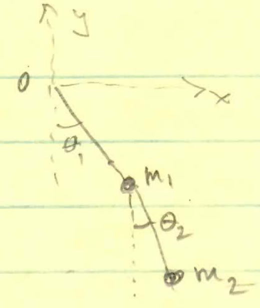
Homework 7

Pb. 1.



Position vectors; small

$\bar{r}_1 = l\sin\theta_1\hat{x} - l\cos\theta_1\hat{y} \simeq l(\theta_1\hat{x} - \hat{y})$

$\bar{r}_2 = (l\sin\theta_1 + l\sin\theta_2)\hat{x} - (l\cos\theta_1 + l\cos\theta_2)\hat{y}$

$\simeq l[(\theta_1+\theta_2)\hat{x} - 2\hat{y}]$

⊛ For the Kinetic energy

$\frac{\partial\bar{r}_1}{\partial t} = \frac{\partial\bar{r}_2}{\partial t} = 0 \Longrightarrow M_0 = M_j = 0$

$\frac{\partial\bar{r}_1}{\partial\theta_1} = l\hat{x} \ : \ \frac{\partial\bar{r}_1}{\partial\theta_2} = 0 \ : \ \frac{\partial\bar{r}_2}{\partial\theta_1} = l\hat{x} \ : \ \frac{\partial\bar{r}_2}{\partial\theta_2} = l\hat{x}$

so

$$T = \frac{1}{2}\left[M_{11}\dot{\theta}_1^2 + 2M_{12}\dot{\theta}_1\dot{\theta}_2 + M_{22}\dot{\theta}_2^2\right]$$

$$M_{11} = m_1\left(\frac{\partial\bar{r}_1}{\partial\theta_1}\right)^2 + m_2\left(\frac{\partial\bar{r}_2}{\partial\theta_1}\right)^2 = m_1 l^2 + m_2 l^2 = l^2(m_1+m_2)$$

$$M_{12} = m_1\frac{\partial\bar{r}_1}{\partial\theta_1}\underbrace{\cancel{\frac{\partial\bar{r}_1}{\partial\theta_2}}}_{0} + m_2\frac{\partial\bar{r}_2}{\partial\theta_1}\frac{\partial\bar{r}_2}{\partial\theta_2} = l^2 m_2$$

$$M_{22} = m_1\Big(\underbrace{\cancel{\frac{\partial\bar{r}_1}{\partial\theta_2}}}_{0}\Big)^2 + m_2\left(\frac{\partial\bar{r}_2}{\partial\theta_2}\right)^2 = m_2 l^2$$

$$T = \frac{l^2}{2}\left[(m_1+m_2)\dot{\theta}_1^2 + 2m_2\dot{\theta}_1\dot{\theta}_2 + m_2\dot{\theta}_2^2\right]$$

$$[T_{ij}] = l^2\begin{pmatrix} m_1+m_2 & m_2 \\ m_2 & m_2 \end{pmatrix}$$

⊛ Potential energy $V = -m_1 g y_1 - m_2 g y_2$

$= -m_1 g l\cos\theta_1 - m_2 g l(\cos\theta_1 + \cos\theta_2)$

(Have to keep higher order in $\cos\theta \simeq 1 - \theta^2/2$) $\simeq -gl\left[m_1(1-\theta_1^2/2) + m_2(2 - \frac{\theta_1^2}{2} - \theta_2^2/2)\right]$

$$V = -gl\left[m_1 + 2m_2 - (m_1+m_2)\frac{\theta_1^2}{2} - m_2\frac{\theta_2^2}{2}\right]$$

