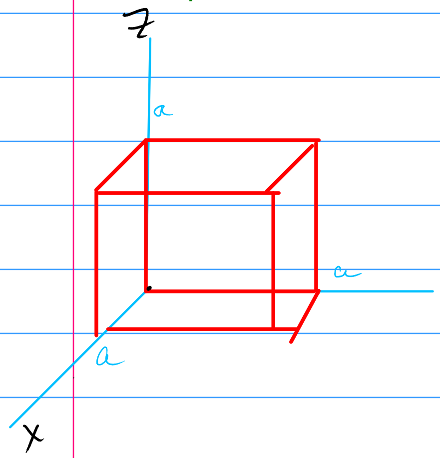
Example, uniform solid cube

$$\rho = \frac{M}{a^3}$$



We want the moment of inertia tensor around the fixed point $\boxed{(a/2, a/2, a/2)}$

$$I_{ij} = \int \rho \left[r^2 \delta_{ij} - x_i x_j \right] dV$$

$$I_{xx} = \iiint_{-a/2}^{a/2} dx\,dy\,dz\, \rho\left[(x^2+y^2+z^2) - x^2\right]$$

$$= \rho a \iint_{-a/2}^{a/2} dy\,dz\,(y^2+z^2)$$

$$= 2\rho a^2 \cdot \frac{1}{3} \cdot \frac{a^3}{4} = \frac{1}{6} M a^2$$

$$I_{yy} = \iiint_{-a/2}^{a/2} dx\,dy\,dz\left[x^2+y^2+z^2-y^2\right]\rho = \iint_{-a/2}^{a/2} dx\,dz\,(x^2+z^2)\rho a$$

$$= \frac{1}{6} M a^2 = I_{zz}$$

off diagonal terms,

odd function

$$I_{ij} = -\rho \iiint dx\,dy\,dz\,(x_i x_j) = 0$$

$$\boxed{I = \frac{1}{6} M a^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

Moment of inertia around x-axis, through the center.

$$\hat{n} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$I_s^x = \hat{n} \cdot I \cdot \hat{n} = \frac{1}{6} M a^2 (1\ 0\ 0) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{6} M a^2$$

$\hat{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\hat{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$I_s^y = I_s^z = I_s^x$$

$$\bar{\omega} = \omega \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\bar{L} = I \bar{\omega} = \frac{1}{6} M a^2 \omega \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{6} M a^2 \bar{\omega}$$

ang momentum points in the same direction as omega

Moment of inertia tensor around the fixed point $\boxed{(0,0,0)}$

$$I_{ij} = \int \rho \left(r^2 \delta_{ij} - x_i x_j \right) dV$$

$$I_{xx} = \rho \iiint_0^a dx\,dy\,dz \left(y^2 + z^2 \right) = \rho a^2 \cdot 2 \cdot \frac{a^3}{3} = \frac{2}{3} M a^2$$

$$= I_{yy} = I_{zz}$$

$$I_{xy} = -\rho \iiint_0^a dx\,dy\,dz \,(xy) = -\rho a \frac{a^4}{4} = -\frac{1}{4} M a^2 = I_{yx}$$

$$= I_{ij} \quad (i \neq j)$$

$$\boxed{I = M a^2 \begin{pmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix}}$$

Moment of inertia around x-axis, through the corner

$$I_s^x = \hat{n} \cdot I \cdot \hat{n} = M a^2 (1\ 0\ 0) \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

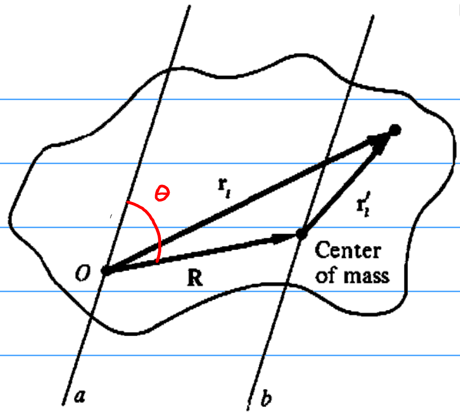
$$= M a^2 (1\ 0\ 0) \begin{pmatrix} 2/3 \\ -1/4 \\ -1/4 \end{pmatrix} = \frac{2}{3} M a^2$$

$$\bar{L} = I \bar{\omega} = \begin{pmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix} \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} = M a^2 \omega \begin{pmatrix} 2/3 \\ -1/4 \\ -1/4 \end{pmatrix}$$

ang mom does not point in the direction of omega

note: if we rotate on the diagonal (1,1,1), L will point in the direction of omega

