CLASSICAL ELECTRODYNAMICS

Lecture 1 – Jan 12

Coulomb's Law:

$$\vec{F} = k \frac{q_1 q_2}{|\vec{x}_1 - \vec{x}_2|^2} \cdot \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|}$$

(force on q_1 due to q_2)

- $F \propto$ mag. charges
- F inversely proportional to square dist.
- direction F along line between charges
- attractive for opp charges, repulsive for like

Electric field due to point charge q_1 at $\vec{x}_1$:

$$\vec{E}(\vec{x}) = \frac{\vec{F}}{q} = k q_1 \cdot \frac{\vec{x} - \vec{x}_1}{|\vec{x} - \vec{x}_1|^3}$$

In SI units: $k = \frac{1}{4\pi\varepsilon_0}$

$\varepsilon_0 = 8.9 \times 10^{12}$ F/m

charge measured in C

$\vec{E}$ field measured in V/m

Superposition of charges:

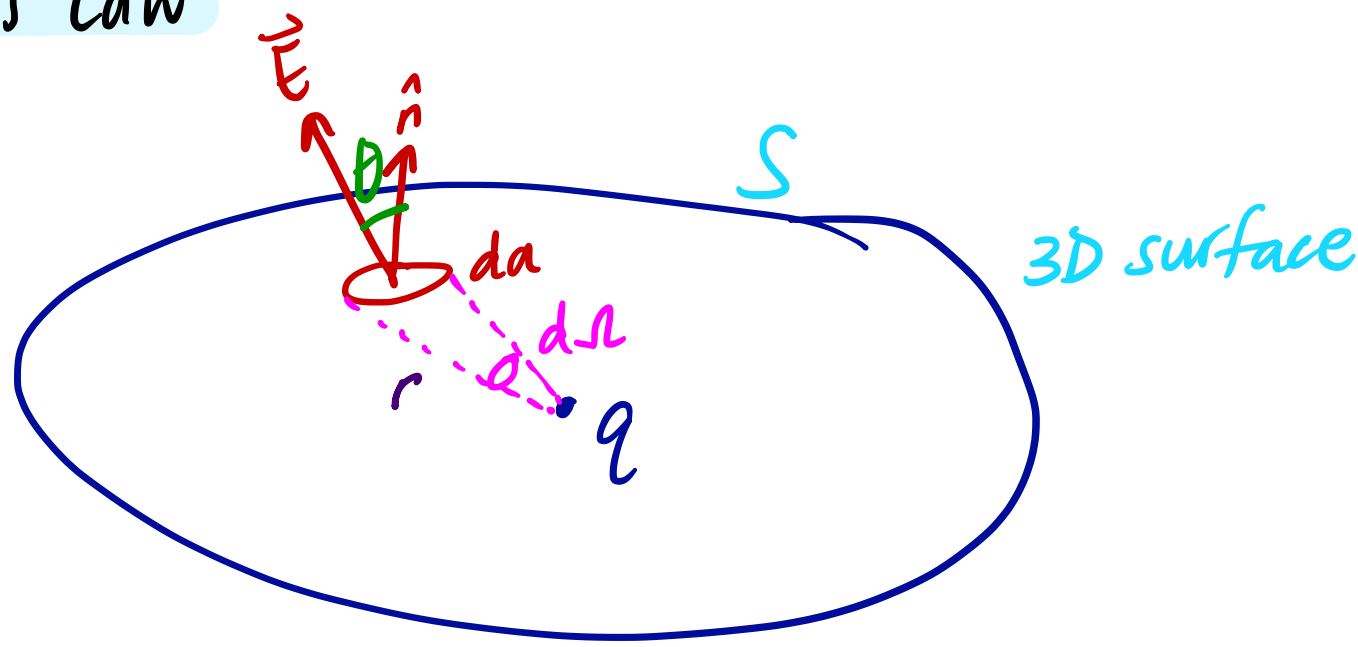
$$\vec{E}(\vec{x}) = k \cdot \sum_{i=1}^{n} q_i \frac{\vec{x} - \vec{x}_i}{|\vec{x} - \vec{x}_i|^3}$$

For charge density (distribution):

$$\vec{E}(\vec{x}) = K \int \rho(\vec{x}') \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} \, d^3x'$$

ρ = volume charge density

Gauss' Law



$$|E| = \frac{q}{4\pi\varepsilon_0} \frac{1}{r^2}$$

normal component of E:

$$\vec{E} \cdot \hat{n} \, da = \frac{q}{4\pi\varepsilon_0} \frac{1}{r^2} \cos\theta \, da$$

$$\cos\theta \, da = r^2 \, d\Omega$$

so $$\vec{E} \cdot \hat{n} \, da = \frac{q}{4\pi\varepsilon_0} \, d\Omega$$

integrate over the surface

$$\oint_S \vec{E}\cdot\hat{n}\,da = \frac{q}{4\pi\varepsilon_0}\cdot 4\pi = \frac{q}{\varepsilon_0}$$

* if q lies within S

for a set of charges inside S:

$$\oint_S \vec{E}\cdot\hat{n}\,da = \frac{1}{\varepsilon_0}\cdot\sum_i q_i$$

for charge distribution:

$$\boxed{\oint_S \vec{E}\cdot\hat{n}\,da = \frac{1}{\varepsilon_0}\int_V \rho(x)\,d^3x}$$

V = volume enclosed by S

(integral form of Gauss' Law
Electric flux through surface = charge enclosed)

Differential form:

obtain from integral form w/ Divergence Theorem

Divergence theorem:
For any vector field $\vec{A}(\vec{x})$, $\oint_S \vec{A}\cdot\hat{n}\,da = \int_V \vec{\nabla}\cdot\vec{A}\,d^3x$

so $\oint_S \vec{E}\cdot\hat{n}\,da = \frac{1}{\varepsilon_0}\int_V \rho(\vec{x})\,d^3x$

can be rewritten:

$$\int_V \vec{\nabla}\cdot\vec{E}\,d^3x = \frac{1}{\varepsilon_0}\int_V \rho(\vec{x})\,d^3x$$

$$\int_V \left(\vec{\nabla}\cdot\vec{E} - \frac{\rho}{\varepsilon_0}\right) d^3x = 0$$

true for any volume V within S enclosed
only if $\boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}}$ everywhere

Differential form

Also need equation for curl of $\vec{E}$

curl of E = $\boxed{\vec{\nabla} \times \vec{E} = 0}$

Why?

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int \rho(x') \frac{x-x'}{|x-x'|^3} d^3x$$

write $$\frac{x-x'}{|x-x'|^3} = -\nabla_{\vec{x}} \left(\frac{1}{|x-x'|} \right)$$

$$\vec{E}(\vec{x}) = -\frac{1}{4\pi\varepsilon_0} \nabla \int \frac{\rho(x')}{|x-x'|} d^3x'$$

since $\nabla \times \nabla_x = 0 \Rightarrow \nabla \times E = 0$

can define a scalar field potential

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(x')}{|x-x'|} d^3x'$$

integral over all space now

then $\boxed{\vec{E} = -\nabla \Phi}$

we can add a constant to Φ and $\vec{E}$ and $\vec{F}$ will stay the same (∇ of constant $= 0$)

so scalar field is arbitrary up to a constant

Interpret scalar field potential as potential energy of charge in $\vec{E}$ field

e.g. $\vec{F} = q\vec{E}$

work to move a charge from point A to B

$$W = -\int_A^B \vec{F} \cdot dl = -q\int_A^B \vec{E} \cdot d\vec{l}$$

$$= q\int_A^B \nabla\Phi \cdot d\vec{l} = q\int_A^B d\Phi$$

$$\boxed{W = q(\Phi_B - \Phi_A)}$$

so $q\Phi$ is potential energy and it is independent of the path

HW1

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1.1 Use Gauss' theorem to prove following:

