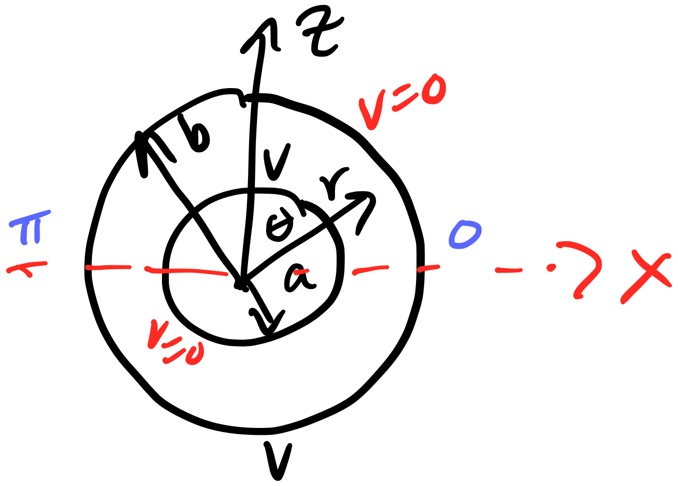
HW 3   3.1, 3.6, 3.19   due 2/23

3.1



Φ in $a \le r \le b$ as series up to $\ell=4$ term?

check case of $b\to\infty$ and $a\to 0$

symmetry about $\hat{z}$ so Φ only depends on r and θ

general solution: $\Phi(r,\theta) = \sum_\ell (A_\ell r^\ell + B_\ell r^{-\ell-1}) P_\ell \cos\theta$

continuity condition says $\Phi(a,\theta) = \Phi(b,\theta)$

BC's: $\Phi(a,\theta) = \begin{cases} V & \text{for } 0<\theta<\pi/2 \\ 0 & \text{for } \pi/2<\theta<\pi \end{cases}$

$\Phi(b,\theta) = \begin{cases} 0 & \text{for } 0<\theta<\pi/2 \\ V & \text{for } \pi/2<\theta<\pi \end{cases}$

so $\Phi(a,\theta) = \sum_\ell A_\ell a^\ell + B_\ell a^{-\ell-1} P_\ell \cos\theta$ (1)

$\Phi(b,\theta) = \sum_\ell A_\ell b^\ell + B_\ell b^{-\ell-1} P_\ell \cos\theta$ (2)

we know $\int_{-1}^{1} P_m(x) P_n(x)\, dx = \frac{2}{2n+1}\delta_{mn}$

from orthogonality of Legendre polynomials

so can get rid of sum by multiply our Φ's by a Legendre polynomial and then integrating both sides of (1) and (2)

Let $x = \cos\theta$

$$\int_{-1}^{1} \Phi(a,x) P_\ell(x)\,dx = V\int_0^1 P_\ell(x)\,dx$$
$$= \int_{-1}^{1} [A_\ell a^\ell + B_\ell a^{-\ell-1}] P_\ell^2(x)\,dx$$

$$\int_{-1}^{1} \Phi(b,x) P_\ell(x)\,dx = V\int_{-1}^{0} P_\ell(x)\,dx$$
$$= \int_{-1}^{1} [A_\ell b^\ell + B_\ell b^{-\ell-1}] P_\ell^2(x)\,dx$$

we know $\int_{-1}^{1} P_\ell^2(x)\,dx = \frac{2}{2\ell+1}$

so comparing coefficients of right hand sides:

$$A_\ell a^\ell + B_\ell a^{-\ell-1} = \frac{2\ell+1}{2} V\int_0^1 P_\ell(x)\,dx$$
$$A_\ell b^\ell + B_\ell b^{-\ell-1} = \frac{2\ell+1}{2} V\int_{-1}^{0} P_\ell(x)\,dx$$
$$= \frac{2\ell+1}{2} V\int_0^1 P_\ell(-x)\,dx$$
$$= \frac{2\ell+1}{2} V (-1)^\ell \int_0^\ell P_\ell(x)\,dx$$

separating out A_ℓ and B_ℓ:

$$A_\ell = \frac{2\ell+1}{2(a^{2\ell+1} - b^{2\ell+1})} V[a^\ell - b^{\ell+1}(-1)^\ell] \int_0^1 P_\ell(x)\,dx \quad (3)$$

