4 + \cdots + \frac{(-1)^m 1.3.5...(2m-1)}{2^m m!} h^{2m} + \cdots$$

Thus $P_{2m+1}(0) = 0$ (no odd powers of h) and

$$P_{2m}(0) \;=\; \frac{(-1)^m 1.3.5...(2m-1)}{2^m m!} \;=\; \frac{(-1)^m (2m-1)!!}{2^m m!} \;=\; \frac{(-1)^m (2m)!}{2^{2m} (m!)^2}.$$

(iii) Differentiating the generating function equation with respect to z gives

$$h\left(1 - 2zh + h^2\right)^{-3/2} \;=\; \sum_{n=0}^{\infty} P_n'(z) h^n,$$

so

$$\begin{aligned}
\sum_{n=0}^{\infty} P_n'(1) h^n \;=\; h(1-h)^{-3} \;&=\; h\left(1 + 3h + \cdots + \frac{(-3).(-4)...(-n-2)}{n!}(-h)^n + \cdots\right) \\
&=\; \sum_{n=0}^{\infty} \tfrac{1}{2}(n+1)(n+2) h^{n+1} \\
&=\; \sum_{m=1}^{\infty} \tfrac{1}{2} m(m+1) h^m
\end{aligned}$$

so

$$P_m'(1) \;=\; \tfrac{1}{2} m(m+1).$$

Since $P_0(z) = 1$, the result is true for $m = 0$ too.

(iv) Similarly,

$$\begin{aligned}\sum_{n=0}^{\infty} P_n'(0)h^n &= h\left(1+h^2\right)^{-3/2} \\ &= h - \tfrac{3}{2}h^3 + \cdots + \frac{(-3)(-5)...(-2m-1)}{2^m m!}h^{2m+1} + \cdots\end{aligned}$$

so $P_{2m}'(0) = 0$ and

$$P_{2m+1}'(0) = \frac{(-1)^m(2m+1)!!}{2^m m!} = \frac{(-1)^m(2m+1)!}{2^{2m}(m!)^2} = (2m+1)P_{2m}(0).$$

(b) Using Stirling's formula, as $m \to \infty$

$$\begin{aligned}P_{2m}(0) &= \frac{(-1)^m(2m)!}{2^{2m}(m!)^2} \\ &\approx \frac{(-1)^m\sqrt{4\pi m}\,(2m)^{2m}e^{-2m}}{2^{2m}\left(\sqrt{2\pi m}\,m^m e^{-m}\right)^2} \\ &= (-1)^m\frac{\sqrt{4\pi m}}{2\pi m} \\ &= \frac{(-1)^m}{\sqrt{\pi m}}.\end{aligned}$$

(c) Since $P_0(x) = 1$, the integral is $\int_{-1}^{1} P_0(x)P_n(x)\,dx = 2\delta_{n0}$ by the orthogonality property of the P_n.

(d) We can use one of the recurrence relations to write

$$P_n(x) = \frac{P_{n+1}'(x) - P_{n-1}'(x)}{2n+1},$$

so

$$\begin{aligned}I_n \equiv \int_0^1 P_n(x)dx &= \frac{1}{2n+1}\left[P_{n+1}(1) - P_{n+1}(0) - (P_{n-1}(1) - P_{n-1}(0))\right] \\ &= \frac{1}{2n+1}\left[P_{n-1}(0) - P_{n+1}(0)\right].\end{aligned}$$

Thus the integral is zero for even n; for odd $n = 2m+1$, using the result from part (a)(ii), we find

$$\begin{aligned}I_{2m+1} &= \frac{1}{4m+3}\left[\frac{(-1)^m(2m)!}{2^{2m}(m!)^2} - \frac{(-1)^{m+1}(2m+2)!}{2^{2m+2}[(m+1)!]^2}\right] \\ &= \frac{(-1)^m(2m)!}{(4m+3)2^{2m}(m!)^2}\left[1 + \frac{2m+1}{2(m+1)}\right] \\ &= \frac{(-1)^m(2m)!}{2^{2m+1}(m+1)(m!)^2}.\end{aligned}$$

2. (a) The general solution to Laplace's equation is

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \left(\alpha_{lm} r^l + \beta_{lm} r^{-l-1}\right) Y_l^m(\theta,\phi).$$

