Applications of Fourier Transforms

1. Solutions of linear differential equations
2. Quantum mechanics
   - energy and time $(e^{-iEt/\hbar})$
   - position and momentum $(e^{ipx/\hbar})$
   - field theory: Feynmann diagrams
3. Signal processing /analysis
   - Periodic signals and chaotic systems
   - Power spectra $[P(\omega) = |F(\omega)|^2]$
   - Numerics: Discrete and Fast Fourier Transforms
4. Deconvolution/noise reduction

Fourier Transforms

- Define Fourier transform and its inverse transform:

$$F(\omega) = \int_{-\infty}^{\infty} dt \;\, f(t)\, e^{-i\omega t}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \;\, F(\omega)\, e^{i\omega t}$$

- Problem: often we can't do the integrals!
- Commonly find integrals of the form

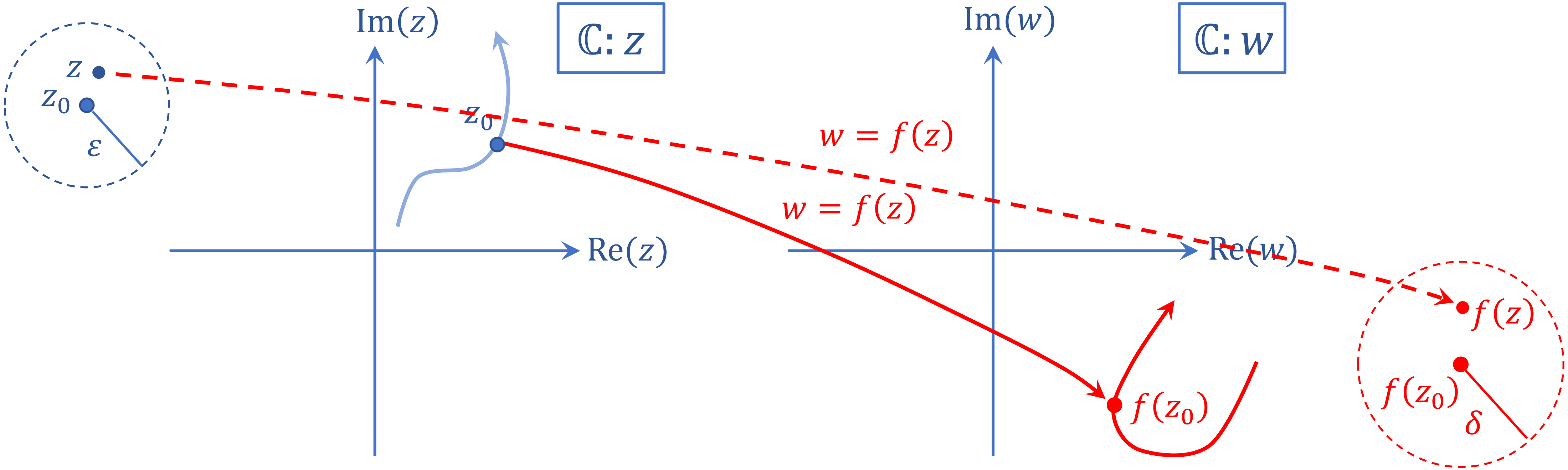
$$\mathrm{I} = \int_C dz \frac{F(z)}{P(z)}, \;\; F \text{ regular (harmless)}, \;\; P \text{ polynomial}$$

- Currently, path C is just (part of) the real axis, but will see others.
- Need a more general method to evaluate such integrals.
- Study complex functions in more detail.

