Example   General 2D oscillator

$$T = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2) \qquad V = \frac{1}{2} V_{ij} x_i x_j$$

$$\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$$

$$T = \frac{1}{2} T_{ij} \dot{x}_i \dot{x}_j$$

$$[T_{ij}] = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

$$= \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$$

$$|V - \lambda T| = 0$$

$$0 = \begin{vmatrix} V_{11} - \lambda T_{11} & V_{12} - \lambda T_{12} \\ V_{21} - \lambda T_{21} & V_{22} - \lambda T_{22} \end{vmatrix} = \begin{vmatrix} V_{11} - \lambda m & V_{12} \\ V_{21} & V_{22} - \lambda m \end{vmatrix}$$

$$(V_{11} - \lambda m)(V_{22} - \lambda m) - V_{21} V_{12} = 0$$

$$\lambda^2 m - \lambda m (V_{11} + V_{22}) + V_{11} V_{22} - V_{12} V_{21} = 0$$

$$\lambda_{1,2} = \frac{1}{2m^2} \left[m(V_{11} + V_{22}) \pm \sqrt{\underbrace{m^2 (V_{11} + V_{22})^2 - 4m^2 (V_{11} V_{22} - V_{12} V_{21})}} \right]$$

$$m^2 (V_{11}^2 + V_{22}^2 + 2V_{11} V_{22} - 4 V_{11} V_{22} + 4 V_{12} V_{21}$$

$$m^2 ((V_{11} - V_{22})^2 + 4 V_{21} V_{12})$$

$$\boxed{\lambda_{1,2} = \frac{1}{2m} \left[(V_{11} + V_{22}) \pm \sqrt{(V_{11} - V_{22})^2 + 4 V_{12} V_{21}} \right]}$$

$*$ $V_{11} > V_{22} > 0$; $0 \neq V_{21} = V_{12} \ll (V_{11} - V_{22})$

define small parameter $\delta \equiv \dfrac{V_{12}}{V_{11} - V_{22}}$

$$\lambda_{1,2} = \frac{1}{2m}\left[(V_{11}+V_{22}) \pm (V_{11}-V_{22})\underbrace{\sqrt{1+\frac{4V_{12}^2}{(V_{11}-V_{22})^2}}}_{\sqrt{1+4\delta^2} \simeq 1+2\delta^2}\right]$$

$$= \frac{1}{2m}\left\{(V_{11}+V_{22}) \pm (V_{11}-V_{22}) \pm \underbrace{(V_{11}-V_{22})}_{V_{12}/\delta} 2\delta^2\right]$$

$$\boxed{\begin{aligned}\lambda_1 &= \frac{1}{2m}\left[2V_{11} + 2V_{12}\delta\right] = \frac{1}{m}(V_{11}+V_{12}\delta) \\ \lambda_2 &= \frac{1}{2m}\left\{2V_{22} - 2V_{12}\delta\right] = \frac{1}{m}(V_{22}-V_{12}\delta)\end{aligned}}$$

Eigenvectors for each eigenvalue

$$(V-\lambda T)a = 0$$

$$\begin{pmatrix} V_{11}-\lambda_1 m & V_{12} \\ V_{21} & V_{22}-\lambda_1 m \end{pmatrix}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = 0$$

a_{ik} — i: row, k: eigenvalue

$$a_1^T T a_1 = 1$$

$$\lambda_1 = \frac{1}{m}(V_{11}+V_{12}\delta) \qquad [V_{11}-(V_{11}+V_{12}\delta)]a_{11} + V_{12}a_{21} = 0$$

$$\boxed{-\delta a_{11} + a_{21} = 0}$$

$$a_1^T T a_1 = (a_{11}\ a_{21})\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = 1 \qquad \boxed{a_{11}^2 m + a_{21}^2 m = 1}$$

$$a_{11}^2 m + \delta^2 a_{11}^2 m = 1$$

$$a_{11}^2 = \frac{1}{m}\cdot\frac{1}{1+\delta^2} \simeq \frac{1}{m}(1-\delta^2) \Rightarrow a_{11} = \frac{1}{\sqrt{m}}\left(1-\frac{\delta^2}{2}\right)$$

$$a_{21} = \delta a_{11} = \frac{1}{\sqrt{m}}\left(\delta - \frac{1}{2}\delta^3\right)$$

$$\boxed{a_1 = \frac{1}{\sqrt{m}}\begin{pmatrix} 1-\delta^2/2 \\ \delta - \delta^3/2 \end{pmatrix}}$$

$$\lambda_2 = \frac{1}{m}\left(V_{22} - V_{12}\delta\right)$$

$$V_{21} a_{12} + \left[V_{22} - \left(V_{22} - V_{12}\delta\right)\right] a_{22} = 0$$

$$\boxed{a_{12} + \delta a_{22} = 0}$$

$$\boxed{a_{12}^2 m + a_{22}^2 m = 1}$$

$$\delta^2 a_{22}^2 m + a_{22}^2 m = 1$$

$$a_{22}^2 = \frac{1}{m} \cdot \frac{1}{1+\delta^2} \simeq \frac{1}{m}\left(1-\delta^2\right) \Rightarrow a_{22} = \frac{1}{\sqrt{m}}\left(1 - \frac{\delta^2}{2}\right)$$

