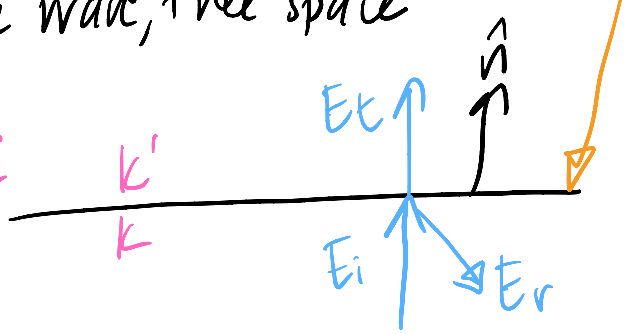
HW 2b - 7.4

plane wave, free space

non-permeable medium w/ σ and ε

$\mu = \mu_0$ here

$k = \frac{\omega}{c}$



a) find amplitude and phase of reflected wave (relative to incident)

boundary conditions:

$E_i + E_r = E_t$ $\quad\quad kE_i - kE_r = k'E_t = k(E_i + E_r)$

so $E_r = E_t - E_i = \frac{k}{k'}E_i - \frac{k}{k'}E_r - E_i$

$E_r + \frac{k}{k'}E_r = \frac{k}{k'}E_i - E_i \quad \Rightarrow \quad \frac{E_r}{E_i} = \frac{k-k'}{k+k'}$

$k'E_r + kE_r = kE_i - k'E_i$

now need to find k'

since k is just $\omega/c = \frac{\omega}{\sqrt{\mu_0 \varepsilon_0}}$

$c = \frac{1}{\sqrt{\mu\varepsilon}}$

so $k' = \frac{\omega}{c} = \omega\sqrt{\mu_0\varepsilon}$

use Maxwell's equations:

$\nabla \times E - i\omega\mu_0 H = 0$

$H = \frac{1}{\mu_0 \omega}\hat{n} \times \frac{\partial E}{\partial \xi}$

$\nabla \times H = J = \sigma E$

for the conducting surface, $\varepsilon = \varepsilon + \frac{i\sigma}{\omega}$

$k' = \omega\sqrt{\mu_0\varepsilon + \mu_0 \frac{i\sigma}{\omega}}$

$$\frac{E_r}{E_i} = \frac{\frac{\omega^2}{\sqrt{\mu_0\varepsilon_0}} - \omega\sqrt{\mu_0\varepsilon + \mu_0 i\sigma/\omega}}{\frac{\omega^2}{\sqrt{\mu_0\varepsilon_0}} + \omega\sqrt{\mu_0\varepsilon + \mu_0 i\sigma/\omega}}$$

divide by $\frac{\omega}{\sqrt{\mu_0\varepsilon_0}}$

$$\frac{E_r}{E_i} = \frac{1-\sqrt{\frac{\varepsilon}{\varepsilon_0}+\frac{i\sigma}{\varepsilon_0\omega}}}{1+\sqrt{\frac{\varepsilon}{\varepsilon_0}+\frac{i\sigma}{\varepsilon_0\omega}}}$$

need to give as amplitude and phase

amplitude $\left|\frac{E_r}{E_i}\right| = \sqrt{\left(\frac{E_r}{E_i}\right)^*\left(\frac{E_r}{E_i}\right)}$

$$= \sqrt{\left(\frac{1-\sqrt{\frac{\varepsilon}{\varepsilon_0}-\frac{i\sigma}{\varepsilon_0\omega}}}{1+\sqrt{\frac{\varepsilon}{\varepsilon_0}-\frac{i\sigma}{\varepsilon_0\omega}}}\right)\left(\frac{1-\sqrt{\frac{\varepsilon}{\varepsilon_0}+\frac{i\sigma}{\varepsilon_0\omega}}}{1+\sqrt{\frac{\varepsilon}{\varepsilon_0}+\frac{i\sigma}{\varepsilon_0\omega}}}\right)}$$

$$\boxed{\left|\frac{E_r}{E_i}\right| = \sqrt{\frac{1-2\sqrt{\frac{\varepsilon}{\varepsilon_0}-\frac{i\sigma}{\varepsilon_0\omega}}+\sqrt{\frac{\varepsilon^2}{\varepsilon_0^2}+\frac{\sigma^2}{\varepsilon_0^2\omega^2}}}{1+2\sqrt{\frac{\varepsilon}{\varepsilon_0}-\frac{i\sigma}{\varepsilon_0\omega}}+\sqrt{\frac{\varepsilon^2}{\varepsilon_0^2}+\frac{\sigma^2}{\varepsilon_0^2\omega^2}}}}}$$

