# GRAVITY

# An Introduction to Einstein's General Relativity

## *SOLUTIONS TO PROBLEMS*

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*Version 1.1*

Addison-Wesley, 2003
(4/8/2003)

ISBN 0-8053-8663-7

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Addison Wesley

# Preface

This manual contains the author's solutions to the 392 problems in *Gravity: An Introduction to Einstein's General Relativity.* I have aimed at explaining the central ideas needed to solve each problem; I have not generally attempted to write out each step of the calculations involved. I hope that the solutions will be clear to instructors teaching from the text. Depending on their level, students may need further discussion or more details.

The text of the problems are provided for convenience. This may differ slightly from the published text because of copy editing changes or subsequent errata.

While a considerable effort has been made to check the solutions, it is inevitable that mistakes remain in a collection of problems of this size and occasional complexity. I invite suggestions for correction and improvement.

The manuscript was typed by Thea Howard who also drew all the line figures. The solutions were checked and corrected by Matt Hansen and Taro Sato. Thea, Matt, and Taro have my gratitude in these regards, as do the many students and teaching assistants who tried out these problems in various courses at Santa Barbara and helped to make them better.

James Hartle
November, 2002

# Chapter 2

# Geometry as Physics

**2-1**. *[B] (a) In a plane, show that a light ray incident from any angle on a right angle corner reflector returns in the same direction from whence it came.*

*(b) Show the same thing in three dimensions with a cubical corner reflector.*

**Solution:**

a) Snell's law of reflection is that the angle the incident ray makes with the normal to the surface is the same as the angle the reflected ray makes with the normal as shown above. Since the sum of the interior angles of the

right triangle OAB is $\pi$, this implies

$$i' = \frac{\pi}{2} - i \ .$$

Equivalently the angle the incident ray makes with $OB$ is $\pi/2$ minus the angle the reflected ray makes with $OA$. The reflected ray is thus parallel to the incident ray.

b) Snell's law may be stated vectorially as follows. Let $\vec{k}$ be a unit vector along a ray incident on a surface with normal $\vec{n}$. $\vec{k}$ can be divided into a component along $\vec{n}$ and a component perpendicular to $\vec{n}$ as follows

$$\vec{k} = \left(\vec{k} \cdot \vec{n}\right) \vec{n} + \left(\vec{k} - \left(\vec{k} \cdot \vec{n}\right) \vec{n}\right) \ .$$

On reflection the component along $\vec{n}$ changes sign while the perpendicular component remains unchanged. Thus, $\vec{k}'$ after reflection is

$$\begin{aligned} \vec{k}' &= -\left(\vec{k} \cdot \vec{n}\right) \vec{n} + \left(\vec{k} - \left(\vec{k} \cdot \vec{n}\right) \vec{n}\right) \qquad (1) \\ \vec{k}' &= \vec{k} - 2\left(\vec{k} \cdot \vec{n}\right) \vec{n} \ . \end{aligned}$$

Consider a ray which reflects off of all three faces of the corner reflector with orthogonal normals $\vec{n}_1, \vec{n}_2, \vec{n}_3$ respectively. Using (1) at each of the three reflections, the output of the previous reflection being the input to the next, one finds for the exiting ray $\vec{k}_{\text{ex}}$ in terms of the incident ray $\vec{k}_{\text{in}}$

$$\begin{aligned} \vec{k}_{\text{ex}} = \vec{k}_{\text{in}} \ &- \ 2\left(\vec{k}_{\text{in}} \cdot \vec{n}_1\right) \vec{n}_1 - 2\left(\vec{k}_{\text{in}} \cdot \vec{n}_2\right) \vec{n}_2 \\ &- \ 2\left(\vec{k}_{\text{in}} \cdot \vec{n}_3\right) \vec{n}_3 \end{aligned}$$

But $\vec{n}_1, \vec{n}_2, \vec{n}_3$ are three orthogonal vectors that form a basis. Thus,

$$\vec{k}_{\text{ex}} = \vec{k}_{\text{in}} - 2\vec{k}_{\text{in}} = -\vec{k}_{\text{in}} \ .$$

so the exit ray leaves in the direction opposite to the incident one.

**2-2.** *[S] The center of the Sun is much further way from a terrestrial measurement of angles than the center of the Earth is. But it is also much more massive. Using* (2.1)*, estimate which would have the greatest effect on a measurement of angles such as is attributed to Gauss.*

**Solution:** For the Earth $GM_\oplus/R_\oplus c^2 \sim 10^{-9}$. The relevant ratio for the Sun is $GM_\odot/c^2 r_\oplus$ where $r_\oplus$ is the distance from the Sun to Earth. From the table of useful constants this is

$$\frac{GM_\odot}{c^2 r_\oplus} \sim \frac{1.48 \text{ km}}{1.4 \times 10^8 \text{ km}} \sim 10^{-8}$$

The effect of the Sun is therefore potentially larger.

**2-3.** *[C] (a) Verify the relation* (2.4) *between the sum of the interior angles of a spherical triangle and its area when two of the angles are right angles.*

*(b) Prove the relation generally.*

**Solution:**

a) Such a triangle can be bounded by the equator and two lines of longitude differing by an angle $\alpha$. The area $A$ is $(\alpha/2\pi) \times$ (area of a hemisphere) $= \alpha a^2$.

$$\begin{pmatrix}\text{sum of the} \\ \text{interior angles}\end{pmatrix} = \pi + \alpha = \pi + \frac{(\alpha a^2)}{a^2} = \pi + \frac{A}{a^2} .$$

b) The three great circles that bound a spherical triangle divide the sphere up into eight triangles. Any two circles divide the sphere into wedges whose opening angle is one of the interior angles of a triangle, and whose area is the sum of the areas of two of the triangles. This gives a set of relations of the form

$$A + A' = \frac{1}{2\pi}\begin{pmatrix}\text{interior} \\ \text{angle}\end{pmatrix} \cdot 4\pi a^2$$

which could be solved for the areas of the triangles in terms of their interior angles.

However, it is not necessary to carry out this solution. Arguments of symmetry and some special cases are enough to find the result. The above relations show that the area of a spherical triangle are *linearly* related to the three interior angles: $\alpha$, $\beta$, $\gamma$. Since, in a general triangle, no one of these angles is preferred over any other, the area must be related linearly and symmetrically to the angles by a relation of the form

$$A = c(\alpha + \beta + \gamma) + d$$

with constants $c$ and $d$ depending on $a$ to be determined. The special case considered in (a) with $\beta = \gamma = \pi/2$ gives

$$A = c(\pi + \alpha) + d = \alpha a^2$$

holding for arbitrary $\alpha$. Thus, $c = a^2$ and $d = -a^2\pi$, giving

$$A = a^2(\alpha + \beta + \gamma - \pi)$$

or, what is the same thing:

$$\alpha + \beta + \gamma = \pi + \frac{A}{a^2} .$$

**2-4**. *Draw examples of a triangle on the surface of a sphere for which:*

a) *the sum of whose interior angles is just slightly greater than $\pi$.*

b) *the sum of whose angles is equal to $2\pi$.*

c) *What is the maximum the sum of angles of a triangle on a sphere can be according to (2.4)? Can you exhibit a triangle where the sum achieves this value?*

**Solution:**

Consider the triangle contained within the equator and two lines of longitude differing by an angle $\alpha$. That triangle has two right interior angles at the

equator and the interior angle $\alpha$ at the pole. The sum of the interior angles is $\pi+\alpha$. By taking $\alpha$ near zero, one has a triangle whose sum of angles is slightly bigger than $\pi$. By taking $\alpha = \pi$, one has a triangle whose sum of angles is $2\pi$.

From (2.4), the maximum sum of interior angles occurs when $A = 4\pi R^2$ — the area of the whole sphere — and is $5\pi$. A triangle which nearly realizes this bound is the *complement* of a small equilateral triangle. The three interior angles are each $(2\pi - \pi/3)$ and add up to $5\pi$.

**2-5**. *Calculate the area of a circle of radius $r$ (distance from center to circumference) in the two - dimensional geometry which is the surface of a sphere of radius $a$. Show that this reduces to $\pi r^2$ when $r \ll a$.*

**Solution:** Refer to Fig.2.6 for the geometry. Consider an element of area at $(\theta, \phi)$ spanned by coordinate intervals $(d\theta, d\phi)$. The length of the edge of size $d\theta$ is $ad\theta$, the length of the edge of size $d\phi$ in $a \sin\theta d\phi$. Since the coordinate lines are orthogonal the area is

$$(ad\theta)(a \sin\theta d\phi) \ .$$

The circle of radius $r$ lies at $\theta = r/a$. Integrating the element of area above

$$A = \int_0^{r/a} d\theta \int_0^{2\pi} d\phi \ a^2 \sin\theta d\theta d\phi$$

gives the result:

$$A = 2\pi a^2[1 - \cos(r/a)] \ .$$

For small $r/a$

$$\begin{aligned} \cos\left(\frac{r}{a}\right) &= 1 - \frac{1}{2}\left(\frac{r}{a}\right)^2 + O\left(\frac{r}{a}\right)^4 \\ A = \pi r^2 &+ O\left(\frac{r}{a}\right)^4 \ . \end{aligned}$$

**2-6**. *[B] Consider a sphere of radius $a$ and on it a segment of length $s$ of a line of latitude that is a distance $d$ from the north pole measured on the sphere. What is the angle between the lines of longitude that this segment spans? Is*

*this angle greater or smaller than the angle the segment would subtend at the same distance on a flat plane?*

**Solution:** This problem is solved in the same way that the ratio of the circumference to radius of a "circle" on the sphere was calculated in (2.17) — (2.19). The answer is

$$s = \Delta\phi\, a \sin\left(\frac{d}{a}\right) .$$

The subtended angle is therefore

$$\Delta\phi = s\left[a \sin\left(\frac{d}{a}\right)\right]^{-1}$$

since $\sin x < x$ this is *more* than the angle $\Delta\phi = s/d$ that would be subtended geometry were flat.

**2-7**. *Consider the following coordinate transformation from familiar rectangular coordinates $(x, y)$ labeling points in the plane to a new set of coordinates $(\mu, \nu)$*

$$\begin{aligned} x &= \mu\nu \\ y &= \frac{1}{2}\left(\mu^2 - \nu^2\right) \end{aligned}$$

*a) Sketch the curves of constant $\mu$ and constant $\nu$ in the $(x, y)$ plane.*

*b) Transform the line element $dS^2 = dx^2 + dy^2$ into $(\mu, \nu)$ coordinates.*

*c) Do the curves of constant $\mu$ and constant $\nu$ intersect at right angles?*

*d) Find the equation of a circle of radius $r$ centered at the origin in terms of $\mu$ and $\nu$.*

*e) Calculate the ratio of the circumference to the diameter of a circle using $(\mu, \nu)$ coordinates. Do you get the correct answer?*

**Solution:**

a)

b)

$$\begin{aligned} dS^2 &= dx^2 + dy^2 \\ &= (\mu d\nu + \nu d\mu)^2 + (\mu d\mu - \nu d\nu)^2 \\ dS^2 &= (\mu^2 + \nu^2)\,(d\mu^2 + d\nu^2) \end{aligned}$$

c) The curves intersect at right angles because there are no cross terms $d\mu d\nu$ in the metric.

d) The equation of a circle is $x^2 + y^2 = r^2$ which becomes

$$\begin{aligned} \mu^2\nu^2 + \frac{1}{4}\left(\mu^2 - \nu^2\right)^2 &= r^2 \\ \frac{1}{4}\left(\mu^2 + \nu^2\right)^2 &= r^2 \\ \mu^2 + \nu^2 &= 2r \end{aligned}$$

e) The circumference $C$ is

$$\begin{aligned} C = \oint dS &= \oint \left(\mu^2 + \nu^2\right)^{\frac{1}{2}} \left(d\mu^2 + d\nu^2\right)^{\frac{1}{2}} \\ C &= (2r)^{\frac{1}{2}} \oint d\mu \left(1 + \left(\frac{d\nu}{d\mu}\right)^2\right)^{\frac{1}{2}} \\ &= (2r)^{\frac{1}{2}} \int_{-\sqrt{2r}}^{+\sqrt{2r}} d\mu \left[1 + \frac{\mu^2}{(2r - \mu^2)}\right]^{\frac{1}{2}} \\ &= 2\pi r \end{aligned}$$

This is, of course, the correct answer. The key step in the above evaluation is recognizing that the whole circle is covered by the coordinate range $\mu = -\sqrt{2r}$ to $\mu = \sqrt{2r}$.

**2-8.** *[A] The surface of an egg is an axisymmetric geometry to a good approximation. In the line element for two-dimensional axisymmetric geometries* (2.21)*, pick an $f(\theta)$ such that the resulting surface would resemble that of an egg. Calculate the ratio of the biggest circle around the axis to the distance from pole to pole.*

**Solution:** There are many solutions to this problem corresponding to the different choices of $f(\theta)$ that make a surface like an egg which is smaller near one pole than the other. A simple linear choice is

$$f(\theta) = 1 - \theta/2\pi$$

which varies between 1 at $\theta = 0$ (the larger end) and $1/2$ at $\theta = \pi$ (the smaller end). The circumference of a circle around the axis at $\theta$ is

$$C(\theta) = 2\pi\, a\, f(\theta).$$

The maximum circumference occurs at $\theta = 0$: $C(0) = 2\pi\, a$. The distance $d$ from pole to pole is

$$\begin{aligned} d &= \int_0^\pi d\theta\; a \\ &= \pi\, a \end{aligned}$$

The ratio $C/d$ is thus 2 for this example.

**2-9.** *The surface of the Earth is not a perfect sphere. The polar radius of the Earth,* 6357 km*, is slightly less than the mean equatorial radius* 6378 km*. Suppose the surface of the Earth is modeled by an axisymmetric surface with a line element of the kind in* (2.21) *with*

$$f(\theta) = \sin\theta(1 + \epsilon \sin^2\theta)$$

*for some small $\epsilon$. What values of $a$ and $\epsilon$ would best fit reproduce the known polar and equatorial radii?*

**Solution:** The line element (2.21) with the given $f(\theta)$ depends on two parameters $a$ and $\epsilon$. We can determine these by fitting to the circumferences of the equator and a great circle through the polar axis. The circumference of the equator $\theta = \pi/2$ from (2.21) is

$$C_{\text{eq}} = \int_0^{2\pi} af(\pi/2)d\phi = 2\pi af(\pi/2) = 2\pi(1+\epsilon)a \ .$$

This must be $2\pi(6378)$ km. The circumference of the great circle $\phi = 0$ from (2.21) is

$$C_{\text{polar}} = 2\int_0^{\pi} ad\theta = 2\pi a \ .$$

This must be $2\pi(6357)$ km. Thus

$$a = 6357 \text{ km}, \qquad \epsilon = .003 \ .$$

**2-10**. *[B] (Equal Area Projections.) An equal area map projection is one for which there is a constant proportionality between areas on the map and areas on the surface of the globe. Given $x = L\phi/2\pi$, what function $y(\lambda)$ would make an equal area map? [Hint: If an infinitesimal area $dxdy$ has the same constant of proportionality to the corresponding infinitesimal area on the sphere wherever it is located, bigger areas will be also proportional.]*

**Solution:** The metric on the sphere (2.24) can be written in the form (2.28) in terms of $x = (L\phi)/2\pi$ and arbitrary $y = y(\lambda)$. The area on the sphere bounded by a small rectangle of coordinate length $dx$ and height $dy$ is thus

$$\left(a\left(\frac{2\pi}{L}\right)\cos\lambda(y)\,dx\right)\left(a\frac{d\lambda}{dy}\,dy\right)$$

If the area $dx\,dy$ on the map is to be proportional to this, then the coefficient of $dx\,dy$ above must be constant. Choosing a convenient constant if proportionally, we have

$$\frac{d\lambda}{dy}\cos\lambda = 1 \ .$$

Integrating this and choosing $y = 0$ to be the equator $\lambda = 0$, we find

$$y(\lambda) = \sin\lambda$$

or

$$\lambda(y) = \sin^{-1}(\lambda)$$

**2-11**. *[B] (Conical Projections.) Conical projections map points on the globe into polar coordinates $(r, \psi)$ in the plane of the map. (We use $\psi$ to avoid confusion with the coordinate $\phi$ on the sphere.) Thus, in general $r = r(\lambda, \phi)$ and $\psi = \psi(\lambda, \phi)$. A particularly simple class of conical projections uses the north pole as the origin of the polar coordinates and has $r = r(\lambda)$ and $\psi = \phi$. For this simple class*

*a) express the line element on the sphere in terms of $r$ and $\psi$.*

*b) find the function $r(\lambda)$ which makes this an equal area projection in which there is a constant proportionality between each area on map and the corresponding area on the sphere. [Hint: See the hint for the previous problem.]*

**Solution:**

(a)

$$\begin{aligned} dS^2 &= a^2\left(d\lambda^2 + \cos^2\lambda d\phi^2\right) \\ &= a^2\left[\left(\frac{d\lambda}{dr}\right)^2 dr^2 + \cos^2\lambda d\psi^2\right] . \end{aligned}$$

(b) The length on the sphere of a line of coordinate length $dr$ extending in the $r$ direction is $a(d\lambda/dr)dr$. The length of a line of coordinate length $d\psi$ extending in the $\psi$ direction is $a\cos\lambda d\psi$. Since $r$ and $\psi$ are orthogonal coordinates the area spanned by $dr$ and $d\psi$ is

$$\left[a\left(\frac{d\lambda}{dr}\right)dr\right]\left[a\cos\lambda d\psi\right] .$$

The area of the corresponding element in the plane is

$$(dr)(rd\psi) .$$

If these are related by a constant of proportionality $L$, we must have

$$a^2 \frac{d\lambda}{dr} \cos\lambda = -Lr \ .$$

(The constant of proportionality must be negative since latitude decreases as $r$ increases from the north pole.) Integrating both sides and choosing the constant so $r = 0$ is $\lambda = \pi/2$ (the north pole) we find

$$a^2 (\sin\lambda - 1) = -\frac{1}{2} L r^2$$

$$r(\lambda) = \sqrt{\frac{2a^2}{L}(1 - \sin\lambda)}.$$

**2-12**. *[B,N]Your Personal World Map The maps in Box 2.3 were made with the Mathematica program* `WorldPlot`. *Make your own projection centered on your home city that uses a radial coordinate that represents your view of the importance of the rest of the world.*

**Solution:** The world map below is a projection which emphasizes the US and, in fact, the neighborhood of New York over other places. It was constructed using the projection

$$\begin{aligned} x &= \phi'/(1 + |\phi'|/6000)^{1.2} \\ y &= \lambda'/(1 + |\lambda'|/1000)^{1.2} \end{aligned}$$

with $\phi' = \phi + 74°$, and $\lambda' = \lambda - 41°$ expressed in minutes of arc.

There are probably many more elegant ways of solving this problem and certainly many more candidates for the most important city.

# Chapter 3

# Space, Time and Gravity in Newtonian Physics

**3-1**. *A free particle is moving in an inertial frame $(x, y, z)$ in the $xy$ -plane on a trajectory $x = d$, $y = vt$ where $d$ and $v$ are constants in time. Consider a rectangular frame $(x', y', z')$ rotating with respect to the inertial frame with an angular velocity $\omega$ about a common $z$-axis $(z' = z)$. What are the equations of motion obeyed by $x'(t)$, $y'(t)$ and $z'(t)$ in the rotating frame? Sketch the trajectory of the particle in the $x'y'$-plane and show explicitly that it satisfies these equations of motion.*

**Solution:** Deriving the equations of motion in a rotating frame is a standard topic in Newtonian mechanics which can be found in almost any text on the subject. If $\vec{x}'(t)$ is the vector with components $(x'(t), y'(t), z'(t))$ in the rotating frame and $\vec{V}'(t)$ is its time derivative, the equation of motion is:

$$\frac{d\vec{V}'}{dt} = 2\vec{V}' \times \vec{\omega} + \vec{\omega} \times (\vec{x}' \times \vec{\omega}) \ .$$

where $\vec{\omega}$ is the angular velocity of the rotating frame. The first term in this expression is the Coriolis force and the second the centrifugal force. The explicit equations for the components $V^{x\prime}$ and $V^{y\prime}$ for an angular velocity of

magnitude $\omega$ pointing along the $z-$axis are:

$$\frac{dV^{x'}}{dt} = +2\omega V^{y'} + \omega^2 x'$$
$$\frac{dV^{y'}}{dt} = -2\omega V^{x'} + \omega^2 y' \ .$$

The given trajectory in the inertial frame

$$x(t) = d \ , \qquad y(t) = vt$$

becomes in the inertial frame

$$x'(t) = +x(t)\cos(\omega t) + y(t)\sin(\omega t) \ ,$$
$$y'(t) = -x(t)\sin(\omega t) + y(t)\cos(\omega t) \ .$$

A plot of the orbit for three periods of rotation with $v = 1$ and $d = 1$ is shown below:

Substituting the $x'(t)$ and $y'(t)$ given above into the equations of motion verifies that they are satisfied.

**3-2**. *Show that Newton's laws of motion are not invariant under a transformation to a frame that is uniformly accelerated with respect to an inertial frames of Newtonian mechanics. What are the equations of motion in the accelerated frame?*

**Solution:** Let $(t, x)$ be the coordinates of an inertial frame and $(t', x')$ the coordinates of a frame accelerating along the $x$-axis with acceleration $g$. Then

$$\begin{aligned} x &= x' + \frac{1}{2}\, g t'^2 \,, \\ t &= t' \,. \end{aligned}$$

Newton's equation of motion for a free particle $d^2x/dt^2 = 0$ implies

$$\frac{d^2x'}{dt'^2} = -g$$

which is not the form it takes in an inertial frame.

**3-3**. *[B,S] How many degrees per hour does the Foucault pendulum described in Box 3.2 precess?*

**Solution:** In this problem it is important to distinguish the Earth's rotation, or revolution about its axis, from its orbital motion around the Sun. The Earth makes one complete revolution with respect to the Sun in 24 hr. The center of the Earth is not fixed in an inertial frame, but orbiting around the Sun. Thus the Earth would make on complete revolution with respect to the Sun in 365 days even if it were not rotating in an inertial frame in which the distant galaxies were at rest. The rotation period of the Earth with respect to inertial frame of the distant galaxies is therefore $(24 - 24/365)$ hr $= 23.93$ hr or 23 hr, 56 min, and 4 s approximately. This is called the *sidereal* rotation period. (There are also negligible corrections for the Sun's rotation around the center of the galaxy, etc. ) The plane of the pendulum makes one complete rotation in 23.93 hr so the angular rate is $360/23.93 = 15.04°/\text{hr}$.

**3-4**. *Find the gravitational potential inside and outside of a sphere of uniform mass density having a radius $R$ and a total mass $M$. Normalize the potential so that it vanishes at infinity.*

**Solution:** The mass density in the sphere is

$$\mu = \frac{M}{(4\pi R^3/3)} = \frac{3M}{4\pi R^3} = \text{const.} \tag{1}$$

The spherical symmetry of the problem implies that $\Phi$ is a function only of the radius $r$. Poisson's equation (3.18) for the gravitational potential is

$$\frac{1}{r^2}\frac{1}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G\mu. \tag{2}$$

The general solution inside ($r < R$) which does not diverge at $r = 0$ is

$$\Phi(r) = Ar^2 + B \tag{3}$$

where $A$ and $B$ are constants. $A$ is determined from (2) and (1) to be

$$A = \frac{2\pi}{3}G\mu = \frac{1}{2}\left(\frac{GM}{R}\right)\frac{1}{R^2}. \tag{4}$$

$B$ is determined by matching to the exterior solution $\Phi(r) = -GM/r$ at $r = R$ to be $B = -3GM/(2R)$. Thus,

$$\Phi(r) = \frac{GM}{2R}\left[\left(\frac{r}{R}\right)^2 - 3\right], \qquad r < R \tag{5}$$

$$= -\frac{GM}{r}, \qquad r > R. \tag{6}$$

**3-5**. *Consider the functional*

$$S[x(t)] = \int_0^T \left[\left(\frac{dx(t)}{dt}\right)^2 + x^2(t)\right] dt\ .$$

*Find the curve $x(t)$ satisfying the conditions*

$$x(0) = 0, \quad x(T) = 1\ ,$$

*which makes $S[x(t)]$ an extremum. What is the extremum value of $S[x(t)]$? Is it a maximum or minimum?*

**Solution:** Lagrange's equations

$$-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)+\frac{\partial L}{\partial x}=0 \tag{1}$$

are the necessary condition for an extremum to functionals of the form

$$S\left[x(t)\right]=\int dt\, L\left(\dot{x},x\right)\ . \tag{2}$$

In the present case, $L=\dot{x}^2+x^2$ and the Lagrange equation (1) is

$$\ddot{x}_{cl}=x_{cl}$$

whose general solution is a linear combination of $\sinh t$ and $\cosh t$. The solution satisfying $x(0)=0$, $x(T)=1$ is

$$x_{cl}(t)=\frac{\sinh t}{\sinh T}\ .$$

The value of the action at this extremum can be found by doing the integral directly, but is most easily computed by integrating (2) by parts to give

$$S[x(t)]=\dot{x}(t)\,x(t)\Big|_0^T+\int_0^T dt\,x(t)\ \left[-\ddot{x}(t)+x(t)\right]\ .$$

The second term vanishes because of Lagrange's equation, so

$$S\left[x_{cl}(t)\right]=\coth T \tag{3}$$

for the extremal path. The argument of the action is positive for any choice of $x(t)$ and can be made arbitrarily big by choosing a wiggly path with big $\dot{x}$. The extremum, therefore, cannot be a maximum but must be a minimum. For example, the simple path

$$x_*(t)=\frac{t}{T}$$

satisfies the boundary conditions, and

$$S\left[x_*(t)\right]=\frac{1}{T}\left(1+\frac{T^2}{3}\right)$$

which is greater than (3), as calculating a few values or a simple plot will show.

**3-6**. *[B,E,C] Estimate the gravitational self-energy of the Moon as a fraction of the Moon's rest mass energy. Is this ratio larger or smaller than the accuracy of the Lunar laser ranging test of the equality of gravitational and inertial mass?*

**Solution:** The gravitational self energy is of order

$$E_{\text{self}} \sim \frac{GM_{\text{moon}}^2}{R_{\text{moon}}}$$

and the ratio to the rest energy is

$$\frac{E_{\text{self}}}{E_{\text{rest}}} \sim \frac{GM_{\text{moon}}}{R_{\text{moon}}c^2} \sim \frac{(6.67 \times 10^{-8})(7.35 \times 10^{25})}{(1.7 \times 10^8)(3 \times 10^{10})^2} \sim 3 \times 10^{-11}$$

which is within the $10^{-13}$ accuracy of lunar laser ranging.

# Chapter 4

# The Principles of Special Relativity

**4-1.** *[B,S] Today a TGV train (train à grande vitesse) leaves Paris (Gare de Lyon) at 8:00 and arrives at Lyon (Part Dieu) at 10:04 (using a 24hr clock). Assuming the train makes no intermediate stops, plot the world line of the train on a copy of the railway spacetime diagram on p. 71. If the distance between Paris and Lyon is 472 km, how fast is the train traveling on average?*

**Solution:**

Fig. 7. Graphique

The average velocity is approximately 228 km/hr. Evidently, from the smaller slope of its world line, this TGV is faster than late nineteenth century trains.

**4-2**. *A rocket ship of proper length $L$ leaves the Earth vertically at speed $(4/5)c$. A light signal is sent vertically after it which arrives at the rocket's tail at $t = 0$ according to both rocket and Earth based clocks. When does the signal reach the nose of the rocket according to (a) the rocket clocks; (b) the Earth clocks?*

**Solution:**

a) In the rocket frame, as in all inertial frames, the velocity of light is $c$, so the time to traverse the proper length $L$ is $t' = L/c$.

b) There are at least two instructive ways of doing this problem:

*Direct calculation in the earth frame;*
Let $t$ be the time as read on earth clocks that the light signal reaches the nose of the rocket. The signal has traveled a distance equal to the *contracted* length of the rocket plus the distance $(4/5)ct$ it traveled in the time $t$. Thus,

$$ct = L\left[1 - (4/5)^2\right]^{1/2} + (4/5)ct$$

Solving for $t$, one finds

$$t = 3L/c$$

*Transforming back from the rocket frame;*
The event of the signal reaching the nose occurs at $t' = L/c$, $z' = L$ in the rocket frame if $c$ is the vertical direction. Therefore in the earth frame

$$t = \frac{t' + (4/5)c\,z'}{\sqrt{1 - (4/5)^2}} = \frac{3L}{c}$$

**4-3**. *A* 20 m *pole is carried so fast in the direction of its length that it appears to be only* 10 m *meters long in the laboratory frame. The runner carries the pole through the front door of a barn* 10 m *long. Just at the instant the head of the pole reaches the closed rear door, the front door can be closed, enclosing the pole within the* 10 m *barn for an instant. The rear door opens and the runner goes through. From the runner's point of view, however, the pole is* 20 m *long and the barn is only* 5 m*! Thus the pole can never be enclosed in*

*the barn. Explain, quantitatively and by means of spacetime diagrams, the apparent paradox.*

**Solution:**

Shown above is a spacetime diagram in the frame where the barn is at rest and the pole is moving. The solid, vertical lines are the world lines of the front and rear barn doors, the heavy parts indicating when the doors are shut. The dotted lines are the world lines of the end of the pole. At the moment $t_*$ the pole is in the barn and both doors are simultaneously shut. The coordinates $(t', x')$ of the frame in which the pole is stationary and the barn is moving are also indicated as well as some lines of constant $t'$.

From this spacetime diagram it is evident that the closing of the front door and the opening of the rear door are not simultaneous in the pole frame. Rather, the front door closes after the rear door opens. This allows the shorter barn time to pass over the longer pole as we now demonstrate quantitatively.

In the pole frame the time difference between the two events simultaneous in the barn frame in [*cf.* (4.24)] is

$$\Delta t' = \gamma v L_* / c^2$$

where $L_* = 10$ m is the proper width of the barn and $v$ is its velocity. This velocity is such that the 20 m pole is contracted to 10 m in the barn frame, i.e., $\gamma = 2$, $v = (\sqrt{3}/2)c$. At the time the rear door opens, 5 m of the pole is within the contracted 5 m length of the barn. Another $v\Delta t' = 2 \cdot (\sqrt{3}/2)^2 \cdot 10 = 15$ m can pass through before the front door closes. The total makes the full 20 m length of the pole. There is no contradiction.

**4-4.** *A satellite orbits the Earth in a circular orbit above the equator a distance of* 200 km *from the surface. By how many seconds per day will a clock on such a satellite run slow compared to a clock on the Earth? (Compute just the special relativistic effects.)*

**Solution:** Neglecting the earth's orbital motion, we can think of the earth as rotating about an axis in an inertial frame. The speed $V_s$ of the satellite is related to the distance $r_s$ from the earth's center by

$$\frac{V_s^2}{r_s} = \frac{GM_\oplus}{r_s^2}$$

where $M_\oplus$ is the mass of the earth. Thus,

$$\frac{V_s}{c} = \left(\frac{GM_\oplus}{c^2 r_s}\right)^{\frac{1}{2}} = 2.6 \times 10^{-5}$$

for $r_s = r_\oplus + 200km$, where $r_\oplus = 6378km$. The speed of the earth at its surface is

$$\frac{V_{\text{surf}}}{c} = \frac{1}{c}\left(\frac{2\pi r_\oplus}{24hrs}\right) = 1.5 \times 10^{-6}$$

The satellite clock is moving faster in the inertial frame and will run slower compared with the clock on the surface. The ratio of rates is

$$\frac{(\text{Rate of sat clock})}{(\text{Rate of surf clock})} = \frac{(1 - V_s^2/c^2)^{\frac{1}{2}}}{(1 - V_{\text{surf}}^2/c^2)^{\frac{1}{2}}} \approx 1 + \frac{1}{2}\left(\frac{V_{\text{surf}}}{c}\right)^2 - \frac{1}{2}\left(\frac{V_s}{c}\right)^2$$

since both velocities are small compared to $c$. The ratio is thus

$$1 + 3.4 \times 10^{-10}$$

So, in one day the clocks will differ by $(3.4 \times 10^{-10}) \times (8.6 \times 10^4 \text{ s}) = 29\ \mu\text{s}$.

**4-5.** *[B,E] The radio source 3C345 is participating in the expansion of the universe and its distance can be determined from the redshift arising from its recession velocity and assumptions about our universe. (Work Problem Chapter 19.1 when you have studied a little cosmology.) However, a rough idea of the distance can be obtained from Hubble's law relating distance $d$ to observed recession velocity $V$:*

$$V = H_0 d$$

*where $H_0 \approx 72$ (km/s)/Mpc is the Hubble constant. (Look at the endpapers for astronomical units like the megaparsec (Mpc).) $V$ for 3C345 is about .6c. Use these facts together with the data in Box 4.3 to roughly estimate the velocity of the cloud C2 assuming (contrary to fact) that it is moving transverse to the line of sight.*

**Solution:** From the figure in Box 4.3 we can roughly estimate that the cloud $C2$ moves about 2 mas (milliarcseconds) in 4.7 yr. To find out how far it moves we need the distance to 3C345. Hubble's law gives

$$.6c = .6(3 \times 10^5 \text{ km/s}) = H_0 d$$

which with the value of $H_0$ specified gives (1 pc $= 3 \times 10^{18}$ cm)

$$d \approx 2.5 \times 10^3 \text{ Mpc} \approx 7.5 \times 10^{22} \text{ km}.$$

Assuming the cloud is moving transversely to the line of sight, the distance it travels is $s = \theta d$ where $\theta$ is 2 mas in radians. Thus,

$$\begin{aligned} s &\approx \frac{2\pi \times 2 \times 10^{-3}}{360 \times 60 \times 60} \times 7.5 \times 10^{22} \text{ km} \\ &\approx 7 \times 10^{14} \text{ km}. \end{aligned}$$

Since 4.7 yr $\approx 1 \times 10^8$ s, the transverse velocity $V_T$is

$$V_T \approx 7 \times 10^6 \text{ km/s} \approx 24c.$$

This is just an estimate but the velocity is larger than $c$. A detailed calculation presented in the solution to Problem 19.1 gives about $13c$.

**4-6**. *Example 2 showed how time dilation in a moving clock could be understood in terms of the working of a model clock consisting of two mirrors oriented along the direction of motion. Show that the same result can be derived using a similar clock oriented perpendicular to the direction of motion.*

**Solution:**

The mirrors are a distance $L$ apart in their rest frame. A light signal starts from $A$, is reflected at $B$, and returns at $C$. We calculate the elapsed time for that in the frame where the mirrors are moving with speed $V$ and compare with the elapsed in $2L/c$ in the rest frame.

With an appropriate choice of origins of $x$ and $t$, the positions of the left ($L$) and right ($R$) mirrors as a function of time are

$$x_L(t) = Vt \tag{1}$$

$$x_R(t) = L\left[1 - \left(\frac{V}{c}\right)^2\right]^{\frac{1}{2}} + Vt \tag{2}$$

where $L[1-(V/c)^2]^{\frac{1}{2}}$ is the Lorentz contract separation between the mirrors. Let $t_A = 0$ be the time the light ray is emitted, $t_B$ the time it is reflected, and $t_C$ the time it returns. Since light travels with speed $c$,

$$ct_B = x_R(t_B) - x_L(0) \tag{3}$$

$$c(t_C - t_B) = x_R(t_B) - x_L(t_C) \tag{4}$$

give the relation between time and distance of the outgoing and reflected ray.

Using (1) and (2) for solving (3) and (4) gives

$$t_B = \frac{L}{c}\left(\frac{1+V/c}{1-V/c}\right)^{\frac{1}{2}}. \tag{5}$$

Substituting this in (4) and solving for $t_C$ gives

$$t_C = \frac{2L}{c}\left[1-\left(\frac{V}{c}\right)^2\right]^{-\frac{1}{2}}. \tag{6}$$

Since $\Delta\tau \equiv 2L/c$ is the interval between ticks in the rest frame, and $\Delta t \equiv t_C$ is the interval in the moving clock frame,

$$\Delta\tau = \Delta t\sqrt{1-\left(\frac{V}{c}\right)^2}. \tag{7}$$

This is (4.15) — time dilation.

**4-7**. *[S,P] In (4.4) we deduced a travel time $\Delta t'$ for a pulse of light traveling between two mirrors that were moving with a speed $V$. This time was different from the travel time $\Delta t$ in the frame in which the mirrors are at rest, (4.3). In Newtonian physics, with its absolute time, these times would necessarily agree. Carry out the analysis that led to $\Delta t'$ in (4.4) using the principles of Newtonian physics and show that this is the case, assuming that the rest frame of the mirrors is the rest frame of the ether.*

**Solution:** Maxwell's equations which govern the propagation of light are valid only in the rest frame of the ether. Suppose this is the frame in which the two mirrors are at rest. The velocity of the light signal is $|\vec{V}| = (0, c, 0)$. In the frame moving with speed $-v$ along the $x$-axis, it is $|\vec{V}'| = (v, c, 0)$ from the Newtonian addition of velocities (4.2). The key point is that the component $V^y$ is the same in both frames so $\Delta t' = 2L/c = \Delta t$.

**4-8**. *[S] Calculate the hyperbolic angle between the sides AC and AB of the triangle ABC illustrated in Figure 4.8.*

**Solution:** From (4.10), the point $(ct, x) = (3, 5)$ makes a hyperbolic angle with the $t = 0$ line of

$$\tanh\theta = \frac{ct}{x} = .6 \ ,$$

so $\theta = \tanh^{-1}(.6) = .69$.

**4-9**. *Consider two twins Joe and Ed. Joe goes off in a straight line traveling at a speed of (24/25)c for seven years as measured on his clock, then reverses and returns at half the speed. Ed remains at home. Make a spacetime diagram showing the motion of Joe and Ed from Ed's point of view. When they return what is the difference in ages between Joe and Ed?*

**Solution:**

The time $t_1$ to the turn around point as measured by Ed is related to Joe's proper time, $\tau_1 = 7$ yr to the same point by

$$t_1 = \tau_1 \left[1 - \left(\frac{24}{25}\right)^2\right]^{-\frac{1}{2}} = 25 \text{ yr}$$

Since the return velocity is half the outbound velocity, it takes twice as long for the return trip (50 yr) according to Ed. Ed has therefore aged by a total

of 75 yr. Joe aged $\tau_1 = 7$ yr on the outbound trip, and

$$\tau_2 = 2t_1 \left[1 - \left(\frac{12}{25}\right)^2\right]^{\frac{1}{2}} = 44 \text{ yr}$$

on the return. The total for Joe is 51 years, so he is younger by 24 years on return.

**4-10**. *In the novel "Return from the Stars" by S. Lem which is concerned with the problems a returning twin in the twin paradox situation might face, there is the following passage:*

> *"Her eyes were shining and attentive. '... I was thirty then'. The expedition ... 'I was a pilot on the expedition to Fomalhaut. That's twenty-three light years away. We flew there and back in a hundred and twenty years ship time. Four days ago we returned ... The* Prometheus *— my ship — remained on Luna. I came from there today. That's all.' "*[1]

*Assuming that all accelerations are instantaneous and the the velocity of the Prometheus was constant in between, with what speed did it travel from the Earth to Fomalhaut?*

**Solution:** We'll present two ways to arrive at the answer.
*First version*:
Let $V$ be the speed of the Prometheus, $d$ the distance it traveled (23 ly), and $\tau$ the total proper time traveled (120 yr). The elapsed time in the earth based frame is

$$T = \frac{\tau}{\sqrt{1 - V^2}}$$

and the speed is $V = (2d)/T$. Thus,

$$V = \frac{2d}{\tau}\sqrt{1 - V^2}$$

[1] S. Lem, *Return from the Stars*, Harcourt Brace Jovanovich, San Diego, 1989.

Solving for $V$ gives,

$$\begin{aligned} V &= \frac{2d/\tau}{\sqrt{1+(2d/\tau)^2}} \\ 2d = 2 \times 23 \text{ ly} &= 4.35 \times 10^{14} \text{ km} \\ \tau = 120 \text{ yr} &= 3.78 \times 10^9 \text{ s} \\ V &= .36 = .36c. \end{aligned}$$

*Second version*:
The spacetime interval is an invariant. For one leg, the twin on the ship measures $\Delta t_{\text{ship}} = 120\,\text{yr}$ and $\Delta x_{\text{ship}} = 0$. The twin on earth measures $\Delta x_{\text{earth}} = 23\,\text{ly}$. Invariance of the interval supplies an equation for $\Delta t_{\text{earth}}$:

$$-\Delta t^2_{\text{ship}} = -\Delta t^2_{\text{earth}} + \Delta x^2_{\text{earth}}$$

which gives $\Delta t_{\text{earth}} = 64\,yr$. The speed in the earth frame is

$$V = \Delta x_{\text{earth}}/\Delta t_{\text{earth}} = .36\,c.$$

**4-11.** *[C] Alice and Bob are moving in opposite directions around a circular ring of radius $R$ which is at rest in an inertial frame. Both move with constant speeds $V$ as measured in that frame. Each carries a clock which they synchronize to zero time at a moment when they are at the same position on the ring. Bob predicts that when next they meet Alice's clock will read less than his because of the time dilation arising because she has been moving with respect to him. Alice predicts that Bob's clock will read less with the same reasoning. They both can't be right. What's wrong with their arguments? What will the clock's really read?*

**Solution:** The problem is most easily analyzed in the inertial frame in which the ring is at rest. In that frame, the time to go once around the ring is $T = 2\pi R/V$. The proper time elapsed for both Alice and Bob is, from (4.14),

$$\Delta\tau_{\text{once around}} = \frac{2\pi R}{V}\sqrt{1-V^2}\,.$$

Alice and Bob agree and their clocks will thus be synchronized when next they meet. Their arguments about moving clocks running slow do not apply

because neither Alice nor Bob, nor their clocks, are at rest in any inertial frame.

**4-12.** *(a) Show explicitly that the straight line path between any two points in flat three-dimensional space ($dS^2 = dx^2 + dy^2 + dx^2$) is the shortest distance between them.*

*(b) Is the straight line path between two spacelike separated points in flat spacetime the shortest distance between them?*

**Solution:**

a) Orient Cartesian coordinates $(x, y, z)$ so that one point is at the origin and the other is a distance $L$ away on the $x$-axis. Any curve connecting the two points can be specified by giving $y(x)$ and $z(x)$. The distance along such a curve is

$$S = \int ds = \int \left[dx^2 + dy^2 + dz^2\right]^{\frac{1}{2}} = \int_0^L dx \left[1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2\right]^{\frac{1}{2}}. \tag{1}$$

The distance is smallest when $dy/dx = dz/dx = 0$. But that is the straight line path along the $x$-axis.

b) In four dimensions, the generalization of (1) would be, from (4.6)

$$s = \int_0^L dx \left[-c^2 \left(\frac{dt}{dx}\right)^2 + 1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2\right]^{\frac{1}{2}}. \tag{2}$$

An argument as in (a) cannot be made because of the minus sign. Indeed the following path has zero distance between the points at $x = 0$ and $x = L$ along the $x-$axis.

**4-13.** *In an inertial frame two events occur simultaneously at a distance of 3 meters apart. In a frame moving with respect to the laboratory frame, one event occurs later than the other by $10^{-8}$ s. By what spatial distance are the two events separated in the moving frame? Solve this problem in two ways: first by finding the Lorentz boost that connects the two frames, and second by making use of the invariance of the spacetime distance between the two events.*

**Solution:** The interval between the two events

$$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2$$

must be the same in both frames. In the laboratory frame

$$(\Delta s)^2 = 0^2 + (3\ \mathrm{m})^2 = 9\ \mathrm{m}^2\ .$$

In the moving frame

$$9\ \mathrm{m}^2 = -\left(3 \times 10^8\ \mathrm{m/s} \cdot 10^{-8}\ \mathrm{s}\right)^2 + (\Delta x)^2$$

which gives

$$\Delta x = \sqrt{18\ \mathrm{m}^2} = 4.24\ \mathrm{m}\ .$$

Let $(t, x)$ be coordinates of the inertial frame in which the events are simultaneous, and $(t', x')$ coordinates of a frame moving with respect to this one along the $x-$axis. The Lorentz boost connecting the two frames implies

$$t_2' - t_1' = \gamma\left[(t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1)\right]$$

In the unprimed frame, the two events are simultaneous ($\Delta t \equiv t_2 - t_1 = 0$) and separated by $\Delta x \equiv x_2 - x_1 = 3$ m. Then

$$\Delta t' \equiv t_2' - t_1' = -\gamma \frac{v}{c^2} \Delta x = 10^{-8} \text{ s}$$

and solving for $\gamma^2$ gives

$$\gamma^2 = 1 + c^2 \left( \frac{\Delta t'}{\Delta x} \right)^2 = 2.$$

Therefore the separation of the two events in the moving frame is

$$\Delta x' = \gamma \Delta x = \sqrt{2}(3) = 4.24 \text{ m}.$$

**4-14.** *[C] This problem concerns the toy model satellite location system discussed in the example on Example 4. Suppose you simultaneously receive broadcasts from two neighboring satellites A and B that report their locations $x'_A$ and $x'_B$ as well as their times of broadcast $t'_A$ and $t'_B$ which are equal $t'_A = t'_B$. The times and positions are in the rest frame of the satellites to which their clocks are all synchronized. Derive a condition that determines your position in $x$. Evaluate it to find your deviation from the midpoint between the satellites to first order in $V/c$ where $V$ is the speed of the satellites.*

**Solution:** Two reference frames are relevant for this problem. The $(t', x', y')$ rest frame of the satellites that is moving with velocity $V$ with respect to the rest frame $(t, x, y)$ of the observer. (The $z$-direction is irrelevant for this problem.) The satellites broadcast their location and the times of the emissions of their signals in their rest frame. Let $(t'_A, x'_A, h)$ and $(t'_B, x'_B, h)$ be the coordinates of the emissions of the two signals that are received simultaneously by the observer in her frame at $(t, x, 0)$. (The problem states $t'_A = t'_B$, but let's keep this general for a moment.) The coordinates of the two events of emission in the observer's rest frame can be found from a Lorentz boost, *e. g.*

$$t_A = \gamma \left( t'_A + V x'_A / c^2 \right) \tag{4.1a}$$

$$x_A = \gamma \left( x'_A + V t'_A \right) \tag{4.1b}$$

and similarly for $(t_B, x_B)$. The events will be received simultaneously by the observer if

$$\left[(x - x_A)^2 + h^2\right]^{\frac{1}{2}} = c(t - t_A) \tag{4.2a}$$

$$\left[(x_B - x)^2 + h^2\right]^{\frac{1}{2}} = c(t - t_B) \tag{4.2b}$$

Subtracting gives the condition

$$\left[(x_B - x)^2 + h^2\right]^{\frac{1}{2}} - \left[(x - x_A)^2 + h^2\right]^{\frac{1}{2}} = c(t_B - t_A) \tag{3}$$

where $t_A$, $x_A$, $t_B$, $x_B$ can be expressed in terms of $t'_A$, $x'_A$, $t'_B$, $x'_B$ by (1). The condition (3) determines $x$.

In particular, using $t'_A = t'_B$

$$t_B - t_A = \gamma\left(\frac{V}{c^2}\right)(x'_B - x'_A) = \gamma\left(\frac{V}{c^2}\right) L_* \tag{4}$$

where $L_*$ is the proper distance between the satellites.

If $t_B - t_A = 0$, the solution to (3) would have the observer at the position $\bar{x} \equiv (x_A + x_B)/2$ equidistant from $x_A$ and $x_B$. But because of the relativity of simultaneity, $t_A - t_B \neq 0$ and the observer is closer to one satellite than to the other. It's messy to solve (3) for $x$, but for $V/c \ll 1$ we can write

$$x = \bar{x} + \delta x$$

and solve for $\delta x$ to first order in $V/c$.

The condition (3) becomes

$$\delta x\, \gamma\, L_* \left[\left(\frac{L}{2}\right)^2 + h^2\right]^{-\frac{1}{2}} = \gamma\left(\frac{V}{c}\right) L_*$$

where $L = L_*\gamma$. The result for $\delta x$ is

$$\delta x = \left(\frac{V}{c}\right)\left[\left(\frac{\gamma L_*}{2}\right)^2 + h^2\right]^{\frac{1}{2}} .$$

**4-15.** *Show that the addition of velocities* (4.28) *implies that (a) if* $|\vec{V}| < c$

*in one inertial frame then $|\vec{V}| < c$ in any other inertial frame, (b) if $|\vec{V}| = c$ in one inertial frame then $|\vec{V}| = c$ in any other inertial frame, and that (c) if $|\vec{V}| > c$ in any inertial frame then $|\vec{V}| > c$ in any other inertial frame.*

**Solution:** Orient coordinates so that the $x$-axis is along $V$. Then the addition of velocities (4.28a) gives

$$V' = \frac{V - v}{1 - Vv/c^2}$$

Plotting $V'(V)$ for $|V| < c$ we see that it ranges between $-c$ for $V = -c$ and $+c$ for $V = +c$. In all cases $|V'| < c$.

The problem can be analyzed algebraically as will as graphically. The addition of velocities (4.28a) can be written

$$V = \frac{V' + v}{1 + V'v/c^2}$$

For (a), set

$$|V| = \left| \frac{V' + v}{1 + V'v/c^2} \right| < c \, .$$

Solving for $V'$ gives $|V'| < c$. (b) and (c) can be done similarly.

**4-16**. *Lengths Perpendicular to Relative Motion are Unchanged*

*Imagine two meter sticks, one at rest, the other moving along an axis perpendicular to the first and perpendicular to its own length, as shown above. There is an observer riding at the center of each meter stick.*

a) *Argue that the symmetry about the $x$-axis implies that both observers will see the ends of the meter sticks cross simultaneously and that both observers will therefore agree if one meter stick is longer than the other.*

b) *Argue that the lengths cannot be different without violating the principle of relativity.*

**Solution:**

a) If either observer saw one end of the other meter stick cross his or hers first that would violate the evident symmetry about the $x$-axis. Both ends must therefore cross simultaneously for both observers.

b) The situation with regard to measuring the length of the moving meter stick is completely symmetric between the two observers. If one measured a shorter length than the other it would distinguish his or her inertial frame from the other one. That would violate the principle of relativity.

**4-17**. *Another derivation of Lorentz contraction. Example 2 showed how the operation of a model clock was consistent with time dilation. This problem aims at showing how Lorentz contraction is consistent with ideal ways of measuring lengths.*

*The length of a rod moving with speed $V$ can be determined from the time it takes to move at speed $V$ past a fixed point (left hand figure above). The length of a stationary rod can also be determined by measuring the time it takes a fixed object to move from end to end at speed $V$ (right hand figure above). Taking account of the time dilation between the two frames, show that the length of the moving rod determined in this way is Lorentz contracted from its stationary length.*

**Solution:** Consider, for example, a rod moving along its own length with speed $v$ past observer $O$ as in the above figure. As measured by Observer $O$, the length will be will be

$$L = v\Delta\tau$$

where $\Delta\tau$ is the time interval between when the nose of the rod coincides with $O$'s position and the time when the tail of the rod is coincident. An observer

$O'$ riding on the rod sees observer $O$ moving in the opposite direction with speed $v$, as illustrated above. The time $\Delta t$ for the observer $O$ to traverse the rod will be

$$\Delta t = L_*/v$$

where $L_*$ is the length measured by $O'$. $L_*$ is the proper length of the rod since it is measured in its rest frame. $\Delta\tau$ is a proper time interval on the clock of $O$ and $\Delta t$ is the corresponding interval in a frame in which that clock is moving with speed $v$. Using the above result for $\Delta t$ and eliminating $\Delta\tau$ from (4.14) one finds

$$L = L_*\sqrt{1 - v^2/c^2}\ . \tag{1}$$

The moving rod is contracted in the direction of its length.

**4-18.** *[S] Show that for two timelike separated events there is some inertial frame in which $\Delta t \neq 0$, $\Delta\vec{x} = 0$. Show that for two spacelike separated events there is an inertial frame where $\Delta t = 0, \Delta\vec{x} \neq 0$*

**Solution:** Two timelike separated events $A$ and $B$, have $\Delta s^2 < 0$. Construct rectangular coordinates by using the straight line through $A$ and $B$ as the time axis and align the spatial axes along three orthogonal spacelike directions. The result is a rectangular system in which evidently $\Delta t' \neq 0$, $\Delta\vec{x}' = 0$.

One can also start with an inertial frame in which none of the $\Delta x^\alpha$ are zero and make a Lorentz transformation to a new frame where $\Delta t' \neq 0$, $\Delta\vec{x}' = 0$. Suppose, for simplicity, $\Delta y = \Delta z = 0$. The required Lorentz transformation is the boost along the $x$-axis such that

$$0 = \Delta x' = \gamma(\Delta x - v\Delta t)\ .$$

The condition that the events are timelike separated $\Delta t > \Delta x$ guarantees that this can be solved with $v < 1$.

The spacelike case is exactly analogous.

**4-19.** *[C] If a photograph is taken of an object moving uniformly with a speed approaching the speed of light parallel to the plane of the film, it does*

*not appear contracted in the photograph, but rather rotated. Explain why. (Assume the object subtends a small angle from the camera lens.)*

**Solution:**

Consider a rectangular object moving parallel to the plane of the film with speed $V$ as shown above. Suppose the long side has a rest length $a$ and the short side a rest length $b$. Because of Lorentz contraction, the image of the long side will have a length $a\sqrt{1-V^2}$. The light from the far side takes a time $b$ ($c = 1$ units) longer to get to the film than the near side. A photo taken at one instant will therefore, show the near side and the far side as it was a time $b$ earlier when it was a distance $Vb$ to the left, as shown. That's just the same as if the object were rotated by an angle $\theta$ with $V = \sin\theta$, since

$$\begin{aligned} a\cos\theta &= a\sqrt{1-V^2} \\ b\sin\theta &= bV \ . \end{aligned}$$

# Chapter 5

# The Spacetime of Special Relativity

**5-1.** *[S] Consider two four-vectors* **a** *and* **b** *whose components are given by*

$$a^\alpha = (-2, 0, 0, 1)$$
$$b^\alpha = (5, 0, 3, 4)\ .$$

*a) Is* **a** *timelike, spacelike, or null? Is* **b** *timelike, spacelike, or null?*

*b) Compute* $\mathbf{a} - 5\mathbf{b}$.

*c) Compute* $\mathbf{a} \cdot \mathbf{b}$.

**Solution:**

a)

$$\begin{aligned} \mathbf{a} \cdot \mathbf{a} &= -2 \cdot 2 + 0 \cdot 0 + 0 \cdot 0 + 1 \cdot 1 = -3 \\ \mathbf{b} \cdot \mathbf{b} &= -5 \cdot 5 + 0 \cdot 0 + 3 \cdot 3 + 4 \cdot 4 = 0 \end{aligned}$$

Thus, **a** is timelike, and **b** is null.

b)

$$\begin{aligned} \mathbf{a} - 5\mathbf{b} &= (-2, 0, 0, 1) + (-25, 0, -15, -20) \\ &= (-27, 0, -15, -19). \end{aligned}$$

c)

$$\mathbf{a} \cdot \mathbf{b} = -(-2 \cdot 5) + 0 \cdot 0 + 0 \cdot 3 + 1 \cdot 4 = 14 \ .$$

**5-2**. *The scalar product between two three-vectors can be written as*

$$\vec{a} \cdot \vec{b} = ab \cos \theta_{ab}$$

*where $a$ and $b$ are the lengths of $\vec{a}$ and $\vec{b}$ respectively and $\theta_{ab}$ is the angle between them. Show that an analogous formula holds for two timelike four-vectors* **a** *and* **b***:*

$$\mathbf{a} \cdot \mathbf{b} = -ab \cosh\theta_{ab}$$

*where $a = (-\mathbf{a} \cdot \mathbf{a})^{1/2}$, $b = (-\mathbf{b} \cdot \mathbf{b})^{1/2}$ and $\theta_{ab}$ is the parameter defined in* (4.18) *that describes the Lorentz boost between the frame where an observer whose world line points along* **a** *is at rest, and the frame where an observer whose world line points along* **b** *is at rest.*

**Solution:** Work in the frame $A$ of the observer whose four-velocity is pointing along **a**, and orient the spatial coordinates so that $x$ points along **b**. Then

$$\mathbf{a} = (a, 0, 0, 0) \tag{1}$$

Similarly in the frame $B$ of an observer whose four-velocity points along **b**, $\mathbf{b} = (\mathbf{b}, 0, 0, 0)$. Suppose $B$ is moving with respect to $A$ with a relative rapidity $\theta_{ab}$. Making a Lorentz transformation from the $B$ to the $A$ frame, the components of $B$ are [*cf* (4.18)]

$$\mathbf{b} = (b \cosh \theta_{ab}, b \sinh \theta_{ab}, 0, 0)$$

in the $A$ frame. Taking the inner product with (2) in the $A$ frame:

$$\mathbf{a} \cdot \mathbf{b} = -ab \cosh \theta_{ab} \ .$$

*Another way of doing the problem is a follows:*

Since $\mathbf{a}_1$ and $\mathbf{a}_2$ are timelike, we can choose a frame so that $\mathbf{a}_1$ lies purely in the time direction

$$\mathbf{a}_1 = (a_1^t, 0, 0, 0)$$

and the spatial part of $\mathbf{a}_2$ points only in the $x$-direction,

$$\mathbf{a}_2 = (a_2^t, a_2^x, 0, 0)$$

Thus, in this frame

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = -a_1^t a_2^t \tag{2}$$

The value of $a_1^t$ in this frame is found from

$$a_1^2 \equiv -\mathbf{a}_1 \cdot \mathbf{a}_1 = -(a_1^t)^2 \tag{3}$$

In this frame the velocity of an observer whose four-velocity points along the direction of $\mathbf{a}_1$ is zero. The velocity of an observer whose four-velocity $\mathbf{u}_2$ points along $\mathbf{a}_2$ is

$$v = \frac{dx}{dt} = \frac{dx/d\tau}{dt/d\tau} = \frac{u_2^x}{u_2^t} = \frac{a_2^x}{a_2^t} .$$

Thus, $a_2^x = v a_2^t$, and

$$a_2^2 = -\mathbf{a}_2 \cdot \mathbf{a}_2 = -\left(a_2^t\right)^2 \left(1 - v^2\right) \tag{4}$$

which determines $a_2^t$ in terms of $a_2$ and $v$. The relative rapidity $\theta_{12}$ is defined in terms of the relative velocity by

$$v = \tanh \theta_{12} \tag{5}$$

Using (3) and (4) to eliminate $a_1^t$ and $a_2^t$ from (2), and (5) to eliminate $v$, gives

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = -a_1 a_2 \cosh \theta_{12} .$$

which is analogous to $\vec{a}_1 \cdot \vec{a}_2 = a_1 a_2 \cos \theta_{12}$ for the three-dimensional scalar product.

**5-3**. *[S] A free particle is moving along the $x-$ axis of an inertial frame with speed $dx/dt = V$ passing through the origin at $t = 0$. Express the particles's world line parametrically in terms of $V$ using the proper time $\tau$ as the parameter.*

**Solution:** The world line of a particle moving along the $x$-axis at constant speed $V$ and passing through the origin at $t = 0$ is written in the form parametric form as

$$\begin{aligned} t &= \tau/\sqrt{1-V^2}\,, \\ x &= V\tau/\sqrt{1-V^2}\,, \\ y &= z = 0\,, \end{aligned}$$

if the starting point for $\tau$ is chosen to be the particle's intersection with the origin. The first equation is the usual relation between proper time $\tau$ and the time of the inertial frame $t$. The second is $x = Vt$ written in terms of proper time using the first. (Remember we are using units where $c = 1$ and three-velocity is dimensionless.)

**5-4**. *Work out the components of the four-acceleration vector* $\mathbf{a} \equiv d\mathbf{u}/d\tau$ *in terms of the three-velocity* $\vec{V}$ *and the three-acceleration* $\vec{A} = d\vec{V}/dt$ *to obtain expressions analogous to* (5.28). *Using this expression and* (5.28) *verify explicitly that* $\mathbf{a} \cdot \mathbf{u} = 0$.

**Solution:** This is an exercise in differentiation

$$a^\alpha \equiv \frac{du^\alpha}{d\tau} = \gamma\,\frac{du^\alpha}{dt}\,.$$

Then inserting (5.28) and noting that

$$\frac{d\gamma}{dt} = \gamma^3 \vec{A}\cdot\vec{V}\,,$$

one finds

$$a^\alpha = \left[\gamma^4 \vec{A}\cdot\vec{V}\,,\ \gamma^2\vec{A} + \gamma^4\vec{V}\left(\vec{A}\cdot\vec{V}\right)\right]\,.$$

Then one can check explicitly that

$$\mathbf{a}\cdot\mathbf{u} = 0$$

where $u^\alpha = (\gamma, \gamma\vec{V})$.

**5-5**. *Make a copy of Figure 5.6 and draw on it the acceleration four vectors* $\mathbf{a}$ *at half-scale. Are these vectors orthogonal to* $\mathbf{u}$*?*

**Solution:** The following spacetime diagram shows both the four-velocities in Figure 5.6 and the four-accelerations with the components of (5.40) at the corresponding points. The accelerations are orthogonal to the four-velocities [*cf.* (5.39)] but in the geometry of spacetime, not in the Euclidean geometry of the page.

**5-6.** *Consider a particle moving along the $x$-axis whose velocity as a function of time is*

$$\frac{dx}{dt} = \frac{gt}{\sqrt{1+g^2t^2}}$$

*where $g$ is a constant.*

*a) Does the particle's speed ever exceed the speed of light?*

*b) Calculate the components of the particle's four velocity.*

*c) Express $x$ and $t$ as a function of the proper time along the trajectory.*

*d) What are the components of the four-force and the three-force acting on the particle?*

**Solution:**

a) $(gt)^2 < 1 + (gt)^2$ so $dx/dt < 1$.

b)

$$\begin{aligned} u^t &= \frac{1}{\sqrt{1-V^2}} = \sqrt{1+(gt)^2} \\ u^x &= \frac{V}{\sqrt{1-V^2}} = gt \\ u^y &= u^z = 0 \end{aligned}$$

c) The clock of an observer riding on the particle reads proper time. The proper time elapsed from $t = 0$ to $t$ is

$$\tau = \int_0^t dt\,\sqrt{1-V^2} = \int_0^t \frac{dt}{\sqrt{1+(gt)^2}} = \frac{1}{g}\,\sinh^{-1}(gt)\ . \tag{1}$$

The particle trajectory is

$$x(t) - x_0 = \int_0^t dt\,\frac{gt}{\sqrt{1+(gt)^2}} = \frac{1}{g}\,\sqrt{1+(gt)^2}\ .$$

Thus the relation between $\tau$ — the time on the observer's clock — and the location $x$ is

$$\begin{aligned} (x - x_0) &= \frac{1}{g}\,\sqrt{1+\sinh^2(g\tau)} \\ &= \frac{1}{g}\,\cosh(g\tau) \end{aligned} \tag{2}$$

d) The four force is $f^\alpha = md^2x^\alpha/d\tau^2$ or

$$f^\alpha = (mg\sinh(g\tau), mg\cosh(g\tau), 0, 0)\ .$$

The three force is given by $\vec{F} = md\vec{u}/dt$

$$F^i = (mg, 0, 0)\ .$$

**5-7.** *[C] A particle is moving along the $x$-axis. It is uniformly accelerated in the sense that the acceleration measured in its instantaneous rest frame is always $g$, a constant. Find $x$ and $t$ as functions of the proper time $\tau$ assuming the particle passes through $x_0$ at time $t = 0$ with zero velocity. Draw the world line of the particle on a spacetime diagram.*

**Solution:** Differentiating the normalization condition for the four-velocity

$$\mathbf{u} \cdot \mathbf{u} = -1 \tag{1}$$

we find that generally

$$\mathbf{a} \cdot \mathbf{u} = 0 \tag{2}$$

where $\mathbf{a}$ is the four-acceleration $\mathbf{a} = d\mathbf{u}/d\tau$.

In the rest frame of the particle therefore $\mathbf{a} = (0, g)$ (listing only $t$ and $x$ components). The invariant $\mathbf{a} \cdot \mathbf{a}$ is thus

$$\mathbf{a} \cdot \mathbf{a} = g^2 \tag{3}$$

When written out in a general frame where the particle is moving, equations (1), (2), and (3) become, respectively

$$-(u^t)^2 + (u^x)^2 = -1 \tag{4}$$

$$-a^t u^t + a^x u^x = 0 \tag{5}$$

$$-(a^t)^2 + (a^x)^2 = g^2 \tag{6}$$

These three equations can be solved for $a^t$ and $a^x$ in terms of $g$ and the components of $\mathbf{u}$. One finds

$$a^t \equiv \frac{du^t}{d\tau} = g u^x \tag{7}$$

$$a^x \equiv \frac{du^x}{d\tau} = g u^t \tag{8}$$

Eqs. (7) and (8) are two coupled differential equations for $u^t$ and $u^x$. They can be solved, for example, by eliminating $u^t$ to find

$$\frac{d^2 u^x}{d\tau^2} = g^2 u^x \ . \tag{9}$$

The solution where the $u^x$ is zero at $\tau = 0$ is

$$u^x = A\sinh(g\tau) \tag{10}$$

for some constant $A$. This leads to $u^t = A\cosh(g\tau)$, but the normalization condition (4) implies $A = 1$. Thus,

$$u^x = \frac{dx}{d\tau} = \sinh(g\tau) \tag{11}$$

$$u^t = \frac{dt}{d\tau} = \cosh(g\tau) \tag{12}$$

These equations are easy to integrate with the given boundary condition to find

$$x(\tau) = x_0 + \frac{1}{g}\left[\cosh(g\tau) - 1\right] \tag{13}$$

$$t(\tau) = \frac{1}{g}\sinh(g\tau) \tag{14}$$

The world line is the hyperbola $t^2 = (x - x_0)^2 + (2/g)(x - x_0)$ shown below:

**5-8.** *[S] A $\pi^0$ meson (rest mass 135 MeV) is moving with a speed (magnitude of the three-velocity) $V = c/\sqrt{2}$ in a direction 45° to the $x$-axis.*

*a) Find the components of the four-velocity of the particle.*

*b) Find the components of the energy momentum four-vector.*

**Solution:** Suppose the meson is moving in the $x$-$y$ plane. The components of the three-velocity are then

$$V^x = V^y = (c/\sqrt{2})\cos(45°) = \frac{c}{2} .$$

Plugging into (5.28) gives

$$u^\alpha = (1.41, .71, .71, 0)$$

in $c = 1$ units. The four-momentum is

$$p^\alpha = (190, 95, 95, 0) \text{ MeV} .$$

**5-9**. *[S] In the now decomissioned Stanford Linear Collider electrons and positrons were accelerated to energies of approximately* 40 GeV *in a beam pipe two miles long but only a few centimeters in diameter. Steering an electron through through such a narrowly defined path over such a distance sounds like a daunting task. But how long is the accelerator in the rest frame of the electron when it has this energy?*

**Solution:** The accelerator will be Lorentz contracted to a length $L = L_*\sqrt{1-V^2}$ where $L_* \approx 2$ mi $\approx 3200$ m. The energy $E = m/\sqrt{1-V^2}$ where $m = .51$ MeV is the rest mass of the electron. Therefore,

$$\sqrt{1-V^2} = m/E = (.51 \text{ MeV})/(40 \text{ GeV}) \approx 1.2 \times 10^{-5}$$

The length of the accelerator in the frame of the electron is $\sim 4$ cm. It's not very hard to steer through that distance.

**5-10**. *In the LEP particle accelerator at CERN, electrons and positrons travel in opposite directions around a circular ring approximately* 10 km *in radius at an energy of* 100 GeV *apiece. (a) How close are these particles to moving at the velocity of light? (b) Electrons and positrons can be stored for two*

*hours. How many turns will an electron or positron make around the ring in this time?*

**Solution:**

a) From (5.44) the energy $E$ of the particles is related to their speed $V$ by

$$E = \frac{m}{\sqrt{1 - V^2}}$$

where $m$ is the rest mass. Inverting gives

$$V = \frac{\sqrt{E^2 - m^2}}{E} \approx 1 - \frac{1}{2}\left(\frac{m}{E}\right)^2 + \ldots .$$

The latter approximation is valid when $m/E$ is small, as here. For electrons and positrons, $m \approx .5$ MeV so $m/E \sim 5 \times 10^{-6}$. The velocity is within a few parts in $10^{11}$ of the velocity of light.

b) The time to make one turn is

$$\frac{2\pi R}{V} \approx \frac{2\pi R}{c} \approx \frac{2\pi(10\text{km})}{3 \times 10^5 \text{km/s}} \approx 2 \times 10^{-4}\text{s}.$$

The number of turns in 2hrs is therefore about 36 million.

**5-11**. *Express the law of addition of parallel velocities in terms of the parameter $\theta$ used to describe Lorentz boosts in* (4.18). *Can you give a geometric interpretation to your result?*

**Solution:** Consider a particle moving with speed $V \equiv \tanh\Theta$ in the $x$ direction in one frame. The velocity of the particle as measured in a frame moving with speed $v$ with respect to the first is

$$V' = \frac{V - v}{1 - Vv}$$

If we put $V' = \tanh\Theta'$ and $v = \tanh\theta$, we find

$$\Theta' = \Theta - \theta$$

so addition of velocities becomes simple addition of Lorentz boost parameters.

**5-12.** *The 2 mile long Stanford linear accelerator accelerates electrons to an energy of* 40 GeV *as measured in the frame of the accelerator. Idealize the acceleration mechanism as a constant electric field $E$ along the accelerator and assume that the equation of motion is*

$$\frac{d\vec{p}}{dt} = e\,E.$$

*where $\vec{p}$ is the spatial part of the relativistic momentum* $\mathbf{p}$.

a) *Assuming that the electron starts from rest, find its position along the accelerator as a function of time in terms of its rest mass $m$ and $F \equiv e|\vec{E}|$.*

b) *What value of $|\vec{E}|$ would be necessary to accelerate it to its final energy.*

**Solution:**

a) The momentum $p$ increases linearly with time. Denoting $e|\vec{E}|$ by $F$, we have

$$p(t) = Ft$$

if $t = 0$ is the time the particle is at rest. From the momentum we can find the three-velocity $V$. From (5.45)

$$\frac{dx}{dt} \equiv V = \frac{p}{E} = \frac{p}{\sqrt{m^2 + p^2}} = \frac{Ft}{\sqrt{m^2 + (Ft)^2}}. \tag{1}$$

Integrating this relation gives

$$x(t) = F^{-1}\sqrt{m^2 + (Ft)^2} \tag{2}$$

for the distance from the starting point as a function of time.

b) The energy reached at time $t$ is

$$E = \left[m^2 + p^2(t)\right]^{\frac{1}{2}} = \sqrt{m^2 + (Ft)^2}. \tag{3}$$

So, from (2), the energy reached at a distance $x$ from the start is

$$E(t) = Fx(t). \tag{4}$$

Evaluating this at the final time when $E = 40$ GeV and $x = 2$ mi gives

$$F = \frac{40\text{GeV} \times (1.6 \times 10^{-10}\ \text{J/GeV})}{2\ \text{mi} \times (1609\text{m/ mi})} = 1.99 \times 10^{-12}\ \text{J/m}. \tag{5}$$

The charge on the electron is $1.6 \times 10^{-19}$Coul. The required field is therefore

$$|\vec{E}| \approx 12 \text{ million Volts/m}. \tag{6}$$

**5-13**. *[B,S] One reaction for photoproducing pions is*

$$\gamma + p \rightarrow n + \pi^+ .$$

*Find the minimum energy (the threshold energy) a photon would have to have to produce a pion in this way in the frame in which the proton is at rest. Is this energy within reach of contemporary accelerators?*

**Solution:** The threshold condition, Ref. (b) in Box 5.1 just needs to be evaluated in the frame in which the proton is at rest. In that frame

$$p_p^\alpha = (m_p, 0, 0, 0)$$

and the condition gives

$$-2E_\gamma m_p - m_p^2 = -\left(m_n + m_\pi\right)^2$$

for the threshold energy $E_\gamma$ of the photon. Solving for $E_\gamma$ gives the approximation for $m_n \approx m_p$

$$E_\gamma = m_\pi \left(1 + \frac{m_\pi}{2m_p}\right) \approx 150\text{MeV}.$$

This is well within the reach of contemporary accelerators although photons are not the particles being accelerated.

**5-14.** *[B] Compare the energy of the highest energy cosmic rays with the energy of a rock thrown energetically by yourself.*

**Solution:** Fun with units. Baseball players are said to pitch at 90mph, so perhaps you can throw at 30mph. Now

$$30 \text{ mph} \approx 15 \text{ m/s} \approx 1500 \text{ cm/s} \quad (1 \text{ m/s} = 2.237 \text{ mph}).$$

The kinetic energy of a 1kg rock at this speed is

$$E = \frac{1}{2}\, 10^3 (1500)^2 \text{ erg} \sim 10^9 \text{ erg}$$

$1\text{erg} = 6.42 \times 10^{11}\text{eV}$, so $E \sim 10^{21}\text{eV}$ which is approximately the energy of the highest energy cosmic rays.

**5-15.** *[C] A source and detector are spaced a certain angle $\phi$ apart on the edge of a rotating disk. The source emits radiation at a frequency $\omega_*$ in its instanteous rest frame. What frequency is the radiation detected at? Hint: Little information is given in this problem because little is needed.*

**Solution:** From the symmetry of the problem the angle the detected radiation makes with the emitter's velocity is the $\pi$ minus the angle it makes with the detectors velocity. The velocity of the emitter and detector are the same. Viewed from the inertial frame in which the center of the disk is at rest, the red shift on emission is canceled by the blue shift on detection from (5.73). The detected frequency is $\omega_*$.

**5-16.** *(Aberration) Consider a star which happens to be directly overhead (the zenith) at midnight in a direction that lies in the plane of the the Earth's orbit. To observe the star through a telescope, the telescope axis must be tilted with respect to the zenith direction by a small angle in the direction the*

*Earth is moving in its orbit. Explain why and calculate the angle. To simplify the situation you may assume that the Earth's orbit is approximately circular and, if necessary, that the rotation axis is perpendicular to the orbital plane.*

**Solution:**

The above figure shows why the telescope must be tilted. Light from a star $S$ falls vertically on the Earth's surface. However, in the time $\Delta t$ it takes for the light to traverse from the top (T) to the bottom (B), the telescope has moved horizontally by $V\Delta t$ where $V$ is the surface velocity of the Earth. The telescope must be tilted so that the bottom is at the right place to receive the signal. Let $\alpha$ be the angle of the tilt and L the length in the frame shown. The travel time from T to B is $\Delta t = L\cos\alpha/c$. The necessary horizontal distance is

$$L\sin\alpha = V\Delta t = (L\cos\alpha)(V/c)$$

so

$$\tan\alpha = V/c \ .$$

The velocity of the Earth at the surface at the equator is about 4km/sec, so

$$\alpha \approx 1.3 \times 10^{-5}\text{radians} \approx 2.7'' \ .$$

**5-17**. *[C] (Relativistic Beaming) A body emits radiation of frequency $\omega_*$ isotropically with a number flux $f_*$ [photons/(cm$^2 \cdot$ s)] in its own rest frame. It moves with speed $V$ along the $x'-$axis in an observer's frame.*

a) *Derive* (5.75) *relating a photon's direction of propagation in the rest frame to the direction of propagation in the observer's frame.*

b) *Find the number flux of photons* $f'(\alpha')$ *as a function of angle* $\alpha'$ *from the* $x'-$*axis in the observer's frame.*

c) *Find the luminosity or energy flux* $L'(\alpha')$ [erg/(cm$^2$ s)] *as a function of the angle* $\alpha'$ *in the observer's frame.*

d) *Discuss the beaming of number and energy in the observer's frame as the velocity of the source approaches the velocity of light.*

**Solution:**

a) Suppose a photon is emitted in the rest frame making an angle $\alpha$ with the $x$-axis. Its four momentum is

$$\mathbf{p} = (p, p\cos\alpha, p\sin\alpha, 0) \ , \ p = \hbar\omega$$

In the observer's frame this is

$$\begin{aligned} p^{t'} &= \gamma(p + Vp\cos\alpha) \\ p^{x'} &= \gamma(p\cos\alpha + Vp) \\ p^{y'} &= p\sin\alpha \ . \end{aligned}$$

The angle $\alpha'$ made by the photon with the $x'$ axis is

$$\cos\alpha' = \frac{p^{x'}}{p^{t'}} = \frac{\cos\alpha + V}{1 + V\cos\alpha}$$

b) $2\pi\sin\alpha' d\alpha' dt' f'(\alpha')$ is the number of photons emitted at angle $\alpha'$ into an annulus of angular width $d\alpha'$ in time $dt'$. This must be the same as the number emitted in the corresponding annulus in the rest frame — $2\pi\sin\alpha d\alpha dt f_*$ in the corresponding time $dt$. The time intervals are connected by time dilation. The connection between angles was derived in part (a) with the result

$$f'(\alpha') = \frac{f_*/\gamma}{\left[\gamma\left(1 - V\cos\alpha'\right)\right]^2}$$

c) The energy of the photons emitted at angle $\alpha'$ is from the inverse of (5.73).

$$\hbar\omega = E'\gamma\,(1 - V\cos\alpha')$$

Solving for $E'$ and using the result of part (b) the luminosity is

$$L'(\alpha') = E'(\alpha')f'(\alpha') = \frac{L_*/\gamma}{\left[\gamma\left(1 - V\cos\alpha'\right)\right]^3}$$

d) The ratio of intensity in the forward to backward direction is

$$\frac{L'(0)}{L'(\pi)} = \left(\frac{1+V}{1-V}\right)^3$$

which as $V \to 1$ becomes very large, meaning most of the radiation is beamed forward.

**5-18.** *Work out the frequency as a function of proper time seen by the observer in Example 5.9 by transforming the components of the wave vector of the photons into the instantaneous rest frame of the observer at proper time $\tau$.*

**Solution:** The three-velocity of the observer in the $x$-direction is

$$V = \frac{u^x}{u^t} = \tanh(a\tau)$$

A Lorentz boost by this velocity in the opposite direction will produce a frame in which the observer is at rest. In this frame, the source of the radiation is moving away from the observer with speed $V$. From (5.73) the observed frequency will be

$$\omega' = \omega_* \frac{\sqrt{1-V^2}}{1+V} = \omega_* \sqrt{\frac{1-V}{1+V}} = \omega_* e^{-a\tau}$$

as obtained by the other method.

**5-19**. *[S] An observer moves with a constant speed $V$ along the $x-$ axis of an inertial frame. Find the components in that frame of orthonormal basis four-vectors $\{\mathbf{e}_{\hat{\alpha}}\}$ to which the observer can refer observations.*

**Solution:** The four-vector $\mathbf{e}_{\hat{0}}$ is the observer's four velocity $\mathbf{u}_{\text{obs}}$ which has components [*cf.* (5.28)]

$$(\mathbf{e}_{\hat{0}})^{\alpha} = u^{\alpha}_{\text{obs}} = (\gamma, V\gamma, 0, 0) \ .$$

The only conditions on the other three four-vectors $\mathbf{e}_{\hat{i}}$ are that they be orthogonal to $\mathbf{e}_{\hat{0}}$, orthogonal to each other, and of unit length. There are many possibilities corresponding to the observer's freedom to orient the spatial axes of the orthonormal frame. The easiest way to satisfy the conditions is to pick $\mathbf{e}_{\hat{2}}$ and $\mathbf{e}_{\hat{3}}$ to be unit four-vectors in the $y$- and $z$-directions respectively. The remaining four-vector $\mathbf{e}_{\hat{1}}$ then has the form $(a, b, 0, 0)$. Orthogonality with $\mathbf{e}_{\hat{0}}$ means

$$\mathbf{e}_{\hat{0}} \cdot \mathbf{e}_{\hat{1}} = -\gamma a + \gamma V b = 0 \ .$$

Unit length means

$$\mathbf{e}_{\hat{1}} \cdot \mathbf{e}_{\hat{1}} = -a^2 + b^2 = 1 \ .$$

These two conditions determine $a$ and $b$. The four-vectors $\mathbf{e}_{\hat{i}}$ that together with (5.77) make up an orthonormal basis for the observer are

$$\begin{aligned}
(\mathbf{e}_{\hat{1}})^{\alpha} &= (\gamma V,\ \gamma,\ 0,\ 0) \ , \\
(\mathbf{e}_{\hat{2}})^{\alpha} &= (\ 0\ ,\ 0,\ 1,\ 0) \ , \\
(\mathbf{e}_{\hat{3}})^{\alpha} &= (\ 0\ ,\ 0,\ 0,\ 1) \ .
\end{aligned}$$

**5-20**. *Consider a particle with four-momentum* $\mathbf{p}$ *and an observer with four-velocity* $\mathbf{u}$. *Show that if the particle goes through the observer's laboratory, the magnitude of the three-momentum measured is*

$$|\vec{p}| = \left[(\mathbf{p} \cdot \mathbf{u})^2 + (\mathbf{p} \cdot \mathbf{p})\right]^{\frac{1}{2}} \ .$$

**Solution:** In any inertial frame we have from (5.43) and (5.17c):

$$\mathbf{p} \cdot \mathbf{p} = -E^2 + \vec{p} \cdot \vec{p} \,. \tag{1}$$

In the inertial frame in which the observer is at rest at the time of the measurement, $E$ is the measured energy, and $|\vec{p}|$ is the measured magnitude of the momentum. Then

$$|\vec{p}| = (E^2 + \mathbf{p} \cdot \mathbf{p})^{1/2} \,. \tag{2}$$

But also from (5.83) the measured energy $E$ is

$$E = -\mathbf{p} \cdot \mathbf{u}_{\text{obs}} \,. \tag{3}$$

Substituting (3) into (2) gives the required result.

