$$\eta_i = C_k a_{ik} e^{-i\omega_k t}$$

$$A^T V A = \lambda$$

$$A^T T A = 1$$

$$a_{ik} \longrightarrow A$$

$$\eta_i = a_{ij} \xi_j$$

"normal coordinates"

principal axis transformation

$$\eta = A \xi \longrightarrow \eta^T = \xi^T A^T$$

$$V = \frac{1}{2} \eta^T V \eta$$

$[V_{ij}]$

$$\lambda_k = \omega_k^2$$

potential energy

$$V = \frac{1}{2} \xi^T \underbrace{(A^T V A)}_{\lambda} \xi = \frac{1}{2} \omega_k^2 \xi_k^2$$

$$T = \frac{1}{2} \dot{\eta}^T T \dot{\eta} = \frac{1}{2} \dot{\xi}^T (A^T T A) \dot{\xi} = \frac{1}{2} \dot{\xi}_k \dot{\xi}_k$$

$[T_{ij}]$

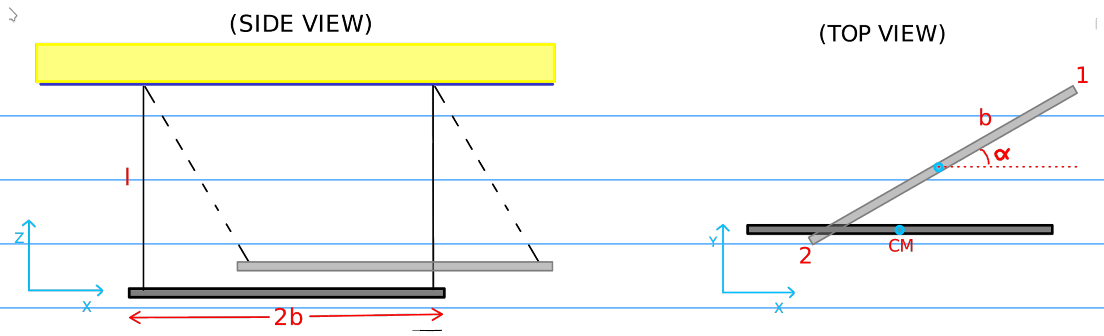
$$L = \frac{1}{2} \dot{\xi}_k \dot{\xi}_k - \frac{1}{2} \omega_k^2 \xi_k^2$$

$$\ddot{\xi}_k + \omega_k^2 \xi_k = 0$$

[no sum]

$$\boxed{\xi_k = C_k e^{-i\omega_k t}}$$

[no sum]



$6\ \text{dof} - 2 - 1 = 3\ \text{dof}$

coordinates are defined as x, y_1, y_2

$$cm = (x, y, z) = \left(x, \tfrac{1}{2}(y_1 + y_2), \tfrac{1}{2}(z_1 + z_2)\right)$$

$$\text{Tip } 1 = (b + x, y_1, z_1)$$

$$\text{tip } 2 = (x - b, y_2, z_2)$$

Potential Energy $U = mgz = mg\,\frac{1}{2}(z_1 + z_2)$

$$l = \sqrt{x^2 + y_1^2 + (l - z_1)^2}$$

$$z_1 = l - \sqrt{l^2 - x^2 - y_1^2} = l - l\sqrt{1 - \frac{x^2 + y_1^2}{l^2}}$$

$x, y_1, y_2 \ll l$

$$\simeq l\left[1 - \left(1 - \frac{x^2 + y_1^2}{2l^2}\right)\right]$$

$$\boxed{z_1 = \frac{1}{2l}\left(x^2 + y_1^2\right)}$$

$$\boxed{z_2 \simeq \frac{1}{2l}\left[x^2 + y_2^2\right]}$$

$$U = \frac{mg}{4l}\left[2x^2 + y_1^2 + y_2^2\right]$$

$$\frac{\partial z_1}{\partial x \partial y_2} = 0$$

$$U = \frac{1}{2}\left(\frac{\partial^2 U}{\partial y_i \partial y_j}\right) \eta_i \eta_j$$

$$[V_{ij}] = \frac{mg}{2\ell}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

