PHYS 501: Mathematical Physics I

Fall 2020, Homework #3

(Due October 21, 2020)

1. (a) Find the series solution of the equation

$$(1-x^2)y'' - xy' + n^2 y = 0$$

that is regular at $x = 0$. Under what circumstances (for what values of n) does the series converge for *all* x?

(b) Find two linearly independent solutions of the equation

$$4x^2 y'' + (1-p^2)y = 0\,.$$

(c) Given that one solution of the differential equation

$$y'' - 2xy' = 0$$

is $y(x) = 1$, use the Wronskian development to find a second, linearly independent solution. Describe its behavior near $x = 0$.

2. A function $f(x)$ is periodic with period 2π, and can be written as a polynomial $P(x)$ for $-\pi < x < a$ and as a polynomial $Q(x)$ for $a < x < \pi$. Show that the Fourier coefficients c_n of f go to zero at least as fast as $1/n^2$ as $n \to \infty$ if $P(a) = Q(a)$ and $P(-\pi) = Q(\pi)$ (i.e. f is continuous), but only as $1/n$ otherwise.

3. (a) Find the Fourier series $\sum_{n=1}^{\infty} b_n \sin(n\pi x)$, for $-1 < x < 1$, for the sawtooth function

$$f(x) = \begin{cases} -1-x & (-1 < x < 0) \\ 1-x & (0 < x < 1)\,. \end{cases}$$

(b) Plot the partial sums $S_N(x) = \sum_{n=1}^{N} b_n \sin(n\pi x)$ of the series for $0 \le x \le 1$, in steps of $\delta x = 0.0005$, and $N = 1$, 5, 10, 20, 50, 100, and 500. What is the maximum overshoot of the Fourier series relative to the original function in the $N = 500$ case, and at what value of x does it occur?

4. The curved surface of a long cylinder of radius b is kept at a constant temperature $T = 0$. Initially the cylinder is at a uniform temperature $T_0 > 0$. Derive an expression for the temperature at the center of the cylinder at any time $t > 0$, and write down a simplified solution (not $T = 0$!) valid in the limit $t \gg b^2/\kappa$, where κ is the heat diffusion coefficient of the cylinder.