Recap 1: Cauchy and Integrals

- Analytic function $f(z) = u(x,y) + iv(x,y)$ must have

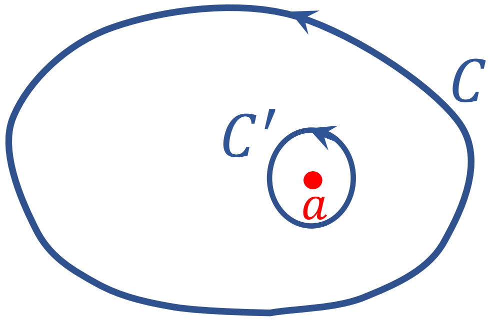
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Cauchy-Riemann conditions

- Then

$$\oint_C f(z)dz = 0$$

Cauchy's theorem



and

$$I = \oint_C \frac{dz}{z-a} = \begin{cases} 2\pi i, & a \text{ lies within } C \\ 0, & \text{otherwise} \end{cases}$$

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{z-a}$$

Cauchy integral formula

Recap 2: Series Expansions

- Taylor series

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

$f(z)$ analytic for $|z-a| < R$

expect a singularity at some point z with $|z-a| = R$

e.g. $f(z) = \frac{1}{z-1}$
Radius of convergence of the Taylor series expansion about $z = 0$ is R =1.

- Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

$f(z)$ analytic for $R_1 < |z-a| < R_2$

expect singularities at points z with $|z-a| = R_1$ and $|z-a| = R_2$

e.g. $f(z) = \frac{1}{(z-1)(z-3i)}$
Radii of convergence of the Laurent series expansion about $z = 0$ are $R_1 = 1, R_2 = 3$.

Construction of Laurent Series

- Several techniques for constructing series

1. Binomial theorem: Taylor series for $|z-a| < R_1$

   Principal Laurent for $|z-a| > R_2$

   Full Laurent for $R_1 < |z-a| < R_2$

   Can always write $\frac{1}{(z-a)(z-b)} = \frac{A}{z-a} + \frac{B}{z-b}$ if $a \neq b$

2. If $a = b$, write $\frac{1}{(z-a)^2} = -\frac{d}{dz}\left(\frac{1}{z-a}\right)$

   expand $\left(\frac{1}{z-a}\right)$ as a Taylor/Laurent series, then differentiate

   (OK because of the Weierstrass theorem – uniform convergence)

3. Also can <u>integrate</u> a known expression, e.g. $f(z) = \log_e(1+z)$

   — write $\frac{df}{dz} = \frac{1}{1+z}$, expand and integrate.

Types of Singularity

- A singularity is any place where a function f is not analytic.
- Our applications and interests point us to a very specific type.
- Suppose $f(z)$ is analytic everywhere in some small region surrounding $z = a$, except at $z = a$

  $\Longrightarrow$ f has an isolated singularity at a
- If $f(z)$ is bounded as $z \to a$, $|f| < B < \infty$, then $\lim_{z \to a} f(z)$ exists, so we can define $f(a)$ as this limit, and then $f(z)$ is analytic at $z = a$

  removable singularity e.g. $f(z) = \dfrac{\sin z}{z}$
- Otherwise, $|f| \to \infty$ "uniformly" as $z \to a$ ($|f| >$ any M for $|z - a| <$ some ε)

  singularity is a pole e.g. $f(z) = \dfrac{1}{z}, \dfrac{1}{\sin z}$

Zeros of Functions

- Taylor series: $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$
- If $c_0 = 0, c_1 \neq 0$, then $f(z) \sim z - a$ near $z = a$ — simple zero

  if $c_0 = c_1 = 0, c_2 \neq 0$, then $f(z) \sim (z-a)^2$ near $z = a$ — zero of order 2

  if $c_0 = c_1 = \cdots = c_{m-1} = 0, c_m \neq 0$, then $z = a$ is a zero or order m
- Order of a zero is lowest m for which $\lim_{z \to a} \frac{f(z)}{(z-a)^m} \neq 0$.

Zeros and Poles

- Similarly, if the Laurent series for f

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

has $c_n = 0$ for $n < -m$, then f has a <u>pole of order m</u> at $z = a$.

- Order of a pole is lowest m for which $|\lim_{z\to a}(z-a)^m f(z)| \neq \infty$.
- Loosely, "a pole of order m is the reciprocal of a zero of order m."

e.g. $f(z) = \operatorname{cosec} z = \frac{1}{\sin z} = \frac{1}{z} + \frac{1}{6}z + \frac{7}{360}z^3 + \cdots$

pole of order 1
simple pole

Other Singularities

- Possible that $f(z)$ is neither bounded nor uniformly bounded as $z \to a$.

  e.g. $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots,$ so $c_n \neq 0, n \to -\infty$

  Wild oscillations as $z \to 0$, and values taken depend on path.

  Can show (Picard's theorem) that $f(z)$ takes on all ("almost all") values in $\mathbb{C}$ infinitely many times in any small region containing a as $z \to a$.

  essential singularity

- Other singularities (non-isolated)
  1. branch point and branch cut
  2. accumulation point of isolated singularities

     e.g. $\operatorname{cosec}\frac{1}{z}$ as $z \to 0$.

