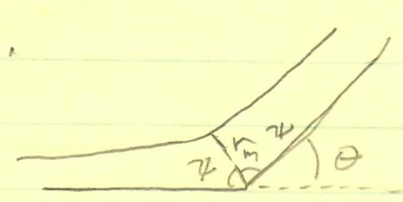
Hwk #4

(3.31) Calculate $\sigma(\theta)d\theta$ for $f = \frac{k}{r^3}$.

First need to calculate the integral



$$\psi = \int_0^{u_m} \frac{s\,du}{\sqrt{1 - \frac{V}{E} - s^2u^2}} \quad \text{where } u = 1/r$$

because $V = \frac{k}{2r^2} = \frac{k}{2}u^2$ then $\psi = \int_0^{u_m} \frac{s\,du}{\sqrt{1 - \frac{ku^2}{2E} - s^2u^2}}$

We can make this look a little nicer if we find the expression for U_m. Using the energy

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V$$

$\dot{r}$: 0 at r_m $\qquad E = \frac{1}{2}m\frac{1}{U_m^2}\dot{\theta}^2 + \frac{k}{2}U_m^2$

Lets eliminate $\dot{\theta}$:

$(E = \frac{1}{2}mv_0^2 \text{ also})$

angular momentum $l = mr_m^2\dot{\theta}$ but $l = mv_0 s = s\sqrt{2Em}$

in E: $\qquad \Rightarrow \dot{\theta} = \frac{s}{r^2}\sqrt{\frac{2E}{m}}$

$$E = \frac{1}{2}m\frac{1}{U_m^2}s^2U_m^4\frac{2E}{m} + \frac{k}{2}U_m^2 = s^2U_m^2E + \frac{k}{2}U_m^2$$

$\Rightarrow \boxed{\frac{1}{U_m^2} = s^2 + \frac{k}{2E}}$ put back in ψ

$$\psi = \int_0^{U_m} \frac{s\,du}{\sqrt{1 - \frac{ku^2}{2E} - u^2\left(\frac{1}{U_m^2} - \frac{k}{2E}\right)}} = \int_0^{U_m} \frac{U_m s\,du}{\sqrt{U_m^2 - u^2}}$$

Changing variables to $u = U_m\sin\varphi$: $du = U_m\cos\varphi\,d\varphi$

$$\boxed{\psi = U_m s\int d\varphi = U_m s\,\varphi = U_m s\,\sin^{-1}\frac{u}{U_m}\Big|_0^{U_m} = \frac{\pi}{2}U_m s}$$

The angle that we want is θ, and

$$\theta = \pi - 2\psi = \pi - 2\left(\frac{\pi}{2}\right)U_m s = \pi - \pi U_m s \quad \text{Use } x = \theta/\pi$$

$$x = 1 - U_m s = 1 - \frac{s}{\sqrt{s^2 + \frac{k}{2E}}}$$

Lets invert this $s^2 = (1-x)^2\left(s^2 + \frac{k}{2E}\right) = s^2(1-x)^2 + \frac{k}{2E}(1-x)^2$

$$S^2 = \frac{\frac{k}{2E}(1-x)^2}{1-(1-x)^2} = \frac{k}{2E}\,\frac{(1-x)^2}{x(2-x)} \qquad \boxed{S = \sqrt{\frac{k}{2E}}\;\frac{(1-x)}{\sqrt{x(2-x)}}}$$

Differentiating

$$\sqrt{\frac{2E}{k}}\cdot\frac{dS}{dx} = \frac{-\sqrt{x(2-x)} - (1-x)\frac{1}{2}[x(2-x)]^{-1/2}(-x+2-x)}{x(2-x)}$$

$$= \frac{1}{x(2-x)\sqrt{x(2-x)}}\left[-x(2-x) - \frac{1}{2}(1-x)(2-2x)\right]$$

$$= \frac{-1}{[x(2-x)]^{3/2}}$$

Changing back to θ:

$$\frac{S}{\sin\theta}\left|\frac{dS}{d\theta}\right| = \frac{k}{2E}\,\frac{(1-x)}{\sqrt{x(2-x)}}\,\frac{1}{\sin\pi x}\cdot\frac{1}{\pi}\cdot\frac{1}{[x(2-x)]^{3/2}}$$

$$\sigma(\theta) = \frac{k}{2E}\,\frac{1}{\sin\pi x}\,\frac{(1-x)}{[x(2-x)]^2}\cdot\frac{1}{\pi}$$

$$\boxed{\sigma(\theta)\,d\theta = \frac{k}{2E}\cdot\frac{1}{\sin\pi x}\cdot\frac{(1-x)}{x^2(2-x)^2}\cdot dx}$$

(4.15) We want $\bar{\omega}$ in Euler angles.