$V_{ij} = \frac{\partial^2 V}{\partial\theta_i\partial\theta_j} \Rightarrow V_{11} = \frac{\partial^2 V}{\partial\theta_1^2} = gl(m_1+m_2) : V_{12} = 0 : V_{22} = \frac{\partial^2 V}{\partial\theta_2^2} = glm_2$

$$[V_{ij}] = gl\begin{pmatrix} m_1+m_2 & 0 \\ 0 & m_2 \end{pmatrix}$$

⊛ Eigenfrequencies satisfy

$$|V-\lambda T| = 0 = \begin{vmatrix} g(m_1+m_2)-l(m_1+m_2)\lambda & -l\lambda m_2 \\ -l\lambda m_2 & g m_2 - l\lambda m_2 \end{vmatrix}$$

$$m_2(m_1+m_2)(g-l\lambda)^2 - l^2\lambda^2 m_2^2 = 0$$

$$\lambda^2 l^2 m_1 m_2 - 2gl m_2(m_1+m_2)\lambda + g^2 m_2(m_1+m_2) = 0$$

$$\boxed{\lambda_\pm = \frac{g}{l}\cdot\frac{1}{m_1}\left[(m_1+m_2) \pm \sqrt{m_2(m_1+m_2)}\right]} \text{ where } \lambda = \omega^2$$

⊛ Eigenvectors; use 1st row $\qquad M \equiv m_1+m_2$

$$(m_1+m_2)(g-l\lambda)a_1 - l m_2 \lambda a_2 = 0$$

$$\lambda\frac{a_2}{a_1} = \frac{M(g-l\lambda)}{lm_2} = \frac{gM}{lm_2}\left[-\frac{m_2}{m_1} \mp \frac{1}{m_1}\sqrt{m_2 M}\right] = \frac{g}{l}\sqrt{\frac{M}{m_2}}\left[-\frac{\sqrt{Mm_2}}{m_1} \mp \frac{M}{m_1}\right]$$

$$= \mp\sqrt{\frac{M}{m_2}}\cdot\frac{g}{l}\cdot\frac{1}{m_1}\left[M \pm \sqrt{Mm_2}\right] = \mp\sqrt{\frac{M}{m_2}}\,\lambda$$

$$\boxed{a_2 = \mp a_1\sqrt{\frac{m_1+m_2}{m_2}}} \equiv \mp a\sqrt{\frac{m_1+m_2}{m_2}} \qquad a_1 \equiv a$$

Use normalization $\quad A^T T A = 1 = l^2(a_1\ a_2)\begin{pmatrix} M & m_2 \\ m_2 & m_2\end{pmatrix}\begin{pmatrix} a_1 \\ a_2\end{pmatrix}$

$$1 = l^2(a_1^2 M + 2a_1 a_2 m_2 + a_2^2 m_2) = 2l^2 a^2(M \mp \sqrt{m_2 M})$$

$$a = \frac{1}{\sqrt{2}\,l}\cdot\frac{1}{\sqrt{M\mp\sqrt{m_2 M}}} \qquad \boxed{A_\pm = \begin{pmatrix} a_1 \\ a_2\end{pmatrix} = \frac{1}{\sqrt{2}\,l}\cdot\frac{1}{\sqrt{M\mp\sqrt{m_2M}}}\begin{pmatrix} 1 \\ \mp\sqrt{\frac{M}{m_2}}\end{pmatrix}}$$

Note that for the high frequency mode, the motion of each mass is opposite each other.

⊛ For $m_1 = m_2$ $\quad \lambda = \frac{g}{l}[2\pm\sqrt{2}]$ and $A = \frac{1}{\sqrt{2m}\,l}\cdot\frac{1}{\sqrt{2\mp\sqrt{2}}}\begin{pmatrix} 1 \\ \mp\sqrt{2}\end{pmatrix}$

$\circledast$ Beats

define $A_\pm \equiv K_\mp \begin{pmatrix} 1 \\ \mp\beta \end{pmatrix}$ where $A_+ = \begin{pmatrix} a_{1+} \\ a_{2+} \end{pmatrix}$ and $A_- = \begin{pmatrix} a_{1-} \\ a_{2-} \end{pmatrix}$ & $\beta = \sqrt{\frac{M}{m_2}}$

