Conservation and Symmetry

1. $p_j = \frac{\partial L}{\partial \dot{q}_j}$ canonical or conjugate momentum

2. Suppose that 'L' does not contain a given generalized coordinate

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \cancel{\frac{\partial L}{\partial q_j}} = 0$$

$$0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{d}{dt} p_j = 0$$

cyclic coordinate $\boxed{p_j = \text{const}}$

Note that this is more general than the conservation theorems from chp 1.
This one provides for conservation even if Newton's 3rd is not valid.

Suppose that you have a particle in an E&M field where neither the potential or the vector potential depend on 'x'

$$U = q\varphi - q\bar{A}\cdot\bar{v}$$

$$L = \frac{1}{2}mv^2 - q\varphi + q\bar{A}\cdot\bar{v}$$

$$\frac{\partial L}{\partial \dot{x}} = P_x = m\dot{x} + qA_x = \text{const}$$

notice that in this case Newton's 3rd is not valid.

There is a connection between generalized coordinates that are related to translation and rotation and that are cyclic
and the symmetries of the problem

Consider a conservative system $\frac{\partial V}{\partial \dot{q}_j} = 0$

if we shift the system along one of the coordinates, velocities will not change their values in that coordinate

$$\frac{\partial T}{\partial q_j} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} + \frac{\partial V}{\partial q_j} = 0$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} = -\frac{\partial V}{\partial q_j} \quad \text{but} \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} = \dot{P}_j$$

$$\Rightarrow \dot{P}_j = -\frac{\partial V}{\partial q_j} = 0$$

if 'qj' is cyclic $V \neq V(q_j)$

$$\dot{P}_j = 0$$

$$\boxed{P_j = \text{const}}$$

If a generalized displacement or rotation coordinate is cyclic, then translation or rotation has no effect on the problem.

More importantly, for a system that is 'invariant' under translation or rotation, then the corresponding linear or rotational momentum is conserved.

i.e. cyclic $\rightarrow$ conjugate momentum is conserved $\rightarrow$ system is symmetric (invariant) in that coordinate

Ex 1: spherically symmetric problem $\rightarrow$ all components of the ang momentum are conserved

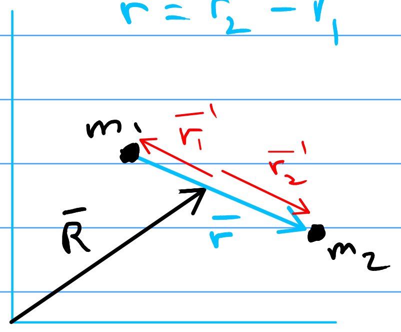
Ex 2: rotationally symmetric pb along the z axis $\rightarrow$ 'Lz' is conserved

Ex 3: planar symmetry in the x-y plane $\longrightarrow$ 'px', 'py', 'Lz' are conserved

The Central Force Problem

How to reduce the problem to a one-body problem

$\bar{r} = \bar{r}_2' - \bar{r}_1'$



$\bar{R} \to cm$

Two masses interation via a potential U.
The system is monogenic.

$$U = U(\bar{r}, \dot{\bar{r}}, \dots)$$

generalized coordinates are the position of CM 'R' and the inter-particle distance 'r'

$$T = T(\dot{\bar{R}}, \dot{\bar{r}})$$

$$T = \frac{1}{2}(m_1 + m_2)\dot{\bar{R}}^2 + T'$$

$$T' = \frac{1}{2} m_1 \dot{\bar{r}}_1'^2 + \frac{1}{2} m_2 \dot{\bar{r}}_2'^2$$

using

$$\left.\begin{aligned} \bar{r} &= \bar{r}_2' - \bar{r}_1' \\ 0 &= m_1 \bar{r}_1' + m_2 \bar{r}_2' \end{aligned}\right\} \quad \bar{r}_1' = -\frac{m_2}{m_1 + m_2}\bar{r}$$

$$\bar{r}_2' = \frac{m_1}{m_1 + m_2}\bar{r}$$

$$T' = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\dot{\bar{r}}^2$$

$$\boxed{L = \frac{1}{2}(m_1 + m_2)\dot{\bar{R}}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\dot{\bar{r}}^2 - U}$$

$\bar{R}$: the three components of 'R' are $\boxed{\text{cyclic}}$ $\longrightarrow$ components of the linear momentum of the CM are conserved!