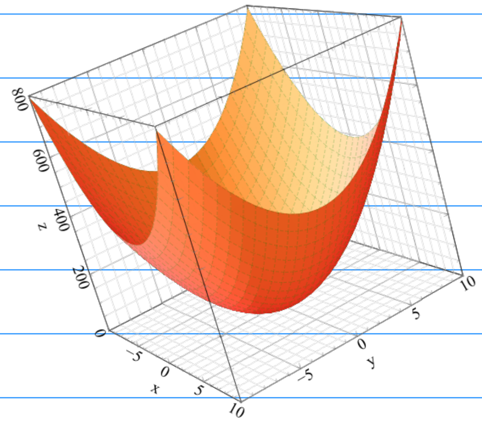
$$a_{12} = -\delta a_{22} = -\frac{1}{\sqrt{m}}\left(\delta - \delta^3/2\right)$$

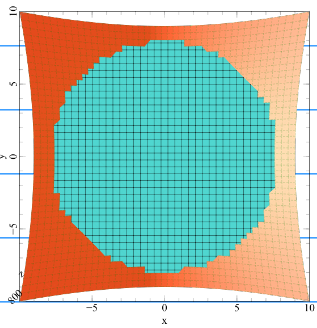
$$\boxed{a_2 = \frac{1}{\sqrt{m}}\begin{pmatrix} -\delta + \delta^3/2 \\ 1 - \delta^2/2 \end{pmatrix}}$$

Graph of potential

$V_{11} \simeq V_{22}$

V_{12} small





limiting vectors

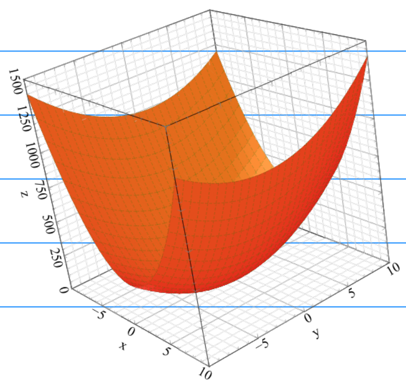
$\delta \to 0$

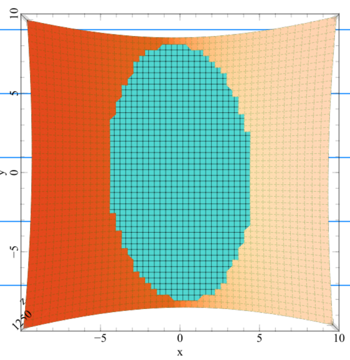
$a_1 \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$a_2 \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$V_{11} \gg V_{22}$

$V_{12} < V_{22}$





Can build a matrix 'A'

$$\boxed{A = \frac{1}{\sqrt{m}} \begin{pmatrix} 1-\delta^2/2 & -\delta+\delta^3/2 \\ \delta-\delta^3/2 & 1-\delta^2/2 \end{pmatrix}}$$

$$A^T T A = \frac{1}{m} \begin{pmatrix} 1-\delta^2/2 & \delta-\delta^3/2 \\ -\delta+\delta^3/2 & 1-\delta^2/2 \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} 1-\delta^2/2 & -\delta+\delta^3/2 \\ \delta-\delta^3/2 & 1-\delta^2/2 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\cancel{\delta^2}+\cancel{\delta^2} & \cdots \\ \cancel{-\delta}+\cancel{\frac{\delta^3}{2}}+\cancel{\frac{\delta^3}{2}}+\cancel{\delta}-\cancel{\frac{\delta^3}{2}}-\cancel{\frac{\delta^3}{2}} & \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^T V A = \frac{1}{m} \begin{pmatrix} \quad \end{pmatrix} \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} \quad \end{pmatrix}$$

using $V_{11} - V_{22} = V_{12}/\delta$

$$A^T V A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \lambda$$

* $V_{12} > V_{22} > 0$; $(V_{11} - V_{22}) \ll V_{12} = V_{21}$

define small parameter $\epsilon \equiv \dfrac{V_{11}-V_{22}}{8V_{12}}$

$$\lambda_{1,2} = \frac{1}{2m}\left[(V_{11}+V_{22}) \pm \underbrace{\sqrt{\underbrace{(V_{11}-V_{22})^2}_{64\epsilon^2 V_{12}^2} + 4V_{12}V_{21}}}_{2V_{12}\sqrt{1+16\epsilon^2}}\right]$$

$$\lambda_{1,2} = \frac{1}{m}\left[\frac{1}{2}(V_{11}+V_{22}) \pm V_{12}\,(1+\underbrace{8\epsilon^2}_{(V_{11}-V_{22})\epsilon/V_{12}})\right]$$

$$\boxed{\lambda_{1,2} = \frac{1}{m}\left[\frac{1}{2}(V_{11}+V_{22}) \pm (V_{12}+(V_{11}-V_{22})\epsilon)\right]}$$

