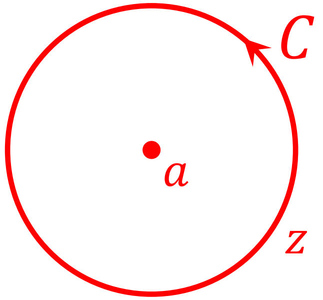
Note: Lost some animations to math editing…

The Residue Theorem

- Define <u>residue</u> of f at a,

  $$\text{Res } f(a) = \frac{1}{2\pi i}\oint_C f(z)dz, \text{ where } C \text{ contains } a$$

  Note: shape of C doesn't matter (Cauchy)

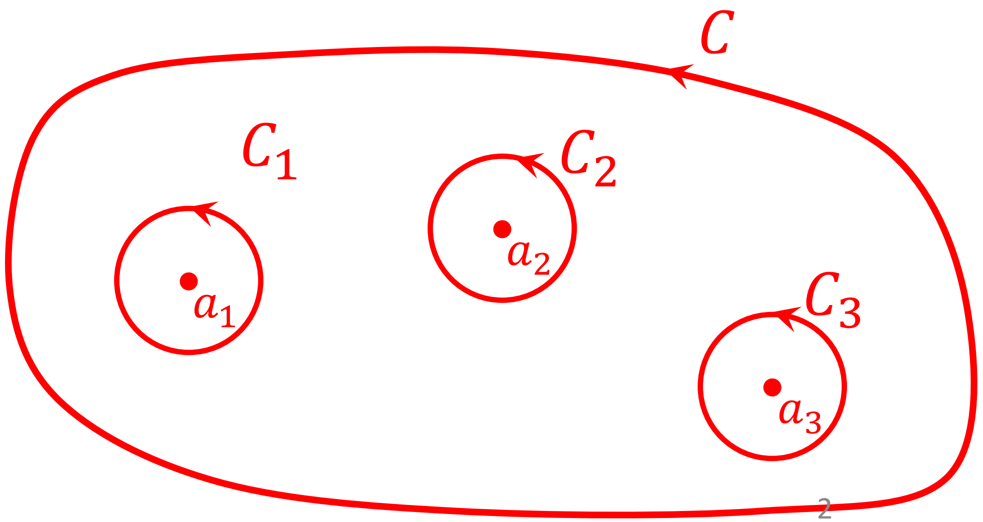


- Then if $f(z)$ is analytic inside and on some closed contour C, except at a finite number of isolated singularities $a_1, a_2, \ldots a_n$ lying within C,

  $$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res } f(a_k)$$

- Again,

  integral is determined by the <u>least</u> singular behavior of f at poles <u>away</u> from C



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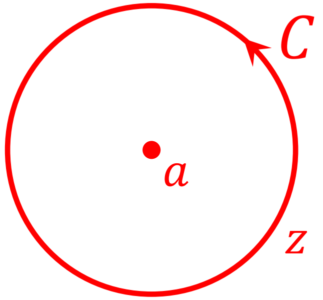

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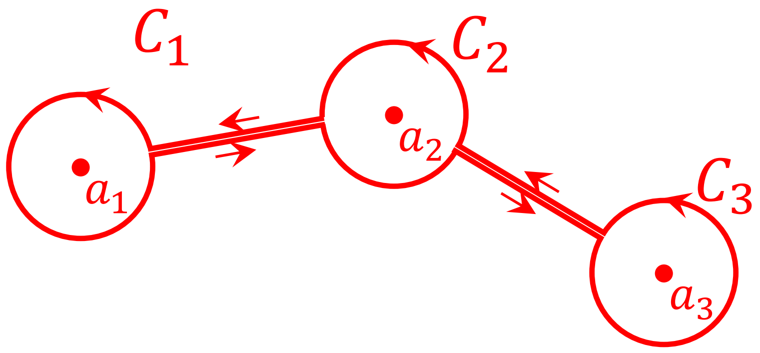


Strategies Using the Residue Theorem

1. Trigonometric polynomials on $[0,2\pi]$

   – maps to integral along the unit circle, rational function $\frac{P(z)}{Q(z)}$ in z