(a) any excess charge placed on a conductor must lie entirely on its surface

Gauss' Law: $\int \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_0}$

no E field within conductor in static equilibrium.
if E field applied parallel to conductor, surface charges move (so not in equilibrium)
only E field perpendicular to E field survive

QED

(b) A closed, hollow conductor shields its interior from fields due to charges outside but does not shield its exterior from fields due to charges placed inside it

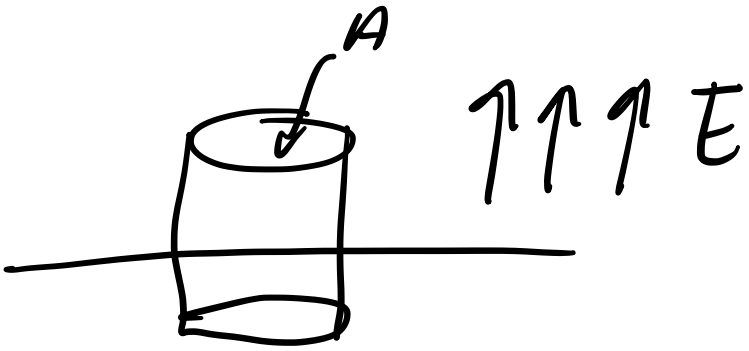
if charges are exterior then $q_{enc} = 0$ so zero flux and interior is shielded

if we have an enclosed charge then flux is non-zero and flux is surface integral so exterior is subject to E field from charge inside

QED

(c) electric field at surface of conductor is normal to the surface and has a magnitude σ/ε_0

can use Gaussian pillbox and Gauss law



$$\int \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_0}$$

$$\sigma = \frac{q}{A}$$

$$A \cdot E = \frac{\sigma A}{\varepsilon_0}$$

so $q = \sigma A$

$$\boxed{E = \sigma / \varepsilon_0}$$

1.3 using Dirac-delta in appropriate, express the following charge distributions as 3D $\rho(x)$

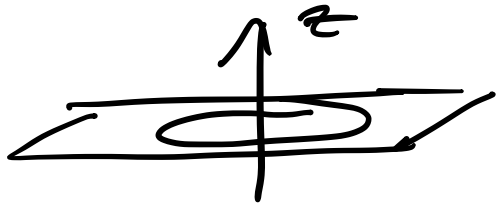
(a) in spherical coords, charge Q uniformly distributed over a radius R

$$\rho(x) = \frac{Q}{4\pi R^2} \delta(r-R)$$

(b) in cylindrical coords, charge λ per unit length uniformly distributed over a cylindrical surface of radius b

$$\rho(x) = \frac{\lambda}{2\pi b} \delta(\rho - b)$$

(c) in cylindrical coordinates, a charge Q spread uniformly over a flat circular disc of negligible thickness and radius R



$$\rho(x) = \frac{Q}{\pi R^2} \delta(z) \delta(R-\rho)$$

(d) same but spherical coords

Lecture 2 - Jan 14

Surface distribution of charges:

Poisson's equation:

divergence $\nabla \cdot E = \rho/\varepsilon_0$ curl $\nabla \times E = 0$ $E = -\nabla \Phi$

$$\nabla \cdot (-\nabla \Phi) = \rho/\varepsilon_0$$

$$\boxed{\nabla^2 \Phi = -\rho/\varepsilon_0}$$ Poisson's equation

compare with integral form:

$$\Phi = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(x')}{|x-x'|} d^3x'$$

Dirac delta function:

$\delta(x-a) = 0$ for $x = a$

$\int \delta(x-a)\,dx = 1$ if region includes a

$\int f(x)\,\delta(x-a)\,dx = f(a)$

$\int f(x)\,\delta'(x-a)\,dx = -f'(a)$

$$\delta(f(x)) = \sum_i \frac{1}{\frac{dt}{dx}(x_i)} \delta(x-x_i)$$

where x_i are zeros of the function $f(x)$

3D: $\delta(\vec{x}-\vec{X}) = \delta(x_1-X_1)\,\delta(x_2-X_2)\,\delta(x_3-X_3)$

$$\int_{\Delta V} \delta(\vec{x}-\vec{X})\,d^3x = 1$$

Dirac delta functions are used with point charges because potential disappears there

Simple example of point charge q at $\vec{x}''$

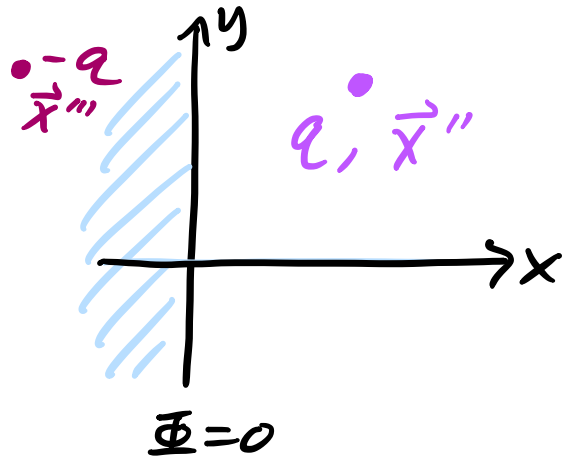
What is $\Phi(\vec{x})$?

$$\rho = q\,\delta(\vec{x}-\vec{x}'')$$

then $$\Phi = \frac{1}{4\pi\varepsilon_0}\int \frac{q\,\delta(\vec{x}'-x'')}{|x-x'|}\,d^3x'$$

$$\boxed{\Phi = \frac{q}{4\pi\varepsilon_0}\,\frac{1}{|x-x''|}}$$

* This doesn't work if boundary is introduced because now we don't know the charge distribution everywhere — would need to use Poisson's equation and method of images



Green's Theorem:

Divergence Theorem: $\int_V \vec{\nabla}\cdot\vec{A}\,d^3x = \oint_S \vec{A}\cdot\hat{n}\,da$

for any vector field $\vec{A}$

Let $\vec{A} = \phi\nabla\psi$ for scalar fields ϕ, ψ

then $\vec{\nabla}\cdot\vec{A} = \nabla\cdot(\phi\nabla\psi)$

use vector calculus formula:

$$\vec{\nabla}\cdot(\psi\cdot\vec{a}) = \vec{a}\cdot\vec{\nabla}\psi + \psi\vec{\nabla}\cdot\vec{a}$$

so $$\nabla\cdot(\phi\nabla\psi) = \nabla\psi\cdot\nabla\phi + \phi\nabla\cdot(\nabla\psi)$$
$$= \phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi$$

$$\vec{A}\cdot\hat{n} = \phi\nabla\psi\cdot\hat{n} = \phi\frac{\partial\psi}{\partial\hat{n}}$$

so $$\boxed{\int_V (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)\, d^3x = \oint_S \phi\frac{d\psi}{dn}\, da}$$

Green's first identity

Interchange ϕ, ψ now:

$$\int_V (\psi\nabla^2\phi + \nabla\psi\cdot\nabla\phi)\, d^3x = \oint_S \psi\frac{d\phi}{dn}\, da$$

subtract:

$$\boxed{\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, d^3x = \oint_S \left(\phi\frac{d\psi}{dn} - \psi\frac{d\phi}{dn}\right) da}$$

