Small Oscillations

Based on the assumption of coupled oscillators.

We define a system to be in equilibrium as that where the generalized forces,

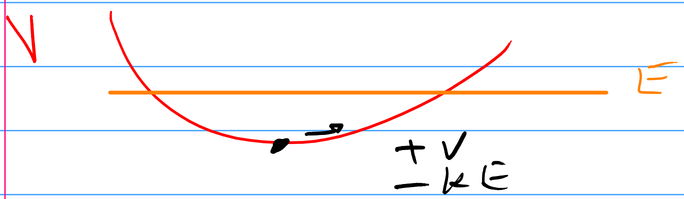
$$Q_i = -\left(\frac{\partial V}{\partial q_i}\right)_0 = 0$$

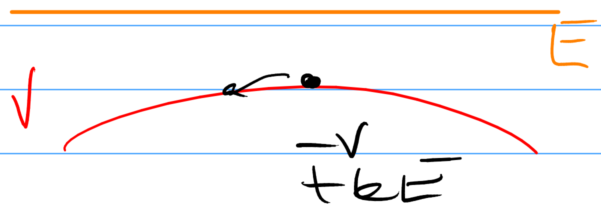
potential energy has an extremum at equilibrium

$$q_{01},\ q_{02},\ q_{03} \ldots$$

Things taken for granted (from previous courses):

1. a system that is initially at equil. with zero initial velocities will stay at equil.
2. "stable" equil. position is that which under small perturbations it results in small bounded motion around the equil. position.
   == e.g. pendulum at rest
3. "unstable" equil. position ... small (infinitesimal) disturbances produces unbounded motion
   == e.g. egg on its tip
4. when extremum of 'V' is a mimum, the equilibrium is "stable"





Assume small displacements away from equil. Do Taylor expansion, and throw away everything that is higher order.

$$q_i = q_{0i} + \eta_i$$

deviation from equil

$$V(q_1 \ldots q_n) = \underbrace{V(q_{01} \ldots q_{0n})}_{\text{shift potential by cnst} \Rightarrow 0} + \underbrace{\cancel{\left(\frac{\partial V}{\partial q_i}\right)_0}}_{0} \eta_i + \frac{1}{2}\left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 \eta_i \eta_j + \ldots$$

$$\boxed{V = \frac{1}{2}\left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 \eta_i \eta_j = \frac{1}{2} \underbrace{V_{ij}}_{\text{constants}} \eta_i \eta_j}$$

in practice you can use the expansion directly rather than calculating the derivatives.

Note that Vij is symmetric $V_{ij} = V_{ji}$

Lets look at the KE. (look into chapter 1)

$$T = \cancel{M}_0 + M_j \cancel{\dot{q}}_j + \frac{1}{2} M_{jk} \dot{q}_j \dot{q}_k$$

0 0

b/c coordinates are not dependent explicitly on time

$q_i \to \eta_i$

$$M_{jk} = m_i \frac{\partial r_i}{\partial q_j} \cdot \frac{\partial r_i}{\partial q_k}$$

functions of coordinates

$$M_{jk}(q_1 \ldots q_n) = \underbrace{M_{jk}(q_{01} \ldots q_{0n})}_{\equiv T_{jk}} + \underbrace{\left(\frac{\partial M_{jk}}{\partial q_i}\right)_0 \eta_i + \cdots}_{\text{drop}}$$

$$\boxed{T = \frac{1}{2} T_{ij} \dot{\eta}_i \dot{\eta}_j}$$

$T_{ij} = T_{ji}$

Lagrangian $L = \frac{1}{2}\left(T_{ij} \dot{\eta}_i \dot{\eta}_j - V_{ij} \eta_i \eta_j\right)$

$$\boxed{T_{ij} \ddot{\eta}_j + V_{ij} \eta_j = 0}$$

A natural solution is an oscillatory solution

$$\eta_i = C a_i e^{-i\omega t}$$

the 'C' is a complex number.
'a' is assumed real.

$\mathrm{Re}\, \eta_i$ is the solution

by substituting $$\boxed{V_{ij} a_j - \omega^2 T_{ij} a_j = 0} \quad (1)$$

'n' linear homogeneous eqns for the 'a's.
Solution only if the determinant of the coefficients is zero.

$$\begin{vmatrix} V_{11} - \omega^2 T_{11} & V_{12} - \omega^2 T_{12} & \cdots \\ V_{21} - \omega^2 T_{21} & V_{22} - \omega^2 T_{22} & \cdots \\ \vdots & & \end{vmatrix} = 0$$

Produces an n-degree polynomial whose roots give the omegas.
For each omega, the 'a's can be found.
Notice that only 'n-1' coefficient 'a's can be determined.

