Bessel Recurrence Relations

- Many recurrences, combine a few here.
- Derived from the generating function for integer m, but in fact true for all real m.

$$J_{m-1} + J_{m+1} = \frac{2m}{x} J_m$$

$$J_{m-1} - J_{m+1} = 2J_m'$$

$$J_{m\pm1} = \frac{m}{x} J_m \mp J_m'$$

$$(x^m J_m)' = x^m J_{m-1}$$

$$(x^{-m} J_m)' = -x^{-m} J_{m+1}$$

(sometimes useful for integration by parts)

Bessel Function Normalization

- Normalization integral

$$B_{mn}^2 = \int_0^a \rho\, J_m\left(\frac{\alpha_{mn}\rho}{a}\right) J_m\left(\frac{\alpha_{mn}\rho}{a}\right)\, d\rho$$

$$= \frac{a^2}{\alpha_{mn}^2}\int_0^{\alpha_{mn}} x\, J_m^2(x)\, dx = \frac{a^2}{\alpha_{mn}^2} I$$

- Recurrence relation: $xJ_m' = mJ_m - xJ_{m+1}$

$$\Longrightarrow\ I = \left[\frac{1}{2}(x^2 - m^2)J_m^2 + \frac{1}{2}(m^2 J_m^2 + x^2 J_{m+1}^2 - 2mxJ_m J_{m+1})\right]_0^{\alpha_{mn}}$$

$$= \frac{1}{2}\left[x^2 J_m^2 + x^2 J_{m+1}^2 - 2mxJ_m J_{m+1}\right]_0^{\alpha_{mn}}$$

$$= \frac{1}{2}{\alpha_{mn}}^2 J_{m+1}^2(\alpha_{mn})$$

- Hence $B_{mn}^2 = \frac{1}{2}a^2 J_{m+1}^2(\alpha_{mn})$

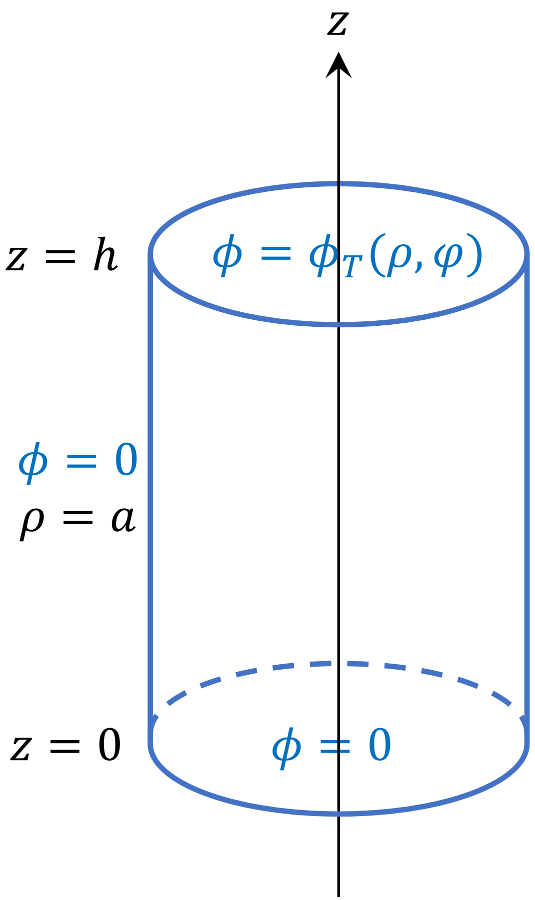
Example (3D): Laplace's Equation in a Cylinder

- BC at $z = h$ gives a Fourier-Bessel series for $\phi_T(\rho, \varphi)$:

$$\phi_T(\rho,\varphi) = \sum_{m,n} J_m\left(\frac{\alpha_{mn}\,\rho}{a}\right) \sinh\left(\frac{\alpha_{mn}\,h}{a}\right)$$
$$\times \left(C_{mn}\cos m\varphi + D_{mn}\sin m\varphi\right)$$

