Partial Differential Equations – Examples

Wave equation, 1D, $u(x,t)$: $$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

c = wave speed
= T/ρ for string, e.g.

Wave equation, 3D, $u(\boldsymbol{x},t)$: $$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Laplace equation, 3D , $u(\boldsymbol{x})$: $$\nabla^2 u = 0$$

Diffusion equation: $$\nabla^2 u = \frac{1}{\kappa}\frac{\partial u}{\partial t}$$

Schrödinger equation: $$-\frac{\hbar^2}{2m}\nabla^2\psi + \mathrm{V}\psi = i\hbar\frac{\partial\psi}{\partial t}$$

Partial Differential Equations – Examples

Wave equation, 1D, $u(x,t)$: $$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = f(x,t)$$

Wave equation, 3D, $u(\boldsymbol{x},t)$: $$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = f(\boldsymbol{x},t)$$

Poisson's equation, 3D: $$\nabla^2 u = 4\pi G\rho \quad (\text{or } -\rho/\varepsilon_0)$$

Diffusion equation: $$\nabla^2 u - \frac{1}{\kappa}\frac{\partial u}{\partial t} = g(\boldsymbol{x},t)$$

Schrödinger equation: $$-\frac{\hbar^2}{2m}\nabla^2\psi + \mathrm{V}\psi = i\hbar\frac{\partial\psi}{\partial t}$$

Nonlinear example: Kortweg-deVries (KdV) Equation

$$\frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0$$

Seek "traveling" solution: $u(x,t) = f(\xi)$, where $\xi = x - ct$

Substitute: $$\frac{d^3 f}{d\xi^3} + (6f - c)\frac{df}{d\xi} = 0$$

Integrate once and reorder: $$\frac{d^2 f}{d\xi^2} = cf - 3f^2$$

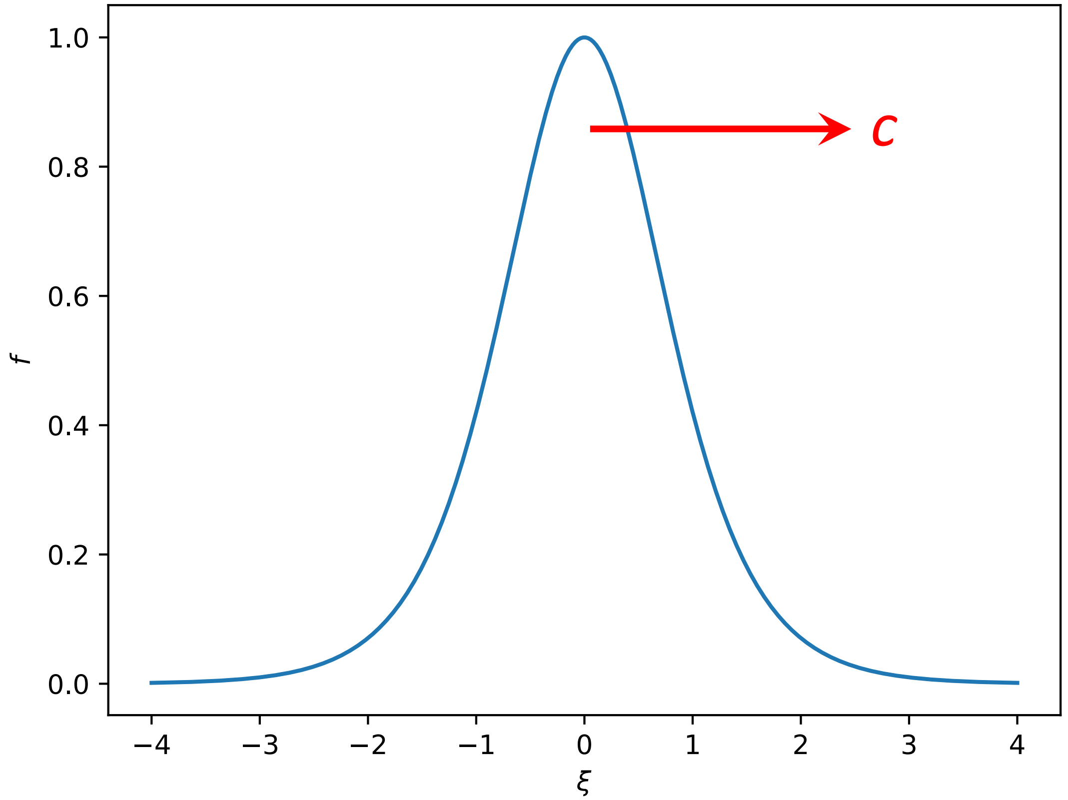
$\times \frac{df}{d\xi}$ and integrate: $$\left(\frac{df}{d\xi}\right)^2 = cf^2 - 2f^3$$

