# PDEs and Coordinate Systems

| | Cartesian | (Cylindrical) Polar | Spherical Polar |
|---|---|---|---|
| **Hyperbolic** | 1D string<br>2D membrane<br>3D volume | Circular membrane<br>Wave in cylinder | Wave in sphere |
| **Elliptic** | Laplace in square<br>Laplace in cube | Laplace in circle<br>Laplace in cylinder | Laplace in sphere<br>Sphere in E-field |
| **Parabolic** | Diffusion in square<br>Diffusion in cube | Diffusion in cylinder | Diffusion in sphere |
| **Schrödinger*** | Particle in line<br>Particle in rectangle<br>Particle in cuboid | Particle in a circle<br>Particle in a cylinder | Particle in sphere<br>Particle in hemisphere |

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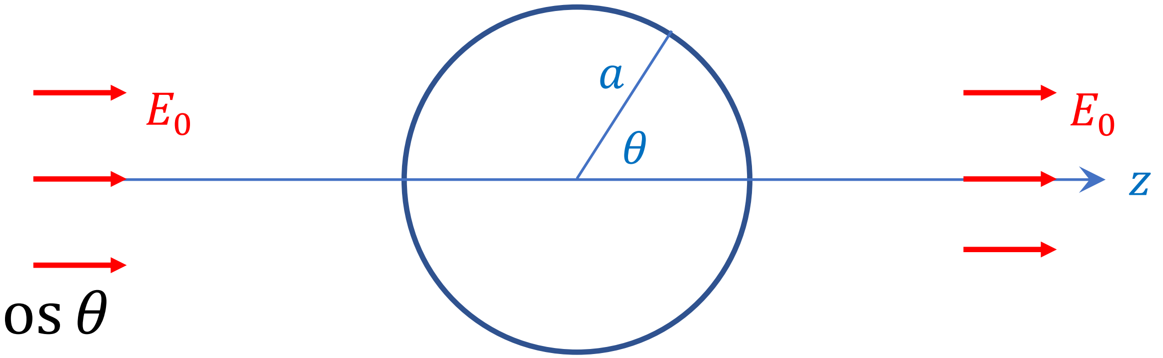
# Conducting Sphere in a Uniform E-field




- Field satisfies $\nabla^2\phi = 0$, $\boldsymbol{E} = -\nabla\phi$
- $\boldsymbol{E}$-field at infinity,

$$\boldsymbol{E} = E_0\hat{z} \Longrightarrow \phi = -E_0 z \; = -E_0 r\cos\theta$$

- Axisymmetric solution, no $\varphi$ dependence, $m = 0$

$$\Longrightarrow \;\; \phi = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-l-1}\right) P_l^0(\cos\theta) \qquad \text{wlog } A_0 = 0 \text{ (constant)}$$

- As $z \longrightarrow \infty$, require $\phi = -E_0 r\cos\theta = -E_0 r P_1^0(\cos\theta)$

$$\Longrightarrow \;\; A_l = 0 \;\text{ for } l > 1, \;\; A_1 = -\mathrm{E}_0$$

- At $r = a$, need $\phi =$ constant, so $A_l a^l + B_l a^{-l-1} = 0$ for all $l$

$$\Longrightarrow \;\; B_l = 0 \;\text{ for } l > 1, \quad A_1 a + B_1 a^{-2} = 0, \text{ so } B_1 = E_0 a^3$$

- Solution for $r > a$ is

$$\phi = -E_0 r\cos\theta\left(1 - \frac{a^3}{r^3}\right)$$

Induced dipole moment

# Laplace's Equation in Polar Coordinates

- In 2D (i.e. Laplace in a circle)

$$r(rR')' - m^2R = 0$$

$$\Longrightarrow \quad r^2R'' + rR' - m^2R = 0$$

$$\Longrightarrow \quad R(r) = r^{\pm m}$$

- True, but there is another solution in the case $m = 0$:

$$r^2R'' + rR' = 0$$

$$\Longrightarrow rR'' + R' = (rR')' = 0$$

$$\Longrightarrow rR' = \text{constant, } a$$

$$\Longrightarrow R(r) = a\log r + b$$

Additional solution to the problem
— dominant solution as $r \longrightarrow \infty$