order

$$\begin{pmatrix} x \\ y_1 \\ y_2 \end{pmatrix}$$

Kinetic Energy

$$T = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I \dot{\alpha}^2$$

$$\bar{r}_{cm} = x\hat{x} + y\hat{y} + z\hat{z} = x\hat{x} + \frac{1}{2}(y_1+y_2)\hat{y} + \frac{1}{2}(z_1+z_2)\hat{z}$$

$$z_1 + z_2 = \frac{1}{2\ell}(2x^2 + y_1^2 + y_2^2)$$

$$\bar{v}_{cm} = \frac{d\bar{r}_{cm}}{dt} = \dot{x}\hat{x} + \frac{1}{2}(\dot{y}_1 + \dot{y}_2)\hat{y} + \frac{1}{4\ell}[4x\dot{x} + 2y_1\dot{y}_1 + 2y_2\dot{y}_2]\hat{z}$$

$$v_{cm}^2 = \dot{x}^2 + \frac{1}{4}(\dot{y}_1^2 + \dot{y}_2^2 + 2\dot{y}_1\dot{y}_2) + O(x^4)$$

$$\tan\alpha = \frac{y_1 - y_2}{2b} \simeq \alpha \Rightarrow \dot{\alpha} = \frac{1}{2b}(\dot{y}_1 - \dot{y}_2)$$

$$\dot{\alpha}^2 = \frac{1}{4b^2}(\dot{y}_1^2 + \dot{y}_2^2 - 2\dot{y}_1\dot{y}_2)$$

$$I = \frac{1}{12} m (2b)^2 = \frac{1}{3} m b^2$$

$$T = \frac{1}{2} m \left[\dot{x}^2 + \frac{1}{4}(\dot{y}_1^2 + \dot{y}_2^2 + 2\dot{y}_1\dot{y}_2)\right] + \frac{1}{6} m b^2 \cdot \frac{1}{4b^2}(\dot{y}_1^2 + \dot{y}_2^2 - 2\dot{y}_1\dot{y}_2)$$

$$T = \frac{1}{2} m \left[\dot{x}^2 + \frac{1}{3}\dot{y}_1^2 + \frac{1}{3}\dot{y}_2^2 + \frac{1}{3}\dot{y}_1 \dot{y}_2 \right]$$

$$[T_{ij}] = m \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 1/6 \\ 0 & 1/6 & 1/3 \end{pmatrix}$$

$\lambda = \omega^2$

$\beta = \frac{mg}{2\ell}$

$$\begin{vmatrix} 2\beta - \lambda m & 0 & 0 \\ 0 & \beta - \frac{\lambda m}{3} & -\frac{\lambda m}{6} \\ 0 & -\frac{\lambda m}{6} & \beta - \frac{\lambda m}{3} \end{vmatrix} = 0$$

$$\boxed{\lambda_1 = \frac{2\beta}{m} = \frac{g}{\ell} \equiv \omega_0^2}$$

$$\left(\beta - \frac{\lambda m}{3}\right)^2 - \frac{\lambda^2 m^2}{36} = 0$$

$$\lambda^2 \frac{m^2}{12} - \frac{2}{3} m \beta \lambda + \beta^2 = 0$$

$$\lambda = \frac{12\beta}{m}\left[\frac{1}{2}, \frac{1}{6}\right] = 6\omega_0^2 \left[\frac{1}{2}, \frac{1}{6}\right]$$

$$\boxed{\lambda_{2,3} = \omega_0^2 \, , \, 3\omega_0^2}$$

$\lambda_{1,2} = \omega_0^2$

$$(V - \lambda T)\, a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{m\omega_0^2}{6} & -\frac{m\omega_0^2}{6} \\ 0 & -\frac{m\omega_0^2}{6} & \frac{m\omega_0^2}{6} \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}$$

$$\boxed{a_{21} = a_{31}}$$

$a_k^T T a_k = 1$

$$1 = m \begin{pmatrix} a_{11} & a_{21} & a_{21} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 1/6 \\ 0 & 1/6 & 1/3 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \\ a_{21} \end{pmatrix}$$

$$\frac{1}{m} = a_{11}^2 + a_{21}^2 \Rightarrow a_{11} = \sqrt{\frac{1}{m} - a_{12}^2}$$

$$a_{21} = \frac{a}{\sqrt{m}} \longrightarrow a_{11} = \frac{1}{\sqrt{m}} \sqrt{1 - a^2}$$

$$\boxed{a_{1,2} = \begin{pmatrix} \sqrt{1-a^2} \\ a \\ a \end{pmatrix} \frac{1}{\sqrt{m}}}$$

