Constraints

Examples:

1. rigid bodies
2. beads in a wire
3. gas molecules in a container
4. a child progressing down a slide

Will classify constraints into two types: holonomic and non-holonomic

Holonomic:

$$f(\bar{r}_1, \bar{r}_2, \bar{r}_3, \ldots, t) = 0$$

e.g. rigid body

$$(\bar{r}_i - \bar{r}_j)^2 - c_{ij}^2 = 0$$

e.g. bodies constrained to surfaces or curves

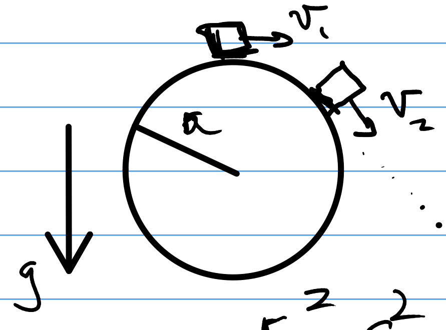
Non-holonomic:

Those that cannot be put in the form above:

1. inequalities
2. contain non-integrable differentials
3. depend on higher order derivatives

$$x^2 + y^2 = l^2$$

holonomic



$$r^2 - a^2 \geq 0$$

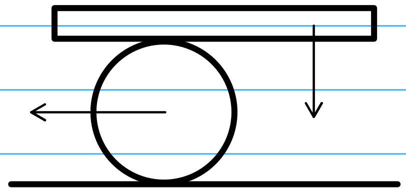
nonholonomic

Constraints are also classified by their time dependence: rheonomous (time-dependent) or scleronomous (time-independent)

Surface or curves that do not move, but restrict motions - scleronomous

e.g. mass down an incline, pendulum, skateboard in a parabolic path.

Bodies on moving or rotating surfaces/curves are constrained in rheonomous constraints.



slab is still in a scleronomous constraint

Pb1: Constraints make some of the coordinates depend on each other.

Soln: Introduce "generalized coordinates".

Given 3N degrees of freedom and k holonomic constraints, then there will be 3N-k independent (generalized) coordinates.

Non-holonomic constraints cannot be used to eliminate dependent variables.

Pb2: The forces of constraint are unknown.

Soln: The problem will be formulated such that the constraint forces do not come into the formulas (disappear).

Lagrange's Equation

Excluding friction, contact forces (those that enforce a constraint) always act perpendicular to the motion.

i.e. constraints do not work! We will use this principle to derive Lagrange's equations.

I. Lets first see how far we get with a problem in equillibrium, i.e. total forces on each particle is zero

$$\bar{F}_i = 0$$

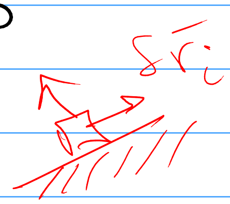
Consider the virtual work of this force in the direction of $\delta\bar{r}_i$

trivially $$\sum_i \bar{F}_i \cdot \delta\bar{r}_i = 0$$

$$\bar{F}_i = \bar{F}_i^{(a)} + \bar{f}_i$$

$$\sum_i \bar{F}_i^{(a)} \cdot \delta\bar{r}_i + \underbrace{\sum_i \bar{f}_i \cdot \delta\bar{r}_i}_{=0} = 0$$

$\perp$



$$\sum_i \bar{F}_i^{(a)} \cdot \delta \bar{r}_i = 0$$

principle of virtual work

gave us an equation that does not depend on the constraint forces f.
However, we cannot obtain equations of motion yet, b/c delta ri
are not independent.

II. Lets consider now the more useful case of a system in motion, and derive the equations of motion.

$$\bar{F}_i = \dot{\bar{P}}_i$$

$$\sum_i \underbrace{(\bar{F}_i - \dot{\bar{P}}_i)}_{=0} \cdot \delta \bar{r}_i = 0$$

$$\sum_i (\bar{F}_i^{(a)} - \dot{\bar{P}}_i) \cdot \delta \bar{r}_i + \overbrace{\sum_i \bar{f}_i \cdot \delta \bar{r}_i}^{=0} = 0$$

$$\boxed{\sum_i (\underbrace{\bar{F}_i^{(a)}}_{(1)} - \underbrace{\dot{\bar{P}}_i}_{(2)}) \cdot \delta \bar{r}_i = 0}$$

