PDE Recap 1

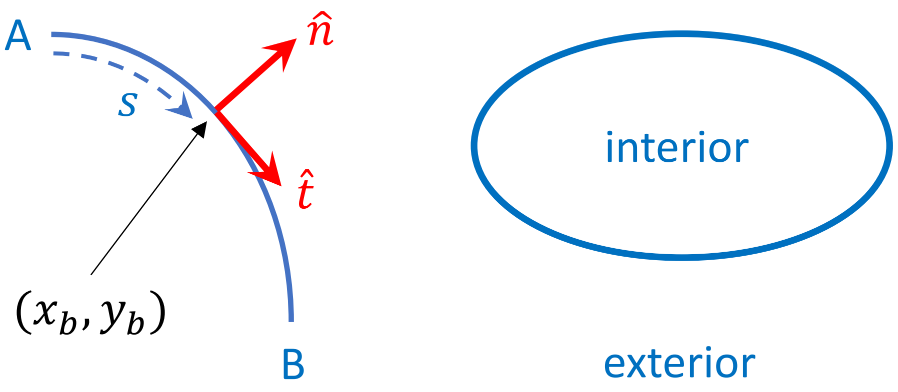
- Setting

$$u_{xx} \equiv \frac{\partial^2 u}{\partial x^2},\ u_{yy} \equiv \frac{\partial^2 u}{\partial y^2},\ u_{xy} \equiv \frac{\partial^2 u}{\partial x \partial y},$$

- we have

$$\begin{aligned}
\frac{dx_b}{ds}u_{xx} + \frac{dy_b}{ds}u_{xy} \qquad\qquad &= \frac{d}{ds}\left(\frac{\partial u}{\partial x}\right)_b \\
\frac{dx_b}{ds}u_{xy} + \frac{dy_b}{ds}u_{yy} &= \frac{d}{ds}\left(\frac{\partial u}{\partial y}\right)_b \\
A u_{xx} \quad + 2B u_{xy} \quad + C u_{yy} \quad &= f
\end{aligned}$$

- Linear third-order simultaneous equation for the second derivatives.



PDE Recap 2

- Equations have a solution <u>unless</u> the determinant of coefficients is zero:

$$\begin{vmatrix} \frac{dx_b}{ds} & \frac{dy_b}{ds} & 0 \\ 0 & \frac{dx_b}{ds} & \frac{dy_b}{ds} \\ A & 2B & C \end{vmatrix} = 0$$

$$\Longrightarrow \quad A\left(\frac{dy_b}{ds}\right)^2 - 2B\frac{dx_b}{ds}\frac{dy_b}{ds} + C\left(\frac{dx_b}{ds}\right)^2 = 0$$

or $$A\left(\frac{dy_b}{dx_b}\right)^2 - 2B\frac{dy_b}{dx_b} + C = 0$$

PDE Recap 3

- Characteristic curves for the PDE are defined by the ODE

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A}$$

- Can show: if we differentiate again, higher derivatives are also subject to the same characteristic equation.
- Cauchy BCs give a solution to the problem (for all higher derivatives) except where the boundary is tangent to a characteristic.
- Classification of solutions based on the discriminant:

$B^2 > AC \quad \Longrightarrow$ 2 real solutions: "hyperbolic equation"

$B^2 < AC \quad \Longrightarrow$ 0 real solutions: "elliptic equation"

$B^2 = AC \quad \Longrightarrow$ 1 real solution: "parabolic equation"

Classification of Linear Equations

- Wave equation standard form: $u_{xx} - \frac{1}{c^2} u_{tt} = 0$

  $\Longrightarrow A = 1,\ B = 0,\ C = -\frac{1}{c^2},\ B^2 > AC,$ so hyperbolic

- Laplace equation standard form: $u_{xx} + u_{yy} = 0$
- $\Longrightarrow A = 1,\ B = 0,\ C = 1,\ B^2 < AC,$ so elliptic
- Diffusion equation standard form: $u_{xx} - \frac{1}{\kappa} u_t = 0$

  $\Longrightarrow A = 1,\ B = C = 0,\ B^2 = AC,$ so parabolic

Wave Equation

- $u_{xx} - \frac{1}{c^2} u_{tt} = 0$

$$A = 1,\ B = 0,\ C = -\frac{1}{c^2}$$

- Characteristic equation is

$$\left(\frac{dt}{ds}\right)^2 - \frac{1}{c^2}\left(\frac{dx}{ds}\right)^2 = 0$$

$$\Longrightarrow \left(\frac{dx}{dt}\right)^2 = c^2$$

$$\Longrightarrow \frac{dx}{dt} = \pm c$$

$$\Longrightarrow x - ct = \xi,\ \text{constant}$$

$$x + ct = \eta,\ \text{constant}$$

Characteristics are straight lines (rays)

