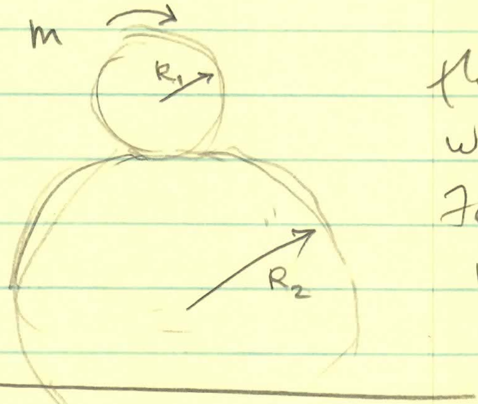
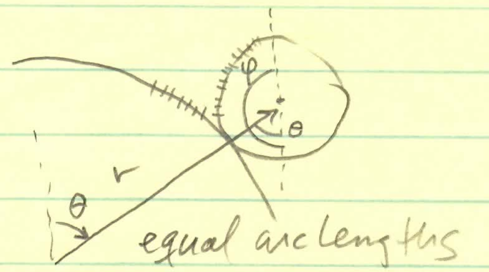
(2.14)



the fall-off point is the point where the normal is zero.
For the normal, use Lagrange multipliers.



equal arclengths mean $R_2\theta = R_1(\varphi-\theta)$

<u>Constraints</u> $r = R_1 + R_2$

(no slipping) $R_1\varphi = R_2\theta$

$f_1 = r - R_1 - R_2 = 0$

$f_2 = R_1(\varphi-\theta) - R_2\theta = 0$

$$T = \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2) + \frac{1}{2}mR_1^2\dot\varphi^2 \quad : V = (R_2 + r\cos\theta)mg$$

$$L = \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2) + \frac{1}{2}mR_1^2\dot\varphi^2 - (R_2 + r\cos\theta)mg$$

<u>Eqns</u> of motion:

(1) r: $\quad m\ddot r - mr\dot\theta^2 + mg\cos\theta = \lambda_1\frac{\partial f_1}{\partial r} + \lambda_2\frac{\partial f_2}{\partial r} = \lambda_1$

(2) θ: $\quad 2mr\dot r\dot\theta + mr^2\ddot\theta - r\sin\theta mg = \lambda_1\frac{\partial f_1}{\partial\theta} + \lambda_2\frac{\partial f_2}{\partial\theta} = -\lambda_2(R_1+R_2)$

(3) φ: $\quad mR_1^2\ddot\varphi = \lambda_1\frac{\partial f_1}{\partial\varphi} + \lambda_2\frac{\partial f_2}{\partial\varphi} = \lambda_2 R_1$

We want to solve for the normal, which is λ_1

Using $r - R_1 - R_2 = 0 \Rightarrow \dot r = \ddot r = 0$ and $R_1\ddot\varphi = (R_1+R_2)\ddot\theta$

$\Rightarrow$ into (3) $\lambda_2 = m(R_1+R_2)\ddot\theta$

$\Rightarrow$ into (2) $m(R_1+R_2)^2\ddot\theta - (R_1+R_2)\sin\theta mg = -m(R_1+R_2)^2\ddot\theta$

$$2(R_1+R_2)\ddot\theta = g\sin\theta$$

multiply by $\dot\theta$ $\quad 2(R_1+R_2)\ddot\theta\dot\theta = g\sin\theta\,\dot\theta$

and factor $\quad (R_1+R_2)\frac{d}{dt}\dot\theta^2 = \frac{d}{dt}(-g\cos\theta)$

$$(R_1+R_2)\dot\theta^2 + g\cos\theta = C$$

the constant can be determined from $\theta(0) = \dot{\theta}(0) = 0$

$$c = g \Rightarrow \dot{\theta}^2 = \frac{1}{R_1+R_2} g(1-\cos\theta)$$

$\Rightarrow$ into (1)

$$-M(R_1+R_2)\left[\frac{1}{R_1+R_2} g(1-\cos\theta)\right] + mg\cos\theta = \lambda_1$$

