Recap 0: Properties of Eigenfunctions

- Close connection between the properties of eigenfunctions of self-adjoint operators and eigenvectors of hermitian matrices.
- Specifically, if $\mathcal{L}$ is self-adjoint and $\mathcal{L}u_i + \lambda_i w(x) u_i = 0$, then
    1. the eigenvalues λ_i are <u>real</u>,
    2. the eigenfunctions u_i are <u>orthogonal</u>,
    3. the eigenfunctions u_i are <u>complete</u>.

$$\int w(x)\, u_i^* u_j \, dx = \delta_{ij}$$

$$f(x) = \sum_i a_i u_i(x)$$

Recap 1: Fourier Series

- Then Fourier series for f is

$$f(x) = \sum_{n=1}^{\infty}\left(\alpha_n \underbrace{\sqrt{\frac{2}{L}}\cos\frac{2\pi n x}{L}}_{\text{normalized}} + \beta_n \underbrace{\sqrt{\frac{2}{L}}\sin\frac{2\pi n x}{L}}_{\text{normalized}}\right) + \alpha_0$$

where

$$\alpha_n = \int_0^L w(x)\sqrt{\frac{2}{L}}\cos\frac{2\pi n x}{L}\, f(x)\, dx\,,\quad \beta_n = \int_0^L w(x)\sqrt{\frac{2}{L}}\sin\frac{2\pi n x}{L}\, f(x)\, dx$$

$$\alpha_0 = \int_0^L w(x)\sqrt{\frac{1}{L}}\, f(x)\, dx$$

- More conventionally,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{2\pi n x}{L} + b_n\sin\frac{2\pi n x}{L}\right),$$

where $\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \frac{2}{L}\int_0^L \begin{pmatrix} \cos\frac{2\pi n x}{L} \\ \sin\frac{2\pi n x}{L} \end{pmatrix} f(x)\, dx$

Recap 2: Legendre Series

- For Legendre's equation, $\mathcal{L}u = (1 - x^2)u'' - 2xu'$, $w(x) = 1$, $\lambda = l(l+1)$

  boundary conditions: any, on $[-1, 1]$

  eigenfunctions: $u_l = P_l(x)$, l integer

  orthogonality: $(P_l, P_m) = A_l \delta_{lm}$ will see $A_l = \frac{1}{l+\frac{1}{2}}$

- Hence

$$f(x) = \sum_{i=0}^{\infty} (u_i, f)\, u_i(x)$$
$$= \sum_{l=0}^{\infty} a_l P_l(x)$$

where

$$a_l = \left(l + \frac{1}{2}\right) \int_{-1}^{1} P_l(x) f(x)\, dx$$

Recap 3: Bessel Series

- For Bessel's equation, $\mathcal{L}^{(m)}u = \rho u'' + u' - \frac{m^2}{\rho}u,\ w(\rho) = \rho,\ \lambda = n^2$

  boundary conditions: u regular at $\rho = 0,\ u(a) = 0$

  eigenfunctions: $u_i = J_m\left(\frac{\alpha_{mi}\rho}{a}\right),\ \ i =$ integer

  orthogonality: $(u_i, u_j) = \int_0^a \rho\, J_m\left(\frac{\alpha_{mi}\rho}{a}\right) J_m\left(\frac{\alpha_{mj}\rho}{a}\right)\, d\rho = B_{mi}^2 \delta_{ij}$

- Hence, can expand for $0 \leq \rho \leq a$

$$f(\rho) = \sum_{i=0}^{\infty} a_i \frac{1}{B_{mi}} J_m\left(\frac{\alpha_{mi}\rho}{a}\right) \qquad f(\rho) = \sum_{i=0}^{\infty} a_i J_m\left(\frac{\alpha_{mi}\rho}{a}\right)$$

where

$$a_i = \int_0^a \frac{1}{B_{mi}} J_m\left(\frac{\alpha_{mi}\rho}{a}\right) f(\rho)\, d\rho \qquad a_i = \int_0^a \frac{1}{B_{mi}^2} J_m\left(\frac{\alpha_{mi}\rho}{a}\right) f(\rho)\, d\rho$$

Application of Fourier Series to PDEs

- String, fixed at $x = 0,\ L$, displacement $u(x,t)$, satisfies wave equation

$$u_{tt} = c^2 u_{xx}$$

- Expand u as a Fourier series in x satisfying the boundary conditions

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{L}$$

- Substitute:

$$\sum_{n=1}^{\infty} \ddot{a}_n(t) \sin \frac{n\pi x}{L} = c^2 \sum_{n=1}^{\infty} a_n(t) \left(-\frac{n^2\pi^2}{L^2}\right) \sin \frac{n\pi x}{L}$$

$$\Longrightarrow \quad \ddot{a}_n + \left(\frac{n\pi c}{L}\right)^2 a_n = 0$$

$$\Longrightarrow \quad a_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t, \quad \text{where } \omega_n = \frac{n\pi c}{L}$$

- A_n and B_n come from the <u>initial conditions</u> $u(x,0) = f(x),\ u_t(x,0) = g(x)$

Application of Fourier Series to PDEs

- Initially at rest, struck at center

$$f(x) = 0, \qquad g(x) = v_0 \delta\left(x - \frac{L}{2}\right)$$

- Then $\quad A_n$ are coefficients in the Fourier series for f

  and $\quad \omega B_n$ are coefficients in the Fourier series for g

  so

$$A_n = 0$$

$$B_n = \frac{1}{\omega_n} \frac{2}{L} \int_0^L \sin \frac{n\pi x}{L} v_0 \delta\left(x - \frac{L}{2}\right) dx$$

$$= \frac{2v_0}{n\pi c} \sin \frac{n\pi}{2}$$

$= 0$, n even

$= (-1)^m, n = 2m + 1$

<u>Spectral</u> method of solution for PDE

Convergence of (Generalized) Fourier Series

- Suppose $\{u_n\}$ is an orthonormal set on $[a, b]$, and define the partial sum

$$p_n(x) = \sum_{i=1}^{n} c_i u_i(x)$$

for <u>any</u> choice of c_i.

