Velocity-dependent potentials

If there is no potential (in the usual sense) we can still obtain the form of Lagrange's equation provided we use a function U where

$$Q = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

$$L = T - U$$

Important example: a charge q moving with v in an EM field

$$\bar{E} = -\bar{\nabla}\varphi - \frac{\partial \bar{A}}{\partial t}$$

$$\bar{B} = \bar{\nabla} \times \bar{A}$$

$\varphi(x,y,z,t)$ scalar potential

$\bar{A}(x,y,z,t)$ vector potential

$$U = q\varphi - q\bar{A}\cdot\bar{v}$$

$$L = T - U$$
$$= \frac{1}{2}mv^2 - q\varphi + q\bar{A}\cdot\bar{v}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

x:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_x}\right) - \frac{\partial L}{\partial x} = 0$$

(1) (2)

(1) $$\frac{d}{dt}\left(\frac{\partial}{\partial v_x}\left[\frac{1}{2}mv^2 - q\varphi + q\bar{A}\cdot\bar{v}\right]\right) =$$

$$= \frac{d}{dt}\left(mv_x + qA_x\right)$$

$$= m\ddot{x} + q\frac{dA_x}{dt}$$

(3)

(3) $$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial A_x}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial A_x}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial A_x}{\partial t}$$

$$= m\ddot{x} +$$

$$+ q\left[\frac{\partial A_x}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial A_x}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial A_x}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial A_x}{\partial t}\right]$$

(2) $$\frac{\partial L}{\partial x} = -q\frac{\partial \varphi}{\partial x} + q\left[v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_y}{\partial x} + \frac{\partial A_z}{\partial x}v_z\right]$$

$$m\ddot{x} \overbrace{- q v_y \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - q v_z \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right)}^{-q(\bar{v} \times \bar{B})_x}$$

$$\underbrace{+ q \frac{\partial A_x}{\partial t} + q \frac{\partial \varphi}{\partial x}}_{-qE_x} = 0$$

$$m\ddot{x} = qE_x + q(\bar{v} \times \bar{B})_x$$

$$\boxed{\bar{F} = q\bar{E} + q(\bar{v} \times \bar{B})}$$