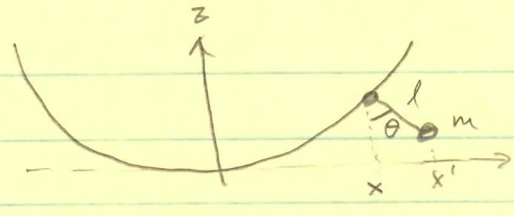
Hmwk #8

Pb. 1



Pendulum on parabola $z = ax^2$

Find Hamiltonian.

Generalised coordinates, x and θ.

Constraints $x' = x + l\sin\theta \qquad z' = z - l\cos\theta = ax^2 - l\cos\theta$

Kinetic Energy

$$T = \tfrac{1}{2}m(\dot{x}'^2 + \dot{z}'^2) = \tfrac{1}{2}m\left[(\dot{x} + l\cos\theta\,\dot\theta)^2 + (2ax\dot{x} + l\sin\theta\,\dot\theta)^2\right]$$

$$= \tfrac{1}{2}m\left[\dot{x}^2 + l^2\cos^2\theta\,\dot\theta^2 + 2l\cos\theta\,\dot{x}\dot\theta + {}\right.$$
$$\left. + 4a^2x^2\dot{x}^2 + l^2\sin^2\theta\,\dot\theta^2 + 4axl\sin\theta\,\dot{x}\dot\theta\right]$$

$$= \tfrac{1}{2}m\left[\dot{x}^2(1 + 4a^2x^2) + l^2\dot\theta^2 + 2l(\cos\theta + 2ax\sin\theta)\dot{x}\dot\theta\right]$$

KE matrix

$$[T_{ij}] = m\begin{pmatrix} 1 + 4a^2x^2 & l(\cos\theta + 2ax\sin\theta) \\ l(\cos\theta + 2ax\sin\theta) & l^2 \end{pmatrix}$$

With inverse

$$[T_{ij}]^{-1} = \frac{1}{ml^2(\sin\theta - 2ax\cos\theta)^2}\begin{pmatrix} l^2 & -l(\cos\theta + 2ax\sin\theta) \\ -l(\cos\theta + 2ax\sin\theta) & 4a^2x^2 + 1 \end{pmatrix}$$

Potential Energy $\quad U = mgz' = mg(ax^2 - l\cos\theta)$

Writing Lagrangian $L = T - U = L_0 + L_1 + L_2$

then $L_0 = -mg(ax^2 - l\cos\theta)$

Hamiltonian $H = \frac{1}{2}p^T T^{-1} p - L_0$ where $p = \begin{pmatrix} p_x \\ p_\theta \end{pmatrix}$

$$H = \frac{1}{2ml^2(\sin\theta - 2ax\cos\theta)^2}(p_x\ p_\theta)\begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} p_x \\ p_\theta \end{pmatrix} + mg(ax^2 - l\cos\theta)$$

Doing the algebra separately

$$(p_x\ p_\theta)\begin{pmatrix} l^2 & -l(2axS\theta + C\theta) \\ -l(2axS\theta + C\theta) & 4a^2x^2 + 1 \end{pmatrix}\begin{pmatrix} p_x \\ p_\theta \end{pmatrix} = (p_x\ p_\theta)\begin{pmatrix} l^2p_x - l(2axS\theta + C\theta)p_\theta \\ -l(2axS\theta + C\theta)p_x + (4a^2x^2 + 1)p_\theta \end{pmatrix}$$

$$= l^2 P_x^2 - l(2ax S\theta + c\theta) P_x P_\theta - l(2ax S\theta + c\theta) P_\theta P_x + (4a^2x^2+1) P_\theta^2$$
$$= l^2 P_x^2 - 2l(2ax S\theta + c\theta) P_x P_\theta + (4a^2x^2+1) P_\theta^2$$