Green's second identity – Green's theorem

set $\phi = \Phi$

$$\psi = \frac{1}{|x-x'|} = \frac{1}{R}$$

use $\nabla^2\Phi = -\rho/\varepsilon_0$ (Poisson's equation)

$$\nabla^2\psi = \nabla^2\left(\frac{1}{R}\right) = -4\pi\,\delta(x-x')$$

$$\Rightarrow \int_V \left[-4\pi\Phi(x')\,\delta(x-x') + \frac{1}{\varepsilon_0 R}\rho(x')\right] d^3x'$$

$$= \oint_S \left[\Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right) - \frac{1}{R}\frac{\partial\Phi}{\partial n'}\right] da'$$

if x is inside V:

$$-4\pi\Phi(x) + \int_V \frac{1}{\varepsilon_0 R}\rho(x')\, d^3x' =$$

$$\oint_S \left(\Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right) - \frac{1}{R}\frac{\partial\Phi}{\partial n'}\right) da'$$

Lecture 3 – Jan 19

Green Functions:

satisfy $\nabla'^2 G(\vec{x}, \vec{x}') = -4\pi\delta(\vec{x}-\vec{x}')$

so $G(x, x') = \dfrac{1}{|x-x'|} + F(x, x')$

where $\nabla'^2 F(x, x') = 0$

For Dirichlet boundary condition:

let $\phi = \Phi$ and $\psi = G(x, x')$ (last lecture $\psi = R$)

then $\Phi = \dfrac{1}{4\pi\varepsilon_0} \int_V \rho(x') G(x, x') \, d^3x'$

$$+ \frac{1}{4\pi} \oint_S G \frac{d\Phi}{dn'} - \Phi \frac{\partial}{\partial n'} G \, da'$$

For Neumann boundary condition:

want $\dfrac{\partial G_N}{\partial n'} = 0$ on surface S

$\nabla'^2 G(x, x') = -4\pi\delta(x, x')$

$\int_V \nabla'^2 G = -4\pi$ rewrite from Poisson and Gauss equations

$\Rightarrow \oint_S \dfrac{\partial G}{\partial n'} da' = -4\pi$

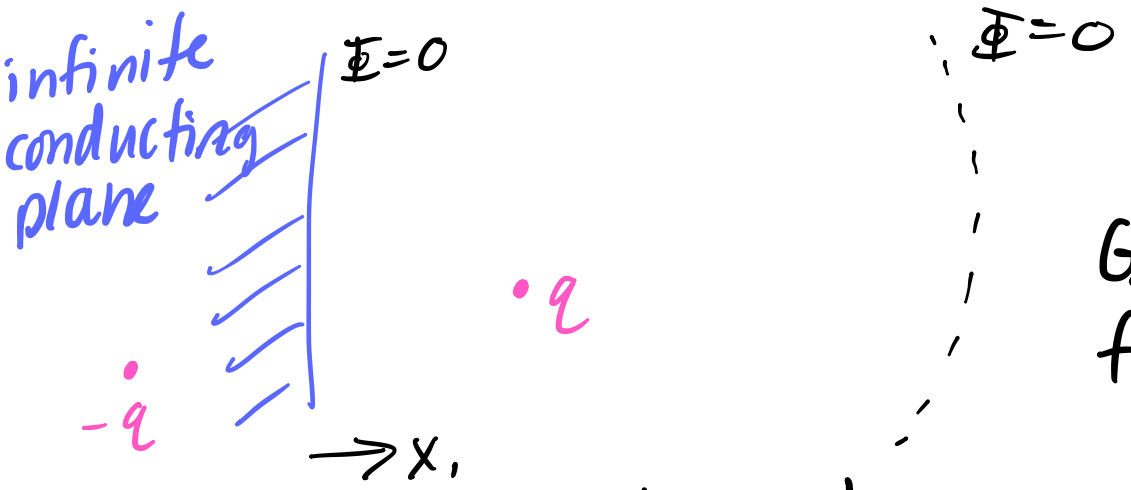
so set $\dfrac{\partial G_N}{\partial n'}(x, x') = \dfrac{-4\pi}{S}$ for x' on S

$\leftarrow$ total area of boundary

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(x')\, G_N(x,x')\, d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial \Phi}{\partial n'} G_N\, da'$$

$$+ \langle \Phi \rangle_S \Longrightarrow \text{average of } \Phi \text{ over } S$$

Method of Images:



$G_D(x,x') = 0$ for x' on S

propose $$G_D(x,x') = \frac{1}{|x-x'|} - \frac{1}{|x-(-x_1', x_2', x_3')|}$$

check condition: For $\vec{x}'$ on S, $x_1' = 0$

$$G_D(x,x') = 0$$

$$\nabla^2 F = \nabla^2\left(\frac{1}{-x-(x_1', x_2', x_3')}\right) = 0$$

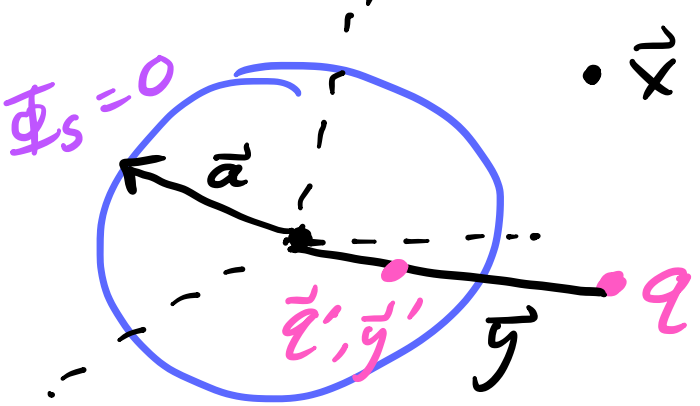
because image charge is outside the volume

now we plug in our Green's function to our problem:

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(x')\, G_D(x,x')\, d^3x' - \frac{1}{4\pi}\oint_S \underbrace{\Phi(x')}_{=0} \frac{\partial G_D}{\partial n'}\, da'$$

for flat conducting plane

another example:



• $\vec{x}$ observation point

q at some point $\vec{y}$

$$\Phi(x) = \frac{q}{4\pi\varepsilon_0 |\vec{x}-\vec{y}|} + \frac{q'}{4\pi\varepsilon_0 |\vec{x}-\vec{y}'|}$$

* want to solve for unknown q' and y'

add image charge along $\vec{y}$

need to satisfy boundary of Φ on $S = 0$

so set $\Phi = 0$ at $|\vec{x}| = a$

$\vec{x} = x\vec{n}$ $\quad \vec{y} = y\vec{n}'$ $\quad \vec{y}' = y'\vec{n}'$

rewrite potential:

$$\Phi(x) = \frac{q}{4\pi\varepsilon_0 |x\vec{n} - y\vec{n}'|} + \frac{q'}{4\pi\varepsilon_0 |x\vec{n} - y'\vec{n}'|}$$

$$\Phi(x=a) = \frac{q}{4\pi\varepsilon_0 a |\vec{n} - \frac{y}{a}\vec{n}'|} + \frac{q'}{4\pi\varepsilon_0 y' |\vec{n}' - \frac{a}{y'}\vec{n}|} = 0$$

if we choose $\frac{q}{a} = -\frac{q'}{y'}$ and $\frac{y}{a} = \frac{a}{y'}$

then $\Phi(x=a) = 0$ like we want

so $y' = \frac{a^2}{y}$ and $q' = -\frac{a}{y} q$

(inside sphere)

now we can write Green's function:

$$G_D(x, x') = 0 \quad \text{for } |x| = a$$

$$G_D(x, x') = \frac{1}{|x - x'|} + F(x, x')$$

where $\vec{\nabla}^2 F = 0$ everywhere outside sphere

$$G_D(x, x') = \frac{1}{|x - x'|} - \frac{a}{x' \left| \vec{x} - \frac{a^2}{x'^2} \vec{x}' \right|}$$