**5-21**. *[P,A] Assume that in all inertial frames the force on a charged particle is given by the usual Lorentz force law:*

$$\vec{F} \equiv \frac{d\vec{p}}{dt} = q(\vec{E} + \vec{V} \times \vec{B})$$

*where $q$ is the charge on the particle, $\vec{V} \equiv d\vec{x}/dt$ is its three-velocity, and $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields as measured in the Lorentz frame. Consider a different inertial frame moving with speed $v$ along the $x-$axis with respect to the first.*

a) *Find the components of the four-force* $\mathbf{f}$ *in terms of $\vec{E}$ and $\vec{B}$ and the components of the particle's four-velocity* $\mathbf{u}$.

b) *Use the transformation law for the components of* $\mathbf{f}$ *and* $\mathbf{u}$ *to find the transformation rules that give the electric and magnetic fields in the new inertial frame for the following special fields in the original inertial frame.*

   i) *An electric field in the $x-$direction.*

   ii) *A magnetic field in the $x-$direction.*

   iii) *An electric field in the $y-$direction.*

   iv) *A magnetic field in the $y-$direction.*

**Solution:**

a) The components of the four-force are

$$\mathbf{f} = q\gamma_V \left(\vec{V} \cdot \vec{E}\ ,\ \vec{E} + \vec{V} \times \vec{B}\right)$$

where $\gamma_V = (1-\vec{V}^2)^{-1/2}$. This can be expressed in terms of the four-velocity components $\mathbf{u} = (\gamma_V\ ,\ \gamma_V \vec{V})$ as

$$\mathbf{f} = q\left(\vec{u} \cdot \vec{E}\ ,\ \ u^t\vec{E} + \vec{u} \times \vec{B}\right) \tag{1}$$

b) The transformation laws for $\mathbf{f}$ are

$$\begin{aligned} f^{t'} &= \gamma_v \left(f^t - v f^x\right) \\ f^{x'} &= \gamma_v \left(f^x - v f^t\right) \\ f^{y'} &= f^y \\ f^{z'} &= f^z \end{aligned}$$

with similar formulae for $\mathbf{u}$

(i) If there is only $E^x$, we have

$$\mathbf{f} = q\left(u^x E^x\ ,\ u^t E^x\ ,\ 0\ ,\ 0\right)$$

and

$$\begin{aligned} f^{x'} &= q\gamma_v \left(u^t - v u^x\right)\ E^x = q u^{t'} E^x \\ f^{y'} &= 0 \\ f^{z'} &= 0 \end{aligned}$$

This is of the form (1) if

$$\vec{E}' = (E^x\ ,\ 0\ ,\ 0)\ \ ,\ \vec{B}' = 0\ .$$

(ii) If there is only $B^x$

$$\begin{aligned}
\mathbf{f} &= q\,(0\,,\ 0\,,\ u^z B^x\,,\ -u^y B^x) \\
f^{x'} &= 0 \\
f^{y'} &= f^y = qu^z B^x = qu^{z'} B^x \\
f^{z'} &= f^z = -qu^y B^x = -qu^{y'} B^x\ .
\end{aligned}$$

This is of the form (1) if

$$\vec{E}' = 0\,,\ \vec{B}' = (B^x\,,\ 0\,,\ 0)\ .$$

Thus, longitudinal fields do not change under a Lorentz transformation.

(iii) If we have only $E^y$

$$\mathbf{f} = q\left(u^y E^y\,,\ 0\,,\ u^t E^y\,,\ 0\right)$$

and as before

$$\begin{aligned}
f^{x'} &= -qv\gamma_v u^y E^y = qu^{y'}\,(-v\gamma_v E^y) \\
f^{y'} &= qu^t E^y = q\gamma_v\left(u^{t'} + vu^{x'}\right) E^y \\
f^{z'} &= 0\ .
\end{aligned}$$

Comparing with (1), we find

$$\begin{aligned}
\vec{E}' &= \gamma_v\,(0\,,\ E_y\,,\ 0) \\
\vec{B}' &= \gamma_v\,(0\,,\ 0\ \ -vE^y)
\end{aligned}$$

(iv) If we have only $B^y$

$$\mathbf{f} = q\,(0\,, -u^z B^y\ \ 0\,,\ u^x B^y)$$

so

$$\begin{aligned}
f^{x'} &= q\gamma_v u^z B^y = qu^{z'}\,(\gamma_v B^y) \\
f^{y'} &= 0 \\
f^{z'} &= qu^x B^y = q\gamma_v\left(u^{x'} + vu^{t'}\right) B^y\ .
\end{aligned}$$

From which we find

$$\begin{aligned}
\vec{E}' &= (0,\ 0,\ \gamma_v v B^y) \\
\vec{B}' &= (0,\ \gamma_v B^y\,,\ 0)\,.
\end{aligned}$$

If $\vec{E}$ and $\vec{B}$ are divided up into components parallel ($\|$) and perpendicular ($\perp$) to the direction of relative motion $\vec{v}$, the general relations are

$$\begin{aligned}
\vec{E}'_{\|} &= \vec{E}_{\|}\ ,\ \vec{B}'_{\|} = \vec{B}_{\|} \\
\vec{E}'_{\perp} &= \gamma_v \left[\vec{E}_{\perp} + \left(\vec{v} \times \vec{B}\right)_{\perp}\right] \\
\vec{B}'_{\perp} &= \gamma_v \left[\vec{B}_{\perp} - \left(\vec{v} \times \vec{E}\right)_{\perp}\right] .
\end{aligned}$$

**5-22.** *[C] (The Relativistic Rocket) A rocket accelerates by ejecting part of its rest mass as exhaust. The speed of the exhaust is a constant value $u$ in the rocket's rest frame. Use the conservation of energy and momentum to find the ratio of final to initial rest mass for a rocket that accelerates from rest to a speed $V$. [Hint: Rest mass is not conserved — energy and momentum are conserved. You might want to start by working the same problem in Newtonian mechanics.]*

**Solution:** *(The Relativistic Rocket)* Let $M$ be the rest mass of the rocket and $V$ its velocity. It's easiest to first work out the conservation of energy and momentum in the rest frame of the rocket and then transform back to a frame where it's moved with speed $V$. In the rest frame, suppose $\Delta M$ of the rest mass (counted negative) is ejected at speed $u$. Let $\Delta V'$ be the resulting change in speed of the rocket and $\Delta M_e$ the rest mass of the ejecta. To first order in small quantities conservation of energy and momentum are:

$$\begin{aligned}
M &= M + \Delta M + \Delta M_e \gamma_u \\
0 &= M \Delta V' - \Delta M_e u \gamma_u
\end{aligned}$$

Eliminating $\Delta M_e$ we get

$$\Delta V' = -(\Delta M/M)\, u\ .$$

To transform back to the charge $\Delta V$ in the frame where the rocket is moving with speed $V$, we use the addition of velocity formula (3.40a) to find

$$V + \Delta V = \frac{\Delta V' + V}{1 + V \Delta V'} \approx V + \Delta V' \left(1 - V^2\right)\ .$$

Then

$$\frac{\Delta V}{1-V^2} = -u\,\frac{\Delta M}{M}\ .$$

Integrating both sides gives

$$\tanh^{-1} V = +u \log\left(\frac{M_0}{M}\right)$$

where $M_0$ is the initial rest mass. With a little algebra one gets

$$\frac{M}{M_0} = \left(\frac{1-V}{1+V}\right)^{\frac{1}{2u}}\ .$$

As the rest mass is used up, $V$ approaches the velocity of light.

**5-23**. *[C] (Tachyons)*

a) *Argue that a kind of particle that always moves faster than the velocity of light would be consistent with Lorentz invariance in the sense that if its speed is greater than light in one frame it will be greater than light in all frames. (Such hypothetical particles are called tachyons.)*

b) *Show that the tangent vector to the trajectory of a tachyon is spacelike, and can be written $u^\alpha = dx^\alpha/ds$, where $s$ is the spacelike interval along the trajectory. Show that $\mathbf{u}\cdot\mathbf{u} = 1$.*

c) *Evaluate the components of a tachyons four-velocity $\mathbf{u}$ in terms of the three-velocity $\vec{V} = d\vec{x}/dt$.*

d) *Define the four-momentum by $\mathbf{p} = m\mathbf{u}$ and find the relation between energy and momentum for a tachyon.*

e) *Show that there is an inertial frame where the energy of any tachyon is negative.*

f) *Show that if tacyhons interact with normal particles, a normal particle could emit a tachyon with total energy and three-momentum being conserved.*

*Comment: The result (f) suggests that a world containing tachyons would be unstable, and there is no evidence for tachyons in nature.*

**Solution:**

a) Tachyons would travel on *spacelike* curves and the distinction between spacelike, timelike, and null is Lorentz invariant.

b) Let $x^\alpha(\lambda)$ be the world line of a tachyon, and $u^\alpha = dx^\alpha/d\lambda = (dx^\alpha/dt)(dt/d\lambda)$. Then

$$\mathbf{u}\cdot\mathbf{u} = \left(\frac{dt}{d\lambda}\right)^2 \left[-1 + \vec{V}^2\right] .$$

This is greater than zero if $|\vec{V}| > 1$. Thus, $s$ is a good parameter and $\mathbf{u}\cdot\mathbf{u} = 1$.

c)

$$\begin{aligned} u^t &\equiv \frac{dt}{ds} = \frac{1}{\sqrt{\vec{V}^2 - 1}} \\ \vec{u} &= \frac{d\vec{x}}{dt}\frac{dt}{ds} = \frac{\vec{V}}{\sqrt{\vec{V}^2 - 1}} \end{aligned}$$

d) If $\mathbf{p} \equiv m\mathbf{u}$, then $\mathbf{p}^2 = m^2 = -E^2 + \vec{p}^{\,2}$, so that

$$E = \pm\sqrt{\vec{p}^{\,2} - m^2} .$$

e) Since the four-momentum is spacelike, there is always some frame in which its $t$ component is negative.

$$E' = \gamma(E - vp)$$

But if $E > 0$, then $|E| < |p|$ from (d) so for a sufficiently large $v$ of the right sign $E' < 0$.

f) Consider a decay of a particle $A$ into a tachyon $T$ and another particle $B$.

$$\mathbf{p}_A = \mathbf{p}_B + \mathbf{p}_T .$$

Examine the decay in the rest frame of the particle $A$ initially. Then from conservation of momentum

$$\vec{p}_B = -\vec{p}_T \equiv \vec{p} .$$

Conservation of energy would mean

$$m_A = \sqrt{p^2 + m_B^2} - \sqrt{p^2 - m_T^2}$$

where we have assumed the tachyon has negative energy. Energy is conserved if this equation can be satisfied for some $p$. At $p = m_T$, the smallest possible value, the right hand side is $\sqrt{m_T^2 + m_B^2}$ which is greater than $m_A$. At $p = \infty$ the right hand side is 0. So somewhere in between there will be a value for which it is equal to $m_A$.

# Chapter 6

# Gravitation as Geometry

**6-1**. *What angle does the fiber of the torsion balance described in Figure 6.1 make with the direction of the local gravitational field $\vec{g}$? What is the value of $g^t$ in (6.2)? Assume that the experiment is carried out at latitude 47°. (This is the latitude of Seattle where the experiment of Su et al. described in the text was carried out.)*

**Solution:** Let $\theta$ be the angle the pendulum makes with the vertical and $\lambda$ the latitude in radians at which the experiment is carried out. The magnitude of the centripetal acceleration is then

$$a = \Omega_\oplus^2 R_\oplus \cos\lambda$$

where $R_\oplus$ is the Earth's radius and $\Omega_\oplus$ is its angular velocity $2\pi/(24\text{hrs})$. It's easiest to resolve the forces and accelerations along the fiber and the perpendicular "twisting direction". A little geometry from Figure 6.1 shows that for small $\theta$ the angle between the centripetal acceleration and the twisting direction is $\pi/2 - (\lambda + \theta)$. Along the twisting direction

$$ma\cos\left(\frac{\pi}{2} - \lambda - \theta\right) = mg\cos\left(\frac{\pi}{2} - \theta\right) .$$

Assuming $\theta \ll \lambda$, this gives

$$\theta = \frac{1}{2}\left(\frac{\Omega_\oplus^2 R_\oplus}{g}\right)\sin(2\lambda) .$$

This vanishes at the pole and the equator as it must by symmetry. For the data given in the problem

$$\theta \sim .1^\circ .$$

The value of $g^t$ is $g \sin\theta \approx .02$ m/s$^2$.

**6-2**. *Suppose any twisting of the torsion balance in the modern versions of the Eötvös experiment was measured by bouncing a light off a mirror attached to the bar and measuring the time dependence of the angle $\theta$ as above. What angular accuracy is needed to test the principle of equivalence to 1 part in $10^{12}$? Assume the bar is 4 cm long and the masses are about 10 gm each, that the torsion constant of the fiber (analogous to the spring constant for linear motion) is $2 \times 10^{-8}$ N-m/radian, and that the acceleration of gravity in the twisting direction is as determined in Problem 1.*

**Solution:** A difference of 1 part in $10^{12}$ between the gravitational and inertial masses of approximately one 10 gm would result in a torque

$$\left(1 \times 10^{-12}\right) \times (.01 \text{ kg}) \times \left(.02 \text{ m/s}^2\right) \times (.04 \text{ m}) \approx 8 \times 10^{-18} \text{N} - \text{m} \ .$$

The angular displacement $\Theta$ of a fiber with a torsion constant of $2 \times 10^{-8}$ N-m/radian is

$$\begin{aligned} \Theta &\approx \frac{8 \times 10^{-18}}{2 \times 10^{-8}} \ \frac{\text{N} - \text{m}}{\text{N} - \text{m/radian}} \\ &\approx 4 \times 10^{-10} \text{radians} \approx 4 \times 10^{-5} \text{arcsec} \ . \end{aligned}$$

**6-3**. *[S] Assuming that the acceleration is the acceleration of gravity at the surface of the Earth, how wide does the elevator in Figure 6.5 have to be for the light ray to fall by 1 mm over the course of its transit? Is this a thought experiment that could be realized on the surface of the Earth?*

**Solution:** To fall 1 mm with the acceleration of gravity at the surface of the Earth, $g = 980$ cm/s$^2$, requires a time $t_f = \sqrt{2(1 \text{ mm})/g} \approx .01$ s. During this time the light ray will travel a horizontal distance of approximately $(.01 \text{ s})\, c \approx$ 3000 km. The laboratory needs to be this wide. Thus it is a significant fraction of the radius of the Earth so the experiment couldn't be carried out there.

**6-4.** *Starting from the equivalence principle in the form given stated in terms of freely falling frames and inertial frames and inertial frames in flat space (as in the boxed statement on p. 159), argue that light must fall in the gravitational field of the Earth.*

**Solution:** Consider a small laboratory falling freely and radially downwards in the gravitational field of the Earth. In the inertial frame in which the center of the Earth is approximately at rest the laboratory is falling with the local acceleration of gravity. A light ray transits the laboratory certain height above the floor. In a laboratory in empty space the light ray would move straight across the lab and exit at the same height. That will be the case in the freely falling lab only if the light ray falls with the same acceleration as the lab in the inertial frame of the Earth. The equivalence principle thus implies that gravity attracts light.

**6-5.** *In Example 3 concerning freely falling pingpong balls, assume that the inner ball is released with just the tangential velocity necessary for a circular orbit about the Earth. The outer ball released with the same velocity will therefore execute an elliptical orbit. What is the eccentricity of this orbit as a function of $s$? Sketch the two orbits. Does your picture support the conclusion of the example that there is significant change in the separation of the particles in one period? Hint: Look up the details of elliptical orbits in your Newtonian mechanics text.*

**Solution:** To make the algebra more manageable, choose $m = 1$ for the mass of the ball, put $G = 1$, and let $M$ be the mass of the Earth. Orbits in Newtonian mechanics are characterized by an energy $\mathcal{E}$ and an angular momentum $\ell$. The ball in question starts a distance $s$ further out than the radius of a circular orbit $R$, but moving with the same velocity

$$V = \left(\frac{M}{R}\right)^{\frac{1}{2}}$$

as a particle in that circular orbit.

The energy $\mathcal{E}$ and angular momentum $\ell$ are therefore

$$\mathcal{E} = \frac{1}{2}\,V^2 - \frac{M}{(R+s)} \quad , \quad \ell = V(R+s). \tag{1}$$

The eccentricity $\epsilon$ of an orbit with these parameters is given by

$$\epsilon = \left[1 + \frac{2\mathcal{E}\ell^2}{M^2}\right]^{\frac{1}{2}} = \frac{s}{R}. \tag{2}$$

The figure below shows the two orbits for $\epsilon = .1$. The distance between the orbits changes significantly over one transversal. That supports the conclusion in Example 6.3 that there is significant change in the distance between the particles over one period.

**6-6**. *(a) Transform the line element of special relativity from the usual $(t, x, y, z)$ rectangular coordinates to new coordinates $(t', x', y', z')$ related by*

$$\begin{aligned} t &= \left(\frac{c}{g} + \frac{x'}{c}\right) \sinh\left(\frac{gt'}{c}\right), \\ x &= c\left(\frac{c}{g} + \frac{x'}{c}\right) \cosh\left(\frac{gt'}{c}\right) - \frac{c^2}{g}, \\ y &= y', \quad z = z'. \end{aligned}$$

*for a constant $g$ with the dimensions of acceleration.*

*(b) For $gt'/c \ll 1$ show that this corresponds to a transformation to a uniformly accelerated frame in Newtonian mechanics.*

*(c) Show that an at-rest clock in this frame at $x' = h$ runs fast compared to a clock at rest at $x' = 0$ by a factor $(1 + gh/c^2)$. How is this related to the equivalence principle idea?*

**Solution:**

a) Writing out the differentials $dt, dx, dy, dz$ in terms of $dt', dx', dy', dz'$ and substituting into the standard line element for flat spacetime gives

$$d\tau^2 = -c^2 \left(1 + \frac{gx'}{c^2}\right)^2 dt'^2 + dx'^2 + dy'^2 + dz'^2 \ . \tag{1}$$

b) Expanding the cosh and sinh we have for small $gt'/c$,

$$\begin{aligned} t &\approx t' \\ x &\approx x' + \frac{1}{2}\, gt'^2 = x' + \frac{1}{2}\, gt^2 \end{aligned}$$

which is the transformation to an accelerated frame.

c) Clocks at rest in an accelerated frame have constant $x'$.

$$\begin{aligned} (d\tau)_{x'=0} &= dt' \\ (d\tau)_{x'=h} &= dt' \left(1 + \frac{gh}{c^2}\right) \end{aligned}$$

so

$$(d\tau)_{x'=h} = (d\tau)_{x'=0} \left(1 + \frac{gh}{c^2}\right) \ .$$

Thus, the clock higher up in the accelerated frame runs faster. This is an expression of the equivalence principle idea.

**6-7.** *(a) An accelerated laboratory has a bottom at $x' = 0$ and a top at $x' = h$, both with extent in the $y'$ and $z'$ direction. Use the line element derived in part (a) of Problem 6 to show that the height of the laboratory remains constant in time, i.e. the laboratory moves rigidly.*

*(b) Compute the invariant acceleration $a \equiv (\mathbf{a} \cdot \mathbf{a})^{1/2}$, where $a^\alpha = d^2x^\alpha/d\tau^2$, and show that it is different for the top and bottom of the laboratory.*

**Solution:**

a) The line element, shown in part (a) of the solution to Problem 6, is independent of $t'$. So the spatial distance $\Delta x'$ between two curves of constant $x'$ remains constant in $t'$.

b) The four-velocity in the $(t, x)$ frame can be calculated from the expression for $x(t')$ and $y(t')$ given in Problem 6, together with the line element derived in part (a) of that problem which gives the connection between $t'$ and proper time $\tau$ along a curve of constant $x'$, $y'$, and $z'$, viz.,

$$d\tau = \left(1 + \frac{gx'}{c^2}\right) dt' .$$

The resulting four-velocity of a curve at constant $x'$ has components

$$\frac{dt}{d\tau} = \cosh\left(\frac{gt'}{c}\right) , \quad \frac{dx}{d\tau} = c \sinh\left(\frac{gt'}{c}\right) .$$

(Components in the $y$- and $z$-direction vanish). The components of the four-acceleration are

$$\begin{aligned} a^t &= \frac{d^2t}{d\tau^2} = \frac{g}{c}\left(1 + \frac{gx'}{c^2}\right)^{-1} \sinh\left(\frac{gt'}{c}\right) , \\ a^x &= \frac{d^2x}{d\tau^2} = g\left(1 + \frac{gx'}{c^2}\right)^{-1} \cosh\left(\frac{gt'}{c}\right) . \end{aligned}$$

Thus,

$$a = (\mathbf{a} \cdot \mathbf{a})^{\frac{1}{2}} = \frac{g}{(1 + gx'/c^2)}$$

which decreases with $x'$.

**6-8.** *[S] It's not legitimate to mix relativistic with non-relativistic concepts, but imagine that that a photon with frequency $\omega_*$ is like a particle with gravitational mass $\hbar\omega_*/c^2$ and kinetic energy $K = \hbar\omega$. Using Newtonian ideas, calculate the "kinetic" energy loss to a photon that is emitted from the surface of a spherical star of radius $R$ and mass $M$ and escapes to infinity. From this calculate the frequency of the photon at infinity. How does this compare with the gravitational redshift in* (6.14) *to first order in $1/c^2$?*

**Solution:** In Newtonian physics, the loss in kinetic energy of a particle that moves from the surface of the star to infinity is equal to the difference in

potential energy between those locations. That difference is $GmM/R$ for a particle of mass $m$. Thus, if $K_R$ is the kinetic energy at $R$ and $K_\infty$ the kinetic energy at infinity,

$$K_R - K_\infty = GmM/R \ .$$

Equating $K = \hbar\omega$ and $m = \hbar\omega/c^2$ for a photon, we find

$$\omega_\infty = \omega_R \left(1 - \frac{GM}{c^2 R}\right) \ .$$

This is exactly the gravitational redshift to first order in $1/c^2$.

**6-9**. *A GPS satellite emits signals at a constant rate as measured by an on board clock. Calculate the fractional difference in the rate at which these are received by an identical clock on the surface of the Earth. Take both the effects of special relativity and gravitation into account to leading order in $1/c^2$. For simplicity assume the satellite is in a circular equatorial orbit, the ground based clock is on the equator, and that the angle between the propagation of the signal and the velocity of the satellite is $90°$ in the instantaneous rest frame of the receiver.*

**Solution:** The parameters of the orbit of a satellite are given in (6.17). The velocity of the Earth's surface, which is also relevant, is $V_\oplus = (2\pi R_\oplus/24 \text{ hr}) = 0.46$ km/s. With these parameters the relativistic effects due to to time dilation and the gravitational potential can be estimated as follows:

*Time dilation.* We are considering the special case when angle between the direction of propagation and the velocity of the satellite is $\alpha' = \pi/2$ in the rest frame of the receiver. Then the Doppler shift is transverse, or, put differently, the difference in rates is due to the time dilation arising from the relative velocity of the emitting and receiving clocks. In the rest frame of the receiver the velocity of the clock is $V_s - V_\oplus$ plus small corrections of order $(V_\oplus/c)^2$ [*cf.* (4.27)]. From (5.73) with $\alpha' = \pi/2$ or from (4.15), the difference in rates is proportional to $1 - (1/2)(V_s - V_\oplus)^2/c^2$ plus corrections that are negligible for the small velocities relevant here. The fractional difference in

rates [(rate of reception) − (rate of emission)]/(rate of emission) is

$$\left(\begin{array}{c}\text{Fractional difference in rates}\\ \text{between emission and reception}\\ \text{due to time dilation.}\end{array}\right) = -\frac{1}{2}\left(\frac{V_s - V_\oplus}{c}\right)^2 \approx -.65 \times 10^{-10} \tag{1}$$

to leading order in $1/c^2$ for the numbers cited above.

*Gravity.* Signals are received at a faster rate by a clock lower down in a gravitational potential than the rate at which they are emitted by a clock higher up. That is the content of (6.12). To a good approximation, the gravitational potential of the Earth is $\Phi(r) = -GM_\oplus/r$ where $r$ is the distance from the center [*cf.* (3.13)]. Thus, from (6.12), the fractional difference between the rates of reception and emission due to the gravitational potential is (to leading order in $1/c^2$)

$$\left(\begin{array}{c}\text{Fractional difference in rates}\\ \text{due to gravitational potential.}\end{array}\right) = -\frac{GM_\oplus}{c^2}\left(\frac{1}{R_s} - \frac{1}{R_\oplus}\right) \approx 5.2 \times 10^{-10} \tag{2}$$

for the numbers cited above.

The relativistic effects due to time dilation and the gravitational potential are of comparable magnitude but in opposite directions. However, the gravitational effect is almost seven times larger than that due to time dilation so they do not cancel. The combined result of (1) and (2) is that signals are received on the ground at a faster rate than they are emitted by the satellites, and the fractional difference in rates is

$$\left(\begin{array}{c}\text{Net fractional}\\ \text{difference in rates}\end{array}\right) = +4.5 \times 10^{-10} \,. \tag{3}$$

This is a small difference but, were it not accounted for, it would take less than a minute to add up to an error which exceeds the few nanosecond accuracy required for the GPS to give 2 m accuracies in location. Both the effects of special relativity and gravity must therefore be accounted for in the operation of the GPS.

**6-10**. *[C,P] The Earth is approximately 5 billion years old. How much younger are rocks at the center of the Earth than at the surface? If equal abundances*

*of a radioactive element with a decay time of 4 billion years were present to start, how much more of that element would be present at the center than the surface? Assume the density of the Earth is constant.*

**Solution:** We give two different approaches to a solution:

*First version.* Suppose for simplicity that the density of the earth $\rho$ is constant over its radius. The gravitational force on a particle of mas $m$ at radius $r$ is

$$F_r = -\frac{GmM(r)}{r^2} = -m\left(\frac{4}{3}\pi G\rho r\right) .$$

The gravitational potential difference between the center and the surface $R_\oplus$ is therefore

$$\Delta\Phi = \Phi(R_\oplus) - \Phi(0) = -\int_0^{R_\oplus} \frac{F_r}{m}\, dr = \frac{2}{3}\pi G\rho R_\oplus^2 = \frac{1}{2}\left(\frac{GM_\oplus}{R_\oplus}\right) .$$

Thus, (see useful constants)

$$\begin{aligned} \Delta\Phi/c^2 &= (.443 \text{ cm})/(2 \times 6.38 \times 10^8 \text{ cm}) \\ &= 3.47 \times 10^{-10} \end{aligned}$$

From (6.23) we can find the difference in elapsed times,

$$\Delta\tau_0 = \left(1 + 3.47 \times 10^{-10}\right) \tau_{R_\oplus} .$$

The abundance of a radioactive species will be

$$N \propto e^{-t/T}$$

where $T$ is the decay time. Thus the ratio of abundances is

$$\frac{N_{\text{center}}}{N_{\text{surface}}} = \frac{e^{-(\tau_0/T)}}{e^{-(\tau_{R_\oplus}/T)}} \approx \exp\left(\frac{\tau_{R_\oplus}}{T}\frac{\Delta\Phi}{c^2}\right)$$

$$N_{\text{center}}/N_{\text{surf}} \approx 1 + \frac{5}{4} \times \left(3.47 \times 10^{-10}\right)$$

— a very small difference!

*Second version*:

We first calculate the gravitational potential difference between the center of the earth and its surface. Let the radius of the surface be $R_\oplus$ and the mass of the earth be $M_\oplus$. The potential difference is

$$\Delta\Phi \equiv \Phi\left(R_\oplus\right) - \Phi(0) = -\int_0^{R_\oplus} \vec{F} \cdot d\vec{r} \,. \tag{1}$$

Here $\vec{F}$ is the gravitational force *per unit mass*, *i.e.*

$$\vec{F} = -\frac{GM(r)}{r^2}\,\vec{e}_r$$

where $M(r)$ is the mass inside a radius $r$. Assuming a constant density $\rho_\oplus$

$$M = \frac{4}{3}\pi\rho_\oplus r^3 = M_\oplus\left(\frac{r}{R_\oplus}\right)^3 \,.$$

so that

$$\vec{F} = -\frac{GM_\oplus}{R_\oplus^2}\left(\frac{r}{R_\oplus}\right)\vec{e}_r \,.$$

Inserting in the integral (1) gives

$$\Delta\Phi = \frac{1}{2}\,\frac{GM_\oplus}{R_\oplus} \,.$$

The surface is at a *higher* in gravitational potential than the center. A clock at the surface therefore runs faster by a factor of

$$\left(1 + \frac{\Delta\Phi}{c^2}\right) = \left(1 + \frac{1}{2}\,\frac{GM_\oplus}{c^2 R_\oplus}\right) \approx 1 + 3.5 \times 10^{-10} \,.$$

The rocks at the center are therefore younger by

$$\left(5 \times 10^9 \text{ yr}\right)\left(3.5 \times 10^{-10}\right) = 1.7 \text{ yr} \,.$$

The abundance of a radioactive element with a half-life $\tau_{1/2} = 4 \times 10^9$yrs decays as

$$N = N_0\, e^{-t/\tau_{1/2}} \,.$$

There will be more of the element at the center than at the surface by

$$e^{\frac{+1.7yrs}{(4\times10^9 yrs)}} \approx 1 + \frac{1.7}{4 \times 10^9} \approx 1 + 4.3 \times 10^{-10} \,.$$

Not much more!

**6-11**. *[E] Aging goes on at a slower rate at the center of a spherical mass than on its surface. Estimate how much mass would need to be assembled in a radius of 10 km such that if you lived at the center for 1 year you would emerge 1 day younger than those who had stayed outside and far away?*

**Solution:** The gravitational potential difference between the center of a sphere of mass $M$ and radius $R$ and infinity is of order

$$\Delta\Phi \sim GM/R \ .$$

The difference in *rates* between a clock at the center and a clock far away is therefore of order

$$\frac{\Delta\Phi}{c^2} \sim \frac{GM}{Rc^2} \ .$$

For a clock to lag behind a clock at infinity by one day in one year, its rate must be slower in rate by 1/365. Thus,

$$\frac{GM}{Rc^2} \sim \frac{1}{365} \ .$$

To express this in solar masses, divide by $GM_\odot/c^2 = 1.5$ km to find

$$\frac{M}{M_\odot} \sim \frac{1}{365}\left(\frac{10 \text{ km}}{1.5 \text{ km}}\right) \sim .02 \ .$$

That's about 20 times the mass of Jupiter!

**6-12**. *[S] In the two-dimensional flat plane, a straight line path of extremal distance is the shortest distance between two points. On a two-dimensional round sphere, extremal paths are segments of great circles. Show that between any two points on the sphere there are extremal paths that provide the shortest distance between them when compared with nearby paths, and also ones that provide the longest distance between them when compared with nearby paths. Are the latter curves of longest distance between the two points?*

**Solution:** There is one great circle through any two points on a sphere but it defines *two* curves of extremal distance connecting the two points. The

shorter segment of the great circle is the path of shortest distance between the two points. But the longer segment around the other way is also an extremal curve. It provides a longer distance than any other *nearby* path. (Imagine two points on the equator and slide the long segment up a bit toward the north pole. It gets shorter.) But it is not the longest path connecting the two points globally. Imagine, for example, a path from one point that circled the globe 10,000 times before ending at the other. That is longer.

**6-13.** *Three observers are standing near each other on the surface of the Earth. Each holds an accurate atomic clock. At time $t = 0$ all the clocks are synchronized. At $t = 0$ the first observer throws his clock straight up so that it returns at time $T$ as measured by the clock of the second observer who holds her clock in their hand for the entire time interval. The third observer carries his clock up to the maximum height the thrown clock reaches and back down moving with constant speed on each leg of the trip and returning in time $T$.*

*Calculate the total elapsed time measured on each clock assuming that the maximum height is much smaller than the radius of the Earth. Include gravitational effects but calculate to order $1/c^2$ only using non-relativistic trajectories. Which clock registers the longest time? Why is this?*

**Solution:** There are two effects: (1) time dilation, and (2) the gravitational effect on clocks. Working to $1/c^2$, and combining these effects, the proper time along any trajectory is

$$\tau = \int dt \left[1 - \frac{1}{c^2}\left(\frac{1}{2}V^2 - \Phi\right)\right] ,$$

or

$$\tau = T - \frac{1}{c^2}\int dt \left[\frac{1}{2}V^2(t) - gh(t)\right]$$

since $\Phi = gh$. The first observer throws the clock upwards from $h = 0$. It reaches a maximum height $h_{\max} = 1/2\, g(T/2)^2 = (1/8)\, gT^2$. Thus,

$$\begin{aligned} h(t) &= h_{\max} - \frac{1}{2}\, gt^2 \\ V(t) &= -gt \end{aligned}$$

assuming $t = 0$ is the time the peak of the trajectory is reached. The elapsed time is $\tau = T - \Delta\tau$, where

$$\Delta\tau \equiv \frac{1}{c^2}\int_{-T/2}^{+T/2} dt\left[\frac{1}{2}\,g^2t^2 - gh_{\max} + \frac{1}{2}\,g^2t^2\right] = -\frac{1}{24}\left(\frac{gT}{c}\right)^2 T$$

For the second observer who holds the clock at rest, $\Delta\tau = 0$. For the third observer

$$V = h_{\max}/(T/2) \;=\; \frac{1}{4}\,gT$$

and

$$h(t) \;=\; h_{\max} - \frac{1}{4}\,gT|t|$$

$$\begin{aligned}\Delta\tau &= \frac{1}{c^2}\int_{-T/2}^{+T/2} dt\left[\frac{1}{32}\,g^2T^2 - gh_{\max} + \frac{1}{16}\,g^2T^2\right]\\ &= -\frac{1}{32}\left(\frac{gT}{c}\right)^2 T\end{aligned}$$

The longest proper time is registered by the path that is thrown because it is the path of a free particle.

**6-14**. *[C] Consider a particle moving in a circular orbit about the Earth of radius $R$. Suppose the geometry of spacetime outside the Earth is given by the static weak field metric (6.20) with $\Phi = -GM_\oplus/r$. Let $P$ be the period of the orbit measured in the time $t$. Consider two events $A$ and $B$ located at the same spatial position on the orbit but separated in $t$ by the period $P$. The particle's world line is a curve of extremal proper time between $A$ and $B$. Analyze the question of whether the world line is a curve of longest, shortest, or just extremal proper time by calculating the proper time to first order in $1/c^2$ along the following curves between $A$ and $B$:*

a) *The orbit of the particle itself.*

b) *The world line of an observer who remains fixed in space between $A$ and $B$.*

c) *The world line of a photon that moves radially away from $A$ and reverses direction in time to return to $B$ in a time $P$.*

*d) Can you find another curve of extremal proper time that connects A and B?*

**Solution:** To order $1/c^2$ accuracy the proper time along any of these curves is given by (6.25) so

$$\Delta\tau = P - \frac{1}{c^2}\int_0^P dt\left[\frac{\vec{V}^2}{2} - \Phi\right]$$

in an inertial frame in which the center of the Earth is approximately at rest.

a) For a circular orbit of period $P$, $\Phi = -GM/R$ where $R$ is related to $P$ by Kepler's law $P^2 = (4\pi^2/GM)R^3$. Further, $V^2/R = GM/R^2$. The net result for the above integral is

$$\Delta\tau = P\left(1 - \frac{3}{2}\frac{GM}{Rc^2}\right)$$

which can be entirely expressed in terms of $P$ and $M$ using Kepler's law.

b) For a stationary observer $\vec{V} = 0$

$$\Delta\tau = P\left(1 - \frac{GM}{Rc^2}\right)$$

which is a longer proper time than a). Therefore, the circular orbit, although an extremal curve, is not a curve of *longest* proper time.

c) There is zero elapsed proper time. A circular orbit is not a curve of shortest proper time either.

d) If the particle is thrown radially outwards with the right velocity so that it returns in time $P$ that is another curve of extremal proper time. [TEXT DELETED] Recall Problem 12.

**6-15.** *[B] Twin Paradox Test (a) Derive the formula for the elapsed difference*

*in proper time between the flying clocks and the surface clock given in Box 6.2.*

*(b) Using typical altitudes and speeds for commercial aircraft, estimate the value of $\Delta\tau$ for both eastward and westward flights around the world.*

**Solution:**

a) Let $\Delta\tau$ be the time interval measured by a clock moving in an inertial frame with speed $v$ in a gravitational potential $\Phi$ corresponding to a time interval $\Delta t$ in an inertial frame. From the combined gravitational and time dilation effects to order $1/c^2$, we have

$$\Delta\tau = \Delta t \left(1 + \frac{\Phi}{c^2} - \frac{1}{2}\frac{V^2}{c^2}\right) .$$

The *difference* in rates between a clock moving with speed $V_\oplus + V_g$ at a height $h$ and a clock moving at speed $V_\oplus$ on the surface is therefore,

$$\Delta\tau = \Delta\tau_g \left(\frac{gh}{c^2} - \frac{1}{2c^2}\left(V_g^2 + 2V_\oplus V_g\right)\right)$$

whence the formula in the box.

b) Typical numbers for airplanes:

$$\begin{aligned} h &= 10,000 \text{ m} . \\ V_g &= 1000 \text{ km/hr} \end{aligned}$$

and a round trip travel line of $2\pi R_\oplus/V_g \approx 1.4 \times 10^5$ s. With these numbers we get $\Delta\tau \approx 302$ nsec one way and $\Delta\tau \approx -111$ nsec the other. These are not so different from Hefele and Keating's predictions.

# Chapter 7

# The Description of Curved Spacetime

**7-1**. *(a) In the singular line element for the plane (7.7) show that the distance between $r' = 0$ and a point with any finite value of $r'$ is infinite.*

*(b) Find the distance between $r' = 5$ and $r' = \infty$ along the line $\phi = 0$.*

**Solution:**

a) The singular line element is

$$dS^2 = \left(\frac{a}{r'}\right)^4 \left(dr'^2 + r'^2 d\phi^2\right) .$$

The distance along a line of constant $\phi$ ($d\phi = 0$) from $r' = 0$ to any finite value of $r'$ is

$$\int dS = \int_0^{r'} \left(\frac{a}{r'}\right)^2 dr' = \infty$$

because the integral diverges at the lower limit.

b) Similarly the distance between $r' = 5$ and $r' = \infty$ along $\phi = 0$ is

$$\int dS = \int_5^{\infty} \left(\frac{a}{r'}\right)^2 dr' = a\left(-\frac{a}{r'}\right)\Bigg|_5^{\infty} = \frac{a^2}{5} .$$

Even though the coordinate range is infinite in $r'$, the distance is finite. Another way to see this is that it's just the distance from $r = 0$ to $r = a^2/5$ in usual polar coordinates.

**7-2**. *The following line element corresponds to flat spacetime*

$$ds^2 = -dt^2 + 2dx\,dt + dy^2 + dz^2 \ .$$

*Find a coordinate transformation which puts the line element in the usual flat space form* (7.1).

**Solution:** Its hard to give a general prescription for solving this kind of problem. Guesswork and trial and error are the main methods. We, therefore, present some solutions without trying to explain exactly how they were arrived at.

$$ds^2 = -dt^2 + 2dx\,dt + dy^2 + dz^2$$

The transformation

$$t = t' + x' \ , \ x = x' \ , \ y = y' \ , \ z = z'$$

leads to

$$\begin{aligned} ds^2 &= -\left(dt' + dx'\right)^2 + 2\,dx'\left(dt' + dx'\right) + dy'^2 + dz'^2 \\ &= -\left(dt'\right)^2 + \left(dx'\right)^2 + \left(dy'\right)^2 + \left(dz'\right)^2 \end{aligned}$$

which is the standard form of the flat space line element.

**7-3**. *[C,P] The Sagnac Effect The Sagnac effect was worked out in an inertial frame in Box 3.1. Two light waves propogate in opposite directions around a rotating ring. The phase of a wave with frequency $\omega$ at time $t$ a distance $S$ around the ring is $\Psi \equiv -\omega(t - S) + const.$. (The speed $v$ of a light wave is 1.) When there is a difference in phase of a multiple of $2\pi$ the waves constructively interfere.*

*It is also possible to work out the Sagnac effect in a frame in rotating with the interferometer. The line element of flat spacetime in that frame can be found by defining defining a new coordinate $\phi = \phi' + \Omega t$. Derive the condition for constructive interference in this frame.*

**Solution:** In the rotating frame the line element for flat spacetime is

$$ds^2 = -dt^2 + dr^2 + r^2\left[d\theta^2 + \sin^2\theta\left(d\phi' + \Omega dt\right)^2\right].$$

In a $t$ = const. slice there is no difference in the distance the waves travel around the ring. It is $2\pi R$ in each case. But there is a difference in the coordinate time $t$ it takes each pulse to travel because the coordinate speed of light is different in the two directions. To see this, restrict the line element to the relevant $r = R$ and $\theta = \pi/2$ to find

$$ds^2 = -dt^2 + R^2 \left(d\phi' + \Omega dt\right)^2 .$$

A light ray travels on a null curve for which $ds^2 = 0$ or

$$R\frac{d\phi'}{dt} = \pm 1 - R\Omega.$$

Thus the *coordinate speed* in the counter-rotating direction is $v_- = 1 + R\Omega$ and the *coordinate speed* in the co-rotating direction is $v_+ = 1 - R\Omega$. The two times (co- and counter- rotating) to complete a circuit of $2\pi$ are therefore

$$t_\pm = 2\pi R(1 \mp \Omega R)^{-1} ,$$

so $t_+ > t_-$, i.e. it takes longer to complete the circuit in the co-rotating direction than the counter-rotating direction. The phase difference $\Delta\psi$ is $\omega$ times the differences in time

$$\Delta\psi = \omega \frac{4\pi\Omega R^2}{1 - (R\Omega)^2}.$$

When $\Delta\psi$ is an integral multiple of $2\pi$ the waves interfere constructively. This is the same as the condition derived in Box 3.1 since $\lambda = 2\pi/\omega$.

**7-4.** *[B] In the Penrose diagram for flat space spanned by the coordinates $(t', r')$ make a rough sketch of the following a curve of constant $r$ and a curve of constant $t$.*

**Solution:**

**7-5**. *Consider the two-dimensional spacetime spanned by coordinates* $(v, x)$ *with the line element*

$$ds^2 = -xdv^2 + 2dvdx \ .$$

*(a) Calculate the light cone at a point* $(v, x)$.

*(b) Draw a* $(v, x)$ *spacetime diagram showing how the light cones change with* $x$.

*(c) Show that a particle can cross from positive* $x$ *to negative* $x$ *but cannot cross from negative* $x$ *to positive* $x$.

*Comment: The light cone structure of this model spacetime is in many ways analogous to that of black hole spacetimes to be considered in Chapter 12, in particular in having a surface like* $x = 0$ *out from which you cannot get.*

**Solution:**

a) Light rays move on curves along which $ds^2 = 0$. These are curves with slopes

$$\frac{dv}{dx} = 0 \, , \quad \frac{dv}{dx} = \frac{1}{(2x)}$$

b)

c) World lines of particles must lie inside the light cone, as the world line moving from positive to negative $x$ shown. But there are none the other way.

**7-6**. *[B] Express the line element for flat spacetime in terms of the coordinates* $(t', r', \theta, \phi)$ *used to construct the Penrose diagram and defined in* (a) *and* (c) *in Box 7.1.*

**Solution:** If the definitions $u' = t' - r'$ and $v' \equiv t' + r'$ are used the line element is:

$$ds^2 = \sec^2(t' + r')\sec^2(t' - r')\left[-dt'^2 + dr'^2 + \sin^2 r' \cos^2 r' \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] .$$

If the definitions $u' = (t' - r')/2$ and $v' \equiv (t' + r')/2$ are used the line element is:

$$ds^2 = \frac{1}{4}\sec^2\left[\left(\frac{t' + r'}{2}\right)\right]\sec^2\left[\left(\frac{t' - r'}{2}\right)\right]\left[-dt'^2 + dr'^2 + \sin^2 r' \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] .$$

**7-7**. *[S] Transformation Law for the Metric. A general coordinate transformation is specified by four functions* $x'^\alpha = x'^\alpha(x^\beta)$.

*(a) Show that the chain rule can be expressed by*

$$dx^\alpha = \frac{\partial x^\alpha}{\partial x'^\gamma} dx'^\gamma \ .$$

*(b) Substitute this into the line element* (7.8) *to show that the transformed metric* $g'_{\gamma\delta}$ *is given by*

$$g'_{\gamma\delta} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^\gamma} \frac{\partial x^\beta}{\partial x'^\delta} \ .$$

*Make sure your answers are consistent with the summation convention.*

**Solution:**

a) Written out for $\alpha = 1$, for example, $dx^\alpha = (\partial x^\alpha / \partial x'^\gamma) dx'^\gamma$ becomes

$$dx^1 = \frac{\partial x^1}{\partial x'^0} dx'^0 + \frac{\partial x^1}{\partial x'^1} dx'^1 + \frac{\partial x^1}{\partial x'^2} dx'^2 + \frac{\partial x^1}{\partial x'^3} dx'^3$$

which is the chain rule.

b) The only thing that needs to be paid attention to here is to substitute $dx^\alpha = (\partial x^\alpha / \partial x'^\gamma) dx'^\gamma$ and $dx^\beta = (\partial x^\beta / \partial x'^\delta) dx'^\delta$ with *different* dummy indices indicating summation. Otherwise, the result would have more than repeated pairs of indices and be inconsistent with the summation convention.

**7-8.** *(a) Use the mathematical fact that any real symmetric matrix can be diagonalized by an orthogonal matrix to show that any metric can be diagonalized at one point* $P$ *by a linear transformation of the form*

$$x'^\alpha = M^\alpha{}_\beta x^\beta \ .$$

*In particular make clear the connection between orthogonal matrix of the theorem and* $g_{\alpha\beta}(x_P)$*, and between* $M^\alpha{}_\beta$ *and the components of the orthogonal diagonalizing matrix.*

*(b) Find the linear transformation that will diagonalize the warp drive metric* (7.25) *at any one point along the trajectory* $x_s(t)$*.*

**Solution:**

a) Substituting the linear transformation into the line element $g'_{\alpha\beta}dx'^{\alpha}dx'^{\beta}$ shows

$$\begin{aligned} g'_{\alpha\beta}dx'^{\alpha}dx'^{\beta} &= g'_{\alpha\beta}\left(M^{\alpha}{}_{\gamma}\,dx^{\gamma}\right)\left(M^{\beta}{}_{\delta}\,dx^{\delta}\right) \\ &= g_{\gamma\delta}\,dx^{\gamma}dx^{\delta}. \end{aligned} \tag{1}$$

This shows that the old metric is related to the new by

$$g_{\gamma\delta} = M^{\alpha}{}_{\gamma}\,g'_{\alpha\beta}\,M^{\beta}{}_{\delta}. \tag{2}$$

In matrix language this is

$$\mathbf{g} = \mathbf{M}^T\mathbf{g}'\,\mathbf{M} \tag{3}$$

where $T$ denotes a transpose. If $\mathbf{O}$ is the orthogonal matrix which diagonalizes $\mathbf{g}$

$$\mathbf{g}' = \mathbf{O}\,\mathbf{g}\,\mathbf{O}^T \tag{4}$$

where $\mathbf{g}'$ is diagonal.

Since $\mathbf{O}\mathbf{O}^T = \mathbf{I}$ for an orthogonal matrix, (4) can be written

$$\mathbf{g} = \mathbf{O}^T\mathbf{g}'\mathbf{O}. \tag{5}$$

Thus, we see $\mathbf{O} = \mathbf{M}$.

b) The linear transformation

$$\begin{aligned} t' &= t \quad , \quad y' = y \\ x' &= x + At\ , \quad z' = z \end{aligned} \tag{6}$$

where $A$ is the constant whose value is $V_s(t)\,f\,(r_s)$ at the point in question will diagonalize the warp drive metric at that point.

**7-9**. *[C] The argument in Section 7.4 shows that at a point $P$ there are coordinates in which the value of the metric takes its flat space form $\eta_{\alpha\beta}$. But are there coordinates in which the first derivatives of the metric vanish at $P$ as they do in flat space? What about the second derivatives? The following counting argument, although not conclusive, shows how far one can go.*

*The rule was for transforming the metric between one coordinate system and another was worked out in Problem 7. This can be expanded as a power (Taylor) series about $x_P$.*

$$\begin{aligned} x^\alpha(x'^\beta) &= x^\alpha(x'^\beta_P) + \left(\frac{\partial x^\alpha}{\partial x'^\beta}\right)_{x_P} (x'^\beta - x'^\beta_P) \\ &+ \frac{1}{2}\left(\frac{\partial^2 x^\alpha}{\partial x'^\beta \partial x'^\gamma}\right)_{x_P} (x'^\beta - x'^\beta_P)(x'^\gamma - x'^\gamma_P) \\ &+ \frac{1}{6}\left(\frac{\partial^3 x^\alpha}{\partial x'^\beta \partial x'^\gamma \partial x'^\delta}\right)_{x_P} (x'^\beta - x'^\beta_P)(x'^\gamma - x'^\gamma_P)(x'^\delta - x'^\delta_P) + \cdots . \end{aligned}$$

*At the point $x^\alpha_P$ there are sixteen numbers $(\partial x^\alpha/\partial x'^\beta)_{x_P}$ to adjust to make the transformed values of the metric $g'_{\alpha\beta}$ equal to $\eta_{\alpha\beta}$. Since there are only ten $g'_{\alpha\beta}$ we can do this and still have six numbers to spare! These six degrees of freedom correspond exactly to the three rotations and three Lorentz boosts which leave $\eta_{\alpha\beta}$ unchanged. Following this line of reasoning fill in the rest of the spaces in the table below to show that there is enough freedom in coordinate transformations to make the first derivatives of the metric vanish in addition to (7.12), but not the second derivatives:*

| | *Conditions* | *Numbers* |
|---|---|---|
| $g'_{\alpha\beta} = \eta_{\alpha\beta}$ | *10* | *16* |
| $\partial g'_{\alpha\beta}/\partial x'^\gamma = 0$ | *?* | *?* |
| $\partial^2 g'_{\alpha\beta}/\partial x'^\gamma \partial x'^\delta = 0$ | *?* | *?* |

*When properly organized, the second derivatives that cannot be transformed away are the measure of spacetime curvature as we shall see in Chapter 22. How many of them are there?*

**Solution:**

a) First derivatives: The number of components of a symmetric $4 \times 4$ matrix is 10. There are therefore $10 \times 4 = 40$ first derivatives of the metric and $4 \times 10 = 40$ second derivatives $\partial^2 x^\alpha/\partial x'^\beta \partial x'^\gamma$ since these are symmetric in $\beta$ and $\gamma$.

b) Second derivatives: $\partial^2 g'_{\alpha\beta}/\partial x'^\gamma \partial x'^\delta$ is symmetric in both $\alpha$ and $\beta$ and in $\gamma$ and $\delta$. There are therefore $10 \times 10 = 100$ conditions. The number of third derivatives $\partial^2 x^\alpha/\partial x'^\beta \partial x'^\gamma \partial x'^\delta$ is $4\times$ (the number of symmetric

combinations of four indices). The pedestrian (but fast) way to find the latter is to list them: 000, 001, 002, 003, 011, 012, 013, 022, . . . , 333.

There are 20 in total. These are therefore $4\times 20 = 80$ disposable coefficients, which leaves 20 second derivatives which cannot be made to vanish. These turn out to be the components of the Riemann curvature.

**7-10**. *An observer moves on a curve $X = 2T$ for $T > 1$ in the two dimensional geometry with metric* (7.20).

*(a) What are the components of the four velocity of this observer? Is the curve a timelike one?*

*(b) Find the components of an orthonormal basis $\mathbf{e}_0$, $\mathbf{e}_1$ for this observer.*

**Solution:**

a) Plugging $X = 2T$ into the line element (7.20) and using $d\tau^2 = -ds^2$ gives

$$d\tau^2 = 4(T^2 - 1)dT^2. \tag{1}$$

Since $d\tau^2 > 0$ the curve is timelike for $T > 1$. The components of the four velocity are

$$u^T = \frac{dT}{d\tau} = \frac{1}{2(T^2-1)^{\frac{1}{2}}}\ , \quad u^X = \frac{dX}{d\tau} = \frac{dX}{dT}\,\frac{dT}{d\tau} = \frac{1}{(T^2-1)^{\frac{1}{2}}} \tag{2}$$

or

$$u^A = \frac{1}{\sqrt{T^2-1}}\left(\frac{1}{2}\,,1\right). \tag{3}$$

b) The first basis vector is $\mathbf{e}_{\hat{0}} = \mathbf{u}$. The basis vector $\mathbf{e}_{\hat{1}}$ is orthogonal to this and of unit length. Let $(\mathbf{e}_{\hat{1}})^A = (F\,,\,G)$. Orthogonality implies

$$\mathbf{e}_{\hat{0}}\cdot\mathbf{e}_{\hat{1}} = -(2T)^2 u^T F + u^X G = 0. \tag{4}$$

Normalization is

$$\mathbf{e}_{\hat{1}}\cdot\mathbf{e}_{\hat{1}} = -(2T)^2(F)^2 + (G)^2 = 1. \tag{5}$$

Solving these two relations for $F$ and $G$ using (2) gives

$$(\mathbf{e}_{\hat{1}})^A = \frac{1}{(T^2-1)^{\frac{1}{2}}}\left(\frac{1}{2T}\,,T\right). \tag{6}$$

**7-11**. *[S] For the warp drive spacetime in Example 4 show that, at every point along the curve $x_s(t)$, the four-velocity of the ship lies inside the forward light cone.*

**Solution:** Assume $V_s > 0$. The slopes of the two curves defining the light cone in the $t-x$ plane are given in (7.26). On the world line of the ship $r_s = 0$ and $f = 1$. The slopes of the two light rays are $V_s \pm 1$. Since

$$V_s - 1 < V_s < V_s + 1$$

the velocity of the ship lies between them. That means the four-velocity is inside the forward light cone.

**7-12**. *In the warp drive spacetime in Example 4 how much ship time elapses on a trip between stations that takes coordinate time $T$?*

**Solution:** The ship time is the proper time along the curve $x = x_s(t)$ that elapses in a coordinate time $t = T$. From (7.25) this is

$$\begin{aligned} \tau &= \int_0^T \left[-dt^2 - (dx - V_s(t)f(0)dt)^2\right]^{\frac{1}{2}} \\ &= \int_0^T dt \left[1 - \left(\frac{dx_s}{dt} - V_s(t)f(0)\right)^2\right]^{\frac{1}{2}} \end{aligned}$$

But since $V_s = dx_s/dt$, and $f(0) = 1$, the elapsed proper time is $\tau = T$.

**7-13**. *[S] Consider two vector fields $\mathbf{a}(x)$ and $\mathbf{b}(x)$ and a world line $x^\alpha(\tau)$ in a spacetime with metric $g_{\alpha\beta}$. Derive an expression for $d(\mathbf{a}\cdot\mathbf{b})/d\tau$ in terms of partial derivatives of the coordinate basis components of $\mathbf{a}$ and $\mathbf{b}$, the partial derivatives of $g_{\alpha\beta}$, and the components of the four velocity $\mathbf{u}$.*

**Solution:**

$$\mathbf{a}(x)\cdot\mathbf{b}(x) = g_{\alpha\beta}(x)a^\alpha(x)b^\beta(x).$$

Evaluated on the world line $x^\alpha(\tau)$, all functions become functions of $\tau$. For example

$$\frac{da^\alpha}{d\tau} = \frac{\partial a^\alpha}{\partial x^\gamma}\frac{dx^\gamma}{d\tau} = \frac{\partial a^\alpha}{\partial x^\gamma}\, u^\gamma .$$

Thus

$$\frac{d}{d\tau}\,(\mathbf{a}\cdot\mathbf{b}) = \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\, a^\alpha b^\beta u^\gamma + g_{\alpha\beta}\,\frac{\partial a^\alpha}{\partial x^\gamma}\, u^\gamma b^\beta + g_{\alpha\beta}\, a^\alpha \frac{\partial b^\beta}{\partial x^\gamma}\, u^\gamma .$$

The only possible difficulty with this problem is keeping the indices straight.

**7-14.** *In a certain spacetime geometry the metric is*

$$ds^2 = -(1-Ar^2)^2 dt^2 + (1-Ar^2)^2 dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\ .$$

a) *Calculate the proper distance along a radial line from the center $r = 0$ to a coordinate radius $r = R$.*

b) *Calculate the area of a sphere of coordinate radius $r = R$.*

c) *Calculate the three volume of a sphere of coordinate radius $r = R$.*

d) *Calculate the four volume of a four dimensional tube bounded by a sphere of coordinate radius $R$ and two $t =$ constant planes separated by a time $T$.*

**Solution:**

a)

$$s = \int_0^R \left(1 - Ar^2\right) dr = R\left(1 - \frac{1}{3}\,\xi^2\right)$$

where $\xi \equiv R\sqrt{A}$.

b)

$$\begin{aligned} A &= \int_{\text{sphere}} (Rd\theta)(R\sin\theta d\phi) \\ &= \int_0^\pi d\theta \int_0^{2\pi} d\phi\, R^2 \sin\theta = 4\pi\, R^2 \end{aligned}$$

c)

$$\begin{aligned} V &= \int_{\text{sphere}} \left[\left(1 - Ar^2\right) dr\right] [r d\theta][r \sin\theta d\phi] \\ &= \int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \; r^2 \left(1 - Ar^2\right) \sin\theta d\theta \\ &= \left(\frac{4\pi R^3}{3}\right) \left(1 - \left(\frac{3}{5}\right)\xi^2\right) \end{aligned}$$

where $\xi$ is as above.

d)

$$\begin{aligned} V_4 &= \int \left[\left(1 - Ar^2\right) dt\right] \left[\left(1 - Ar^2\right) dr\right] [r d\theta][r \sin\theta d\phi] \\ &= \int_0^T dt \int_0^R dr \left(1 - Ar^2\right)^2 r^2 \int_0^\pi d\theta \int_0^{2\pi} d\phi \sin\theta \\ &= \left(\frac{4}{3}\pi R^3 T\right) \left(1 - \frac{6}{5}\xi^2 + \frac{3}{7}\xi^4\right) \end{aligned}$$

**7-15**. *[S] Calculate the area of the peanut illustrated in Figure 2.7.*

**Solution:** From the line element (2.21) and (7.28), an element of area is

$$dA = (a d\theta)\,(a\, f(\theta) d\phi).$$

The area of the peanut is

$$A = \int_0^\pi d\theta \int_0^{2\pi} d\phi \; a^2 f(\theta)$$

for

$$f(\theta) = \sin\theta \left(1 - \left(\frac{3}{4}\right)\sin^2\theta\right),$$

$$A = 2\pi a^2.$$

**7-16**. *[B] Suppose that you have a map of the world in the Mercator projection as described in Box 2.3. The map is is 1m wide. You use the Cartesian coordinates $(x, y)$ described in the box to locate points on the map. Greenland is approximated by a rectangle extending from $x = -5$ cm to $x = -14$ cm and $y = 21$ cm to $y = 38$ cm. The US is approximated by a rectangle extending from $x = -21$ cm to $x = -34$ cm and $y = 8$ cm to $y = 12$ cm. On the map, therefore, Greenland has an area about 3 times that of the US. Use the line element specified in these coordinates by equations* (f) *and* (i) *in the box find the true ratio of areas of these rectangles. Caution: These rectangles do not represent the actual areas of Greenland and the US very accurately.*

**Solution:** We calculate the area of a rectangle extending from $x_1$ to $x_2$ and $y_1$ to $y_2$. Using the metric specified in the box and (7.28), an element of area is

$$dA = \Omega^2(x, y)\, dx\, dy$$

and the area of a rectangle is

$$A = \int dA = \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} dy\ \Omega^2(x, y).$$

Using the form of $\Omega$ given in the box, this is (the substitution $z = e^{2\pi y/L}$ helps to do the integral)

$$A = 8\pi a^2 \left(\frac{x_2 - x_1}{L}\right) \left[\tan^{-1}\left(e^{2\pi y_2/L}\right) - \tan^{-1}\left(e^{2\pi y_1/L}\right)\right]$$

Evaluating for the specific rectangles gives

$$\frac{A_{\text{Greenland}}}{A_{\text{US}}} = .45.$$

**7-17**. *[S] Calculate the three dimensional volume on a $t =$ const. slice of the wormhole geometry* (7.39) *bounded by two spheres of coordinate radius $R$ on each side of the throat.*

**Solution:** The metric on a $t =$ const. spacelike slice is

$$dS^2 = dr^2 + \left(b^2 + r^2\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

From (7.33)

$$d\mathcal{V} = (dr)\left(\sqrt{b^2+r^2}\,d\theta\right)\left(\sqrt{b^2+r^2}\,\sin\theta\,d\phi\right)$$

The three-volume between $-R$ and $R$ is

$$\begin{aligned}\mathcal{V} &= \int_{-R}^{R} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\,\left(b^2+r^2\right)\sin\theta \\ &= 4\pi\left(b^2+\frac{r^3}{3}\right)\Bigg|_{-R}^{R} = 8\pi\left(b^2+\frac{R^3}{3}\right).\end{aligned}$$

Note that for very large $R$, this goes to the volume of two spheres as one would expect.

**7-18**. *Consider the three dimensional space with the line element*

$$dS^2 = \frac{dr^2}{(1-2M/r)} + r^2\left(d\theta^2+\sin^2\theta d\phi^2\right)$$

*a) Calculate the radial distance between the sphere $r = 2M$ and the sphere $r = 3M$.*

*b) Calculate the spatial volume between the two spheres in part (a).*

**Solution:**

a)

$$d = \int dS = \int_{2M}^{3M} dr\,\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}}.$$

Let $r = 2Mx$. Then,

$$d = 2M\int_1^{3/2} dx\,\left(\frac{x}{x-1}\right)^{\frac{1}{2}} \approx 3.05M.$$

b)

$$\begin{aligned}V &= \int dV = \int_{2M}^{3M} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\, r^2\sin\theta\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}} \\ &= 4\pi\int_{2M}^{3M} dr\, r^2\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}}\end{aligned}$$

Making the same substitution as above,

$$V = 4\pi(2M)^3 \int_1^{3/2} dx \, \frac{x^{\frac{5}{2}}}{(x-1)^{\frac{1}{2}}} \approx 215M^3.$$

**7-19.** *The surface of a sphere of radius $R$ in four, flat, Euclidean dimensions is given by*

$$X^2 + Y^2 + Z^2 + W^2 = R^2$$

*a) Show that points on the sphere may be located by coordinates $(\chi, \theta, \phi)$ where:*

$$X = R\sin\chi\sin\theta\cos\phi \ , \quad Z = R\sin\chi\cos\theta,$$
$$Y = R\sin\chi\sin\theta\sin\phi \ , \quad W = R\cos\chi.$$

*b) Find the metric describing the geometry on the surface of the sphere in these coordinates.*

**Solution:** To show that the angles $(\chi, \theta, \phi)$ are coordinates on the three-sphere, substitute the expressions for $W, X, Y, Z$ into

$$X^2 + Y^2 + Z^2 + W^2 = R^2 \ .$$

It will be satisfied identically. To find the metric on the three-sphere, work out

$$dX = R\left[(\sin\chi\sin\theta\cos\phi)d\phi + (\sin\chi\cos\theta\sin\phi)d\theta + (\cos\chi\sin\theta\sin\phi)d\chi\right]$$

etc. and substitute into

$$dS^2 = dX^2 + dY^2 + dZ^2 + dW^2 \ .$$

The result is

$$dS^2 = R^2\left[d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \ .$$

**7-20**. *(Make the cover.) Consider the two dimensional geometry which has the line element:*

$$d\Sigma^2 = \frac{dr^2}{(1-2M/r)} + r^2 d\phi^2 \ .$$

*Find a two-dimensional surface in three-dimensional flat space which has the same intrinsic geometry as this slice. Sketch a picture of your surface. [Comment: This is a slice of the Schwarzschild black hole geometry to be discussed in Chapter 12. It is also the surface on the cover of this book.]*

**Solution:** It's clear that the surface must be symmetric about an axis corresponding to the symmetry in $\phi$. Therefore, write the flat space metric in cylindrical coordinates $(\rho, \psi, z)$ as

$$dS^2 = d\rho^2 + \rho^2 d\psi^2 + dz^2 \ .$$

An axisymmetric surface can be specified by giving its height $z(\rho)$ above the $(x, y)$ or $(\rho, \psi)$ plane. The induced metric on the surface is

$$d\Sigma^2 = \left[\left(\frac{dz}{d\rho}\right)^2 + 1\right] d\rho^2 + \rho^2 d\psi^2$$

This will match up with the given geometry

$$d\Sigma^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\phi^2 \ .$$

provided $\rho$ is identified with $r$, $\phi$ with $\psi$, and

$$\left(\frac{dz}{d\rho}\right)^2 + 1 = \left(1 - \frac{2M}{\rho}\right)^{-1} \ .$$

This is easily integrated to give

$$z(\rho) = 2\sqrt{2M(\rho - 2M)} \ .$$

The figures below illustrate $z(r)$ which is rotated about the $z$-axis to give the surface itself:

Comment: The two-dimensional geometry given is a slice of the Schwarzschild geometry of a black hole to be studied in Chapter 12. It represents half of a *Schwarzschild throat*, a wormhole bridge between two asymptotically flat regions as will be described in Box 12.4.

**7-21**. *Consider a two-dimensional flat space with a skew coordinate system, the $x^1, x^2$ axes making an angle of 45° with each other.*

a) *Reproduce the coordinate grid below and draw on it the basis vectors $\mathbf{e}_1, \mathbf{e}_2$ of a coordinate basis associated with $x^1, x^2$.*

b) *Calculate the components of the metric $g_{AB}$ ($A, B$ range over 1,2) from the scalar product of the basis vectors.*

c) *Draw on the coordinate grid a vector $\mathbf{V}$ of length 2 making an angle of 30° with the $x$-axis. Calculate the components $V^A$ for this vector. Can you give a geometric construction for finding $V^A$.*

**Solution:**

a)

b) The metric is given in terms of the inner product of the coordinate basis vectors shown above by

$$g_{AB} = \mathbf{e}_A \cdot \mathbf{e}_B$$

Therefore,

$$g_{AB} = \begin{pmatrix} 1 & 2^{-\frac{1}{2}} \\ 2^{-\frac{1}{2}} & 1 \end{pmatrix}$$

c) In the above figure (enlarged for clarity over the scale of the previous one) the vector $\mathbf{V}$ is shown, as well as two vectors $\mathbf{V}^1 = V^1\mathbf{e}_1$ and $\mathbf{V}^2 = V^2\mathbf{e}_2$ that lie along the coordinate axes and sum to $\mathbf{V}$. Since the coordinate basis vectors are unit vectors, the lengths of these vectors $\mathbf{V}^1$ and $\mathbf{V}^2$ are the values of the components. The values follow from a little geometry $V^1 = \sqrt{3} - 1$, $V^2 = \sqrt{2}$. Transforming to the rectangular frame, we have

$$V'^A = \frac{\partial x'^A}{\partial x^B} V^B \ .$$

Using

$$\begin{aligned} x^1 &= x'^1 - x'^2 \\ x^2 &= \sqrt{2}x'^2, \end{aligned}$$

one finds $V'^1 = \sqrt{3}$, $V'^2 = 1$ which is also obvious from the geometry of the figure. Transforming back from these values is another way of doing the problem. The dotted lines give a geometrical construction for $(V^1, V^2)$.

**7-22.** *[S] (a) Find the coordinate basis components of an orthonormal basis for the wormhole metric* (7.39) *that is oriented along the coordinate lines.*

*(b) Find the components of the coordinate basis vectors in this orthonormal basis.*

**Solution:**

a) Since the wormhole metric is diagonal it's easy to construct an orthonormal basis with vectors pointing along the coordinate directions using the prescription given in (7.61). The coordinate basis components of these vectors are

$$\begin{aligned}
(\mathbf{e}_{\hat{t}})^\alpha &= (1,0,0,0)\ , \\
(\mathbf{e}_{\hat{r}})^\alpha &= (0,1,0,0)\ , \\
(\mathbf{e}_{\hat{\theta}})^\alpha &= \left(0,0,(b^2+r^2)^{-\frac{1}{2}},0\right)\ , \\
(\mathbf{e}_{\hat{\phi}})^\alpha &= \left(0,0,0,(b^2+r^2)^{-\frac{1}{2}}(\sin\theta)^{-1}\right)\ .
\end{aligned}$$

b) The coordinate basis vectors satisfy $\mathbf{e}_\alpha \cdot \mathbf{e}_\beta = g_{\alpha\beta}$. In the orthonormal basis the inner product of any two vectors $\mathbf{a}$ and $\mathbf{b}$ is [*cf.* (7.51)]

$$\mathbf{a}\cdot\mathbf{b} = -a^{\hat{0}}b^{\hat{0}} + a^{\hat{1}}b^{\hat{1}} + a^{\hat{2}}b^{\hat{2}} + a^{\hat{3}}b^{\hat{3}}\ .$$

The following will therefore do it.

$$\begin{aligned}
(\mathbf{e}_t)^{\hat{\alpha}} &= (1,0,0,0)\,, \\
(\mathbf{e}_r)^{\hat{\alpha}} &= (0,1,0,0)\,, \\
(\mathbf{e}_\theta)^{\hat{\alpha}} &= \left(0,0,(b^2+r^2)^{\frac{1}{2}},0\right)\ , \\
(\mathbf{e}_\phi)^{\hat{\alpha}} &= \left(0,0,0,(b^2+r^2)^{\frac{1}{2}}\sin\theta\right)\ .
\end{aligned}$$

**7-23**. *Show that any two orthonormal bases are related by a Lorentz transformation. More precisely, show that the vectors in one basis are linear combinations of the vectors in another with a matrix of coefficients that define a Lorentz transformation.*

**Solution:** Let $\{\mathbf{e}_{\hat{\alpha}}\}$ and $\{\mathbf{e}'_{\hat{\alpha}}\}$ be two orthonormal bases. Since they are bases, the vectors of one are linear combinations of the vectors of the other, *e.g.*

$$\mathbf{e}'_{\hat{\alpha}} = \Lambda^{\hat{\gamma}}_{\hat{\alpha}}\mathbf{e}_{\hat{\gamma}}.$$

Eq. (7.50) is the defining property of an orthonormal basis. Thus

$$\eta_{\hat{\alpha}\hat{\beta}} = \mathbf{e}'_{\hat{\alpha}}\cdot\mathbf{e}'_{\hat{\beta}} = \Lambda^{\hat{\gamma}}_{\hat{\alpha}}\Lambda^{\hat{\delta}}_{\hat{\beta}}\mathbf{e}_{\hat{\gamma}}\cdot\mathbf{e}_{\hat{\delta}} = \Lambda^{\hat{\gamma}}_{\hat{\alpha}}\Lambda^{\hat{\delta}}_{\hat{\beta}}\eta_{\hat{\gamma}\hat{\delta}}.$$

This shows that $\Lambda^{\hat{\gamma}}_{\hat{\alpha}}$ defines a linear transformation that preserves the metric of flat spacetime in an inertial frame. But that is the definition of a Lorentz transformation.

**7-24.** *In an inertial frame $(t, x, y, z)$ consider the spacelike hypersufaces of constant time $t'$ of another frame moving along the $x-$axis with a velocity $v$ with respect to the first.*

*a) Make a rough graph in a $(t, x)$ spacetime diagram of the family of surfaces separated by equal values of $t'$. Does every point in flat spacetime lie on one of these surfaces?*

*b) Find the $(t, x, y, z)$ coordinate components of a unit normal vector to these spacelike surfaces.*

**Solution:**

a) Since $t' = \gamma(t - vx)$, the equation of the spacelike surfaces is $t = vx + \text{const.}$. These look as follows:

The slope is $v$. Clearly, every point lies on one of these surfaces.

b) Tangent vectors that lie *in* the surfaces have the form

$$t^\alpha = (vt^x, t^x, t^y, t^z)$$

for any choice of $t^x$, $t^y$, and $t^z$. The normal vector must be orthogonal to any $t^\alpha$ of this form. It therefore has the form

$$n^\alpha = \left(n^t, n^x, 0, 0\right)$$

where

$$\mathbf{n} \cdot \mathbf{t} = -vt^x n^t + t^x n^x = 0 \, .$$

Thus, $n^t = n^x/v$ and one normal is

$$n^\alpha = (1\,, v\,, 0\,, 0) \; .$$

This can be made into a *unit* normal by dividing by $|\mathbf{n} \cdot \mathbf{n}|^{1/2}$ to find

$$n^\alpha = (\gamma,\ v\gamma,\ 0,\ 0)$$

where as usual $\gamma = (1 - v^2)^{-1/2}$. *Note*: this reduces correctly when $v = 0$ and the surfaces coincide with $t =$ const. surfaces.

**7-25.**

*[C] A toy model of a wormhole connecting two regions of space. Take a plane and delete two disks of equal radius R whose centers are separated by a distance d. Identify points on the edges of one disk with points on the edge of the other as shown, so that all points labeled 1 are identified, all points labeled 2, etc. A free particle or light ray whose straight line path intersects a point on the left*

*hand disk would emerge from the identified point on the right hand disk as shown making the same angle with the normal as it went in with.*

*a) Provide an argument based on the identification that straight line particle trajectories behave as shown.*

*b) Two points lie on the $x$-axis at locations $x = +L$ and $x = -L$, $L > R + d/2$. A particle starts moving along the x-axis from one point toward the other. What distance has it traveled when it reaches the other point?*

*c) Find a closed orbit for a free particle in this geometry. Is your orbit stable against small perturbations?*

*b) Suppose two spheres were deleted from three-dimensional flat space and identified in an analogous way. What kind of scene would an observer some distance out along the $x-$axis see when looking back towards the wormhole mouth?*

**Solution:**

a) The path must be continuous across the identification. Imagine the plane was a rubber sheet. Cut out a small piece containing the incident path at left, and join it up to the part at right matching identified points. If the path was moving toward 2 (as shown) when it was incident from the left, it must be moving toward 2 when it emerges on the right. Put differently, the normal and tangential components of the velocity must be continuous.

b) The path along the $x$-axis that goes in at 3 on the left and out at 3 on the right is the shortest. Its length is $2[L - (R + d/2)]$.

c) The straight line path from 1 to 1 and back is closed. It is unstable because a small error in direction will lead the path to move away from the $x$-axis.

d) Light rays bounce off a spherical mirror making equal angles with the normal — reversing the normal components of their velocities while preserving their tangential ones. An observer a distance $L$ out on the $x$-axis looking back through the wormhole would see the same thing as an observer at $-L$ looking at a spherical mirror replacing the wormhole. The M.C. Escher drawing "Hand with Reflecting Sphere," available on a number of websites, gives you an idea of what this would be like.

**7-26**. *(Another division into space and time.) Show that each point inside the forward light cone of the origin* $(-t^2+r^2<0)$ *lies on some Lorentz hyperboloid of the form* (7.74) *for some value of* $a$. *Points inside can be labeled using* $a$ *as a time coordinate and* $(\chi,\theta,\phi)$ *as spatial coordinates as in* (7.75). *Find the line element of flat spacetime in these new coordinates. Sketch the family of spacelike surfaces in a* $(t,r)$ *spacetime diagram.*

**Solution:** Eq. (7.75) provides the coordinate transformation between $(t,r)$ and $(a,\chi)$. For any point inside the forward light cone, where $r<t$, the value of $a$ can be computed from (7.74). The metric can be found by substituting (7.75) into (7.4) to find

$$ds^2 = -da^2 + a^2\left[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2\right].$$

The family looks as follows:

8
6
4
2
0
$\chi^2$
(a)
(b)
0.342
0.3425
0.343
0.3435
0.344
T / J

20
15
10
5
$(\rho_y^W/T_{KT}-2/\pi)^{-1}$
1
2
3
4
5

# Chapter 8

# Geodesics

The first two problems concern the geometry in a two-dimensional sphere. In usual spherical coordinates the metric can be written

$$ds^2 = R^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

where $R$ is a constant. We define $x^1 = \theta$, $x^2 = \varphi$ and let indices $A, B, C, \cdots$ range over 1,2.

**8-1**. *[S] Use Cartesian coordinates to write out and solve the geodesic equations for a two-dimensional flat plane and show that the solutions are the straight lines.*

**Solution:** The line element is

$$dS^2 = dx^2 + dy^2.$$

Since the metric coefficients are constant, either 1 or 0, all the Christoffel symbols vanish [*cf.* (8.19)]. The geodesic equation (8.14) becomes

$$\frac{d^2x}{dS^2} = 0 \quad , \quad \frac{d^2y}{dS^2} = 0.$$

The general solution is

$$x = AS + B \quad , \quad y = CS + D$$

for constants $A$, $B$, $C$, and $D$. Eliminating $S$ gives a linear relation

$$y = mx + b$$

for constants $m$ and $b$. This is the equation of all straight lines.

**8-2**. *In usual spherical coordinates the metric on a two-dimensional sphere is [cf. (2.15)]*

$$dS^2 = a^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

*where $a$ is a constant.*

*(a) Calculate the Christoffel symbols "by hand".*

*(b) Show that a great circle is a solution of the geodesic equation. (Hint: Make use of the freedom to orient the coordinates so the equation of a great circle is simple.)*

**Solution:**

a)

$$\begin{aligned} g_{AB} &= \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix} \\ g^{AB} &= \begin{pmatrix} R^{-2} & 0 \\ 0 & (R\sin\theta)^{-2} \end{pmatrix} \end{aligned}$$

$$\Gamma^1_{22} = -\sin\theta\cos\theta \quad , \quad \Gamma^2_{12} = \cot\theta$$

all the rest are zero.

b) Orient coordinates so that the great circle lies along the equator $\theta = \pi/2$. The equation of the great circle is then $\theta = \pi/2, \phi = S/a$ where $S$ is the distance around and $dx^A/dS = (0, 1/a)$. The geodesic equation is then

$$\frac{d^2x^A}{dS^2} = -\Gamma^A_{22}\frac{1}{a^2}$$

Evidently the left hand side vanishes, and the right hand side vanishes because the relevant Christoffel symbols vanish at $\theta = \pi/2$.

