## Benefits of the Lagrangian formalism, holonomic systems

1. No explicit forces of constraint in the eqns of motion.
2. Not vectorial, we only deal with quantities such as 'T' and 'V'.
3. Transformation of 'T' and 'V' into generalized coordinates can be done through:

$$T = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i \left( \sum_j \frac{\partial \bar{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \bar{r}_i}{\partial t} \right)^2$$

$$T = M_0 + \sum_j M_j \dot{q}_j + \frac{1}{2} \sum_{jk} M_{jk} \dot{q}_j \dot{q}_k$$

where $$M_0 = \sum_i \frac{1}{2} m_i \left( \frac{\partial \bar{r}_i}{\partial t} \right)^2$$

$$M_j = \sum_i m_i \frac{\partial \bar{r}_i}{\partial t} \cdot \frac{\partial \bar{r}_i}{\partial q_j}$$

$$M_{jk} = \sum_i m_i \frac{\partial \bar{r}_i}{\partial q_j} \cdot \frac{\partial \bar{r}_i}{\partial q_k}$$

note that if there is no explicit 't' dependence on 'r', $M_0 = M_j = 0$

## simple examples

1. Single particle, Cartesian coordinates

$$\bar{r} = x\hat{x} + y\hat{y} + z\hat{z} = \sum_{s=1}^{3} x_s \hat{e}_s$$

$$M_0 = M_j = 0$$

$$\frac{\partial x_s}{\partial x_j} = \delta_{sj}$$

$$M_{jk} = m\, \delta_{sj}\, \delta_{s'k}\, \hat{e}_s \cdot \hat{e}_{s'}$$

$$= m\,\delta_{sj}\,\delta_{sk} = m\,\delta_{jk}$$

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

$$\frac{\partial T}{\partial x_s} = 0 \qquad \frac{\partial T}{\partial \dot{x}_s} = m\dot{x}_s \qquad \frac{d}{dt}\frac{\partial T}{\partial \dot{x}_s} = m\ddot{x}_s$$

$$Q_j = \bar{F}\cdot\frac{\partial \bar{r}}{\partial x_j} = \bar{F}\cdot\hat{e}_j = F_j$$

$$\boxed{m\ddot{x} = F_x \qquad m\ddot{y} = F_y \qquad m\ddot{z} = F_z}$$

2. Single particle in  two dimensions, polar coordinates

$$x = r\cos\theta \qquad \bar{r} = r\cos\theta\,\hat{x} + r\sin\theta\,\hat{y}$$

$$y = r\sin\theta \qquad \frac{\partial \bar{r}}{\partial r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$

$$\frac{\partial \bar{r}}{\partial \theta} = -r\sin\theta\,\hat{x} + r\cos\,\hat{y}$$

$$M_0 = M_j = 0 \qquad M_{rr} = m(\cos\theta^2 + \sin^2\theta) = m$$

$$M_{\theta\theta} = mr^2$$

$$M_{r\theta} = 0$$

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$$

$$\frac{\partial T}{\partial r} = m r \dot{\theta}^2 \qquad \frac{\partial T}{\partial \theta} = 0$$

$$\frac{\partial T}{\partial \dot{r}} = m \dot{r} \qquad \frac{\partial T}{\partial \dot{\theta}} = m r^2 \dot{\theta}$$

$$Q_r = \bar{F} \cdot \frac{\partial \bar{r}}{\partial r} = \bar{F} \cdot (\underbrace{\cos\theta \hat{x} + \sin\theta \hat{y}}_{\hat{r}}) = \bar{F} \cdot \hat{r}$$

$$= F_r$$

$$Q_\theta = \bar{F} \cdot \frac{\partial \bar{r}}{\partial \theta} = \bar{F} \cdot (\underbrace{-r\sin\theta \hat{x} + r\cos\theta \hat{y}}_{r\hat{\theta}})$$

$$= r F_\theta$$

---

$$r: \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = \boxed{m\ddot{r} - m r \dot{\theta}^2 = F_r}$$

centripetal acceleration

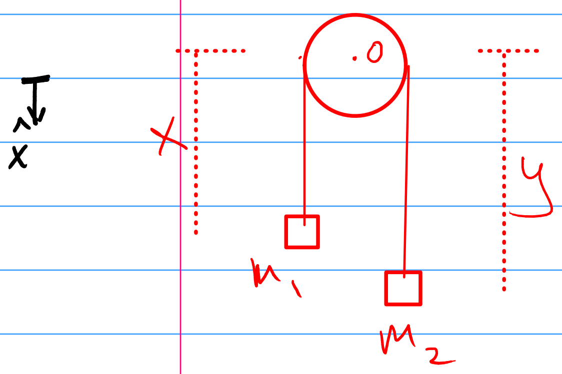
$$\theta: \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{d}{dt}(m r^2 \dot{\theta}) = r F_\theta$$

angular momentum

torque

$$\boxed{2 m \dot{r} r \dot{\theta} + m r^2 \ddot{\theta} = r F_\theta}$$

3. Atwood's machine: conservative system w/ holonomic scleronomous constraints




$$\bar{r}_1 = x\,\hat{x} \qquad \bar{r}_2 = y\,\hat{x}$$

constraint $y = l - x$

$$f(q) = y + x - l = 0$$

$$\bar{r}_2 = (l - x)\,\hat{x}$$

$$\frac{\partial \bar{r}_1}{\partial x} = \hat{x} \qquad \frac{\partial \bar{r}_2}{\partial x} = -\hat{x}$$