- Invert

$$\begin{pmatrix} C_{mn} \\ D_{mn} \end{pmatrix} = \frac{2}{\pi a^2 J_{m+1}^2\,(\alpha_{mn})} \sinh\left(\frac{\alpha_{mn}\,h}{a}\right)$$
$$\times \int_0^a \rho d\rho \int_0^{2\pi} d\varphi\, J_m\left(\frac{\alpha_{mn}\rho}{a}\right) \begin{pmatrix} \cos m\varphi \\ \sin m\varphi \end{pmatrix} \phi_T(\rho,\varphi)$$



Laplace Equation in a Cylinder, v.2

- Modify the BCs slightly
- Solution is a sum of terms of the form

$$\phi_{ml}(\rho, \varphi, z) = J_m(\lambda\rho)\, e^{\pm im\varphi} e^{\pm\lambda z}$$

- But the boundary condition at $z = 0, h$

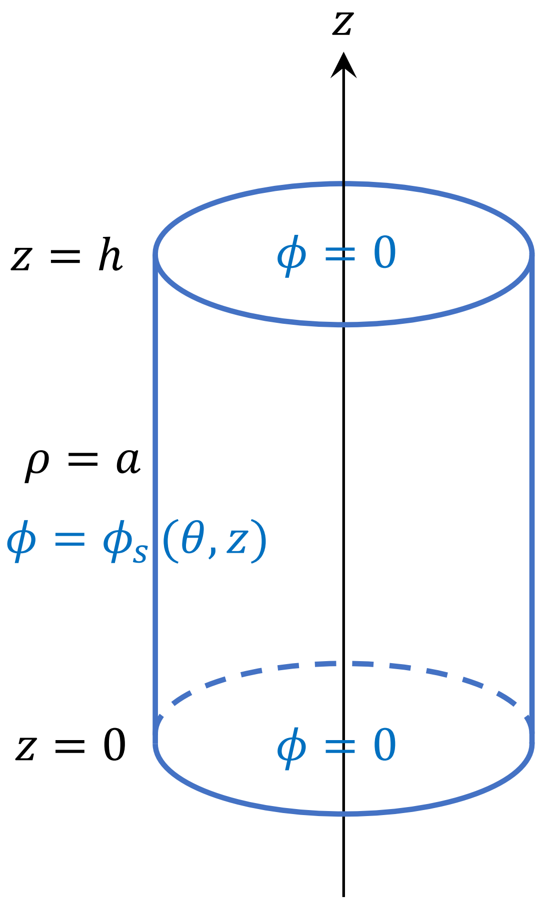
$$\Longrightarrow \lambda = il,\ \ l \text{ real},\ \ lh = n\pi$$