Interpretation of Characteristics

- $u_{xx} - \frac{1}{c^2} u_{tt} = 0$
- Seek traveling solution $u(x,t) = f(x - ct)$, as before, $= f(\xi)$

  then $u_{xx} = f''(\xi)$, $u_{tt} = c^2 f''(\xi)$, so $u = f(\xi)$ is a solution
- Similarly, $u = g(\eta)$ is a solution, $\eta = x + ct$
- General solution then is

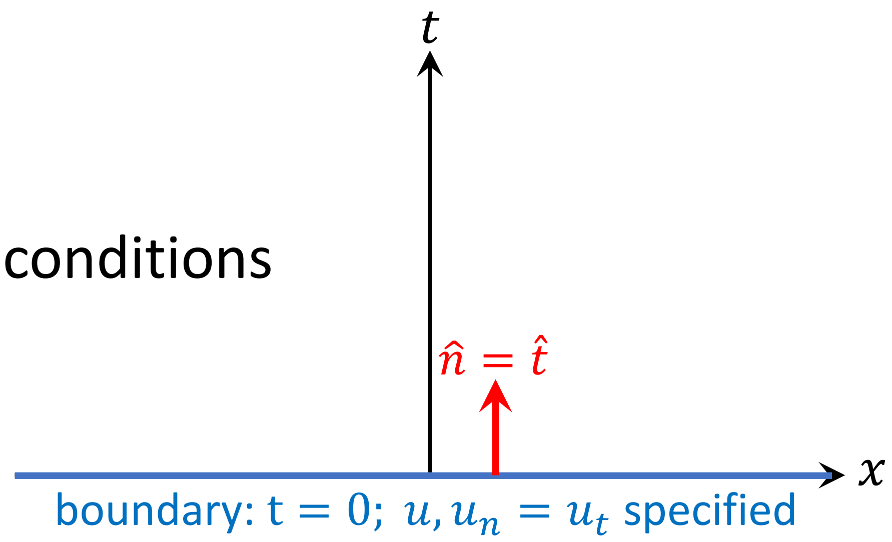
  $$u(x,t) = f(\xi) + g(\eta)$$

  for <u>any</u> functions f and g.
- Can determine f and g from the boundary conditions.

Method of Characteristics

- General solution is

$$u(x,t) = f(\xi) + g(\eta)$$



- "Boundary conditions" in this case are initial conditions

$$u(x,0) = f(x) + g(x)$$

$$u_t(x,0) = -cf'(x) + cg'(x)$$

so

$$f(x) + g(x) = u(x,0)$$

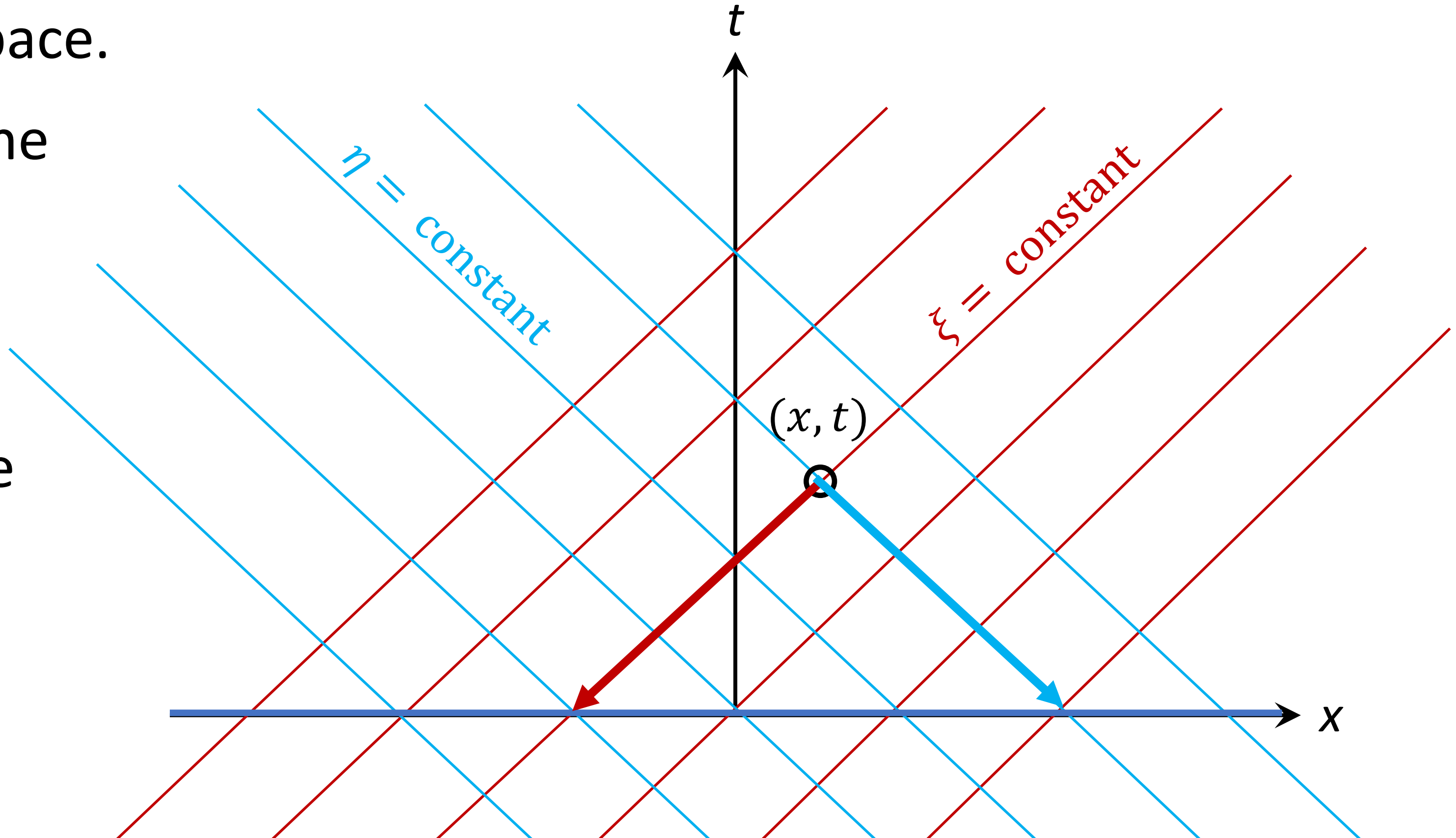
$$-f'(x) + g'(x) = \frac{1}{c} u_t(x,0)$$