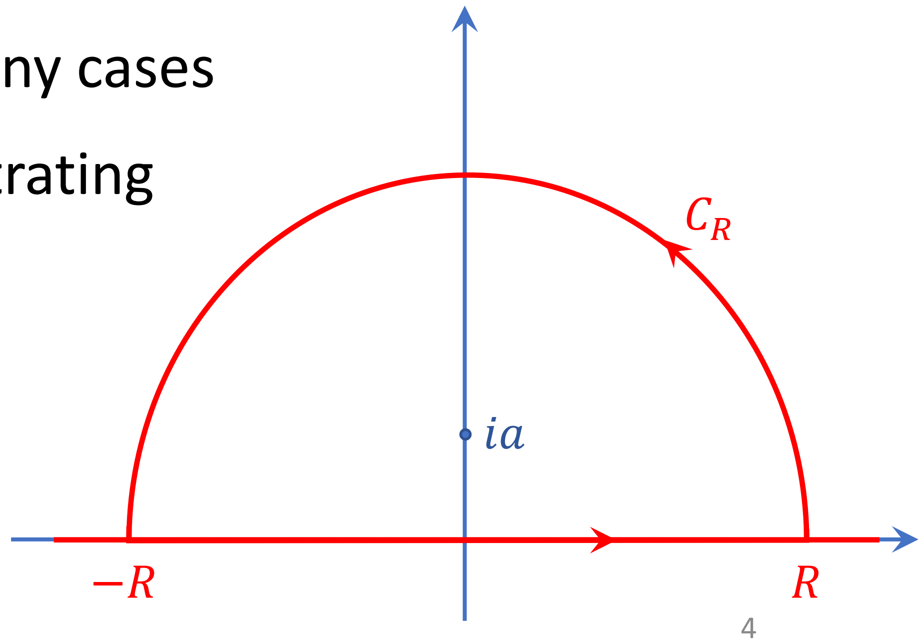
$$z = e^{i\theta}, \quad d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+1/z}{2}, \sin\theta = \frac{z-1/z}{2i}$$

2. The "big semicircle at infinity"

   – convenient way to close the contour in many cases

3. Estimating integrals: important for demonstrating unwanted terms $\to 0$

$$\left|\int_C f\right| \leq \max_C |f| \; L(C)$$



Strategies Using the Residue Theorem

4. Problems with sines and cosines

$$\text{e.g. } \cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = \frac{1}{2}\left(e^{ix-y} + e^{-ix+y}\right)$$

– better to work with exponential form

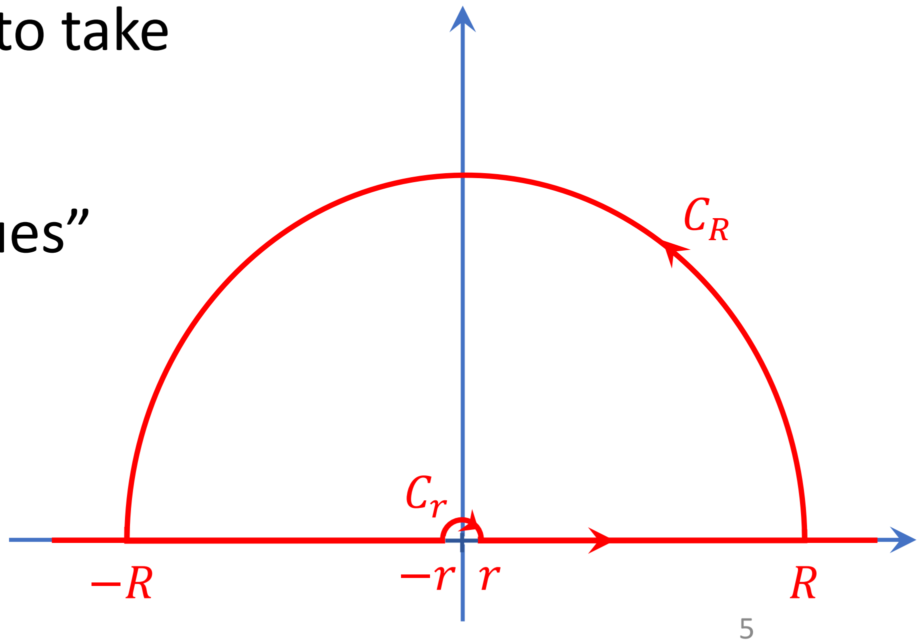
$$\text{e.g. } \left|e^{iz}\right| = \left|e^{ix}\right|\left|e^{-y}\right| = \left|e^{-y}\right| < 1 \text{ for } y > 0$$

