## Euler's theorem on the motion of a rigid body

"The general displacement of a rigid body with one point fixed is a rotation about some axis"

GOAL: find an axis at theta and phi such that a rotation by psi will render the actual rotation of the rigid body

We will take the origin of the "body set" of axes (that attached to the body) coinciding with the fixed point - no translation, only rotation

Corollary, Chasles' Theorem:

"The most general displacement of a rigid body is a translation plus a rotation."

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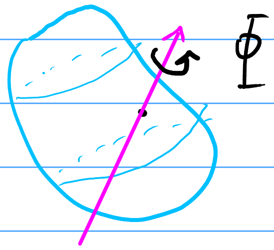


$$A' = \begin{pmatrix} \cos\Phi & \sin\Phi & 0 \\ -\sin\Phi & \cos\Phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\varphi$

$$\mathbf{A} = \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}$$

Trace of a transformation matrix is invariant under a similarity transformation.

$$\mathrm{Tr}\, A' = 1 + 2\cos\Phi = \mathrm{Tr}\, A$$

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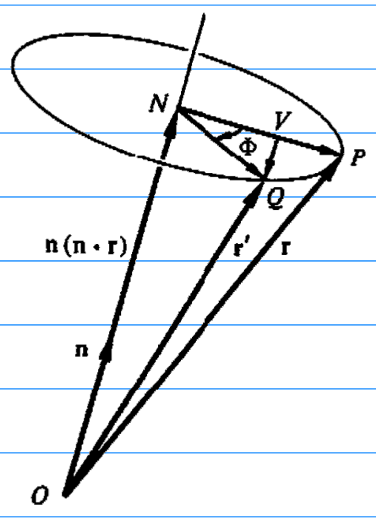
## Finite Rotations

GOAL: Find a vectorial transformation that takes r into an r' that is equivalent to a rotation around an axis.

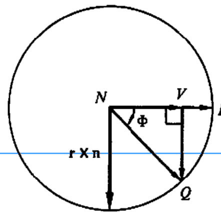



$$\bar{r}' = \overline{ON} + \overline{NV} + \overline{VQ}$$

$\overline{ON}$: $\hat{n}$ unit vector in the direction of ON

$$|\overline{ON}| = \hat{n}\cdot\bar{r} \Rightarrow \overline{ON} = \hat{n}(\hat{n}\cdot\bar{r})$$

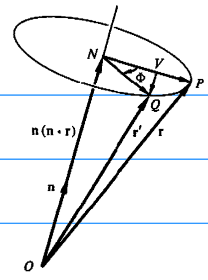
$\overline{NV}$ :




$$|\overline{NQ}|\cos\phi = |\overline{NV}|$$

$$|\overline{NQ}| = |\overline{NP}|$$

$$|\overline{NV}| = |\overline{NP}|\cos\phi$$




$$\Rightarrow \quad \overline{NV} = [\underbrace{\bar{r} - \hat{n}(\hat{n}\cdot\bar{r})}_{\overline{NP}}]\cos\phi$$

$\overline{VQ}$ : $|\overline{VQ}| = |\overline{NQ}|\sin\phi = |\bar{r}\times\hat{n}|\sin\phi$

$$\overline{VQ} = (\bar{r}\times\hat{n})\sin\phi$$

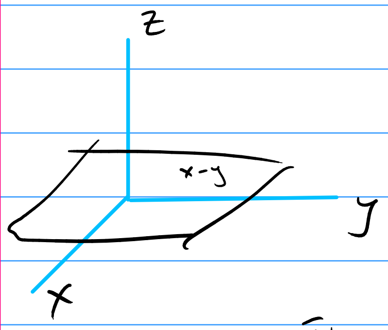
$$\bar{r}' = \hat{n}(\hat{n}\cdot\bar{r}) + [\bar{r} - \hat{n}(\hat{n}\cdot\bar{r})]\cos\phi + (\bar{r}\times\hat{n})\sin\phi$$

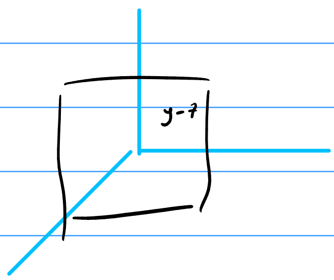
$$\boxed{\bar{r}' = \bar{r}\cos\phi + \hat{n}(\hat{n}\cdot\bar{r})(1-\cos\phi) + (\bar{r}\times\hat{n})\sin\phi}$$

This is the "rotation formula" - it is valid for any angle, thus it is a finite-rotation formula (clockwise).