Continuous Functions

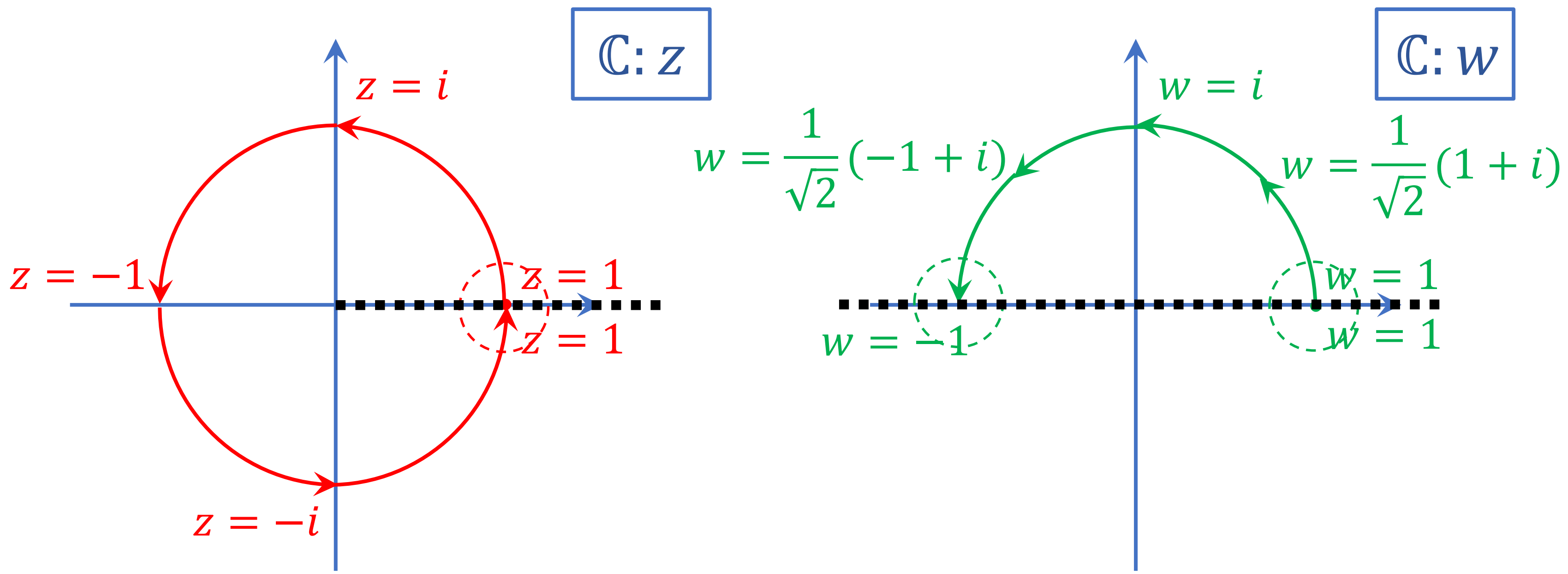
- Expect w to move "smoothly" along the image curve as z moves along its curve.
- Say f is <u>continuous</u> at z_0 if all points "close" to z_0 map to points "close" to $f(z_0)$.
- Formally, can say $f(z)$ is continuous at $z = z_0$ if

  for any $\delta > 0$ there exists $\varepsilon > 0$ such that $|f(z) - f(z_0)| < \delta$ if $|z - z_0| < \varepsilon$



Continuous Functions

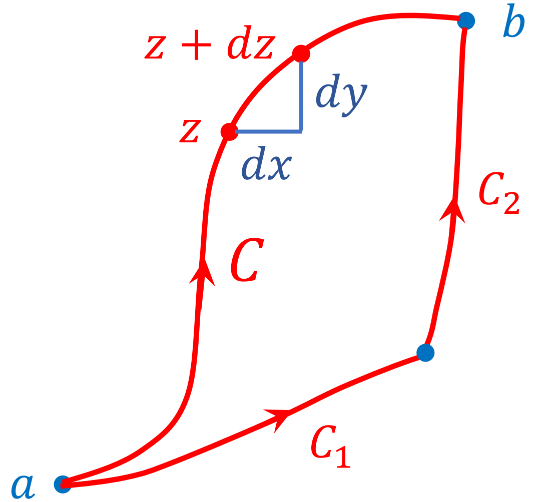
- Some simple functions are not continuous – consider $f(z) = z^{1/2}$ as z traverses a unit circle around the origin: $z = e^{i\theta}, \quad w = z^{1/2} = e^{i\theta/2}$
- Fix by introducing branch cuts – prosaic but works!
    - path not allowed to cross a branch cut (but can run along either side)
    - function is continuous so long as we obey this rule



Contour Integration

- From here on, assume that functions are continuous, maybe with suitable cuts.
- Steal knowledge of integration from calculus on the real plane $\mathbb{R}^2$.
- End points of the integral are no longer sufficient.
- Integration path is a <u>contour</u> in the complex plane.
- Suppose $z = x + iy,\ f(z) = u(x, y) + iv(x, y)$, where $z,\ x$ and y are constrained to lie on the contour
- Then $dz = dx + idy$ and we can <u>define</u> the integral as

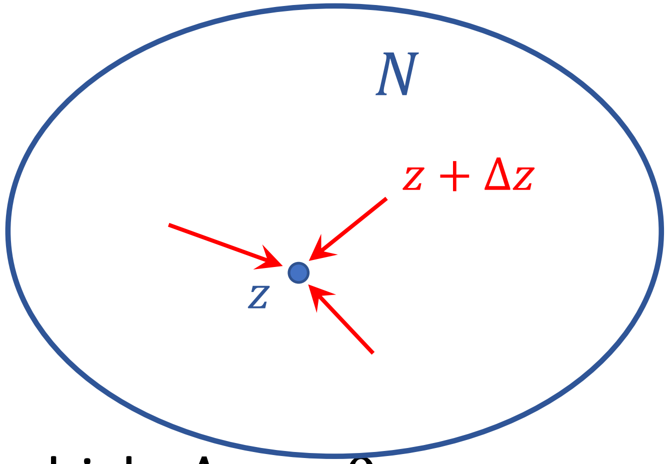
$$\int_C f(z)\,dz = \int_C (u + iv)(dx + idy)$$
$$= \int_C (udx - vdy) + i\int_C (vdx + udy)$$



- Everything you know about real integrals works for complex contour integrals…

$$\int_C (f + g) = \int_C f + \int_C g,\quad \int_C kf = k\int_C f,\quad \int_{C_1+C_2} f = \int_{C_1} f + \int_{C_2} f$$

Differentiation



- Derivative is defined exactly as in elementary calculus

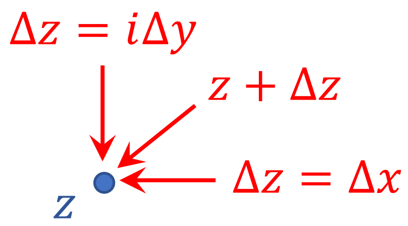
$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

but must be well-defined: <u>independent</u> of the path along which $\Delta z \to 0$.

