Recap: Fourier Transforms

- Double integral

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, e^{i\omega x} \int_{-\infty}^{\infty} dt \;\, f(t)\, e^{-i\omega t}$$

- Defines <u>Fourier transform</u> and its inverse transform:

$$F(\omega) = \int_{-\infty}^{\infty} dt \;\, f(t)\, e^{-i\omega t}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \;\, F(\omega)\, e^{i\omega t}$$

Applications of Fourier Transforms

1. Solutions of linear differential equations

Solving ODES

- Let

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \;\, f(t)\, e^{-i\omega t}$$

$$F_1(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \;\, f'(t)\, e^{-i\omega t}$$

$$= i\omega\, F(\omega)$$

- Assuming the boundary conditions cooperate, a Fourier transform
  <u>simplifies</u> the problem: $F_n(\omega) = (i\omega)^n\, F(\omega)$

  converts an ODE to an algebraic (polynomial) equation

  converts a PDE to an ODE

Applications of Fourier Transforms

1. Solutions of linear differential equations
2. Quantum mechanics
   - energy and time $(e^{-iEt/\hbar})$
   - position and momentum $(e^{ipx/\hbar})$
   - field theory: Feynmann diagrams
3. Signal processing /analysis
   - Periodic signals and chaotic systems
   - Power spectra $[P(\omega) = |F(\omega)|^2]$
   - Numerics: Discrete and Fast Fourier Transforms
4. Deconvolution/noise reduction

Fourier Transforms Example 1: Square Pulse

- Pulse

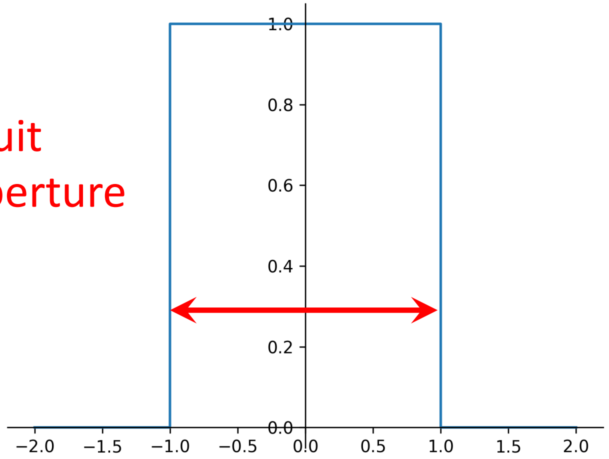
$$f(t) = \begin{cases} 1, & |t| < a \\ 0, & |t| > a \end{cases}$$

$$\Longrightarrow \quad F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} dt\ e^{-i\omega t}$$

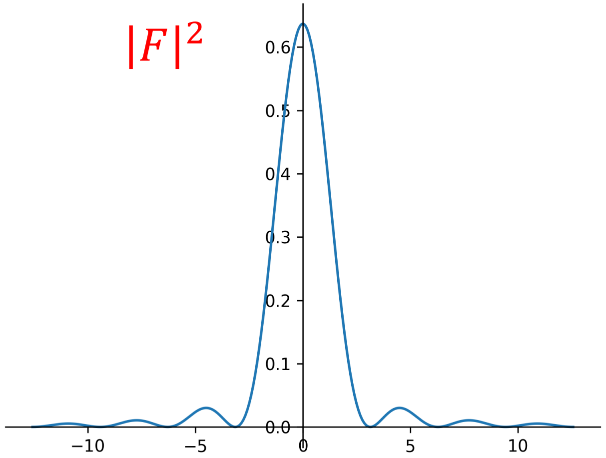
$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\omega t}}{-i\omega} \right]_{-a}^{a}$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{\sin \omega a}{\omega}$$

e.g. rectangular pulse in circuit
uniformly illuminated aperture



$|F|^2$



- Note: width of $f(t)$ is a
  width of $F(\omega)$ is $2\pi/a$

- Inverse relation is a generic feature of transforms.

Fourier Transforms Example 1: Square Pulse

- Inverse transform

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \sqrt{\frac{2}{\pi}}\, \frac{\sin \omega a}{\omega}\, e^{i\omega t}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega\, \frac{\sin \omega a}{\omega}\, e^{i\omega t}$$

$$= ?$$

Note: regular function divided by a polynomial

Fourier Transforms Example 2: Gaussian

- Gaussian

$$f(t) = e^{-\alpha t^2}$$

Gaussian, width $1/\sqrt{\alpha}$

$$\Longrightarrow \quad F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \; e^{-\alpha t^2 - i\omega t}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \, e^{-\left(t\sqrt{\alpha}+\frac{i\omega}{2\sqrt{\alpha}}\right)^2 - \frac{\omega^2}{4\alpha}}$$

$u = t\sqrt{\alpha} + \frac{i\omega}{2\sqrt{\alpha}}$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{4\alpha}} \int_{-\infty}^{\infty} \frac{du}{\sqrt{\alpha}} e^{-u^2}$$

$$= \frac{1}{\sqrt{2\pi\alpha}} e^{-\frac{\omega^2}{4\alpha}} \underbrace{\int_{-\infty}^{\infty} du \, e^{-u^2}}_{\sqrt{\pi}}$$

$$= \frac{1}{\sqrt{2\alpha}} e^{-\frac{\omega^2}{4\alpha}}$$

Gaussian, width $2\sqrt{\alpha}$
"uncertainty principle"
in f and F: $\Delta f \Delta F \sim 1$