- Then the mean-square error in approximating function $f(x)$ by $p_n(x)$ is

$$E_n = \int_a^b w(x)[f(x) - p_n(x)]^2 \, dx \;\; \geq 0$$

- Expand:

$$\sum_{i=1}^{n} \int_a^b w(x) f(x) c_i u_i(x) \, dx$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \int_a^b w(x) c_i c_j u_i(x) u_j(x) \, dx$$

$$0 \leq E_n = \int_a^b w(x) f^2(x) \, dx$$
$$-2 \int_a^b w(x) f(x) p_n(x) \, dx$$
$$+ \int_a^b w(x) p_n^2(x) \, dx$$

Convergence of (Generalized) Fourier Series

- Now let

$$a_i = \int_a^b w(x)\, u_i^*(x)\, f(x)\, dx$$

(i.e. the generalized Fourier coefficient).

- Then

$$0 \leq E_n = \int_a^b w(x) f^2(x)\, dx \; - 2\sum_{i=1}^n a_i c_i + \sum_{i=1}^n c_i^2$$
$$= \int_a^b w(x) f^2(x)\, dx \; - \sum_{i=1}^n a_i^2 + \sum_{i=1}^n (c_i - a_i)^2$$

- Clearly minimized by choosing $c_i = a_i$

$\Longrightarrow$ best approximation by a partial sum for given n is the truncated Fourier series.

Convergence of (Generalized) Fourier Series

- Smallest mean square error comes from the truncated Fourier series ($c_i = a_i$) and

$$\sum_{i=1}^{n} a_i^2 \le \int_a^b w(x) f^2(x)\, dx$$ Bessel Inequality

- Also implies $a_i \to 0$ as $i \to \infty$ (convergence).
- For a <u>complete</u> set of basis functions, $E_n \to 0$ as $n \to \infty$ and the Bessel Inequality becomes

$$\sum_{i=1}^{\infty} a_i^2 = \int_a^b w(x) f^2(x)\, dx$$ Parseval Identity

Gibbs Phenomenon

- Fourier series converges in mean square. Can show (Homework 3) that we can do better.
- Any partial sum is a sum of continuous functions and therefore must be continuous. What happens when f is <u>discontinous</u>?
- Proceed by example. Look at a square-wave function

$$f(x) = \begin{cases} -1, -\pi < x < 0 \\ +1,\ 0 < x < \pi \end{cases}$$

- Odd function, so expect sine Fourier series

$$f(x) = \sum_{n=1}^{\infty} a_n \sin nx,$$

with
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx\, f(x)\, dx = \frac{2}{\pi}\int_0^{\pi} \sin nx\, dx$$
$$= \frac{2}{\pi n}[1 - (-1)^n]$$