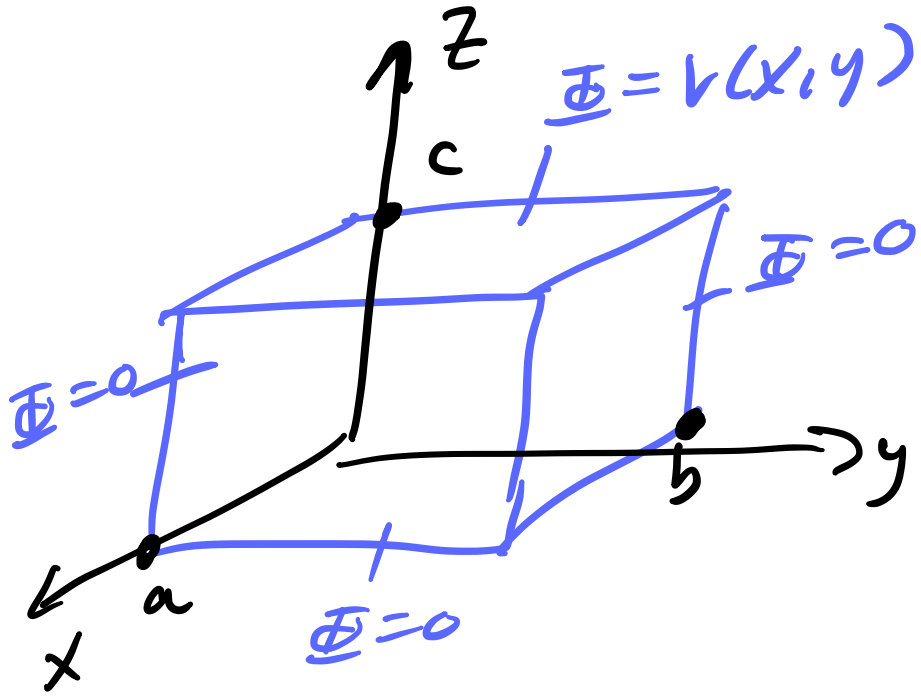
note magnitude of $\frac{a^2}{x'^2} \vec{x}' = \frac{a^2}{x'}$

Lecture 4 - Riya notes

Lecture 5 - Jan 26

Sec 2

HW: 2.1, 2.13, 2.15



$\Phi = 0$ everywhere except top surface

What is potential everywhere in the box?

$$X = e^{\pm i\alpha x} \qquad X(0) = 0$$

constant is implicit

$$X = \cos(\alpha x) + i\sin(\alpha x)$$

$$X(0) = 0 \rightarrow X = \sin(\alpha x)$$

$$Y(0) = 0 \rightarrow Y = \sin(\beta y)$$

$$Z(0) = 0 \rightarrow Z = \sinh\left(\sqrt{\alpha^2 + \beta^2}\, z\right)$$

$\Phi = 0$ at $x = a$ $\quad \sin(\alpha a) = 0 \quad \alpha a = n\pi$

$\Phi = 0$ at $y = b$ $\quad \beta b = m\pi$

$$\alpha_n = \frac{n\pi}{a} \qquad \beta_m = \frac{m\pi}{b} \qquad \gamma_{nm} = \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$$

$$\Phi_{nm} = \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z)$$

gives us solutions to all BC's except $\Phi(z=0) = V(x,y)$

more generally:

$$\Phi = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z)$$

now consider final boundary condition:

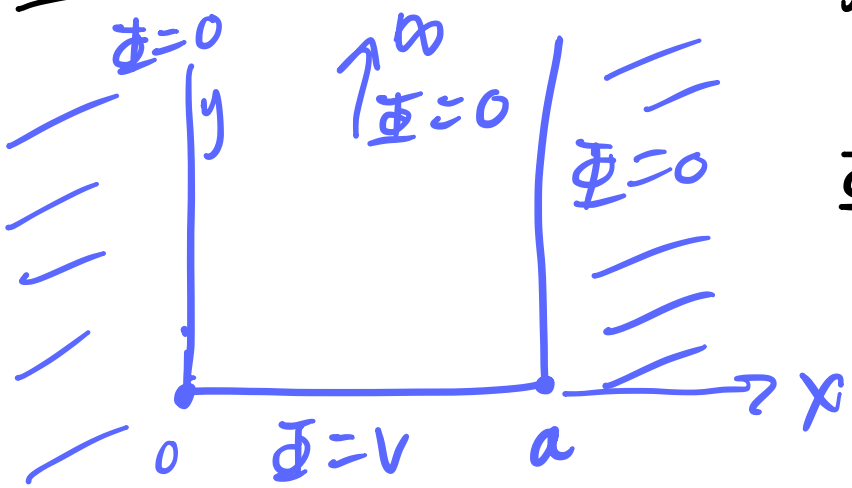
$$V(x,y) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} c)$$

double Fourier series

$$\Rightarrow A_{nm} = \frac{4}{ab \sinh(\gamma_{nm} c)} \int_0^a dx \int_0^b dy \, V(x,y) \sin(\alpha_n x) \sin(\beta_m y)$$

* For non-zero potentials on all sides, solve for each side separately then superimpose

2D example



solutions will be of form $e^{\pm i\alpha x} e^{\pm \alpha y}$

$\Phi = 0$ at $y = \infty$ means we need α to be real and exponential decay (y part $= e^{-\alpha y}$)

Then $\Phi = 0$ at $x=0$ and $x=a$ will give sine term

$$\Phi(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \exp\left(-\frac{n\pi y}{a}\right)$$

$\Phi = V$ at $y = 0$ so

$$A_n = \frac{2}{a}\int_{-a}^{a} \Phi(x,0) \sin\left(\frac{n\pi x}{a}\right) dx$$

and $\Phi(x,0) = V$

$$A_n = \frac{2V}{a}\left(\frac{-\cos\left(\frac{n\pi x}{a}\right)}{\frac{n\pi}{a}}\right)\Bigg|_0^a = 0 \text{ for } n \text{ even}$$

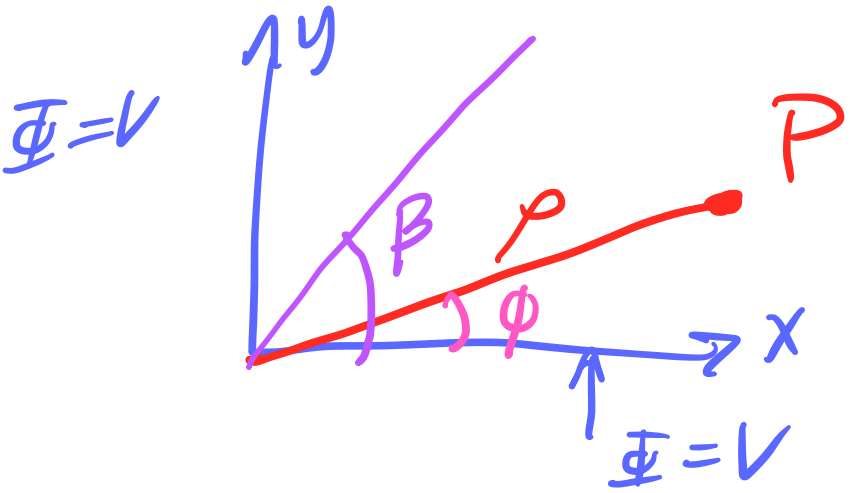
$$= \frac{4V}{n\pi} \text{ for } n \text{ odd}$$

$$\Phi(x,y) = \frac{4V}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin\left(\frac{n\pi x}{a}\right) \exp\left(\frac{-n\pi y}{a}\right)$$

for $y >> a$ the first term $(n=1)$ dominates

$$\Phi(x,y) \longrightarrow \frac{4V}{\pi} \sin\left(\frac{\pi x}{a}\right) \exp\left(-\frac{\pi y}{a}\right)$$