$\Delta E \Delta t \geq \frac{1}{2}\hbar$

$\Delta x \Delta p \geq \frac{1}{2}\hbar$

Diffusion Equation

- Diffusion equation (1-D)

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = f(x), \qquad -\infty < x < \infty, \qquad t \geq 0$$

- FT with respect to x

$$U(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \;\, u(x,t)\, e^{ikx}$$

$$\Longrightarrow \qquad \frac{\partial U}{\partial t} = -\kappa k^2 U$$

$$\Longrightarrow \qquad U(k,t) = U(k,0) e^{-\kappa k^2 t}$$

- Suppose $f(x) = \delta(x)$ (sharp spike at a point)

$$\Longrightarrow \qquad U(k,0) = \frac{1}{\sqrt{2\pi}}$$

$$\Longrightarrow \qquad U(k,t) = \frac{1}{\sqrt{2\pi}} e^{-\kappa k^2 t}$$

Diffusion Equation

$$f(t) = e^{-\alpha t^2}$$

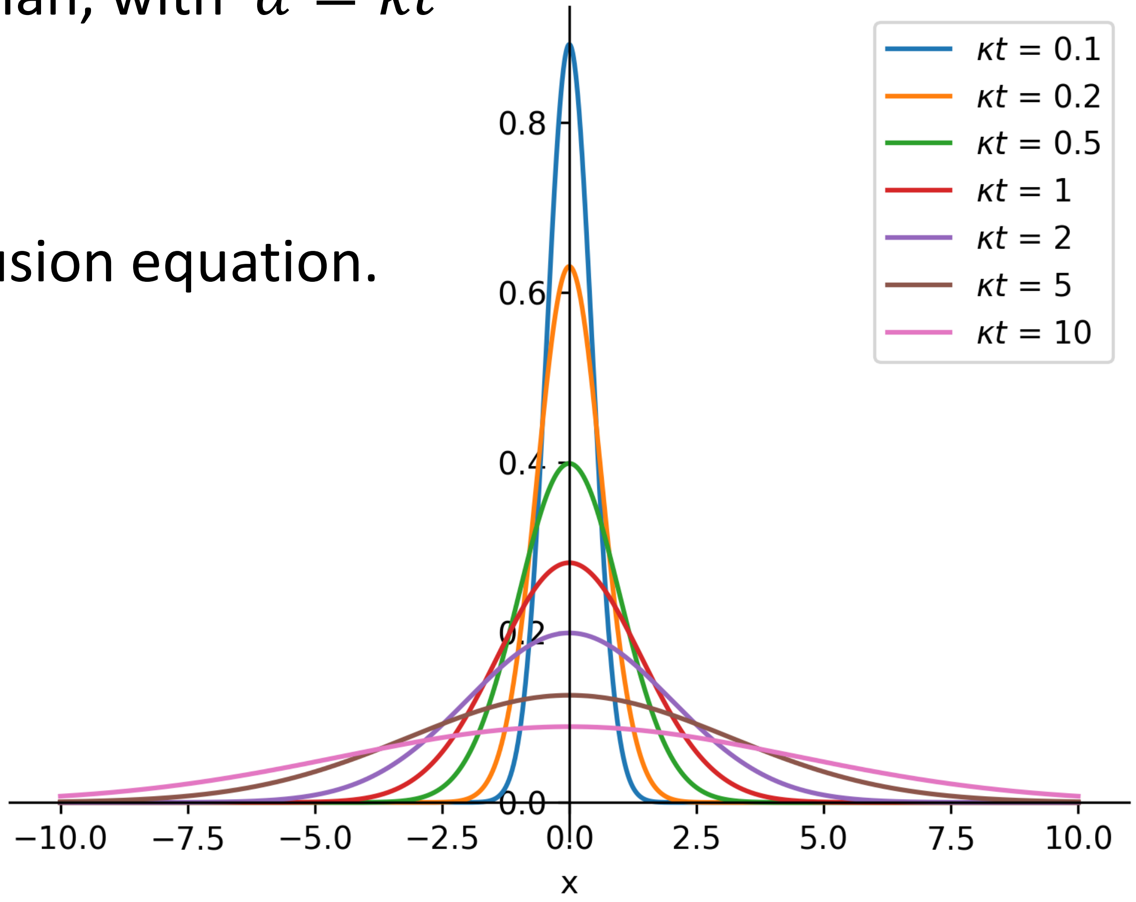
$$F(\omega) = \frac{1}{\sqrt{2\alpha}} e^{-\frac{\omega^2}{4\alpha}}$$

$$U(k,t) = \frac{1}{\sqrt{2\pi}} e^{-\kappa k^2 t}$$

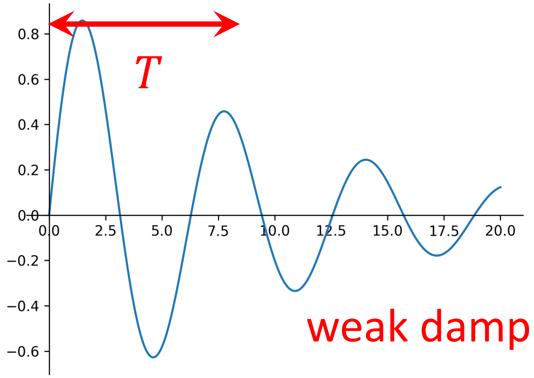
- Transform back: U is just a Gaussian, with $\alpha = \kappa t$

$$\Longrightarrow \qquad U(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} e^{-\frac{x^2}{4\kappa t}}$$

<u>fundamental solution</u> to the diffusion equation.