- If $f'(z)$ exists in some finite neighborhood N around a point z, then the function $f(z)$ is <u>analytic</u> at z (= regular = holomorphic = …).
- Quite restrictive
  - existence of a derivative in N means that $f'(z)$ is continuous
  - in fact, analytic f is continuously differentiable
- Most of our favorite functions are in fact analytic:

  z^2, z^n (integer n), $\sin z$, e^z, …
- Good news: Rules for differentiation are what we learned in high school.

Differentiation



- Consider

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

for 2 particular paths with $\Delta z \to 0$: (1) $\Delta z = \Delta x$ and (2) $\Delta z = i\Delta y$

- Write

$$f(z) = u(x, y) + iv(x, y)$$

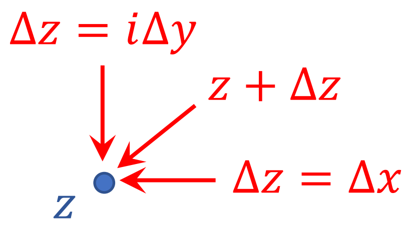
Then limit 1 is

$$\lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - u(x, y) - iv(x, y)}{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

and limit 2 is

$$\lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + iv(x, y+\Delta y) - u(x, y) - iv(x, y)}{i\Delta y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Differentiation



- Hence

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \qquad (1)$$

$$= \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y} \qquad (2)$$

- Consistency requires

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Cauchy-Riemann conditions

- Note that

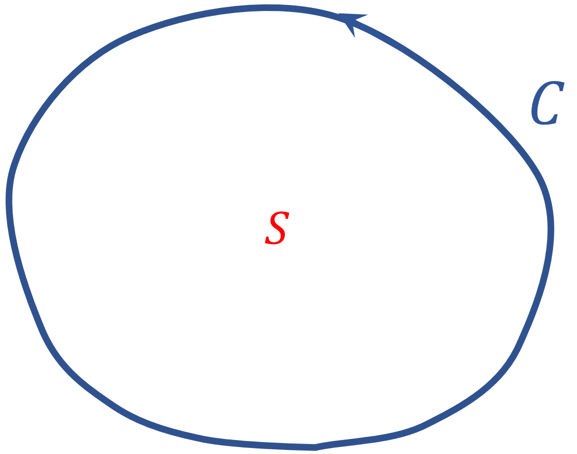
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2} \implies \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \implies \nabla^2 u = 0$$

and similarly

$$\nabla^2 v = 0$$

Not just any old u and v will do!

Cauchy's Theorem



- Cauchy-Riemann

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

- Consider the integral of $f = u + iv$ around a <u>closed</u> contour C:

$$\oint_C f(z)dz = \oint_C (udx - vdy) + i\oint_C (vdx + udy)$$

$$= \iint_S \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy$$

$$+ i\iint_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy$$

$= 0$, by Cauchy-Riemann

Stokes's theorem:

$$\oint_c \boldsymbol{W}.\boldsymbol{dx} = \iint_S \nabla\times\boldsymbol{W}.\boldsymbol{dS}$$

$\Longrightarrow$ Green's theorem (2-D):

$$\oint_c (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

- <u>Cauchy's theorem</u>: the integral of <u>any</u> analytic function around a closed contour is zero.

Cauchy's Theorem

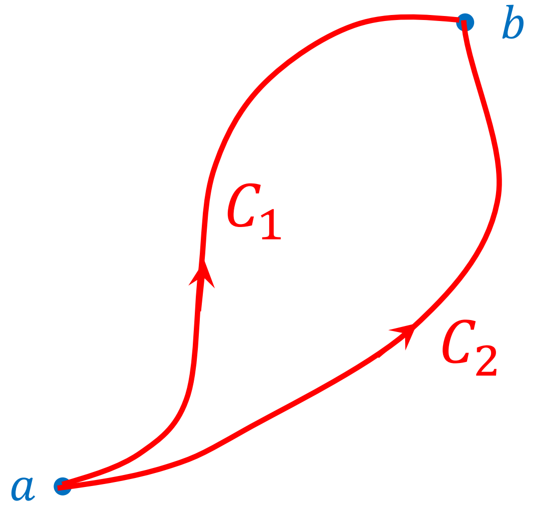
- Important corollary:

  The integral of an analytic function between two points a and b is <u>independent</u> of the contour joining the points.

  Proof: $C_1 - C_2$ is a closed curve, so

  $$\int_{C_1 - C_2} f = \int_{C_1} f - \int_{C_2} f = 0$$

  $$\Longrightarrow \quad \int_{C_1} f = \int_{C_2} f$$



- NB close connection with potential theory
- Means we can <u>deform</u> contours as we see fit, so long as the function remains analytic in the domain of interest.