So that

$$\boxed{H = \frac{1}{2m(\sin\theta - 2ax\cos\theta)^2}\left[P_x^2 - \frac{2}{l}(2ax\sin\theta + \cos\theta) P_x P_\theta + \frac{1}{l^2}(4a^2x^2+1) P_\theta^2\right] + mg(ax^2 - l\cos\theta)}$$

Hamilton's eqns of motion

$$\boxed{\dot{x} = \frac{\partial H}{\partial P_x} = \frac{1}{m(\sin\theta - 2ax\cos\theta)^2}\left[P_x - \frac{1}{l}(2ax\sin\theta + \cos\theta) P_\theta\right]}$$

$$\boxed{\dot{\theta} = \frac{\partial H}{\partial P_\theta} = \frac{1}{m(\sin\theta - 2ax\cos\theta)^2}\left[-\frac{1}{l}(2ax\sin\theta + \cos\theta) P_x + \frac{1}{l^2}(4a^2x^2+1) P_\theta\right]}$$

$\dot{P}_x = -\frac{\partial H}{\partial x}$: Doing this in steps

$$\frac{d}{dx}\left\{\frac{1}{2m(S\theta - 2axC\theta)^2}\left[P_x^2 - \frac{2}{l}(2axS\theta + C\theta)P_x P_\theta + \frac{1}{l^2}(4a^2x^2+1)P_\theta^2\right]\right\} =$$

$$= \frac{1}{2ml^2(S\theta - 2aC\theta x)^3}\left[4al^2 C\theta P_x^2 + 4a[C\theta + 2aS\theta x]P_\theta^2 - 4alP_xP_\theta[2ax\cos\theta + 2 - S\theta^2]\right]$$

$$= \frac{2a}{m(S\theta - 2aC\theta x)^3}\left[C\theta P_x^2 + \frac{1}{l^2}(C\theta + 2aS\theta x)P_\theta^2 - \frac{1}{l}(2ax\cos\theta + 2 - S\theta^2)P_x P_\theta\right]$$

$$\boxed{\dot{P}_x = -\frac{2a}{m(\sin\theta - 2a\cos\theta x)^3}\left[\cos\theta P_x^2 + \frac{1}{l^2}(\cos\theta + 2ax\sin\theta)P_\theta^2 - (2ax\cos\theta\sin\theta + 2 - \sin^2\theta)\frac{P_x P_\theta}{l}\right] - 2mgax}$$

Now for the last one

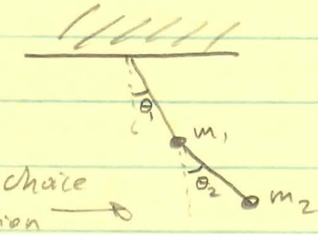
$$\dot{P}_\theta = -\frac{\partial H}{\partial \theta}$$

$$\frac{\partial H}{\partial \theta} = -\frac{1}{m(s\theta - 2axc\theta)^3}\left\{(2axs\theta + c\theta)\left[p_x^2 - \frac{2}{\ell}(2axs\theta + c\theta)p_x p_\theta + \frac{p_\theta^2}{\ell^2}(4a^2x^2+1)\right]\right\}$$

$$+ \frac{(s\theta - 2axc\theta)}{m\ell(s\theta - 2axc\theta)^2}p_x p_\theta + mg\ell s\theta$$

$$\boxed{\begin{aligned}\dot{p}_\theta = \frac{1}{m(\sin\theta - 2ax\cos\theta)^3}\Big\{&(2ax\sin\theta + \cos\theta)\left[p_x^2 + \frac{1}{\ell^2}(4a^2x^2+1)p_\theta^2\right] \\ &-\frac{1}{\ell}\left[2(2ax\sin\theta + \cos\theta)^2 + (\sin\theta - 2ax\cos\theta)^2\right]p_x p_\theta\Big\} \\ &- mg\ell\sin\theta\end{aligned}}$$

Pb. 2 Double pendulum



notice my choice of direction for θ_2.