# Special Functions in Physics

- Harmonic Oscillator
  $$y'' + k^2 y = 0$$ all
- Bessel
  $$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$ polar, cyl. polar, sph. polar
- Legendre ($m = 0$)
  $$(1 - x^2) y'' - 2 x y' + l(l+1) y = 0$$ spherical polar
- Legendre (general)
  $$(1 - x^2) y'' - 2 x y' + \left[ l(l+1) - \frac{m^2}{1 - x^2} \right] y = 0$$ spherical polar
- Hermite
  $$y'' - 2 x y' + 2 n y = 0$$ QM harmonic oscillator
- Laguerre
  $$x y'' - (1 - x) y' + n y = 0$$ QM hydrogen atom

# Second-Order Linear (Ordinary) Differential Equations

- General form of SOLDE

$$y'' + P(x)y' + Q(x)y = 0$$

- Some definitions, at some point $x_0$

  $P(x_0)$ and $Q(x_0)$ finite $\Longrightarrow$ $x = x_0$ is an ordinary point

  $P(x_0)$ or $Q(x_0)$ infinite

  $(x - x_0)P(x_0)$, $(x - x_0)^2 Q(x_0)$ finite

  $\Longrightarrow$ $x = x_0$ is a regular singular point

  otherwise, $x = x_0$ is an essential singularity

- All of the equations just cited fall into the first two categories
  - all points are ordinary or regular singular
- Why worry about singular points?
  - they tell us about whether we can construct a power series solution

# Series Solutions to SOLDEs

- In the vicinity of some point (here $x = 0$), seek a power-series solution of the form (power series with an $x^k$ multiplier)

  $$y(x) = x^k \sum_{i=0}^{\infty} a_i x^i$$

- Notes: $k$ and $a_i$ are formally undetermined, no constraint on $k$, $a_0 \neq 0$.
- Basic approach: assume convergence and substitute the series into the ODE, then compare powers of $x$.
- Start with the equation of the Harmonic Oscillator as a familiar starting point:

  $$y'' + y = 0$$

  In our terminology, $P(x) = 0, \;\; Q(x) = 1$
- Note that $x = 0$ is an ordinary point.
- Solutions $\cos x$ and $\sin x$ and their series are all well known.

# Harmonic Oscillator

- Differential equation:

$$y'' + y = 0$$

(solutions $y = \cos x,\ \sin x$)

- Substitute the assumed solution

$$y(x) = x^k \sum_{i=0}^{\infty} a_i x^i = \sum_{i=0}^{\infty} a_i x^{k+i}$$

into the equation

$$y' = \sum_{i=0}^{\infty} a_i (k+i) x^{k+i-1}$$

$$y'' = \sum_{i=0}^{\infty} a_i (k+i)(k+i-1) x^{k+i-2}$$

$$\Longrightarrow \sum_{i=0}^{\infty} a_i (k+i)(k+i-1) x^{k+i-2} + \sum_{i=0}^{\infty} a_i x^{k+i} = 0$$

- Equation must be satisfied by the coefficient of <u>every</u> power of $x$.

- Start by rewriting to equalize the powers of $x$ in both sums.