Gibbs Phenomenon

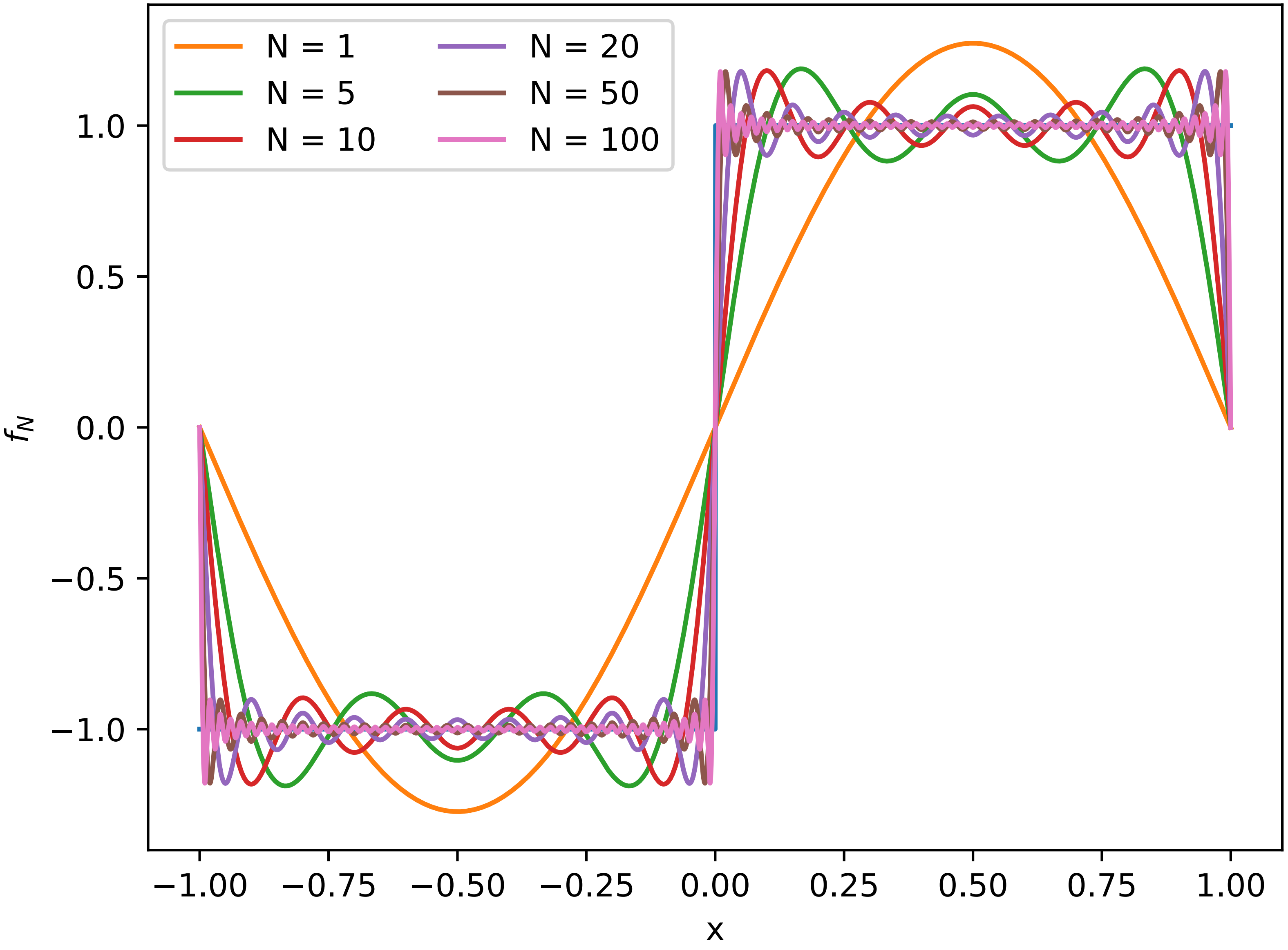
- Fourier series of the square wave is

$$f(x) = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin(2m+1)x}{2m+1}$$

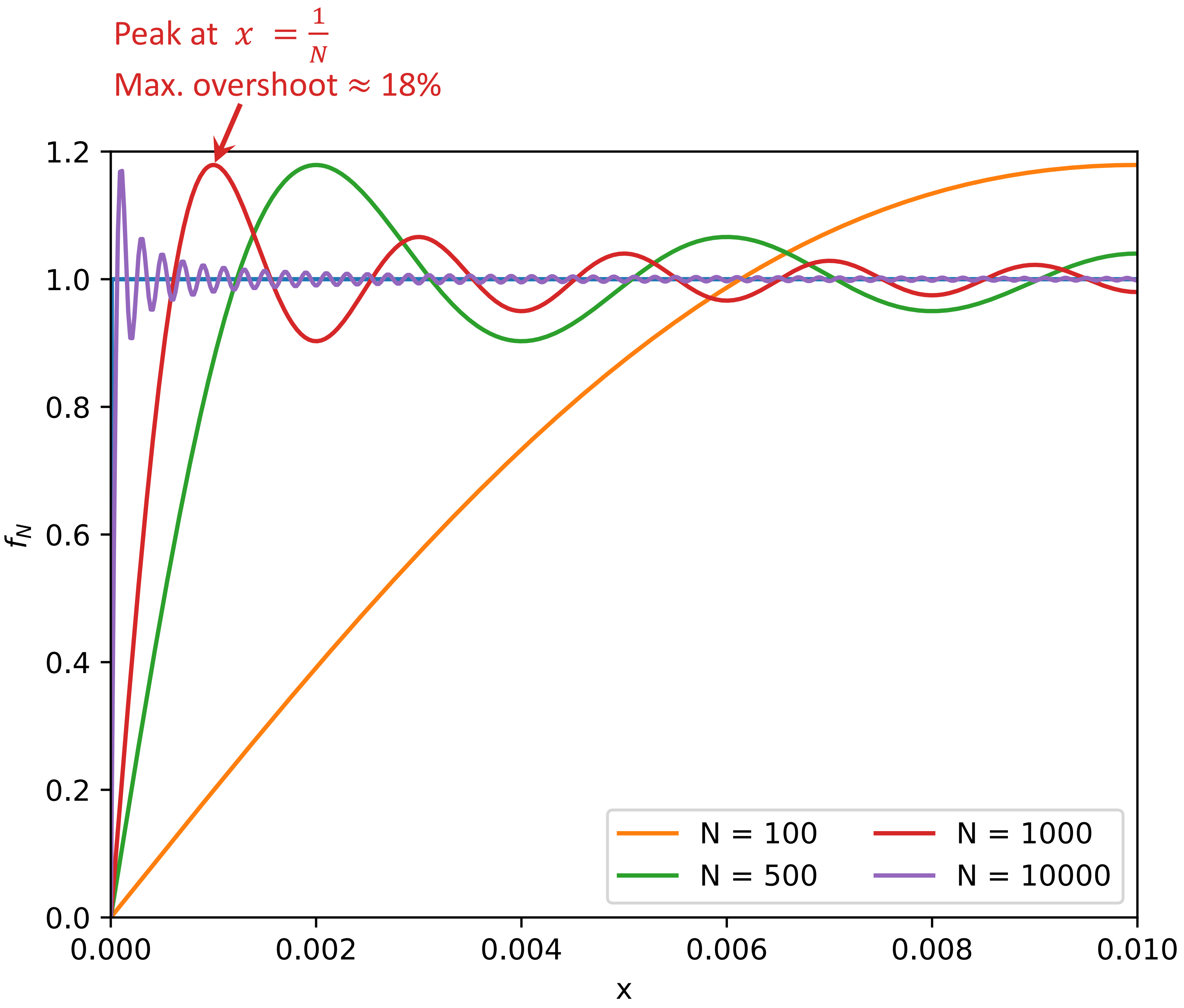
- Illustrate by looking at some partial sums

$$f_M(x) = \frac{4}{\pi}\sum_{m=0}^{M}\frac{\sin(2m+1)x}{2m+1}$$

- Plot the sums for $N = 2M + 1 = 1, 5, 10, 20, 50\ 100, 500, 1000, \ldots$



N = 1
N = 5
N = 10
N = 20
N = 50
N = 100
1.0
0.5
0.0
−0.5
−1.0
f_N
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
x