Positions of masses

$$\bar{r}_1 = l\sin\theta_1\hat{x} - l\cos\theta_1\hat{y}$$

$$\bar{v}_1 = l\cos\theta_1\dot{\theta}_1\hat{x} + l\sin\theta_1\dot{\theta}_1\hat{y}$$

$$v_1^2 = l^2(\cos^2\theta_1\dot{\theta}_1^2 + \sin^2\theta_1\dot{\theta}_1^2) = l^2\dot{\theta}_1^2$$

$$\bar{r}_2 = (l\sin\theta_1 + l\sin\theta_2)\hat{x} - (l\cos\theta_1 + l\cos\theta_2)\hat{y}$$

$$\bar{v}_2 = l(\cos\theta_1\dot{\theta}_1 + \cos\theta_2\dot{\theta}_2)\hat{x} + l(\sin\theta_1\dot{\theta}_1 + \sin\theta_2\dot{\theta}_2)\hat{y}$$

$$v_2^2 = l^2(\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2$$

Kinetic Energy $T = \frac{1}{2}m_1 l^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l^2(\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2$

Matrix $[T_{ij}] = l^2\begin{pmatrix} m_1+m_2 & m_2\cos(\theta_1-\theta_2) \\ m_2\cos(\theta_1-\theta_2) & m_2 \end{pmatrix}$ following order $\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}$

Inverse $$[T_{ij}]^{-1} = \frac{1}{l^2[m_2\cos^2(\theta_1-\theta_2) - (m_1+m_2)]}\begin{pmatrix} -1 & \cos(\theta_1-\theta_2) \\ \cos(\theta_1-\theta_2) & -\frac{m_1+m_2}{m_2}\end{pmatrix}$$

Potential energy
$$\begin{aligned} V &= -m_1 g y_1 - m_2 g y_2 \\ &= -m_1 g l\cos\theta_1 - m_2 g l(\cos\theta_1 + \cos\theta_2) \\ &= -gl(m_1+m_2)\cos\theta_1 - m_2 g l\cos\theta_2 \end{aligned}$$

Note $V = -L_0$ and $a = 0$

So Hamiltonian

$$H = \frac{-1}{2}(P_1 P_2)\frac{1}{l^2(M_1+m_2 s^2\theta)}\begin{pmatrix} -1 & C\theta \\ C\theta & -M/m_2\end{pmatrix}\begin{pmatrix}P_1\\ P_2\end{pmatrix} + V$$

where $s\theta = \sin(\theta_1-\theta_2)$, $C\theta = \cos(\theta_1-\theta_2)$, $M = m_1+m_2$, $P_1 = P_{\theta_1}$, $P_2 = P_{\theta_2}$

$$\boxed{H = \frac{1}{2l^2 D}\left[P_1^2 - 2C\theta P_1 P_2 + \frac{M}{m_2}P_2^2\right] - gl\left[MC\theta_1 + m_2 C\theta_2\right]}$$

where $D = m_1 + m_2 s^2\theta$, $C\theta_1 = \cos\theta_1$, $C\theta_2 = \cos\theta_2$

Hamilton's Eqns of motion

$$\boxed{\begin{aligned}\dot\theta_1 &= \frac{\partial H}{\partial P_1} = \frac{P_1}{l^2 D} - \frac{C\theta}{l^2 D} P_2 = \frac{1}{l^2(m_1+m_2\sin^2(\theta_1-\theta_2))}\left[P_1 - \cos(\theta_1-\theta_2)P_2\right]\\ \dot\theta_2 &= \frac{\partial H}{\partial P_2} = \frac{M}{m_2 l^2 D} P_2 - \frac{C\theta}{l^2 D} P_1 = \frac{-1}{l^2\left[m_1+m_2\sin^2(\theta_1-\theta_2)\right]}\left[\cos(\theta_1-\theta_2)P_1 - \frac{m_1+m_2}{m_2}P_2\right]\end{aligned}}$$

