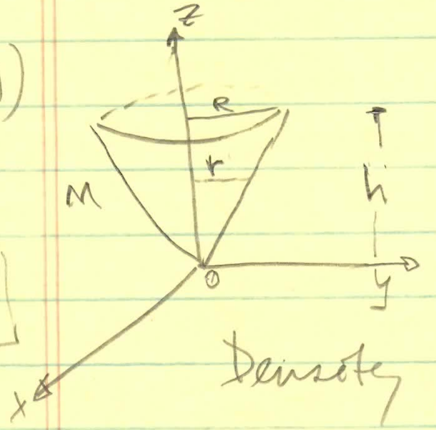
(Pb. 1)



$\boxed{\frac{z}{r} = \frac{h}{R}}$

Moment of Inertia around tip.

$$I_{ij} = \int \rho \,(r^2 \delta_{ij} - x_i x_j)\, dV$$

We will be using cylindrical coordinates.

Density $\rho = \frac{M}{\frac{1}{3}\pi R^2 h} = \frac{3M}{\pi R^2 h}$

Equation for the cone $x^2 + y^2 = r^2 = \left(\frac{R}{h}\right)^2 z^2 \equiv \alpha^2 z^2$

$$\boxed{I_{zz}} = \int \rho (x^2 + y^2)\, dV = \rho \int_0^h \int_0^{2\pi} \int_0^r (r'^2)\, r' dr' d\theta dz$$

$$= \rho\, 2\pi \int_0^h \frac{r^4}{4}\, dz = \frac{2\pi\rho}{4} \int_0^h \alpha^4 z^4\, dz = \frac{\pi}{2} \rho \alpha^4 \frac{h^5}{5}$$

$$= \frac{\pi}{2} \frac{3M}{\pi R^2 h} \cdot \frac{R^4}{h^4} \frac{h^5}{5} = \boxed{\frac{3}{10} M R^2}$$

Note: $\int \rho (x^2 + y^2)\, dV = \int \rho x^2 dV + \int \rho y^2 dV = \frac{3}{10} M R^2$

by symmetry $\int \rho x^2 dV = \int \rho y^2 dV \Rightarrow \int \rho x^2 dV = \int \rho y^2 dV = \frac{3}{20} M R^2$

$$I_{xx} = \int \rho (y^2 + z^2)\, dV = \int \rho y^2 dV + \underbrace{\int \rho z^2 dV}$$

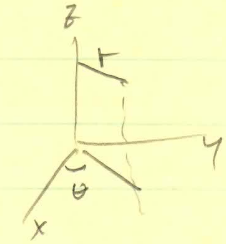
$$\int \rho z^2 dV = \rho \int_0^h \int_0^{2\pi} \int_0^r z^2 r' dr' d\theta dz = 2\pi\rho \int_0^h z^2 \frac{r^2}{2}\, dz$$

$$= \pi \rho \int_0^h z^2 \alpha^2 z^2\, dz = \pi \frac{3M}{\pi R^2 h} \cdot \frac{R^2}{h^2} \cdot \frac{h^5}{5} = \frac{3}{5} M h^2$$

$$\boxed{I_{xx} = \frac{3}{20} M R^2 + \frac{3}{5} M h^2 = \frac{3}{5} M \left(\frac{R^2}{4} + h^2\right)}$$

$$\boxed{I_{yy} = \int \rho (x^2 + z^2)\, dV = \int \rho x^2 dV + \int \rho z^2 dV = I_{xx}}$$

For the off-diagonal terms



$x = r\cos\theta$

$y = r\sin\theta$

$$I_{xy} = I_{yx} = -\int \rho (xy)\, dV$$
$$= -\int_0^h\int_0^{2\pi}\int_0^r \rho r'^2 \cos\theta \sin\theta \, r' dr' d\theta dz$$
$$= -\int_0^h \int_0^r \rho r'^3 dr' dz \underbrace{\int_0^{2\pi} \cos\theta \sin\theta d\theta}_{=0} = 0$$

$$I_{xz} = I_{zx} = -\int \rho (xz)\, dV = -\int_0^h \int_0^r \rho r'^2 dr' \, z dz \underbrace{\int_0^{2\pi} \cos\theta d\theta}_{=0} = 0$$

Similarly $I_{yz} = I_{zy} = 0$

$$\boxed{I = \begin{pmatrix} \frac{3}{5} M\left(\frac{R^2}{4} + h^2\right) & 0 & 0 \\ 0 & \frac{3}{5} M\left(\frac{R^2}{4} + h^2\right) & 0 \\ 0 & 0 & \frac{3}{10} MR^2 \end{pmatrix}}$$

Moment of inertia around z:

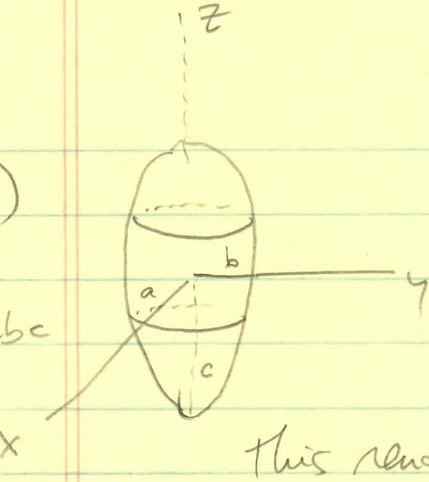
(i) $\boxed{I_s^z} = \hat{n} \cdot I \cdot \hat{n} = (0\ 0\ 1)\begin{pmatrix} & \end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = (0\ 0\ 1)\begin{pmatrix}0\\0\\ \frac{3}{10}MR^2\end{pmatrix} = \boxed{\frac{3}{10}MR^2}$

(ii) Around x:

$\boxed{I_s^x} = \hat{n} \cdot I \cdot \hat{n} = (1\ 0\ 0)\begin{pmatrix} & \end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \boxed{\frac{3}{5} M\left(\frac{R^2}{4} + h^2\right)}$

For $\bar{\omega} = \begin{pmatrix}\omega_1\\ \omega_2 \\ \omega_3\end{pmatrix}$ $\quad \bar{L} = \begin{pmatrix} \frac{3}{5} M\omega_1 \left(\frac{R^2}{4} + h^2\right) \\ \frac{3}{5} M\omega_2 \left(\frac{R^2}{4} + h^2\right) \\ \frac{3}{10} M\omega_3 R^2 \end{pmatrix}$

(Pb.2)



$V = \frac{4}{3}\pi abc$

Because of symmetry, all off-diagonal terms of moment of inertia tensor are zero $I_{ij} = 0 \quad i \neq j$.

This renders a diagonal matrix. So moment of inertia along z

$$I_s^z = I_{zz} = \int \rho (x^2+y^2)\, dV = \rho \iiint (x^2+y^2)\, dx\, dy\, dz$$

Change variables to $\xi = \frac{x}{a}$, $\eta = y/b$, $\nu = z/c$ $\quad \xi^2+\eta^2+\nu^2 = 1$

$$I_s^z = \rho abc \iiint (\underset{①}{a^2\xi^2} + \underset{②}{b^2\eta^2})\, d\xi\, d\eta\, d\nu$$

Lets look at ① $\displaystyle\int_{-1}^{1} d\xi \int_{-\sqrt{1-\xi^2}}^{\sqrt{1-\xi^2}} d\eta \int_{-\sqrt{1-\xi^2-\eta^2}}^{\sqrt{1-\xi^2-\eta^2}} d\nu\, \xi^2 = \iint \xi^2 d\xi\, d\eta\, 2\sqrt{1-\xi^2-\eta^2}$

But first $\displaystyle\int_{-\sqrt{1-\xi^2}}^{\sqrt{1-\xi^2}} d\eta \sqrt{1-\xi^2-\eta^2} = \int_{-\pi/2}^{\pi/2} \sqrt{1-\xi^2}\cos\theta\, d\theta \sqrt{1-\xi^2}\cos\theta\, d\theta$

using $\eta = \sqrt{1-\xi^2}\sin\theta$, $d\eta = \sqrt{1-\xi^2}\cos\theta\, d\theta$ $\rightarrow -\sqrt{1-\xi^2} \leq \eta \leq \sqrt{1-\xi^2} \rightarrow -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$$\frac{1}{2}(1-\xi^2)\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = \frac{\pi}{2}\left[1-\xi^2\right]$$

$$= 2\int_{-1}^{1} \xi^2 d\xi\, \frac{\pi}{2}\left[1-\xi^2\right] = \pi\left[\frac{\xi^3}{3} - \frac{\xi^5}{5}\right]_{-1}^{1} = \frac{4}{15}\pi$$

the other one ② gives the same by reordering variables:

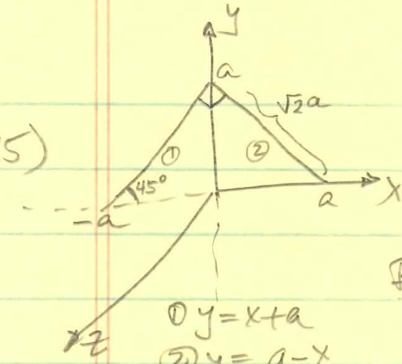
$$\iiint \eta^2 d\eta\, d\xi\, d\nu = 4\pi/15$$

So that $I_s^z = \rho abc\left[a^2+b^2\right]\frac{4\pi}{15} = \frac{3M}{4\pi abc} \cdot abc(a^2+b^2)\frac{4\pi}{15}$

$$\boxed{I_s^z = \frac{1}{5} M (a^2+b^2)}$$

for a sphere $a=b$ $\quad I_s^z = \frac{2}{5} Ma^2$

(5.15)



Consider a right triangle with sides $\sqrt{2}a$, $\sqrt{2}a$, $2a$ and with its long side on the x-axis.

Because the problem wants I around CM, we will first calculate its location.