$\lambda_3 = 3\omega_0^2$

$$\begin{pmatrix} -2\omega_0^2 m & 0 & 0 \\ 0 & \frac{m}{2}\omega_0^2 - \omega_0^2 m & -\omega_0^2 m/2 \\ 0 & -\frac{\omega_0^2 m}{2} & \frac{m}{2}\omega_0^2 - \omega_0^2 m \end{pmatrix} \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = 0$$

$$\boxed{a_{23} = - a_{33}}$$

$$a_k^T T a_k = 1$$

$$1 = m \begin{pmatrix} a_{13} & a_{23} & -a_{23} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 1/6 \\ 0 & 1/6 & 1/3 \end{pmatrix} \begin{pmatrix} a_{13} \\ a_{23} \\ -a_{23} \end{pmatrix}$$

$$a_{13} = \sqrt{\frac{1}{m} - \frac{a_{23}^2}{3}} = \frac{1}{\sqrt{m}} \sqrt{1 - c^2/3}$$

$$a_{23} = \frac{c}{\sqrt{m}}$$

$$\boxed{a_3 = \frac{1}{\sqrt{m}} \begin{pmatrix} \sqrt{1 - c^2/3} \\ c \\ -c \end{pmatrix}}$$

x

y_1

y_2

$$a_1 = \frac{1}{\sqrt{m}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad a_2 = \frac{1}{\sqrt{m}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad a_3 = \sqrt{\frac{3}{m}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

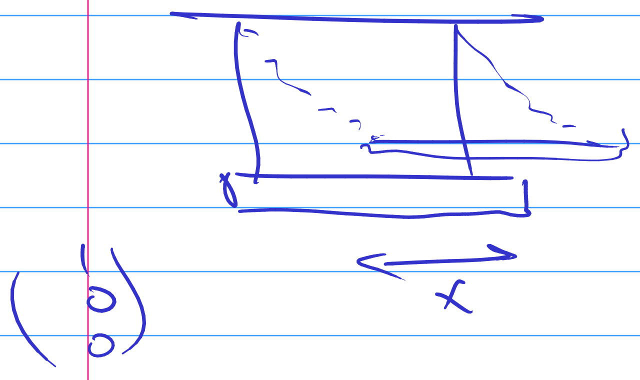
$a = 0$ $\qquad$ $a = 1$ $\qquad$ $c = \sqrt{3}$

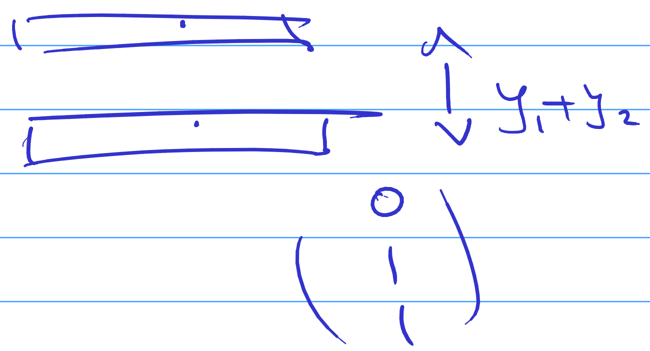
they also satisfy normalization condition

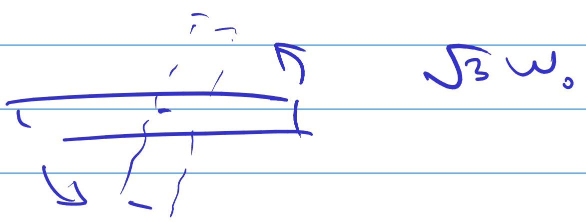
$$a_k^T T a_l = \delta_{kl}$$

$$\eta_i = C_k a_{ik} e^{-i\omega_k t}$$

$$\boxed{\begin{pmatrix} x \\ y_1 \\ y_2 \end{pmatrix} = \frac{c_1}{\sqrt{m}}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-i\omega_0 t} + \frac{c_2}{\sqrt{m}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{-i\omega_0 t} + c_3\sqrt{\frac{3}{m}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-i\sqrt{3}\omega_0 t}}$$







