Euler's Eqns of Motion

We are going to explore the equations of motion for a rotating rigid body.

- Freely rotating body - easier to use Euler's equations.
- Rigid body rotating with one point fixed - easier with Lagrange's equations.

We will work with the body set axes, with axes coinciding with the principal axes.

$$\bar{L} = (\lambda_1 \omega_1, \lambda_2 \omega_2, \lambda_3 \omega_3)$$

Use Newton's 2nd law

$$\left(\frac{d\bar{L}}{dt}\right)_s = \bar{N}$$

$$= \left(\frac{d\bar{L}}{dt}\right)_r + \bar{\omega} \times \bar{L}$$

$$\left(\frac{d\bar{L}}{dt}\right)_r + \bar{\omega} \times \bar{L} = \bar{N}$$

Lets express everything in terms of the body set axes with principal axes as the coordinate axes.

Principal axes direction $\hat{e}_1, \hat{e}_2, \hat{e}_3$

$$\omega_i = \bar{\omega} \cdot \hat{e}_i \qquad L_i = \bar{L} \cdot \hat{e}_i \qquad N_i = \bar{N} \cdot \hat{e}_i$$

$$\left(\frac{d\bar{L}}{dt}\right)_r = \left(\frac{d}{dt} I \bar{\omega}\right)_r = I_i \dot{\omega}_i \hat{e}_i$$

$$\boxed{\begin{aligned} I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3) &= N_1 \\ I_2 \dot{\omega}_2 - \omega_3 \omega_1 (I_3 - I_1) &= N_2 \\ I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) &= N_3 \end{aligned}}$$

Euler's eqns of motion for a rigid body w/ one point fixed.

Problems with these equations:

1. omega_i and N_i are projected into the body set axes.
   They could be functions of time, while in the space axes they may not.
2. solution only gives what an observer in the body set sees.
   For the inertial system soln, further transformations are necessary.

Only really good for a couple of cases:

1. principal axes are partially constrained (reduced # of degrees of freedom)
2. torque-free motion

Torque-free motion

$$I_1\dot{\omega}_1 = \omega_2\omega_3(I_2 - I_3)$$
$$I_2\dot{\omega}_2 = \omega_3\omega_1(I_3 - I_1)$$
$$I_3\dot{\omega}_3 = \omega_1\omega_2(I_1 - I_2)$$

(i) initial rotation is around a principal axis

$$t=0, \quad \omega_1 = \omega_2 = 0 \quad \omega_3 \neq 0$$

$$\left.\begin{array}{l} I\dot{\omega}_1 = 0 \\ I\dot{\omega}_2 = 0 \\ I\dot{\omega}_3 = 0 \end{array}\right\} \begin{array}{l} \omega_1 = c \Rightarrow \omega_1(t) = 0 \\ \omega_2 = d \Rightarrow \omega_2(t) = 0 \\ \omega_3 = e \Rightarrow \omega_3(t) = \text{const} \end{array}$$

$$\bar{L} = I_3\omega_3\,\hat{e}_3 = \text{const} \qquad \bar{\omega} = \text{const}$$

L is then constant in both, body set and space set axes.

(ii) rotation is not around a principal axis

$$t=0 \quad \omega_1 \neq 0, \ \omega_2 \neq 0, \ \omega_3 = 0$$

$$I_1\dot{\omega}_1 = 0$$
$$I_2\dot{\omega}_2 = 0$$
$$I_3\dot{\omega}_3 = \omega_1\omega_2(I_1 - I_2)$$

$\dot{\omega}_3 \neq 0$

$t > 0 \quad \omega_3 \neq \text{const}$

$\omega_1(t) \neq 0$

$\omega_2(t) \neq 0$

$L = I\vec{\omega}$ $\Longleftarrow$ $\vec{\omega}$ not constant

Question: is (i) stable? Submit it to a small perturbation.

$$\dot{\omega}_3 \sim \omega_1\omega_2 \ll 1 \Rightarrow \omega_3 \sim \text{const}$$

$$I_1\dot{\omega}_1 = [(I_2 - I_3)\,\omega_3]\,\omega_2$$ [ ] ~ const

$$I_2\dot{\omega}_2 = [(I_3 - I_1)\,\omega_3]\,\omega_1$$

$$\ddot{\omega}_1 = \left[\frac{I_2 - I_3}{I_1}\,\omega_3\right]\dot{\omega}_2$$

$$\ddot{\omega}_1 = -\left[\frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}\,\omega_3^2\right]\omega_1$$

similar equation for ω_2

only if $I_3 > I_1, I_2$

or $I_3 < I_1, I_2$

oscillatory solutions

omega_3 is softly perturbed by smooth oscillatory rocking of omega_1 and omega_2

motion is stable

if $I_1 < I_3 < I_2$ v.v.