$\lambda = \omega^2$

(1) can be rewritten in matrix notation

$$V \bar{a} = \lambda T \bar{a}$$

Not your typical ordinary eigenvalue problem since 'V' gives not a number but a number times the result of 'T' acting on 'a'.

Assume without proof:
1. eigenvalues 'lambda' are all real positive
2. eigenvectors 'a' are also real and orthogonal

To be shown below:
1. matrix of eigenvectors 'A' diagonalizes both 'T' and 'V'
   'T' is transformed into the unit matrix
   'V' is transformed into a diagonal matrix with values 'lambda'

Diagonalization of 'T'

k vector $\bar{a}_k$

$$V \bar{a}_k = \lambda_k T \bar{a}_k \quad (2)$$

no sum

$$\bar{a}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}$$

$$\bar{a}_\ell^T V \bar{a}_k = \lambda_k \bar{a}_\ell^T T \bar{a}_k$$

$$(V\bar{a}_\ell)^T = \lambda_\ell (T\bar{a}_\ell)^T$$

$$\bar{a}_\ell^T V = \lambda_\ell \bar{a}_\ell^T T$$

$$\bar{a}_\ell^T V \bar{a}_k = \lambda_\ell \bar{a}_\ell^T T \bar{a}_k$$

$$(\lambda_k - \lambda_\ell)\, \bar{a}_\ell^T T \bar{a}_k = 0$$

If all eigenvalues are different,

$$\bar{a}_\ell^T T \bar{a}_k = 0 \qquad \ell \neq k$$

$$A = (\bar{a}_1, \bar{a}_2 \ldots) \qquad \bar{a}_\ell^T T \bar{a}_k = 1 \qquad \ell = k$$

$$(3) \quad \boxed{A^T T A = 1}$$

along with (1), these completely determine the 'a's

In chapter 4 we had a similarity transformation $C' = BCB^{-1}$

We define now a "congruence" transformation as $C' = A^T C A$

If 'A' is orthogonal, there is no difference between the two, using $A^{-1} = B$

So, we can say that in (3), 'A' transforms 'T' by a congruence transformation into a diagonal (unit) matrix.

Diagonalization of 'V'

Define a diagonal matrix λ $\qquad \lambda_{ik} = \lambda_k \delta_{ik}$

then in (2)

$$V\bar{a}_k = \lambda_k T \bar{a}_k \Rightarrow V_{ij} a_{jk} = T_{ij} a_{j\ell} \lambda_{\ell k}$$

$$\Rightarrow \quad VA = TA\lambda$$

$$A^T VA = A^T TA\lambda = \lambda$$

$$\boxed{A^T VA = \lambda}$$

Thus, a congruence transformation of 'V' by 'A' changes it into a diagonal matrix with elements being the eigenvalues λ_k

Multiple roots

Lets consider the case of double roots.

For multiple roots, there are less eqns than variables, so we have freedom to choose arbitrary eigenvectors.

If 'lambda' is a double root, any two of the components of 'a_i' may be freely chosen.

However, we want the freely-chosen 'a_i' to still be orthogonal.

a_k', a_l' are two allowable eigenvectors for a given double root λ

also normalized according to $a_k^T T a_k = 1$

Linear combination of a_k', a_l' is also an eigenvector for the root λ

$$a_l = c_1 a_k' + c_2 a_l'$$

$$a_l^T T a_k' = 0 = c_1 a_k'^T T a_k' + c_2 a_l'^T T a_k'$$

$$0 = c_1 + c_2 a_l'^T T a_k'$$

$$\Rightarrow \frac{c_1}{c_2} = -a_l'^T T a_k' \equiv -\tau_l \qquad (4)$$

$$a_l^T T a_l = 1 = (c_1 a_k' + c_2 a_l')^T T (c_1 a_k' + c_2 a_l')$$

$$= c_1^2 + c_1 c_2 a_k'^T T a_l' + c_2 c_1 a_l'^T T a_k' + c_2^2$$

$$1 = c_1^2 + c_2^2 + 2 c_1 c_2 \tau_l \qquad (5)$$

Together, (4) and (5) fix the constants 'c_1' and 'c_2', thus completely specifies 'a_l'

a_l , $a_k = a_k'$