Recap 1: Boundary Conditions and Domains

- Hyperbolic, elliptic, and parabolic equations significantly different mathematical properties, and generally require different combinations of boundary conditions on geometrically different (open/closed) boundaries.
- Rule of thumb:

Type	Typical Variables	Boundary Conditions	Domain
Hyperbolic	Space+time	Cauchy/mixed	Open
Elliptic	Space	Dirichlet/Neumann	Closed
Parabolic	Space+time	Dirichlet/Neumann	Open

Recap 2: Separation of Variables

$$\frac{\nabla^2 \chi}{\chi} = \frac{1}{c^2}\frac{1}{T}\frac{d^2 T}{dt^2} = \text{constant}, \; -k^2$$

separation constant form is conventional

function of $\boldsymbol{x}$ only function of t only

- Effectively splits the PDE into an ODE and a lower-dimensional PDE.
- Time dependence

  $T'' + k^2 c^2 T = 0$, and define $\omega = kc$
- Solutions $T = e^{\pm i\omega t}$
- Spatial dependence

  $\nabla^2 \chi + k^2 \chi = 0$ Helmholtz equation
- Wave solution: expect $k^2 > 0$

Recap 3: All Roads Lead to Helmholtz

- Wave equation, $e^{\pm i\omega t}$ time dependence

  $\Longrightarrow \quad \nabla^2 u + k^2 u = 0, \quad$ where $k^2 = \omega^2 c^2 > 0$

- Laplace equation

  $\Longrightarrow \quad \nabla^2 \phi = 0, \quad$ so $k^2 = 0$

- Diffusion equation, $e^{-l^2 \kappa t}$ time dependence

  $\Longrightarrow \quad \nabla^2 u + k^2 u = 0, \quad$ where $k^2 = l^2 > 0$

- Schrödinger equation, particle in a box

$\Longrightarrow \quad \nabla^2 \psi + k^2 \psi = 0, \quad$ where $k^2 = \frac{2mE}{\hbar} > 0$

Separation of Variables

- Seek separable solutions of the Helmholtz equation.

$$\nabla^2\chi + k^2\chi = 0$$

where we anticipate $k^2 > 0$, but should not always assume so.

- Start in <u>Cartesian</u> coordinates in 3D: x, y, z.
- Write

$$\chi(x,y,z) = X(x)Y(y)Z(z)$$

then

$$\nabla^2\chi = X''YZ + XY''Z + XYZ''$$

$$\Longrightarrow \underbrace{\frac{X''}{X}}_{\text{function of } x} + \underbrace{\frac{Y''}{Y}}_{\text{function of } y} + \underbrace{\frac{Z''}{Z}}_{\text{function of } z} + k^2 = 0$$

All must be <u>constant</u>!

Separation of Variables

- Set $$\frac{X''}{X} = -\lambda^2$$ $$\frac{Y''}{Y} = -\mu^2$$ $$\frac{Z''}{Z} = -\nu^2$$

  Again, conventional — no assumptions about signs

  with the constraint
  $$\lambda^2 + \mu^2 + \nu^2 = k^2$$
- Individual solutions are
  $$X_\lambda(x) = e^{\pm i\lambda x}, \quad Y_\mu(y) = e^{\pm i\mu y}, \quad Z_\nu(z) = e^{\pm i\nu z}$$

Separation of Variables

- General solution is a sum of all possible terms solving the equation subject to the constraints:

$$\chi(x,y,z) = \sum_{\lambda^2+\mu^2+\nu^2=k^2} X_\lambda(x) Y_\mu(y) Z_\nu(z)$$

where

$$\begin{aligned} X_\lambda(x) &= A_\lambda e^{i\lambda x} + B_\lambda e^{-i\lambda x} && \text{or} && A_\lambda \sin\lambda x + B_\lambda \cos\lambda x \\ Y_\mu(y) &= C_\mu e^{i\mu y} + D_\mu e^{-i\mu y} && \text{or} && C_\mu \sin\mu y + D_\mu \cos\mu y \\ Z_\nu(z) &= E_\nu e^{i\nu z} + F_\nu e^{-i\nu z} && \text{or} && E_\nu \sin\nu z + F_\nu \cos\nu z \end{aligned}$$

- Note: sum may be a sum over integers, or it may be an integral over real values, depending on circumstances TBD.
- λ, μ, ν values and coefficients are determined by matching the boundary and initial conditions.