Another 2D example:



Separation of variables in Polar:

$$\Phi(\rho, \phi) = R(\rho)\, \Phi(\phi)$$

Laplacian: $$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{d\Phi}{d\rho}\right) + \frac{1}{\rho^2}\frac{d^2\Phi}{d\phi^2} = 0$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right)\Phi + \frac{1}{\rho^2}\frac{d^2\Phi}{\partial \phi^2} R = 0$$

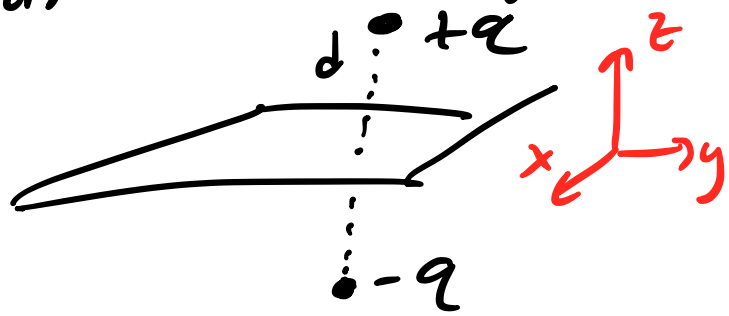
get $$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) = \nu^2$$

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -\nu^2$$

HW 2 - 2.1, 2.13, 2.15

2.1 Point charge q is brought to position d away from infinite conductor at $\Phi = 0$. use method of images to find

(a) surface charge density induced on plane



$$\sigma = -\varepsilon_0 \frac{\partial \Phi}{\partial z}$$

consider potential of each charge

$$\Phi = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(z-d)^2+x^2+y^2}} - \frac{1}{\sqrt{(z+d)^2+x^2+y^2}}\right]$$

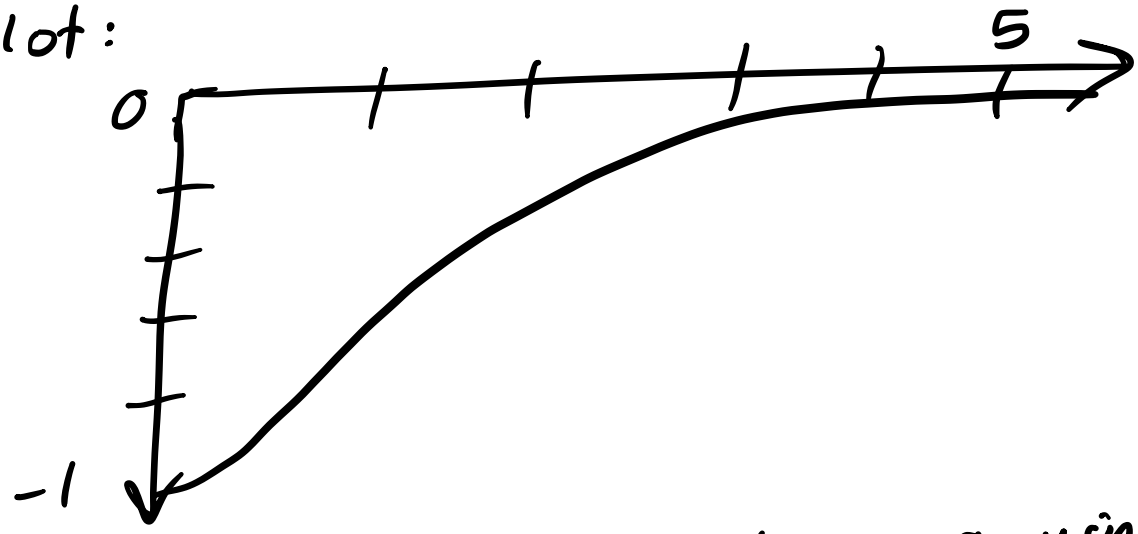
$$\sigma = -\frac{q}{4\pi}\left[\frac{-\frac{1}{\cancel{2}}(\cancel{2}(z-d))}{[(z-d)^2+x^2+y^2]^{3/2}} - \frac{-\frac{1}{\cancel{2}}(\cancel{2}(z+d))}{[(z+d)^2+x^2+y^2]^{3/2}}\right]_{z=0}$$

$$\sigma = -\frac{q}{4\pi}\left[\frac{d}{[d^2+x^2+y^2]^{3/2}} + \frac{d}{[d^2+x^2+y^2]^{3/2}}\right]$$

evaluate σ at $z=0$ since we want surface σ

$$\boxed{\sigma = -\frac{q}{2\pi}\left[\frac{d}{[d^2+x^2+y^2]^{3/2}}\right]}$$

plot:



(b) Force between plane and charge using Coulomb

Coulomb's law: $F = \dfrac{q_1 q_2}{4\pi\varepsilon_0 r^2}$

$q_1 = q$
$q_2 = -q$
$r = 2d$

$$F = \frac{-q^2(-\hat{z})}{4\pi\varepsilon_0 (2d)^2} = \boxed{\frac{q^2}{16\pi\varepsilon_0 d^2}\hat{z}}$$

(c) total force acting on plane (integrate $\sigma^2/2\varepsilon_0$)

so $F = \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\sigma^2}{2\varepsilon_0}\, dx\, dy$

$$\sigma^2 = \frac{q^2}{4\pi^2} \frac{d^2}{[d^2+x^2+y^2]^3}$$

$$F = \frac{q^2}{4\pi^2\varepsilon_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d^2}{[d^2+x^2+y^2]}\, dx\, dy$$

write in polar: $F = \dfrac{q^2 d^2}{4\pi^2\varepsilon_0}\displaystyle\int_{-\infty}^{\infty} \frac{r\, dr}{[r^2+d^2]^3}$

$$= \frac{q^2 d^2}{4\pi\varepsilon_0}\left(\frac{1}{2}\right)\int_{d^2}^{\infty} \frac{1}{u^3}\, du$$

$$= \frac{q^2 d^2}{8\pi\varepsilon_0}\left[-\frac{1}{2u^2}\right]_{d^2}^{\infty} = \frac{q^2 d^2}{16\pi\varepsilon_0}\left(\frac{1}{d^4}\right)$$

$$\boxed{F = \frac{q^2}{16\pi\varepsilon_0 d^2}}$$

(d) work needed to remove charge q to infinity

$$W = \int_d^{\infty} F\, dz$$

$$= \int_d^{\infty} \frac{q^2}{16\pi\varepsilon_0 z^2}\, dz = \frac{q^2}{16\pi\varepsilon_0}\int_d^{\infty} \frac{1}{z^2}\, dz$$

$$= \frac{q^2}{16\pi\varepsilon_0}\left[-\frac{1}{z}\right]_d^{\infty} = \boxed{\frac{q^2}{16\pi\varepsilon_0 d} = W}$$

(e) potential energy between q and image

$KE + PE = W$ so PE at rest $= W$

$$W = \int_{-d}^{d} F\,dz = \frac{q^2}{16\pi\varepsilon_0}\int_{-d}^{d}\frac{1}{z^2}\,dz$$

$$= \frac{q^2}{16\pi\varepsilon_0}\left[\frac{1}{-d}\right]_{-d}^{d} = \frac{-q^2}{16\pi\varepsilon_0 d} - \frac{q^2}{16\pi\varepsilon_0 d}$$

$$= -\frac{2q^2}{16\pi\varepsilon_0 d} \qquad \boxed{W = \frac{-q^2}{8\pi\varepsilon_0 d}}$$