Parallel axis theorem



Lets relate moments of inertia between parallel axes of rotation, one of which is through the CM

$$I_s^a = m_i(\bar{r}_i \times \hat{n})^2$$

$$\bar{r}_i = \bar{R} + \bar{r}_i'$$

$$I_s^a = m_i\left[(\bar{R} + \bar{r}_i') \times \hat{n}\right]^2$$

$$= \sum_i m_i(\bar{R} \times \hat{n})^2 + \overbrace{m_i(\bar{r}_i' \times \hat{n})^2}^{I_s^b} + 2m_i(\bar{R} \times \hat{n}) \cdot (\bar{r}_i' \times \hat{n})$$

$$-2(\bar{R} \times \hat{n}) \cdot (\hat{n} \times \underbrace{m_i \bar{r}_i'}_{=0})$$

$$I_s^a = M(\bar{R} \times \hat{n})^2 + I_s^b$$

$|\bar{R} \times \hat{n}| = R\sin\theta$ perp distance between axes 'a' and 'b'

Moment of inertia about a given axis is the moment of inertia about a parallel axis through the CM plus the moment of inertia of the body, as if concentrated at the CM, with respect to the original axis.

Principal Axes of Inertia

For the cases that L and omega are parallel we this axis a "principal axis".

$\bar{a} = \lambda \bar{b}$ lambda is a real number

$\bar{L} = \lambda \bar{\omega}$ lambdas are related to moments of inertia

If we find axes through which I is diagonal, these axes will be principal axes.

If we have principal axes, then the matrix of I will be diagonal.

There always exist 3 perpendicular axes through a point O for any rigid body at which I is diagonal, and L points along omega, when omega points in the direction of a principal axis.

The proof lies in the fact that we can always find a transformation of a real symmetric matrix into a diagonal.

$$\bar{L} = I\bar{\omega} \qquad L_1 = I_1\omega_1$$
$$L_2 = I_2\omega_2$$
$$L_3 = I_3\omega_3$$

$$T = \frac{1}{2}\bar{\omega}\cdot I\cdot\bar{\omega} = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2$$

Finding the Principal Axes

We want $\bar{L} = I\bar{\omega} = \lambda\bar{\omega} \Rightarrow \boxed{(I - \lambda)\bar{\omega} = 0}$

We just need to solve this eigenvalue equation.
There is a non-zero solution only if the determinant of the matrix on the lhs is zero.

characteristic equation $|I - \lambda| = 0$

roots of this cubic equation will be the "principal moments".
We can solve for the principal axes by using the eqn above

Example: cube wrt point (0,0,0). Find the principal moments and axes.

$$I = Ma^2\begin{pmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix} = \frac{1}{12}Ma^2\begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}$$

$c = \frac{1}{12}Ma^2$

$$\begin{vmatrix} 8c-\lambda & -3c & -3c \\ -3c & 8c-\lambda & -3c \\ -3c & -3c & 8c-\lambda \end{vmatrix} = 0$$

$$(8c-\lambda)\left[(8c-\lambda)^2 - 9c^2\right] + 3c\left[-3c(8c-\lambda) - 9c^2\right] - 3c\left[9c^2 + 3c(8c-\lambda)\right] = 0$$

$$(2c-\lambda)(11c-\lambda)^2 = 0$$

$$\boxed{\lambda_1 = 2c} \qquad \boxed{\lambda_2, \lambda_3 = 11c}$$

$(I - \lambda)\bar{w} = 0$

λ_1:

$$c\begin{pmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = 0$$

$$\left.\begin{array}{l} 6w_1 - 3w_2 - 3w_3 = 0 \\ -3w_1 + 6w_2 - 3w_3 = 0 \end{array}\right\} w_1 = w_2$$

$$\left.\begin{array}{l} -3w_1 + 6w_2 - 3w_3 = 0 \\ -3w_1 - 3w_2 + 6w_3 = 0 \end{array}\right\} w_3 = w_1$$

$w_1 = w_2 = w_3$

eigenvector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\boxed{\hat{e}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$

$\bar{L} = I\bar{w} = \lambda_1 \bar{w}$

λ_2, λ_3:

$$c\begin{pmatrix} -3 & -3 & -3 \\ -3 & -3 & -3 \\ -3 & -3 & -3 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = 0 \Rightarrow w_1 + w_2 + w_3 = 0$$

$\boxed{\bar{w} \cdot \hat{e}_1 = 0}$

$\boxed{\hat{e}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}}$

$\boxed{\hat{e}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}$ $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$

The similarity transformation that will transform the coordinate system from the original to the final (in which the axes are principal axes) is given by:

$$I' = S^{-1} I S \qquad S = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & 0 \end{pmatrix}$$

$$I' = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$