Example (1D): Vibration of a String

- Underlying equation is the 1D wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \qquad\qquad c^2 = T/\sigma$$

with $u(0,t) = u(a,t) = 0.$

- Seek $e^{i\omega t}$ behavior (normal mode), gives Helmholtz equation with $k = \omega/c$

$$\frac{d^2\chi}{dt^2} + k^2\chi = 0$$

- Solution with $\chi(0) = 0$

$$\chi(x) = \sin kx$$

- BC $\chi(a) = 0 \implies \sin ka = 0 \implies ka = l\pi \implies k_l = \frac{l\pi}{a}$

Example (1D): Vibration of a String

- General solution is a sum of normal modes of the form

$$u_l(x) = \sin\frac{l\pi x}{a}$$

$$\Longrightarrow\ u(x,t) = \sum_l A_l \sin\frac{l\pi x}{a} e^{ick_l t}$$

- Complete the solution by looking at the initial conditions

$$u(x,0) = \sum_l A_l \sin\frac{l\pi x}{a}$$

- This is a 1D Fourier series to determine the coefficients A_l

Reminder: Fourier Series

- Going to see a lot of Fourier series in this context!
- Will see this material later as part of a much more general result, but review the basics here.
- See R&H Sec. 4.1, 4.2
- A Fourier series is an expansion of a function defined in some interval $0 \le x < L$ in terms of trigonometric functions or complex exponentials.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}\right)$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi i n x}{L}}$$

Recall: $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$

$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

Fourier Series

- Temporarily setting aside mathematical rigor, we can multiply the first form by $\cos\left(\frac{2\pi nx}{L}\right)$ or $\sin\left(\frac{2\pi nx}{L}\right)$ and integrate to find

$$a_n = \frac{2}{L}\int_0^L dx\ f(x)\cos\left(\frac{2\pi nx}{L}\right)$$

$$b_n = \frac{2}{L}\int_0^L dx\ f(x)\sin\left(\frac{2\pi nx}{L}\right)$$

Works because

$$\int_0^L dx\ \cos\left(\frac{2\pi mx}{L}\right)\cos\left(\frac{2\pi nx}{L}\right) = \frac{1}{2}L\delta_{mn}$$

$$\int_0^L dx\ \sin\left(\frac{2\pi mx}{L}\right)\sin\left(\frac{2\pi nx}{L}\right) = \frac{1}{2}L\delta_{mn}$$

$$\int_0^L dx\ \cos\left(\frac{2\pi mx}{L}\right)\sin\left(\frac{2\pi nx}{L}\right) = 0$$

- Similarly,

$$c_n = \frac{1}{L}\int_0^L dx\ f(x)\, e^{-\frac{2\pi inx}{L}}$$

$$\int_0^L dx\ e^{\frac{2\pi imx}{L}}\ e^{-\frac{2\pi inx}{L}} = L\delta_{mn}$$

- Sums converge to $f(x)$ for $0 \le x < \mathrm{L}$, periodic outside that range.

- Convergence: $\int_0^L dx\ \left|f(x) - \sum_{n=-N}^{N} c_n e^{\frac{2\pi inx}{L}}\right|^2 \longrightarrow 0$ as $N \longrightarrow \infty$

Example (1D): Vibration of a String

- Solution is

$$u(x,t) = \sum_l A_l \sin\frac{l\pi x}{a} e^{i\omega_l t}$$

where $k_l = \frac{l\pi}{a}$, $\omega_l = ck_l$

- Initial conditions

$$u(x,0) = \sum_l A_l \sin\frac{l\pi x}{a}$$

so

$$A_l = \frac{2}{L}\int_0^L dx \sin\frac{l\pi x}{a}\, u(x,0)$$

- BCs set the allowed l, ICs determine the coefficients—<u>always</u> enough information to do this in a well-posed problem.

Example (2D): Vibration of a Membrane

- Underlying equation is the 2D wave equation

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \qquad\qquad c^2 = T/\sigma$$

- Seek $e^{i\omega t}$ behavior (normal mode), gives Helmholtz equation with $k = \omega/c$

$$\nabla^2 u + k^2 u = 0$$

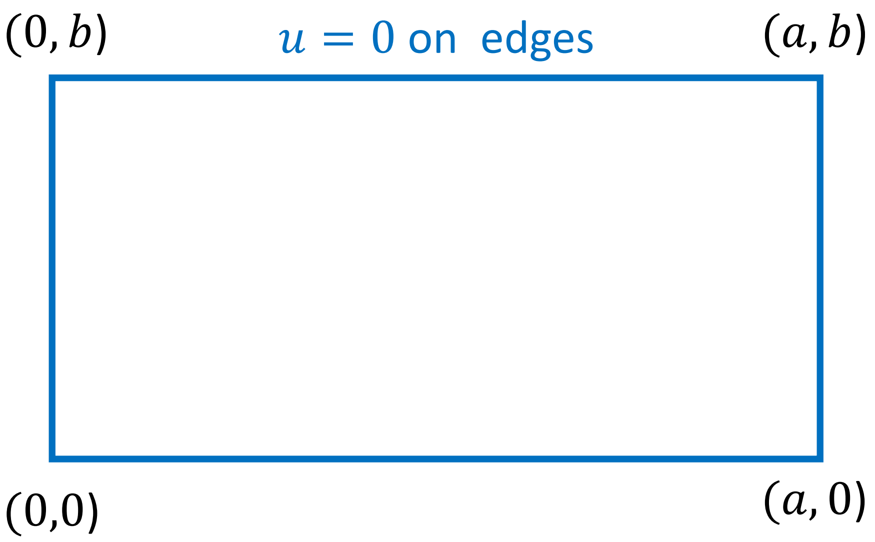
- Find

$$X'' + \lambda^2 X = 0$$

$$Y'' + \mu^2 Y = 0, \quad \text{where} \quad \lambda^2 + \mu^2 = k^2$$

- Solutions with $u(0, y) = u(x, 0) = 0$

$$X = \sin \lambda x, \qquad Y = \sin \mu y$$



Example (2D): Vibration of a Membrane

- Solutions satisfying BCs at $x = 0, y = 0$ are

$$X = \sin \lambda x, \qquad Y = \sin \mu y$$

- At $x = a$, must have $\sin \lambda a = 0$, so $\lambda a = l\pi, l$ integer.
- At $y = b$, must have $\sin \mu b = 0$, so $\mu b = m\pi, m$ integer.