So that $a_{1+} = K_-$, $a_{2+} = -K_-\beta$, $a_{1-} = K_+$, $a_{2-} = K_+\beta$

Initial conditions: $\eta(0) = \begin{pmatrix} \theta_1(0) \\ \theta_2(0) \end{pmatrix} = \begin{pmatrix} \theta_0 \\ 0 \end{pmatrix}$ & $\dot{\eta}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\omega = \sqrt{\lambda}$

General solution $\eta_i = C_k a_{ik} e^{-i\omega_k t}$

For the coefficients we know that $\operatorname{Re} C_k = a_{jk} T_{jl} \eta_l(0)$

and $\operatorname{Im} C_k = \frac{1}{\omega_k} a_{jk} T_{jl} \dot{\eta}_l(0) = 0$

Each one is:

$$\begin{cases} \operatorname{Re} C_+ = a_{1+} T_{11} \eta_1(0) + a_{1+} T_{12} \cancel{\eta_2(0)} + a_{2+} T_{21} \eta_1(0) + a_{2+} T_{22} \cancel{\eta_2(0)} \\ \qquad = \theta_0 l^2 (a_{1+} M + a_{2+} m_2) \\ \operatorname{Re} C_- = \theta_0 l^2 (a_{1-} M + a_{2-} m_2) \end{cases}$$

$$\Rightarrow \theta_1(t) = \operatorname{Re}\left[C_+ a_{1+} e^{-i\omega_+ t} + C_- a_{1-} e^{-i\omega_- t} \right]$$

$$= (\operatorname{Re} C_+) a_{1+} \cos\omega_+ t + (\operatorname{Re} C_-) a_{2-} \cos\omega_- t$$

$$\theta_2(t) = (\operatorname{Re} C_+) a_{2+} \cos\omega_+ t + (\operatorname{Re} C_-) a_{2-} \cos\omega_- t$$

Now $(\operatorname{Re} C_+) a_{1+} = \theta_0 l^2 K_-^2 (M - \beta m_2) = \theta_0 l^2 \frac{1}{2l^2} \frac{1}{(M - \beta m_2)} (M - \beta m_2) = \theta_0/2$

$$(\operatorname{Re} C_-) a_{1-} = \theta_0 l^2 K_+^2 (M + \beta m_2) = \theta_0/2$$

Likewise $(\operatorname{Re} C_+) a_{2+} = -\frac{\theta_0}{2}\beta$ and $(\operatorname{Re} C_-) a_{2-} = \frac{\theta_0}{2}\beta$

$$\Rightarrow \theta_1(t) = \frac{\theta_0}{2}(\cos\omega_+ t + \cos\omega_- t)$$

$$\theta_2(t) = -\frac{\theta_0}{2}\beta(\cos\omega_+ t - \cos\omega_- t)$$

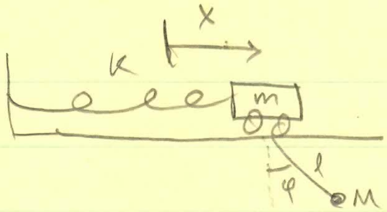
Defining $\bar{\omega} \equiv \frac{1}{2}(\omega_+ + \omega_-)$ and $\Delta\omega \equiv \frac{1}{2}(\omega_+ - \omega_-)$ then $\omega_\pm = \bar{\omega} \pm \Delta\omega$

where $\bar{\omega} > \Delta\omega$. then solutions become

$$\Rightarrow \boxed{\theta_1(t) = \theta_0 \cos\bar{\omega}t \cos\Delta\omega t \quad \text{and} \quad \theta_2(t) = \theta_0 \beta \sin\bar{\omega}t \sin\Delta\omega t}$$

These solutions have oscillations at $\bar{\omega}$ with a periodic amplitude modulation with freq $\Delta\omega$. the modulation is 90° out of phase such that when $\theta_1(t) = 0$ at $t = (2n+1)\pi/2\Delta\omega$, $\theta_2(t)$ is a maximum. these form "beats".

