QUANTUM

Lecture 1 - Jan 12 reading 1.1-1.3

1.1 Stern-Gerlach experiment

Ag →

* one measurement destroys information

Ag → into S.G. $\hat{z}$ → S_{z+} → $\boxed{S.G.}_{\hat{z}}$ → S_{z+} ⋯> nothing

Ag → into S.G. $\hat{z}$ → S_{z-} → $\boxed{S.G.}_{\hat{z}}$

Ag → into S.G. $\hat{z}$ → S_{z+} → $\boxed{S.G.}_{\hat{x}}$ → S_{x+} → $\boxed{S.G.}_{\hat{z}}$ → S_{z+}

$\boxed{S.G.}_{\hat{z}}$ → S_{z-}

$\boxed{S.G.}_{\hat{x}}$ → S_{x-}

Ag → into S.G. $\hat{z}$ → S_{z-} → $\boxed{S.G.}_{\hat{z}}$

1.2 Kets, bras, operators → Dirac notation

<u>a physical state</u> → state vector ket $|\alpha\rangle$

i.e. which way spin is pointing in $\hat{y}$

kets can be added: $|\alpha\rangle + |\beta\rangle = |\gamma\rangle$

multiplied by a complex number: $c|\alpha\rangle = |\alpha\rangle c$

when $c = 0$ → null ket

when $c \neq 0$ then $c|\alpha\rangle$ and $|\alpha\rangle$ are the same physical state

<u>operator A</u> → measures observables

$A|\alpha\rangle$ measures the observable A of state α

$A|\alpha\rangle = |\beta\rangle$ new state is created because measurements destroy information

<u>eigenkets</u> $A|a'\rangle = a'|a'\rangle$

↳ complex eigenvalue

<u>eigenstates</u> are physical states of eigenkets

$$\underset{\text{operator}}{S_z}\ \underbrace{|S_z;+\rangle}_{\text{this ket describes the state}} = \frac{\hbar}{2}|S_z;+\rangle$$

gives us the measurement and same state back

$$\underset{\text{operator}}{S_z}\ \underset{\text{eigenket}}{|S_z;-\rangle} = \underset{\text{eigenvalue}}{-\frac{\hbar}{2}}\ \underset{\text{eigenket}}{|S_z;-\rangle}$$

contracted notation: $S_x \, | S_x ; \pm \rangle = \pm \frac{\hbar}{2} | S_x ; \pm \rangle$

Dimensionality of eigenspace:

$$|\alpha\rangle = \sum_{a'}^{1 \sim N} c_{a'} \, |a'\rangle$$

any state α can be described as sum of eigenstates with an appropriate complex number in front

Dual correspondance (mirror images):

$$|\alpha\rangle \xleftrightarrow{DC} \langle\alpha|$$

ket bra

$$|\alpha\rangle + |\beta\rangle \longleftrightarrow \langle\alpha| + \langle\beta|$$

$$c|\alpha\rangle \longleftrightarrow c^* \langle\alpha|$$

$$c_\alpha |\alpha\rangle + c_\beta |\beta\rangle \longleftrightarrow c_\alpha^* \langle\alpha| + c_\beta^* \langle\beta|$$

Inner Product:

$$(\langle\alpha|) \cdot (|\beta\rangle) = \langle\alpha|\beta\rangle$$

complex conjugate of α

postulate 1: $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$

$\langle\alpha|\alpha\rangle$ is real

proof: $\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^*$ (both are numbers)

$\langle\alpha|\alpha\rangle = \langle\alpha|\alpha\rangle^*$

$\langle\alpha|\alpha\rangle =$ real

Postulate 2: $\langle\alpha|\alpha\rangle = 0$ where $\langle\alpha|\alpha\rangle \geq 0$ for non-null ket $|\alpha\rangle$

$\langle\alpha|\alpha\rangle = 0$ for null ket $|\alpha\rangle$

$|\alpha\rangle$ and $|\beta\rangle$ are orthogonal if $\langle\alpha|\beta\rangle = 0$ and thus $|\beta|\alpha\rangle = 0$

<u>Normalized kets:</u> $\langle\tilde{\alpha}|\tilde{\alpha}\rangle = 1$

notation is $|\tilde{\alpha}\rangle = \dfrac{1}{\sqrt{\langle\alpha|\alpha\rangle}} |\alpha\rangle$

physical state is same but with multiplicative factor in front

<u>Observables + operators</u>

A
B
C $|\alpha\rangle$ observables

operators $X(|\alpha\rangle) = X|\alpha\rangle = |\beta\rangle$

operators $X = Y$ if $X|\alpha\rangle = Y|\alpha\rangle$

$X =$ null operator if $X|\alpha\rangle = 0$

(erases physical state)

operators are cumulative and associative

$X + Y = Y + X$ implied: $X|\alpha\rangle + Y|\alpha\rangle$

$X + (Y+Z) = (X+Y) + Z$

operators don't exist on their own

operators are linear

$$X\,(c_\alpha|\alpha\rangle + c_\beta|\beta\rangle) = c_\alpha \underbrace{X|\alpha\rangle}_{|\gamma\rangle} + c_\beta \underbrace{X|\beta\rangle}_{|\delta\rangle}$$

$$(\langle\alpha|)X = \langle\alpha|X = \langle\beta|$$

direction matters in terms of bras + kets

$$X|\alpha\rangle \longleftrightarrow \langle\alpha|X \quad \Rightarrow \text{WRONG}$$

$$X|\alpha\rangle \longleftrightarrow \langle\alpha|X^\dagger \quad \Rightarrow \text{CORRECT}$$

i.e. $|\gamma\rangle \longleftrightarrow \langle\gamma|$ because new states created

$X^\dagger$ is the Hermitian adjoint of X

a Hermitian operator is if $X = X^\dagger$

* for kets, operate left to right

$$X|\alpha\rangle$$

for bras, operate right to left

$$\langle\alpha|X$$

* only kets are states, <u>not</u> bras

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$$X|\alpha\rangle \longleftrightarrow \langle\alpha|X^{\dagger}$$

$$|\gamma\rangle \longleftrightarrow \langle\gamma|$$

"new states" are also dual correspondances

operator multiplication is non-commutative

$$XY \neq YX$$

operators <u>are</u> associative

$$X(YZ) = (XY)Z$$

$$X(Y|\alpha\rangle) = (XY)\underbrace{|\alpha\rangle}_{\text{ket}}$$

$$(\underbrace{\langle\beta|}_{\text{bra}}X)Y = \langle\beta|XY$$

$$\boxed{\begin{aligned} XY|\alpha\rangle &\longleftrightarrow \langle\alpha|(XY)^{\dagger} \\ (XY)^{\dagger} &= Y^{\dagger}X^{\dagger} \end{aligned}}$$

$$(XY)|\alpha\rangle = X(Y|\alpha\rangle)$$

$$\updownarrow \qquad\qquad \updownarrow$$

$$\langle\alpha|(XY)^{\dagger} = (\langle\alpha|Y^{\dagger})X^{\dagger}$$

$$\langle\alpha|(XY)^{\dagger} = \langle\alpha|Y^{\dagger}X^{\dagger}$$

$$(\underbrace{\langle\alpha|Y^{\dagger}})X^{\dagger}$$

do this first then operate $X^{\dagger}$ by it

Outer Product: an operator

$$(|\beta\rangle)(\langle\alpha|) = |\beta\rangle\langle\alpha|$$

so $$(|\beta\rangle\langle\alpha|)|\gamma\rangle = \underbrace{|\beta\rangle}_{\text{ket}}\underbrace{\langle\alpha|\gamma\rangle}_{\text{complex number}}$$

$$= \langle\alpha|\gamma\rangle\,|\beta\rangle$$

our operator turned state $|\gamma\rangle$ into new state $|\beta\rangle$
(operators change kets most of the time)

$$X = |\beta\rangle\langle\alpha| \quad \text{then} \quad X^\dagger = |\alpha\rangle\langle\beta|$$

prove:

$$\underbrace{\langle\beta|X|\alpha\rangle}_{\text{a number}} = \langle\alpha|X^\dagger|\beta\rangle^*$$

let $X|\alpha\rangle = |\gamma\rangle$

then $\langle\beta|\gamma\rangle = \langle\gamma|\beta\rangle^*$

if $|\gamma\rangle = X|\alpha\rangle$ then $\langle\gamma| = \langle\alpha|X^\dagger$

so $\langle\beta|\gamma\rangle = \langle\beta|X|\alpha\rangle = \langle\alpha|X^\dagger|\beta\rangle$

1.3 Base kets + Matrix Operation

A Hermitian operator A is an observable

Theorem: then eigenvalues of A are real and eigen kets of A are orthogonal

Proof: $A|a'\rangle = a'|a'\rangle$

operator ket num ket

so $\langle a''|A = a''^* \langle a''|$

$$c|\alpha\rangle \xleftrightarrow{DC} c^*\langle\alpha|$$

$$\langle a''|A|a'\rangle = a'\langle a''|a'\rangle \quad (1)$$

$$\langle a''|A|a'\rangle = a''^*\langle a''|a'\rangle \quad (2)$$

complex num complex num

(1) − (2): $0 = (a' - a''^*)\langle a''|a'\rangle$ if A Hermitian

- if $a'' = a'$ then $0 = (a' - a'^*)\langle a'|a'\rangle$

  then $a' = a'^*$ $\rightarrow$ $\boxed{a' \text{ is real}}$

- if $a'' \neq a'$ then $\boxed{\langle a''|a'\rangle = 0}$ eigenkets are orthogonal

$$* \quad A|a'\rangle = a'|a'\rangle$$

$$\updownarrow \qquad\qquad \updownarrow$$

$$\langle a'|A = \langle a'|a'^*$$

can take the DC of both sides to rewrite equation — if 2 things equal in normal space, they also equal in DC space

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eigenkets (cont)

$\{a'\} = a', a'', a''' \ldots$ eigenvalues are real

$|a'\rangle$ are orthogonal

$|\alpha\rangle$ can be spanned by $\{|a'\rangle\}$ orthogonal

and eigenkets span the whole ket space

$$|\alpha\rangle = \sum_{a'=1}^{N} C_{a'} |a'\rangle$$

$$= C_{a'}|a'\rangle + C_{a''}|a''\rangle + C_{a'''}|a'''\rangle \ldots$$

$$\langle a''|\alpha\rangle = \sum_{a'=1}^{N} \langle a''| C_{a'} |a'\rangle$$

$$= \sum_{a'=1}^{N} C_{a'} \underbrace{\langle a''|a'\rangle}_{\delta_{a''a'}} = C_{a''}$$

sub in $C_{a'} = \langle a'|\alpha\rangle$

$$|\alpha\rangle = \sum_{a'=1}^{N} \langle a'|\alpha\rangle |a'\rangle = \underbrace{\sum_{a'=1}^{N} |a'\rangle\langle a'|}_{\text{identity operator}} |\alpha\rangle$$

completeness relation:

$$\boxed{\sum_{a'=1}^{N} |a'\rangle\langle a'| = \mathbb{1}}$$

example of why it's useful:

$$\langle\alpha|\alpha\rangle = \langle\alpha|\left(\sum_{a'} |a'\rangle\langle a'|\right)|\alpha\rangle$$

complex number

$$= \sum_{a'} \langle\alpha|a'\rangle\langle a'|\alpha\rangle$$

$$= \sum_{a'} \langle a'|\alpha\rangle^* \langle a'|\alpha\rangle$$

$$= \sum_{a'} |\langle a'|\alpha\rangle|^2$$

$$= \sum_{a'} |\underbrace{c_{a'}}_{\text{expansion coefficients}}|^2 = 1$$

if $|\alpha\rangle$ is normalized!

$$A = \sum_{a'}\sum_{a''} \underbrace{|a''\rangle\langle a''|}_{\mathbb{1}} A \underbrace{|a'\rangle\langle a'|}_{\mathbb{1}}$$

= operator · operator · operator

= $|a''\rangle \cdot \langle a''|A|a'\rangle \cdot \langle a'|$

complex number

outer product = operator

so $A = \sum_{a'} \sum_{a''} |a'\rangle \underbrace{\langle a''|A|a'\rangle}_{a'\langle a''|a'\rangle \,=\, a'\delta_{a''a'}} \langle a'|$

$$= \sum_{a'} \sum_{a''} |a''\rangle \left(a'\delta_{a''a'}\right) \langle a'|$$

$$= \sum_{a'} \sum_{a''} a' \, \delta_{a''a'} \, |a''\rangle\langle a'|$$

$$\boxed{A = \sum_{a'} a' \, |a'\rangle\langle a'|}$$

Spin 1/2 system:

$|S_z; \pm\rangle \Rightarrow |+\rangle, |-\rangle$ in S_z basis

$$\sum_{a'=1}^{N} |a'\rangle\langle a'| = \mathbb{1} = \underbrace{|+\rangle\langle +|}_{\text{operator}} + \underbrace{|-\rangle\langle -|}_{\text{operator}}$$

$$A = \sum_{a'} a' \, |a'\rangle\langle a'|$$

$$S_z = \frac{\hbar}{2} |+\rangle\langle +| + \frac{-\hbar}{2} |-\rangle\langle -|$$

$$S_z |+\rangle = \left(\frac{\hbar}{2}|+\rangle\langle +| - \frac{\hbar}{2}|-\rangle\langle -|\right)|+\rangle$$

$$= \frac{\hbar}{2}|+\rangle \underbrace{\langle +|+\rangle}_{1} - \frac{\hbar}{2}|-\rangle \underbrace{\langle -|+\rangle}_{0}$$

$$= \frac{\hbar}{2}|+\rangle$$

Matrix Representation:

$$X = \sum_{a'=1}^{N} \sum_{a''=1}^{N} |a''\rangle \underbrace{\langle a''|X|a'\rangle}_{\text{total of } N^2 \text{ numbers}} \langle a'|$$

represented by

$$X \doteq \begin{pmatrix} \langle a_1|X|a_1\rangle & \langle a_1|X|a_2\rangle & \cdots \\ \langle a_2|X|a_1\rangle & \ddots & \\ \vdots & & \ddots \end{pmatrix}$$

where $a' = a_1$
$a'' = a_2$
$a''' = a_3$

$\langle a_1|X|a_2\rangle$
"row" "column"

if $X = X^\dagger$ (X is Hermitian), $\langle a''|X|a'\rangle = \langle a'|X|a''\rangle^*$

$Z = XY$

$$\langle a''|Z|a'\rangle = \langle a''|XY|a'\rangle$$

$$= \sum_{a'''} \underbrace{\langle a''|X|a'''\rangle}_{N \times N} \underbrace{\langle a'''|Y|a'\rangle}_{N \times N}$$

$$|\alpha\rangle = \sum_{a'}^{N} |a'\rangle \underbrace{\langle a'|\alpha\rangle}_{N \text{ total numbers}} \doteq \begin{pmatrix} \langle a_1|\alpha\rangle \\ \langle a_2|\alpha\rangle \\ \vdots \end{pmatrix}$$

$$|\gamma\rangle = X|\alpha\rangle$$

$$\underbrace{|\gamma\rangle}_{N\times 1} = \sum_{a'}\sum_{a''} |a''\rangle \underbrace{\langle a''|X|a'\rangle}_{N\times N} \underbrace{\langle a'|\alpha\rangle}_{N\times 1}$$

$$\langle\alpha| \doteq (\langle\alpha|a_1\rangle, \langle\alpha|a_2\rangle \ldots)$$

$$\langle\beta|\alpha\rangle = \sum_{a'} \underbrace{\langle\beta|a'\rangle}_{(1\times N)} \underbrace{\langle a'|\alpha\rangle}_{\begin{pmatrix} N \times 1 \end{pmatrix}} = (1\times 1)$$

number

$$|\beta\rangle\langle\alpha| = \begin{pmatrix} N \times 1 \end{pmatrix} (1\times N) = (N\times N)$$

operator

for our spin 1/2 system:

$$|+\rangle = \begin{pmatrix} \langle +|+\rangle \\ \langle -|+\rangle \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|-\rangle = \begin{pmatrix} \langle +|-\rangle \\ \langle -|-\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$S_z = \frac{\hbar}{2}|+\rangle\langle +| - \frac{\hbar}{2}|-\rangle\langle -|$$

$$= \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}(1\ 0) - \frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}(0\ 1)$$

$$= \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

HW1 Show eq 1.50, 1.5, 1.7

eq 1.50: Show $X^\dagger = |\alpha\rangle\langle\beta|$ if $X = |\beta\rangle\langle\alpha|$

$$X|\gamma\rangle = |\beta\rangle\langle\alpha|\gamma\rangle$$

DC $\updownarrow$ DC $\updownarrow$

$$\langle\gamma|X^\dagger = \langle\beta|\langle\alpha|\gamma\rangle^* = \langle\beta|\langle\gamma|\alpha\rangle$$

$$\langle\gamma|X^\dagger = \langle\gamma|\alpha\rangle\langle\beta|$$

$$\boxed{X^\dagger = |\alpha\rangle\langle\beta|}$$

1.7 Consider ket space spanned by $\{|a'\rangle\}$ of Hermitian operator A

a. Show $\prod_{a'}(A-a')$ is null operator

$$A|a'\rangle = a'|a'\rangle$$

$$\prod_{a'}(A-a')|\alpha\rangle = \prod_{a'}(A-a')\sum_{a''}|a''\rangle\langle a''|\alpha\rangle$$

$$= \sum_{a''} C_{a''}\prod_{a'}(A-a')|a''\rangle$$

$$= \sum_{a''} C_{a''}\prod_{a'}(a''-a')|a''\rangle = 0$$

for $a'' \in \{a'\}$

b. Explain $\prod_{a''\neq a'} \frac{A-a''}{a'-a''}$

$$\left(\prod_{a''\neq a'} \frac{A-a''}{a'-a''}\right)|\alpha\rangle$$

$$= \left(\prod_{a''\neq a'} \frac{A-a''}{a'-a''}\right) \sum_{a'''} |a'''\rangle \underbrace{\langle a'''|\alpha\rangle}_{\text{num}}$$

$$= \sum_{a'''} \langle a'''|\alpha\rangle \prod_{a''\neq a'} \frac{a'''-a''}{a'-a''} |a'''\rangle$$

$$= \sum_{a'''} \langle a'''|\alpha\rangle \underbrace{\delta_{a'''a'}}_{\text{gives 0 unless } a'''=a'} |a'''\rangle$$

$$= \langle a'|\alpha\rangle |a'\rangle = |a'\rangle\langle a'|\alpha\rangle$$

so $\boxed{\prod_{a''\neq a'} \frac{A-a''}{a'-a''} = |a'\rangle\langle a'|}$

just projects to eigenket $|a'\rangle$

c. Illustrate a) and b) using A set equal to S_z

$$a' = \frac{\hbar}{2} \qquad a'' = -\frac{\hbar}{2} \qquad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\prod_{a'} (A - a') = \left(S_z + \frac{\hbar}{2}\right)\left(S_z - \frac{\hbar}{2}\right)$$

$$= \frac{\hbar}{2}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \frac{\hbar}{2}\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \frac{\hbar^2}{4}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = 0 \quad \checkmark$$

pick $a' = \frac{\hbar}{2}$

operator from (b):

$$\prod_{a'' \neq a'} \frac{A - a''}{a' - a''} = \frac{S_z - \frac{\hbar}{2}}{\frac{\hbar}{2} - \left(-\frac{\hbar}{2}\right)}$$

$$= \frac{S_z - \frac{\hbar}{2}}{\hbar} = \frac{S_z}{\hbar} + \frac{1}{2}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

since we picked $a' = \frac{\hbar}{2}$, our operator projected onto state $|S_z; +\rangle = |S_z = \hbar/2\rangle$