– form of the integrand dictates which path to take

5. Poles on the integration path

– can bypass and incorporate as "half residues"

$$\int_{C_r} \frac{dz}{z} = \int_{\pi}^{0} i d\theta = -\pi i$$

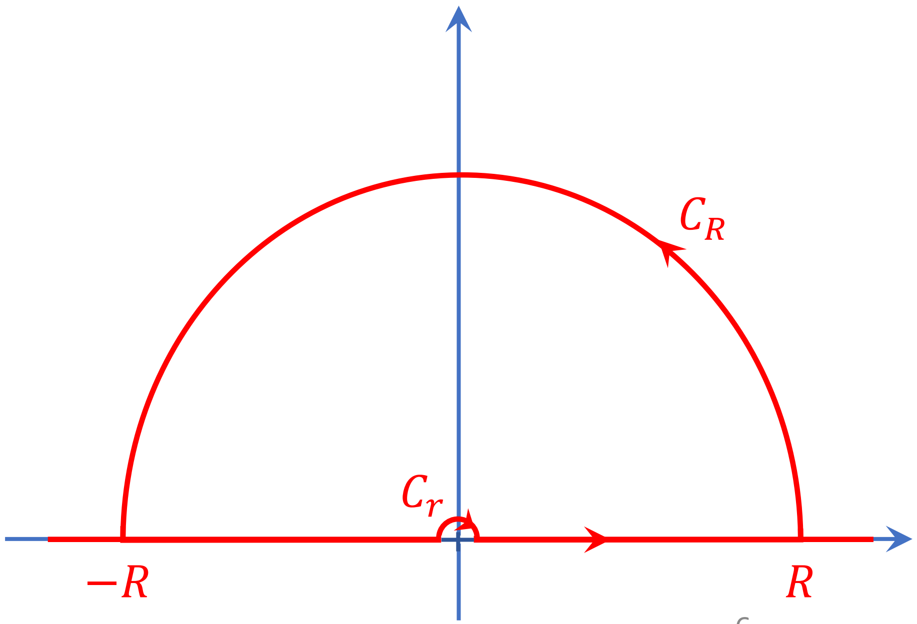


Applications of the Residue Theorem

- Possible problem with estimating $\int_{C_R} \frac{e^{iz}}{z}\,dz$
  - previous estimate would just say $\left|\int_{C_R} \frac{e^{iz}}{z}\,dz\right| \leq \frac{\pi R}{R}$, which does not $\to 0$
  - integrate by parts:

$$\int_{C_R} \frac{e^{iz}}{z}\,dz = \underbrace{\left[\frac{e^{iz}}{iz}\right]_{z=-R}^{z=R}}_{\to 0} + \underbrace{\int_{C_R} \frac{e^{iz}}{iz^2}\,dz}_{\to 0\text{, by earlier argument}}$$

$$= 0$$



Jordan's Lemma

- Can generalize the earlier statement about $\int_{C_R} \frac{e^{iz}}{z} dz$
- If $f(z)$ converges uniformly to 0 on C_R as $|z| \to \infty$ (i.e. if $|f| < \varepsilon$ for any z with $|z| > R$), then

$$\lim_{R\to\infty} \int_{C_R} f(z)\, e^{i\lambda z}\, dz = 0$$

  for any $\lambda > 0$.

- Previously, had $f(z) = \frac{1}{z}$, $\lambda = 1$.
- Now have a general result applicable to Fourier-transform-type integrals.