⊛ One way to do this is to follow Goldstein section 4.9 and argue the <u>reverse</u> operations that lead to equation (4.87).

⊛ Another way is to consider equation (4.87) that gives $\bar{\omega}'$ and obtain $\bar{\omega}$ by inverse transformation

$$\bar{\omega}' = A^{-1}\bar{\omega} = A^{T}\bar{\omega}$$

Using the notation $\cos\varphi \to c\varphi$, $\sin\varphi \to s\varphi$ [shorthand notation] then from equation (4.47)

$$A^{-1} = \begin{pmatrix} c\psi c\varphi - c\theta s\varphi s\psi & -s\psi c\varphi - c\theta s\varphi c\psi & s\theta s\varphi \\ c\psi s\varphi + c\theta c\varphi s\psi & -s\psi s\varphi + c\theta c\varphi c\psi & -s\theta c\varphi \\ s\psi s\theta & c\psi s\theta & c\theta \end{pmatrix}$$

Each component separately substituting equation (4.87)

$$\omega_x = (c\psi c\varphi - c\theta s\varphi s\psi)\,\omega_x' - (s\psi c\varphi + c\theta s\varphi c\psi)\,\omega_y' + s\theta s\varphi\,\omega_z'$$

$$= (c\psi c\varphi - c\theta s\varphi s\psi)(\dot{\varphi} s\psi s\theta + \dot{\theta} c\psi) - (s\psi c\varphi + c\theta s\varphi c\psi)(\dot{\varphi} c\psi s\theta - \dot{\theta} s\psi)$$

$$+ s\theta s\varphi(\dot{\varphi} c\theta + \dot{\psi})$$

$$= \dot{\varphi}[\cancel{c\psi c\varphi s\psi s\theta} - c\theta s\varphi s^2\psi s\theta \cancel{- s\psi c\varphi c\psi s\theta} - c\theta s\varphi c^2\psi s\theta + s\theta s\varphi c\theta]$$

$$+ \dot{\theta}[c^2\psi c\varphi - c\theta s\varphi \cancel{s\psi c\psi} + s^2\psi c\varphi + c\theta s\varphi \cancel{c\psi s\psi}]$$

$$+ \dot{\psi} s\theta s\varphi$$

$$= \dot{\varphi}[-c\theta \cancel{s\varphi} s\theta + s\theta \cancel{s\varphi} c\theta] + \dot{\theta} c\varphi + \dot{\psi} s\theta s\varphi$$

$$\boxed{\omega_x = \dot{\theta}\cos\varphi + \dot{\psi}\sin\theta\sin\varphi}$$

Now for y:

$$\omega_y = (c\psi s\varphi + c\theta c\varphi s\psi)\omega_x' + (-s\psi s\varphi + c\theta c\varphi c\psi)\,\omega_y' - s\theta c\varphi\,\omega_z'$$

$$= (c\psi s\varphi + c\theta c\varphi s\psi)(\dot{\varphi} s\theta s\psi + \dot{\theta} c\psi) + (-s\psi s\varphi + c\theta c\varphi c\psi)(\dot{\varphi} c\psi s\theta - \dot{\theta} s\psi)$$

$$- s\theta c\varphi(\dot{\varphi} c\theta + \dot{\psi})$$

$$\omega_y = \dot\varphi\left[c\psi s\varphi \cancel{s\theta s\psi} + c\theta c\varphi s^2\psi s\theta - \cancel{s\psi s\varphi c\psi s\theta} + c\theta c\varphi c^2\psi s\theta - s\theta c\varphi c\theta\right]$$
$$+ \dot\theta\left[c^2\psi s\varphi + \cancel{c\theta c\varphi s\psi c\psi} + s^2\psi s\varphi - \cancel{c\theta c\varphi c\psi s\psi}\right]$$
$$+ \dot\psi\left[-s\theta c\varphi\right]$$
$$= \dot\varphi\left[\cancel{c\theta c\varphi s\theta - s\theta c\varphi c\theta}\right] + \dot\theta\left[s\varphi\right] - s\theta c\varphi\,\dot\psi$$