**8-3**. *A certain three-dimensional spacetime has the line element*

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\phi^2$$

a) *Find the explicit Lagrangian for the variational principle for geodesics in this spacetime in these coordinates.*

b) *Using the results of (a) write out the components of the geodesic equation by computing them from the Lagrangian.*

c) *Read off the non-zero Christoffel symbols for this metric from your results in (b).*

**Solution:**

a) The Lagrangian $L(\dot{x}^\alpha, x^\alpha) \equiv [-g_{\alpha\beta}(x)\dot{x}^\alpha \dot{x}^\beta]^{1/2}$ defined by the principle of extremal proper time [*cf.* (8.10)] is for this problem

$$L\left(\dot{t}, \dot{r}, \dot{\phi}, r\right) = \left[\left(1 - \frac{2M}{r}\right)(\dot{t})^2 - \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 - r^2\dot{\phi}^2\right]^{\frac{1}{2}}$$

where $\dot{x}^\alpha = dx^\alpha/d\sigma$.

b) The components of the geodesic equations are [*cf.* (8.9)]

$$\begin{aligned}
&- \frac{d}{d\tau}\left[\left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}\right] = 0 \\
&- \frac{d}{d\tau}\left[-r^2\frac{d\phi}{d\tau}\right] = 0 \\
&- \frac{d}{d\tau}\left[-\left(1 - \frac{2M}{r}\right)^{-1}\frac{dr}{d\tau}\right] + \frac{M}{r^2}\left(\frac{dt}{d\tau}\right)^2 \\
&+ \left(1 - \frac{2M}{r}\right)^{-2}\frac{M}{r^2}\left(\frac{dr}{d\tau}\right)^2 - r\left(\frac{d\phi}{d\tau}\right)^2 = 0
\end{aligned}$$

c) From these we find the following non-vanishing Christoffel symbols

$$\begin{aligned}
\Gamma^t_{tr} &= \left(1-\frac{2M}{r}\right)^{-1}\frac{M}{r^2}\,, \qquad \Gamma^r_{\phi\phi} = -(r-2M)\,,\\
\Gamma^r_{tt} &= \left(1-\frac{2M}{r}\right)\frac{M}{r^2}\,, \qquad \Gamma^\phi_{\phi r} = \frac{1}{r}\,,\\
\Gamma^r_{rr} &= -\left(1-\frac{2M}{r}\right)^{-1}\frac{M}{r^2}\,.
\end{aligned}$$

**8-4.** *[A] (Rotating Frames) The line element of flat spacetime in a frame $(t, x, y, z)$ that is rotating with an angular velocity $\Omega$ about the $z-$axis of an inertial frame is*

$$ds^2 = -[1-\Omega^2(x^2+y^2)]dt^2 + 2\Omega(ydx - xdy)dt + dx^2 + dy^2 + dz^2\,.$$

*(a) Verify this by transforming to polar coordinates and checking that the line element is (7.4) with the substitution $\phi \to \phi - \Omega t$.*

*(b) Find the geodesic equations for $x$, $y$, and $z$ in the rotating frame.*

*(c) Show that in the non-relativistic limit these reduce to the usual equations of Newtonian mechanics for a free particle in a rotating frame exhibiting the centrifugal force and the Coriolis force.*

**Solution:**

a) Self explanatory.

b) The geodesic equations can be derived from the conditions for extremal proper time. The equations for $x$, $y$, and $z$ (respectively) are:

$$\begin{aligned}
-\frac{d^2x}{d\tau^2} - 2\Omega\,\frac{dy}{d\tau}\,\frac{dt}{d\tau} + \Omega^2 x\left(\frac{dt}{d\tau}\right)^2 &= 0\\
-\frac{d^2y}{d\tau^2} + 2\Omega\,\frac{dx}{d\tau}\,\frac{dt}{d\tau} + \Omega^2 y\left(\frac{dt}{d\tau}\right)^2 &= 0\\
\frac{d^2z}{d\tau^2} &= 0.
\end{aligned}$$

c) For example, in the non-relativistic limit, the $x$ equation becomes

$$\frac{d^2x}{dt^2} = -2\Omega\,\frac{dy}{dt} + \Omega^2 x\ .$$

The second term on the right hand side is the $x$-component centrifugal force $\vec{\Omega}\times(\vec{\Omega}\times\vec{x}))$ when $\vec{\Omega} = \Omega\vec{e}_z$. The first term is the Coriolis force $2\vec{\Omega}\times(d\vec{x}/dt)$.

**8-5**. *Derive the Christoffel symbols $\Gamma^{\phi}_{r\phi}$ and $\Gamma^{\theta}_{\phi\phi}$ for the wormhole metric* (7.39) *directly from the general formula* (8.19) *and not starting from variational principle of extremal proper time.*

**Solution:** There is not much to say about the solution to this problem except to evaluate (8.19). The diagonal metric simplifies the sums, *e.g.*

$$g_{r\alpha}\Gamma^{\alpha}{}_{\phi\phi} = g_{rr}\Gamma^{r}{}_{\phi\phi}.$$

We write out the two equations analogous to (8.20):

$$\begin{aligned} g_{\phi\phi}\Gamma^{\phi}_{r\phi} &= \frac{1}{2}\left(\frac{\partial g_{\phi r}}{\partial\phi} + \frac{\partial g_{\phi\phi}}{\partial r} - \frac{\partial g_{r\phi}}{\partial\phi}\right) = r\sin^2\theta \\ g_{\theta\theta}\Gamma^{\theta}_{\phi\phi} &= \frac{1}{2}\left(\frac{\partial g_{\theta\phi}}{\partial\phi} + \frac{\partial g_{\theta\phi}}{\partial\phi} - \frac{\partial g_{\phi\phi}}{\partial\theta}\right) = -\left(b^2+r^2\right)\sin\theta\cos\theta \end{aligned}$$

which gives

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta\ , \qquad \Gamma^{\phi}_{r\phi} = \frac{r}{(b^2+r^2)}.$$

**8-6**. *Show by direct calculation from* (8.15) *that the norm of the four velocity* $\mathbf{u}\cdot\mathbf{u}$ *is a constant along a geodesic.*

**Solution:** Let $N \equiv \mathbf{u}\cdot\mathbf{u} = g_{\alpha\beta}u^\alpha u^\beta$, then

$$\begin{aligned} \frac{dN}{d\tau} &= 2g_{\alpha\beta}\frac{du^\alpha}{d\tau}\,u^\beta + \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\,u^\alpha u^\beta u^\gamma \\ &= -2g_{\alpha\beta}\,\Gamma^{\alpha}_{\gamma\delta}\,u^\beta u^\gamma u^\delta + \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\,u^\alpha u^\beta u^\gamma. \end{aligned}$$

Relabeling dummy indices so that both terms have the same factor of $u^\alpha u^\beta u^\gamma$ this becomes

$$\frac{dN}{d\tau} = \left(-2g_{\beta\delta}\,\Gamma^{\delta}_{\gamma\alpha} + \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right) u^\alpha u^\beta u^\gamma \,.$$

Using (8.19) this is

$$\frac{dN}{d\tau} = \left(-\frac{\partial g_{\beta\gamma}}{\partial x^\alpha} - \frac{\partial g_{\beta\alpha}}{\partial x^\gamma} + \frac{\partial g_{\gamma\alpha}}{\partial x^\beta} + \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right) u^\alpha u^\beta u^\gamma$$

By relabeling the dummy indices and using $g_{\alpha\beta} = g_{\beta\alpha}$, one can see that all these terms are equal up to sign. They cancel and $dN/d\tau = 0$. Thus $(\mathbf{u}\cdot\mathbf{u})$ is conserved along a geodesic.

**8-7**. *[S] Consider a particle of mass $m$ moving in a central potential $V(r)$ in non-relativistic Newtonian mechanics. Write down the Lagrangian for this system in polar coordinates. Using the method of Section 8.2, show that invariance under rotations about the $z$-axis implies the conservation of the $z-$component of the angular momentum.*

**Solution:** The Lagrangian for the particle is

$$L\left(\dot{\vec{x}}\,,\,\vec{x}\right) = \frac{1}{2}\,m\,\left(\dot{\vec{x}}\right)^2 - V(r)$$

which in polar coordinates reads

$$L = \frac{1}{2}\,m\,\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - V(r)$$

taking polar axis to be the $z$-axis. Rotations about the $z$-axis are generated by $\phi \to \phi+$ const. $L$ is independent of $\phi$ and invariant under these rotations. Since $L$ doesn't depend on $\phi$

$$\frac{\partial L}{\partial\dot{\phi}} = m\,r^2\sin\theta\,\dot{\phi} = m\,\left(\vec{r}\times\dot{\vec{x}}\right)^z$$

is conserved.

**8-8**. *Verify the claim in Example 6 that the Killing vector corresponding to the rotational symmetry of flat space about the $z-$axis has components $(-y, x, 0)$ in Cartesian coordinates. In the same coordinates find the components of the Killing vectors corresponding to the rotational symmetry of flat space around the $y-$ and $x-$axes.*

**Solution:** The Killing vector is $\vec{\xi} = (0, 0, 1)$ in $(r, \theta, \phi)$ spherical coordinates which means that it points in the $\phi$-direction and has squared length

$$\vec{\xi} \cdot \vec{\xi} = g_{AB}\xi^A\xi^B = g_{\phi\phi}(\xi^\phi)^2 = r^2 \sin^2\theta \ .$$

Then just a little geometry gives the $(x, y, z)$ components of $\vec{\xi}$ as

$$(-r\sin\theta\sin\phi, r\sin\theta\cos\phi, 0) = (-y, x, 0) \ .$$

Cyclic permutation of the axes gives the other two Killing vectors

$$(0, -z, y) \ , \ (-z, 0, x) \ .$$

(Note that the overall sign is arbitrary since it wasn't specified which way the rotation went.)

**8-9**. *Consider the two-dimensional spacetime with the line element*

$$ds^2 = -X^2 dT^2 + dX^2$$

*Find the shapes $X(T)$ of all the timelike geodesics in this spacetime.*

**Solution:** There are several ways to do this problem. The most direct is to note that the metric is independent of $T$ and therefore has a Killing vector $\xi^A = (1, 0)$. The quantity

$$-\boldsymbol{\xi} \cdot \mathbf{u} = X^2 \left(\frac{dT}{d\tau}\right) \equiv e \tag{1}$$

is conserved and is a first integral of the geodesic equations. Another first integral is supplied by $\mathbf{u} \cdot \mathbf{u} = -1$ which is

$$-X^2 \left(\frac{dT}{d\tau}\right)^2 + \left(\frac{dX}{d\tau}\right)^2 = -1$$

Substituting for $dT/d\tau$ from (1) gives

$$\frac{dX}{d\tau} = \pm\left(\frac{e^2}{X^2} - 1\right)^{\frac{1}{2}}$$

To compute the shape $t(x)$ note that

$$\frac{dT}{dX} = \frac{dT/d\tau}{dX/d\tau} = \pm\frac{e}{X^2}\left(\frac{e^2}{X^2} - 1\right)^{-\frac{1}{2}}$$

This can be integrated to find

$$T(X) = \pm\ \cosh^{-1}\left(\frac{e}{X}\right) + T_0\ .$$

This is a two parameter family of all timelike geodesics.

Another way of doing this problem is to write out the geodesic equations and deduce the integral (1). Yet another is to note that $x = X\cosh T$, $t = X\sinh T$ transforms flat spacetime into this form, and transforms back the straight lines in flat spacetime.

**8-10**. *Show that any one of the four rectangular coordinates is an affine parameter for a light ray in flat spacetime.*

**Solution:** Light rays move on straight lines at the speed 1 in flat spacetime. When this straight line is expressed in parametric form $x^\alpha = x^\alpha(\lambda)$, the parameter $\lambda$ is affine if

$$\frac{d^2x^\alpha}{d\lambda^2} = 0 \tag{1}$$

Consider a light ray moving just along the $x$-axis. If we use $x$ as a parameter, the curve is

$$x^0 = x\ , \quad x^1 = x$$

and (1) is satisfied. If we use $t$ as the parameter

$$x^0 = t\ , \quad x^1 = t$$

and it's also satisfied.

**8-11.** *Solve for the null geodesics in three dimensional flat spacetime using polar coordinates so the line element is* $ds^2 = -dt^2 + dr^2 + r^2 d\phi^2$. *Do light rays move on straight lines?*

**Solution:** Corresponding to the $t$-displacement symmetry, there is a Killing vector $\boldsymbol{\xi} = (1, 0, 0)$. Corresponding to the $\phi$-displacement symmetry, there is a Killing vector $\boldsymbol{\eta} = (0, 0, 1)$. There are therefore first integrals of the geodesic equation

$$-\boldsymbol{\xi} \cdot \mathbf{u} = \frac{dt}{d\lambda} \equiv e \ , \ \boldsymbol{\eta} \cdot \mathbf{u} = r^2 \frac{d\phi}{d\lambda} = \ell \tag{1}$$

where $u^\alpha = dx^\alpha/d\lambda$. A further first integral is provided by the condition for a *null* geodesic

$$\mathbf{u} \cdot \mathbf{u} = -\left(\frac{dt}{d\lambda}\right)^2 + \left(\frac{dr}{d\lambda}\right)^2 + r^2 \left(\frac{d\phi}{d\lambda}\right)^2 = 0 \ . \tag{2}$$

Using (1) to eliminate $d\phi/d\lambda$ and $dt/d\lambda$ from (2) gives the radial equation

$$\frac{dr}{d\lambda} = \left[e^2 - \frac{\ell^2}{r^2}\right]^{\frac{1}{2}}$$

The shape of the light ray orbits in space $\phi(r)$ can be compiled from

$$\frac{d\phi}{dr} = \frac{d\phi/d\lambda}{dr/d\lambda} = \frac{1}{r^2}\left(\frac{1}{b^2} - \frac{1}{r^2}\right)^{-\frac{1}{2}}$$

where $b \equiv \ell/e$. This is just like (8.36) with the $\ell$ of that equation being replaced by $b$. The solution for the orbits is therefore [*cf.* (8.38)]

$$r\cos(\phi - \phi_0) = b$$

which is the equation of a straight line.

**8-12.** *(The Hyperbolic Plane.) The hyperbolic plane defined by the metric*

$$dS^2 = y^{-2}\left(dx^2 + dy^2\right) \ , \quad y \geq 0$$

*is a classic example of a two-dimensional surface.*

a) *Show that points on the $x$-axis are an infinite distance from any point $(x, y)$ in the upper half-plane.*

b) *Write out the geodesic equations.*

c) *Show that the geodesics are semi-circles centered on the $x$-axis or vertical lines.*

d) *Solve the geodesic equations to find $x$ and $y$ as functions of the length $S$ along these curves.*

*Remark: This example was important in the history of geometry. Euclid's fifth postulate for Euclidean geometry states that for a straight line $L$ and a point $P$ there is only one straight line (a geodesic) through $P$ that does not intersect $L$. (That straight line is the one parallel to $L$.) The sphere is an example for which there are* no *such straight lines through $P$ (all great circles intersect.) The hyperbolic plane is a constant negative curvature example (see Chapter 21) where there are an infinite number of straight lines through $P$ that do not intersect $L$ (see the example in the figure above).*

**Solution:**

a) The distance along an $x$-constant line to a point on the $x$-axis is $\int_0 dy/y$ which diverges. Similarly, the distance along any other curve to a point on the $x$-axis will diverge.

b) The non-vanishing Christoffel symbols are:

$$-\Gamma^x_{xy} = \Gamma^y_{xx} = -\Gamma^y_{yy} = 1/y \ .$$

The geodesic equations are:

$$\begin{aligned} \frac{d^2x}{dS^2} &= \frac{2}{y}\frac{dx}{dS}\frac{dy}{dS} \\ \frac{d^2y}{dS^2} &= -\frac{1}{y}\left(\frac{dx}{dS}\right)^2 + \frac{1}{y}\left(\frac{dy}{dS}\right)^2 \ . \end{aligned}$$

c) The first geodesic equation in (b) can be written

$$y^2\frac{d}{dS}\left(\frac{1}{y^2}\frac{dx}{dS}\right) = 0 \ .$$

This can be integrated to give

$$\frac{dx}{dS} = \frac{y^2}{r} \tag{1}$$

for any constant $r$. This is one integral of the geodesic equations. Another is supplied by the normalization condition

$$\frac{1}{y^2}\left[\left(\frac{dx}{dS}\right)^2 + \left(\frac{dy}{dS}\right)^2\right] = 1 \ .$$

Using (1) we have,

$$\frac{dy}{dS} = \pm\left[y^2 - \frac{y^4}{r^2}\right]^{\frac{1}{2}} \ . \tag{2}$$

To find the shape of the geodesics we compute (for $dy/dS > 0$)

$$\frac{dx}{dy} = \frac{dx/dS}{dy/dS} = \frac{y}{\sqrt{r^2 - y^2}} \ .$$

The integral of this is

$$(x - x_0)^2 + y^2 = r^2$$

where $x_0$ is a constant. This is a circle of radius $r$ centered at $(x_0, 0)$ on the $x$-axis. Vertical lines correspond to the limit in which $r$ becomes infinite.

d) Eq (2) can be solved to yield

$$y(S) = \frac{r}{\cosh(S)}$$

and then using this in (1) gives

$$x(S) = r\tanh(S) \ .$$

You can check that $x^2 + y^2 = r^2$.

**8-13**. *[S] Construct Riemann normal coordinates for flat space by the procedure discussed in Section 8.4 using the origin of an inertial frame as the point $P$, and four unit vectors pointing along its axes. Do the resulting coordinates coincide with the inertial frame coordinates?*

**Solution:** Denote the coordinates of the inertial frame by $X^\alpha$ to keep them distinct from the Riemann normal coordinates $x^\alpha$ under construction. Take a point $Q$ located by $X^\alpha$ and spacelike separated from the origin $P$. (The case of timelike separation is similar.) $Q$ lies on a spacelike, straight-line geodesic from the origin. The distance $s$ along this geodesic is given by

$$s^2 = \eta_{\alpha\beta} X^\alpha X^\beta . \tag{1}$$

The unit vector pointing along this straight line has components

$$N^\alpha = X^\alpha / s \tag{2}$$

so that $\mathbf{N} \cdot \mathbf{N} = 1$.

Riemann normal coordinates are constructed by choosing an orthonormal basis at $P$ and labeling each point by coordinates

$$x^\alpha = s\, n^\alpha \tag{3}$$

where $n^\alpha$ are the orthonormal basis components of the unit vector at the origin pointing along the geodesic connecting $P$ to $Q$. But coordinate basis vectors of a Cartesian coordinate are an orthonormal basis. Therefore, $n^\alpha = N^\alpha$ and (3) becomes

$$x^\alpha = X^\alpha . \tag{4}$$

The Riemann normal coordinates coincide with the coordinates of the inertial frame.

**8-14.** *[C] (Fermat's Principle of Least Time.) Consider a medium with an index of refraction $n(x^i)$ that is a function of position. The velocity of light in the medium varies with position and is $c/n(x^i)$. Fermat's principle states that light rays follow paths between two points in space (not spacetime!) which take the least travel time.*

a) *Show that the paths of least time are geodesics in three dimensional space with the line element*

$$dS^2_{\text{fermat}} = n^2(x^i)\ dS^2$$

*where $dS^2$ is the usual line element for flat three-dimensional space, e.g.*

$$dS^2 = dx^2 + dy^2 + dz^2 \ .$$

b) *Write out the geodesic equations for the extremal paths in $(x, y, z)$ rectangular coordinates.*

**Solution:**

a) The travel time between two points $A$ and $B$ in space is

$$\Delta\tau = \int_A^B \frac{n(x^i(s))}{c}\ dS$$

where $S$ is the usual distance along the path. Evidently curves of extremal $\Delta\tau$ are curves of extremal

$$\int n dS = \int dS_{\text{fermat}} \ .$$

b) An easy way to get the geodesic equations is as Lagrange's equations from the variational principle for extremal time. Using $\lambda$ as a parameter along the paths

$$\Delta\tau = \int_0^1 d\lambda\ n(x^i)\ \left[\left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2 + \left(\frac{dz}{d\lambda}\right)^2\right]^{\frac{1}{2}} \ .$$

Thus for example, the equation for $x$ is

$$\frac{d}{dS}\left[n(x^i)\,\frac{dx}{dS}\right] - \frac{\partial n(x^i)}{\partial x} = 0$$

and similar equations for $y$ and $z$.

**8-15.** *[C] (The Lunenberg Lens) A sphere of radius $R$ with an index of refraction that varies with radius as*

$$n(r) = \left[2 - \left(\frac{r}{R}\right)^2\right]^{\frac{1}{2}}$$

*is called a Lunenberg lens. Use the results of Problem 14 to show that it has the property that any bundle of parallel rays incident from one direction is focused on one point on the surface of the sphere.*

**Solution:** Orient the $x$-axis along the direction of the incoming rays as shown below.

Use polar coordinates $(r, \phi)$ in the plane to describe the trajectory of the light ray as $\phi = \phi(r)$. From Problem 14 the path is an extremum of

$$\Delta\tau = \int n(r)dS = \int dr\; n(r)\left[1 + r^2\left(\frac{d\phi}{dr}\right)^2\right]^{\frac{1}{2}} .$$

Writing out Langrange's equations for $\phi(r)$, one finds

$$\frac{d}{dr}\left[n(r)r^2\frac{d\phi}{dr}\left(1 + r^2\left(\frac{d\phi}{dr}\right)^2\right)^{-\frac{1}{2}}\right] = 0 .$$

This implies

$$n(r)r^2\frac{d\phi}{dr}\left(1+r^2\left(\frac{d\phi}{dr}\right)^2\right)^{-\frac{1}{2}} \equiv \ell$$

is constant along the geodesic. (A conserved quantity can be expected because of the spherical symmetry.) Solving for $d\phi/dr$ one finds

$$\frac{d\phi}{dr} = \frac{\ell}{r^2\left(n^2-\ell^2/r^2\right)^{1/2}} \ .$$

(Alternatively one can follow Example 8.7 and use $S$ as the parameter along the geodesic and construct $d\phi/dr$ as in (8.36).)

The value of $\ell$ is set by the value of $\phi_*$ when the ray first crosses the surface. There $n(R) = 1$, so after computing the relevant derivatives

$$nr^2\frac{d\phi}{dS} \equiv \ell = R\sin\phi_* \ .$$

($\ell = 0$ when $\phi_* = 0$ as expected.)

The place $\phi_{\text{image}}$ where the ray crosses the surface again is

$$\phi_{\text{image}} = \phi_* + \Delta\phi$$

where $\Delta\phi$ is the angle swept out between the first crossing and the second. This is

$$\Delta\phi = 2\int_{TP}^{R} dr\ \frac{\ell}{r^2\left(n^2-\ell^2/r^2\right)^{1/2}} \ .$$

Here $TP$ is the turning point closest to the center where the denominator vanishes.

A few hints on carrying out the integral are given below, but for the given $n^2(r) = 2-(r/R)^2$ one finds

$$\Delta\phi = \pi - \phi_* \ .$$

Thus,

$$\phi_{\text{image}} = \pi \ .$$

All rays arrive at the same point; they are therefore focussed.

*Hints on the integral:* In dimensionless variables $\xi = r/R$ and $\lambda = \ell/R$, the relevant integral is

$$2\int_{TP}^{1} d\xi\ \lambda\xi^{-2}\left(2-\xi^2-\frac{\lambda^2}{\xi^2}\right)^{-\frac{1}{2}}$$

Making the substitution $u = 1/\xi^2$ transforms this into

$$\lambda\int_{1}^{TP} du\ \left(2u-1-\lambda^2u^2\right)^{-\frac{1}{2}}$$

which is a standard integral that can be looked up in tables.

# Chapter 9

# The Geometry Outside a Spherical Star

**9-1**. *[S] An advanced civilization living outside a spherical neutron star of mass $M$ constructs a massless shell concentric with the star such that the area of the inner surface is $144\pi M^2$ and the area of the outer surface is $400\pi M^2$. What is the physical thickness of the shell?*

**Solution:** The area of a sphere of Schwarzschild radius $r$ is $4\pi r^2$. The Schwarzschild radii of the inside and outside of the shell are therefore $6M$ and $10M$. The radial distance between these shells is thus,

$$\begin{aligned} d = \int ds &= \int_{6M}^{10M} dr \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} \\ &= \left\{ r\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} + M \log\left[ r - M + r\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} \right] \right\} \Bigg|_{6M}^{10M} \\ &= 4.64M \ . \end{aligned}$$

**9-2**. *In the dense plasma surrounding a neutron star which is accreting material from a binary companion positrons are produced, and electrons and positrons annihilate to produce $\gamma$-rays. Assuming the neutron star has a mass of $2.5M_\odot$ (solar masses) and a radius of 10 km, at what energy should a distant observer look for the $\gamma$-rays being emitted from the star by this process?*

*Assume the center of mass of the electron and positron is at rest with respect to the star when they annihilate.*

**Solution:** Electrons and positrons annihilate to produce two gamma rays. (Producing just one is forbidden by conservation of linear momentum.) Therefore, in the center of mass frame of the electron and positron each gamma ray will have an energy of $m_e c^2 \approx .5MeV$, the rest energy of an electron. The gamma ray is redshifted when it reaches infinity by an amount

$$\begin{aligned} E_\infty &= m_e c^2 \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} \\ &\approx .5MeV \left[1 - \frac{2(2.5M_\odot)(1.474 \text{ km}/M_\odot)}{10 \text{ km}}\right]^{\frac{1}{2}} \\ &\approx .26 \text{ MeV} \end{aligned}$$

**9-3**. *An observer is stationed at fixed radius $R$ in the Schwarzschild geometry produced by a spherical star of mass $M$. A proton moving radially outward from the star traverses the observer's laboratory. Its energy $E$ and momentum $|\vec{P}|$ are measured.*

*a) What is the connection between $E$ and $|\vec{P}|$?*

*b) What are the components of the four-momentum of the proton in the Schwarzschild coordinate basis in terms of $E$ and $|\vec{P}|$?*

**Solution:**

a) From the principle of equivalence, the relationship between $E$ and $P$ must be exactly the same as it is for a flat space:

$$E^2 - P^2 = m^2$$

where $m$ is the proton mass.

b) Measured quantities correspond to projections on the orthonormal basis $\mathbf{e}_{\hat{\alpha}}$ associated with the observers' laboratory. If one of the spacelike vectors $\mathbf{e}_{\hat{r}}$ is oriented in the $r$ direction

$$E = -\mathbf{p} \cdot \mathbf{e}_{\hat{0}} \quad , \quad P = \mathbf{p} \cdot \mathbf{e}_{\hat{r}} \ .$$

The Schwarzschild coordinate components of $\mathbf{e}_{\hat{0}}$ and $\mathbf{e}_{\hat{r}}$ are easily worked out from $\mathbf{e}_{\hat{\alpha}} \cdot \mathbf{e}_{\hat{\beta}} = \eta_{\alpha\beta}$ and the requirement that $\mathbf{e}_{\hat{0}}$ point in the $t$ direction (stationary) and $\mathbf{e}_{\hat{r}}$ in the $r$ direction

$$\begin{aligned} \mathbf{e}_{\hat{0}} &= \left( \left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}}, 0, 0, 0 \right) , \\ \mathbf{e}_{\hat{r}} &= \left( 0, \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}}, 0, 0 \right) . \end{aligned}$$

Thus

$$E = -\mathbf{p} \cdot \mathbf{e}_{\hat{0}} = \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} p^t$$

and

$$P = (1 - 2M/R)^{-\frac{1}{2}} p^r \ .$$

From these two relations $p^t$ and $p^r$ can be calculated in terms of $P$ and $E$.

**9-4**. *[B,E] Suppose the shell discussed in Box 9.1 is to be designed so the g-forces experienced by an observer falling into the shell are to be less than* $20g$ *where* $g = 9.8$ m/s$^2$. *If the observer falls feet first into the shell these g-forces are the difference between the force per unit mass at the observer's head and feet. Estimate, using Newtonian theory, how massive and how big would the shell have to be to meet this design criterion.*

**Solution:** Let $h$ be the height of the observer and $g_{\max} = 20g$ the maximum difference in accelerations between her head and feet. The design criterion is

$$\frac{GM}{r^2} - \frac{GM}{(r+h)^2} \sim \frac{2GM}{r^2} \left(\frac{h}{r}\right) \lesssim g_{\max} \tag{1}$$

for any $r$ through which the observer falls. The largest value occurs at the radius of the shell $R$. As described in the box (with $c$ and $G$ restored)

$$\frac{2GM}{c^2R} \approx 1. \tag{2}$$

Using this to eliminate $M$ from (1) gives

$$R > \left(\frac{c^2h}{g_{\max}}\right)^{\frac{1}{2}}. \tag{3}$$

Putting in the numbers gives from (3) and (2)

$$R \gtrsim 20,000\text{km} \quad , \quad M \gtrsim 13,000\ M_\odot. \tag{4}$$

Not on this planet!!

**9-5**. *Sketch the qualitative behavior of a particle orbit that comes in from infinity with a value of $\mathcal{E}$ exactly equal to the maximum of the effective potential $V_{\text{eff}}$. How does the picture change is the value of $\mathcal{E}$ is a little bit larger than the maximum or a little bit smaller.*

**Solution:** Were $\mathcal{E}$ *exactly* equal to the maximum value of the potential, the orbit would come in from infinity, spiral around the black hole, approaching ever closer to the circular orbit at the potential maximum. The figure below shows what happens if the value of $\mathcal{E}$ is slightly below the maximum.

This figure was made with the *Mathematica* program, *Shape of Orbits in the Schwarzschild Geometry*, on the book website. The value for $\ell$ is 4.3, the

maximum of the effective potential is $V_{\max} = 0.0401391$, and $\mathcal{E} = 0.0401300$. The orbit spirals the center and then returns to infinity. Were $\mathcal{E}$ slightly greater than the maximum of the potential, the orbit would spiral around the center, and then plunges into it.

**9-6**. *[S] An observer falls radially inward towards a black hole of mass $M$ (exterior geometry the Schwarzschild geometry) starting with zero kinetic energy at infinity. How much time does it take, as measured on the observer's clock, to pass between the radii $6M$ and $2M$?*

**Solution:** From (9.37) we have for the inward a radial orbit ($\ell = 0$) starting from zero initial velocity ($e = 1$)

$$\left(\frac{dr}{d\tau}\right) = -\left(\frac{2M}{r}\right)^{\frac{1}{2}}$$

Therefore the proper time to pass between $6M$ and $2M$ is

$$\begin{aligned} \mathcal{T} &= \int d\tau = \int_{6M}^{2M} dr \left(\frac{d\tau}{dr}\right) = -\int_{6M}^{2M} dr \left(\frac{r}{2M}\right)^{\frac{1}{2}} \\ &= M\,\frac{2^{\frac{1}{2}}}{3}\left(6^{\frac{3}{2}} - 2^{\frac{3}{2}}\right) = 5.59M\ . \end{aligned}$$

**9-7**. *Two particles fall radially in from infinity in the Schwarzschild geometry. One starts with $e = 1$, the other with $e = 2$. A stationary observer at $r = 6M$ measures the speed of each when they pass by. How much faster is the second particle moving at that point?*

**Solution:** The energy $E$ seen by a stationary observer at radius $r$ is

$$E = -\mathbf{p}\cdot\mathbf{u}_{\text{obs}} = \frac{m}{\sqrt{1-V^2}} \tag{1}$$

where $V$ is the speed of the particle determined by the observer and $\mathbf{u}_{\text{obs}}$ is the observer's four-velocity. The last equality holds because in an orthonormal

basis where $\mathbf{e}_{\hat{\alpha}} \cdot \mathbf{e}_{\hat{\beta}} = \eta_{\alpha\beta}$, the usual kinematic relationships of special relativity holds. For a stationary observer at radius $r$, $u^r_{\text{obs}} = u^\theta_{\text{obs}} = u^\phi_{\text{obs}} = 0$ and

$$u^t_{\text{obs}} = \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} \tag{2}$$

so that $\mathbf{u}_{\text{obs}} \cdot \mathbf{u}_{\text{obs}} = -1$.

From (1) and (9.9)

$$\begin{aligned} E = -\mathbf{p} \cdot \mathbf{u}_{\text{obs}} &= -m\left(\mathbf{u} \cdot \mathbf{u}_{\text{obs}}\right) = -m g_{\alpha\beta} u^\alpha u^\beta_{\text{obs}} \\ &= m\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} u^t \end{aligned}$$

But $e = (1 - 2M/r)u^t$ is conserved so

$$\frac{1}{\sqrt{1 - V^2}} = \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} e$$

From this we can solve for $V$ to find

$$V(e) = \frac{1}{e}\left(e^2 - 1 + \frac{2M}{r}\right)^{\frac{1}{2}}$$

which is less than 1 as it must be. In particular for $r = 6M$,

$$\frac{V(2)}{V(1)} = \frac{\sqrt{10}}{2} = 1.58 \ .$$

**9-8.** *A spaceship is moving without power in a circular orbit about a black hole of mass $M$. (The exterior geometry is the Schwarzschild geometry.) The Schwarzschild radius of the orbit is $7M$.*

*a) What is the period of the orbit as measured by an observer at infinity?*

*b) What is the period of the orbit as measured by a clock in the spaceship?*

**Solution:**

a) From (7.48) $\Omega = d\phi/dt = (M/r^3)^{1/2}$. A clock at infinity measures $t$, so that $\Omega$ is the angular velocity as measured from infinity. The period measured at infinity is

$$P_\infty = \frac{2\pi}{\Omega} = 2\pi\left(\frac{r^3}{M}\right)^{\frac{1}{2}} = 2\pi 7^{\frac{3}{2}} M = 116.M$$

for $r = 7M$.

b) To get the period as measured on the spaceship we need

$$\frac{d\phi}{d\tau} = \frac{d\phi}{dt}\frac{dt}{d\tau} = \Omega\,\frac{dt}{d\tau}\,.$$

To find $dt/d\tau$ note that for a circular orbit in the equatorial plane

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{u} = -1 &= -\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2 \\
-1 &= -\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + r^2\Omega^2\left(\frac{dt}{d\tau}\right)^2 \\
-1 &= -\left(1-\frac{2M}{r}-r^2\Omega^2\right)\left(\frac{dt}{dr}\right)^2 \\
&= -\left(1-\frac{3M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 = -\frac{4}{7}\left(\frac{dt}{dr}\right)^2\,.
\end{aligned}$$

Thus $dt/d\tau = (7/4)^{1/2}$ and

$$\frac{d\phi}{d\tau} = \left(\frac{7}{4}\right)^{\frac{1}{2}}\Omega$$

and the period $P_{\text{spaceship}} = (4/7)^{1/2}P_\infty \approx 88\,M$.

**9-9**. *Find the relation between the rate of change of angular position of a particle in a circular orbit with respect to proper time and the Schwarzschild radius of the orbit. Compare with* (9.46).

**Solution:** For a circular orbit of radius $R$, the proper angular velocity is

$$\frac{d\phi}{d\tau} = \frac{\ell}{R^2} .$$

We therefore only have to determine $\ell$ from the condition that a circular orbit lies at the minimum of the effective potential

$$\left.\frac{\partial V_{\text{eff}}}{\partial r}\right|_R = 0 .$$

This gives

$$\ell^2 = \frac{MR^2}{R - 3M}$$

so that

$$\frac{d\phi}{d\tau} = \left(\frac{M}{R^3}\right)^{\frac{1}{2}} \left(1 - \frac{3M}{R}\right)^{-\frac{1}{2}} .$$

This is faster than $d\phi/dt$, given in (9.46). A clock on the circulating particle runs slow compared to a clock at infinity both because it is moving (time dilation) and because it is in a lower gravitational potential. The proper period is therefore *less* than the period in $t$. The proper speed must therefore be *greater*. Note that there can be no circular orbits with $R < 3M$.

The problem can also be done by computing

$$\frac{d\phi}{d\tau} = \frac{d\phi}{dt}\frac{dt}{d\tau}$$

and computing $d\phi/dt$ from (9.46) and $dt/d\tau$ from (9.21).

**9-10**. *Find the linear velocity of a particle in a circular orbit of radius $R$ in the Schwarzschild geometry that would be measured by a stationary observer stationed at one point on the orbit. What is its value at the ISCO?*

**Solution:** There are several ways of doing this problem, each involving projecting the four-velocity of the particle onto the orthonormal basis of the observer. It's perhaps simplest to calculate the energy the observer measures and relate that to the velocity. From (7.53) and (5.44)

$$E = -\mathbf{p} \cdot \mathbf{u}_{\text{obs}} = \frac{m}{\sqrt{1 - V^2}} \tag{1}$$

where $\mathbf{p} = m\mathbf{u}$ is the particle's four-momentum, $\mathbf{u}_{\text{obs}}$ is the stationary observer's four-velocity, and we have assumed a rest mass $m$ for the particle. The components of $\mathbf{u}_{\text{obs}}$ are in (9.16). The components of the four-velocity of the particle are in (9.47) and (9.48). The result is

$$E = -g_{tt} m\, u^t u^t_{\text{obs}} = m\left(1 - \frac{3M}{R}\right)^{\frac{1}{2}}\left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}} . \tag{2}$$

Then computing $V$ from (1) gives

$$V = \left(\frac{M}{R}\right)^{\frac{1}{2}}\left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}} . \tag{3}$$

At the ISCO, $V = 1/2$.

**9-11** *A small perturbation of an unstable circular orbit will grow exponentially in time. Show that a displacement $\delta r$ from the unstable maximum of the Schwarzschild will grow initially as*

$$\delta r \propto e^{\tau/\tau_*}$$

*where $\tau$ is the proper time along the particle's trajectory and $\tau_*$ is a constant. Evaluate $\tau_*$. Explain its behavior as the radius of the orbit approaches $6M$.*

**Solution:** In the neighborhood of its maximum at radius $r_{\text{max}}$, the effective potential $V_{\text{eff}}(r)$ behaves

$$V_{\text{eff}}(r) = V_{\text{eff}}(r_{\text{max}}) + \frac{1}{2}\left(\frac{d^2 V_{\text{eff}}}{dr^2}\right)_{r_{\text{max}}} (\delta r)^2 + \cdots$$

where $\delta r \equiv r - r_{\text{max}}$. Denote $(d^2V_{\text{eff}}/dr^2)_{r_{\text{max}}}$ by $-K^2$ since it is negative at the maximum of the potential. Eq. (9.29) becomes

$$\frac{1}{2}\left(\frac{d(\delta r)}{d\tau}\right)^2 - \frac{1}{2}K^2(\delta r)^2 = 0 .$$

There are growing and decaying solutions to this upside down harmonic oscillation. The growing solution behaves as

$$\delta r \propto \exp(K\tau) .$$

The time constant $\tau_*$ is thus $1/K$. Carrying out the derivatives explicitly, one finds

$$\tau_* = \frac{1}{K} = \sqrt{\frac{r_{\max}^5}{2\ell^2(r_{\max} - 6M)}}$$

where $r_{\max}$ is given by (9.34). This diverges as $r$ approaches $6M$ because the stable and unstable circular orbits are coalescing.

**9-12**. *A comet orbit starts at infinity, goes around a relativistic star and goes out to infinity. The impact parameter at infinity is $b$. The Schwarzschild radius of closest approach is $R$. What is the speed of the comet at closest approach as measured by a stationary observer at that point?*

**Solution:** Two ways to arrive at the answer are presented.
*Version one*:
At the radius $R$ of closest approach, the Schwarzschild coordinate components of the four-velocity are ($\theta = \pi/2$):

$$\frac{dr}{d\tau} = 0 \ , \ \frac{d\phi}{d\tau} = \frac{\ell}{R^2} \tag{1}$$

We can find the speed $V$ from the energy measured by the stationary observer, which is

$$\frac{m}{\sqrt{1-V^2}} = E = -\mathbf{p} \cdot \mathbf{u}_{\text{obs}} \tag{2}$$

where $\mathbf{p} = m\mathbf{u}$ is the four-momentum of the comet and $\mathbf{u}_{\text{obs}}$ is the four-velocity of the stationary observer

$$\mathbf{u}_{\text{obs}} = \left( \left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}}, 0, 0, 0 \right)$$

Thus,

$$E = \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} m\, u^t \tag{3}$$

$u^t$ can be calculated from $\mathbf{u} \cdot \mathbf{u} = -1$ and (1) with the result

$$u^t = \left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}} \left(\frac{\ell^2}{R^2} + 1\right)^{1/2}$$

Combining (1), (2), and (3) we have

$$V = \frac{\ell/R}{\sqrt{1 + (\ell/R)^2}} \tag{4}$$

It remains to connect $\ell$ to $b$ and $R$.

At large $r$, $\phi \approx b/r$, where $b$ is the impact parameter,

$$\ell \equiv r^2 \frac{d\phi}{d\tau} = -b\left(\frac{dr}{d\tau}\right) = +b\sqrt{e^2 - 1} \tag{5}$$

because $dr/d\tau$ is negative. The energy per unit mass $e$ is determined by the turning point condition

$$e^2 = \left(1 - \frac{2M}{R}\right)\left(1 + \frac{\ell^2}{R^2}\right) . \tag{6}$$

Solving (5) and (6) for $\ell$ and inserting (4) gives the speed in terms of $R$ and $b$:

$$V = \left(1 - \frac{2M}{R}\right)^{1/2} \frac{b}{R} .$$

*Version two*:
Orient the observers $\hat{\phi}$ direction along the coordinate $\phi$ direction. In accord with (7.50) two of the observers orthonormal basis vectors are

$$\mathbf{e}_{\hat{\mathbf{t}}} = \left(1 - \frac{2M}{r}\right)^{-1/2} \mathbf{e}_{\mathbf{t}} \tag{7}$$

$$\mathbf{e}_{\hat{\phi}} = \frac{1}{r}\mathbf{e}_{\phi}. \tag{8}$$

The conserved quantities $e$ and $\ell$ are as defined above, which makes the impact parameter

$$b = \ell/e = \frac{r^2}{1 - \frac{2M}{r}} u^{\phi}/u^t. \tag{9}$$

At the turn-around point ($r = R$) the speed $V$ measured by the observer is $V = u^{\hat{\phi}}/u^{\hat{t}}$. Now we just need to write the coordinate basis components $u^t$ and $u^\phi$ in terms of the orthonormal basis components $u^{\hat{t}}$ and $u^{\hat{\phi}}$. Using the change of basis identity$\mathbf{v} = v^\alpha \mathbf{e}_\alpha = v^{\hat{\alpha}} \mathbf{e}_{\hat{\alpha}}$ and (7), (8) above one finds

$$\begin{aligned}
u^t &= \frac{1}{g_{tt}} \mathbf{e_t} \cdot \mathbf{u} = \frac{1}{g_{tt}} \left(1 - \frac{2M}{r}\right)^{1/2} \mathbf{e}_{\hat{\mathbf{t}}} \cdot \mathbf{u} = \frac{g_{\hat{t}\hat{t}}}{g_{tt}} \left(1 - \frac{2M}{r}\right)^{1/2} u^{\hat{t}} \\
&= +\left(1 - \frac{2M}{r}\right)^{-1/2} u^{\hat{t}} \\
u^\phi &= \frac{1}{g_{\phi\phi}} \mathbf{e}_\phi \cdot \mathbf{u} = r \frac{1}{g_{\phi\phi}} \mathbf{e}_{\hat{\phi}} \cdot \mathbf{u} = r \frac{g_{\hat{\phi}\hat{\phi}}}{g_{\phi\phi}} u^{\hat{\phi}} \\
&= \frac{1}{r} u^{\hat{\phi}} \, .
\end{aligned}$$

Plugging these into (9) and solving for $V$ gives

$$V = \left(1 - \frac{2M}{R}\right)^{1/2} \frac{b}{R} \, .$$

**9-13**. *[N,C] Particle orbits in the Schwarzschild geometry generally do not close after one turn. Explain why there should be a set of values $\mathcal{E}(\ell)$ for which orbits close for a given number of turns greater than one. Using the Mathematica program on the book website or otherwise find a value of $\mathcal{E}$ for which the orbit closes after four turns when $\ell = 4.6$ making a kind of clover leaf pattern.*

**Solution:** For the orbit to close after four turns the precession angle $\delta\phi_{\text{prec}}$ in one turn must equal $\pi/2$. The condition $\delta\phi_{\text{prec}}(e, \ell) = \pi/2$ is one condition on two variables so we expect a solution of the form $\mathcal{E} = \mathcal{E}(\ell)$. This is the curve defined by the intersection of the surface in Figure 9.5 with the plane $\delta\phi_{\text{prec}}/\pi = .5$. Trial and error with the *Mathematica* program *Shape of Orbits in the Schwarzschild Geometry* shows this happens for $\ell = 4.6$ when $\mathcal{E} = -0.015$. The resulting clover leaf shape is shown below:

**9-14**. *In Newtonian mechanics one of Kepler's laws says that equal areas are swept out in equal time as a particle moves around an elliptical orbit in a $1/r$ potential. Consider the area outside a radius $R > 2M$ that is swept out by an orbit in the Schwarzschild geometry that stays outside this radius. Does Kepler's area law hold true using either proper time or Schwarzschild time?*

**Solution:** In the equatorial plane an element of area in the Schwarzschild geometry is

$$\left[(1-2M/r)^{-\frac{1}{2}}dr\right]\,[rd\phi]$$

for $r > 2M$. Consider the area $A(\sigma)$ between $r = 2M$ and the position of the particle in its orbit as a function of a parameter $\sigma$. The rate at which $A(\sigma)$ changes is

$$\frac{dA}{d\sigma} = \left(\int_{2M}^{r(\sigma)}(1-2M/r')^{-\frac{1}{2}}r'dr\right)\left(\frac{d\phi}{d\sigma}\right) .$$

When $\sigma$ is Schwarzschild time $t$, $d\phi/dt = \ell^2/r^2$. When its proper time $\tau$

$$\frac{d\phi}{d\tau} = \frac{d\phi}{dt}\frac{dt}{d\tau} = \frac{\ell^2}{r^2}\frac{e}{(1-2M/r)}.$$

When $M = 0$ the result is $dA/dt = r^2 d\phi/dt = \ell^2$ which is Kepler's equal area in equal times result. However when $M \neq 0$ neither of these makes $dA/dt$ or $dA/d\tau$ constant. The integral is nothing that simple.

**9-15**. *[A] Precession of the Perihelion of a Planet: To find the first order in $1/c^2$ relativistic correction to the angle $\Delta\phi$ swept out by in one bound orbit one might be be tempted to expand the integrand in (9.52) in the small quantity $2GM\ell^2/c^2r^3$ and keep only the first two terms. This would be a mistake because the resulting integral would diverge near a turning point like $\int^{r_2} dr/(r_2 - r)^{3/2}$ whereas the original integral is finite. There are several ways of rewriting the integrand so it can be expanded. One trick is to factor $(1 - 2GM/c^2r)$ out of the denominator so that it can be written*

$$\Delta\phi = 2\ell \int_{r_1}^{r_2} \frac{dr}{r^2}\left(1 - \frac{2GM}{c^2r}\right)^{-1/2} \left[c^2e^2\left(1 - \frac{2GM}{c^2r}\right)^{-1} - \left(c^2 + \frac{\ell^2}{r^2}\right)\right]^{-1/2}.$$

*The factor in the brackets is then still the square root of a quantity quadratic in $1/r$ to order $1/c^2$. To derive the expression (9.55) evaluate this expression as follows:*

*a) Expand the factors of $(1 - 2GM/c^2r)$ in the above equation in powers of $1/c^2$ keeping only the $1/c^2$ corrections to Newtonian quantities and using (9.53).*

*b) Introduce the integration variable $u = 1/r$, and show that the integral can be put in the form*

$$\begin{aligned}\Delta\phi &= \left[1 + 2\left(\frac{GM}{c\ell}\right)^2\right] 2\int_{u_2}^{u_1} \frac{du}{[(u_1 - u)(u - u_2)]^{\frac{1}{2}}} \\ &+ \frac{2GM}{c^2}\int_{u_2}^{u_1} \frac{udu}{[(u_1 - u)(u - u_2)]^{\frac{1}{2}}} + \begin{pmatrix}\text{higher} \\ \text{order in } 1/c^2\end{pmatrix}\end{aligned}$$

c) *The first integral (including the 2) is just the one in (9.54) and equals* $2\pi$. *Show that the second integral gives* $(\pi/2)(u_1 + u_2)$ *and that this equals* $\pi\, GM/\ell^2$ *to lowest order in* $1/c^2$.

d) *Combine these results to derive* (9.55).

**Solution:** There is no need for a solution to this problem since the problem itself describes how to do it.

**9-16.** *A beam of photons with a circular cross section of radius* $a$ *is aimed towards a black hole of mass* $M$ *from far away. The center of the beam is aimed at the center of the black hole. What is the largest radius* $a = a_{max}$ *of the beam such that all the photons in the beam are captured by the black hole? The capture cross section is* $\pi a_{max}^2$.

**Solution:** From Figure 9.8 we learn that wherever $1/b^2$ is greater than the height of the barrier the photon will be captured. This is the condition that the impact parameter be less than $\sqrt{27}\,M$ photon will be captured. The cross section is

$$\sigma = \pi(\sqrt{27}\,M)^2 = 27\pi M^2 \,.$$

**9-17.** *Calculate the deflection of light in Newtonian gravitational theory assuming that the photon is a "non-relativistic" particle which moves with speed* $c$ *when far from all sources of gravitational attraction. Compare your answer to the general relativistic result.*

**Solution:** In Newtonian physics the energy integral for a particle of mass $m$ is [*cf.* (9.31)]

$$\frac{E}{m} = \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{\ell^2}{2r^2} - \frac{M}{r}$$

with $\ell \equiv L/m$ and $G = 1$. The angular momentum integral for an orbit in the

equatorial plane is

$$\ell = r^2 \frac{d\phi}{dt} \ .$$

An incoming "photon" with speed $c = 1$ and impact parameter $b$ has

$$E = mc^2/2 \ , \quad L = mcb \ .$$

Then for the shape of the orbit

$$\frac{d\phi}{dr} = \frac{d\phi/dt}{dr/dt} = \frac{\ell}{r^2} \left[ 2 \left( \frac{E}{m} - \frac{\ell^2}{2r^2} + \frac{M}{r} \right) \right]^{-\frac{1}{2}} \ .$$

Introducing $u \equiv b/r$ and following the discussion that led to (9.77) gives the following discussion for the total deflection angle $\Delta\phi$ is one pass.

$$\Delta\phi = 2 \int_0^{u_1} du \left[ 1 - u^2 + \frac{2M}{c^2 b} u \right]^{-\frac{1}{2}}$$

where $u_1$ is the first turning point (zero of the denominator) in $u$.

This is

$$\Delta\phi = 2 \int_0^{u_1} \left[ (u_1 - u) \ (u + u_2) \right]^{-\frac{1}{2}}$$

where $u_1 u_2 = 1$ and $u_1 - u_2 = 2M/c^2 b$. The result (*e.g.* using *Mathematica*) for small $2M/c^2 b$ is

$$\Delta\phi = \pi + (2M/c^2 b) \ .$$

This gives $\delta\phi = 2M/c^2 b$ — half the general relativistic result.

**9-18**. *Suppose in another theory of gravity (not Einstein's general relativity) the metric outside a spherical star was given by*

$$ds^2 = (1 - \frac{2M}{r}) \left[ -dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right] \ .$$

*Calculate the deflection of light by a spherical star in this theory assuming that photons move along null geodesics in this geometry and following the steps that led to* (9.78). *When you get the answer see if you can find a simpler way to do the problem.*

**Solution:** We just trace through the steps leading to (9.78) using the new metric. In the given metric outside the star, the normalization integral of the equations of motion $\mathbf{u} \cdot \mathbf{u} = 0$ becomes

$$\left(1 - \frac{2M}{r}\right) \left[ -\left(\frac{dt}{d\lambda}\right)^2 + \left(\frac{dr}{d\lambda}\right)^2 + r^2 \left(\frac{d\phi}{d\lambda}\right)^2 \right] = 0 \tag{1}$$

for an orbit in the equatorial plane $\theta = \pi/2$. The metric is independent of $t$ and $\phi$, so $\boldsymbol{\xi} \cdot \mathbf{u}$ and $\boldsymbol{\eta} \cdot \mathbf{u}$ are conserved, but now there are new expressions analogous to (9.58) and (9.59)

$$\begin{aligned} e &\equiv -\boldsymbol{\xi} \cdot \mathbf{u} = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\lambda} , \\ \ell &\equiv \boldsymbol{\eta} \cdot \mathbf{u} = \left(1 - \frac{2M}{r}\right) r^2 \frac{d\phi}{d\lambda} . \end{aligned}$$

Eliminating $dt/d\lambda$ and $d\phi/d\lambda$ in (1) in favor of $e$ and $\ell$ gives

$$\frac{1}{\ell} \frac{dr}{d\lambda} = \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{1}{b^2} - \frac{1}{r^2}\right)^{\frac{1}{2}} .$$

Thus, constructing $d\phi/dr$ from this and $d\phi/d\lambda$, one has

$$\frac{d\phi}{dr} = \frac{1}{r^2} \left(\frac{1}{b^2} - \frac{1}{r^2}\right)^{-\frac{1}{2}} .$$

Now note that this is the same as (9.76) except with $M = 0$. The deflection $\delta\phi_{\text{def}}$ is therefore 0. (One can work through the analogs of (9.76) – (9.84) to verify this explicitly.)

There is a reason underlying this simple answer. Light rays move along null curves for which $ds^2 = 0$. But the null curves for the metric given are the same as the null curves for flat space, because $ds^2 = (\text{factor}) \times (ds^2)_{\text{flat}}$.

**9-19**. *[N] Write a Mathematica program analogous to the one on the website for the null geodesics in the Schwarzschild geometry. Use this program to illustrate the orbits with impact parameters a little above and a little below the critical impact parameter for a circular orbit.*

**Solution:** A sample of a program is in the Appendix.

**9-20**.

a) *What is the speed of a particle in the smallest possible unstable circular orbit in the Schwarzschild geometry as measured by a stationary observer at that radius?*

b) *What is the connection of this orbit to the unstable circular orbit of a photon in the Schwarzschild geometry?*

**Solution:**

a) The maximum of the effective potential occurs at a radius of [*cf.* (9.34)]

$$r_{\max} = \frac{\ell^2}{2M}\left[1 - \left(1 - 12\left(\frac{M}{\ell}\right)^2\right)^{\frac{1}{2}}\right].$$

This gets smaller for large $\ell$ and in the limit of infinite $\ell$ is

$$r_{\max} = \frac{\ell^2}{2M}\left[1 - 1 + 6\left(\frac{M}{\ell}\right)^2 + \cdots\right] = 3M$$

which is the minimum possible radius.

To calculate the speed measured by a stationary observer we need to project the four-velocity of the particle $\mathbf{u}$ on the orthonormal basis of the observer. The components of orthonormal basis vectors pointing in the $t$ and $\phi$ directions (all that will be needed) are

$$\begin{aligned}
\mathbf{e}_{\hat{t}} &= \left[\left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}}, 0, 0, 0\right] \\
\mathbf{e}_{\hat{\phi}} &= \left[0, 0, 0, r^{-1}\right]
\end{aligned}$$

The observed speed will be

$$\begin{aligned}
V &= \frac{u^{\hat{\phi}}}{u^{\hat{t}}} = \frac{\mathbf{u}\cdot\mathbf{e}_{\hat{\phi}}}{-\mathbf{u}\cdot\mathbf{e}_{\hat{t}}} = \frac{r\,(d\phi/d\tau)}{(1-2M/r)^{\frac{1}{2}}(dt/d\tau)} \\
&= \frac{r}{(1-2M/r)^{\frac{1}{2}}}\,\frac{d\phi}{dt} = \frac{r}{(1-2M/r)^{\frac{1}{2}}}\left(\frac{M}{r^3}\right)^{\frac{1}{2}}
\end{aligned}$$

where the last equality comes from (9.46). Evaluating this at $r = 3M$ we find

$$V = 1$$

the particle is moving at the speed of light!

b) Clearly this particle orbit coincides with the unstable photon orbit in the Schwarzschild geometry.

**9-21.** *[E] Suppose a neutron star were luminous so that features on its surface could be viewed with a telescope. The gravitational bending of light means that not only the hemisphere pointing towards us could be seen but also part of the far hemisphere. Explain why and estimate the latitude above which the far side could be seen. A typical neutron star has a mass of* $\sim M_\odot$ *and a radius of* $\sim 10$ km.

**Solution:**

For a neutron star $M_\odot/R \sim (1.5km)/(10km) \sim .15$. This is a small enough angle that a reasonable estimate for the latitude $\lambda$ that can be obtained from (9.77) for the deflection angle $\delta\phi$ and the geometry of the above figure. Note that by symmetry light rays at constant azimuthal angle will illuminate the lowest latitude of the backside. Evidently $2\lambda + \delta\phi = \pi$ so $\lambda = \pi/2 - \delta\phi = \pi/2 - 4M/R \approx .97$ radians $\approx 56°$.

**9-22**. *[N,C] Looking for Black Holes with Lasers. Suppose primordial black holes of mass* $\sim 10^{15}$ g *were made in the early universe and are now distributed throughout space. If an observer shines a laser on a black hole some of the light is backscattered to the observer. A search for such primordial black holes could in principle be carried out by shining lasers into space and looking for the backscattered radiation.*

a) *Explain why some light is backscattered.*

b) *Suppose the flux of photons* $[(number)/\ \mathrm{m}^2 - \mathrm{s}]$ *in the laser beam is* $f_*$, *the mass of the black hole is* $M$, *and it is a distance* $R$ *away. Derive a formula for the number of photons per second that will be returned to a collecting area of radius* $d$ *at the origin of the beam. Assume that the width of the beam is much larger than the size of the black hole. [Hint: A little numerical integration is required to get an accurate answer for this problem.]*

c) *Could the lasers described in Box 2.1 on p. 16 hope to detect such a black hole?*

**Solution:**

a) Some light is back-scattered because a light ray can be deflected by an angle $\delta\phi = \pi$ as it orbits the black hole and can return in the direction from whence it came.

b) The crude schematic diagram above shows the geometry of the situation. Light incident at impact parameter $b$ is deflected by an angle $\phi$ (called $\delta\phi_{\text{def}}$ in the text) giving a relation $\phi(b)$ or $b(\phi)$ defined in (9.78) via (9.82). Light with impact parameter $b(\pi)$ will be directly back-scattered. Light emitted with impact parameters in the range $b(\pi - \psi)$ to $b(\pi + \psi)$ will be received in a detector that subtends a $\psi = d/R$ at the black hole (the contributions from $b(3\pi - \psi)$ to $b(3\pi + \psi)$, etc., will be negligible). Since the situation is symmetric about the (observer)—(black hole) axis, light in an angular ring between these impact parameters will be detected. The detected number per second is therefore

$$\begin{aligned}\left(\frac{\text{Number}}{\text{sec}}\right) &= \left[\pi b^2(\pi + \psi) - \pi b^2(\pi - \psi)\right] f_* \\ &\approx 2\pi \left| b \frac{db}{d\phi}\right|_{\phi=\pi} 2\psi\, f_* \\ &\approx 4\pi \left| b \frac{db}{d\phi}\right|_{\phi=\pi} \frac{d}{R}\, f_*.\end{aligned}$$

A little numerical investigation of (9.78) shows

$$b(\pi) = 5.35M$$

where $M$ is the mass of the black hole and

$$\frac{db}{d\phi} = .17M.$$

This gives

$$\left(\frac{\text{Number}}{\text{sec}}\right) = 11.4\ \left(f_* M^2\right)\ \left(\frac{d}{R}\right).$$

c) The key number in estimating feasibility is the size of a $10^{15}$g black hole. In geometrized units this is $\sim 10^{-15}$m. If the lunar laser ranging experiment detects only one photon every few seconds from a retroreflector $\sim$ .5m it will take an impossibly long time to detect photons back-scattered from such small black holes even assuming perfect conditions.

# Chapter 10

# Solar System Tests

**10-1**. *[E] Estimate the gravitational redshift of light from the surface of the Sun. Discuss the possibility of measuring this effect given that the velocities of matter in convection cells at the surface of the Sun is of order 1 km/s. Is there one part of the surface which is better than another for making the observation?*

**Solution:** The redshift from the surface of the sum is

$$\frac{\Delta\omega}{\omega} = \frac{GM_\odot}{R_\odot c^2} \sim 10^{-6} \ .$$

The competing Doppler shift from solar convection is

$$\frac{\Delta\omega}{\omega} = \left(\frac{V}{c}\right) \sim \frac{1 \text{ km/s}}{(3 \times 10^5 \text{ km/s})} \sim 10^{-6} \ .$$

Thus, it will be difficult to separate the Doppler shift from the gravitational redshift. However, by looking at the limb of the sun (near the edge), where the connective motions are largely transverse, the Doppler shift is diminished, $(\Delta\omega/\omega) \sim (V/c)^2$. It then becomes possible to make a measurement.

**10-2**. *Is the experiment of Vessot and Levine sensitive enough go say anything about the parameters $\beta$ and $\gamma$? Is the third order Doppler effect important in analyzing the experiment?*

**Solution:** Working through the derivation of the gravitational redshift in Section 9.2 for a PPN metric of the form (10.4) gives

$$\frac{\omega_\infty}{\omega_\sigma} = \sqrt{A(r)} = 1 - \frac{GM}{c^2 R} + \left(\beta - \gamma - \frac{1}{2}\right)\left(\frac{GM}{c^2 R}\right)^2 + \ldots.$$

to quadratic order in $(GM/c^2R)^2$. For the Earth $GM_\oplus/c^2R_\oplus \sim 10^{-9}$. Corrections to the gravitational redshift from terms involving $\beta$ and $\gamma$ are unimportant.

The third order Doppler shift is a factor $V/c \sim 10^{-5}$ from the second order effects. This is just below the $10^{-4}$ accuracy of Vessot & Levine.

**10-3**. *Evaluate the maximum deflection of light by the Sun predicted by general relativity in seconds of arc.*

**Solution:** The maximum deflection occurs when $b$ is approximately a solar radius and is $1.75''$.

**10-4**. *Derive* (10.8) *for the deflection of light as a function of the parametrized post-Newtonian parameters.*

**Solution:** The idea of this problem is to follow through the steps that led to (9.63) and (9.64) and then from (9.76) to (9.81) with the PPN metric (10.4) instead of the Schwarzschild metric. We present only the major steps.

The conserved quantities $e$ and $\ell$ are

$$e = A\,u^t \quad , \quad \ell = r^2 u^\phi. \tag{1}$$

The condition $\mathbf{u}\cdot\mathbf{u} = 0$ can be re-written as an effective equation for radial motion

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{B}\left(\frac{e^2}{A} - \frac{\ell^2}{r^2}\right). \tag{2}$$

The equation for the shape of the orbit $d\phi/dr = u^\phi/u^r$ is

$$\frac{d\phi}{dr} = \frac{B^{\frac{1}{2}}}{r^2}\left(\frac{1}{b^2 A} - \frac{1}{r^2}\right)^{-\frac{1}{2}} \tag{3}$$

where $b^2 = (\ell/e)^2$.

The angle $\Delta\phi$ swept out in the passage by the light ray is

$$\Delta\phi = 2\int_{r_1}^{\infty} dr\, \frac{B^{\frac{1}{2}}}{r^2}\left(\frac{1}{b^2 A} - \frac{1}{r^2}\right)^{-\frac{1}{2}} \tag{4}$$

where $r_1$ is the turning point where the denominator vanishes. Insert (10.6) for $A$ and $B$ into (4), and transform to the dimensionless variable $w \equiv b/r$. Then expand both $B$ in the numerator and $1/A$ in the denominator in powers of $M/b$ keeping only the leading correction in both places. The result is

$$\Delta\phi = 2\int_0^{w_1} dw\, \left(1 + \frac{\gamma M}{b} w\right)\left(1 + \frac{2M}{b} w - w^2\right)^{-\frac{1}{2}} \tag{5}$$

where $w_1 = b/r_1$. This is in a form where it can easily be looked up. The result expanded to leading order in $M/b$ is

$$\Delta\phi = \pi + \frac{1}{2}(1+\gamma)\,(M/b). \tag{6}$$

**10-5.** *Evaluate, in seconds of arc per century, the precession of the perihelion of Mercury, Venus and the Earth as predicted by general relativity.*

| | *Semimajor axis* $10^6$ *(km)* | *Eccentricity* | *Mass/*$M_\oplus$ | *Period (yr)* |
|---|---|---|---|---|
| *Mercury* | *57.91* | *.2056* | *.054* | *.241* |
| *Venus* | *108.21* | *.0068* | *.815* | *.615* |
| *Earth* | *149.60* | *.0167* | *1.00* | *1.000* |

**Solution:** The precession for Mercury is $42.98''$/century. So the precession for any other planet is given by

$$\delta\phi = (42.98''/\text{century})\,\frac{a_{\text{merc}}(1-\epsilon_{\text{merc}}^2)}{a(1-\epsilon^2)}\left(\frac{P_{\text{Merc}}}{P}\right)$$

where $a, \epsilon$ and $P$ are the semimajor axis, eccentricity, and period for the planet and $a_{\text{Merc}}$, $\epsilon_{\text{Merc}}$ and $P_{\text{Merc}}$ are the same quantities for Mercury.

**10-6**. *Evaluate the precession of the perihelion of Mercury caused by a Newtonian quadrupole potential of the form given in* (10.20), *and show that with the observed value of* $J_{2\odot}$ *it is too small to correct the determined value of the PPN parameter* $\beta$.

**Solution:** From (10.20) the quadrupole term in the Newtonian potential has the same form as the relativistic correction $GM\ell^2/(c^2r^3)$. The ratio

$$\frac{\text{(quadrupole)}}{\text{(relativistic)}} = J_{2\odot}\,\frac{R_\odot^2}{2(\ell/c)^2} = J_{2\odot}\left(\frac{c^2R_\odot}{2GM_\odot}\right)\left(\frac{R_\odot}{a}\right)\frac{1}{1-\epsilon^2}$$

where $R_\odot$ is the solar radius and $a$ and $\epsilon$ are the semimajor axis and eccentricity of Mercury's orbit respectively. The last equality follows from (9.56). Putting in the numbers (see Table in Problem 5) this ratio is

$$\frac{\text{(quadrupole)}}{\text{(relativistic)}} \sim 10^{-7}\left(\frac{6.96\times10^5 km}{2\times1.48 km}\right)\left(\frac{6.96\times10^5 km}{5.79\times10^7 km}\right) \sim 3\times10^{-4}\,.$$

The precession due to the quadrupole moment will therefore be about .02″/century. That's getting close to the error on the measured precession but still within it.

**10-7**. *Solar Oblateness and the Precession of the Perihelion. Measuring the shape of the solar surface is an alternative way of determining the solar quadrupole moment. Optical measurements can determine the solar oblateness defined by*

$$\Delta = \frac{\text{(radius at equator)} - \text{(radius at pole)}}{\text{(mean radius)}}$$

*If the surface of the Sun is a surface of equal gravitational potential, this oblateness can be used to determine the solar mass quadrupole moment. Early measurements gave values for* $\Delta$ *as large as* $5\times10^{-5}$.

a) *Explain why the surface of the Sun is a surface of equal gravitational potential if the centripetal accelerations due to the rotation at the surface are a negligible contribution to the Sun's distortion (contrary to fact).*

b) *Calculate the value of $J_2$ from the oblateness using* (10.20) *and assuming that $\Phi$ is constant on the surface of the Sun.*

c) *Calculate the magnitude of the precession of the perihelion of Mercury that would result from $\Delta \sim 10^{-5}$.*

*Later measurements gave a much lower value for $\Delta$.*

**Solution:**

a) Were the surface not an equipotential there would be a component of $\vec{\nabla}\Phi$ tangent to the surface. The resulting force on a surface fluid element [*cf.* (3.12)] would cause the fluid to flow. In equilibrium, therefore, $\vec{\nabla}\Phi$ must be perpendicular to the surface and $\Phi$ must be constant along it.

b) The gravitational potential is given by (10.19). To be constant on the surface $\Phi$ must have the same value at the pole ($\theta = 0$) as at the equator ($\theta = \pi/2$). Let $r_p$ be the polar radius and $r_e$ the equatorial one. Then (in $G = 1$ units)

$$-\frac{M_\odot}{r_p} + J_2\frac{M_\odot}{r_p}\left(\frac{R_\odot}{r_p}\right)^2 = -\frac{M_\odot}{r_e} - \frac{1}{2}J_2\frac{M_\odot}{r_e}\left(\frac{R_\odot}{r_e}\right)^2$$

Here $R_\odot$ is the mean solar radius $(r_p + r_e)/2$. This can be solved for $J_2$ to give to first order in the difference $r_e - r_p = \Delta R_\odot$ to give

$$J_2 = \frac{2}{3}\Delta$$

c) The solution to this part is the same as the solution to Problem 6 except that here $J_2$ is approximately 300 times larger leading to a precession of about 6"/century. This is 14 % of the general relativistic effect. The predictions of general relativity would be in conflict with experiment.

*Comment:* It is only a modest extension of this calculation to include the effects of rotation. The surface of fluid star rotating with angular velocity $\Omega$ lies on a surface of constant $\Psi \equiv \Phi + (1/2)\Omega^2 r^2 \sin^2\theta$ where the second term is a potential for centrifugal force. This is called van Zeipel's theorem in astronomy. However, the additional rotational distortion is negligible compared to that

corresponding to a $\Delta$ as large as $\sim 10^{-5}$.

**10-8**. *[P, E] Starting from* (10.12), *make a rough estimate of the angular accuracy that could be expected in the NRAO experiment to detect the deflection of light. Under ideal circumstances, what size optical telescope above the atmosphere in space would be needed to achieve the same accuracy?*

**Solution:** If a phase difference of .01 in (10.12) is detectable then an angular accuracy of

$$\delta\theta \sim 10^{-3}\,\frac{\lambda}{B}$$

can be expected. For the NRAO experiment $B \sim 35\text{km}$, $\lambda \sim 2\pi c/(3\text{ GHz})$ gives

$$\delta\theta \sim .001''.$$

To get the same resolution from an optical telescope operating at visible wavelengths would require

$$\begin{aligned} 10^{-3}\frac{\lambda}{B} &\sim .001'' \\ B &\sim \frac{(10^{-3})(5\times 10^{-7}\text{m})}{(.001'')\ (5\times 10^{-6}\text{rad}/'')} \sim 10\text{cm}. \end{aligned}$$

**10-9**. *[E] Estimate the amount by which radio signals used in the quasar bending of light observation would be bent by the solar corona. The corona is reasonably well modeled by a free electron gas whose index of refraction is (SI units)*

$$n(r) = 1 - \frac{e^2 N(r)}{2\epsilon_0 m\omega^2}$$

*where the electron density $N(r)$ may be taken to be $10^8$ cm$^{-3}$ out to twice the solar radius. The frequencies used in the NRAO experiment were 8.1 GHz and 2.7 GHz.*

**Solution:** Imagine the electron gas to be spread out over a sphere of radius $2R_\odot$.

A little geometry (see above figure) shows that the total deflection angle of a ray passing through the sphere is

$$\delta\phi = 2(i - r) \tag{1}$$

Snell's law gives $\sin i = n \sin r$ or, for small angles, $r = i/n$. Inserting in (1), and assuming $n$ is close to 1 gives

$$|\delta\phi| \sim i \, \frac{e^2 N}{\epsilon_0 m \omega^2}.$$

(This formula is in SI electromagnetic units. For Gaussian units replace $e^2/(4\pi\epsilon_0)$ by $e^2$.) Putting in the numbers gives

$$|\delta\phi| \sim 10^4 \text{ (arc seconds)} \left(\frac{i}{1 \text{ radian}}\right) \left(\frac{2.7GHz}{\omega}\right)^2$$

This is larger or competitive with the deflection predicted by general relativity.

**10-10**. *[E] In the analysis of the radar ranging time delay experiment in Section 9.4 the time interval between the emission and return of a radar pulse was equated with the Schwarzschild coordinate time interval* $(\Delta t)_{\text{total}}$ *between these events. Clocks on the surface of the Earth measure proper time along their trajectories not necessarily Schwarzschild coordinate time. Estimate whether the corrections due to the spacetime curvature produced by the Earth and the Sun are important corrections in the Viking experiment.*

**Solution:** The proper time interval $\Delta\tau_{\text{excess}}$ measured by clocks on Earth that corresponds to the to the Schwarzschild coordinate time interval $\Delta t$ predicted by (9.91) is [*cf.* (9.20)]

$$\Delta\tau_{\text{excess}} = \left(1 - \frac{2GM_\oplus}{c^2 R_\oplus}\right)^{1/2} \Delta t_{\text{excess}}$$

neglecting the rotation of the Earth. Since $GM_\oplus/c^2R_\oplus \sim 10^{-9}$, this is comparable to the accuracy needed to measure the Shapiro effect to 1%.

The relevant factor for curvature produced by the Sun is

$$\left(1 - \frac{2GM_\odot}{c^2 r_\oplus}\right)^{1/2}$$

where $r_\oplus$ is the radius of the Earth's orbit. This is $\sim 10^{-8}$. As discussed in the text an accuracy of one part in $10^9$ is needed to reach 1% accuracy. The effect must therefore be taken into account.

**10-11**. *Assuming that general relativity correctly predicts the excess time delay measured in the Viking experiment, what can you infer from the data in Figure 10.7 about the closest a radar pulse involved in the experiment came to the Sun. Express your answer in solar radii from the center.*

**Solution:** The graph in Figure 10.7 shows that the maximum "excess" time delay in the Viking experiment was 247 $\mu$s. Equating this to the predicted delay (9.92) gives the value $r_1$ of closest approach to the center of the Sun. The rest is algebra but we quote a few intermediate steps:

$$247 \times 10^{-6}\ \text{s} = \frac{4 \cdot 1.47\ \text{km}}{3 \times 10^5\ \text{km/s}} \left[\log\left(\frac{4r_R r_\oplus}{r_1^2}\right) + 1\right]$$

where $r_R$ is the orbital radius of Mars, approximately $1.52r_\oplus$. This gives

$$\frac{r_1}{R_\odot} = \frac{2r_\oplus}{R_\odot}\left(\frac{r_R}{r_\oplus}\right)^{1/2} e^{11.6/2} \approx 1.6$$

where $R_\odot$ is the solar radius.

# Chapter 11

# Relativistic Gravity in Action

**11-1**. *At what radius would an observatory have to orbit the Sun in order to use it as a gravitational lens to image more distant objects?*

**Solution:** To use the sun as a gravitational lens, the observatory must be placed at a distance $R$ such that the angle subtended by the solar radius $R_\odot$ is the deflection of light at the solar limb $1.75''$. That is

$$\frac{R_\odot}{R} = 1.75''$$

or

$$\begin{aligned} R &= \left(\frac{360 \times 60 \times 60}{2\pi \times 1.75}\right) R_\odot \\ &= 1.2 \times 10^5 R_\odot \approx 550 \text{ AU} . \end{aligned}$$

**11-2**. *An odd number of gravitational lens images. Realistic gravitational lenses are not point sources as assumed in the discussion in Section 11.1 but rather are a mass distribution. A lens that is a distribution of mass produces an odd number of images. For a simple model, assume that the gravitational lens is a transparent disk of radius $r_*$ and constant surface mass density $\sigma$ oriented perpendicularly to the line of sight. Using the thin lens approximation show that, in addition to the two images given by* (11.6), *there is a third image inside the angle subtended by the disk and find its angular position $\theta$. Assume only the mass inside the deflection radius affects the bending of light.*

**Solution:** The geometry of the lens is the same as in Figure XI.4 except that the disk extends from the axis to $\xi = r_*$. The deflection angle $\alpha$ for $\xi < r_*$ is therefore [*cf.* (11.1)]

$$\alpha = \frac{4GM_<(\xi)}{c^2\xi}$$

where $M_<(\xi)$ is the mass of the disk interior to radius $\xi$. This is

$$M_<(\xi) = \pi\sigma\xi^2 .$$

The lens equation (11.3) then leads to

$$\theta = \beta + \frac{4\pi G\sigma}{c^2}\left(\frac{D_L D_{LS}}{D_S}\right)\theta$$

instead of (11.4). The solution for $\theta$ is

$$\theta = \beta\left[1 - \frac{4\pi G\sigma}{c^2}\left(\frac{D_L D_{LS}}{D_S}\right)\right]^{-1} .$$

There will be this image *in* the disk in addition to the two already derived *outside* the disk at $\theta_\pm$ given in (11.6).

**11-3**. *When the line of sight to a star is far from the line of sight to a gravitational lens the effects of lensing should become negligible. Show that when $\beta >> \theta_E$, $\theta_+ \approx \beta$, $\theta_- \approx 0$, $I_+/I_* \approx 1$, and $I_- \approx 0$. Explain why these results mean that gravitational lensing is negligible.*

**Solution:** The properties mentioned are easily verified from (11.6) and (11.10). One image moves to the center and vanishes. The other has the same position and intensity as if the lens were not there at all.

**11-4**. *Derive the path length difference in* (11.12).

**Solution:** The assumptions are $D_L = D_{LS} = D_S/2$ and

$$\beta \ll \theta_E \ll 1.$$

The figure below shows this lensing situation and the two paths that light can take from the source $S$ to the observer $O$.

This figure is slightly different from Figure 11.2 in the definitions of $D_S$ and the lensing plane. However, these differences will not affect results calculated to first order in $\beta$, as here. For example, $D_S^{(11.2)} = D_S \cos\beta \approx D_S$ to first order in $\beta$. The symmetry in the lens plane means that there are two equal segments with length $D_-$ and two with length $D_+$. The distances $D_+$ and $D_-$ are related to $D_S/2$ by

$$D_+ \cos\,(\theta_+ - \beta) = \frac{D_S}{2} \quad , \quad D_- \cos\,(\theta_- + \beta) = \frac{D_S}{2}.$$

The difference in lengths of the two paths $\Delta D$ is

$$\Delta D = 2(D_+ - D_-) = D_S \left[ \frac{1}{\cos\,(\theta_+ - \beta)} - \frac{1}{\cos\,(\theta_- + \beta)} \right].$$

Using the approximation that all angles are small and $\cos x \approx 1 - x^2/2$ for small $x$, we have

$$\Delta D \approx \frac{D_S}{2} \left[ (\theta_+ - \beta)^2 - (\theta_- + \beta)^2 \right].$$

Using (11.6) this is to first order in $\beta$.

$$\Delta D \approx 2\beta\theta_E D_S$$

which is (11.12).

**11-5.** *[E] Equation (11.12) estimates the path length difference traveled by light making up two images in a gravitational lens. The difference in arrival times of the light from the two images due to this effect is $\Delta D/c$. Estimate whether the Shapiro time delay discussed in Section 9.4 is a competitive effect.*

**Solution:** Eq. (9.92) can be used to estimate the Shapiro time delay in the lensing by a galaxy of a source at cosmological distances. For simplicity consider the case $D_L = D_{LS} = D_S/2$ that was used to derive the geometric delay (11.12). The estimate of the Shapiro delay $t_{\text{Shapiro}}$ can then be made with $r_R = r_\oplus = D_S/2$ and $r_1 = \theta_\pm D_S/2$ depending on which path is taken. Of interest here is the *difference* in the time delays along the two paths:

$$\begin{aligned}\Delta t_{\text{Shapiro}} &\sim \frac{4GM}{c^3}\left[\log\left(\frac{4}{\theta_-^2}\right) - \log\left(\frac{4}{\theta_+^2}\right)\right] \\ &\sim \frac{8GM}{c^3}\log\left|\frac{\theta_+}{\theta_-}\right| \sim \frac{8GM}{c^3}\left(\frac{\beta}{\theta_E}\right)\end{aligned}$$

when $\beta \ll \theta_E$. Using (11.2), this gives

$$\Delta t_{\text{Shapiro}} \sim 10^6\left(\frac{\beta}{\theta_E}\right)\text{s} \sim 2\left(\frac{\beta}{\theta_E}\right)\text{ weeks.}$$

Therefore, for say $\beta/\theta_E \sim .1$, the Shapiro delay will be competitive with the geometric effect.

**11-6.** *In a typical microlensing event, a moving gravitational lens passes close to the line of sight to a distant source. The magnification $I_{\text{tot}}/I_*$ defined by (11.11) increases in time and then decreases. Express the predicted ratio in terms of time measured in units of the time $t_{\text{var}}$ to cross the Einstein angle $\theta_E$, and $p$ the ratio $\beta_{closest}/\theta_E$ where $\beta_{closest}$ is the smallest angular separation between lens and source. Plot the ratio $I_{\text{tot}}/I_*$ as a function of time in these units for $p = .1$, $p = .3$, and $p = .7$. Do your curves look like the one in Figure 11.6?*

**Solution:** Clearing fractions in (11.11) and writing $u \equiv \beta/\theta_E$ gives

$$\frac{I_{\text{tot}}}{I_*} = \frac{u^2+2}{u(u^2+4)^{\frac{1}{2}}}.$$

The figure below shows the trajectory of the lens *as projected onto the sky.* For example, the angle "$u$" in the diagram refers to the angle subtended (in units of $\theta_E$) by the projected line connecting $L$ and $S$, as seen by the observer. The projected distances corresponding to the angles $u$, $p$, and $x$ form a right triangle on a small patch of the sky. Since the distance to the lens is large, the Pythagorean theorem will then hold for these subtended angles as well.

$p$ is the angle of closest approach and $x$ is the angle between the current location of the lens and the location of closest approach. $x$ can be written $t/t_{\text{var}}$, where $t = 0$ is the time of closest approach, and $t_{\text{var}}$ is the time to move $\theta_E$. Then from the figure

$$u \equiv \frac{\beta}{\theta_E} = \sqrt{p^2 + \left(\frac{t}{t_{\text{var}}}\right)^2}.$$

The figure below shows $R \equiv I_{\text{tot}}(t)/I_*$ as a function of $T \equiv t/t_{\text{tot}}$ for several different $p = .1$, $.3$, and $.7$.