1.5

a) consider $|\alpha\rangle$ and $|\beta\rangle$ and form set of base kets $\{|a'\rangle\}$. Find matrix representation of $|\alpha\rangle\langle\beta|$

so expand $|\alpha\rangle = \langle a'|\alpha\rangle|a'\rangle + \langle a''|\alpha\rangle|a''\rangle \ldots$

$|\beta\rangle = \langle a'|\beta\rangle|a'\rangle + \langle a''|\beta\rangle|a''\rangle \ldots$

$\langle\beta| = \langle a'|\beta\rangle\langle a'| + \langle a''|\beta\rangle\langle a''|$

now $|\alpha\rangle\langle\beta| = \langle a'|\alpha\rangle\langle a'|\beta'\rangle|a'\rangle\langle a'|$

$+ \langle a'|\alpha\rangle\langle a''|\beta'\rangle|a'\rangle\langle a''|$

$+ \langle a''|\alpha\rangle\langle a'|\beta\rangle|a''\rangle\langle a'|$

$+ \langle a''|\alpha\rangle\langle a''|\beta\rangle|a''\rangle\langle a''| \ldots$

b) consider spin 1/2 system. $|\alpha\rangle = |S_z = \hbar/2\rangle$

$|\beta\rangle = |S_x = \hbar/2\rangle$

write square matrix of $|\alpha\rangle\langle\beta|$ in S_z basis

$|S_z = \hbar/2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|S_x = \hbar/2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\langle S_x = \hbar/2| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}$

$|S_z = \hbar/2\rangle\langle S_x = \hbar/2| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\boxed{|\alpha\rangle\langle\beta| = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}}$$

Lecture 4 - Jan 21

reading: 1.4

HW 2: 2nd ed: explain eq 1.4.49, 1.9, 1.10

1.4 Measurements, observables, uncertainty

Dirac: measurements cause system to "jump" to an eigenstate of the dynamical variable being measured

(think of Stern-Gerlach experiment)

↳ not just any eigenstate observed, only in the direction of measurement

Before $|\alpha\rangle = \sum_{a'} c_{a'} |a'\rangle$

$$= \sum_{a'} \langle a'|\alpha\rangle |a'\rangle$$

measure $|\alpha\rangle \longrightarrow |a'\rangle$

$|a'\rangle \longrightarrow |a'\rangle$

the probability to measure a' is $|\langle a'|\alpha\rangle|^2$

Fundamental Postulate of Quantum Mechanics

expectation value

$\equiv$ means define

$$\langle A\rangle \equiv \langle\alpha|A|\alpha\rangle$$

expectation value is state-dependent!

and is a number

$$\langle A\rangle \equiv \langle\alpha|\overset{\mathbb{1}}{\downarrow}A\overset{\mathbb{1}}{\downarrow}|\alpha\rangle$$ insert one's

$$= \sum_{a'}\sum_{a''} \langle\alpha|\left(|a'\rangle\langle a'|\right)A\left(|a''\rangle\langle a''|\right)|\alpha\rangle$$

$$= \sum_{a'}\sum_{a''} \langle\alpha|a'\rangle\langle a'|A|a''\rangle\langle a''|\alpha\rangle$$

$$= \sum_{a'}\sum_{a''} \langle\alpha|a'\rangle\underbrace{\langle a'|a''|a''\rangle}_{a''\underbrace{\langle a'|a''\rangle}_{\delta_{a'a''}}}\langle a''|\alpha\rangle$$

now a number

$$= \sum_{a'} \underbrace{\langle\alpha|a'\rangle}_{\langle a'|\alpha\rangle^*} a' \langle a'|\alpha\rangle$$

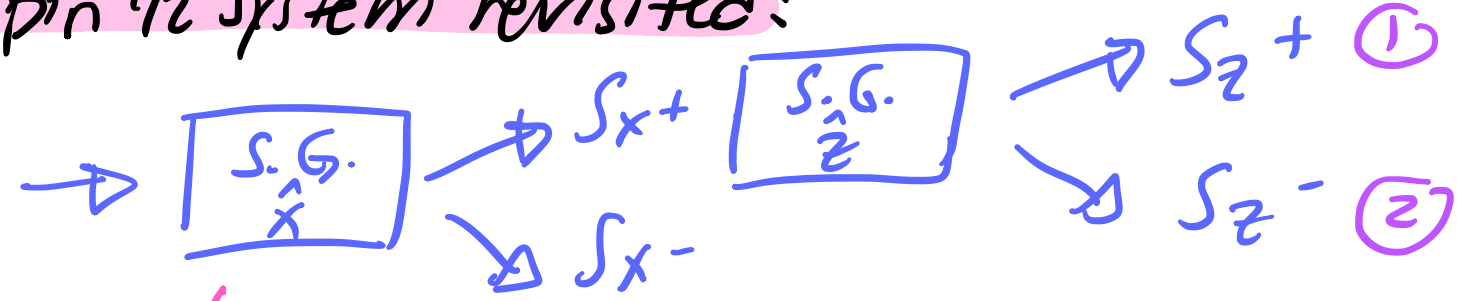
$$\boxed{\langle A\rangle = \sum_{a'} \underset{\text{eigenvalue}}{a'}\ \underbrace{|\langle a'|\alpha\rangle|^2}_{\text{probability}}}$$

expectation value isn't just an eigenvalue

$\downarrow$ quantized

can be anything in between min a' and max a'

spin 1/2 system revisited:



$$|\langle \overbrace{S_{z+}}^{a'} | \underbrace{S_{x+}}_{\alpha} \rangle|^2 = \frac{1}{\sqrt{2}}$$

$$= |\langle S_{z-} | S_{x+} \rangle|^2$$

means given I started with state α, the probability of measuring a'

$|\langle S_{z+} | S_{x+} \rangle|^2$ is in S_z eigenbasis so need to rewrite $\alpha = S_{x+}$

complex phase

$$|S_{x+}\rangle = \frac{1}{\sqrt{2}} |S_{z+}\rangle + \frac{1}{\sqrt{2}} \boxed{e^{i\delta}} |S_{z-}\rangle$$

* $|e^{i\delta}|$ goes away

$$|S_{x-}\rangle = \frac{1}{\sqrt{2}} |S_{z+}\rangle - \frac{1}{\sqrt{2}} e^{i\delta} |S_{z-}\rangle$$

* $|S_{x+}\rangle$ and $|S_{x-}\rangle$ are orthogonal (inner product is zero)

$$\langle S_x+|S_x-\rangle = \left[\frac{1}{\sqrt{2}}\langle S_x+| + \frac{1}{\sqrt{2}}e^{-i\delta}\langle S_z-|\right]$$

$$\left[\frac{1}{\sqrt{2}}|S_z+\rangle - \frac{1}{\sqrt{2}}e^{i\delta}|S_z-\rangle\right]$$

$$= \frac{1}{2} - \frac{1}{2} = 0$$ * orthogonal!

Since we chose S_z eigenbasis, can write in shorthand:

$$|S_x+\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}e^{i\delta}|-\rangle$$

$$|S_x-\rangle = \frac{1}{\sqrt{2}}|-\rangle - \frac{1}{\sqrt{2}}e^{i\delta}|+\rangle$$

can construct S_x operator:

* $(e^{i\delta})^* = e^{-i\delta}$

$$S_x = \sum_{a'} a'|a'\rangle\langle a'|$$

$$= \frac{\hbar}{2}|S_x+\rangle\langle S_x+| - \frac{\hbar}{2}|S_x-\rangle\langle S_x-|$$

now need to plug in our $|S_x+\rangle$ and $|S_x-\rangle$

$$S_x = \frac{\hbar}{2}\left[\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}e^{i\delta}|-\rangle\right]\left[\frac{1}{\sqrt{2}}\langle+| + \frac{1}{\sqrt{2}}e^{-i\delta}\langle-|\right]$$

$$-\frac{\hbar}{2}\left[\frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}e^{i\delta}|-\rangle\right]\left[\frac{1}{\sqrt{2}}\langle+| - \frac{1}{\sqrt{2}}e^{i\delta}\langle-|\right]$$

$$S_x = \frac{\hbar}{2}\left[\frac{1}{2}e^{i\delta}|-\rangle\langle+| + \frac{1}{2}e^{-i\delta}|+\rangle\langle-|\right]$$

$$-\frac{\hbar}{2}\left[-\frac{1}{2}e^{-i\delta}|+\rangle\langle-| - \frac{1}{2}e^{i\delta}|-\rangle\langle+|\right]$$

$$S_x = \frac{\hbar}{2}\left[e^{i\delta}|-\rangle\langle+| + e^{-i\delta}|+\rangle\langle-|\right]$$

- only cross terms survive, Hermitian operator

in its own S_x basis:

$$S_x = \frac{\hbar}{2}\left[\qquad\qquad\right] \quad ??$$

proving $S_x = S_x{}^\dagger$ (Hermitian)

$$S_x{}^\dagger = \frac{\hbar}{2}\left[|+\rangle\langle-|e^{-i\delta} + |-\rangle e^{i\delta}\langle+|\right] = S_x$$

find δ in S_z basis:

$$S_x = \frac{\hbar}{2}\left[e^{i\delta_1}|-\rangle\langle+| + e^{-i\delta_1}|+\rangle\langle-|\right]$$

$$S_y = \frac{\hbar}{2}\left[e^{i\delta_2}|+\rangle\langle-| + e^{-i\delta_2}|-\rangle\langle+|\right]$$

then $|\langle S_y \pm | S_x ; + \rangle| = |\langle S_y \pm | S_x ; - \rangle| = \frac{1}{\sqrt{2}}$

$$\left|\left(\frac{1}{\sqrt{2}}\langle+| \pm \frac{1}{\sqrt{2}}e^{-i\delta_2}\langle-|\right)\left(\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}e^{i\delta_1}|-\rangle\right)\right|$$

$$= \left|\frac{1}{2} + \frac{1}{2}e^{i(\delta_1-\delta_2)}\right| = \frac{1}{\sqrt{2}}$$

$$= \left|1 \pm e^{i(\delta_1-\delta_2)}\right| = \sqrt{2}$$

$$e^{i(\delta_1-\delta_2)} = i$$

$$\boxed{\delta_1 - \delta_2 = \frac{\pi}{2} \text{ or } -\frac{\pi}{2}}$$

* check

$|1 \pm i| = \sqrt{2}$

by convention, we choose S_x real so $\delta_1 = 0$ and then $\delta_2 = \frac{\pi}{2}$ (gives us a right hand coordinate system)

now can plug in δ_1 and δ_2:

$$|S_x; \pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{1}{\sqrt{2}}|-\rangle$$

$$|S_y; \pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{i}{\sqrt{2}}|-\rangle$$

$$S_x = \frac{\hbar}{2}\left[|+\rangle\langle-| + |-\rangle\langle+|\right]$$

$$S_y = \frac{\hbar}{2}\left[-i|+\rangle\langle-| + i|-\rangle\langle+|\right]$$

$$S_z = \frac{\hbar}{2}\left[|+\rangle\langle+| - |-\rangle\langle-|\right]$$

* all states + operators in S_z basis

* check: $e^{i(\pi/2)} = i$

with $|S_z; \pm\rangle \equiv |\pm\rangle$

$$S_\pm \equiv S_x \pm i S_y$$

commutation:

$$[A, B] = AB - BA$$ operator!

anti-commutation:

$$\{A, B\} \equiv AB + BA$$ also operator!

HW 2 eq 1.4.49, 1.9, 1.10 2nd ed

1) Explain eq 1.4.49:

$[A,B]=0$ $\qquad$ $[A,B]=AB-BA$

if $AB|\psi\rangle = BA|\psi\rangle$ then $AB|\psi\rangle - BA|\psi\rangle = 0$

and the operators A and B commute

and are compatible observables with

simultaneous eigenkets $\{K'\}$

with $|K'\rangle = |a',b',c'\rangle \ldots$

apply orthonormality:

$$\langle K''|K'\rangle = \delta_{K'K''} = \delta_{aa'}\delta_{bb'}\delta_{cc'}\ldots$$

then $\sum_{b'} \langle c'|b'\rangle\langle b'|a'\rangle\langle a'|b'\rangle\langle b'|c'\rangle$

$$= \sum_{b'}\sum_{b''} \langle c'|b'\rangle\langle b'|a'\rangle\langle a'|b''\rangle\langle b''|c'\rangle$$

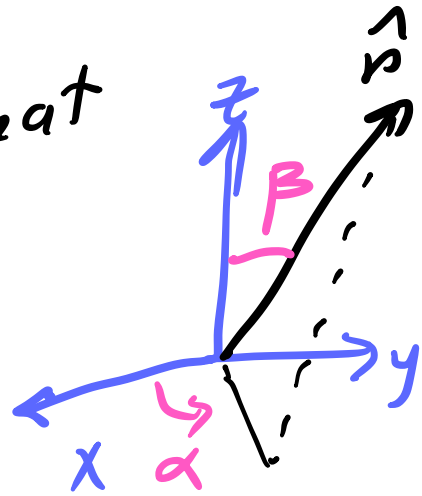
when $[A,B]=0$ or $[B,C]=0$ and

we can use orthonormality $\langle K''|K'\rangle$ of

simultaneous eigenkets to turn b'' to b'

and sum over only b'

<u>1.9</u> construct $|S\cdot\hat{n};+\rangle$ such that



$$S\cdot\hat{n}\,|S\cdot\hat{n};+\rangle = \frac{\hbar}{2}|S\cdot\hat{n};+\rangle$$

$$|S\cdot\hat{n};+\rangle = a|+\rangle + b|-\rangle$$

$$\hat{n} = n_x\hat{x} + n_y\hat{y} + n_z\hat{z}$$

$$= \cos\alpha\sin\beta\,\hat{x} + \sin\alpha\sin\beta\,\hat{y} + \cos\beta\,\hat{z}$$

then let $S\cdot\hat{n}\,|S\cdot\hat{n};+\rangle = \frac{\hbar}{2}|S\cdot\hat{n};+\rangle$

$$S\cdot\hat{n} = S_x n_x + S_y n_y + S_z n_z$$

$$S\cdot\hat{n} = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\cos\alpha\sin\beta + \frac{\hbar}{2}\begin{pmatrix}0&i\\-i&0\end{pmatrix}\sin\alpha\sin\beta + \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\cos\beta$$

$$S\cdot\hat{n} = \frac{\hbar}{2}\begin{pmatrix}\cos\beta & e^{-i\alpha}\sin\beta\\ e^{i\alpha}\sin\beta & -\cos\beta\end{pmatrix}$$

plug into: $S\cdot\hat{n}\,|S\cdot\hat{n};+\rangle = \frac{\hbar}{2}|S\cdot\hat{n};+\rangle$

$$\cancel{\frac{\hbar}{2}}\begin{pmatrix}\cos\beta & e^{-i\alpha}\sin\beta\\ e^{i\alpha}\sin\beta & -\cos\beta - 1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \cancel{\frac{\hbar}{2}}\begin{pmatrix}a\\b\end{pmatrix}$$

$$a\cos\beta + b e^{-i\alpha}\sin\beta = a$$

$$a(1-\cos\beta) = b e^{-i\alpha}\sin\beta$$

* trig identities

$1-\cos\alpha = 2\sin^2(\beta/2)$

$\sin\beta = 2\sin(\beta/2)\cos(\beta/2)$

so $a \cdot 2\sin^{\cancel{2}}(\beta/2) = b\, e^{-i\alpha}\, \cancel{\sin(\beta/2)}\cos(\beta/$

$a\sin(\beta/2) = b\, e^{-i\alpha}\cos(\beta/2)$

normalization says $|a^2| + |b^2| = 1$

so up to a phase, $a = \cos(\beta/2)$

$b = e^{i\alpha}\sin(\beta/2)$

then $|S\cdot\hat{n};+\rangle = a|+\rangle + b|-\rangle$

$$\boxed{|S\cdot\hat{n};+\rangle = \cos(\beta/2)|+\rangle + e^{i\alpha}\sin(\beta/2)|-\rangle}$$