Applications of the Residue Theorem

- Return to an earlier problematic result: The finite pulse

$$f(t) = \begin{cases} 1, & |t| < a \\ 0, & |t| > a \end{cases}$$

$$\Longrightarrow \qquad F(\omega) = \sqrt{\frac{2}{\pi}}\,\frac{\sin \omega a}{\omega}$$

- Problem: we had no way of doing the inverse integral.
- Now we do.

Applications of the Residue Theorem

5. Inversion integral is

$$I = \frac{1}{\pi}\int_{-\infty}^{\infty} d\omega\ \frac{\sin \omega a}{\omega}\ e^{i\omega t}$$

Recognize this as typical of the integrals just considered.

- Expand $\sin \omega a = (e^{i\omega a} - e^{-i\omega a})/2i$ and write

$$I = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\omega}{2i\omega}\ e^{i\omega(a+t)} - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\omega}{2i\omega}\ e^{-i\omega(a-t)}$$

$$= I_1 - I_2$$

- Introduced a pole into both integrals.
- Manage it with a small semicircle above, as just described.
- (Can verify that placing the semicircle below leads to the same conclusions.)

Applications of the Residue Theorem

- For integral $I_1 = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\omega}{\omega}\; e^{i\omega(a+t)}$

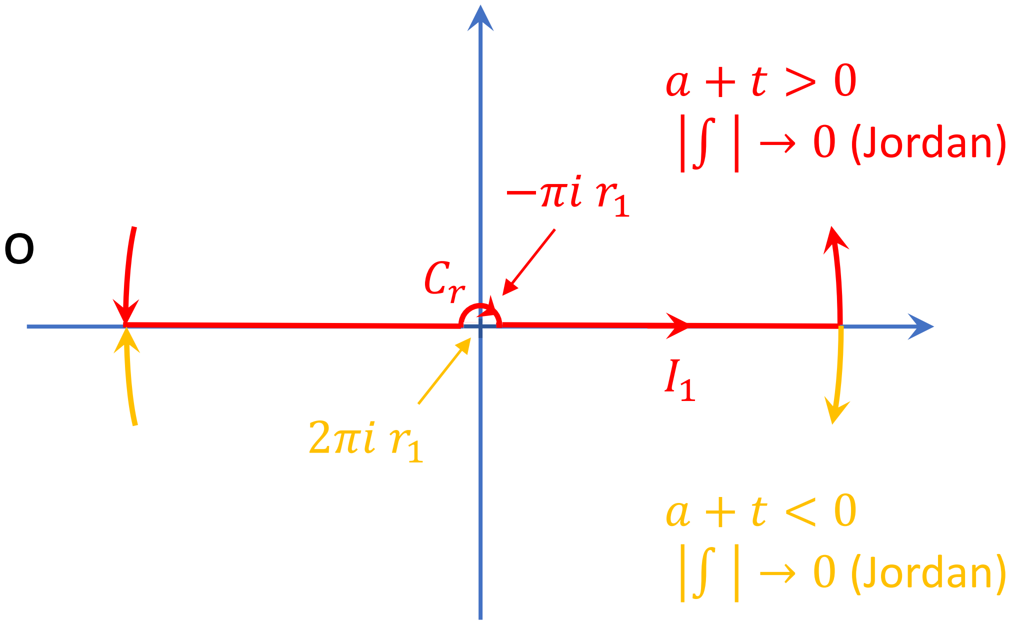
  – pole at $\omega = 0$, residue $r_1 = \frac{1}{2\pi i}$

- Choose semicircular completions at infinity so that their contribution to the integral is zero, by Jordan's lemma.

- For $t > -a$, …

- For $t < -a$, …

See HW5, problem 3!



Applications of the Residue Theorem

- Turn instead to verifying the Parseval Identity for this problem.

$$f(t) = \begin{cases} 1, & |t| < a \\ 0, & |t| > a \end{cases}$$

$$\Longrightarrow \qquad F(\omega) = \sqrt{\frac{2}{\pi}}\,\frac{\sin \omega a}{\omega}$$

- Parseval says

$$\int_{-\infty}^{\infty} dt\,|f(t)|^2 = \int_{-\infty}^{\infty} d\omega\,|F(\omega)|^2$$

- Can easily compute

$$\int_{-\infty}^{\infty} dt\,|f(t)|^2 = \int_{-a}^{a} dt = 2a$$

- Now do the integral in ω using contour integration.

