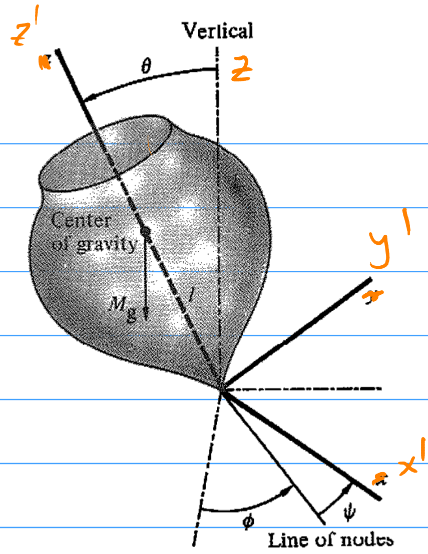
The spinning top with one point fixed



The rate of change of the Euler angles mean:

$\dot{\psi}$ – spinning of the top around z'

$\dot{\varphi}$ – precession around the vertical

$\dot{\theta}$ – nutation (bobbing up and down motion)

Using the body set axes and the components of the angular velocity derived for these axes, from chapter 4 we have:

$$\omega_{x'} = \dot{\varphi}\sin\psi\sin\theta + \dot{\theta}\cos\psi \qquad \omega_{x'} = \omega_1$$

$$\omega_{y'} = \dot{\varphi}\cos\psi\sin\theta - \dot{\theta}\sin\psi \qquad \omega_{y'} = \omega_2$$

$$\omega_{z'} = \dot{\varphi}\cos\theta + \dot{\psi} \qquad \omega_{z'} = \omega_3$$

$\sin\psi = s\psi$
$\cos\theta = c\theta$

$$T = \frac{1}{2}I_1\left(\omega_{x'}^2 + \omega_{y'}^2\right) + \frac{1}{2}I_3\,\omega_{z'}^2$$

$$= \frac{1}{2}I_1\left(\dot{\varphi}^2 s^2\psi\, s^2\theta + \dot{\theta}^2 c^2\psi + 2\dot{\varphi}\dot{\theta} s\psi\, \cancel{s\theta\dot{\theta} c\psi} \right.$$
$$\left. + \dot{\varphi}^2 c^2\psi\, s^2\theta + \dot{\theta}^2 s^2\psi - 2\dot{\varphi}\dot{\theta}\cancel{c\psi}\, s\theta\, s\psi\right)$$
$$+ \frac{1}{2}I_3\left(\dot{\varphi}c\theta + \dot{\psi}\right)$$

$$\boxed{T = \frac{1}{2}I_1\left(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}I_3\left(\dot{\varphi}\cos\theta + \dot{\psi}\right)^2}$$

$$\boxed{V = mgl\cos\theta}$$

$$L = \tfrac{1}{2} I_1 (\dot\varphi^2 \sin^2\theta + \dot\theta^2) + \tfrac{1}{2} I_3 (\dot\varphi \cos\theta + \dot\psi)^2 - mg\ell\cos\theta$$

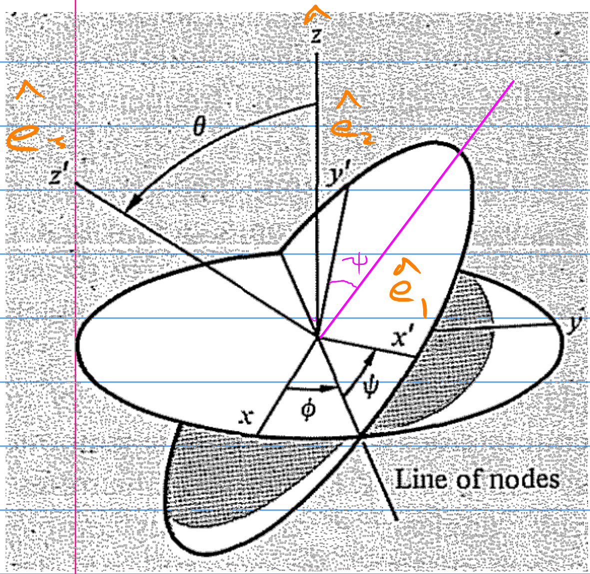
$$\varphi: \quad P_\varphi = \frac{\partial L}{\partial \dot\varphi} = I_1 \dot\varphi \sin^2\theta + I_3(\dot\varphi\cos\theta + \dot\psi)\cos\theta = \text{const}$$

$$\psi: \quad P_\psi = \frac{\partial L}{\partial \dot\psi} = I_3(\dot\varphi\cos\theta + \dot\psi) = \text{const}$$

