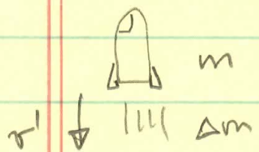
(1.13)



at t : $p = mv$

at $t+\Delta t$: $p(t+\Delta t) = (m-\Delta m)(v+\Delta v) + \Delta m v_e$

where v_e is escape velocity of gases relative to earth

Def. of derivative

$$\frac{dP}{dt} = \lim_{t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t} = m\frac{dv}{dt} + (v_e - v)\frac{dm}{dt} + \cancel{O(\Delta m \Delta v)}^{0}$$

Now $v_e - v = v'$ relative to rocket.

Also $\frac{dP}{dt} = -mg$ $\Rightarrow$ $\boxed{m\frac{dv}{dt} = -v'\frac{dm}{dt} - mg}$

To obtain $v(m)$: $dv = -\frac{v'}{m}dm - g\,dt$

$= -\frac{v'}{m}dm - \frac{g}{\dot{m}}dm$

using $\frac{dm}{dt} = \dot{m}$ $\Rightarrow dt = \frac{dm}{\dot{m}}$

Integrating $v(m) = -v'\ln m - \frac{g}{\dot{m}}m + C$

using initial condition at $t=0$, $m = m_0$ and $v = 0$

$$v(m_0) = 0 = -v'\ln m_0 - \frac{g}{\dot{m}}m_0 + C$$

$$C = v'\ln m_0 + \frac{g}{\dot{m}}m_0$$

$$\Rightarrow \boxed{v(m) = -\frac{g}{\dot{m}}(m - m_0) - v'\ln\frac{m}{m_0}}$$

Now assume: m_e = mass empty rocket, m_f = mass fuel $\Big\}$ $m_0 = m_e + m_f$

$$v(m_e) = v_{esc} = -\frac{g}{\dot{m}}\left(m_e - (m_e + m_f)\right) - v'\ln\frac{m_e}{m_e + m_f}$$

escape speed

$$v_{esc} = + \frac{g}{\dot{m}} m_f - v' \ln \frac{1}{1 + m_f/m_e}$$

Further $\dot{m} = -\frac{1}{60[s]} \cdot m_0 = \frac{1}{60}(m_e + m_f)$

and $m_e/m_f \ll 1$

$$v_{esc} = \frac{-60 \cdot g}{(m_e + m_f)} m_f - v' \ln \frac{1}{1 + \frac{m_f}{m_e}} \simeq -60g - v' \ln \frac{m_e}{m_f}$$

$$\frac{m_e}{m_f} = \exp\left[-(60g + v_{esc})\frac{1}{v'}\right]$$

Substituting $v_{esc} = 11.2$ km/s and $v' = 2.1$ km/s

$$\frac{m_e}{m_f} = \exp\left[-\left(60[s] \cdot 9.8\frac{[m]}{[s^2]} + 11.2 \times 10^3 \frac{[m]}{[s]}\right)\frac{1}{2.1 \times 10^3 \frac{[m]}{[s]}}\right]$$

$$\frac{m_e}{m_f} \simeq e^{-5.61\bar{3}} \simeq \frac{1}{274} \quad (\overset{*}{\text{not}} \text{ too close})$$

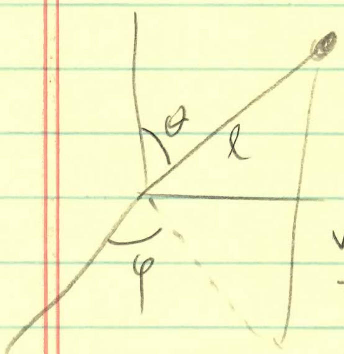
* the discrepancy comes from Goldstein 2nd edition where $v' = 6800$ ft/s $\simeq 2.07264$ km/s

and with this v'

$$\frac{m_e}{m_f} \simeq \frac{1}{295}$$

Closer to the desired answer!

(1.19) Motion is constrained to a spherical shell.



$\hat{r} = \sin\theta\cos\varphi\,\hat{\imath} + \sin\theta\sin\varphi\,\hat{\jmath} + \cos\theta\,\hat{k}$

$\hat{\theta} = \cos\theta\cos\varphi\,\hat{\imath} + \cos\theta\sin\varphi\,\hat{\jmath} - \sin\theta\,\hat{k}$

$\hat{\varphi} = -\sin\varphi\,\hat{\imath} + \cos\varphi\,\hat{\jmath}$

velocity

$\vec{r} = l\hat{r}$

$\bar{v} = \dot{\vec{r}} = l\dot{\hat{r}}$

Now $\dot{\hat{r}} = (\cos\theta\cos\varphi\dot{\theta} - \sin\theta\sin\varphi\dot{\varphi})\hat{\imath}$

$\quad + (\sin\theta\cos\varphi\dot{\varphi} + \cos\theta\sin\varphi\dot{\theta})\hat{\jmath} - \sin\theta\dot{\theta}\hat{k}$

$\quad = \dot{\theta}\hat{\theta} + \sin\theta\dot{\varphi}\hat{\varphi}$

So velocity $\bar{v} = l(\dot{\theta}\hat{\theta} + \sin\theta\dot{\varphi}\hat{\varphi}$