$$-f(x) + g(x) = \frac{1}{c} \int dx\, u_t(x,0)$$

$$\Longrightarrow \begin{cases} f(x) = \dfrac{1}{2} u(x,0) - \dfrac{1}{2c} \displaystyle\int dx\, u_t(x,0) \\[2ex] \mathrm{g}(x) = \dfrac{1}{2} u(x,0) + \dfrac{1}{2c} \displaystyle\int dx\, u_t(x,0) \end{cases}$$

Method of Characteristics

- Method works on an infinite domain because every point (x, t) lies on two characteristics that both cross the x-axis where BCs are specified.
- $\xi = x - ct$ and $\eta = x + ct$ effectively define an alternative coordinate system that spans the space.
- Characteristics sample the boundary conditions at two distinct points.
- Characteristics through any given point cross the boundary at a point <u>where the boundary conditions are specified</u>.



Characteristic Coordinates

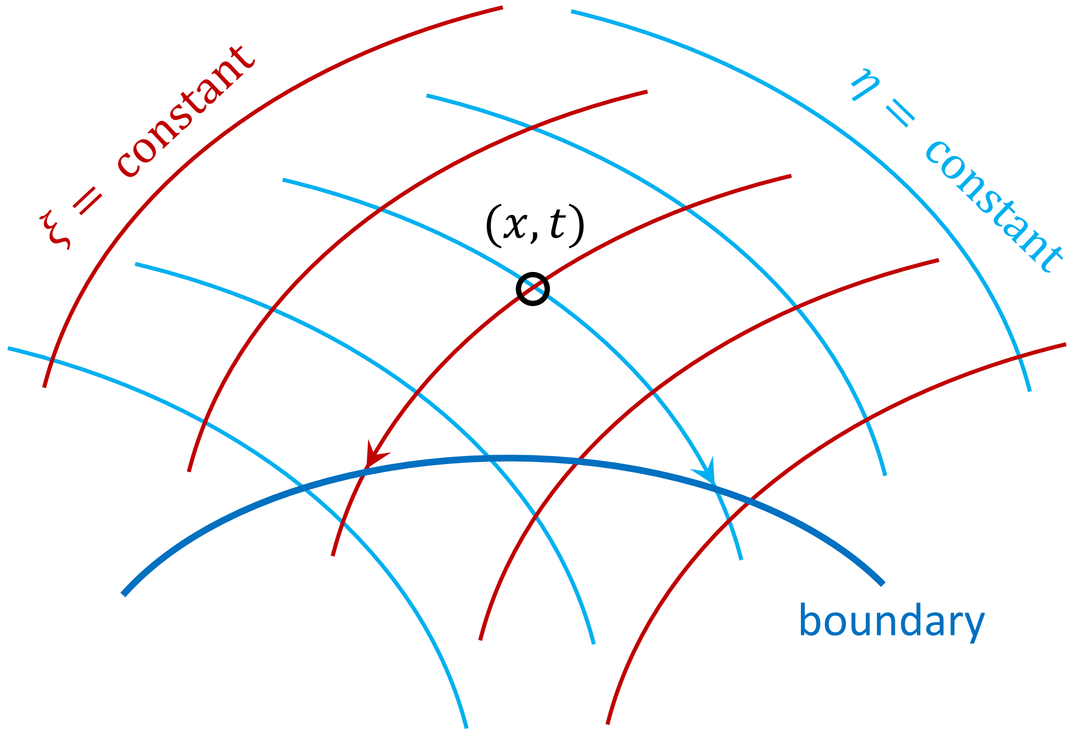
- Given $u_{xx} - \frac{1}{c^2} u_{tt} = 0,\ \xi = x - ct,\ \eta = x + ct$
- Write $u_x = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\xi + u_\eta$