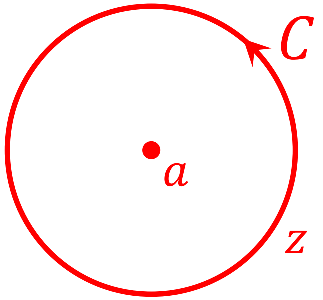
- Often of great interest to mathematicians
- Don't turn up often in "our" applications

The Residue Theorem

- The point!
- Our primary interest is in poles (regular functions divided by polynomials).
- Recall

$$\oint_C \frac{dz}{(z-a)^n} = \int_0^{2\pi} \frac{re^{i\theta} id\theta}{r^n e^{in\theta}} = \frac{i}{r^{n-1}} \int_0^{2\pi} e^{i(1-n)\theta} id\theta$$

$$= \begin{cases} 0, & n \neq 1 \\ 2\pi i, & n = 1 \end{cases}$$



$z = a + re^{i\theta}$

$dz = re^{i\theta} id\theta$

- Then if $f(z) = \sum_{n=-m}^{\infty} c_n (z-a)^n$, substitute in and find that

$$\oint_C f(z) dz = 2\pi i \, c_{-1}$$

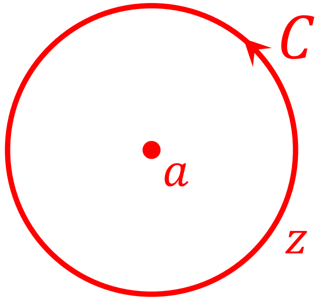
1. Integral along C depends on behavior of function away from C
2. Integral doesn't depend on the most singular part of the function, but on the least singular part of its Laurent expansion.

The Residue Theorem

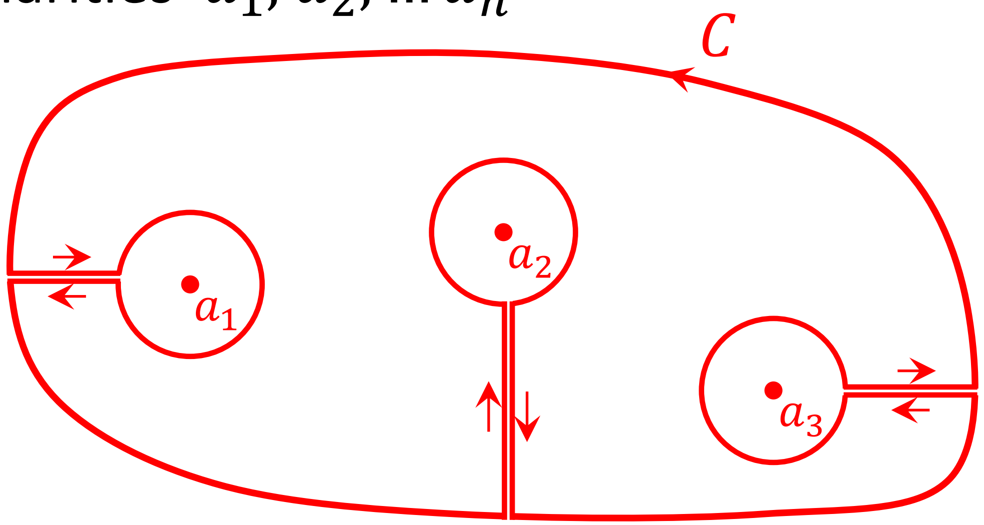
- Turn all this into a formal statement of the theorem:
- Define residue of f at a,

$$\text{Res}\, f(a) = \frac{1}{2\pi i}\oint_C f(z)dz, \text{ where } C \text{ contains } a$$

Note: shape of C doesn't matter (Cauchy)



- Then if $f(z)$ is analytic inside and on some closed contour C, except at a finite number of isolated singularities $a_1, a_2, \ldots a_n$ lying within C



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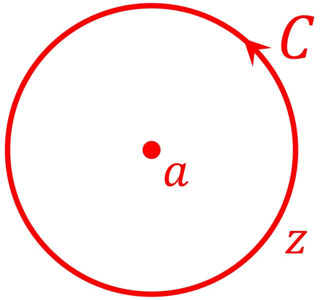
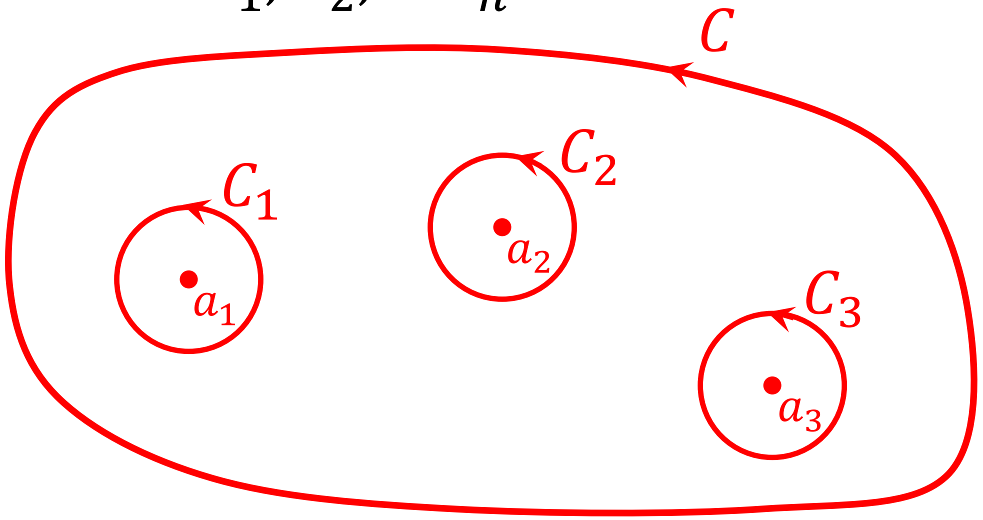


- Then if $f(z)$ is analytic inside and on some closed contour C, except at a finite number of isolated singularities $a_1, a_2, \ldots a_n$ lying within C,

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res } f(a_k)$$



- Again,

  integral is determined by the <u>least</u> singular behavior of f at poles <u>away</u> from C