Fourier Transforms Example 3: Damped Oscillator



weak damping: $\omega_0 T \gg 1$

- Damped oscillator

$$f(t) = \begin{cases} 0, & t < 0 \\ e^{-t/T} \sin \omega_0 t\,, & t \geq 0 \end{cases}$$

$$\Longrightarrow \quad F(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty dt\, e^{-t/T} e^{-i\omega t} \sin \omega_0 t$$

$\sin \omega_0 t = (e^{i\omega_0 t} - e^{-i\omega_0 t})/2i$

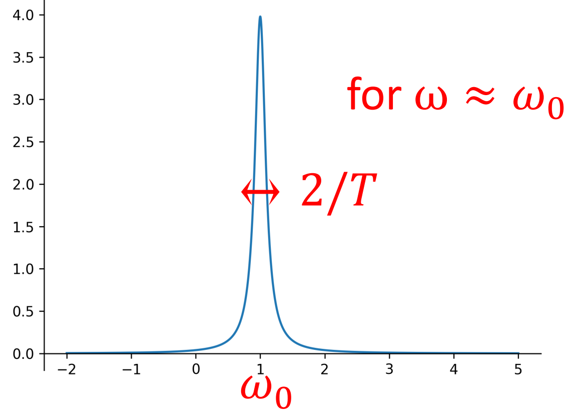
$$= \frac{1}{\sqrt{2\pi}} \frac{1}{2i} \int_0^\infty dt \left(e^{-t/T - i\omega t + i\omega_0 t} - e^{-t/T - i\omega t - i\omega_0 t} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{2i} \left[\frac{-1}{-1/T - i(\omega - \omega_0)} + \frac{1}{-1/T - i(\omega + \omega_0)} \right]$$

narrow peaks near $\pm\omega_0$ if $\omega_0 T \gg 1$
height $\sim T$, width $\Delta\omega \sim 1/T$

$$F(\omega) \approx \frac{1}{2\sqrt{2\pi}} \cdot \frac{1}{\omega - \omega_0 - i/T}$$

$$\Longrightarrow \quad |F(\omega)| \approx \frac{1}{2\sqrt{2\pi}} \cdot \frac{1}{\sqrt{(\omega - \omega_0)^2 + 1/T^2}}$$



Solving an Inhomogeneous ODE

- Equation

$$y'' - \lambda^2 y = f(x)$$

Let $\quad Y(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\ y(x)\, e^{-ikx}$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\ f(x)\, e^{-ikx}$$

$\Longrightarrow \quad -k^2 Y - \lambda^2 Y = F$

$\Longrightarrow \quad Y(k) = \frac{-F(k)}{k^2+\lambda^2}$

$\Longrightarrow \quad y(x) = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\ \frac{F(k)}{k^2+\lambda^2}\ e^{ikx}$