**11-7**. *(a) For the lensing event in Figure 11.6, what is the ratio $\beta/\theta_E$ when the lens comes closest to intersecting the line of sight to the lensed star. (Working Problem 6 may help with this.)*

*(b) What is the value of $t_{\rm var}$ — the time for the angular position of the lens to move by an amount $\theta_E$. (You can make a rough estimate or fit the data with the results of Problem 6.)*

*(c) Assuming that the lens is moving with a velocity of $V = 200$ km/ s transverse to the line of sight, and located half way between the Earth and the center of the galaxy, estimate the mass of the lens. (The distance between the Sun and the galactic center is approximately 8.5 kpc.)*

**Solution:**

a) From the solution to the previous problem, at $t = 0$ — the time of closest approach of lens to source — the lensed intensity is

$$R \equiv \left(\frac{I_{\rm tot}}{I_*}\right)_{\rm max} = \frac{p^2+2}{p(p^2+4)^{\frac{1}{2}}}.$$

From Figure 11.6, the maximum of $R$ is about 8.2 which corresponds (by trial and error) to

$$(\beta/\theta_E)_{\rm closest} \equiv p = .12.$$

b) Roughly estimating the characteristic time of variation from the graph, $t_{\rm var} \sim 30$ days.

c) If $V$ is the velocity of the lens transverse to the line-of-sight and $D_L$ is the distance to the lens, then

$$\dot{\beta} = \frac{V}{D_L} \approx \frac{\theta_E}{t_{\text{var}}}$$

since $t_{\text{var}}$ is the time to cross the Einstein angle. Using (11.5) for $\theta_E$, we find

$$t_{\text{var}} \sim \frac{D_L}{V}\theta_E \sim \frac{D_L}{V}\left[4M\left(\frac{D_{LS}}{D_S D_L}\right)\right]^{\frac{1}{2}}.$$

Assuming $D_L \sim D_S \sim D_{LS}$ this works out to be

$$t_{\text{var}} \sim .2 \text{ yr}\left(\frac{M}{M_\odot}\right)^{\frac{1}{2}}\left(\frac{D_L}{10 \text{ kpc}}\right)^{\frac{1}{2}}\left(\frac{200 \text{ km/s}}{V}\right).$$

Taking $V \sim 200$ km/s and $D_L \sim 5$ kpc, a $t_{\text{var}} \sim 30$ days $\sim .08$yr gives

$$M \sim .3 M_\odot .$$

**11-8.** *[E,P] Estimate the energy in eV necessary to pull the last electron off of an Fe atom. Above what temperatures (in keV and °K) will Fe atoms be completely ionized? At a temperature of* 2 keV *how many of an Fe atom's electrons would you expect to remain?*

**Solution:** The energy of the ground state of a single electron in the field of an atom of charge $Ze$ is

$$E_1 = \frac{(Ze)^2}{2a} = Z^2 \text{Ry} = Z^2(13.6\text{eV}).$$

Here, $a$ is the Bohr radius, and Ry is the unity of energy called the Rydberg. (See your elementary quantum mechanics for these facts.)

For iron with $Z = 26$ we have

$$E_1 = 9.2 \text{ keV}.$$

Above this energy and the corresponding temperature $k_B T_1 = E_1$, we expect Fe to be completely ionized. $T_1$ is $1.1 \times 10^8$ K.

At a temperature of 2keV we would expect the screened charge to be given by

$$Z^2(13.6\text{eV}) = 2\text{keV}.$$

That is $Z = 12$. So we would expect $26 - 12 = 14$ electrons to remain. That's plenty to exhibit discrete transitions.

**11-9**. *[B,E] An X-ray source with a luminosity $L = 3 \times 10^{36}$ erg/s is powered by accretion onto black hole with mass $6M_\odot$. Assuming all the radiation is released at the innermost stable circular orbit, estimate the rate $\dot{M}$ at which mass is being accreted by the black hole in $M_\odot$/yr.*

**Solution:** The energy released is the gravitational binding energy. The luminosity is therefore roughly

$$L \sim \frac{G\dot{M}M}{R_{\text{ISCO}}} .$$

Since $R_{ISCO} = 6GM/c^2$ we have

$$L \sim \frac{\dot{M}c^2}{6}$$

or

$$\begin{aligned} \dot{M} = \frac{6L}{c^2} &\sim 2 \times 10^{16} \text{ g/s} \\ &\sim 3 \times 10^{-10} M_\odot/\text{yr} . \end{aligned}$$

**11-10**.

a) *From the data on PSR B1913+16 given in the text, determine the semi-major axis of the orbit $a$ and its angle of inclination $i$ with respect to the line of sight. (Three significant figures is adequate.)*

b) *What does this tell you about the companion star? Could it be a normal star like the Sun?*

**Solution:**

a) Use the values of the period $P_b$ and total mass $M_{\text{tot}}$ given in the text to solve Kepler's law (11.15) for $a$. The result is $1.95 \times 10^6$km or about 3 solar radii. Also given in the text is $a_{\text{pulsar}} \sin i$ and the masses of the two stars. Since $a_{\text{pulsar}} = (M_{\text{comp}}/M_{\text{tot}})a$, this gives $a_{\text{pulsar}} = 9.77 \times 10^5 km =$ 3.25 light-s and $\sin i = .73$.

b) If the Sun were in orbit with another solar mass object only three solar radii away, it would be tidally distorted in a way that would affect the orbit. The companion therefore can't be a normal star.

# Chapter 12

# Black Holes

**12-1**. *[P] How many protons must combine to make He nuclei every second to provide the luminosity of the Sun? Estimate how long the Sun could go on at this rate before all its protons were used up.*

**Solution:** The atomic mass of He is 4.0026. That of Hydrogen is 1.0079. The difference between 4 Hydrogens and 1 He is $2.9 \times 10^{-2}$ atomic mass units or $4.326 \times 10^{-5}$ erg. The luminosity of the Sun is $3.8 \times 10^{33}$ erg/s. Approximately $3 \times 10^{38}$ protons must be converted to He every second to sustain the luminosity of the Sun.

The mass of the Sun is $1.99 \times 10^{33}$ g. It therefore has approximately $10^{57}$ protons. At the above rate it would take $10^{11}$ years to exhaust them — about the age of the universe. A lot will happen before that.

**12-2**. *[E,B] Follow through the order of magnitude estimate for the maximum mass of white dwarfs in Box 12.1 without assuming that the electrons are necessarily relativistic.*

*(a) Sketch the behavior of the total energy $E_{TOT}(A, R) = E_F(A, R) + E_G(A, R)$ as a function of $R$ both for values of $A$ greater and less than $A_{crit}$ defined in* (d) *of the box.*

*(b) Find the radius $R_*(A)$ for which $E_{TOT}(A, R)$ has a minimum in $R$. This is an estimate of the radius of the equilibrium star where gravitational and Fermi pressure forces balance. How does $R_*$ compare to the radii of white dwarf stars quoted in the text.*

*(c) Show that there is a value $A_{crit}$ above which no equilibrium is possible.*

*Find its value and compare with $A_{crit}$ estimated in the text.*

*(d) Are the electrons relativistic at the equilibrium?*

**Solution:**

(a) Keeping the general expression for energy as a function of momentum the Fermi energy $E_F(A\,,R)$ is:

$$E_F(A\,,R) = A\left[(p_F c)^2 + (m_e c)^2\right]^{\frac{1}{2}} = A\left[\left(\frac{A^{\frac{1}{3}}\hbar c}{R}\right)^2 + \left(m_e c^2\right)^2\right]^{1/2}$$

The total energy $E_{\rm TOT}(A\,,R) = E_F(A\,,R) + E_G(A\,,R)$ then behaves as sketched schematically below.

,

(b) The radius $R_*(A)$ is

$$R_*(A) = \left(\frac{\hbar}{m_e c}\right) A^{\frac{1}{3}}\left[\left(\frac{A_{\rm crit}}{A}\right)^{\frac{4}{3}} - 1\right]^{\frac{1}{2}} .$$

(The factor $\hbar/m_e c$ is the electron Compton wavelength.) For $A \sim A_{\rm crit}$, $R_* \sim 10^3$km which is roughly comparable to the few thousand km quoted in the text.

(c) Evidently for $A > A_{\rm crit}$ no equilibrium is possible so $A_{*\rm crit} = A_{\rm crit}$.

(d) At any reasonable fraction of $A_{\text{crit}}$

$$cp_F \sim \frac{A_{\text{crit}}^{\frac{1}{3}}\hbar c^2}{R} \sim m_e c^2$$

so the electrons are indeed relativistic.

**12-3**. *[S] Carry out the transformation from Schwarzschild to Eddington-Finkelstein coordinates defined by* (12.1) *to get the line element* (12.2).

**Solution:** Just compute the derivatives and do the algebra.

**12-4**. *Consider the spacetime specified by the line element*

$$ds^2 = -\left(1-\frac{M}{r}\right)^2 dt^2 + \left(1-\frac{M}{r}\right)^{-2} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ .$$

*Except for $r = M$, the coordinate $t$ is always timelike and the coordinate $r$ is spacelike.*

*(a) Find a transformation to new coordinates $(v, r, \theta, \phi)$ analogous to* (12.1) *that sets $g_{rr} = 0$ and shows that the geometry is not singular at $r = M$.*

*(b) Sketch a $(\tilde{t}, r)$ diagram analogous to Figure 12.2 showing the world lines of ingoing and outgoing light rays and the light cones.*

*(c) Is this the geometry of a black hole?*

**Solution:**

a) Put $t = v - F(r)$. The metric becomes

$$ds^2 = -\left(1-\frac{M}{r}\right) dv^2 + 2F'\left(1-\frac{M}{r}\right)^2 dv\, dr$$

$$+\left[\left(1-\frac{M}{r}\right)^{-2} - (F')^2\left(1-\frac{M}{r}\right)^2\right] dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

where $F' = dF/dr$. The choice

$$F' = \left(1-\frac{M}{r}\right)^{-2} \tag{1}$$

makes $g_{rr} = 0$ and gives the metric

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dv^2 + 2dv\,dr + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) . \qquad (2)$$

This is non-singular at $r = M$.

b) Integrating (1) the explicit form for the coordinate transformation is

$$t = v - (r - M) - 2M \log|r/M - 1| + \frac{M^2}{r - M} ,$$

where we've chosen the constant of integration so that $v = 0$ at $t = r = 0$. Light rays move on curves of $v =$ const. and, from (2), curves of

$$\frac{dv}{dr} = \frac{2}{(1 - M/r)^2} .$$

Equivalently, in terms of the variable $\tilde{t} = v - r$ used in the text to plot Eddington-Finkelstein diagrams

$$\frac{d\tilde{t}}{dv} = -1 ,$$

and

$$\frac{d\tilde{t}}{dv} = -1 + \frac{2}{(1 - M/r)^2} .$$

This information on slopes is enough to give the following $(\tilde{t}, r)$ spacetime diagram. (Three light rays and some light cones are sketched qualitatively.)

c) This is a black hole. No light rays from inside $r = M$ escape to outside $r = M$. Put differently, $r = M$ is a stationary null surface generated by the light rays at $r = M$.

**12-5.** *An observer falls radially into a spherical black hole of mass $M$. The observer starts from rest relative to a stationary observer at a Schwarzschild coordinate radius of $10M$. How much time elapses on the observer's own clock before hitting the singularity?*

**Solution:** Geodesics in the Schwarzschild geometry are characterized by the two integrals $e$ and $\ell$. For radial geodesics, such as the one in this problem, $\ell = 0$. The value of $e$ is determined by the condition that the particle starts from rest as seen by a stationary observer at $r = 10M$. The energy measured by this observer is therefore just the rest energy $m$. Thus,

$$m = E = -\mathbf{p} \cdot \mathbf{U} \tag{1}$$

where $\mathbf{p}$ is the particle's four momentum $m\mathbf{u}$, and $\mathbf{U}$ is the four velocity of the stationary observer

$$\mathbf{U} = \left[\left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}}, 0, 0, 0\right] . \tag{2}$$

Here, $R = 10M$.

Evaluating (1) with (2) one finds for the component $u^t$ of the four velocity of the particle at $r = 10M$.

$$u^t = \left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}} = \left(1 - \frac{2M}{10M}\right)^{-\frac{1}{2}} = \frac{\sqrt{5}}{2} .$$

We could obtain the same result by arguing that the radial velocity seen by the observer will be zero implying $dr/d\tau = 0$, so that the *particle* is stationary at the radius $R = 10M$, and then using (2) for $\mathbf{u}$.

Then for $e$ [*cf.* (9.21)]

$$e = \left(1 - \frac{2M}{R}\right) u^t = \frac{2}{\sqrt{5}} .$$

At other radii, using $\mathbf{u} \cdot \mathbf{u} = -1$,

$$\left(\frac{dr}{d\tau}\right)^2 = e^2 - \left(1 - \frac{2M}{r}\right) = \frac{2M}{r} - \frac{1}{5}$$

which vanishes at $r = 10M$ as it must. The proper time to go from $r = 10M$ to $r = 0$ is thus

$$\tau = \int_0^{10M} dr \left(\frac{2M}{r} - \frac{1}{5}\right)^{-\frac{1}{2}} .$$

Writing $r = 10M\,\xi$, this is

$$\tau = 10\sqrt{5}\,M \int_0^1 d\xi \left(\frac{1}{\xi} - 1\right)^{-\frac{1}{2}}$$

which works out to be $\tau = 5\sqrt{5}\,\pi M$.

**12-6**. *An observer decides to explore the geometry outside a Schwarzschild black hole of mass $M$ by starting with an initial velocity at infinity and then falling freely on an orbit that will come close to the black hole and then move out to infinity again. What is the closest that the observer can come to the black hole on an orbit of this kind? How can the observer arrange to have a long time to study the geometry between crossing the radius $r = 3M$ and crossing it again?*

**Solution:** Orbits in the Schwarzschild geometry are characterized by values of $e$ and $\ell$. For a given $\ell$ the orbit which comes closest to the black hole is the one which has an energy just below the maximum of the potential. The radius of closest approach is just slightly larger than the radius of the maximum which is [*cf.* (9.34)]

$$r_{\max} = \frac{M}{2}\left(\frac{\ell}{M}\right)^2 \left\{1 - \left[1 - 12\left(\frac{M}{\ell}\right)^2\right]^{\frac{1}{2}}\right\} .$$

As $\ell/M$ becomes large, this radius becomes smaller approaching $r = 3M$ as

$$r_{\max} \approx 3M\left[1 + 6\left(\frac{M}{\ell}\right)^2 + \cdots\right] . \tag{1}$$

In order not to fall into the black hole, $(e^2 - 1)/2$ should be just below the maximum of the potential at this radius which is

$$V_{\text{eff}}(r_{\max}) \approx \frac{1}{54}\left(\frac{\ell}{M}\right)^2 + \cdots . \tag{2}$$

The observer should therefore start with a very large $\ell$, with a large energy given by (2) — very nearly at the speed of light. He or she will then be able to explore down to the radius $r = 3M$. Further, since the energy is close to the maximum of the potential the observer will spend a very long time in the vicinity of $r = 3M$, orbiting the black hole many times before escaping to infinity. Put differently, the observer should come as close as possible to the circular light ray orbit at $r = 3M$.

**12-7**. *[E] A meter stick falls radially into a center of Newtonian gravitational attraction produced by one solar mass located at a point. Using Newtonian physics estimate the distance from the point at which the meter stick would break or be crushed.*

**Solution:** When the meter stick is a distance $R$ from the center of attraction, the difference in gravitational force on its ends is approximately

$$\frac{GM_\odot m}{R^2} - \frac{GM_\odot m}{(R+d)^2} \approx \frac{GM_\odot md}{R^3}$$

where $m$ is the mass of the meter stick ($\sim$ 100 g) and $d$ is its length (1 m). The meter stick will break when this is comparable to the breaking force $F_{\text{break}}$ at a radius

$$R \sim \left(\frac{GM_\odot md}{F_{\text{break}}}\right)^{\frac{1}{3}} .$$

Guessing $F_{\text{break}} \sim 1000$ N $\sim 200$ lbs, one finds $R \sim 200$ km.

**12-8**. *Can an observer who falls into a spherical black hole receive information about events which take place outside? Is there any region of spacetime outside the black hole which an interior observer cannot eventually see? Analyze these questions using diagram like the one in Figure 12.2.*

**Solution:**

An observer who falls into the black hole can, in principle, receive information in the shaded region above, between crossing the horizon at $r = 2M$ and destruction at $r = 0$. This includes information about events outside $r = 2M$ but not all of the region outside $r = 2M$.

**12-9**. *[S] Darth Vader is pursuing some Jedi knights. The Jedi knights plunge into a large black hole seeking the source of the force. Darth Vader knows that, once inside, any light emitted from his light-ray gun move to smaller and smaller Schwarzschild radii. He decides to try it by firing in the radial directions. Should he worry that light from his gun will fall back on him before his destruction in the singularity?*

**Solution:**

The Kruskal diagram above shows a possible world line for Vader as well

as two dotted world lines for the radial light rays emitted by his light sword. Although all three fall into the singularity, they never intersect. Vader does not have to worry.

**12-10**. *Show that the the slopes of the curves of $\tilde{t}$ vs. $r$ of non-radial light rays in an Eddington-Finkelstein diagram like Figure 12.2 must lie within the light cones defined by the radial light rays.*

**Solution:** Without loss of generality, we can take the non-radial light rays to lie in the equatorial plane. Then the condition $ds^2 = 0$ for light rays becomes

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\phi^2 = 0 \ .$$

Divide by $dr^2$ and solve for $dv/dr$ to find

$$\frac{dv}{dr} = \left(1 - \frac{2M}{r}\right)^{-1} \left\{1 \pm \left[1 + \left(1 - \frac{2M}{r}\right) r^2 \left(\frac{d\phi}{dr}\right)^2\right]^{\frac{1}{2}}\right\}$$

In the radial case the + sign corresponds to outgoing light rays $r > 2M$ and the − to ingoing, $v =$ const. one. The quantity inside the square bracket is always larger for non-radial light rays $d\phi/dr \neq 0$ than for radial ones. This can be seen to mean that the trajectories of non-radial light rays always lie *within* the light cones of the radial ones in the $(\tilde{t}, r)$ plane.

**12-11**. *Negative mass does not occur in nature. But just as an excercise analyze the behavior of radial light rays in a Schwarzschild geometry with a negative value of $M$. Sketch the Eddington-Finkelstein diagram showing these light rays. Is the negative mass Schwarzschild geometry a black hole?*

**Solution:** When $M$ is negative (12.5) implies

$$\frac{dr}{dv} = \frac{1}{2}\left(1 + \frac{2|M|}{r}\right) > 0.$$

The light rays are always outgoing. The $v$ = const. light rays are ingoing. Eq. (12.6) is replaced by

$$v - 2\left[r - 2|M|\log\left(\frac{r}{2|M|} + 1\right)\right] = \text{const.}$$

The resulting Eddington-Finkelstein plot of $T \equiv \tilde{t} = v - r$ vs. $r$ is shown below.

From each point there is an outgoing light ray which escapes to infinity. The negative mass Schwarzschild geometry is therefore not a black hole.

**12-12.** *[S] Check that the normal vector to the horizon three-surface of a Schwarzschild black hole is a null vector.*

**Solution:** It's essential to work this problem in a system of coordinates that is non-singular at the horizon like the Eddington-Finkelstein coordinates. A normal to a surface $\mathbf{n}$ is perpendicular to three independent directions $\mathbf{t}$ that lie in the surface [*cf.* (7.68)]. These can be taken to be the $v$, $\theta$, and $\phi$ directions:

$$t_1^\alpha = (1,0,0,0)\,, \quad t_2^\alpha = (0,0,1,0)\,, \quad t_3^\alpha = (0,0,0,1)\,.$$

Orthogonality to $\mathbf{t}_2$ and $\mathbf{t}_3$ means $\mathbf{n}$ must be of the form $(n^v, n^r, 0, 0)$. Then,

$$\begin{aligned} \mathbf{n}\cdot\mathbf{t}_1 &= g_{\alpha\beta}n^\alpha t_1^\beta = g_{\alpha v}n^\alpha \\ &= g_{vv}n^v + g_{rv}n^r = 0\,. \end{aligned}$$

But since $g_{vv} = 0$ at $r = 2M$ and $g_{rv} = 1$, we conclude $n^r = 0$. A normal vector is therefore,

$$n^\alpha = (1, 0, 0, 0) \ .$$

This is null because

$$\mathbf{n} \cdot \mathbf{n} = g_{vv}(n^v)^2 = 0$$

at $r = 2M$.

**12-13.**

a) *An observer falls feet first into a Schwarzschild black hole looking down at her feet. Is there ever a moment when she cannot see her feet? For instance, can she see her feet when her head is crossing the horizon? If so what radius does she see them at? Does she ever see her feet hit the singularity at $r = 0$ assuming she remains intact until her head reaches that radius? Analyze these questions with and Eddington-Finkelstein or Kruskal diagram.*

b) *(Is it dark inside a black hole?) An observer outside sees a star collapsing to a black hole become dark. But would it be dark inside a black hole assuming a collapsing star continues to radiate at a steady rate as measured by an observer on its surface?*

**Solution:**

a) The figure above shows an Eddington-Finkelstein diagram with schematic world lines of the observer's head and feet. At a given $\tilde{t}$ her feet are at a smaller radius than her head because she is falling in feet first. Radial rays emitted originating at her feet are shown. (There segments of the light rays illustrated in Figure 12.2.)

   There is no instant when she is not receiving a light ray from her feet. She sees them always. When her head crosses the horizon she sees her feet at the same radius, because the horizon is generated by light rays. When her head hits the singularity she still sees light from her feet that was emitted earlier but is falling into the singularity as well. (Some students interpret this question to ask if she sees her feet *when* they hit the singularity. But there is no invariant meaning to "when".)

b) It will be light inside a black hole for the same reason. Just replace the world line of feet in the above picture with the world line of the star's surface.

**12-14.** *[C] Once across the event horizon of a black hole, what is the longest proper time the observer can spend before being destroyed in the singularity?*

**Solution:** The path of longest proper time should be a geodesic from $r = 2M$ to $r = 0$. Using (9.26) we have for the elapsed proper time

$$\begin{aligned}\tau &= -\int_{2M}^{0} dr/(dr/d\tau) = -\int_{2M}^{0} dr \left[e^2 - \left(1 + \frac{\ell^2}{r^2}\right)\left(1 - \frac{2M}{r}\right)\right]^{-\frac{1}{2}} \\ &= \int_0^{2M} dr \left[e^2 - \left(1 + \frac{\ell^2}{r^2}\right)\left(\frac{2M}{r} - 1\right)\right]^{-\frac{1}{2}} .\end{aligned}$$

Written this way it is clear that non-zero values of $\ell$ and $e$ only decrease the proper time from $\ell = e = 0$. That geodesic therefore gives the longest time

$$\tau = \int_0^{2M} dr \left(\frac{2M}{r} - 1\right)^{-\frac{1}{2}} = 2M \int_0^1 d\xi \, \frac{\xi^{\frac{1}{2}}}{\sqrt{1-\xi}} = \pi M .$$

One of the author's students characterized this result as "The more you struggle, the shorter your life."

**12-15**. *[C] A spaceship whose mission is to study the environment around black holes is hovering at a Schwarzschild coordinate radius $R$ outside a spherical black hole of mass $M$. To escape back to infinity, crew must eject part of the rest mass of the ship to propel the remaining fraction to escape velocity. What is the largest fraction $f$ of the rest mass that can escape to infinity? What happens to this fraction as $R$ approaches $2M$?*

**Solution:** The key to this problem is to recognize that in the ejection event, energy-momentum is conserved, but rest mass is not necessarily conserved. Let $m$ and $\mathbf{u}$ be the rest mass and the four-velocity of the rocket hovering at radius $R$.

$$u^\alpha = \left[\left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}}, 0, 0, 0\right] .$$

Let $m_{\text{esc}}$, $\mathbf{u}_{\text{esc}}$, and $m_{\text{ej}}$, $\mathbf{u}_{\text{ej}}$ be the corresponding quantities for the escaping and ejected fragment. The minimum four-velocity for escape corresponds to an orbit with $e = 1$ and $\ell = 0$ and is [*cf.* (9.35)]

$$u^\alpha_{\text{esc}} = \left[\left(1 - \frac{2M}{R}\right)^{-1}, + \left(\frac{2M}{R}\right)^{\frac{1}{2}}, 0, 0\right] .$$

Conservation of three-momentum implies

$$0 = m_{\text{esc}} \left(\frac{2M}{R}\right)^{\frac{1}{2}} + m_{\text{ej}} u^r_{\text{ej}}$$

or

$$u^r_{\text{ej}} = - \left(\frac{m_{\text{esc}}}{m_{\text{ej}}}\right) \left(\frac{2M}{R}\right)^{\frac{1}{2}}$$

assuming that the fragment is ejected in the radial direction. Then imposing $\mathbf{u} \cdot \mathbf{u} = -1$, the four-velocity of the ejected fragment has a time component

$$u^t_{\text{ej}} = \left(1 - \frac{2M}{R}\right)^{-1} \left\{1 - \frac{2M}{R} \left[1 - \left(\frac{m_{\text{esc}}}{m_{\text{ej}}}\right)^2\right]\right\}^{\frac{1}{2}} .$$

Conservation of energy

$$mu^t = m_{\text{esc}} u^t_{\text{esc}} + m_{\text{ej}} u^t_{\text{ej}}$$

then gives

$$\left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} = \left(\frac{m_{\text{esc}}}{m}\right) + \left(\frac{m_{\text{ej}}}{m}\right) \left\{1 - \frac{2M}{R}\left[1 - \left(\frac{m_{\text{esc}}}{m_{\text{ej}}}\right)^2\right]\right\}^{\frac{1}{2}} .$$

The largest fragment that can escape is the largest value of $m_{\text{esc}}/m$ that can satisfy this relation as $m_{\text{ej}}/m$ ranges from zero to 1. Plotting the function for a few cases shows that $m_{\text{esc}}/m$ is maximized when $m_{\text{ej}} = 0$, i.e. all the rest mass is turned into energy. Then

$$\frac{m_{\text{esc}}}{m} = \frac{(1 - 2M/R)^{\frac{1}{2}}}{1 + (2M/R)^{\frac{1}{2}}} .$$

This vanishes when $R = 2M$.

**12-16**. *In Section 9.2 we derived the formula* (9.20) *for gravitational redshift from a stationary observer. We started from the conservation law* (8.32) *arising from the time translation symmetry of the Schwarzschild geometry. Use similar techniques to derive for the red-shift of light emmitted radially from a star in free fall collapse as a function of the time $t_R$ the radiation is received by a distant observer. Compare your result with* (12.11). *Eq.* (9.20) *held for non-radial radiation as well. Do you expect that for radiation from the surface of a collapsing star? (Note that in* (9.20) *$R$ was the radius of the stationary observer emitting the radiation, while in the example that led to* (12.11), *$R$ was the location of the observer receiving the radiation.)*

**Solution:** It's simplest to use Schwarzschild coordinates to work out this problem, although the computation in Eddington-Finkelstein coordinates is similar. Suppose that $\mathbf{p}$ is the energy momentum four vector of a photon emitted radially by the freely falling observer with frequency $\omega_*$. Thus [*cf.* (5.83)]

$$\omega_* = -\mathbf{p} \cdot \mathbf{u}/\hbar \tag{1}$$

where $\mathbf{u}$ is the falling observer's four velocity — the same as the four-velocity of the surface of the star. The components of $\mathbf{u}$ in Schwarzschild coordinates were given in (9.36). There are only time and radial components since the observer is falling radially. Writing out (1), we have

$$\hbar\omega_* = p^t(r) + \left(\frac{2M}{r}\right)^{1/2}\left(1 - \frac{2M}{r}\right)^{-1} p^r(r) \tag{2}$$

for photons emitted when the surface of the star crosses the radius $r$. For a radially moving photon $p^t$ and $p^r$ are not independent but are connected by the relation $\mathbf{p}\cdot\mathbf{p} = 0$. Written out this gives

$$-\left(1 - \frac{2M}{r}\right)(p^t(r))^2 + \left(1 - \frac{2M}{r}\right)^{-1}(p^r(r))^2 = 0\ , \tag{3}$$

so that

$$p^r(r) = \left(1 - \frac{2M}{r}\right)p^t(r) \tag{4}$$

and

$$\hbar\omega_* = p^t(r)\left[1 + \left(\frac{2M}{r}\right)^{1/2}\right]\ . \tag{5}$$

The energy at $\infty$ as measured by a stationary observer is according to (9.19)

$$\hbar\omega_\infty = p^t(\infty)\ . \tag{6}$$

The quantities $p^t(r)$ and $p^t(\infty)$ are related by the conservation law in (8.32) giving the connection between $\omega_\infty$ and $\omega_*$

$$\omega_\infty = \omega_*\left(1 - \frac{2M}{r}\right)\left[1 + \left(\frac{2M}{r}\right)^{1/2}\right]^{-1} \approx \frac{1}{2}\omega_*\left(\frac{r}{2M} - 1\right) \tag{7}$$

for $r \approx 2M$. Then using (12.9) we have

$$\omega_\infty(t_R) \approx \frac{1}{2}\omega_* e^{-(t_R - r_R)/4M}\ . \tag{8}$$

This is exactly the same as (12.11) if the freely falling four velocity (9.36) is used in (12.10).

The same result will not hold for non-radial photons. Eq. (3) will have more terms.

**12-17**. *[B] Derive the rocket thrust* (b) *in Box 12.2.*

**Solution:**

(a) When $u^\alpha = [(1-2M/R)^{-1/2}, 0, 0, 0]$, the $d^2x^\alpha/d\tau^2$ term in expression (a) for the four-force vanishes and we are left with

$$f^r = m\Gamma^r_{tt}(u^t)^2 = mM/R^2$$

since $\Gamma^r_{tt} = (1-2M/r)\,(M/r^2)$.

(b) Using (7.61) we have

$$(e_{\hat{r}})^\alpha = \left[0\,,\,\left(1-\frac{2M}{R}\right)^{\frac{1}{2}}\,,0\,,0\right]\,.$$

Then (suspending the rule for balancing indices)

$$f^{\hat{r}} = \mathbf{e}_{\hat{r}}\cdot\mathbf{f} = m\left(1-\frac{2M}{R}\right)^{-\frac{1}{2}}\frac{M}{R^2}\,.$$

**12-18**. *[C] The Horizon Inside a Collapsing Shell. Consider the collapse of a spherical shell of matter of very small thickness and mass $M$. The shell describes a spherical three-surface in spacetime. Outside this surface the geometry is the Schwarzschild geometry with this mass. Inside make the following assumptions: i) The world line of the shell is known as a function $r(\tau)$ going to zero at some finite proper time. ii) The geometry inside the shell is flat. iii) The geometry of the three-surface of the collapsing shell is the same inside as outside.*

*a) Draw two spacetime diagrams: one like that in Figure 12.2 and another corresponding to the spacetime inside the shell in a suitable set of coordinates. Draw the world line of the shell in both and indicate how points on the inside and outside correspond. Locate the horizon inside the shell as well as outside.*

*b) How does the area of horizon inside the shell change moving along the light rays which generate it?*

**Solution:**

a) If the metric inside is flat there are coordinates $(T, R, \Theta, \Phi)$ in which the metric can be written

$$ds^2 = -dT^2 + dR^2 + R^2(d\Theta^2 + \sin^2\Theta\, d\Phi^2) \ .$$

The world line of the shell will be a curve $R(\tau)$. The metric in the three-surface of the collapsing shell is then,

$$(ds^2) = -d\tau^2 + R^2(\tau)\,(d\Theta^2 + \sin^2\Theta\, d\Phi^2) \ .$$

Outside, the story is similar using Schwarzschild $(t, r, \theta, \phi)$ coordinates. The metric on the surface of the collapsing shell is

$$(ds^2)_{\text{outside}} = -d\tau^2 + r^2(\tau)\,(d\theta^2 + \sin^2\theta\, d\phi^2) \ .$$

The inside and outside geometries will match if $r = R$, $\theta = \Theta$, $\phi = \Phi$. In particular, $R(\tau) = r(\tau)$. The function $T(\tau)$ can then be computed from the normalization condition

$$-\left(\frac{dT}{d\tau}\right)^2 + \left(\frac{dr}{d\tau}\right)^2 = -1$$

and the given $r(\tau)$. We can then plot a $(T, r)$ diagram analogous to the $(\tilde{t}, r)$ diagram in figure 12.2.

The figure at left refers to the geometry inside the shell. The figure at right is an Eddington-Finkelstein diagram referring to the outside.

The world line of the collapsing shell is shown schematically in both diagrams. It's not the same curve because, although $R(\tau) = r(\tau)$, $T(\tau)$ and $\tilde{t}(\tau)$ will be different. That's why two separate plots are required. Only the unshaded part of each plot is physical. They match up across the surface of the shell. For example, points A, B, and C match.

b) The horizon $H$ is the surface generated by light rays that neither escape to infinity nor collapse to the singularity. Inside the shell those are the radial light rays that just make it to the surface at $r = 2M$ as shown. After passing through the shell they remain stationary at $r = 2M$. The area increases from zero inside the shell and remains stationary outside.

**12-19**. *Figure 12.5b illustrates area of the horizon of a spherical black hole if a shell of mass $M_{shell}$ later fell into it. The discussion assumed that the shell was made of usual matter with $M_{shell} > 0$. What would happen if it were negative? Would the area of the horizon always increase? Illustrate the qualitative behavior of the horizon as well as that of light rays for the case of a negative mass shell with a diagram like Figure 12.5b. Also show the behavior of the light rays emitted from the center that would have generated the horizon if the shell had not crossed.*

**Solution:** The final radius of the horizon will be $2(M - |M_{\text{shell}}|)$ where $M$ is the mass of the spherical collapsing star. This is less than the $2M$ radius the horizon would have had in the absence of the shell. The following figure shows what happens:

The horizon is generated by light rays just inside the $r = 2M$ old horizon radius that hit the shell just at the radius $2(M - |M_{\text{shell}}|)$ of the new horizon outside the shell. In the absence of the shell these light rays would have fallen back into the singularity. Before the shell crosses, when they are moving in the geometry of the mass $M$, they are moving to smaller radii, and the area is decreasing. You can think of the negative energy shell as giving them a repulsive "kick" which leaves them hovering at the final radius.

The dotted line shows the light rays which would have made up the horizon had the shell not crossed. They hover at $r = 2M$ before the shell crosses but escape afterwards, being less attracted by the smaller final mass.

**12-20**. *[S,A] Explicitly carry out the transformation from Schwarzschild to Kruskal coordinates defined in* (12.13). *Find the metric in Kruskal coordinates for both $r > 2M$ and $r < 2M$.*

**Solution:** This is a straightforward problem in virtuous algebra. The transformation formulae (12.13) are differentiated to give $dU$ and $dV$ in terms of $dr$ and $dt$. When the resulting expressions are substituted in (9.9), the Schwarzschild metric results.

**12-21**. *Show that in a Kruskal diagram $|dV/dU|$ must be greater than unity for a timelike particle world line even if it is moving non-radially.*

**Solution:** Particle trajectories must always be timelike, $\mathbf{u} \cdot \mathbf{u} = -1$. This means

$$\frac{32M^3 e^{-r/2M}}{r}\left[-\left(\frac{dV}{d\tau}\right)^2 + \left(\frac{dU}{d\tau}\right)^2\right] + r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2 = -1 \ .$$

In particular, for the sign of the left hand side to be negative one must have

$$\left(\frac{dV}{d\tau}\right)^2 > \left(\frac{dU}{d\tau}\right)^2$$

or

$$\left(\frac{dV}{dU}\right)^2 > 1 \ .$$

Thus at any one point on the Kruskal diagram the world line of a particle must have a slope which is greater in magnitude than unity.

**12-22**. *Two observers in two rockets are hovering above a Schwarzschild black hole of mass $M$. They hover at a fixed radius $R$ such that*

$$\left(\frac{R}{2M} - 1\right)^{\frac{1}{2}} e^{R/4M} = \frac{1}{2}$$

*and fixed angular position. (In fact, $R \approx 2.16M$). The first observer leaves this position at $t = 0$ and travels into the black hole on a straight line in a Kruskal diagram until destroyed in the singularity at the point where the singularity crosses the line $U = 0$. The other observer continues to hover at $R$.*

*a) On a Kruskal diagram sketch the world lines of the two observers.*

*b) Is the observer who goes into the black hole following a timelike world line?*

*c) What is the latest Schwarzschild time after the first observer departs that the other observer can send a light signal which will reach the first before being destroyed in the singularity?*

**Solution:**

a)

b) The straight line has a slope of 2 which means the observer is within the 45° lines which are the light cones.

c) The latest time is the value of $t$ at which the 45° dotted line from $U = 0$, $V = 1$ intersects the curve $r = R$. The equation of the 45° line is $V = 1 - U$ so,

$$\left(\frac{R}{2M} - 1\right)^{\frac{1}{2}} e^{\frac{R}{4M}} \sinh\left(\frac{t}{4M}\right) = 1 - \left(\frac{R}{2M} - 1\right)^{\frac{1}{2}} e^{\frac{R}{4M}} \cosh\left(\frac{t}{4M}\right)$$

or

$$\frac{1}{2}\sinh\left(\frac{t}{4M}\right) = 1 - \frac{1}{2}\cosh\left(\frac{t}{4M}\right)$$

so that

$$t = 4M\log(2)\ .$$

**12-23**. *The formula* (12.11) *is for the red-shift of light from a collapsing star as a function of the time $t_R$ it is received by a distant stationary observer. It could have been worked out in any non-singular coordinate system for the Schwarzschild geometry. Derive the same result using Kruskal coordinates.*

**Solution:** The essential point of this problem is to use Kruskal coordinates to derive the relation (12.9) between the time $t_R$ a signal is received by the distant observer and the radius $r_E$ of the surface at the time it was emitted. The rest

of the derivation of the redshift follows (12.10) to (12.11). In the Kruskal geometry outgoing radial light rays propagate along the paths $V = U+$ const. Thus if $(V_E, U_E)$ are the coordinates of the light ray when it was emitted and $(V_R, U_R)$ the coordinates when it is detected we have

$$V_R - V_E = U_R - U_E \tag{1}$$

as the statement that it travels on a 45° line. Equivalently

$$V_R - U_R = V_E - U_E \quad . \tag{2}$$

Using (12.13) this can be rewritten in terms of the Schwarzschild coordinates $(r_E, t_E)$ of the point where the ray was emitted and the coordinates $(r_R, t_R)$ of the point where it is received:

$$\left(\frac{r_R}{2M} - 1\right)^{\frac{1}{2}} e^{(r_R - t_R)/4M} = \left(\frac{r_E}{2M} - 1\right)^{\frac{1}{2}} e^{(r_E - t_E)/4M} \ . \tag{3}$$

In this equation $t_E$ is given as a function of $r_E$ by (12.7b) and (12.1). The connection between the time $t_R$ and the radius $r_E$ is obtained by solving (3). A short calculation shows that for $r_E$ very close to $2M$

$$r_E = 2M(1 + e^{-(t_R - r_R)/4M}) \ . \tag{4}$$

exactly as in (12.9).

**12-24.** *[B,N] Construct embedding diagrams for slices of the Kruskal extension of the Schwarzschild geometry for the values $V = .9$ and $V = .999$ that are analogous to that for $V = 0$ in Box 12.4. You may exhibit the cross section of the axisymmetric two-dimensional surface if it's easier. How to these embedding diagrams support the statement that the wormhole of the Kruskal extension is not constant in time? What happens when $V > 1$?*

**Solution:** The geometry of a $V =$ const., $\theta = \pi/2$ two-dimensional slice of Kruskal metric (12.14) is given by

$$d\Sigma^2 = \frac{32M^2}{r} \ e^{-(r/2M)} dU^2 + r^2 d\phi^2. \tag{1}$$

Eq. (12.15) determines $U(r, V)$ as a function of $r$ for a given $V$:

$$U^2(r, V) = \left(\frac{r}{2M} - 1\right) e^{r/2M} + V^2. \tag{2}$$

Inserting (2) into (1) gives

$$d\Sigma^2 = \left(\frac{r}{2M}\right) \frac{e^{r/2M}}{U^2(r, V)} dr^2 + r^2 d\phi^2. \tag{3}$$

The construction of an embedding surface now proceeds as in Example 7.7. For the three functions (7.43) giving the cylindrical coordinates $(z, \rho, \psi)$ in flat space locating the point at $(r, \phi)$ on the surface in the flat embedding space, we take $\rho = r$, $\psi = \phi$, and

$$\left(\frac{dz}{dr}\right)^2 \equiv \left(\frac{r}{2M}\right) \frac{e^{r/2M}}{U^2(r, V)} - 1 \equiv F^2(r, V). \tag{4}$$

Then

$$z(r, V) = \int_{r_{\min}(V)}^{r} F(r', V)\, dr'$$

where $r_{\min}(V)$ is the minimum value of $r$ found by solving (2) when $U = 0$.

The figures below show the embedded surfaces and their cross sections for (left to right) $V = 0$, $V = .9$, and $V = .999$. For the graphs $r$ and $z$ are in units of $M$. (The accuracy of these integrations near the minimum radius could be improved.)

Evidently the wormhole is getting narrower and narrower as $V$ increases and pinches off when $V = 1$. After that it is not possible to embed $V > 1$ surfaces in flat three-dimensional space because $F^2$ as defined above becomes negative for some $r$.

**12-25.** *[B,S] Suppose that the black hole in the center of our galaxy were really described by the maximal Kruskal extension instead of having been produced by collapsing stars. Using a Kruskal diagram, explain why it would not be possible to traverse from one asymptotic region of the Kruskal extension to the other. (The question in Box 12.4.) But could an observer see light from stars on the other side of the extension even if they could not travel there? If so what would they look like?*

**Solution:** The world line of particles must all have slopes greater than 45° in a Kruskal diagram. There is no world line with this property going from the region $V > -U$ to the region $V < -U$. In fact, they are separated by the null surface $V = -U$.

An observer outside the black hole could never see stars from the other side of the extension. But an observer who falls into the black hole could in principle receive light from the other side as the diagram below makes clear. The light would be very blue-shifted, but stars could be seen over a fraction of the sky.

However, if the future on the other side is in the direction of $t = +\infty$, then the stars would have to emit light in their past light cone which they do not do on our side.

The example is too idealized to reach a realistic physical conclusion.

**12-26**. *[B] Show that the boundaries of the Penrose diagram for the Kruskal extension of the Schwarzschild geometry are as given in Box 12.5.*

**Solution:** The coordinates $U$ and $V$ range over all possible values such that $r(U,V)$ is greater than zero. Eq. (12.15) shows that the $r=0$ boundaries are

$$U^2 - V^2 = -1 \quad , \quad \text{or} \quad uv = +1. \tag{1}$$

We check that this is the line $V' = \pi/4$. When $V' = \pi/4$

$$\begin{aligned} u &= \tan(u') = \tan\left(\frac{\pi}{4} - U'\right) \\ v &= \tan(v') = \tan\left(\frac{\pi}{4} + U'\right). \end{aligned} \tag{2}$$

The standard formula for $\tan(A+B)$ and $\tan(\pi/4)$ shows that (1) is satisfied for any $U'$.

The infinite values of $U$ and $V$ are mapped onto finite values of $U'$ and $V'$. For instance positive infinite values of $U$ are mapped onto $u' = \pi/2 = V' - U'$ and similarly for the other infinite values. These 45° lines are the various parts of null infinity.

There are two event horizons for the two asymptotic regions which are the lines $V = \pm U$. These are $u = 0$ or $v = 0$, which are the 45° lines $V' = \pm U'$.

**12-27**. *If the area of a black hole must always increase, show that a black hole can never bifurcate into two black holes preserving total mass.*

**Solution:** Let $M$ be the mass of the initial black hole and $M_1$ and $M_2$ be the masses of the black holes into which it hypothetically bifurcates. Preserving total mass means

$$M = M_1 + M_2 \ . \tag{1}$$

Increasing area means

$$4\pi\,(2M)^2 \leq 4\pi\,(2M_1)^2 + 4\pi\,(2M_2)^2 \tag{2}$$

or

$$M^2 \leq M_1^2 + M_2^2 \ . \tag{3}$$

Using (1), the relation (3) can be rewritten

$$\begin{aligned} M^2 &\leq (M - M_1)^2 + M_1^2 \\ &\leq M^2 - 2M_1(M - M_1) \ . \end{aligned} \tag{4}$$

But since $M \geq M_1$ this can never be satisfied.

# Chapter 13

# Astrophysical Black Holes

**13-1**. *Figure 13.2 shows the radial velocity curve of the black hole X-ray binary Nova Muscae. The symmetrical form of the curve indicates that the mutual orbit is close to circular. Estimate the value of the mass function for this system. The period of the orbit is .423 days.*

**Solution:** From (13.6), the mass function $f$ is

$$f = \frac{V_{\text{max}}^3 P}{2\pi G}$$

where $P$ is the period of .4 days and $V_{\text{max}}$ is the maximum radial velocity of 420 km/s read off the graph. This gives

$$f \approx 3M_\odot$$

**13-2**. *(The Roche Lobe) (a) Consider two point masses $M_1$ and $M_2$ held at fixed positions in space a distance $d$ apart. Sketch contour lines of constant total Newtonian gravitational potential in a plane through the axis connecting the masses. Find the position between the stars at which the Newtonian gravitational force on a test particle vanishes.*

*(b) Suppose the star with mass $M_1$ is surrounded by a fluid envelope whose mass contributes negligibly to the gravitational potential. Explain why the boundary of the envelope must lie on an equipotential. Sketch the shape of that boundary when material from the envelope is just about to flow onto the second mass. That is the Roche lobe. Compare with Figure 13.1.*

*Comment: In this problem the masses were imagined to be fixed in space. In a model of a binary star system they would be rotating around one another. For a harder problem work out the shape of the Roche lobe taking proper account of this rotation.*

**Solution:**

a) The figure below shows a plot of the equipotentials when $M_2/M_1 = .3$ and $d = 1$. $M_1$ is located at the origin. Mass $M_2$ is located at $(0, 1)$. The four equipotentials shown are all separated by the same interval $\Delta\Phi$.

The equilibrium position is located on the axis between the two masses a distance

$$x_{\text{eq}} = d\left[1 + \left(\frac{m_2}{m_1}\right)^{\frac{1}{2}}\right]^{-1}$$

from $m_1$.

b) When the surface is an equipotential the gravitational force is normal to the surface. Were the surface not an equipotential there would be a component of gravitational force along the surface and fluid would flow along it.

Small envelopes will have nearly circular boundaries around $m_1$. Matter will flow from the envelope to the second star if the potential energy becomes larger than that at the equilibrium position. The equipotential has the figure eight form shown above and the Roche lobe is the shaded part of it.

**13-3.** *In the image of the radio source Cygnus A in Figure 13.5 one jet is much brighter than the other. Rotating black hole models of the source suggest that the two jets emerge in opposite directions along the rotation axis. What famous effect of special relativity could contribute to an explanation of the difference in brightness? Assuming the intensities differ by a factor of 100, and that the axis makes an angle of 45° with respect to the line of sight, what can you say about the velocity of the sources of the visible radiation in the jets?"*

**Solution:** Relativistic beaming would make the jet moving toward us brighter than the jet moving away as discussed in Section 5.5. The computation of the ratio of intensities is given in part (d) of the solution to Problem 5.17. The ratio for sources in the jets moving with speed $V$ at angles $\alpha' = \pi/4$ and $\alpha' = 5\pi/4$ is

$$\left(\frac{1+V/\sqrt{2}}{1-V/\sqrt{2}}\right)^3 = 100 \ .$$

This gives a velocity $V = .9$.

**13-4.** *[E] Figure 13.4 shows the orbits of stars around the $3 \times 10^6 M_\odot$ black hole at the center of our galaxy approximately 9 kpc (kpc=kiloparsec) away. Make a rough estimate of the predicted linear orbital velocities as a function of angular separation from the center by assuming that the stars are in circular, Newtonian orbits whose plane is perpendicular to the line of sight. How do your results compare with the velocities that can be estimated from the angular positions that are shown over several years?*

**Solution:** Equating the centripetal acceleration to the gravitational acceleration gives the usual Newtonian relation for a circular orbit of radius $R$:

$$\frac{V^2}{R} = \frac{GM}{R^2}$$

where $V$ is the linear orbital velocity and $M$ is the mass of the black hole. The radius $R$ is related to the angular size of the orbit $\theta$ by $R = d\theta$ where $d$ is the distance to the center of the galaxy and $\theta$ is in radians. Combining this with

the Newtonian result gives for the predicted linear velocity $V_{\text{pred}}$:

$$V_{\text{pred}} = \frac{c}{\theta^{1/2}} \left(\frac{GM}{c^2 d}\right)^{1/2} = 542 \left(\frac{1''}{\theta}\right)^{1/2} \frac{\text{km}}{\text{s}} .$$

Here the velocity of light has been introduced in a way to make computation in geometrized units easy and the final result has been evaluated with the data given in the problem.

A rough estimate of an angular radius of the orbit of the star SO-1 in Figure 13.4 is .15 ″. The resulting predicted linear velocity is

$$V_{\text{pred}} \approx 1400 \text{ km/s} .$$

The observations show that SO-1 moved an angular distance $\alpha \approx .15''$ over a time $T$ of four years. Assuming the orbit is transverse to the line of sight the observed angular velocity is

$$V_{\text{obs}} = \frac{\alpha d}{T}$$

where $\alpha$ is measured in radians. Evaluating this gives

$$V_{\text{obs}} \approx 1600 \text{ km/s} .$$

This is unexpectedly good agreement for such a rough estimate.

**13-5.** *What is the mass of a black hole formed at the beginning of the universe that would explode by the Hawking process at the time the universe becomes transparent to radiation — approximately 400,000 years after the big bang.*

**Solution:** From (13.19) a black hole which evaporates at a time $t_*$ has a mass at an earlier time of

$$M(t) = \left[\frac{3}{15,360\pi}\, \hbar\, (t_* - t)\right]^{\frac{1}{3}} .$$

The problem asks for the mass at $t = 0$ with $t_* = 3 \times 10^5$ yr. The above expression is in units where $M$ and $t_*$ are measured in units of cm, and $\hbar =$

$\ell_{Pl}^2 \approx 10^{-66}$ cm$^2$. The conversion factors are $3 \times 10^{10}$ cm/s and $3.15 \times 10^7$ s/yr. Putting these together, we have

$$M(0) = 3 \times 10^{-16} \text{ cm} = 4 \times 10^{12} \text{ g}$$

where we have used [*cf.* Appendix A] $.742 \times 10^{-28}$ cm/g. $M(0)$ is about the mass in a rock cube with side 100 m.

**13-6**. *[E] Estimate how long an electron-positron pair created in a vacuum fluctuation can last assuming that the fluctuation can violate energy conservation for a time $\Delta t$ consistent with the energy-time uncertainty principle $\Delta E \Delta t > \hbar$*

**Solution:** The fluctuation in energy is $\Delta E = 2m_e c^2$. The time is

$$\begin{aligned} \Delta t &\sim \frac{\hbar}{2m_e c^2} = \frac{(1.05 \times 10^{-27} \text{erg} \cdot \text{s})}{2(9.11 \times 10^{-28}\text{g})\,(3 \times 10^{10} \text{ cm/s})^2} \\ &\sim 6 \times 10^{-22} \text{ s (!)} \end{aligned}$$

**13-7**. *[E] Estimate the distance at which the energy received at Earth from an exploding primordial black hole in the last one second of its life would be comparable to that received from a nearby star in the same period. (For definiteness take the star to have the luminosity of the Sun and be 10 pc away.)*

**Solution:** Let $M_0$ and $T_0$ denote the present mass and temperature of a black hole going to explode 1 sec after the present time $t_0$. As Example 13.2 showed, to have survived from the beginning of the universe to explode now $M_0 \sim 10^{14}$ g. Correspondingly, from (13.18), $T_0 \sim 10^{12}$ K. The present luminosity of the black hole $L_0$ can be roughly estimated by

$$L_0 \approx \left(\sigma T_0^4\right) \left(4\pi R_0^2\right)$$

where $\sigma$ is the Stephan-Boltzmann constant and $R_0 \sim 2GM_0/c^2$ is the present size of the black hole. Putting in the numbers gives

$$R_0 \sim 10^{-14} \text{ cm} \quad , \quad L_0 \sim 10^{17} \text{ erg/s}.$$

Over the course of the second after $t_0$ the mass will vary as in (13.19) and the temperature inversely as this, from (13.16). Thus the luminosity $L \propto M^2 T^4$ varies as

$$L(t) = L_0 \left( \frac{t_* - t_0}{t_* - t} \right)^{\frac{2}{3}}$$

where $t_* = t_0 + 1$ s. The total energy $\Delta E$ emitted from $t_0$ to $t_*$ is the integral of this which is

$$\Delta E = 3L_0 \left( t_* - t_0 \right).$$

This is just a factor of 3 larger than the energy emitted in that second at the present luminosity.

Let $d$ be the distance to the exploding black hole. For the energy emitted in a second to be comparable to a star with a solar luminosity $L_\odot$ a distance 10 pc away the inverse square law implies

$$\frac{3L_0 \left( t_* - t_0 \right)}{4\pi d^2} = \frac{L_\odot \left( t_* - t_0 \right)}{4\pi (10 \text{ pc})^2}.$$

Thus

$$d \approx (10 \text{ pc}) \left( \frac{3L_0}{L_\odot} \right)^{\frac{1}{2}} \sim 10^{-7} \text{ pc} \sim 10^6 \text{ km}.$$

That's just a few times the distance to the Moon. The black hole will have to be very close on astronomical scales.

# Chapter 14

# A Little Rotation

**14-1.** *Show that the gyroscope equation (14.6) implies* $\mathbf{s} \cdot \mathbf{s}$ *and* $\mathbf{s} \cdot \mathbf{u}$ *are constant along the geodesic followed by a gyro. Show that for any two gyros* $A$ *and* $B$ *moving along the same geodesic* $\mathbf{s}_A \cdot \mathbf{s}_B$ *is constant.*

**Solution:** Differentiate $\mathbf{s} \cdot \mathbf{s}$ with respect to the proper time along the gyro's trajectory to find

$$\begin{aligned} \frac{d}{d\tau}\,(\mathbf{s} \cdot \mathbf{s}) &= \frac{d}{d\tau}\left(g_{\alpha\beta}\; s^\alpha s^\beta\right) \\ &= 2\; g_{\alpha\beta}\; \frac{ds^\alpha}{d\tau}\; s^\beta + \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\; \frac{dx^\gamma}{d\tau}\; s^\alpha s^\beta \;. \end{aligned}$$

Then use the equation for the spin (14.6) to eliminate $ds^\alpha/d\tau$ in the first term to get

$$\frac{d}{d\tau}\,(\mathbf{s} \cdot \mathbf{s}) = -2\; g_{\alpha\beta}\; \Gamma^\alpha_{\delta\gamma}\; s^\delta s^\beta u^\gamma + \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\; s^\alpha s^\beta u^\gamma \;.$$

Utilizing the definition of the Christoffel symbols (8.19), and relabeling some dummy indices we find

$$\frac{d}{d\tau}\,(\mathbf{s} \cdot \mathbf{s}) = \left(-\frac{\partial g_{\beta\gamma}}{\partial x^\alpha} - \frac{\partial g_{\beta\alpha}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right)\; s^\alpha s^\beta u^\gamma = 0 \;.$$

**14-2.** *Check explicitly from the gyroscope equation* (14.6) *that, if the spatial*

*part of the spin initially points in the equatorial plane (so that $s^\theta = 0$), it remains pointing in the equatorial plane for a circular orbit lying in that plane.*

**Solution:** We first calculate the rate of change of $s^\theta$:

$$\begin{aligned}\frac{ds^\theta}{d\tau} &= -\Gamma^\theta_{\beta\gamma}\, s^\beta u^\gamma \\ &= -\Gamma^\theta_{r\theta}(s^r u^\theta + s^\theta u^r) - \Gamma^\theta_{\phi\phi} s^\phi u^\phi \ .\end{aligned}$$

We assume that the trajectory is in the equatorial plane, $u^\theta = 0$, and that initially $s^\theta = 0$. Then the initial rate of change of $s^\theta$ is

$$\frac{ds^\theta}{d\tau} = -\Gamma^\theta_{\phi\phi} s^\phi u^\phi \ .$$

Since $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$ vanishes on the equatorial plane we find that $ds^\theta/d\tau$ vanishes initially. Therefore $s^\theta$ remains zero throughout the trajectory.

**14-3.** *[S] Check that for the solution to the gyroscope equation given in* (14.16) *the magnitude of the spin* $(\mathbf{s}\cdot\mathbf{s})^{1/2}$ *remains constant in time and equal to the* $s_*$ *specified by the solution.*

**Solution:**

$$\mathbf{s}\cdot\mathbf{s} = -\left(1-\frac{2M}{R}\right)\left(s^t\right)^2 + \left(1-\frac{2M}{R}\right)^{-1}(s^r)^2 + R^2\left(s^\phi\right)^2 .$$

Using (14.9), (14.12), (14.15), and (14.16) this becomes

$$\mathbf{s}\cdot\mathbf{s} = s_*^2.$$

**14-4.** *[A] (a) Consider the gyroscope in circular orbit about a non-rotating body discussed in Section 14.3. Find the coordinate components of the orthonormal basis* $\mathbf{e}_{\hat{\alpha}}$ *of an observer at* $\phi = 0$ *who is moving with the spin, assuming that the spatial parts of two of the spacelike vectors* $\mathbf{e}_{\hat{1}}$ *and* $\mathbf{e}_{\hat{3}}$ *point along the* $r$ *and* $\phi$ *directions respectively.*

*(b) Project the spin vector of* (14.16) *onto the orthonormal basis constructed in part (a) to obtain*

$$s^{\hat{1}} = s_* \cos(\Omega' t) \ , \quad s^{\hat{3}} = s_* \sin(\Omega' t) \ ,$$

*showing very explicitly how the spin precesses in a comoving frame.*

**Solution:** (a) Let's construct the four orthogonal unit vectors at the point $\phi = 0$ in the gyro's orbit. An observer moving with the spin has a four-velocity that lies along that of the gyro $\mathbf{u}$. Thus,

$$(\mathbf{e}_{\hat{0}})^\alpha = u^t(1, 0, 0, \Omega)$$

where the order of the components is $(t, r, \theta, \phi)$ and $u^t$ is given by (9.48). The vectors $\mathbf{e}_{\hat{1}}$ and $\mathbf{e}_{\hat{2}}$ can be chosen to lie in the $r$ and $\theta$ directions with components

$$\begin{aligned} (\mathbf{e}_{\hat{1}})^\alpha &= \left(0, \left(1 - 2M/R\right)^{\frac{1}{2}}, 0, 0\right) \ , \\ (\mathbf{e}_{\hat{2}})^\alpha &= (0, 0, 1/R, 0) \ . \end{aligned}$$

The remaining vector, $\mathbf{e}_{\hat{3}}$, must have unit norm and be orthogonal to $\mathbf{e}_{\hat{0}}$, $\mathbf{e}_{\hat{1}}$, and $\mathbf{e}_{\hat{2}}$. It can have both $\phi$ and $t$ components, which makes orthogonality with $\mathbf{e}_{\hat{1}}$ and $\mathbf{e}_{\hat{2}}$ automatic assuming it has *only* these components. The first two conditions are

$$\mathbf{e}_{\hat{3}} \cdot \mathbf{e}_{\hat{3}} = 1 \ , \quad \mathbf{e}_{\hat{3}} \cdot \mathbf{e}_{\hat{0}} = 0 \ .$$

Assuming the form

$$(\mathbf{e}_{\hat{3}})^\alpha = (A, 0, 0, B) \ ,$$

after a little algebra, one finds

$$\begin{aligned} A &= \left(1 - \frac{3M}{R}\right)^{-\frac{1}{2}} \left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}} (R\Omega) \ , \\ B &= \left(1 - \frac{3M}{R}\right)^{-\frac{1}{2}} \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} \frac{1}{R} \ . \end{aligned}$$

(b) The spin can be projected onto the orthonormal basis using (5.82) as follows

$$s^{\hat{1}} = \mathbf{e}_{\hat{1}} \cdot \mathbf{s} = g_{\alpha\beta}(\mathbf{e}_{\hat{1}})^\alpha s^\beta \ , \qquad s^{\hat{2}} = \mathbf{e}_{\hat{2}} \cdot \mathbf{s} = g_{\alpha\beta}(\mathbf{e}_{\hat{e}})^\alpha s^\beta \ ,$$

(summation convention suspended as in (5.82)). It's just a matter of using (14.16), the results from part (a), and doing the algebra to reach the result quoted in the problem.

**14-5.** *[S] Plug in the numbers in* (14.19) *to evaluate the total geodetic precession of a gyroscope in orbit around the Earth and the rate of geodetic precession.*

**Solution:** Self-explanatory.

**14-6.** *[S] What is the largest possible geodetic precession for a stable circular orbit in the Schwarzschild geometry?*

**Solution:** The radius of the innermost stable circular orbit is 6M (Section 9.3). From (14.18) therefore

$$\Delta\phi_{\text{geodetic}} = 2\pi\left(1 - \frac{1}{\sqrt{2}}\right) = 1.84 \text{ radians/orbit} = 105^\circ/\text{orbit} .$$

**14-7.** *Work through the derivation of geodetic precession again using the PPN metric given in* (10.4) *and* (10.6). *Show that*

$$\Delta\phi_{\text{geodetic}} \approx \left(\gamma + \frac{1}{2}\right) 2\pi \left(\frac{GM}{Rc^2}\right)$$

*so that a measurement of the geodetic precession is another way to determine the PPN parameter* $\gamma$.

**Solution:** Using the PPN metric of eqs (10.6) and (10.4) produces the following modifications of eqs. (14.12) — (14.14) leading to the geodetic precession (14.18):

Eq. (14.12) is unchanged to leading order in $1/c^2$ but the Christoffel symbols become

$$\Gamma^r_{tt} = \frac{M}{r^2}\left(1 - \frac{2\gamma M}{r}\right) , \quad \Gamma^r_{\theta\theta} = \Gamma^r_{\phi\phi} = -(r - 2\gamma M)$$

and are otherwise unmodified.

Eq. (14.13a) becomes

$$\frac{ds^r}{dt} - [r - (2\gamma + 1)M]\,\Omega s^\phi = 0$$

while eq. (14.13b) remains unchanged.

The frequency $\Omega'$ is

$$\Omega' = \left[1 - (2\gamma + 1)\,\frac{M}{R}\right]^{\frac{1}{2}} \approx 1 - \left(\gamma + \frac{1}{2}\right)\,\frac{M}{R}$$

from which (14.19) follows.

**14-8**. *Consider the thought experiment described in Section 14.5 concerning a gyro freely falling along the rotation axis of a slowly rotating body. Show from the gyroscope equation* (14.6) *that if the spatial part of the spin starts out in the $x - y$ plane it remains in the $x - y$ plane to leading $1/c^3$ order.*

**Solution:** Suppose at one time the spatial part spin is in the $x$-$y$ plane and orient the $x$, $y$ coordinates so it points along the $x$-axis. The gyro equations (14.6) and (14.27) for $u^\alpha$ give for the time rate of change of $s^z$.

$$\frac{ds^z}{d\tau} = -\Gamma^z_{tx} u^t s^x - \Gamma^z_{zx} u^z s^x - \Gamma^z_{tt} u^t s^t - \Gamma^z_{zt} u^t s^t$$

where $s^t$ is determined by (14.12). But, following the argument just below (14.25), to leading $1/c^3$ order, all the Christoffel symbols in the above expression vanish. That is because there are no terms $g_{z\alpha}$ in (14.25) and the metric coefficients in $(ds^2)_{\text{flat}}$ are all constant. Thus,

$$\frac{ds^z}{d\tau} = 0$$

so $s^z$ remains zero if it starts from zero.

**14-9**. *[C] General relativity predicts that, because the Sun is rotating, a light ray passing by will be deflected slightly by an amount additional to the*

*deflection of light in the Schwarzschild geometry considered in Section 9.4. Calculate the amount and direction of this deflection to lowest non-vanishing order in $1/c$ assuming that the orbit is in the equatorial plane perpendicular to the axis of rotation. Estimate the magnitude of this effect for the Sun. Is it an important correction to the results of the observations discussed in Section 10.3? [Hint: Before doing any algebra think about what terms in the metric will contribute to the final answer in leading order in $1/c$.]*

**Solution:** The key to this problem is to note that the leading order in $1/c$ answer will be proportional to the rotation term in the metric (14.22) and $GM/rc^2$ terms will not enter at all. The leading order deflection can therefore be calculated in the metric ($G = c = 1$)

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 - \frac{4J}{r}\, d\phi\, dt$$

assuming the orbit is in the equatorial plane $\theta = \pi/2$. The rest follows the calculation of the deflection of light in the Schwarzschild geometry (9.4). We quote just a few important steps:

The conserved quantities are

$$\begin{aligned} e = -\boldsymbol{\xi}\cdot\mathbf{u} &= g_{tt}u^t + g_{t\phi}u^\phi = \frac{dt}{d\lambda} - \frac{2J}{r}\frac{d\phi}{d\lambda}\,, \\ \ell = \boldsymbol{\eta}\cdot\mathbf{u} &= g_{\phi t}u^t + g_{\phi\phi}u^\phi = -\frac{2J}{r}\frac{dt}{d\lambda} + r^2\frac{d\phi}{d\lambda}\,. \end{aligned}$$

Solving then for $dt/d\lambda$ and $d\phi/d\lambda$ neglecting terms of order $J^2$ and above gives

$$\frac{dt}{d\lambda} \approx e - \frac{2J}{r^3}\,\ell\,, \qquad r^2\frac{d\phi}{d\lambda} = \ell + \frac{2Je}{r}\,.$$

The radial equation found from $\mathbf{u}\cdot\mathbf{u} = 0$ by eliminating $dt/d\lambda$ and $d\phi/d\lambda$ in favor of $\ell$ and $e$ is

$$\left(\frac{dr}{d\lambda}\right)^2 = e^2 - \frac{1}{r^2}\left(\ell^2 + \frac{4J}{r}\,\ell\, e\right)$$

to linear order in $J$. For the shape of the orbit $\phi(r)$

$$\frac{d\phi}{dr} = \frac{d\phi/d\lambda}{dr/d\lambda} = \frac{\ell + 2J\,e/r}{r^2}\left[e^2 - \frac{1}{r^2}\left(\ell^2 + \frac{4J\,\ell\, e}{r}\right)\right]^{-\frac{1}{2}}$$

The integral for the deflection can then be carried out like that deflection of light in Section 9.4. The result is

$$(\delta\phi)_{\text{rot}} = -\frac{8J}{b^2} = -(\delta\phi)_{\text{def}} \left(\frac{2J}{Mb}\right) .$$

where $(\delta\phi)_{\text{def}}$ is deflection of light (9.83) and $b$ is the impact parameter $\ell/e$. The total angle $\Delta\phi$ swept out in one passage is less than $\pi$. In order of magnitude the ratio is

$$\frac{J}{Mb} \sim \frac{MbV}{Mb} \sim V$$

where $V$ is the velocity at the solar surface in $c = 1$ units. Since $V \sim 2$ km/s, the effect is negligible compared to the bending in the spherical approximation.

**14-10**. *[B] The figure in Box 14.1 shows schematically the shift of the spin of a gyro due to the geodetic and frame dragging after one orbit around the rotating Earth. Explain the directions of the shifts of the gyro and calculate the magnitude of the two effects using* (14.34) *for the Lense-Thirring part of the precession.*

**Solution:** The geodetic precession is in the plane of the orbit in the direction of motion as shown in Figure 14.2. The angle of displacement is given in (14.18).

The Lense-Thirring precession $\vec{\Omega}_{\text{LT}}$ varies with the position of gyro along the orbit according to (14.34). Let $\theta$ be the polar angle from the Earth's rotation axis locating the gyro, and $\Omega_{\text{orbit}}$ the orbital angular velocity. According to Kepler's law $\Omega_{\text{orbit}} = (GM_\oplus/R^3)^{1/2}$ where $R$ is the radius of the orbit. Since the rate of precession depends on position, it is necessary to use a basis which doesn't change with the position of the gyro (as the coordinate basis for polar coordinates does). A Cartesian coordinate basis centered on the Earth with $z$ along its rotation axis and $x$ in the plane of the orbit is convenient. The shift in angle in the plane of the orbit is

$$(\Delta\phi_{\text{LT}})_{\text{in plane}} = \int_{\text{orbit}} \Omega^x_{\text{LT}} dt = \int \left(\frac{d\theta}{\Omega_{\text{orbit}}}\right) \Omega^x_{\text{LT}}(\theta).$$

Since $\Omega^x_{\text{LT}}(\theta) = (GJ/c^2r^3)\cos\theta$, this will cancel in the integral between the northern and southern hemispheres. *The displacement due to the Lense-Thirring precession is therefore in a plane perpendicular to the rotation axis.*

This result also is evident from the symmetry of the problem. The result for the angular shift $\Delta\phi_{\rm LT}$ in a plane perpendicular to the rotation axis is

$$\Delta\phi_{\rm LT} = \int_{\rm orbit} \Omega^z_{\rm LT} dt = \int_{\rm orbit} \left(\frac{d\theta}{\Omega_{\rm orbit}}\right) \Omega^z_{\rm LT}.$$

where

$$\Omega^z_{\rm LT} = \frac{GJ}{c^2 R^3} \left(3\cos^2\theta - 1\right).$$

Then

$$\Delta\phi_{\rm LT} = \frac{\pi GJ}{c^2 R^3} \frac{1}{\Omega_{\rm orbit}} .$$

This is in the direction of rotation as is shown in the figure in Box 14.1.

# Chapter 15

# Rotating Black Holes

**15-1**. *[E] Estimate the Kerr parameter $a$ for the Sun and the Earth. Are they bigger or smaller than their rest masses?*

**Solution:** The angular momentum of a rotating body of mass $M$ and radius $R$ is

$$J \sim MVR \sim MR^2/P$$

where $V$ is the surface velocity and $P$ the rotational period. It's convenient to use geometrized units where $M_\odot = 1.48$ km and $M_\oplus = .443$ cm. Periods have to be converted to units of length by multiplying with $c$. The result is

$$J_\odot \sim 1 \text{ km}^2 \ , \quad J_\oplus \sim 70 \text{ cm}^2$$

and for $a \equiv J/M$

$$a_\odot \sim .7 \text{ km} \quad a_\oplus \sim 200 \text{ cm} \ .$$

For the Earth $a \gg M$. If the Earth were to suddenly collapse it would have to get rid of a lot of angular momentum before forming a black hole. For the Sun the estimate is not accurate enought to be conslusive. If the actual angular momenta of the Sun, $J_\odot = 1.63 \times 10^48$ g $\cdot$ cm$^2$/s, is used, then $a_\odot = .3$ km which is smaller than $M_\odot$.

**15-2**. *[S] Reversing the direction of time reverses the angular momentum and the direction of rotation of a rotating body. Show that the action of $t \to -t$ on* (15.1) *is the same as sending $J \to -J$. What happens when $\phi \to -\phi$?*

**Solution:** The transformations $t \to -t$ and $J \to -J$ (or $a \to -a$) leave the Kerr metric unchanged. So do $\phi \to -\phi$, $J \to -J$, and $\phi \to -\phi$, $t \to -t$. These are all symmetries one would expect of a rotating body.

**15-3**. *[A] Show that the transformation*

$$dt = dv - \frac{r^2 + a^2}{\Delta}\, dr, \quad d\phi = d\psi - \frac{a}{\Delta}\, dr$$

*applied to the Kerr metric in Boyer-Lindquist coordinates leads to a coordinate system for the Kerr geometry which is non-singular at $r = r_\pm$.*

*Comment: These are the generalization of the Eddington-Finkelstein coordinates for spherical black holes discussed in Section 12.1 as can be seen by comparing the above transformation formulas with* (12.1) *when $a = 0$.*

**Solution:** This is an exercise in virtuous algebra. Your result at the end should be

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2Mr}{\rho^2}\right) dv^2 + 2dr dv + \rho^2 d\theta^2 \\ &+ \frac{1}{\rho^2}\left[\left(r^2 + a^2\right)^2 - \Delta\, a^2 \sin^2\theta\right] \sin^2\theta\, d\psi^2 \\ &- 2a \sin^2\theta\, d\psi dr - \frac{4Mra}{\rho^2} \sin^2\theta\, d\psi dv\ . \end{aligned}$$

Here $\rho^2 = r^2 + a^2 \cos^2\theta$. Note that this is not singular at $r_\pm$, but still singular where $\rho = 0$.

**15-4**. *Show that when starting at $r = r_+$, all future-directed light rays in the Kerr geometry move to smaller values of $r$. Use a non-singular coordinate system such as that given in Problem 3.*

**Solution:** The idea of this problem is to find the future light cone on the horizon $r = r_+$ and show that it points to the horizon's interior. This means

finding the null directions $\boldsymbol{\ell}$ on the horizon and showing for all future directions $\ell^r < 0$. We can do this in a non-singular coordinate system such as that worked out in the previous problem. Carry out the following steps:

a) Write out the condition $\boldsymbol{\ell} \cdot \boldsymbol{\ell} = 0$ in terms of the components $\ell^v, \ell^r, \ell^\theta, \ell^\psi$.

b) Evaluate this at $r = r_+$.

c) Solve for $\ell^r$ in terms of the other components.

After a little algebra the result is

$$\ell^r = -\frac{1}{2Q}\left\{\rho_+^2(\ell^\theta)^2 + \frac{\sin^2\theta}{\rho_+^2}\left[a\ell^v - (2Mr_+)\,\ell^\psi\right]^2\right\} \tag{1}$$

where

$$\rho_+^2 \equiv \rho(r_+,\theta) = r_+^2 + a^2\cos^2\theta$$

and

$$Q = \ell^v - a\sin^2\theta\,\ell^\psi\ .$$

The term in braces is a sum of squares and so is always positive, which means $\ell^r$ is positive if $Q$ is negative and vice versa. Note that $\ell^r = 0$ when $\ell^\theta = 0$ and $\ell^\psi = (a/2Mr_+)\ell^v$ as expected because this is the null direction lying in the horizon. If we change these values of $\ell^\theta$,$\ell^\psi$, $\ell^v$ by any small amount,

$$\begin{aligned}\ell^\theta &= 0 + \delta\\ \ell^\psi &= (a/2Mr_+)\ell^v \pm \epsilon\ ,\end{aligned}$$

then

$$Q = \ell^v\left(1 - \frac{a^2\sin^2\theta}{2Mr_+} \pm \epsilon\right)\ .$$

As $Mr_+ = r_+^2 + a^2 > a^2\sin^2\theta$, we see that for any sufficiently small $\epsilon$, $Q$ is positive for positive $\ell^v$. By definition future directed light rays have $\ell^v > 0$. Therefore $\ell^r < 0$ for all future-directed rays lying near the null direction in the horizon.

If (1) were inverted to solve for $a\ell^v - (2Mr_+)\ell^\psi$, there would be a $\pm$ ambiguity corresponding to the future and past light cone. If we interpret positive $\ell^v$ as the future direction at large $r$ for positive $\ell^\psi$ then the "+"

corresponds to future directed light rays and the "−" to past directed light rays. Future-directed light rays have

$$\ell^v > \left(\frac{a}{2Mr_+}\right)\ell^\psi$$

and this is enough to guarantee that $Q > 0$. Therefore $\ell^r < 0$ for future-directed null vectors.

**15-5.** *[A] The null directions on the horizon of a rotating black hole were identified in* (15.10). *But does a light ray that starts out in one of these directions remain on the horizon? Use the geodesic equation for light rays in the Kerr geometry to show that it does. Show also that the light ray remains at a fixed value of $\theta$.*

**Solution:** The null generators are set up on

$$\ell^\alpha = (1, 0, 0, \Omega_H) \tag{1}$$

and they obey the geodesic equation

$$\frac{d\ell^\alpha}{d\lambda} = -\Gamma^\alpha_{\beta\gamma}\ell^\beta\ell^\gamma \tag{2}$$

where $\lambda$ is an affine parameter along the geodesic. To show that the light rays remain stuck on the horizon we have to show that

$$\frac{d\ell^\theta}{d\lambda} = 0\,, \qquad \text{and} \qquad \frac{d\ell^r}{d\lambda} = 0\,, \tag{3}$$

when the right hand side of (2) is evaluated with (1). You can regard this as a virtuous exercise in algebra or note the following tricks. A little consideration on the form of the Christoffel symbols shows

$$\begin{aligned}\frac{d\ell^\theta}{d\lambda} &= \frac{1}{2}\,g^{\theta\theta}\left(\frac{\partial g_{tt}}{\partial\theta} + 2\Omega_H\frac{\partial g_{t\phi}}{\partial\theta} + \Omega_H^2\frac{\partial g_{\phi\phi}}{\partial\theta}\right)\\ &= \frac{1}{2}\,g^{\theta\theta}\frac{\partial}{\partial\theta}(\boldsymbol{\ell}\cdot\boldsymbol{\ell}) = 0\,.\end{aligned}$$

The same works even more easily for $d\ell^r/d\lambda$ because $g^{rr}$ vanishes on the horizon.

**15-6**. *Show explicitly that the two vectors* $(0,0,1,0)$ *and* $(0,0,0,1)$ *on the horizon* $r = r_+$ *are (a) spacelike and (b) orthogonal to each other and to the null generator* $\boldsymbol{\ell}$ *[(15.10)].*

**Solution:** Let $v^\alpha = (0,0,1,0)$ and $w^\alpha = (0,0,0,1)$. The problem asks for a demonstration that $\mathbf{v}\cdot\mathbf{v} > 0$, $\mathbf{w}\cdot\mathbf{w} > 0$, $\mathbf{v}\cdot\mathbf{w} = 0$, $\mathbf{v}\cdot\boldsymbol{\ell} = 0$, and $\mathbf{w}\cdot\boldsymbol{\ell} = 0$ when $r = r_\pm$. Evidently

$$\begin{aligned}
\mathbf{v}\cdot\mathbf{v} &= g_{\theta\theta} > 0, \\
\mathbf{w}\cdot\mathbf{w} &= g_{\phi\phi} > 0, \\
\mathbf{v}\cdot\mathbf{w} &= g_{\theta\phi} = 0, \\
\mathbf{v}\cdot\boldsymbol{\ell} &= g_{\theta v} + \Omega_H\, g_{\theta\phi} = 0 \\
\mathbf{w}\cdot\boldsymbol{\ell} &= g_{\phi v} + \Omega_H\, g_{\phi\phi} = 0.
\end{aligned}$$

The last requires a little algebra in which $r_+^2 - 2Mr_+ + a^2 = 0$ is important.

**15-7**. *Show that the distance around the poles in the horizon geometry* (15.12) *is always less than the distance around the equator.*

**Solution:** The distance around the equator $\theta = \pi/2$ is

$$d_{\text{equator}} = \int_0^{2\pi} d\phi \left(\frac{2Mr_+}{\rho_+}\right)_{\theta=\pi/2} = 2\pi\left(\frac{2Mr_+}{\sqrt{r_+^2 + a^2}}\right) = 2\pi(2Mr_+)^{\frac{1}{2}}.$$

The distance around the poles is

$$d_{\text{poles}} = 2\int_0^{\pi} d\theta\, \rho_+ = 2\int_0^{\pi} d\theta\, \sqrt{r_+^2 + a^2\cos^2\theta} \le 2\int_0^{\pi} d\theta\, \sqrt{r_+^2 + a^2} = 2\pi(2Mr_+)^{\frac{1}{2}}.$$

Here, $r_+^2 + a^2 = 2Mr_+$ has been used.

**15-8**. *[N] Construct the embedding diagram for a* $t =$ *const. slice of the horizon of a Kerr black hole for values of* $a/M$ *equal to* 0, .5 *and* .86. *The intrinsic*

*geometry is given by* (15.12). *Figure 15.2 shows the result for* $a/M = .86$. *Does your construction explain why there is a maximum value of* $a/M$ *for which such an embedding is possible?*

**Solution:** The metric on the surface is given by (15.12). We follow the procedure for constructing an embedding diagram described in Example 7.7. Use cylindrical flat space coordinates $(z, R, \psi)$. (We use $R$ here rather than the $\rho$ of (7.42) to avoid confusion with the $\rho(r,\theta)$ in the Kerr metric.) The embedding can be achieved with the identifications $\psi = \phi$, and

$$R(\theta) = \frac{2Mr_+}{\rho_+(\theta)} \sin\theta \ , \quad \frac{dz}{d\theta} = \left[\rho_+^2(\theta) - \left(\frac{dR}{d\theta}\right)^2\right]^{1/2} .$$

The resulting line element (7.45) then agrees with (15.12).