1.10 $H = a(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|)$

Find energy eigenvalues and corresponding energy eigenkets (as a linear combo)

want to solve $H|E_i\rangle = E|E_i\rangle$

$$H = a\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

so $$a\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_i \\ y_i \end{pmatrix} = a\lambda_i \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

$$a\underbrace{\begin{pmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix}}_{\det = 0}\begin{pmatrix} x_i \\ y_i \end{pmatrix} = 0$$

$$(1-\lambda)(-1-\lambda) - 1 = 0$$

$$-1 + \lambda^2 = 1 \quad \text{so } \lambda^2 = 2 \qquad \lambda_i = \pm\sqrt{2}$$

then $\boxed{E = a\lambda_i = \pm a\sqrt{2}}$ eigenvalues

$$\cancel{a}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_i \\ y_i \end{pmatrix} = \pm\cancel{a}\sqrt{2}\begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

$$x_i + y_i = \sqrt{2}\,x_i \qquad y_i = x_i(\sqrt{2}-1)$$

$$x_i - y_i = -\sqrt{2}\,y_i \qquad x_i = y_i(1-\sqrt{2})$$

so $|E_1\rangle = x_1 \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}$

$|E_2\rangle = x_2 \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix}$

need to normalize: $1 = \langle E_1|E_1\rangle = \langle E_2|E_2\rangle$

$$1 = \langle E_1|E_1\rangle = |x_1|^2 (1+2+1-2\sqrt{2})$$
$$= |x_1|^2 (4-2\sqrt{2}) = 1$$

then $x_1 = \sqrt{\frac{1}{4-2\sqrt{2}}}$

$$1 = \langle E_2|E_2\rangle = |x_2|^2 (1+2+1+2\sqrt{2})$$
$$= |x_2|^2 (4+2\sqrt{2}) = 1$$

$$x_2 = \sqrt{\frac{1}{4+2\sqrt{2}}}$$

so
$$\boxed{\begin{aligned} |E_1\rangle &= \frac{1}{\sqrt{4-2\sqrt{2}}} \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix} \\ |E_2\rangle &= \frac{1}{\sqrt{4+2\sqrt{2}}} \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix} \end{aligned}}$$

eigenkets

4. Prove $[\vec{S}^2, S_i] = 0$ let $S_i = S_x$

$$(S_xS_x + S_yS_y + S_zS_z)S_x - S_x(S_xS_x + S_yS_y + S_zS_z)$$

$$= \cancel{S_x^3} + S_y^2S_x + S_z^2S_x - \cancel{S_x^3} - S_xS_y^2 - S_xS_z^2$$

$$= S_y^2S_x + S_z^2S_x - S_xS_y^2 - S_xS_z^2$$

$$= [S_y^2, S_x] + [S_z^2, S_x]$$

$$= S_y[S_y, S_x] + S_z[S_z, S_x]$$

we know $[S_i, S_j] = i\varepsilon_{ijk}\hbar S_k$

xyzxyz

$$= S_y \cdot i\hbar(-1)S_z + S_z \cdot (i\hbar)S_y = 0$$

Prove $\vec{S}^2 = \frac{3}{4}\hbar \mathbb{1}$

rewrite $\vec{S}^2$ using anti commutator:

$$\vec{S}^2 = S_xS_x + S_yS_y + S_zS_z$$

$$= \frac{1}{2}\{S_x, S_x\} + \frac{1}{2}\{S_y, S_y\} + \frac{1}{2}\{S_z, S_z\}$$

we know $\{S_x, S_x\} = 2S_x = \frac{\hbar^2}{2}$

$$\vec{S}^2 = \frac{\hbar^2}{4} + \frac{\hbar^2}{4} + \frac{\hbar^2}{4} = \frac{3}{4}\hbar^2 \mathbb{1}$$

Lecture 5 - Jan 26

reading: 1.4 and 1.5

$$|S_x; \pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{1}{\sqrt{2}}|-\rangle$$

$$|S_y; \pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{i}{\sqrt{2}}|-\rangle$$

$$|S_z; \pm\rangle = |\pm\rangle$$

$$S_x = \frac{\hbar}{2}\left[|+\rangle\langle -| + |-\rangle\langle +|\right]$$

$$S_y = \frac{\hbar}{2}\left[-i|+\rangle\langle -| + i|-\rangle\langle +|\right]$$

$$S_z = \frac{\hbar}{2}\left[|+\rangle\langle +| - |-\rangle\langle -|\right]$$

$$[A,B] \equiv AB - BA \qquad [S_i, S_j] = i\,\varepsilon_{ijk}\,\hbar\, S_k$$

$$\{A,B\} = AB + BA \qquad \{S_i, S_j\} = \frac{\hbar^2}{2}\delta_{ij}\,\mathbb{1}$$

$\varepsilon_{ijk} = 1$ if ijk in order i.e. $xyzxyz$

$= -1$ if not in order

$= 0$ if $i, j,$ or k repeated

$S_\pm \equiv S_x \pm i S_y \rightarrow$ non-Hermitian (no observable in this basis)

$$\vec{S}^2 \equiv \vec{S}\cdot\vec{S} = S_x S_x + S_y S_y + S_z S_z$$

$$\boxed{\begin{aligned} \vec{S}^2 &= \frac{3}{4}\hbar^2\,\mathbb{1} \\ [\vec{S}^2, S_i] &= 0 \end{aligned}}$$

$$(S_xS_x + S_yS_y + S_zS_z)S_x$$

Prove $[\vec{S}^2, S_i] = 0$

$$- S_x(S_xS_x + S_yS_y + S_zS_z)$$

$$= \cancel{S_x^3} + S_y^2S_x + S_z^2S_x - \cancel{S_x^3} - S_xS_y^2 - S_xS_z^2 = 0$$

$$S_y^2S_x + S_z^2S_x - S_xS_y^2 - S_xS_z^2 = 0$$

$$= [S_y^2, S_x] + [S_z^2, S_x] = 0$$

compatible observables:

if $[A,B] = 0$ then A and B are compatible

if $[A,B] \neq 0$ then they are incompatible

Theorem: if A and B are compatible and $\{a'\}$ are non-degenerate (each is a diff number) then matrix w/ elements $\langle a''|B|a'\rangle$ is diagonal

Proof: $\langle a''|[A,B]|a'\rangle$

$$= \langle a''|AB|a'\rangle - \langle a''|BA|a'\rangle$$

$$= a''\langle a''|B|a'\rangle - a'\langle a''|B|a'\rangle$$

$$= (a'' - a')\langle a''|B|a'\rangle = 0$$

when $a'' = a'$ then $\langle a''|B|a'\rangle \neq 0$

when $a'' \neq a'$ then $\langle a''|B|a'\rangle = 0$

↳ this case is for non-degenerate $\{a'\}$

then matrix $\begin{pmatrix} \langle a'|B|a'\rangle & \langle a''|B|a'\rangle & \cdots \\ \langle a'|B|a'\rangle & \langle a''|B|a''\rangle & \cdots \\ \vdots & & \end{pmatrix}$

is just $\begin{pmatrix} \langle a'|B|a'\rangle & 0 & \\ 0 & \langle a''|B|a''\rangle & \\ \vdots & & \ddots \end{pmatrix}$

diagonal!

so $\langle a''|B|a'\rangle = \delta_{a'a''} \langle a'|B|a'\rangle$

remember $\langle a''|A|a'\rangle$ A is diagonal in its own basis

then A and B can be represented as diagonal with the same base kets

$$B = \sum_{a''} \sum_{a'} |a''\rangle\langle a''|B|a'\rangle\langle a'|$$

$$= \sum_{a'} |a'\rangle\langle a'|B|a'\rangle\langle a'|$$

then $B|a''\rangle = \sum_{a'} |a'\rangle\langle a'|B|a'\rangle\langle a'|a''\rangle$

$$= \underbrace{\langle a''|B|a''\rangle}_{\text{num}} \underbrace{|a''\rangle}_{\text{same state out}}$$

so $\{a''\}$ are eigenkets of B!

$$B|b'\rangle = b'|b'\rangle$$
$$\| \qquad\qquad \|$$
$$B|a'\rangle = \langle a'|B|a'\rangle\, |a'\rangle$$

So $|a'\rangle$ is <u>simultaneous eigenket of A + B</u>!

can write as: $A|a',b'\rangle = a'|a',b'\rangle$

$B|a',b'\rangle = b'|a',b'\rangle$

physical importance: measuring doesn't destroy the state

$$|\alpha\rangle \xrightarrow{\text{measure } A} |a',b'\rangle \xrightarrow{\text{measure } B} |a',b'\rangle$$

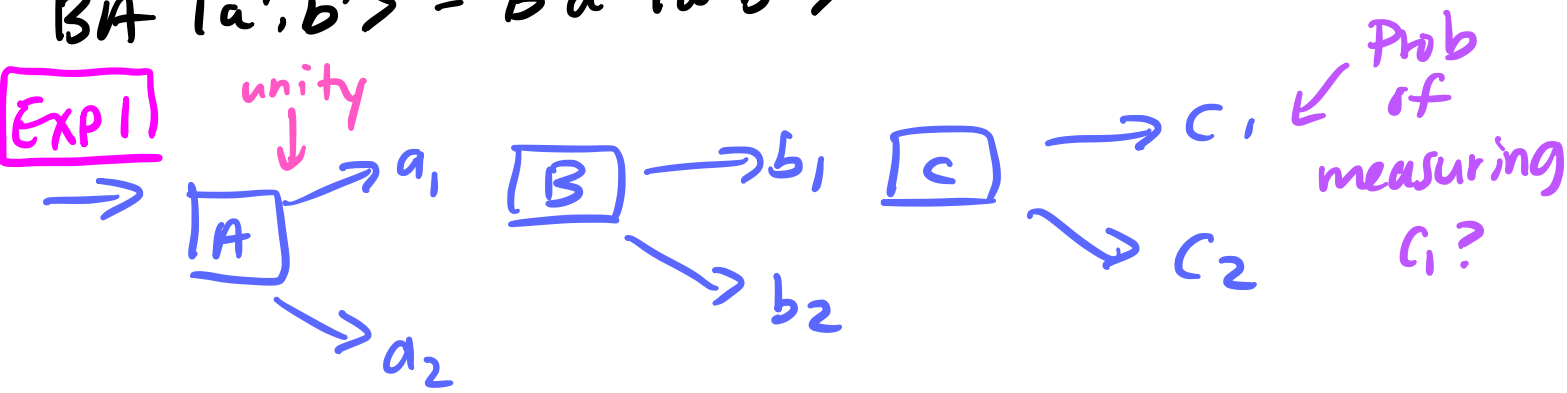
Incompatible observables:

if $[A,B] \neq 0$ then A and B incompatible

$\rightarrow$ no simultaneous eigenkets

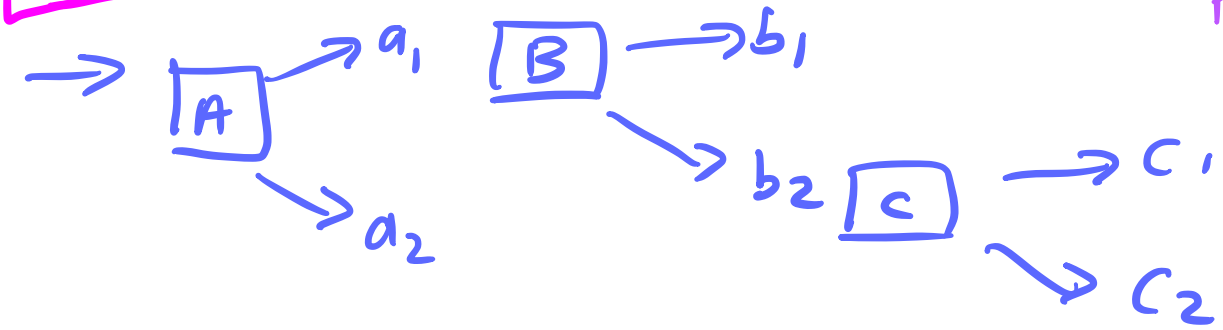
$$AB\,|a',b'\rangle = A\,b'\,|a',b'\rangle = a'b'\,|a',b'\rangle$$
$$BA\,|a',b'\rangle = B\,a'\,|a'b'\rangle =$$

Exp 1



Probability: $|\langle c_1|b_1\rangle|^2\, |\langle b_1|a_1\rangle|^2$

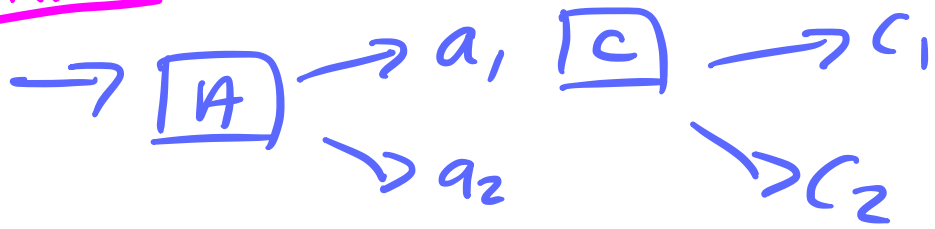
EXP 2



Prob of measuring?

Probability: $|\langle c_1|b_2\rangle|^2 \, |\langle b_2|a_1\rangle|^2$

EXP 3



Prob of measuring?

Probability: $|\langle c_1|a_1\rangle|^2$

* probability is different with/without B device
B measuring devices interferes w/ the probability, even if you consider both B outcomes!

Proof: $|\langle c_1|a_1\rangle|^2$

$$= \left|\sum_{b'=b_1,b^2} \langle c_1|b'\rangle\langle b'|a_1\rangle\right|^2$$

$$= |\langle c_1|b_1\rangle\langle b_1|a_1\rangle + \langle c_1|b_2\rangle\langle b_2|a_1\rangle|^2$$

$$= \sum_{b'}\sum_{b''} \langle c_1|b'\rangle\langle b'|a_1\rangle\langle a_1|b''\rangle\langle b''|c_1\rangle$$

* Adding probabilities of all routes B can take only add to probability of Exp 3 if A and B are compatible, B and C are compatible, or all are

$$\boxed{A} \rightarrow \boxed{C} \rightarrow$$

$$\neq$$

$$\boxed{A} \rightarrow \boxed{B} \rightarrow \boxed{C} \rightarrow$$

$$\boxed{B} \rightarrow \boxed{C} \rightarrow$$

this one physically measures B which changes the math

Lecture 6 - Jan 26

HW3: 1.12, 1.13, 1.16 (2nd edition)

uncertainty relations:

define operator $\Delta A \equiv A - \langle A \rangle$

$\equiv A - \langle A \rangle \mathbb{1}$ in reality

$\langle A \rangle = \langle \alpha | A | \alpha \rangle$ a number

dispersion of A $\rightarrow$ broadness of measurement

$$\langle (\Delta A)^2 \rangle = \langle (A - \langle A \rangle)^2 \rangle$$

num

$$= \langle AA - A\langle A \rangle - A\langle A \rangle + \langle A \rangle^2 \rangle$$

$$= \langle A^2 - 2\langle A \rangle A + \langle A \rangle^2 \rangle$$

$$= \langle A^2 \rangle - 2\langle A \rangle \langle A \rangle + \langle A \rangle^2$$

$$\boxed{\langle (\Delta A)^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2}$$

$$= \langle \alpha | A^2 | \alpha \rangle - \langle \alpha | A | \alpha \rangle^2$$

if take $|\alpha\rangle$ to be $|a'\rangle$ then dispersion $= 0$

dispersion is fuzziness of measurement

and $A|a'\rangle = a'|a'\rangle$ every time

so no broadness of measurement

ex: Dispersion of S_x for $|+\rangle$

$$\langle S_x \rangle = (\langle +|)\left(\tfrac{\hbar}{2}|+\rangle\langle -| + |-\rangle\langle +|\right)(|+\rangle)$$

$$= \tfrac{\hbar}{2}\left(\langle +|+\rangle \underbrace{\langle -|+\rangle}_{0} + \underbrace{\langle +|-\rangle}_{0}\langle +|+\rangle\right) = 0$$

$$\langle S_x^2 \rangle = \langle +| \frac{\hbar}{2} (|+\rangle\langle -| + |-\rangle\langle +|)^2 |+\rangle$$
$$= \left(\frac{\hbar}{2}\right)^2 \langle +| (|+\rangle\langle +| + |-\rangle\langle -|) |+\rangle$$
$$= \frac{\hbar^2}{4}$$

so $\langle (\Delta S_x)^2 \rangle = \langle S_x^2 \rangle - \langle S_x \rangle^2$
$$= \frac{\hbar^2}{4} - 0^2 = \frac{\hbar^2}{4}$$

Generalized uncertainty Relation:

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} |\langle [A, B] \rangle|^2$$

Lemmas:

(1) $\langle \alpha | \alpha \rangle \langle \beta | \beta \rangle \geq |\langle \alpha | \beta \rangle|^2$

(2) if X is Hermitian $(X = X^\dagger)$, $\langle X \rangle$ = real

(3) if Y is anti-Hermitian $(Y = -Y^\dagger)$, $\langle Y \rangle$ = imaginary

1.5 Change of Basis

Theorem: There exists a unitary operator U such that $|b^1\rangle = U|a^1\rangle$ for a set of
$|b^2\rangle = U|b^2\rangle$
etc.
incompatible observables A and B with their separate $\{a^1\}$ and $\{b^1\}$