regular function divided by a polynomial

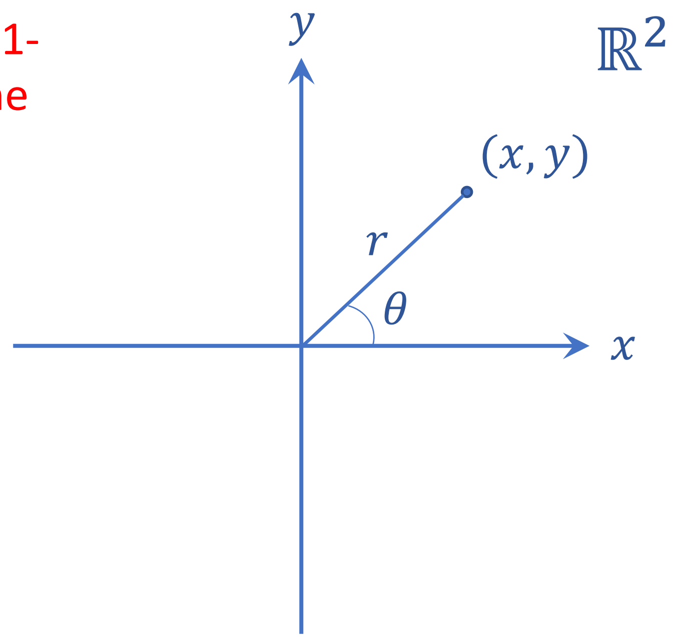
$$= ?$$

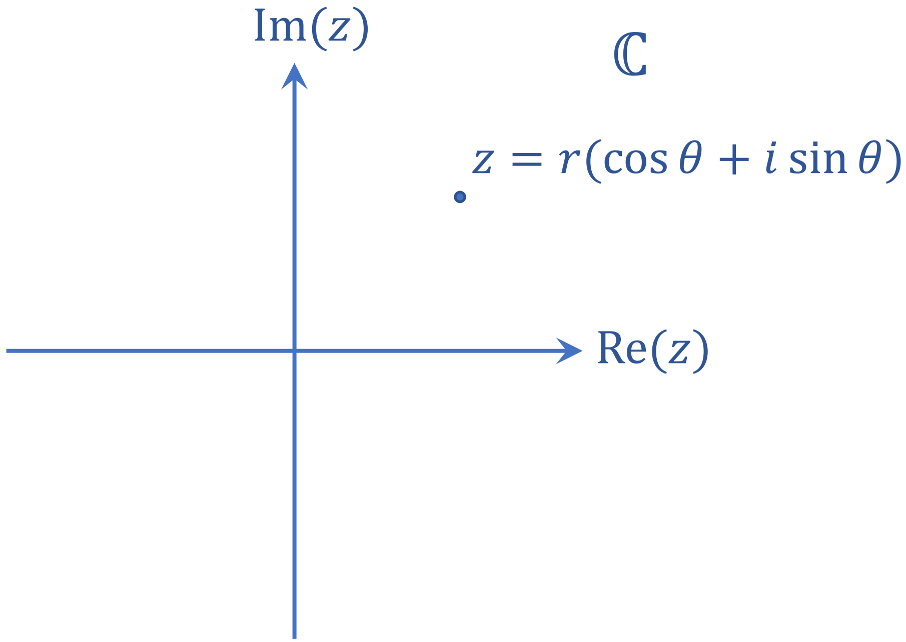
Functions in the Complex Plane

- Need to make a significant digression to address integrals like these.
- FT inherently <u>complex</u>, integrals often involve paths in the complex plane.
- Start by considering complex <u>functions</u>: mappings $\mathbb{C} \to \mathbb{C}$
- Notation: $z = x + iy = re^{i\theta}$, $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$, $r = |z|$, $\theta = \arg(z)$

cartesian polar

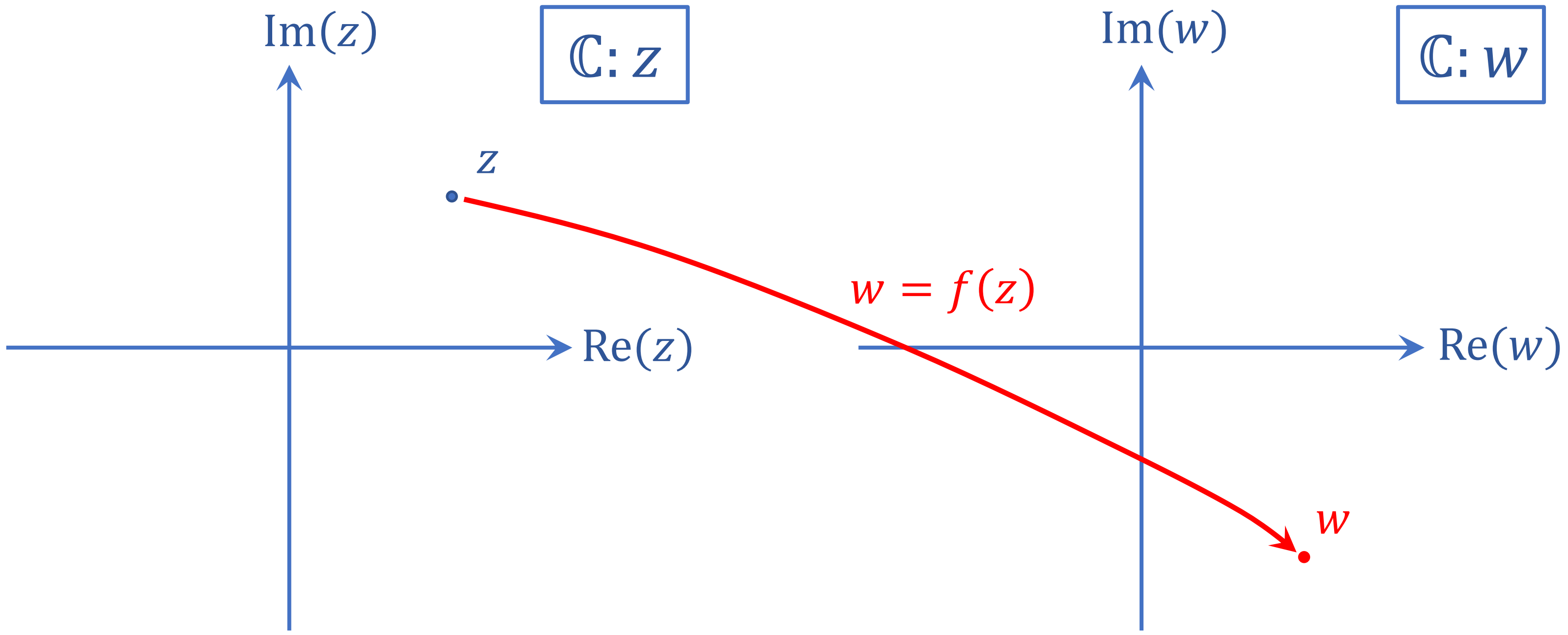
Relying heavily on the 1-1 mapping between the 2-D real plane and the complex plane.





