Chasles' theorem - general displacement of a rigid body can be described by a translation + a rotation

3 coordinates for the fixed point
3 coordinates for the rotation description

if the fixed point coincides with the CM,
then we know that many physical quantities naturally split into properties of the CM + props about the CM

e.g. Kinetic energy $T = \frac{1}{2} M v^2 + T'(\varphi, \theta, \psi)$

e.g. gravitational field $V = V(r)$ only on position

magnetic moment $V \sim \bar{M} \cdot \bar{B}$ only rotation

Langrangian also typically splits into two parts.

$\bar{r}_i, \bar{v}_i$ $\quad$ $\bar{r}_i$ body set of axes $\quad$ $\bar{v}_i$ space set axes

$$\bar{L} = m_i(\bar{r}_i \times \bar{v}_i)$$

$$\bar{v}_i = \left(\frac{d}{dt}\bar{r}_i\right)_s = \left(\frac{d}{dt}\bar{r}_i\right)_r + \bar{\omega} \times r_i$$

$$0$$

$$\bar{v}_i = \bar{\omega} \times \bar{r}_i$$

$$\bar{L} = m_i \bar{r}_i \times (\bar{\omega} \times \bar{r}_i) = m_i\left[\bar{\omega} r_i^2 - \bar{r}_i(\bar{r}_i \cdot \bar{\omega})\right]$$

$$a \times (b \times c) = (a \cdot c) b - (a \cdot b) c$$

Look at only one component,

$$L_x = \omega_x m_i(r_i^2 - x_i^2) - \omega_y m_i x_i y_i - \omega_z m_i x_i z_i$$

$$L_x = I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z$$

$$L_y = I_{yx} \omega_x + I_{yy} \omega_y + I_{yz} \omega_z$$

$$L_z = I_{zx} \omega_x + I_{zy} \omega_y + I_{zz} \omega_z$$

$$\boxed{\bar{L} = I \bar{\omega}}$$

Note that I is composed of 9 elements, forming a transformation matrix.

diagonal terms

e.g. $I_{xx} = m_i (r_i^2 - x_i^2)$

off-diagonal terms

e.g. $I_{xy} = -m_i (x_i y_i)$

for a continuous description $I_{xx} = \int_V \rho(\bar{r}) (r^2 - x^2) dV$

in general $I_{jk} = \int_V \rho(\bar{r}) [r^2 \delta_{jk} - x_j x_k] dV$

it is symmetric

$j, k = x, y, z$

$$\bar{L} = I \bar{\omega}$$

I matrix transforms the vector omega into a vector L,

1. L and omega are two different vectors,
2. they have different units,
3. I also has units in itself, and is not necessarily orthogonal,
4. 'operator' I acts on vector 'omega' resulting in a new physical vector 'L'

$$I = \frac{L}{\omega}$$

$$\frac{4}{2} = 2 \qquad \frac{\pi}{2} \neq int \qquad \frac{5}{2} \neq int$$

I is a tensor

General remarks about tensors.

In a 3D Cartesian space, an N-th rank tensor has 3^N components

$T_{ijk\ldots}$ (N indices)

0th rank	$3^0 = 1$	T
1th rank	$3^1 = 3$	T_i
2nd rank	$3^2 = 9$	T_{ij}

Under an orthogonal transformation A, they transform as

$$T'_{ijk\ldots}(\bar{x}') = a_{il} a_{jm} a_{kn} \ldots T_{lmn\ldots}(\bar{x})$$

Tensor of rank 0 - 1 component $\quad T' = T$ (scalar)

Tensor of rank 1 - 3 components $\quad T'_i = a_{ij} T_j$

mathematically, this tensor is equivalent to a vector

Tensor of rank 2 - 9 components $\quad T'_{ij} = a_{il} a_{jm} T_{lm}$

Matrix representation $\quad T' = A T A^{-1} = A T A^T$

similarity transformation

$$T'_{ij} = a_{il} (T A^T)_{lj}$$

$$= a_{il} T_{lm} \tilde{a}_{mj} = a_{il} T_{lm} a_{jm}$$

Other properties of tensor

The unit tensor $I_{ij} = \delta_{ij}$

The dot product of a tensor with a vector (from either side) is a vector

$$D = T \cdot C \Longrightarrow D_i = T_{ij} C_j$$

A double dot product gives a scalar - this is called a "contraction"

$$S = F \cdot T \cdot C = F_i (TC)_i = F_i T_{ij} C_j$$

(no surviving indices)

$$M \cdot N = M_i N_i = \#$$

Kinetic energy $T = \frac{1}{2} m_i v_i^2$

r_i - relative to the body set
v_i - relative to the space

$$T = \frac{1}{2} m_i \bar{v}_i \cdot (\bar{\omega} \times \bar{r}_i)$$

$a \cdot (b \times c) = b \cdot (c \times a)$

$$= \frac{1}{2} m_i \bar{\omega} \cdot (\bar{r}_i \times \bar{v}_i)$$

$$= \frac{1}{2} \bar{\omega} \cdot (m_i \bar{r}_i \times \bar{v}_i) = \frac{1}{2} \bar{\omega} \cdot \bar{L}$$