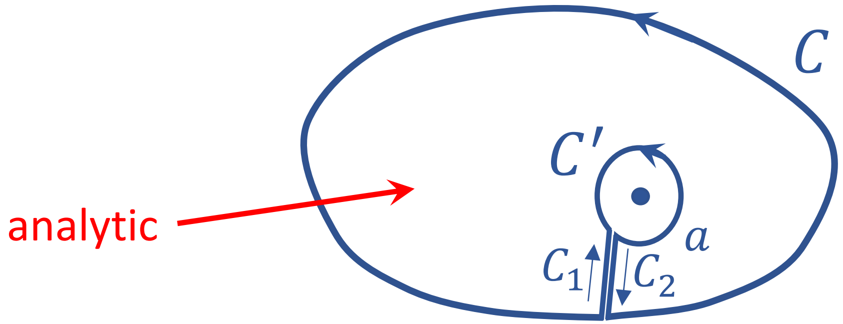
Also means that $F(z) = \int_a^z f(z)dz$ is well defined, and $F'(z) = f(z)$.

Cauchy's Integral Formula

- Now consider

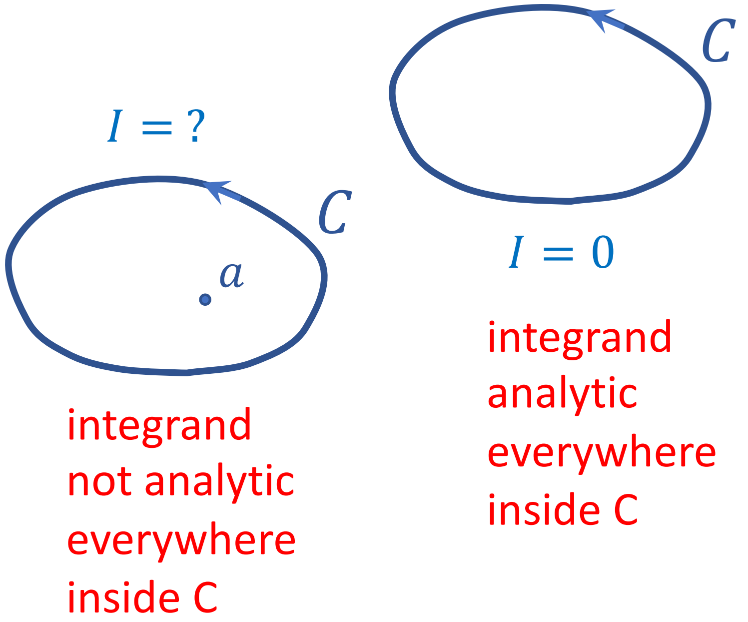
$$I = \oint_C \frac{dz}{z-a}$$

- If a lies inside C, use Cauchy's theorem to deform C to C', a circle centered on a.



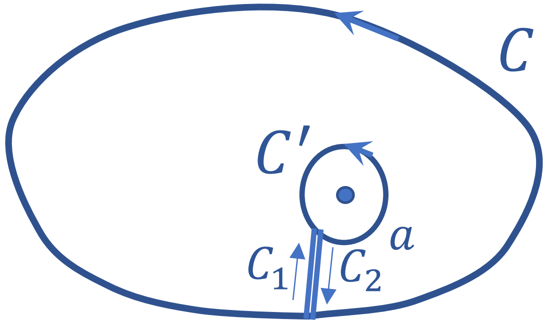
- Cauchy $\Longrightarrow \oint_{C+C_1-C'+C_2} \frac{dz}{z-a} = 0$

$$\Longrightarrow \mathrm{I} = \oint_C \frac{dz}{z-a} = \oint_{C'} \frac{dz}{z-a}$$



Idea: Construct a closed contour with the integrand analytic everywhere inside by connecting C and C' with 2 equal and opposite contours C_1 and C_2. Net effect: go around C counterclockwise, C' clockwise, C_1 and C_2 cancel.

Cauchy's Integral Formula



- The point: We can do the integral!

$$I = \oint_C \frac{dz}{z-a} = \oint_{C'} \frac{dz}{z-a}$$

- On C' let $\zeta = z - a = re^{i\theta}$, $r =$ constant

  Then $dz = re^{i\theta} i d\theta$, so

$$I = \oint_{C'} \frac{ire^{i\theta}\, d\theta}{re^{i\theta}}$$

$$= \oint_{C'} i d\theta$$

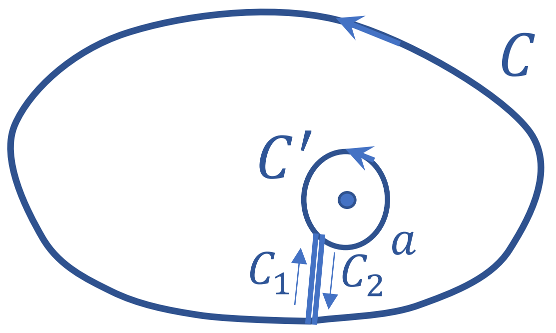
$$= 2\pi i$$

- General statement:

$$I = \oint_C \frac{dz}{z-a} = \begin{cases} 2\pi i, \text{ if } a \text{ lies within } C \\ 0, \text{ otherwise} \end{cases}$$