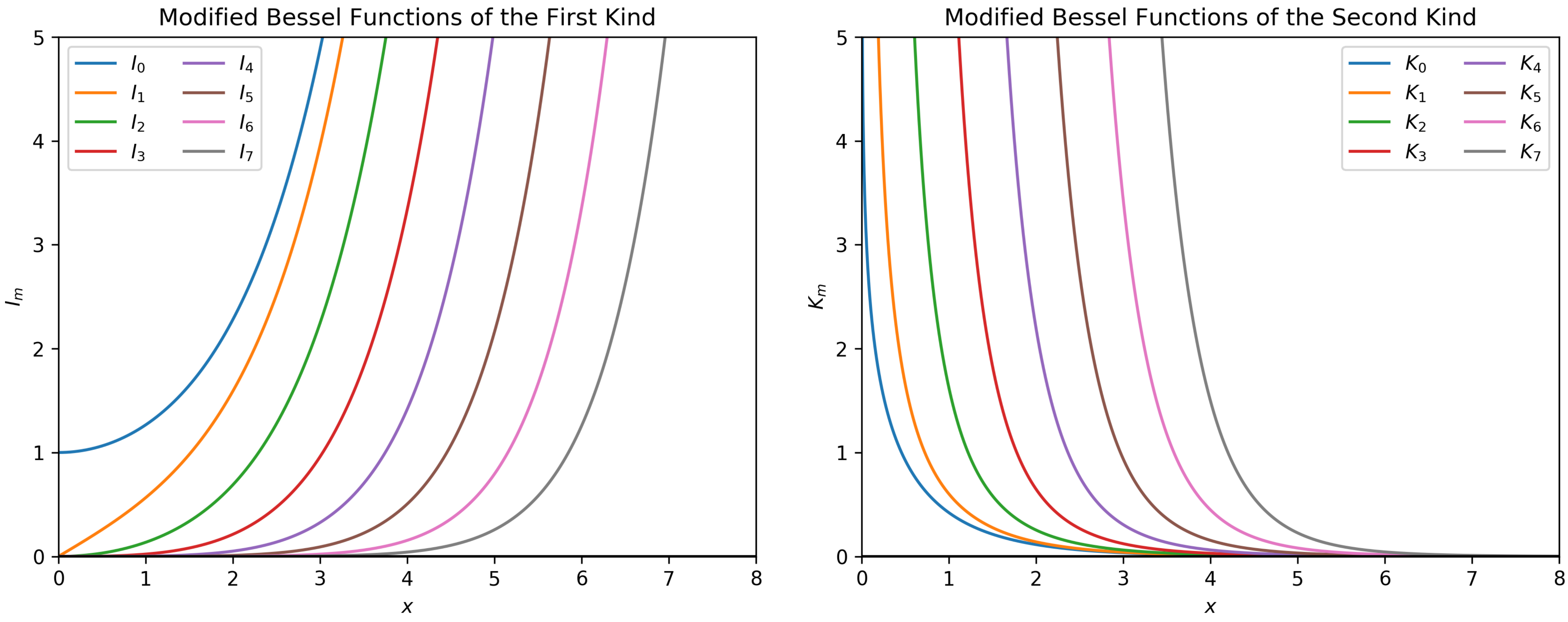
$$\Longrightarrow z\text{-dependence is } \sin\frac{n\pi z}{h}$$

- $\Longrightarrow$ <u>New</u> radial dependence, λ now pure imaginary
- New radial equation is

$$\rho^2 u'' + \rho u' - (l^2\rho^2 + m^2)u = 0$$

Modified Bessel function





Modified Bessel Functions of the First Kind
I_m
x
I_0
I_1
I_2
I_3
I_4
I_5
I_6
I_7
0
1
2
3
4
5
6
7
8
Modified Bessel Functions of the Second Kind
K_m
K_0
K_1
K_2
K_3
K_4
K_5
K_6
K_7

Spherical Bessel Functions

- Appear in the spherical polar problem (next topic), $x = kr$.
- Address here for completeness (l is integer, coupling with P_l^m).

$$j_l(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} J_{l+1/2}(x)$$

$$y_l(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} Y_{l+1/2}(x)$$

$$h_l^{(1,2)}(x) = j_l(x) \pm i y_l(x)$$

- Can infer all properties from those of the regular Bessel functions, but convenient to recast the recurrence and other relations.

- First few: $j_0(x) = \dfrac{\sin x}{x}$ $\qquad j_1(x) = \dfrac{\sin x}{x^2} - \dfrac{\cos x}{x}$

$$n_0(x) = -\frac{\cos x}{x} \qquad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

Spherical Bessel Functions

- Asymptotic behavior

$$j_l(x) \sim \frac{1}{x}\sin\left(x - \frac{l\pi}{2}\right)$$
$$y_l(x) \sim \frac{1}{x}\cos\left(x - \frac{l\pi}{2}\right)$$

- Recurrence relations (same for y_l, h_l)

$$j_{l-1} + j_{l+1} = \frac{2l+1}{x} j_l$$
$$l j_{l-1} - (l+1) j_{l+1} = (2l+1) j_l'$$
$$\left(x^{l+1} j_l\right)' = x^{l+1} j_{l-1}$$
$$\left(x^{-l} j_l\right)' = x^{-l} j_{l+1}$$

Asymptotic Behavior

- Rigorous derivation will take us too far afield, but here's a quick and dirty version.
- Interested in solutions to Bessel's equation for $x \gg 1$.