$U^\dagger U = \mathbb{1}$
$UU^{-1} = \mathbb{1}$

$$U = \sum_k |b^k\rangle\langle a^k|$$

check unitarity of operator:

$$U^\dagger U = \left(\sum_k |a^k\rangle\langle b^k|\right)\left(\sum_\ell |b^\ell\rangle\langle a^\ell|\right)$$

$$= \sum_k\sum_\ell |a^k\rangle\underbrace{\langle b^k|b^\ell\rangle}_{\delta_{k\ell}}\langle a^\ell|$$

$$= \sum_k |a^k\rangle\langle a^k| = \mathbb{1}$$

matrix representation of U:

let's go from $\{a^i\}$ basis to $\{b^i\}$ basis

elements of matrix: $\langle a^k|U|a^\ell\rangle = \langle a^k|b^\ell\rangle$

i.e.
$$\begin{pmatrix} \langle a^1|b^1\rangle & \langle a^2|b^1\rangle & \langle a^3|b^1\rangle \\ \langle a^1|b^2\rangle & \langle a^2|b^2\rangle & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

since $|\alpha\rangle = \sum_{a^i} |a^i\rangle\langle a^i|\alpha\rangle$

$* \langle b^k| = \langle a^k|U^\dagger$

$$\langle b^k|\alpha\rangle = \sum_\ell \langle b^k|a^\ell\rangle\langle a^\ell|\alpha\rangle$$

$$= \sum_\ell \langle a^k|U^\dagger|a^\ell\rangle\langle a^\ell|\alpha\rangle$$

$$\overset{\text{new}}{\begin{pmatrix} N \\ \times \\ 1 \end{pmatrix}} = \begin{pmatrix} N & \times & N \end{pmatrix} \overset{\text{old}}{\begin{pmatrix} N \\ \times \\ 1 \end{pmatrix}}$$

given operator X is known in $\{a'\}$, what is X in $\{b'\}$?

$$\langle b^k|X|b^l\rangle = \sum_m \sum_n \langle b^k|a^m\rangle\langle a^m|X|a^n\rangle\langle a^n|b^l\rangle$$

$$= \sum_m \sum_n \langle a^k|U^\dagger|a^m\rangle\langle a^m|X|a^n\rangle\langle a^n|U|a^l\rangle$$

$$X' = U^\dagger X U$$

Unitary Equivalent Observables:

Unitary transform of A is UAU^{-1}

because $UU^{-1} = \mathbb{1}$

$$A|a^l\rangle = a^l|a^l\rangle$$

$$UA|a^l\rangle = Ua^l|a^l\rangle \qquad \text{(1) act } U \text{ on } A$$

$$UAU^{-1}U|a^l\rangle = a^l U|a^l\rangle \qquad \text{(2) insert } \mathbb{1}\ (U^{-1}U)$$

$$\underbrace{UAU^{-1}}_{\text{unitary transform}}\underbrace{|b^l\rangle}_{\text{new eigenket}} = \underbrace{a^l}_{\text{old eigenvalue}}\underbrace{|b^l\rangle}_{\text{same eigenket}}$$

Trace of an operator:

$$\mathrm{tr}(X) = \sum_{a'} \langle a'|X|a'\rangle$$

i.e. $X \equiv$

trace is Σ of diagonal elem.

$\mathrm{tr}(X)$ is the same for any base kets!

$$\sum_{a'} \langle a'|X|a'\rangle = \sum_{b'} \langle b'|X|b'\rangle$$

HW 3 1.12, 1.13, 1.16

1.12 Spin 1/2 system is eigenstate of $S\cdot\hat{n}$ w/ eigenvalue $\hbar/2$. $\hat{n}$ makes angle γ w/ $\hat{z}$

(a) Prob of measuring $|S_x; \hbar/2\rangle$?

we have $S\cdot\hat{n}\,|\hat{n};+\rangle = \frac{\hbar}{2}|\hat{n};+\rangle$

$$|\hat{n};+\rangle = \cos(\gamma/2)|+\rangle + \sin(\gamma/2)|-\rangle$$

so $$|S_x;+\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

then prob $= |\langle S_x;+|\hat{n};+\rangle|^2$ scalar product

$$= \frac{1}{2}\left(\cos^2(\gamma/2) + 2\sin(\gamma/2)\cos(\gamma/2) + \cos^2(\gamma/2)\right)$$

$$= \frac{1}{2}\left[2\cos^2(\gamma/2) + 2\sin(\gamma/2)\cos(\gamma/2)\right]$$

$$= \boxed{\frac{1+\sin(\gamma)}{2}}$$

(b) Evaluate dispersion in S_x: $\langle (S_x - \langle S_x\rangle)^2\rangle$

check cases of $\gamma = 0, \pi/2, \pi$

$$\langle(\Delta S_x)^2\rangle = \langle S_x^2\rangle - \langle S_x\rangle^2$$

$$\langle S_x^2\rangle = \langle +|\,\frac{\hbar}{2}\left(|+\rangle\langle -| + |-\rangle\langle +|\right)^2|+\rangle$$

$$= \left(\frac{\hbar}{2}\right)^2\langle +|\left(\underbrace{|+\rangle\langle +|}_{1} + \underbrace{|-\rangle\langle -|}_{1}\right)|+\rangle$$

$$= \frac{\hbar^2}{4}$$

$$\langle S_x \rangle = \frac{\hbar}{2} \langle \hat{n}; + | \mathbb{1} | \hat{n}; + \rangle = \frac{\hbar}{2} \sin(\gamma)$$

$$\langle S_x \rangle^2 = \frac{\hbar^2}{4} \sin^2(\gamma)$$

so $$(\langle \Delta S_x \rangle^2) = \langle S_x^2 \rangle - \langle S_x \rangle^2$$

$$= \frac{\hbar^2}{4} - \frac{\hbar^2}{4} \sin\gamma$$

$$= \frac{\hbar^2}{4}(1 - \sin^2\gamma)$$

$$\boxed{= \frac{\hbar^2}{4} \cos^2\gamma}$$

1.13

A beam of spin ½ atoms go through SG exps

$$\longrightarrow \boxed{S_z} \longrightarrow \frac{\hbar}{2} \xrightarrow[\text{projected to } |S_n;+\rangle]{1} \boxed{S_n;+} \xrightarrow[\text{project to } |S_z;-\rangle]{2} \boxed{}$$

we want projector operators:

$$P_{n+} = |\hat{n};+\rangle\langle\hat{n};+| \quad (1)$$

$$= [\cos(\beta/2)|+\rangle + \sin(\beta/2)|-\rangle][\cos(\beta/2)\langle+| + \sin(\beta/2)\langle-|]$$

$$P_- = |-\rangle\langle-| \quad (2)$$

so our $|+\rangle$ gets operated on by $P_{n+} P_-$

then final state $|\alpha\rangle = P_- P_{n+} |+\rangle$

$\neq$ order matters!

$|\hat{n}; +\rangle = \cos(\gamma/2)|+\rangle + \sin(\gamma/2)|-\rangle$

$$|\alpha\rangle = |-\rangle \underbrace{\langle -|\hat{n}; +\rangle}_{\text{num}} \underbrace{\langle \hat{n}; +|+\rangle}_{\text{num}}$$

$$|-\rangle \left[\cos(\beta/2)\underbrace{\langle -|+\rangle}_{0} + \sin(\beta/2)\underbrace{\langle -|-\rangle}_{1}\right]$$

$$\cdot \left[\cos(\beta/2)\underbrace{\langle +|+\rangle}_{1} + \sin(\beta/2)\underbrace{\langle -|+\rangle}_{0}\right]$$

$$|\alpha\rangle = \sin(\beta/2)\cos(\beta/2)|-\rangle$$

$$\text{final intensity} = \langle\alpha|\alpha\rangle$$

$$= |\sin(\beta/2)\cos(\beta/2)|^2$$

$$= \frac{1}{4}\sin^2(\beta)$$

<u>final intensity is $\frac{1}{4}$ of intensity at start</u>

for max intensity, we want $\sin(\beta) = 1$ so must orient second apparatus at angle $\boxed{\beta = 90°}$

1.16 if $\{A,B\} = AB + BA = 0$

Is it possible to have simultaneous eigenkets?

simultaneous eigenkets would be $|a,b\rangle$

$$\{A,B\}|a,b\rangle = 0$$

$$= AB|a,b\rangle + BA|a,b\rangle$$

$$= ab|a,b\rangle + ba|a,b\rangle$$

$$= 2ab|a,b\rangle = 0$$

ab just numbers so rearrange

$$2ab = 0$$

means $a=0$ and/or $b=0$

so $\boxed{\text{yes Hermitian commutators that anti-commute have simultaneous eigenkets}}$

Lecture 7 - Feb 2

$$tr(X) = \sum_{a'} \langle a'|X|a'\rangle$$

$tr(X)$ is same in any base ket

$$\sum_{a'} \langle a'|X|a'\rangle = \sum_{a'}\sum_{b'}\sum_{b''} \langle a'|\overbrace{b'\rangle\langle b'}^{\mathbb{1}}|X|\overbrace{b''\rangle\langle b''}^{\mathbb{1}}|a'\rangle$$

$$= \sum_{a'}\sum_{b'}\sum_{b''} \langle b''|a'\rangle\langle a'|b'\rangle\langle b'|X|b''\rangle$$

$$= \sum_{b'}\sum_{b''} \underbrace{\langle b''|b'\rangle}_{\delta_{b''b'}} \langle b'|X|b'\rangle$$

$$= \sum_{b'} \langle b'|X|b'\rangle$$

other properties of traces:

$$tr(XY) = tr(YX)$$

$$tr(U^\dagger X U) = tr(X)$$

$$tr(|a'\rangle\langle a''|) = \delta_{a'a''}$$

$$tr(|b'\rangle\langle a'|) = \langle a'|b'\rangle$$

Breakout Room Proofs:

$$\mathrm{tr}\left(|b'\rangle\langle a'|\right) = \langle a'|b'\rangle \qquad (4)$$

using $\mathrm{tr}(X) = \sum_{a'} \langle a'|X|a'\rangle$

$$\mathrm{tr}\left(|b'\rangle\langle a'|\right) = \sum_{a'} \langle a'|b''\rangle\langle a''|a'\rangle$$

$$= \sum_{a'} \langle a''|a'\rangle\langle a'|b''\rangle$$

$$= \langle a''|b''\rangle$$

$$\mathrm{tr}(XY) = \sum_{a'} \langle a'|XY|a'\rangle \qquad (3)$$

$$= \sum_{a'}\sum_{a''} \langle a'|X|a''\rangle\langle a''|Y|a'\rangle$$

$$= \sum_{a'}\sum_{a''} \langle a''|Y|a'\rangle\langle a'|X|a''\rangle$$

$$= \sum_{a''} \langle a''|YX|a''\rangle$$

② $\mathrm{tr}(U^{+}XU) = \mathrm{tr}(X)$

$$\sum_{c'}\sum_{k}\sum_{l} \langle c'|a^{k}\rangle\langle b^{k}|X|b^{l}\rangle\langle a^{l}|c'\rangle$$

$$\sum_{c'}\sum_{k}\sum_{l} \langle a^{l}|c'\rangle\langle c'|a^{k}\rangle\langle b^{k}|X|b^{l}\rangle$$

$$\sum_{k}\sum_{l} \underbrace{\langle a^{l}|a^{k}\rangle}_{\delta_{kl}} \langle b^{k}|X|b^{l}\rangle$$

$$= \sum_{l} \langle b^{l}|X|b^{l}\rangle = \mathrm{tr}(X)$$

①

1.6 Position, Momentum, and Translation:

eigenkets represent a continuous spectra

$$\xi |\xi'\rangle = \xi' |\xi'\rangle$$

discrete basis	continuous basis
$\langle a'|a''\rangle = \delta_{a'a''}$	$\langle \xi'|\xi''\rangle = \delta(\xi''-\xi')$
$\sum_{a'} |a'\rangle\langle a'| = \mathbb{1}$	$\int d\xi' |\xi'\rangle\langle\xi'| = \mathbb{1}$
$|\alpha\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle$	$|\alpha\rangle = \int d\xi' |\xi'\rangle\langle\xi'|\alpha\rangle$

Position $X|x'\rangle = x'|x'\rangle$

$$|\alpha\rangle = \int_{-\infty}^{\infty} dx' \, |x'\rangle\langle x'|\alpha\rangle$$

X operator measures position

experimentally, apparatus will click when $|\alpha\rangle$ is around x' (in reality, x' has a width)

$x' \rightarrow x' - \frac{\Delta}{2} \sim x' + \frac{\Delta}{2}$ is range

when $|\alpha\rangle$ is measured, it becomes

$$|\alpha\rangle = \int_{x'-\frac{\Delta}{2}}^{x'+\frac{\Delta}{2}} dx'' \, |x''\rangle\langle x''|\alpha\rangle$$

measurement still disturbs the state!

Probability that $|\alpha\rangle$ is at x':

$$= |\langle x'|\alpha\rangle|^2 \, dx'$$

integrating probability over all position space:

$$\int_{-\infty}^{\infty} |\langle x'|\alpha\rangle|^2 \, dx' = 1$$

particle has to be somewhere

position space is 3D so more generally:

$$|\alpha\rangle = \int d^3x' \, |\vec{x}'\rangle\langle\vec{x}'|\alpha\rangle$$

$|\vec{x}'\rangle$ represents a simultaneous eigenket of x', y', z' : $|\vec{x}'\rangle = |x', y', z'\rangle$

simultaneous eigenkets

x', y', z' commute:

$$[x_i, x_j] = 0$$

components of position space commute w/ each other

Translation

let x' be initial state and say we want to be at $x' + dx'$ without changing the state

every perturbation is due to an operator

translation operator: $\mathcal{J}(d\vec{x}')$

$$\mathcal{J}(d\vec{x}')|x'\rangle = |x' + dx'\rangle$$

what is $\mathcal{T}(dx')|\alpha\rangle$?

$$= \mathcal{T}(dx') \int dx' \, |x'\rangle\langle x'|\alpha\rangle$$

$$= \int dx' \, \mathcal{T}(dx') \, |x'\rangle\langle x'|\alpha\rangle$$

$$= \int dx' \, |x' + dx'\rangle \langle x'|\alpha\rangle$$

in class assignment 2:

use operator identity $[\vec{x}, \mathcal{P}(d\vec{x}')] = d\vec{x}'$

to show $[x_i, K_j] = i\delta_{ij}\mathbb{1}$

$$\mathcal{P}(dx') = (1 - i\hat{K})d\vec{x}$$

K operator = related to momentum operator

$$[x_j, \mathcal{P}_j(d\vec{x}')] = d\vec{x}'$$

$$= [x_j, 1 - i\hat{K}_j\, d\vec{x}']$$

$$= [x_j, -i\hat{K}_j]\, d\vec{x}' = d\vec{x}'$$

$$= [x_j, -i\hat{K}_j] = 1$$

$$= [x_j, \hat{K}_j] = \frac{1}{-i} = i$$

$$= [x_j, \hat{K}_j] = i$$

then $[x_i, \hat{K}_j] = i\delta_{ij}\mathbb{1}$

$$\boxed{\text{From } [\vec{x}, \mathcal{T}(d\vec{x}')] = d\vec{x}' \text{ show } [\vec{x}_i, \vec{K}_j] = i\delta_{ij}}$$

infinitesimal translation operator:

$$\mathcal{T}(d\vec{x}') = 1 - i\vec{K}\cdot d\vec{x}'$$

$$[\vec{x}, \mathcal{T}(d\vec{x}')] = d\vec{x}' \cdot \mathbb{1}$$

$$= [\vec{x}, 1 - i\vec{K}\cdot d\vec{x}']$$

$$= [\vec{x}, 1] - i[\vec{x}, \vec{K}\cdot d\vec{x}']$$

$\vec{x}(1) - (1)\vec{x} = 0$ $\qquad$ $-i\vec{x}\cdot\vec{K}d\vec{x}' + i\vec{K}d\vec{x}'\cdot\vec{x}$

$$= -i\vec{x}\cdot\vec{K}\,d\vec{x}' + i\vec{K}\cdot d\vec{x}'\,\vec{x} = d\vec{x}'$$

choose $\vec{x}$ and $\vec{K}$ to be in $\hat{j}$

$$-i\vec{x}_j\vec{K}_j\,d\vec{x}'_j + i\vec{K}_j\,d\vec{x}'_j\,\vec{x}_j = d\vec{x}'_j$$

$$-i\,d\vec{x}'_j\,(\vec{x}_j\vec{K}_j - \vec{K}_j\vec{x}_j) = d\vec{x}'_j$$

$$-i\,d\vec{x}'_j\,[\vec{x}_j, \vec{K}_j] = d\vec{x}'_j$$

$$[\vec{x}_j, \vec{K}_j] = \frac{d\vec{x}'_j}{d\vec{x}'_j}\frac{1}{-i} = i$$

$$\boxed{[\vec{x}_i, \vec{K}_j] = i\delta_{ij}}$$

Lecture 8 - Feb 4

HW 4: 1.19, 1.26, 1.31
(due Feb 18)

continuous eigenstates

$$\underbrace{x}_{\text{operator}}\overbrace{|x'\rangle}^{\text{ket}} = \underbrace{x'}_{\text{number}}\overbrace{|x'\rangle}^{\text{ket}}$$

infinitesimal translation operator

$$\mathcal{J}(dx')|\vec{x}'\rangle = |\vec{x}'+d\vec{x}'\rangle$$

$$\mathcal{J}(d\vec{x}')|\alpha\rangle = \int d^3\vec{x}''\, \mathcal{J}(d\vec{x}')|\vec{x}''\rangle\langle\vec{x}''|\alpha\rangle$$

$$= \int_{-\infty}^{\infty} d^3\vec{x}'\, |\vec{x}''+d\vec{x}'\rangle\langle\vec{x}''|\alpha\rangle$$

$$= \int_{-\infty}^{\infty} d^3\vec{x}'\, |\vec{x}''\rangle\langle\vec{x}''-d\vec{x}'|\alpha\rangle$$

$$\langle\vec{x}'|\alpha\rangle \rightarrow \langle\vec{x}'|\mathcal{J}(d\vec{x}')|\alpha\rangle = \underbrace{\langle\vec{x}'-d\vec{x}'|\alpha\rangle}_{\text{redefines axis}}$$

$$\int d^3\vec{x}'\, \langle\vec{x}'|\vec{x}''\rangle\langle\vec{x}''-d\vec{x}'|\alpha\rangle$$

$$= \int d^3\vec{x}'\, \delta(\vec{x}'-\vec{x}'')\langle\vec{x}''-d\vec{x}''|\alpha\rangle$$

$$= \langle\vec{x}'-d\vec{x}'|\alpha\rangle$$

Requirements for $\mathcal{J}(d\vec{x}')$

(1) unitarity $\langle\alpha|\alpha\rangle = 1$

$$\underbrace{\langle\alpha|\mathcal{J}(d\vec{x}')^{+}}_{\langle\gamma|}\ \underbrace{\mathcal{J}(d\vec{x}')|\alpha\rangle}_{|\gamma\rangle} = 1$$

so $\mathcal{J}(d\vec{x}')^{+}\,\mathcal{J}(d\vec{x}') = 1$

unitary operator

(2) successive translations make sense

$$\mathcal{J}(d\vec{x}'')\mathcal{J}(d\vec{x}') = \mathcal{J}(d\vec{x}''+d\vec{x}')$$