① $y = x + a$

② $y = a - x$

$\sigma = M/a^2$

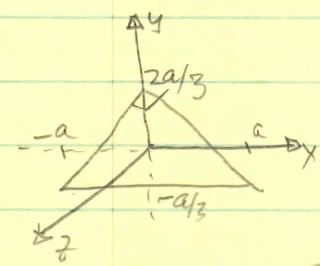
$$MR_x^{cm} = \sigma \int_0^a \int_{y-a}^{a-y} x\,dx\,dy = \frac{\sigma}{2}\int dy\left[(a-y)^2 - (y-a)^2\right] = 0$$

$$MR_y^{cm} = \sigma \int_0^a \int_{y-a}^{a-y} dx\, y\,dy = \sigma \int y\,dy\left[(a-y) - (y-a)\right]$$

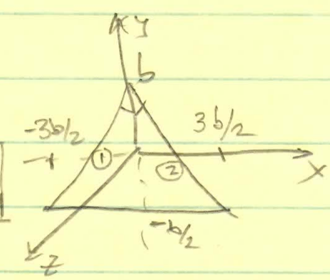
$$= 2\sigma \int y\,dy\,(a-y) = 2\sigma\left(\frac{a^3}{2} - \frac{a^3}{3}\right) = \frac{1}{3}Ma \quad : R_z^{cm} = 0$$

$$\boxed{\bar{R}^{cm} = \left(0, \tfrac{1}{3}a, 0\right)}$$

Now we shift the center to the CM



to simplify the math rescale the sides $\boxed{a = \frac{3}{2}b}$



$\sigma = 4M/9b^2$

① $y = x + b$ ② $y = b - x$

Lets build the moment of inertia tensor

$$I_{xx} = \iint \sigma\,(y^2 + \underset{0}{\cancel{z^2}})\,dx\,dy = \int_{-b/2}^{b} \int_{y-b}^{b-y} \sigma\,dx\,y^2 dy = \sigma \int y^2 dy\left[b - y - (y - b)\right]$$

$$= 2\sigma \int_{-b/2}^{b} y^2(b-y)\,dy = 2\sigma\left[\frac{b^4}{3} - \frac{b^4}{4} + \frac{b^4}{24} + \frac{b^4}{64}\right] = \frac{9}{32}b^4\sigma = \frac{1}{8}Mb^2$$

$$I_{yy} = \iint \sigma\,(x^2 + \underset{0}{\cancel{z^2}})\,dx\,dy = \sigma \int_{-b/2}^{b} dy\, \frac{x^3}{3}\Big|_{y-b}^{b-y} = \frac{\sigma}{3}\int_{-b/2}^{b} dy\left[-2y^3 + 6by^2 - 6b^2y + 2b^3\right]$$

$$= \frac{\sigma}{3}\cdot\frac{81}{32}b^4 = \frac{1}{3}\,\frac{4M}{9b^2}\cdot\frac{81}{32}\cdot b^4 = \frac{3}{8}Mb^2$$

$$I_{zz} = \iint \sigma(x^2 + y^2)\,dx\,dy = \sigma \underbrace{\iint x^2 dx\,dy}_{I_{yy}} + \overbrace{\sigma \iint dx\,y^2 dy}^{I_{xx}}$$

$$= I_{yy} + I_{xx} = \frac{1}{2}Mb^2$$

Because $z=0$, $I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0$

$$I_{xy} = I_{yx} = -\sigma \iint xy\,dx\,dy = -\sigma \int_{-b/2}^{b} y\,dy \left.\frac{x^2}{2}\right|_{y-b}^{b-y}$$

$$= -\frac{\sigma}{2}\int dy\, y \underbrace{\left[(b-y)^2 - (y-b)^2\right]}_{0} = 0$$

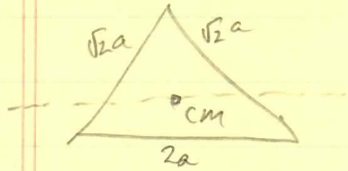
Moment of inertia tensor

$$I = \begin{pmatrix} \frac{1}{8}Mb^2 & 0 & 0 \\ 0 & \frac{3}{8}Mb^2 & 0 \\ 0 & 0 & \frac{1}{2}Mb^2 \end{pmatrix} = \frac{1}{8}Mb^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

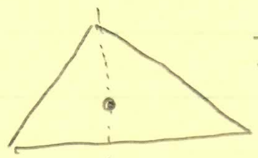
using $b = \frac{2}{3}a$

$$\boxed{I = \frac{1}{18}Ma^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}}$$

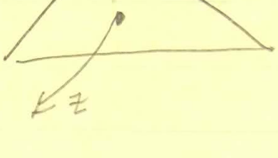
Because the matrix is diagonal, the moments of inertia are the diagonal elements with principal axes $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$



$I_s^x = \frac{1}{18}Ma^2 \; : \; \bar{\omega}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$



$I_s^y = \frac{1}{6}Ma^2 : \bar{\omega}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$



$I_s^z = \frac{2}{9}Ma^2 \; : \; \bar{\omega}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$