For $r < b$, the solution must satisfy the boundary condition that $\Phi = \Phi_0$ when $r = a$. Hence

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \left(\alpha_{lm} a^l + \beta_{lm} a^{-l-1}\right) Y_l^m(\theta,\phi) \;=\; \Phi_0 \;=\; \sqrt{4\pi}\, Y_0^0(\theta,\phi)\Phi_0,$$

(since $\sqrt{4\pi}\, Y_0^0 = 1$). For equality to hold, it must do so for every (l, m) pair. Hence

$$\begin{aligned} \alpha_{00} + \beta_{00}\, a^{-1} &= \sqrt{4\pi}\Phi_0 \\ \alpha_{lm} a^l + \beta_{lm}\, a^{-l-1} &= 0, \end{aligned}$$

so

$$\begin{aligned} \alpha_{00} &= \sqrt{4\pi}\Phi_0 - \beta_{00}\, a^{-1} \\ \alpha_{lm} &= -\beta_{lm}\, a^{-2l-1} \qquad (l \neq 0, m \neq 0). \end{aligned}$$

(b) For $r > b$ (replacing α and β inside by γ and δ outside) the boundary condition at infinity implies $\gamma_{lm} = 0$.

(c) The solutions then are

$$\Phi(r,\theta,\phi) \;=\; \begin{cases} \displaystyle\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \left(\alpha_{lm} r^l + \beta_{lm} r^{-l-1}\right) Y_l^m(\theta,\phi) & (r < b), \\ \displaystyle\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \delta_{lm} r^{-l-1}\, Y_l^m(\theta,\phi) & (r > b). \end{cases}$$

Continuity at $r = b$ implies

$$\alpha_{lm}\, b^l + \beta_{lm}\, b^{-l-1} \;=\; \delta_{lm}\, b^{-l-1}.$$

for all l, m.

(d) Due to the surface charge on the shell, the radial component of the electric field $-\partial\Phi/\partial r$ has a discontinuous jump at $r = b$, as described by Gauss's Law:

$$\left(-\frac{\partial\Phi}{\partial r}\right)_{r=b+} - \left(-\frac{\partial\Phi}{\partial r}\right)_{r=b-} \;=\; \frac{\sigma}{\epsilon_0}.$$

If σ is expressed as a spherical harmonic expansion

$$\sigma(b,\theta,\phi) \;=\; \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \sigma_{lm}\, Y_l^m(\theta,\phi),$$

then, applying this condition term by term, we must have

$$(l+1)\,\delta_{lm}\, b^{-l-2} - \left[-l\,\alpha_{lm}\, b^{l-1} + (l+1)\,\beta_{lm}\, b^{-l-2}\right] \;=\; \frac{\sigma_{lm}}{\epsilon_0}.$$

Note that the given expression for σ consists of just two modes (using the conventional definitions of Y_l^m given in Riley & Hobson, p. 340):

$$\begin{aligned} \sigma_{21} &= -\sqrt{\frac{8\pi}{15}}\,\sigma_0 \\ \sigma_{2,-1} &= -i\sigma_{21}. \end{aligned}$$

We can now solve for α_{lm}, β_{lm}, and δ_{lm}. Dropping the subscripts to avoid clutter, we have two cases:

(i) For $l = m = 0$, we have

$$\begin{aligned} \alpha + \beta\, a^{-1} &= \sqrt{4\pi}\Phi_0 \\ \alpha + \beta\, b^{-1} &= \delta\, b^{-1} \\ \delta\, b^{-2} - \beta\, b^{-2} &= \frac{\sigma}{\epsilon_0} \quad = \quad 0 \text{ here.} \end{aligned}$$

The solutions are easily shown to be

$$\begin{aligned} \alpha &= 0 \\ \beta \;=\; \delta &= \sqrt{4\pi}\, a\Phi_0. \end{aligned}$$

(ii) For other l and m, we have

$$\begin{aligned} \alpha + \beta\, a^{-2l-1} &= 0 \\ \alpha\, b^l + \beta\, b^{-l-1} &= \delta\, b^{-l-1} \\ (l+1)\delta\, b^{-l-2} + l\alpha b^{l-1} - (l+1)\beta\, b^{-l-2} &= \frac{\sigma}{\epsilon_0}. \end{aligned}$$