(3) translation in opposite direction makes sense

$$\mathcal{J}(-d\vec{x}') = \mathcal{J}(d\vec{x}')^{-1}$$

(4) $\lim_{d\vec{x}\to 0} \mathcal{J}(d\vec{x}') = \mathbb{1}$

number vector

$$\mathcal{J}(d\vec{x}') = \mathbb{1} - i\vec{K}\cdot d\vec{x}'$$

operator scalar

number scalar

$\vec{K} = K_x, K_y, K_z$ (operator vector)

all Hermitian operators

$\mathcal{J}(d\vec{x}')$ is a scalar operator
(because vector dot product is scalar)

What is $\vec{K}$?

$$\mathcal{J}(d\vec{x}') = 1 - i\vec{K}\cdot d\vec{x}'$$

$$\vec{x}\,\mathcal{J}(d\vec{x}')|\vec{x}'\rangle = \vec{x}\,|\vec{x}' + d\vec{x}'\rangle$$

$$= (\vec{x}' + d\vec{x}')|\vec{x}' + d\vec{x}'\rangle$$

$$\mathcal{J}(d\vec{x}')\,\vec{x}|\vec{x}'\rangle = \mathcal{J}(d\vec{x}')\,\vec{x}'|\vec{x}'\rangle$$

$$= \vec{x}'(\vec{x}' + d\vec{x}')$$

$$[\vec{x}, \mathcal{J}(d\vec{x}')]\,|\vec{x}'\rangle = d\vec{x}'\,|\vec{x}' + d\vec{x}'\rangle$$

$$\cong d\vec{x}'\,|\vec{x}'\rangle$$

$$[\vec{x}, \mathcal{J}(d\vec{x}')] = d\vec{x}'\,\mathbb{1}$$

x and $\mathcal{J}$ operators don't commute - only when they are orthogonal $[x_i, K_j] = 0$

$$\boxed{\vec{K} = \frac{\vec{p}}{\hbar}}$$

$$\vec{p}\,|\vec{p}'\rangle = \vec{p}'\,|\vec{p}'\rangle$$

p parallels x

$$\boxed{\begin{aligned} \mathcal{J}(d\vec{x}') &= 1 - \frac{i\,\vec{p}\cdot d\vec{x}}{\hbar} \\ [x_i, p_j] &= i\hbar\,\delta_{ij} \end{aligned}}$$

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} |[A,B]|^2$$

$$\langle (\Delta X)^2 \rangle \langle (\Delta P)^2 \rangle \geq \frac{\hbar^2}{4}$$

$$[P_i, P_j] = 0 \qquad \mathcal{J}(d\vec{x}')\,|\vec{P}'\rangle$$

Lecture 9 - Feb 9

reading: 1.6 – 1.7

$$[\vec{x}, \mathcal{J}(d\vec{x}')] = d\vec{x}'$$

$$\Downarrow$$

$$[x_i, K_j] = i\,\delta_{ij}$$

$$[x_i, P_j] = i\hbar\,\delta_{ij}$$

* don't commute along the same axis!
* no simultaneous eigenket in same axis

Finite translations (non-infinitesimal):

we want $\Delta x\,\hat{x}$ now (vs. $d\vec{x}'$)

need translation operator:

$$\mathcal{J}(\Delta x\,\hat{x})\,|\vec{x}'\rangle = |\vec{x} + \Delta x'\,\hat{x}\rangle$$

$$\mathcal{J}(\Delta x\,\hat{x}) = \lim_{N\to\infty}\left(1 - \frac{iP_x}{\hbar}\left(\frac{\Delta x'}{N}\right)\right)^N$$

$$= e^{\frac{-iP_x\Delta x}{\hbar}}$$ operator

$$e^x = \mathbb{1} + x + \frac{x^2}{2!} + \dots$$ addition of operators

Fundamental Property of Translation:

Translations in different directions commute

$$\mathcal{J}(\Delta y'\hat{y})\,\mathcal{J}(\Delta x'\hat{x}) = \mathcal{J}(\Delta x'\hat{x})\,\mathcal{J}(\Delta y'\hat{y})$$

$$[\mathcal{J}(\Delta y'\hat{y}), \mathcal{J}(\Delta x'\hat{x})] =$$

$$\left[\left(1 - \frac{iP_y\Delta y'}{\hbar} - \frac{P_y^2\Delta y'^2}{\hbar^2}\ldots\right), \left(1 - \frac{iP_x\Delta x'}{\hbar} - \frac{P_x^2\Delta x'^2}{\hbar^2}\ldots\right)\right]$$

$$\cong \frac{\Delta x'\Delta y'}{\hbar^2}[P_y, P_x] + \underbrace{\cancel{\mathcal{O}(\Delta x'^2)}}_{0} = 0$$

so $[P_y, P_x] = 0$

$$\boxed{[P_i, P_j] = 0 = [X_i, X_j]}$$

eigenstate of $\vec{P}$:

* operators don't have prime, numbers do

$$|\vec{p}'\rangle = |P_x', P_y', P_z'\rangle$$

$$P_x|\vec{P}'\rangle = P_x'|\vec{P}'\rangle$$

$$\mathcal{J}(d\vec{x}')|\vec{P}'\rangle = \left(1 - \frac{i\vec{P}\cdot d\vec{x}'}{\hbar}\right)|\vec{P}'\rangle$$

$$= \left(1 - \frac{i\vec{P}'\cdot d\vec{x}'}{\hbar}\right)|\vec{P}'\rangle$$

if you are in P eigenstate, you are also in $\mathcal{J}$ eigenstate!

$$\boxed{[\vec{P}, \mathcal{J}(d\vec{x}')] = 0}$$

$|\vec{p}'\rangle$ is eigenstate of $\mathcal{J}$ but $|\vec{x}'\rangle$ is NOT!

1.7 Wavefunctions in Position + Momentum:

$$|\alpha\rangle = \int_{-\infty}^{\infty} dx' \, |x'\rangle \underbrace{\langle x'|\alpha\rangle}_{\text{wavefunction}}$$

wavefunction $\langle x'|\alpha\rangle = \psi_\alpha(x')$

$$\langle\beta|\alpha\rangle = \int dx' \, \langle\beta|x'\rangle\langle x'|\alpha\rangle$$

$$= \int dx' \, \underbrace{\psi_\beta(x')^* \, \psi_\alpha(x')}_{\text{overlap between wavefunctions}}$$

$$\langle x'|\alpha\rangle = \sum_{a'} \underbrace{\langle x'|a'\rangle}\underbrace{\langle a'|\alpha\rangle}$$

$$\psi_\alpha(x') = \sum_{a'} c_{a'} \, u_{a'}(x')$$

$c_{a'}$ → coefficients

$u_{a'}(x')$ → eigenfunctions = wavefunction of eigenstate $= \langle x'|a'\rangle$

$$\langle\beta|A|\alpha\rangle = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx'' \, \langle\beta|x'\rangle\langle x'|A|x''\rangle\langle x''|\alpha\rangle$$

$$= \int dx' \int dx'' \, \psi_\beta^*(x') \, \langle x'|A|x''\rangle \, \psi_\alpha(x'')$$

ex: for $A = x^2$

$$\langle x'|A|x''\rangle = \langle x'|x^2|x''\rangle = x''\langle x'|x''\rangle$$

$$= x'' \, \delta(x''-x')$$

$$\langle\beta|x^2|\alpha\rangle = \int dx' \, \psi_\beta^*(x') \, \psi_\alpha(x') \, x'^2$$

in general:

$$\langle\beta|f(x)|\alpha\rangle = \int dx' \, \psi_\beta^*(x') \, \psi_\alpha(x') \, f(x')$$

= number

Position basis: $\langle x'|\alpha\rangle$
Momentum basis: $\langle p'|\alpha\rangle$

$$\left(1 - \frac{ip\Delta x'}{\hbar}\right)|\alpha\rangle = \int dx' \left(1 - \frac{ip\Delta x'}{\hbar}\right)|x'\rangle\langle x'|\alpha\rangle$$

$$= \int dx' \, |x' + \Delta x'\rangle\langle x'|\alpha\rangle$$

$$= \int dx' \, |x'\rangle \underbrace{\langle x' - \Delta x'|\alpha\rangle}_{\psi_\alpha(x' - \Delta x')}$$

Taylor expansion:

$$= \underbrace{\int dx' \, |x'\rangle \left(\langle x'|\alpha\rangle}_{|\alpha\rangle} - \Delta x' \frac{\partial}{\partial x'}\langle x'|\alpha\rangle + \dots\right)$$

partial deriv because α could be fcn of other var.

$$-\frac{ip\Delta x'}{\hbar}|\alpha\rangle = \int dx' \, |x'\rangle \left(-\Delta x' \frac{\partial}{\partial x'}\langle x'|\alpha\rangle\right)$$

$$p|\alpha\rangle = \int dx' \, |x'\rangle \frac{\partial}{\partial x'}\langle x'|\alpha\rangle (-i\hbar)$$

$$\langle x''|p|\alpha\rangle = \int dx' \, \langle x''|x'\rangle \frac{\partial}{\partial x'}\langle x'|\alpha\rangle (-i\hbar)$$

$$= \int dx' \, \delta(x'' - x') \frac{\partial}{\partial x'}\langle x'|\alpha\rangle (-i\hbar)$$

$$= \frac{\partial}{\partial x''}\langle x''|\alpha\rangle (-i\hbar)$$

can just call our x'' as x'

$$\langle x'|p|\alpha\rangle = -i\hbar \frac{\partial}{\partial x'} \langle x'|\alpha\rangle$$

matrix elements of P in x basis:

$$\langle x'|p|x''\rangle = -i\hbar \frac{\partial}{\partial x'} \langle x'|x''\rangle$$

$$= i\hbar \frac{\partial}{\partial x'} \delta(x'-x'')$$

Lecture 10 - Feb 16

reading 1.7, 2.1

1.7 cont.

$$\langle x'|p|\alpha\rangle = -i\hbar \frac{\partial}{\partial x'}\langle x'|\alpha\rangle$$

$$\langle\beta|p|\alpha\rangle = \int dx' \langle\beta|x'\rangle\langle x'|p|\alpha\rangle$$

$$= \int dx' \, \psi_\beta^*(x') \, -i\hbar \frac{\partial}{\partial x'}\langle x'|\alpha\rangle$$

$$= i\hbar \int dx' \, \psi_\beta^*(x') \frac{\partial}{\partial x'} \psi_\alpha(x')$$

momentum-space wavefunction:

$$p|p'\rangle = p'|p'\rangle$$

$$|\alpha\rangle = \int dp' \, |p'\rangle \underbrace{\langle p'|\alpha\rangle}_{\phi_\alpha(p')}$$

$\phi_\alpha(p')$ analogous to $\psi_\alpha(x')$

normalization

$$\langle\alpha|\alpha\rangle = \int dp' \langle\alpha|p'\rangle\langle p'|\alpha\rangle$$

$$= \int dp' \, \phi_\alpha^*(p') \, \phi_\alpha(p')$$

$$= \int_{-\infty}^{\infty} dp' \, |\phi_\alpha(p')|^2 = 1$$

* random state $|\alpha\rangle$ can be described in position space or momentum space

recall change of basis u matrix elements:

$$\langle \overset{\text{old}}{a^k} | \overset{\text{new}}{b^\ell} \rangle$$

Change of basis between x' and p':

$\langle x'|p'\rangle$ means $\psi_\alpha \leftrightarrow P_\alpha$

$$\langle x'|P|\alpha\rangle = -i\hbar \frac{\partial}{\partial x'} \langle x'|\alpha\rangle$$

so $\langle x'|P|p'\rangle = -i\hbar \frac{\partial}{\partial x'} \langle x'|p'\rangle$

$$p' \langle x'|p'\rangle = -i\hbar \frac{\partial}{\partial x'} \langle x'|p'\rangle$$

$$\langle x'|p'\rangle = N e^{\frac{ip'x'}{\hbar}}$$

N = normalization (needs to be calculated)

meaning of $\langle x'|p'\rangle$: in a momentum eigenstate, what does wave function look like in position space?

Solving for N: $\langle x'|x''\rangle$

$$\langle x'|x''\rangle = \int dp' \langle x'|p'\rangle \langle p'|x''\rangle$$

$$= \int dp' \, N e^{\frac{ip'x'}{\hbar}} N^* e^{-\frac{ip'x''}{\hbar}}$$

$$= \int dp' \, |N|^2 e^{\frac{ip'(x'-x'')}{\hbar}}$$

$$\delta(x'-x'') = |N|^2 2\pi\hbar \, \delta(x'-x'')$$

$|N|^2 = \frac{1}{2\pi\hbar}$ then $N = \frac{1}{\sqrt{2\pi\hbar}}$

so $\boxed{\langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ip'x'}{\hbar}}}$

$\psi \longleftrightarrow \phi$:

$$\psi_\alpha(x') = \langle x'|\alpha\rangle$$
$$= \int dp' \langle x'|p'\rangle\langle p'|\alpha\rangle$$
$$= \int dp' \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ip'x'}{\hbar}} \phi(p')$$

$\phi \longleftrightarrow \psi$:

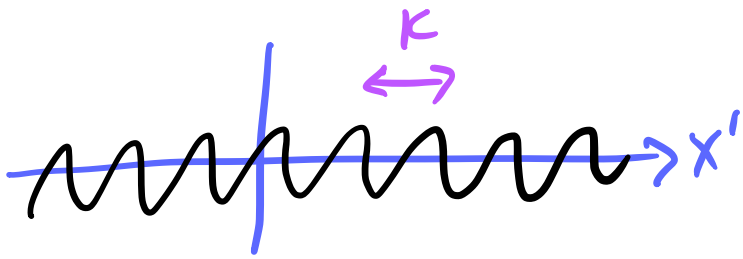
flipped so we get $-i$ in exp

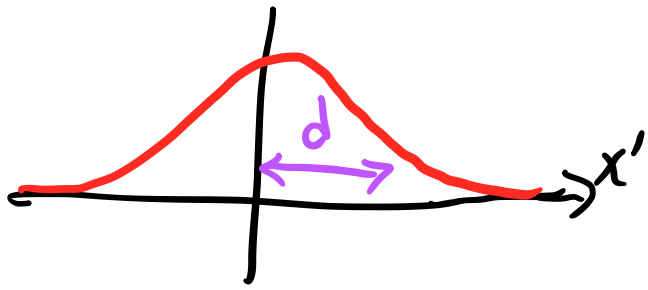
$$\phi_\alpha(p') = \langle p'|\alpha\rangle$$
$$= \int dx' \langle p'|x'\rangle\langle x'|\alpha\rangle$$
$$= \int_{-\infty}^{\infty} dx' \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{-ip'x'}{\hbar}} \psi_\alpha(x')$$

example: Gaussian wave packet

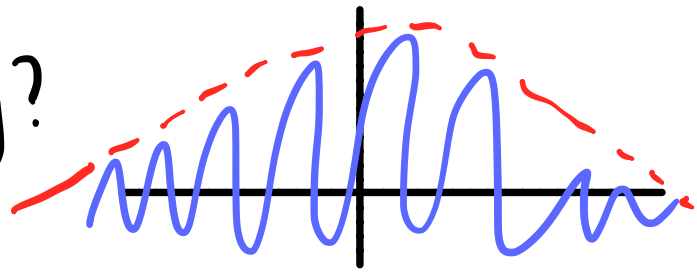
$$\psi_\alpha(x') = \langle x'|\alpha\rangle = \frac{1}{\pi^{1/4}\sqrt{d}} e^{ikx' - \frac{x'^2}{2d}}$$

k, d are constants





uncertainty? of x'



$$\langle (\Delta x)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$$

$$\langle x \rangle = \langle \alpha | x | \alpha \rangle$$

$$= \iint dx' dx'' \langle \alpha | x' \rangle \langle x' | x | x'' \rangle \langle x'' | \alpha \rangle$$

$$= \iint dx' dx'' \, \psi_\alpha^*(x') \, x'' \langle x' | x'' \rangle \langle x'' | \alpha \rangle$$

$$= \iint dx' dx'' \, \psi_\alpha^*(x') \, x'' \underbrace{\delta(x' - x'')}_{\text{0 unless } x' = x''} \psi_\alpha(x'')$$

$$= \int dx' \, \psi_\alpha^*(x') \, x' \, \psi_\alpha(x')$$

$$= \int_{-\infty}^{\infty} dx' \, \underbrace{x'}_{\text{odd}} \, \frac{1}{\sqrt{\pi d}} \, \underbrace{e^{-\frac{x'^2}{d}}}_{\text{even}} = 0$$