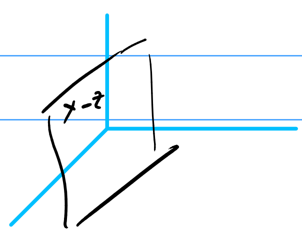
Trace $\quad 1+2\cos\phi = \mathrm{Tr}\,A$

$$\boxed{\cos\frac{\Phi}{2} = \cos\tfrac{1}{2}(\varphi+\psi)\cos\frac{\theta}{2}}$$




$x, z \quad \pi/2$




$z, x \quad \pi/2$

$$AB \neq BA$$

Infinitesimal rotations - or in search of the rotation vector through the fixed point

$$x_i' = x_i + \epsilon_{ij} x_j = (\delta_{ij} + \epsilon_{ij}) x_j$$

$$\Rightarrow \quad \boxed{x' = (1+\epsilon)\, x} \qquad 1 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

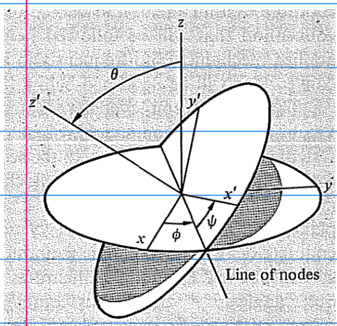
$$(1+\epsilon_1)(1+\epsilon_2) = 1 + \epsilon_1 + \epsilon_2 + \underbrace{\epsilon_1 \epsilon_2}_{0}$$

$$(1+\epsilon_2)(1+\epsilon_1) = 1 + \epsilon_2 + \epsilon_1 + \underbrace{\epsilon_2 \epsilon_1}_{0}$$

so the order of rotations is not important - they commute

Example, use Euler transformation matrix

$$A = \begin{pmatrix} 1 & (d\varphi + d\psi) & 0 \\ -(d\varphi + d\psi) & 1 & d\theta \\ 0 & -d\theta & 1 \end{pmatrix}$$




$$d\bar{\Omega} = \hat{i}\, d\theta + \hat{k}\,(d\varphi + d\psi)$$

Inverse operation

$$B = 1 + \epsilon \qquad B^{-1} = 1 - \epsilon$$

$$BB^{-1} = 1 - \epsilon^2 = 1$$

Also, because of orthogonality

$$B^{-1} = B^T$$

$$(1-\epsilon) = 1 + \epsilon^T \Rightarrow -\epsilon = \epsilon^T$$

antisymmetric with zero diagonal elements

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$a = e = i = 0$$
$$b = -d$$
$$c = -g \qquad h = -f$$

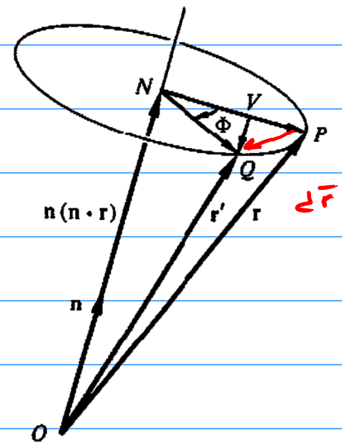
only three independent parameters

$$\epsilon = \begin{pmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{pmatrix}$$

$$x' = (1+\epsilon)x \Rightarrow \bar{r}' - \bar{r} = \epsilon\bar{r} = d\bar{r}$$

$$\epsilon\bar{r} = \begin{pmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ & \cdots\cdots & \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \begin{pmatrix} x_2 d\Omega_3 - x_3 d\Omega_2 \\ -x_1 d\Omega_3 + x_3 d\Omega_1 \\ x_1 d\Omega_2 - x_2 d\Omega_1 \end{pmatrix} = \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$$




$$dx_1 = x_2 d\Omega_3 - x_3 d\Omega_2$$
$$dx_2 = x_3 d\Omega_1 - x_1 d\Omega_3$$
$$dx_3 = x_1 d\Omega_2 - x_2 d\Omega_1$$

elements of a cross product

$$\boxed{d\bar{r} = \bar{r} \times d\bar{\Omega}}$$

$$\bar{r}' = \bar{r}\cos\phi + \hat{n}(\hat{n}\cdot\bar{r})(1-\cos\phi) + (\bar{r}\times\hat{n})\sin\phi$$

$\phi \rightarrow d\phi$

$$\bar{r}' = \bar{r} + (\bar{r}\times\hat{n})d\phi \Rightarrow d\bar{r} = (\bar{r}\times\hat{n})d\phi$$

$$\boxed{d\bar{\Omega} = \hat{n}\, d\phi}$$

from now on, we want to treat transformations of vectors in the counter-clockwise sense

$$\bar{r}' = r\cos\phi + \hat{n}(\hat{n}\cdot\bar{r})(1-\cos\phi) + (\hat{n}\times\bar{r})\sin\phi$$

$$d\bar{r} = d\bar{\Omega}\times\bar{r}$$

$$(dG)_s = (dG)_{body} + d\bar{\Omega}\times G$$

$$(d\bar{r})_s = (d\bar{r})_r + d\bar{\Omega}\times\bar{r}$$

$$\left(\frac{d}{dt}\right)_s = \left(\frac{d}{dt}\right)_r + \bar{\omega}\times$$