Pb.2.



Two degrees of freedom x, φ

$\bar{r}_1 = x\hat{x}$ and $\bar{r}_2 = (x + l\sin\varphi)\hat{x} - l\cos\varphi\,\hat{y}$

By calculating the M_{jk} and using the small angle approx

$$KE = \frac{1}{2}(m+M)\dot{x}^2 + Ml\dot{x}\dot{\varphi} + \frac{1}{2}Ml^2\dot{\varphi}^2$$

$$\Rightarrow [T_{ij}] = \begin{pmatrix} m+M & Ml \\ Ml & Ml^2 \end{pmatrix}$$

Potential energy: $V \cong \frac{1}{2}kx^2 + \frac{1}{2}Mgl\varphi^2 - Mgl \Rightarrow [V_{ij}] = \begin{pmatrix} k & 0 \\ 0 & Mgl \end{pmatrix}$

Characteristic equation

$$\begin{vmatrix} k - \lambda(m+M) & -\lambda Ml \\ -\lambda Ml & Mgl - \lambda Ml^2 \end{vmatrix} = 0 = [k - \lambda(m+M)][Mgl - \lambda Ml^2] - \lambda^2 M^2 l^2$$

$$\Rightarrow \lambda^2 ml - \lambda[(m+M)g + kl] + kg = 0$$

* roots $\boxed{\lambda_\pm = \frac{1}{2}\left[\left(1+\frac{M}{m}\right)\omega_0^2 + \omega_1^2 \pm \sqrt{\left[\left(1+\frac{M}{m}\right)\omega_0^2 + \omega_1^2\right]^2 - 4\omega_0^2\omega_1^2}\,\right]}$

where $\omega_0^2 \equiv g/l$ and $\omega_1^2 \equiv k/m$ and $\omega_\pm = \sqrt{\lambda_\pm}$

For the eigenvectors, use 1st row

$$[k - \lambda(m+M)]a_1 - \lambda Ml a_2 = 0$$

that have to satisfy $A^T T A = 1 \Rightarrow (m+M)a_1^2 + 2Ml\,a_1 a_2 + Ml^2 a_2^2 = 1$

* Using algebra software $\quad a_{1\pm} = (\lambda_\pm)\sqrt{\frac{M}{B_\pm}}\ ;\ a_{2\pm} = \frac{-1}{Ml}\left(\lambda_\pm(m+M) - k\right)\sqrt{\frac{M}{B_\pm}}$

where $B_\pm \equiv \lambda_\pm^2 m^2 + (\lambda_\pm^2 M - 2\lambda_\pm k)m + k^2$ eigen vectors $\boxed{A_\pm = \begin{pmatrix} a_{1\pm} \\ a_{2\pm} \end{pmatrix}}$

* Take the limit $\frac{M}{m} \ll 1$:

$$\lambda_\pm \cong \frac{1}{2}\left[\omega_0^2 + \omega_1^2 \pm \sqrt{[\omega_0^2 + \omega_1^2]^2 - 4\omega_0^2\omega_1^2}\,\right]$$

$$= \frac{1}{2}\left[\omega_0^2 + \omega_1^2 \pm (\omega_0^2 - \omega_1^2)\right]$$

$\boxed{\begin{matrix} \lambda_+ = \omega_0^2 \\ \lambda_- = \omega_1^2 \end{matrix}}$ } the two oscillators behave as if they were decoupled.

p6.3. $T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2)$ & $V = b x_1 x_2$

$$[T_{ij}] = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \qquad [V_{ij}] = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$$

Characteristic equation $|V - \lambda T| = 0$ $\quad \begin{vmatrix} -\lambda m & b \\ b & -\lambda m \end{vmatrix} = 0$

$$\lambda^2 m^2 - b^2 = 0 \Rightarrow \boxed{\lambda_\pm = \pm b/m}$$

For eigenvectors, solve

$$-\lambda m a_1 + b a_2 = 0 \quad \text{and} \quad a_1^2 m + a_2^2 m = 1$$

$$a_1 = +b \frac{1}{\sqrt{m}\sqrt{\lambda^2 m^2 + b^2}} \quad \text{and} \quad a_2 = \frac{\lambda \sqrt{m}}{\sqrt{\lambda^2 m^2 + b^2}}$$