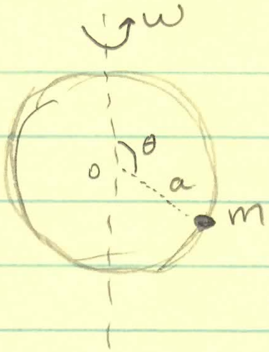
$$-mg(1-\cos\theta) + mg\cos\theta = \lambda_1$$

$$\boxed{\lambda_1 = -mg(1-2\cos\theta)} \quad \text{normal}$$

and this is zero when $\cos\theta = 1/2$

$$\boxed{\theta = 60^\circ}$$

(2.18)



From pb.(1.19) $\bar{v} = a(\dot\theta\hat\theta + \sin\theta\dot\varphi\hat\varphi)$
so that $\bar{v}\cdot\bar{v} = a^2(\dot\theta^2 + \sin^2\theta\dot\varphi^2)$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}ma^2(\dot\theta^2 + \sin^2\theta\dot\varphi^2)$$
$$V = mga\cos\theta$$
$$L = \frac{1}{2}ma^2(\dot\theta^2 + \sin^2\theta\dot\varphi^2) - mga\cos\theta$$

$\boxed{\text{but } \dot\varphi = \omega}$

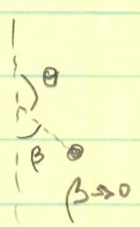
only one independent coordinate,
Eqn of motion in θ:

$$ma\ddot\theta - \frac{1}{2}ma^2\omega^2\, 2\sin\theta\cos\theta - mga\sin\theta = 0$$
$$\boxed{\ddot\theta - (\omega^2\cos\theta + g/a)\sin\theta = 0}$$

Equillibrium points satisfy $\ddot\theta = 0$
* Two equillibrium points at $\sin\theta = 0$, $\theta = 0, \pi$
** Another equil point at $\theta = \cos^{-1}(-g/\omega^2 a)$
that also require $\omega^2 \geq g/a$ (for $\pi/2 \leq \theta \leq \pi$)

$\boxed{\omega_0^2 = g/a}$

But want stable points. Close to these, $\ddot\theta < 0$ so that perturbations are pulled back to equillibrium



Case $\theta_0 = 0$: $\ddot\theta \simeq (\omega^2 + g/a)\theta > 0$ $\boxed{\text{unstable}}$

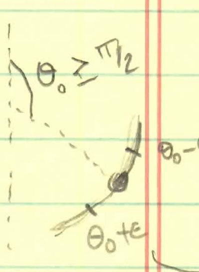
Case $\theta_0 = \pi$: $\theta = \pi - \beta$ substitute

$+\ddot\beta \simeq (+\omega^2 - g/a)\beta$ (i) $\omega_0^2 < g/a$, $\ddot\beta < 0$ $\boxed{\text{stable}}$

(ii) $\omega_0^2 > g/a$, $\ddot\beta > 0$ $\boxed{\text{unstable}}$

Case $\theta_0 = \cos^{-1}(-g/\omega^2 a)$ and $\omega^2 \geq g/a$:

Is it stable? Check: [$\sin\theta > 0$ for $\pi/2 \leq \theta \leq \pi$]



→ if $\theta' < \theta_0$: $\omega^2\cos\theta' = \omega^2\cos(\theta_0 - \epsilon) = -\frac{g}{a}(1 - \epsilon^2)$

$\Rightarrow \ddot\theta > 0$ pushing it to θ_0

→ if $\theta' > \theta_0$: $\omega^2\cos\theta' = \omega^2\cos(\theta_0 + \epsilon) = -\frac{g}{a}(1 + \epsilon^2)$

$\Rightarrow \ddot\theta < 0$ pulling it back to θ_0

$\boxed{\text{Stable}}$

For the conserved quantity we notice that $L \neq L(t)$ thus the Jacobi integral $\underline{h}$ is conserved. This is an energy function that in many cases coincides with the total energy