D'Alembert's principle

$$\bar{F}_i^{(a)} \longrightarrow \bar{F}_i$$

To obtain eqns of motion, we will now express the ri in terms of generalized coordinates (thar are independent) and then equate the coefficients to zero.

$$\bar{r}_i = \bar{r}_i(q_1, q_2, \dots q_n, t)$$

n degrees of freedom

to change coordinates, we will use the chain rule

velocity $\bar{v}_i = \frac{d}{dt}\bar{r}_i = \sum_k \frac{\partial \bar{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \bar{r}_i}{\partial t}$

$$\frac{\partial \bar{v}_i}{\partial \dot{q}_j} = \frac{\partial \bar{r}_i}{\partial q_j}$$

$$\delta \bar{r}_i = \sum_j \frac{\partial \bar{r}_i}{\partial q_j}\delta q_j$$

note: no time dependence

(1) $\sum_i \bar{F}_i \cdot \delta \bar{r}_i = \sum_{ij} \bar{F}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}\delta q_j = \sum_j Q_j \delta q_j$

$$\boxed{Q_j = \sum_i \bar{F}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}}$$

generalized force

(2) $\sum_i \dot{\bar{p}}_i \cdot \delta \bar{r}_i = \sum_i m_i \ddot{\bar{r}}_i \cdot \delta r_i$

$$= \sum_{ij} m_i \ddot{\bar{r}}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}\delta q_j$$

mini-goal: put (2) in terms of kinetic energy

$$\sum_i m_i \ddot{\bar{r}}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left(m_i \dot{\bar{r}}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j}\right) - m_i \dot{\bar{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right)\right]$$

$$\frac{\partial \bar{v}_i}{\partial \dot{q}_j} = \frac{\partial \bar{r}_i}{\partial q_j} \qquad \frac{d}{dt}\left(\frac{\partial \bar{r}_i}{\partial q_j}\right) = \frac{\partial \bar{v}_i}{\partial q_j}$$

$$\sum_i m_i \ddot{\bar{r}}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left(m_i \bar{v}_i \cdot \frac{\partial \bar{v}_i}{\partial \dot{q}_j}\right) - m_i \bar{v}_i \cdot \frac{\partial \bar{v}_i}{\partial q_j} \right]$$

$$= \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m_i v_i^2\right)\right] - \frac{\partial}{\partial q_j}\sum_i \frac{1}{2} m_i v_i^2$$

$$T = \sum_i \frac{1}{2} m_i v_i^2$$

(2)

$$\sum_i \dot{\bar{p}}_i \cdot \delta \bar{r}_i = \sum_j \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right] \delta q_j$$

$$\sum_j \left\{ \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right] - Q_j \right\} \delta q_j = 0$$

b/c all delta qj are independent of each other, then the argument has to be separately zero.

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j}$$

there are n of these equations.

IF $\bar{F}_i = -\bar{\nabla}_i V$

$$Q_j = -\frac{\partial V}{\partial q_j}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial (T-V)}{\partial q_j} = 0$$

and IF

$V \neq V(\dot{q}_j)$

$$\frac{d}{dt}\frac{\partial (T-V)}{\partial \dot{q}_j} - \frac{\partial (T-V)}{\partial q_j} = 0$$

$L \equiv T - V$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}$$

Lagrange's equation

after-class questions

$$\bar{r}_i = \bar{r}_i(q_i, t)$$

$$\bar{v}_i = \frac{d}{dt}\bar{r}_i = \sum_k \frac{\partial \bar{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \bar{r}_i}{\partial t}$$

$$\frac{\partial \bar{v}_i}{\partial \dot{q}_j} =$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial t}$$

$$\checkmark \; \frac{df}{dt} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial z}\dot{z} + \frac{\partial f}{\partial t}$$

$$\left.\frac{d}{dx}\sin 2x = 2\cos 2x\right\} = \bar{\nabla} f \cdot \bar{v}$$

$$\sum_k \frac{\partial \bar{r}_i}{\partial q_k}\dot{q}_k =$$