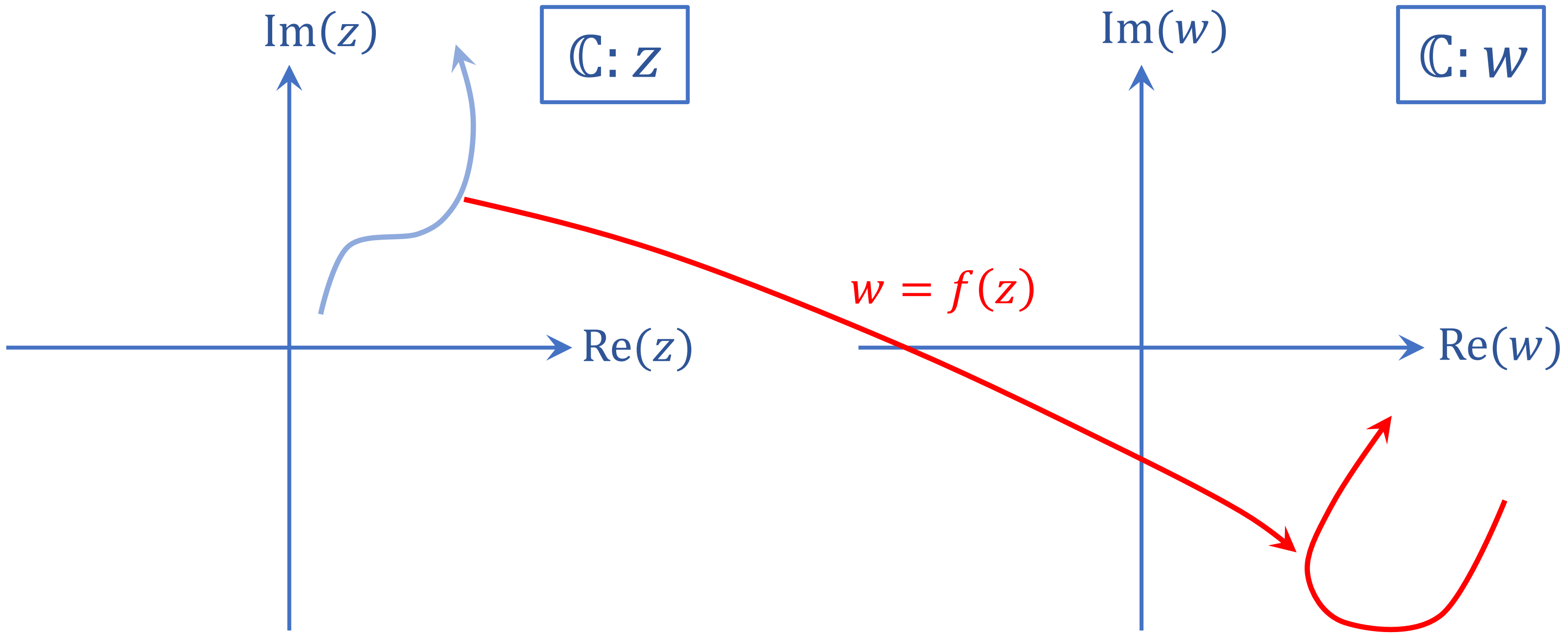
Functions in the Complex Plane

- A function f is a mapping from $\mathbb{C}$ to $\mathbb{C}$
- Notation: $w = f(z) = u + iv = u(x, y) + iv(x, y)$



Functions in the Complex Plane

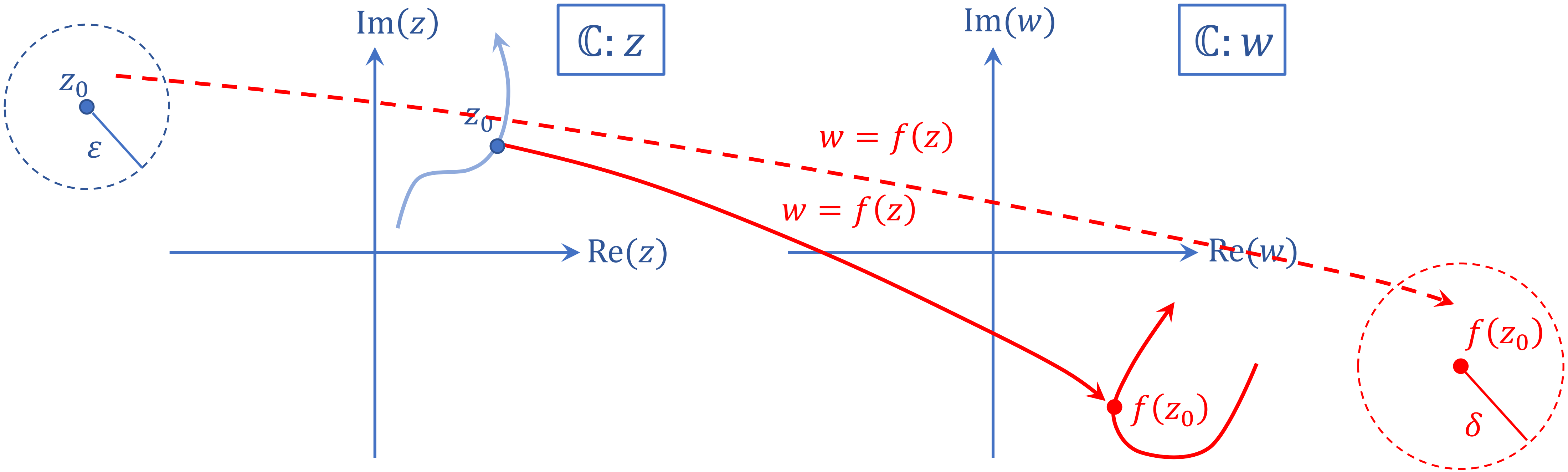
- A function f is a mapping from $\mathbb{C}$ to $\mathbb{C}$
- Notation: $w = f(z) = u + iv = u(x,y) + iv(x,y)$