Peak at $x = \frac{1}{N}$
Max. overshoot ≈ 18%
1.2
1.0
0.8
0.6
0.4
0.2
0.0
f_N
0.000
0.002
0.004
0.006
0.008
0.010
x
N = 100
N = 500
N = 1000
N = 10000

Gibbs Phenomenon

- Fourier series for the square wave is

$$f(x) = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin(2m+1)x}{2m+1}$$

- Illustrate by looking at some partial sums

$$f_M(x) = \frac{4}{\pi}\sum_{m=0}^{M}\frac{\sin(2m+1)x}{2m+1}$$

- Plot the sums for $M = 1, 5, 10, 20, 50\ 100, 500, 1000, \ldots$
- Clear illustration of the difference between mean-square and pointwise convergence!
- At point of discontinuity, Fourier series converges to $\frac{1}{2}[f(x_-) + f(x_+)]$.
- If f is continuous, then Gibbs doesn't arise, and Fourier series is pointwise and actually <u>uniformly</u> convergent.
- Generic behavior for <u>all</u> generalized Fourier series.

Example (3D): Laplace's Equation in a Cylinder

- Finite uniform cylinder, radius a, potential ϕ on all surfaces except the top is 0, potential on top ($z = h$) is ϕ_T.
- As before, solution is a sum of terms of the form

$$\phi_{ml}(\rho, \varphi, z) = J_m(\lambda\rho)e^{\pm im\varphi}e^{\pm\lambda z}$$

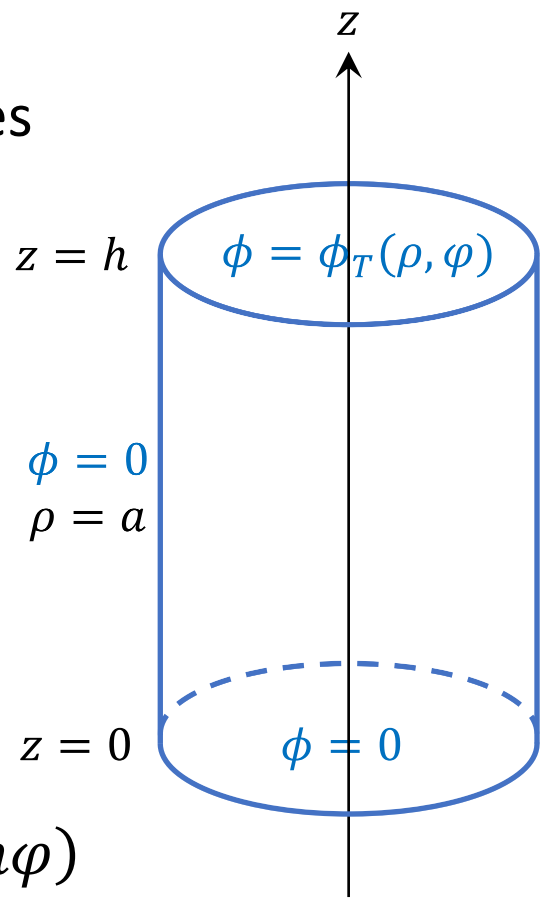
shorthand

- Boundary condition at $\rho = a \implies \lambda a = \alpha_{mn}$, n integer.
- Boundary condition at $z = 0 \implies \sinh \lambda z$ solution.

$$\implies \phi_{ml}(\rho, \varphi, z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\left(\frac{\alpha_{mn}\,\rho}{a}\right)\sinh\left(\frac{\alpha_{mn}\,z}{a}\right)$$

$$\times\left(C_{mn}\cos m\varphi + D_{mn}\sin m\varphi\right)$$

- BC at $z = h$ gives a Fourier-Bessel series for $\phi_T(\rho, \varphi)$.



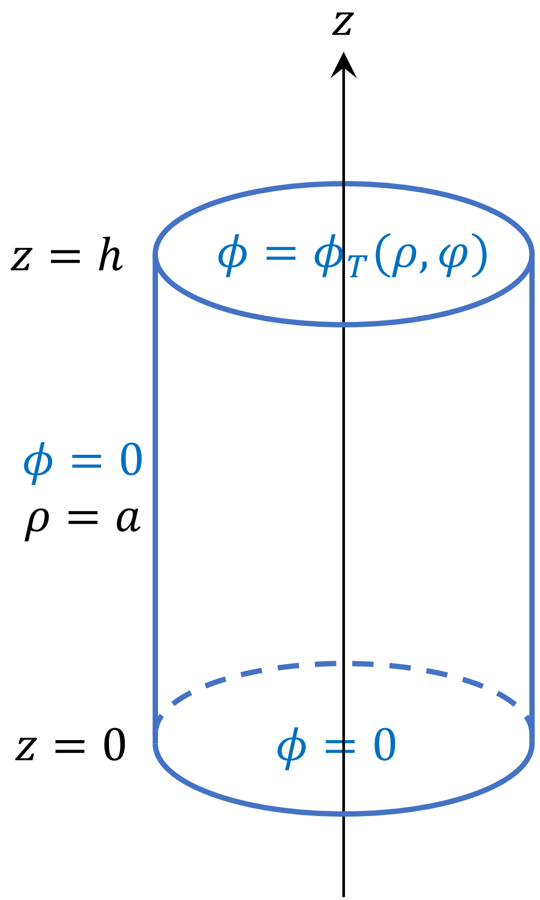
Example (3D): Laplace's Equation in a Cylinder