# Harmonic Oscillator

$$\sum_{i=0}^{\infty} a_i (k+i)(k+i-1)x^{k+i-2} + \sum_{i=0}^{\infty} a_i x^{k+i} = 0$$

- Write $j = i - 2$ in the first sum, $j = i$ in the second

$$\Longrightarrow \sum_{j=-2}^{\infty} a_{j+2}(k+j+2)(k+j+1)x^{k+j} + \sum_{j=0}^{\infty} a_j x^{k+j} = 0$$

$$\Longrightarrow a_0 k(k-1)x^{k-2} + a_1(k+1)kx^{k-1}$$
$$+\sum_{j=0}^{\infty} \left[a_{j+2}(k+j+2)(k+j+1) + a_j\right]x^{k+j} = 0$$

- Equation must be satisfied term by term, and $a_0 \neq 0$, so

$x^{k-2}$ $\quad a_0 k(k-1) = 0 \Longrightarrow k = 0$ or $1$ $\quad$ Indicial Equation

$x^{k-1}$ $\quad a_1 k(k+1) = 0 \Longrightarrow$ no new constraint if $k = 0$

$a_1 = 0$ if $k = 1$

# Harmonic Oscillator

- Indicial equation <u>always</u> gives algebraic constraint on $k$.
- Second equation <u>may</u> constrain coefficients.
- Note that in either case, the solution is a <u>polynomial</u> in $x$.
- Now consider the remaining sum:

$$\sum_{j=0}^{\infty} \left[a_{j+2}(k+j+2)(k+j+1) + a_j\right] x^{k+j} = 0$$

- Coefficient of $x^{k+j}$ must be zero, so

$$a_{j+2}(k+j+2)(k+j+1) + a_j$$

- A <u>recurrence relation</u> for the coefficients:

$$a_{j+2} = \frac{-1}{(k+j+2)(k+j+1)} a_j$$

# Harmonic Oscillator

- Thus,

$$a_{j+2} = \frac{-1}{(k+j+2)(k+j+1)} a_j$$

- Note that even couples with even, odd with odd.
- For $k = 1$, we know $a_1 = 0$, so the solution consists only of terms <u>odd</u> in $x$:

$$a_0 x,\ \ a_2 x^3,\ \ a_4 x^5,\ \text{etc.}$$

$$a_{2n} = \frac{-1}{(2n+1)(2n)} a_{2n-2} = \frac{(-1)^2}{(2n+1)(2n)(2n-1)(2n-2)} a_{2n-4}$$

$$\vdots$$

$$= \frac{(-1)^n}{(2n+1)!} a_0,$$

so $\quad y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = a_0 \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right) \qquad = \sin x$

# Harmonic Oscillator

- If $k = 1$, the polynomial consists of only <u>odd</u> powers of $x$.
- If $k = 0$, $a_1, a_3, a_5, \cdots$ are not determined by the recurrence, but conventional to take them to be zero, so the polynomial has only <u>even</u> powers of $x$.

  (Recall that any linear combination of two solutions is a solution, so this is equivalent to absorbing the odd powers for $k = 0$ into the $k = 1$ solution.)
- For $k = 0$, we now have

$$a_{2n} = \frac{-1}{(2n)(2n-1)}\, a_{2n-2} = \frac{(-1)^2}{(2n)(2n-1)(2n-2)(2n-3)}\, a_{2n-4}$$

$$\vdots$$

$$= \frac{(-1)^n}{(2n)!}\, a_0,$$

so $\quad y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = a_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) \qquad = \cos x$

# Series Solutions to Legendre's Equation

- Now look at the Legendre equation (just the associated Legendre equation with $m = 0$):

$$(1 - x^2)y'' - 2xy' + l(l + 1)y = 0$$

In our terminology, $P(x) = \frac{-2x}{1-x^2}$, $Q(x) = \frac{l(l+1)}{1-x^2}$

- Again, $x = 0$ is an ordinary point.
- Substitute the assumed solution

$$y(x) = x^k \sum_{i=0}^{\infty} a_i x^i = \sum_{i=0}^{\infty} a_i x^{k+i}$$

with $a_0 \neq 0$ into the equation and again note

$$y' = \sum_{i=0}^{\infty} a_i (k + i) x^{k+i-1}$$

$$y'' = \sum_{i=0}^{\infty} a_i (k + i)(k + i - 1) x^{k+i-2}$$

# Series Solutions to Legendre's Equation

- Substitute the assumed solution

$$y(x) = x^k \sum_{i=0}^{\infty} a_i x^i = \sum_{i=0}^{\infty} a_i x^{k+i}$$

into the Legendre ODE: $(1 - x^2)y'' - 2xy' + l(l+1)y = 0$

- Result:

$$(1 - x^2) \sum_{i=0}^{\infty} a_i (k+i)(k+i-1) x^{k+i-2}$$
$$-2x \sum_{i=0}^{\infty} a_i (k+i) x^{k+i-1} + l(l+1) \sum_{i=0}^{\infty} a_i x^{k+i} = 0$$

$$\Rightarrow \sum_{i=0}^{\infty} a_i (k+i)(k+i-1) x^{k+i-2}$$
$$- \sum_{i=0}^{\infty} [(k+i)(k+i-1) + 2(k+i) - l(l+1)] a_i x^{k+i} = 0$$

$$\Rightarrow \sum_{j=-2}^{\infty} a_{j+2} (k+j+2)(k+j+1) x^{k+j} \qquad j = i-2$$
$$- \sum_{j=0}^{\infty} a_j [(k+j)(k+j+1) - l(l+1)] x^{k+j} = 0 \qquad j = i$$

# Series Solutions to Legendre's Equation

- Combine the sums:

$$a_0 k(k-1)x^{k-2} + a_1(k+1)kx^{k-1}$$
$$+\sum_{j=0}^{\infty} a_{j+2}(k+j+2)(k+j+1)x^{k+j}$$
$$-\sum_{j=0}^{\infty} a_j[(k+j)(k+j+1) - l(l+1)]x^{k+j} = 0$$

- Expect every coefficient of $x^{k+j}$ to sum to zero individually, and $a_0 \neq 0$
- Leading terms are

$x^{k-2}$ $\quad a_0 k(k-1) = 0 \implies k = 0$ or $1$

$x^{k-1}$ $\quad a_1 k(k+1) = 0 \implies a_1 = 0$ if $k = 1$

- Again, indicial equation gives the allowed values of $k$, second equation constrains the coefficients.

# Series Solutions to Legendre's Equation

- In either case, the solution is a <u>Legendre polynomial</u>, $P_l(x)$.
- Now consider the remaining sum.

$$\sum_{j=0}^{\infty} a_{j+2}(k+j+2)(k+j+1)x^{k+j}$$
$$-\sum_{j=0}^{\infty} a_j[(k+j)(k+j+1)-l(l+1)]\,x^{k+j}=0$$

- Coefficient of $x^{k+j}$ must be zero, so

$$a_{j+2}(k+j+2)(k+j+1)=a_j[(k+j)(k+j+1)-l(l+1)]$$

- Recurrence relation for the coefficients.
- Again connects even to even, odd to odd.

# Series Solutions to Legendre's Equation

- Recurrence relation is

$$a_{j+2}(k+j+2)(k+j+1) = a_j[(k+j)(k+j+1) - l(l+1)]$$

- In the case $k = 1, a_0 \neq 0, a_1 = 0$, so

$$a_{j+2} = \frac{(j+1)(j+2)-l(l+1)}{(j+3)(j+2)} a_j$$

and all even terms are determined (odd terms are zero):

$$a_{2n} = \frac{(2n)(2n-1)-l(l+1)}{(2n+1)(2n)} a_{2n-2}$$

$$\vdots$$

$$= \frac{[(2n)(2n-1)-l(l+1)]\,[(2n-2)(2n-3)-l(l+1)] \cdots [6-l(l+1)][-l(l+1)]}{(2n+1)!} a_0$$

(recall in this case, $a_{2n}$ is the coefficient of $x^{2n+1}$)