Applications of the Residue Theorem

6. Want to compute

$$\mathrm{I} = \int_{-\infty}^{\infty} d\omega\, |F(\omega)|^2$$

with $\quad F(\omega) = \sqrt{\frac{2}{\pi}}\, \frac{\sin \omega a}{\omega}$

- Integral is

$$\begin{aligned}\mathrm{I} &= \frac{2}{\pi} \int_{-\infty}^{\infty} d\omega\, \frac{\sin^2 \omega a}{\omega^2} \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega\, \frac{1-\cos 2\omega a}{\omega^2} \\ &= \frac{1}{\pi} \operatorname{Re} \underbrace{\int_{-\infty}^{\infty} d\omega\, \frac{1-e^{2i\omega a}}{\omega^2}}_{J}\end{aligned}$$

Applications of the Residue Theorem

- Just defined

$$J = \int_{-\infty}^{\infty} d\omega \, \frac{1-e^{2i\omega a}}{\omega^2}$$

Near $\omega = 0$

$$1 - e^{2i\omega a} = -2i\omega a + O(\omega^2)$$

so integrand has a <u>simple</u> pole and
$\mathrm{Res}(0) = -2ia$

- Pole is on the axis, so use our second favorite contour.
- If $a > 0$, OK to close with C_R in the upper half plane.
- <u>No</u> poles inside the contour, and the C_r integral yields

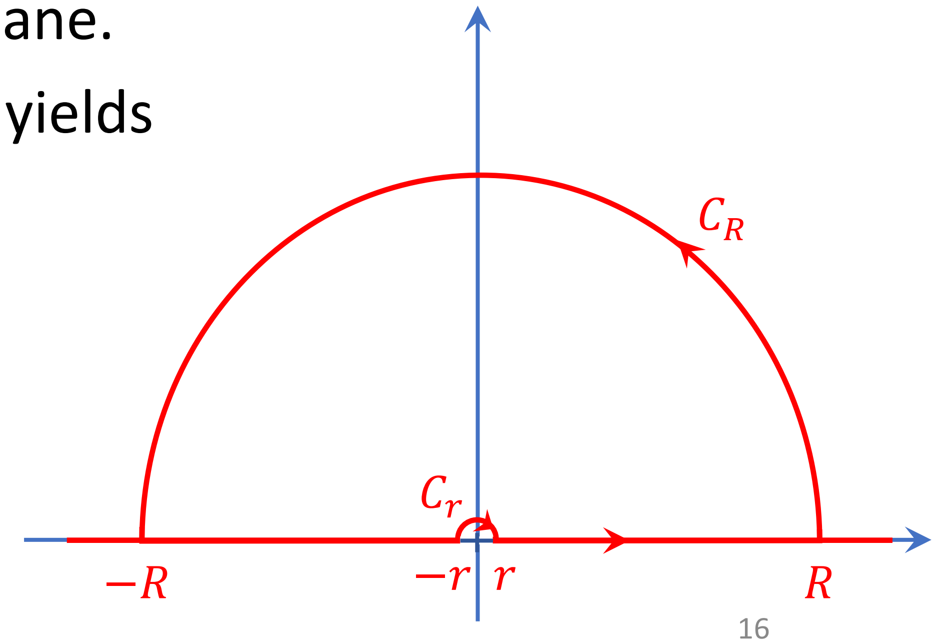
$$-\pi i\,(-2ia) = \ -2\pi a$$

clockwise, half way   residue

- Hence $J - 2\pi a = 0$

$$J = 2\pi a$$

$$I = 2a \quad ✅$$



Applications of the Residue Theorem

7. Other contours may be preferred in other circumstances

e.g. $\mathrm{I} = \int_{-\infty}^{\infty} \frac{e^{az}\, dz}{\cosh \pi z} \qquad (-\pi < a < \pi)$