I_3 is intermediate

exponential solutions for omega_2, omega_1

they will push away $\vec{\omega}$ from $\hat{e}_3$

motion is unstable

Special case: two equal principal moments $I_1 = I_2$

$$\begin{cases} I_1 \dot{\omega}_1 = \omega_2 \omega_3 (I_1 - I_3) \\ I_1 \dot{\omega}_2 = \omega_3 \omega_1 (I_3 - I_1) \end{cases}$$

$$I_1 \dot{\omega}_3 = 0 \longrightarrow \omega_3 = \text{const}$$

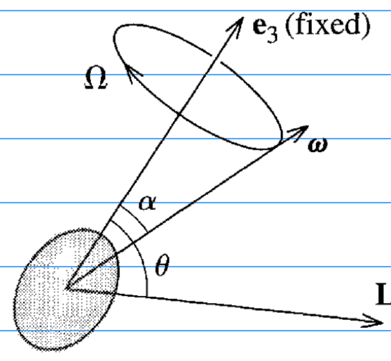
$$\dot{\omega}_1 = \left[\frac{I_1 - I_3}{I_1} \omega_3 \right] \omega_2 = \Omega \omega_2$$

$$\dot{\omega}_2 = -\left[\frac{I_1 - I_3}{I_1} \omega_3 \right] \omega_1 = -\Omega \omega_1$$

$$\ddot{\omega}_1 = -\Omega^2 \omega_1 \longrightarrow \boxed{\omega_1 = A \cos \Omega t}$$

$$\dot{\omega}_2 = -\Omega A \cos \Omega t \longrightarrow \boxed{\omega_2 = -A \sin \Omega t}$$

$$\boxed{\bar{\omega} = \begin{pmatrix} A \cos \Omega t \\ -A \sin \Omega t \\ \omega_3 \end{pmatrix}}$$



$A, \Omega, \omega_3 \to \text{const}$

$\Rightarrow \alpha \to \text{const}$

$$\bar{L} = I \bar{\omega} = \begin{pmatrix} I_1 A \cos \Omega t \\ -I_1 A \sin \Omega t \\ I_3 \omega_3 \end{pmatrix}$$

$$(\bar{\omega} \times \hat{e}_3) = (-A\sin\Omega t, -A\cos\Omega t, 0)$$

$$\bar{L} \times \hat{e}_3 = (-I_1 A\sin\Omega t, -I_1 A\cos\Omega t, 0)$$

$$(\bar{\omega}\times\hat{e}_3)\times(\hat{L}\times\hat{e}_3) = 0$$

Earth is an example of this particular case. We know that I_3 > I_1

$$\frac{I_3 - I_1}{I_1} = 0.00327 \Rightarrow \Omega \simeq \frac{\omega_3}{305.81}$$

$$\omega_3 \sim 1\text{rev}/\text{day} \qquad = \frac{1\text{rev}}{306\text{days}}$$

This has beenn roughly observed and called the Chandler wooble (~400 days).

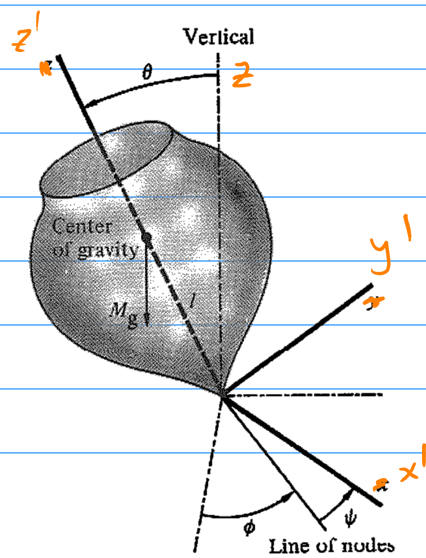
This is a separate phenomena than the precession of the equinoxes, that arises from the small torque produced by the Sun and Moon, with a period of about 26, 000 yrs

The constant 'A' can be determined from the KE

$$T = \frac{1}{2}I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2}I_3\omega_3^2 = \frac{1}{2}I_1 A^2 + \frac{1}{2}I_3\omega_3^2$$

$$A = \sqrt{\frac{2}{I_1}\left(T - \frac{1}{2}I_3\omega_3^2\right)}$$

The spinning top with one point fixed



The rate of change of the Euler angles mean:

$\dot{\psi}$ - spinning of the top around z'

$\dot{\varphi}$ - precession around the vertical

$\dot{\theta}$ - nutation (bobbing up and down motion)

Using the body set axes and the components of the angular velocity derived for these axes, from chapter 4 we have:

$$\omega_{x'} = \dot{\varphi} \sin\psi \sin\theta + \dot{\theta} \cos\psi \qquad \omega_{x'} = \omega_1$$

$$\omega_{y'} = \dot{\varphi} \cos\psi \sin\theta - \dot{\theta} \sin\psi \qquad \omega_{y'} = \omega_2$$

$$\omega_{z'} = \dot{\varphi} \cos\theta + \dot{\psi} \qquad \omega_{z'} = \omega_3$$

$$T = \frac{1}{2} I_1 \left(\omega_{x'}^2 + \omega_{y'}^2\right) + \frac{1}{2} I_3 \omega_{z'}^2$$