# Series Solutions to Legendre's Equation

- In the case $k = 0$,

$$a_{j+2} = \frac{j(j+1) - l(l+1)}{(j+2)(j+1)} a_j$$

- Odd terms are undetermined; make the same arguments as before to absorb them into the $k = 1$ solution, so just look at the even terms.
- Then all even terms are determined:

$$a_{2n} = \frac{(2n-2)(2n-1) - l(l+1)}{(2n)(2n-1)} a_{2n-2}$$

$$\cdots \quad = \frac{[(2n-2)(2n-1) - l(l+1)]\,[(2n-4)(2n-3) - l(l+1)] \cdots [6 - l(l+1)][-l(l+1)]}{(2n)!} a_0$$

- Note: as before, if $k = 1$, polynomial consists of only <u>odd</u> powers of $x$.
- If $k = 0$, polynomial has only <u>even</u> powers of $x$.

# Convergence of the Series

- Note that since

$$\frac{a_{j+2}}{a_j} = \frac{(k+j)(k+j+1)-l(l+1)}{(k+j+2)(k+j+1)} \to 1 \text{ as } j \to \infty,$$

ratio test implies that the series is <u>divergent</u> at $x = \pm 1$.

Ratio of $j+2$ and $j$ terms is

$$\frac{a_{j+2}}{a_j} x^2$$

- The only way the solution can converge everywhere is if <u>terminates</u>.
- Require $a_{j+2} = 0$ for some $j$

$$\Longrightarrow (k+j)(k+j+1) - l(l+1) = 0 \text{ for some } j$$

$$\Longrightarrow l = k + j \text{ for some } j$$

$$\Longrightarrow l \text{ must be an } \underline{\text{integer}}$$

- (Recall $P_l(x) = P_l^0(x)$ previously, $x = \cos\theta$, want solution finite everywhere on the sphere.)