The result of numerically integrating the above to find $z(\theta)$ is the following three embedding diagrams for $a/M = 0$, .5, and .86. At $a/M = .86$ the surface becomes flat at the poles, and beyond that $a$ it becomes negatively curved, no longer embeddable in flat space.

**15-9**. *Show that the surface* $r = r_-$ *is another stationary axisymmetric null surface inside the Kerr black hole.*

**Solution:** The demonstration that $r = r_-$ is a stationary null surface follows that for $r = r_+$ given in the text following (15.7). Indeed all the equations (15.7) to (15.11) hold just by changing + into −.

**15-10**. *(Surrounded by a Horizon!) Consider the metric*

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right) dt^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) .$$

*This metric is not asymptotically flat, but imagine that we were living at the center near $r = 0$. Show that were we to cross the radius $r = R$ we could never return.*

**Solution:** The surface $r = R$ is generated by light rays whose tangent vectors $\mathbf{u}$ point along the Killing vector $\boldsymbol{\xi} = (1, 0, 0, 0)$. To show this note that the geodesic equation for radial motion of a light ray which initially starts along $\boldsymbol{\xi}$ is (*cf.* (8.42)):

$$\frac{du^r}{d\lambda} = -\Gamma^r_{\alpha\beta} u^\alpha \xi^\beta = -\Gamma^r_{tt} u^t = \frac{1}{2} g^{rr} \frac{\partial g_{tt}}{\partial r} = \frac{r}{R^2}\left(1 - \frac{r^2}{R^2}\right) u^t .$$

This vanishes at $r = R$ showing that if $u^r$ is initially zero it remains zero, and light rays that start along $\boldsymbol{\xi}$ remain along $\boldsymbol{\xi}$. The three-surface $r = R$ is thus a null surface which has the "one-way" property that once crossed it is not possible to return. (See the discussion in Section 7.9.)

**15-11**. *Show that in the geometry of an extremal Kerr black hole of mass $M$ there are circular light ray orbits in the equatorial plane at Boyer-Lindquist radii $r = M$ rotating with the black hole (corotating) and $r = 4M$ in the opposite direction (counter-rotating).*

**Solution:** The conditions for a circular orbit are, from (15.21) and (15.22)

$$\frac{1}{b^2} = W_{\text{eff}}(r, b, \sigma) \quad , \quad \frac{\partial W_{\text{eff}}}{\partial r} = 0.$$

These give two algebraic equations to solve for the allowed values of $b$ and $r$. Since you are given the radii, use one of the equations to solve for $b$, and then check that the other is satisfied. The values of $b$ are $b = 2M$ (co-rotating) and $b = 7M$ (counter rotating).

**15-12**. *The angular velocity $\Omega = d\phi/dt$ of circular orbits of Boyer-Lindquist radius $r$ in the Kerr geometry is given by the simple formula:*

$$\Omega = \pm \frac{M^{1/2}}{r^{3/2} \pm aM^{1/2}} .$$

*Here the upper sign refers to corotating orbits and the lower one to counter rotating orbits. Explain how to derive this formula, and exhibit the algebraic equations from which it follows. However, don't try and solve the equations unless you really like algebra!*

**Solution:** The conditions for a circular orbit are (15.24a) and (15.24b). These two relations can be solved for $e$ and $\ell$ in terms of $r$ and $a$. The angular velocity for an orbit with these parameters, $\Omega = d\phi/dt = u^\phi/u^t$, is determined from (15.17). Specifically,

$$-\frac{e}{\ell} = \frac{g_{tt} + \Omega\, g_{t\phi}}{g_{\phi t} + \Omega\, g_{\phi\phi}}.$$

If you did decide to do the algebra, the following solutions for $e$ and $\ell$ may help:

$$\begin{aligned} e &= \frac{r^2 - 2Mr \pm a\sqrt{Mr}}{r\left(r^2 - 3Mr \pm 2a\sqrt{Mr}\right)^{1/2}} \\ \ell &= \pm\frac{\sqrt{Mr}\left(r^2 \mp 2a\sqrt{Mr} + a^2\right)}{r\left(r^2 - 3Mr \pm 2a\sqrt{Mr}\right)^{1/2}} \end{aligned}$$

where the top sign corresponds to corotating orbits and the bottom one to counter rotating orbits.

**15-13**. *[S] Just because the Boyer-Lindquist radii of the co-rotating innermost stable circular orbits in the Kerr geometry are less than the corresponding radius $r = 6M$ in the Schwarzschild geometry [cf. Figure 15.3] doesn't mean that that those orbits are closer to the black hole. After all, these are just coordinate radii in different geometries. The circumference is one invariant*

*measure of the size of the orbit. Use figure Figure 15.3 to plot the circumference of the innermost stable co-rotating orbit in the Kerr geometry for the values* 0, .2, .4, .6, .8 *and* 1 *of* $a/M$. *Is the circumference of an innermost stable co-rotating circular orbit in the Kerr geometry always bigger or smaller than the innermost stable circular orbit in the Schwarzschild geometry? Can you explain what happens when* $a/M = 1$*?*

**Solution:** The circumference of an equatorial, co-rotating, circular orbit in the Kerr geometry is given by

$$C = \int_0^{2\pi} \sqrt{g_{\phi\phi}}\, d\phi = 2\pi r \left[1 + \frac{a^2}{r^2}\left(1 + \frac{2M}{r}\right)\right]^{\frac{1}{2}} \tag{1}$$

where $r$ is the radius of the orbit. The figure below shows a plot of $C/2\pi$ vs. $a/M$ where the relation for $r$ vs. $a$ from Figure 15.3 has been substituted into (1). In all cases $C/2\pi < 6M$. $C$ vanishes at $a = M$ because the orbit is becoming null.

**15-14.** *Work out the range of angular velocities* $\Omega_{\text{obs}}$ *allowed an observer inside the ergosphere who remains at a fixed value of* $r$. *Show that this range becomes increasingly limited as the observer is located closer and closer to the horizon and is eventually limited to the single value* $\Omega_H$.

**Solution:** The four-velocity of an observer rotating with angular velocity $\Omega$ at a fixed radius $r$ has the form

$$u^\alpha_{\text{obs}} = u^t_{\text{obs}}(1, 0, 0, \Omega).$$

The condition that this be timelike, $\mathbf{u} \cdot \mathbf{u} = -1$, reads

$$\mathbf{u} \cdot \mathbf{u} = \left(u^t_{\text{obs}}\right)^2 \left(g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi}\right) = -1.$$

The combination in brackets must be negative which means

$$\Omega_- < \Omega < \Omega_+$$

where

$$\Omega_\pm = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \left[\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \left(\frac{g_{tt}}{g_{\phi\phi}}\right)\right]^{\frac{1}{2}}$$

As $r \to r_+$,

$$g_{tt} \to \frac{a^2 \sin^2\theta}{\rho_+^2} \quad , \quad g_{\phi\phi} \to \frac{(2Mr_+)^2}{\rho_+^2} \sin^2\theta.$$

as a consequence of $\Delta = r_+^2 - 2Mr_+ + a^2 = 0$. As a result the term in brackets vanishes, and

$$\Omega_+ = \Omega_- = -\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)_{r=r_+} = \Omega_H.$$

**15-15.** *[C] Temperature of a Rotating Black Hole. (a) An axisymmetric body is spinning about its symmetry axis with angular velocity $\Omega$ and angular momentum $J$ along the axis. Show that, in Newtonian mechanics, the work required to increase the angular momentum by a small amount $\Delta J$ is $\Omega \Delta J$.*

*(b) Reorganize* (15.36) *for the change in area $A$ of a rotating black hole given changes in its mass and angular momentum into a form like the first law of thermodynamics assuming that the entropy of the black hole is $k_B A/4\hbar$ like as in the Schwarzschild case [cf.* (13.22)*]. Find the Hawking temperature of a Kerr black hole.*

*(c) Show that the temperature of an extreme ($a = M$) black hole is zero and explain this fact from properties of the Kerr geometry.*

**Solution:**

a) The energy $E$ of an axisymmetric body rotating about its symmetry axis is

$$E = \frac{J^2}{2I}$$

where $J = I\Omega$ is the angular momentum and $I$ is the moment of inertia about the axis. Evidently for a small change in angular momentum $\Delta J$

$$\Delta E = \left(\frac{J}{I}\right) \Delta J = \Omega \Delta J.$$

b) Eq. (15.36) can be rewritten

$$\Delta M = T\Delta S + \Omega \Delta J$$

with

$$S = \frac{k_B A}{4\hbar} \quad , \quad T = \frac{\kappa\hbar}{2\pi k_B}.$$

This is the form of the first law of thermodynamics with $S$ being the entropy and $T$ the temperature of the black hole.

c) Eq. (15.37) shows that the temperature $T$ vanishes when $a = M$. As discussed in Section 13.3, a black hole radiates because of the behavior of a Killing vector $\boldsymbol{\zeta}$ which defines a conserved quantity $\boldsymbol{\zeta} \cdot \mathbf{u}$. When $\boldsymbol{\zeta}$ is timelike $\boldsymbol{\zeta} \cdot \mathbf{u}$ is an energy and must be positive. However, if $\boldsymbol{\zeta}$ is timelike outside the horizon but spacelike inside, a pair can be created respecting conservation with a positive value of $\boldsymbol{\zeta} \cdot \mathbf{u}$ outside and a negative value inside. In the case of the Schwarzschild geometry, the relevant Killing vector was $\boldsymbol{\xi}$ arising from invariance under $t$ translations. In the case of rotating black holes it is

$$\boldsymbol{\zeta} = \boldsymbol{\xi} + \Omega \boldsymbol{\eta}$$

because that is the form of the timelike four-velocity of fixed $r$ observers inside the ergosphere [*cf.* (15.29)]. In particular, for $r$ close to the horizon we should take $\Omega = \Omega_H$ since the world lines of observers approach the generators of the horizon (Problem 14). Now,

$$\boldsymbol{\zeta} \cdot \boldsymbol{\zeta} = g_{tt} + 2\Omega_H g_{t\phi} + g_{\phi\phi}.$$

On the horizon $\boldsymbol{\zeta} = \boldsymbol{\ell}$ so $\boldsymbol{\zeta} \cdot \boldsymbol{\zeta} = 0$. Writing $r = M(1 + W)$ and expanding in $W$ gives

$$\boldsymbol{\zeta} \cdot \boldsymbol{\zeta} = -W^2 \frac{[3 + 28\cos(2\theta) + \cos(4\theta)]}{16[3 + \cos(2\theta)]} + \dots .$$

Therefore, on both sides of the horizon $\boldsymbol{\zeta}$ is timelike. Pairs could not be produced without violating energy conservation. That is why the temperature is zero.

**15-16**. *[B,E] An active galactic nucleus with a luminosity of* $10^{46}$ ergs *is powered by the rotational energy of an extreme rotating black hole as described in Box 15.1. Estimate how long the active galactic nucleus can radiate in this way? Compare your answer to the present age of the universe, approximately 15 billion years.*

**Solution:** The area $A$ of an extreme ($a = M$) Kerr black hole and its irreducible mass are from (15.13) and (15.39)

$$A = 8\pi\, M^2 = 16\pi\, M_{\rm irr}^2 \ .$$

Thus, $M_{\rm irr} = M/\sqrt{2}$. The maximum rotational energy which can be extracted is therefore $M_{\rm rot} = M(1 - 1/\sqrt{2}) \approx .293M$. Assuming the black hole has a mass of $3 \times 10^9 M_\odot$, this is

$$M_{\rm rot} \approx 1.6 \times 10^{63}{\rm erg} \ .$$

If this is radiated away at a rate of $10^{46}$ erg/s, the radio source would last $\sim 10^{17}$ s or $\sim 3$ Gyr.

**15-17**. *[B,P,S] Show that* (b) *in Box 15.1 gives the voltage developed across any axi-symmetric conductor rotating around its symmetry axis.*

**Solution:** One way to think about this problem is to note that any axisymmetric shape can be approximated by concentric cylinders as shown below.

A path connecting the two contacts can be made up of "horizontal" segments along the ends of the cylinders and "vertical" segments along their length. The horizontal segments contribute the as in (b). The vertical ones contribute nothing.

**15-18**. *[B,E,P] Consider a rotating black hole with* $M \sim 10^9 M_\odot$, $\Omega_H M \sim 1$ *immersed in a magnetic field* $B \sim 10^4$ gauss *as described in Box 15.1. Estimate how far an electron in the vicinity of the black hole has to move in the electric field there before it acquires enough energy to make a further electron-positron pair in a collision with a similar electron or positron.*

**Solution:** From (e) in Box 15.1, the voltage drop across the characteristic distance scale $M \sim 10^9$ km is $10^{20}$ V. The electric field is therefore of order $\mathcal{E} = 10^8$ V/m. To make an electron position pair in a collision with a similar electron a kinetic energy of order the rest energy is required ($\sim .5$ MeV). An electron has to move a distance $10^{-2}$ m $\sim 1$ cm to acquire such an energy. (One electron volt is the energy an electron acquires in a voltage change of 1 V).

# Chapter 16

# Gravitational Waves

**16-1**. *Show that the gravitational wave spacetime* (16.2) *has three Killing vectors:* $(0,1,0,0)$, $(0,0,1,0)$, *and* $(1,0,0,1)$.

**Solution:** The metric (16.2) is a function only of $z-t$. It is therefore unchanged by displacements in $x$, displacements in $y$, and simultaneous displacements by the same amount in $z$ and $t$. These correspond to the three Killing vectors given.

**16-2**. *Consider a Gaussian wave packet with* $f(t-z) = a\exp[-(t-z)^2/\sigma^2]$.

*(a) Draw a spacetime diagram showing a* $z-t$ *slice of spacetime with* $x=y=0$. *Shade the region where the wave packet has a size greater that* $a/2$. *Show the world line of the test mass at the origin.*

*(b) Draw a graph of the distance between the two test masses initially at rest in the given frame, at the origin and at a distance* $L_*$ *along the* $x-$*axis. What is the maximum value of the change in that distance?*

**Solution:**

a) See below, $z_1 = \sigma[\log(a/2)]^{1/2}$.

b) See Figure 16.1; the maximum separation is $a/2$.

**16-3**. *Consider the gravitational wave in* (16.2) *and two test masses, one a the origin and the other at a location* $(X, Y, Z)$ *in the Cartesian coordinates used in* (16.2). *Show that the change in distance between the masses produced by the wave is given by*

$$\delta L(t) = \frac{1}{2} \int_0^{L_*} d\lambda \; h_{ij}\,(t - n^z \lambda)\, n^i n^j \;,$$

*Here* $n^i = (X/L_*, Y/L_*, Z/L_*)$ *is the unit tangent vector to the straight line path between the test masses and* $L_*$ *is the unperturbed distance between them.*

**Solution:** Denote by $\gamma$ the "straight line" path that connects $(0,0,0)$ with $(X, Y, Z)$ in the flat spacetime unperturbed by the gravitational wave. Its length $L_*$ is $L_* = (X^2 + Y^2 + Z^2)^{1/2}$ and $\vec{n} = (X/L_*, Y/L_*, Z/L_*)$ the unit tangent vector along it. A parametric equation for the curve $\gamma$ is then

$$x^i_{(\gamma)}(\lambda) = n^i \lambda$$

where $\lambda$ varies from 0 to $L_*$. (There are three different paths at play in this discussion: the paths $x^\alpha_{(A)}(\tau)$, $x^\alpha_{(B)}(\tau)$ of the two test masses, and the spatial path $x^i_{(\gamma)}(\lambda)$ connecting them.) The spatial distance along the curve $\gamma$ is

$$\begin{aligned} S_\gamma &= \int dS = \int_\gamma \left[ (\delta_{ij} + h_{ij}(t - z(\lambda)))\; dx^i dx^j \right]^{\frac{1}{2}} \;, \\ &= \int_0^{L_*} d\lambda \left\{ [\delta_{ij} + h_{ij}(t - n^z \lambda)] \; \frac{dx^i_{(\gamma)}}{d\lambda} \; \frac{dx^j_{(\gamma)}}{d\lambda} \right\}^{\frac{1}{2}} \end{aligned}$$

This expression can be expanded in powers of the amplitude of the gravitational wave. The lowest term with no perturbation reproduces the distance $L_*$. The first order *change* in that distance along the curve $\gamma$.

$$\delta L(t) = \frac{1}{2} \int_0^{L_*} d\lambda \; h_{ij}(t - n^z \lambda) n^i n^j \; .$$

**16-4**. *[C] Calculate the displacement $\delta x^i_{(\ell)}(\lambda)$ in the path of a light ray between two test masses from that of a flat space straight line. Assume a gravitational wave of the form* (16.2) *having a definite frequency $\omega$.*

**Solution:** Consider, for simplicity, the case when initially one test mass is at the origin and the other is located a coordinate distance $L$ along the $x$-axis. (This is the situation described in example 1 with $X = L$, $Y = Z = 0$.) These coordinate positions do not change as the wave passes according to (16.8).

Suppose the light ray starts at the origin at $t = 0$. Its path in spacetime is a null geodesic that intersects the world line of the second test mass at $x = L$ some time later. In the absence of the wave, this null geodesic is the world line $t = x$, $0 \leq x \leq L$. The coordinate $x$ is an affine parameter, so the unperturbed value of $dx^\alpha/d\lambda$ is

$$dx^\alpha/d\lambda = (1, 1, 0, 0) \tag{1}$$

with $\lambda = x$.

A null geodesic generally solves the geodesic equation (8.42)

$$\frac{d^2 x^\alpha}{d\lambda^2} = -\Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} \; . \tag{2}$$

The passing gravitational wave will cause a deviation $\delta x^\alpha(\lambda)$ in the world line of the light ray from its unperturbed value (1). Evaluating (2) to first order in the amplitude of the wave using the unperturbed (1) gives

$$\frac{d^2 \delta x^i}{dx^2} = -\left(\delta\Gamma^i_{tt} + 2\delta\Gamma^i_{xt} + \delta\Gamma^i_{xx}\right) \tag{3}$$

where $x$ is the affine parameter and $\delta\Gamma^\alpha_{\beta\gamma}$ is first order in $h_{\alpha\beta}$. Evaluating the changes in the Christoffel symbols to first order in $h_{\alpha\beta}$, one finds

$$\delta\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} n^{\alpha\rho} \left(\partial_\beta h_{\rho\gamma} + \partial_\gamma h_{\rho\beta} - \partial_\rho h_{\beta\gamma}\right) \; .$$

This gives

$$\frac{d^2\delta x}{d\tau^2} = 0 \tag{4}$$

$$\frac{d^2\delta y}{d\tau^2} = 0 \tag{5}$$

$$\frac{d^2\delta z}{d\tau^2} = \frac{1}{2}\frac{\partial h_{xx}}{\partial z}\bigg|_{t=x} = \frac{a\omega}{2}\cos(\omega x) \tag{6}$$

for $f(z) = a\sin[\omega(t-z)]$. The boundary conditions for solving these equations are

$$\delta\vec{x}(0) = 0\ ,\ \delta\vec{x}(L) = 0 \tag{7}$$

so the light ray starts on one test mass and ends on the other. Evidently $\delta x(x) = \delta y(x) = 0$. The solution to (6) with the boundary condition (7) is

$$\delta z(x) = \frac{a}{2\omega}\left\{1 - \cos(\omega x) - \frac{x}{L}[1 - \cos(\omega L)]\right\}\ .$$

**16-5**. *An observer is riding on one of the test particles discussed in Section 16.2 holding a cup of coffee filled to the brim. The size of the coffee cup is much less that the wavelength of the gravitational wave. Is there any danger that the coffee will spill because of the passage of the gravitational wave? If so, estimate how close to the top the observer can fill the cup and not have it spill.*

**Solution:** If the cup is sufficiently small, the equivalence principle can be invoked. The behavior of the cup and coffee will be the same as in an inertial frame in flat space and the coffee will not slosh.

A more quantitative estimate can be obtained from the equation of geodesic deviation (21.19). The right hand side involving the Riemann curvature competes with the pressure forces in the fluid to determine its motion. For a gravitational wave of amplitude $a$ and wavelength $\lambda$, typical curvature components are of order $a/\lambda^2$. The effective force acceleration between particles on opposite sides of the cup size $L$ is therefore $(a/\lambda)(L/\lambda)$. The expected displacement of any sloshing is therefore $\delta x \sim aL$ which will be very small, $a \sim 10^{-21}$, $L \sim$ cm.

**16-6**. *The equation for an ellipse is $x^2/a^2 + y^2/b^2 = 1$ where $a$ is the semi-major axis and $b$ the semi-minor axis if $a > b$. Show that an initial circle of test particles distorts into an ellipse according to* (16.13) *to lowest order in $a$ and compute the semi-major and semi-minor axes as a function of time.*

**Solution:** Two ways to arrive at the answer are presented.

*Version one:* Suppose the axes are just slightly different

$$a = L(1+\epsilon) \ , \quad b = L(1-\epsilon) \ . \tag{1}$$

Then to lowest order in $\epsilon$ the ellipse equation is

$$L^{-2}\left[x^2 + y^2 - 2\epsilon\left(x^2 - y^2\right)\right] = 1 \ .$$

In polar coordinates this is

$$L^{-2}\left[r^2 - 2\epsilon(\cos 2\phi)\right] = 1 \ .$$

The solution is of the form $r = L + \delta L$ with

$$\frac{\delta L}{L} = \epsilon \ \cos 2\phi \ .$$

Thus $\epsilon = \frac{1}{2}\ a \sin \omega t$, and the axes are given by (1).

*Version two:* From (16.13)

$$\begin{aligned} x &= X\left(1 + \frac{1}{2}a\sin\omega t\right)^{-1} \\ y &= Y\left(1 - \frac{1}{2}a\sin\omega t\right)^{-1} \ . \end{aligned}$$

The coordinate positions form a circle of radius $L$ and remain unchanged by the wave, so $x^2 + y^2 = L^2$. In terms of the proper distances this is

$$\frac{X^2}{\left(1 + \frac{1}{2}a\sin\omega t\right)^2} + \frac{Y^2}{\left(1 - \frac{1}{2}a\sin\omega t\right)^2} = L^2 \ ,$$

which makes the semi-major axes $L\left(1+\frac{1}{2}a\sin\omega t\right)$, and the semi-minor axes $L\left(1-\frac{1}{2}a\sin\omega t\right)$.

**16-7**. *In Section 16.3 we produced a gravitational wave with $\times$ polarization by rotating the $+$ polarization (16.2) by 45°. Show that a rotation by an arbitrary angle $\theta$ doesn't give another independent solution but rather one that could be written as a superposition of $+$ and $\times$. This is one way of seeing that there are only two linearly independent polarizations of a gravitational wave.*

**Solution:** Consider just the $2\times 2$, $x$-$y$ submatrix of (16.2a):

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Utilizing (3.9), a rotation by an angle $\theta$ would produce

$$\begin{pmatrix} \cos 2\theta & 2\sin\theta\,\cos\theta \\ 2\sin\theta\,\cos\theta & -\cos 2\theta \end{pmatrix}$$

which can be written as a superposition of $+$ and $\times$.

$$\cos 2\theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2\sin\,\cos\theta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

**16-8**. *[P]*

a) *In a linearly polarized electromagnetic wave the electric field oscillates along one fixed direction in space. What pattern of motion is produced in a ring of test charges like those in Figure 16.2 by a electromagnetic wave propagating in the $z$-direction that is polarized in the $x$-direction and normally incident on the plane of the ring. (Neglect the magnetic forces on the charges.)*

b) *Is there a combination of the two gravitational wave polarizations that would produce the same motion?*

**Solution:**

a) The circular pattern maintains its circular shape but oscillates on the $x$-axis.

b) The above motion would be reproduced if there were a gravity wave which had the form

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

in the $x - y$ submatrix. But no combination of $+$ and $\times$ polarization will do this.

**16-9**. *[C] Circularly Polarized Gravitational Waves. If a linearly polarized electromagnetic wave with a given frequency is added to a wave of the same amplitude and frequency propagating in the same direction but polarized in a perpendicular direction and 90° out of phase, the result is a circularly polarized wave in which the tip of the electric field vector moves in a circle at any one position in space. By analogy, the superposition of a + polarized gravitational plane wave with another of the same amplitude, frequency and propagation direction but with $\times$ polarization and 90° out of phase is called a "circularly polarized" gravitational wave. Show that a circularly polarized plane gravitational wave with frequency $\omega$ that is normally incident on an ellipse of test particles causes each test particle to rotate in a small circle such that the elliptical pattern rotates with a constant angular frequency. What is that angular frequency?*

**Solution:** The gravitational wave amplitudes $h_{AB}$ in the two transverse directions at one point in space have the time dependence

$$h_{AB} = a \begin{pmatrix} C & 0 \\ 0 & -C \end{pmatrix} + a \begin{pmatrix} 0 & S \\ S & 0 \end{pmatrix} = a \begin{pmatrix} C & S \\ S & -C \end{pmatrix} \tag{1}$$

where $C \equiv \cos(\omega t)$ and $S \equiv \sin(\omega t)$ with a simple choice of overall phase.

To see that each particle moves on a circle follow the strategy that led to (16.13). Introduce new coordinates $(X, Y)$ in the $x - y$ plane defined by ($A, B$ range over 1, 2):

$$X^A = (\delta^A_B + \frac{1}{2}\delta^{AC} h_{CB}(0,t))x^B$$

There are no metric perturbations *to first order in* $h_{AB}$ in the $(X, Y)$ coordinates. Distances can therefore be computed to this order in the usual Euclidean way.

Consider a particle located $(L, 0)$ in $x$ and $y$. These values of $x$ and $y$ don't change with time, but the values of $X$ and $Y$ do. In particular we have, from the form of $h_{AB}$ above,

$$X = (1 + aC/2)L \ , \qquad Y = aS/2)L.$$

Evidently

$$\left(\frac{X - L}{2aL}\right)^2 + \left(\frac{Y}{2aL}\right)^2 = 1$$

so the particle moves on a circle of radius $2aL$. The discussion of test particles at other positions is similar.

We seek to show that this motion in circles is a fixed pattern of distortion rotating with some angular frequency $\Omega$. A fixed elliptical distortion would correspond to a *time-independent* distortion. By an appropriate rotation of the $x - y$ coordinates this can be chosen to be the $+$ polarization. The question then is whether the matrix $h_{AB}$ of wave amplitudes can be written

$$h_{AB} = \begin{pmatrix} C & -S \\ S & C \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} C & S \\ -S & C \end{pmatrix}$$

with $C = \cos(\Omega t)$ and $S = \sin(\Omega t)$. This corresponds to a steadily rotating pattern. Multiplying the above matrix product out shows it coincides with (1) provided $\Omega = \omega/2$.

**16-10**. *(Interference Pattern.) Suppose at the detector $D$ in Figure 16.3 the electric field of the two light waves that have traveled along the different arms of the interferometer have the forms $a \sin[\omega(t - L_{(x)})]$ and $a \sin[\omega(t - L_{(y)})]$ respectively. Show that if these are combined (added) the intensity of the resulting wave (proportional to the square of the amplitude) has the form of the interference pattern discussed in Figure 16.4.*

**Solution:** The intensity of the field which is the sum of the two amplitudes is proportional to

$$|a \ \sin\omega\,(t - L_x) + a \sin\omega\,(t - L_y)|^2 \ .$$

Using the standard trigonometric identity for $\sin x + \sin y$, this is

$$|a|^2 \left| 2\sin\left[\omega\left(t - \frac{L_x + L_y}{2}\right)\right] \cos\left[\omega\left(\frac{L_x - L_y}{2}\right)\right]\right|^2$$

The time average of the first sine over a period is $1/2$, giving the result for the intensity

$$2|a|^2 \cos^2\left[\omega\left(\frac{L_x - L_y}{2}\right)\right] .$$

This is precisely the interference pattern illustrated in Figure 16.4.

**16-11**. *[S] Energy Density in a Newtonian Gravitational Field. Consider assembling a system of $N$ particles of mass $M_A$ at assigned positions $\vec{x}_A$, $A = 1, \cdots, N$. The potential energy of the system $W$ is the total potential energy of of all the particles found by bringing them one by one from infinity in potential of the particles already assembled. Show that this is:*

$$W = -\frac{1}{2} \sum_{A \neq B} \frac{G M_A M_B}{|\vec{x}_A - \vec{x}_B|}$$

*and that corresponding formula for a continuum distribution of mass with density $\mu(\vec{x})$ is*

$$W = -\frac{1}{2} \int d^3x \int d^3x' \, \frac{G\mu(\vec{x})\mu(\vec{x}')}{|\vec{x} - \vec{x}'|} = \frac{1}{2} \int d^3x \; \mu(\vec{x})\Phi(\vec{x})$$

*Use the Newtonian field equation (3.18) to eliminate $\mu(\vec{x})$ from this expression and then the divergence theorem to write this as*

$$W = -\frac{1}{8\pi G} \int d^3x \; [\vec{\nabla}\Phi(\vec{x})]^2 = -\frac{1}{8\pi G} \int d^3x \; [\vec{g}(\vec{x})]^2 \equiv \int d^3x \; \epsilon_{Newt}(\vec{x})$$

*where $\epsilon_{Newt}(\vec{x})$ is the energy density of a Newtonian gravitational field.*

**Solution:** Suppose particles $A = 1, \cdots, B-1$ are already in place. The energy $W_B$ required to bring particle $B$ in from infinity is $M_B$ times the gravitational potential of particles $1, \cdots, B-1$ that are already there. This is

$$W_B = -M_B \sum_{A=1}^{B-1} \frac{G M_A}{|\vec{x}_B - \vec{x}_A|}. \tag{1}$$

(Since $W_B$ is negative, it is really the energy *released* in bringing $B$ in, not an energy *expended.*) Summing (1) over all particles gives

$$W = \sum_{B=1}^{N} W_B = -\sum_{A<B}^{N} \frac{GM_A M_B}{|\vec{x}_A - \vec{x}_B|}. \tag{2}$$

However, since the summand is symmetric in $A$ and $B$, this can be written as half the unrestricted sum

$$W = -\frac{1}{2} \sum_{A \neq B}^{N} \frac{GM_A M_B}{|\vec{x}_A - \vec{x}_B|}. \tag{3}$$

The analogous continuum expression is

$$W = -\frac{1}{2} \int d^3x \int d^3x' \, \frac{G\mu(\vec{x})\mu(\vec{x}')}{|\vec{x} - \vec{x}'|} = \frac{1}{2} \int d^3x \mu(\vec{x}) \, \Phi(\vec{x}). \tag{4}$$

The last equality follows from (3.15) for the gravitational potential

Using the Newtonian field equation (3.18) to eliminate $\mu(\vec{x})$ from (4) gives

$$\begin{aligned} W &= \frac{1}{8\pi G} \int d^3x \Phi \nabla^2 \Phi \\ &= \frac{1}{8\pi G} \int d^3x \left[ \vec{\nabla} \cdot \left( \Phi \vec{\nabla} \Phi \right) - \left( \vec{\nabla} \Phi \right)^2 \right]. \end{aligned} \tag{5}$$

Suppose for a moment that the integral is over a sphere of very large radius $R$. The divergence theorem can be used to convert the first term into a surface integral over the sphere

$$\frac{1}{8\pi G} \int_R d\vec{A} \cdot (\Phi \vec{\nabla} \Phi).$$

Since $\Phi \sim M_{\text{tot}}/R$ at large $R$, the integrand decreases like $1/R^3$. The area increases like $R^2$, so the net behavior is $\sim 1/R$. The surface term therefore vanishes as $R$ approaches infinity, leaving

$$W = -\frac{1}{8\pi G} \int d^3x (\vec{\nabla} \Phi)^2.$$

**16-12.** *Show that for a wave traveling at the speed of light the flux of energy*

*across a surface is the momentum density multiplied by c. Show that the magnitude of the momentum density is the energy density divided by $c^2$.*

**Solution:** Suppose the surface has area $A$ denote the flux of energy by $f$ and the momentum density by $\pi$. In a time $\Delta t$ an energy $f\Delta t$ crosses the surface. That is the energy contained in a volume $c\Delta t$ behind the surface. But for zero rest mass particles $E = pc$. Energy density is therefore $c\pi$. The energy contained in a volume of cross-sectional area $A$ and length $c\Delta t$ is therefore $c^2\pi A\Delta t$. The flux across the surface is then $c^2\pi$.

**16-13**. *The LIGO gravitational wave detector expects to detect gravitational waves at frequencies of $\sim 200Hz$ that cause a dimensionless strain of $\delta L/L \sim 10^{-21}$ What is the flux of energy of such waves incident on Earth? If they come from 20 Mpc away how fast was their source losing energy to gravitational waves when they were emitted? How far away would the Sun have to be to produce the same flux in electromagnetic radiation?*

**Solution:** Gravitational waves propagate at the speed of light. Therefore, as discussed on in Section 16.16.5, in geometrized units

$$f \equiv (\text{energy flux}) = (\text{energy density}) = \frac{\omega^2 a^2}{32\pi}$$

each term in the above relation having the dimension $\mathcal{L}^{-2}$. The dimensionless amplitude $a$ is comparable to $\delta L/L$ [*cf.* (16.19)]. The only part of this problem that requires care is converting the geometrized units $\text{cm}^{-2}$ for flux back to $\text{erg}/(\text{cm}^2 \cdot \text{s})$. To do this, note

$$\frac{1}{\mathcal{L}^2} \sim \frac{\mathcal{L}}{\mathcal{L}^2(c\mathcal{T})} \sim \frac{GM}{c^2}\frac{1}{\mathcal{L}^2(c\mathcal{T})} \sim \frac{G}{c^5}\frac{\mathcal{E}}{\mathcal{L}^2\mathcal{T}}$$

so that

$$(f \text{ in cm}^{-2}) = \frac{G}{c^5}\,(f \text{ in erg}/(\text{cm}^2 \cdot \text{s})) .$$

Recognizing also that $(\omega \text{ in cm}^{-1}) = ((\omega \text{ in Hz})/c))$ one has

$$f \sim \frac{c^3}{32\pi G}\left(\frac{\omega}{1\text{ Hz}}\right)^2\left(\frac{\delta L}{L}\right)^2 \frac{\text{erg}}{\text{cm}^2 \cdot \text{s}} .$$

Taking $\omega \sim 200$ Hz, $\delta L/L \sim 10^{-21}$ gives

$$f \sim 2 \times 10^{-1} \text{ erg/(cm}^2 \cdot \text{s)} .$$

A star at distance $r$ away with the luminosity of the sun, $L_\odot \sim 4 \times 10^{33}$erg/s, would give a flux

$$f = \frac{L_\odot}{4\pi r^2} .$$

A star of the sun's luminosity would give the same flux in electromagnetic radiation at a distance of $\sim 0.01$ pc.

**16-14**. *[E] The binary star system $\iota$ Boo is located about 11.7 parsecs from Earth in the direction of the constellation Boötes. (1 parsec $= 3.09 \times 10^{18}$ cm.) The two stars orbit each other with a period of approximately 6.5 hours. A gravitational wave detector in the vicinity of Earth detects gravitational radiation from this source with a strain of $\delta L/L \sim 10^{-21}$. Estimate the energy flux in this radiation at the Earth and compare to that of the Sun in electromagnetic radiation if it were located the same distance away. (Gravitational wave detectors contemplated on Earth can't make this detection because the frequency of the wave is too low, but detectors in space might be able to do it.)*

**Solution:** We estimate the amplitude of the gravitational wave as $a \sim \delta L/L$ from (16.11) or similar equations. The flux from (16.22) is

$$f = \frac{\omega^2 a^2}{32\pi} \sim \frac{\pi}{8} \left(\frac{a}{P}\right)^2$$

where $P$ is the period of the orbit and assuming $\omega \sim 2\pi/P$. (It's actually twice that value as we'll see in Section 23). Using

$$P = 6.5 \text{ hr} \sim 2 \times 10^4 \text{ s} \sim 7 \times 10^{14} \text{ cm}$$

and $a \sim 10^{-21}$ we find

$$f_{GW} \sim 10^{-72} \text{ cm}^{-2}$$

in geometrized units. Multiplying by $c^5/G$ converts this to usual $\mathcal{MLS}$ units

$$f_{GW} \sim 10^{-13} \text{ erg/(cm}^2 \cdot \text{s)}$$

The flux from the Sun were it $d = 11.7$ pc away would be

$$f_\odot = \frac{L_\odot}{4\pi d^2} = \frac{3.8 \times 10^{33}\ \mathrm{erg/s}}{4\pi\,(11.7\ \mathrm{pc} \times 3 \times 10^{18}\ \mathrm{cm/pc})^2} \approx 2 \times 10^{-7}\ \mathrm{erg/(cm^2 \cdot s)}.$$

The flux in gravitational waves is a million times smaller than from a star like the Sun at that distance.

# Chapter 17

# The Universe Observed

**17-1**. *A distant galaxy has a redshift $z = \Delta\lambda/\lambda$ of .2. According to Hubble's law how far away was the galaxy when the light was emitted if the Hubble constant is 72 (km/s)/Mpc?*

**Solution:** A redshift of $z = .2$ corresponds to a recession velocity of $V/c = .2$ or $V = 6 \times 10^4$ km/s [*cf.* (17.4)]. According to Hubble's law, the present distance is thus

$$d = \frac{V}{H_0} = \frac{6 \times 10^4 (\text{km/s})}{72\ (\text{km/s})/\text{Mpc}} = 8.3 \times 10^2\ \text{Mpc}\ .$$

The distance $d_0$ when the light was emitted was smaller because the galaxies have been moving apart with a speed $V$ since then. If $\Delta t$ is the light travel time

$$c\Delta t = d = d_0 + V\Delta t\ .$$

Thus,

$$d_0 = d\left(1 - \frac{V}{c}\right) = 6.6 \times 10^2\ \text{Mpc}\ .$$

**17-2**. *[E] Parallaxes greater than .005″ can be measured for about 120,000 stars in the immediate solar neighborhood. Estimate the number of stars per cubic pc in the solar neighborhood.*

**Solution:** The smallest parallax corresponds to the greatest distance. This distance in pc is

$$d(pc) = \frac{1}{p} \approx 10^2 pc \ .$$

If there are $10^5$ stars in a volume $10^2$ pc on a side, the density of stars is

$$(\text{density}) \sim \frac{10^4}{(10^2)^3} \sim 10^{-3} pc^{-3} \ .$$

**17-3**. *[P] Planck's radiation law specifies the energy $dE$ in a black body gas at temperature $T$ that is incident in a small time $dt$, on a small area $dA$, from a small solid angle $d\Omega$ about the normal direction, in a small frequency range $d\omega$, as*

$$dE = \frac{\hbar\omega^3}{4\pi^3 c^2} \frac{1}{\exp(\hbar\omega/k_B T) - 1} dt dA d\Omega d\omega$$

*Assuming that the data in the top part of Figure 17.3 fit a black body spectrum, calculate the temperature of the radiation. (Note that the frequency plotted in Figure 17.3 is $\omega/2\pi$.) (Hint: There are several ways to do this, some easier than others.)*

**Solution:** The simplest way to determine the temperature from the top part of Figure 17.3 is to focus on the maximum intensity and the frequency $\omega_m$ at which it occurs. The peak of the function $x^3/(e^x - 1)$ occurs at $x = 2.822$. Therefore

$$\frac{\hbar\omega_m}{k_B T} = 2.822$$

a relation called *Wien's displacement law.* Estimating $f_m = 5.3 \text{ cm}^{-1}$ from the graph gives

$$\omega_m = 2\pi c f_m = 2\pi \, (3 \times 10^{10} \text{ cm/s})(5.3 \text{ cm}^{-1}) = 9.98 \times 10^{11} \text{ Hz}$$

This gives $T = 2.7$ K.

**17-4**. *[E] The distance to the Andromeda galaxy (M31) is 725 kpc. Use the*

*data in Figure 17.4 to estimate the mass in $M_\odot$'s inside a sphere extending 150′ from Andromeda's center.*

**Solution:** In geometrized units (17.3) gives

$$M = r(V/c)^2 \tag{1}$$

where $r$ is the distance from the center of M31 at 150′. This is (150′ in radians) $\times$ 725kpc = (.044) $\times$ (725kpc) = 32kpc = $9.7 \times 10^{17}$km. Using $V = 200$ km/s, (1) gives

$$M = 4.32 \times 10^{11}\text{km} = 2.9 \times 10^{11} M_\odot \ .$$

**17-5**. *[E,S] Use the data from the 2dF Survey in Figure 17.14 to estimate the distance scale above which the universe is approximately homogeneous.*

**Solution:** Very roughly the scales of the larger voids and filaments in Figure 17.14 correspond to redshift intervals $\Delta z \sim .03$. Hubble's law (17.5) connects intervals in $z = V/c$ to intervals in $d$ according to

$$c\Delta z = H_0 \Delta d \ .$$

Assuming $H_0 \approx 72$ (km/s)/Mpc gives

$$\Delta d \sim 100 \text{ Mpc}$$

(to one significant figure). Above that the galaxy distribution is roughly homogeneous.

**17-6**. *Expansion by Copy Machine. For a more compelling demonstration that the expansion of the universe doesn't have a center, try the following experiment related to Figure 17.6. Take a transparency such as used with overhead projectors and cover it approximately uniformly with dots representing galaxies as in one of the boxes in Figure 17.6. Xerox these dots onto another transparency at 20% expansion. Line up the transparencies so the position of one galaxy after the expansion is on top of its position before. The rest will be seen to be moving outwards in all directions. Try the same thing with different galaxy. What do you see?*

**Solution:** The same thing — all galaxies moving away from the new center.

**17-7**. *Radio signals are received from the vicinity of a star exactly like our Sun that has an apparent magnitude of 3.9. How long ago were these signals sent to us?*

**Solution:** Eq. (17.10) shows that the absolute magnitude of the Sun is $M = 4.74$. The connection

$$m - M = 5\log_{10}(d/10 \text{ pc})$$

with $m = 3.9$ gives $d = 6.8$ pc. Since 1 pc = 3.26 ly, the signals were emitted 22 years ago.

**17-8**. *The main sequence of a distant cluster of stars is approximately fit by the relation (apparent magnitude) = 6 (color index) +18. Assuming stars are of the same kind as those in the solar neighborhood, approximately how far away is this cluster?*

**Solution:** Plotting the given relation on a copy of Figure 17.8 shows that it roughly coincides with the main sequence if the difference between apparent magnitudes $m$ and absolute magnitudes $M$ is about 17. The distance modulus connection

$$m - M = 5\log_{10}(d/10\text{pc})$$

gives a distance of about 25kpc.

**17-9**. *A Cepheid variable star is observed with an apparent magnitude of 22 and a period of 25 days. How far away is this star?*

**Solution:** From Figure 17.9, the absolute peak magnitude is $-6.6$. The distance modulus connection

$$m - M = 5\log_{10}(d/10 \text{ pc})$$

gives $d = 5.2$ Mpc.

# Chapter 18

# Cosmological Models

**18-1.** *[S] Initially the raisins in Example 1 can be located by Cartesian coordinates $(x, y, z)$ in the flat Euclidean space occupied by the dough. Continue to label the points occupied by the raisins with the same coordinates they started with so the coordinates $(x, y, z)$ are co-moving. Express the line-element of flat Euclidean space in terms of these comoving coordinates at and a scale factor $a(t)$ assuming the expansion is homogeneous and isotropic at all times. Sketch the qualitative behavior of $a(t)$ between the start of baking and its completion. How would $a(t)$ look if the dough were a badly behaved spherical soufflé? If the dough were contained in a spherical boundary of radius $R$ initially, what would be the equation of the boundary as a function of time? What would be the area of the boundary at the end of baking?*

**Solution:** The line element in rectangular coordinates comoving with the raisins would have the form

$$dS^2 = a^2(t)\ \left(dx^2 + dy^2 + dz^2\right).$$

The figure below shows a possible behavior for $a(t)$:

Since the coordinates are co-moving, the equation of the boundary would be

$$r \equiv \left(x^2 + y^2 + z^2\right)^{\frac{1}{2}} = R$$

for all time. The area of the boundary would scale with $a^2(t)$. The area $A_{\text{end}}$ would be related to the area $A_{\text{start}}$ at the start by

$$A_{\text{end}} = \left(\frac{a(t_{\text{end}})}{a(t_{\text{start}})}\right)^2 A_{\text{start}}.$$

**18-2**. *Suppose that the scale factor describing the expansion of the universe is*

$$a(t) = (t/t_*)^{\frac{1}{2}}$$

*where $t_*$ is a constant and $t$ is the proper time from the singularity. Suppose that the present age of the universe is 14 Gyr.*

*a) What would be the value (in $\text{yr}^{-1}$) of the Hubble constant observed today?*

*b) At what age in years would the temperature of the microwave background be 3000 K?*

**Solution:**

a) If $a(t) = (t/t_*)^{\frac{1}{2}}$ then the Hubble "constant" at any time $t$ is

$$H = \frac{\dot{a}}{a} = \frac{1}{2t} .$$

At the present age, $t_0 = 14 \times 10^9$ years, we have

$$H_0 = \left(28 \times 10^9 \text{ yr}\right)^{-1} = 3.6 \times 10^{-11} \text{ yr}^{-1} .$$

b) The temperature of radiation varies with $a$ as

$$T = T_0 \left(\frac{a_0}{a}\right) = T_0 \left(\frac{t_0}{t}\right)^{\frac{1}{2}} .$$

Let $t_{3k}$ be the time of decoupling when $T = 3000°K$. The present temperature is $3°K$, so we have

$$3000°K = 3°K \left(\frac{14 \times 10^9 \text{ yr}}{t_{3k}}\right)^{\frac{1}{2}}$$

so

$$t_{3k} = 14 \times 10^3 \text{ yr} .$$

**18-3**. *Consider a flat FRW model whose metric is given by* (18.1). *Show that, if a particle is shot from the origin at time $t_*$ with a speed $V_*$ as measured by a co-moving observer (constant $x, y, z$), then asymptotically it comes to rest with respect to a co-moving frame. Express the co-moving coordinate radius at which it comes to rest as an integral over $a(t)$.*

**Solution:** Orient coordinates so that the particle is moving along the $x$-axis and restrict attention to the two relevant dimensions $(t, x)$. The metric is [*cf.* (18.1)]

$$ds^2 = -dt^2 + a^2(t)\, dx^2 . \tag{1}$$

This is unchanged under displacements in $x$. There is thus a Killing vector $\xi^\alpha = (0, 1)$ and a conserved quantity $\boldsymbol{\xi} \cdot \mathbf{u} \equiv q$ which is

$$\boldsymbol{\xi} \cdot \mathbf{u} = a^2(t) \frac{dx}{d\tau} \equiv q \tag{2}$$

where $\mathbf{u}$ is the particle's four-velocity. The value of $q$ is determined by the initial velocity $V_*$ (see below). Another integral is supplied by the normalization condition

$$\mathbf{u}\cdot\mathbf{u} = -\left(\frac{dt}{d\tau}\right)^2 + a^2(t)\left(\frac{dx}{d\tau}\right)^2 = -1\ . \tag{3}$$

Eqs. (2) and (3) can be solved for the components of the four-velocity

$$u^x = \frac{dx}{d\tau} = \frac{q}{a^2(t)}\quad,\quad u^t = \frac{dt}{d\tau} = \left[1+\frac{q^2}{a^2(t)}\right]^{\frac{1}{2}}\ . \tag{4}$$

The three-velocity $dx/dt$ is then

$$\frac{dx}{dt} = \frac{u^x}{u^t} = \frac{q}{a^2(t)}\left[1+\frac{q^2}{a^2(t)}\right]^{-\frac{1}{2}}\ . \tag{5}$$

As the universe expands, $a(t)$ grows and $dx/dt$ tends to zero. The particle therefore comes to rest at a coordinate $x_f$ which can be found by integrating (5):

$$x_f = \int_{t_*}^{\infty} dt\,\frac{q}{a^2(t)}\left[1+\frac{q^2}{a^2(t)}\right]^{-\frac{1}{2}}\ . \tag{6}$$

It remains to express $q$ in terms of the initial velocity $V_*$ measured by a co-moving observer. Orthonormal basis vectors for such an observer are $\mathbf{e}_{\hat{t}} = (1\,,0)$ and $\mathbf{e}_{\hat{x}} = [0\,,1/a(t_*)]$. We have, for instance [*cf.* (5.82)],

$$u^{\hat{x}} = \mathbf{e}_{\hat{x}}\cdot\mathbf{u} = a(t_*)\,\frac{dx}{d\tau} = \frac{q}{a(t_*)}\,, \tag{7}$$

where the last equality follows from (2). Thus,

$$q = a(t_*)\,u^{\hat{x}} = a(t_*)\,\frac{V_*}{\sqrt{1-V_*^2}}\ . \tag{8}$$

**18-4**. *[S] Suppose the present value of the Hubble constant is* 72 (km/s)/Mpc *and that the universe is at critical density. A photon is emitted from our galaxy*

*now. What is the redshift of this photon when it is received in another galaxy 10 billion years in the future, assuming it continues to be matter dominated?*

**Solution:** For a matter-dominated universe at critical density $a(t) \propto t^{2/3}$. The Hubble constant today is therefore

$$H_0 = \frac{\dot{a}(t_0)}{a(t_0)} = \frac{2}{3t_0}$$

which gives the present age $t_0$. For $H_0 = 65[km/sec/Mpc]$ the present age is $1 \times 10^{10}$yrs. (See the entry for *Hubble time* in Appendix A). Therefore at $2 \times 10^{10}$yrs, the photon will have redshifted by

$$\frac{\omega_{\text{received}}}{\omega_{\text{emitted}}} = \frac{a(t_0)}{a(t_0+10)} = \left(\frac{1}{2}\right)^{\frac{2}{3}} = .63\ .$$

**18-5.** *[S] The cosmic background radiation has been propagating to us since the universe became transparent at a temperature of approximately* 3000 K. *Its temperature today is* 2.73 K. *What is the redshift $z$ of the radiation?*

**Solution:** From (18.26) the ratio of the scale factor at emission $t_e$ to the scale factor now is related to the temperature $T$ at those two times by

$$\frac{a(t_0)}{a(t_e)} = \frac{T(t_e)}{T(t_0)} = \frac{3000}{2.73} \approx 1000.$$

But from (18.11) that ratio equals $1+z$. Thus, $z \approx 1000$.

**18-6.** *[S] A type Ia supernova has a redshift of $z = 1.1$. The observed brightness rises and falls on a timescale of two months. (More precisely let's say the difference in times between when the supernova is at half peak brightness is two months.) What is the timescale for the rise and fall in the supernova's rest frame as would be seen by a hypothetical observer close to the supernova and at rest with respect to it?*

**Solution:** Eq. (18.9) relates the two time intervals and shows that the time measured by the nearby observer will be a factor of $a(t_e)/a(t_0) = 1/(1+z)$

*smaller* than the time we observe. The time seen by the nearby observer is therefore $.48 \times 2\text{months} = .96$ of a month.

**18-7**. *Consider a galaxy whose light we see today at time $t_0$ was emitted at time $t_e$. Show that the present proper distance to the galaxy (along a curve of constant $t_0$) is*

$$d = a(t_0) \int_{t_e}^{t_0} dt/a(t) \ .$$

**Solution:** Use $(t, \chi, \theta, \phi)$ coordinates for the FRW models and assume we are located at $\chi = 0$. The coordinates are comoving with the matter, so the coordinate of the galaxy is a fixed value $\chi = \chi_*$ for all time. Since the light ray propagates radially to us along a curve of $ds^2 = 0$, we have

$$0 = -dt^2 + a^2(t) d\chi^2 \ . \tag{1}$$

This expression holds in open, closed, and flat cases. Integrating (1) we find

$$\chi_* = \int_{t_e}^{t_0} \frac{dt}{a(t)} \ .$$

The distance between $\chi = 0$ and $\chi_*$ at $t = t_0$ is from the metric

$$\begin{aligned} d &= \int_0^{\chi_*} a(t_0) d\chi = a(t_0)\chi_* \ , \\ &= a(t_0) \int_{t_e}^{t_0} \frac{dt}{a(t)} \ . \end{aligned}$$

**18-8**. *In Section 9.2 the red-shift of a photon in the Schwarzschild geometry was derived using the conservation law arising from time-translation symmetry. Show that the cosmological red shift* (18.10) *can be derived from the space translation symmetry of the metric* (18.1) *in a similar way.*

**Solution:**

The metric coefficients in (18.1) are independent of $x, y$, and $z$. There are three corresponding Killing vectors. Let us suppose the photon's direction of

propagation is along $x$ and consider the corresponding Killing vector $\boldsymbol{\zeta}$ with components (0,1,0,0).

If $\mathbf{p}$ is the four-momentum of the photon, $\boldsymbol{\zeta} \cdot \mathbf{p}$ is conserved. A brief calculation shows

$$\boldsymbol{\zeta} \cdot \mathbf{p} = g_{\alpha\beta}\zeta^\alpha p^\beta = g_{xx}p^x = a^2(t)p^x(t) \ .$$

Thus $p^x(t)$ varies inversely with $a^2(t)$ as the photon propagates to later times.

Because $\mathbf{p}$ is null we can easily deduce $p^t(t)$ as well

$$\mathbf{p} \cdot \mathbf{p} = -(p^t)^2 + a^2(t)(p^x)^2 = 0 \ .$$

Thus, $p^t$ varies inversely with $a(t)$ as the photon propagates.

We can relate these components of $\mathbf{p}$ to the energies measured by the observers in the usual way. The energy measured by an observer moving with four-velocity $\mathbf{u}$ is $-\mathbf{u} \cdot \mathbf{p}$. For observers at rest in the surfaces of homogeneity, the four-velocity is just

$$\mathbf{u} = (1, 0, 0, 0)$$

(Check $\mathbf{u} \cdot \mathbf{u} = -1$ using (18.1).) Thus

$$E = -\mathbf{p} \cdot \mathbf{u} = -g_{\alpha\beta}u^\alpha p^\beta = p^t$$

Noting that $E = \hbar\omega$, and that $p^t$ varies inversely with $a(t)$, we have

$$\frac{\omega_2}{\omega_1} = \frac{a(t_1)}{a(t_2)}$$

**18-9**. *[E] Estimate in centimeters the size of the universe visible today at the time the CMB radiation last interacted with matter at a temperature of approximately* 3000 *K.*

**Solution:** The present size of the visible universe can be roughly estimated as the Hubble distance

$$d_H \equiv ct_H = \frac{c}{H_0} = 3 \times 10^3 h^{-1} \text{ Mpc} = 9.25 \times 10^{27} h^{-1} \text{ cm}$$

where $h \equiv H_0/(100\ (\mathrm{km/s})/\mathrm{Mpc})$. For $H_0 \sim 72\ (\mathrm{km/s})/\mathrm{Mpc}$ this gives

$$d_H \sim 10^{28}\ \mathrm{cm}\ .$$

The universe becomes transparent at a temperature of $\sim 3000°$ K. The scale factor now $a(t_0)$ and at decoupling a $(t_{dec})$ are therefore related by [*cf.* (18.26)]

$$\frac{a(t_{\mathrm{dec}})}{a(t_0)} = \frac{2.73}{3000} \sim 9 \times 10^{-4}\ .$$

The visible universe at decoupling will be smaller by just this factor so it had a size

$$d_{\mathrm{dec}} \sim 10^{25}\ \mathrm{cm} \approx 10\ \mathrm{Mpc}\ .$$

**18-10**. *[E,C] As the universe expands the horizon grows. Estimate the time it has to grow for one new galaxy to come within the horizon, assuming the universe was matter dominated over the whole of its history.*

**Solution:** To make an estimate, let's assume that the universe is spatially flat with $\Omega_m \sim .3$. The amount of mass contained within the present horizon of radius $d_{\mathrm{horiz}}(t)$ is

$$M_{\mathrm{horiz}}(t) = \frac{4}{3}\,\pi\, d^3_{\mathrm{horiz}}(t)\,\rho_m(t). \tag{1}$$

We seek the time it takes for $M_{\mathrm{horiz}}(t)$ to increase by the mass of one large galaxy, $\sim 10^{12} M_\odot$. To find that, we need the *rate* of increase in $M_{\mathrm{horiz}}(t)$ at the present time $t_0$. Using (18.47), eq. (1) for $M_{\mathrm{horiz}}(t)$ can be rewritten as

$$M_{\mathrm{horiz}}(t) = \frac{4}{3}\,\pi\,\rho_m(t)\,a^3(t)\,\left(\int_0^t \frac{dt'}{a(t')}\right)^3. \tag{2}$$

Since $\rho_m a^3$ is constant in time,

$$\frac{dM_{\mathrm{horiz}}}{dt} = 4\pi\,\rho_m(t)\,a^2(t)\,\left(\int_0^t \frac{dt'}{d(t')}\right)^2. \tag{3}$$

Again, using (18.47)

$$\frac{dM_{\mathrm{horiz}}}{dt} = 4\pi\,\rho_m(t)\,d^2_{\mathrm{horiz}}(t). \tag{4}$$

Dimensionally at the present moment $d_0 \sim 1/H_0$. The factor in front is 2 for a matter-dominated universe as (18.49) shows. Inclusion of the cosmological constant brings this up to 3.3 as working through Problems 32 and 33 will show. Taking $d_{\text{horiz}}(t_0) \sim 2/H_0$ for a rough estimate and using

$$\rho_m(t_0) = \Omega_m \, \rho_{\text{crit}} = \Omega_m \, \frac{3H_0^2}{8\pi} \tag{5}$$

we find (geometrized units)

$$\frac{dM_{\text{horiz}}}{dt} \sim 6\Omega_m \sim 1.8. \tag{6}$$

The time $\Delta t_{\text{gal}}$ for $M_{\text{gal}} \sim 10^{12} M_\odot$ to come across the horizon is therefore $(M_{\text{gal}}/1.8)$ or

$$\Delta t \sim \frac{10^{12} M_\odot \times 1.5 \text{ km}/M_\odot}{(1.8)\,(3 \times 10^5 \text{ km/s})} \sim 10^6 \text{ s} \sim 1 \text{ month}. \tag{7}$$

Every month a new galaxy with $10^{12}$ stars and who knows how many intelligent species becomes observable — in principle.

**18-11.**

a) *Eq.* (18.69) *gives the scale factor as a function of time for closed, matter dominated, FRW models. Show that, if the parameter* $\eta$ *which occurs there is used as a time coordinate, the FRW metric takes the form:*

$$ds^2 = a^2(\eta)[-d\eta^2 + d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)] \, .$$

b) *Draw an* $\eta$*-*$\chi$ *spacetime diagram indicating the big bang, the big crunch, and the past light cone of a comoving observer at the origin at the moment of maximum expansion.*

c) *Is there time before the big crunch for the observer to receive information from all parts of this spatially finite universe, or are there parts of it he or she is doomed never to see?*

d) *Could an observer traverse the entire circumference of the universe in the time between the big bang and the big crunch?*

**Solution:**

a) From (18.69), $dt = a(\eta)d\eta$, which gives the form of the metric displayed.

b)

c) At maximum expansion, the observer can receive information from anywhere in the shaded region above whose spatial extent is the whole closed universe. At any later moment the observer can receive information from any spatial location. (Of course, signals received at a given time will be from an earlier time that depends on the speed they traveled.)

d) A light ray (dotted curve) can just make it from the north ($\chi = 0$) to the south ($\chi = \pi$) pole and back to the north pole in the time between big bang and big crunch. An observer therefore cannot do it.

**18-12.** *(Legislating the value of $\pi$.) There is a story that a bill was introduced in a state legislature to declare the value of $\pi$ to be some constant other than 3.14159... Could the bill's author have been correct? Is there some other geometry of three-dimensional space where the ratio of the circumference, $C$,*

*to the radius $R$ has a constant value for all circles that is different from $2\times$ 3.14159... ? (A circle in this context means the locus of points a given distance — the radius — from a given point called the center.) [Hint: Think why this problem is in a chapter on cosmology.]*

**Solution:** The spatial geometry must be homogeneous and isotropic for the ratio of circumference $C$ to radius $R$ to have the same value for *all* circles. Were it not isotropic, differently oriented circles would have different ratios. Were it not homogeneous, circles in different locations would have different ratios.

But we know all homogeneous isotropic spatial geometries. Their line elements are summarized by the spatial part of (18.62)

$$dS^2 = a^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

for the three different values of $k$. Let's compute the ratio of $C$ to $R$ for the circle in the equatorial plane centered at $r = 0$. Evidently, $C = 2\pi ar$ and

$$R = a\int_0^r \frac{dr}{\sqrt{1-kr^2}}.$$

For $k \neq 0$, $R(r) \neq ar$. The ratio $C/R$ for open and closed geometries is not constant. Only for $k = 0$ is the ratio constant, but for that value $C/R = 2\pi$.

There is no geometry for which the bill could have been correct.

**18-13**. *Suppose the total spatial volume of a closed, matter dominated, FRW model is $10^{12}$ Mpc$^3$ at its moment of maximum expansion. What is the duration of this universe from the big bang to the big crunch in years?*

**Solution:** For a closed matter dominated universe [*cf.* (18.69)]

$$\begin{aligned} a(\eta) &= (a_*/2)(1-\cos\eta)\ , \\ t(\eta) &= (a_*/2)(\eta - \sin\eta)\ . \end{aligned}$$

Evidently $a_*$ is the scale factor at maximum expansion, $\eta = \pi$. The volume at maximum expansion is [*cf.* (18.70)] is

$$V_* = 2\pi^2 a_*^3\ .$$

The lifetime is the time $t$ between $\eta = 0$ and $\eta = 2\pi$. $T = \pi a_*$. Thus

$$\begin{aligned} T &= \frac{\pi}{2}\left(\frac{V}{2\pi^2}\right)^{\frac{1}{3}} \\ T &= 1.16 \times 10^{10} pc \end{aligned}$$

In more usual units, since $1pc = 3.26\ell y$ we have

$$T = 3.8 \times 10^{10} yrs \ .$$

**18-14.** *[B] Box 2.2 described how objects of a given size would subtend different angles at a given distance in positively curved, flat, and negatively curved spatial geometries. Calculate these the angle subtended by an object a size s a distance d away in each of the FRW spacetimes, assuming for simplicity that the scale factor is independent of time. (The next chapter will deal with the expansion.) Do your results confirm the statements in the Box?*

**Solution:** If the scale factor is independent of time, only the spatial part of the FRW metrics (18.60) are relevant for finding the relation between size, distance, and subtended angle. Suppose we as observers are located at $\chi = 0$. Suppose an object of size $s$ is located a coordinate distance $\chi_s$ away in the equatorial plane $\theta = \pi/2$ and oriented transversely to the line-of-sight extending over an angle $\Delta\phi$. By spherical symmetry the light rays from the ends of the object propagate along curves of constant $\phi$ to the observer. The angle subtended at the observer is therefore $\Delta\phi$. The distance $d$ of the object is $a\chi_s$. The size $s$, for given $\Delta\phi$, is

$$s = a \sin \chi_s \Delta\phi \ , \quad s = a\chi_s \Delta\phi \ , \quad s = a \sinh \chi_s \Delta\phi$$

for the closed, flat, and open geometries respectively. The subtended angle is therefore related to size and distance by

$$\begin{aligned} (\Delta\phi)_{\text{closed}} &= s\left[a \sin\left(\frac{d}{a}\right)\right]^{-1} \quad \text{(closed)} \ , \\ (\Delta\phi)_{\text{flat}} &= s/d \quad \text{(flat)} \ , \\ (\Delta\phi)_{\text{open}} &= s\left[a \sinh\left(\frac{d}{a}\right)\right]^{-1} \quad \text{(open)} \ . \end{aligned}$$

Note that this correctly reproduces the elementary flat space result and that the curved cases reduce to this when $d/a \ll 1$. Since $\sinh x > x > \sin x$ for any $x$,

$$(\Delta\phi)_{\text{open}} < (\Delta\phi)_{\text{flat}} < (\Delta\phi)_{\text{closed}}$$

which is the result anticipated in Box 2.2.

**18-15**. *[S] Consider a homogeneous, isotropic, cosmological model described by the line element*

$$ds^2 = -dt^2 + (t/t_*)\left[dx^2 + dy^2 + dz^2\right]$$

*where $t_*$ is a constant*

*a) Is this model open, closed, or flat?*

*b) Is this a matter-dominated universe? Explain.*

*c) Assuming the Friedman equation holds for this universe, find $\rho(t)$.*

**Solution:**

a) The model is evidently flat ($k = 0$).

b) This is not a matter-dominated universe for which $a(t) \propto t^{2/3}$. Here, $a \propto t^{1/2}$.

c) Using $k = 0$ in the basic equation for the FRW models (18.30) one finds

$$\rho = \frac{3}{8\pi}\left(\frac{\dot{a}}{a}\right)^2 .$$

But $a(t) \propto t^{1/2}$ from the given metric giving

$$\rho(t) = \frac{3}{32\pi t^2} .$$

Are the dimensions right?

**18-16**. *The scale factor $a(t)$ of any FRW model can be expanded about the present moment in the form*

$$a(t) = a(t_0)[1 + H_0(t - t_0) - (1/2)q_0 H_0^2 (t - t_0)^2 + \cdots] ,$$

*where $q_0$ is called the deceleration parameter. Explain why the coefficient of the first term is the Hubble constant and evaluate $q_0$ in terms of the cosmological parameters.*

**Solution:** The Hubble constant is given by (18.14) which coincides with the $H_0$ in the first term of the expansion.

To evaluate $q_0$, first rewrite the expansion in terms of the rescaled variables $\tilde{a}$ and $\tilde{t}$ defined by (18.72) and (18.73) respectively. The result is

$$\tilde{a}(\tilde{t}) = 1 + \left(\tilde{t} - \tilde{t}_0\right) - \frac{1}{2}\, q_0 \left(\tilde{t} - \tilde{t}_0\right)^2 + \ldots \,. \tag{1}$$

Differentiating the rescaled Friedman equation (18.77) gives

$$\frac{d^2\tilde{a}}{d\tilde{t}^2} = \Omega_v \tilde{a} - \frac{\Omega_m}{2\tilde{a}^2} - \frac{\Omega_r}{\tilde{a}^3} \,.$$

Evaluating this at $\tilde{t} = t_0$ (or $t = t_0$) using (1) and $\tilde{a}(t_0) = 1$ gives

$$q_0 = \Omega_r + \frac{\Omega_m}{2} - \Omega_v \,.$$

**18-17**. *[S] Verify that* (18.69) *and* (18.71) *solve the Friedman equation* (18.63) *for a matter dominated universe.*

**Solution:** This problem is essentially a matter of accurate algebra. The matter density can be written

$$\rho_m(t) = \Omega_m\, \rho_{\text{cut}} \left(\frac{a_0}{a(t)}\right)^3 = \frac{3\Omega_m H_0^2}{8\pi} \left(\frac{a_0}{a(t)}\right)^3 . \tag{1}$$

The Friedman equation (18.63) for a closed, matter-dominated universe becomes (with $\Omega_m \equiv \Omega$)

$$\dot{a}^2 - \frac{H_0^2 a_0^3 \Omega}{a} = -1 . \tag{2}$$

Clearly, we will need to find $a_0$ to check whether (18.69) satisfies this equation. We can do that by dividing (2) by $a_0^2$, evaluating at the present moment, and by using $H_0 = (\dot{a}/a)_0$. The result is

$$H_0 a_0 = (\Omega - 1)^{-\frac{1}{2}}. \tag{3}$$

With this, (2) becomes

$$\dot{a}^2 - \frac{H_0 \Omega}{(\Omega - 1)^{3/2}} \frac{1}{a} = -1. \tag{4}$$

It's then a simple matter to check that (18.69) satisfies this equation. For example,

$$\dot{a} = \frac{da/d\eta}{dt/d\eta} = \frac{\sin\eta}{1 - \cos\eta}. \tag{5}$$

The open case is similar.

**18-18.** *(Radiation Dominated FRW Models.) Solve the Friedman equation to exhibit the scale factor as a function of time for FRW models that are radiation dominated from start to finish. Express your answers in terms of $H_0$ and $\Omega = \Omega_r$.*

**Solution:** For a FRW model with only radiation, the rescaled Friedman equation (18.77) reads

$$\left(\frac{d\tilde{a}}{d\tilde{t}}\right)^2 = \Omega_c + \frac{\Omega_r}{\tilde{a}^2}$$

where $\Omega_c = 1 - \Omega_r$. This leads to the relation

$$d\tilde{t} = \frac{\tilde{a} d\tilde{a}}{(\Omega_c \tilde{a}^2 + \Omega_r)^{\frac{1}{2}}} .$$

This is easily integrated to find

$$-\Omega_c \tilde{a}^2 + (\Omega_c \tilde{t})^2 = \Omega_r .$$

The constant of integration fixing the zero of $t$ has been chosen arbitrarily to give a simple formula. This is the equation of an ellipse in the $(\tilde{a}, \tilde{t})$ plane

when $\Omega_c < 0$ (closed) and a hyperbola when $\Omega_c > 0$ (open). The solutions are sketched below. The behavior $a(t)$ can be found from (18.79).

**18-19.** *(de Sitter Space) Solve the Friedman equation* (18.63) *for the scale factor as a function of time for closed FRW models that have only vacuum energy $\rho_v$. Do these models have an initial big-bang singularity?*

**Solution:** Defining $H$ in terms of the vacuum energy of (18.40), the Friedman equation (18.63) for closed FRW models ($k = 1$) is

$$\dot{a}^2 - H^2 a^2 = -1.$$

Reorganizing, this is

$$\frac{da}{(H^2a^2 - 1)^{1/2}} = dt.$$

Integrating both sides gives

$$a(t) = \frac{1}{H}\cosh(Ht)$$

with suitable choice for the origin of $t$.

Evidently there is no big bang singularity since $a(t)$ never vanishes over the whole range of $t$. Rather the universe starts from large values of $a$ at large negative times, collapses, reaches a minimum value of $a$ at $t = 0$, and re-expands. This solution is called De Sitter space.

**18-20**. *[C] Find a closed form solution to the dynamical equation for the flat FRW models* (18.35) *in the case when there is no radiation* $\Omega_r = 0$ *but both vacuum energy and matter are present. Express your answer in terms of* $H_0$, $\Omega_m$, *and* $\Omega_v = 1 - \Omega_m$. *How large would* $\Omega_v$ *have to be for the universe to be accelerating* ($\ddot{a} > 0$) *at the present time? Find an explicit expression for the age of the universe* $t_0$ *as a function of* $H_0$ *and* $\Omega_v$.

**Solution:** When $\Omega_r = 0$ (no radiation) and $\Omega_c = 0$ (no spatial curvature), the Friedman equation (18.77) is

$$\left(\frac{d\tilde{a}}{d\tilde{t}}\right)^2 = \Omega_v \tilde{a}^2 + \frac{\Omega_m}{\tilde{a}} \tag{1}$$

where $\Omega_v = 1 - \Omega_m$. This implies

$$\frac{\tilde{a}^{\frac{1}{2}} d\tilde{a}}{(\Omega_v \tilde{a}^3 + \Omega_m)^{\frac{1}{2}}} = d\tilde{t} \ . \tag{2}$$

The substitution $y = \tilde{a}^{3/2}$ allows this to be straightforwardly integrated. Adjusting the origin of time so that $\tilde{a}(0) = 0$, we find

$$\tilde{a}(\tilde{t}) = \left(\frac{\Omega_m}{\Omega_v}\right)^{\frac{1}{3}} \sinh^{\frac{2}{3}}\left(\frac{3}{2}\sqrt{\Omega_v}\tilde{t}\right) \tag{3}$$

where $\tilde{t} = H_0 t$. The present moment $\tilde{t}_0 = H_0 t_0$ occurs when $\tilde{a}(\tilde{t}_0) = 1$. This is

$$t_0 = \frac{2}{3H_0\sqrt{\Omega_v}} \sinh^{-1}\left[\left(\frac{\Omega_v}{\Omega_m}\right)^{\frac{1}{2}}\right] \ . \tag{4}$$

Computing the acceleration from (1) gives

$$\frac{d^2\tilde{a}}{d\tilde{t}^2} = \Omega_v \tilde{a} - \frac{\Omega_m}{2\tilde{a}^2} \ . \tag{5}$$

The acceleration will be negative for small $\tilde{a}$ and positive for large values. The crossover occurs at the value $\tilde{a}_{\rm acc}$ when the right hand side of (5) vanishes

$$\tilde{a}_{\rm acc} = \left(\frac{\Omega_m}{2\Omega_v}\right)^{\frac{1}{3}} \ .$$

For this to happen at the present moment requires $\tilde{a}_{\rm acc} = 1$ or

$$\Omega_m = 2\Omega_v \ .$$

**18-21**. *[S] Show that the formula for the cosmological redshift* (18.10) *holds in non-spatially flat FRW models.*

**Solution:** Consider the general FRW line element in the form (18.60). Radial light rays will satisfy

$$0 = -dt^2 + a^2(t)\, d\chi^2.$$

The argument from (18.6) to (18.10) is exactly the same in the general case with the comoving coordinate $\chi$ of the emitting galaxy replacing $R$.

**18-22**. *[S] Equation* (18.47) *for the present size of the cosmological horizon was derived for for a flat FRW model. Show that the same formula holds for all FRW models.*

**Solution:** Consider the general FRW line element in the form (18.60). Radial light rays will satisfy

$$0 = -dt^2 + a^2(t)\, d\chi^2.$$

The argument from (18.46) to (18.47) is exactly the same in the general case with $\chi_{\rm horiz}$ replacing $r_{\rm horiz}$.

**18-23**. *(a) Show that for FRW models with any combination of matter and radiation but no vacuum energy, the curve of $a(t)$ curves downward, i.e. has negative second derivative. Show that this means that $1/H_0$ is always larger than the age $t_0$.*

*(b) Show that this is not always the case if there is a non-zero vacuum energy.*

**Solution:** (a) Solving for $d\tilde{a}/d\tilde{t}$ in (18.77) and differentiating with respect to

$\tilde{t}$ gives

$$\frac{d^2\tilde{a}}{d\tilde{t}^2} = -\frac{\Omega_r}{\tilde{a}^2} - \frac{\Omega_m}{2\tilde{a}} + \Omega_v \tilde{a} \,. \tag{1}$$

The $\Omega$'s are positive because the densities of matter and radiation are positive. If $\Omega_v = 0$, the right hand side of the above relation is negative, whence $\ddot{a} < 0$. The universe *de*cellerates. As Figure 18.2 makes clear, when the curve of $a(t)$ curves downward, $1/H_0$ is an upper bound on the age of the universe.

(b) If $\Omega_m = \Omega_r = 0$, and $\Lambda > 0$, then $\Omega_v$ is positive, $\ddot{a}$ is positive and the universe accelerates.

**18-24**. *(The Einstein Static Universe) Consider a closed ($k = +1$) FRW model containing a matter density $\rho_m$, a vacuum energy density corresponding to a positive cosmological constant $\Lambda$, and no radiation.*

*(a) Show that for a given value of $\Lambda$ there is a critical value of $\rho_m$ for which the scale factor does not change with time. Find this value.*

*(b) What is the spatial volume of this universe in terms of $\Lambda$?*

*(c) If $\rho_m$ differs slightly from this value the scale for factor will vary in time. Does the evolution remain close to the static universe or diverge from it.*

*Comment: This is the Einstein static universe for which Einstein originally introduced the cosmological constant.*

**Solution:**

a) There is a stationary solution for $a(t)$ at the maximum of the effective potential $U_{\text{eff}}(a)$ defined in (18.78). This occurs when

$$-\frac{\Omega_m}{\tilde{a}^2} + 2\Omega_v \tilde{a} = 0$$

or

$$\rho_m = 2\rho_v = \frac{\Lambda}{4\pi} \,.$$

b) The Friedman equation (18.77) implies

$$U_{\text{eff}}(\tilde{a}) = \frac{\Omega_c}{2}$$

or

$$-\frac{8\pi}{3}\ (\rho_m + \rho_v)\ a^2 = -1$$

or

$$a = \frac{1}{\sqrt{\Lambda}}\ .$$

So the volume is [*cf.* (18.55)]

$$V = 2\pi^2 a^3 = 2\pi^2 \Lambda^{-\frac{3}{2}}\ .$$

c) As the plot of $U_{\text{eff}}(a)$ in Figure 18.9 shows, the static universe is unstable. A small change in $\rho_m$ will either cause it to expand to infinite volume or collapse to a singularity.

**18-25**. *[E] Estimate the smallest value of the $\Omega_v$ that would allow the universe to bounce at a small radius, but still reach a temperature $T \sim 10^{10}$ K such that nucleosynthesis could occur. Assume $\Omega_r = 8 \times 10^{-5}$ and $\Omega_m = .3$.*

**Solution:** For the temperature at the bounce to be greater than $\sim 10^{10}$K, $\tilde{a}_{\text{bounce}}$ must be less than $\sim 10^{-10}$ [*cf.* (18.26)]. Evidently, from Figure 18.9 the maximum of $U_{\text{eff}}(\tilde{a})$ must occur at a value of $\tilde{a}$ less than the value at the bounce. At such early times the matter can be neglected and the maximum of $U_{\text{eff}}(\tilde{a})$ occurs at

$$\tilde{a}_{\max} = \left(\frac{\Omega_r}{\Omega_v}\right)^{\frac{1}{4}}\ .$$

Thus, a bounce would require $\Omega_0/\Omega_r > 10^{40}$ or $\Omega_v > 10^{44}$ since $\Omega_r \approx 10^{-4}$ [*cf.* (17.2), (18.32)]. That kind of value is wildly inconsistent with present observations as will be seen in the next chapter.

**18-26**. *[A!] Assuming that $\Omega_r = 0$ as is approximately true for our universe, find the algebraic relations between $\Omega_v$ and $\Omega_r$ that determine the boundaries in Figure 18.10 dividing the various behaviors of the FRW models.*

**Solution:** As Figure 18.9 illustrates, the qualitative behavior of the FRW models is determined by the relationship between $\Omega_c$ and the maximum of the

effective potential $U_{\text{eff}}(\tilde{a})$. When $\Omega_r = 0$

$$U_{\text{eff}}(\tilde{a}) = -\frac{1}{2}\left(\Omega_v\, \tilde{a}^2 + \frac{\Omega_m}{\tilde{a}}\right).$$

The maximum occurs at $\tilde{a}_{\max} = (\Omega_m/2\Omega_v)^{1/3}$ and is

$$U_{\max} = -\frac{1}{2}\left(2^{1/3} + \frac{1}{2^{2/3}}\right)\, \Omega_m^{2/3}\, \Omega_v^{1/3} \equiv -\frac{q}{2}\, \Omega_m^{2/3}\, \Omega_v^{1/3}$$

where $q \approx 1.89$. Models which start at a big bang and expand forever have

$$\Omega_c/2 > U_{\max} \qquad \text{(big bang \& expand forever)}.$$

The borderline is defined by the algebraic equation

$$\Omega_c = 1 - \Omega_v - \Omega_m = -q\, \Omega_m^{2/3}\, \Omega_v^{1/3}.$$

This equation can be rewritten as a cubic equation in $y = \Omega_v^{1/3}$ and $x = \Omega_m^{1/3}$. There are therefore a total of three roots for $y(x)$ or $\Omega_v(\Omega_m)$. Some understanding of these can be found from limiting cases: When $\Omega_m = 0$ the solution is $\Omega_v = 1$. That is a case where the universe has no big bang. The root $\Omega_v(\Omega_m)$ which includes this solution is the curve in the upper left of Figure 18.10. There is another root for $\Omega_v = 0$ which has $\Omega_m = 1$. That is a case where the universe recollapses. That is the curve that lies almost along the $\Omega_v = 0$ axis in Figure 18.10.

