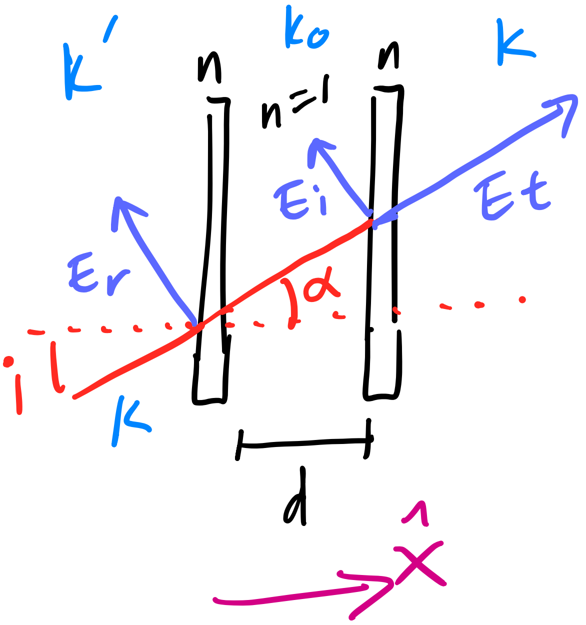
HW 2a: #7.3

Two plane semi-infinite slabs are parallel with n and width d. Plane EM wave is incident on gap w/ $\theta = i$. For linear polarization both parallel and perpendicular to plane of incidence:

a) calculate ratio of power transmitted into 2nd slab and reflected/incident power



$$n \sin(i) = \sin \alpha$$

$$n^2 \sin^2 i + \cos^2 i = 1$$

$$= \cos^2 \alpha + \sin^2 \alpha$$

$$\text{so } \cos\alpha = \sqrt{1 - \sin^2\alpha}$$

$$= \sqrt{1 - n^2 \sin^2 i}$$

define $\vec{E}_r e^{ik' \cdot \vec{x}}$ on incident side

$\vec{E}_i e^{ik_0 \cdot \vec{x}}$ in air gap

$\vec{E}_t e^{ik \cdot (\vec{x} - \vec{d})}$ on transmitted

for E perpendicular to plane of incidence:

slab 1

$$E_i + E_r = E_+ + E_- \qquad (E)$$

$$n(E_i - E_r) \cos i = (E_+ + E_-) \cos\alpha$$

$$\frac{n \cos i}{\cos\alpha}(E_i - E_r) = E_+ + E_- \qquad (H)$$

$E_i + E_r = E_+ + E_-$ $\quad \frac{n\cos i}{\cos\alpha}(E_i - E_r) = E_+ + E_-$

so $E_i + E_r = \frac{n\cos i}{\cos\alpha} E_i - \frac{n\cos i}{\cos\alpha} E_r$

let $\frac{n\cos i}{\cos\alpha} = \frac{n\cos i}{\sqrt{1-n^2\sin^2 i}} = \beta$

$E_i - \beta E_i = -E_r - \beta E_r$

$$\left.\begin{aligned} E_i + E_r &= -\beta E_r + \beta E_i \\ E_i + E_r &= E_+ + E_- \end{aligned}\right\} \text{system}$$

so $E_+ = \frac{E_i}{2}(1+\beta) + \frac{E_r}{2}(1-\beta)$

$E_- = \frac{E_i}{2}(1-\beta) + \frac{E_r}{2}(1+\beta)$

Slab 2

$E_+ e^{i\vec{k}_0\cdot\vec{d}} + E_- e^{-i\vec{k}_0\cdot\vec{d}} = E_t$ $\qquad E$