* when q moves, image charge moves too

(f) Find (d) in eV for electron 1 ang from surface

↳ 1×10^{-10} m

$q = 1.6\times10^{-19}$

$\varepsilon_0 = 8.85\times10^{-12}$

$$W = \frac{q^2}{16\pi\varepsilon_0 d}$$

$$= \frac{(1.6\times10^{-19})^2}{16\pi(8.85\times10^{-12})(1\times10^{-10})}$$

$$= \boxed{3.6\text{ eV}}$$

2.13

(a) Two halves of long hollow conducting cylinder of inner radius b are separated by small lengthwise gaps and are at V_1 and V_2

Show $\Phi(\rho,\phi) = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right)$

want to use Laplace in polar (ρ,ϕ)

$$\Phi(\rho,\phi) = a_0 + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n (a_n\cos n\phi + b_n \sin n\phi)$$

initial conditions

$$\Phi(b,\phi) = V_1 \text{ for } [0,\pi]$$
$$= V_2 \text{ for } [\pi, 2\pi]$$

solve for Fourier coefficients a_n and b_n

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \cos n\phi \, V_0 \, d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos n\phi \, V_1 \, d\phi + \frac{1}{\pi}\int_{\pi}^{2\pi} \cos n\phi \, V_2 \, d\phi$$

$$= \frac{V_1}{\pi}\left(\frac{\sin n\pi}{n}\right)\Big|_0^{\pi} + \frac{V_2}{\pi}\left(\frac{\sin n\pi}{n}\right)\Big|_{\pi}^{2\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \sin n\phi \, V_0 \, d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin n\phi \, V_1 \, d\phi + \frac{1}{\pi}\int_{\pi}^{2\pi} \sin n\phi \, V_2 \, d\phi$$

$$= \frac{V_1}{\pi}\left(-\frac{\cos n\phi}{n}\right)\Big|_0^{\pi} + \frac{V_2}{\pi}\left(-\frac{\cos n\phi}{n}\right)\Big|_{\pi}^{2\pi}$$

$$= -\frac{V_1}{n\pi}\left[(-1)^n - 1\right] + \frac{V_2}{n\pi}\left[1-(-1)^n\right]$$

$$= \frac{V_1}{n\pi}\left[1-(-1)^n\right] + \frac{V_2}{n\pi}\left[1-(-1)^n\right]$$

$$= \frac{V_1+V_2}{n\pi}\left[1-(-1)^n\right]$$

2.15

(a) Show that the Green function $G(x,y,x',y')$ for Dirichlet boundary conditions for 2D region $0 \le x \le 1$; $0 \le y \le 1$ has expansion

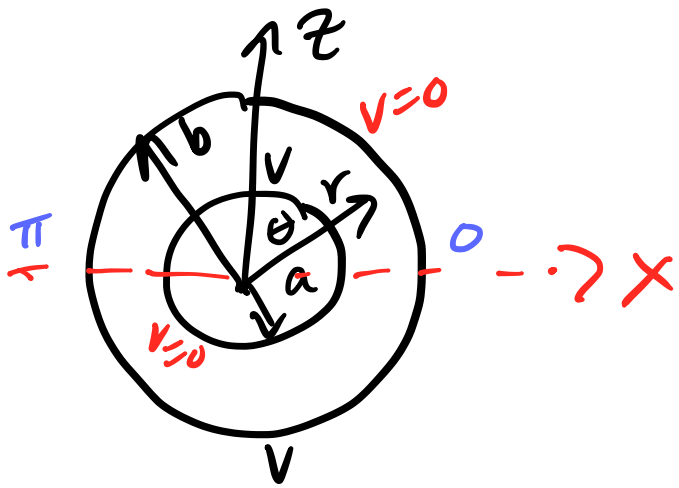
$$G(x,y;x',y') = 2\sum_{n=1}^{\infty} g_n(y,y') \sin(n\pi x)\sin(n\pi x')$$

for $g_n(y,y')$ satisfying $\left(\frac{\partial^2}{\partial y'^2} - n^2\pi^2\right) g_n(y,y') = -4\pi\delta(y'-y)$

$g_n(y,0) = g_n(y,1) = 0$

HW 3 3.1, 3.6, 3.14 due 2/23

3.1



Φ in $a \le r \le b$ as series up to $\ell = 4$ term?

Check case of $b \to \infty$ and $a \to 0$

symmetry about $\hat{z}$ so Φ only depends on r and θ

general solution: $\Phi(r,\theta) = \sum_\ell (A_\ell r^\ell + B_\ell r^{-\ell-1}) P_\ell \cos\theta$

continuity condition says $\Phi(a,\theta) = \Phi(b,\theta)$

BC's: $\Phi(a,\theta) = \begin{cases} V & \text{for } 0 < \theta < \pi/2 \\ 0 & \text{for } \pi/2 < \theta < \pi \end{cases}$

$\Phi(b,\theta) = \begin{cases} 0 & \text{for } 0 < \theta < \pi/2 \\ V & \text{for } \pi/2 < \theta < \pi \end{cases}$

so $\Phi(a,\theta) = \sum_\ell A_\ell a^\ell + B_\ell a^{-\ell-1} P_\ell \cos\theta$ (1)

$\Phi(b,\theta) = \sum_\ell A_\ell b^\ell + B_\ell b^{-\ell-1} P_\ell \cos\theta$ (2)

we know $\int_{-1}^{1} P_m(x) P_n(x)\, dx = \frac{2}{2n+1} \delta_{mn}$

from orthogonality of Legendre polynomials

so can get rid of sum by multiply our Φ's by a Legendre polynomial and then integrating both sides of (1) and (2)

Let $x = \cos\theta$

$$\int_{-1}^{1} \Phi(a,x) P_\ell(x)\, dx = V \int_0^1 P_\ell(x)\, dx$$
$$= \int_{-1}^{1} [A_\ell a^\ell + B_\ell a^{-\ell-1}] P_\ell^2(x)\, dx$$

$$\int_{-1}^{1} \Phi(b,x) P_\ell(x)\, dx = V \int_{-1}^{0} P_\ell(x)\, dx$$
$$= \int_{-1}^{1} [A_\ell b^\ell + B_\ell b^{-\ell-1}] P_\ell^2(x)\, dx$$

we know $\int_{-1}^{1} P_\ell^2(x)\, dx = \frac{2}{2\ell+1}$

so comparing coefficients of right hand sides:

$$A_\ell a^\ell + B_\ell a^{-\ell-1} = \frac{2\ell+1}{2} V \int_0^1 P_\ell(x)\, dx$$

$$A_\ell b^\ell + B_\ell b^{-\ell-1} = \frac{2\ell+1}{2} V \int_{-1}^{0} P_\ell(x)\, dx$$
$$= \frac{2\ell+1}{2} V \int_0^1 P_\ell(-x)\, dx$$
$$= \frac{2\ell+1}{2} V (-1)^\ell \int_0^\ell P_\ell(x)\, dx$$

separating out A_ℓ and B_ℓ:

$$A_\ell = \frac{2\ell+1}{2(a^{2\ell+1} - b^{2\ell+1})} V [a^\ell - b^{\ell+1}(-1)^\ell] \int_0^1 P_\ell(x)\, dx \quad (3)$$

$$B_\ell = (a^{\ell+1}) \frac{2\ell+1}{2} V \int_0^1 P_\ell(x)\, dx - A_\ell a^{2\ell+1} \quad (4)$$

