PHYS 501: Mathematical Physics I

Fall 2020

Solutions to Homework #1

1. (a) Writing $\xi = x - ct$ and seeking solutions $\psi(\xi)$, we have

$$\begin{aligned}
\frac{\partial\psi}{\partial t} &= \frac{d\psi}{d\xi}\frac{\partial\xi}{\partial t} = -c\psi'(\xi) \\
\frac{\partial\psi}{\partial x} &= \psi'(\xi) \\
\frac{\partial^3\psi}{\partial x^3} &= \psi'''(\xi),
\end{aligned}$$

so the equation becomes

$$(6\psi - c)\psi' + \psi''' = 0.$$

Integrating once, we have

$$3\psi^2 - c\psi + \psi'' = 0$$

so

$$\psi'' = c\psi - 3\psi^2.$$

(b) Multiplying by ψ' and integrating again, we have

$$\left(\psi'\right)^2 = c\psi^2 - 2\psi^3$$

or

$$\psi' = \psi\,(c - 2\psi)^{1/2}.$$

Hence, writing $u = 2\psi/c$, we have

$$\begin{aligned}
\xi &= \int \frac{d\psi}{\psi(c-2\psi)^{1/2}} \\
&= \frac{1}{\sqrt{c}}\int \frac{du}{u(1-u)^{1/2}} \\
&= \frac{1}{\sqrt{c}}\log_e\left(\frac{1-\sqrt{1-u}}{1+\sqrt{1-u}}\right).
\end{aligned}$$

We can invert this as follows:

$$\begin{aligned}
\frac{1-\sqrt{1-u}}{1+\sqrt{1-u}} &= e^{\sqrt{c}\xi} \\
1-\sqrt{1-u} &= \left(1+\sqrt{1-u}\right)e^{\sqrt{c}\xi} \\
1-e^{\sqrt{c}\xi} &= \sqrt{1-u}\left(1+e^{\sqrt{c}\xi}\right) \\
\sqrt{1-u} &= \frac{1-e^{\sqrt{c}\xi}}{1+e^{\sqrt{c}\xi}} = \tanh\tfrac{1}{2}\sqrt{c}\xi \\
1-u &= \tanh^2\tfrac{1}{2}\sqrt{c}\xi \\
u &= 1-\tanh^2\tfrac{1}{2}\sqrt{c}\xi = \operatorname{sech}^2\tfrac{1}{2}\sqrt{c}\xi
\end{aligned}$$

Thus we find

$$\psi = \frac{c}{2\cosh^2 \sqrt{c}\xi/2},$$

which represents a non-dispersive, traveling nonlinear wave.

2. For the PDE

$$A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy} = 0,$$

the two solutions of the characteristic equation

$$A\left(\frac{dy}{dx}\right)^2 - 2B\frac{dy}{dx} + C = 0$$

are

$$\begin{aligned}
\xi(x, y) &= \text{constant}, \\
\eta(x, y) &= \text{constant}.
\end{aligned}$$

Hence, along a characteristic,

$$\frac{dy}{dx} = -\frac{\partial\xi}{\partial x} \bigg/ \frac{\partial\xi}{\partial y} = -\xi_x/\xi_y,$$

so ξ satisfies

$$A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2 = 0, \tag{1}$$

and similarly for η. We want to use ξ and η as coordinates and write the PDE in terms of them. We assume that the functions A, B, and C can always be written explicitly in terms of ξ and η (which is in principle true, but often difficult in practice!).

We start by expanding

$$\begin{aligned}
\psi_x &= \psi_\xi\xi_x + \psi_\eta\eta_x, \\
\psi_{xx} &= (\psi_{\xi\xi}\xi_x + \psi_{\xi\eta}\eta_x)\xi_x + \psi_\xi\xi_{xx} + (\psi_{\xi\eta}\xi_x + \psi_{\eta\eta}\eta_x)\eta_x + \psi_\eta\eta_{xx} \\
&= \psi_{\xi\xi}\xi_x^2 + 2\psi_{\xi\eta}\xi_x\eta_x + \psi_{\eta\eta}\eta_x^2 + \psi_\xi\xi_{xx} + \psi_\eta\eta_{xx}.
\end{aligned}$$

Similarly, we find

$$\begin{aligned}
\psi_{yy} &= \psi_{\xi\xi}\xi_y^2 + 2\psi_{\xi\eta}\xi_y\eta_y + \psi_{\eta\eta}\eta_y^2 + \psi_\xi\xi_{yy} + \psi_\eta\eta_{yy}, \\
\psi_{xy} &= \psi_{\xi\xi}\xi_x\xi_y + \psi_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + \psi_{\eta\eta}\eta_x\eta_y + \psi_\xi\xi_{xy} + \psi_\eta\eta_{xy}.
\end{aligned}$$

Combining terms, the coefficients of $\psi_{\xi\xi}$ and $\psi_{\eta\eta}$ are, respectively, $A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2$ and $A\eta_x^2 + 2B\eta_x\eta_y + C\eta_y^2$, which are both zero, by Equation (1), so

$$\begin{aligned}
A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy} &= 2[A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + C\xi_y\eta_y]\psi_{\xi\eta} + D(\xi, \eta, \psi_\xi, \psi_\eta) \\
&= 0,
\end{aligned}$$

where the function D involves only first derivatives of ψ (and in fact is linear in them). Dividing through by the coefficient of $\psi_{\xi\eta}$ brings the equation into the desired form.