$$\begin{pmatrix} V_{11}-\lambda_i m & V_{12} \\ V_{21} & V_{22}-\lambda_i m \end{pmatrix}\begin{pmatrix} a_{1i} \\ a_{2i} \end{pmatrix} = 0$$

$$a_i^{\dagger} T a_i = 1$$

$$\lambda_1 = \frac{1}{m}\left[\frac{1}{2}(V_{11}+V_{22}) + V_{12} + (V_{11}-V_{22})\epsilon\right]$$

$$\{\underbrace{V_{11}-[\tfrac{1}{2}(V_{11}+V_{22})+V_{12}+(V_{11}-V_{22})\epsilon]}_{\frac{1}{2}(V_{11}-V_{22})\;-V_{12}\;-(V_{11}-V_{22})\epsilon}\}a_{11}+V_{12}a_{21}=0$$

$$4V_{12}\epsilon \quad -V_{12} \quad -8\cancel{V_{12}}\,\epsilon^2$$

$$(4\epsilon-1)a_{11} + a_{21} = 0$$

$$\frac{1}{m} = a_{11}^2 + (4\epsilon-1)^2 a_{11}^2 \doteq a_{11}^2(2-8\epsilon)$$

$$a_{11} \simeq \frac{1}{\sqrt{2m}}(1+2\epsilon)$$

$$a_{21}^2 = \frac{1}{m} - a_{11}^2 = \frac{1}{m}\left[1-\underbrace{\tfrac{1}{2}(1+4\epsilon)}_{1/2\,(1-4\epsilon)}\right]$$

$$a_{21} = \frac{1}{\sqrt{2m}}(1-2\epsilon)$$

$$\boxed{a_1 = \frac{1}{\sqrt{2m}}\begin{pmatrix} 1+2\epsilon \\ 1-2\epsilon \end{pmatrix}}$$

$$\lambda_2 = \frac{1}{m}\left[\frac{1}{2}(V_{11}+V_{22}) - V_{12} - (V_{11}-V_{22})\epsilon\right]$$

$$V_{21}a_{12} + \Big\{\underbrace{V_{22} - \left[\frac{1}{2}(V_{11}+V_{22}) - V_{12} - (V_{11}-V_{22})\epsilon\right]}_{-4V_{12}\epsilon + V_{12} + 8V_{12}\epsilon^2}\Big\} a_{22} = 0$$

$$a_{12} + (1-4\epsilon)a_{22} = 0 \Rightarrow a_{12}^2 m + a_{22}^2 m = 1$$

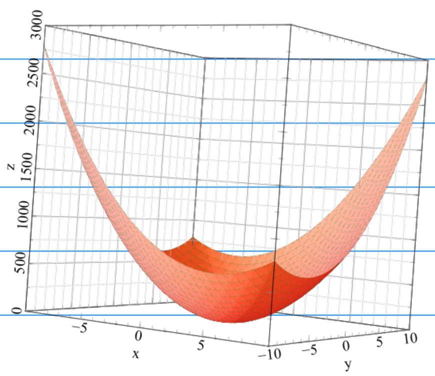
$$a_{22} = \frac{1}{\sqrt{2m}}(1+2\epsilon)$$

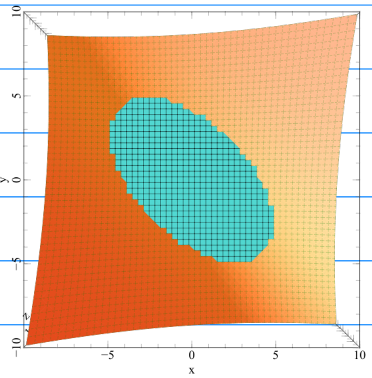
$$a_{12} = -(1-4\epsilon)a_{22}$$

$$a_{12} = -\frac{1}{\sqrt{2m}}(1-2\epsilon)$$

$$\boxed{a_2 = \frac{1}{\sqrt{2m}}\begin{pmatrix} -(1-2\epsilon) \\ 1+2\epsilon \end{pmatrix}}$$

Graph potential



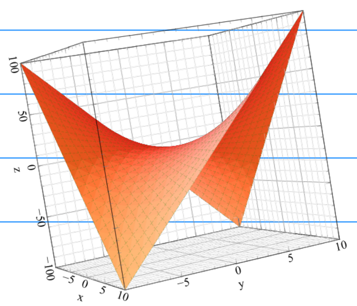


limit of $\epsilon \to 0$

$$a_1 \sim \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$a_2 \sim \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Purely off-diagonal



the point (0,0,0) is called a "saddle" point.

The characteristic eqn is derived using solutions like

$$\eta_i = c a_i e^{-i\omega t}$$

we actually obtain several omegas. $\lambda_k = \omega_k^2$

A general solution is a superposition of all these frequencies

$$\eta_i = C_k a_{ik} e^{-i\omega_k t}$$

C_k are complex scale factors for each resonant frequency, and these are determined from initial conditions.

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = c_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \end{pmatrix} e^{-i\omega_1 t} + C_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \end{pmatrix} e^{-i\omega_2 t}$$

$$+ \cdots$$