so $$\frac{df}{d\xi} = f(c - 2f)^{1/2}$$

Kortweg-deVries (KdV) Solution

$$f(\xi) = \frac{c/2}{\cosh^2(\sqrt{c}\xi/2)}$$

- Non-linear, non-dispersive traveling wave
- Wave amplitude is proportional to the wave velocity c
- “Soliton” solution
- Observed in fluid flows, rogue waves, etc.



Hydrodynamics

Continuity equation, 1D, $\rho(x,t)$, $u(x,t)$:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0$$

Euler equation, 1D (Newton II):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

Nonlinear terms generally require numerical solution

shock waves
turbulence
etc.

General Relativity

$$G = 8\,\pi\,T$$

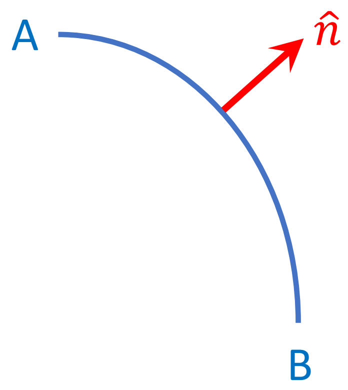
$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

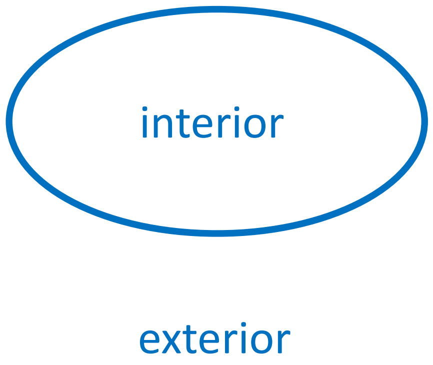
$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = \frac{\partial \Gamma^{\alpha}_{\nu\mu}}{\partial x_\alpha} - \frac{\partial \Gamma^{\alpha}_{\alpha\mu}}{\partial x_\nu} + \Gamma^{\beta}_{\beta\alpha}\,\Gamma^{\alpha}_{\nu\mu} - \Gamma^{\alpha}_{\nu\beta}\,\Gamma^{\beta}_{\alpha\mu}$$

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x_\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x_\beta} + \frac{\partial g_{\beta\gamma}}{\partial x_\alpha}\right)$$

Boundary Conditions

- 2-D, for simplicity: x, y
- Open and closed boundaries
- Standard boundary conditions:
  - **Dirichlet**: u specified everywhere on the boundary
  - **Neumann**: $\frac{\partial u}{\partial n} = \hat{n} \cdot \nabla u$ specified everywhere on the boundary
  - **Cauchy**: both Dirichlet and Neumann