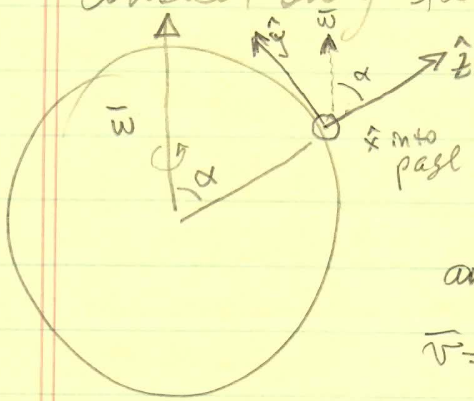
$$\boxed{\omega_y = \dot\theta \sin\varphi - \dot\psi \sin\theta \cos\varphi}$$

and finally

$$\omega_z = (s\psi s\theta)\omega_x' + c\psi s\theta\,\omega_y' + c\theta\,\omega_z'$$
$$= s\psi s\theta(\dot\varphi s\psi s\theta + \dot\theta c\psi) + c\psi s\theta(\dot\varphi c\psi s\theta - \dot\theta s\psi)$$
$$+ c\theta(\dot\varphi c\theta + \dot\psi)$$
$$= \dot\varphi\left[s^2\psi s^2\theta + c^2\psi s^2\theta + c^2\theta\right] + \dot\theta\left[\cancel{s\psi s\theta c\psi - c\psi s\theta s\psi}\right]$$
$$+ \dot\psi\, c\theta$$
$$= \dot\varphi\left[s^2\theta + c^2\theta\right] + \dot\psi\, c\theta$$

$$\boxed{\omega_z = \dot\psi \cos\theta + \dot\varphi}$$

(4.21) Consider only the Coriolis term on the force.



In Cartesian coordinates

$$\bar{\omega} = \omega_y \hat{y} + \omega_z \hat{z}$$
$$= \omega \sin\alpha \, \hat{y} + \omega \cos\alpha \, \hat{z}$$

and velocity is all in the vertical

$$\bar{v} = v(t)\,\hat{z}$$

Force relative to earth $\bar{F} = -mg\,\hat{z} - 2m\,\bar{\omega}\times\bar{v}$

$$= -mg\,\hat{z} - 2m\,\omega \sin\alpha \, v(t)\,\hat{x}$$

the Coriolis deflection only happens on the x-axis

$$m\ddot{x} = -2m\omega \sin\alpha \, v(t)$$

the movement in z is given by

$$m\ddot{z} = -mg \quad \Rightarrow \quad v = -gt + v_0$$

$$\ddot{x} = -2\omega \sin\alpha\,(-gt + v_0)$$

$$\dot{x} = -2\omega \sin\alpha\,(-gt^2/2 + v_0 t) \qquad \dot{x}(0) = 0$$

$$x(t) = -2\omega \sin\alpha\,(-gt^3/6 + v_0 t^2/2) \qquad x(0) = 0$$

⊛ Case1: $v_0 \hat{z}$, $z(0) = 0$: $z(t) = -g t^2/2 + v_0 t$

time of flight given by roots of $z(t) = 0$, $\boxed{t = \frac{2v_0}{g}}$

$$\boxed{x_1(tof)} = -2\omega \sin\alpha\left(-\frac{g}{6}\frac{8v_0^3}{g^3} + \frac{v_0}{2}\frac{4v_0^2}{g^2}\right) = \boxed{-\frac{4}{3}\omega \sin\alpha \frac{v_0^3}{g^2}}$$

⊛ Case2: $v_0 = 0$, $z(0) = z_0$: $z(t) = -gt^2/2 + z_0$; $t = \sqrt{\frac{2z_0}{g}}$

but z_0 is z_{max} from Case1, $\frac{dz}{dt} = 0 = -gt + v_0 \rightarrow \boxed{t = v_0/g}$

$$\boxed{x_2(tof)} = -2\omega \sin\alpha\left(-\frac{g}{6}\frac{v_0^3}{g^3} + \frac{\cancel{v_0} t^2}{2}\right) = \boxed{+\frac{1}{3}\omega \sin\alpha \frac{v_0^3}{g^2}}$$

So that $\boxed{x_1 = -4x_2}$