Computation of Residues

- Integral of a function around a contour boils down to locating its poles inside the contour and computing the residues at each.
- Formal expressions:

  for a simple pole at $z = a$, $\text{Res}\, f(a) = \lim_{z \to a} (z - a) f(z)$

  for a pole of order m at $z = a$, $\text{Res}\, f(a) = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dx^{m-1}} [(z - a)^m f(z)]$

  — follows from definition of residue in terms of Laurent series

  — basically, getting rid of leading terms to expose c_{-1}

  e.g. $f(z) = e^z/z^4, m = 4, \text{Res}\, f(0) = \frac{1}{3!} \lim_{z \to 0} \frac{d^3}{dx^3} [e^z] = \frac{1}{6}$
- Simple pole, with $f(z) = \frac{\phi(z)}{\psi(z)}$, $\phi(a) \neq 0$, $\psi(a) = 0 \implies \text{Res}\, f(a) = \frac{\phi(a)}{\psi'(a)}$
- But usually, just expanding out the Laurent series is the best way to get there!

Computation of Residues

- Examples

1. $f(z) = e^z/z^4 = \frac{1}{z^4}\left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots\right) = \frac{1}{z^4} + \frac{1}{z^3} + \frac{1}{2z^2} + \frac{1}{6z} + \cdots$

   residue is $\frac{1}{6}$

2. $f(z) = \sin z\,/z^4 = \frac{1}{z^4}\left(z - \frac{z^3}{6} + \frac{z^5}{120}\ldots\right) = \frac{1}{z^3} - \frac{1}{6z} + \frac{z}{120}\ldots$

   residue is $-\frac{1}{6}$

3. $f(z) = \cos z\,/z^4 = \frac{1}{z^4}\left(1 - \frac{z^2}{2} + \frac{z^4}{24}\ldots\right) = \frac{1}{z^4} - \frac{1}{2z^2} + \frac{1}{24}\ldots$

   residue is 0

4. $f(z) = \cos z\,/z^3 = \frac{1}{z^3}\left(1 - \frac{z^2}{2} + \frac{z^4}{24}\ldots\right) = \frac{1}{z^3} - \frac{1}{2z} + \frac{z}{24}\ldots$

   residue is $-\frac{1}{2}$

Applications of the Residue Theorem

1. Real integral

$$I = \int_0^{2\pi} \frac{d\theta}{1 - 2p\cos\theta + p^2} \qquad (|p| \neq 1)$$

Turns up quite frequently in optics, transforms, …

Standard "trick": convert to a contour integral by writing

$$z = e^{i\theta},\ dz = ie^{i\theta}d\theta = izd\theta,\ \text{so}\ d\theta = dz/iz$$

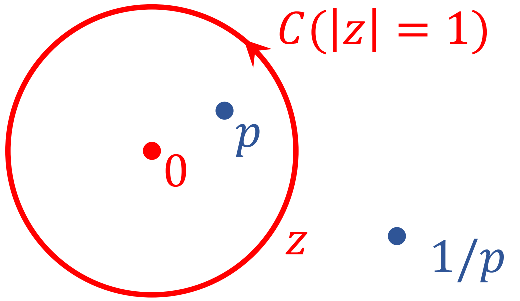
$$\cos\theta = \left(e^{i\theta} + e^{-i\theta}\right)/2 = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\Longrightarrow\ I = \oint_C \frac{dz}{iz\left[1 - p\left(z + \frac{1}{z}\right) + p^2\right]}$$

$$\begin{aligned} z\left[1 - p\left(z + \tfrac{1}{z}\right) + p^2\right] &= z - pz^2 - p + p^2 z \\ &= z - p - pz(z - p) \\ &= (z - p)(1 - pz) \\ &= -p(z - p)(z - 1/p) \end{aligned}$$

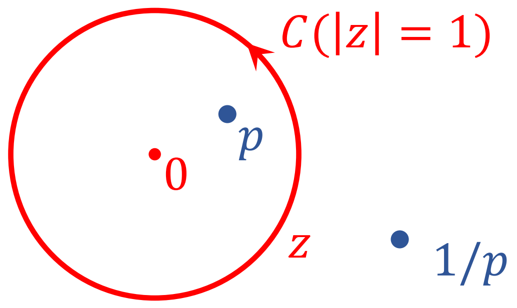
Two poles, at $z = p,\ 1/p$

Only one lies inside the contour.



Applications of the Residue Theorem

1. Integral



$$I = \oint_C \frac{dz}{iz\left[1-p\left(z+\frac{1}{z}\right)+p^2\right]}$$
$$= \oint_C \frac{dz}{-ip(z-p)(z-1/p)}$$

- For $|p| < 1$, the residue at $z = p$ is the coefficient of $(z-p)^{-1}$ in the integrand evaluated at $z = p$:

$$\text{Res}\, f(a) = \frac{1}{-ip(p-1/p)} = \frac{i}{p^2-1} = \frac{-i}{1-p^2}$$

$$\Longrightarrow \quad I = 2\pi i \,\text{Res}\, f(a) = \frac{2\pi}{1-p^2}$$

Typical candidate integral is

$$I = \int_0^{2\pi} \frac{P(\cos\theta, \sin\theta)\, d\theta}{Q(\cos\theta, \sin\theta)},$$

where P and Q are polynomials in $\cos\theta$ and/or $\sin\theta$.