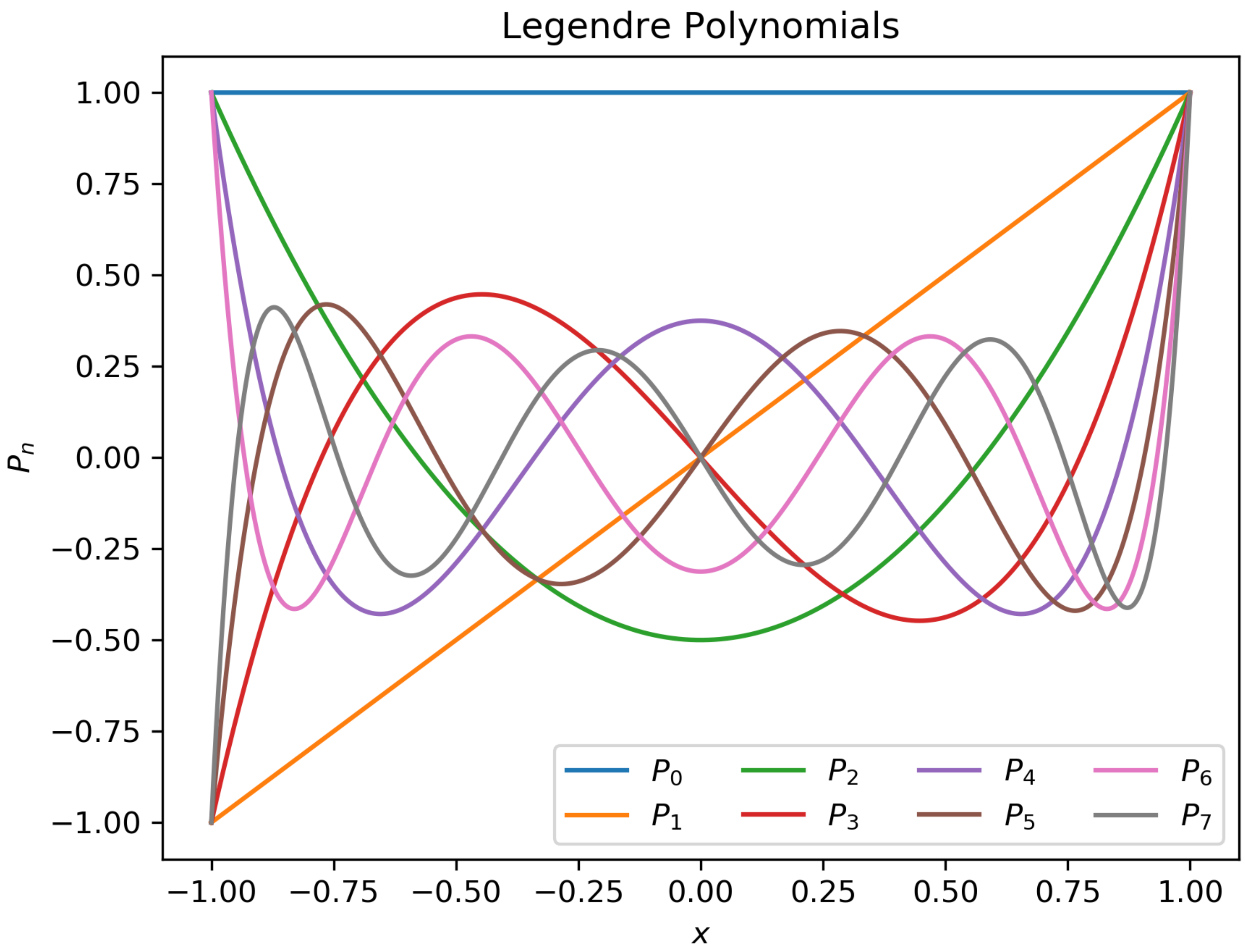
# Legendre Polynomials

- $l$ must be an integer for convergence
- Even solution ($k = 0$) stops at $j = l$
- Odd solution ($k = 1$) stops at $j = l - 1$

  $P_l(x)$ is a polynomial of order $l$

  even powers of $x$ only if $l$ is even: $0, 2, 4, \dots, l$

  odd powers of $x$ only if $l$ is odd: $1, 3, 5, \dots, l$
- Already seen the first few ($P_l = P_l^0$).
- Convention: $P_l(1) = 1$
- Parity: $P_l(-x) = (-1)^l \, P_l\,(x)$
- Standard function in `Python` and other languages.


Legendre Polynomials
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
$P_n$
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
$x$
$P_0$
$P_1$
$P_2$
$P_3$
$P_4$
$P_5$
$P_6$
$P_7$

# Series Solutions to Bessel's Equation

- The Bessel Equation is:

$$x^2y'' + xy' + (x^2 - m^2)y = 0, \ \ m \text{ \underline{integer}}$$

$$P(x) = \frac{1}{x}, \quad Q(x) = 1 - \frac{m^2}{x^2}$$

- Now $x = 0$ is a <u>regular singular point</u>.
- Substitute the assumed solution

$$y(x) = x^k \sum_{i=0}^{\infty} a_i x^i = \sum_{i=0}^{\infty} a_i x^{k+i}$$

$$\Longrightarrow \sum_{i=0}^{\infty} a_i (k+i)(k+i-1)x^{k+i}$$

Leading term is $x^k$

$$+ \sum_{i=0}^{\infty} a_i (k+i)x^{k+i} + \sum_{i=0}^{\infty} a_i x^{k+i+2} = 0$$

$$-m^2 \sum_{i=0}^{\infty} a_i (k+i)x^{k+i} = 0$$

- Indicial equation: $a_0[(k(k-1) + k - m^2] = 0 \ \ \Longrightarrow \ \ k^2 = m^2, \ k = \pm m$

# Series Solutions to Bessel's Equation

- Indicial equation: $a_0[(k(k-1)+k-m^2]=0 \quad \Longrightarrow \quad k^2=m^2,\ k=\pm m$

