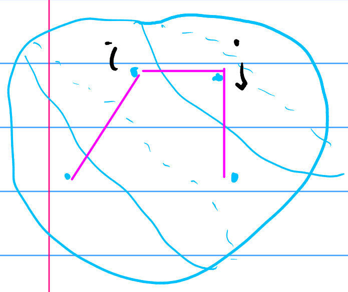
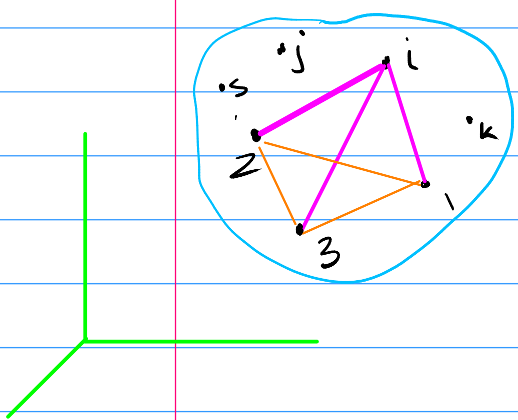
How many independent coordinates are necessary to specify its configuration



N particles $\quad 3N \quad \sim N$

$r_i - r_j = c_{ij} \quad \dfrac{N(N-1)}{2} \quad \sim N^2$

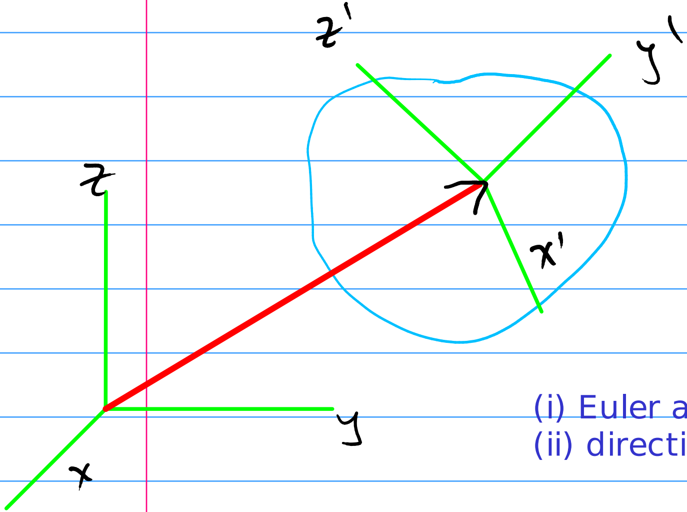
not independent



9 ind. degrees of freedom

-3

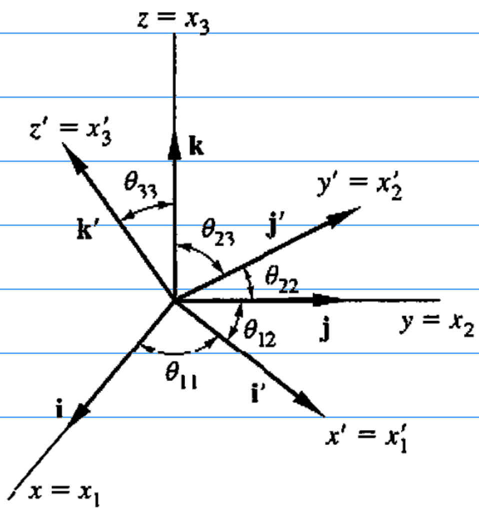
6



3 coordinates for specifying the position of the prime system.

3 coordinates to specify the orientation of the prime system.