$(E_+ e^{i\vec{k}_0\cdot\vec{d}} - E_- e^{-i\vec{k}_0\cdot\vec{d}})\cos\alpha = E_t n\cos i$ $\qquad H$

can simplify $\vec{k}_0\cdot\vec{d} = k_0 d\cos\alpha = \frac{wd}{c}\cos\alpha$

let $\frac{wd}{c}\cos\alpha = \phi$

$$\left.\begin{aligned} E_+e^{i\phi} - E_-e^{-i\phi} &= E_t\frac{n\cos i}{\cos\alpha} = E_t\beta \\ E_+e^{i\phi} + E_-e^{-i\phi} &= E_t \end{aligned}\right\} \text{solve as system}$$

$2E_+ e^{i\phi} = E_t(1+\beta)$

$-2E_- e^{-i\phi} = E_t\beta - E_t$

so $E_+ = \frac{E_t}{2}(1+\beta)e^{-i\phi}$

$E_- = \frac{E_t}{2}(1-\beta)e^{+i\phi}$

E_+ and E_- are same at each interface because continuity

so $\frac{E_i}{2}(1+\beta) + \frac{E_r}{2}(1-\beta) = \frac{1}{2}e^{-i\phi}E_+(1+\beta)$ E_+

$\frac{E_i}{2}(1-\beta) + \frac{E_r}{2}(1+\beta) = \frac{1}{2}e^{i\phi}E_t(1-\beta)$ E_-

ratios we want are $\left|\frac{E_t}{E_i}\right|^2$ and $\left|\frac{E_r}{E_i}\right|^2$

$$E_t = \frac{e^{-i\phi}}{(1-\beta)}\left[E_i(1-\beta) + E_r(1+\beta)\right]$$

$$E_i = e^{i\phi}E_t(1-\beta) - E_r(1+\beta)$$

$$\frac{E_t}{E_i} = \frac{e^{-i\phi}E_i(1-\beta)}{(1-\beta)e^{i\phi}\left[E_t(1-\beta) - E_r(1+\beta)\right]} + \frac{e^{-i\phi}E_r(1+\beta)}{(1-\beta)e^{i\phi}\left[E_t(1-\beta) - E_r(1+\beta)\right]}$$

$$= \frac{e^{-i\phi}\left[e^{i\phi}E_t(1-\beta) - E_r(1+\beta)\right] + e^{-i\phi}E_r(1+\beta)}{(1-\beta)e^{i\phi}\left[E_t(1-\beta) - E_r(1+\beta)\right]}$$

$$= \frac{E_t(1-\beta)}{(1-\beta)e^{i\phi}\left[E_t(1-\beta) - E_r(1+\beta)\right]}$$

$$= \frac{4\beta}{e^{-i\phi}(1+\beta)^2 - e^{i\phi}(1-\beta)^2} = \frac{2\beta}{2\beta\cos\phi - i\sin\phi\,(1+\beta^2)}$$

so $$\boxed{\left|\frac{E_t}{E_i}\right|^2 = \frac{4\beta^2}{4\beta^2\cos^2\phi + \sin^2\phi\,(1+\beta^2)^2}}$$

$$\frac{E_r}{E_i} = \frac{E_i - e^{i\phi} E_t(1-\beta)}{e^{i\phi} E_t(1-\beta) - E_r(1+\beta)}$$

$$= \frac{E_i - e^{i\phi} E_t(1-\beta)}{e^{i\phi} E_t(1-\beta) - \left[\frac{(1-\beta) E_t}{e^{-i\phi}} - E_i(1-\beta)\right]}$$

$$= \frac{(e^{i\phi} - e^{-i\phi})(1-\beta)}{e^{-i\phi}(1+\beta)^2 - e^{i\phi}(1-\beta)^2}$$

$$= \frac{i \sin\phi (1-\beta^2)}{2\beta\cos\phi - i\sin\phi(1+\beta^2)}$$

so $$\boxed{\left|\frac{E_r}{E_i}\right|^2 = \frac{\sin^2\phi\,(1-\beta^2)^2}{4\beta^2\cos^2\phi + \sin^2\phi\,(1+\beta^2)^2}}$$

now $\vec{E}$ parallel:

slab 1:

$$E_i \cos i - E_r \cos i = E_+ \cos\alpha + E_- \cos\alpha \qquad E$$

$$n E_i + n E_r = E_+ + E_- \qquad \text{so } E_i + E_r = \frac{E_+}{n} + \frac{E_-}{n} \qquad H$$

slab 2:

$$E_t e^{i\phi} \cos\alpha - E_- e^{i\phi} \cos\alpha = E_+ \cos i \qquad G$$

$$E_t e^{i\phi} - E_- e^{-i\phi} = n E_+ \qquad H$$

match the conditions because continuity:

$$\frac{nE_i}{2}\left(1+\frac{\cos i}{n\cos\alpha}\right)+\frac{nE_r}{2}\left(1-\frac{\cos i}{n\cos\alpha}\right)=\frac{ne^{-i\phi}}{2}E_t\left(1+\frac{\cos i}{n\cos\alpha}\right)$$

$$\frac{nE_i}{2}\left(1-\frac{\cos i}{n\cos\alpha}\right)+\frac{nE_r}{2}\left(1+\frac{\cos i}{n\cos\alpha}\right)=\frac{ne^{i\phi}}{2}E_t\left(1-\frac{\cos i}{n\cos\alpha}\right)$$

since we said $\beta=\frac{n\cos i}{\cos\alpha}$

$$\frac{\cos i}{n\cos\alpha}=\frac{\beta}{n^2}=\gamma$$

so $$\frac{nE_i}{2}(1+\gamma)+\frac{nE_r}{2}\left(1-\frac{\cos i}{n\cos\alpha}\right)=\frac{ne^{-i\phi}}{2}E_t(1+\gamma)$$

$$\frac{nE_i}{2}(1-\gamma)+\frac{nE_r}{2}(1+\gamma)=\frac{ne^{i\phi}}{2}E_t(1-\gamma)$$

$$\frac{E_t}{E_i}=\frac{2n^2\gamma}{2n^2\gamma\cos\phi-i\sin\phi\,(n^4+\gamma^2)}$$

so $$\boxed{\left|\frac{E_t}{E_i}\right|^2=\frac{4n^4\gamma^2}{4n^4\gamma^2\cos^2\phi+\sin^2\phi\,(n^4+\gamma^2)^2}}$$

$$\frac{E_r}{E_i}=\frac{i\sin\phi\,(n^4-\gamma^2)}{2n^2\gamma\cos\phi-i\sin\phi\,(n^4+\alpha^2)}$$

so $$\boxed{\left|\frac{E_r}{E_i}\right|^2=\frac{\sin^2\phi\,(n^4-\gamma^2)}{4n^2\alpha^2\cos^2\phi+\sin^2\phi\,(n^4+\gamma^2)^2}}$$

b) for $i >$ critical angle i_0, plot E_t/E_i as function of d

in this limit, trig functions become hyperbolic because β and ϕ become imag. (i.e. $\beta = i\beta'$ and $\phi = i\phi'$)

perpendicular polarization:

$$\left|\frac{E_t}{E_i}\right|^2 = \frac{4\beta^2}{4\beta^2\cos^2\phi + \sin^2\phi\,(1+\beta^2)^2}$$

$$\rightarrow \frac{4\beta^2}{4\beta^2\cosh^2\phi + \sinh^2\phi\,(1+\beta^2)^2}$$

parallel polarization:

$$\left|\frac{E_t}{E_i}\right|^2 = \frac{4n^4\gamma^2}{4n^4\gamma^2\cos^2\phi + \sin^2\phi\,(n^4+\gamma^2)^2}$$

$$\rightarrow \frac{4n^4\gamma^2}{4n^4\gamma^2\cosh^2\phi + \sinh^2\phi\,(n^4+\gamma^2)^2}$$

both polarizations go as $\frac{1}{\cosh^2\phi + \sinh^2\phi}$:

where $\phi \propto d$