$x^{k+1}$: $\qquad a_1[(k+1)k+k+1-m^2]=0$

$$\Longrightarrow \quad a_1\underbrace{(k+1+m)(k+1-m)}_{\neq 0 \text{ for } k=\pm m}=0$$

$$\Longrightarrow \quad a_1=0$$

- General term for $k=+m$ is

$$a_i[(m+i)(m+i-1)+(m+i)-m^2]+a_{i-2}=0$$

$$a_i[(m+i)^2-m^2]+a_{i-2}=0$$

$$\Longrightarrow \qquad a_i=\frac{-1}{i(i+2m)}a_{i-2} \quad \text{"etc."}$$

- Solution is $J_m(x)$

# Series Solutions to Bessel's Equation

- Expect a second-order differential equation to have two linearly independent solutions.
- If m is <u>not</u> an integer, turns out that $J_{-m}(x)$ is <u>independent</u> of $J_m(x)$.
- But if m is an <u>integer</u> (as in the case of separation in cylindrical polars), can show (see R&H 9.5.2) that

$$J_{-m}(x) = (-1)^m J_m(x)$$

- The two solutions for $k$ do <u>not</u> necessarily yield independent solutions to the problem.
- But sometimes they do: e.g. $\sin x$ and $\cos x$…
- How do we know when we actually have two independent solutions?

# Fuch's Theorem

- Classifies behavior of SOLDE solutions.
- Expand about an ordinary point or a regular singular point

  $\Longrightarrow$ indicial equation has $\geq 1$ solutions

  - if roots of indicial equation are equal, then they provide <u>only 1</u> solution
  - if roots of indicial equation differ by a non-integer, then they represent 2 independent solutions
  - if roots of indicial equation differ by an integer, then the larger of the two gives a solution; the lower may or may not — <u>must check</u>
- Expand about an essential singularity $\Longrightarrow$ all bets are off!
- SOLDEs of interest all fall into the former category

# Second Solutions

- Expect all SOLDEs to have 2 independent solutions.
- Power series development gives us a first solution in all cases.
- Some SOLDEs naturally generate 2 independent series solutions.
- Others do not.
- How can we determine if two solutions are independent?
- How can we generate a second solution, given a first series solution?
- Wronskian development shows the way

# Wronskians

- Suppose we have 2 solutions $y_1$ and $y_2$
- Define Wronskian:

$$W(x) = y_1 y_2' - y_1' y_2$$

- Then

$$\begin{aligned} W = 0 \quad &\Leftrightarrow \quad \frac{y_1'}{y_1} = \frac{y_2'}{y_2} \\ &\Leftrightarrow \quad \frac{d(\log y_1)}{dx} = \frac{d(\log y_2)}{dx} \\ &\Leftrightarrow \quad \log y_1 = \log y_2 + \log A \\ &\Leftrightarrow \quad y_1 = A y_2 \end{aligned}$$

- Non-zero Wronskian means <u>linear independence</u>.

# Second Solutions

- Can use the Wronskian to develop a second solution given the first:

$$y_1'' + P(x)y_1' + Q(x)y_1 = 0$$

$$W(x) = y_1 y_2' - y_1' y_2$$

- Then

$$\begin{aligned} W' &= y_1' y_2' + y_1 y_2'' - y_1'' y_2 - y_1' y_2' \\ &= y_1(-P y_2' - Q y_2) - y_2(-P y_1' - Q y_1) \\ &= -P(x)(y_1 y_2' - y_1' y_2) \\ &= -P(x) W \end{aligned}$$

- Simple differential equation for $W$: $W(x) = \cancelto{1}{W(x_0)} e^{-\int_{x_0}^{x} P(x') dx'}$

# Second Solutions

- But we can also write

$$W(x) = y_1 y_2' - y_1' y_2 = y_1^2 \frac{d}{dx}\left(\frac{y_2}{y_1}\right)$$

- so

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{e^{-\int_{x_0}^{x} P(x')dx'}}{y_1^2}$$

$$\Longrightarrow y_2(x) = y_1(x) \int_{x_0}^{x} \frac{e^{-\int_{x_0}^{x''} P(x')dx'}}{y_1^2(x'')} dx''$$

- Can <u>always</u> generate a second solution.