$T = \frac{1}{2}m\bar{v}\cdot\bar{v} = \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2)$

Potential

$V = mg\,l\cos\theta$

Lagrangian $L = \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) - mgl\cos\theta$

Eqns of motion

θ: $ml^2\ddot{\theta} - ml^2\sin\theta\cos\theta\dot{\varphi}^2 - mgl\sin\theta = 0$

$$\boxed{\ddot{\theta} - \sin\theta\cos\theta\dot{\varphi}^2 - \frac{g}{l}\sin\theta = 0}$$

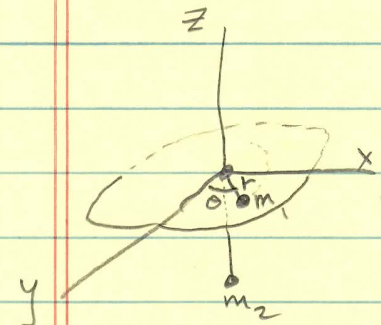
φ: $\left.\frac{d}{dt}(ml^2\sin^2\theta\dot{\varphi}) = 0\right\} \longrightarrow (L_\varphi = \text{const})$

$$\boxed{\sin^2\theta\ddot{\varphi} + 2\sin\theta\cos\theta\dot{\theta}\dot{\varphi} = 0}$$

▷ Note: if $\varphi = \text{const}$ then get $\ddot{\theta} - \frac{g}{l}\sin\theta = 0$ normal pendulum

▷ if $\dot{\theta} = 0$, $\ddot{\varphi} = 0 \Rightarrow \dot{\varphi} = c \Rightarrow \varphi = ct + d$ is a linear function of time

(1.21)



Generalized coordinates $\boxed{r, \theta, z}$

Constraint $s = r + z$ (s - length of string)

then only have two independent coordinates

Positions:

$\bar{r}_1 = r\sin\theta\,\hat{\imath} + r\cos\theta\,\hat{\jmath}$

$\bar{r}_2 = -z\,\hat{k} = (r-s)\,\hat{k}$

Velocities $\bar{v}_1 = \dot{\bar{r}}_1 = (r\cos\theta\,\dot\theta + \dot r\sin\theta)\,\hat{\imath} + (\dot r\cos\theta - r\sin\theta\,\dot\theta)\,\hat{\jmath}$

$$\bar{v}_1\cdot\bar{v}_1 = r^2\cos^2\theta\,\dot\theta^2 + \dot r^2\sin^2\theta + \cancel{2r\dot r\sin\theta\cos\theta\,\dot\theta}$$
$$+ r^2\sin^2\theta\,\dot\theta^2 + \dot r^2\cos^2\theta - \cancel{2r\dot r\sin\theta\cos\theta\,\dot\theta}$$
$$= \dot r^2 + r^2\dot\theta^2$$

$$\bar{v}_2 = \dot{\bar{r}}_2 = \dot r\,\hat{k}$$

$$\bar{v}_2\cdot\bar{v}_2 = \dot r^2$$

Potential $V = -m_2 g z$

Lagrangian $L = \frac{1}{2}m_1(\dot r^2 + r^2\dot\theta^2) + \frac{1}{2}m_2\dot r^2 + m_2 g(s-r)$

Eqns of motion

r: $(m_1+m_2)\ddot r - m_1 r\dot\theta^2 + m_2 g = 0$

θ: $\frac{d}{dt}(m_1 r^2\dot\theta) = 0 \Longrightarrow$ Conservation of angular momentum

$\boxed{l \equiv m_1 r^2\dot\theta} = \text{constant}$

$$\boxed{(m_1+m_2)\ddot r - \frac{l^2}{m_1 r^3} + m_2 g = 0}$$

For the 1st integral, multiply by $\dot r$: $\ddot r\dot r = \frac{1}{2}\frac{d}{dt}\dot r^2$

$$\frac{d}{dt}\Big[\underbrace{\tfrac{1}{2}(m_1+m_2)\dot r^2 + \tfrac{1}{2}\cdot\frac{l^2}{m_1 r^2} + m_2 g r + C}_{\text{energy } E}\Big] = 0$$

$C \rightarrow$ constant

constant can be chosen so that $E = T + V$

$$\boxed{E = \frac{1}{2}(m_1+m_2)\dot r^2 + \frac{1}{2}\frac{l^2}{m_1 r^2} + m_2 g(r-s)}$$

first integral

(1.23) We use the form $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \frac{\partial f}{\partial \dot{q}_j} = 0$

where $L = \frac{1}{2} m \dot{z}^2 + mgz$

and $f = \frac{1}{2} k v^2$

Eqn of motion $\boxed{m\ddot{z} - mg + k\dot{z} = 0}$

$$\ddot{z} + \frac{k}{m}\dot{z} = g$$

because $\dot{z} = v$ then $\dot{v} + \frac{k}{m} v = g$

$v(t) = Be^{-kt/m} + \frac{mg}{k}$ || Initial conditions $v(0) = v_0 = B + \frac{mg}{k}$

$B = v_0 - mg/k$

$$\boxed{v(t) = \left(v_0 - \frac{mg}{k}\right) e^{-kt/m} + \frac{mg}{k}}$$

as $t \to \infty$ $\boxed{v(\infty) \to mg/k}$