BCs constrain the separation constants

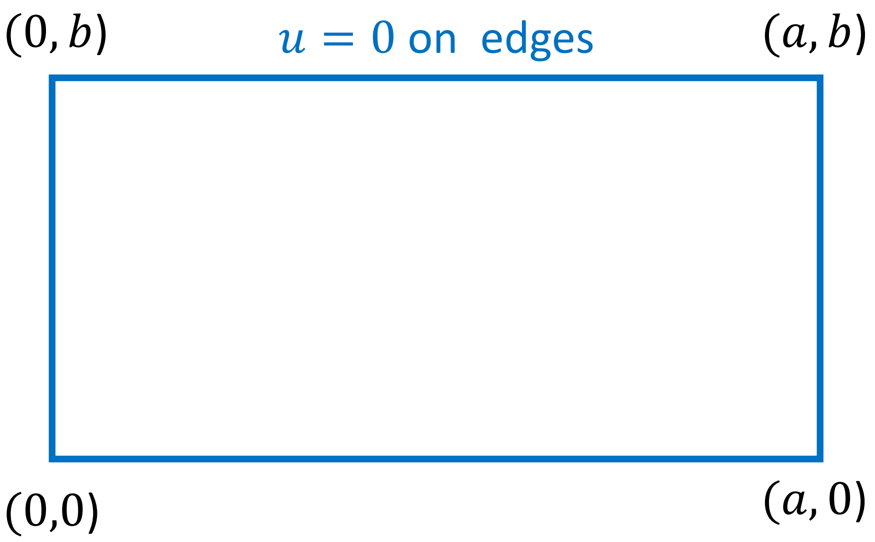
- Solution is a sum of normal modes of the form

$$u_{lm}(x, y) = \sin \frac{l\pi x}{a} \sin \frac{m\pi y}{b}$$

$$u(x, y, t) = \sum_{l,m} A_{lm} \sin \frac{l\pi x}{a} \sin \frac{m\pi y}{b} e^{i\omega_{lm} t}$$

where $k_{lm}^2 = \frac{l^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}, \;\; \omega_{lm}^2 = c^2 \, k_{lm}^2$

- 2D Fourier series for ICs, much as before



Example (2D): Vibration of a Membrane

- Solution is

$$u(x, y, t) = \sum_{l,m} A_{lm} \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b} e^{i\omega_{lm} t}$$

where $k_{lm}^2 = \frac{l^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}$, $\omega_{lm}^2 = c^2\, k_{lm}^2$

- Initial conditions

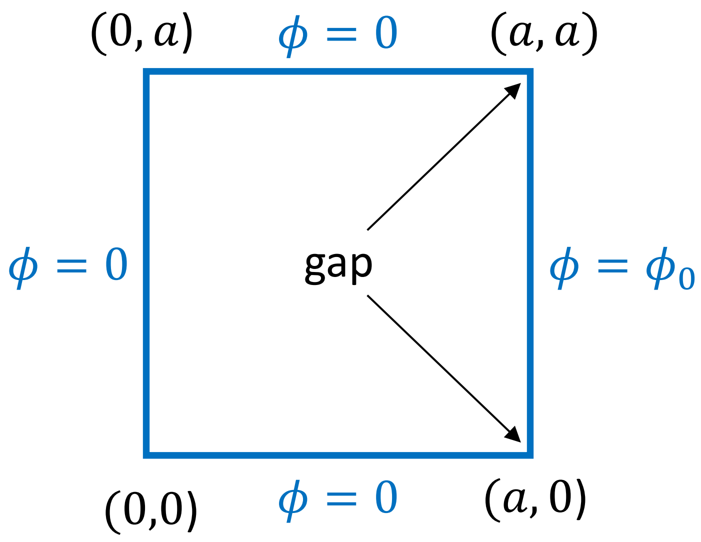
$$u(x, y, 0) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} A_{lm} \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b}$$

so

$$A_{lm} = \frac{4}{L^2} \int_0^L dx \int_0^L dy\ \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b}\ u(x, y, 0)$$

- Again, BCs set the allowed l, m, ICs determine the coefficients.