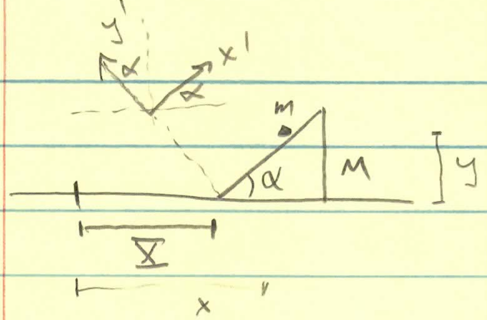
$$h = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L$$

$$= ma^2 \dot{\theta}^2 - \frac{1}{2} ma^2 (\dot{\theta}^2 + \sin^2\theta \, \omega^2) + mga \cos\theta$$

$$h = \frac{1}{2} ma^2 \dot{\theta}^2 - \frac{1}{2} ma^2 \sin^2\theta \, \omega^2 + mga \cos\theta$$

is conserved

(2.20)



Conversion between coordinates

$$x = X + x'\cos\alpha - y'\sin\alpha$$
$$y = x'\sin\alpha + y'\cos\alpha$$

Time derivatives $\dot{x} = \dot{X} + \dot{x}'\cos\alpha - \dot{y}'\sin\alpha$

$$\dot{y} = \dot{x}'\sin\alpha + \dot{y}'\cos\alpha$$

Squares of derivatives

$$\dot{x}^2 = \dot{x}'^2\cos^2\alpha + \dot{y}'^2\sin^2\alpha - 2\dot{x}'\dot{y}'\sin\alpha\cos\alpha + \dot{X}^2 + 2\dot{X}(\dot{x}'\cos\alpha - \dot{y}'\sin\alpha)$$
$$\dot{y}^2 = \dot{x}'^2\sin^2\alpha + \dot{y}'^2\cos^2\alpha + 2\dot{x}'\dot{y}'\sin\alpha\cos\alpha$$

Kinetic energy of m:

$$T = \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2) = \tfrac{1}{2}m(\dot{x}'^2+\dot{y}'^2) + \tfrac{1}{2}m\dot{X}^2 + m\dot{X}(\dot{x}'\cos\alpha - \dot{y}'\sin\alpha)$$

Potential $V = mgy = mg(x'\sin\alpha + y'\cos\alpha)$

Lagrangian

$$L = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m(\dot{x}'^2+\dot{y}'^2) + \tfrac{1}{2}m\dot{X}^2 + m\dot{X}(\dot{x}'\cos\alpha - \dot{y}'\sin\alpha) - mg(x'\sin\alpha + y'\cos\alpha)$$

Constraint $\boxed{y' = 0} = f(x', y', X)$

Eqns of motion

X: $$\frac{d}{dt}\Big[\underbrace{(M+m)\dot{X} + m(\dot{x}'\cos\alpha - \dot{y}'\sin\alpha)}\Big] = 0$$

constant of the motion, conservation of momentum

$$\boxed{(M+m)\dot{X} + m(\dot{x}'\cos\alpha - \dot{y}'\sin\alpha) = \text{constant}}$$

$$(M+m)\ddot{X} + m(\ddot{x}'\cos\alpha - \ddot{y}'\sin\alpha) = Q_X = \lambda\frac{\partial f}{\partial X} = 0$$

x': $m\ddot{x}' + m\ddot{X}\cos\alpha + mg\sin\alpha = Q_{x'} = \lambda\frac{\partial f}{\partial x'} = 0$

y': $m\ddot{y}' - m\ddot{X}\sin\alpha + mg\cos\alpha = Q_{y'} = \lambda\frac{\partial f}{\partial y'} = \lambda$

Eqns of motion then

$$\boxed{\begin{aligned} &\ddot{X} + \frac{m}{M+m}\left(\ddot{x}'\cos\alpha - \ddot{y}'\sin\alpha\right) = 0 \\ &\ddot{x}' + \ddot{X}\cos\alpha + g\sin\alpha = 0 \\ &\ddot{y}' - \ddot{X}\sin\alpha + g\cos\alpha = \frac{\lambda}{m} \end{aligned}}$$