- BC at $z = h$ gives a Fourier-Bessel series for $\phi_T(\rho, \varphi)$:

$$\phi_T(\rho,\varphi) = \sum_{m,n} J_m\!\left(\frac{\alpha_{mn}\,\rho}{a}\right) \sinh\left(\frac{\alpha_{mn}\,h}{a}\right) \times \left(C_{mn}\cos m\varphi + D_{mn}\sin m\varphi\right)$$

where

$$\begin{pmatrix} C_{mn} \\ D_{mn} \end{pmatrix} = \frac{1}{\pi B_{mn}^2} \sinh\left(\frac{\alpha_{mn}\,h}{a}\right) \times \int_0^a \rho d\rho \int_0^{2\pi} d\varphi\, J_m\!\left(\frac{\alpha_{mn}\rho}{a}\right) \begin{pmatrix} \cos m\varphi \\ \sin m\varphi \end{pmatrix} \phi_T(\rho,\varphi)$$



- Need to know a little more about Bessel functions to
  1. do the integral,
  2. determine the normalization B_{mn}^2.

Bessel Generating Function

- A generating function is a convenient mathematical device for encoding an infinite sequence of numbers by treating them as the coefficients of a formal power series.
- Consider

$$g(x,t) = \exp\left[\frac{1}{2}x(t - t^{-1})\right] = 1 + \frac{1}{2}x(t - t^{-1}) + \frac{1}{8}x^2(t - t^{-1})^2 \ldots$$

- Plausible to write

$$g(x,t) = \sum_{m=-\infty}^{\infty} g_m(x)\, t^m$$

- $g(x,t)$ is the generating function of the sequence $\{g_m(x)\}$.
- Can use the properties of $g(x,t)$ to explore the behavior of $g_m(x)$.
- Properties of $g(x,t)$ define the $g_m(x)$.

Bessel Generating Function

Let $g(x,t) = \exp\left[\frac{1}{2}x(t - t^{-1})\right] = \sum_m g_m(x)\, t^m$

$$\Rightarrow \quad \frac{\partial g}{\partial t} = \frac{1}{2}x(1 + t^{-2}) \sum_m g_m(x)\, t^m = \sum_m m g_m(x)\, t^{m-1}$$

$$\Rightarrow \quad \sum_m g_m(x)\, t^m + \sum_m g_m(x)\, t^{m-2} = \sum_m \frac{2m}{x} g_m(x) t^{m-1}$$

$$\Rightarrow \quad \sum_m g_{m-1}(x)\, t^{m-1} + \sum_m g_{m+1}(x)\, t^{m-1} = \sum_m \frac{2m}{x} g_m(x) t^{m-1}$$

$$\Rightarrow \quad g_{m-1}(x) + g_{m+1}(x) = \frac{2m}{x} g_m(x) \qquad (\dagger)$$

- recurrence relation
- can use to compute $g_m(x)$ given (say) $g_0(x)$ and $g_1(x)$.

Bessel Generating Function

Also,

$$\frac{\partial g}{\partial x} = \frac{1}{2}(t - t^{-1}) \sum_m g_m(x)\, t^m = \sum_m g_m'(x)\, t^{m-1}$$

$$\Longrightarrow \quad \sum_m g_m(x)\, t^{m+1} - \sum_m g_m(x)\, t^{m-1} = 2\sum_m g_m'(x) t^m$$

$$\Longrightarrow \quad \sum_m g_{m-1}(x)\, t^m - \sum_m g_{m+1}(x)\, t^m = 2\sum_m g_m'(x) t^m$$

$$\Longrightarrow \quad g_{m-1}(x) - g_{m+1}(x) = 2g_m'(x) \qquad (\dagger\dagger)$$

$$g_{m-1}(x) + g_{m+1}(x) = \frac{2m}{x} g_m(x) \qquad (\dagger)$$

- Combine: $(\dagger) \mp (\dagger\dagger)$

$$\Longrightarrow \quad g_{m\pm 1}(x) = \frac{m}{x} g_m(x) \mp g_m'(x)$$

Bessel Generating Function

- Given $\quad g_{m\pm1} = \frac{m}{x} g_m \mp g_m'$

$\Longrightarrow \quad x g_m' + m g_m - x g_{m-1} = 0 \qquad (\ddagger)$

$x(\ddagger)'$: $\quad x^2 g_m'' + x g_m' + m x g_m' - x^2 g_{m-1}' - x g_{m-1} = 0$

$m(\ddagger)$: $\quad m x g_m' + m^2 g_m - m x g_{m-1} = 0$

$- \quad \Longrightarrow \quad x^2 g_m'' + x g_m' - m^2 g_m - x[x g_{m-1}' - (m-1) g_{m-1}] = 0$

$-x g_m$

$\Longrightarrow \quad x^2 g_m'' + x g_m' + (x^2 - m^2) g_m = 0$

Bessel's equation!

- $g_m(x) = J_m(x)$ for integer m

- Many ways to define functions
  - ODE
  - series solution
  - generating function
  - complex integral

Bessel Recurrence Relations

- Many recurrences, combine a few here.
- Derived from the generating function for integer m, but in fact true for all real m.

$$J_{m-1} + J_{m+1} = \frac{2m}{x} J_m$$

$$J_{m-1} - J_{m+1} = 2J_m'$$

$$J_{m\pm1} = \frac{m}{x} J_m \mp J_m'$$

$$(x^m J_m)' = x^m J_{m-1}$$

$$(x^{-m} J_m)' = -x^{-m} J_{m+1}$$

(sometimes useful for integration by parts)

Defining the Bessel Functrions

- Define $J_0(x)$ from the series solution, conventionally set $J_0(0) = 1$, and then <u>all</u> the other functions are defined by the recurrence relations:

$$J_1 = -J_0'$$

$$J_2 = \frac{2}{x} J_1 - J_0$$

$$J_3 = \frac{4}{x} J_2 - J_1$$

$$J_4 = \frac{6}{x} J_3 - J_2$$

"etc."