- For $|p| > 1$, the residue at $z = 1/p$ is the coefficient of $(z-1/p)^{-1}$ in the integrand evaluated at $z = 1/p$:

$$\text{Res}\, f(a) = \frac{1}{-ip(1/p-p)} = \frac{i}{1-p^2}$$

$$\Longrightarrow I = 2\pi i \,\text{Res}\, f(a) = \frac{2\pi}{p^2-1}$$

Applications of the Residue Theorem

2. A well-known integral

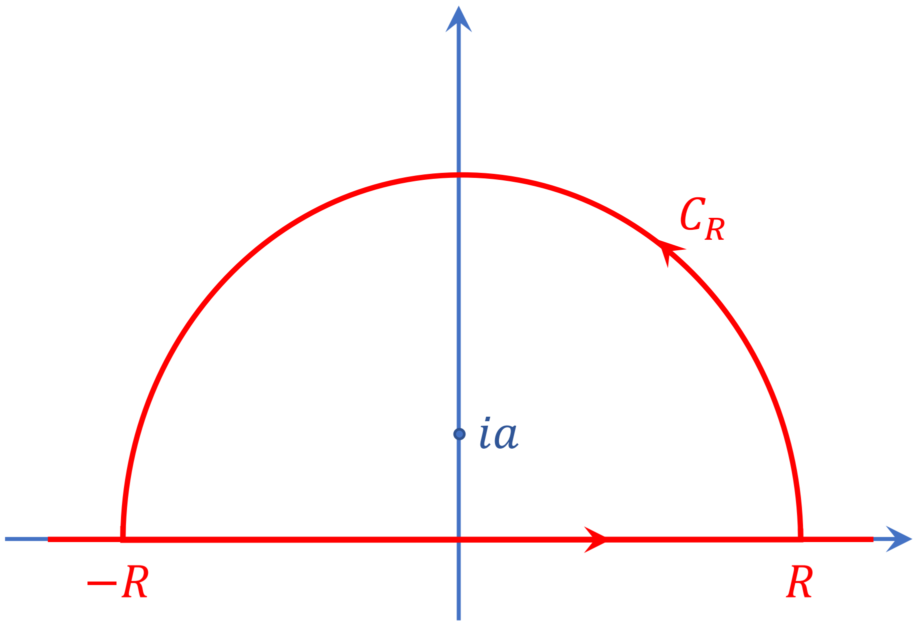
$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2+a^2}$$

Easy to show:
$I \;= \pi/a$

Integration path is the real axis.

Convert to a closed contour integral as follows:

1. Write as finite integral $\int_{-R}^{R} f(z)$.
2. Create closed contour C with a large semicircle C_R with $|z| = R > a$.
3. Evaluate using residue theorem
4. Take limit $R \to \infty$.
5. Demonstrate that the extra integral along the curved contour goes to 0.



Applications of the Residue Theorem

2. Residue theorem

$$\Longrightarrow \oint_C \frac{dz}{z^2+a^2} = I_R + \oint_{C_R} \frac{dz}{z^2+a^2} = 2\pi i \,\mathrm{Res}(ia)$$

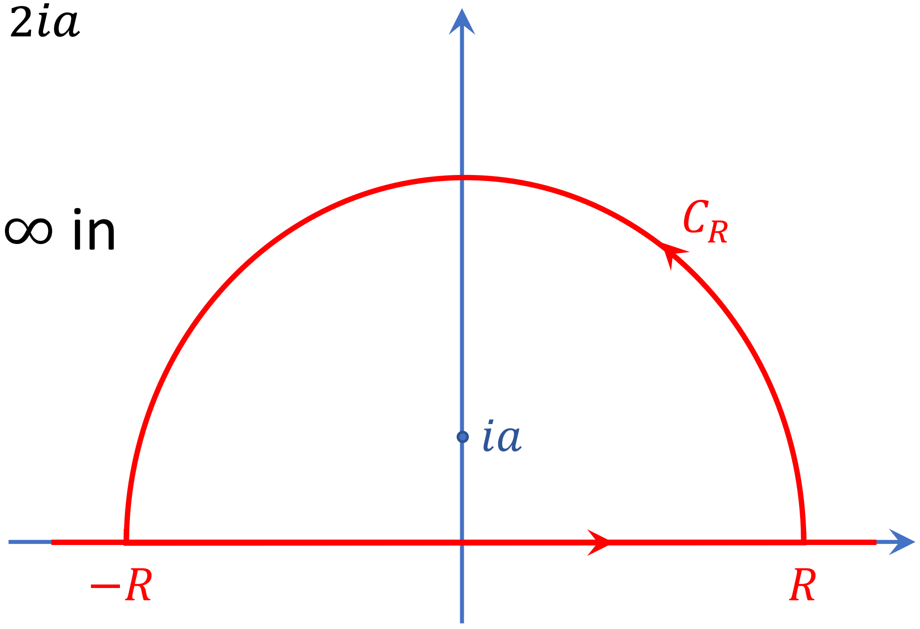
$z^2 + a^2 = (z - ia)(z + ia)$
poles of order 1 at $z = \pm ia$
only ia lies inside C

- Calculate the residue at ia

$$f(z) = \frac{1}{(z-ia)(z+ia)} \Longrightarrow \mathrm{Res}\, f(ia) = \frac{1}{2ia}$$

$$2\pi i \,\mathrm{Res}\, f(ia) = \pi/a$$

- Note that $I_R = \int_{-R}^{R} f(z)\, dz \to I$ as $R \to \infty$ in a very specialized sense
  - Cauchy Principal Value
  - possible for $\int_{-R_1}^{R_2}$ to diverge as R_1 and $R_2 \to \infty$ separately