Cauchy's Integral Formula

$$I = \oint_C \frac{dz}{z-a} = \begin{cases} 2\pi i, & \text{if } a \text{ lies within } C \\ 0, & \text{otherwise} \end{cases}$$



- Now let

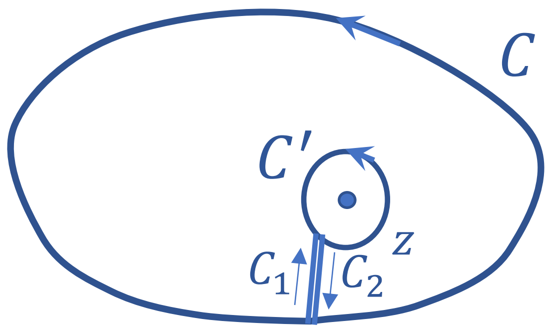
$$I_n = \oint_{C'} \frac{dz}{(z-a)^n}$$

$$= \oint_{C'} \frac{ire^{i\theta}\, d\theta}{r^n e^{in\theta}}$$

$$= ir^{1-n} \oint_{C'} e^{i(1-n)\theta}\, d\theta$$

$$= 0, \quad n \neq 1$$

Cauchy's Integral Formula



- The result generalizes further:

$$I = \oint_C \frac{f(\zeta)\,d\zeta}{\zeta - z}$$
$$= 2\pi i\, f(z)$$

- Can write

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

Cauchy Integral Formula

$$\Longrightarrow \quad \frac{d^n f(z)}{dz^n} = \frac{n!}{2\pi i}\oint_C \frac{f(\zeta)\,d\zeta}{(\zeta - z)^{n+1}}$$

- If f is defined on C, it is defined at <u>every</u> point inside C.
- Close connection to boundary problems for Laplace's equation.

Taylor Series

- Power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

- If the series converges for some $z = z_0$, then it is absolutely convergent within a circle of radius $R = |z_0 - a|$ around a and $f(z)$ is analytic inside the circle.
- Any analytic function near $z = a$ can be expanded as a unique power series in $z - a$.

  Taylor series

- Can prove using Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)\, d\zeta}{\zeta - z}$$

Taylor Series

- Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

$\zeta - z = (\zeta - a) - (z - a)$

$|z - a| < |\zeta - a|$

$$= \frac{1}{2\pi i}\oint_C d\zeta\, f(\zeta)\, \frac{1}{\zeta - a}\,\frac{1}{1 - \frac{z-a}{\zeta - a}}$$

$$= \frac{1}{2\pi i}\oint_C d\zeta\, f(\zeta)\, \frac{1}{\zeta - a}\sum_{n=0}^{\infty}\left(\frac{z-a}{\zeta - a}\right)^n$$

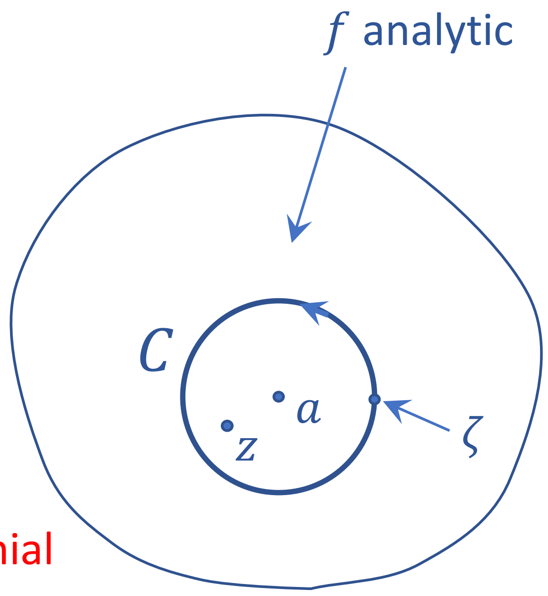
binomial theorem

$$= \frac{1}{2\pi i}\oint_C d\zeta \sum_{n=0}^{\infty}\frac{f(\zeta)(z-a)^n}{(\zeta - a)^{n+1}}$$

$$= \sum_{n=0}^{\infty}(z - a)^n\,\frac{1}{2\pi i}\oint_C d\zeta\,\frac{f(\zeta)}{(\zeta - a)^{n+1}}$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(a)(z - a)^n$$

Taylor series



Singularities

- Any convergent power series is an analytic function inside the radius of convergence.
- Any analytic function can be represented by a power (Taylor) series.
- Any point where the function is not analytic is a singularity.
- If a power series diverges at $|z-a|=R$, it means that the function has a singularity somewhere on that circle.

  e.g. $f(z)=(1+3z)^{-1}=1-3z+9z^2-27z^3\ldots+(-1)^n(3z)^n\ldots$

  radius of convergence of the series is $R=1/3$

  singularity is at $z=-\frac{1}{3}$, but the function is actually analytic everywhere except at $z=-\frac{1}{3}$
- Taylor series representation has broken down, but other Traylor expansions may exist, relative to some other point.