The solutions are

$$\begin{aligned} \alpha &= \frac{\sigma b^{-l+1}}{(2l+1)\epsilon_0} \\ \beta &= -\frac{\sigma b^{l+2}}{(2l+1)\epsilon_0}\left(\frac{a}{b}\right)^{2l+1} \\ \delta &= \frac{\sigma b^{l+2}}{(2l+1)\epsilon_0}\left[1 - \left(\frac{a}{b}\right)^{2l+1}\right]. \end{aligned}$$

Putting together the pieces, the complete solution for $r < a$ is

$$\begin{aligned} \Phi(r,\theta,\phi) &= \beta_{00} r^{-1}\, Y_0^0 + \alpha_{21} r^2\, Y_{21} + \beta_{21} r^{-3}\, Y_{21} + \alpha_{2,-1} r^2\, Y_{2,-1} + \beta_{2,-1} r^{-3}\, Y_{2,-1} \\ &= \Phi_0\left(\frac{a}{r}\right) + \frac{\sigma_0 b}{5\epsilon_0}\left[\left(\frac{r}{b}\right)^2 - \left(\frac{a}{b}\right)^5\left(\frac{b}{r}\right)^3\right]\sin 2\theta\cos\phi. \end{aligned}$$

For $r > b$,

$$\begin{aligned} \Phi(r,\theta,\phi) &= \delta_{00} r^{-1}\, Y_0^0 + \delta_{21} r^{-3}\, Y_{21} + \delta_{2,-1} r^{-3}\, Y_{2,-1} \\ &= \Phi_0\left(\frac{a}{r}\right) + \frac{\sigma_0 b}{5\epsilon_0}\left[1 - \left(\frac{a}{b}\right)^5\right]\left(\frac{b}{r}\right)^3\sin 2\theta\cos\phi. \end{aligned}$$

3. (a) Inside the spherical cavity formed by the two hemispheres, the general solution of Laplace's equation is

$$\phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(b_{lm} r^{-l-1} + c_{lm} r^{l} \right) P_l^m(\cos\theta)\, e^{im\phi},$$

where r, θ, ϕ are spherical polar coordinates and the line of contact between the hemispheres is at $\theta = \frac{\pi}{2}$. For ϕ to be regular at $r = 0$ we must have $b_{lm} = 0$; axial symmetry implies that $c_{lm} = 0$ for $m \neq 0$. Thus the solution is of the form

$$\phi(r, \theta) = \sum_{l=0}^{\infty} c_l\, r^l\, P_l(\cos\theta).$$

The boundary condition at $r = a$ is

$$\phi(a, \theta) = \sum_{l=0}^{\infty} c_l\, a^l\, P_l(\cos\theta) = \begin{cases} +V_0, & 0 \le \theta < \frac{\pi}{2}, \\ -V_0, & \frac{\pi}{2} < \theta \le \pi. \end{cases}$$

Inverting this Legendre series, we find

$$\begin{aligned} c_l\, a^l \left(\frac{2}{2l+1} \right) &= \int_0^{\pi} \phi(a, \theta) P_l(\cos\theta) d(\cos\theta) \\ &= V_0 \left[\int_{-1}^{0} -P_l(\mu) d\mu + \int_0^1 P_l(\mu) d\mu \right] \\ &= \begin{cases} 0 & (l \text{ even}) \\ 2V_0 \displaystyle\int_0^1 P_l(\mu) d\mu & (l \text{ odd}) \end{cases} \end{aligned}$$

Hence, for $l = 2m+1$, we can evaluate the integral using the solution to problem 1(d), to find

$$\phi(r, \theta) = V_0 \sum_{m=0}^{\infty} \frac{(-1)^m (4m+3)(2m)!}{2^{2m+1}(m+1)(m!)^2} \left(\frac{r}{a} \right)^{2m+1} P_{2m+1}(\cos\theta)\,.$$

(b) Now we are seeking a solution to the wave equation

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = 0,$$

with boundary conditions specified on the same hemispheres as in part (a), except that now the boundary values are variable,

$$\Phi(t, r = a) = \pm V_0 e^{-i\omega t},$$

and we want the solution for $r > a$.