we want $\ell = 0, 1, 2, 3, 4$ terms so find A_ℓ's first and plug into B_ℓ equation

we know for even ℓ that $\int_0^1 P_\ell(x)\,dx = \frac{1}{2}\int_{-1}^{1} P_\ell(x)\,dx$
$= 0$

so only odd terms ($\ell > 0$) will survive
and only need P_0, P_1, P_3

$$\int_0^1 P_0(x)\,dx = 1$$
$$\int_0^1 P_1(x)\,dx = \frac{1}{2}$$
$$\int_0^1 P_3(x)\,dx = -\frac{1}{8}$$

$$B_3 = \frac{-7Va^4 - a^7}{16} - \left[\frac{-7V(a^4+b^4)}{16(a^7-b^7)}a^7\right]$$
$$= -\frac{7}{16}Va^4b^4\left[\frac{b^3+a^3}{b^7-a^7}\right]$$

plugging into (3):

$$A_0 = V/2$$
$$A_1 = \frac{3V(a^2 - b^2(-1)^1)}{2(a^3-b^3)} \cdot \frac{1}{2} = \frac{3V(a^2+b^2)}{4(a^3-b^3)}$$
$$A_3 = \frac{7V(a^4 - b^4(-1)^3)}{2(a^7-b^7)} \cdot \left(-\frac{1}{8}\right) = \frac{-7V(a^4+b^4)}{16(a^7-b^7)}$$

now plug A_0, A_1 and A_3 into (4):

$$B_0 = \frac{a}{2}V - \frac{V}{2}a = 0$$
$$B_1 = a^2\left(\frac{3}{2}\right)V\left(\frac{1}{2}\right) - a^2\left[\frac{3V(a^2+b^2)}{4(a^3-b^3)}\right]$$
$$= \frac{3}{4}Va^2 - \frac{3}{4}Va^3\frac{(a^2+b^2)}{a^3-b^3} = \frac{3}{4}Va^2b^2\left[\frac{b+a}{b^3-a^3}\right]$$
$$B_3 = a^4\left(\frac{7}{2}\right)V\left(-\frac{1}{8}\right) - a^7\left[\frac{-7V(a^4+b^4)}{16(a^7-b^7)}\right]$$
$$= -\frac{7}{16}Va^4b^4\left[\frac{b^3+a^3}{b^7-a^7}\right]$$

so plugging these back in:

$$\Phi(r,\theta) = A_0 P_0(x) + \underbrace{B_0 P_0(x)}_{0} + A_1 r P_1(x) + \frac{B_1}{r^2} P_1(x)$$

$$+ A_3 r^3 P_3(x) + \frac{B_3}{r^3} P_3(x)$$

but $\Phi(r,\theta)$ needs to go to 0 as $r \to \infty$
so this means only B terms survive (neg exp)
besides A_0 term

$$\Phi(r,\theta) = A_0 P_0(x) + \frac{B_1}{r^2} P_1(x) + \frac{B_3}{r^4} P_3(x)$$

$$= \frac{V}{2} P_0(x) + \frac{3}{4} \frac{V a^2 b^2}{r^2} \left[\frac{b+a}{b^3 - a^3}\right] P_1(x) - \frac{7}{16} \frac{V a^4 b^4}{r^4} \left[\frac{b^3 + a^3}{b^7 - a^7}\right] P_3(x)$$

$$\boxed{= \frac{V}{2}\left[P_0(x) + \frac{3}{2} \frac{a^2 b^2}{r^2} \left(\frac{b+a}{b^3 - a^3}\right) P_1(x) - \frac{7}{8} \frac{a^4 b^4}{r^4} \left[\frac{b^3 + a^3}{b^7 - a^7}\right] P_3(x)\right]}$$

so at $b \to \infty$

$$\Phi(r,\theta) = \frac{V}{2}\left[P_0(x) + \frac{3}{2} \frac{a^3}{r^2} P_1(x) - \frac{7}{8} \frac{a^4}{r^4} P_3(x)\right]$$

for $a \to 0$

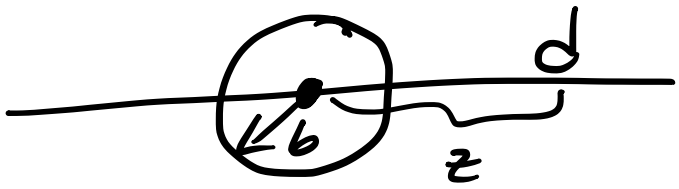
$$\Phi(r,\theta) = \frac{V}{2}\left[P_0(x)\right]$$

Since we want Φ at large distance to vanish, only B_ℓ terms (and A_0) survive because of negative exponent

so $\Phi(r,\theta) = \frac{V}{2}\left[P_0(x) + \frac{3}{2}\frac{a^3}{r^2}P_1(x) - \frac{7}{8}\frac{a^4}{r^4}P_3(x) \ldots\right]$

as $b \to \infty$

3.14



$\lambda = d^2 - z^2$

(lecture 10)
p. 124

a) potential inside sphere?

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\rho(x')\,G_D - \underbrace{\frac{1}{4\pi}\oint\left(\phi\,\frac{\partial G_D}{\partial x'}\right)dx'}_{=0 \text{ at } x=b}$$

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\rho(x')\,G_D \quad (1)$$

consider $\vec{\rho}(x')$ and integrate over space

$$\vec{\rho}(x') = \frac{Q}{2b}\,\frac{1}{2\pi r'^2}\left[\delta(\cos\theta'-1) + \delta(\cos\theta'+1)\right]$$

azimuthal symmetry so $m=0$ for Green's function

$$G_D = \sum_{\ell=0}^{\infty} P_\ell\cos\theta\, P_\ell\cos\theta'\left[\frac{r_<^{\ell}}{r_>^{\ell+1}} - \frac{(rr')^{\ell}}{b^{2\ell+1}}\right]$$

Plug in ρ and G_D to (1):

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\rho(x')\,G_D\,r'^2\sin\theta'\,dr'\,d\theta'\,d\phi$$

$$\Phi(x) = \frac{Q}{8\pi\varepsilon_0 b}\sum_{\ell=0}^{\infty} P_\ell\cos\theta\left[P_\ell(1) + P_\ell(-1)\right]$$

$$\cdot\int_0^b d^2 - r'^2\,\frac{1}{r_<^{\ell}}\left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^{\ell}}{b^{2\ell+1}}\right)dr'$$

3.6

Feb 23 - Dielectrics

Define the electric displacement $\vec{D}$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

$$\nabla \cdot \vec{D} = \rho - \nabla \cdot \vec{P} + \nabla \cdot \vec{P}$$

$$\nabla \cdot \vec{D} = \rho$$ (1st Maxwell equation)

$$\nabla \times E = 0$$ (same as when there is no medium)

In many materials we can assume $\vec{P}$ is parallel to $\vec{E}$ and proportional

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

χ_e = electric susceptibility

$$\vec{D} = \varepsilon_0 \vec{E} +$$

assume uniform ε: $\nabla \cdot \vec{E} = \rho / \varepsilon$

So we can solve problems the same way (as long as material is uniform) by replacing the ε_0 with ε, i.e. electric fields are reduced by a factor of $\varepsilon_0 / \varepsilon$

electric field in a dielectric is always smaller than in a vacuum

and capacitance ($C = Q/V$) increases by a factor of $\varepsilon / \varepsilon_0$

in a vacuum, $(\vec{E}_2 - \vec{E}_1) \cdot \hat{n} = \sigma / \varepsilon_0$
and tangential component is continuous across boundary