(i) Euler angles -
(ii) direction cosines -



Define:

$\cos\theta_{11} = \hat{i}' \cdot \hat{i}$

$\cos\theta_{12} = \hat{i}' \cdot \hat{j}$

$\vdots$

$\cos\theta_{32} = \hat{k}' \cdot \hat{j}$

$\vdots$

$$\hat{i}' = \cos\theta_{11}\,\hat{i} + \cos\theta_{12}\,\hat{j} + \cos\theta_{13}\,\hat{k}$$

$$\hat{j}' = \cos\theta_{21}\,\hat{i} + \cos\theta_{22}\,\hat{j} + \cos\theta_{23}\,\hat{k} \qquad (1)$$

$$\hat{k}' = \cos\theta_{31}\,\hat{i} + \cos\theta_{32}\,\hat{j} + \cos\theta_{33}\,\hat{k}$$

$$\bar{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k} = x'\,\hat{i}' + y'\,\hat{j}' + z'\,\hat{k}'$$

$$x' = \bar{r}\cdot\hat{i}' = \cos\theta_{11}\,x + \cos\theta_{12}\,y + \cos\theta_{13}\,z$$

$$y' = \bar{r}\cdot\hat{j}' = \cos\theta_{21}\,x + \cos\theta_{22}\,y + \cos\theta_{23}\,z \qquad (2)$$

$$z' = \bar{r}\cdot\hat{k}' = \cos\theta_{31}\,x + \cos\theta_{32}\,y + \cos\theta_{33}\,z$$

solve for the $\hat{i}, \hat{j}, \hat{k}$ in (1)

then we use $\hat{i}\cdot\hat{j} = \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = 0$

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$$

to obtain $\displaystyle\sum_{l=1}^{3} \cos\theta_{lm'}\cos\theta_{lm} = 0 \qquad m \neq m'$

$$\sum_{l=1}^{3} \cos^2\theta_{lm} = 1$$

these two eqns will reduce from 9 to 3 the # of independent variables

$$\boxed{\sum_{l=1}^{3} \cos\theta_{lm'}\cos\theta_{lm} = \delta_{m'm}}$$

Condense the notation

$x \to x_1$

$y \to x_2$

$z \to x_3$

$\cos\theta_{ij} \equiv a_{ij}$

from (2)

$$x_1' = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$$
$$x_2' = a_{21}x_1 + a_{22}x_2 + a_{23}x_3$$
$$x_3' = a_{31}x_1 + a_{32}x_2 + a_{33}x_3$$

$$\boxed{x_i' = a_{ij}x_j} \qquad i = 1,2,3 \qquad (4)$$

$[a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3]$

$$x_i' x_i' = x_i x_i \qquad x_i x_i = \sum_1^3 x_i^2$$
$$= (a_{ij}x_j)(a_{ik}x_k)$$
$$= (a_{ij}a_{ik})(x_j x_k)$$

$$\boxed{a_{ij}a_{ik} = \delta_{jk}} \qquad (5) \qquad j,k = 1,2,3$$

Any linear transformation (4) that satisfies (5) is called an "orthogonal" transformation.

(5) is called an "orthogonality" condition

The matrix A of the elements aij is called the "matrix of transformation"

a_{11}

$$A = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

a_{21}

$$(\bar{r})' = A\bar{r}$$

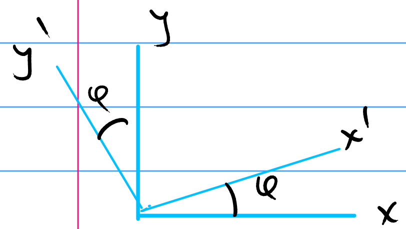
operator A transforms the components from the unprimed to the primed system. The vector is the same.

$$\bar{r}' = A\bar{r}$$

operator A is rotating the vector r, and both r' and r are being represented by the same coordinate system.

The math is the same whether we rotate the coordinate system, or rotate the vector.

However, if we rotate the coordinate system - rotate "counterclockwise"



"passive" interpretation

if we rotate rather the vector, the we are rotating the vector "clockwise"

"active" interpretation

Formalism behind the Transformation Matrix

I. Lets apply two sequential rotations

from $\bar{r}$ to $\bar{r}'$ $\qquad x'_k = b_{kj} x_j$

from $\bar{r}'$ to $\bar{r}''$ $\qquad x''_i = a_{ik} x'_k$

$$x''_i = a_{ik} b_{kj} x_j = c_{ij} x_j$$

b/c A and B are orthogonal, matrix C will also be an orthogonal matrix

$$C = AB$$

Note that if we do the reverse $b_{ik} a_{kj}$ in general will not get c_{ij}

$$\boxed{AB \neq BA}$$

order of application of transformations is important

II. The inverse operation from r to r' is $\bar{r}' = A\bar{r} \Rightarrow \bar{r} = A^{-1}\bar{r}'$

$$AA^{-1} = 1 \qquad \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ \vdots & & & \end{pmatrix}$$