$$x^2 y'' + xy' + (x^2 - \cancel{m^2})y = 0$$

(can't throw away the xy' term – would lead to $y = e^{ix}$)

- Look for a solution $y = x^\alpha e^{ix}$ (for large x)

$$\Longrightarrow\; y' = x^{\alpha-1}(\alpha + ix)e^{ix}, \qquad y'' = x^{\alpha-2}[\alpha(\alpha-1) + 2i\alpha x - x^2]e^{ix}$$

$$\Longrightarrow\; x^2 y'' + xy' + x^2 y = x^\alpha e^{ix}[\cancel{\alpha^2} + (2\alpha+1)ix]$$

$$= 0 \;\text{ if }\; 2\alpha + 1 = 0$$

$$\Longrightarrow\; \alpha = -\frac{1}{2}$$

- Solution varies as $x^{-1/2}e^{ix}$ for large x.

Spherical Bessel Functions

- Recurrence relations lead to the <u>Rayleigh formulae</u>

$$j_l(x) = (-1)^l x^l \left(\frac{1}{x}\frac{d}{dx}\right)^l j_0(x)$$

$$n_l(x) = (-1)^l x^l \left(\frac{1}{x}\frac{d}{dx}\right)^l n_0(x)$$

  easiest proof: <u>induction</u> on l

- Orthogonality:

$$\int_0^a j_l\left(\alpha_{lm}\frac{r}{a}\right) j_l\left(\alpha_{ln}\frac{r}{a}\right) r^2 dr = \frac{1}{2}a^2[j_{l+1}(\alpha_{ln})]^2 \delta_{mn}$$

Example: Spherical Piston

- Sphere of radius a oscillates radially (no angular dependence), spatial dependence of the surrounding pressure satisfies the Helmholtz equation

$$\nabla^2 u + k^2 u = 0.$$

- For $r > a$, solution is $(l = m = 0)$

$$u(r) = A j_0(kr) + B n_0(kr)$$

- Outgoing wave with $e^{-i\omega t}$ time dependence has

$$u(r) = C h_0^{(1)}(kr) = -C \frac{i}{kr} e^{ikr}$$

$$\Longrightarrow \quad P(r,t) = \frac{-iC}{kr} e^{ikr - i\omega t}$$

amplitude $\propto \frac{1}{r}$

power $\propto \frac{1}{r^2}$

Legendre Polynomials

- Recall that the series solution to Legendre's equation with $k = 0$ gives the recurrence relation

$$a_j = \frac{(j-1)(j-2)-l(l+1)}{j(j-1)} a_{j-2} = -\frac{(l+j-1)(l+2-j)}{j(j-1)} a_{j-2}$$

- Series terminates at $j = l$.
- For <u>even</u> l have only even j, so write $l = 2L,\ j = 2m$, for $m = 0, \ldots, L$.
- Recurrence relation connects a_j to a_0 in $m = j/2$ steps.
- Then

$$a_{2m} = -\frac{\overset{1}{(2L+2m-1)}\overset{2}{(2L+2-2m)}}{2m(2m-1)} a_{2m-2} = \ \ldots = \frac{(-1)^m}{(2m)!} c_0 \overset{1}{A_{Lm}} \overset{2}{B_{Lm}}$$

Legendre Polynomials

- Expanding

$$A_{Lm} = (2L + 2m - 1)(2L + 2m - 3) \dots (2L + 1)$$

$$= \frac{(2L+2m)(2L+2m-1)(2L+2m-2)(2L+2m-3)\dots(2L+1)}{(2L+2m)(2L+2m-2)(2L+2m-4)\dots(2L+2)}$$

$$= \frac{(2L+2m)!}{(2L)!} \cdot \frac{1}{2^m(L+m)(L+m-1)(L+m-2)\dots(L+1)}$$

$$= \frac{(2L+2m)!}{(2L)!} \cdot \frac{L!}{2^m(L+m)!}$$

$$B_{Lm} = (2L + 2 - 2m)(2L + 4 - 2m) \dots (2L)$$

$$= 2^m(L + 1 - m)(L + 2 - m) \dots (L)$$

$$= \frac{2^m L!}{(L-m)!}$$

Legendre Polynomials

- Thus

$$a_{2m} = \frac{(-1)^m}{(2m)!} c_0 \, A_{Lm} B_{Lm}$$

$$= \frac{(-1)^m c_0}{(2m)!} \, \frac{(2L+2m)!}{(2L)!} \, \frac{L!}{2^m (L+m)!} \, \frac{2^m L!}{(L-m)!}$$