**18-27**. *[C] Bouncing Universes*

a) *Show that for any form of the effective potential $U_{\text{eff}}(a)$ defined in* (18.77) *there is an equation of state $p = p(\rho)$ that will produce it. Find (parametric) expressions for $p$ and $\rho$ in terms of $U_{\text{eff}}(a)$.*

b) *Sketch a potential $U_{\text{eff}}(a)$ that would give rise to a closed bouncing universe — one which eternally oscillates between a maximum and minimum volume. What properties does your potential have to have so that it has no detectable effect on the past evolution of the universe between today and say a radiation temperature of $k_B T \sim 10 Mev$ just above that when nuclei were synthesized in the big bang. (See Box 19.1 for more on that, but that information is not necessary to work the problem.)*

c) *Show that generally the combination $\rho + 3p$ for this hypothetical matter will be negative at very high densities.*

*(Comment: The result in part (c) can be turned around to say that if $\rho+3p$ is always positive there is a big bang singularity — an example of a singularity theorem. (Problem 28). No known form of matter has even negative pressure below nuclear densities.)*

**Solution:**

(a) From (18.63) and (18.77) the density is related to the effective potential $U_{eff}(a)$ by

$$\rho(a) = -\frac{3}{4\pi a^2}\, U_{eff}(a)\ . \tag{1}$$

The pressure can be found from the first law of thermodynamics (18.20).

$$p(a) = -\frac{d(\rho a^3)}{d(a^3)} = +\frac{1}{4\pi a^2}\,\frac{d[a\, U_{eff}(a)]}{da} \tag{2}$$

Eqs. (1) and (2) provide a parametric representation of the equation of state $p = p(\rho)$ that would produce any given $U_{eff}(a)$.

(b) The figure below is a sketch of a potential $U_{eff}(a)$ that would produce a bouncing universe that oscillates between $a_{\min}$ and $a_{\max}$. To reproduce the usual evolution between $k_B T \sim 10 Mev$ and now, the potential should be well approximated by (18.78) between $a_{ns}$ and $a_0$ where these are the scale factors when the temperatures are $k_B T_{ns} \sim 10 Mev$ and $T_0 \sim 3°K$, i.e. over $a$ range $(a_0/a_{ns}) \sim (T_{ns}/T_0) \sim 10^{11}$. The figure below is therefore not to scale!

(c) From equations (1) and (2)

$$\rho + 3p = \frac{3}{4\pi a^2} \frac{dU_{eff}}{da} .$$

The upwards turn in the potential necessary for the universe to bounce means $\rho + 3p$ must be negative for $a$ between $a_{\min}$ and the value which minimizes $U_{\text{eff}}$.

**18-28**. *FRW Singularity Theorem. Show in the context of the FRW models that if the combination $\rho + 3p$ is always positive then there will be a big bang singularity sometime in the past.*

**Solution:** Extrapolating backwards from the present there will be a big bang singularity in the past if $a(t)$ crosses zero at a finite time. To avoid crossing the $a = 0$ axis, $\ddot{a}$ must be positive somewhere along the curve $a(t)$. Therefore if $\ddot{a} < 0$ always, there will be a big bang singularity.

To compute $\ddot{a}$, differentiate (18.77) with respect to time. One finds

$$\frac{d^2\tilde{a}}{d\tilde{t}^2} = \frac{1}{2}\left(2\Omega_v\tilde{a} - \frac{\Omega_m}{\tilde{a}^2} - \frac{2\Omega_r}{\tilde{a}^3}\right) .$$

Converting this to an equation for $\ddot{a}$ using (18.72), (18.74), (18.32), (18.25), and (18.22) gives

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p)$$

Thus $\ddot{a}$ is always negative if $\rho + 3p > 0$. In that situation there is always a big bang singularity.

**18-29**. *[S] We don't know much about the cosmological constant. Suppose it were negative. Show that then every FRW model would recollapse to end in a big crunch.*

**Solution:** If $\Lambda < 0$ the effective potential $U_{\text{eff}}(\tilde{a})$ defined in (18.78) *grows* at large $\tilde{a}$ like $|\Omega_v|\tilde{a}^2/2$. As the analog of Figure 18.9 makes clear, there is no FRW evolution that reaches arbitrarily large $\tilde{a}$. All turn around and recollapse to a big crunch.

**18-30**. *[N] Embedding A Slice of an Open FRW Universe. (a) Show that a $t = \text{const.}$, $\theta = \pi/2$ slice of the open FRW metric in* (18.60) *can't be embedded as an axisymmetric surface in flat-three dimensional space.*

*(b) The following is a simple axisymmetric metric with constant negative curvature:*

$$d\Sigma^2 = du^2 + \cosh^2 u d\phi^2 \ .$$

*Show that this can be embedded as an axisymmetric surface in flat three-dimensional space but only for a limited range of $u$ starting at $u = 0$. Find the upper limit of this range, and exhibit the embedding diagram.*

*Comment: Minding's theorem in differential geometry says that all constant curvature surfaces have the same local geometry. The surface in (b) is therefore an embedding of a piece of the surface discussed in (a). It doesn't matter which piece since the geometry is homogeneous. This is the surface shown in Figure 18.6.*

**Solution:**

(a) The geometry of a $t = \text{const.}$, $\theta = \pi/2$ slice of an open FRW model is summarized by

$$d\Sigma^2 = \left(d\chi^2 + \sinh^2 \chi d\phi^2\right) \ ,$$

taking $a = 1$ for simplicity. Following the discussion in Example Chapter 7.7, we seek functions $z(\chi)$ and $\rho(\chi)$ that will reproduce this line element

when substituted into

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2 \ .$$

Evidently $\rho(\chi) = \sinh(\chi)$ and

$$\left(\frac{dz}{d\chi}\right)^2 + \left(\frac{d\rho}{d\chi}\right)^2 = 1 \ .$$

But this implies

$$\left(\frac{dz}{d\chi}\right)^2 = \left(1 - \cosh^2\chi\right)$$

which can't be satisfied because $\cosh\chi$ is always greater than one.

(b) If we repeat the above process for

$$d\Sigma^2 = du^2 + \cosh^2 u d\phi^2$$

we find

$$\begin{aligned} \rho(u) &= \cosh u \\ z(u) &= \int_0^u du' \left(1 - \sinh^2 u'\right)^{\frac{1}{2}} \ . \end{aligned}$$

This does define an embeddable surface but only up to $u = \sinh^{-1}(1) = .88$.

**18-31**. *Evaluate* (18.82) *to find the age of a FRW model that is matter dominated from start to finish as a function of* $H_0$ *and* $\Omega = \Omega_m$. *For given* $H_0$, *which are older — open models or closed models? Does your analytic answer agree with Figure 18.8?*

**Solution:** When $\Omega_v = \Omega_r = 0$ and $\Omega = \Omega_m$, (18.82) becomes

$$H_0 t_0(\Omega) = \int_0^1 d\tilde{a} \left[(1 - \Omega) + \frac{\Omega}{\tilde{a}}\right]^{-\frac{1}{2}} .$$

The explicit result is

$$H_0 t_0(\Omega) = \frac{\Omega}{2(\Omega-1)^{3/2}}\left[\cos^{-1}\left(\frac{2}{\Omega}-1\right)-\frac{2}{\Omega}(\Omega-1)^{\frac{1}{2}}\right], \Omega > 1$$

$$H_0 t_0(\Omega) = \frac{\Omega}{2(\Omega-1)^{3/2}}\left[\frac{2}{\Omega}(1-\Omega)-\cosh^{-1}\left(\frac{2}{\Omega}-1\right)\right], \Omega < 1.$$

The following plot shows $H_0 t_0(\Omega)$ as a function of $\Omega$

Evidently the open models ($\Omega < 1$) are always older than the closed ones ($\Omega > 1$). That's as one might expect physically. Low density universes decellerate less quickly than high density ones. Note also that $H_0 t_0 < 1$ as is consistent with the argument given in Figure 18.2. The plot is in good agreement with Figure 18.2 when $\Omega_v = 0$.

**18-32**. *Express the present distance to the particle horizon $d_{horiz}(t_0)$ in terms of the cosmological parameters by an integral formula analogous to* (18.82) *for the age.*

**Solution:** The distance to the present horizon $d_{\text{horiz}}(t_0)$ is given by (18.47)

$$d_{\text{horiz}}(t_0) = a(t_0)\int_0^{t_0}\frac{dt}{a(t)}\,. \tag{1}$$

Although this formula was derived for flat FRW models, it holds for curved ones as well. (See the solution to Problem 22.) In terms of the rescaled

variables in (18.72) and (18.73) this becomes

$$H_0\, d_{\text{horiz}}(t_0) = \int_0^{H_0 t_0} \frac{d\tilde{t}}{\tilde{a}(\tilde{t})} = \int_0^1 \frac{1}{\tilde{a}} \frac{d\tilde{a}}{(d\tilde{a}/d\tilde{t})} \tag{2}$$

Then solving (18.77) for $d\tilde{a}/d\tilde{t}$ gives

$$H_0\, d_{\text{horiz}}(t_0) = \int_0^1 \frac{d\tilde{a}}{\tilde{a}\,[\Omega_c - 2U_{\text{eff}}(\tilde{a})]^{\frac{1}{2}}} \,. \tag{3}$$

Explicitly,

$$H_0\, d_{\text{horiz}}(t_0) = \int_0^1 \frac{d\tilde{a}}{\tilde{a}} \left[\Omega_c\,(\Omega_r,\, \Omega_m,\, \Omega_v) + \Omega_v \tilde{a}^2 + \Omega_m \tilde{a}^{-1} + \Omega_r \tilde{a}^{-2}\right]^{-\frac{1}{2}} \,. \tag{4}$$

Here $\Omega_c = 1 - \Omega_r - \Omega_m - \Omega_v$. This expresses $d_{\text{horiz}}(t_0)$ in terms of $H_0$ and the $\Omega$'s.

**18-33**. *[N] Evaluate the formula for the present distance to the horizon obtained in Problem 32 for the cosmological parameters* $\Omega_v = .7$, $\Omega_r = 8\times10^{-5}$, $\Omega_m = .3$, $H_0 = 72$ (km/s)/Mpc *which best characterize our universe at the time of writing. Express your answer in Mpc.*

**Solution:** Evaluating the integral numerically for the given parameters gives ($c = 1$ units)

$$H_0 d_{\text{horiz}} = 3.27 \,.$$

Since $c/H_0 = 2998h^{-1}$ Mpc this yields

$$d_{\text{horiz}} = 14 \text{ Gpc}$$

for $h = .72$.

# Chapter 19

# Which Universe and Why?

**19-1**. *[B] The radio source 3C345 discussed in Box 4.3 has a redshift of* $z = .595$. *The angular velocity of the outward moving cloud C2 is approximately* .47 mas/yr. *Assuming (contrary to fact) that the cloud is moving transverse to the line of sight, what velocity would be seen by an observer on station at 3C345 assuming a flat* ($k = 0$)*, matter dominated FRW model of the universe with* $h = .72$*?*

**Solution:** As usual we denote quantities at the present by a subscript 0 and quantities at the event of emission by a subscript $e$. Assume we are at the origin ($r = 0$) of the spherical coordinates in the flat FRW line element (18.5) and that 3C345 is at a co-moving radius $R$. In these coordinates, transverse motion means that the cloud is moving on a circle of constant $R$. The angle $\Delta\phi$ moved in a time $\Delta t_e$ is connected to the distance moved $\Delta s$ by (19.11) (evaluated at $t_e$ not $t_{ls}$)

$$\Delta\phi = (1+z)\,\frac{\Delta s}{d_{\text{eff}}}\,. \tag{1}$$

The angle $\Delta\phi$ is the same as the angular separation observed by us because light rays from the cloud and source to us propagate along $\phi =$ const. curves by the symmetry of the geometry. Thus, the velocity $V_e = ds/dt_e$ observed by a nearby observer is

$$V_e = \frac{d_{\text{eff}}}{(1+z)}\,\frac{d\phi}{dt_e} = \frac{d_{\text{eff}}}{(1+z)}\left(\frac{dt_0}{dt_e}\right)\left(\frac{d\phi}{dt_0}\right). \tag{2}$$

This relates $V_e$ to our observed rate of change of angle. The quantity $dt_0/dt_e$ is just $1+z$ [*cf.* (18.9), (18.11)]. The rate is slower now than then because of

the redshift. The effective distance, for a matter-dominated, flat FRW model is given by (19.5). Thus,

$$V_e = \frac{2}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right)\left(\frac{d\phi}{dt_0}\right). \tag{3}$$

Evaluating with $1/H_0 = 9.8 \times 10^9 h^{-1}$ yr, $z = .595$, $d\phi/dt_0 = .47$ mas/yr, and $h = .72$ gives (in $c \neq 1$ units)

$$\frac{V_e}{c} = 9.3\ h^{-1} \approx 13. \tag{4}$$

Superluminal motion indeed!

**19-2**. *[S] Could the observed vacuum mass-energy density in the universe be a consequence of quantum gravity? One obstacle to such an explanation is the great difference in scale between observed vacuum mass density $\rho_v$ and the Planck mass density $\rho_{Pl} \equiv c^5/\hbar G^2$ [cf. (1.6)] that might be expected on dimensional grounds to characterize quantum gravitational phenomena (Chapter 1).*

*(a) Show that $\rho_{Pl}$ is the correct combination of $\hbar$, $G$, and $c$ with the dimensions of mass density.*

*(b) Evaluate the ratio $\rho_v/\rho_{Pl}$.*

**Solution:**

a) The dimensions of $G$ are $\mathcal{L}^3/\mathcal{M}\mathcal{T}^2$, those of $\hbar$ are $\mathcal{M}\mathcal{L}^2/\mathcal{T}$, and those of $c$ are $\mathcal{L}/\mathcal{T}$ in usual $\mathcal{M}\mathcal{L}\mathcal{T}$ units. The dimensions of $\rho_{Pl} \equiv c^5/\hbar G^2$ are therefore $\mathcal{M}/\mathcal{L}^3$ showing it is a mass density.

b) The value $\Omega_v \sim .7$ corresponds to $\rho_v \sim .7\rho_{\text{crit}} \sim 6 \times 10^{-30}$ g/cm$^3$ from (18.32) with $h \sim .7$. The ratio is therefore

$$\frac{\rho_v}{\rho_{Pl}} \sim \frac{6 \times 10^{-30} g/\text{cm}^3}{5 \times 10^{93} g/\text{cm}^3} \sim 10^{-123}(!)$$

**19-3**. *[S] Show that the expression (19.5) for the effective distance $d_{\text{eff}}$ in a flat, matter dominated FRW model follows from (19.7) and (19.9)*

**Solution:** Combining (19.7) and (19.9) gives the following expression for $H_0 d_{\text{eff}}$

$$H_0 d_{\text{eff}} = \int_{(1+z)^{-1}}^{1} \frac{d\tilde{a}}{\tilde{a}\left[\Omega_c - 2U_{\text{eff}}(\tilde{a})\right]^{1/2}} .$$

Evaluating this for $\Omega_m = 1$, $\Omega_v = \Omega_r = 0$ using (18.76) and (18.78) gives

$$H_0 d_{\text{eff}} = \int_{(1+z)^{-1}}^{1} \frac{d\tilde{a}}{\tilde{a}^{1/2}} = 2\left[1 - \frac{1}{\sqrt{1+z}}\right] .$$

This is (19.5).

**19-4**. *For small $z$ the redshift-magnitude relation is given by the inverse square law* (19.3). *This is the first term in an expansion in $z$ of the form*

$$\frac{f}{L} = \frac{H_0^2}{4\pi z^2}(1 + const.z + \cdots) .$$

*Find the constant and express it in terms of the cosmological parameters. Sketch redshift-magnitude curves which both have $\Omega_m = .3$ and $\Omega_r = 0$ for the two values $\Omega_v = 0$ and $\Omega_v = .7$.*

**Solution:** The red-shift magnitude relation is given generally by (19.4) so the key to this problem is to expand $d_{\text{eff}}(z)$ in powers of $z$. The leading term will be $d_{\text{eff}}(z) \approx z/H_0$ to reproduce (19.3). We seek the coefficient of the next term proportional to $z^2$ in the effective distance $d_{\text{eff}}(z)$ is defined by (19.7) and (19.9). When $z$ is small $\chi$ will be small. For small $\chi$

$$\sin\chi = \chi - \frac{\chi^3}{3} + \cdots \quad , \quad \sin\chi = \chi + \frac{\chi^3}{3} + \cdots .$$

The $\chi^3$ corrections will not affect $d_{\text{eff}}$ calculated to quadratic order in $z$. Therefore, combining (19.7) and (19.9)

$$H_0 d_{\text{eff}}(z) \approx \int_{(1+z)^{-1}}^{1} \frac{d\tilde{a}}{\tilde{a}[\Omega_c - 2U_{\text{eff}}(\tilde{a})]^{\frac{1}{2}}} \equiv I(z) . \quad \text{(small z)}$$

The rest is the algebra expanding out the right-hand side.

$$I(z) = I'(0)z + \frac{1}{2} I''(0)z^2 + \cdots .$$

Calculating the derivatives is straightforward but the following facts may be helpful

$$\begin{aligned} \frac{d}{dx}\int_x^1 dy\, f(y) &= -f(x)\,, \\ (1+z)^{-1} = 1 - z &+ z^2 + \cdots\,. \end{aligned}$$

The result is

$$H_0 d_{\text{eff}}(z) = z - \frac{1}{2}\left[1 + \frac{1}{2}(2\Omega_r + \Omega_m - 2\Omega_v)\right] z^2 + \cdots\,.$$

Inserting this into (19.4) gives

$$\frac{f}{L} = \frac{H^2}{4\pi z^2}\left[1 - (1 - \Omega_r + \Omega_v - \Omega_m/2)\, z + \cdots\right]\,.$$

When $\Omega_m = 1$, $\Omega_v = \Omega_r = 0$ this reduces to the expansion of (19.6).

Your sketch should look like the theoretical curves in Figure 19.2.

**Comment:** At this order of $z$, measurements of the redshift-magnitude relation determine the combination $\Omega_r - \Omega_v - \Omega_m/2$.

**19-5.** *[S] Show that the effective distance $d_{\text{eff}}$ defined in (19.7) of a galaxy in a spatially flat universe at redshift $z$ can be written as*

$$d_{\text{eff}} = \int_0^z dz'/H(z')\,.$$

*where $H(z')$ is the value of the Hubble constant at the time at when light from a galaxy with red shift $z'$ was emitted.*

**Solution:** Light emitted at time $t$ in the universe reaches us at today's time $t_0$ with redshift $z$ given by

$$1 + z = \frac{a(t_0)}{a(t)} \tag{1}$$

according to (18.11). The Hubble constant at the time of emission was

$$H(t) = \frac{\dot{a}(t)}{a(t)} \tag{2}$$

according to (18.14). The effective distance of the galaxy emitting light at time $t_e$ in a spatially flat universe is given by (19.7) and (19.8)

$$d_{\text{eff}} = a(t_0) \int_{t_e}^{t_0} \frac{dt'}{a(t')}. \tag{3}$$

Eq. (1) can be used to trade $t'$ for $z'$ as an integration parameter and (2) can be used to re-express the integrand. The result is as quoted in the problem.

**19-6**. *Standard Rulers: Suppose a certain kind of galaxy always had a fixed size. It then could be used as a standard ruler — from its angular size its distance could be computed. Derive the (redshift)-(angular size) relation that is analogous to the (redshift)-(magnitude) relation for a flat, matter dominated FRW universe. Show that there is a certain redshift beyond which the angular size of the object increases with redshift and find its value. Does this mean that objects will get brighter the further they are from us?*

**Solution:** Combining (19.11) and (19.5) gives the following relation for the angular size as a function of redshift.

$$\Delta\phi = (1+z)\frac{\Delta s}{d_{\text{eff}}(z)} = \frac{\Delta s H_0}{2}\frac{(1+z)^{\frac{3}{2}}}{\sqrt{1+z}-1}$$

For small $z$, $\Delta\phi \approx (H_0 s)/z$ which is decreasing with increasing $z$. For large $z$, $\Delta\phi \approx \Delta s z/H_0$ which is increasing with increasing $z$. There is therefore a minimum angular size which occurs at $z = 5/4$.

This does not mean that galaxies get brighter as they move further away because (19.6) shows that they get steadily dimmer with increasing $z$. They get bigger and dimmer at very large $z$.

**19-7**. *[C] (Number Counts of Galaxies.) Suppose a census was taken of the number of galaxies $N_{\text{gal}}(Z)$ with a redshift less than a particular value $Z$. Assume that the number density of galaxies $n_{\text{gal}}(t)$ is uniform in space but changing in time. What is the prediction of a flat, matter dominated, FRW model for how $N_{\text{gal}}(Z)$ depends on $Z$? Express your answer in terms of $Z$, the Hubble constant, and the present density of galaxies $n_{\text{gal}}(t_0)$. Comment: Counting galaxies is another route to determining cosmological parameters,*

*but the further away they are, the dimmer they are, and the harder to count.*

**Solution:** In a flat, matter-dominated FRW model, galaxies at redshift $z$ are located at a coordinate radius $\chi$ which is [*cf.* (19.5)]

$$\chi(z) = \frac{2}{H_0}\left[1 - \frac{1}{\sqrt{1+z}}\right] \tag{1}$$

(The normalization $a(t_0) = 1$ has been adopted.) The number of galaxies at redshift $z$ in an interval $dz$ is

$$n_{\rm gal}(t_e)\, 4\pi a^3(t_e)\chi^2 d\chi \tag{2}$$

where $t_e$ is the time the light at redshift $z$ was emitted, and $d\chi$ is the interval in $\chi$ corresponding to $dz$ through (1). Now,

$$n_{\rm gal}(t_e) = n_{\rm gal}(t_0)/a^3(t_e)$$

so that the total number with a redshift $z < Z$ is

$$\begin{aligned} N_{\rm gal}(Z) &= \int_0^Z dz\, 4\pi\, n_{\rm gal}(t_0)\chi^2(z)\, \frac{d\chi}{dz} \\ &= \frac{16\pi\, n_{\rm gal}(t_0)}{H_0^3}\int_0^Z \frac{dz}{(1+z)^{\frac{3}{2}}}\left(1 - \frac{1}{\sqrt{1+z}}\right)^2 \\ &= \frac{32\pi\, n_{\rm gal}(t_0)}{3H_0^3}\left[1 + \frac{3\sqrt{1+Z} - 3Z - 4}{(1+Z)^{\frac{3}{2}}}\right]. \end{aligned}$$

For small $Z$

$$N_{\rm gal}(Z) = \frac{4}{3}\,\pi\, n_{\rm gal}(t_0)\,\left(\frac{Z}{H_0}\right)^3$$

which is the correct relationship given Hubble's law.

**19-8**. *Assume that our universe is characterized by the cosmological parameters* $H_0 = 72$ (km/s)/Mpc, $\Omega_m = .3$, $\Omega_r = 8 \times 10^{-5}$, $\Omega_v = .7$. *Also assume that last scattering occurs at a redshift of 1100.*

*(a) Compare the temperature at last scattering with the temperature $T$ at which $k_BT$ is equal to the binding energy of hydrogen.*

*(b) Before what redshift $z$ is the universe radiation dominated?*

**Solution:**

a) The binding energy of hydrogen is 13.6 eV. The energy $k_B T$ has this value when $T \approx 1.6 \times 10^5$ K. At a redshift of $z = 1100$ the temperature of the photon gas is, from (18.26) and (18.11)

$$T_{ls} = T_0 \left( \frac{a(t_0)}{a(t_{ls})} \right) = T_0(1+z) = 2.73 \text{ K} \times 1100 \approx 3000 \text{ K}.$$

Evidently, the surface of last scattering is at a much lower temperature than a naive estimate based on the binding energy of hydrogen. For detailed calculations of the ionization as a function of redshift see, *e.g.* Kolb and Turner (1990).

b) The energy density in matter is $\Omega_m \rho_{\text{crit}}$ now and increases to the past like $(1+z)^3$ [*cf.* (18.11) and (18.22)]. Similarly, the energy density in radiation is $\Omega_r \rho_{\text{crit}}$ now and increases to the past like $(1+z)^4$ [*cf.* (18.11) and (18.25)]. They are equal at the redshift $z_{\text{eq}}$ when

$$\Omega_m \rho_{\text{crit}} (1 + z_{\text{eq}})^3 = \Omega_r \rho_{\text{crit}} (1 + z_{\text{eq}})^4 \ .$$

Taking $\Omega_m = .3$ and $\Omega_r = 8 \times 10^{-5}$ gives

$$z_{\text{eq}} \approx 3700.$$

For large $z$ we have $\tilde{a} \sim 1/z$ [equation (18.11)], which means $\tilde{a}_{\text{eq}}$ is about 3.4 times smaller than $\tilde{a}_{ls}$. Since $\tilde{a}_{ls}$ is $1,100$ times smaller than the present scale factor of $\tilde{a}_0 = 1$, we see that on cosmological scales last scattering and matter-radiation equality happen almost at the same moment.

**19-9**. *Calculate the age of our universe at the time of last scattering.*

**Solution:** The age of the universe at a given redshift $z$ is determined in the same way as the present age of the universe when $z = 0$. The age $t(z)$ is given by expressions like (18.81) and (18.82) except that the upper limit is given by

the value of $\tilde{a}$ at redshift $z$ rather than 0. That value from (18.11) and (18.72) is just $\tilde{a} = 1/(1+z)$. Thus,

$$H_0 t(z) = \int_0^{(1+z)^{-1}} \frac{d\tilde{a}}{[\Omega_c - 2U_{\text{eff}}(\tilde{a})]^{1/2}}.$$

We evaluate this for the cosmological parameters of our universe where $\Omega_r \approx 0$, $\Omega_c \approx 0$ to find

$$H_0 t(z) = \int_0^{(1+z)^{-1}} \frac{d\tilde{a}}{[\Omega_m/\tilde{a} + \Omega_v \tilde{a}^2]^{1/2}}.$$

This can be evaluated numerically for $z = 1100$ even including the small effect of the radiation. However, it's clear that over the range $\tilde{a} = 0$ to $1/(1101)$, the vacuum energy term is much less than the matter energy. Therefore, to a good approximation, $\Omega_v$ can be neglected in the integrand and the resulting integral done immediately. The result is

$$t(z) \approx \frac{2}{3\Omega_m^{1/2}} \left(\frac{1}{1+z}\right)^{3/2} t_H, \qquad \text{(large } z\text{)}$$

where $t_H = H_0^{-1} = 9.78h^{-1}$ Gyr. Evaluating this for $z = 1100$, $\Omega_m = .3$ we find

$$t(1100) \approx 450,000 \text{ yr}.$$

**19-10**. *Measure the Cosmological Constant in the Laboratory? It's not possible to measure the matter density $\rho_m$ in a laboratory of the typical size found on Earth. The FRW approximation that the matter is smoothly distributed breaks down on those scales. But a fundamental vacuum energy could be exactly uniform and therefore in principle detectable in a laboratory experiment. Calculate how two test particles in a freely falling laboratory would move relative to each other in the presence of a vacuum energy corresponding to $\Omega_v = 1$. Estimate the time scale for significant relative motion and the size of their relative acceleration assuming they start 1 cm apart. Is laboratory detection feasible?*

**Solution:** Inside the laboratory the geometry will be well approximated by a flat FRW model with a scale factor evolving according to (18.39). Consider

two test particles initially at rest at time $t_0$, at the origin and at radius $r_0$ in the coordinates in which the metric has the form (18.5). The test particles remain at rest in these coordinates, but the distance $d(t)$ between them changes. The distance as a function of $t$ is

$$d(t) = e^{H(t-t_0)} r_0$$

or

$$d(t) = e^{Ht} d(0) .$$

The characteristic time for evolution is therefore $1/H$ which is the Hubble time (18.15) when $\Omega_v = 1$. This is of order 10 Gyr.

The characteristic acceleration if the particles are initially 1 cm apart is $d(0)H^2 \sim 10^{-35}$ cm/s$^2$. A 200 kg experimenter standing a distance of 1 m away would produce an acceleration of

$$G(200 \text{ kg})/(1 \text{ m})^2 \sim 10^{-6} \text{ cm/s}^2 .$$

Since gravity cannot be shielded, such perturbations would make a detection of vacuum energy unfeasible.

# Chapter 20

# A Little More Math

**20-1**. *[S] Show explicitly that the transformation rule* (20.6a) *leads to the transformation of vector components under a Lorentz boost* (4.33) *given in* (5.9).

**Solution:** From (20.6a) the transformed coordinates of a vector **a** are

$$a'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} a^{\beta}. \tag{1}$$

Evaluating the partial derivatives from (4.33) we have for example

$$\frac{\partial x'^0}{\partial x^0} = \frac{\partial t'}{\partial t} = \gamma, \quad \frac{\partial x'^0}{\partial x^1} = \frac{\partial t'}{\partial x} = \upsilon\gamma, \quad \text{etc.} \tag{2}$$

Inserting (2) into (1) and carrying out the implied sum gives (5.9).

**20-2**. *[S] (a) Evaluate*

$$\frac{\partial x^{\beta}}{\partial x'^{\alpha}} \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} .$$

*(b) Use this result to show explicitly that the transformation law* (20.6b) *is the inverse of* (20.6a).

**Solution:**
a)

$$\frac{\partial x^{\beta}}{\partial x'^{\alpha}} \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} = \delta^{\beta}_{\gamma'} .$$

b) Multiply both sides of (20.6a) by $\partial x^\gamma/\partial x'^\beta$ to find

$$\frac{\partial x^\gamma}{\partial x'^\beta}\, a'^\beta = \frac{\partial x^\gamma}{\partial x'^\beta}\,\frac{\partial x'^\beta}{\partial x^\alpha}\, a^\alpha = \delta^\gamma_\alpha a^\alpha = a^\gamma \ ,$$

which is (20.6b) with the free index relabeled.

**20-3**. *[S] Use the transformation (7.2) connecting rectangular coordinates $(t, x, y, z)$ for flat space to polar coordinates $(t, r, \theta, \phi)$, to find explicitly the transformation laws giving the components $(a^t, a^x, a^y, a^z)$ of a vector* **a** *in terms of the components $(a^t, a^r, a^\theta, a^\phi)$ and the components $(a_t, a_x, a_y, a_z)$ in terms of $(a_t, a_r, a_\theta, a_\phi)$.*

**Solution:** Since $t$ is the same in both cases, $a^t$ doesn't change and neither does $a_t$. Writing out the transformation rule (20.6a) gives, for example,

$$a^x = \frac{\partial x}{\partial t}\, a^t + \frac{\partial x}{\partial r}\, a^r + \frac{\partial x}{\partial \theta}\, a^\theta + \frac{\partial x}{\partial \phi}\, a^\phi \ .$$

Then using $x = r\sin\theta\cos\phi$ gives

$$a^x = \sin\theta\cos\phi\, a^r + r\cos\theta\cos\phi\, a^\theta - r\sin\theta\sin\phi\, a^\phi \ .$$

The results for $a^y$ and $a^z$ are then obtained similarly:

$$\begin{aligned} a^y &= \sin\theta\sin\phi\, a^r + r\cos\theta\sin\phi\, a^\theta + r\sin\theta\cos\phi\, a^\phi \ , \\ a^z &= \cos\theta\, a^r - r\sin\theta\, a^\theta \ . \end{aligned}$$

The transformations of the inverse metric are obtained most efficiently just by lowering indices

$$a_x = a^x \ , \quad a_y = a^y \ , \quad a_z = a^z$$

and

$$a_r = a^r \ , \quad a_\theta = r^2 a^\theta \ , \quad a_\phi = r^2\sin^2\theta a^\phi \ .$$

Then, for example

$$a_x = \sin\theta\cos\phi\, a_r + r^{-1}\cos\theta\cos\phi\, a_\theta - r^{-1}\sin^{-1}\theta\sin\phi\, a_\phi \ ,$$

etc.

**20-4**. *In the Schwarzschild geometry consider the following function:*

$$f(x) = (5t^2 - 2r^2)/(2M)^2$$

*where $t$ and $r$ are the usual Schwarzschild coordinates in which the metric has the form* (9.9). *Find the coordinate basis components $(\boldsymbol{\nabla} f)^\alpha$ of the gradient of $f$.*

**Solution:** The downstairs components of the gradient are

$$\nabla_\alpha f = \frac{\partial f}{\partial x^\alpha} = (10t\,, -4r\,, 0\,, 0)/(2M)^2$$

The upstairs components are

$$\begin{aligned} \nabla^\alpha f &= g^{\alpha\beta}\nabla_\alpha f \\ &= \frac{1}{(2M)^2}\left[-\left(1-\frac{2M}{r}\right)^{-1} 10t\,, \quad -\left(1-\frac{2M}{r}\right)(4r)\,, 0\,, 0\right] . \end{aligned}$$

**20-5**. *Eq.* (20.81) *gives the upstairs coordinate basis components of a set of four vectors $\{\mathbf{e}_{\hat\alpha}\}$ constituting an orthonormal frame in the Schwarzschild geometry.*

*(a) Verify explicitly that this is an orthonormal set of vectors.*

*(b) Find the downstairs coordinate basis components of each of these vectors.*

*(c) Find the upstairs coordinate basis components of the basis $\mathbf{e}^\alpha$ that is dual to the given set of basis vectors*

*(d) Consider a vector* **a** *with upstairs coordinate basis components*

$$a^\alpha = (4, 3, 0, 0)$$

*at the point $(0, 3M, 0, 0)$. Find the components $a^{\hat\alpha}$ and $a_{\hat\alpha}$ of this vector in the given orthonormal frame.*

**Solution:**

a) The scalar products can be computed from the given coordinate basis components as

$$\mathbf{e}_{\hat{\alpha}} \cdot \mathbf{e}_{\hat{\beta}} = g_{\alpha\beta}(\mathbf{e}_{\hat{\alpha}})^{\alpha}(\mathbf{e}_{\hat{\beta}})^{\beta}$$

and should equal $\eta_{\hat{\alpha}\hat{\beta}}$. For example

$$\begin{aligned}\mathbf{e}_{\hat{t}} \cdot \mathbf{e}_{\hat{r}} = \; &+ \left(1-\frac{2M}{r}\right)\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{2M}{r}\right)^{\frac{1}{2}}\left(1-\frac{2M}{r}\right)^{-1} \\ &+ \left(1-\frac{2M}{r}\right)^{-1}\left[-\left(\frac{2M}{r}\right)^{\frac{1}{2}}\right]\cdot 1 = 0 \; .\end{aligned}$$

b) The downstairs components are

$$(\mathbf{e}_{\hat{\alpha}})_{\alpha} = g_{\alpha\beta}(\mathbf{e}_{\hat{\alpha}})^{\beta} \; .$$

The result is

$$\begin{aligned}(\mathbf{e}_{\hat{t}})_{\alpha} &= \left[-1\,,-\left(\frac{2M}{r}\right)^{\frac{1}{2}}\left(1-\frac{2M}{r}\right)^{-1},0\,,0\right] \; , \\ (\mathbf{e}_{\hat{r}})_{\alpha} &= \left[\left(\frac{2M}{r}\right)^{\frac{1}{2}},\left(1-\frac{2M}{r}\right)^{-1},0\,,0\right] \; , \\ (\mathbf{e}_{\hat{\theta}})_{\alpha} &= (0\,,0\,,r\,,0) \; , \\ (\mathbf{e}_{\hat{\phi}})_{\alpha} &= (0\,,0\,,0\,,r\sin\theta) \; .\end{aligned}$$

c) Since this is an orthonormal basis, the dual vectors are given by

$$e^{\hat{0}} = -e_{\hat{0}} \; , e^{\hat{\imath}} = e_{\hat{\imath}} \; .$$

d) The components $a_{\hat{\alpha}} = \mathbf{e}_{\alpha} \cdot \mathbf{a}$ are

$$a_{\hat{\alpha}} = \left[-3\left(\frac{4}{3}+3\left(\frac{2}{3}\right)^{\frac{1}{2}}\right),3\left(\frac{4}{3}\left(\frac{2}{3}\right)^{\frac{1}{2}}+3\right),0\,,0\right] \; .$$

Then $a^{\hat{\alpha}}$ is given by (20.19).

**20-6**. *For the basis of dual vectors* $\{\mathbf{e}^\alpha\}$ *that is dual to a basis of vectors* $\{\mathbf{e}_\alpha\}$ *work out* $e^\alpha(\mathbf{a})$ *and* $a(\mathbf{e}^\alpha)$ *in terms of the components of the vector* $\mathbf{a}$ *in the basis* $\{\mathbf{e}_\alpha\}$ .

**Solution:** Writing $\mathbf{a} = a^\alpha \mathbf{e}_\alpha$, the linearity of dual vectors and the defining connection in (20.11) can be used to find

$$e^\alpha(\mathbf{a}) = e^\alpha(a^\beta \mathbf{e}_\beta) = a^\beta e^\alpha(\mathbf{e}_\beta) = a^\beta \delta^\alpha_\beta = a^\alpha.$$

Similarly, starting from the expansion of the dual vector $\mathbf{a} = a_\beta \mathbf{e}^\beta$ we find

$$a(\mathbf{e}_\alpha) = a_\beta e^\beta(\mathbf{e}_\alpha) = a_\beta \delta^\beta_\alpha = a_\alpha = g_{\alpha\beta} a^\alpha.$$

The last equality follows from (20.16).

**20-7**. *Consider a set of coordinate basis vectors* $\{\mathbf{e}_\alpha\}$ *and the associated dual basis* $\{\mathbf{e}^\alpha\}$ *defined by the relations* (20.11) *or* (20.23).

*(a) Show that basis vectors* $\{\mathbf{e}_\alpha\}$ *and dual basis vectors* $\{\mathbf{e}^\alpha\}$ *are related to each other by* $\mathbf{e}_\alpha = g_{\alpha\beta}\mathbf{e}^\beta$ *and* $\mathbf{e}^\alpha = g^{\alpha\beta}\mathbf{e}_\beta$.

*(b) Show that* $\mathbf{e}^\alpha \cdot \mathbf{e}^\beta = g^{\alpha\beta}$.

**Solution:** The starting points for this problem are the connection between a basis $\{\mathbf{e}_\alpha\}$ and its dual $\{\mathbf{e}^\alpha\}$

$$\mathbf{e}^\alpha \cdot \mathbf{e}_\beta = \delta^\alpha_\beta \tag{1}$$

and the defining relation for a coordinate basis

$$\mathbf{e}_\alpha \cdot \mathbf{e}_\beta = g_{\alpha\beta} \ . \tag{2}$$

a) Since the $\{\mathbf{e}_\alpha\}$ and $\{\mathbf{e}^\alpha\}$ are both bases, any vector can be expanded in terms of them. Thus,

$$\mathbf{e}_\alpha = c_{\alpha\beta}\, \mathbf{e}^\beta \tag{3}$$

$$\mathbf{e}^\alpha = d^{\alpha\beta}\, \mathbf{e}_\beta \tag{4}$$

for some set of coefficients $c_{\alpha\beta}$ and $d^{\alpha\beta}$. Taking the scalar product of (3) with $\mathbf{e}_\gamma$ gives

$$\mathbf{e}_\gamma \cdot \mathbf{e}_\alpha = c_{\alpha\beta}\, \mathbf{e}_\gamma \cdot \mathbf{e}^\beta = c_{\alpha\beta}\, \delta^\beta_\gamma = c_{\alpha\gamma} \ .$$

Using (2) shows $c_{\alpha\gamma} = g_{\alpha\gamma}$. Taking the scalar product of (3) and (4) and using (1) shows $d^{\alpha\beta} = g^{\alpha\beta}$.

b) Multiplying (1) by $g^{\beta\gamma}$ gives

$$\mathbf{e}^\alpha \cdot \left(g^{\gamma\beta}\, \mathbf{e}_\beta\right) = g^{\alpha\gamma} \ .$$

Then using (4) with $d^{\alpha\beta} = g^{\alpha\beta}$ gives the desired result.

**20-8**. *[S] At a point, the coordinate basis vectors $\{\mathbf{e}_\alpha\}$ in one system of coordinates $x^\alpha$ must be linear combinations of the coordinate basis vectors $\{\mathbf{e}'_\alpha\}$ in another system of coordinates $x'^\alpha$. Find the explicit transformation rule.*

**Solution:** An arbitrary vector $\mathbf{a}$ can be expressed as a linear combination of both sets of basis vectors

$$\mathbf{a} = a^\alpha \mathbf{e}_\alpha = a'^\alpha \mathbf{e}'_\alpha .$$

Eq. (20.6b) connects the components $a^\alpha$ to the components $a'^\alpha$ giving

$$a'^\alpha \mathbf{e}'_\alpha = a^\alpha \mathbf{e}_\alpha = a^\beta \mathbf{e}_\beta = \frac{\partial x^\beta}{\partial x'^\alpha}\, a'^\alpha \mathbf{e}_\beta .$$

Since this must hold for arbitrary components $a'^\alpha$, we have the connection

$$\mathbf{e}'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha}\, \mathbf{e}_\beta .$$

Note how the indices balance in this relation.

**20-9**. *Show that the operation of contraction as exemplified by (20.40) is basis independent by showing that if carried out in a another system of coordinates $x'^\alpha = x'^\alpha(x^\beta)$ the components of $w^\alpha$ transform correctly as a consequence of the transformation law for tensors.*

**Solution:** The transformation rule for a tensor $t_{\alpha\beta}{}^{\gamma}$ is

$$t'_{\alpha\beta}{}^{\gamma} = \frac{\partial x^{\delta}}{\partial x'^{a}} \frac{\partial x^{\epsilon}}{\partial x'^{\beta}} \frac{\partial x'^{\gamma}}{\partial x^{\zeta}} t_{\delta\epsilon}{}^{\zeta} \tag{1}$$

Contracting $\beta$ and $\gamma$ as in (20.40) gives

$$w'_{\alpha} \equiv t'_{\alpha\beta}{}^{\beta} = \frac{\partial x^{\delta}}{\partial x'^{\alpha}} \frac{\partial x^{\epsilon}}{\partial x'^{\beta}} \frac{\partial x'^{\beta}}{\partial x^{\zeta}} t_{\delta\epsilon}{}^{\zeta}. \tag{2}$$

But from the chain rule

$$\frac{\partial x^{\epsilon}}{\partial x'^{\beta}} \frac{\partial x'^{\beta}}{\partial x^{\zeta}} = \frac{\partial x^{\epsilon}}{\partial x^{\zeta}} = \delta^{\epsilon}_{\zeta}. \tag{3}$$

The transformation (2) becomes

$$w'_{\alpha} \equiv t'_{\alpha\beta}{}^{\beta} = \frac{\partial x^{\delta}}{\partial x'^{\alpha}} \delta^{\epsilon}_{\zeta} t_{\delta\epsilon}{}^{\zeta} = \frac{\partial x^{\delta}}{\partial x'^{\alpha}} t_{\delta\epsilon}{}^{\epsilon} = \frac{\partial x^{\delta}}{\partial x'^{\alpha}} w_{\delta}. \tag{4}$$

But that is the correct transformation law for a vector [*cf.* (20.43)]. The contraction transforms correctly as a vector.

**20-10**. *[A] Eq.* (20.48) *gives the expression for the components of the second rank tensor that results from covariant differentiation in local inertial frame where all the* $\widetilde{\Gamma}$*'s vanish. Use the transformation law for tensors* (20.45) *to obtain an expression for the* $\widetilde{\Gamma}$*'s in a general coordinate system. Use this result to show that* $\widetilde{\Gamma}^{\alpha}_{\beta\gamma}$ *is symmetric in* $\beta$ *and* $\gamma$*. (This problem fills in a gap in the argument identifying the* $\widetilde{\Gamma}^{\alpha}_{\beta\gamma}$ *with the Christoffel symbols.)*

**Solution:** Consider a point $P$, let $\bar{x}^{\alpha}$ be the coordinates of a local inertial frame at $P$, and $x^{\alpha}$ a general coordinate system whose connection $x^{\alpha}(\bar{x}^{\beta})$ is known. (We use a bar rather than a prime to avoid a *debauche d'indices.*) All of the following equations should be considered as evaluated at $P$.

The components of the covariant derivative in the local inertial frame are

$$\bar{\nabla}_{\alpha}\bar{v}^{\beta} = \frac{\partial \bar{v}^{\beta}}{\partial \bar{x}^{\alpha}}. \tag{1}$$

In the general coordinates therefore [*cf.* (20.45)]

$$\nabla_\alpha v^\beta = \frac{\partial \bar{x}^\gamma}{\partial x^\alpha} \frac{\partial x^\beta}{\partial \bar{x}^\delta} \bar{\nabla}_\gamma \bar{v}^\delta . \tag{2}$$

But also [*cf.* (20.6)]

$$\bar{v}^\beta = \frac{\partial \bar{x}^\beta}{\partial x^\epsilon} v^\epsilon . \tag{3}$$

Substitute (3) into (1), the result into (2), and expand the derivatives to find

$$\nabla_\alpha v^\beta = \frac{\partial \bar{x}^\gamma}{\partial x^\alpha} \frac{\partial x^\beta}{\partial \bar{x}^\delta} \frac{\partial \bar{x}^\delta}{\partial x^\epsilon} \frac{\partial v^\epsilon}{\partial \bar{x}^\gamma} + \frac{\partial \bar{x}^\gamma}{\partial x^\alpha} \frac{\partial x^\beta}{\partial \bar{x}^\delta} \frac{\partial^2 \bar{x}^\delta}{\partial \bar{x}^\gamma \partial x^\epsilon} v^\epsilon . \tag{4}$$

This can be simplified by using, *e.g.*

$$\frac{\partial x^\beta}{\partial \bar{x}^\delta} \frac{\partial \bar{x}^\delta}{\partial x^\epsilon} = \frac{\partial x^\beta}{\partial x^\epsilon} = \delta^\beta_\epsilon \tag{5}$$

to find

$$\nabla_\alpha v^\beta = \frac{\partial v^\beta}{\partial x^\alpha} + \tilde{\Gamma}^\beta_{\alpha\epsilon} v^\epsilon \tag{6}$$

with

$$\tilde{\Gamma}^\beta_{\alpha\epsilon} = \frac{\partial x^\beta}{\partial \bar{x}^\delta} \frac{\partial^2 \bar{x}^\delta}{\partial x^\alpha \partial x^\epsilon} . \tag{7}$$

Eq. (7) is explicitly symmetric in $\alpha$ and $\epsilon$.

**20-11**. *Work out the expression for the covariant derivative* $\nabla_\gamma t_{\alpha\beta}$ *analogous to* (20.66) *and* (20.69).

**Solution:** One way of constructing a tensor $t_{\alpha\beta}$ is to take the product of two vectors $t_{\alpha\beta} = u_\alpha v_\beta$. Employing the Leibniz rule

$$\nabla_\gamma t_{\alpha\beta} = (\nabla_\gamma u_\alpha)\ v_\beta + u_\alpha \nabla_\gamma v_\beta \ .$$

Then using (20.68) one finds

$$\nabla_\gamma t_{\alpha\beta} = \frac{\partial}{\partial x^\gamma}\ (u_\alpha v_\beta) - \Gamma^\delta_{\gamma\alpha} u_\delta v_\beta - \Gamma^\delta_{\gamma\beta} u_\alpha v_\delta \ .$$

Thus

$$\nabla_\gamma t_{\alpha\beta} = \frac{\partial t_{\alpha\beta}}{\partial x^\gamma} - \Gamma^\delta_{\gamma\alpha} t_{\delta\beta} - \Gamma^\delta_{\gamma\beta} t_{\alpha\delta}$$

which is a general formula for this covariant derivative.

**20-12**. *[A] Following Example 9 work out all the components of* $\nabla_A w^B$ *and* $\nabla_A \nabla_B w^C$ *for the vector* $w^A = (1, 0)$.

**Solution:** The Christoffel symbols are

$$\Gamma^\theta_{\theta\theta} = 0 \qquad \Gamma^\phi_{\phi\phi} = 0$$

$$\Gamma^\theta_{\theta\phi} = \Gamma^\theta_{\phi\theta} = 0 \qquad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta$$

$$\Gamma^\theta_{\phi\phi} = \sin\theta\cos\theta \qquad \Gamma^\phi_{\phi\phi} = 0.$$

For $w = (1, 0)$ the only non-vanishing component of $\nabla_A w^B$ is

$$\nabla_\phi w^\phi = \cot\theta.$$

The only non-vanishing components of $\nabla_A \nabla_B w^C$ are

$$\begin{aligned} \nabla_\phi \nabla_\phi w^\theta &= +\cos^2\theta \\ \nabla_\theta \nabla_\phi w^\phi &= -1/\sin^2\theta \\ \nabla_\phi \nabla_\theta w^\phi &= -\cot^2\theta \end{aligned}$$

Note that

$$\nabla_\theta \nabla_\phi w^\phi \neq \nabla_\phi \nabla_\theta w^\phi.$$

Covariant derivatives generally do not commute in curved spacetimes!

**20-13**. *In Example 8 the acceleration four-vector of a stationary observer in the Schwarzschild geometry could have been computed using*

$$a_\alpha = u^\beta \nabla_\beta u_\alpha$$

*and the formula* (20.68). *Show that the same result,* (20.61) *could have been obtained this way.*

**Solution:** Using (20.68) and

$$u_\alpha = \left[-\left(1-2M/r\right)^{\frac{1}{2}}, 0, 0, 0\right]$$

we compute

$$a_\alpha \equiv u^\beta \nabla_\beta u_\alpha = u^t \nabla_t u_\alpha = u^t \left(\partial_t u_\alpha - \Gamma^\gamma_{\alpha t} u_\gamma\right) = -\Gamma^t_{\alpha t} u_t u^t \ .$$

Only for $\alpha = r$ do the relevant Christoffel symbols have a non-zero value, which is $\Gamma^t_{rt} = (1-2M/r)^{-1}(M/r^2)$. Since $u_t = -(1-2M/r)^{1/2}$ we find

$$a_\alpha = \left(0, \frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1}, 0, 0\right) \ .$$

Raising the index using $a^\alpha = g^{\alpha\beta} a_\beta$ we recover (20.61).

**20-14**. *Covariant Derivative in an Arbitrary Basis. Let $x^\alpha(\sigma)$ be a curve and $\mathbf{t}(\sigma)$ the unit tangent vector to the curve at $\sigma$. Show that the components of the covariant derivative $\boldsymbol{\nabla}_\mathbf{t}\mathbf{v}$ of a vector $\mathbf{v}$ in the direction $\mathbf{t}$ can be written in an arbitrary basis $\{\mathbf{e}_\alpha\}$ as*

$$(\boldsymbol{\nabla}_\mathbf{t}\mathbf{v})^\alpha = \frac{dv^\alpha}{d\sigma} + \widetilde{\Gamma}^\alpha_{\beta\gamma} v^\beta t^\gamma \ .$$

*where*

$$\widetilde{\Gamma}^\alpha_{\beta\gamma} = \mathbf{e}^\alpha \cdot \boldsymbol{\nabla}_{\mathbf{e}_\gamma} \mathbf{e}_\beta \ .$$

*These are called Ricci rotation coefficients. Show that they reduce to the Christoffel symbols when $\{\mathbf{e}_\alpha\}$ is a coordinate basis.*

**Solution:** The vector $\mathbf{v}$ can be expanded in the basis $\{\mathbf{e}_\alpha\}$ as

$$\mathbf{v} = v^\beta \mathbf{e}_\beta. \tag{1}$$

Thus, thinking of $v^\beta$ as a function,

$$\boldsymbol{\nabla}_\mathbf{t}\mathbf{v} = \left(\boldsymbol{\nabla}_\mathbf{t} v^\beta\right) \mathbf{e}_\beta + v^\beta \left(\boldsymbol{\nabla}_\mathbf{t}\, \mathbf{e}_\beta\right) \tag{2}$$

where $\nabla_{\mathbf{t}} v^\beta = \partial v^\beta / \partial \sigma$ since the derivative is along the curve. Also since $\mathbf{t} = t^\gamma \mathbf{e}_\gamma$

$$\nabla_{\mathbf{t}} \mathbf{e}_\beta = t^\gamma \nabla_{\mathbf{e}_\gamma} \mathbf{e}_\beta . \tag{3}$$

To find the $\alpha^{\text{th}}$ component of (2), take the inner product of both sides with $\mathbf{e}^\alpha$ [*cf.* (20.24)] and use (20.23) with the result

$$(\nabla_{\mathbf{t}} \mathbf{v})^\alpha = \frac{dv^\alpha}{d\sigma} + \left(\mathbf{e}^\alpha \cdot \nabla_{\mathbf{e}_\gamma} \mathbf{e}_\beta\right) \; v^\beta t^\gamma \tag{4}$$

as advertised. To check that this reduces correctly in a coordinate basis, use the fact that $(\mathbf{e}_1)^\alpha = (0,1,0,0)$, etc. and $(\mathbf{e}^1)_\alpha = (0,1,0,0)$, to find

$$\mathbf{e}^\alpha \cdot \left(\nabla_{\mathbf{e}_\gamma} \mathbf{e}_\beta\right) = (\nabla_\gamma \mathbf{e}_\beta)^\alpha = \frac{\partial \left(\mathbf{e}_\beta\right)^\alpha}{\partial x^\gamma} + \Gamma^\alpha_{\gamma\delta} (\mathbf{e}_\beta)^\delta = \Gamma^\alpha_{\gamma\beta} . \tag{5}$$

**20-15**. *[S] (Null Geodesics with Non-Affine Parametrization) As we showed in Section 8.3, when the tangent vector to a null geodesic* $\mathbf{u}$ *is parametrized with an affine parameter* $\lambda$*, it obeys the geodesic equation*

$$\nabla_{\mathbf{u}} \mathbf{u} = 0 \; .$$

*Show that even if a non-affine parameter is used*

$$\nabla_{\mathbf{u}} \mathbf{u} = -\kappa \, \mathbf{u}$$

*for some function* $\kappa$ *of the parameter* $\lambda$.

**Solution:** Let $\lambda$ be the affine parameter so that $u^\alpha = dx^\alpha / d\lambda$ satisfies $\nabla_{\mathbf{u}} \mathbf{u} = 0$. If $\sigma$ is a different, non-affine parameter so that $\lambda = \lambda(\sigma)$, then

$$u'^\alpha = \frac{dx^\alpha}{d\sigma} = \frac{dx^\alpha}{d\lambda} \frac{d\lambda}{d\sigma} \equiv f(\lambda) u^\alpha \; .$$

Then,

$$\begin{aligned} (\nabla_{\mathbf{u}'} \mathbf{u}')^\alpha &= u'^\beta \nabla_\beta u'^\alpha = f \, u^\beta \nabla_\beta (f u^\alpha) \\ &= f \left(u^\beta \nabla_\beta f\right) u^\alpha + f^2 u^\beta \nabla_\beta u^\alpha \\ &= \frac{1}{2} \frac{df^2}{d\lambda} \, u^\alpha \; . \end{aligned}$$

This exhibits the form of $\kappa$ explicitly.

**20-16**. *(Surface Gravity of a Black Hole.) In the geometry of a spherical black hole, the Killing vector $\boldsymbol{\xi} = \partial/\partial t$ corresponding to time translation invariance is tangent to the null geodesics that generate the horizon. If you have happened to work problem 15 you will know that this means*

$$\boldsymbol{\nabla}_{\boldsymbol{\xi}}\boldsymbol{\xi} = -\kappa\boldsymbol{\xi}$$

*for a constant of proportionally $\kappa$ which is called the it surface gravity of the black hole. Evaluate this relation to find the value of $\kappa$ for a Schwarzschild black hole in terms of its mass $M$. Be sure to use a coordinate system which is non-singular on the horizon such as the Eddington-Finkelstein coordinates discussed in Section 12.1.*

**Solution:** The essence of the problem is to use (20.54) to evaluate $\boldsymbol{\nabla}_{\boldsymbol{\xi}}\boldsymbol{\xi}$ on the horizon of a Schwarzschild black hole and show it has the form $-\kappa\boldsymbol{\xi}$ for some constant $\kappa$.

In Eddington-Finkelstein coordinates $(v, r, \theta, \phi)$ we can take $\xi^\alpha = (1, 0, 0, 0)$. The partial derivative term in (20.54) therefore vanishes, and we find

$$(\boldsymbol{\nabla}_{\boldsymbol{\xi}}\boldsymbol{\xi})^\alpha = \xi^\beta\nabla_\beta\xi^\alpha = \nabla_v\xi^\alpha = -\Gamma^\alpha_{v\beta}\xi^\beta = -\Gamma^\alpha_{vv}.$$

The Christoffel symbols can be evaluated from the Lagrangian for geodesics, from the general formula (20.53), or using the *Mathematica* program *Christoffel Symbols and Geodesic Equation* on the book website. In using the formula it's important to remember that the metric is off-diagonal and the universe metric $g^{\alpha\beta}$ must be computed. It's $v, r$ components are

$$g^{vv} = 0 \quad , \quad g^{vr} = g^{rv} = 1 \quad , \quad g^{rr} = \left(1 - \frac{2M}{r}\right).$$

The only non-zero ones are

$$\Gamma^v_{vv} = \frac{M}{r^2} \quad , \quad \Gamma^r_{vv} = -\frac{2M}{r^2}\left(1 - \frac{2M}{r}\right) .$$

On the horizon only $\Gamma^v_{vv}$ is non-zero. This shows that $\boldsymbol{\nabla}_{\boldsymbol{\xi}}\boldsymbol{\xi}$ is of the form $-\kappa\boldsymbol{\xi}$ with a constant $\kappa$ that has the value

$$\kappa = \frac{1}{4M}$$

on the horizon $r = 2M$.

**20-17.** *Show explicitly that the covariant derivative of the metric vanishes by working it out using the expression* (20.66) *or analogous expressions for other components of the covariant derivative (for example that worked out in Problem 11) and the explicit expression for the* $\Gamma$*'s in* (20.53).

**Solution:** From the results of Problem 11

$$\nabla_\gamma g_{\alpha\beta} = \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} - \Gamma^\delta_{\gamma\alpha} g_{\delta\beta} - \Gamma^\delta_{\gamma\beta} g_{\delta\alpha} .$$

Using the definition of the Christoffel symbol (20.53)

$$g_{\beta\delta}\Gamma^\delta_{\gamma\alpha} = \frac{1}{2}\left(\frac{\partial g_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial g_{\beta\alpha}}{\partial x^\gamma} - \frac{\partial g_{\alpha\gamma}}{\partial x^\beta}\right) .$$

Inserting this into the expression above one finds all terms cancel to give

$$\nabla_\gamma g_{\alpha\beta} = 0 .$$

**20-18.** *(Killing's equation.) In Section 8.2 a Killing vector corresponding to a symmetry of a metric was defined in a coordinate system in which the metric was independent of one coordinate* $x^1$. *The components of the corresponding Killing vector* $\boldsymbol{\xi}$ *is then*

$$\xi^\alpha = (0,\ 1,\ 0,\ 0,) .$$

*By explicit calculation show that*

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0 .$$

*This is Killing's equation. It is a general characterization of Killing vectors in the sense that any solution corresponds to a symmetry of the metric.*

**Solution:** Eq. (20.68) gives the formula for the covariant derivative

$$\nabla_\beta \xi_\alpha = \frac{\partial \xi_\alpha}{\partial x^\beta} - \Gamma^\gamma_{\beta\alpha} \xi_\gamma \tag{1}$$

But we also have

$$\xi_\alpha = g_{\alpha\beta}\, \xi^\beta = g_{\alpha\beta}\, \delta^\beta_A = g_{\alpha A} \ .$$

Substituting this in (1) and using the definition for the Christoffel symbols, one finds

$$\nabla_\beta \xi_\alpha = \frac{1}{2} \left( \frac{\partial g_{\alpha A}}{\partial x^\beta} - \frac{\partial g_{\beta A}}{\partial x^\alpha} + \frac{\partial g_{\alpha\beta}}{\partial x^A} \right) \ .$$

The last term vanishes because the metric is independent of $x^A$. The remaining expression is antisymmetric under interchange of $\alpha$ and $\beta$. Thus

$$\nabla_\beta \xi_\alpha + \nabla_\alpha \xi_\beta = 0 \ .$$

**20-19**. *A three-surface $f(x^\alpha) = 0$ is null if its normal $\ell_\alpha = \partial f / \partial x^\alpha$ is a null vector (Section 7.9). Show that these normal vectors are tangent to null geodesics that satisfy $\boldsymbol{\nabla}_{\boldsymbol{\ell}} \boldsymbol{\ell} = 0$.*

**Solution:** If the normal $\ell^\alpha$ is the generator of a light ray, it must solve the geodesic equation $\boldsymbol{\nabla}_{\boldsymbol{\ell}} \boldsymbol{\ell} = 0$. Using (20.68) to evaluate the components of the covariant derivative, the left hand side of the geodesic equation is

$$\begin{aligned} \ell^\beta \nabla_\beta \ell_\alpha &= \ell^\beta \left[ \frac{\partial \ell_\alpha}{\partial x^\beta} - \Gamma^\gamma_{\alpha\beta}\, \ell_\gamma \right] \\ &= \ell^\beta \left[ \frac{\partial^2 f}{\partial x^\beta \partial x^\alpha} - \Gamma^\gamma_{\alpha\beta}\, \ell_\gamma \right] \\ &= \ell^\beta \left[ \frac{\partial \ell_\beta}{\partial x^\alpha} - \Gamma^\gamma_{\beta\alpha}\, \ell_\gamma \right] \ . \end{aligned}$$

Here the symmetry of the $\Gamma$'s and the fact that partial derivatives commute has been used. Continuing on, the result can be written

$$\ell^\beta \nabla_\beta \ell_\alpha = \ell^\beta \nabla_\alpha \ell_\beta = \frac{1}{2}\, \nabla_\alpha \left( \ell^\beta \ell_\beta \right) = 0$$

because $\ell_\beta \ell^\beta = 0$. Thus, the geodesic equation is satisfied.

**20-20.** *(a) Show that the three Killing vectors $\partial/\partial x$, $\partial/\partial y$, and $\boldsymbol{\eta} \equiv -y(\partial/\partial x) + x(\partial/\partial y)$ in Example 8.6 satisfy Killing's equation from Problem 18.*

*(b) Show that in polar coordinates on the plane, $\boldsymbol{\eta} = \partial/\partial\phi$.*

*(c) Show that the rotational symmetry about a point which is not the origin corresponds to a Killing vector which is a linear combination of $\partial/\partial x$, $\partial/\partial y$, and $\boldsymbol{\eta}$.*

**Solution:**

a) In rectangular $(x, y)$ coordinates of the plane, eq. (20.54) shows that Killing's equation becomes

$$\nabla_A \xi_B + \nabla_B \xi_A = \frac{\partial \xi_B}{\partial x^A} - \frac{\partial \xi_A}{\partial x^B} \tag{1}$$

because the Christoffel symbols vanish. The components of the three Killing vectors are $(1, 0)$, $(0, 1)$, $(-y, x)$. Eq. (1) is easily seen to vanish for all three.

b) Let $(r, \phi)$ be the standard polar coordinates on the plane. Then

$$\begin{aligned} \boldsymbol{\eta} &= -y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y} \\ &= -y\left(\frac{\partial r}{\partial x}\,\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial x}\,\frac{\partial}{\partial \phi}\right) + x\left(\frac{\partial r}{\partial y}\,\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial y}\,\frac{\partial}{\partial \phi}\right) \\ &= \frac{\partial}{\partial \phi}. \end{aligned}$$

c) Rotational symmetry about a point $(X, Y)$ would be represented by a Killing vector

$$\boldsymbol{\eta}' = -y'\,\frac{\partial}{\partial x'} + x'\,\frac{\partial}{\partial y'}$$

in terms of rectangular coordinates $(x', y')$ centered on $(X, Y)$. Since

$$x' = x - X \quad , \quad y' = y - Y$$

we have

$$\begin{aligned} \boldsymbol{\eta}' &= -y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y} - X\,\frac{\partial}{\partial y} + Y\,\frac{\partial}{\partial x} \\ &= \boldsymbol{\eta} - \boldsymbol{\eta}_P\ , \end{aligned}$$

where $\boldsymbol{\eta}_P$ is the rotational Killing vector of the point $P = (X, Y)$:

$$\boldsymbol{\eta}_P = -Y\,\frac{\partial}{\partial y} + X\,\frac{\partial}{\partial x}\ . \tag{2}$$

**20-21**. *[B] Derive the formula* (g) *in Box 20.1. For simplicity you can just consider the case of a diagonal metric although the result is general.*

**Solution:** Consider the case of a diagonal metric

$$dS^2 = \sum_i h_{(i)}^2 (dx^i)^2$$

where the three $h_{(i)}$ are functions of $(x^1, x^2, x^3)$. (We suspend the summation convention for this problem.) The inverse metric $g^{ij}$ is also diagonal with elements $1/h_{(i)}^2$, $i = 1, 2, 3$. Eq. (20.53) for $\Gamma^j_{ki}$ becomes

$$\begin{aligned} \Gamma^i_{ki} &= \frac{1}{2}\sum_i g^{ii}\left[\frac{\partial g_{ik}}{\partial x^i} + \frac{\partial g_{ii}}{\partial x^k} - \frac{\partial g_{ki}}{\partial x^i}\right] \\ &= \frac{1}{2}\sum_i \frac{1}{h_{(i)}^2}\,\frac{dh_{(i)}^2}{dx^k} = \sum_i \frac{d}{dx^k}\log h_{(i)} \\ &= \frac{d}{dx^k}\log\left[h_{(1)}h_{(2)}h_{(3)}\right] = \frac{d}{dx^k}\log(\sqrt{g}) \end{aligned}$$

where $g = \det(g_{ij}) = h_{(1)}h_{(2)}h_{(3)}$. From which it follows that

$$\Gamma^i_{ki} = \frac{1}{\sqrt{g}}\,\frac{\partial\sqrt{g}}{\partial x^k}$$

as advertised.

To do the general case efficiently use the identity

$$\det \mathbf{M} = e^{tr\log(\mathbf{M})}$$

for any matrix **M** and work backwards.

**20-22**. *[B,A] Demonstrate that the alternating tensor defined in* (k) *in Box 20.1 transforms correctly as a third rank tensor under coordinate transformations. Hint: The definition of the determinant of a* $3 \times 3$ *matrix* $A_{ij}$ *is*

$$det(A) = \epsilon^{ijk} A_{1i} A_{2j} A_{3k} \ .$$

**Solution:** Denote the tensor defined in $(k)$ by

$$v^{ijk} = g^{-\frac{1}{2}} \ \epsilon^{ijk} \tag{1}$$

If these are the components of a third-rank tensor, they should transform like the analog of (20.45), namely

$$v'^{ijk} = \frac{\partial x'^i}{\partial x^\ell} \frac{\partial x'^j}{\partial x^m} \frac{\partial x'^k}{\partial x^n} \ v^{lmn} \tag{2}$$

The components $v'^{ijk}$ are proportional to $\epsilon^{ijk}$ because interchanging any two indices changes the sign of (2), for example

$$\frac{\partial x'^j}{\partial x^\ell} \frac{\partial x'^i}{\partial x^m} \ \epsilon^{\ell mn} = \frac{\partial x'^j}{\partial x^m} \frac{\partial x'^i}{\partial x^\ell} \ \epsilon^{m\ell n} = -\frac{\partial x'^i}{\partial x^\ell} \frac{\partial x'^j}{\partial x^m} \ \epsilon^{\ell mn} \tag{3}$$

from the antisymmetry of $\epsilon^{ijk}$. It remains only to check the value of one of the components. The definition of the determinant of a matrix **A** is

$$\det \mathbf{A} = A^1_i A^2_j A^3_k \epsilon^{ijk}. \tag{4}$$

Using this (2) gives

$$v'^{123} = \det \left( \frac{\partial x'^i}{\partial x^j} \right) \ g^{-\frac{1}{2}} \tag{5}$$

where the $\partial x'^i / \partial x^j \equiv M^i_j$ are thought of as the components of a $3 \times 3$ matrix. But the transformation law for the metric (20.44) can also be represented in matrix form

$$\mathbf{g}' = \mathbf{M}^{-1} \, \mathbf{g} \left( \mathbf{M}^{-1} \right)^t \tag{6}$$

where $t$ denotes transpose and the components of the inverse matrix are

$$(\mathbf{M}^{-1})^i_j = \frac{\partial x^i}{\partial x'^j}.$$

Thus,

$$\det(\mathbf{g}') = \left[\det(\mathbf{M}^{-1})\right]^2 \det(\mathbf{g}). \tag{7}$$

So using

$$\det(\mathbf{M}^{-1}) = [\det(\mathbf{M})]^{-1}$$

we find

$$(g')^{-\frac{1}{2}} = [\det(\mathbf{M})]\, g^{-\frac{1}{2}} \tag{8}$$

which, together with (4), shows

$$v'^{123} = (g')^{-\frac{1}{2}}.$$

The quantity $v^{ijk}$ thus transforms correctly as a third-rank tensor.

**20-23**. *[B,A] In three-dimensional flat space parabolic coordinates $(\mu, \nu, \phi)$ are defined by*

$$\begin{aligned} x &= \mu\nu\cos\phi\ , \\ y &= \mu\nu\sin\phi\ , \\ z &= (\mu^2 - \nu^2)/2\ . \end{aligned}$$

*(a) Sketch the lines of constant $\mu$ and constant $\nu$ in the $\phi = 0$ plane.*
*(b) Find the flat space line element in the coordinates $(\mu, \nu, \phi)$.*
*(c) Work out the expressions for grad, div, curl and the Laplacian in these coordinates.*

**Solution:**

a)

b)

$$\begin{aligned} dx &= (\nu\cos\phi)d\mu + (\mu\cos\phi)d\nu - \mu\nu\sin\phi d\phi \\ dy &= (\nu\sin\phi)d\mu + (\mu\sin\phi)d\nu + \mu\nu\cos\phi d\phi \\ dz &= \mu d\mu - \nu d\nu \end{aligned}$$

Inserting these into

$$ds^2 = dx^2 + dy^2 + dz^2$$

gives, in the parabolic cylindrical coordinates,

$$ds^2 = (\mu^2+\nu^2)(d\mu^2 + d\nu^2) + \mu^2\nu^2 d\phi^2 \ .$$

c) The formulas for grad, div, and curl are respectively (e), (i), and (n) in Box 20.1. Using the line element above and defining $q \equiv (\mu^2+\nu^2)^{1/2}$ for convenience, we find

$$\begin{aligned} \vec{\nabla} f &= \frac{1}{q}\frac{\partial f}{\partial \mu}\,\vec{e}_{\hat{\mu}} + \frac{1}{q}\frac{\partial f}{\partial_\gamma}\,\vec{e}_{\hat{\nu}} + \frac{1}{\mu\nu}\frac{\partial f}{\partial \phi}\,\vec{e}_{\hat{\phi}} \\ \vec{\nabla}\cdot\vec{V} &= \frac{1}{q^2}\left[\frac{1}{\mu}\frac{\partial}{\partial\mu}\left(\mu q V^{\hat{\mu}}\right) + \frac{1}{\nu}\frac{\partial}{\partial\nu}\left(\nu q V^{\hat{\nu}}\right)\right] + \frac{1}{\mu\nu}\frac{\partial V^{\hat{\phi}}}{\partial\phi} \\ \left(\vec{\nabla}\times\vec{V}\right)^{\hat{\mu}} &= \frac{1}{\nu q}\frac{\partial\left(\nu V^{\hat{\phi}}\right)}{\partial\nu} - \frac{1}{\mu\nu}\frac{\partial V^{\hat{\nu}}}{\partial\phi}\ , \text{ etc.} \end{aligned}$$

**20-24**. *[S] Use formulas* (20.72) *and* (20.73) *to show that, if the spin* $\mathbf{s}$ *of a free gyro starts out orthogonal to its four-velocity* $\mathbf{u}$, *it remains orthogonal.*

**Solution:** The derivative of $\mathbf{s} \cdot \mathbf{u}$ along the geodesic followed by the gyro is

$$\boldsymbol{\nabla}_{\mathbf{u}}(\mathbf{s} \cdot \mathbf{u}) = (\boldsymbol{\nabla}_{\mathbf{u}}\mathbf{s}) \cdot \mathbf{u} + \mathbf{s} \cdot \boldsymbol{\nabla}_{\mathbf{u}}\mathbf{u}$$

which vanishes as a consequence of (20.72) and (20.73).

**20-25**. *[A] Show that the basis vectors of the freely falling frame* (20.81) *in the Schwarzschild geometry are indeed parallelly propagated along the geodesic of the freely falling observer, by showing explicitly that they each satisfy* (20.79).

**Solution:** Any one of the basis vectors $\mathbf{e}_{\hat{\alpha}}$ is parallel propagated along $\mathbf{u}$ if [*cf.* (20.79)]

$$\boldsymbol{\nabla}_{\mathbf{u}}\mathbf{e}_{\hat{\alpha}} = 0 \ .$$

This condition can be written out in the Schwarzschild coordinate basis using (20.54)

$$u^{\beta}\nabla_{\beta}(\mathbf{e}_{\hat{\alpha}})^{\alpha} = u^{\beta}\left(\frac{\partial(\mathbf{e}_{\hat{\alpha}})^{\alpha}}{\partial x^{\beta}} + \Gamma^{\alpha}_{\beta\gamma}(\mathbf{e}_{\hat{\alpha}})^{\gamma}\right) \ .$$

To verify these equations explicitly, they can be written out using the components in (20.81), the four-velocity in (20.80), and the Christoffel symbols from the Mathematica note book. For example,

$$\nabla_{t}(\mathbf{e}_{\hat{\alpha}})^{\alpha} = \frac{\partial(\mathbf{e}_{\hat{\alpha}})^{\alpha}}{\partial t} + \Gamma^{\alpha}_{t\gamma}(\mathbf{e}_{\hat{\alpha}})^{\gamma} \ .$$

The first term vanishes because nothing depends on time. The second gives, for example,

$$\begin{aligned}
\nabla_{t}(\mathbf{e}_{\hat{r}})^{t} &= \Gamma^{t}_{tt}(\mathbf{e}_{\hat{r}})^{t} + \Gamma^{t}_{tr}(\mathbf{e}_{\hat{r}})^{r} \\
&= \frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1}(\mathbf{e}_{\hat{r}})^{r} \\
&= -\frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{2M}{r}\right)^{1/2} \ .
\end{aligned}$$

Proceeding in this way, one finds

$$\begin{aligned}
\nabla_t(\mathbf{e}_{\hat{r}})^\alpha &= \left(-\frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{2M}{r}\right)^{1/2}, \frac{M}{r^2}, 0, 0\right) \\
\nabla_r(\mathbf{e}_{\hat{r}})^\alpha &= \left(-\left(\frac{M}{r^2}\right)\left(1-\frac{2M}{r}\right)^{-2}, \frac{M}{r^2}\left(\frac{2M}{r}\right)^{-1/2}\left(1-\frac{2M}{r}\right)^{-1}, 0, 0\right).
\end{aligned}$$

Then, forming the combinations $u^\beta \nabla_\beta(\mathbf{e}_{\hat{r}})^\alpha = 0$ we find that all four, $\alpha = 0, 1, 2, 3$ vanish identically.

The other three basis vectors are done similarly.

**20-26.** *Show that the orthonormal basis of the freely falling frame* (20.81) *at Schwarzschild radius $r$ is connected to the orthonormal basis of a stationary observer at that point by a Lorentz boost. Find the velocity of that boost. Comment: This is a special case of the general result that any two orthonormal bases are connected by a Lorentz transformation, cf. Problem 7.23*

**Solution:** Denote the orthonormal basis vectors of a stationary observer at Schwarzschild radius $r$ by $\{\mathbf{e}'_{\hat{\alpha}}\}$. They can be constructed by following Example 7.9 and are

$$\begin{aligned}
(\mathbf{e}'_{\hat{t}})^\alpha &= \left[(1-2M/r)^{-1/2}, 0, 0, 0\right] \\
(\mathbf{e}'_{\hat{r}})^\alpha &= \left[0, (1-2M/r)^{1/2}, 0, 0, 0\right] \\
(\mathbf{e}'_{\hat{\theta}})^\alpha &= [0, 0, 1/r, 0] \\
(\mathbf{e}'_{\hat{\phi}})^\alpha &= [0, 0, 0, 1/(r\sin\theta)].
\end{aligned}$$

Evidently $\mathbf{e}_{\hat{\theta}} = \mathbf{e}'_{\hat{\theta}}$ and $\mathbf{e}_{\hat{\phi}} = \mathbf{e}'_{\hat{\phi}}$ and

$$\begin{aligned}
\mathbf{e}_{\hat{r}} &= \left(1-\frac{2M}{r}\right)^{-1/2}\mathbf{e}'_{\hat{t}} - \left(\frac{2M}{r}\right)^{1/2}\left(1-\frac{2M}{r}\right)^{-1/2}\mathbf{e}'_{\hat{r}}, \\
\mathbf{e}_{\hat{t}} &= -\left(\frac{2M}{r}\right)^{1/2}\left(1-\frac{2M}{r}\right)^{-1/2}\mathbf{e}'_{\hat{t}} + \left(1-\frac{2M}{r}\right)^{-1/2}\mathbf{e}'_{\hat{r}}.
\end{aligned}$$

Defining $v = -(2M/r)^{1/2}$ and $\gamma = (1-v^2)^{-1/2}$ this can be written

$$\begin{aligned}
\mathbf{e}_{\hat{r}} &= \gamma\mathbf{e}'_{\hat{t}} + v\gamma\mathbf{e}'_{\hat{r}}, \\
\mathbf{e}_{\hat{t}} &= v\gamma\mathbf{e}'_{\hat{t}} + \gamma\mathbf{e}'_{\hat{r}}.
\end{aligned}$$

which is a Lorentz boost. The velocity $v$ is the velocity of the falling observer measured by the stationary observer [*cf.* (9.42)].

# Chapter 21

# Curvature and the Einstein Equation

**21-1.** *Why do Newtonian tidal gravitational forces outside a mass distribution always squeeze in some directions and expand in others? [Hint: The Newtonian gravitational potential satisfies Laplace's equation.]*

**Solution:** Newtonian tidal gravitational forces are represented by the derivatives [*cf.* (21.5)]

$$\frac{\partial^2 \Phi}{\partial x^i \partial x^j} \ .$$

This can be thought of as a $3 \times 3$ matrix which can be diagonalized at a point by a rotation to new Cartesian coordinates $(x', y', z')$. (But not necessarily diagonalized at every point because it's generally a different matrix at different points.) The three diagonal components have to add to zero — that's Laplace's equation

$$\frac{\partial^2 \Phi}{\partial x'^2} + \frac{\partial^2 \Phi}{\partial y'^2} + \frac{\partial^2 \Phi}{\partial z'^2} = 0 \ .$$

Therefore, at least one term has to be positive (squeezing) and at least one term has to be negative (expanding).

**21-2.** *[B,C] (The Shape of the Tides) This problem concerns the shape of the tides raised by the Moon in Newtonian gravity. (See Box 21.1.) Consider the*

*freely falling frame following the center of mass of the Earth in its mutual orbit with the Moon. (Neglect the slower motion of the Earth around the Sun and the rotation of the Earth.) Assume the surface of the solid Earth is a sphere which is covered with a worldwide ocean.*

a) *Explain why the surface of the ocean should be at an equal total gravitational potential.*

b) *Find a gravitational potential $\Phi_{tidal}$ that will reproduce the tidal gravitational force of the Moon given in* (c) *of the Box 21.1 and the gravitational force of the Earth on an ocean fluid element of mass $m$ according to :*

$$\vec{F}_{tidal} = -m\vec{\nabla}\Phi_{tidal} .$$

c) *Find the difference $\delta h(\theta, \phi)$ between the depth of the ocean in the presence of the Moon and in its absence caused by the tidal gravitational force of the Moon. Use the usual polar angles with the $z$-axis pointing towards the Moon. Express your answer in terms of the mass of the Earth, the mass of the Moon, the distance between them, and the distance from the center of the Earth to the surface of the ocean were the Moon not present.*

d) *Estimate the expected height of the ocean tides from your result in part (c).*

e) *Answer the question at the end Box 21.1.*

**Solution:**

(a) Consider a cubical fluid element of mass $m$ at the surface of the ocean. The gravitational force on it is $-m\vec{\nabla}\Phi$. The pressure force results from the *difference* between the pressure on opposite sides. There is therefore no pressure force *along* surfaces of constant pressure. In fact, the pressure force is $\vec{\nabla}p$ which is normal to surfaces of constant pressure. If the fluid element is not to accelerate, $\vec{\nabla}p$ must lie along $-m\vec{\nabla}\Phi$ which means that surfaces of constant pressure are surfaces of constant potential. But the surface of the ocean is a constant pressure ($p = 0$) surface, so it must be an equipotential.