Continuous Functions

- Expect w to move "smoothly" along the image curve as z moves along its curve.
- Say f is <u>continuous</u> at z_0 if all points "close" to z_0 map to points "close" to $f(z_0)$.
- Formally, can say $f(z) \to f(z_0)$ as $z \to z_0$ if

  for any $\delta > 0$ there exists $\varepsilon > 0$ such that $|f(z) - f(z_0)| < \delta$ if $|z - z_0| < \varepsilon$



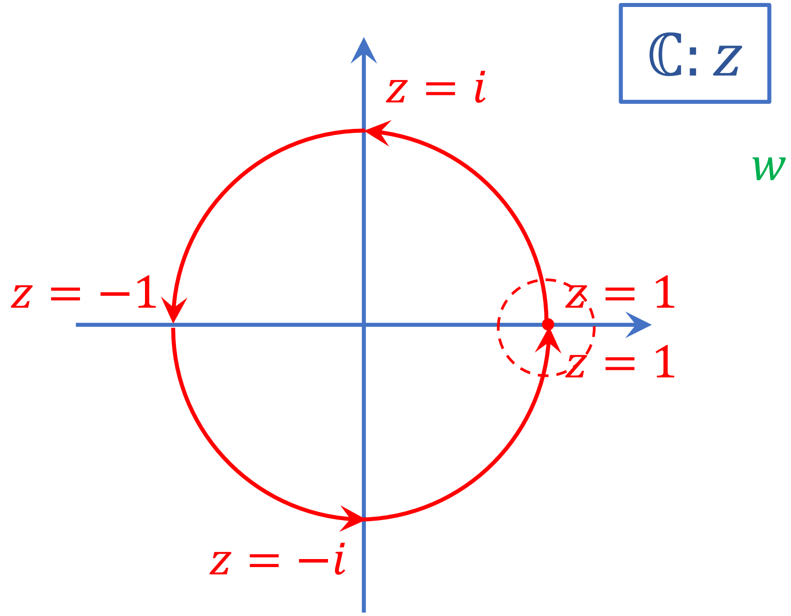
Continuous Functions

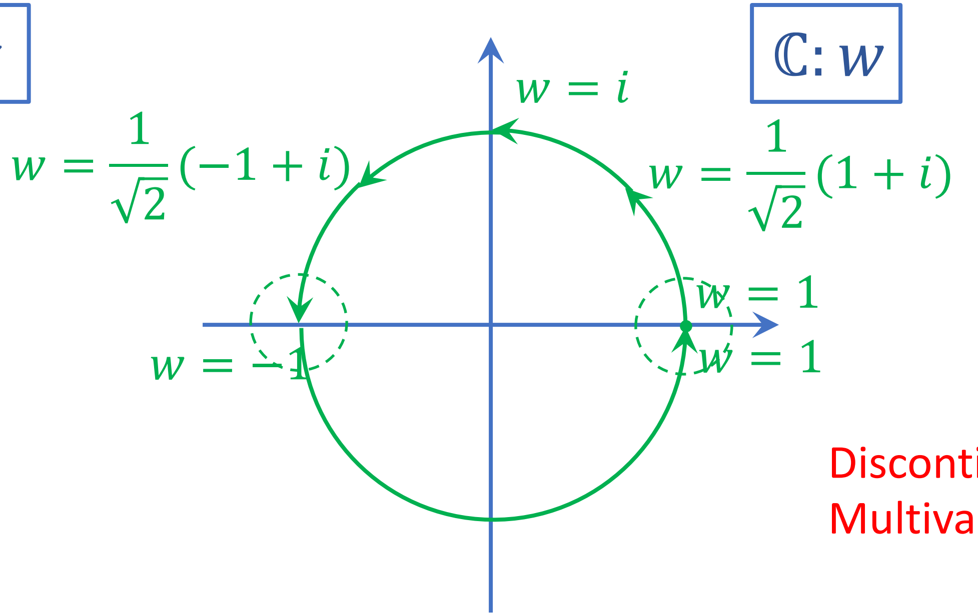
- Many standard functions are continuous

  e.g. z, z^2, z^n (integer n), e^z, $\sin z$ etc.

- But some simple functions are not – consider $f(z) = z^{1/2}$ as z traverses a unit circle around the origin