Half-odd Integer Bessel Functions

- Can easily show from the series solution (or directly from the ODE)

$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x$$

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x$$

- Then $J_{m+1} = \frac{2m}{x} J_m - J_{m-1}$, so

$$J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)$$

$$= \left(\frac{2}{\pi x}\right)^{1/2} \left[\frac{\sin x}{x} - \cos x\right]$$

$$J_{5/2}(x) = \frac{3}{x} J_{3/2}(x) - J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \left[\left(\frac{3}{x^2} - 1\right) \sin x - \frac{3 \cos x}{x}\right]$$

"etc."

Bessel Function Normalization

- Inversion of the Bessel series requires the normalization integral

$$B_{mn}^2 = \int_0^a \rho\, J_m\left(\frac{\alpha_{mn}\rho}{a}\right) J_m\left(\frac{\alpha_{mn}\rho}{a}\right)\, d\rho$$
$$= \frac{a^2}{\alpha_{mn}^2} \int_0^{\alpha_{mn}} x\, J_m^2(x)\, dx$$

- Let $I = \int_0^a x\, J_m^2(x)\, dx$

$$= \left[\frac{1}{2} x^2 J_m^2(x)\right]_0^a - \int_0^a x^2\, J_m(x)\, J_m'(x)\, dx$$

ODE: $\quad x^2 J_m = m^2 J_m - x J_m' - x^2 J_m''$

$$\Longrightarrow\ I = \left[\frac{1}{2} x^2 J_m^2\right]_0^a - \int_0^a \left(m^2 J_m J_m' - x {J_m'}^2 - x^2 J_m'' J_m'\right) dx$$
$$= \left[\frac{1}{2}(x^2 - m^2) J_m^2 + \frac{1}{2} x^2 {J_m'}^2\right]_0^a$$

Bessel Function Normalization

- Recurrence relation: $xJ'_m = mJ_m - xJ_{m+1}$

$$\Longrightarrow I = \left[\frac{1}{2}(x^2 - m^2)J_m^2 + \frac{1}{2}(m^2J_m^2 + x^2J_{m+1}^2 - 2mxJ_mJ_{m+1})\right]_0^a$$

$$= \frac{1}{2}[x^2J_m^2 + x^2J_{m+1}^2 - 2mxJ_mJ_{m+1}]_0^a \qquad (a = \alpha_{mn})$$

$$= \frac{1}{2}\alpha_{mn}{}^2J_{m+1}^2(\alpha_{mn})$$

- Hence

$$B_{mn}^2 = \frac{a^2}{\alpha_{mn}^2}\int_0^{\alpha_{mn}} x\,J_m^2(x)\,dx$$

$$= \frac{1}{2}a^2J_{m+1}^2(\alpha_{mn})$$

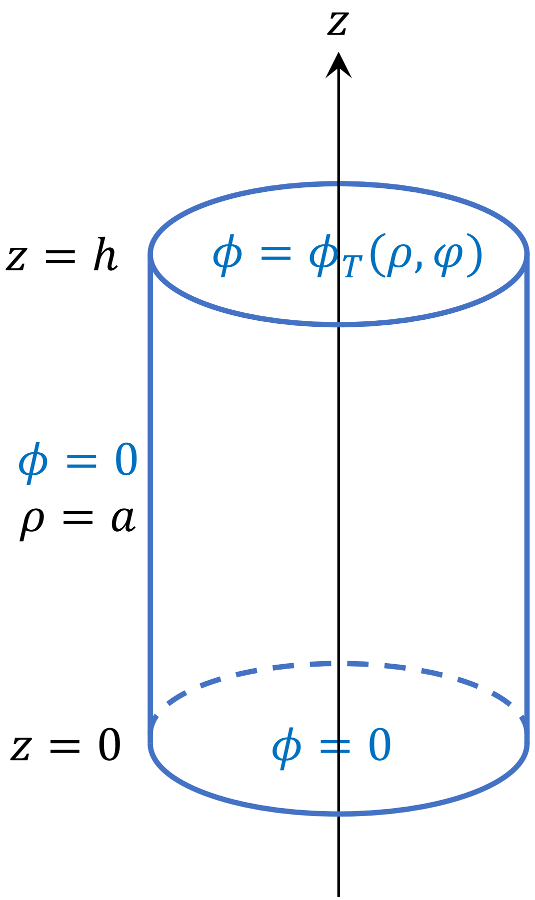
Example (3D): Laplace's Equation in a Cylinder

- Can fill in the details on the earlier problem.
- BC at $z = h$ gives a <u>Fourier-Bessel series</u> for $\phi_T(\rho, \varphi)$:

$$\phi_T(\rho,\varphi) = \sum_{m,n} J_m\left(\frac{\alpha_{mn}\,\rho}{a}\right) \sinh\left(\frac{\alpha_{mn}\,h}{a}\right)$$
$$\times\left(C_{mn}\cos m\varphi + D_{mn}\sin m\varphi\right)$$

where

$$\begin{pmatrix} C_{mn} \\ D_{mn} \end{pmatrix} = \frac{2}{\pi a^2 J_{m+1}^2\left(\alpha_{mn}\right)} \sinh\left(\frac{\alpha_{mn}\,h}{a}\right)$$
$$\times \int_0^a \rho d\rho \int_0^{2\pi} d\varphi\, J_m\left(\frac{\alpha_{mn}\rho}{a}\right) \begin{pmatrix} \cos m\varphi \\ \sin m\varphi \end{pmatrix} \phi_T(\rho,\varphi)$$