$A^{-1} \rightarrow a'_{ij}$

III. Consider the double sum $a_{k\ell}\, a_{ki}\, a'_{ij}$

$(a_{k\ell} a_{ki})\, a'_{ij}$

$\delta_{\ell i}\, a'_{ij}$

$a'_{\ell j}$

$x_i = a'_{ij} x'_j$

in

$x_k = a_{ki} x_i$

$x'_k = \underbrace{a_{ki} a'_{ij}}_{\delta_{kj}} x'_j$

$a_{k\ell}(a_{ki} a'_{ij})$

$a_{k\ell}\, \delta_{kj}$

$a_{j\ell}$

$$\Rightarrow \quad a'_{\ell j} = a_{j\ell}$$

$$\boxed{A^{-1} = A^T}$$

for orthogonal matrices, the inverse is equal to the transposed matrix.

IV. Other properties

$(Ax)_i = a_{ij} x_j$

$\begin{pmatrix} a_{11} & a_{12} & \cdots \\ \vdots & & \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ x'_i \\ \vdots \end{pmatrix}$

$(x^T A^T)_i = x_j \tilde{a}_{ji} = x_j a_{ij}$

$(x_1 \cdots)\begin{pmatrix} a_{11} & a_{21} & \cdots \\ \vdots & & \end{pmatrix} = (\cdots x'_i \cdots)$

$$\boxed{Ax = x^T A^T}$$

valid for all square matrices

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if $a_{ij} = a_{ji} \Rightarrow A = A^T$ called symmetric

if $a_{ij} = -a_{ji} \Rightarrow A = -A^T$ called antisymmetric or skew symmetric.

$a_{ii} = 0$

V. Lets go back to the interpretations (passive vs active) and see how they can work together

Let A operate on F to produce G $$G = AF$$

A as transforming the vector

Now, lets transform the coordinate system

$$\underbrace{BG}_{\text{G in new coordinate system}} = BAF = \underbrace{(BAB^{-1})}_{\text{A in the new coordinate system, acting on F that is also in the new coordinate system}} \underbrace{BF}_{\text{F in new coordinate system}}$$

$$A' = BAB^{-1}$$ called a similarity transformation

VI. Look at determinants of square matrices

1. We start by saying $$|AB| = |A| \cdot |B|$$

$$|A| = |A^T|$$

2. determinant of orthogonal matrix

$$|A^2| = |AA| = |A| \cdot |A| = |A^T| \cdot |A|$$
$$= |A^{-1}| \cdot |A| = |A^{-1}A| = 1$$

the determinant of an orthogonal matrix is either+1 or -1.

3. For any square matrix

$$A' = BAB^{-1}$$
$$A'B = BA$$

$$|A'| \cdot |B| = |B| \cdot |A|$$

$$|A'| = |A|$$

determinant of a matrix is the same even after a similarity transformation

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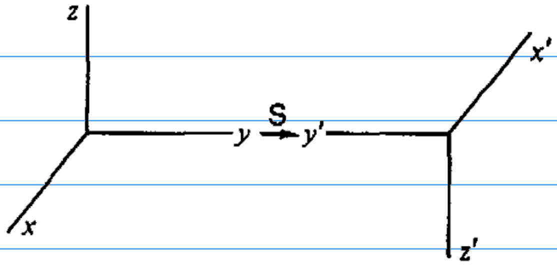
The determinant of an orthogonal matrix can only be +1 in order to represent a physical displacement of a rigid body

$$S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$



$$ab = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = S$$

This operation changes from a right-handed to a left-handed system

Cannot be attained by any rigid physical change, hence it is not physical for rigid bodies.

Thus, physical transformations of rigid bodies must have determinant +1, called a "proper" transformation.