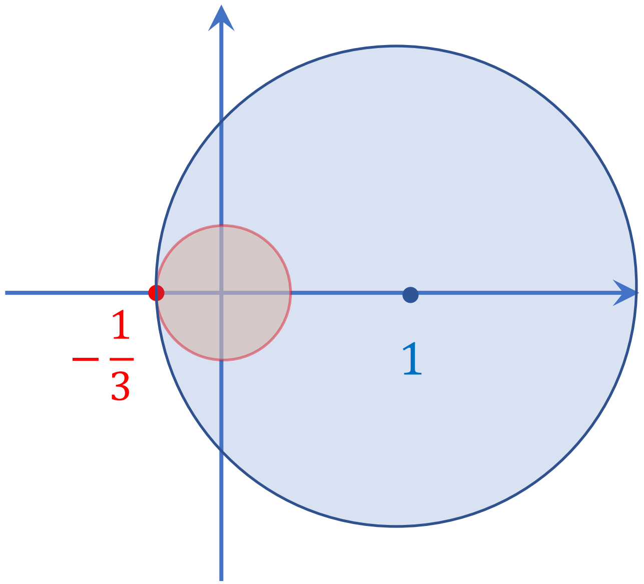
Singularities

- Example: $f(z) = \frac{1}{1+3z} = \frac{1}{4+3(z-1)} = \frac{\frac{1}{4}}{1+\frac{3}{4}s}$, where $s = z - 1$
- Expand in s (i.e. as a Taylor series about $z = 1$):

$$f(z) = \frac{1}{4}\left[1 - \frac{3}{4}s + \left(\frac{3}{4}s\right)^2 - \left(\frac{3}{4}s\right)^3 + \cdots\right]$$

- New series converges for $\left|\frac{3}{4}s\right| < 1$, or $|s| < \frac{4}{3}$
- Diverges at $|s| = \frac{4}{3}$, which corresponds to $z = -\frac{1}{3}$.
- New expansion agrees with the old one where they overlap, but extends the definition to larger domain

  Analytic continuation

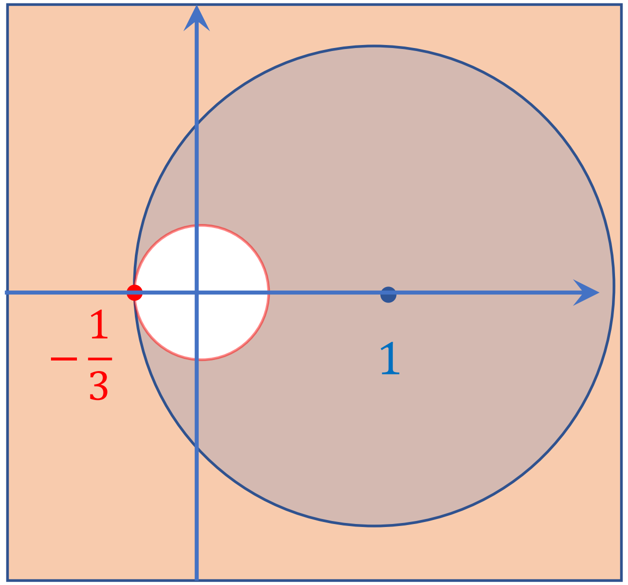


- Any two analytic functions that agree in some region represent the <u>same</u> analytic function in all continuations.

Singularities

- In our example: $f(z) = \frac{1}{1+3z}$ has a singularity at $z = -\frac{1}{3}$, which limits the convergence of the Taylor series.
- But it is analytic for $|z| > \frac{1}{3}$, and can be expanded using the binomial theorem:

$$(1+3z)^{-1} = \frac{1}{3z}\left(1+\frac{1}{3z}\right)^{-1}$$
$$= \frac{1}{3z}\left[1 - \frac{1}{3z} + \left(\frac{1}{3z}\right)^2 - \left(\frac{1}{3z}\right)^3 \ldots\right]$$



- Series of entirely <u>negative</u> powers of z, convergent for $|z| > \frac{1}{3}$.
- New domain overlaps with the expansion about $z = 1$.
- Analytic continuation to the <u>entire</u> complex plane.