Example (2D): Laplace's Equation in a Square



- Underlying equation for potential ϕ:
$$\nabla^2\phi = 0$$
- Separation
$$\phi(x, y) = X(x)Y(y)$$
$$\Longrightarrow X''Y + XY'' = 0$$
$$\Longrightarrow \frac{X''}{X} + \frac{Y''}{Y} = 0$$
$$\Longrightarrow X'' + \lambda^2 X = 0, \quad Y'' + \mu^2 Y = 0, \quad \lambda^2 + \mu^2 = 0$$
- BCs: ϕ is periodic in y for all x, so $\mu^2 > 0$, $\lambda^2 = -\mu^2 < 0$
- $\phi = 0$ for $y = 0$, so y solution is $\sin \mu y$
- $\phi = 0$ for $y = a$, so $\mu a = m\pi$, m <u>integer</u>
- $\phi = 0$ for $x = 0$, so x solution is $\sinh \lambda x$

Example (2D): Laplace's Equation in a Square

- General solution is

$$\phi(x,y) = \sum_{m=1}^{\infty} a_m \sinh\frac{m\pi x}{a} \sin\frac{m\pi y}{a}$$

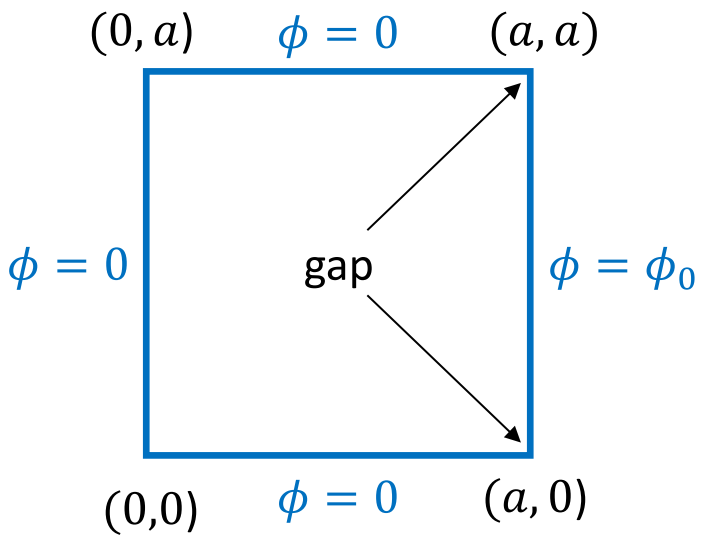
- Satisfies 3 of 4 BCs
- BC at $x = a$:

$$\phi_0 = \sum_{m=1}^{\infty} a_m \sinh m\pi \sin\frac{m\pi y}{a}$$

- Another Fourier series for a_m

$$a_m \sinh m\pi = \frac{2}{a}\int_0^a dy\, \phi_0 \sin\frac{m\pi y}{a} = \frac{2\phi_0}{m\pi}\left[1 - (-1)^m\right]$$

so $$\phi(x,y) = \sum_{\substack{m=1 \\ m\ \text{odd}}}^{\infty} \frac{4\phi_0}{m\pi} \frac{\sinh\frac{m\pi x}{a}}{\sinh m\pi} \sin\frac{m\pi y}{a}$$



BCs set the allowed l, m, ICs determine the coefficients.

Separation of Variables in Cylindrical Polar Coordinates

- Cylindrical polars: $u(\rho, \varphi, z, t) = \chi(\rho, \varphi, z)e^{\pm ickt}$
- Helmholtz: $\nabla^2\chi + k^2\chi = 0$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\chi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\chi}{\partial\varphi^2} + \frac{\partial^2\chi}{\partial z^2} + k^2\chi = 0$$