Applications of the Residue Theorem

2. Still have to show that

$$\oint_{C_R} \frac{dz}{z^2+a^2} \to 0 \text{ as } R \to \infty$$

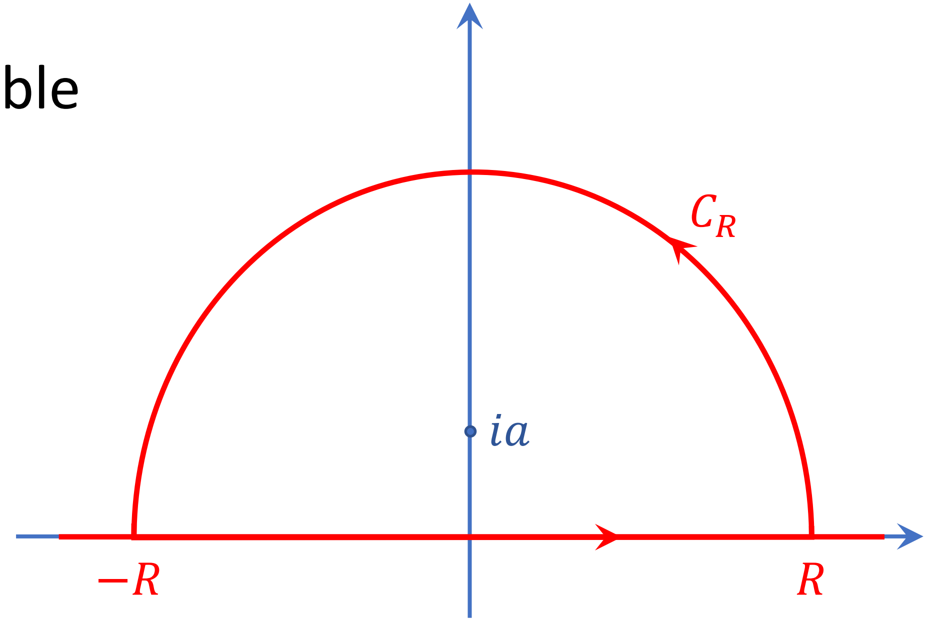
- Strategy: place a limit on the integral and show that the limit goes to zero as $R \to \infty$.

- Important result (straightforwardly provable from the definition of the integral):

$$\left|\int_C f\right| \leq \max_C |f| \; L(C)$$

max. value of $|f|$ on C

length of C



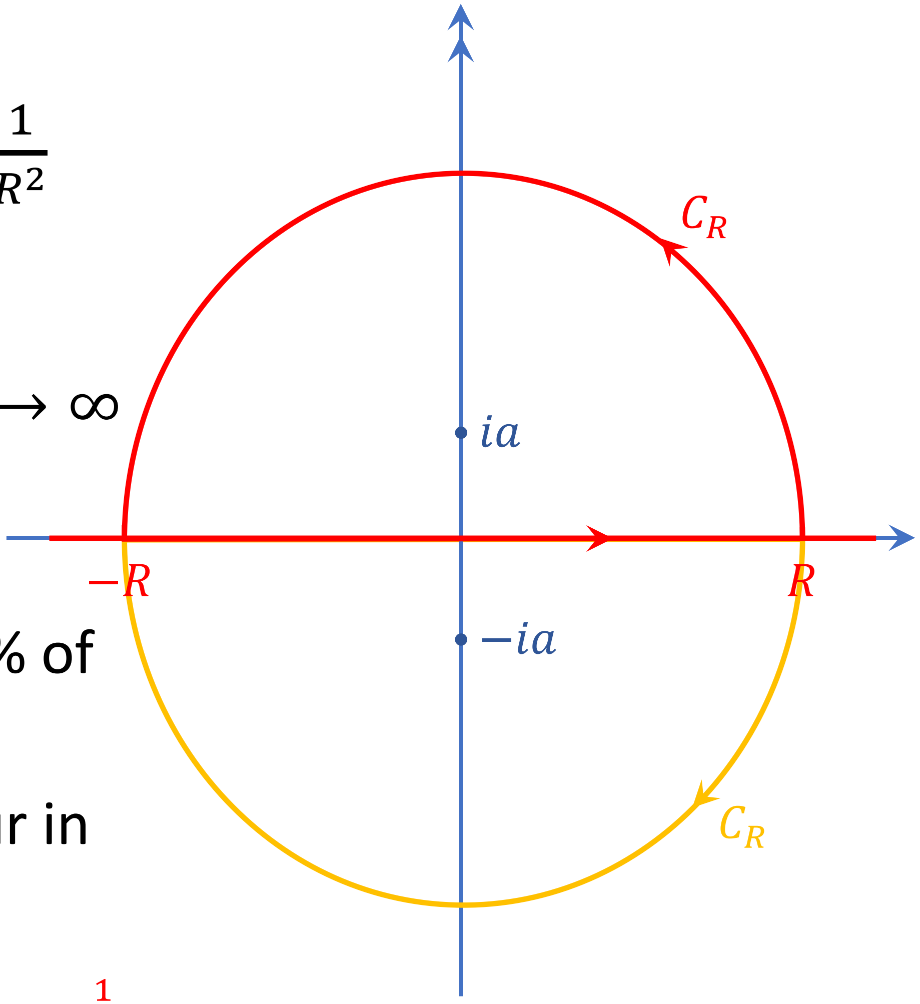
Applications of the Residue Theorem

2. In this case, on C_R, $|f| \sim \frac{1}{R^2}$, so $\max_{C_R} |f| \sim \frac{1}{R^2}$

   Clearly, $L(C_R) = \pi R$

   $$\Rightarrow \left|\int_{C_R} f\right| \leq \max_{C_R}|f| \; L(C) \sim \frac{\pi}{R} \to 0 \text{ as } R \to \infty$$

- Notes:
  1) Standard method of solution for >90% of complex integrals.
  2) What if I close the loop with a contour in the <u>lower</u> half plane?
     - pick up residue at $-ia$: $\frac{-1}{2ia}$ $\quad f(z) = \frac{1}{(z-ia)(z+ia)}$
     - but clockwise, so $\oint = -2\pi i \operatorname{Res}(-ia)$



Result is <u>unchanged</u>.