3. (a) In this case, $A = 1, B = 0, C = -c(x)^2$, and the characteristic equation is

$$\left(\frac{dx}{dt}\right)^2 = c(x)^2,$$

the solutions to which are

$$t = \pm \int_{x_0}^{x} \frac{ds}{c(s)}.$$

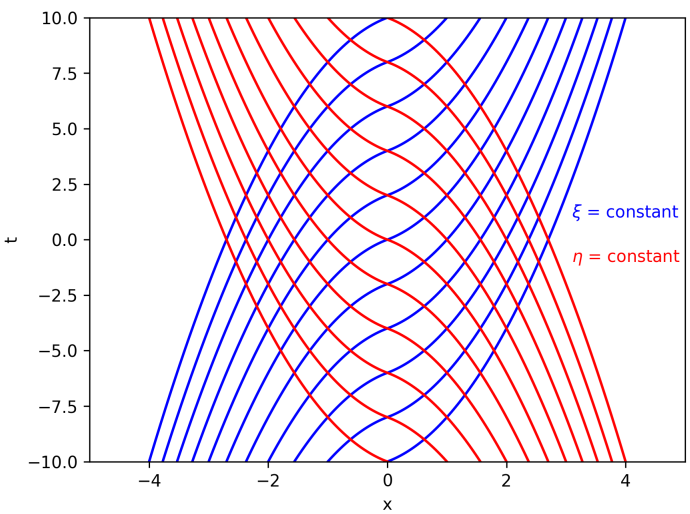
For $c(x) = c_0(1 + |x|/a)^{-1}$, we find

$$c_0 t = \pm \int^x ds\,(1 + |s|/a) = \pm \left[x + \operatorname{sign}(x)\frac{x^2}{2a}\right] + \text{constant}.$$

In the language of the previous question, we have

$$\xi, \eta = x + \operatorname{sign}(x)\frac{x^2}{2a} \pm c_0 t.$$

Some typical characteristic curves are shown in the figure below (for $c = 1, a = 2$).



(b) For $a \to \infty$, we have $c(x) = c_0$, and the characteristics are simply given by $x \pm c_0 t =$ constant. As discussed in class, the solution is $\psi(x,t) = f(\xi) + g(\eta)$, where $\xi = x + c_0 t, \eta = x - c_0 t$. Applying the initial conditions at $t = 0$, we have

$$\begin{aligned} f(x) + g(x) &= 0, \\ c_0 f'(x) - c_0 g'(x) &= e^{-|x|}, \end{aligned}$$

so

$$\begin{aligned} -g'(x) = f'(x) &= e^{-|x|}/2c_0, \\ -g(x) = f(x) &= \frac{1}{2c_0}\int^x e^{-|s|} ds = -\operatorname{sign}(x) e^{-|x|}/2c_0 + \text{constant}, \end{aligned}$$

and hence

$$\psi(x,t) = f(x + c_0 t) - f(x - c_0 t) = \frac{1}{2c_0}\int_{x - c_0 t}^{x + c_0 t} e^{-|s|} ds\,.$$

4. In terms of $T' = T - T_0$,

$$\nabla^2 T' = \frac{1}{\kappa}\frac{\partial T'}{\partial t}\,,$$

with $T' = -T_0$ initially inside the cube and $T' = 0$ on the surface. As usual, we separate out the time dependence $e^{-\alpha\kappa t}$, so the spatial part of the solution $\chi(x, y, z)$ satisfies

$$\nabla^2\chi + \alpha\chi = 0.$$

Separating in x, y, and z, we find that, to satisfy the boundary conditions at $x, y, z = 0$, χ must be a sum of terms of the form

$$\chi \sim \sin ax\,\sin by\,\sin cz\,.$$

Applying the boundary conditions at $x, y, z = L$ gives

$$a = \frac{k\pi}{L}, \quad b = \frac{l\pi}{L}, \quad c = \frac{m\pi}{L},$$

and

$$\alpha = \alpha_{klm} = a^2 + b^2 + c^2 = \frac{\pi^2}{L^2}\left(k^2 + l^2 + m^2\right).$$

Thus the general solution satisfying the differential equation and the boundary conditions is

$$T = T_0 + \sum_{k,l,m} a_{klm}\,\sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{l\pi y}{L}\right)\sin\left(\frac{m\pi z}{L}\right)\,e^{-\alpha_{klm}\kappa t}\,.$$

We determine the coefficients a_{klm} by enforcing the initial condition, $T = 0$, or $T' = -T_0$, so

$$\begin{aligned}
a_{klm} &= \frac{8}{L^3}\int_0^L dx\int_0^L dy\int_0^L dz\,(-T_0)\,\sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{l\pi y}{L}\right)\sin\left(\frac{m\pi z}{L}\right) \\
&= -\frac{8T_0}{L^3}\left[\frac{L}{k\pi}\left\{1-(-1)^k\right\}\right]\left[\frac{L}{l\pi}\left\{1-(-1)^l\right\}\right]\left[\frac{L}{m\pi}\left\{1-(-1)^m\right\}\right] \\
&= \begin{cases} -\dfrac{64T_0}{klm\pi^3} & (k, l, m \text{ all odd}) \\ 0 & (\text{otherwise}) \end{cases}
\end{aligned}$$

and hence

$$T(x, y, z, t) = T_0\left[1 - \frac{64}{\pi^3}\sum_{\substack{k,l,m\\ \text{odd}}}\frac{1}{klm}\,\sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{l\pi y}{L}\right)\sin\left(\frac{m\pi z}{L}\right)\,e^{-\alpha_{klm}\kappa t}\right].$$