phase of complex number z: $\tan^{-1}\left(-i\frac{(z-z^*)}{(z+z^*)}\right)$

$$\phi = \tan^{-1}\left[-i\,\frac{\left(\frac{1-\sqrt{\frac{\varepsilon}{\varepsilon_0}+\frac{i\sigma}{\varepsilon_0\omega}}}{1+\sqrt{\frac{\varepsilon}{\varepsilon_0}+\frac{i\sigma}{\varepsilon_0\omega}}} - \left(\frac{1-\sqrt{\frac{\varepsilon}{\varepsilon_0}-\frac{i\sigma}{\varepsilon_0\omega}}}{1+\sqrt{\frac{\varepsilon}{\varepsilon_0}-\frac{i\sigma}{\varepsilon_0\omega}}}\right)\right)}{\left(\frac{1-\sqrt{\frac{\varepsilon}{\varepsilon_0}+\frac{i\sigma}{\varepsilon_0\omega}}}{1+\sqrt{\frac{\varepsilon}{\varepsilon_0}+\frac{i\sigma}{\varepsilon_0\omega}}} + \left(\frac{1-\sqrt{\frac{\varepsilon}{\varepsilon_0}-\frac{i\sigma}{\varepsilon_0\omega}}}{1+\sqrt{\frac{\varepsilon}{\varepsilon_0}-\frac{i\sigma}{\varepsilon_0\omega}}}\right)\right)}\right]$$

$$\phi = \tan^{-1}\left[-i \frac{\left(\frac{1-\sqrt{\frac{\varepsilon}{\varepsilon_0}+\frac{i\sigma}{\varepsilon_0\omega}}}{1+\sqrt{\frac{\varepsilon}{\varepsilon_0}+\frac{i\sigma}{\varepsilon_0\omega}}}\right) - \left(\frac{1-\sqrt{\frac{\varepsilon}{\varepsilon_0}-\frac{i\sigma}{\varepsilon_0\omega}}}{1+\sqrt{\frac{\varepsilon}{\varepsilon_0}-\frac{i\sigma}{\varepsilon_0\omega}}}\right)}{\left(\frac{1-\sqrt{\frac{\varepsilon}{\varepsilon_0}+\frac{i\sigma}{\varepsilon_0\omega}}}{1+\sqrt{\frac{\varepsilon}{\varepsilon_0}+\frac{i\sigma}{\varepsilon_0\omega}}}\right) + \left(\frac{1-\sqrt{\frac{\varepsilon}{\varepsilon_0}-\frac{i\sigma}{\varepsilon_0\omega}}}{1+\sqrt{\frac{\varepsilon}{\varepsilon_0}-\frac{i\sigma}{\varepsilon_0\omega}}}\right)} \right]$$

$$\boxed{\phi = \tan^{-1}\left[\frac{2\sqrt{\frac{\varepsilon}{\varepsilon_0}-\frac{i\sigma}{\varepsilon_0\omega}}}{1-\sqrt{\frac{\varepsilon^2}{\varepsilon_0^2}+\frac{\sigma^2}{\varepsilon_0^2\omega^2}}}\right]}$$

b) Discuss limiting cases of poor and good conductor show for latter that $R \sim 1-2\frac{\omega}{c}\delta$.

$\sigma \ll \varepsilon_0\omega$ for very poor conductor: $\frac{\sigma}{\varepsilon_0\omega} \ll 1$

$$\text{so } R = \left|\frac{E_r}{E_i}\right| = \sqrt{\frac{1-2\sqrt{\frac{\varepsilon}{\varepsilon_0}-\cancel{\frac{i\sigma}{\varepsilon_0\omega}}^{\to 0}}+\sqrt{\frac{\varepsilon^2}{\varepsilon_0^2}+\cancel{\frac{\sigma^2}{\varepsilon_0^2\omega^2}}^{\to 0}}}{1+2\sqrt{\frac{\varepsilon}{\varepsilon_0}-\cancel{\frac{i\sigma}{\varepsilon_0\omega}}^{\to 0}}+\sqrt{\frac{\varepsilon^2}{\varepsilon_0^2}+\cancel{\frac{\sigma^2}{\varepsilon_0^2\omega^2}}^{\to 0}}}}$$

$$= \sqrt{\frac{1-2\sqrt{\frac{\varepsilon}{\varepsilon_0}}+\frac{\varepsilon}{\varepsilon_0}}{1+2\sqrt{\frac{\varepsilon}{\varepsilon_0}}+\frac{\varepsilon}{\varepsilon_0}}}$$

$$= \frac{1-\sqrt{\frac{\varepsilon}{\varepsilon_0}}}{1+\sqrt{\frac{\varepsilon}{\varepsilon_0}}}$$