$$= \frac{(-1)^m c_0}{(2m)!} \, \frac{(2L+2m)!}{(L-m)!(L+m)!} \, \frac{(L!)^2}{(2L)!}$$

$= K_l$, independent of m

- Conventional to write $r = L - m$ (i.e. count down, not up)

$$\Longrightarrow \; P_l(x) = \sum_{m=0}^{L} a_{2m} x^{2m} = K_l \sum_{r=0}^{L} (-1)^r \frac{(2l-2r)!}{r!(l-r)!} \frac{x^{l-2r}}{(l-2r)!}$$

$$\begin{aligned} 2L + 2m &= 4L - 2r \\ &= 2l - 2r \\ L - m &= r \\ L + m &= 2L - r \\ &= l - r \end{aligned}$$

- Can show result holds for odd l also.

Legendre Polynomials

- Can write

$$P_l(x) = K_l \sum_{r=0}^{L} \frac{(-1)^r}{r!(l-r)!} \left(\frac{d}{dx}\right)^l x^{2l-2r}$$

$$= \frac{K_l}{l!} \left(\frac{d}{dx}\right)^l \sum_{r=0}^{L} (-1)^r \frac{l!}{r!(l-r)!} x^{2l-2r}$$

$$= \frac{K_l}{l!} \left(\frac{d}{dx}\right)^l (x^2-1)^l$$

$$(-1)^r \frac{(2l-2r)!}{r!\,(l-r)!} \frac{x^{l-2r}}{(l-2r)!}$$

$$\left(\frac{d}{dx}\right)^l x^{2l-2r}$$

$$= (2l-2r)(2l-2r-1)$$

$$\ldots (l-2r+1)\, x^{l-2r}$$

$$= \frac{(2l-2r)!}{(l-2r)!} x^{l-2r}$$

- Conventional to take $P_l(1) = 1$ for all l.

$$\Rightarrow \; P_l(1) = \frac{K_l}{l!} \frac{d^l}{dx^l} \Big[\underbrace{(x-1)\ldots(x-1)}_{l \text{ terms}} \cdot \underbrace{(x+1)\ldots(x+1)}_{l \text{ terms}}\Big]\Big|_{x=1}$$

$$= 2^l K_l$$

- nonzero only when all $x-1$ factors are removed
- $l!$ ways to do this
- result is 2^l

Legendre Polynomials

- Hence

$$K_l = 2^{-l}$$

and

$$P_l(x) = \frac{1}{2^l l!} \left(\frac{d}{dx}\right)^l (x^2 - 1)^l$$

$$= \frac{(-1)^l}{2^l l!} \left(\frac{d}{dx}\right)^l (1 - x^2)^l$$

Rodrigues formula

Legendre Normalization

- Can use Rodrigues to evaluate the normalization integral for P_l:

$$I_{lm} = \int_{-1}^{1} dx\, P_l(x) P_m(x), \quad m \leq l$$
$$= \frac{(-1)^{l+m}}{2^{l+m}\, l!\, m!} \int_{-1}^{1} dx \left(\frac{d}{dx}\right)^{l} (1-x^2)^l \left(\frac{d}{dx}\right)^{m} (1-x^2)^m$$

- Integrate by parts

$$I_{lm} = -\frac{(-1)^{l+m}}{2^{l+m}\, l!\, m!} \int_{-1}^{1} dx \left(\frac{d}{dx}\right)^{l-1} (1-x^2)^l \left(\frac{d}{dx}\right)^{m+1} (1-x^2)^m$$
$$\vdots$$
$$= \frac{(-1)^{l}}{2^{l+m}\, l!\, m!} \int_{-1}^{1} dx\, (1-x^2)^l \left(\frac{d}{dx}\right)^{l+m} (1-x^2)^m$$
$$= 0 \text{ if } m \neq l$$