$$B_\ell = (a^{\ell+1}) \frac{2\ell+1}{2} V\int_0^1 P_\ell(x)\,dx - A_\ell a^{2\ell+1} \quad (4)$$

we want $\ell = 0, 1, 2, 3, 4$ terms so find A_ℓ's first and plug into B_ℓ equation

we know for even l that $\int_0^1 P_l(x)\,dx = \frac{1}{2}\int_{-1}^1 P_l(x)\,dx$
$= 0$

so only odd terms ($l > 0$) will survive
and only need P_0, P_1, P_3

$$\int_0^1 P_0(x)\,dx = 1$$
$$\int_0^1 P_1(x)\,dx = \frac{1}{2}$$
$$\int_0^1 P_3(x)\,dx = -\frac{1}{8}$$

$$B_3 = \frac{-7Va^4 - a^7}{16} - \left[\frac{-7V(a^4+b^4)}{16(a^7-b^7)}a^7\right]$$
$$= -\frac{7}{16}Va^4b^4\left[\frac{b^3+a^3}{b^7-a^7}\right]$$

plugging into (3):

$$A_0 = V/2$$
$$A_1 = \frac{3V(a^2 - b^2(-1)^1)}{2(a^3-b^3)} \cdot \frac{1}{2} = \frac{3V(a^2+b^2)}{4(a^3-b^3)}$$
$$A_3 = \frac{7V(a^4 - b^4(-1)^3)}{2(a^7-b^7)} \cdot \left(-\frac{1}{8}\right) = \frac{-7V(a^4+b^4)}{16(a^7-b^7)}$$

now plug A_0, A_1 and A_3 into (4):

$$B_0 = \frac{a}{2}V - \frac{V}{2}a = 0$$
$$B_1 = a^2\left(\frac{3}{2}\right)V\left(\frac{1}{2}\right) - a^2\left[\frac{3V(a^2+b^2)}{4(a^3-b^3)}\right]$$
$$= \frac{3}{4}Va^2 - \frac{3}{4}Va^3\frac{(a^2+b^2)}{a^3-b^3} = \frac{3}{4}Va^2b^2\left[\frac{b+a}{b^3-a^3}\right]$$
$$B_3 = a^4\left(\frac{7}{2}\right)V\left(-\frac{1}{8}\right) - a^7\left[-\frac{7V(a^4+b^4)}{16(a^7-b^7)}\right]$$
$$= -\frac{7}{16}Va^4b^4\left[\frac{b^3+a^3}{b^7-a^7}\right]$$

so plugging these back in:

$$\Phi(r,\theta) = A_0 P_0(x) + \underbrace{B_0 P_0(x)}_{0} + A_1 r P_1(x) + \frac{B_1}{r^2} P_1(x)$$

$$+ A_3 r^3 P_3(x) + \frac{B_3}{r^3} P_3(x)$$

but $\Phi(r,\theta)$ needs to go to 0 as $r \to \infty$
so this means only B terms survive (neg exp)
besides A_0 term

$$\Phi(r,\theta) = A_0 P_0(x) + \frac{B_1}{r^2} P_1(x) + \frac{B_3}{r^4} P_3(x)$$

$$= \frac{V}{2} P_0(x) + \frac{3}{4} \frac{V a^2 b^2}{r^2} \left[\frac{b+a}{b^3-a^3}\right] P_1(x) - \frac{7}{16} \frac{V a^4 b^4}{r^4} \left[\frac{b^3+a^3}{b^7-a^7}\right] P_3(x)$$

$$\boxed{= \frac{V}{2}\left[P_0(x) + \frac{3}{2} \frac{a^2 b^2}{r^2} \left(\frac{b+a}{b^3-a^3}\right) P_1(x) - \frac{7}{8} \frac{a^4 b^4}{r^4} \left[\frac{b^3+a^3}{b^7-a^7}\right] P_3(x)\right]}$$

so at $b \to \infty$

$$\Phi(r,\theta) = \frac{V}{2}\left[P_0(x) + \frac{3}{2} \frac{a^3}{r^2} P_1(x) - \frac{7}{8} \frac{a^4}{r^4} P_3(x)\right]$$

for $a \to 0$

$$\Phi(r,\theta) = \frac{V}{2}\left[P_0(x)\right]$$