- Big semicircle won't work in this case

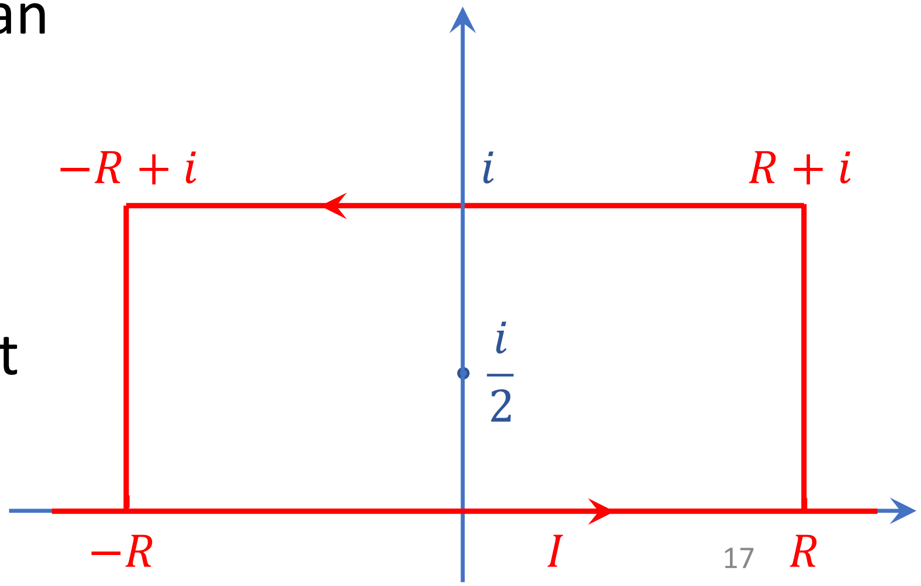
  on imaginary axis, $\dfrac{e^{iay}}{\cosh i\pi y} = \dfrac{e^{iay}}{\cos \pi y}$

  - Doesn't go to zero as $y \to \infty$
  - Has simple poles at $z = (n + \frac{1}{2})i$

  but the denominator's periodicity in y suggests an alternative:

  $$\cosh \pi(x + i) = -\cosh \pi x$$

- periodicity of $\cos \pi y$ allows us to relate the integrand on the top side of the rectangle to that on the bottom
- single simple pole at $z = \frac{i}{2}$



Applications of the Residue Theorem

- Write

$$\mathrm{I} + \underset{I_1}{\int_R^{R+i}} + \underset{I_2}{\int_{R+i}^{-R+i}} + \underset{I_3}{\int_{-R}^{-R+i}} = 2\pi i \,\mathrm{Res}\left(\frac{i}{2}\right)$$

strategy

i. Argue I_1 and I_3 away.

ii. Relate I_2 to I.

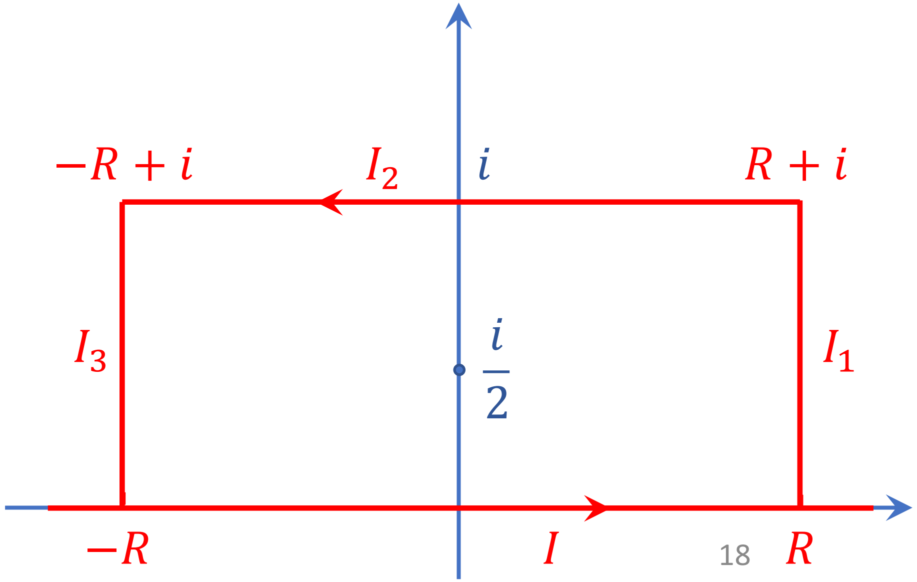
iii. Evaluate the residue at $z = \frac{i}{2}$.