$$z = e^{i\theta}, \qquad w = z^{1/2} = e^{i\theta/2}$$

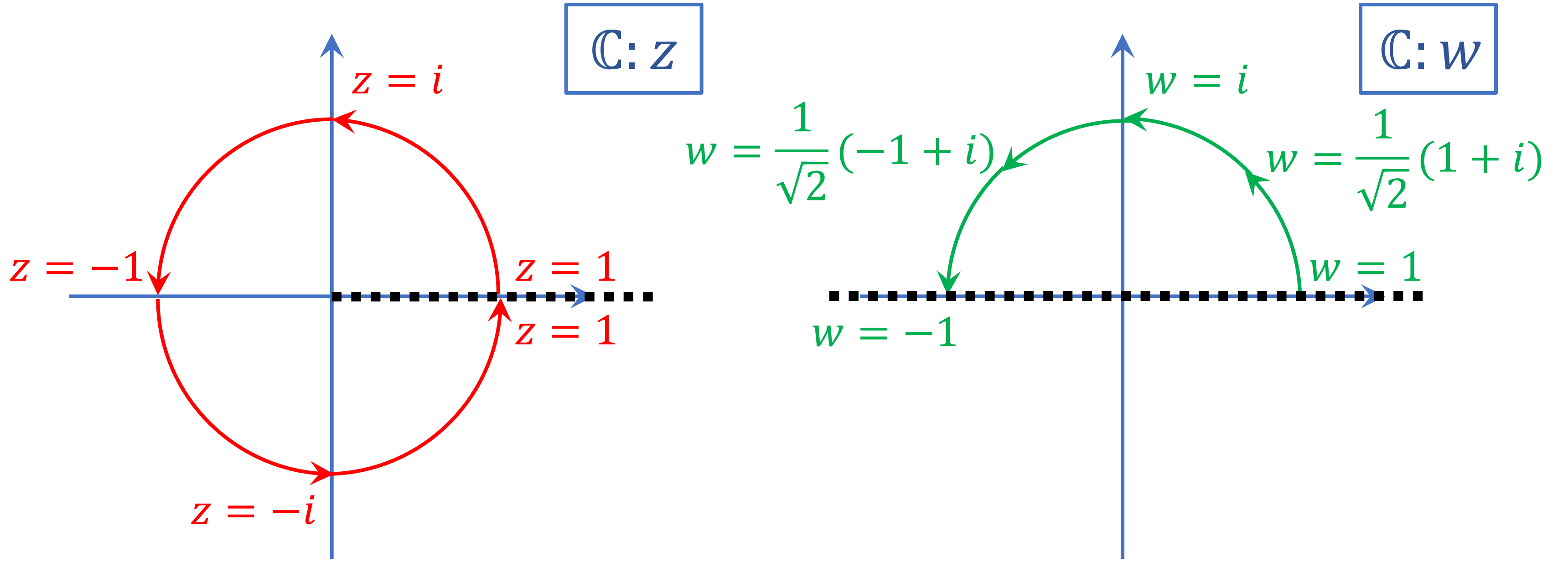




Discontinuous function
Multivalued function

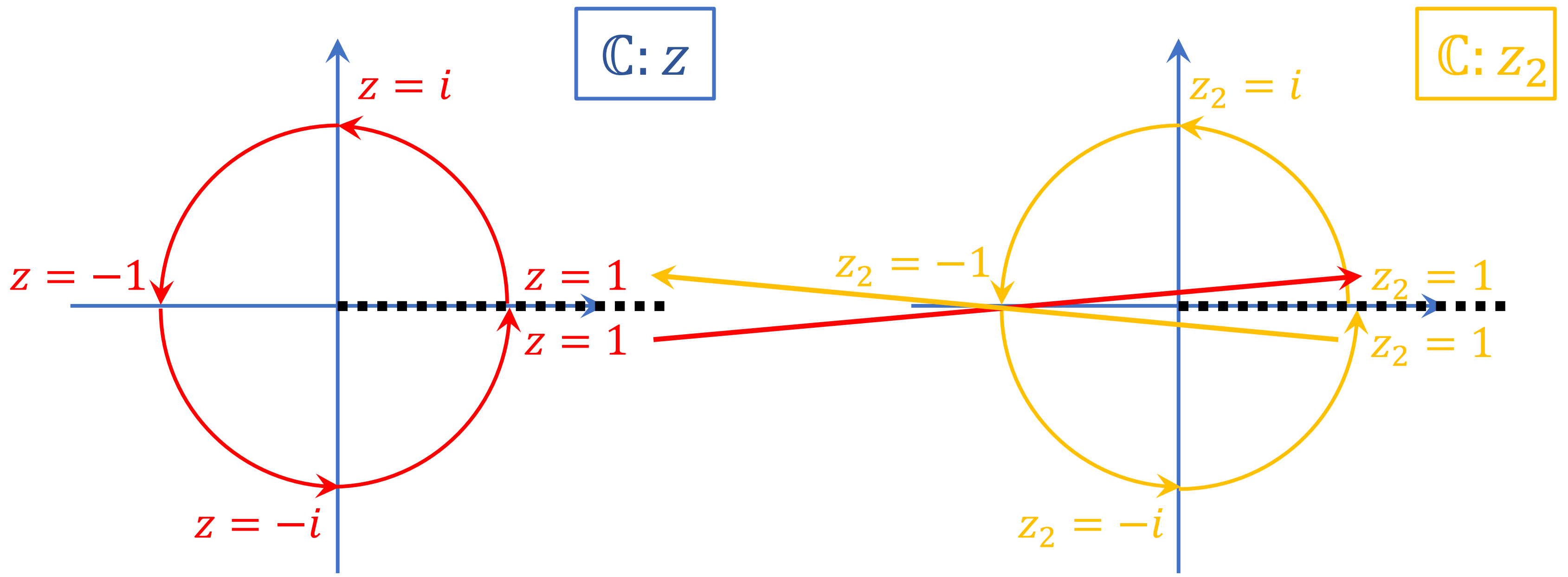
Discontinuous Functions

- Two ways to deal with discontinuity
  1. "Double" the original (z) space – "Riemann surface" – elegant but difficult.
  2. Restrict the image (w) space – introduce <u>branch cuts</u> – prosaic but works!
     - not allowed to cross certain curves in the z-plane
     - function is continuous so long as our path never crosses a cut



Riemann Sheets

- Riemann surfaces – two (or more) copies of the z-plane grafted together along a branch cut.