then $\boxed{\phi = \tan^{-1}\left(\dfrac{2\sqrt{\frac{\varepsilon}{\varepsilon_0} - \frac{i\sigma}{\varepsilon_0\omega}}}{1-\sqrt{\frac{\varepsilon^2}{\varepsilon_0^2}+\frac{\sigma^2}{\varepsilon_0^2\omega^2}}}\right)}$

let $x = \varepsilon/\varepsilon_0$

$y = \dfrac{\sigma}{\varepsilon_0\omega}$

we know $\sqrt{x+iy} = (x^2+y^2)^{1/4} e^{\frac{i}{2}\tan^{-1}(y/x)}$

$= (x^2+y^2)^{1/4} \sin(\tfrac{1}{2}\tan^{-1}(y/x))$

$x \gg y$ since $y \ll 1$

then can approximate trig functions:

$\tan^{-1}(y/x) \sim y/x$

and $\sin(\frac{1}{2}\tan^{-1}(y/x)) = \sin(\frac{1}{2}(y/x)) = \dfrac{y}{2x}$

so $\sqrt{x+iy} = (x^2+y^2)^{1/4}(y/2x) = \dfrac{2y\sqrt{x}}{x}$

gives $\phi = \tan^{-1}\left(\dfrac{2(2)(\sigma/\varepsilon_0\omega)\sqrt{\varepsilon/\varepsilon_0}(\varepsilon/\varepsilon_0)}{1-\sqrt{\frac{\varepsilon^2}{\varepsilon_0^2}+\frac{\sigma^2}{\varepsilon_0^2\omega^2}}}\right)$

$$\phi = \tan^{-1}\left(\frac{\sqrt{\frac{\varepsilon}{\varepsilon_0}}\frac{\sigma}{\varepsilon_0\omega}}{\frac{\varepsilon}{\varepsilon_0}-1}\right)$$

since we expect ϕ to $= \pi$ as $\sigma \to 0$ we have

$$\boxed{\phi = \pi + \frac{\sqrt{\frac{\varepsilon}{\varepsilon_0}}\frac{\sigma}{\varepsilon_0\omega}}{\frac{\varepsilon}{\varepsilon_0}-1}}$$

now for very good conductor: $\sigma \gg \epsilon_0 \omega$ so $\frac{\sigma}{\epsilon_0 \omega} \gg 1$

and $\frac{\epsilon}{\epsilon_0} \ll 1$

$\frac{\epsilon_0 \omega}{\sigma} \ll 1$

$$\left|\frac{E_r}{E_i}\right| = \sqrt{\frac{1 - 2\sqrt{\frac{\epsilon}{\epsilon_0} - \frac{i\sigma}{\epsilon_0 \omega}} + \sqrt{\frac{\epsilon^2}{\epsilon_0^2} + \frac{\sigma^2}{\epsilon_0^2 \omega^2}}}{1 + 2\sqrt{\frac{\epsilon}{\epsilon_0} - \frac{i\sigma}{\epsilon_0 \omega}} + \sqrt{\frac{\epsilon^2}{\epsilon_0^2} + \frac{\sigma^2}{\epsilon_0^2 \omega^2}}}}$$

$$\Rightarrow \sqrt{\frac{1 - \sqrt{\frac{2\sigma}{\epsilon_0 \omega}} + \frac{\sigma}{\epsilon_0 \omega}}{1 + \sqrt{\frac{2\sigma}{\epsilon_0 \omega}} + \frac{\sigma}{\epsilon_0 \omega}}}$$

drop small terms: $= \sqrt{\frac{1 - \sqrt{\frac{2\epsilon_0 \omega}{\sigma}}}{1 + \sqrt{\frac{2\epsilon_0 \omega}{\sigma}}}}$

binomial expansion:

$$= \frac{1 - \frac{1}{2}\sqrt{\frac{2\epsilon_0 \omega}{\sigma}}}{1 + \frac{1}{2}\sqrt{\frac{2\epsilon_0 \omega}{\sigma}}} = \frac{1 + \frac{\epsilon_0 \omega}{2\sigma} - \sqrt{\frac{2\epsilon_0 \omega}{\sigma}}}{1 - \frac{\epsilon_0 \omega}{2\sigma}}$$

$$= 1 - \sqrt{\frac{2\epsilon_0 \omega}{\sigma}}$$

$$R = \left|\frac{E_r}{E_i}\right|^2 = \left(1 - \sqrt{\frac{2\epsilon_0 \omega}{\sigma}}\right)^2 = 1 + \cancelto{0}{\frac{2\epsilon_0 \omega}{\sigma}} - 2\sqrt{\frac{2\epsilon_0 \omega}{\sigma}}$$

$$\boxed{R = 1 + \frac{2\omega\delta}{c}}$$