Solutions for the 'R' are either that it stays at rest or moves uniformly.

Done with 'R'!

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$ reduced mass

Drop the 'R' from the 'L' since it does not appear in any of the 'r' eqns

$$\boxed{L = \frac{1}{2}\mu \dot{\bar{r}}^2 - U(\bar{r}, \dot{\bar{r}}, \ldots)}$$

we have reduce the pb of two masses to an equivalent single mass 'mu'
pb under a potential 'U' with a distance 'r' from the origin

GOAL: obtain eqns of motion in terms of first integrals

Further assumption: consider now 'U' conservative and only a function of 'r'

$$\| \quad U \to V \qquad V = V(r)$$

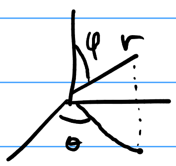
$$\mu \to m$$

Suddenly now this pb is spherically symmetric $\longrightarrow$ angular coordinates are $\boxed{\text{cyclic}}$

$$\bar{L} = \bar{r} \times \bar{p}$$ is conserved

1. This means that 'L' (ang momentum) points always in the same direction.
2. b/c 'r' is perpendicular to 'L', then 'r' will always lie on a plane.

$\longrightarrow$ motion is restricted to a plane



$$\boxed{\varphi = \pi/2}$$

pb is reduce to only two generalized coordinates θ, r

Can write the Lagrangian in polar coordinates as

$$\boxed{L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)}$$

b/c 'theta' is cyclic, then its conjugate momentum is conserved

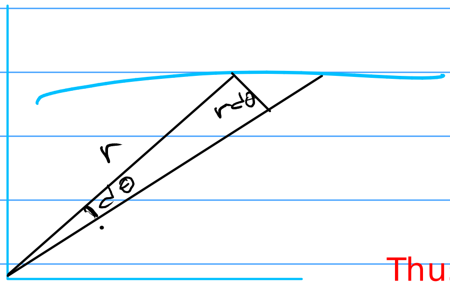
$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = \boxed{m r^2 \dot{\theta} \equiv \ell} \qquad \ell = |\bar{L}|$$

this is our 1st first integral

* Fun fact, this also means

$$\frac{d}{dt}\Big(\underbrace{\tfrac{1}{2} r^2 \dot{\theta}}\Big) = 0$$

area swept out by the radius vector per unit time



$$dA = \frac{1}{2} r (r\, d\theta)$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

Thus, conservation of ang momentum leads to Kepler's 2nd law: equal areas in equal times.

NOTE: this law is independent on the form of 'V'

r:
$$m\ddot{r} - m r \dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$$

$$\boxed{m\ddot{r} - \frac{\ell^2}{m r^3} + \frac{\partial V}{\partial r} = 0} \qquad (a)$$

$$m\ddot{r} = -\frac{d}{dr}\left(V + \frac{1}{2}\cdot\frac{\ell^2}{m r^2}\right) \qquad (b)$$

multiply by $\dot{r}$

(lhs) $$m\ddot{r}\,\dot{r} = \frac{d}{dt}\left(\frac{1}{2} m \dot{r}^2\right)$$

(rhs) $$-\frac{d}{dr}\left(V + \frac{1}{2}\frac{\ell^2}{m r^2}\right)\dot{r} = -\frac{d}{dt}\left(V + \frac{1}{2}\frac{\ell^2}{m r^2}\right)$$

$$\frac{d}{dt}\left(\frac{1}{2} m \dot{r}^2 + \frac{1}{2}\frac{\ell^2}{m r^2} + V\right) = 0$$

$$\boxed{E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2}\frac{\ell}{m r^2} + V} = \text{const}$$

(c)

this is our 2nd first integral

substituting for 'l' $E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$

$$\dot{r} = \sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}$$

$$dt = \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}}$$

$$\boxed{t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}}}$$

$$mr^2\dot{\theta} = l \quad \Rightarrow \quad d\theta = \frac{l}{mr^2}\,dt$$

$$\boxed{\theta = l\int_0^t \frac{dt}{mr^2} + \theta_0}$$

$$r = r(t)$$

Givens of the problem are $r_0,\ \theta_0,\ E,\ l$

equiv $r_0,\ \theta_0,\ \dot{r}_0,\ \dot{\theta}_0$

two 2nd order diff eqns need four initial conditions