* For $\lambda_+ = b/m$ $\quad a_1 = \frac{1}{\sqrt{2m}} \quad a_2 = \frac{1}{\sqrt{2m}}$

* For $\lambda_- = -b/m$ $\quad a_1 = \frac{1}{\sqrt{2m}} \quad a_2 = -\frac{1}{\sqrt{2m}}$

$$\boxed{\begin{aligned} A_+ &= \frac{1}{\sqrt{2m}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ A_- &= \frac{1}{\sqrt{2m}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}}$$

* General Solution $\eta_i = c_k a_{ik} e^{-i\omega_k t}$

$$x_1(t) = c_+ a_{1+} e^{-i\omega_+ t} + c_- a_{1-} e^{-i\omega_- t}$$

$$x_2(t) = c_+ a_{2+} e^{-i\omega_+ t} + c_- a_{2-} e^{-i\omega_- t}$$

$\gamma = \sqrt{b/m}$

Now $\omega_\pm = \sqrt{\lambda_\pm} = \sqrt{\pm b/m} \to \omega_+ = \gamma, \ \omega_- = i\gamma$

$$\boxed{\begin{aligned} x_1(t) &= \frac{1}{\sqrt{2m}}\left[c_+ e^{-i\gamma t} + c_- e^{\gamma t}\right] \\ x_2(t) &= \frac{1}{\sqrt{2m}}\left[c_+ e^{-i\gamma t} - c_- e^{\gamma t}\right] \end{aligned}}$$

Solutions are the real part of these expressions.

* Find normal coordinates from $\zeta = A^{-1}\eta$

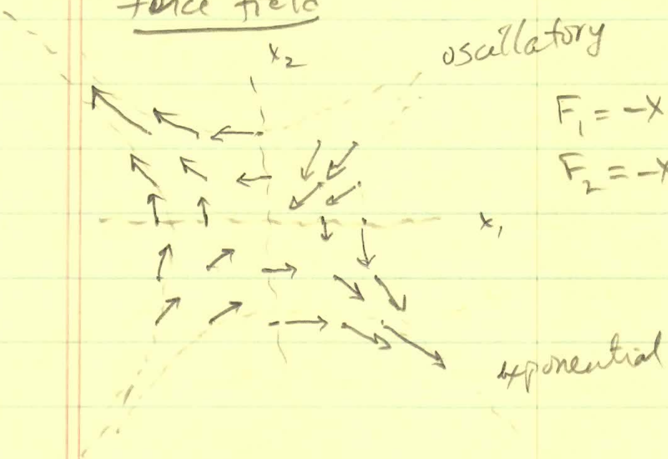
$$A = \frac{1}{\sqrt{2m}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow A^{-1} = \sqrt{\frac{m}{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \sqrt{\frac{m}{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \sqrt{\frac{m}{2}}\begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$$

$$\Rightarrow \boxed{\zeta_1(t) = c_+ e^{-i\gamma t} \qquad \zeta_2(t) = c_- e^{\gamma t}}$$

oscillatory $\qquad$ exponential growth

the solution can be understood by looking at the Force field



$F_1 = -x_2$
$F_2 = -x_1$ } forces $\vec{F} = (-x_2, -x_1)$

the particle will perform oscillatory motion in the $x_1 = x_2$ line and go to infinity on the $x_1 = -x_2$ line. But the purely osillatory motion is unstable as any perturbation away from $x_1 = -x_2$ will push the particle to $\pm\infty$.

Note that the normal coordinates are a 45° rotation of the $x_1 - x_2$ axis. the A matrix rotates the coordinates