* odd · even function is odd function, integral from $-\infty$ to ∞ of odd $= 0$

$$\langle x^2 \rangle = \langle \alpha | x^2 | \alpha \rangle$$

do same steps w/ x^2

$$= \int dx' \, \underbrace{x'^2}_{\text{even}} \, \frac{1}{\sqrt{\pi d}} \, \underbrace{e^{-\frac{x'^2}{d}}}_{\text{even}} = \frac{d^2}{2}$$

$$\boxed{\langle (\Delta x)^2 \rangle = 0 - \frac{d^2}{2} = \frac{d^2}{2}}$$

uncertainty on x doesn't depend on k!

$$\boxed{\langle x'|p|\alpha\rangle = -i\hbar\frac{\partial}{\partial x'}\langle x'|\alpha\rangle} \quad *$$

$$\langle(\Delta p)^2\rangle = \langle p^2\rangle - \langle p\rangle^2 = \frac{\hbar^2}{2d^2} + \hbar^2k^2 - \hbar^2k^2$$

$$\boxed{\langle(\Delta p^2)\rangle = \frac{\hbar^2}{2d^2}}$$

minimum uncertainty: $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \frac{\hbar^2}{4}$

as $d \to \infty$, $\langle(\Delta p)^2\rangle \to 0$
$\langle(\Delta x)^2\rangle \to \infty$

Gaussian wave gets wider

as $d \to 0$, $\langle(\Delta p)^2\rangle \to \infty$
$\langle(\Delta x)^2\rangle \to 0$

wave gets narrower

$$\psi_\alpha(x') = \frac{1}{\pi^{1/4}d^{1/2}}\, e^{ikx' - \frac{x'^2}{2d^2}}$$

$$\phi_\alpha(p') = \langle p'|\alpha\rangle$$

add to equation sheet!

$$= \int dx' \langle p'|x'\rangle\langle x'|\alpha\rangle$$

insert $\mathbb{1}$ in basis we know (x')

$$= \frac{1}{\sqrt{2\pi\hbar}}\int dx'\, e^{-\frac{ip'x'}{\hbar}}\psi_\alpha(x')$$

math $\Rightarrow$

$$= \frac{1}{\sqrt{2\pi\hbar}\sqrt{\pi}\, d} \int dx' \, e^{-\frac{ip'x'}{\hbar}} \, e^{ikx' - \frac{x'^2}{2d^2}}$$

$$= \frac{1}{\sqrt{2\pi\hbar}\sqrt{\pi}\, d} \int dx' \, e^{-\frac{x'^2}{2d^2} + \left(ik - \frac{ip'}{\hbar}\right)x'}$$

$$\boxed{\phi_\alpha(p') = \sqrt{\frac{d}{\hbar\sqrt{\pi}}} \, e^{\frac{-(p' - \hbar k)^2 d^2}{2\hbar^2}}}$$

hw4:

$$= \frac{\hbar^2}{4}\Big[|+\rangle\langle -|+\rangle\langle -| + |-\rangle\langle -|-\rangle\langle +| + |-\rangle\langle +|+\rangle\langle -| + |-\rangle\langle +|-\rangle\langle +| \Big]$$

$$= \frac{\hbar^2}{4}\Big[\cancel{\langle -|+\rangle\langle -|+\rangle} + \cancel{\langle +|-\rangle}\langle -|-\rangle + \langle -|-\rangle\langle +|+\rangle + \cancel{\langle +|-\rangle\langle +|-\rangle} \Big]$$

$$= \frac{\hbar^2}{4}\Big[\langle -|-\rangle\langle +|+\rangle \Big] =$$

HW4 1.19, 1.26, 1.31

1.19 a) compute $\langle(\Delta S_x)^2\rangle = \langle S_x^2\rangle - \langle S_x\rangle^2$ in S_z+ check generalized relation

$$S_x = \frac{\hbar}{2}\left[|+\rangle\langle -| + |-\rangle\langle +|\right]$$

$$S_x^2 = \frac{\hbar^2}{4}\left[|+\rangle\langle -| + |-\rangle\langle +|\right]\left[|+\rangle\langle -| + |-\rangle\langle +|\right]$$

$$= \frac{\hbar^2}{4}\left[|+\rangle\langle +| + |-\rangle\langle -|\right]$$

$$\langle +|(\Delta S_x)^2|+\rangle = \langle +|S_x^2|+\rangle - \langle +|S_x|+\rangle^2$$

$$\langle a|S_x^2|\alpha\rangle = \sum_{a'}\sum_{a''} \langle\alpha|a'\rangle\langle a'|S_x^2|a''\rangle\langle a''|\alpha\rangle$$

$$= \sum_{a'} |\langle\alpha|a'\rangle|^2 \langle a'|S_x^2|a'\rangle$$

so $\langle +|S_x^2|+\rangle = |\langle +|+\rangle|^2 \langle +|S_x^2|+\rangle$

$$= \frac{\hbar^2}{4}\langle +|\ |+\rangle\langle +| + |-\rangle\langle -|\ |+\rangle$$

$$= \frac{\hbar^2}{4}\left[\langle +|+\rangle\langle +|+\rangle + \underbrace{\cancel{\langle +|-\rangle\langle -|+\rangle}}_{=0}\right] = \frac{\hbar^2}{4}$$

$$\langle +|S_x|+\rangle = |\langle +|+\rangle|^2 \langle +|S_x|+\rangle$$

$$= \frac{\hbar}{2}\langle +|\ |+\rangle\langle -| + |-\rangle\langle +|\ |+\rangle$$

$$= \frac{\hbar}{2}\left[\underbrace{\cancel{\langle +|+\rangle\langle -|+\rangle}}_{=0} + \underbrace{\cancel{\langle +|-\rangle\langle +|+\rangle}}_{=0}\right] = 0$$

$$\boxed{\langle(\Delta S_x)^2\rangle = \frac{\hbar^2}{4} - 0 = \frac{\hbar^2}{4}}$$

Check generalized relation:

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}\left|\langle[A,B]\rangle\right|^2$$

let $A = S_x$ and $B = S_y$

do same for $\langle(\Delta S_y)^2\rangle = \langle S_y^2\rangle - \langle S_y\rangle^2$

$$= \langle +|S_y^2|+\rangle - \langle +|S_y|+\rangle$$

$$S_y = -\frac{i\hbar}{2}\left[|+\rangle\langle -| + |-\rangle\langle +|\right]$$

$$S_y^2 = \frac{\hbar^2}{4}\left[|+\rangle\langle -| + |-\rangle\langle +|\right]\left[|+\rangle\langle -| + |-\rangle\langle +|\right]$$

$$= \frac{\hbar^2}{4}\left[|+\rangle\langle +| + |-\rangle\langle -|\right]$$

$$\langle +|S_y^2|+\rangle = \frac{\hbar^2}{4}\langle +|\,|+\rangle\langle +| + |-\rangle\langle -|\,|+\rangle$$

$$= \frac{\hbar^2}{4}\left[\langle +|+\rangle\langle +|+\rangle + \underbrace{\cancel{\langle +|-\rangle\langle -|+\rangle}}_{0}\right] = \frac{\hbar^2}{4}$$

$$\langle +|S_y|+\rangle = -\frac{i\hbar}{2}\langle +|\,|+\rangle\langle -| + |-\rangle\langle +|\,|+\rangle$$

$$= -\frac{i\hbar}{2}\left[\underbrace{\cancel{\langle +|+\rangle\langle -|+\rangle}}_{0} + \underbrace{\cancel{\langle +|-\rangle\langle +|+\rangle}}_{0}\right] = 0$$

so $\langle(\Delta S_y)^2\rangle = \langle +|S_y^2|+\rangle - \langle +|S_y|+\rangle^2$

$$= \frac{\hbar^2}{4} - 0 = \frac{\hbar^2}{4}$$

also need $[S_x, S_y] = i\hbar S_z = i\hbar|+\rangle$

then $\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle \geq \frac{1}{4}\left|\langle[S_x, S_y]\rangle\right|^2$

$$\left(\frac{\hbar^2}{4}\right)\left(\frac{\hbar^2}{4}\right) \geq \frac{1}{4}\left|-\hbar^2\right|$$

$$\frac{\hbar^2}{2} \geq \frac{\hbar^2}{4}$$ ✓ checks out

b) repeat check w/ S_x and S_y in S_x+ state

we know $|S_x;+\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$

then $\langle S_x;-| = \frac{1}{\sqrt{2}}\langle +| + \frac{1}{\sqrt{2}}\langle -|$

so $\langle(\Delta S_x)^2\rangle =$

$\langle S_x;+|S_x^2|S_x;+\rangle - \langle S_x;+|S_x|S_x;+\rangle$

$S_x = \frac{\hbar}{2}[|+\rangle\langle -| + |-\rangle\langle +|]$

$S_x^2 = \frac{\hbar^2}{4}[|+\rangle\langle +| + |-\rangle\langle -|]$

$\langle S_x;+|S_x^2|S_x;+\rangle =$

$\left(\frac{1}{\sqrt{2}}\langle +| + \frac{1}{\sqrt{2}}\langle -|\right)\frac{\hbar^2}{4}|+\rangle\langle +| + |-\rangle\langle -|\left(\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\right)$

$= \quad '' \quad \frac{\hbar}{4\sqrt{2}}|+\rangle\langle +|+\rangle + \underbrace{|+\rangle\langle +|-\rangle}_{0} + \underbrace{|-\rangle\langle -|+\rangle}_{0} + |-\rangle\langle -|-\rangle$

$= \frac{1}{\sqrt{2}}\langle +| + \frac{1}{\sqrt{2}}\langle -| \left(\frac{\hbar}{4\sqrt{2}}\right)(|+\rangle + |-\rangle)$

$= \frac{\hbar^2}{8}[\langle +|+\rangle + \underbrace{\langle +|-\rangle}_{0} + \underbrace{\langle -|+\rangle}_{0} + \langle -|-\rangle]$

$= \frac{\hbar^2}{8}[2] = \frac{\hbar^2}{4}$

$\langle S_x;+|S_x|S_x;+\rangle =$

$\frac{1}{\sqrt{2}}(\langle +| + \langle -|)\frac{\hbar}{2}[|+\rangle\langle -| + |-\rangle\langle +|]\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$

$= \frac{\hbar}{4}(\langle +| + \langle -|)[\underbrace{|+\rangle\langle -|+\rangle}_{0} + |+\rangle\langle -|-\rangle + |-\rangle\langle +|+\rangle + \underbrace{|-\rangle\langle +|-\rangle}_{0}]$

$$= \frac{\hbar}{4}\left(\langle +| + \langle -|\right)\left[\,|+\rangle\langle -|+\rangle + |+\rangle\langle -|-\rangle + |-\rangle\langle +|+\rangle + |-\rangle\langle +|-\rangle\,\right]$$

$$= \frac{\hbar}{4}\left(\langle +|+\rangle + \langle +|-\rangle + \langle -|+\rangle + \langle -|-\rangle\right] = \frac{\hbar}{2}$$

so $\boxed{\langle (\Delta S_x)^2\rangle = \frac{\hbar^2}{4} - \left(\frac{\hbar}{2}\right)^2 = 0}$

now $\langle (\Delta S_y)^2\rangle = \langle S_x;+|S_y^2|S_x;+\rangle - \langle S_x;+|S_y|S_x;+\rangle$

$S_y = -\frac{i\hbar}{2}\left[\,|+\rangle\langle -| + |-\rangle\langle +|\,\right]$ $\quad S_y^2 = \frac{\hbar^2}{4}\left[\,|+\rangle\langle +| + |-\rangle\langle -|\,\right]$

$\langle S_x;+|S_y^2|S_x;+\rangle$

$$= \frac{1}{\sqrt{2}}\left(\langle +| + \langle -|\right)\frac{\hbar^2}{4}\left[\,|+\rangle\langle +| + |-\rangle\langle -|\,\right]\frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$$

$$= \frac{\hbar^2}{8}\left(\langle +| + \langle -|\right)\left[\,|+\rangle\langle +|+\rangle + |+\rangle\langle +|-\rangle + |-\rangle\langle -|+\rangle + |-\rangle\langle -|-\rangle\,\right]$$

$$= \frac{\hbar^2}{8}\left(\langle +| + \langle -|\right)\left(|+\rangle + |-\rangle\right)$$

$$= \frac{\hbar^2}{8}\left[\langle +|+\rangle + \langle +|-\rangle + \langle -|+\rangle + \langle -|-\rangle\right] = \frac{\hbar^2}{8}(2) = \frac{\hbar^2}{4}$$

$\langle S_x;+|S_y|S_x;+\rangle$

$$= \frac{1}{\sqrt{2}}\left(\langle +| + \langle -|\right)\frac{-i\hbar}{2}\left[\,|+\rangle\langle -| + |-\rangle\langle +|\,\right]\frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$$

$$= \frac{i\hbar}{4}\left(\langle +| + \langle -|\right)\left[\,|+\rangle\langle -|+\rangle + |+\rangle\langle -|-\rangle + |-\rangle\langle +|+\rangle + |-\rangle\langle +|-\rangle\,\right]$$

$$= \frac{i\hbar}{4}\left(\langle +| + \langle -|\right)\left(|+\rangle + |-\rangle\right)$$

$$= \frac{i\hbar}{4}\left(\langle +|+\rangle + \langle +|-\rangle + \langle -|+\rangle + \langle -|-\rangle\right] = \frac{i\hbar}{2}$$

then $\boxed{\langle (\Delta S_y)^2\rangle = 0 - \left(-\frac{i\hbar}{2}\right)^2 = \frac{\hbar^2}{4}}$

$[S_x, S_y] = i\hbar S_z$ but now S_z in $S_x;+$

$\langle S_x;+ | S_z | S_x;+ \rangle = 0$

so $\langle (\Delta S_x)^2 \rangle \langle (\Delta S_y)^2 \rangle \geq \frac{1}{4} | \langle [S_x, S_y] \rangle^2$

$$(0)\left(\frac{\hbar^2}{4}\right) \geq \frac{1}{4}(0)$$

$$0 = 0$$ ✓ checks out

<u>1.26</u> Construct transformation matrix $S_z \to S_x$

Show consistency w/ $u = \sum_r |b^r\rangle\langle a^r|$

$$|S_x;+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$
$$|S_x;-\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$$

so $u \cdot S_z = S_x$

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{12} & u_{22} \end{pmatrix} \begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix} = \begin{pmatrix} |S_x;+\rangle \\ |S_x;-\rangle \end{pmatrix}$$

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{12} & u_{22} \end{pmatrix} \begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix} = \begin{matrix} \frac{1}{\sqrt{2}} |+\rangle + |-\rangle \\ \frac{1}{\sqrt{2}} |+\rangle - |-\rangle \end{matrix}$$

$\Downarrow$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = U$$

check $U = \sum_r |b^r\rangle\langle a^r|$

$|S_x;+\rangle = U_{11}|+\rangle + U_{12}|-\rangle$

$|S_x;-\rangle = U_{12}|+\rangle + U_{22}|-\rangle$

then $|S_x;+\rangle = \sum_{a'} \langle a'|S_x;+\rangle |a'\rangle$

$|S_x;-\rangle = \sum_{a''} \langle a''|S_x;-|a''\rangle$

so $U|+\rangle = \sum_{a'} \langle a'|S_x;+\rangle |a'\rangle$

$U|+\rangle = \sum_{a'} |a'\rangle\langle a'|S_x;+\rangle$

then $U = \sum_{a'} |a'\rangle\langle a'|$

1.31 use 1.6.25 : $[X, \mathcal{J}(dx')] = dx'$

and 1.6.45 : $[P, \mathcal{J}(dx')] = 0$

and $|\alpha\rangle \to \mathcal{J}(dx')|\alpha\rangle$ so $\langle\alpha| = \langle\alpha|\mathcal{J}^\dagger(dx')$

Prove $\langle x\rangle \to \langle x\rangle + dx'$ under infinitesimal translation

$\langle p\rangle \to \langle p\rangle$

$\langle x\rangle = \langle\alpha|X|\alpha\rangle$

$= \langle\alpha|\mathcal{J}^\dagger(dx')|X|\mathcal{J}(dx')|\alpha\rangle$

$= \langle\alpha|\mathcal{J}^\dagger(dx')X\mathcal{J}(dx')|\alpha\rangle$

need to make use of commutators:

to get $\mathcal{J}^\dagger(dx')X\mathcal{J}(dx')$

$$= \mathcal{J}^{+}(dx') \cdot \left[X\mathcal{J}(dx') - \mathcal{J}(dx')X + \mathcal{J}(dx')X \right]$$

$$= \mathcal{J}^{+}(dx') \left[X, \mathcal{J}(dx') \right] + \mathcal{J}(dx')X$$

so $\langle \alpha | \mathcal{J}^{+}(dx') [X, \mathcal{J}(dx')] | \alpha \rangle$
$+ \langle \alpha | \underbrace{\mathcal{J}^{+}(dx') \mathcal{J}(dx')}_{=1} X | \alpha \rangle$

$$= \langle \alpha | \mathcal{J}^{+}(dx') [X, \mathcal{J}(dx')] | \alpha \rangle + \langle \alpha | X | \alpha \rangle$$

$$= \langle \alpha | \mathcal{J}^{+}(dx') \underbrace{dx'}_{\text{num}} | \alpha \rangle + \langle \alpha | X | \alpha \rangle$$

$$= \langle \alpha | \underbrace{\mathcal{J}^{+}(dx') dx'}_{\sim dx'} | \alpha \rangle + \langle X \rangle$$

$$= \langle \alpha | dx' | \alpha \rangle + \langle X \rangle$$

so $\boxed{\langle X \rangle = \langle X \rangle + dx'}$

now $\langle P \rangle = \langle \alpha | P | \alpha \rangle$

$$= \langle \alpha | \mathcal{J}^{+}(dx') | P | \mathcal{J}(dx') | \alpha \rangle$$

$$= \langle \alpha | \mathcal{J}^{+}(dx') P \mathcal{J}(dx') | \alpha \rangle$$

using commutators similarly:

$\langle P \rangle = \langle \alpha | \mathcal{J}^{+}(dx') [P, \mathcal{J}(dx')] | \alpha \rangle$
$+ \langle \alpha | \mathcal{J}^{+}(dx') \mathcal{J}(dx') P | \alpha \rangle$

$$\langle p \rangle = \langle \alpha | \mathcal{J}^{+}(dx') \cancel{[P, \mathcal{J}(dx')]}^{\,0} | \alpha \rangle$$
$$+ \langle \alpha | \mathcal{J}^{+}(dx') \mathcal{J}(dx') P | \alpha \rangle$$

$$= \langle \alpha | \underbrace{\mathcal{J}^{+}(dx') \mathcal{J}(dx')}_{\mathbb{1}} P | \alpha \rangle$$

$$= \langle \alpha | P | \alpha \rangle$$

so $\boxed{\langle p \rangle = \langle p \rangle}$

Lecture 11 - Feb 18

HW 5: 1.31, 1.32, 1.33, 2.3

$$\psi_\alpha(x') = \frac{1}{\pi^{1/4}\sqrt{d}} e^{-ikx' - \frac{x^2}{2d^2}}$$

$$\phi_\alpha(p') = \frac{\sqrt{d}}{\sqrt{\hbar}\,\pi^{1/4}} e^{-\frac{(p'-\hbar k)^2 d^2}{2\hbar^2}}$$

2.1 Time Evolution + Schrodinger's Eqn

time is a parameter in QM
- neither an operator nor an observable

Time evolution of state $|\alpha\rangle$:

$|\alpha, t_0; t\rangle$ for $t > t_0$

require $\lim_{t \to t_0} |\alpha, t_0; t\rangle = |\alpha\rangle$

i.e. $|\alpha, t_0\rangle = |\alpha\rangle$ goes to $|\alpha, t_0; t\rangle$

time-evolved state $|\alpha, t_0; t\rangle = \mathcal{U}(t, t_0)\,|\alpha, t_0\rangle$

requirements of time-evolution operator $\mathcal{U}$:

(i) unitary $\quad |\alpha, t_0\rangle = \sum_{a'} \underbrace{c_{a'}(t_0)}_{\text{changes in time}} |a'\rangle$

$$|\alpha, t_0; t\rangle = \sum_{a'} c_{a'}(t)\,|a'\rangle$$

$$\boxed{\sum_{a'} |c_{a'}(t_0)|^2 = \sum_{a'} |c_{a'}(t)|^2}$$

$c_{a'}(t_0) \neq c_{a'}t$ and $|c_{a'}(t)| \neq |c_{a'}(t_0)|$

(2) composition $\quad t_2 > t_1 > t_0$

$$\mathcal{U}(t_2, t_0) = \mathcal{U}(t_2, t_1)\,\mathcal{U}(t_1, t_0)$$

(order matters!)

(3) infinitesimal evolution makes sense

$$|\alpha, t_0; t_0 + dt\rangle = \mathcal{U}(t_0 + dt, t_0)\,|\alpha\rangle$$

$$\lim_{dt \to 0} \mathcal{U}(t_0 + dt, t_0) = \mathbb{1}$$

putting these requirements together:

$$\mathcal{U}(t_0 + dt, t_0) = \mathbb{1} - i\,\underbrace{\Omega}_{\text{Hermitian operator}}\,dt$$

$\mathcal{U}$ is unitless!

$$[\Omega] = \frac{1}{[\text{time}]} = [\text{freq}]$$

Ω must have [freq] so that $\mathcal{U}$ is unitless

since $E = \hbar\omega$, $\Omega = \dfrac{H}{\hbar}$ $\rightarrow$ Hamiltonian operator

$$\boxed{\mathcal{U}(t_0 + dt, t_0) = 1 - \frac{iH\,dt}{\hbar}}$$

$\leftarrow$ check left hand side

now can apply to Schrodinger Eqn

$$\underbrace{\mathcal{U}(t + dt, t_0)}_{\text{not infinitesimal!}} = \mathcal{U}(t + dt, t)\,\underbrace{\mathcal{U}(t, t_0)}$$

$$\mathcal{U}(t+dt, t_0) = \left(1 - \frac{iH\,dt}{\hbar}\right)\mathcal{U}(t, t_0)$$

$$= \mathcal{U}(t, t_0) - \frac{iH\,\mathcal{U}(t, t_0)\,dt}{\hbar}$$

$$\frac{\mathcal{U}(t+dt, t_0) - \mathcal{U}(t, t_0)}{dt} = -\frac{i}{\hbar} H\,\mathcal{U}(t, t_0)$$

↳ this is just the form of a derivative

$$\frac{f(x+dx) - f(x)}{dx} \text{ as } dx \to 0 \text{ is } \frac{df(x)}{dx}$$

so $$\frac{\partial}{\partial t}\mathcal{U}(t, t_0) = -\frac{i}{\hbar} H\,\mathcal{U}(t, t_0)$$

$$\underbrace{i\hbar}_{\text{constants}} \frac{\partial}{\partial t} \underbrace{\mathcal{U}(t, t_0)}_{\text{operator}} = \underbrace{H}_{\text{operator}}\,\underbrace{\mathcal{U}(t, t_0)}_{\text{operator}}$$

num that can vary (parameter)

this is Schrodinger's eqn w/ time evolution!

$$\boxed{i\hbar \frac{\partial}{\partial t}\mathcal{U}(t, t_0) = H\,\mathcal{U}(t, t_0)}$$

so $$i\hbar \frac{\partial}{\partial t}\mathcal{U}(t, t_0)|\alpha, t_0\rangle = H\,\mathcal{U}(t, t_0)|\alpha, t_0\rangle$$

$$i\hbar \frac{\partial}{\partial t}|\alpha, t_0; t\rangle = H|\alpha, t_0; t\rangle$$

$$\underbrace{i\hbar \frac{\partial}{\partial t}\psi_\alpha(x', t) = H\,\psi_\alpha(x', t)}_{\text{time-dependent Schrodinger eqn}}$$

note that the right hand side is really:

$$= \langle x' | H | \alpha, t_0; t \rangle$$

$$= \sum_{a'} \langle x' | a' \rangle \langle a' | H | \alpha, t_0; t \rangle$$

now consider non-infinitesimal cases:

Case 1: Hamilton is time-independent

if $H \neq H(t)$ then $U(t, t_0) = e^{-\frac{iH(t-t_0)}{\hbar}}$

$$\boxed{U(t, t_0) = \mathbb{1} + \frac{-iH(t-t_0)}{\hbar} + \frac{(-iH(t-t_0))^2}{\hbar^2 \, 2!} \cdots}$$

Taylor expansion

Case 2: Hamiltonian does depend on time but Hamiltonians at 2 diff times commute

$H = H(t)$ but $[H(t_1), H(t_2)] = 0$

$$\boxed{U(t, t_0) = e^{-\frac{i}{\hbar} \int_{t_0}^{t} dt' \, H(t')}}$$

Case 3: Hamiltonian is time-dependent, but Hamiltonians at 2 diff times DON'T commute

$H = H(t)$ and $[H(t_1), H(t_2)] \neq 0$

Dyson series solution (series of infinite integrals)

$$U(t, t_0) = \mathbb{1} + \sum_{n=1}^{\infty} \left(-\frac{1}{\hbar}\right)^n \int_{t_1}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_{n-1}} dt_n \, H(t_1) H(t_2)$$

* we won't be covering this case

energy eigenkets (case 1)

How does $\mathcal{U}(t,t_0)$ affect $|\alpha\rangle$? –

consider $|\alpha\rangle$ expanded in base kets $\{|a'\rangle\}$

$\{|a'\rangle\}$ = complete set of base kets

if $[A,H]=0$ then $H|a'\rangle = E_{a'}|a'\rangle$

so $$e^{-\frac{iHt}{\hbar}} = \sum_{a'}\sum_{a''} |a''\rangle\langle a''| e^{\frac{-iHt}{\hbar}} |a'\rangle\langle a'|$$

* case 1 is Taylor series so H in exp acts as operator not in exp

$$= \sum_{a'}\sum_{a''} |a''\rangle \underbrace{e^{\frac{-i}{\hbar}E_{a'}t}}_{\text{num}} \underbrace{\langle a''|a'\rangle}_{\delta_{a''a'}} \langle a'|$$

$$= \sum_{a'} |a'\rangle e^{\underbrace{\frac{-i}{\hbar}E_{a'}t}_{\text{frequency term}}} \langle a'|$$

* phase is oscillatory because frequency depends on a'
higher energy = higher f

$$\boxed{\begin{aligned}\mathcal{U}(t,t_0)|\alpha,t_0\rangle &= |\alpha,t_0;t\rangle \\ &= \sum_{a'} |a'\rangle \underbrace{e^{\frac{-i}{\hbar}E_{a'}t}}_{\text{oscillatory phase}} \underbrace{\langle a'|\alpha,t_0\rangle}_{c_{a'}(t_0)}\end{aligned}}$$

so $$\boxed{e^{\frac{-i}{\hbar}E_{a'}t}\langle a'|\alpha\rangle = c_{a'}(t)}$$

→ just gives a relative phase shift to $c_{a'}(t_0)$

remember: $|\alpha\rangle = \sum_{a'} |a'\rangle \underbrace{\langle a'|\alpha\rangle}_{c_{a'}}$

shorthand: $|\alpha,t_0\rangle = |\alpha\rangle$

Lecture 11 - Feb 23

$$\mathcal{U}(t,t_0) = 1 - \frac{i}{\hbar} H(t-t_0)$$ infinitesimal case

for finite cases:

$$[A,H] = 0 \quad \text{in} \quad \{|a'\rangle\}$$

$$H|a'\rangle = E_{a'}|a'\rangle$$

so $$|\alpha, t_0; t\rangle = \sum_{a'} c_{a'}(t)\,|a'\rangle$$

can always decompose state in basis kets

where $$c_{a'}(t) = \underbrace{\langle a'|\alpha\rangle}_{c_{a'}(t_0)}\, e^{-\frac{iE_{a'}t}{\hbar}}$$

if $|\alpha\rangle = |a'\rangle$ then $$|\alpha, t_0; t\rangle = \underbrace{e^{-\frac{iE_{a'}t}{\hbar}}}_{\text{phase}}|a'\rangle$$

* if more than one operator commutes with H (say $[A,H]=0=[B,H]$) then you need $|a',b'\rangle$ to be in a complete set

Expectation values

Find $\langle B\rangle$ if $[B,A]\neq 0$ and $[B,H]\neq 0$ for state $|a'\rangle$

$$\langle B\rangle = \underbrace{\langle a'|\mathcal{U}^{\dagger}(t,0)}_{\text{dual correspondance of}}\ \underbrace{B}_{\text{operator}}\ \underbrace{\mathcal{U}(t,0)|a'\rangle}_{\text{state } |a'\rangle \text{ is in after time } t}$$

because at $t=0$, $\langle B\rangle = \langle a'|B|a'\rangle$
but at $t\neq 0$, need to evolve the $\langle a'|$ and $|a'\rangle$

$$\langle B\rangle = \langle a'|\mathcal{U}^{\dagger}(t,0)\, B\, \mathcal{U}(t,0)|a'\rangle$$

$$= \langle a'|\underbrace{e^{-i\frac{E_{a'}t}{\hbar}}}_{\text{num}}\, \underbrace{B}_{\text{operator}}\, \underbrace{e^{i\frac{E_{a'}t}{\hbar}}}_{\text{num}}|a'\rangle$$

$$= \langle a'|B|a'\rangle$$ time independent!

$$\boxed{\text{if state is in an energy eigenstate, all expectation values (even ones that don't commute) are stationary}}$$

* this is why energy eigenstates $|a'\rangle$ are called stationary states

What if $|\alpha\rangle$ not in energy eigenstate?

$$|\alpha, t_0=0\rangle = \sum_{a'} c_{a'}|a'\rangle$$ have to decompose our random state

$$\langle B\rangle = \underbrace{\Big(\underbrace{\sum_{a'} c_{a'}^{*}\langle a'|}_{\langle\alpha|}\Big)\mathcal{U}^{\dagger}(t,0)}_{\langle\alpha, t_0=0; t|}\, B\, \underbrace{\mathcal{U}(t,0)\Big(\underbrace{\sum_{a''} c_{a''}|a''\rangle}_{|\alpha\rangle}\Big)}_{|\alpha, t_0=0; t\rangle}$$

$$\langle B \rangle = \sum_{a'} \sum_{a''} c_{a'}^* c_{a''} \langle a' | B | a'' \rangle e^{-i \frac{(E_{a''} - E_{a'})}{\hbar} t}$$

at $t=0$

*expectation value oscillates with $\omega = \frac{E_{a''} - E_{a'}}{\hbar}$

Example: Spin Precession

electron in $\vec{E}$ field w/ spin 1/2 and mag. moment of $\frac{e\hbar}{2m_e c}$ in the presence of an external $\vec{B}$

$$H = -\frac{e}{m_e c} \vec{S} \cdot \vec{B}$$

$\vec{S}$: operator vector

$\vec{B}$: number vector

→ case 1 since H is time-independent

let $\vec{B} = B\hat{z}$ then $H = \frac{-e}{m_e c} S_z B$

note that $[H, S_z] = 0$ so want to be in S_z basis

define $\omega \equiv \frac{|e| B}{m_e c}$ then $H = \omega S_z$

$$\mathcal{U}(t,0) = e^{-i \frac{\omega S_z}{\hbar} t}$$ → Taylor series

at $t=0$, electron in $|\alpha\rangle = c_+ |+\rangle + c_- |-\rangle$

since picking $\{|a'\rangle\} = |+\rangle, |-\rangle$ (S_z basis)

if $c_+ = c_- = \frac{1}{\sqrt{2}}$ (i.e. $|\alpha\rangle = |S_x; +\rangle$)

Probability to measure $S_x \pm$ at t

$$P = |\langle S_x \pm | \alpha, t_0; t\rangle|^2$$

$$= \left|\left\langle \left(\frac{1}{\sqrt{2}}\langle +| \pm \frac{1}{\sqrt{2}}\langle -|\right) | \mathcal{U}(t,0) \left(\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\right)\right\rangle\right|^2$$

$$= \left| \frac{1}{2}\left(\langle +| \pm \langle -|\right) \left| \left(e^{-\frac{i\omega t}{\hbar}}|+\rangle + e^{\frac{i\omega t}{\hbar}}|-\rangle\right)\right|^2$$

* implicit step:

$$\mathcal{U}(t) = e^{-\frac{i\omega S_z t}{\hbar}}$$

$$\mathcal{U}(t)|+\rangle = e^{-\frac{i\omega t}{\hbar}}|+\rangle$$

can turn $\mathcal{U}(t)|\pm\rangle$ into number

since $S_z|+\rangle = \frac{\hbar}{2}|+\rangle$

$$= \left|\frac{1}{2}\left(e^{-\frac{i\omega t}{2}} \pm e^{\frac{i\omega t}{2}}\right)\right|^2$$

$$= \left|\frac{1}{2}\left(\cos\frac{\omega t}{2} - \sin\frac{\omega t}{2} \pm \cos\frac{\omega t}{2} + \sin\frac{\omega t}{2}\right)\right|^2$$

$$\boxed{P = \begin{cases} \cos^2\frac{\omega t}{2} & \text{for } S_x{}^+ \\ \sin^2\frac{\omega t}{2} & \text{for } S_x{}^- \end{cases}}$$

in $S_x{}^+$ state, let it go over time, probability to return to $S_x{}^+$ state oscillates between 0 and 1, same w/ $S_x{}^-$

Sum of these probabilities $= 1$

$\langle S_x \rangle$ will be a function of time then
$\langle S_y \rangle$

$$\frac{1}{2}|S_x^+|^2 - \frac{1}{2}|S_x^-|^2$$

derived from 1.4.6

$$\langle A \rangle = \sum a' p(a')$$

HW 5 1.32, 1.33, 2.3

1.32 a) verify 1.7.39a and b for expectation value of p and p^2 for Gaussian wave packet (1.7.35)