Example (3D): Laplace's Equation in a Cylinder

- Wimp out and choose a simple problem: $\phi_T\,(\rho,\varphi) = \phi_0 \cos\varphi$
- Then

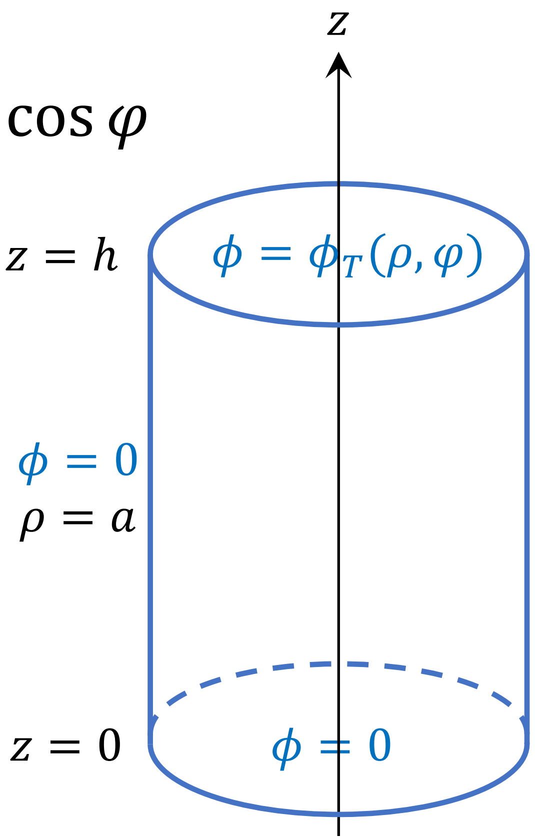
$$\begin{pmatrix} C_{mn} \\ D_{mn} \end{pmatrix} = \frac{2\phi_0}{\pi a^2 J_{m+1}^2\,(\alpha_{mn})} \sinh\left(\frac{\alpha_{mn}\, h}{a}\right)$$

$$\times\ \int_0^a \rho\, J_m\left(\frac{\alpha_{mn}\rho}{a}\right) d\rho \int_0^{2\pi} d\varphi \cos\varphi \begin{pmatrix} \cos m\varphi \\ \sin m\varphi \end{pmatrix}$$

- φ integral is zero except for the $m = 1$ cosine solution
- ρ integral for J_1 is

$$\int_0^{\alpha_{0n}} x\, J_1(x)\, dx = -\int_0^{\alpha_{0n}} x\, J_0'(x)\, dx$$

$$= -[x\cancel{J_0}]_0^{\alpha_{0n}} + \int_0^{\alpha_{0n}} J_0(x)\, dx$$



- Not exactly a closed-form solution, but calculable!
- Expect Gibbs effects at $\rho = a$!
- Tools for managing Bessel integrals much more limited than for sin/cos.

Laplace Equation in a Cylinder, v.2

- Modify the BCs slightly
- Solution is a sum of terms of the form

$$\phi_{ml}(\rho, \varphi, z) = J_m(\lambda\rho)\, e^{\pm im\varphi} e^{\pm\lambda z}$$

- But the boundary condition at $z = 0, h$

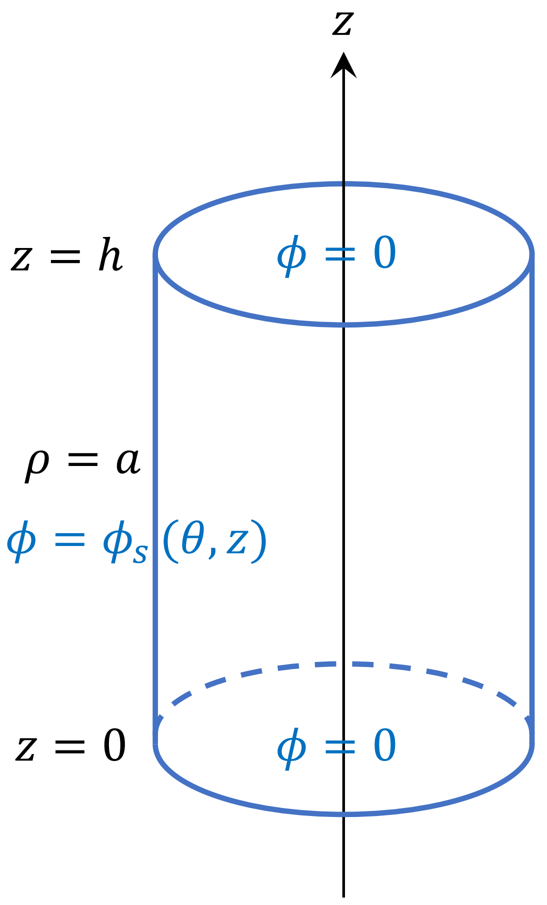
$$\Longrightarrow \lambda = il,\ \ l \text{ real},\ \ lh = n\pi$$

$$\Longrightarrow z\text{-dependence is } \sin\frac{n\pi z}{h}$$

- $\Longrightarrow$ <u>New</u> radial dependence, λ now pure imaginary
- New radial equation is

$$\rho^2 u'' + \rho u' - (l^2\rho^2 + m^2)u = 0$$

Modified Bessel function



Modified Bessel Functions

- Modified radial equation
$$\rho^2 u'' + \rho u' - (l^2\rho^2 + m^2)u = 0$$
- Set $x = l\rho$ and find
$$x^2 u'' + x u' - (x^2 + m^2)u = 0$$

Modified Bessel equation

- Formal first solution is $I_m(x) = i^{-m} J_m(ix)$
  - modified Bessel function of the first kind.