For the others

$$\frac{\partial H}{\partial \theta_1} = \frac{1}{l^2 D} P_1 P_2 S\theta + \frac{1}{l^2 D^2}\left[-m_2 C\theta S\theta\left[P_1^2 + \frac{M}{m_2}P_2^2 - 2P_1P_2 C\theta\right]\right] + glMS\theta_1$$

$$= \frac{S\theta}{l^2 D^2}\left[\left[m_1+m_2(1+C^2\theta)\right]P_1P_2 - m_2 C\theta\left[P_1^2 + \frac{M}{m_2}P_2^2\right]\right] + glMS\theta_1$$

$$= \frac{S\theta}{l^2 D^2}\left[(m_2 C\theta P_1 - MP_2)(P_2 C\theta - P_1)\right] + glMS\theta_1$$

$$\boxed{\dot P_1 = -\frac{\partial H}{\partial \theta_1} = \frac{S\theta}{l^2 D^2}\left[(P_1 - P_2 C\theta)(m_2 C\theta P_1 - MP_2)\right] - gl(m_1+m_2)\sin\theta_1}$$

$$\frac{\partial H}{\partial \theta_2} = \frac{1}{l^2 D^2}\left\{-S\theta\, D\cdot P_1P_2 + m_2 C\theta S\theta\left[-2P_1P_2 C\theta + \frac{M}{m_2}P_2^2 + P_1^2\right]\right\} + glm_2 S\theta_2$$

$$= \frac{S\theta}{l^2 D^2}\left\{-\left[m_1+m_2(1+C^2\theta)\right]P_1P_2 + m_2 C\theta\left(P_1^2 + \frac{M}{m}P_2^2\right)\right\} + glm_2 S\theta_2$$

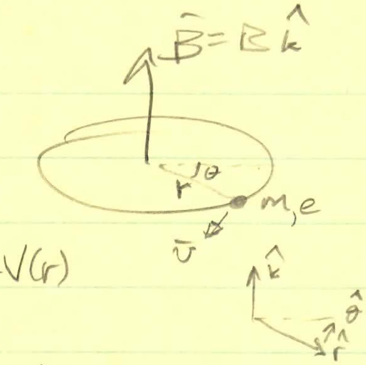
$$= \frac{S\theta}{l^2 D^2}\left\{-(m_2 C\theta P_1 - MP_2)(P_2 C\theta - P_1)\right\} + glm_2 S\theta_2$$

$$\boxed{\dot P_2 = -\frac{\partial H}{\partial \theta_2} = -\frac{S\theta}{l^2 D^2}\left[(P_1 - P_2 C\theta)(m_2 C\theta P_1 - MP_2)\right] - glm_2\sin\theta_2}$$

$S\theta = \sin(\theta_1-\theta_2)$ $\qquad$ $C\theta = \cos(\theta_1-\theta_2)$

$D = m_1 + m_2\sin^2(\theta_1-\theta_2)$ $\qquad$ $M = m_1 + m_2$

defines a uniform B field



Pb. 3 Particle in magnetic field $\bar{A} = \frac{1}{2}\bar{B}\times\bar{r}$

As done in lecture, the Lagrangian is $L = T - V = \frac{1}{2}mv^2 + e\bar{A}\cdot\bar{v} - V(r)$

taking $\bar{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$ and $\bar{r} = r\hat{r}$

then $\bar{A}\cdot\bar{v} = \frac{1}{2}(B\hat{k}\times r\hat{r})\cdot(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$

$$= \frac{1}{2}Br(\hat{\theta})\cdot(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) = \frac{1}{2}Br^2\dot{\theta}$$

$$\Rightarrow \quad L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}eBr^2\dot{\theta} - V(r)$$

Generalized momenta

$$\boxed{p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}} \qquad \boxed{p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} + \frac{1}{2}eBr^2}$$