Function Analytic in an Annulus

- Consider

$$f(z) = \frac{1}{(z-i)(z+2)}, \quad \text{singularities at } z = i, -2$$

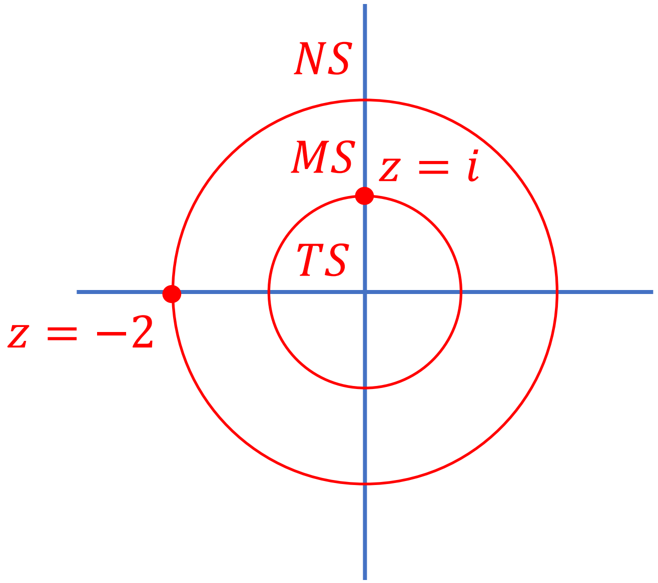
  - exactly the sort of function we saw in our 2nd order ODE solution!

- Expand about $z = 0$
  - expect Taylor series for $|z| < 1$
  - expect series in negative powers of z for $|z| > 2$
  - expect <u>mixed</u> series for $1 < |z| < 2$

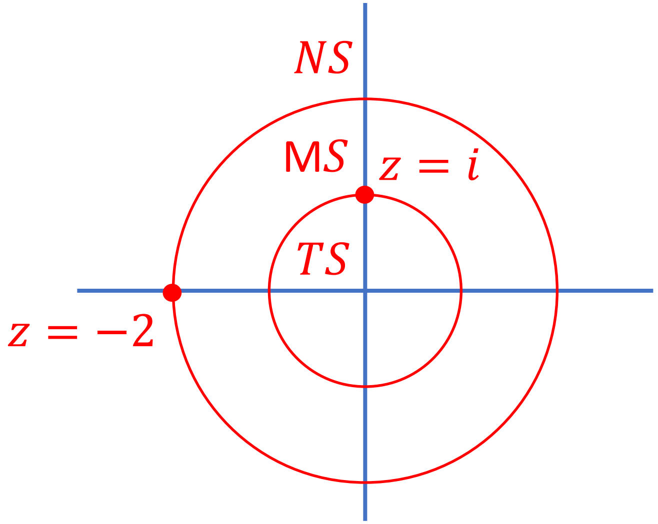
- Explore each by decomposing $f(z)$

$$f(z) = \frac{A}{z-i} + \frac{B}{z+2}$$

(where $A = i + 2, \; B = 2 - i$)



Example



$$f(z) = \frac{A}{z-i} + \frac{B}{z+2}$$

- Expand:

$$(z-i)^{-1} = i(1+iz)^{-1} = i[1 - iz + (iz)^2 - (iz)^3 \ldots] \text{ for } |z| < 1$$

$$(z+2)^{-1} = \frac{1}{2}\left(1+\frac{z}{2}\right)^{-1} = \frac{1}{2}\left[1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^3 \ldots\right] \text{ for } |z| < 2$$

- Outside the radii of convergence, rewrite

$$(z-i)^{-1} = \frac{1}{z}\left(1-\frac{i}{z}\right)^{-1} = \frac{1}{z}\left[1 + \frac{i}{z} + \left(\frac{i}{z}\right)^2 + \left(\frac{i}{z}\right)^3 \ldots\right] \text{ for } |z| > 1$$

$$(z+2)^{-1} = \frac{1}{z}\left(1+\frac{2}{z}\right)^{-1} = \frac{1}{z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 \ldots\right] \text{ for } |z| > 2$$

- pure TS for $|z| < 1$, MS for $1 < |z| < 2$, NS for $|z| > 2$

Laurent Series

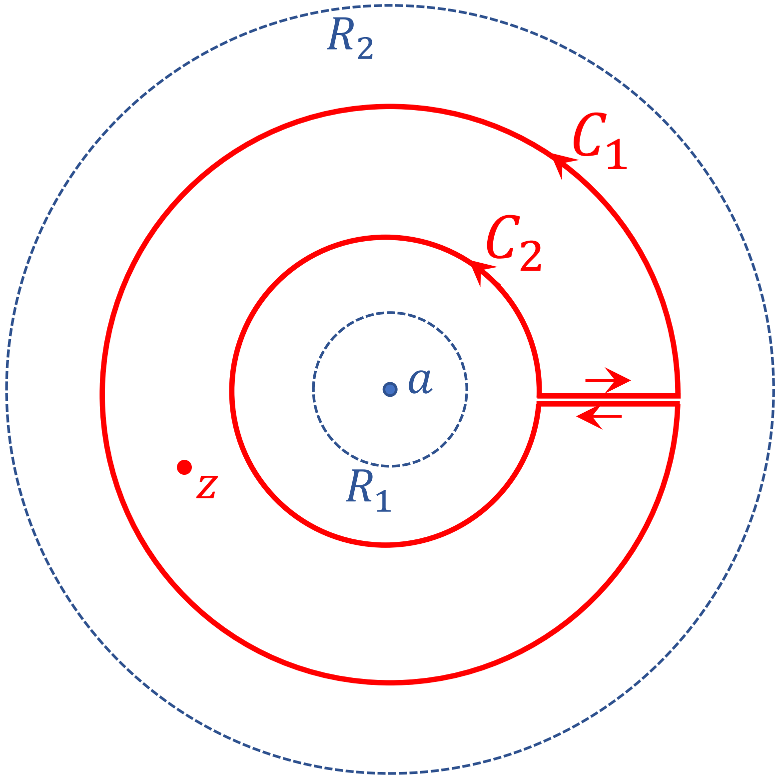
- Power series with positive powers of $z - a$ converges inside some circle.
- Power series with negative powers of $z - a$ converges outside some circle.
- In general, a series with both positive and negative powers of $z - a$

