GOAL: to see how much information we can get without explicitly solving the integrals from previous class

1. $E = \frac{1}{2} m v^2 + V(r) \Rightarrow v = \sqrt{\frac{2}{m}(E - V(r))}$

$\bar{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

$\dot{r} = \sqrt{\frac{2}{m}\left(E - V - \ell^2/2mr^2\right)}$

2. $m\ddot{r} - \frac{\ell^2}{mr^3} + \frac{\partial V}{\partial r} = 0$

$m\ddot{r} = -\frac{\partial V}{\partial r} + \frac{\lambda^2}{mr^3} \equiv f'$

this is 1D b/c it only depends on one variable, and describes the movement of a particle under the 'f^prime' force

$m\ddot{r} = -\frac{d}{dr}\left(V + \frac{1}{2} \cdot \frac{\ell^2}{mr^2} \right) \equiv -\frac{d}{dr} V'$

it is an effective potential

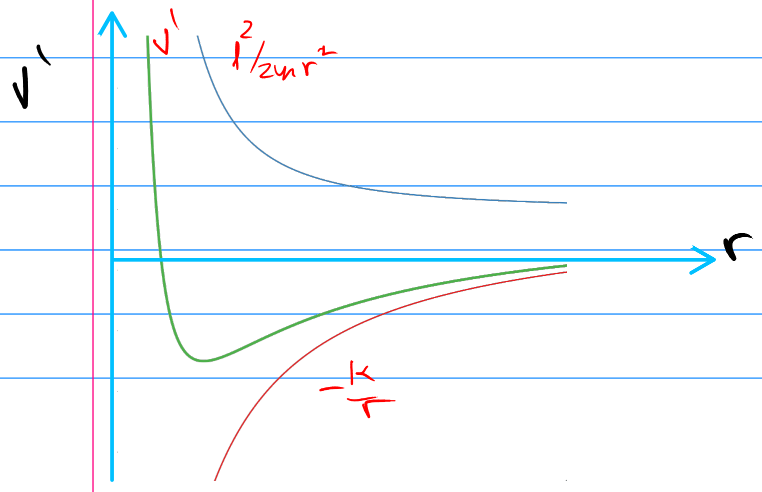
use the same definition in the previous expression of the energy

$E = \frac{1}{2} m\dot{r}^2 + V'$

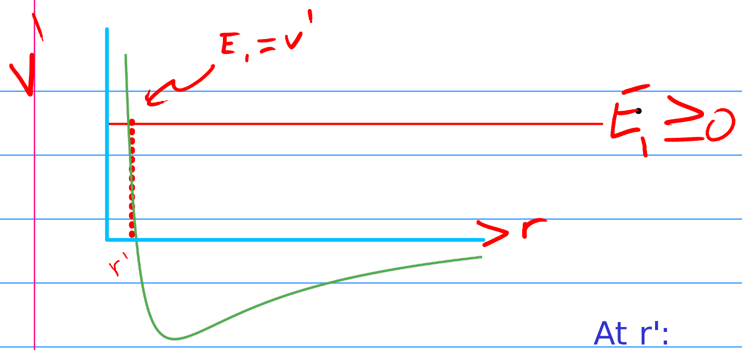
Special case: where the true force is an inverse-square force

$f = -\frac{\partial V}{\partial r} = \frac{-k}{r^2}$ $\qquad k > 0$

$V = -\frac{k}{r} \Longrightarrow \boxed{V' = -\frac{k}{r} + \frac{1}{2} \cdot \frac{\ell^2}{mr^2}}$



I.



for a particle that is approaching the origin, there is a minimum r' at which the particle will reverse its r behavior and never come back.
This is an unbounded orbit.

At r': 1. radial KE = 0 $\rightarrow \frac{1}{2}m\dot{r}^2 = 0$
2. angular KE is not 0 $\rightarrow \frac{1}{2}mr^2\dot{\theta}^2 \neq 0$

will see shortly:
hyperbola — $E > 0$
parabola — $E = 0$

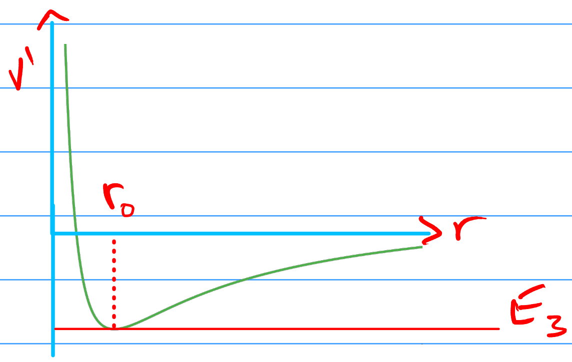
II.

if the total energy is negative, there is a minimum r1 and max r2 distances.
This is a bounded orbit.
Both distances are turning points.

r1 and r2 are usually called "apsidal" distances

ellipse — $E < 0$

III.