Constraint

$$y = \dot{y} = \ddot{y} = 0$$

Applying the constraint

$$\left.\begin{aligned} &\ddot{X} + \frac{m}{m+M}\ddot{x}'\cos\alpha = 0 \\ &\ddot{x}' + \ddot{X}\cos\alpha + g\sin\alpha = 0 \\ &\ddot{X}\sin\alpha - g\cos\alpha = -\lambda/m \end{aligned}\right\}$$

Solving:

$$\ddot{X} = \frac{mg\sin\alpha\cos\alpha}{M+m\sin^2\alpha}$$

$$\ddot{x}' = -\frac{(M+m)g\sin\alpha}{M+m\sin^2\alpha}$$

b/c it is a conservative system, the energy is conserved

$$\boxed{E = \frac{1}{2}(M+m)\dot{X}^2 + \frac{1}{2}m\left(\dot{x}'^2 + \dot{y}'^2\right) + m\dot{X}\dot{x}'\cos\alpha + mgx'\sin\alpha}$$

(this can also be derived from the Jacobi integral)

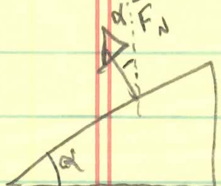
Forces of constraint $Q_{y'} = \lambda$ *magnitude*

$$\lambda = -m\sin\alpha\,\frac{mg\sin\alpha\cos\alpha}{M+m\sin^2\alpha} + mg\cos\alpha$$

$$\boxed{\lambda = mg\cos\alpha\left(\frac{M}{M+m\sin^2\alpha}\right)}$$

For the work of the constraint forces per unit time, this is the power $\frac{dW}{dt} = \text{power} = \bar{F}_N \cdot \bar{v}$

where $\bar{F}_N = \lambda\left(-\sin\alpha\,\hat{\imath} + \cos\alpha\,\hat{\jmath}\right)$



Lets look at work on m:

$\bar{v}_m = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} = (\dot{X} + \dot{x}'\cos\alpha)\hat{\imath} + \dot{x}'\sin\alpha\,\hat{\jmath}$

Now integrating the above equations for $\ddot{X}$ and $\ddot{x}'$ and using $\dot{X}(t=0)=0$, $\dot{x}'(t=0)=0$ we get

$$\dot{x}' = -\frac{(M+m)\,g\sin\alpha}{M+m\sin^2\alpha}\,t \quad ; \quad \dot{X} = \frac{mg\sin\alpha\cos\alpha}{M+m\sin^2\alpha}\,t$$

$$\Longrightarrow \quad \bar{v}_m = -\frac{g\sin\alpha}{M+m\sin^2\alpha}\left[M\cos\alpha\,\hat{\imath} + (M+m)\sin\alpha\,\hat{\jmath}\right]t$$

$$\frac{dW_m}{dt} = \bar{F}_N\cdot\bar{v}_m = -\frac{\lambda g\sin\alpha}{M+m\sin^2\alpha}(m\sin\alpha\cos\alpha)\,t$$

Now force on M: $\quad \bar{v}_M = \dot{X}\,\hat{\imath} = \frac{mg\sin\alpha\cos\alpha}{M+m\sin^2\alpha}\,t\,\hat{\imath}$

$$\frac{dW_M}{dt} = -\bar{F}_N\cdot\bar{v}_M = \frac{\lambda g\sin\alpha}{M+m\sin^2\alpha}(m\sin\alpha\cos\alpha)\,t$$

Note that total work by constraint force $\frac{dW_m}{dt} + \frac{dW_M}{dt} = 0$

* $\boxed{\text{Constraint forces do no work.}}$

$\Longrightarrow$ We can obtain the case of the fixed wedge by considering $\boxed{M\to\infty}$.

In this case $\ddot{X}\to 0$, $\dot{X}\to 0$ $\quad \lambda \to mg\cos\alpha$

from previous expressions

$$\bar{v}_m \to -g\sin\alpha[\cos\alpha\,\hat{\imath} + \sin\alpha\,\hat{\jmath}]$$

$$E \to \tfrac{1}{2}m\dot{x}'^2 + mgx'\sin\alpha$$