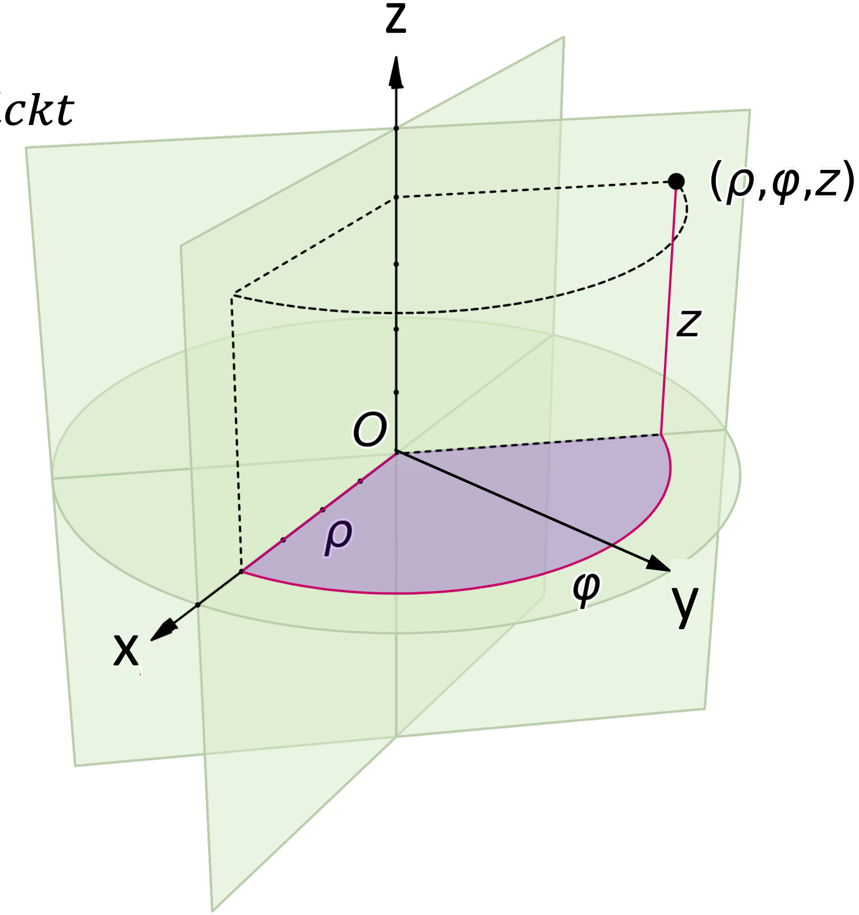
- Separation: assume

$$\chi(\rho, \varphi, z) = \mathrm{P}(\rho)\Phi(\varphi)Z(z)$$

- Substitute and divide by χ:

$$\frac{1}{\rho\mathrm{P}}(\rho\mathrm{P}')' + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} + \frac{Z''}{Z} + k^2 = 0$$

$$\underbrace{\frac{1}{\rho\mathrm{P}}(\rho\mathrm{P}')' + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} + k^2}_{\text{function of } \rho,\ \varphi} = \underbrace{-\frac{Z''}{Z}}_{\text{function of } z} = -l^2$$



Separation of Variables in Cylindrical Polar Coordinates

$$Z'' - l^2 Z = 0$$ solution $Z_l(z)$

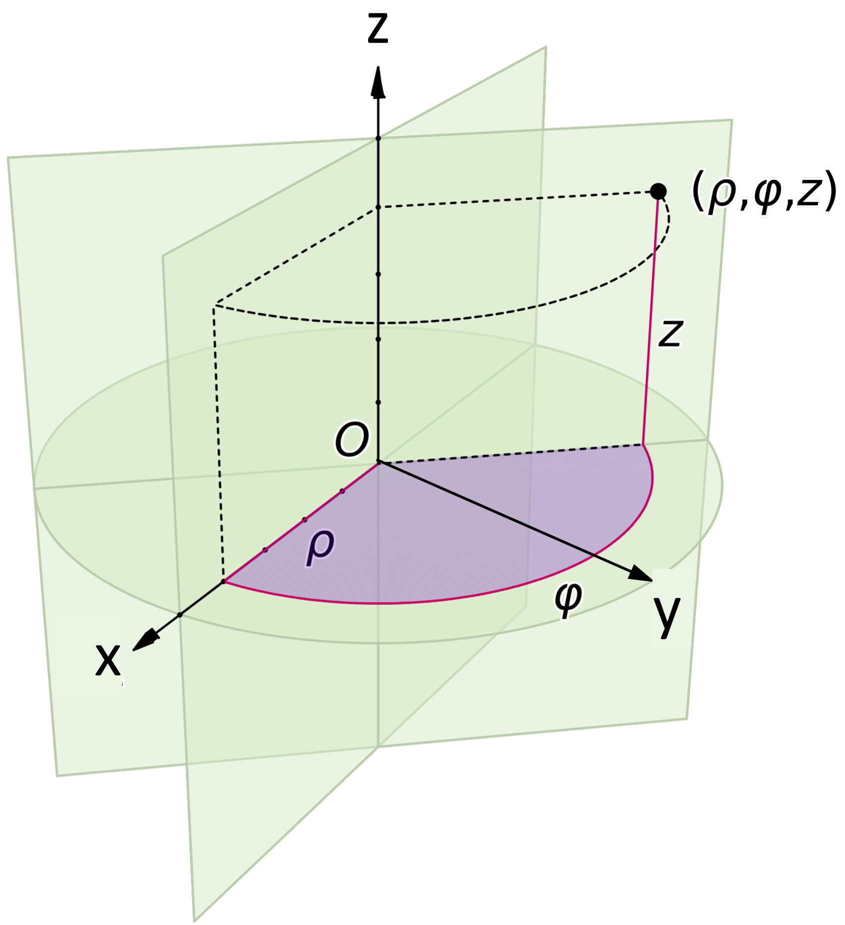
$$\frac{1}{\rho \mathrm{P}}(\rho \mathrm{P}')' + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} + k^2 = -l^2$$

$$\Longrightarrow \quad \frac{\rho}{\mathrm{P}}(\rho \mathrm{P}')' + (k^2 + l^2)\rho^2 = -\frac{\Phi''}{\Phi} = m^2$$

so $\Phi'' + m^2 \Phi = 0$ solution $\Phi_m(\varphi)$

and $\rho(\rho \mathrm{P}')' + \underbrace{(k^2 + l^2)}_{\text{call this } n^2}\rho^2 \mathrm{P} - m^2 \mathrm{P} = 0$

solution $\mathrm{P}_{nm}(\rho)$



- As usual, the general solution is

$$\chi(\rho, \varphi, z) = \sum_{lmn} a_{lmn}\, \mathrm{P}_{nm}(\rho)\Phi_m(\varphi)Z_l(z), \text{ with } n^2 = k^2 + l^2$$