- Look first at the vertical contours.

i. Estimate $|I_1| \leq \dfrac{e^{aR}}{\frac{1}{2}e^{\pi R}}.1 = 2e^{(a-\pi)R}$

$\rightarrow 0$ as $R \rightarrow \infty$ if $a < \pi$

Similarly, $|I_3| \leq 2e^{-(a+\pi)R} \rightarrow 0$ as $R \rightarrow \infty$



Applications of the Residue Theorem

- ii. Now for I_2, write $z = x + i$, so

$$\mathrm{I}_2 = \int_{R+i}^{-R+i} \frac{e^{az}\,dz}{\cosh \pi z} = \int_{R}^{-R} dx \frac{e^{ai} e^{ax}}{-\cosh \pi x} = -e^{ai}(-I),$$

so $I + I_2 = \left(1 + e^{ai}\right) I$

iii. Finally, residue at $z = \frac{i}{2}$ is

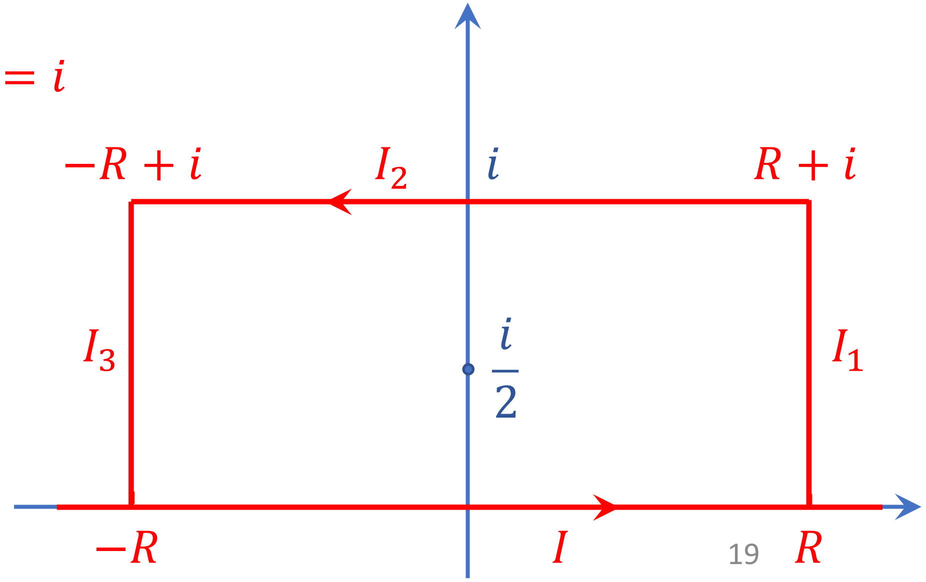
$$r = \frac{e^{ai/2}}{(\cosh \pi z)'|_{z=i/2}} = \frac{e^{ai/2}}{\pi i}$$

$$\sinh \frac{\pi i}{2} = i \sin \frac{\pi}{2} = i$$

so

$$\left(1 + e^{ai}\right) I = 2e^{ai/2}$$

$$\Longrightarrow I = \frac{2e^{ai/2}}{1+e^{ai}} = \frac{2}{e^{ai/2}+e^{-ai/2}} = \sec \frac{a}{2}$$

