PHYS 501: Mathematical Physics I

Fall 2020, Homework #4

(Due November 9, 2020)

1. (a) Use the Legendre polynomial generating function

$$\left(1 - 2zh + h^2\right)^{-1/2} = \sum_{n=0}^{\infty} P_n(z) h^n,$$

together with the binomial theorem, to derive expressions for the following quantities: (i) $P_n(1)$, (ii) $P_n(0)$, (iii) $P_n'(1)$, (iv) $P_n'(0)$.

(b) Use Stirling's formula

$$n! = \sqrt{2\pi n}\, n^n e^{-n} \left[1 + \mathcal{O}(1/n)\right]$$

to simplify the expression for $P_n(0)$ for even n as $n \to \infty$.

(c) Evaluate the integral $\int_{-1}^{1} P_n(x)dx$.

(d) Use a recurrence relation to evaluate the integral $\int_0^1 P_n(x)dx$.

2. A conducting sphere of radius a is surrounded by charged spherical shell of radius b. The inner sphere is held at potential Φ_0. The outer shell has surface charge density

$$\sigma(\theta, \phi) = \sigma_0 \sin 2\theta \cos\phi$$

(in spherical coordinates r, θ, ϕ). The potential Φ satisfies Laplace's equation:

$$\nabla^2 \Phi = 0$$

for $a < r < b$ and $r > b$, and is constrained by the requirements that

(i) $\Phi = \Phi_0$ at $r = a$,

(ii) $\Phi \to 0$ as $r \to \infty$,

(iii) Φ is continuous at $r = b$, and

(iv) from Gauss's Law, the radial electric field $E_r = -\partial\Phi/\partial r$ has a discontinuity at $r = b$:

$$\left[-\frac{\partial \Phi}{\partial r}\right]_{b-}^{b+} = \frac{\sigma}{\epsilon_0}.$$

(a) Write down a general expression for Φ for $a < r < b$ that satisfies Laplace's equation and condition (i).

(b) Write down a general expression for Φ for $r > b$ that satisfies Laplace's equation and condition (ii).

(c) These expressions should take the form of sums over spherical harmonics $Y_l^m(\theta, \phi)$, for $-l \le m \le l$ and $l \ge 0$. The surface density σ can similarly be expressed as:

$$\sigma(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sigma_{lm} Y_l^m(\theta, \phi).$$

Use the given expression for $\sigma(\theta, \phi)$ to determine the σ_{lm}.

(d) Conditions (iii) and (iv) must apply for each term in the sum over l and m. Apply them term by term at $r = b$ to obtain the complete solution to the problem.

3. (a) Two hemispherical shells each of radius a are fitted together, insulated around their circle of contact, and kept at potentials $\pm V_0$, respectively. Find the potential $\Phi(r, \theta)$ inside the resulting sphere, where $\nabla^2\Phi = 0$ inside the sphere, $\Phi = \pm V_0$ on the two hemispheres, and the polar axis $\theta = 0$ is the axis of symmetry.

   (b) Now suppose that the potentials of the hemispheres in part (a) oscillate in time, with $V(t) = \pm V_0 e^{-i\omega t}$. Find the *exterior* solution $\Phi(r, \theta, t)$ to the wave equation

$$\nabla^2\Phi = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}$$

   for $r > a$, with $\Phi = \pm V(t)$ on the hemispheres, that describes an *outgoing* wave as $r \to \infty$.

   [Recall that the asymptotic forms of the spherical Bessel functions $j_l(x)$ and $n_l(x)$ as $x \to \infty$ are $j_l(x) \sim \frac{1}{x}\cos\{x - \frac{\pi}{2}(l+1)\}$ and $n_l(x) \sim \frac{1}{x}\sin\{x - \frac{\pi}{2}(l+1)\}$.]

4. Each of the two 1S electrons in a helium atom may be described by a hydrogenic wave function

$$\psi(\mathbf{r}) = \left(\frac{8}{\pi a_0^3}\right)^{1/2} e^{-2r/a_0}$$

   in the absence of the other electron. Here, $a_0 = \hbar^2/me^2$ is the Bohr radius. Use the expansion

$$\frac{1}{r_{12}} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{4\pi}{2n+1}\,[Y_n^m(\theta_1,\phi_1)]^*\,Y_n^m(\theta_2,\phi_2)\begin{cases} \frac{r_1^n}{r_2^{n+1}}, & |\mathbf{r}_1| < |\mathbf{r}_2| \\ \frac{r_2^n}{r_1^{n+1}}, & |\mathbf{r}_1| > |\mathbf{r}_2| \end{cases}$$

   to find the classical mutual electrostatic potential energy of the two electrons, treating each as a charge density distribution $\rho = -e|\psi|^2$:

$$U = \int \psi^*(\mathbf{r}_1)\psi^*(\mathbf{r}_2)\frac{e^2}{r_{12}}\psi(\mathbf{r}_1)\psi(\mathbf{r}_2)\,d^3r_1 d^3r_2,$$

   where $r_i = |\mathbf{r}_i|$, $d^3r_i = r_i^2 dr_i\,\sin\theta_i d\theta_i\,d\phi_i$ and $r_{12} = |\mathbf{r}_1 - \mathbf{r}_1|$.