Applications of the Residue Theorem

3. Now consider

$$I = \int_{-\infty}^{\infty} \frac{\cos x \, dx}{x^2 + a^2}$$

Almost a Fourier transform…

This time, just replacing x by z and closing with a large semicircle won't work, because

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = \frac{1}{2}\left(e^{ix-y} + e^{-ix+y}\right)$$

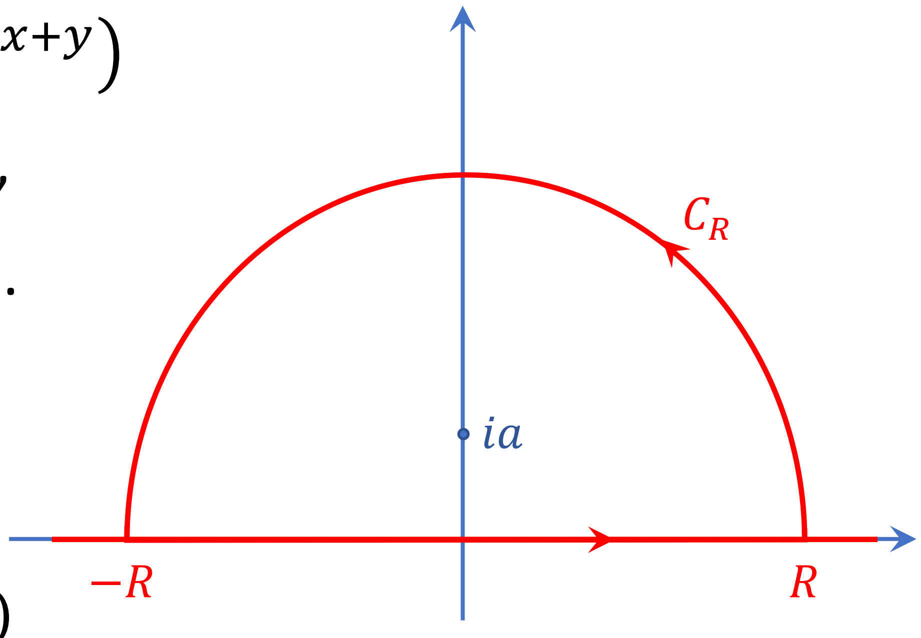
$$\to \infty \text{ as } |y| \to \infty,$$

so $\left|\int_{C_R} f\right|$ does not go to zero as $R \to \infty$.

Instead (another standard trick), write

$$\cos x = \mathrm{Re}\left(e^{ix}\right)$$

and consider $J = \int_{-\infty}^{\infty} \frac{e^{ix} dx}{x^2 + a^2}$, so $I = \mathrm{Re}(J)$



Applications of the Residue Theorem

3. Now considering

$$J = \int_{-\infty}^{\infty} \frac{e^{iz}\,dz}{z^2+a^2}$$

- Now

$$e^{iz} = e^{ix-y} \to 0 \text{ as } y \to +\infty,$$

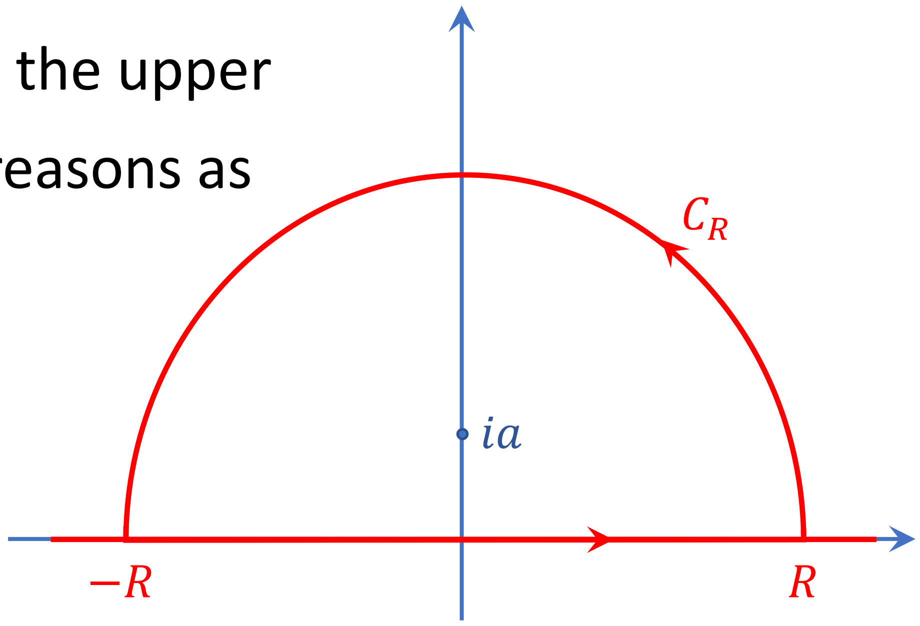
so OK to close with the large semicircle in the upper half plane, and $\left|\int_{C_R} f\right| \to 0$ for the same reasons as before as $R \to \infty$.

- Residue at ia is $\frac{e^{-a}}{2ia}$,

so $J = 2\pi i \frac{e^{-a}}{2ia} = \frac{\pi}{a} e^{-a}$

$$I = J = \frac{\pi}{a} e^{-a}$$

Note: J is real because sine is an odd function, so the imaginary part of the integral = 0.