$$M_0 = M_j = 0 \qquad M_{xx} = m_1 + m_2$$

$$T = \frac{1}{2}(m_1 + m_2)\,\dot{x}^2$$

$$V = -m_1 g x - m_2 g (l - x)$$

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_1 g x + m_2 g (l - x)$$

$$\frac{\partial L}{\partial x} = (m_1 - m_2) g$$

$$\frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

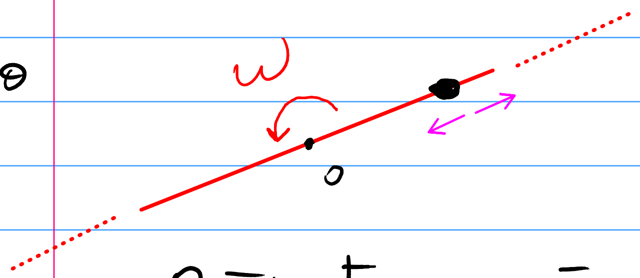
$$\boxed{(m_1+m_2)\ddot{x} = (m_1-m_2)g}$$

Note that the force of constraint (tension) does not appear anywhere in the eqns.

Thus, if you want the tension, you have to use other equations.

4. A bead sliding on a rotating straight wire (holonomic, rheonomous) in outer space

$r, \theta$




$$x = r\cos\theta = r\cos\omega t$$

$$y = r\sin\theta = r\sin\omega t$$

$$\theta = \omega t$$

$$\theta - \omega t = 0$$

$$\bar{r} = r\cos\omega t\,\hat{x} + r\sin\omega t\,\hat{y}$$

$$\frac{\partial \bar{r}}{\partial r} = \cos\omega t\,\hat{x} + \sin\omega t\,\hat{y}$$

$$\rightarrow \quad \frac{\partial \bar{r}}{\partial t} = -r\omega\sin\omega t\,\hat{x} + r\omega\cos\omega t\,\hat{y}$$

$$M_o = \frac{1}{2} m r^2 \omega^2 \qquad // r^2\omega^2(\sin^2 + \cos^2)$$

$$M_r = m(-r\omega\cos\omega t\sin\omega t + r\omega\sin\omega t\cos\omega t) = 0$$

$$M_{rr} = m$$

$$T = \frac{1}{2} m r^2 \omega^2 + \frac{1}{2} m \dot{r}^2 = \frac{1}{2} m(\dot{r}^2 + r^2\omega^2)$$

$Q_r = 0$ (no applied forces)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot r}\right) = m\ddot r \qquad \frac{\partial T}{\partial r} = m r \omega^2$$

$$m\ddot r - m r \omega^2 = 0$$

$$\boxed{\ddot r - r\omega^2 = 0} \qquad r = Ae^{\omega t}$$

ang mom $\frac{dL}{dt} = \frac{d}{dt}(m r^2 \omega) = rF$

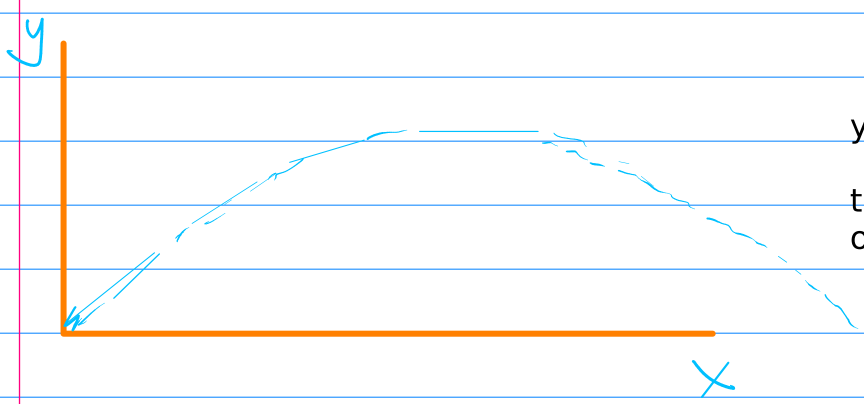
unknown constraint force

---

trajectory in coordinate space

$$I = \int L\,dt$$

Hamilton's principle: the particles will follow a path that extremizes that integral.




y and x are the coordinates.

time 't' parameterizes the values of y and x.

trajectory is the path in coordinate space

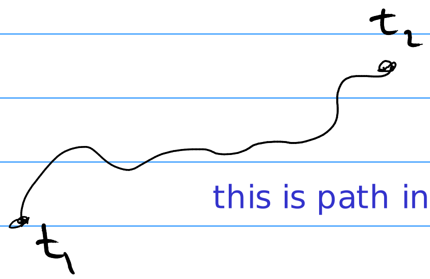
3 marbles rolling down an incline plane

x,y,z each -> 9 degrees of freedom

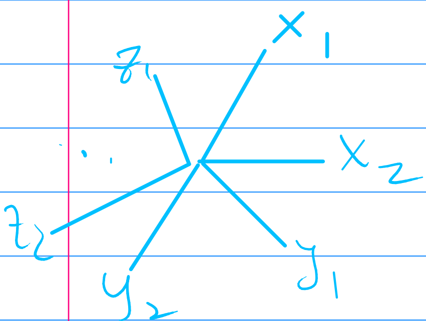
$$(x_1, y_1, z_1, x_2, y_2 \dots z_3)$$

$$\downarrow$$

$$(x_1', y_1' \dots z_3')$$




this is path in coordinate space




$$I = \int_{t_1}^{t_2} L \, dt$$

$$\delta I = \delta \int_{t_1}^{t} L(q_1 \dots q_n, \dot{q}_1 \dots \dot{q}_n, t) \, dt$$

$$= 0$$