Total Angular Momentum for a System of Particles

Start by adding up each individual torques

$$\sum_i \bar{r}_i \times \dot{\bar{p}}_i = \sum_i \frac{d}{dt}(\bar{r}_i \times \bar{p}_i) = \frac{d}{dt}\bar{L} = \dot{\bar{L}}$$

$$\frac{d}{dt}(\bar{r}_i \times \bar{p}_i) = \bar{v}_i \times \bar{p}_i + \bar{r}_i \times \dot{\bar{p}}_i$$

$$= \underbrace{\bar{v}_i \times m_i \bar{v}_i}_{=0} + \bar{r}_i \times \dot{\bar{p}}_i$$

$$= \bar{r}_i \times \dot{\bar{p}}_i$$

substitute $\dot{\bar{p}}_i = \sum_j \bar{F}_{ji} + \bar{F}_i^{(e)}$

(1)

$$\sum_i \bar{r}_i \times \dot{\bar{p}}_i = \sum_i \bar{r}_i \times \bar{F}_i^{(e)} + \sum_{ij}{}' \bar{r}_i \times \bar{F}_{ji}$$

(1) $$\sum_{ij}{}' \bar{r}_i \times \bar{F}_{ji} = \sum_{j>i}(\bar{r}_i \times \bar{F}_{ji} + \bar{r}_j \times \bar{F}_{ij})$$

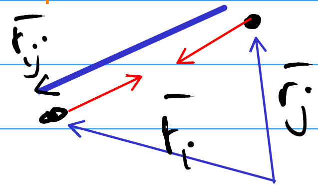
"weak" law of action/reaction $\bar{F}_{ij} = -\bar{F}_{ji}$

$$= \sum_{j>i}(\bar{r}_i - \bar{r}_j) \times \bar{F}_{ji}$$

$$= \sum_{j>i} \bar{r}_{ij} \times \bar{F}_{ji}$$

additionally require the forces to obey the "strong" law of action/reaction - which basically means that the interal forces lie along the line joining the particles

$$\bar{F}_{ji} = \bar{r}_{ij} \cdot f$$



$$\bar{r}_{ij} \parallel \bar{F}_{ji} \qquad \bar{r}_{ij} \times \bar{F}_{ji} = 0$$

$$(1) = 0$$

$$\dot{\bar{L}} = \sum_i \bar{r}_i \times \bar{F}_i^{(e)} = \bar{N}^{(e)}$$

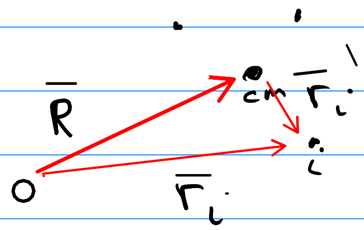
If there are no external torques, then the total angular momentum of the SYSTEM is conserved.

Notes:

1. while the conservation of total P requires the "weak" law, the total L requires the "strong" law to apply.
2. The strong law means that the forces are "central".
3. Moving charges with corresponding E&M forces do not necessarily obey either of these. There are, however, generalizations of P and L that these charged systems will conserve.

Rephrase all of this in terms of the CM.

$$\bar{L} = \sum_i \bar{r}_i \times \bar{P}_i$$



$$\bar{r}_i = \bar{r}_i' + \bar{R}$$

$$\bar{v}_i = \bar{v}_i' + \bar{v}$$

$$\bar{L} = \sum_i (\bar{r}_i' + \bar{R}) \times m_i (\bar{v}_i' + \bar{v})$$

$$= \sum_i \bar{R} \times m_i \bar{v} + \sum_i \bar{r}_i' \times m_i \bar{v}_i'$$

$$+ \underbrace{\sum_i m_i \bar{r}_i' \times \bar{v}}_{(2)} + \underbrace{\sum_i \bar{R} \times m_i \bar{v}_i'}_{(3)}$$

(2) $\left(\sum_i m_i \bar{r}_i'\right) \times \bar{v}$

0 $\qquad$ 0

(3) $\bar{R} \times \left(\sum_i m_i \bar{v}_i'\right) = \bar{R} \times \frac{d}{dt} \sum_i m_i \bar{r}_i'$

$$\Rightarrow \boxed{\bar{L} = \bar{R} \times M\bar{v} + \sum_i \bar{r}_i' \times \bar{P}_i'}$$

Total Angular Momentum of a System of Particles about point O is the Angular Momentum about the CM + Angular Momentum of the CM.