  $$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

  will converge in an annulus: $R_1 < |z - a| < R_2$
- In fact, <u>any</u> function analytic in an annulus can be expanded as a series with both positive and negative powers of $z - a$ — <u>Laurent series</u>
- Positive powers in the Laurent series are called the <u>regular</u> part
- Negative powers in the Laurent series are called the <u>principal</u> part

Laurent Series

- Function $f(z)$ analytic in an annulus: expect Laurent series expansion about $z = a$ to converge for $R_1 < |z - a| < R_2$ (must be singularities on each circle…)
- Construct contours C_1 and C_2 inside the annulus, surrounding a.
- Can't apply Cauchy's theorem or CIF to either contour since the function is not analytic inside $|z - a| = R_1$.
- Can add connectors and apply the theorem to $C' = C_1 - C_2$ (i.e. C_1 counterclockwise, C_2 clockwise, connectors cancel).
- Now the function is analytic everywhere inside and on C'!



Laurent Series

- Now apply the CIF on the contour C':

$$f(z) = \frac{1}{2\pi i}\oint_{C'} \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

$$= \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)\,d\zeta}{\zeta - z} - \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

- Write $\zeta - z = (\zeta - a) - (z - a)$, so on C_1

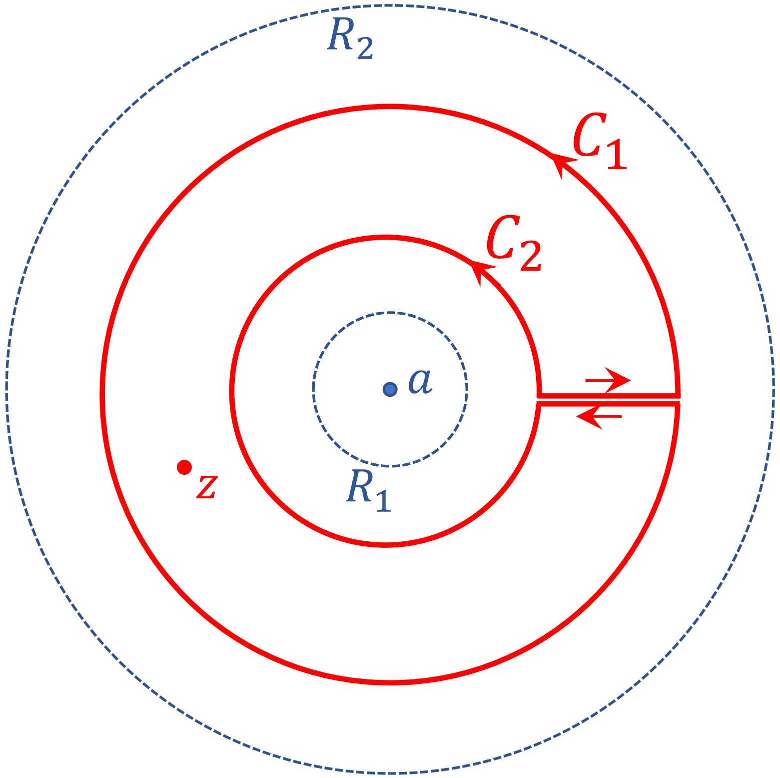
$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a}\left(1 - \frac{z-a}{\zeta - a}\right)^{-1} \quad |\cdots| < 1$$

$$= \frac{1}{\zeta - a}\sum_{n=0}^{\infty}\left(\frac{z-a}{\zeta - a}\right)^{n}$$

- On C_2,

$$\frac{1}{\zeta - z} = \frac{-1}{z - a}\left(1 - \frac{\zeta - a}{z - a}\right)^{-1} = \frac{-1}{z-a}\sum_{n=0}^{\infty}\left(\frac{\zeta - a}{z - a}\right)^{n}$$

$$|\cdots| < 1$$



Laurent Series

- Bring the results together

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-a)^n \oint_{C_1}\frac{f(\zeta)\,d\zeta}{(\zeta-a)^{n+1}}$$

$$+\frac{1}{2\pi i}\sum_{n=1}^{\infty}(z-a)^{-n}\oint_{C_2}(\zeta-a)^{n-1}f(\zeta)\,d\zeta$$

$$= \sum_{m=-\infty}^{\infty} a_m(z-a)^m$$

where

$$a_m = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)\,d\zeta}{(\zeta-a)^{m+1}} \quad \text{for } \underline{\text{all}}\ m$$

<u>any</u> C in the annulus enclosing a

<u>Laurent series</u> for $f(z)$

- Statement of the result is much more important than the proof.