  so $u_{xx} = u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x + u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x = u_{\xi\xi} + 2u_{\xi\eta}\eta_x + u_{\eta\eta}$

  similarly $u_{tt} = c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta}\eta_x + c^2 u_{\eta\eta}$
- Thus $u_{xx} - \frac{1}{c^2} u_{tt} = 4u_{\xi\eta} = 0 \implies u_{\xi\eta} = 0$ <u>Normal form</u>
- In general (HW1), a hyperbolic PDE can be written in the form

$$u_{\xi\eta} = f(\xi, \eta, u, u_\xi, u_\eta)$$

  where $\xi(x,y) = \text{constant}$ and $\eta(x,y) = \text{constant}$ are characteristics.
- Effectively, separate into ODEs in ξ and η.



$\xi =$ constant
$\eta =$ constant
(x, t)
boundary

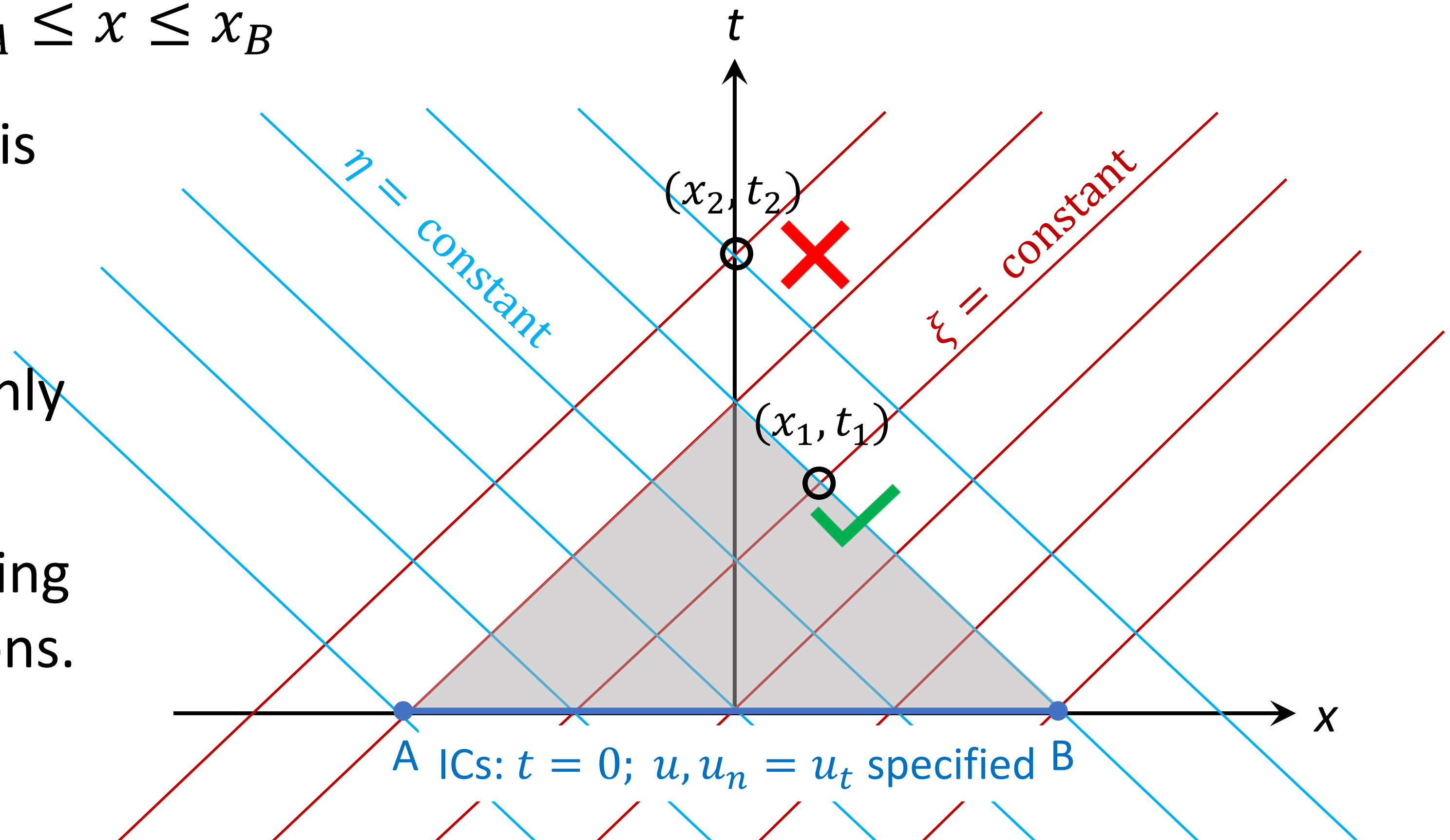
Finite Domain

- What if the region where the boundary conditions are specified is <u>finite</u>?

$$u(x,0) = f(x),\ \ x_A \le x \le x_B$$

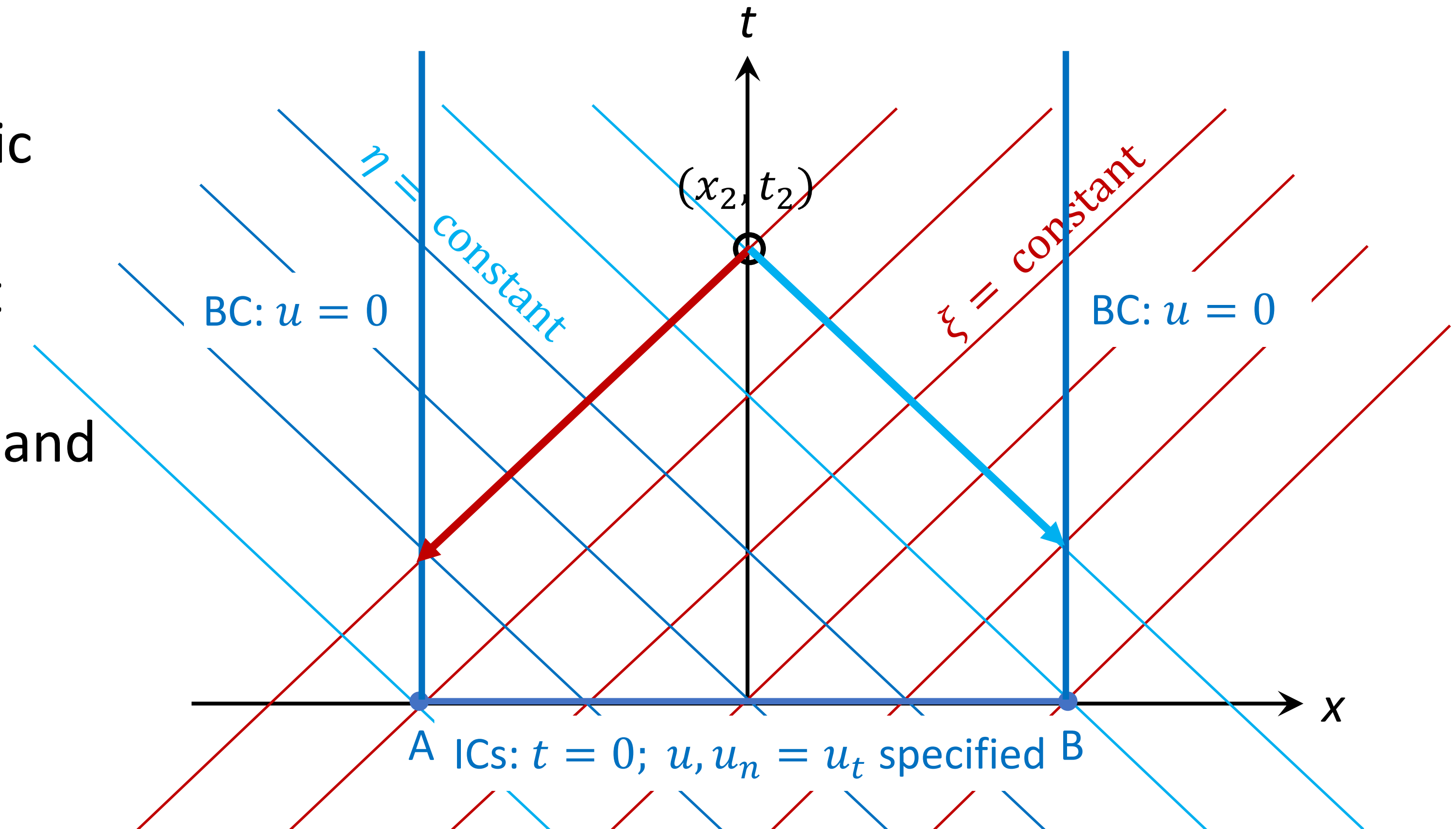
$$\frac{\partial u}{\partial t}(x,0) = g(x), x_A \le x \le x_B$$