(again, generalized sum — not clear yet what l, m, n really are)

Separation of Variables in Cylindrical Polar Coordinates

- z and φ equations are easy to solve:

$$Z'' - l^2 Z = 0 \implies Z_l(z) = e^{\pm lz} \quad (l \text{ may be real or complex})$$

$$\Phi'' + m^2\Phi = 0 \implies \Phi_m(\varphi) = e^{\pm im\varphi}$$

- φ is an angular coordinate, so the solution must be periodic

$$\implies m \text{ must be an \underline{integer}}$$

- Left with the radial equation

$$\rho(\rho \mathrm{P}')' + (n^2\rho^2 - m^2)\mathrm{P} = 0$$

$$\implies \rho^2 \mathrm{P}'' + \rho \mathrm{P}' + (n^2\rho^2 - m^2)\mathrm{P} = 0$$

$$\implies x^2 P'' + xP' + (x^2 - m^2)P = 0$$

Define $x = n\rho$, so

$$x\frac{d\mathrm{P}}{dx} = n\rho\frac{d\mathrm{P}}{nd\rho} = \rho \mathrm{P}'$$

and redefine ʻ to mean $\frac{d}{dx}$

<u>Bessel's Equation</u>

Bessel's Equation of Integer Order

- For integral m, the solutions to Bessel's equation are very well studied

$$x^2y'' + xy' + (x^2 - m^2)y = 0$$

- The regular (i.e. non-singular) solutions are called $J_m(x)$, so the general solution to the radial equation is

$$\mathrm{P}_{nm}(\rho) = J_m(n\rho)$$

- Note how the separation constants couple — the integer angular constant m couples to the order of the Bessel function, while k and l (via n) are attached to the argument.

- General solution is

$$\chi(\rho, \varphi, z) = \sum_{lmn} a_{lmn}\, J_m(n\rho)\;\; e^{\pm im\varphi}e^{\pm lz}, \quad \text{where } \; n^2 = k^2 + l^2$$

shorthand, again

Bessel Functions of Integer Order

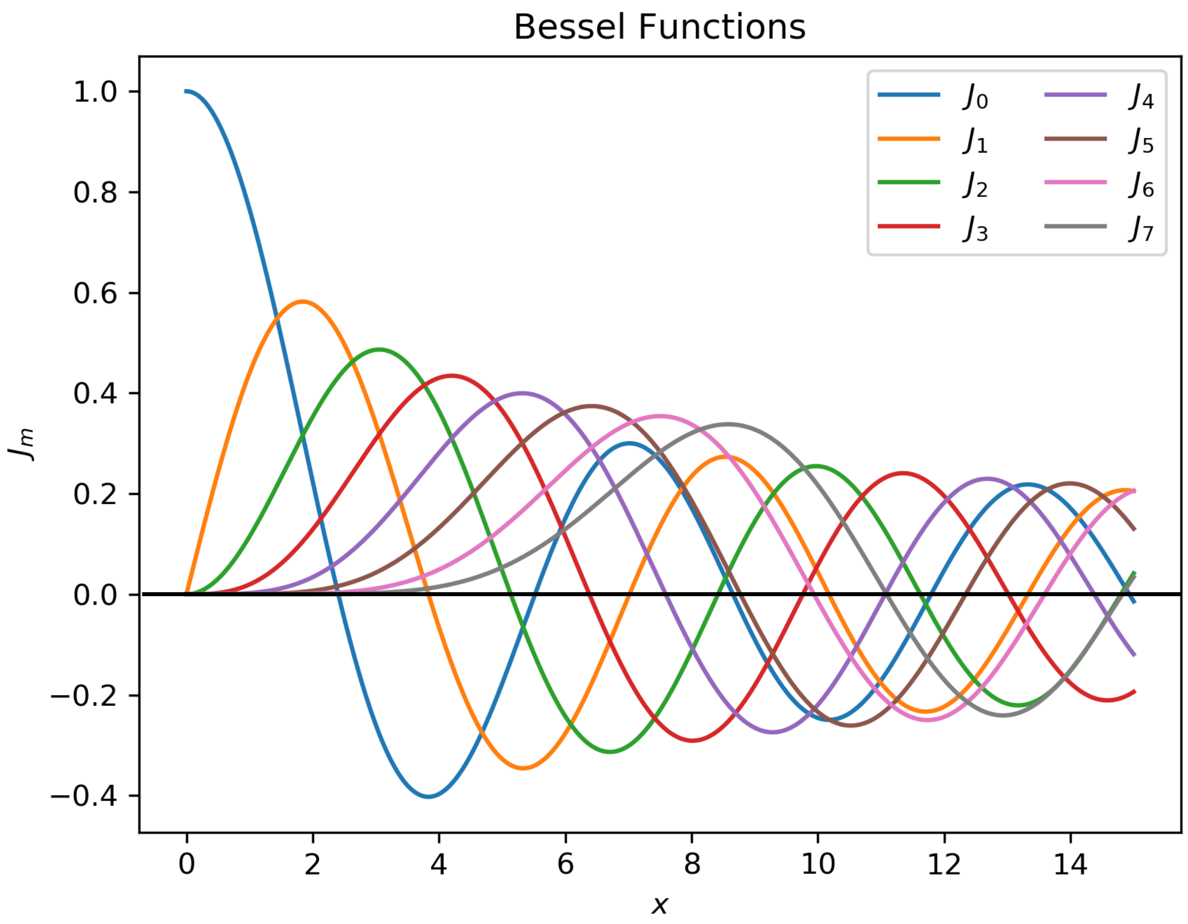
- Interested in Bessel's equation for integral m