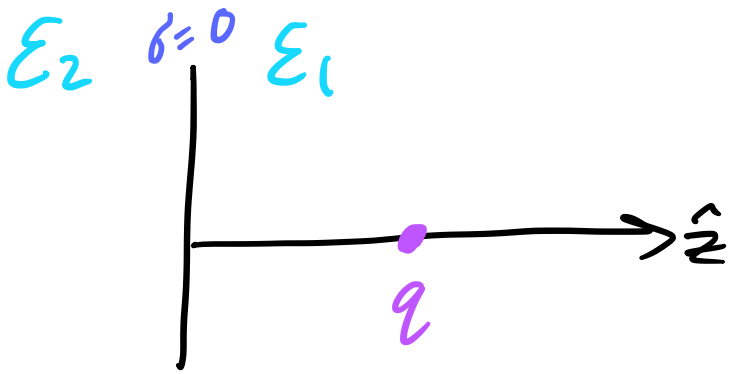
$\hat{n}_{21}$ pointing from $1 \to 2$

for a dielectric boundary, $\boxed{(\vec{D}_2 - \vec{D}_1) \cdot \hat{n}_{21} = \sigma}$
and tangential component of E is still continuous (E is a conservative force)

surface charge density

$\hookrightarrow \boxed{(\vec{E}_2 - \vec{E}_1) \times \hat{n}_{21} = 0}$

example: two different materials w/ a boundary



$\nabla \cdot E = \rho / \varepsilon_1$ for $z > 0$ (region 1)

$\nabla \cdot E = 0$ for $z < 0$ (region 2)

$\hookrightarrow$ since no charge in region 2

$\nabla \cdot E = 0$ $z < 0$

$\nabla \times E = 0$ everywhere

boundary conditions:

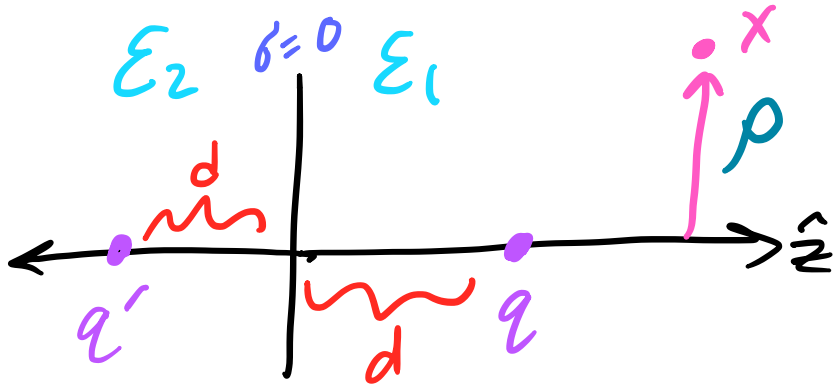
$(\vec{D}_2 - \vec{D}_1) \cdot \hat{n}_{21} = 0$

$\varepsilon_2 E_{2z} = \varepsilon_1 E_{1z}$ for $z \to 0$

$(E_2 - E_1) \times \hat{n}_{21} = 0$ for $z \to 0$

$E_{2x} = E_{1x}$
$E_{2y} = E_{2y}$

use method of images to find potential:



$$\Phi_1 = \frac{1}{4\pi\varepsilon_1}\left(\frac{q}{\sqrt{\rho^2+(d-z)^2}} + \frac{q'}{\sqrt{\rho^2+(d+z)^2}}\right) \quad \text{for } z>0$$

(in cylindrical)

to apply boundary conditions, need potential for $z<0$ $\rightarrow$ must have a solution to Laplace equation w/ no charges in the region but possibly charges in other region

try $$\Phi_2 = \frac{1}{4\pi\varepsilon_2}\frac{q''}{\sqrt{\rho^2+(d-z)^2}}$$

i.e. charge at position of actual charge but multiplied by scale factor

boundary condition:

$$\varepsilon_2 E_{2z} = \varepsilon_1 E_{1z} \quad \text{for } z=0 \qquad E = -\nabla\Phi$$

$$\varepsilon_2\left(-\frac{\partial\Phi_2}{\partial z}\right) = \varepsilon_1\left(-\frac{\partial\Phi_1}{\partial z}\right)$$

$$\frac{\varepsilon_2}{4\pi\varepsilon_2}\, q''\frac{d}{(\rho^2+d^2)^{3/2}} = \frac{\varepsilon_1}{4\pi\varepsilon_1}\, q\,\frac{d}{(\rho^2+d^2)^{3/2}}$$

then $E_{2\rho} = E_{1\rho}$ at $z=0$

$$\frac{1}{4\pi\varepsilon_2} q''\left(\frac{-\rho}{(\rho^2+d^2)^{3/2}}\right) = \frac{1}{4\pi\varepsilon_1}\left[q\frac{(-\rho)}{(\rho^2+d^2)^{3/2}} + \frac{q'(-\rho)}{(\rho^2+d^2)^{3/2}}\right]$$

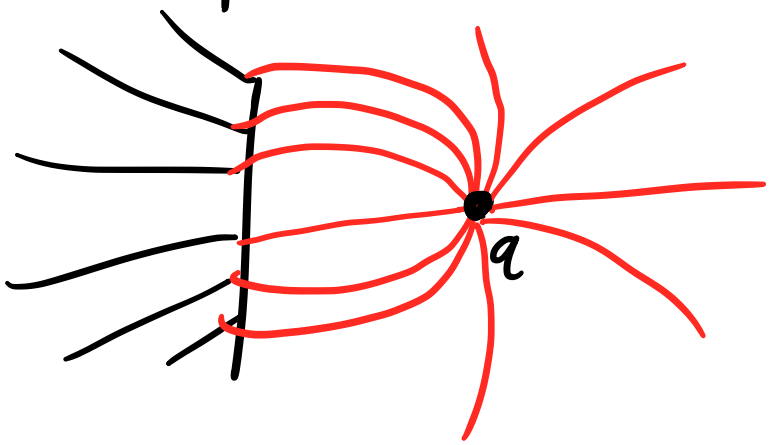
$$\frac{q''}{\varepsilon_2} = \frac{q+q'}{\varepsilon_1}$$

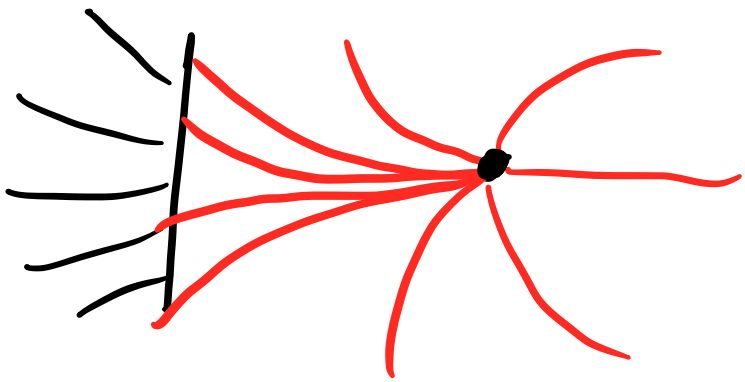
solve: $$q' = -\left(\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}\right) q$$

$$q'' = \left(\frac{2\varepsilon_2}{\varepsilon_2+\varepsilon_1}\right) q$$

▷ field lines:

if $\varepsilon_2 > \varepsilon_1$:



in limit $\varepsilon_2 >> \varepsilon_1$ field lines bend around to be completely normal to the surface, i.e. acts like a conductor ($q' = -q$)

if $\varepsilon_2 < \varepsilon_1$