Legendre Normalization

- If $m = l$,

$$I_{ll} = -\frac{(-1)^{l+m}}{2^{l+m}\, l!\, m!} \int_{-1}^{1} dx \left(\frac{d}{dx}\right)^{l-1} (1-x^2)^l \left(\frac{d}{dx}\right)^{m+1} (1-x^2)^m$$

$$\vdots$$

$$= \frac{(-1)^l}{2^{2l}\,(l!)^2} \int_{-1}^{1} dx\, (1-x^2)^l \left(\frac{d}{dx}\right)^{2l} \underbrace{(1-x^2)^l}_{\text{leading term is } (-1)^l x^{2l}}$$

$$= \frac{(2l)!}{2^{2l}\,(l!)^2} \underbrace{\int_{-1}^{1} dx\, (1-x^2)^l}_{= \frac{2^{2l+1}(l!)^2}{(2l+1)!}}$$

$$= \frac{(2l)!}{2^{2l}\,(l!)^2} \frac{2^{2l+1}(l!)^2}{(2l+1)!}$$

$$= \frac{2}{2l+1}$$

Legendre Generating Function

- The generating function for P_l is

$$F(h,z) = (1-2hz+h^2)^{-1/2} = \sum_{l=0}^{\infty} P_l(z)h^l$$

$$\frac{\partial}{\partial z}: \quad h(1-2hz+h^2)^{-3/2} = \sum_{l=0}^{\infty} P_l'(z)h^l \qquad (1-2hz+h^2)^{-3/2} = \frac{1}{h}\sum_{l=0}^{\infty} P_l'(z)h^l$$

$$\Longrightarrow \quad \sum_{l=0}^{\infty} P_l(z)h^{l+1} = (1-2hz+h^2)\sum_{l=0}^{\infty} P_l'(z)h^l$$

$$\Longrightarrow \quad P_l = P_{l+1}' - 2zP_l' + P_{l-1}' \qquad (\dagger)$$

$$\frac{\partial}{\partial h}: \quad (z-h)(1-2hz+h^2)^{-3/2} = \sum_{l=0}^{\infty} lP_l(z)h^{l-1}$$

$$\Longrightarrow \quad (z-h)\sum_{l=0}^{\infty} P_l'(z)h^l = \sum_{l=0}^{\infty} lP_l(z)h^l$$

$$\Longrightarrow \quad zP_l' - P_{l-1}' = lP_l \qquad (\ddagger)$$

$$+: \quad (l+1)P_l = P_{l+1}' - zP_l'$$

Legendre Generating Function

$$lP_{l-1} = P_l' - zP_{l-1}'$$

$$z(\ddagger): \quad z^2P_l' - zP_{l-1}' = lzP_l$$

$$\Longrightarrow \quad (1-z^2)P_l' = l(P_{l-1} - zP_l)$$

$$\frac{\partial}{\partial z}: \quad (1-z^2)P_l'' - 2zP_l' = l(\underbrace{P_{l-1}' - zP_l'}_{-lP_l\ (\ddagger)} - P_l)$$

$$\Longrightarrow \quad (1-z^2)P_l'' - 2zP_l' + l(l+1)P_l = 0$$

Legendre Generating Function

- Generating function for P_l is

$$F(h,z) = (1-2hz+h^2)^{-1/2} = \sum_{l=0}^{\infty} P_l(z) h^l$$

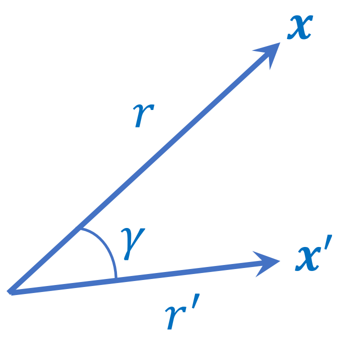
- This time, has a physical interpretation!
- Consider points $\boldsymbol{x}$ and $\boldsymbol{x}'$ in 3D space, with $r = |\boldsymbol{x}| > r' = |\boldsymbol{x}'|$