Recall: Solving an Inhomogeneous ODE

- Equation

$$y'' - \lambda^2 y = f(x)$$

Let $\quad Y(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\ y(x)\, e^{-ikx}$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\ f(x)\, e^{-ikx}$$

$\Longrightarrow \quad -k^2 Y - \lambda^2 Y = F$

$\Longrightarrow \quad Y(k) = \frac{-F(k)}{k^2+\lambda^2}$

$\Longrightarrow \quad y(x) = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\ \frac{F(k)}{k^2+\lambda^2}\ e^{ikx}$

- Start simple: suppose $f(x) = \delta(x)$, so $F(k) = \frac{1}{\sqrt{2\pi}}$

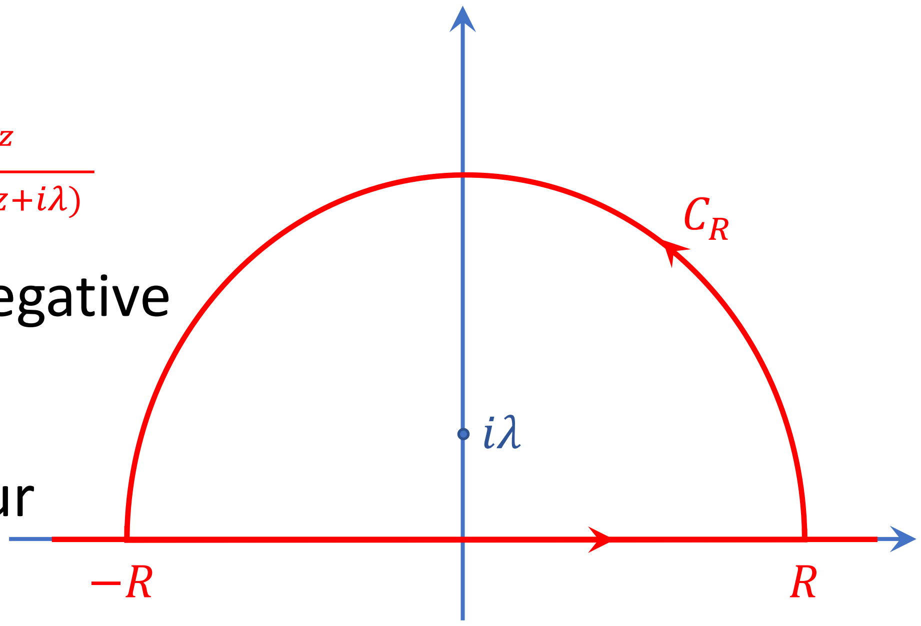
Applications of the Residue Theorem

4. Now consider an actual Fourier transform (λ real):

$$I = \int_{-\infty}^{\infty} \frac{e^{ikx}\,dx}{x^2+\lambda^2}$$

- If $k > 0$, then the same arguments as before mean that the integrand goes to zero as $y \to +\infty$, so OK to close with a semicircle in the upper half plane.
- As we just saw, $I = \frac{\pi}{\lambda} e^{-k\lambda}$ $\quad$ $f(z) = \frac{e^{ikz}}{(z-i\lambda)(z+i\lambda)}$
- If $k < 0$, close with a semicircle in the negative half plane, pick up residue at $-i\lambda$: $-\frac{e^{\kappa\lambda}}{2i\lambda}$ and a negative sign for a clockwise contour

$$\Longrightarrow \qquad I = \frac{\pi}{\lambda} e^{-|k\lambda|}$$



Applications of the Residue Theorem

5. What if the pole is on the integration path (e.g. finite pulse transform)?

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$$

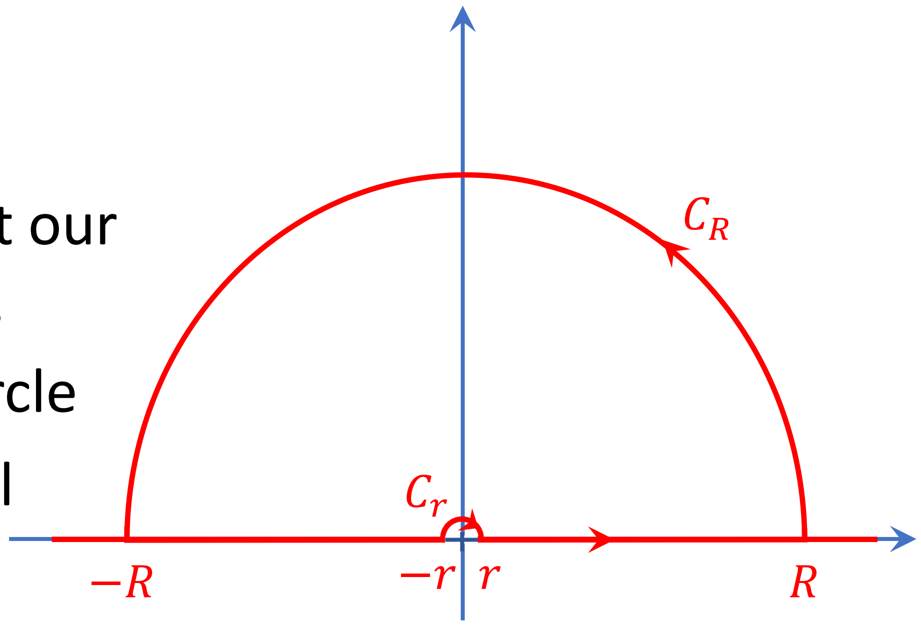
- Removable singularity at $x = 0$, so the integral is valid, but the behavior of the sine function is such that we can't use our favorite contour.
- Again, look at

$$J = \int_{-\infty}^{\infty} \frac{e^{iz}}{z}\, dz, \text{ so } I = \mathrm{Im}(J)$$

Now we can use the upper semicircle, but our removable singularity has become a <u>pole</u>.