"Bessel functions of imaginary argument"

- Second solution $K_m(x) = \frac{\pi}{2}\frac{I_{-m}(x) - I_m(x)}{\sin m\pi}$
  - modified Bessel function of the second kind.

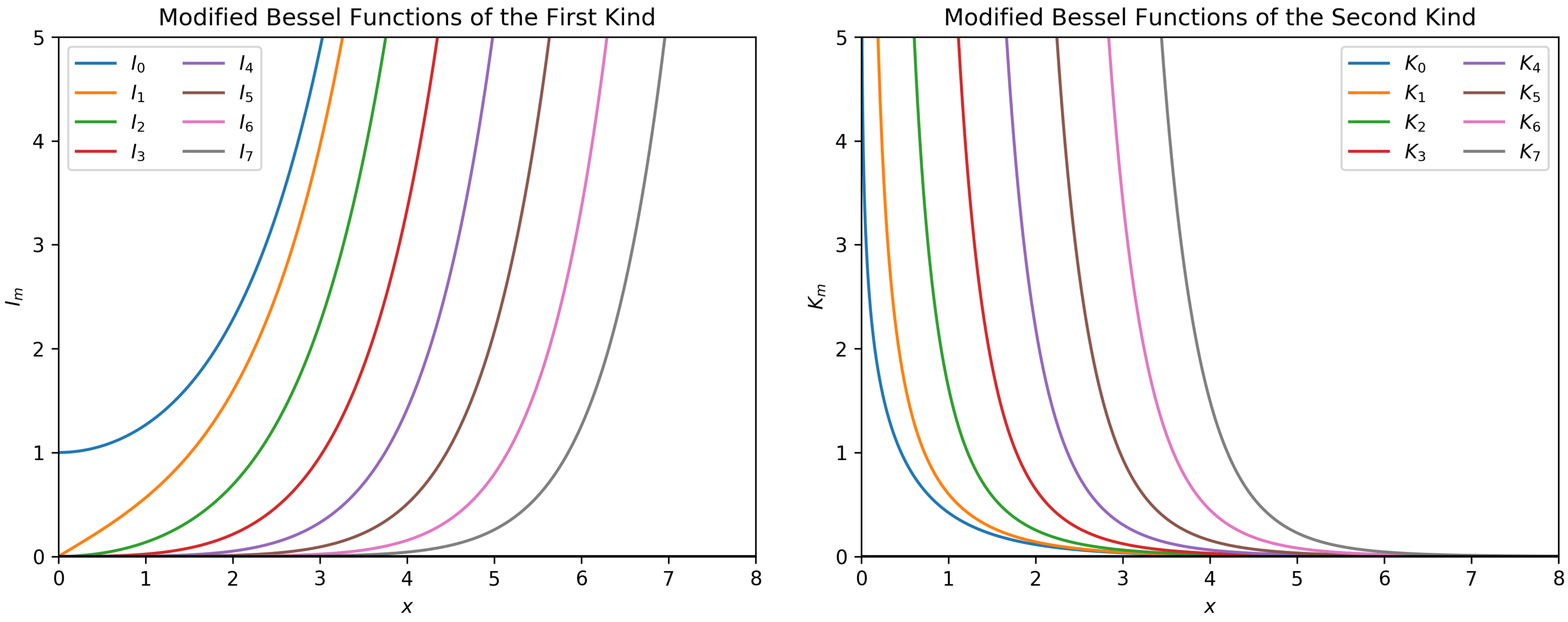
Modified Bessel Functions

- Properties and recurrence relations follow those for $J_m\ /Y_m$.
- $I_m(x)$ is finite at at $x = 0$; non-oscillatory

  asymptotically, $I_m(x) \sim \left(\frac{2}{\pi x}\right)^{1/2} e^{x}$ $\qquad \left(\frac{2}{\pi k\rho}\right)^{1/2} e^{k\rho}$
- Second solution $K_m(x)$

  singular at at $x = 0$; non-oscillatory

  asymptotically, $K_m(x) \sim \left(\frac{2}{\pi x}\right)^{1/2} e^{-x}$ $\qquad \left(\frac{2}{\pi k\rho}\right)^{1/2} e^{-k\rho}$
- Note
  - Solutions exponential in z are oscillatory in ρ
  - Solutions exponential in ρ are oscillatory in z



Modified Bessel Functions of the First Kind
Modified Bessel Functions of the Second Kind
I_m
K_m
x
I_0 I_1 I_2 I_3 I_4 I_5 I_6 I_7
K_0 K_1 K_2 K_3 K_4 K_5 K_6 K_7
0 1 2 3 4 5
0 1 2 3 4 5 6 7 8

Laplace Equation in a Cylinder, v.2

- Write down the solution to this problem…
- Solution regular at ρ = 0 is a sum of terms of the form

$$\phi_{ml}(\rho,\varphi,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} a_{mn} I_m\left(\frac{n\pi\rho}{h}\right)\ e^{\pm im\varphi}\sin\frac{n\pi z}{h}$$

- BC at $\rho = a$

$$\phi_{ml}(a,\varphi,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} a_{mn} I_m\left(\frac{n\pi a}{h}\right)\ e^{\pm im\varphi}\sin\frac{n\pi z}{h}$$

- another double Fourier series for the coefficients …