Then the Coulomb potential is

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}}$$

$$= \frac{1}{r}\left[1 - 2\frac{r'}{r}\cos\gamma + \left(\frac{r'}{r}\right)^2\right]^{-1/2}$$

$$= \sum_{l=0}^{\infty} \frac{r'^l}{r^{l+1}} P_l(\cos\gamma)$$



$h = \frac{r'}{r}$

$z = \cos\gamma$

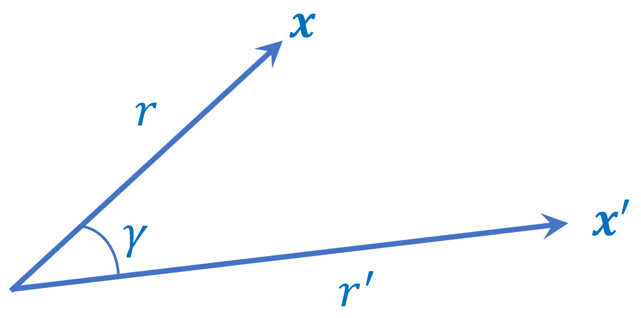
Legendre Generating Function

- If $r < r'$, the Coulomb potential is

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{1}{\sqrt{r^2+{r'}^2-2rr'\cos\gamma}}$$

$$= \frac{1}{r'}\left[1 - 2\frac{r}{r'}\cos\gamma + \left(\frac{r}{r'}\right)^2\right]^{-1/2}$$

$$= \sum_{l=0}^{\infty} \frac{r^l}{(r')^{l+1}} P_l(\cos\gamma)$$



- Sometimes write $r_< = \min(r, r')$, $r_> = \max(r, r')$, so

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$$

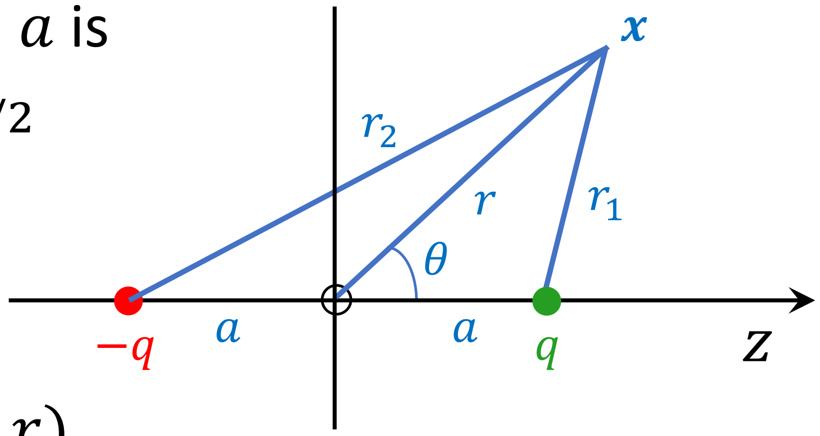
Example: Electrostatic Potential

- Potential at location $\boldsymbol{x}$ due to a charge q at $z = a$ is

$$\phi(\boldsymbol{x}) = \frac{kq}{r_1} = kq(r^2 + a^2 - 2ar\cos\theta)^{-1/2}$$

$$= kq\sum_{l=0}^{\infty} \frac{a^l}{r^{l+1}} P_l(\cos\theta)$$

$$= \frac{kq}{r}\sum_{l=0}^{\infty} \left(\frac{a}{r}\right)^l P_l(\cos\theta) \qquad (a < r)$$



- Now add a charge $-q$ at $z = -a$ (<u>dipole</u>), so

$$\phi(\boldsymbol{x}) = \frac{kq}{r_1} - \frac{kq}{r_2} = \frac{kq}{r}\left[\sum_{l=0}^{\infty}\left(\frac{a}{r}\right)^l P_l(\cos\theta) - \sum_{l=0}^{\infty}\left(\frac{-a}{r}\right)^l P_l(\cos\theta)\right]$$

even terms <u>cancel</u>

$$= \frac{2kq}{r}\sum_{m=0}^{\infty}\left(\frac{a}{r}\right)^{2m+1} P_{2m+1}(\cos\theta)$$