Hamiltonian $H = \dot{r}p_r + \dot{\theta}p_\theta - L$

$$= m\dot{r}^2 + \dot{\theta}(mr^2\dot{\theta} + \tfrac{1}{2}eBr^2) - \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \tfrac{1}{2}eBr^2\dot{\theta} + V$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + V$$

$\dot{r} = p_r/m$ and $\dot{\theta} = (p_\theta - \frac{1}{2}eBr^2)/mr^2$

so that $H = \dfrac{p_r^2}{2m} + \dfrac{1}{2}\cdot\dfrac{1}{mr^2}(p_\theta - \frac{1}{2}eBr^2)^2 + V$

$$\boxed{H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{eB}{2m}p_\theta + \frac{e^2B^2}{8m}r^2 + V}$$

Note that since $r, \theta \neq x(t), \theta(t)$ and $V = V(r)$

H is total energy, and $H \neq H(t)$ it is conserved

Eqns of motion $\boxed{\dot{r} = \frac{\partial H}{\partial p_r} = p_r/m}$ $\boxed{\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} - \frac{eB}{2m}}$

$$\boxed{\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \left(\frac{eB}{2m}\right)^2 mr - \frac{\partial V}{\partial r}} \qquad \boxed{\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0}$$

Now, lets change coordinates to an axis rotating with $\omega = -\dfrac{eB}{2m}$

$\omega = -\frac{eB}{2m}$ and $\theta' = \theta - \omega t \Rightarrow \theta = \theta' + \omega t$ and $\dot\theta = \dot\theta' + \omega$

Lagrangian in the new coordinate system

$$L = \frac{1}{2}m\left(\dot r^2 + r^2(\dot\theta'^2 + \omega^2 + 2\dot\theta'\omega)\right) + \frac{1}{2}eBr^2(\dot\theta' + \omega) - V$$

$$= \frac{1}{2}m(\dot r^2 + r^2\dot\theta'^2) + \frac{1}{2}m\omega^2 r^2 + m\dot\theta'\omega r^2 - \frac{1}{2}(2m\omega r^2)\dot\theta'$$

$$-\frac{1}{2}\omega 2mr^2\omega - V$$

$$L = \frac{1}{2}m(\dot r^2 + r^2\dot\theta'^2) - \frac{1}{2}m\omega^2 r^2 - V$$

Generalized momenta $\boxed{p_r = \frac{\partial L}{\partial \dot r} = m\dot r}$ $\boxed{p_{\theta'} = \frac{\partial L}{\partial \dot\theta'} = mr^2\dot\theta'}$

Hamiltonian

$$H' = \dot r p_r + \dot\theta' p_{\theta'} - L$$

$$= m\dot r^2 + mr^2\dot\theta'^2 - \frac{1}{2}m(\dot r^2 + r^2\dot\theta'^2) + \frac{1}{2}m\omega^2 r^2 + V$$

$\dot r = p_r/m$ and $\dot\theta' = p_{\theta'}/mr^2$

$$H' = m\frac{p_r^2}{m^2} + mr^2\frac{p_{\theta'}^2}{m^2r^4} - \frac{1}{2}m\left(\frac{p_r^2}{m^2} + r^2\frac{p_{\theta'}^2}{m^2r^4}\right) + \frac{1}{2}m\omega^2 r^2 + V$$

$$\boxed{H' = \frac{p_r^2}{2m} + \frac{p_{\theta'}^2}{2mr^2} + \frac{1}{2}m\omega^2 r^2 + V}$$

Eqns of motion

$$\dot r = \frac{\partial H'}{\partial p_r} = \frac{p_r}{m} \qquad \dot\theta' = \frac{\partial H'}{\partial p_{\theta'}} = \frac{p_{\theta'}}{mr^2}$$

$$\dot p_r = -\frac{\partial H'}{\partial r} = \frac{p_{\theta'}^2}{mr^3} - m\omega^2 r - \frac{\partial V}{\partial r}$$

$$\dot p_{\theta'} = -\frac{\partial H'}{\partial \dot\theta'} = 0$$