$$x^2y'' + xy' + (x^2 - m^2)y = 0$$

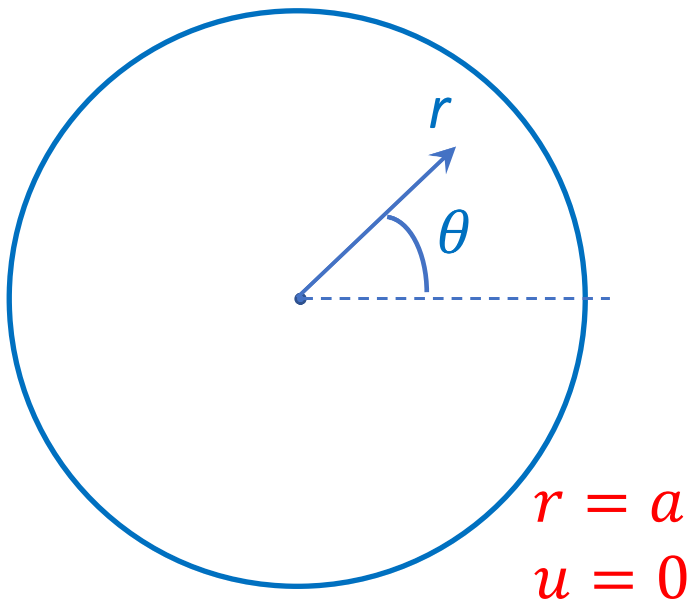
- Specifically interested in the regular (non-singular) solutions $J_m(x)$.
- Think of these as fundamental solutions in the same league as $\sin$ and $\cos$, but possibly not as well known (to the student).
- Will see parts of this theory (as with Fourier series) in a more general context later, but let's spell out some basic properties here.
- Easy to solve this equation, solutions are built into many programming languages (e.g. Python: `scipy.special.jn()`)
- Focus on general properties here.

Bessel Functions of Integer Order

- First few Bessel functions of integral order:
- Functions are oscillatory.
- Functions are damped.
- Zeroes and maxima/minima are well documented.
- $J_m \sim x^m$ near $x = 0$
- Regard these as as "known" functions, in the same category as the trigonometric functions taught in high school.



Vibration of a Circular Membrane



- Same problem as before, but in a different geometry.
- Derivation is the same as cylindrical polars, but just neglect the z term.
- Conventionally, $\rho, \varphi \longrightarrow r, \theta$ here
- Seek separable solution

$$u(r,\theta) = R(r)\Theta(\theta)$$

- where

$$\Theta'' + m^2\Theta = 0$$

solution $e^{\pm im\theta}$, as before, m integer

$$r^2R'' + rR' + (k^2r^2 - m^2)R = 0$$

solution $J_m(kr)$

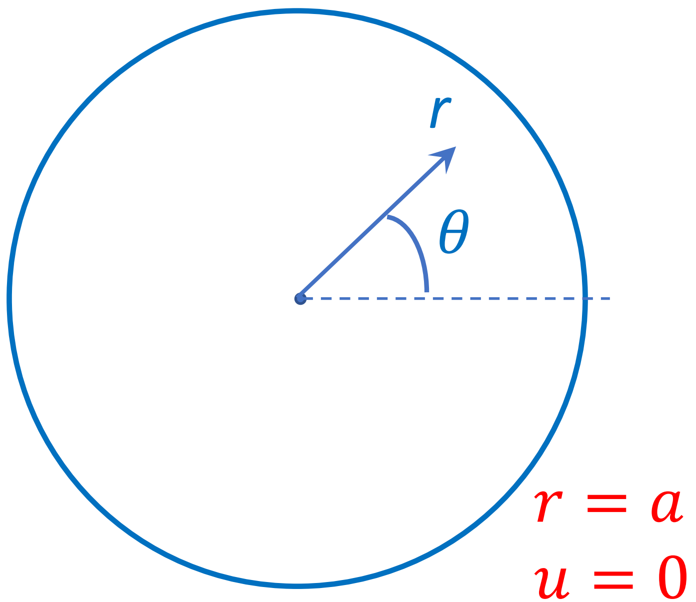
- Look for normal modes, with $R(a) = 0$