$$\bar{L} = I_1 \omega_1 \hat{e}_1 + I_1 \omega_2 \hat{e}_2 + \underbrace{I_3 \omega_3}_{L_3} \hat{e}_3$$

$$L_3 = I_3 \omega_3 = I_3(\dot\varphi\cos\theta + \dot\psi) = \text{const}$$

$$\boxed{L_3 = \text{const}} \qquad \omega_3 \text{ const}$$



$$\hat{e}_1 \cdot \hat{z} = \sin\psi \sin\theta$$

$$\hat{e}_2 \cdot \hat{z} = \cos\psi \sin\theta$$

$$\hat{e}_3 \cdot \hat{z} = \cos\theta$$

$$\bar{L} \cdot \hat{z} = L_z = I_1 \omega_1 \hat{e}_1 \cdot \hat{z} + I_1 \omega_2 \hat{e}_2 \cdot \hat{z} + I_3 \omega_3 \hat{e}_3 \cdot \hat{z}$$

$$= I_1(\dot\varphi s\psi s\theta + \dot\theta \cancel{c\psi}) s\psi s\theta$$

$$+ I_1(\dot\varphi c\psi s\theta - \dot\theta \cancel{s\psi}) c\psi s\theta$$

$$+ I_3(\dot\varphi c\theta + \dot\psi) c\theta$$

$$L_z = I_1 \dot\varphi \sin^2\theta + I_3(\dot\varphi\cos\theta + \dot\psi)\cos\theta$$

$$\boxed{L_z = \text{const}}$$

$$\theta: \quad \frac{\partial L}{\partial \theta} = I_1 \dot\varphi^2 \sin\theta\cos\theta - I_3(\dot\varphi\cos\theta + \dot\psi)\dot\varphi\sin\theta$$
$$+ mgl\sin\theta$$

$$\frac{\partial L}{\partial \dot\theta} = I_1 \dot\theta$$

$$\boxed{I_1\ddot\theta - I_1\dot\varphi^2\sin\theta\cos\theta + I_3(\dot\varphi\cos\theta + \dot\psi)\dot\varphi\sin\theta - mgl\sin\theta = 0}$$

$$L_z = I_1\dot\varphi\sin^2\theta + I_3(\dot\varphi\cos\theta + \dot\psi)\cos\theta$$
$$= I_1\dot\varphi\sin^2\theta + L_3\cos\theta$$

$$\boxed{\dot\varphi = \frac{L_z - L_3\cos\theta}{I_1\sin^2\theta}}$$

Steady Precession $\quad \theta = \theta_0 \Longrightarrow \dot\varphi = \Omega$ const

$$L_3 = I_3(\Omega\cos\theta + \underbrace{\dot\psi}_{\text{const}}) = \text{Const}$$

$$\cancel{I_1\ddot\theta} - I_1\underbrace{\dot\varphi}_{\Omega}{}^2\sin\theta\cos\theta + I_3\underbrace{(\dot\varphi\cos\theta+\dot\psi)}_{\omega_3}\underbrace{\dot\varphi}_{\Omega}\sin\theta - mgl\sin\theta = 0$$

$$I_1\Omega^2\cos\theta - I_3\omega_3\Omega + mgl = 0$$

$$\boxed{\Omega = \frac{1}{2I_1\cos\theta}\left[I_3\omega_3 \pm \sqrt{I_3^2\omega_3^2 - 4I_1 mgl\cos\theta}\right]}$$

Case $\theta > \pi/2$ (CM below the fixed point)

any omega_3 will give rise to a uniform precession

Case $\theta < \pi/2$ (CM above fixed points)

omega_3 has to be above a certain value

$$I_3^2 \omega_3^2 > 4 I_1 m g l \cos\theta$$

$$\omega_3 > \frac{2}{I_3}\sqrt{I_1 m g l \cos\theta}$$

A special case is when the KE of rotation is much larger than the potential energy

$$\frac{1}{2} I_3 \omega_3^2 \gg m g l \qquad I_1 \sim I_3$$

$$\Omega = \frac{I_3 \omega_3}{2 I_1 \cos\theta}\left[1 \pm \sqrt{1 - \frac{4 I_1 m g l \cos\theta}{I_3^2 \omega_3^2}}\right]$$

$$\Omega^- \simeq \frac{\cancel{I_3}\cancel{\omega_3}}{\cancel{2}\cancel{I_1}\cancel{\cos\theta}} \cdot \frac{1}{\cancel{2}} \cdot \frac{\cancel{4}\cancel{I_1} m g l \cancel{\cos\theta}}{I_3^{\cancel{2}} \omega_3^{\cancel{2}}} = \frac{m g l}{I_3 \omega_3} \qquad \text{slow}$$

$$\Omega^+ \simeq \frac{I_3 \omega_3}{I_1 \cos\theta} \qquad \text{fast}$$

does not depend on gravity.
This one actually corresponds to the torque-free precession.

can show that L is very close to the vertical.