(b) Use $(x, y, z)$ coordinates as the figure in the box Box 21.1. Let $R$ be the radius of the earth, $d$ be the distance from the earth to the moon, and $\delta h$ be the difference between the height of the ocean and it's mean height. Assume (correctly) that $\delta h/R \ll 1$. The fluid element experiences forces from the gravitational field of both the earth and the moon. The tidal acceleration of the moon (b) can be reproduced by the potential (per unit mass)

$$a_{\text{moon}} = \frac{1}{2}\left(\frac{\partial^2 \Phi_{\text{moon}}}{\partial x^i \partial x^j}\right)_0 x^i x^j = \frac{GM_{\text{moon}}}{2d^3}\left(x^2 + y^2 - 2z^2\right) .$$

The earth's potential requires more care since it is singular at the origin of the coordinates. The gravitational force from a fluid element at $(x, y, z)$ is

$$\vec{F}_{\text{earth}} = -m\frac{GM_\oplus}{R^3}(x, y, z) .$$

For small variations in $x, y, z$ near the surface this can be reproduced by the potential (per unit mass)

$$\Phi_{\text{earth}} = \frac{GM_\oplus}{2R^3}\left(x^2 + y^2 + z^2\right) .$$

The combined potential $\Phi_{\text{earth}} + \Phi_{\text{moon}}$ gives the total force per mass on a fluid element.

(c) The difference $\delta h$ between the height of the ocean and its mean height $\bar{h}$ is a function of $\theta$ from axisymmetry. The difference is also proportional to $M_{\text{moon}}$. To first order in $\delta h$ the total potential of a fluid element at $r = R + \delta h(\theta)$

$$\frac{GM_\oplus}{2R^3} R^2 \left(1 + \frac{2\delta h(\theta)}{R}\right) + \frac{GM_{\text{moon}}}{2d^3} R^2 \left(1 - 3\cos^2\theta\right) .$$

If this is to be a constant the angular dependence of $h(\theta)$ must cancel that in the second term. A possible constant in $\delta h(\theta)$ is fixed by the requirement that it averages to zero. Thus

$$\frac{\delta h(\theta)}{R} = \frac{M_{\text{moon}}}{M_\oplus}\left(\frac{R}{d}\right)^3 \left(\frac{3}{2}\cos^2\theta - 1\right) .$$

This is positive in the direction of the moon and negative 90° away as expected.

[*Remark:* The problem can be worked from the full potentials rather than the tidal ones used here, but then the fact that the earth is accelerating towards the moon (and vice versa) must be taken into account.]

(d) The difference between the maximum and minimum heights is of order

$$\Delta h \sim \frac{3}{2} R_\oplus \left(\frac{M_{\text{moon}}}{M_\oplus}\right) \left(\frac{R_\oplus}{d}\right)^3 .$$

For $M_{\text{moon}}/M_\oplus \approx 1/81$, $R_\oplus \approx 6378\text{km}$, $d \approx 384,404\text{km}$, this works out to be $\Delta h \sim 1m$ (Not bad!)

(e) As the earth rotates under the tides, it loses rotational energy to friction. The rotation rate of the earth is therefore slowing down. This requires that a torque be exerted on the tidally distorted earth by the moon. A little geometry convinces one that the tidal bulge must not point exactly at the moon but so that the high point of the tide *lags* the time the moon is at the zenith.

**21-3**. *[E] A meter stick falls radially into the center of Newtonian gravitational attraction produced by one solar mass located at a point. Estimate the distance from the point that the meter stick would break or be crushed.*

**Solution:** The tidal gravitational force between two particles of mass $m$ separated by a displacement $n^i$ is given in Newtonian gravity by [*cf.* (21.5)]

$$m\Sigma_k \frac{\partial^2 \Phi}{\partial x^i \partial x^k} n^k .$$

Outside a point solar mass $\Phi = -GM_\odot/r$, so the order of magnitude for the tidal force will be

$$\sim \frac{GM_\odot m}{r^3} |\vec{n}| .$$

Guessing that the meter stick will break if this force exceeds 100 N and noting that $|\vec{n}| = 1$ m and the mass of the meter stick $\sim 50$ g, we have

$$\begin{aligned} r &\sim \left(\frac{GM_\odot m\,|\vec{n}|}{100\text{ N}}\right)^{\frac{1}{3}} \\ &\sim \left(\frac{7\times 10^{-8}\times 2\times 10^{33}\times 50\times 10^{2}}{1\times 10^{5}}\right)^{\frac{1}{3}} \sim 400\text{ km}\,. \end{aligned}$$

**21-4**. *Show that, if $\boldsymbol{\chi}$ is a separation vector obeying the equation of geodesic deviation* (21.19), *then $\boldsymbol{\chi} + C\mathbf{u}$ is another separation vector also obeying the equation of geodesic deviation where $C$ is any constant.*

**Solution:** Substitute $\boldsymbol{\chi} + C\mathbf{u}$ into the equation of geodesic deviation (21.19). The change proportional to $C$ vanishes on the left hand side because $\nabla_{\mathbf{u}}\mathbf{u} = 0$. The change on the right hand side is

$$-CR^{\alpha}{}_{\beta\gamma\delta}\,u^{\beta}u^{\gamma}u^{\delta}$$

which vanishes because the Riemann tensor is antisymmetric in $\gamma$ and $\delta$ but $u^{\gamma}u^{\delta}$ is symmetric.

**21-5**. *Fill in the details in the derivation of* (21.27) *for the Riemann curvature component $R^{i}{}_{tjt}$ in the Newtonian limit.*

**Solution:** Starting from (21.26), we next have to evaluate $R^{i}_{tjt}$ to leading order; leading order is $1/c^2$ or in $c = 1$ units to order $\Phi$. Since the flat space Christoffel symbols vanish, (21.20) tells us that

$$R^{i}_{tjt} = \frac{\partial\Gamma^{i}_{tt}}{\partial x^{j}} - \frac{\partial\Gamma^{i}_{tj}}{\partial x^{t}},$$

i.e. the $\Gamma\Gamma$ terms vanish to leading order. Looking up the Christoffel symbols in Appendix C, or calculating by hand from the metric (21.25), we see that

$$\Gamma^{i}_{tj} = 0 \quad \Gamma^{i}_{tt} = \frac{\partial\Phi}{\partial x^{i}}\,.$$

Putting these into the above expression for the Riemann tensor gives (21.27).

**21-6**. *Derive the expression* (21.28) *for the Riemann curvature in a local inertial frame from its definition* (21.20).

**Solution:** In a local inertial frame at a point [Chapter 7, Section 7.4] the metric is $g_{\alpha\beta} = \eta_{\alpha\beta}$. Its first derivatives vanish but not its second derivatives. Thus, the $\Gamma$'s vanish but not the derivatives of the $\Gamma$'s. The $\Gamma\Gamma$ terms in the definition of the Riemann tensor (21.20) therefore vanish. Then,

$$R_{\alpha\beta\gamma\delta} = \eta_{\alpha\epsilon} R^{\epsilon}{}_{\beta\gamma\delta} = \frac{\partial \Gamma_{\alpha\beta\delta}}{\partial x^{\gamma}} - \frac{\partial \Gamma_{\alpha\beta\gamma}}{\partial x^{\delta}} \tag{1}$$

where we defined

$$\Gamma_{\alpha\beta\delta} \equiv \eta_{\alpha\epsilon} \Gamma^{\epsilon}_{\beta\delta} \ .$$

The formula for the Christoffel symbols (8.19) gives

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2} \left( \frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{g_{\beta\gamma}}{\partial x^{\alpha}} \right) \ .$$

Inserting this into (1) gives six terms, two of which cancel, resulting in (21.28).

**21-7**. *(a) Derive the symmetries* (21.29) *from the form of the Riemann curvature in a local inertial frame* (21.28).

*(b) Use these symmetries to show that the Riemann curvature has twenty independent components.*

**Solution:**

a) The symmetries (21.29a) – (21.29b) follow from inspection of (21.28). Eq. (21.29d) follows from adding the three expressions arising from permuting the indices on (21.28).

b) There are six antisymmetric pairs $(\alpha\beta)$. The Riemann tensor has the same number of components as a symmetric [from (21.29c)] matrix — $6(6+1)/2 = 21$. But equation (21.29d) is one further condition reducing the number of independent components to twenty.

**21-8**. *[A] Calculate $R_{\hat{r}\hat{r}r\hat{r}}$ for the Schwarzschild metric in the frame of the freely falling observer described in* (20.81). *To do this first calculate $R_{\alpha\beta\gamma\delta}$ in the Schwarzschild coordinate basis, and then use* (21.24) *to get the component in the freely falling frame. Does your answer agree with* (21.30a)*?*

**Solution:** The table below gives the output of the *Mathematica* program for the components $R^{\alpha}{}_{\beta\gamma\delta}$ of the Riemann tensor in the Schwarzschild coordinate basis:

```
Out[13]//TableForm=
```

| | |
|---|---|
| R[1, 2, 2, 1] | $\frac{m}{r}$ |
| R[1, 3, 3, 1] | $\frac{m\,\mathrm{Sin}[\theta]^2}{r}$ |
| R[1, 4, 4, 1] | $\frac{2\,m\,(-2\,m+r)}{r^4}$ |
| R[2, 1, 2, 1] | $\frac{m}{(2\,m-r)\,r^2}$ |
| R[2, 3, 3, 2] | $-\frac{2\,m\,\mathrm{Sin}[\theta]^2}{r}$ |
| R[2, 4, 4, 2] | $\frac{m\,(2\,m-r)}{r^4}$ |
| R[3, 1, 3, 1] | $\frac{m}{(2\,m-r)\,r^2}$ |
| R[3, 2, 3, 2] | $\frac{2\,m}{r}$ |
| R[3, 4, 4, 3] | $\frac{m\,(2\,m-r)}{r^4}$ |
| R[4, 1, 4, 1] | $\frac{2\,m}{r^2\,(-2\,m+r)}$ |
| R[4, 2, 4, 2] | $-\frac{m}{r}$ |
| R[4, 3, 4, 3] | $-\frac{m\,\mathrm{Sin}[\theta]^2}{r}$ |

Here $x^4 = t$, $x^1 = r$, $x^2 = \theta$, and $x^3 = \phi$. You can also find them in the copy of this program reproduced in Appendix C. It's then just a matter of carrying out the algebra by lowering the first index on $R^{\alpha}{}_{\beta\gamma\delta}$ and then using (20.81) together with (21.24) to obtain (21.30).

**21-9**. *[C] Are we already in a black hole? At the time of writing, measurements*

*of the velocities of galaxies indicate that the Milky Way (our own galaxy) is falling towards the Andromeda galaxy and that these two together with other members of the local group of galaxies are falling towards a "great attractor" in the direction of the Hydra-Centaurus supercluster of galaxies. What observations would be necessary to determine whether or not we are already in a black hole falling towards its center? To discuss this question you may assume that the "great attractor" is spherically symmetric.*

**Solution:** If it is assumed that the geometry is spherically symmetric about the "great attractor", then we need to determine our Schwarzschild coordinate radius from it. That could be done by studying the relative motion of nearby galaxies in principle [see equations (21.31)]. The same observations could also determine the attracting mass. If $r < 2M$ we are already inside a black hole.

**21-10**. *(A Uniform Gravitational Field) Calculate the Riemann curvature for the metric*

$$ds^2 = -(1+gx)^2 dt^2 + dx^2 + dy^2 + dz^2$$

*thereby showing this spacetime is flat. Find a coordinate transformation that puts this metric into the usual Minkowski form.*

**Solution:** The Riemann curvature is most easily shown to vanish by using the *Mathematica program Curvature and the Einstein Equation* on the course website.

The coordinate transformation to usual flat coordinates is the inverse of the one given in Problem 6.6, namely

$$\begin{aligned} t' &= \left(g^{-1} + x\right) \sinh(gt) \\ x' &= \left(g^{-1} + x\right) \cosh(gt) \\ y' &= y\,, \quad z' = z\,. \end{aligned}$$

Explicit computation starting from

$$ds^2 = -dt'^2 + dx'^2 + dy'^2 + dz'^2$$

will show this.

**21-11**. *Problem 8.12 introduced the two-dimensional hyperbolic plane and claimed it has constant negative curvature. Does it? Calculate* $R \equiv R^{\alpha}_{\alpha}$ *for this two-dimensional geometry to find out.*

**Solution:** Running the *Mathematica* program *Curvature and the Einstein Equation* for the metric given in Problem 8.12 gives $R = -2$.

**21-12**. *[C,A] (a) For the wormhole metric* (7.39)*, calculate the components of the Riemann curvature in an orthonormal basis whose vectors point along the* $(t, r, \theta, \phi)$ *coordinate axes.*

*(b) Show that a stationary observer at the wormhole throat feels no tidal gravitational forces.*

*(c) Show that an observer moving radially through the throat with speed* $V$ *as measured by a stationary observer at any point along its trajectory, experiences tidal gravitational forces proportional to* $V^2$.

*(d) How do these tidal forces depend on the radius of the throat? What combination of* $b$ *and* $V$ *would make for a survivable trip through the wormhole?*

**Solution:**

a) The components of the Riemann curvature in an orthonormal basis pointing along the $(t, r, \theta, \phi)$ coordinate directions can be found with the following steps:

(1) Use the *Mathematica* program *Curvature and the Einstein Equation* on the website to calculate $R^{\alpha}{}_{\beta\gamma\delta}$ in the coordinate basis of the wormhole metric (7.39).

(2) Lower the first index to get the coordinate basis components $R_{\alpha\beta\gamma\delta}$ [*cf.* (20.35)].

(3) Construct the components of an orthonormal basis according to (7.61).

(4) Project the Riemann tensor into this basis [*cf.* (20.41)]. The result is the following non-vanishing components

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = -R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = -R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = \frac{b^2}{(r^2 + b^2)^2}$$

and the others related to these by the symmetries of the Riemann tensor.

b) Stationary observers have only $t$-components of their four-velocities. But there are no $t$-components of the Riemann tensor. The right hand side of the equation of geodesic deviation (21.19) therefore vanishes.

c) The four-velocity of an observer falling radially is given by (8.21). Expressed in terms of the velocity $V$ measured by a stationary observer this is

$$u^{\hat{\alpha}} = (\gamma\,, \gamma V, 0\,, 0)$$

where $\gamma = (1 - V^2)^{-1/2}$. (A simple way to see this is to note that the $(t, r)$ part of the metric (7.39) is like flat spacetime.) Then, for instance,

$$(\boldsymbol{\nabla}_{\mathbf{u}}\boldsymbol{\nabla}_{\mathbf{u}}\boldsymbol{\chi})^{\hat{\theta}} = -R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}}(u^{\hat{r}})^2\chi^{\hat{\theta}} = (\gamma V)^2\frac{b^2}{(r^2+b^2)^2}\,\chi^{\hat{\theta}}$$

is proportional to $V^2$ as advertised.

d) The tidal forces are largest at the smallest $r$ which is 0. To survive traversing the wormhole $(V^2/b^2)\times$ (length of the observer)$\times$ (mass of the observer) must be less than the maximum tolerable force. For survival it's best to go through a large wormhole and to go through *slowly*.

For more on traversable wormholes, see M.S. Morris and K.S. Thorne (1998), *Wormholes in Spacetime and Their Use for Interstellar Travel: A Tool for Teaching General Relativity*, Am. J. Phys. **395**.

**21-13**. *[S] The Ricci curvature was defined by a particular sum of the components of the Riemann curvature in* (21.32). *Show that if any pair of indices of* $R_{\alpha\beta\gamma\delta}$ *are summed using the inverse metric (e.g.* $g^{\alpha\delta}R_{\alpha\beta\gamma\delta}$*) the result is either zero or a multiple of the Ricci curvature.*

**Solution:** There are six ways of contracting two indices of the Riemann tensor which can be grouped as follows

$$g^{\gamma\delta}R_{\gamma\delta\alpha\beta} \quad , \quad g^{\gamma\delta}R_{\alpha\beta\gamma\delta} \tag{1}$$

$$g^{\gamma\delta}R_{\gamma\alpha\delta\beta} \quad , \quad g^{\gamma\delta}R_{\alpha\gamma\beta\delta} \tag{2}$$

$$g^{\gamma\delta}R_{\gamma\alpha\beta\delta} \quad , \quad g^{\gamma\delta}R_{\alpha\gamma\delta\beta} \tag{3}$$

The combination in group (1) vanish because of (21.29a) and (21.29b). The combinations in group (2) are equal to $R_{\alpha\beta}$ because of (21.32) and (21.29a) and (21.29b) together. The combinations in group (3) are equal to $-R_{\alpha\beta}$ because of (21.32) and (21.29a) and (21.29b).

**21-14**. *[S] Starting from its definition* (21.31) *or from* (21.33) *show that the Ricci curvature $R_{\alpha\beta}$ is symmetric in $\alpha$ and $\beta$.*

**Solution:** The symmetries of $R_{\alpha\beta}$ must follow from its definition (21.32) and the symmetries of the Riemann tensor (21.29). Using the definition (21.32), raising indices (20.18), the symmetry (21.29c) we have

$$R_{\alpha\beta} \equiv R^{\gamma}{}_{\alpha\gamma\beta} = g^{\gamma\delta}R_{\delta\alpha\gamma\beta} = g^{\gamma\delta}R_{\gamma\beta\delta\alpha} = R^{\delta}{}_{\beta\delta\alpha} = R_{\beta\alpha}$$

**21-15**. *Calculate $R_{AB}$ for the metric on the sphere*

$$ds^2 = a^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

$x^1 = \theta, x^2 = \varphi, A, B, \cdots = 1, 2$ *and* $a =$ *constant. This problem can easily be done using the Mathematica program for computing curvature on the book website. However, work through this problem by hand to make sure you understand what the Mathematica program is doing.*

**Solution:** This is a problem in virtuous algebra that can be checked using the Mathematica program in the Appendix. We quote a few answers here. The non-vanishing Christoffel symbols are

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\ \cos\theta \qquad \Gamma^{\phi}_{\phi\theta} = \cot\theta\ .$$

The components of the Ricci curvature are

$$R_{\theta\theta} = 1\ , \quad R_{\theta\phi} = 0\ , \quad R_{\phi\phi} = \sin^2\theta\ .$$

There is no dependence on the radius $a$, but there is for the scalar curvature $R$

$$R = 2/a^2 \ .$$

**21-16.** *Check three of the non-vanishing Christoffel symbols for the Schwarzschild metric given in Appendix B.*

**Solution:** The basic formula is (20.53) evaluated by any of the methods in Section 8.1. The rest is algebra.

**21-17.** *[A] (The Schwarzschild geometry satisfies the Einstein equation.) Insert the Christoffel symbols for the Schwarzschild geometry in given Appendix B into* (21.33) *and evaluate. You should find find $R_{\alpha\beta} = 0$ identically for the each of the ten possible combinations of $\alpha$ and $\beta$ thus proving that the Schwarzschild geometry is a solution of the empty space Einstein equation. For a shorter problem do just the diagonal cases.*

**Solution:** You just have to do the algebra.

**21-18.** *[C] (Birkhoff's Theorem) (a) In Example 4 invariance under $t \to -t$ was used to exclude a $g_{rt}drdt$ from the line element representing the geometry outside a static spherically symmetric distribution of stress energy. However, in a dynamical situation such as spherically symmetric collapse, that argument no longer holds. Then the most general spherically symmetric line element is of the form*

$$ds^2 = -A(r,t)\,dt^2 + 2B\,(r,t)\,drdt + C(r,t)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right) \ .$$

*Show that a transformation of the form*

$$t \to t + f(r,t) \ ,$$

*for some $f(r,t)$, can be used to eliminate the $drdt$ term leaving the most general spherically symmetric metric in the form*

$$ds^2 = -e^{\nu(r,t)}dt^2 + e^{\lambda(r,t)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right) \ .$$

*(b) Using the expressions for the Einstein tensor in Appendix B, show that the equation*

$$G_{\hat{r}\hat{t}} = 0$$

*implies that $\lambda(r,t)$ is independent of time and use the remaining components of the Einstein equation to show*

$$\nu(r,t) = -\lambda(r) + f(t)$$

*for some $f(t)$.*

*(c) Use these results to conclude that the Schwarzschild geometry is the most general asymptotically flat, spherically symmetric solution of the Einstein equation, dynamical or not. That is called Birkhoff's theorem.*

**Solution:**

a) Write $t = t' + f(r,t')$ and substitute into the given metric. The $drdt'$ cross term is

$$2drdt\left[A\frac{\partial f}{\partial r} + B\left(1 + \frac{\partial f}{\partial t}\right)\right] .$$

Appropriate choice of $f$ will therefore make this term vanish. Relabeling $t'$ as $t$ gives the desired form of the metric.

b) Running the *Mathematica* program *Curvature and the Einstein Equation* with $\lambda$ and $\nu$ functions of both $r$ and $t$ gives the components of the Einstein tensor $G_{\alpha\beta}$. The $G_{\hat{r}\hat{t}}$ component is

$$e^{-(\nu+\lambda)/2}(\partial\lambda/\partial t)/r .$$

This vanishes if the Ricci curvature does so $\lambda$ is independent of $t$, i.e. $\partial\lambda/\partial t = 0$. The two equations (21.36) turn out to hold whether or not $\nu$ and $\lambda$ are functions of $t$. Subtracting the two gives $\nu' = -\lambda'$, so that

$$\nu(r,t) = -\lambda(r) + h(t)$$

for arbitrary $h(t)$.

c) The first of Eqs. (21.36) gives

$$e^{-\lambda(r)} = 1 - 2M/r$$

as before. That makes the $dt^2$ term in the line element

$$-(1-2M/r)\,(e^{h/2}dt)^2 \ .$$

But since $h$ is a function only of $t$, we can define a new time variable $\tilde{t}$ by

$$d\tilde{t} = e^{h(t)/2}dt \ .$$

Relabeling $\tilde{t}$ as $t$ after the transformation gives the Schwarzschild line element.

**21-19**. *[C] (Static Weak Field Metric Derived.) This problem aims at showing that the static, weak field metric* (21.25) *is the most general such solution of the linearized, vacuum Einstein equation.*

*(a) Argue that the metric perturbations $h_{\alpha\beta}$ for a time-independent source should be unchanged by $t \to -t$ and that this means $h_{it} = h_{ti} = 0$.*

*(b) Show that the residual gauge freedom analogous to that discussed in the subsection "More Gauge" can be used to make $h_{ij}$ diagonal without affecting either $h_{it} = 0$ or the Lorentz gauge condition.*

*(c) Show that then* (21.25) *is the unique asymptotically flat solution of the equations of linearized gravity.*

**Solution:**

a) Assume the general metric perturbations are independent of $t$. Then $h_{it}$ has to vanish if the metric is to be unchanged under $t \to -t$.

b) The general gauge freedom can be used to impose the four Lorentz gauge conditions $V_\alpha = 0$. For a static metric this means that [*cf.* (21.55)]

$$\vec{\nabla}^2 h_{\alpha\beta}(\vec{x}) = 0 \ .$$

Lorentz gauge will be preserved by further gauge conditions based on $\xi^\alpha = (0, \vec{\xi})$ with $\xi^i(\vec{x})$ independent of time. These will not affect $h_{it}$ as (21.53) shows. These three functions can be used to make the three off-diagonal components of $h_{ij}$ vanish.

c) With the reduced form of the metric the Lorentz gauge condition $V_1 = 0$ becomes

$$\partial_x \left[-h^x_x + h^y_y + h^z_z + h^t_t\right] = 0 \ .$$

This says the combination in brackets is a function independent of $x$. But that function must vanish if the metric is to be asymptotically flat. Thus,

$$-h^x_x + h^y_y + h^z_z + h^t_t = 0 \ .$$

$V_2 = V_3 = 0$ give the same equation with $(x, y, z)$ cyclically permuted. These three equations can only be satisfied by

$$h^x_x = h^y_y = h^z_z = -h^t_t \ .$$

or

$$h_{xx} = h_{yy} = h_{zz} = h_{tt} \equiv -\Phi \ .$$

That's the general static, linearized solution in the form (21.25).

**21-20**. *[S] Calculate the Ricci tensor for the five-dimensional metric* (a) *in Box 7.3. (Hint: No computation is needed.)*

**Solution:** By writing $w = R\psi$ the metric is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + dw^2$$

which is obviously flat. The Riemann curvature vanishes.

**21-21**. *Carry out the steps leading to the expression* (21.44) *for the perturbation of the Ricci curvature in linearized gravity.*

**Solution:** Inserting (21.42) into (21.43) gives (using $\partial_\alpha$ for $\partial/\partial x^\alpha$)

$$\begin{aligned} \delta R_{\alpha\beta} &= \frac{1}{2}\eta^{\gamma\delta}\left(\partial_\gamma\partial_\alpha h_{\delta\beta} + \partial_\gamma\partial_\beta h_{\delta\alpha} - \partial_\gamma\partial_\delta h_{\alpha\beta}\right) \\ &- \frac{1}{2}\eta^{\gamma\delta}\left(\partial_\beta\partial_\alpha h_{\delta\gamma} + \partial_\beta\partial_\gamma h_{\delta\alpha} - \partial_\beta\partial_\delta h_{\alpha\gamma}\right) \ . \end{aligned}$$

The middle terms in brackets cancel against each other because $\partial_\beta\partial_\gamma = \partial_\gamma\partial_\beta$. The last term in the first bracket is $-(1/2)\Box h_{\alpha\beta}$ since $\Box = \eta^{\gamma\delta}\partial_\gamma\partial_\delta$. The first

term in the second bracket is $-(1/2)\partial_\alpha\partial_\beta(h^\gamma_\gamma)$. With these identifications, the expression easily organizes itself into (21.44).

**21-22**. *[S] Eq.* (21.40) *gives the metric in linearized gravity. Work out the inverse metric to first order in* $h_{\alpha\beta}$.

**Solution:** The inverse metric can be written as

$$g^{\alpha\beta} = \eta^{\alpha\beta} + q^{\alpha\beta}$$

where $q^{\alpha\beta}$ is the first order correction proportional to $h_{\alpha\beta}$. The inverse metric is related to the metric by (20.17) which reads

$$\left(\eta^{\alpha\gamma} + q^{\alpha\gamma}\right)\left(\eta_{\gamma\beta} + h_{\gamma\beta}\right) = \delta^\alpha_\beta \ .$$

Evaluating this to first order in $h_{\alpha\beta}$ gives

$$q^{\alpha\gamma}\eta_{\gamma\beta} = -\eta^{\alpha\gamma}h_{\gamma\beta} \ .$$

Multiplying both sides by $\eta^{\beta\delta}$ gives

$$q^{\alpha\delta} = -\eta^{\alpha\gamma}\eta^{\beta\delta}h_{\gamma\beta} \equiv -h^{\gamma\beta} \ .$$

**21-23**. *Evaluate all components of the linearized Riemann curvature for the gravitational wave metric (16.2a).*

**Solution:** The linearized Riemann curvature can be found by inserting (21.40) into the definition (21.20) and expanding to first order in $h_{\alpha\beta}$. The result is of the same form as (21.28) with $g_{\alpha\beta}$ replaced by $h_{\alpha\beta}$,

$$\delta R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\frac{\partial^2 h_{\alpha\delta}}{\partial x^\beta \partial x^\gamma} - \frac{\partial^2 h_{\alpha\gamma}}{\partial x^\beta \partial x^\delta} - \frac{\partial^2 h_{\beta\delta}}{\partial x^\alpha \partial x^\gamma} + \frac{\partial^2 h_{\beta\gamma}}{\partial x^\alpha \partial x^\delta}\right) \ .$$

The linearized wave (16.2) depends on $z$ and $t$ so only these derivatives occur in the above expression. The only non-vanishing components of $h_{\alpha\beta}$ are $h_{xx}$

and $h_{yy}$. The only non vanishing components of $\delta R_{\alpha\beta\gamma\delta}$ are then

$$\begin{aligned}
\delta R_{txtx} &= -\frac{1}{2}\frac{\partial^2 h_{xx}}{\partial t^2} = -\frac{1}{2}\frac{\partial^2}{\partial t^2} f(t-z) = -\frac{1}{2} f''(t-z) \ , \\
\delta R_{zxzx} &= -\frac{1}{2}\frac{\partial^2 h_{xx}}{\partial z^2} = -\frac{1}{2}\frac{\partial^2}{\partial z^2} f(t-z) = -\frac{1}{2} f''(t-z) \ , \\
\delta R_{tyty} &= -\frac{1}{2}\frac{\partial^2 h_{yy}}{\partial t^2} = +\frac{1}{2}\frac{\partial^2}{\partial t^2} f(t-z) = +\frac{1}{2} f''(t-z) \ , \\
\delta R_{zyzy} &= -\frac{1}{2}\frac{\partial^2 h_{yy}}{\partial z^2} = +\frac{1}{2}\frac{\partial^2}{\partial z^2} f(t-z) = +\frac{1}{2} f''(t-z) \ , \\
\delta R_{txzx} &= -\frac{1}{2}\frac{\partial^2 h_{xx}}{\partial t \partial z} = -\frac{1}{2}\frac{\partial^2}{\partial t \partial z} f(t-z) = +\frac{1}{2} f''(t-z) \ , \\
\delta R_{tyzy} &= -\frac{1}{2}\frac{\partial^2 h_{yy}}{\partial t \partial z} = +\frac{1}{2}\frac{\partial^2}{\partial t \partial z} f(t-z) = -\frac{1}{2} f''(t-z) \ ,
\end{aligned}$$

and components related to these by symmetry. Here, $f''(u) = d^2 f/du^2$ — the second derivative of $f$ with respect to its argument.

**21-24.** *A linearized gravitational wave in the + polarization is normally incident on a plane containing a circle of test particles such as shown in Figure 16.2. Work out what happens to the test particles when the gravitational wave passes by using the equation of geodesic deviation.*

**Solution:** Let $\chi^{\hat{\alpha}}(t)$ be the components of the displacement vector between the observer and one of the test particles. Let $X^{\hat{\alpha}}$ be this displacement vector before the wave passes. This is constant in time. The wave will produce a change in this displacement $\delta\chi^{\hat{\alpha}}(t)$ that is first order in the gravitational wave perturbation. Evaluating the equation of geodesic deviation in the freely-falling frame (21.23) one has

$$\frac{d^2 \delta\chi^{\hat{\alpha}}}{dt^2} = -\delta R^{\hat{\alpha}}{}_{\hat{t}\hat{\beta}\hat{t}} X^{\hat{\beta}} \ . \tag{1}$$

Here $\tau$ has been replaced by $t$ consistent with the first order accuracy required of the equation. To the zeroth order, the basis vectors of the freely-falling frame may be taken to be the coordinate basis vectors $(1,0,0,0)$, $(0,1,0,0)$, etc. Therefore the form in the coordinate basis will be the same as in the

freely-falling frame. This is similar to (21.26) and gives

$$\frac{d^2\delta\chi^i}{dt^2} = -\delta R^i{}_{tjt} X^j \ . \tag{2}$$

(The $\beta = t$ term in the sum over $\beta$ vanishes since $\delta R^i{}_{\hat{t}\hat{t}\hat{t}} = 0$.) $\delta R^i{}_{tjt}$ can be evaluated by using (21.42) in (21.20). To first order in $h_{\alpha\beta}$, the result has the same form as (21.28) with $g_{\alpha\beta}$ replaced by $h_{\alpha\beta}$. Since only the transverse components $h_{xx}$, $h_{yy}$, and $h_{xy}$ are non-zero, one finds only one of the terms in (21.28) survives

$$\delta R^A{}_{tBt} = -\frac{1}{2}\,\frac{\partial^2 h_{CB}}{\partial t^2}\,\delta^{CA} \tag{3}$$

where $A, B$ range over the transverse directions. Eq. (2) is then straightforwardly integrated for a wave of the form (16.2) to find

$$\begin{aligned}
\chi^x(t) &= X^x\left(1 + \frac{1}{2}f(t)\right) \\
\chi^y(t) &= X^y\left(1 - \frac{1}{2}f(t)\right) \\
\chi^z(t) &= 0 \ .
\end{aligned}$$

This is essentially (16.12).

Although this is the same as (16.13), the derivation given in Section 16.2 is more general. The equation of geodesic deviation assumes that $\boldsymbol{\chi}$ is small, in the present case small compared to the wavelength $\lambda$ characterizing the curvature produced by the gravitational wave. But no such assumption was made in deriving (16.13).

**21-25**. *Show explicitly using (21.28) and (21.53) that the linearized Riemann curvature found in Problem 23 is invariant under gauge transformations.*

**Solution:** The linearized Riemann curvature can be formed by inserting (21.40) into the definition (21.20). The last two terms are quadratic in the perturbation. The first two give the same formula as (21.28) with $g_{\alpha\beta}$ replaced by $h_{\alpha\beta}$ everywhere:

$$\delta R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\frac{\partial^2 h_{\alpha\delta}}{\partial x^\beta \partial x^\gamma} - \frac{\partial^2 h_{\alpha\gamma}}{\partial x^\beta \partial x^\delta} - \frac{\partial^2 h_{\beta\delta}}{\partial x^\alpha \partial x^\gamma} + \frac{\partial^2 h_{\beta\gamma}}{\partial x^\alpha \partial x^\delta}\right) \ .$$

Replacing $h_{\alpha\beta}$ by $h_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha$, one finds that all the terms involving $\xi_\alpha$ cancel because partial derivatives commute. Thus, $\delta R_{\alpha\beta\gamma\delta}$ is invariant under gauge transformations.

**21-26.** *Consider the following metric perturbation*

$$h_{\alpha\beta}(x) = \begin{array}{c} \\ t \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} t & x & y & z \end{array} \\ \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & -c & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & -a \end{pmatrix} \end{array} \sin k(x-t)$$

*for arbitrary constants $c$ and $a$.*

*a) Show how this solves the linearized Einstein equation $\delta R_{\alpha\beta} = 0$.*

*b) Find the functions $\xi_\alpha(x)$ that transform it into a form where $c = 0$. What is the new value of $a$?*

**Solution:**

a) The linearized vacuum equation (21.44) is

$$\delta R_{\alpha\beta} = \frac{1}{2}\left[-\Box h_{\alpha\beta} + \partial_\alpha V_\beta + \partial_\beta V_\alpha\right] \tag{1}$$

where [*cf.* (21.46)]

$$V_\alpha \equiv \partial_\beta h^\beta_\alpha - \frac{1}{2}\,\partial_\alpha h^\beta_\beta\,.$$

Evidently the first term in (1) vanishes because $\sin k(x-t)$ is a solution of the flat space wave equation. The trace $h^\beta_\beta = 2c \sin k(x-t)$. (Remember $h^t_t = -h_{tt}$). Thus, evaluating the components, we find $V_\alpha = 0$, and equation (1) is satisfied.

b) Since the Lorentz gauge freedom is satisfied, the only remaining gauge freedom arises from $\xi_\alpha(x)$ satisfying the wave equation. A little trial and

error shows that

$$\begin{aligned}
\xi_t &= \frac{c}{2k}\cos[k(x-t)] \\
\xi_x &= \frac{c}{2k}\cos[k(x-t)] \\
\xi_y &= \xi_z = 0
\end{aligned}$$

generates a gauge transformation (21.53) that cancels the $c$ terms in the metric, while not producing $h_{tx}$ terms, etc. Further, the value of $a$ is unchanged. That is a special case of the discussion leading to (21.70).

# Chapter 22

# The Source of Curvature

**22-1.** *(Four dimensional divergence theorem and conservation laws.) Consider a cube in flat spacetime with sides oriented along the $(t, x, y, z)$ axes of an inertial frame. Suppose that the cube's dimensions are $\Delta t$, $\Delta x$, $\Delta y$ and $\Delta z$ in the respective directions. Show that for any vector $\mathbf{v}(x)$*

$$\int_{\mathcal{V}_4} d^4x(\partial v^\alpha/\partial x^\alpha) = \int_{\partial\mathcal{V}_4} d^3x(n_\alpha v^\alpha)$$

*where $\mathcal{V}_4$ is the four-volume of the cube, $\partial\mathcal{V}_4$ is the three-surface boundary, and $\mathbf{n}$ is an appropriately chosen normal. Discuss how the normal should be chosen on the spacelike and timelike parts of the cube so that above relation is true. Show that when $\partial v^\alpha/\partial x^\alpha = 0$ , and the cube extends over all space, the right hand side of the above relation implies a conservation law, and find the conserved quantity that is the same on the two bounding spacelike surfaces.*

**Solution:** Let $(0, 0, 0, 0)$ be one corner of the cube. Then,

$$\int_{\mathcal{V}_4} d^4x \left(\frac{\partial v^\alpha}{\partial x^\alpha}\right) = \int_0^{\Delta t} dt \int_0^{\Delta x} dx \int_0^{\Delta y} dy \int_0^{\Delta z} dz \left[\frac{\partial v^t}{\partial t} + \frac{\partial v^x}{\partial x} + \frac{\partial v^y}{\partial y} + \frac{\partial v^z}{\partial z}\right]$$

$$= \int_0^{\Delta x} dx \int_0^{\Delta y} dy \int_0^{\Delta z} dz\, v^t \Big|_{t=0}^{t=\Delta t} + \int_0^{\Delta t} dt \int_0^{\Delta y} dy \int_0^{\Delta z} dz\, v^x \Big|_{x=0}^{x=\Delta x}$$
$$+ \begin{pmatrix} \text{three other} \\ \text{terms} \end{pmatrix} . \qquad (1)$$

The first integral is

$$\int d^3x\, v^t \Big|_{t=0}^{t=\Delta t} = \int_{t=\Delta t} d^3x \left(v^t n_t\right) + \int_{t=0} d^3x \left(v^t n_t\right) . \tag{2}$$

Here we understand the outward-pointing normal to have components $n_t = (1,0,0,0)$ on the three-surface $t = \Delta t$ and $n_t = (-1,0,0,0)$ on $t = 0$. Proceeding similarly for the remaining contributions to (1) gives the surface integral over the eight faces of the hypercube.

Eq. (1) shows that the left hand side of (2) vanishes when $\partial v^\alpha/\partial x^\alpha = 0$. On the right hand side the normals of the two three surfaces are pointing in opposite directions. If the normals are chosen consistently on both surfaces, then (2) becomes

$$\int_{t=\Delta t} d^3x \left(v^t n_t\right) = \int_{t=0} d^3x \left(v^t n_t\right) . \tag{3}$$

This says that the integral is the same at all times — a conservation law!

**22-2.** *A cube of mass $M$ with sides of length $L$ is at rest on an inclined plane whose surface makes an angle $\theta$ with the horizontal. What are the components of the stress $T^{ij}$ exerted by the cube.*

*(a) in rectangular coordinates oriented along the plane and perpendicular to it?*

*(b) in rectangular coordinates oriented horizontally and vertically?*

*(c) Does the stress you calculated in (b) give the correct force in the plane?*

**Solution:**

a) We can confine attention to the two dimensions $(x, y)$ or $(x', y')$ as shown. The stresses exerted by the cube are all across its interface with the plane. Therefore, $T^{x'x'}L^2$ vanishes because it is force exerted across a surface with a normal in the $x'$ direction.

To evaluate the remaining two components, $T^{x'y'}$ and $T^{y'y'}$, take the normal $\vec{n}$ to be as shown with $(x', y')$ components $(0, -1)$. The components of the force exerted by the cube are

$$F^{x'} = Mg\sin\theta \quad , \quad F^{y'} = -Mg\cos\theta \ . \tag{1}$$

These are related to the stress by (22.21)

$$F^{x'} = T^{x'y'}n_{y'}L^2 = -T^{x'y'}L^2 \ , \ F^{y'} = T^{y'y'}n_{y'}L^2 = -T^{yy'}L^2 \tag{2}$$

because there is only a $y'$ component to the normal. Combining (1) and (2) gives

$$T^{x'y'} = -K\sin\theta \quad , \quad T^{y'y'} = K\cos\theta \tag{3}$$

where $K \equiv Mg/L^2$.

b) The coordinates $(x, y)$ are related to $(x', y')$ by the rotation

$$\begin{aligned} x &= x'\cos\theta + y'\sin\theta \ , \\ y &= -x'\sin\theta + y'\cos\theta \ . \end{aligned}$$

The stress-tensor transforms like [*cf.* (20.44)]

$$T^{ij} = \frac{\partial x^i}{\partial x^{k'}}\frac{\partial x^j}{\partial x^{\ell'}}T^{k'\ell'} \ .$$

The result is

$$\begin{aligned} T^{xx} &= -K\cos\theta\sin^2\theta \ , \\ T^{xy} &= +K\sin^3\theta \ , \\ T^{yy} &= +K\cos\theta(1+\sin^2\theta) \ . \end{aligned}$$

c) Using (22.21) in the $(x, y)$ coordinates with $n^i = -(\sin\theta, \cos\theta)$ and the components of the stress from (b) gives correctly

$$F^y = -Mg \quad , \quad F^x = 0 \ .$$

**22-3**. *(The Law of Atmospheres.) Assume the atmosphere is a perfect fluid gas of molecules with mass $m$, where the pressure $p$, number density $n$, and temperature $T$ are related by*

$$p = nk_BT$$

*where $k_B$ is Boltzmann's constant. Using Newtonian gravity, find how the pressure $p$ varies with height $z$ when pressure forces are in equilibrium with gravitational forces. Assume the atmosphere has a constant temperature $T$, that the pressure at sea level is $p_{sea}$, that the heights of interest are small compared to the Earth's radius, and that the Earth's gravitation supplies all the force on the particles of the gas.*

**Solution:** Let $z$ be the vertical direction measured from the earth's surface. The pressure $p(z)$ and number density $n(z)$ are functions of $z$ alone. The pressure force on a little cube of side $L$ is given by (22.35). This must be balanced by the gravitational force. This is given to an excellent approximation by the gravitational force of the earth on the cube

$$-(\text{mass of the cube})\, g = -mn(z)\, L^3 g$$

where $g = GM_\oplus/R_\oplus^2$. Balancing pressure and gravitational forces gives

$$-\frac{dp}{dz}\, L^3 = mn(z)\, L^3 g\ .$$

Using $p = nk_BT$ gives

$$\frac{dn}{dz} = -\frac{mg}{k_BT}\, n(z)\ .$$

Thus $n$ and therefore $p$ vary with height $z$ as $\exp(-mgz/k_BT)$. Explicitly,

$$p(z) = p_0\ \exp(-mgz/k_BT)$$

**22-4**. *(The stress tensor is symmetric.) Calculate the torque about its center exerted on a small cube of side $L$ assuming that $T^{xy}$ and $T^{yx}$ are the only*

*non-zero components of the stress tensor but that the stress tensor is not symmetric, $T^{xy} \neq T^{yx}$. Consider smaller and smaller cubes made of the same density material. How does the net torque vary with smaller and smaller pieces? Can you see any reason from this variation why the stress tensor has to be symmetric?*

**Solution:**

$T^{ij}n_jL^2$ is the $i^{\text{th}}$ component of the force exerted across the face with normal $\vec{n}$. Assuming that $T^{xy}$ and $T^{yx}$ are the only components of the stress tensor the forces acting on the cube are as shown above. The net torque about the $z$-axis is

$$(\text{torque})^z = \frac{L}{2}\left(T^{yx}L^2\right) - \frac{L}{2}\left(T^{xy}L^2\right) + \frac{L}{2}\left(T^{yx}L^2\right) - \frac{L}{2}\left(T^{xy}L^2\right)$$

where successive terms correspond to forces in the above picture counted clockwise starting from the top. The net torque is

$$(\text{torque})^z = L^3\left(T^{yx} - T^{xy}\right) .$$

The moment of inertia of the cube around the $z$-axis $I_z$ is a dimensionless geometrical factor $f$ times $L^2M$ where $M$ is the mass of the cube given in terms of its density by $\rho L^3$. The angular acceleration $\alpha$ is found from

$$I_z\alpha = (\text{torque})^z$$

or

$$\alpha = \frac{1}{f\rho L^2}\,(T^{yx} - T^{xy})\ .$$

If $T^{yx} \neq T^{xy}$ this equation of motion would predict larger and larger accelerations as $L \to 0$ for smaller and smaller pieces of the material — clearly an unphysical solution. That is a general argument why the stress tensor is symmetric.

**22-5.** *A box of gas is at rest. The molecules of the gas are uniformly distributed throughout the box and are moving with a distribution of momenta $f(\vec{p})$ so that $f(\vec{p})d^3p$ is the number of molecules per unit volume with momentum in the range $d^3p$ centered on $\vec{p}$. Suppose $f(\vec{p})$ is isotropic meaning it depends only on $|\vec{p}|$.*

*(a) Argue that the stress tensor for the gas is*

$$T^{\alpha\beta} = \int d^3p\,\frac{f(\vec{p})}{m}\,p^\alpha p^\beta$$

*where $p^\alpha$ is the four-momentum considered as a function of $\vec{p}$ and $m$ is the rest mass of the molecule.*

*(b) Calculate $T^{ij}$ and show it is diagonal with all diagonal entries equal.*

*(c) Find the pressure and energy density in the gas. Assuming the distribution is peaked about ultrarelativistic momenta, find the equation of state of the gas.*

**Solution:**

a) Consider all the particles in the gas moving with momentum $\mathbf{p}$. Since the distribution in space is uniform, the contribution of these molecules to the stress energy will be $m\,u^\alpha u^\beta$ from (22.17). Expressed in terms of the momenta $p^\alpha = m\,u^\alpha$ this is $p^\alpha p^\beta/m$. Summed over the distribution this is the quoted result.

b) Since the distribution function depends only on $|\vec{p}|$, integrals over it weighted by an odd number of factors of $p^x$, for example, will vanish. Similarly, for $p^y$ and $p^z$. This shows that $T^{ij}$ is diagonal

$$T^{ij} = A\delta^{ij} = \int d^3p\,\frac{f(\vec{p})}{m}\,p^i p^j\ .$$

Contracting over $i$ and $j$ gives

$$3A = \int d^3p \, \frac{f(|\vec{p}|)}{m} \, \vec{p}^2 \, .$$

c) The result from (b) shows that the stress energy is in perfect fluid form with

$$\rho = m \int d^3p \, f(|\vec{p}|)$$

and

$$p = \frac{1}{3} \int d^3p \, \frac{f(|\vec{p}|)}{m} \, \vec{p}^2 \, .$$

If the distribution is peaked around very relativistic momenta, $|\vec{p}| \gg m$, and the equation of state will be $p = \rho/3$.

**22-6**. *[P] (The Stress Energy for Electromagnetism.) This problem concerns the stress-energy tensor for the electromagnetic field in flat spacetime. Standard results are quoted in $c = 1$, SI units which involve the defined factors $\mu_0 \equiv 4\pi \times 10^{-7}$ and $\epsilon_0 \equiv 1/\mu_0$. You can also use Gaussian units simply by making the replacements $\mu_0 \to 4\pi$ and $\epsilon_0 \to 1/(4\pi)$.*

*In electrodynamics the energy density $\epsilon$ is given in vacuum by*

$$\epsilon = \frac{1}{2} \left( \epsilon_0 \vec{E}^2 + \frac{1}{\mu_0} \vec{B}^2 \right) ,$$

*the energy flux is given by the Poynting vector*

$$\vec{S} = \frac{1}{\mu_0} \left( \vec{E} \times \vec{B} \right) ,$$

*and the stress by the Maxwell stress-tensor:*

$$T^{ij} = \epsilon_0 \left( -E^i E^j + \frac{1}{2} \, \delta^{ij} \vec{E}^2 \right) + \frac{1}{\mu_0} \left( -B^i B^j + \frac{1}{2} \, \delta^{ij} \vec{B}^2 \right) .$$

*(In comparing with other possible formulas you may have seen, remember that $c^2 = 1 = 1/(\epsilon_0 \mu_0)$. Also watch for sign changes in the definition of the stress tensor. )*

a) *Put these together to form the stress energy tensor $T^{\alpha\beta}$ for the electromagnetic field.*

b) *Show explicitly from Maxwell's equations that this $T^{\alpha\beta}$ is conserved, i.e. satisfies the four equations (22.31).*

**Solution:**

a) The answer is in (22.27).

b) It's easiest to use $c = 1$ units. Then, since generally $c^2 = 1/(\mu_0\epsilon_0)$, the relation $\mu_0 = 1/\epsilon_0$ holds and the vacuum Maxwell's equations take the form

$$\begin{aligned} \vec{\nabla} \times \vec{H} &= \epsilon_0 \left(\partial\vec{E}/\partial t\right) \qquad \epsilon_0 \left(\vec{\nabla} \times \vec{E}\right) = -\partial\vec{H}/\partial t \\ \vec{\nabla} \cdot \vec{E} &= 0 \,, \quad \vec{\nabla} \cdot \vec{H} = 0 \,, \quad \vec{B} = \vec{H}/\epsilon_0 \end{aligned}$$

The equation $\partial_\beta T^{t\beta} = 0$ is

$$\frac{\partial\epsilon}{\partial t} + \vec{\nabla} \cdot \vec{S} = 0 \,. \tag{1}$$

To check this, evaluate $\partial\epsilon/\partial t$ from the expression provided in the problem

$$\frac{\partial\epsilon}{\partial t} = \epsilon_0 \vec{E} \cdot \frac{\partial\vec{E}}{\partial t} + \mu_0 \vec{B} \cdot \frac{\partial\vec{B}}{\partial t}$$

and use Maxwell's equations to eliminate the time derivatives. The result is

$$\begin{aligned} \frac{\partial\epsilon}{\partial t} &= \vec{E} \cdot \left(\vec{\nabla} \times \vec{H}\right) - \vec{H} \cdot \vec{\nabla} \times \vec{E} \,, \\ &= -\vec{\nabla} \cdot \left(\vec{E} \times \vec{H}\right) = -\vec{\nabla} \cdot \vec{S} \,. \end{aligned} \tag{2}$$

The second equality follows from a standard vector identity. (2) shows (1) is satisfied.

The equation $\partial_\beta T^{x\beta} = 0$ is handled similarly. It is

$$\frac{\partial\pi^x}{\partial t} + \frac{\partial T^{xj}}{\partial x^j} = 0 \tag{3}$$

where $\pi^x$ is the $x$-component of the momentum density equal to $S^x = E^y H^z - E^z H^y$. Thus,

$$\frac{\partial \pi^x}{\partial t} = \frac{\partial E^y}{\partial t} H^z + \frac{\partial H^z}{\partial t} E^y - \frac{\partial E^z}{\partial t} H^y - \frac{\partial H^y}{\partial t} E^z .$$

Again, using Maxwell's equations to eliminate the time derivatives, we find

$$\begin{aligned} \frac{\partial \pi^x}{\partial t} &= \frac{1}{\epsilon_0}\left(\frac{\partial H^z}{\partial x} - \frac{\partial H^x}{\partial z}\right) H^z - \epsilon_0 \left(\frac{\partial E^x}{\partial y} - \frac{\partial E^y}{\partial x}\right) E^y \\ &- \frac{1}{\epsilon_0}\left(\frac{\partial H^x}{\partial y} - \frac{\partial H^y}{\partial x}\right) H^y + \epsilon_0 \left(\frac{\partial E^z}{\partial x} - \frac{\partial E^x}{\partial z}\right) E^z . \end{aligned}$$

Further algebra gives

$$\begin{aligned} \frac{\partial \pi^x}{\partial t} &= \epsilon_0 \left\{ \frac{1}{2}\frac{\partial}{\partial x}\left[(E^y)^2 + (E^z)^2 + (B^y)^2 + (B^z)^2\right] \right. \\ &- \left. \frac{\partial E^x}{\partial y} E^y - \frac{\partial E^x}{\partial z} E^z - \frac{\partial B^x}{\partial z} B^z - \frac{\partial B^x}{\partial y} B^y \right\} \end{aligned} \tag{4}$$

The quantity

$$\epsilon_0 \left[E^x \left(\vec{\nabla} \cdot \vec{E}\right) + B^x \left(\vec{\nabla} \cdot \vec{B}\right)\right]$$

vanishes identically because of the remaining Maxwell's equations. When subtracted from (4) it verifies (3).

**22-7**. *[S] Consider the cube described in Example 5. Show that the surface integral in* (22.34) *gives the net pressure force acting on the cube.*

**Solution:** With $T^{ij}$ of the form (22.25) the surface integral in (22.34) becomes

$$F^i = \int dA n_j^{(\text{in})} T^{ij} = \int dA n_j^{(\text{in})} \delta^{ij} p = \int dA n^i_{(\text{in})} p. \tag{1}$$

where $n^i_{(\text{in})}$ are the components of the inward pointing unit normal. First consider $F^x$. There are 6 faces of the cube contributing to the surface integral in (1). But only the two lying in $y$-$z$ planes have non-zero $x$-components of the normal. The face at $x$ has $n^x_{(\text{in})} = 1$, that at $x + L$ has $n^x_{(\text{in})} = -1$. Thus,

$$F^x = \int_0^L dy \int_0^L dz \, [p(x, y, z) - p(x + L, y, z)]. \tag{2}$$

When $L$ is sufficiently small that the pressure varies little over $y$ and $z$, this is

$$F^x = -L^3 \frac{\partial p}{\partial x} \tag{3}$$

as in (22.35). The other directions are similar.

**22-8**. *[S] Show that the stress-energy of the vacuum defined by* (22.38) *satisfies the local conservation law* (22.40).

**Solution:** The vacuum stress energy tensor (22.38) automatically satisfies the local conservation law (22.40) because all covariant derivatives of the metric vanish [*cf.* (20.70)].

**22-9**. *[S] Show that all observers measure the same energy density of the vacuum no matter how they are moving through spacetime.*

**Solution:** The energy measured by an observer is given by (20.30). The vacuum stress tensor is given by (22.38). The result is

$$(\text{energy density}) = -\frac{\Lambda}{8\pi G}\, \mathbf{u}_{\text{obs}} \cdot \mathbf{u}_{\text{obs}} = \frac{\Lambda}{8\pi G}$$

which is independent of observer.

**22-10**. *The Weak Energy Condition. Consider stress-energy tensors which in a local inertial frame have the form* $T_{\alpha\beta} = \text{diag}(A, B, C, D)$.

*(a) What condition on $A$, $B$, $C$, and $D$ must be satisfied so that any observer will see positive energy density no matter how fast that observer is moving with respect to the frame in which the stress-energy tensor is given.*

*(b) The vacuum stress-energy tensor* (22.39) *is of this form with negative values of $B$, $C$, and $D$. Is there some frame where an observer would see negative energy density?*

**Solution:** (a)The energy density measured by an observer is given by (22.30). Since there is an inertial frame, the observer's four velocity is given by (5.28).

The energy density measured by the observer is therefore

$$\gamma^2 \left[A + B(V^x)^2 + C(V^y)^2 + D(V^z)^2\right] .$$

Since each component of $\vec{V}$ could possibly range from -1 to 1, the conditions for positive energy density are

$$A \geq |B| , \; A \geq |C| , \; A \geq |D| .$$

(b) The vacuum stress energy saturates these bounds so no observer will see negative energy density.

**22-11**. *Reinforce the argument given in Section 16.5 that there is no local gravitational energy by showing that there is no stress energy tensor that can be constructed from the metric and its first derivatives that reduces to zero when space is flat.*

**Solution:** The covariant derivative of the metric vanishes. Therefore, the only second rank symmetric tensor that can be constructed from the metric and its first derivatives is the metric itself,

$$T_{\alpha\beta} = C g_{\alpha\beta}$$

for some constant $C$. But $C = 0$ if $T_{\alpha\beta}$ vanishes in flat space.

**22-12**. *[C] (Symmetry Implies Conservation.) The results of the following problem are general but to keep the algebra manageable restrict attention to metrics of the form:*

$$ds^2 = -A^2 dt^2 + B^2 dx^2 + C^2 dy^2 + D^2 dz^2$$

*where $A$, $B$, $C$, and $D$ are functions of $(t, x, y, z)$.*

*(a) Show that a relation like*

$$\nabla_\alpha J^\alpha = 0$$

*can be written in the form*

$$\frac{\partial(fJ^\alpha)}{\partial x^\alpha} = 0$$

*for some function $f$ specified by the metric, and therefore implies a conservation law for the current $J^\alpha$.*

*(b) When spacetime has a symmetry there is an associated Killing vector satisfying*

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0 \ .$$

*(See Problem 20.18 to be led through a demonstration.) Show that*

$$J^\alpha \equiv \xi_\beta T^{\alpha\beta}$$

*is a conserved current.*

**Solution:**

a) Writing out $\nabla_\alpha J^\alpha = 0$ using (20.54) gives

$$\nabla_\alpha J^\alpha = \frac{\partial J^\alpha}{\partial x^\alpha} + \Gamma^\alpha_{\alpha\beta} J^\beta = 0 \ . \tag{1}$$

To keep the algebra at a manageable level, it's convenient to restrict to the special case of diagonal metrics

$$ds^2 = -A^2 dt^2 + B^2 dx^2 + C^2 dy^2 + D^2 dz^2$$

where $A, B, C, D$ are all functions of the coordinates $(t, x, y, z)$. Explicit calculation then shows the Christoffel symbol in (1) to be

$$\Gamma^\alpha_{\alpha\beta} = \frac{1}{ABCD} \frac{\partial}{\partial x^\beta} (ABCD) \ , \tag{2}$$

$$= \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial x^\beta} \tag{3}$$

where $g \equiv -ABCD = \det(g_{\alpha\beta})$. Written in the form (3), the result is, in fact, valid for a general metric. This results shows (1) is equivalent to

$$\frac{\partial(\sqrt{-g}\, J^\alpha)}{\partial x^\alpha} = 0 \tag{4}$$

This is in the form of a conservation law. Going one step further, the conserved quantity follows from (22.8), (22.9) and (4) above. Define a quantity $Q(t)$ by

$$Q(t) = \int_t d^3x \sqrt{-g}\, J^t$$

where the integral extends over the spacelike surface $x^0 = t$. Since $J^i$ vanishes at spatial infinity, we see from (22.8) that $\frac{d}{dt} Q(t) = 0$, i.e. $Q(t)$ is constant in time — conserved. As an aside, we note this can be written in the form

$$Q(t) = \int dV\, n_\alpha J^\alpha$$

where $\mathbf{n} \cdot \mathbf{n} = -1$ since $n_\alpha = (A, 0, 0, 0)$ and $dV = (BCD)d^3x$, [*cf.* (7.29)]

b)

$$\nabla_\alpha \left(\xi_\beta T^{\alpha\beta}\right) = \nabla_\alpha \xi_\beta\, T^{\alpha\beta} + \xi_\beta \nabla_\alpha T^{\alpha\beta}\ .$$

The second term vanishes because $\nabla_\alpha T^{\alpha\beta} = 0$ generally. The first term vanishes because $\nabla_\alpha \xi_\beta$ is antisymmetric (the condition satisfied by $\xi_\alpha$ given in the problem) and $T^{\alpha\beta}$ is symmetric. Thus,

$$\int d^3x \sqrt{-g}\, \xi_\beta T^{t\beta}$$

is conserved.

**22-13.** *[A,C] (Proving the Bianchi Identity) In a local inertial frame the Bianchi identities* (22.50) *read*

$$\partial_\alpha R^\alpha_\beta - \frac{1}{2}\, \partial_\beta R = 0$$

*where* $\partial_\alpha = \partial/\partial x^\alpha$. *Use* (21.20) *and* (21.28) *to demonstrate these identities as follows:*

*a) Use* (21.20) *to demonstrate that only terms containing third derivatives of the metric survive in the local inertial frame.*

*b) Use* (21.28) *to evaluate the combinations of third derivatives of the metric that occur in the above expression for the Bianchi identities and show that they cancel.*

**Solution:**

a) In a local inertial frame (LIF) the Γ's vanish but derivatives of the Γ's do not. The expression above involves first derivatives of

$$g^{\alpha\gamma} R_{\gamma\beta} \tag{1}$$

where $R_{\gamma\beta}$ is given by (21.33). Clearly, first derivatives of $g^{\alpha\gamma}$ in (1) vanish in an LIF. Derivatives of the second two terms in (21.33) are of the form (Γ) × (derivative of Γ) and vanish in an LIF because the Γ's vanish there. Similar arguments show that only the third derivatives of the metric from the first two terms in (21.33) survive.

b) In view of (a) the Bianchi identity quantity

$$B_\beta \equiv \partial_\alpha R^\alpha_\beta - \frac{1}{2}\,\partial_\beta R$$

can be written in an LIF as

$$B_\beta = \eta^{\alpha\gamma}\left(\partial_\alpha R_{\beta\gamma} - \frac{1}{2}\,\partial_\beta R_{\alpha\gamma}\right) .$$

Then using the definition of the Ricci curvature in terms of the Riemann curvature, this becomes

$$B_\beta = \eta^{\alpha\gamma}\eta^{\delta\epsilon}\left(\partial_\alpha R_{\delta\beta\epsilon\gamma} - \frac{1}{2}\,\partial_\beta R_{\delta\alpha\epsilon\gamma}\right) .$$

Substituting (21.28) into this gives all the third derivatives. There are a total of eight terms. By relabeling the dummy indices and using the fact that partial derivatives commute, they can all be seen to cancel with the result

$$B_\beta = 0 .$$

**22-14.** *[C,N] Warp drive requires negative energy density. This problem concerns the Alcubierre warp drive spacetime discussed in Section 7.4 whose line element is given in* (7.24).

*(a) Calculate the components of the normal $n_\alpha$ to a surface of constant $t$.*

*(b) Modify the Mathematica program "Curvature and the Einstein Equation" available on the book website to show that*

$$T_{\alpha\beta}n^\alpha n^\beta = -\frac{1}{8\pi}\frac{V_s^2(y^2+z^2)}{(2r_s)^2}\left(\frac{df}{dr_s}\right)^2 .$$

*This is the energy density measured by observers at rest with respect to the surfaces of constant $t$. The fact that it is negative means that the warp drive spacetime can't be supported by classical matter with positive energy density.*

**Solution:**

a) A normal to the surface $t =$ const. has components $n_\alpha = (1,0,0,0)$ so that the components

$$n^\alpha = g^{\alpha 0} .$$

The components of $g^{\alpha 0}$ can be found from the *Mathematica* program described below.

b) Choose $(x^1, x^2, x^3, x^4)$ to be $(t, x, y, z)$. Then input the metric (7.24) into the program "Curvature and the Einstein Equation", say as follows

```
warpdrive.nb                                                    1

In[4]:= rs[t, x, y, z] := ((x - vs t)^2 + y^2 + z^2)^(1/2)

In[5]:= metric = {{-1 + (vs f[rs[t, x, y, z]])^2, -vs f[rs[t, x, y, z]], 0 , 0},
          {-vs f[rs[t, x, y, z]], 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}

Out[5]= {{-1 + vs^2 f[Sqrt[(-t vs + x)^2 + y^2 + z^2]]^2, -vs f[Sqrt[(-t vs + x)^2 + y^2 + z^2]], 0, 0},
         {-vs f[Sqrt[(-t vs + x)^2 + y^2 + z^2]], 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}
```

Append to the bottom, a line to calculate $G_{nn} = G_{\alpha\beta}n^\alpha n^\beta$

```
warpdrive.nb                                                    1

In[24]:= Gnn = Simplify[Sum[
           inversemetric[[1, i]] einstein[[i, j]] inversemetric[[1, j]], {i, 1, 4}, {j, 1, 4}]]

Out[24]= -(vs^2 (y^2 + z^2) f'[Sqrt[t^2 vs^2 - 2 t vs x + x^2 + y^2 + z^2]]^2) / (4 (t^2 vs^2 - 2 t vs x + x^2 + y^2 + z^2))
```

giving the result. Since

$$T_{\alpha\beta} = \frac{1}{8\pi} G_{\alpha\beta}$$

this is the result quoted in the problem.

**22-15.** *Wormholes require negative energy density. Recall the wormhole geometry* (7.39). *Calculate the components of the stress-energy* $T^{\hat{\alpha}\hat{\beta}}$ *that would be needed for this geometry to be a solution of the Einstein equation in the orthonormal basis used in that example. Show that the energy density (as measured by a stationary observer) required is negative. Since all realistic matter described classically has positive energy density, this means it is impossible to construct a wormhole like* (7.39) *by classical means.*

**Solution:** The components of $T^{\hat{\alpha}\hat{\beta}}$ can be found by the following steps:

a) Use the *Mathematica* program *Curvature and the Einstein Equation* on the book website to calculate $G_{\alpha\beta}$ in the coordinate basis associated with the wormhole metric (7.39).

b) Use $G_{\alpha\beta} = 8\pi G T_{\alpha\beta}$ to find the coordinate basis components of $T_{\alpha\beta}$.

c) Construct an orthonormal basis according to (7.61) and project $T_{\alpha\beta}$ onto this basis according to (20.41) and raise indices.

The result is

$$T^{\hat{t}\hat{t}} = T^{\hat{r}\hat{r}} = -T^{\hat{\theta}\hat{\theta}} = -T^{\hat{\phi}\hat{\phi}} = -\frac{1}{8\pi G}\frac{b^2}{(b^2+r^2)^2}$$

with all other components vanishing. Evidently $T^{\hat{t}\hat{t}}$ is negative.

**22-16.** *[A,N] Embedding Constant Negative Curvature Surfaces in Euclidean Space. (a) Compute the scalar curvature of of the two-surface which is a* $\theta = \pi/2$ *slice of the spatial geometry of an open universe as represented in* (18.60). *Show that the curvature scalar curvature is constant over the surface and negative.*

*(b) Show that the two-dimensional geometry*

$$d\Sigma^2 = du^2 + \cosh^2 u d\phi^2$$

*Has constant curvature as well. By Minding's theorem in differential geometry this surface must have the same local geometry as the slice of an open universe in (a).*

*(c) Find an embedding of the surface in (b) in three-dimensional flat space. Does it look like a potato chip? (This part is the same as part (b) of Problem 18.30 if you worked that.)*

**Solution:**

a) The metric on a $t = \text{const.}$, $\theta = \pi/2$ slice of (18.60) in the open case is

$$d\Sigma^2 = a^2(d\chi^2 + \sinh^2\chi\, d\phi^2)\ . \tag{1}$$

Using the *Mathematica* program *Curvature and the Einstein Equation*, we find

$$R = -\frac{2}{a^2}$$

which is constant and negative.

b) Again, using the *Mathematica* program *Curvature and the Einstein Equation* gives

$$R = -2$$

for

$$d\Sigma^2 = du^2 + \cosh^2 u\, d\phi^2.$$

For the rest, see (b) of the solution to Problem 18.30.

# Chapter 23

# Gravitational Wave Emission

**23-1**. *In Example 21.5 we showed that the equations of vacuum linearized gravity were satisfied for the time-independent static, weak field, metric* (21.25) *when* $\Phi$ *satisfies the vacuum Newtonian equation* $\nabla^2\Phi = 0$. *Show the same thing for the equations of linearized gravity with non-relativistic sources when* $\Phi$ *satisfies the Newtonian equation with sources* $\nabla^2\Phi = 4\pi\mu$ *(geometrized units).*

**Solution:** The two equations of linearized gravity are (23.5) and (23.6). The gauge condition (23.5) was shown to be satisfied in Example 21.6. From the form of the perturbations (21.25) for the Newtonian metric, one finds the only non-vanishing component of $\bar{h}^{\alpha\beta}$ is

$$\bar{h}^{tt} = -4\Phi \ . \tag{1}$$

The rest energy dominates the stress energy of non-relativistic matter so the only significant component of $T^{\alpha\beta}$ is

$$T^{tt} = \mu \ . \tag{2}$$

Putting (1) and (2) in the linearized gravity equation (23.6) gives the Newtonian equation

$$\vec{\nabla}^2\Phi = 4\pi\mu \ .$$

**23-2.** *[C] Equation (23.28) for the metric perturbation produced by the slow and uniform rotation of a spherical body was derived only for large values of $r$ compared to the size of the source. Show that it holds for all values of $r$ outside the rotating body.*

**Solution:** The metric perturbation is a solution of the differential equation (23.6). Consider the equation for $\bar{h}^{tx}$ with the source $T^{tx}$ given in (23.24) and (23.26). Since $\bar{h}^{tx}$ is assumed to be independent of time, (23.6) becomes

$$\vec{\nabla}^2 \bar{h}^{tx} = 16\pi\, \mu\,(r)\, \Omega\, y. \tag{1}$$

(If you are familiar with electromagnetism you will recognize this as Poisson's equation.) We will have found the solution if we can find an $h^{tx}$ which (i) vanishes at infinity, (ii) is regular at the origin, (iii) is continuous across the boundary of the star, and (iv) satisfies the Lorentz gauge condition. The form of the answer at large $r$ in (23.28) suggests the ansatz

$$h^{tx} = (y/r)\, \omega(r) = \sin\theta \sin\phi\, \omega(r) \tag{2}$$

in usual polar coordinates $(r, \theta, \phi)$. To see if there is a solution of (1) in this form, first recall the form of the operator $\vec{\nabla}^2$ in polar coordinates

$$\vec{\nabla}^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}\right]. \tag{3}$$

Inserting (2) into (1) and using (3), we find that there is a solution of this form which satisfies conditions (i)-(iii) above provided $\omega(r)$ satisfies the differential equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\omega}{dr}\right) - \frac{2\omega}{r^2} = 16\pi\, \mu\,(r)\, \Omega. \tag{4}$$

Outside the star where the right-hand side vanishes, the solution which vanishes at infinity is

$$\omega(r) = \frac{A}{r^2} \tag{5}$$

for some constant $A$ — *everywhere* outside the star. The value of $A$ could be determined by solving for the solution inside that is regular at the origin and

matching across the boundary. But there is no need to do this. The asymptotic form already derived in (23.28) shows that $A = -2J$. Thus, $h^{tx} = -2Jy/r^3$ everywhere outside the star. Similarly, we'll also find that $h^{ty} = 2Jx/r^3$ and $h^{tz} = 0$ everywhere outside the star.

Finally, we need to check that these $h^{ti}$ are in the Lorentz gauge [see (23.5)]. Since $\bar{h}^{ti} = h^{ti}$,

$$\begin{aligned}\frac{\partial \bar{h}^{ti}}{\partial x^i} &= \frac{\partial h^{tx}}{\partial x} + \frac{\partial h^{ty}}{\partial y} \\ &= \frac{\partial}{\partial x}\left(\frac{y\omega(r)}{r}\right) - \frac{\partial}{\partial y}\left(\frac{x\omega(r)}{r}\right) \\ &= 0\,,\end{aligned}$$

and so the Lorentz gauge condition is satisfied.

If you are familiar with spherical harmonics you can also do this problem by expanding $|\vec{x} - \vec{x}'|^{-1}$ in powers of $(r'/r)$ and doing the angular integrals in (23.25).

**23-3**. *Work through the details of deriving the Lorentz gauge condition* (23.23) *from* (23.22).

**Solution:** The only subtlety in this problem is keeping track of the derivatives of $t_{\text{ret}} \equiv t - |\vec{x} - \vec{x}'|$ with respect to $x^i$ or $x'^i$. For instance,

$$\frac{\partial T^{\alpha i}\left(t_{\text{ret}}, x'^i\right)}{\partial x'^i} = \left(\frac{\partial T^{\alpha i}\left(t', x'^i\right)}{\partial x'^i}\right)_{t'=t_{\text{ret}}} - \left(\frac{\partial T^{\alpha i}}{\partial t'}\right)_{t'=t_{\text{ret}}} \frac{\partial}{\partial x'^i}\left(|\vec{x} - \vec{x}'|\right). \tag{1}$$

The right-hand side of (23.22) can be written out explicitly taking account of the dependence of $t_{\text{ret}}$ on $x^i$ as follows:

$$4\int d^3x' \left[\frac{\partial T^{\alpha t}}{\partial t}\frac{1}{|\vec{x} - \vec{x}'|} - \frac{\partial T^{\alpha i}}{\partial t}\left(\frac{\partial}{\partial x^i}|\vec{x} - \vec{x}'|\right)\frac{1}{|\vec{x} - \vec{x}'|} + T^{\alpha i}\frac{\partial}{\partial x^i}\frac{1}{|\vec{x} - \vec{x}'|}\right]_{t=t_{\text{ret}}}. \tag{2}$$

Next, make use of the fact that

$$\frac{\partial}{\partial x^i}|\vec{x} - \vec{x}'| = -\frac{\partial}{\partial x'^i}|\vec{x} - \vec{x}'|$$

to find

$$4\int d^3x' \left[\frac{\partial T^{\alpha t}}{\partial t}\frac{1}{|\vec{x}-\vec{x}'|} + \frac{\partial T^{\alpha i}}{\partial t}\left(\frac{\partial}{\partial x'^i}|\vec{x}-\vec{x}'|\right)\frac{1}{|\vec{x}-\vec{x}'|} - T^{\alpha i}\frac{\partial}{\partial x'^i}\frac{1}{|\vec{x}-\vec{x}'|}\right]_{t=t_{\rm ret}} . \tag{3}$$

Integrate the last term by parts. If the source is bounded, the surface term vanishes. The result is

$$4\int d^3x' \left[\frac{\partial T^{\alpha t}}{\partial t}\frac{1}{|\vec{x}-\vec{x}'|} + \frac{\partial T^{\alpha i}}{\partial t}\left(\frac{\partial}{\partial x'^i}|\vec{x}-\vec{x}'|\right)\frac{1}{|\vec{x}-\vec{x}'|} + \frac{\partial T^{\alpha i}}{\partial x'^i}\frac{1}{|\vec{x}-\vec{x}'|}\right]_{t=t_{\rm ret}} . \tag{4}$$

Then, using (1) we get

$$4\int d^3x' \left[\frac{\partial T^{\alpha t}}{\partial t} + \left(\frac{\partial T^{\alpha i}}{\partial x'^i}\right)_t\right]_{t=t_{\rm ret}} \frac{1}{|\vec{x}-\vec{x}'|} \tag{5}$$

where $(\partial T^{\alpha i}/\partial x'^i)_t$ means the partial derivative with respect to $x'^i$ keeping $t$ constant. This is the result quoted in (23.23).

**23-4**. *Spell out all the steps in the derivation of the metric outside a slowly rotating body* (23.28) *from* (23.25).

**Solution:** Let's consider the calculation of $h^{tx}$. The other components are similar. Inserting (23.27) and (23.26) into (23.25) we have

$$h^{tx} = -\frac{4}{r}\int d^3x' \mu(r')\ (\vec{x}\cdot\vec{x}')\ y'\ \Omega + \cdots .$$

The integrand of the first term is odd under $y' \to -y'$, but the integration range is symmetric. The first term therefore vanishes. In the second term

$$(\vec{x}\cdot\vec{x}')\ y' = (xx' + yy' + zz')\ y'.$$

The first and last terms are odd under $y' \to -y'$ and vanish. The second gives

$$h^{tx} = -\frac{4y}{r^3}\int d^3x' \mu(r')\, y'^2\Omega$$

But from (23.29)

$$J = \int d^3x' \mu(r') \left(y'^2 + z'^2\right) \Omega.$$

The $y'^2$ and $z'^2$ terms contribute equally in this expression. Therefore,

$$h^{tx} = -\frac{2yJ}{r^3}.$$

which is (23.28).

**23-5**. *[E] Would a nuclear explosion half way around the Earth produce a gravitational wave of sufficient amplitude to be detected by the LIGO gravitational wave receiver? To answer this question, estimate the amplitude $h$ that might be expected from such an explosion and compare with the rough sensitivity $h \sim 10^{-22}$ expected of the advanced LIGO detectors. (A large nuclear explosion is 20 megatons of TNT. 1 megaton of TNT= $4.2 \times 10^{22}$ erg).*

**Solution:** From (23.35) we have (putting back the factors of $G$ and $c$ so $h_{ij}$ is dimensionless)

$$\bar{h}_{ij}(t,r) = \frac{2G\ddot{I}_{ij}}{c^4 r}.$$

Dimensionally $[I] \sim \mathcal{M}\mathcal{L}^2$, so $[\ddot{I}] \sim \mathcal{M}(\mathcal{L}/\mathcal{T})^2$ which is the dimensions of energy. Thus, very roughly we estimate

$$h \sim \frac{GE}{c^4 R_\oplus}$$

where $E$ is the energy of the explosion. Specifically,

$$h \sim \frac{(6.7 \times 10^{-8}\ \mathrm{cm}^3/(\mathrm{g}\cdot\mathrm{s}^2))(20 \times 4.2 \times 10^{22}\ \mathrm{erg})}{(3 \times 10^{10}\ \mathrm{cm/\ s})^4 (6.4 \times 10^8\ \mathrm{cm})} \sim 10^{-34}\ (!)$$

which is evidently many orders of magnitude below what is detectable by LIGO.

**23-6**. *[C] (No dipole gravitational radiation.) (a) In Section 23.4 we did not*

*discuss the large $r$ behavior of the $\bar{h}^{t\alpha}$ parts of the metric perturbations. Show that in the long wavelength approximation these are given by*

$$\bar{h}^{t\alpha} = 4P^{\alpha}/r$$

*where $P^{\alpha}$ is the total energy momentum four-vector of the matter. To simplify your discussion you may assume that the stress-energy tensor of the matter has the non-relativistic form $T^{\alpha\beta} = \mu\, u^{\alpha}u^{\beta}$ with all velocities much less than unity.*

*(b) Show that*

$$\bar{h}^{ti} = 4\dot{p}^{i}/r$$

*where $\vec{p}$ is the mass dipole moment*

$$\vec{p} = \int d^3x\, \mu\,(x)\, \vec{x}\ .$$

*What other important quantity in Newtonian mechanics is the mass dipole moment connected to?*

*(c) Argue that by a Lorentz transformation to a new inertial frame the mass dipole term can be made to vanish, and that therefore there is no contribution to gravitational radiation. Find the velocity and direction of the Lorentz transformation.*

**Solution:** (a) The result is general and follows from (23.30) and the identification of the components $T^{t\alpha}$ in terms of energy-momentum density [*cf.* (22.27)]. However, it can be seen explicitly in the non-relativistic approximation where

$$T^{tt} = \mu\ , \qquad T^{ti} = \mu V^{i}$$

neglecting terms of order $V^2$. The integrals over these components in (23.30) are just the expressions for the total rest energy and momentum respectively.

(b) There are two ways of making this demonstration. The way analogous to the discussion leading to (23.35) starts from the conservation of rest mass in the non-relativistic approximation

$$\frac{\partial \mu}{\partial t} + \vec{\nabla}\cdot\left(\mu\vec{V}\right) = 0\ .$$

Multiply this equation by $x^i$ and integrate over all space. The first term gives the time derivative of $p^i$. After integration by parts (or use of the divergence

theorem) the second gives $-P^i = -\int \mu V^i d^3x$ for a bounded source. Thus,

$$\vec{P} = d\vec{p}/dt \tag{1}$$

and the result follows.

In Newtonian mechanics the center of the mass position is $\vec{X}_{cm} = \vec{p}/M$ where $M$ is the total mass. The relation (1) is just

$$\vec{P} = M\dot{\vec{X}}_{\text{cm}} \tag{2}$$

and recognizing this is another way to derive the result.

(c) In a Lorentz frame where the center of mass is at rest, the total momentum $\vec{p}$ vanishes. The velocity of the necessary transformation is from (2)

$$\vec{v} = \vec{P}/M \ .$$

**23-7**. *Spell out the detailed steps in the derivation of the the large distance gravitational wave amplitude* (23.35) *from* (23.30).

**Solution:** The only part of the derivation that needs to be explicitly spelled out is that from (23.32) to (23.33). Multiplying (23.32) by $x^i x^j$ and integrating, we have

$$\int d^3x\, x^i x^j \frac{\partial^2 T^{tt}}{\partial t^2} = \int d^3x\, x^i x^j \frac{\partial^2 T^{k\ell}}{\partial x^k \partial x^\ell} \ . \tag{1}$$

On the left take the time derivation outside the integral. On the right integrate the $\partial/\partial x^k$ by parts (or alternatively use Gauss' theorem). The surface terms vanish if the volume contains all the source. The result is

$$\ddot{I}^{ij}(t) = -\int d^3x \frac{d(x^i x^j)}{dx^k} \frac{\partial T^{k\ell}}{\partial x^\ell}. \tag{2}$$

Now,

$$\frac{d(x^i x^j)}{dx^k} = \frac{dx^i}{dx^k} x^j + x^i \frac{dx^j}{dx^k} = \delta^i_k x^j + x^i \delta^j_k. \tag{3}$$

Inserting (3) into (2) and carrying out the sum over $k$ gives

$$\ddot{I}^{ij}(t) = -\int d^2x \left( x^j \frac{\partial T^{i\ell}}{\partial x^\ell} + x^i \frac{\partial T^{j\ell}}{\partial x^l} \right). \tag{4}$$

Again integrate by parts obtaining $\delta_\ell^j$ in the first term and $\delta_\ell^i$ in the second. The result is

$$\ddot{I}^{ij}(t) = 2\int d^3x\, T^{ij}. \tag{5}$$

This is (23.33).