$\forall r, \quad E_3 = V', \quad \frac{1}{2}m\dot{r}^2 = 0$

from above

$m\ddot{r} = 0 = f' \Rightarrow$

$$f = -\frac{l^2}{mr^3} = -mr\dot{\theta}^2$$

centripetal force

circle

The classification of open, bounded, and circular will work out for any 'V' that is attractive and with exponent

$$V \sim \frac{-1}{r^n} \qquad n < 2$$

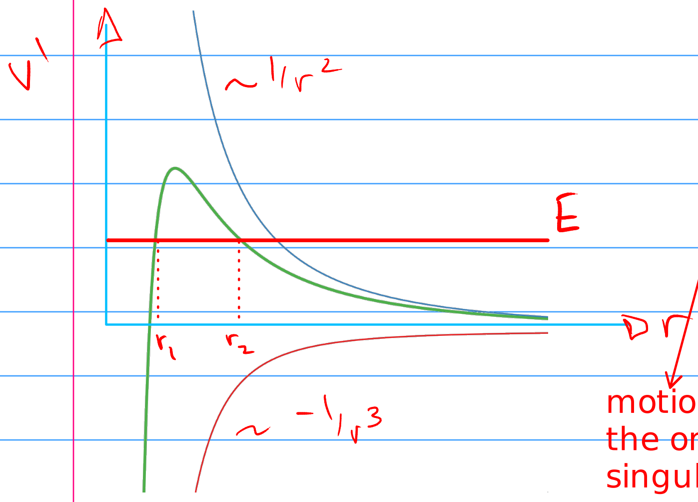
b/c of the competing term of the angular momentum

The actual shape of the orbits might not be the same as above.

Examples outside of this n<2 case.

n=3 $\quad V(r) = -\frac{a}{r^3} \Rightarrow f = -\frac{3a}{r^4}$

$$V' = -\frac{a}{r^3} + \frac{1}{2} \cdot \frac{\ell^2}{mr^2}$$



For this E, either motion is bounded with

$r < r_1$

or unbounded with $\quad r > r_2$

motion may pass through the origin, but this is singular

motion never passes through origin

$r_1 < r < r_2$ is physically impossible

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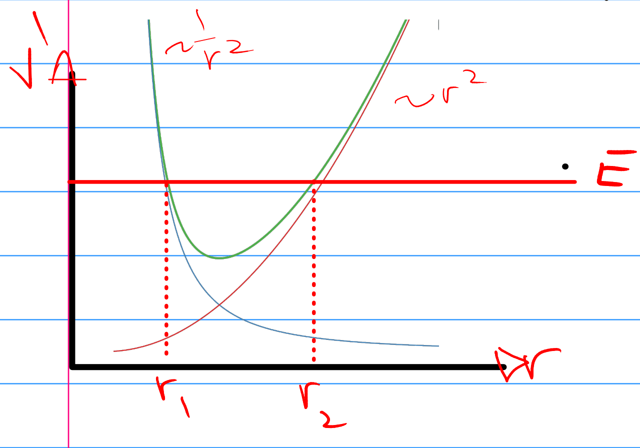
n=-2 $\quad V(r) = \frac{1}{2}kr^2 \Rightarrow f = -kr$

$$V' = \frac{1}{2}kr^2 + \frac{1}{2}\frac{\ell^2}{mr^2}$$



bounded motion between r1 and r2

Differential eqns of motion,

Instead of solving r(t) or theta(t), we will solve r(theta).

This involves eliminating the 't' variable from the equations.

$$l = mr^2\dot{\theta}$$

$$\frac{d}{dt} = \frac{l}{mr^2}\frac{d}{d\theta}$$

$$l\,dt = mr^2 d\theta$$

recall from before $m\ddot{r} - \frac{l^2}{mr^3} + \frac{\partial V}{\partial r} = 0$

$$m\frac{d^2}{dt^2}r - \frac{l^2}{mr^3} = -\frac{\partial V}{\partial r}$$

$$m\frac{l}{mr^2}\frac{d}{d\theta}\left(\frac{l}{mr^2}\frac{d}{d\theta}r\right) - \frac{l^2}{mr^3} = -\frac{dV}{dr}$$

use $u = \frac{1}{r}$: $\frac{du}{d\theta} = \frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{-1}{r^2}\frac{dr}{d\theta}$

$$-lu^2\frac{d}{d\theta}\left(\frac{l}{m}\frac{du}{d\theta}\right) - \frac{l^2}{m}u^3 = -\frac{dV}{du}\frac{du}{dr}$$

$\left\{\frac{du}{dr} = \frac{-1}{r^2} = -u^2\right.$

$$\boxed{\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\frac{dV}{du}}$$

Alternatively, we can use $dt = \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - l^2/2mr^2\right)}}$

$$\frac{mr^2}{l}d\theta = \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - l^2/2mr^2\right)}}$$

$$d\theta = \frac{l\,dr}{mr^2\sqrt{\frac{2}{m}\left(E - V - l^2/2mr^2\right)}}$$

$$\theta = \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2m}{l^2}(E-V) - \frac{1}{r^2}}} + \theta_0$$

$$\boxed{\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2m}{l^2}(E-V) - u}}}$$

In general, these integrals and diff eqns do not have analytical solutions.
Only certain power-law functions of 'r' can be solved.

$$V = a r^{n+1} \Rightarrow \text{force } f \sim r^n$$

Note: n=-1 is excluded as that would be a constant potential, or no force.
Also, if n=-1 directly in 'f', then the potential would be logarithmic - not a power-law.

Using the form of theta

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2m}{l^2}\left(E - \frac{a}{u^{n+1}}\right) - u^2}}$$

trigonometric solutions using n = 1, -2, -3
elliptic function solutions using n = 5, 3, 0, -4, -5, -7

The Kepler problem: 1/r^2 force

Lets now specialize to this important force

$$f = -\frac{k}{r^2} \Rightarrow V = -\frac{k}{r}$$

$$\theta = \theta' - \int \frac{du}{\sqrt{\frac{2m}{l^2}(E + ku) - u^2}}$$

where theta' is not necessarily the initial angle, but a constant that is obtained after inserting in the solution the initial conditions.