- Cut angle is arbitrary – just has to run from $z = 0$ to infinity.
- Too complicated for most physics applications – focus on branch cuts.

Branch Cuts

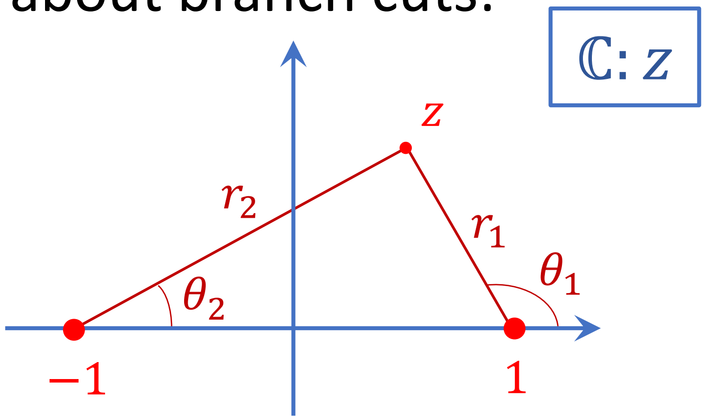
- Back in the Physics complex analysis world, branch cuts rule.
- More complex discontinuities require more thought about branch cuts:

  consider $f(z) = \sqrt{(z-1)(z+1)}$

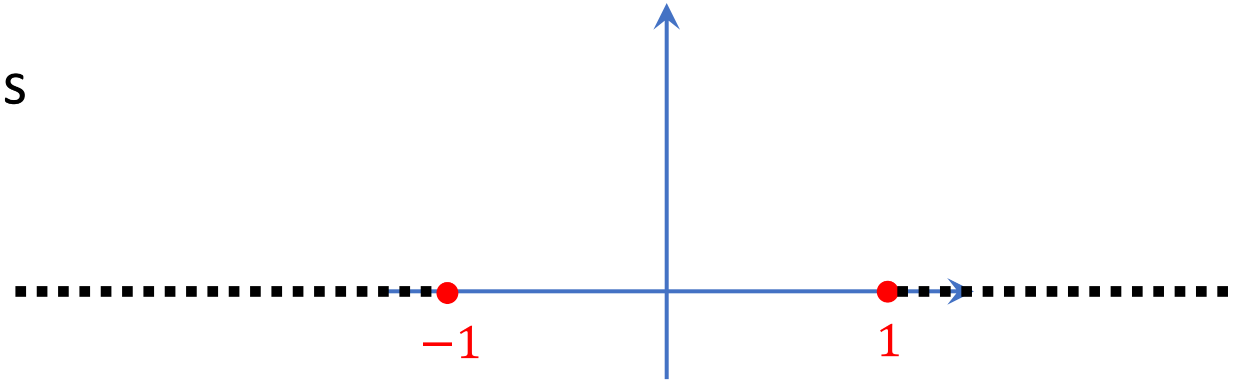
  potential problems at $z = \pm 1$

  write $z - 1 = r_1 e^{i\theta_1}, \quad z + 1 = r_2 e^{i\theta_2}$

  $$f(z) = r_1^{\frac{1}{2}} r_2^{\frac{1}{2}} e^{\frac{1}{2} i(\theta_1 + \theta_2)}$$



- Can't circle $z = \pm 1$, so 2 basic choices
  1. circle neither: branch cuts from $z = 1$ and $z = -1$ to infinity
  2. circle <u>both</u>



Branch Cuts

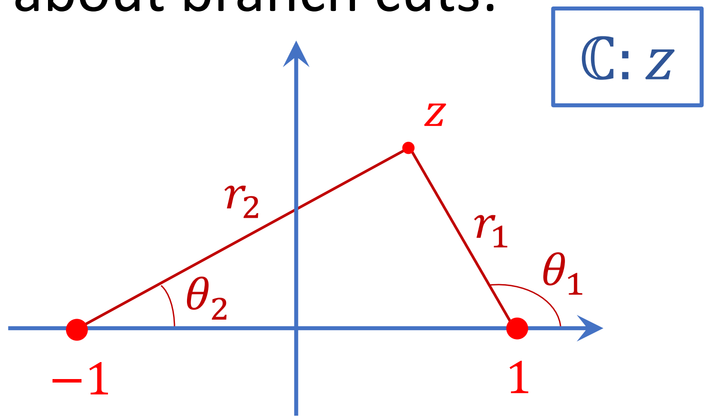
- Back in complex analysis world, branch cuts rule.
- More complex discontinuities require more thought about branch cuts:

  consider $f(z) = \sqrt{(z-1)(z+1)}$

  potential problems at $z = \pm 1$

  write $z - 1 = r_1 e^{i\theta_1}, \quad z + 1 = r_2 e^{i\theta_2}$

  $$f(z) = r_1^{\frac{1}{2}} r_2^{\frac{1}{2}} e^{\frac{1}{2} i(\theta_1 + \theta_2)}$$



- Can't circle $z = \pm 1$, so 2 basic choices
  1. circle neither: branch cuts from $z = 1$ and $z = -1$ to infinity
  2. circle <u>both</u>: branch cut from $z = -1$ to $z = 1$, $f(z)$ is continuous