Vibration of a Circular Membrane

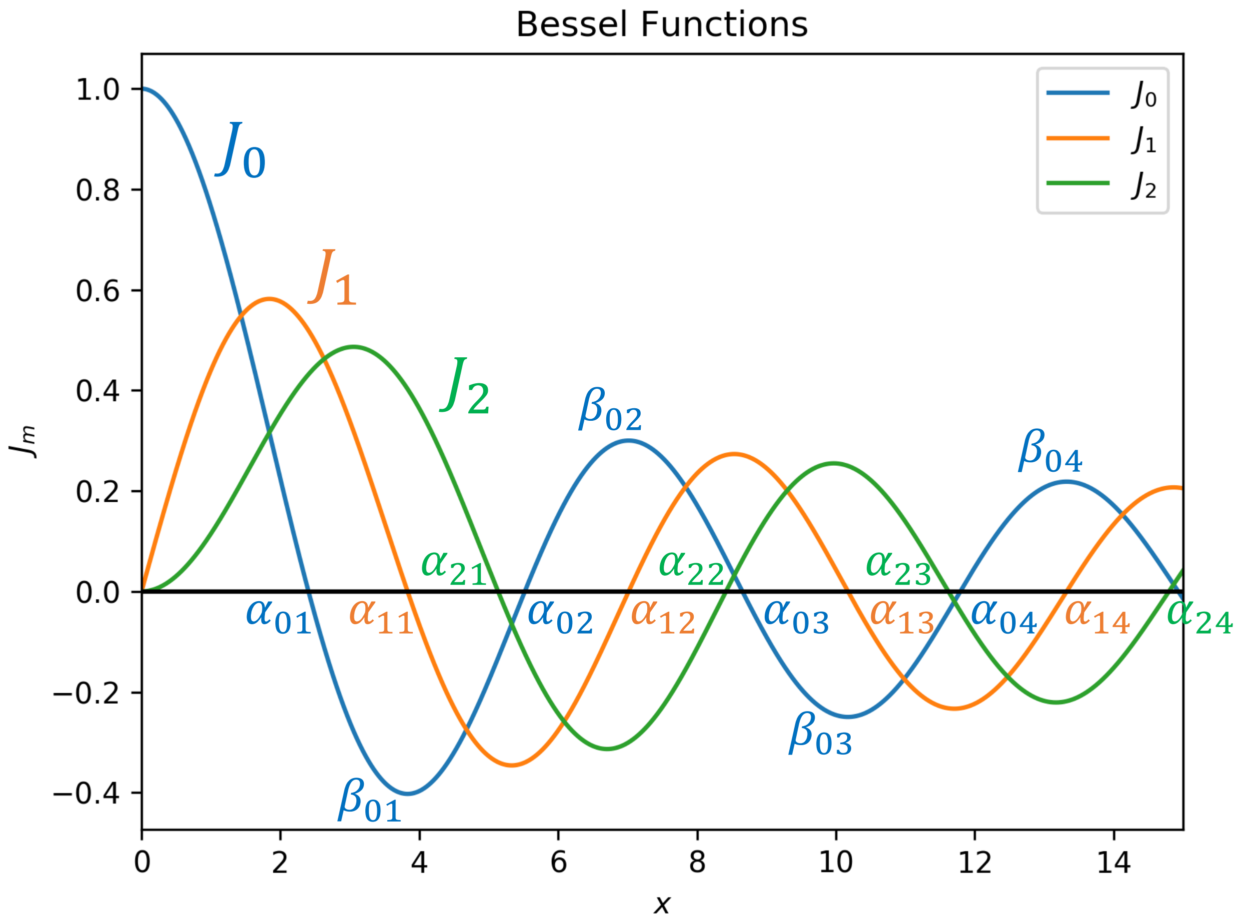
- Normal modes of vibration have the form

$$u(r,\theta) = \begin{cases} J_m(kr)\cos m\theta \\ J_m(kr)\sin m\theta \end{cases}$$

- Boundary condition at $r = a$ implies $J_m(ka) = 0$
  $\Longrightarrow ka$ is a zero of J_m
- Recall nearly identical wording for the vibrating string, with $J_m(kr)$ replaced by $\sin kx$; BCs in that case $\Longrightarrow kx$ is a zero of the sine function.
- Main difference here is that the zeros of $\sin\theta$ are well known: $\theta_n = n\pi$
- Zeros of $J_m(t)$ are not as easy to write down, but they can be calculated and tabulated as α_{mn} .



Zeros of Bessel Functions



	n=1	n=2	n=3	n=4
m=0	2.40	5.52	8.65	11.79
m=1	3.83	7.02	10.17	13.32
m=2	5.14	8.42	11.62	14.80
m=3	6.38	9.76	13.02	16.22

- zeros of J_m interleave those of J_m
- ordering is irregular
- turning points β_{mn} also tabulated

m	n
0	1
1	1
2	1
0	2
3	1
1	2
4	1
2	2
0	3
5	1
3	2
6	1
1	3
4	2
7	1
2	3
0	4
8	1
5	2
3	3