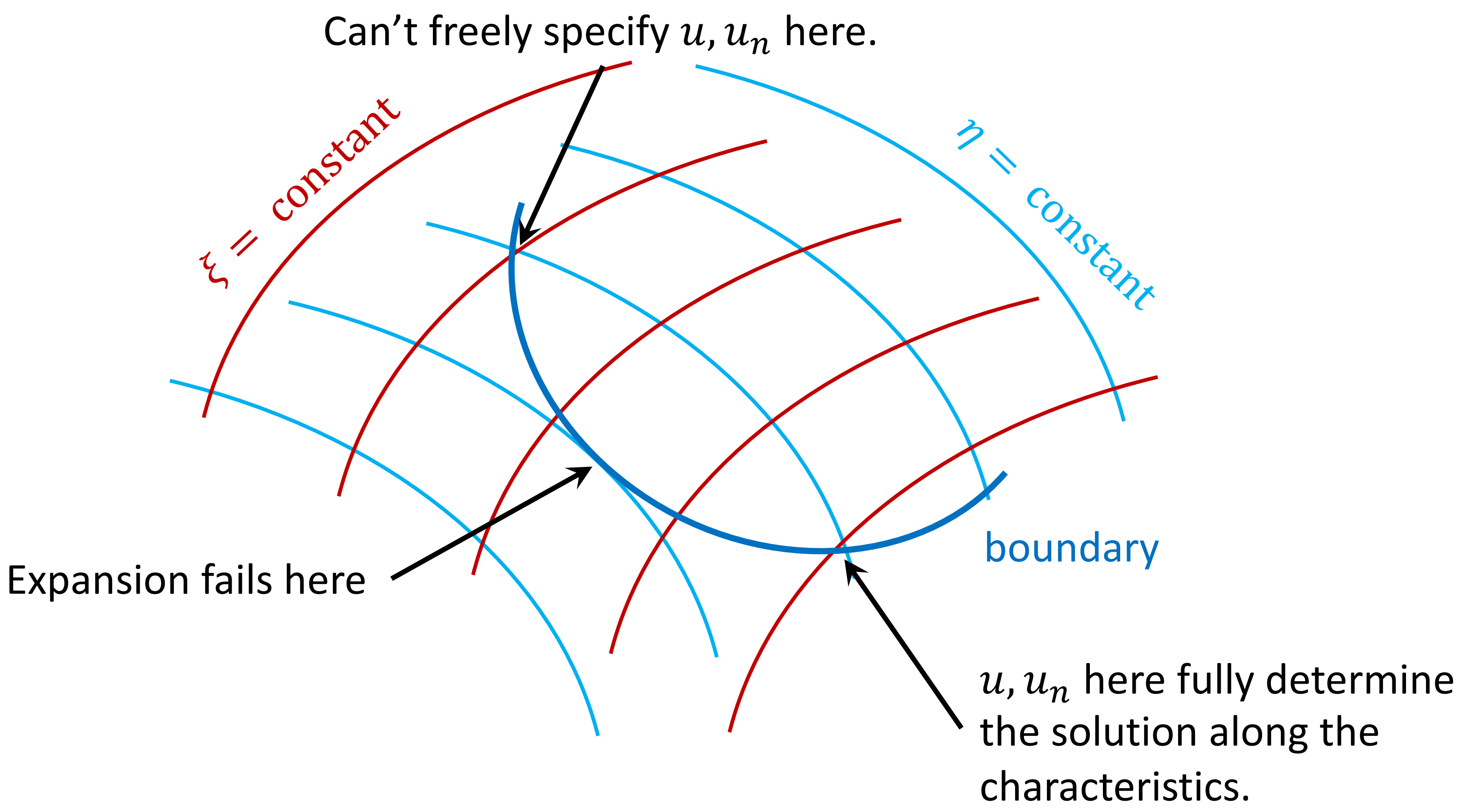
- <u>Domain of dependence</u> is defined by the characteristic structure.
- Explicit solution exists only in this <u>finite</u> domain.
- Partly remedy by providing more boundary conditions.