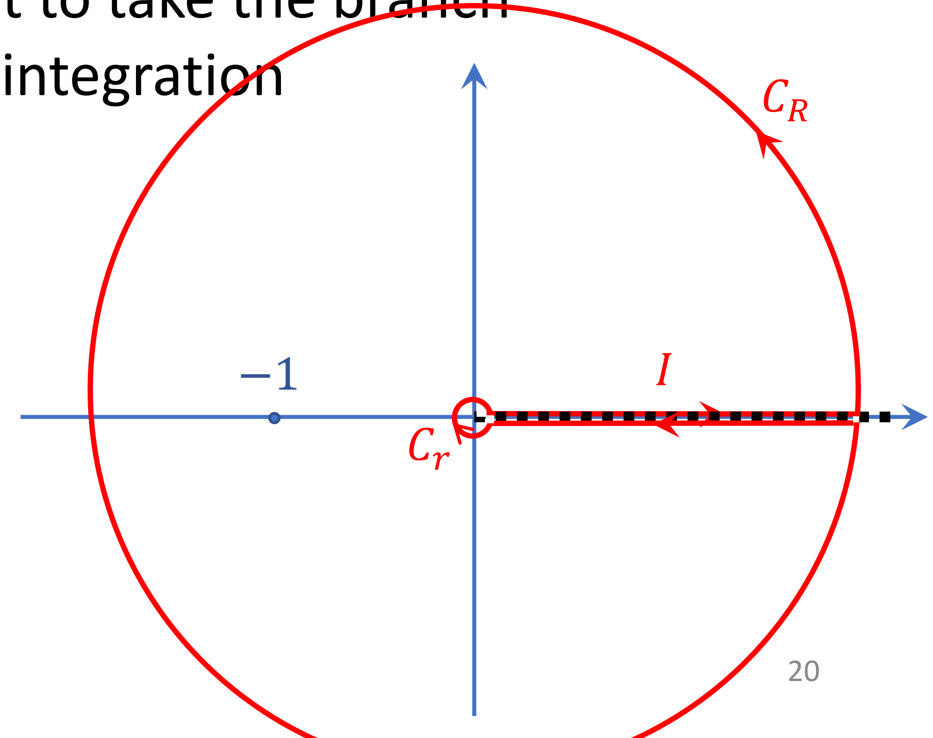


Applications of the Residue Theorem

8. Can also handle integrals with branch cuts

e.g. $\mathrm{I} = \int_0^\infty \frac{x^{\alpha-1}\,dx}{1+x} \qquad (0 < \alpha < 1)$

- Simple pole at $z = -1$, but we also have a <u>branch point</u> at $z = 0$.
- Somewhat counterintuitively, most convenient to take the branch cut to run along (well, just below) the path of integration
- How to close the contour?
  - – big circle works!
  - – but with qualifications…
  - – can't cross the branch cut
  - – still have to deal with the branch point
- Again, take the pieces in turn.



Applications of the Residue Theorem

$\mathrm{I} = \int_0^{\infty} \frac{x^{\alpha-1}\,dx}{1+x}$

- Residue at $z = -1$ is $(-1)^{\alpha-1} = e^{\pi i(\alpha-1)}$
- $\left|\int_{C_R}\right| \leq \frac{R^{\alpha-1}}{R} 2\pi R = 2\pi R^{\alpha-1} \to 0$ as $R \to \infty$ if $\alpha < 1.$
- $\left|\int_{C_r}\right| \leq r^{\alpha-1} 2\pi r = 2\pi r^{\alpha} \to 0$ as $r \to 0$ if $\alpha > 0.$
- On the lower contour, instead of $x^{\alpha-1}$ on the top, we now have $x^{\alpha-1}e^{2\pi i(\alpha-1)}$, so the integral is

$$I' = \int_R^0 \frac{x^{\alpha-1}e^{2\pi i(\alpha-1)}\,dx}{1+x}$$
$$= e^{2\pi i(\alpha-1)}(-I)$$

- Hence

$$I + I' = \left[1 - e^{2\pi i(\alpha-1)}\right]I = 2\pi i\, e^{\pi i(\alpha-1)}$$

$$\Longrightarrow \quad \mathrm{I} = \frac{-2\pi i\, e^{\pi i\alpha}}{1-e^{2\pi i\alpha}} = \frac{-2\pi i}{e^{-\pi i\alpha}-e^{\pi i\alpha}} = \frac{\pi}{\sin \pi\alpha}$$