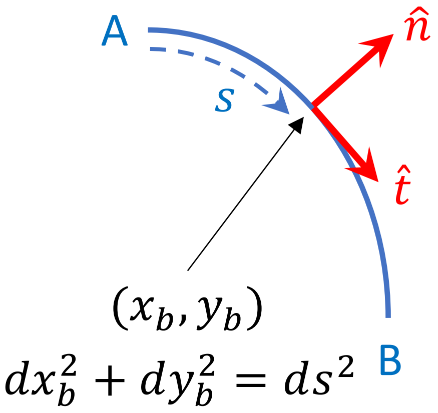


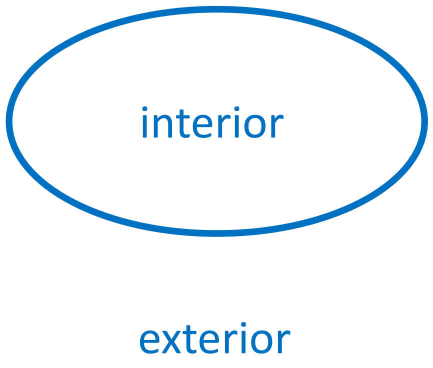


- Which are necessary?
- Which are sufficient?
- Which are too much?
- Is the solution unique?

Boundary Conditions

- Specify the boundary parametrically: $x = x_b(s),\ y = y_b(s)$
- Unit vectors are $\hat{t} = \left(\frac{dx_b}{ds}, \frac{dy_b}{ds}\right)$ and $\hat{n} = \left(-\frac{dy_b}{ds}, \frac{dx_b}{ds}\right)$
- Suppose we have Cauchy boundary conditions, so we are given the field on the boundary, $u = u_b(s)$, and its normal derivative, $\partial u/\partial n = N_b(s)$
- Can we use this information to construct the solution away from the boundary?
- If so, repeat to generate the solution.





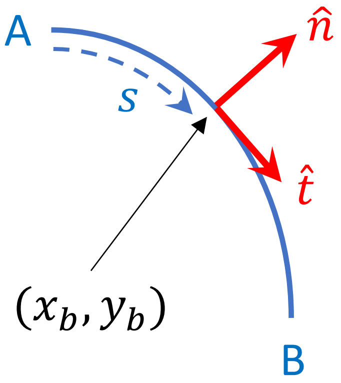
Reminder: 1-D Taylor Series

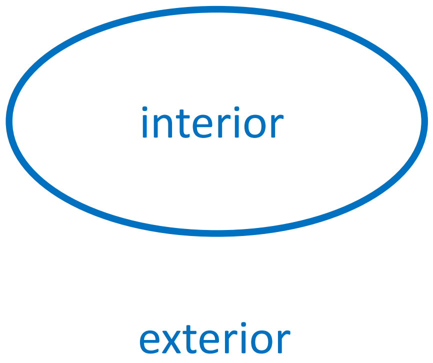
- Expand function
- Let $\delta x = x - x_0$, so

$$f(x) = f(x_0) + \delta x\, f'(x_0) + \frac{1}{2}\delta x^2 f''(x_0) + \ldots$$

Boundary Conditions

- Want to write down a Taylor expansion of the solution in the vicinity of the boundary point $[x_b(s), y_b(s)]$





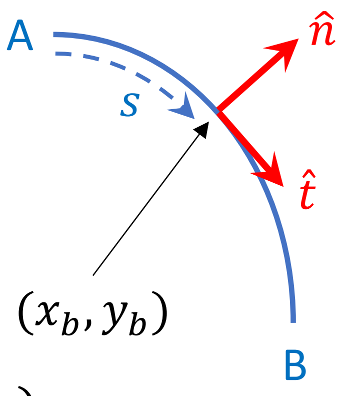
- Let $\delta x = x - x_b,\ \delta y = y - y_b$, so