Nutation

Lets look at the energy. $E = T + V$

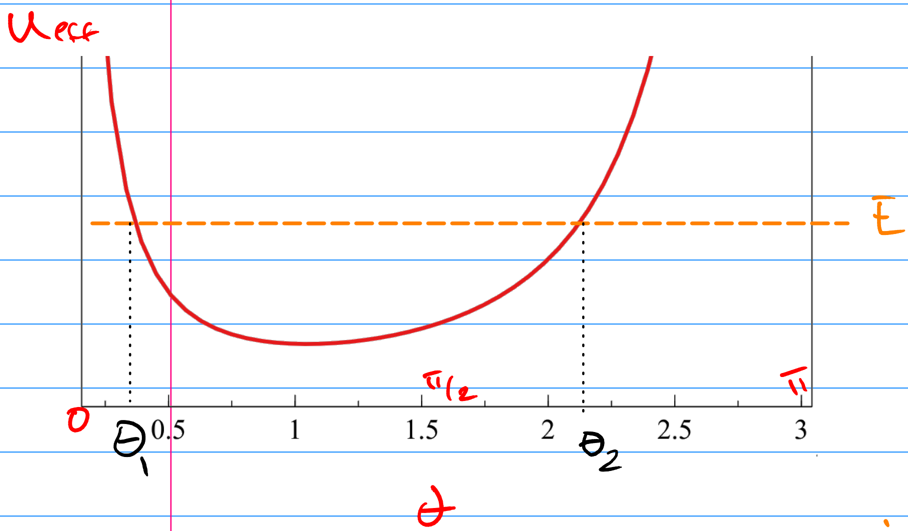
$$T = \frac{1}{2} I_1 (\dot\varphi^2 \sin^2\theta + \dot\theta^2) + \frac{1}{2} I_3 (\dot\varphi \cos\theta + \dot\psi)^2$$

$$V = m g l \cos\theta$$

$$L_3 = I_3 \omega_3 = I_3 (\dot{\varphi} \cos\theta + \dot{\psi})$$

$$\dot{\varphi} = \frac{L_z - L_3 \cos\theta}{I_1 \sin^2\theta}$$

$$E = \frac{1}{2} I_1 \dot{\theta}^2 + \underbrace{\frac{(L_z - L_3 \cos\theta)^2}{2 I_1 \sin^2\theta} + \frac{1}{2}\frac{L_3^2}{I_3} + mgl\cos\theta}_{\text{effective potential} \quad U_{eff}(\theta)}$$



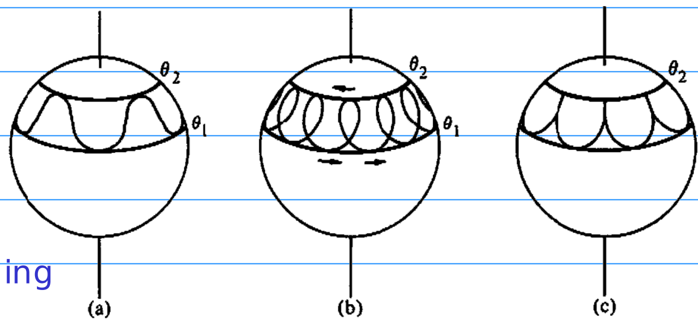
theta "nutates" between theta_1 and theta_2 for a given energy E

$$\dot{\varphi} = \frac{L_z - L_3 \cos\theta}{I_1 \sin^2\theta}$$

Case L_z > L_3:

$\dot{\varphi} \neq 0$ never changes sign φ keeps increasing (or decreasing)

get motion (a)



Case L_z < L_3: phi-dot can vanish, thus can change sign.

phi can increase or decrease depending on the sign

get motion (b)

Case L_z = L_3: phi-dot can vanish, but does not change sign.
phi can stop changing, but keeps increase (or decreasing) after that.

get motion (c)

Small Oscillations

If deviations of a system away from a stable equilibrium are small enough, the motion can generally be described as that of a system of couple linear harmonic oscillators.

Assumptions of the problem:

1. conservative system with potential only dependent on position
2. generalized coordinates do not explicitly depend on time
3. no time-dependent constraints