As usual, we seek $e^{-i\omega t}$ time dependence, so the spatial part χ of the solution satisfies the Helmholtz equation

$$\nabla^2 \chi + k^2 \chi = 0,$$

with $k = \omega / c$. We require axisymmetry, so the general solution is

$$\Phi(r, \theta, t) = e^{-i\omega t} \sum_l \left[A_l j_l(kr) + B_l n_l(kr) \right] P_l(\cos\theta),$$

where we must retain both the j_l and the n_l solutions for $r > a$. Since the asymptotic forms are

$$j_l(x) \sim \frac{1}{x} \cos\left[x - \tfrac{\pi}{2}(l+1)\right], \quad n_l(x) \sim \frac{1}{x} \sin\left[x - \tfrac{\pi}{2}(l+1)\right],$$

as $x \to \infty$, the combination $h_l^{(1)} = j_l + i n_l$ clearly satisfies the "outgoing wave" condition as $r \to \infty$ and the solution takes the form

$$\Phi(r,\theta,t) = e^{-i\omega t} \sum_l C_l h_l^{(1)}(kr)\, P_l(\cos\theta).$$

At $r = a$,

$$\Phi(a,\theta,t) = e^{-i\omega t} \sum_l C_l h_l^{(1)}(ka)\, P_l(\cos\theta) = \pm V_0 e^{-i\omega t},$$

and the solution is essentially the same as in part (a), with $l = 2m+1$ and a^l replaced with $h_l^{(1)}(ka)$:

$$c_{2m+1}\, h_{2m+1}^{(1)}(ka) = V_0\, \frac{4m+3}{I_{2m+1}}.$$

(again using the terminology of Problem 1d). The complete solution therefore is

$$\phi(r,\theta,t) = V_0 e^{-i\omega t} \sum_{m=0}^{\infty} \frac{(-1)^m (4m+3)(2m)!}{2^{2m+1}(m+1)(m!)^2}\, \frac{h_{2n+1}^{(1)}(kr)}{h_{2n+1}^{(1)}(ka)}\, P_{2m+1}(\cos\theta)\,.$$

4. We wish to evaluate

$$\phi = \int d^3r_1 \int d^3r_2\ \psi^*(\mathbf{r}_1)\psi^*(\mathbf{r}_2)\, \frac{e^2}{r_{12}}\, \psi(\mathbf{r}_1)\psi(\mathbf{r}_2)\,,$$

where $\psi(\mathbf{r}) = \left(\frac{Z^3}{\pi a_0^3}\right)^{1/2} e^{-Zr/a_0}$, $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$, and $Z = 2$ here. Expand the r_{12}^{-1} term in spherical harmonics:

$$\frac{1}{r_{12}} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1}\ Y_l^m(\theta_1,\phi_1)^*\, Y_l^m(\theta_1,\phi_2)\ \frac{r_<^l}{r_>^{l+1}}\,,$$

where $r_< = \min(r_1,r_2)$ and $r_> = \max(r_1,r_2)$, with $r = |\mathbf{r}|$. Writing $d^3r = r^2 dr d\Omega$, the angular (Ω_1 and Ω_2) integrals give zero except for $l = m = 0$, and

$$\int d\Omega_1\, Y_0^0(\theta_1,\phi_1)^* = \int d\Omega_2\, Y_0^0(\theta_2,\phi_2) = \sqrt{4\pi}\,,$$

so

$$\phi = 16\pi^2 e^2 \left(\frac{Z^3}{\pi a_0^3}\right)^2 \int r_1^2 dr_1 \int r_2^2 dr_2\ e^{-2Zr_1/a_0} e^{-2Zr_2/a_0}\ \frac{1}{\max(r_1,r_2)}\,.$$

Splitting the r_2 integral into two parts ($0 < r_2 < r_1$ and $r_1 < r_2 < \infty$), we have

$$\phi = \frac{16Z^6e^2}{a_0^6} \int_0^{\infty} dr_1\, r_1\, e^{-2Zr_1/a_0}\ \left[\int_0^{r_1} dr_2\, r_2^2 e^{-2Zr_2/a_0} + r_1 \int_{r_1}^{\infty} dr_2\, r_2 e^{-2Zr_2/a_0}\right]\,.$$

After some algebra (or application of Maple), this yields the desired result

$$\phi = \frac{5Ze^2}{8a_0} = \frac{5e^2}{4a_0}\,.$$