Find normal coordinates

$$A = \frac{1}{\sqrt{m}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \sqrt{3} \\ 0 & 1 & -\sqrt{3} \end{pmatrix} \quad A^{-1} = \sqrt{m}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & \frac{1}{2\sqrt{3}} & \frac{-1}{2\sqrt{3}} \end{pmatrix}$$

$$\bar{\zeta} = A^{-1}\bar{\eta} = \sqrt{m}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2\sqrt{3} & \frac{-1}{2\sqrt{3}} \end{pmatrix}\begin{pmatrix} x \\ y_1 \\ y_2 \end{pmatrix}$$

$$\zeta = \sqrt{m}\begin{pmatrix} x \\ (y_1+y_2)/2 \\ (y_1-y_2)/2 \end{pmatrix}$$

$$\begin{cases} \zeta_1 = c_1 e^{-i\omega_0 t} & \text{translation in x} \\ \zeta_2 = c_2 e^{-i\omega_0 t} & \text{translation in y} \\ \zeta_3 = c_3 e^{-i\sqrt{3}\omega_0 t} & \text{rotation} \end{cases}$$

---

x, y, α

$$u = \frac{1}{2}mg(z_1+z_2) = \frac{1}{2}mg\left[\frac{x^2+y_1^2}{2\ell} + \frac{x^2+y_1^2}{2\ell}\right]$$

$$= \frac{mg}{2\ell}\left[x^2 + \frac{1}{4}\left[(y_1+y_2)^2 + (y_1-y_2)^2\right]\right]$$

$y = \frac{1}{2}(y_1+y_2)$

$\alpha = \frac{1}{2b}(y_1-y_2)$

$$u = \frac{mg}{2\ell}\left(x^2+y^2+b^2\alpha^2\right)$$

$$T = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\frac{1}{3}b^2\dot{\alpha}^2\right)$$

$$[V_{ij}] = \frac{mg}{\ell}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b^2 \end{pmatrix}$$

$$[T_{ij}] = m\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b^2/3 \end{pmatrix}$$

$$|V - \lambda T| = 0 = \begin{vmatrix} \omega_0^2 - \lambda & 0 & 0 \\ 0 & \omega_0^2 - \lambda & 0 \\ 0 & 0 & b^2\omega_0^2 - \lambda b^2/3 \end{vmatrix}$$

$$\lambda_i = \omega_0^2,\ 3\omega_0^2$$

$$\eta_i = \text{Re}\, C_k a_{ik} e^{-i\omega_k t}$$

involves all of the Ck

$$\eta_i(0) = \text{Re}\, C_k a_{ik}$$

$$\eta(0) = A\,\text{Re}\, C \longrightarrow A^T T \eta(0) = A^T T A\, \text{Re}\, C$$

$$= \text{Re}\, C$$

$$\boxed{\text{Re}\, C_k = a_{jk} T_{j\ell}\, \eta_\ell(0)}$$

involves only one Ck

$$\dot{\eta}_i = \mathrm{Re}\left[-i\,\omega_k C_k a_{ik} e^{-i\omega_k t} \right]$$

$$\dot{\eta}_i(0) = \mathrm{Im}\left[C_k \omega_k a_{ik} \right]$$

$$\dot{\eta}(0) = A\,\mathrm{Im}\left[\omega C \right] \qquad \omega = \begin{pmatrix} \omega_1 & & 0 \\ & \omega_2 & \\ 0 & & \omega_3 \\ & & & \ddots \end{pmatrix}$$

$$A^T T \dot{\eta}(0) = \mathrm{Im}\left[\omega C \right]$$

if ω real

$$\boxed{\mathrm{Im}\, C_k = \frac{1}{\omega_k} a_{jk} T_{jl} \dot{\eta}_l(0)}$$

[no sum in 'k']