- Avoid the pole with another small semicircle

(Note: now we have <u>two</u> Cauchy Principal Value integrals, as $R \to \infty$, $r \to 0$)



Applications of the Residue Theorem

5. Now the integrand in the J integral

$$J = \int_{-\infty}^{\infty} \frac{e^{iz}}{z}\, dz$$

by construction is analytic <u>everywhere</u> inside the closed contour, so

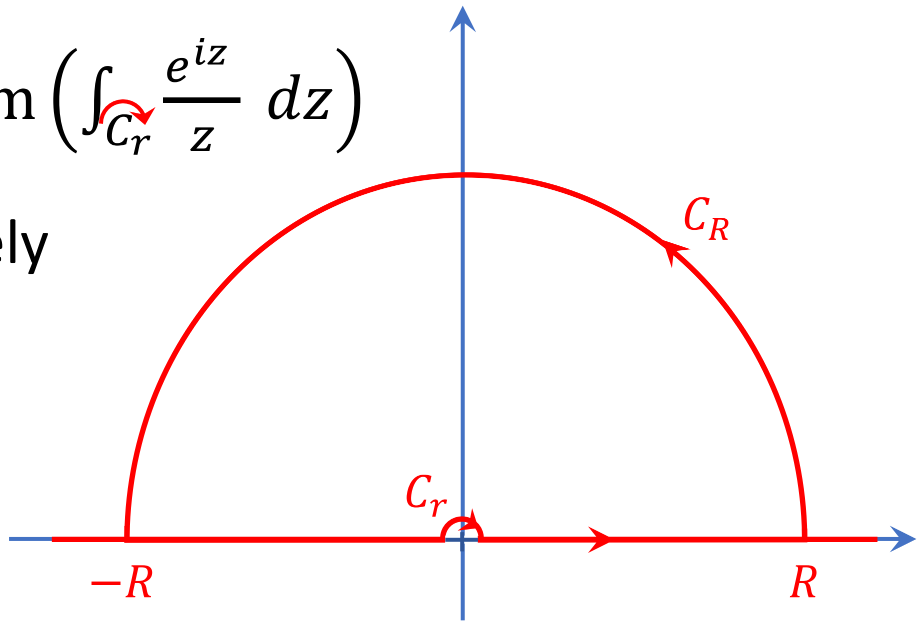
$$\operatorname{Im}\left(\oint_C \frac{e^{iz}}{z}\, dz\right) = 0$$

$$= I + \operatorname{Im}\left(\int_{C_R} \frac{e^{iz}}{z}\, dz\right) + \operatorname{Im}\left(\int_{C_r} \frac{e^{iz}}{z}\, dz\right)$$

Need to look at the two semicircles separately

1) $\int_{C_R} \frac{e^{iz}}{z} dz$ is not so obviously 0

2) how to handle $\int_{C_r} \frac{e^{iz}}{z} dz$?



Applications of the Residue Theorem

5. Look first at $\int_{C_R} \frac{e^{iz}}{z} dz$
   - previous estimate would just say $\left|\int_{C_R} \frac{e^{iz}}{z} dz\right| \leq \frac{\pi R}{R}$, which does not $\to 0$
   - integrate by parts:

$$\int_{C_R} \frac{e^{iz}}{z} dz = \underbrace{\left[\frac{e^{iz}}{iz}\right]_{z=-R}^{z=R}}_{\to 0} + \underbrace{\int_{C_R} \frac{e^{iz}}{iz^2} dz}_{\to 0\text{, by earlier argument}}$$

$$= 0$$

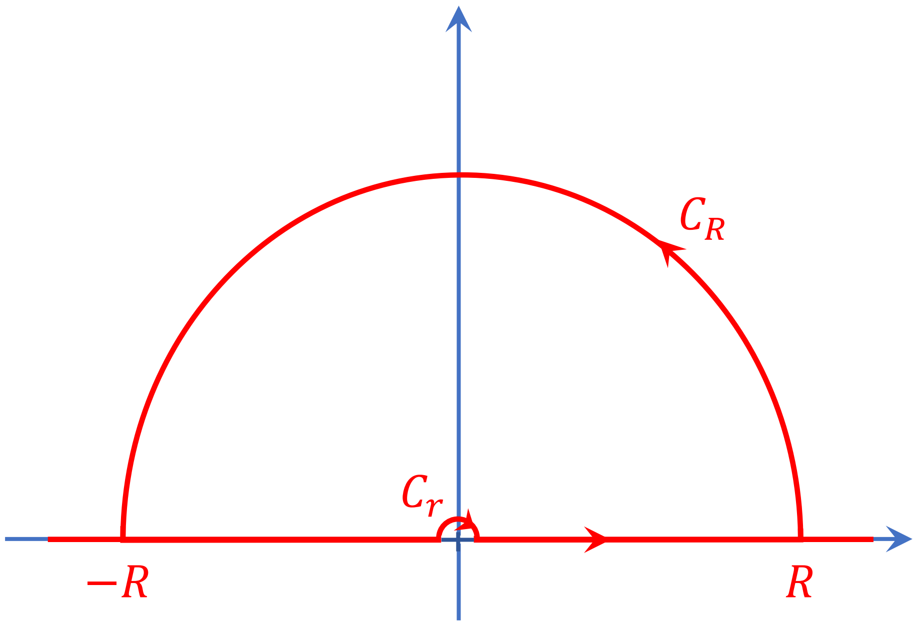
Now for $\int_{C_r} \frac{e^{iz}}{z} dz$

- write $e^{iz} = 1 + (e^{iz} - 1)$

$$\Longrightarrow \quad \int_{C_r} \frac{e^{iz}}{z} dz = \underbrace{\int_{C_r} \frac{dz}{z}}_{= -\pi i} + \int_{C_r} \frac{e^{iz}-1}{z} dz$$

"half a residue"

$\frac{e^{iz}-1}{z} \approx i$



Applications of the Residue Theorem

- End result:

$$I = -\mathrm{Im}\left(\lim_{r\to 0}\int_{C_r}\frac{e^{iz}}{z}dz\right)$$
$$= \pi$$