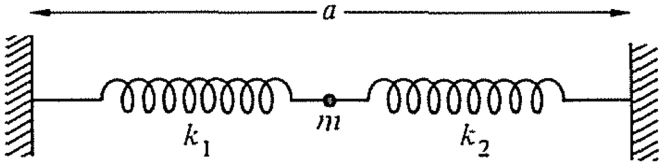
Problems

Solve the following problems.

1. A rigid body consists of three masses fastened as follows: *m* at (*a, 0, 0*), *2m* at (*0, a, a*), and *3m* at (*0, a, -a*).
   - Using $I_{ij}=m_\alpha(\delta_{ij}r_\alpha^2-x_{\alpha i}x_{\alpha j})$ find the moment of inertia tensor $\boldsymbol{I}$.
   - Find the principal moments and a set of orthogonal principal axes.
2. Consider a particle of mass *m* with two degrees of freedom (x_1, x_2) that obeys the Lagrangian given by $L=\frac{1}{2}m(\dot{x}_i\dot{x}_i)-\frac{1}{2}V_{ij}x_ix_j$ where the V_{ij} are constants satisfying $V_{11}>V_{22}>0$ and $V_{12}=V_{21}=0$.
   - Find the normal frequencies and eigenvectors. Make sure the eigenvectors satisfy the corresponding orthogonality condition.
   - Starting from $\eta_i=C_k a_{ik}e^{-i\omega_k t}$ where $\boldsymbol{\eta}=\begin{pmatrix}x_1\\x_2\end{pmatrix}$ derive the explicit solutions (real part of η_i) given initial conditions $x_1(0)=x_2(0)=a$ and $\dot{x}_1(0)=\dot{x}_2(0)=v$.
3. Consider a particle of mass *m* constrained to move on a frictionless cylinder of radius *R*, given by the equation $\rho=R$ in cylindrical coordinates (ρ,ϕ,z). The mass is subjected to an external force $\boldsymbol{F}=-kr\hat{\boldsymbol{r}}$, where *k* is a positive constant, *r* is its distance from the origin, and $\hat{\boldsymbol{r}}$ is the radial unit vector pointing away from the origin.
   - Find the generalized momenta and Hamiltonian.
   - Write down Hamilton's equations of motion.
   - Find general solutions to all the generalized coordinates and describe the motion of each one as a function of time.
4. A particle of mass *m* can move in one dimension under the influence of two springs connected to fixed points a distance *a* apart, as shown in the figure. The springs obey Hooke's law and have zero unstretched lengths and force constants k_1 and k_2, respectively. Using the position of the particle from one fixed point as the generalized coordinate, answer the following questions.



   - Calculate the equilibrium position of the particle by using the extrema of the potential energy.
   - Find the Lagrangian.
   - Find the conjugate momenta and Hamiltonian.
   - Is the Hamiltonian conserved? Is the total energy conserved? Justify your answers.