- leading term for $r \gg a$ is $\phi \sim \frac{2kqa}{r^2} P_1(\cos\theta)$

dipole potential

$D = 2qa$ is the <u>dipole moment</u>

Example: Electrostatic Potential

- Now imagine 3 charges as indicated

$$\phi(\boldsymbol{x}) = \frac{2kq}{r} - \frac{kq}{r}\left[\sum_{l=0}^{\infty}\left(\frac{a}{r}\right)^{l} P_l(\cos\theta\,) \;+\; \sum_{l=0}^{\infty}\left(\frac{-a}{r}\right)^{l} P_l(\cos\theta\,)\right]$$

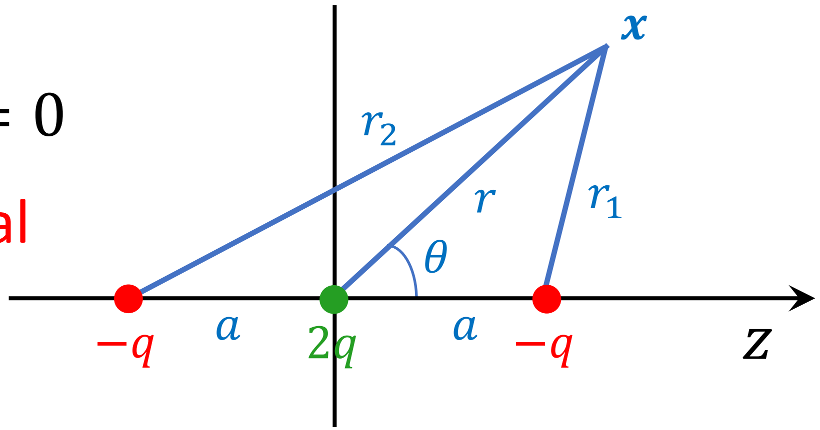
$$l = 0 \;\Longrightarrow \frac{2kq}{r} - \frac{2kq}{r} = 0$$

$$l = 1 \;\Longrightarrow -\frac{kqa}{r^2} P_1(\cos\theta) - \frac{kq(-a)}{r^2} P_1(\cos\theta) = 0$$

- Leading term is the $l = 2$ quadrupole potential

$$\phi \sim -\frac{kq}{r^3}(2qa^2)P_2(\cos\theta)$$

- $Q = 2qa^2$ is the <u>quadrupole moment</u>



Multipole Moments

- Derivation so far only works in axisymmetric distributions, but can already see a trend
  - total charge is the <u>monopole moment</u> M

    leading term for $r \gg a$ is $\quad \phi \sim \phi_M = \frac{kM}{r}\cancelto{1}{P_0}(\cos\theta)$
  - if $M = 0$, next term is the dipole (D)

    leading term is $\quad \phi \sim \phi_D = \frac{kD}{r^2}P_1(\cos\theta)$
  - if $d = 0$, next term is the quadrupole (Q)

    leading term is $\quad \phi \sim \phi_Q = \frac{kQ}{r^3}P_2(\cos\theta)$
- Note: r and θ have to do with the <u>field</u> point $\boldsymbol{x}$, while the moments $M, d, Q, \ldots$ have to do with the distribution of charge within the <u>source</u> region.

Multipole Moments

- Form of the next term:
  - expect scaling appropriate for $l = 3$

    $\Longrightarrow r^{-4} P_3(\cos\theta)$
  - moment should scale as qa^3, exact result depend on details of the geometry
  - called the <u>octupole moment</u>, $\mathcal{O}$
  - octupole potential $\phi_O = \frac{k\mathcal{O}}{r^4} P_3(\cos\theta)$
  - next is <u>hexadecapole</u>, $\phi_H = \frac{kH}{r^5} P_4(\cos\theta)\,,\ H \sim qa^4$
  - etc.