$$T = \frac{1}{2} \bar{\omega} \cdot I \cdot \bar{\omega}$$

$\bar{\omega} = \omega \hat{n}$

$I_s = \frac{1}{\omega^2} \bar{\omega} \cdot \bar{L}$

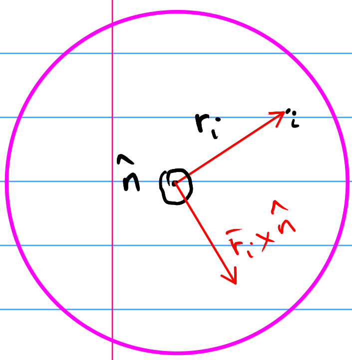
$$\boxed{T = \frac{1}{2} \omega^2 \hat{n} \cdot I \cdot \hat{n} = \frac{1}{2} I_s \omega^2} \quad (1)$$

$$I_s = \hat{n} \cdot I \cdot \hat{n} = m_i \left[r_i^2 - (\bar{r}_i \cdot \hat{n})^2 \right]$$

I_s is a scalar resulting from the contraction of the I tensor

I - moment of inertia tensor
I_s - moment of inertia

"Sum over particles of the product of mass and perpendicular distance square from the rotation axis"



$$r_i = |\bar{r}_\perp| = |\bar{r}_i \times \hat{n}|$$

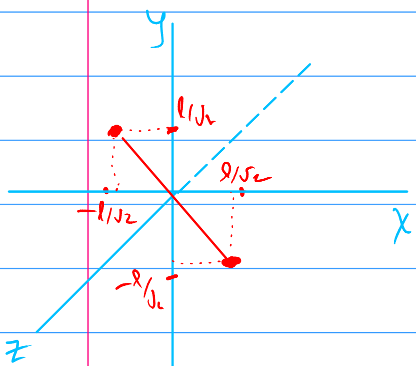
$$I_s = m_i (\bar{r}_i \times \hat{n}) \cdot (\bar{r}_i \times \hat{n})$$

$$= \frac{m_i}{\omega^2} (\bar{r}_i \times \bar{\omega}) \cdot (\bar{r}_i \times \bar{\omega})$$

$$= \frac{m_i}{\omega^2} \bar{v}_i \cdot \bar{v}_i = \frac{m_i v_i^2}{\omega^2} = \frac{2T}{\omega^2}$$

$$T = \frac{1}{2} I_s \omega^2$$

Example



Moment of inertia tensor $I_{ij} = m_i(\delta_{ij} r^2 - x_i x_j)$

$$I_{xx} = m\left[l^2 - \left(\frac{l}{\sqrt{2}}\right)\left(-\frac{l}{\sqrt{2}}\right)\right] + m\left[l^2 - \left(-\frac{l}{\sqrt{2}}\right)\left(\frac{l}{\sqrt{2}}\right)\right]$$

$$= m l^2$$

$$I_{yy} = m l^2$$

$$I_{zz} = m(l^2 - 0) + m(l^2 - 0) = 2ml^2$$

$$I_{xy} = -m \frac{l}{\sqrt{2}}\left(-\frac{l}{\sqrt{2}}\right) - m\left(-\frac{l}{\sqrt{2}}\right)\left(\frac{l}{\sqrt{2}}\right) = ml^2 = I_{yx}$$

$$I_{xz} = 0 = I_{zx} \qquad I_{yz} = I_{zy} = 0$$

$$I = \begin{pmatrix} ml^2 & ml^2 & 0 \\ ml^2 & ml^2 & 0 \\ 0 & 0 & 2ml^2 \end{pmatrix}$$

x: $\hat{n} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad I_s^x = \hat{n} \cdot I \cdot \hat{n} = (1 \;\; 0 \;\; 0) \begin{pmatrix} ml^2 \\ ml^2 \\ 0 \end{pmatrix} = ml^2$

y: $\hat{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad I_s^y = (0 \;\; 1 \;\; 0) \begin{pmatrix} ml^2 \\ ml^2 \\ 0 \end{pmatrix} = ml^2$

z: $\hat{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad I_s^z = (0 \;\; 0 \;\; 1) \begin{pmatrix} 0 \\ 0 \\ 2ml^2 \end{pmatrix} = 2ml^2$

Angular momentum $\quad \bar{L} = I \bar{\omega}$

for rotations around z-axis: $\quad \bar{L} = I \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} = 2ml^2 \omega \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

for rotations around x-axis: $\quad \bar{L} = I \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} = ml^2 \omega \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$