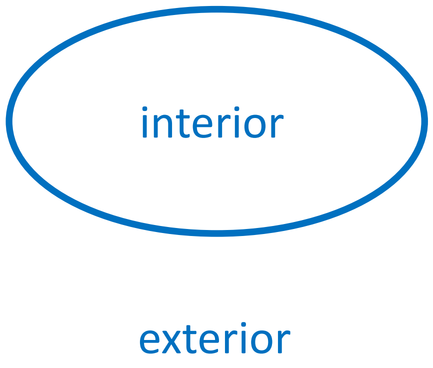
$$u(x,y) = u(x_b, y_b) + \delta x \left.\frac{\partial u}{\partial x}\right|_b + \delta y \left.\frac{\partial u}{\partial y}\right|_b + \frac{1}{2}\left(\delta x^2 \left.\frac{\partial^2 u}{\partial x^2}\right|_b + 2\delta x \delta y \left.\frac{\partial^2 u}{\partial x \partial y}\right|_b + \delta y^2 \left.\frac{\partial^2 u}{\partial y^2}\right|_b \right) + \ldots$$

- Now we can systematically write down the derivatives in this expansion.

Boundary Conditions

- Boundary conditions give us the first term: $u(x_b, y_b) = u_b(s)$





- They also give us the first derivatives of the field:

$$\hat{t} \cdot \nabla u = \frac{dx_b}{ds} \left.\frac{\partial u}{\partial x}\right|_b + \frac{dy_b}{ds} \left.\frac{\partial u}{\partial y}\right|_b = \frac{du_b}{ds} \text{ (known)}$$

$$\hat{n} \cdot \nabla u = -\frac{dy_b}{ds} \left.\frac{\partial u}{\partial x}\right|_b + \frac{dx_b}{ds} \left.\frac{\partial u}{\partial y}\right|_b = N_b(s)$$

- Linear equations; solve for the derivatives:

$$\left.\frac{\partial u}{\partial x}\right|_b = -N_b(s) \frac{dy_b}{ds} + \frac{du_b}{ds} \frac{dx_b}{ds}$$

$$\left.\frac{\partial u}{\partial y}\right|_b = N_b(s) \frac{dx_b}{ds} + \frac{du_b}{ds} \frac{dy_b}{ds}$$

- All terms on the RHS are known.
- Now we need the second derivatives.

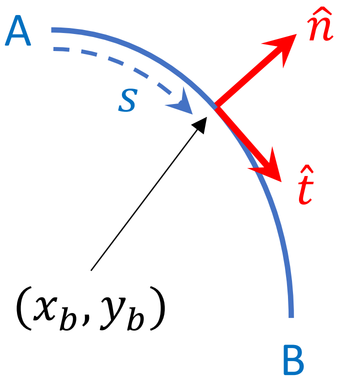
Boundary Conditions

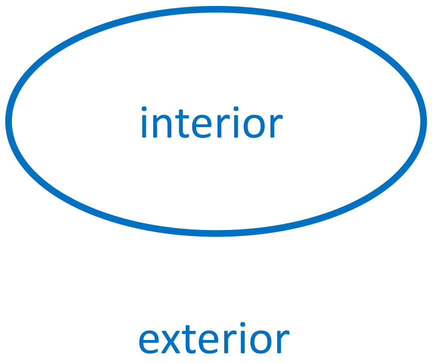
- For the second derivatives, differentiate the first derivative expressions:

known

$$\frac{d}{ds}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}\frac{dx_b}{ds} + \frac{\partial^2 u}{\partial x \partial y}\frac{dy_b}{ds}$$

$$\frac{d}{ds}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2}{\partial x \partial y}\frac{dx_b}{ds} + \frac{\partial^2}{\partial y^2}\frac{dy_b}{ds}$$





- Need a third equation — PDE provides it:

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + 2B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} = f\left(x, y, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

known

Boundary Conditions

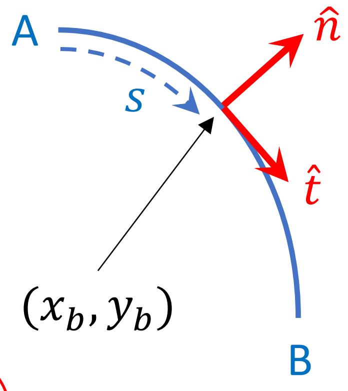
- Setting

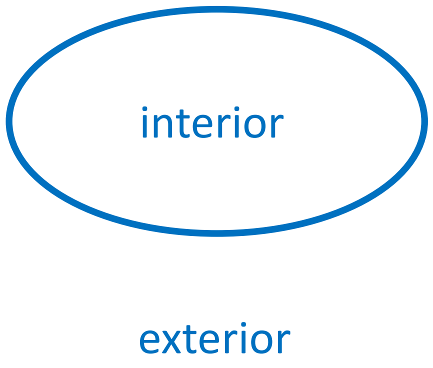
$$u_{xx} \equiv \frac{\partial^2 u}{\partial x^2},\ u_{yy} \equiv \frac{\partial^2 u}{\partial y^2},\ u_{xy} \equiv \frac{\partial^2 u}{\partial x \partial y},$$

- we have

$$\begin{aligned}
\frac{dx_b}{ds}u_{xx} + \frac{dy_b}{ds}u_{xy} \qquad\qquad &= \frac{d}{ds}\left(\frac{\partial u}{\partial x}\right)_b \\
\frac{dx_b}{ds}u_{xy} + \frac{dy_b}{ds}u_{yy} &= \frac{d}{ds}\left(\frac{\partial u}{\partial y}\right)_b \\
Au_{xx} + 2Bu_{xy} + Cu_{yy} &= f
\end{aligned}$$

- Linear third-order simultaneous equation for the second derivatives.





Boundary Conditions

- Equations have a solution <u>unless</u> the determinant of coefficients is zero:

$$\begin{vmatrix} \frac{dx_b}{ds} & \frac{dy_b}{ds} & 0 \\ 0 & \frac{dx_b}{ds} & \frac{dy_b}{ds} \\ A & 2B & C \end{vmatrix} = 0$$

$$\Longrightarrow \qquad A\left(\frac{dy_b}{ds}\right)^2 - 2B\,\frac{dx_b}{ds}\frac{dy_b}{ds} + C\left(\frac{dx_b}{ds}\right)^2 = 0$$

or

$$A\left(\frac{dy_b}{dx_b}\right)^2 - 2B\,\frac{dy_b}{dx_b} + C = 0$$

- Quadratic equation for the shape of the boundary $y_b(x_b)$ that permits/denies solutions (nonlinear ODE, in general):

$$\frac{dy_b}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

Boundary Conditions

- Characteristic curves for the PDE are defined by the ODE

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

- Can show: if we differentiate again, higher derivatives are also subject to the same characteristic equation.
- Cauchy BCs give a solution to the problem (for all higher derivatives) except where the boundary is tangent to a characteristic.
- Classification of solutions based on the discriminant:

$B^2 > AC \implies$ 2 real solutions: "hyperbolic equation"

$B^2 < AC \implies$ 0 real solutions: "elliptic equation"

$B^2 = AC \implies$ 1 real solution: "parabolic equation"

Boundary Conditions

- Bottom line: Cauchy boundary conditions, specified along a curve that doesn't coincide with a characteristic, allow us to solve the PDE in some neighborhood of the boundary

  $\Longrightarrow$ <u>existence</u> of a solution

- How do our "standard" equations stack up?

Wave Equation

- Standard form: $u_{xx} - \frac{1}{c^2} u_{tt} = 0$

  $\Longrightarrow A = 1,\ B = 0,\ C = -\frac{1}{c^2},\ B^2 > AC,$ so hyperbolic

- Characteristic equation is

$$\left(\frac{dt}{ds}\right)^2 - \frac{1}{c^2}\left(\frac{dx}{ds}\right)^2 = 0$$

$$\Longrightarrow \left(\frac{dx}{dt}\right)^2 = c^2$$

$$\Longrightarrow \frac{dx}{dt} = \pm c$$

$$\Longrightarrow x - ct = \xi, \text{ constant}$$

$$x + ct = \eta, \text{ constant}$$

Characteristics are straight lines (rays)