**23-8**. *What combination of the two polarizations + and × is the gravitational radiation emerging at an angle of 45° with respect to the axis perpendicular to the plane of the circular orbit of two equal mass stars.*

**Solution:** The gravitational wave amplitude a large distance from the binary system is given by (23.46). However, to analyze the polarization it is necessary to transform it to the $TT$-gauge associated with the direction of interest. Take that direction to be in the $y$-$z$ plane making an angle of 45° with the $z$-axis. As a first step we rotate the coordinates about the $x$-axis by 45° so the direction of interest is along the new $z$-axis.

The amplitude matrix in (23.46) that is relevant for the polarization can be written

$$\mathbf{A} = \begin{pmatrix} c & s & 0 \\ s & -c & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $c = \cos[2\Omega(t-r)]$ and $s = \sin[2\Omega(t-r)]$. A rotation by 45° about the $x$-axis is represented by the rotation matrix

$$R = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

The rotated amplitude is

$$\mathbf{A} = \mathbf{RAR}^{-1} = \begin{pmatrix} c & s/\sqrt{2} & -s/\sqrt{2} \\ s/\sqrt{2} & -c/2 & c/2 \\ -s/\sqrt{2} & c/2 & -c/2 \end{pmatrix}.$$

Using the algorithm for passing $TT$ gauge described in eq. (21.65) we cross out the $z$ rows and columns and make the remaining $x$-$y$ matrix traceless to

find

$$\mathbf{A}'^{TT} = \begin{pmatrix} \frac{3}{4}c & s/\sqrt{2} & 0 \\ s/\sqrt{2} & -\frac{3}{4}c & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

Thus, the polarization is 3/4 + and $1/\sqrt{2}$ $\times$ these being out of phase with each other by $\pi/2$.

**23-9**. *[C] (Angular Distribution of Radiated Gravitational Wave Power From a Binary Star System) In Example 3, the time-averaged power radiated in gravitational waves was calculated for a binary star system for two directions — one normal to and one in the plane of the orbit. This problem aims at calculating the complete angular distribution (the "antenna pattern"). The time-averaged distribution will be symmetric about the axis of rotation because the time-averaged source is axisymmetric. It is, therefore, necessary only to calculate the power radiated in a direction making an angle $\theta$ with the $z-$axis which can be conveniently taken to lie in the $y-z$ plane. Proceed as follows:*

*(a) Rotate the spatial coordinates about the $x-$axis by an angle $\theta$ so that the new $z-$axis makes an angle $\theta$ with the old one. Transform the gravitational wave amplitude* (23.46) *to this coordinate system.*

*(b) Put the approximate plane wave propagating in the new $z-$direction in TT gauge.*

*(c) Calculate the power radiated in the new $z-$direction thereby getting the radiated power as a function of $\theta$. Check that your answer agrees with the two special cases in Example 3. Draw a rough plot of the antenna pattern.*

*(d) If you integrate the angular distribution of radiated power to get the total radiated power do you get the answer from the quadrupole formula quoted in* (23.53)*?*

**Solution:** A few abbreviations will make this solution more compact. Define $C \equiv \cos[2\Omega(t-r)]$, $S \equiv \sin[2\Omega(t-r)]$, $c \equiv \cos\theta$, and $s \equiv \sin\theta$.

a) The rotation matrix for a rotation about the $x$-axis by an angle $\theta$ is

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix}$$

If $\mathbf{H}$ is the matrix whose elements are $\bar{h}^{ij}$ in (23.46), the transformed matrix is

$$\mathbf{H}' \equiv \mathbf{R}^t \mathbf{H}\, \mathbf{R} = -\frac{8\Omega^2 M R^2}{r} \begin{pmatrix} C & Sc & Ss \\ Sc & -Cc^2 & -Ccs \\ Ss & -Ccs & -Cs^2 \end{pmatrix} .$$

b) To put this matrix into $TT$ form for the $z'$-axis, cross out the $\bar{h}'_{z\mu}$ components and subtract the trace from the remainder as in (21.70). The result is

$$\mathbf{H}'_{TT} = -\frac{8\Omega^2 M R^2}{r} \begin{pmatrix} C(1+c^2)/2 & Sc & 0 \\ Sc & -C(1+c^2)/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

c) The above is a superposition of two linear perturbations, so the radiated power is the sum of the plane wave expression for each as described in Example 23.3. The result is

$$\frac{dL}{d\Omega_{sa}} = \frac{2}{\pi} \left(\Omega^3 M R^2\right)^2 \left(1 + 6\cos^2\theta + \cos^4\theta\right) .$$

When $\theta = 0$ and $\pi/2$, this answer coincides with those derived in the example. A plot is shown below.

d) Integrating

$$L = \int \left(\frac{dL}{d\Omega_{sa}}\right) \sin\theta d\theta d\phi$$

over the whole sphere does give the correct answer.

**23-10**. *Two equal masses $M$ are at the ends of a massless spring of unstretched length $L$ and spring constant $k$. The masses are started oscillating in line with the spring with an amplitude $A$ so that their center of mass remains fixed. Calculate the amplitude of gravitational radiation a long distance away from the center of mass of the spring as a function of the angle $\theta$ from the axis of the spring to lowest non-vanishing order in $A$. Analyze the polarization of the radiation. Calculate the angular distribution of power radiated in gravitational waves.*

**Solution:** Take the center of mass to be the origin of a Cartesian reference frame whose $z$-axis is oriented along the spring. If the mass at $z = +L/2$ is displaced by an amount $\delta z$, and the mass at $z = -L/2$ by $-\delta z$, then the center of mass is unchanged, and the magnitude of the restoring force on either mass is

$$|F_z| = 2k\delta z \ .$$

The masses thus oscillate with a frequency $\omega = (2k/M)^{1/2}$ according to

$$\delta z(t) = A\cos\omega t \ .$$

The only non-vanishing component of the second mass moment $I_{ij}$ is $I_{zz}$ because the masses never stray from the $z$-axis. This is

$$I_{zz} = 2M\left[L/2 + \delta z(t)\right]^2 \ .$$

From this one finds to lowest non-vanishing order in $A$

$$\ddot{I}_{zz} = 2MAL\omega^2\cos(\omega t) \ .$$

Eq. (23.35) then gives the gravitational wave amplitude far from the source, but not in transverse-traceless gauge.

To put the result (23.35) into transverse-traceless gauge for a direction making an angle $\theta$ with the $z$-axis, in the $y-z$ plane, make a rotation about the $x$-axis to a new set of axes in which the $z'$-axis makes an angle $\theta$ with respect to the $z$-axis. The rotation is given by the analog of (3.4) and $I_{ij}$ transforms like (20.44). The result is

$$I'_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sin^2\theta & -\cos\theta\sin\theta \\ 0 & -\cos\theta\sin\theta & \cos^2\theta \end{pmatrix} I_{zz} \ .$$

Using the algorithm for transforming to transverse-traceless gauge discussed in Section 21.5, one finds

$$\bar{h}'_{ij} \to \frac{2}{r} \begin{pmatrix} -\frac{1}{2}\sin^2\theta & 0 & 0 \\ 0 & \frac{1}{2}\sin^2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \ddot{I}_{zz} \ .$$

Evidently the wave is linearly polarized. The time-averaged luminosity radiated into a solid angle $d\Omega_{sa}$ is from (16.22)

$$\frac{dL}{d\Omega_{sa}} == \frac{1}{8\pi} M^2 A^2 L^2 \omega^6 \sin^4\theta \ .$$

**23-11**. *A particle of mass $m$ moves along the $z-$axis according to $z(t) = (1/2)gt^2$ ($g$ is a constant) between times $t = -T$ and $t = +T$ and is otherwise moving with constant speed. Calculate the gravitational wave metric perturbations at a large distance $L$ along the positive $z-$axis.*

**Solution:** We first calculate the gravitational wave amplitude from [*cf.* (23.35)]

$$\bar{h}^{ij} = \frac{2\ddot{I}^{ij}(t-r)}{r} \ .$$

The only component of the second mass moment is

$$I^{zz}(t) = mz^2(t) = \frac{1}{4} mg^2t^4$$

so

$$\ddot{I}^{zz} = 3\,mg^2t^2 \ .$$

Therefore the only component of $\bar{h}^{ij}$ a distance $L$ from the center is

$$\bar{h}^{zz} = \frac{6mg^2(t-L)^2}{L}$$

between $t = L - T$ and $t = L + T$ and zero for other times.

**23-12**. *What is the longest period a binary consisting of two neutron stars in*

*circular orbit, each with $1.4M_\odot$, could have now and coalesce before the end of the universe (assuming that it has about 15 billion more years to go)?*

**Solution:** The rate of decrease in period due to gravitational radiation (23.59) can be written

$$\frac{dP}{dt} = -\left(\frac{\tau}{P}\right)^{\frac{5}{3}} \tag{1}$$

where (putting back factors of $G$ and $c$)

$$\tau = \frac{2\pi GM}{c^3}\left(\frac{96}{5}\pi 4^{\frac{1}{3}}\right)^{\frac{3}{5}} \approx 5\times 10^{-4}\left(\frac{M}{M_\odot}\right) sec\ .$$

Solving (1) with the boundary condition that $P = P_{in}$ at the present time, $t = 0$, gives

$$P^{\frac{8}{3}}(t) = P_{in}^{\frac{8}{3}} - \frac{8}{3}\tau^{\frac{5}{3}}t\ .$$

The binary coalesces when the period goes to zero which is at a time

$$t_{\text{coal}} = \frac{3}{8}\tau\left(\frac{P_{in}}{\tau}\right)^{\frac{8}{3}}\ .$$

If $t_{\text{coal}} \lesssim 10^{10}$ yrs, $P_{in}$ must be less than about 20 hrs.

**23-13**. *(Angular Momentum Loss through Gravitational Radiation.) In Newtonian physics an axisymmetric body rotating rigidly about a principal axis with an angular velocity $\Omega$ has a kinetic energy $E$ and angular momentum along the axis $J$ given by:*

$$E = \frac{1}{2}I\Omega^2\ , \qquad J = I\Omega$$

*where $I$ is the moment of inertia about that axis. Assuming that this is true for linearized gravity (it is), calculate the average rate over a period at which the binary star system discussed in Section 23.5 is losing angular momentum in gravitational radiation.*

**Solution:** From the assumed relation $E = J^2/(2I)$

$$\frac{dJ}{dt} = \frac{I}{J}\frac{dE}{dt} .$$

The moment of inertia about the axis of rotation is $I = 2MR^2$. Using $L_{GW}$ from (23.53) for $dE/dt$

$$\frac{dJ}{dt} = \frac{128}{5} M^2 R^4 \Omega^5 .$$

**23-14**. *[C] (Gravitational Radiation Reaction.) A particle of mass $m$ moves because of an applied force and radiates gravitational radiation. Suppose velocity of the particle is much less than the velocity of light so that non-relativistic kinematics applies. Show that the rate at which the particle loses energy in gravitational waves is the same, in a time averaged sense, as if it were acted on by a gravitational radiation reaction force*

$$F_i^{\text{rad. react.}}(t) = -\frac{2}{5} m \frac{d^5 I_{ik}(t)}{dt^5} x^k(t)$$

*where $x^i$ are usual rectangular coordinates giving the particle's position and $I_{ij}$ is the quadrupole moment defined in (23.50). If you are familiar with electromagnetism, compare this force with the radiation reaction force in electrodynamics.*

**Solution:** The rate of energy loss in gravitational radiation is

$$\frac{dE}{dt} = \frac{1}{5}\left\langle \dddot{I}_{ij}\dddot{I}^{ij}\right\rangle .$$

Since $dE/dt = \vec{F}\cdot\vec{v}$, we will have found a radiation reaction force if

$$\left\langle F_i \dot{x}^i\right\rangle = \frac{1}{5}\left\langle \dddot{I}_{ij}\dddot{I}^{ij}\right\rangle = \frac{1}{5}\left\langle \dddot{I}_{ij}\dddot{I}^{ij}\right\rangle$$

since $I_{ij}$ is traceless. Integrate two of the derivatives by parts and neglect the surface terms to find

$$\left\langle F_i \dot{x}^i\right\rangle = -\frac{1}{5}\left\langle \frac{d^5 I_{ij}}{dt^5}\frac{dI^{ij}}{dt}\right\rangle$$

For a point particle $I^{ij} = mx^i x^j$ so

$$\left\langle F_i \dot{x}^i \right\rangle = -\frac{1}{5} \frac{d^5 I_{ij}}{dt^5} \, m \left( \dot{x}^i x^j + x^j \dot{x}^i \right) .$$

Making use of the symmetry of $I_{ij}$, this can be written

$$\left\langle F_i \dot{x}^i \right\rangle = -\frac{2}{5} m \, \frac{d^5 I_{ij}}{dt^5} \, \dot{x}^i x^j .$$

From which we read off

$$F_i = -\frac{2}{5} m \, \frac{d^5 I_{ij}}{dt^5} \, x^j .$$

This is analogous to the radiation reaction force in electrodynamics

$$\vec{F}_{\text{elect}} = -\frac{2}{3} \frac{e^2}{c^3} \frac{d^3 \vec{x}}{dt^3} .$$

**23-15.** *[E] Lunar laser ranging measurements of the position of the Moon relative to the Earth lead to the inference that the length of the day is increasing by 2 millisec per century. Estimate whether gravitational radiation from the Earth is an important or negligible contribution to this slow down in the Earth's rotation rate.*

**Solution:** Let $P$ be the earth's rotational period. The decrease in rotational kinetic energy $E_K$ due to gravitational radiation is

$$\dot{E}_K = \frac{1}{5} \frac{G}{c^5} \left\langle \dddot{I}_{ij} \dddot{I}^{ij} \right\rangle .$$

The important point is that the second mass moment changes with time only because of *non-axisymmetric* parts of the mass distribution, *e.g.* mountains. Let $M_{\text{irreg}}$ be the mass in these irregularities. We can estimate

$$\begin{aligned} M_{\text{irreg}} &\sim \begin{pmatrix} \text{fraction of surface} \\ \text{with mountains} \end{pmatrix} \frac{\begin{pmatrix} \text{hgt. typical} \\ \text{mountain} \end{pmatrix}}{R_\oplus} M_\oplus \\ &\sim (.05) \left( \frac{2 \text{ km}}{6 \times 10^3 \text{ km}} \right) M_\oplus \sim 2 \times 10^{-5} M_\oplus \end{aligned}$$

where $M_\oplus$ and $R_\oplus$ are the mass and radius of the earth. (The author just *guessed* 5% for the fraction of the surface with mountains from thinking about a map.)

Then

$$\dddot{I}_{ij} \sim M_{\text{irreg}} \frac{R_\oplus^2}{P^3}$$

where $P$ is the earth's rotational period. The rotational kinetic energy is

$$E_K \sim M_{\text{irreg}} \left(\frac{R_\oplus}{P}\right)^2$$

so that $\dot{E}_K \sim E_K(\dot{P}/P)$. Putting this all together gives

$$\dot{P} \sim \left(\frac{M_{\text{irreg}}}{M_\oplus}\right) \left(\frac{GM_\oplus}{c^2 R_\oplus}\right) \left(\frac{R_\oplus}{cP}\right)^3$$

both sides being dimensionless.

Numerically,

$$\dot{P} \sim \left(2 \times 10^{-5}\right) \left[\frac{.443 \text{ cm}}{6 \times 10^8 \text{ cm}}\right] \left[\frac{6 \times 10^8 \text{ cm}}{(3 \times 10^{10} \text{ cm})\ (60 \times 60 \times 24 \text{ s})}\right]^3 \sim 10^{-34} .$$

The change in the length of the day in a century is

$$\dot{P} \times (100 \text{ yr}) \sim 10^{-34} \times 10^{10} \text{ s} \sim 10^{-24} \text{ s} .$$

**23-16**. *A steel beam of mass $M$ and length $L$, much longer than it is wide, rotates about an axis through its center of mass perpendicular to its length with an angular frequency $\Omega$. Under what conditions is the quadrupole formula for the total power applicable? Assuming it is, use it to calculate the power radiated in gravitational radiation. If the beam were contained in a drag free satellite, what would be the predicted decrease in angular frequency in one year of rotation?*

**Solution:** The quadrupole formula should be valid if the source is weak $M/L \ll 1$ (geometrized units) and its velocity is small $\Omega L \ll 1$. Two methods

of using it to derive the luminosity are presented.

*Version one*:
The radiated power is given by the quadrupole formula (23.51). To calculate the reduced quadrupole moment, begin by calculating the second mass moment in a frame $(x', y', z')$ frame rotating with the beam. If the beam is oriented along the $x'$-axis, the only non-vanishing component is

$$I^{x'x'} = \int_{-\frac{L}{2}}^{+\frac{L}{2}} dx' \left(\frac{M}{L}\right) x'^2 = \frac{1}{12} ML^2 \equiv I_0 \ .$$

The transformation to the frame in which the beam is rotating is

$$\begin{aligned} x &= \cos(\Omega t)\, x' \quad + \sin(\Omega t)\, y' \ , \\ y &= -\sin(\Omega t)\, x' + \cos(\Omega t)\, y' \ , \\ z &= z \ . \end{aligned}$$

In this frame $I^{zi} = 0$ and

$$\begin{aligned} I^{xx} &= \cos^2(\Omega t)\, I_0 = (I_0/2)\ [1 + \cos(2\Omega t)] \ , \\ I^{xy} &= -\sin(\Omega t)\cos(\Omega t)\, I_0 = -\,(I_0/2)\ \sin(2\Omega t) \ , \\ I^{yy} &= \sin^2(\Omega t)\, I_0 = (I_0/2)\ [1 - \cos(2\Omega t)] \ . \end{aligned}$$

Thus, $\dddot{I}^{zi} = 0$ and

$$\begin{aligned} \dddot{I}^{xx} &= (I_0/2)\ (2\Omega)^3 \sin(2\Omega t) \ , \\ \dddot{I}^{xy} &= (I_0/2)\ (2\Omega)^3 \cos(2\Omega t) \ , \\ \dddot{I}^{yy} &= -\,(I_0/2)\ (2\Omega)^3 \sin(2\Omega t) \ . \end{aligned}$$

It follows that the trace of $\dddot{I}^{ij}$ vanishes so that $\dddot{I}^{ij} = \dddot{I}^{ij}$. The quadrupole formula (23.51) then gives

$$L = \frac{1}{5}\ (I_0/2)^2\ (2\Omega)^6 \left\langle \sin^2(2\Omega t) + 2\cos^2(2\Omega t) + \sin^2(2\Omega t) \right\rangle$$

or

$$L = \frac{2}{45} M^2 L^4 \Omega^6 \ .$$

*Version two*:
Break the beam into infinitesimal mass ($dM = M/L\,dr$) binary pairs, where $r$ is the radius at which the $dM$ is located. Use (23.45) to find

$$dI^{xx} = 2(dM)r^2\cos^2(\Omega t) \tag{1}$$

and likewise for $dI^{xy}$ and $dI^{yy}$. Integrate from $r = 0$ to $r = L/2 = R$:

$$I^{xx} = \frac{1}{6}\cdot 2MR^2\cos^2(\Omega t) \tag{2}$$

and likewise for the other two. The beam has 1/6 the mass moment of a single binary pair. Since the luminosity depends on two factors of the mass moment the beam will have 1/36 the luminosity of a binary pair. Dividing (23.53) by 36 gives

$$\begin{aligned} L_{GW} &= \frac{32}{45}M^2R^4\Omega^6 \\ &= \frac{2}{45}M^2L^4\Omega^6 . \end{aligned}$$

Following section 23.7, a Newtonian analysis is sufficient to find the change in angular frequency in one year. Let $E$ be the total Newtonian energy of the beam, $E = K + U$. The power lost is $L_{GW}$: $dE/dt = L_{GW}$. A beam has a moment of inertia of $1/12\,ML^2$, which makes the kinetic energy $K = 1/24\,ML^2\Omega^2$. The gravitational potential of the beam, i.e. the gravitational self-energy, will not affect the calculation at this level of analysis: dimensionally we know it must look like $U \sim -M^2/L\,f(M/L)$, where $f(\cdot)$ is some function. Since the beam is rigid $L$ can't change (in contrast to the example in Section 23.7). The energy radiated is small compared to the mass, so to this order we take $M$ constant. Therefore

$$dE \approx \frac{1}{12}ML^2\Omega\,d\Omega.$$

Over the course of one year we approximate $L_{GW}$ to be constant. Plugging in for $L_{GW}$, $dE$, and solving for $d\Omega$ gives us the approximate change in frequency,

$$\Delta\Omega \approx \frac{12}{45}ML^2\Omega^5\Delta t .$$

Suppose the beam represents a small rotating piece on a satellite, so say it rotates at 60 Hz, is 1 mm long, and weighs $10^{-2}$ g. One year is $9.5 \times 10^{15}$ m. This gives

$$\Delta\Omega \approx 6 \times 10^{-57}\ \mathrm{m}^{-1} = 10^{-48}\ \mathrm{Hz}\ .$$

It goes without saying that this is a negligible effect.

**23-17**. *[E] When a small body of mass $m$ falls from rest into a large black hole of mass $M$ there is a burst of gravitational radiation. Estimate the duration of the burst, the peak gravitational wave luminosity, and also the total power radiated as a fraction of the small body's rest mass. What is is the peak gravitational wave luminosity of produced by a $10M_\odot$ black hole falling into the $\sim 10^6 M_\odot$ black hole at the center of our galaxy (Section 13.2)? How does this compare with the optical luminosity of the whole galaxy?*

**Solution:** The key is to estimate the gravitational wave luminosity from the quadrupole formula (23.51) which means estimating the second mass moment and its time evolution. The black hole remains approximately fixed as the small mass falls into it. The second mass moment is therefore of order

$$I \sim mr^2$$

when the small mass is at distance $r$ from the center of the large one. The free-fall time $\tau$ is of order [cf. (9.40)]

$$\tau \sim M$$

which will give the duration of the burst. The third time derivative of the quadrupole moment is therefore

$$\dddot{I} \sim I/\tau^3 \sim \frac{(mr^2)}{M^3}\ .$$

The peak of the burst should occur at the greatest acceleration when $r$ is the smallest radius outside the black hole. Thus, $r \sim M$, and the peak radiated power from the quadrupole formula is

$$L_{GW} \sim \left(\frac{m}{M}\right)^2\ .$$

The total energy radiated in the burst is

$$E_{\text{burst}} \sim L_{GW}\tau \sim m\left(\frac{m}{M}\right) .$$

Thus, a fraction $(m/M)$ of the rest mass of the small body is radiated in gravitational radiation.

For a $10M_\odot$ star falling into the $10^6 M_\odot$ black hole at the center of the galaxy, the peak luminosity in gravitational waves will be

$$L_{GM} \sim 10^{-10}\left(\frac{c^5}{G}\right) \sim 10^{49} \text{ erg/s} .$$

That's approximately 100,000 times brighter than the optical luminosity of the entire galaxy, but all in gravitational waves.

**23-18.** *[E] (Maximum Luminosity.) Imagine a point source of radiation at the center of a spherical star of radius $R$. Suppose that the luminosity of the source is $L$ (energy/time) and is steady in time. Using Newtonian physics to estimate the maximum luminosity as follows:*

*(a) Calculate the energy density in radiation in the interior of the star.*

*(b) Estimate the maximum $L$ above which the star would be inside its Schwarzschild radius.*

*(c) Compare the maximum luminosity to the estimate of the luminosity from two merging black holes* (23.61).

**Solution:**

a) Since the radiation moves outwards with the speed of light, the energy density $\rho_{\text{rad}}$ in geometrized units is

$$\rho_{\text{rad}} = \frac{L}{4\pi r^2}$$

b) The mass in radiation can be estimated as

$$M_{\text{rad}} = \int_0^R dr 4\pi r^2 \rho_{\text{rad}}(r) = LR .$$

For the star to be outside its Schwarzschild radius, $2M/R < 1$ which implies

$$L < \left(\frac{1}{2}\right)$$

or in $\mathcal{MLS}$ units

$$L < \frac{c^5}{2G} \sim 10^{59} \text{ erg/s} .$$

c) The luminosity in gravitational waves from two merging black holes could be close to the maximum possible luminosity.

**23-19**. *[E,C] Gravitational Waves from Merging Supermassive Black Holes. Suppose for simplicity that i) every galaxy contains a $10^9 M_\odot$ black hole, ii) that every galaxy merges once in its lifetime, and iii) that when they do the black holes in their cores coalesce. Consider a detector built to detect the gravitational waves from such events. Even though they do not really apply, use the results of linearized gravity to:*

*a) Estimate the frequency range the detector would have to operate in.*

*b) Estimate the strain sensitivity would be necessary to see mergers out to edge of the visible universe, $\sim 1$ Gpc in radius.*

*c) Estimate the duration of such events in usual time units.*

*d) Estimate the rate should such events would be detected.*

**Solution:** There is only one scale in this problem — the $M \sim 10^9 M_\odot$ mass of the merging black holes.

a) The time scale for the merger is $\tau \sim M$. The corresponding frequency $\omega$ is $1/M$. In usual units this is $\sim 10^{-4}$ Hz. Only space-based detectors seem likely to be sensitive enough in this frequency range.

b) From (23.35) and (16.12), the required strain sensitivity is very roughly

$$\frac{\delta L}{L} \sim \frac{\ddot{I}}{r} \sim \frac{(M \cdot M^2)/\tau^2}{r} \sim \frac{M}{r} .$$

For $M \sim 10^9 M_\odot$ and $r \sim 4,000$ Mpc, this is approximately

$$\frac{\delta L}{L} \sim 10^{-14} .$$

c) From (a) the duration is $\sim 10^4$ s or about 3 hr.

d) If every galaxy participates in one merger during the lifetime of the universe, the event rate is very roughly

$$\begin{pmatrix} \text{Event} \\ \text{rate} \end{pmatrix} \sim \frac{\begin{pmatrix} \text{number of galaxies} \\ \text{in visible universe} \end{pmatrix}}{(\text{age of universe})}$$

The relevant numbers for evaluating this can be found in Chapters 16–19.

The radius of the visible universe is about 4,000 Mpc. The density of visible matter is $\sim 10^{-31}\text{g/cm}^3$. If a typical galaxy has $10^{11}$ stars weighing a solar mass a piece, it has $\sim 10^{44}$g of visible matter. The number of galaxies in the visible universe is therefore $\sim 10^{10}$. The age is 14 Gyr, giving an event rate

$$\begin{pmatrix} \text{Event} \\ \text{rate} \end{pmatrix} \sim 1/\text{ yr} .$$

# Chapter 24

# Relativistic Stars

**24-1.** *[S] White dwarfs can have surface temperatures of* $10^5$ K *which is hot by everyday measures. Is this temperature large enough that approximation of a degenerate gas of electrons at zero temperature will break down?*

**Solution:** A Fermi gas is effectively degenerate as long as the thermal energy is much less than the Fermi energy. Per particle, this is the requirement

$$k_B T \ll \sqrt{p_F^2 c^2 + m_e^2 c^4} \ . \tag{1}$$

The estimates in Box 12.1 give

$$p_F \sim A^{\frac{1}{3}} \hbar / R \tag{2}$$

where $A$ is the number of electrons and $R$ is the radius of the star. Taking $R \sim 10^3$ km, $A \sim M_\odot / m_p \sim 10^{57}$ from (2), we find

$$p_F c \sim 2 \text{ MeV} \tag{3}$$

so the electrons are moderately relativistic. By contrast,

$$k_B \left(10^5 \text{ K}\right) \sim 10^{-5} \text{ MeV} \tag{4}$$

so the thermal energy is negligible compared to the Fermi energy. White dwarfs are cold objects that are white hot.

**24-2**. *(The One-Dimensional Gas of Degenerate Fermions.) Suppose that we lived in one-dimension not three. A gas of $\mathcal{N}$ degenerate fermions in a "box" of length $\mathcal{L}$ would be characterized by an energy per unit length $\rho$, a force on the walls $p$, and a number per unit length $n$. (The notation stresses the analogy with three dimensions.)*

*(a) Evaluate the sum in (24.3) to find the total energy in the box for large values of $\mathcal{N}$. Find $\rho$ as a function of $n$.*

*(b) Find the force on the walls $p$ as a function of $n$.*

**Solution:**

(a) The sum in (24.4) is:

$$\mathcal{E} = \sum_{n=1}^{\mathcal{N}/2} (2e_n) = \frac{1}{m}\left(\frac{\pi\hbar}{\mathcal{L}}\right)^2 \sum_{n=1}^{\mathcal{N}/2} n^2 \ . \tag{1}$$

For the very large values of $\mathcal{N}$ that might characterize a realistic gas or star the total energy in the box the sum can be replaced by an integral to an excellent approximation, giving:

$$\mathcal{E} = \frac{\mathcal{N}^3}{24m}\left(\frac{\pi\hbar}{\mathcal{L}}\right)^2 \ . \tag{2}$$

The energy per unit length $\rho \equiv mn + \mathcal{E}/\mathcal{L}$ is then

$$\rho = mn + \frac{(\hbar\pi)^2}{24m} n^3 \tag{3}$$

(b) The force $p$ is given by $p = -d\mathcal{E}/d\mathcal{L}$ keeping $\mathcal{N}$ fixed. From (2) this is

$$p = \frac{(\hbar\pi)^2}{12m}\left(\frac{\mathcal{N}}{\mathcal{L}}\right)^3 = \frac{(\hbar\pi)^2}{24m} n^3 \ . \tag{4}$$

**24-3**. *(No Equation of State for Bosons.) Continuing the one-dimensional example from the previous problem, what would be the ground state energy of $\mathcal{N}$ bosons (particles not restricted by the Pauli principle) in a box of length $\mathcal{L}$? Would there be a relation between $p$ and $n$ that is independent of $\mathcal{N}$ when it is large?*

**Solution:** The ground state of $\mathcal{N}$ bosons would have all $\mathcal{N}$ particles in the lowest ($n = 1$) single particle energy state. The total energy would therefore be:

$$\mathcal{E} = \frac{\mathcal{N}}{2m}\left(\frac{\hbar\pi}{\mathcal{L}}\right)^2$$

The energy per unit length $\rho = mn + \mathcal{E}/\mathcal{L}$ is then

$$\rho = mn + \frac{n}{2m}\left(\frac{\hbar\pi}{\mathcal{L}}\right)^2$$

in which the second term vanishes as $\mathcal{L} \to \infty$. There is thus no equation of state.

**24-4**. *[A] (Deriving the Equations of Structure from the Einstein Equation.) Use the the metric* (24.11) *and the stress-energy of a perfect fluid* (22.37) *to derive the three equations of hydrostatic equilibrium (24.13) from Einstein equation $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ and the local conservation of stress-energy $\nabla_\beta T^{\alpha\beta} = 0$ which follows from it. (Hint: You can use any combination of the equations you choose but the equations $\nabla_\beta T^\beta_r = 0$, $G_{\hat{t}\hat{t}} = 8\pi T_{\hat{t}\hat{t}}$, and $G_{\hat{r}\hat{r}} = 8\pi T_{\hat{r}\hat{r}}$ in the orthonormal basis pointing along the coordinate axes involve the least algebra. Those components of the Einstein tensor are given in Appendix B for the metric* (24.11)*. )*

**Solution:** First write out the equation $\nabla_\alpha T^\alpha_r = 0$. Using the form of the perfect fluid stress tensor (22.39) this is,

$$u_r u^\alpha \partial_\alpha(\rho + p) + (\rho + p)\left[(\nabla_\alpha u^\alpha)\, u_r + u^\alpha \nabla_\alpha u_r\right] + \partial_r p = 0 \tag{1}$$

where for this static star

$$u^\alpha = \left(e^{-\frac{\nu}{2}}, \vec{0}\right) . \tag{2}$$

The first term vanishes because of this form for $u^\alpha$, and the fact that all quantities are independent of $t$. The divergence $\nabla_\alpha u^\alpha$ vanishes for a similar reason. There remains

$$-\frac{dp}{dr} = (\rho + p)\, e^{-\frac{\nu}{2}} \nabla_t u_r . \tag{3}$$

Writing out the covariant derivative using (20.68) and the form of the metric gives

$$-\frac{dp}{dr} = \frac{1}{2}\,(\rho + p)\,\nu' \tag{4}$$

which is one of the equations of structure (24.13c).

When the Einstein equation

$$G_{\hat{t}\hat{t}} = 8\pi\, T_{\hat{t}\hat{t}} = 8\pi\,\rho \tag{5}$$

is rewritten in terms of $m(r)$ using the definition (24.12) using the $G_{\hat{t}\hat{t}}$ found in Appendix (C) we find,

$$\frac{dm}{dr} = 4\pi r^2 \rho\ . \tag{6}$$

This is (24.13a).

The final equation is

$$G_{\hat{r}\hat{r}} = 8\pi\, T_{\hat{r}\hat{r}} = 8\pi\, p\ . \tag{7}$$

This gives

$$\nu' = -\frac{1}{r} + \left(1 - \frac{2m}{r}\right)^{-1}\left(8\pi\, p\, r + \frac{1}{r^2}\right) \tag{8}$$

which when combined with (3) gives the third equation of structure, (24.13b).

**24-5**. *Argue that the metric*

$$d\Sigma^2 = \alpha^2 dr^2 + r^2 d\phi^2,$$

*where $\alpha$ is a constant greater than 1, represents the geometry on the two-dimensional surface of a cone.*

*What is the opening angle of the cone?*

*Comment: This is an example of a geometry which is not locally flat at $r = 0$.*

**Solution:** Introducing new coordinates $R \equiv \alpha r$ and $\Phi \equiv \phi/\alpha$, the line element becomes

$$d\Sigma^2 = dR^2 + R^2 d\Phi^2\ .$$

This shows the geometry is flat except possibly at $R = 0$. But it is not the geometry of a plane. The circumference of a circle a distance $d$ from the vertex is $2\pi d/\alpha$ since $\Phi$ ranges only from 0 to $2\pi/\alpha$ if $\phi$ ranges from 0 to $2\pi$. The ratio $C/d$ is $2\pi/\alpha$ rather than $2\pi$ of a plane.

A cone has a locally flat geometry except at its vertex. (It can be made by rolling up a flat piece of paper.) If the cone has opening angle $\beta$, the ratio $C/d$ is $2\pi \sin(\beta/2)$. Thus, a cone with opening angle $\beta = 2\sin^{-1}(1/\alpha)$ will reproduce the given geometry.

**24-6**. *[C] Incompressible matter with a constant fixed density $\rho$ is inconsistent with special relativity because it could be used to send signals faster than the speed of light. (How?) But it does provide a simple example of relativistic hydrostatic equilibrium. Integrate the equations of structure (24.13a) and (24.13b) to find the pressure as a function of radius for a star of total mass $M$ made out of material for which $\rho$ is a given constant. Plot the mass vs. radius relation for these stars. Is there a maximum mass?*

**Solution:** This is a straightforward exercise in solving first order differential equations. If the density $\rho$ is constant (23.7) can be integrated to give

$$M(r) = \frac{4}{3}\pi r^3 \rho \ .$$

The integration constant has been fixed so that $M = 0$ at $r = 0$ — a necessary condition for spacetime to be regular there. Inserting this result in (23.7) one finds

$$-\frac{dp}{(\rho + p)(\rho/3 + p)} = \frac{4\pi\, r^3 dr}{r^2(1 - 8\pi\rho r^2/3)} \ .$$

Both sides can be integrated and the integration constant fixed by the requirement that

$$p(R) = 0 \ .$$

The result is

$$p(r) = \rho \left[ \frac{(1 - 8\pi\,\rho\, r^2/3)^{\frac{1}{2}} - (1 - 8\pi\,\rho\, R^2/3)^{\frac{1}{2}}}{3(1 - 8\pi\,\rho\, R^2/3)^{\frac{1}{2}} - (1 - 8\pi\,\rho\, r^2/3)^{\frac{1}{2}}} \right] \ .$$

The total mass $M$ is given by $M = \frac{4}{3}\pi\rho R^3$. There is a maximum mass because at a radius $R = R_{\max}$ the pressure at the center center becomes infinite. That happens when $M/R = 4/9$ or when $R = R_{\max} = (8/9)^{1/2}a$ where $a$ is a convenient length defined to be $a \equiv (3/8\pi\rho)^{1/2}$. The maximum mass is

$$M_{max} = \frac{1}{2}\left(\frac{8}{9}\right)^{3/2} a = .41a.$$

A plot of $M$ vs $R$ is shown below:

**24-7.** *[A] (a) Find the metric in the interior of the family of spherical, constant density stars whose structure was solved for in Problem 6. Discuss the junction conditions between the interior of the star and the exterior carefully.*

*(b) Show that the geometry of a $t$ = const surface inside a constant density star is the same locally as that of a homogeneous spatial surface of a closed FRW universe discussed in Chapter 18.*

*(c) Is the volume inside the star bigger or smaller for a given surface area than it would be in flat space?*

**Solution:**

a) There are two unknown functions in the metric of the interior geometry — $\nu(r)$ and $\lambda(r)$. $\lambda(r)$ is given by (24.12) with $m(r) = 4\pi\rho r^3/3$. Eq. (24.13c) must be solved to find $\nu(r)$. Working out the right hand side of this equation we find

$$\frac{1}{2}\frac{d\nu}{dr} = \frac{8\pi\rho r}{3}\left(1 - \frac{8\pi\rho r^2}{3}\right)^{-\frac{1}{2}}\left[3\left(1 - \frac{8\pi\rho R^2}{3}\right)^{\frac{1}{2}} - \left(1 - \frac{8\pi\rho r^2}{3}\right)^{\frac{1}{2}}\right]^{-1}$$

whose integral is

$$\nu(r) = \log\left\{\left[\frac{3}{2}\left(1-\frac{8\pi\rho R^2}{3}\right)^{\frac{1}{2}} - \frac{1}{2}\left(1-\frac{8\pi\rho r^2}{3}\right)^{\frac{1}{2}}\right]^2\right\} + \text{const.}$$

The constant has to be adjusted to match onto the Schwarzschild geometry at $r = R$ where

$$\nu(R) = \log\left(1-\frac{2M}{R}\right) = \log\left(1-\frac{8\pi\rho R^2}{3}\right) .$$

Evidently the constant is zero. So the metric inside is

$$\begin{aligned} ds^2 &= -\left[\frac{3}{2}\left(1-\frac{8\pi\rho R^2}{3}\right)^{\frac{1}{2}} - \frac{1}{2}\left(1-\frac{8\pi\rho r^2}{3}\right)^{\frac{1}{2}}\right]^2 dt^2 \\ &+ \left(1-\frac{8\pi\rho r^2}{3}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) . \end{aligned}$$

b) Defining $r'^2 = (8\pi\rho/3)r^2$, the metric of a $t =$ const. spatial slice becomes

$$ds^2 = \frac{3}{8\pi\rho}\left[\frac{dr'^2}{1-r'^2} + r'^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

which is a slice of a spatially closed FRW model with $a^2 = 3/8\pi\rho$.

c) The volume inside the star is

$$V = 4\pi\int_0^R \frac{dr\, r^2}{(1-8\pi\rho r^2/3)^{1/2}} \geq \frac{4}{3}\pi R^3 .$$

**24-8**. *Using the results of Problem 7, find and plot an embedding diagram for a $t =$ const., $\theta =$ const., two-dimensional slice of the spacetime geometry of a constant density star.*

**Solution:** The geometry of the surface inside the radius $R$ follows from the result of Problem 7 and is

$$d\Sigma^2 = \left(1-\frac{8\pi\rho r^2}{3}\right)^{-1} dr^2 + r^2 d\phi^2. \tag{1}$$

Outside it follows from the Schwarzschild geometry that

$$d\Sigma^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\phi^2 . \tag{2}$$

The geometries match at the radius of the star $R$ where $2M/R = 8\pi\rho R^2/3$ because $M = 4\pi R^3\rho/3$. Following Example 7-7, the embedding surface can be described by cylindrical coordinates $(\rho, \psi, z)$ in three-dimensional flat space. The embedding functions

$$z = z(r) \quad , \quad \rho = r \quad , \quad \psi = \phi \tag{3}$$

give the location in flat space of the points in the surface labeled by $(r, \phi)$. The function $z(r)$ satisfies

$$\frac{dz}{dr} = [g_{rr}(r) - 1]^{\frac{1}{2}} \tag{4}$$

where $g_{rr}(r)$ is given by (1) or (2) depending on whether $r$ is less than or greater than $R$.

Integrating the differential equation (4) assuming $z(0) = 0$ gives

$$z(r) = \xi^{-\frac{1}{2}} R \left\{ 1 - \left[ 1 - \xi \left(\frac{r}{R}\right)^2 \right]^{\frac{1}{2}} \right\} \quad , \quad r < R \tag{5}$$

and

$$z(r) = 2\,\xi^{\frac{1}{2}} R \left[ \left(\frac{r}{R}\right) - \xi \right]^{\frac{1}{2}} + C \quad , \quad r > R \tag{6}$$

where $\xi \equiv 2M/R$ and the constant $C$ is determined by matching at $r = R$.

The resulting surface and cross-section are plotted below for $\xi = 8/9$ [its maximum possible value (Problem 19)].

**24-9**. *[C] (a) In the text we took the central density $\rho_c$ to parametrize families of non-rotating stars, as for example in Figure 24.7. But they could equally well have been parametrized by the central pressure $p_c$. Stars made from the hypothetical constant density material discussed in Problem 6 all have the same, given density.*

*(a) For these constant density stars, show that for each value of the central pressure $p_c$ there is a unique mass $M$ and radius $R$.*

*(b) What is the largest redshift [cf. (9.20)] from the surface that is exhibited in this family of spherical stars with constant density $\rho$. To what central pressure $p_c$ does it correspond.*

**Solution:**

a) From the solution to Problem 6 we have for the central pressure $p_c$

$$p_c = \rho \left( \frac{1-\xi}{3\xi - 1} \right) \tag{1}$$

where

$$\xi \equiv \left( 1 - \frac{8\pi}{3} \rho R^2 \right)^{\frac{1}{2}} = \left( 1 - \frac{2M}{R} \right)^{\frac{1}{2}} . \tag{2}$$

Eq. (1) can be solved for $\xi$ to find

$$\xi = \frac{\rho + p_c}{\rho + 3p_c} .$$

As $p_c$ ranges from 0 to $\infty$, $\xi$ ranges from 0 to 1/3. For each of these values there is an $R$ from (2) and an $M$ from $M = (4/3)\pi\rho R^3$.

b) The maximum redshift at the surface $\omega_\infty/\omega_*$ is the maximum value of $\xi$ [*cf.* (9.20)] which is 1/3.

**24-10**. *A spherical distribution of matter with a $p = \rho$ equation of state is contained within a spherical shell of area $4\pi R^2$ and negligible mass. (The shell is not made of realistic matter.)*

*(a) Find a simple solution of the equations of hydrostatic equilibrium* (24.13) *in which the distributions of pressure and density are inverse powers of the radius.*

*(b) What is the total mass of this distribution?*

*(c) What pressure does the shell have to exert?*

**Solution:** (a) Plugging $p = \rho = K/r^n$ into the equations of structure (24.13) shows that there is a solution with $n = 2$ and $k = 1/8\pi$.

(b)The mass $m(R)$ is found by integrating (24.13a) to give

$$m(R) = \int_0^R 4\pi r^2 \rho \, dr = R/2 .$$

(c)The pressure at $R$ is evidently $p = 1/(8\pi R^2)$.

**24-11**. *[E,B] Estimate the densities in* g/cm$^3$ *in which, for matter in its ground state, the energy of a typical electron (i) exceeds typical atomic binding energies* $\sim$ 10 keV, *(ii) exceeds the electron rest mass* $\sim$ .5 MeV, *and (iii) exceeds the neutron-proton mass difference* $\sim$ 1.3 MeV.

**Solution:** As discussed in the text, below $4 \times 10^{11}$ g/cm$^3$ matter in its ground state can be roughly thought of as a lattice of $^{56}$Fe nuclei immersed in a free,

degenerate electron gas. The mass density is dominated by the rest mass density and therefore approximately

$$\rho \approx (56m_p)\, n \approx (56m_p)\, \frac{p_F^3}{3\pi^2\hbar^3}$$

where $n$ is the number density of protons — the same as the number density of electrons — and $p_F$ is the electrons' Fermi momentum. The Fermi momentum is related to the Fermi energy of $E_F = (m_e^2c^4 + p_F^2c^2)^{\frac{1}{2}}$. The rest is multiplication and converting to a consistent set of units. (In the latter connection note that $1\ \text{eV} = 1.6 \times 10^{-12}$ erg.)

$$\begin{aligned} E_F &\sim 10\ \text{keV} + .5\ \text{MeV}\ , & \rho &\sim 10^5\ \text{g/cm}^3\ , \\ E_F &\sim .5\ \text{MeV} + .5\ \text{MeV}\ , & \rho &\sim 10^8\ \text{g/cm}^3\ , \\ E_F &\sim 1.3\ \text{MeV} + .5\ \text{MeV}\ , & \rho &\sim 10^9\ \text{g/cm}^3\ . \end{aligned}$$

**24-12.** *[B] An electrically neutral gas of highly relativistic free electrons, protons, and neutrons is maintained in equilibrium by the reactions*

$$e^- + p \leftrightarrow n + \nu$$

*so that no electrons are absorbed on the average and no neutrons decay. How are the number densities of electrons, protons, and neutrons related to each other?*

**Solution:** In equilibrium the energies of the highest energy states occupied by the neutrons (their Fermi energy) must be equal to the sum of the Fermi energies of the electrons and protons

$$(E_F)_n = (E_F)_e + (E_F)_p\ .$$

That way the neutron cannot decay because the states of the decay products are already occupied. Similarly the reaction cannot go the other way and conserve energy.

For relativistic particles the energy is proportional to momentum, $E = pc$. Thus,

$$(p_F)_n = (p_F)_e + (p_F)_p\ . \tag{1}$$

The number density $n$ of each species is proportional to its Fermi momentum cubed, explicitly

$$n = \frac{p_F^3}{3\pi^2\hbar^3} \ . \tag{2}$$

Electrical neutrality means

$$n_p = n_e$$

so that $(p_F)_e = (p_F)_p$. This, together with eqns. (1) and (2) imply

$$n_p = n_e = \frac{1}{8} n_n \ .$$

The neutrons predominate.

**24-13**. *[S] Evaluate $\gamma$ defined by* (24.18) *for the equation of state of degenerate fermions given in* (24.7) *and* (24.9) *for both the non-relativistic and relativistic limits.*

**Solution:** This problem is a straightforward exercise in differentiation. Consider, for example, the equation of state for non-relativistic fermions given in (24.7b) and (24.9b). This is

$$\rho = an + bn^{5/3} \quad , \quad p = \frac{2}{3}\, bn^{5/3}$$

for constants $a$ and $b$. We have

$$\begin{aligned} \gamma &= \frac{\rho + p}{p} \frac{dp}{d\rho} = \frac{\rho + p}{p} \frac{dp/dn}{d\rho/dn} \\ &= \frac{an + \frac{5}{3}\, bn^{5/3}}{\frac{2}{3}\, bn^{5/3}} \cdot \frac{\frac{5}{3} \cdot \frac{2}{3}\, bn^{2/3}}{a + \frac{5}{3}\, bn^{2/3}} = \frac{5}{3}. \end{aligned}$$

Similarly, in the relativistic case $\gamma = 4/3$.

**24-14**. *[E] (Estimating the Radius of Neutron Stars.) Estimate the radius of a neutron star by assuming that the degeneracy pressure of free neutrons supplies the pressure that holds it up and then work Problem 12.2.*

**Solution:** See the solution to Problem 12.2. The neutron is approximately $10^3$ times heavier than the electron, its Compton wavelength $10^3$ times smaller, and the radius of neutron stars approximately $10^3$ times smaller than those of white dwarfs. The estimate thus gives $\sim 1$ km which is not too bad considering that nuclear forces are important.

**24-15**. *[S] Starting from the wave equation* (24.21) *derive the frequencies and shapes of the normal modes of a vibrating string that are given in* (24.22) *and* (24.24) *respectively.*

**Solution:** The wave equation (24.21) can be solved by making the ansatz of harmonic time dependence

$$\xi(t,x) = e^{\pm i\omega t}\xi(x). \tag{1}$$

The amplitude $\xi(x)$ then solves the equation

$$\frac{d^2\xi}{dx^2} + \omega^2\left(\frac{\sigma}{T}\right)\xi = 0 \tag{2}$$

whose general solution is

$$\xi(x) = A\sin\left[\omega\left(\frac{\sigma}{T}\right)^{\frac{1}{2}} x\right] + B\cos\left[\omega\left(\frac{\sigma}{T}\right)^{\frac{1}{2}} x\right] \tag{3}$$

for arbitrary constants $A$ and $B$.

The boundary conditions that the string remain fixed at the ends are $\xi(0) = \xi(L) = 0$. This requires $B = 0$ and

$$\omega_n\left(\frac{\sigma}{T}\right)^{\frac{1}{2}} L = (n+1)\,\pi \quad , \quad n = 0, 1, 2, \ldots \tag{4}$$

Condition (4) gives the allowed frequencies (24.22). Eq. (3) then becomes (24.24).

**24-16**. *[S] Work through the changes in stability of the modes of the that occur at the extrema of the mass vs. radius relation for the family of stars in Figure 24.11. Assume that the curves of squared frequency vs. central density*

*never cross. Show that the changes in stability are as illustrated in Figure 24.10, and that there are only two ranges of stable equilibrium stars as shown.*

**Solution:** The analysis follows that given in the text for extrema $A$ and $B$. Consider, for example, the question of what happens at the next extremum $C$. Just before $C$ the lowest two modes $n = 0$ and $n = 1$ are unstable. The question at $C$ is whether the $n = 2$ mode becomes unstable or the $n = 1$ mode becomes stable. The $M$ vs. $R$ curve in Figure 24.7 shows that the radius $R$ is increasing with increasing central density at $C$. Therefore, a mode with an odd number of nodes changes stability. The $n = 1$ mode therefore becomes stable again as shown in Figure 24.11. Proceeding in this way the whole of Figure 24.11 can be mapped out and the two regimes of stable stars identified.

**24-17**. *Stable Equilibria Beyond Neutron Stars? A theorist proposes a new equation of state for matter above nuclear densities and wonders whether it might lead to a new kind of ultra-high-density endstates to stellar evolution beyond neutron and white dwarf stars. You use the equation of state and the equations of this chapter to calculate the mass-radius relationship of stars with central density greater than nuclear density that is shown at right. The curve represents stable neutron stars (NS) at the lowest densities but then spirals around at higher densities. Assuming the lowest density, largest radius stars shown are stable, will there be a new family of stable equilibria?*

**Solution:** The radius is decreasing at the first maximum meaning that the lowest mode (zero nodes) is becoming unstable. At the next extremum — the first minimum — the radius is increasing. Then the lowest mode cannot be becoming stable. Rather, the next highest mode (one node) must be becoming unstable. And so it goes with the successive extrema. The modes changing stability alternate between even and odd number of nodes. The behavior of

the squared frequencies is schematically as shown below.

Once unstable, no mode ever recovers stability and the sequence of stars above the first maximum is unstable.

**24-18.** *[E] Using Newtonian physics, estimate the ratio of centrifugal forces to gravitational forces at the surface of a neutron star rotating with a period of one second.*

**Solution:** At the surface of a star of mass $M$ and radius $R$, the centrifugal force on a mass $m$ is

$$\begin{pmatrix}\text{centrifugal}\\ \text{force}\end{pmatrix} = \frac{mV^2}{R}$$

where $V$ is the surface velocity. This velocity is related to the period $P$ by $V = 2\pi R/P$. The gravitational force on the mass $m$ can be estimated from its Newtonian value

$$\begin{pmatrix}\text{gravitational}\\ \text{force}\end{pmatrix} = \frac{GmM}{R^2}.$$

The ratio of centrifugal to gravitational force is therefore

$$(\text{ratio}) = \frac{4\pi^2 R^3}{GMP^2}.$$

For $R \sim 10$ km, $M \sim M_\odot$, $P = 1$ s, this is

$$\begin{aligned}(\text{ratio}) &\sim \frac{4\pi^2(10 \text{ km} \times 10^5 \text{ cm/km})^3}{(6.67 \times 10^{-8} \text{ dyn} \cdot \text{cm}^2/\text{g}^2)\ (2 \times 10^{33} \text{ g})\ (1 \text{ s})} \\ &\sim 3 \times 10^{-7}.\end{aligned}$$

By this measure, a solar mass neutron star rotating with a period of 1 s is *slowly* rotating.

**24-19**. *(Refining the Upper Bound on the Maximum Mass of Neutron Stars.) Use the results of Problem 9 to show that, in the notation of Section 24.6, the mass and radius of the core must satisfy $M_0/r_0 < 4/9$. Assuming that $\rho_0$ is nuclear density, find a bound on the mass of the core for stars with central densities that are above this value.*

**Solution:** The maximum redshift occurs at the maximum value of $M_0/r_0$ according to (9.20). From the solution to Problem 9, this occurs at $\xi = 1/3$ giving

$$\left(1 - \frac{2M_0}{r_0}\right)^{\frac{1}{2}} > \frac{1}{3}$$

or

$$\frac{M_0}{r_0} < \frac{4}{9}.$$

Repeating the arguments that led to (24.7) gives the improved bound

$$M_0 < \frac{1}{2}\left(\frac{8}{9}\right)^{\frac{3}{2}}\left(\frac{3}{8\pi\rho_0}\right)^{\frac{1}{2}} = 6.7 M_\odot \left(\frac{2.9 \times 10^{14} \text{ g/cm}^3}{\rho_0}\right)^{\frac{1}{2}}.$$

# Appendix: Light Orbits in the Schwarzschild Geometry

# Computes Light Ray Orbits

## ■ Defines Effective Potential and Deflection Angle

We put M=1, and use u=1/r as a coordinate. The effective potential W is defined by (9.65)

```
W[u_] := u^2 (1 - 2 u)
```

The following function computes the deflection angle from a point u1 to u . This is (9.78) of the text.

```
phi[u_, e_, u1_] := NIntegrate[ (e - W[w]) ^ (-1 / 2), {w, u1, u}]
```

The following computes the turning point of the orbit if $e < W_{max}$=(1/27)=.37037

```
soln[e_] := NSolve[W[tp] == e, tp]
```

The following define ranges of r and u for plots.

```
rmax = 6
```

6

```
umax = 1 / 2
```

$\frac{1}{2}$

```
umin = 1 / ((1.414) rmax)
```

0.117869

```
N[1 / 27]
```

0.037037

## ■ Plunge Orbit (e just above maximum)

```
e = .0371
```

0.0371

```
0.03704
```

0.03704

In[10]:= 
```
Plot[phi[u, e, umax], {u, 1 / 2, umin}]
```

Out[10]= - Graphics -

In[11]:= 
```
frticks = {-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5}
```

Out[11]= {-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5}

In[12]:= 
```
orbit = ParametricPlot[{Cos[phi[u, e, umax]] / u, Sin[phi[u, e, umax]] / u}, {u, .5, umin},
  PlotRange → {{-rmax, rmax}, {-rmax, rmax}}, AspectRatio → 1, Frame → True,
  Ticks → None, Ticks → {frticks, frticks, frticks, frticks}, DisplayFunction → Identity]
```

Out[12]= - Graphics -

In[13]:= 
```
circle2 := Graphics[{GrayLevel[.95], Disk[{0, 0}, 2]}]
```

In[14]:= 
```
graph = Show[orbit, circle2, DisplayFunction → $DisplayFunction, Axes → None]
```

Out[14]= - Graphics -

In[15]:= 
```
Display["lgtplunge.tmp", graph, "EPS"]
```

Out[15]= - Graphics -

## ■ Scattering Orbit (e just below maximum)

```
In[16]:= e = .0369

Out[16]= 0.0369
```

The next steps determine the turning point of the orbit:

```
In[17]:= temp = soln[e]

Out[17]= {{tp → -0.166392}, {tp → 0.321486}, {tp → 0.344906}}

In[18]:= tp1 = tp /. temp[[1]]

Out[18]= -0.166392

In[19]:= tp2 = tp /. temp[[2]]

Out[19]= 0.321486

In[20]:= tp3 = tp /. temp[[3]]

Out[20]= 0.344906
```

Starts the integration a small distance away from the turning point.

```
In[21]:= eps = .00005

Out[21]= 0.00005

In[22]:= 1 / tp2

Out[22]= 3.11056

In[23]:= utp = tp2 (1 - eps)

Out[23]= 0.32147
```

The next steps compute the two halves of the orbit --going into the turning point and coming out.

```
In[24]:= orbit1 = ParametricPlot[{Cos[phi[u, e, utp]] / u, Sin[phi[u, e, utp]] / u}, {u, utp, umin},
           PlotRange → {{-rmax, rmax}, {-rmax, rmax}}, AspectRatio → 1, Frame → True,
           Ticks → None, Ticks → {frticks, frticks, frticks, frticks}, DisplayFunction → Identity]

Out[24]= - Graphics -

In[25]:= orbit2 = ParametricPlot[{Cos[phi[u, e, utp]] / u, -Sin[phi[u, e, utp]] / u},
           {u, utp, umin}, PlotRange → {{-rmax, rmax}, {-rmax, rmax}},
           AspectRatio → 1, Frame → True, Ticks → None,
           Ticks → {frticks, frticks, frticks, frticks}, DisplayFunction → Identity]

Out[25]= - Graphics -

In[26]:= graph = Show[orbit1, orbit2, circle2, DisplayFunction → $DisplayFunction, Axes → None]
```

```
Out[26]= - Graphics -

In[27]:= Display["lgtscat.tmp", graph, "EPS"]

Out[27]= - Graphics -
```

# Errata for Instructor's Solutions Manual for Gravity, An Introduction to Einstein's General Relativity *Versions 1.0 (first printing) and 1.1*

Updated 11/20/2006

(Thanks to Ted Jacobson, John Friedman, Don Page, and Mario Serna who provided most of these.)

*Problem 2.2:*
Replace Solution with the following:

Gauss' triangle was located near the surface of the Earth. The relevant radius in the expression (2.1) is the distance of the triangle from the center of attraction. Eq (2.1) gives the approximate size of the effect of the Earth where $R_\oplus = 6378$ km is the radius of the Earth. However for Sun the relevant radius is the distance of the triangle from the center of the Sun, which approximately the size of the Earth's orbit $r_\oplus \approx 1.4 \times 10^8$ km. Then, from (2.1) the ratio of the effect of the Sun to that for the Earth can be written

$$(\text{ratio})_{\text{SuntoEarth}} \sim \left(\frac{GM_\odot}{c^2}\right)\left(\frac{c^2}{GM_\oplus}\right)\left(\frac{R_\oplus}{r_\oplus}\right)^3 .$$

For the Earth $GM_\oplus/c^2 = .443$ cm and for the Sun $GM_\odot/c^2 = 1.48$ km. For the ratio we get

$$(\text{ratio})_{\text{SuntoEarth}} \sim 10^{-19} \quad (!) .$$

The effect of the Sun is therefore much smaller than the effect of the Earth.

*Problem 2.9:*
In the 5th printing and later the solution has to be revised in an obvious way to reflect the minor changes in the data in the problem.

*Problem 4.4:*
In the last equation it should be $-3.4$ instead of $+3.4$.

*Problem 4.14:*

The solution given is when $\vec{V}$ and $\vec{v}$ are colinear. The general case is not too difficult and will appear in the next version of this manual.

*Problem 5.17:*
In the fourth printing and onward the problem is replaced with the following clearer statement and with the corresponding changes in notation in the solution:

[C] *(Relativistic Beaming)* A body emits photons of frequency $\omega_*$ at equal rates in all directions in its rest frame. A detector at rest in this frame a large distance away (compared to the size of the body) receives photons at a rate per unit solid angle $(dN/dtd\Omega)_*$ [photons/(sec·sr)] that is independent of direction. In an inertial frame $(t', x', y', z')$ in which an observer is at rest the body is moving with speed $V$ along the $x'-$axis.

1. Derive (5.75) relating a photon's direction of propagation in the rest frame to the direction of propagation in the observer's frame.

2. Find the rate at with photons are received per unit solid angle $dN/dt'd\Omega'$ a large distance away in the observer's frame as a function of angle $\alpha'$ from the $x'-$axis. [Hint: Remember that the time interval between the reception of two photons by a stationary observer is not the same as the time interval between their emission if the source is moving.]

3. Find the luminosity per unit solid angle $dL'/d\Omega'$ [erg/(sec·sr)] a large distance away as a function of the angle $\alpha'$ in the observer's frame.

4. Discuss the beaming of number and energy in the observer's frame as the velocity of the source approaches the velocity of light.

Solution

1. Suppose a photon is emitted in the rest frame making an angle $\alpha$ with the $x$-axis. The components of its four momentum $\mathbf{p}$ in this frame are:

$$\mathbf{p} = (p, p\cos\alpha, p\sin\alpha, 0)\ ,\ p = \hbar\omega_*\ .$$

In the observer's frame the components of **p** are:

$$p^{t'} = \gamma(p + Vp\cos\alpha) \ , \tag{1}$$
$$p^{x'} = \gamma(p\cos\alpha + Vp) \ , \tag{2}$$
$$p^{y'} = p\sin\alpha \ . \tag{3}$$

The angle $\alpha'$ made by the photon with the $x'$ axis is then

$$\cos\alpha' = \frac{p^{x'}}{p^{t'}} = \frac{\cos\alpha + V}{1 + V\cos\alpha} \ .$$

The inverse of this obtained by replacing $V$ by $-V$ is also useful:

$$\cos\alpha = \frac{\cos\alpha' - V}{1 - V\cos\alpha'} \ . \tag{4}$$

2. The number of photons emitted in in a time $dt$ and solid angle $d\Omega$ in the rest frame must be the same as the number emitted in a corresponding interval $dt'_e$ and solid angle $d\Omega'$ in the frame in which the source is moving. We use the notation $dt'_e$ for the time interval between photons at *emission* to reserve $dt'$ for the time interval between photons when they are *received*. $2\pi\sin\alpha' d\alpha' dt'(dN/dt'_e d\Omega')$ is the number of photons emitted at angle $\alpha'$ into an annulus of angular width $d\alpha'$ in time $dt'_e$. This must be the same as the number emitted in the corresponding annulus in the rest frame — $2\pi\sin\alpha d\alpha dt(dN/dt d\Omega)$ in the corresponding time $dt$. The time intervals $dt$ and $dt'_e$ are connected by time dilation — $dt'_e = \gamma dt$ The connection between angles (4) allows us to compute:

$$\frac{d(\cos\alpha)}{d(\cos\alpha')} = \frac{1}{\gamma^2(1 - V\cos\alpha')^2}$$

The result is:

$$\frac{dN}{dt'_e d\Omega'} = \left(\frac{dN}{dt d\Omega}\right) \frac{1}{\left[\gamma\left(1 - V\cos\alpha'\right)\right]^2} \tag{5}$$

Now we connect the time interval for emission $dt'_e$ with the time interval for reception $dt'$ — *both in the same frame where the source is moving*. Suppose one photon is emitted at an angle $\alpha'$ to the motion and travels a distance $d$ to reception. When the second photon is emitted a time $dt'_e$

later the source has travelled a distance $V dt'_e$ in the direction of motion. A little geometry shows that the distance this photon travels is shorter by $(V dt'_e)\cos\alpha'$ when $d$ is large. Thus

$$dt' = dt'_e(1 - V\cos\alpha') \tag{6}$$

Combining this result with (5) we find finally:

$$\frac{dN}{dt' d\Omega'} = \left(\frac{dN}{dt d\Omega}\right) \frac{1}{\left[\gamma\left(1 - V\cos\alpha'\right)\right]^3} \tag{7}$$

3. The energy of the photons emitted at angle $\alpha'$ is, from the inverse of (5.73) or from (1):

$$\hbar\omega_* = E'\gamma\left(1 - V\cos\alpha'\right)$$

Solving for $E'$ and using the result of part (b) the luminosity per unit solid angle is

$$\frac{dL'}{d\Omega'}(\alpha') = E'(\alpha')\frac{dN}{dt' d\Omega'}(\alpha') = \frac{(dL/d\Omega)}{\left[\gamma\left(1 - V\cos\alpha'\right)\right]^4}$$

4. The ratio of luninosity in the forward to backward direction is

$$\frac{(dL'/d\Omega')(0)}{(dL'/d\Omega')(\pi)} = \left(\frac{1+V}{1-V}\right)^4 \tag{8}$$

which as $V \to 1$ becomes very large, meaning most of the radiation is beamed forward.

For a solution to the version of this problem in printings 1-3 replace $(dN/dt d\Omega)_*$ with $f_*$, $dN/dt' d\Omega'$ by $f'(\alpha')$, and $dL'/d\Omega'$ by $L'(\alpha')$.

Further comments: An easy way to remember this result is that the combination:

$$\omega^{-3}\frac{dN}{dt d\Omega}$$

is the same in both frames — an invariant.

As $V \sim 1$, $1/(1-V) \sim 2\gamma^2$ and the forward/backward ratio (8) becomes $\sim 256\gamma^8$. Since $\gamma \sim 10$ for some matter in active galactic nuclei jets the forward/backward difference in luminosity can be very large.

Solution:

*Solution to Problem 6.9:*
In (2) change "5.2" to "5.3" and in (3) change "4.5" to "4.6".

*Solution to Problem 6.10:* In the 5th printing the problem was revised so that the radioactive species had an *exponential decay time* of 6.5 billion years like $^{238}U$ implying obvious changes in the solution. In at the end of the problem the exponential decay time is erronously labeled the half-life although the calculations are o.k.

*Solution to Problem 6.12:* In the 5th printing this problem was modified slightly. Replace the solution with the following:

There is one great circle through any two points on a sphere but it defines *two* curves of extremal distance connecting the two points. The shorter segment of the great circle is the path of shortest distance between the two points. But the longer segment around the other way is also an extremal curve. However, it provides neither the longest or shortest distance when compared with other nearby paths. To see that there is a *shorter* nearby path imagine the two points are on the equator and slide the long segment up a bit toward the north pole (left figure below). It gets shorter. To see that there is a *longer* nearby path, imagine a path which wiggles a little up above the long segment and below it many many times (right figure below). That will be a longer path. There is no longest path connecting the two points. Imagine for example taking paths that start at one endpoint and circle the globe 10 times, 1000 times, 10,000 times, etc before connecting to the other endpoint. Those are a sequence of increasingly longer paths, and there is no limit to how long they can be.

*Solution to Problem 6.14:*
Replace with following which has more comments than the earlier solution:

To order $1/c^2$ accuracy the proper time along any of these curves is given by (6.25) so

$$\Delta\tau = P - \frac{1}{c^2}\int_0^P dt\left[\frac{\vec{V}^2}{2} - \Phi\right]$$

in an inertial frame in which the center of the Earth is approximately at rest.

1. For a circular orbit of period $P$, $\Phi = -GM/R$ where $R$ is related to $P$ by Kepler's law $P^2 = (4\pi^2/GM)R^3$. Further, $V^2/R = GM/R^2$. The net result for the above integral is

   $$\Delta\tau = P\left(1 - \frac{3}{2}\frac{GM}{Rc^2}\right)$$

   which can be entirely expressed in terms of $P$ and $M$ using Kepler's law.

2. For a stationary observer $\vec{V} = 0$

   $$\Delta\tau = P\left(1 - \frac{GM}{Rc^2}\right)$$

   which is a longer proper time than a). Therefore, the circular orbit, although an extremal curve, is not a curve of *longest* proper time.

3. There is zero elapsed proper time. A circular orbit is not a curve of shortest proper time either.

There are many other extremal world lines connecting the two points. For instance, there is the world line followed when a ball is thrown radially outwards with the right velocity so that it falls back in time $P$. More generally the elliptical orbits with the same period $P$ that pass through the radius $R$ will be alternate extremal curves.

**Comment**: By calculating the second variation of the proper time the circular orbit can be shown to have the longest proper time with respect to *nearby* world lines connecting $A$ and $B$ but part (b) shows that the proper time is not the longest when compared to *any* world line connecting the two points. The elliptical orbits mentioned above with the same period and semi-major axes close to $R$ will be nearby the circular orbit. They are therefore extremal world lines connecting $A$ and $B$ with *shorter* proper time. Conversely for any one of these elliptical orbits, the circular one is a nearby world line with *longer* proper time. There is always nearby world line of *shorter* proper time made up of small lightlike segments. The elliptical orbits are therefore examples of saddle points, extremal but neither the longest or shortest when compared with nearby world lines. The problem can be extended along these lines.

*Solution to Problem 7.5:*
Change $1/(2x)$ to $2/x$.

*Solution to Problem 8.1:*
Strictly speaking the case $A = 0$ should be considered separately. It is a vertical straight line.

*Solution to Problem 9.11:*
In the denominator the factor $2\ell^2(r_{\max} - 6M)$ should be $\ell^2(6M - r_{\max})$.

*Solution to Problem 9.12:* Replace (7.48) with (9.46).

*Solution to Problem 9.12:*
Delete version two. Its incorrect. In version one replace the equation and text

after "Solving (5) and (6)" with:

$$V = \left( \frac{2M}{R} \frac{1}{1 - R^2/b^2} \right)^{1/2} .$$

This is coincidently the same as the formula for $V$ in Newtonian theory.

You can also derive the same relation for $V$ by starting, not from (2), but from the relation that at the turning point $V = u^{\hat{\phi}}/u^{\hat{t}}$ in the orthonormal basis associated with the stationary observer.

*Problem and Solution to 9.13:* Replace $\ell = 4.6$ with $\ell/M = 4.6$. (Correct units.)

*Problem and Solution to 9.21:* The reference to "latitude" can confuse. Replace with the following"

[E] Suppose a neutron star were luminous so that features on its surface could be viewed with a telescope. The gravitational bending of light means that, not only could the hemisphere facing us be seen, but also a part of the far hemisphere. Explain why and *estimate* the angle measured from the line of sight on the far side above which the surface could be seen. This would be $\pi/2$ if there were no bending, but less than that because of the bending. A typical neutron star has a mass of $\sim M_\odot$ and a radius of $\sim 10$ km.

Solution:

The above figure shows the geometry relevant to the problem. The telescope and observer are off to the far left along the line of sight. The solid line is the trajectory of the light ray that leaves the surface almost tangent to it, but reaches the observer because of light bending. The observer can see features on the part of the surface surface bounding the unshaded part of the figure, and cannot see features on the the part of the surface bounding the shaded part. The angle $\lambda$ defined in the problem, and dividing the seen from unseen parts, is shown together with its connection to the deflection angle $\delta\phi_{\rm def}$. For a neutron star $M_\odot/R \sim (1.5{\rm km})/(10{\rm km}) = .15$. This is small enough that a reasonable estimate for the angle $\lambda$ can be obtained from (9.83) for the deflection angle and the geometry of the above figure. Note that by symmetry light rays from the smallest $\lambda$ visible on the far side will be ones moving at constant azimuthal angle with respect to the line of sight. Evidently $2\lambda + \delta\phi_{\rm def} = \pi$ so $\lambda = (\pi - \delta\phi_{\rm def})/2 = \pi/2 - 2M/R \approx$ 1.3 radians $\approx 73^\circ$.

For a further problem try and figure out what such an image would look like.

*Solution to Problem 11.4:* Delete a factor of 2 from the final result.

*Solution to Problem 12.4:*
On p. 158 in two the two equations specifying $d\tilde{t}/dv$ the left hand sides should be $d\tilde{t}/dr$ not $d\tilde{t}/dv$. The text beginning "This information..." and the figure can be

replaced by the following improved versions

This information on slopes is enough to give the following $(\tilde{t}, r)$ spacetime diagram. Three radial light rays and some light cones are sketched qualitatively. The size of the cones is abitrary. One light ray is at $r = M$, another escaping to infinity is at $r > M$, and one confined inside the horizon is at $r < M$.

*Solution to Problem 12.8:* Improved text and corrected figure:

An observer who falls into the black hole can in principle receive information from any point in the shaded region between crossing the horizon at $r = 2M$ and destruction at $r = 0$. This is the region of points which can be connected to the world line of the observer between $r = 2M$ and $r = 0$ by null or timelike world lines. (Not necessarily radial ones.) This includes information about events out-

side $r = 2M$ but not all of the region outside $r = 2M$.

*Solution to Problem 12.13:* Improved figure:

Replace the solution to (a) with the following, its a bit clearer.

The figure above shows an Eddington-Finkelstein diagram with schematic world lines of the observer's head and feet. At a given $\tilde{t}$ her feet are at a smaller radius than her head because she is falling in feet first. Radial rays emitted originating at her feet are shown. (These are segments of light rays illustrated in Figure 12.2.)

There is no instant when she is not receiving a light ray from her feet. She sees them always. When her head crosses the horizon she sees her feet at the same radius, because the horizon is generated by light rays.

When her head hits the singularity she still sees light from her feet that was emitted earlier but is falling into the singularity as well. But, she never sees her feet hit the singularity because her head and feet meet the singularity at spacelike separated points. When close to the singularity light rays from her feet fall into the singularity before they intersect her world line, as the figure shows. (Some students interpret this question to ask if she sees her feet *when* they hit the singularity. But there is no invariant meaning to "when". For some it is made clearer in a Kruskal diagram.)

*Solution to Problem 12.14:* In the second line of the expression for $\tau$ the sign following $e^2$ should be a $+$.

*Solution to Problem 12.19:* Improved figure:

*Solution to Problem 13.5:* Change $3 \times 10^5$yr to $4 \times 10^5$yr, $3 \times 10^{-16}$cm to $4 \times 10^{-16}$cm and $4 \times 10^{12}$g to $5 \times 10^{12}$g. This makes the solution consistent with the statement of the problem.

*Solution to Problem 13.7:* Replace with the following simpler solution:

Let $M_0$ denote the present mass of a black hole going to explode 1s after the present time. From (13.19) we can find its mass by putting $t_* - t = 1$s with the result

$$M_0 = 1.7 \times 10^{-20}\text{cm} = 2.3 \times 10^8\text{g}.$$

If the black hole evaporates completely the energy emitted in the next second will be $M_0 c^2 = 4.2 \times 10^{29}$erg. Therefore, the total energy per unit area received at Earth in that 1s will be $M_0 c^2/(4\pi r^2)$ where $r$ is the distance to the evaporating black hole. In the same 1s the energy per unit area recived from a star with the solar luminosity $L_\odot$ a distance 10 pc away will be $(L_\odot \cdot (1\,\text{sec}))/(4\pi(10\text{pc})^2)$.

These will be equal when

$$\left(\frac{r}{10\text{pc}}\right) = \frac{M_0 c^2}{L_\odot \cdot 1\text{s}}$$

Thus $r$ is 0.007 of 10 pc or $2.2 \times 10^{17}$cm or $1.5 \times 10^4$AU. The black hole would be within the region called the "Oort cloud" containing the debris from the formation of the solar system that supplies some of the comets.

*Solution to Problem 18.6:* In the final sentence replace $\tilde{t} = t_0$ with $\tilde{t} = \tilde{t}_0$.

*Problem 18.25:* In the 5th printing the problem was modified slightly as follows. The solution here is for that, but is also a better solution to the version that appeared in earlier printings:

**Problem:**[C] Is there a value of $\Omega_v$ that would allow the universe to bounce at a small radius, but still reach a temperature $T \sim 10^{10}$ K such that nucleosynthesis could occur? Assume $\Omega_r = 8 \times 10^{-5}$ and $\Omega_m = .3$.

**Solution:** For the temperature at the bounce to be greater than $\sim 10^{10}$K, the value of $\tilde{a}$ at the bounce $\tilde{a}_b$ must be less than $\sim 10^{-10}$ [cf. (18.26)]. For simplicity let us suppose that the bounce occurs when the temperature is exactly $10^{10}$K corresponding to a redshift $z$ of approximately $3 \times 10^9$ and an $\tilde{a}_b \approx 3 \times 10^{-10}$.

A necessary condition for a bounce [cf. Fig 18.9] and (18.78)] is

$$\frac{1}{2}\Omega_c = U_{\text{eff}}(\tilde{a}_b) = -\frac{1}{2}\left[\Omega_v \tilde{a}_b^2 + \frac{\Omega_m}{\tilde{a}_b} + \frac{\Omega_r}{\tilde{a}_b^2}\right]. \tag{9}$$

Using $\Omega_c = 1 - \Omega_v - \Omega_m - \Omega_r$ from (18.76) this can be solved for $\Omega_v$ in terms of the other cosmological parameters and $\tilde{a}_b$. The result is

$$\Omega_v = \frac{1}{1-\tilde{a}_b^2}\left[1 + \Omega_m\left(\frac{1}{\tilde{a}_b} - 1\right) + \Omega_r\left(\frac{1}{\tilde{a}_b^2} - 1\right)\right]. \tag{10}$$

For the given values of $\Omega_m$ and $\Omega_m$ and $\tilde{a}_b \approx 3 \times 10^{-10}$ this evaluates to

$$\Omega_v \sim 10^{15} \quad (!), \tag{11}$$

the radiation term in (10) being the dominent one. (This value is wildly inconsistent with current observations as we will see in Chapter 19, but let's first see if it even corresponds to a bounce.)

While the condition (9) is necessary for a bounce, it is not sufficient. That is because, as Figure 18.9 makes clear, (9) is satisfied both by cosmological models where $\tilde{a}_b$ is a minimum of $\tilde{a}(\tilde{t})$ (bounces) and where it is a maximum of the expansion. For a bounce solution, $\tilde{a}_b$ lies above the value of the maximum of $U_{\text{eff}}(\tilde{\ })$. Let's check whether this is satisfied for (11).

At the time of the putative bounce when $T \sim 10^{10}$K the matter can be neglected. The maximum of $U_{\text{eff}}(\tilde{a})$ occurs at

$$\tilde{a}_{\text{max}} = \left(\frac{\Omega_r}{\Omega_v}\right)^{\frac{1}{4}} \sim 10^{12} \tag{12}$$

for (11). This is vastly bigger than $\tilde{a}_b \approx 3 \times 10^{-10}$. The maximum of $U_{\text{eff}}(\tilde{a})$ is above the value of $\tilde{a}_b$, not below it as would be required for a bounce.

We conclude that there is *no* value of $\Omega_v$ that is that would lead to a bounce at a temperature higher than $10^{10}$K.

*Solution to Problem 15.10:* Replace with the following (simpler, more closely related to the discussion in the text):

There are a number of different ways of solving this problem. We give two.

1) We can follow the demonstration given in Section 15.3 that the surface $r = R$ is a stationary null surface. Tangent vectors $\mathbf{t}$ in the surface have the general form

$$t^\alpha = (t^t, 0, t^\theta, t^\phi) \ . \tag{13}$$

The surface is null if, at each point, a null tangent vector $\ell$ can be found along with two orthogonal (to $\ell$ and each other) spacelike tangent vectors. (See Section 7.9.) The condition $\ell \cdot \ell$ for a tangent vector reads

$$\ell \cdot \ell = \left(1 - \frac{r^2}{R^2}\right)(\ell^t)^2 + r^2(\ell^\theta)^2 + r^2 \sin^2\theta(\ell^\phi)^2 \ .$$

On $r = R$ we see that $\ell^\alpha = (1,0,0,0)$ is a null tangent vector, (indeed its the unique one up to a multiplicative constant). Further, $(0,0,1,0)$ and $(0,0,0,1)$ are two spacelike tangent vectors orthogonal to each other and to $\ell$

The null surface at $r = R$ has the one-way property discussed in Section 7.9. Once through it you cannot come back.

2) Its possible to worry about the above solution because the $t$ coordinate is singular at $r = R$ in much the same way that the Schwarzschild $t$ coordinate is singular in the Schwarzschild metric. We can both demonstrate this and understand the nature of the $r = R$ three-surface by transforming to coordinates analogous to Eddington-Finkelstein coordinates. Following (12.1) we transform from $t$ to a new coordinate $u$ defined by

$$t = u + F(r) \tag{14}$$

where

$$\frac{dF}{dr} = \left(1 - \frac{r^2}{R^2}\right)^{-1}, \quad F(r) = \frac{R}{2}\log\left(\frac{R+r}{R-r}\right) . \tag{15}$$

Then the metric takes the form:

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right) du^2 - 2dudr + r^2(d\theta^2 + \sin^2\theta d\phi^2) . \tag{16}$$

In this non-singular form of the line element, its clear that $r = R$ is a null surface. Furthermore, because of the choice of signs in (14), its an outgoing null surface — fixed $u$ means $r$ has to get larger as $t$ gets larger. It therefore has the property that once crossed its impossible to return. (See the discussion in Section 7.9.)

*Comment:* The given metric is one form of the metric of deSitter space which is the maximally symmetric solution of Einstein's equation with a cosmological constant. Another form, covering a different patch, is given by (18.1) and (18.39). The surface $r = R$ is called the "deSitter horizon".

*Solution to Problem 15.13:* Delete the sentence "$C$ vanishes at .... null.' Its meaningless.