1.7.39a $\langle p \rangle = \hbar k$

1.7.39b $\langle p^2 \rangle = \frac{\hbar^2}{2d^2} + \hbar^2 k^2$

1.7.35: $\langle x' | \alpha \rangle = \psi_\alpha(x') = \frac{1}{\pi^{1/4}\sqrt{d}} e^{-ikx' - \frac{x'^2}{2d^2}}$

$$\langle p \rangle = \int \langle \alpha | x' \rangle P \langle x' | \alpha \rangle \, dx'$$

$$= \int \langle \alpha | x' \rangle \left(-i\hbar \frac{\partial}{\partial x'}\right) \langle x' | \alpha \rangle \, dx'$$

operator

$$\frac{\partial}{\partial x} \psi_\alpha(x') = \frac{1}{\pi^{1/4}\sqrt{d}} \left(ik - \frac{x'}{d^2}\right) e^{-ikx' - \frac{x'^2}{2d^2}}$$

$$= \int \frac{1}{\pi^{1/4}\sqrt{d}} e^{-ikx' - \frac{x'^2}{2d^2}} \left[\frac{-i\hbar}{\pi^{1/4}\sqrt{d}} \left(ik - \frac{x'}{d^2}\right) e^{-ikx' - \frac{x'^2}{2d^2}} \right] dx'$$

$$= \frac{-i\hbar}{\pi^{1/2} d} \int \left(ik - \frac{x'}{d^2}\right) e^{-ikx' - \frac{x'^2}{2d^2}} e^{-ikx' - \frac{x'^2}{2d^2}} \, dx'$$

$$= \frac{-i\hbar}{\pi^{1/2} d} \int \left(ik - \frac{x'}{d^2}\right) e^{-\frac{x'^2}{d^2}} \, dx'$$

$$= \frac{\hbar k}{\pi^{1/2} d} \int e^{-\frac{x'^2}{d^2}} \, dx' = \frac{\hbar k}{\pi^{1/2} d} \left(d\pi^{1/2}\right) = \boxed{\hbar k}$$

$$\langle p^2 \rangle = \int \langle \alpha | x' \rangle P^2 \langle x' | \alpha \rangle \, dx'$$

$$= \int \langle \alpha | x' \rangle \left(-\hbar^2 \frac{\partial^2}{\partial x'} \right) \langle x' | \alpha \rangle \, dx'$$

operator

$$\frac{\partial}{\partial x} \psi_\alpha(x') = \frac{1}{\pi^{1/4}\sqrt{d}} \left(ik - \frac{x'}{d^2} \right) e^{-ikx' - \frac{x'^2}{2d^2}}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} \psi_\alpha(x') \right) =$$

$$\frac{1}{\pi^{1/4}\sqrt{d}} \left(-k^2 - \frac{1}{d^2} - \frac{2ikx'}{d^2} + \frac{x'^2}{d^4} \right) e^{-ikx' - \frac{x'^2}{2d^2}}$$

so $$\langle p^2 \rangle = \int \frac{1}{\pi^{1/4}\sqrt{d}} e^{-ikx' - \frac{x'^2}{2d^2}}$$

$$\cdot \frac{-\hbar^2}{\pi^{1/4}\sqrt{d}} \left(-k^2 - \frac{1}{d^2} - \frac{2ikx'}{d^2} + \frac{x'^2}{d^4} \right) e^{-ikx' - \frac{x'^2}{2d^2}} \, dx'$$

$$= \frac{-\hbar^2}{\pi^{1/2} d} \int e^{-\frac{x'^2}{d^2}} \left(-k^2 - \frac{1}{d^2} - \frac{x^2}{d^4} \right) dx'$$

$$= \frac{-\hbar^2}{\pi^{1/2} d} \left[d\pi^{1/2} \right] \left[-k^2 + \frac{1}{2d^2} \right]$$

$$= \boxed{\hbar^2 k^2 + \frac{\hbar^2}{2d^2}}$$

b) Evaluate $\langle p \rangle$ and $\langle p^2 \rangle$ using ϕ function (1.7.42)

$$\langle p' | \alpha \rangle = \frac{\sqrt{d}}{\sqrt{\hbar}\,\pi^{1/4}} e^{-\frac{(p' - \hbar k)^2 d^2}{2\hbar^2}}$$

so $\langle p \rangle = \int \langle \alpha | p \rangle \, P \, \langle p | \alpha \rangle$

leave as P since only know P in x-basis?

$$= \int \frac{\sqrt{d}}{\sqrt{\hbar}\,\pi^{1/4}} e^{-\frac{(p - \hbar k)^2 d^2}{2\hbar^2}} \, p \, \frac{\sqrt{d}}{\sqrt{\hbar}\,\pi^{1/4}} e^{-\frac{(p - \hbar k)^2 d^2}{2\hbar^2}} \, dp$$

$$= \frac{d}{\hbar\sqrt{\pi}} \int p \, e^{-\frac{(p - \hbar k)^2 d^2}{\hbar^2}} \, dp$$

let $u = p - \hbar k$ then $p = u + \hbar k$

$$= \frac{d}{\hbar\sqrt{\pi}} \int (u + \hbar k) \, e^{-\frac{u^2 d^2}{\hbar^2}} \, du$$

drop odd terms

$$= \frac{d}{\hbar\sqrt{\pi}} (\hbar k) \left(\frac{\hbar \pi^{1/2}}{d} \right) = \boxed{\hbar k}$$

then $\langle p^2 \rangle = \int \langle \alpha | p \rangle \, p^2 \, \langle p | \alpha \rangle$

$$= \frac{d}{\hbar\sqrt{\pi}} \int p^2 \, e^{-\frac{(p - \hbar k)^2 d^2}{\hbar^2}} \, dp$$

again let $u = p - \hbar k$

if $p = u + \hbar k$ then $p^2 = u^2 + 2u\hbar k + \hbar^2 k^2$

$$= \frac{d}{\hbar\sqrt{\pi}} \int (u^2 + 2u\hbar k + \hbar^2 k^2) e^{-\frac{u^2 d^2}{\hbar^2}} du$$

$$= \frac{d}{\hbar\sqrt{\pi}} \left(\frac{\hbar^3\sqrt{\pi}}{2d^3} + \frac{\hbar^3\sqrt{\pi}\, k^2}{d} \right) = \boxed{\frac{\hbar^2}{2d^2} + \hbar^2 k^2}$$

1.33 a)

i) Prove $\langle p'|x|\alpha\rangle = i\hbar \frac{\partial}{\partial p'} \langle p'|\alpha\rangle$

$$\langle p'|x|\alpha\rangle = \int dp'' \underbrace{\langle p'|x|p''\rangle}_{\text{rewrite}} \langle p''|\alpha\rangle \quad \text{insert } \mathbb{1} \quad (1)$$

$$\langle p'|x|p''\rangle = \int \langle p'|x|x'\rangle\langle x'|p''\rangle\, dx'$$

$$= \int x' \underbrace{\langle p'|x'\rangle}_{\phi_{x'}(p')} \underbrace{\langle x'|p''\rangle}_{\phi^*_{x'}(p'')} dx'$$

$$= \frac{1}{2\pi\hbar} \int x' \cdot e^{-\frac{ix'(p'-p'')}{\hbar}} dx' \quad (2)$$

we know $\frac{1}{2\pi\hbar} \int e^{-\frac{ix'(p'-p'')}{\hbar}} dx' = \delta(p'-p'')$

So to get x' out of exp to match (2) take $\frac{\partial}{\partial p'}$

$$\frac{\partial}{\partial p'}\delta(p'-p'') = \frac{\partial}{\partial p'}\left[\frac{1}{2\pi\hbar}\int e^{-ix'\frac{(p'-p'')}{\hbar}}dx'\right]$$

$$= \frac{1}{2\pi\hbar}\int \frac{x'}{i\hbar} e^{-ix'\frac{(p'-p'')}{\hbar}} dx'$$

so $\frac{1}{2\pi\hbar}\int x' \cdot e^{-ix'\frac{(p'-p'')}{\hbar}} dx' = i\hbar\frac{\partial}{\partial p'}\delta(p'-p'')$

now plug back into (i):

$$\langle p'|X|\alpha\rangle = \int dp''\; i\hbar\frac{\partial}{\partial p'}\delta(p'-p'')\langle p''|\alpha\rangle$$

can get rid of integral by changing p'' to p'

$$\boxed{\langle p'|X|\alpha\rangle = i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle}$$

(ii) Prove $\langle\beta|X|\alpha\rangle = \int dp'\, \phi_\beta^*(p')\, i\hbar\frac{\partial}{\partial p'}\phi_\alpha(p')$

where $\phi_\alpha(p') = \langle p'|\alpha\rangle$ and $\phi_\beta(p') = \langle p'|\beta\rangle$

$$\langle\beta|X|\alpha\rangle = \int dp'\, \langle\beta|p'\rangle\langle p'|X|\alpha\rangle$$ insert $\mathbb{1}$ in momentum space

use our results from (i)

$$= \int dp'\, \langle\beta|p'\rangle\, i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle$$

$$\boxed{= \int dp'\, \phi_\beta^*(p')\, i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle}$$

b) What is physical significance of $e^{\frac{ix\Xi}{\hbar}}$?

* Ξ is some number w/ dimension of momentum

$e^{\frac{ix\Xi}{\hbar}}$ is operator because of x so let's see what $\underbrace{P}_{\text{operator}}\, \underbrace{e^{\frac{ix\Xi}{\hbar}}}_{\text{operator}} |p'\rangle$ does:

$$P e^{\frac{ix\Xi}{\hbar}} |p'\rangle = \left(e^{\frac{ix\Xi}{\hbar}} p + [p, e^{\frac{ix\Xi}{\hbar}}]\right) |p'\rangle$$

$$[p, e^{\frac{ix\Xi}{\hbar}}] = -i\hbar \frac{\partial}{\partial x}\left(e^{\frac{ix\Xi}{\hbar}}\right)$$

$$= -i\hbar \left(\frac{i\Xi}{\hbar}\right) e^{\frac{ix\Xi}{\hbar}} = \Xi e^{\frac{ix\Xi}{\hbar}}$$

$$= \left(e^{\frac{ix\Xi}{\hbar}} p + \Xi e^{\frac{ix\Xi}{\hbar}}\right) |p'\rangle$$

$$= e^{\frac{ix\Xi}{\hbar}} P|p'\rangle + \Xi e^{\frac{ix\Xi}{\hbar}} |p'\rangle$$

$$= e^{\frac{ix\Xi}{\hbar}} p' |p'\rangle + \Xi e^{\frac{ix\Xi}{\hbar}} |p'\rangle$$

Ξ is num so regroup $= \underbrace{(p' + \Xi)}_{\text{this looks like eigenvalue}} e^{\frac{ix\Xi}{\hbar}} |p'\rangle$

so $e^{\frac{ix\Xi}{\hbar}} |p'\rangle$ must be an eigenstate of P

i.e. $P e^{\frac{ix\Xi}{\hbar}} |p'\rangle = (p' + \Xi) e^{\frac{ix\Xi}{\hbar}} |p'\rangle$

$$P|p' + \Xi\rangle = (p' + \Xi)|p' + \Xi\rangle$$

this means that momentum is getting translated so $e^{\frac{ix\Xi}{\hbar}}$ is an operator that translates momentum

2.3 Electron is subject to uniform time-independent $\vec{B}$ in $\hat{z}$. At $t=0$, electron in eigenstate $|S\cdot\hat{n};+\rangle = \frac{\hbar}{2}$ ($\hat{n}$ is unit vector in x-z plane making angle β with $\hat{z}$)

(a) $P(S_x = \frac{\hbar}{2})$ as function of time?

we know $|S\cdot\hat{n};+\rangle = \cos(\frac{\beta}{2})|+\rangle + \sin(\frac{\beta}{2})|-\rangle$

will need to use Hamiltonian and $\mathcal{U}(t,t_0)$

$$\mathcal{U}(t) = e^{-\frac{i\omega S_z t}{\hbar}}$$

since $H = \omega S_z$ and $S_z|+\rangle = \frac{\hbar}{2}|+\rangle$

plug in $S_z = \hbar/2$: $\mathcal{U}(t) = e^{-\frac{i\omega t}{2}}$

prob at $t \neq 0$ is $|\langle S_x;+|S\cdot\hat{n};+\rangle(t)|^2$

so $|S\cdot\hat{n};+;t\rangle = \mathcal{U}(t)|S\cdot\hat{n};+\rangle$

$$= e^{-\frac{i\omega t}{2}}\cos(\tfrac{\beta}{2})|+\rangle + e^{\frac{i\omega t}{2}}\sin(\tfrac{\beta}{2})|-\rangle \quad *$$

in S_z basis, $|S_x;+\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$

so $|\langle S_x;+|S\cdot\hat{n};+;t\rangle|^2$

$$= \left|\frac{1}{\sqrt{2}}\langle +| + \frac{1}{\sqrt{2}}\langle -|\left[e^{-\frac{i\omega t}{2}}\cos(\tfrac{\beta}{2})|+\rangle + e^{\frac{i\omega t}{2}}\sin(\tfrac{\beta}{2})|-\rangle\right]\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}e^{-\frac{i\omega t}{2}}\cos(\tfrac{\beta}{2})\langle +|+\rangle + \frac{1}{\sqrt{2}}e^{\frac{i\omega t}{2}}\sin(\tfrac{\beta}{2})\langle -|-\rangle\right|^2$$

$$= \left[\frac{1}{\sqrt{2}}e^{-\frac{i\omega t}{2}}\cos(\tfrac{\beta}{2}) + \frac{1}{\sqrt{2}}e^{\frac{i\omega t}{2}}\sin(\tfrac{\beta}{2})\right]$$
$$\left[\frac{1}{\sqrt{2}}e^{-\frac{i\omega t}{2}}\cos(\tfrac{\beta}{2}) + \frac{1}{\sqrt{2}}e^{\frac{i\omega t}{2}}\sin(\tfrac{\beta}{2})\right]$$

$$= \left[\frac{1}{\sqrt{2}} e^{-\frac{i\omega t}{2}} \cos\left(\frac{\beta}{2}\right) + \frac{1}{\sqrt{2}} e^{\frac{i\omega t}{2}} \sin\left(\frac{\beta}{2}\right)\right]$$

$$\left[\frac{1}{\sqrt{2}} e^{-\frac{i\omega t}{2}} \cos\left(\frac{\beta}{2}\right) + \frac{1}{\sqrt{2}} e^{\frac{i\omega t}{2}} \sin\left(\frac{\beta}{2}\right)\right]$$

$$= \frac{1}{2} e^{-2i\omega t} \cos^2\left(\frac{\beta}{2}\right) + \frac{1}{2} \sin\left(\frac{\beta}{2}\right) \cos\left(\frac{\beta}{2}\right) + \frac{1}{2} e^{2i\omega t} \sin^2\left(\frac{\beta}{2}\right)$$

$$\boxed{P(S_x{}^{+}, t) = \frac{1}{2}\left[1 + \sin(\beta)\cos(\omega t)\right]}$$

(b) $\langle S_x \rangle$ as function of time?

$$S_x = \frac{\hbar}{2}\left[|+\rangle\langle -| + |-\rangle\langle +|\right]$$

in $|S\cdot\hat{n}; +\rangle$ state, expectation value is

$$\langle S\cdot\hat{n}; +, t | S_x | \underbrace{S\cdot\hat{n}; +, t}_{\text{from part a}} \rangle$$

$$= e^{-\frac{i\omega t}{2}} \cos\left(\frac{\beta}{2}\right)|+\rangle + e^{\frac{i\omega t}{2}} \sin\left(\frac{\beta}{2}\right)|-\rangle$$

$$\langle S_x \rangle = \left(e^{+\frac{i\omega t}{2}} \cos\left(\frac{\beta}{2}\right)\langle +| + e^{-\frac{i\omega t}{2}} \sin\left(\frac{\beta}{2}\right)\langle -|\right)$$

$$\cdot \frac{\hbar}{2}\left[|+\rangle\langle -| + |-\rangle\langle +|\right]\left(e^{-\frac{i\omega t}{2}} \cos\left(\frac{\beta}{2}\right)|+\rangle + e^{\frac{i\omega t}{2}} \sin\left(\frac{\beta}{2}\right)|-\rangle\right)$$

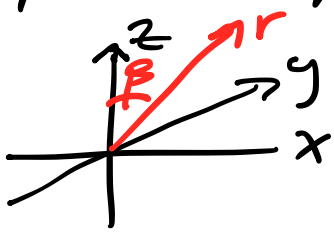
$$= \left(e^{+\frac{i\omega t}{2}} \cos\left(\frac{\beta}{2}\right)\langle +| + e^{-\frac{i\omega t}{2}} \sin\left(\frac{\beta}{2}\right)\langle -|\right)$$

$$\cdot \frac{\hbar}{2}\left[e^{-\frac{i\omega t}{2}} \cos\left(\frac{\beta}{2}\right)|-\rangle\langle +|+\rangle + e^{\frac{i\omega t}{2}} \sin\left(\frac{\beta}{2}\right)|+\rangle\langle -|-\rangle\right]$$

$$= \frac{\hbar}{2}\left[e^{i\omega t} \cos\left(\frac{\beta}{2}\right)\sin\left(\frac{\beta}{2}\right) + e^{-i\omega t} \sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\beta}{2}\right)\right]$$

$$\boxed{\langle S_x, t \rangle = \frac{\hbar}{2}\cos(\omega t)\sin(\beta)}$$

c) Show answers make sense in limits $\beta \to 0$ and $\beta \to \frac{\pi}{2}$



for $\beta \to 0$:

$$|S\cdot\hat{n};+\rangle = e^{-\frac{i\omega t}{2}}\cos\left(\frac{\beta}{2}\right)|+\rangle + e^{\frac{i\omega t}{2}}\sin\left(\frac{\beta}{2}\right)|-\rangle$$

$$= e^{-\frac{i\omega t}{2}}|+\rangle$$

makes sense $|+\rangle$ would point fully in $\hat{z}$ direction if $\beta = 0$

and $\langle S_x\rangle = \frac{\hbar}{2}\cos(\omega t)\sin(\beta) = 0$

if pointing in $\hat{z}$, this makes sense

for $\beta \to \pi/2$:

$$|S\cdot\hat{n};+\rangle = e^{-\frac{i\omega t}{2}}\cos\left(\frac{\pi}{4}\right)|+\rangle + e^{\frac{i\omega t}{2}}\sin\left(\frac{\pi}{4}\right)|-\rangle$$

$$= \frac{1}{\sqrt{2}}\left[e^{-\frac{i\omega t}{2}}\cos|+\rangle + e^{\frac{i\omega t}{2}}\sin|-\rangle\right]$$

and $\langle S_x\rangle = \frac{\hbar}{2}\cos(\omega t)\sin(\pi/2)$

$$= \frac{\hbar}{2}\cos(\omega t)$$

both these make sense since they show spin precession