Finite Domain

- Other boundary conditions:
  - consider vibrating string: initial conditions determine initial motion, but extra (non-Cauchy) boundary conditions are essential (e.g. $u = 0$ at ends)
- Now <u>every</u> characteristic ends on a well-defined boundary, but we don't have Cauchy boundary conditions everywhere and the previous derivation may fail to uniquely determine f and g.





Can't freely specify u, u_n here.
$\xi =$ constant
$\eta =$ constant
boundary
Expansion fails here
u, u_n here fully determine the solution along the characteristics.

Finite Domain, Worked Example

- Guitar string $u(x,t)$

$$u_{xx} - u_{tt} = 0 \quad (c = 1)$$

$$u(x,0) = \sin \pi x$$

$$u_t(x,0) = 0$$

$$u(\pm 1, t) = 0$$

- The solution inside the grey triangular region is

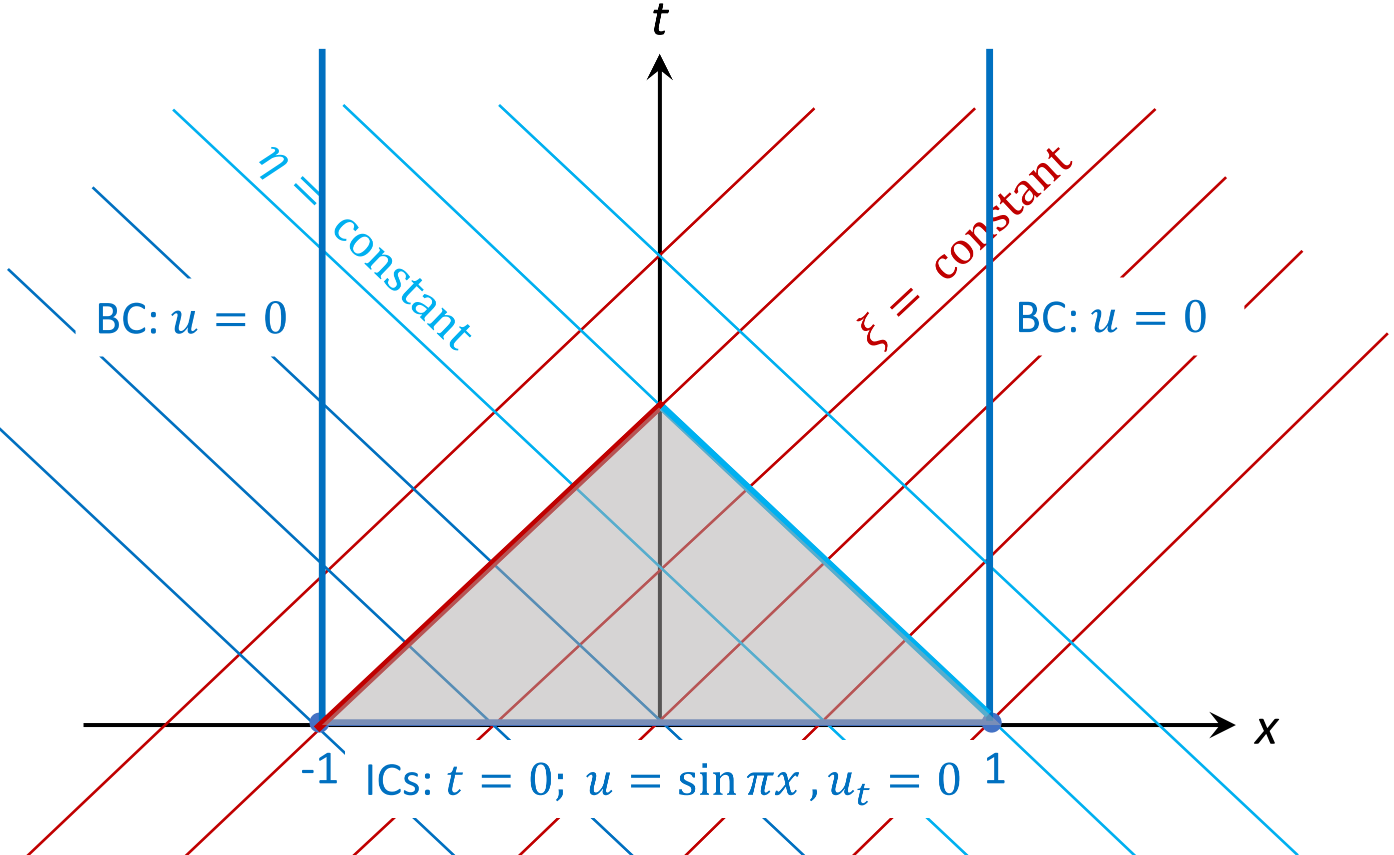
$$u = f(x-t) + g(x+t)$$

where

$$f(x) = g(x) = \frac{1}{2}\sin \pi x$$

$$f(x) = \frac{1}{2}u(x,0) - \frac{1}{2c}\int dx\, u_t(x,0)$$

$$g(x) = \frac{1}{2}u(x,0) + \frac{1}{2c}\int dx\, u_t(x,0)$$



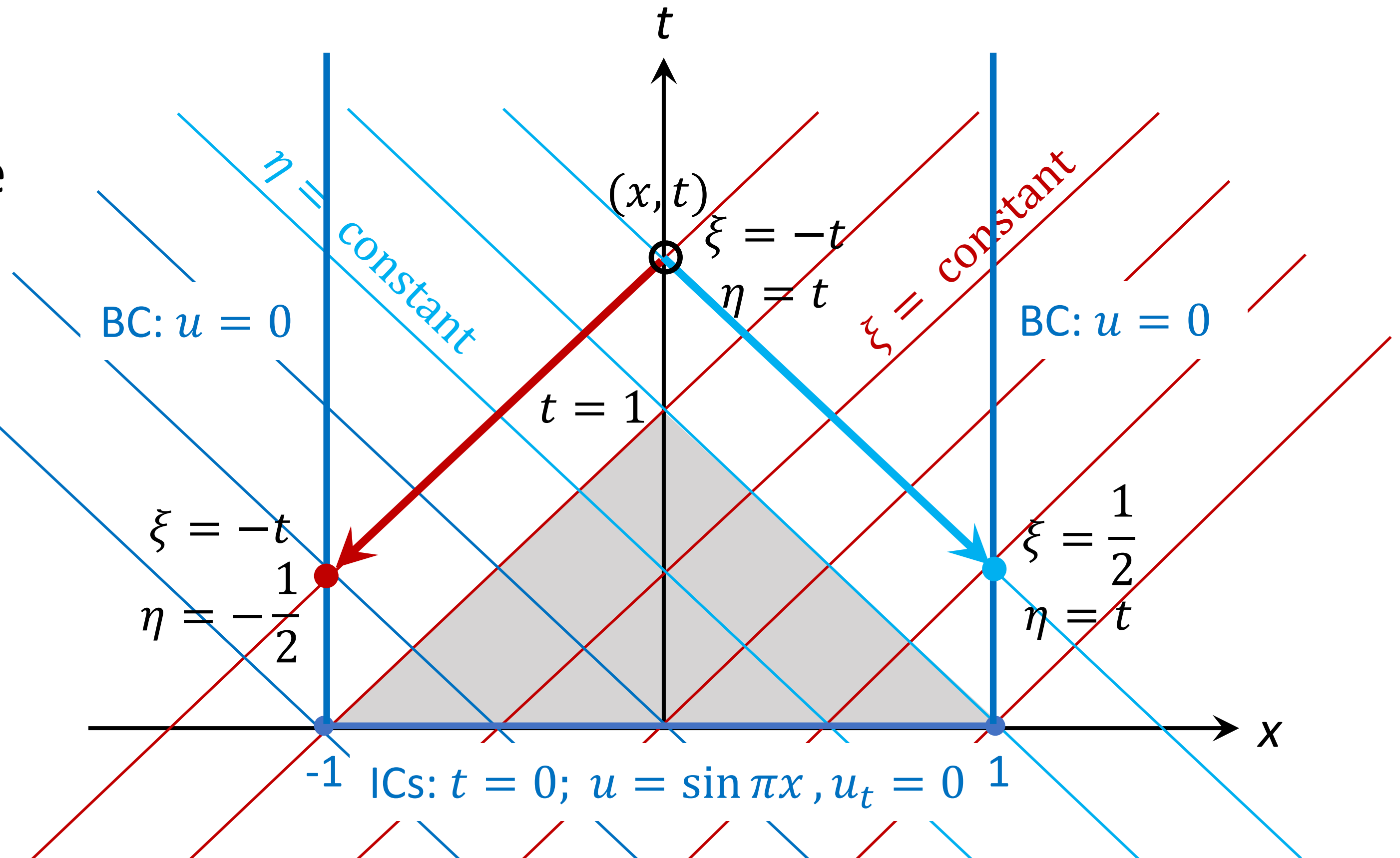
Finite Domain, Worked Example

- Thus $u(x,t) = \frac{1}{2}[\sin \pi(x-t) + \sin \pi(x+t)]$
  $= \sin \pi x \cos \pi t$

$$f(-t) + g\left(-\frac{1}{2}\right) = 0$$

$$f\left(\frac{1}{2}\right) + g(t) = 0$$

- Outside the triangle, the same solution still satisfies the BCs and therefore is the solution to this problem.
- More generally, may not be the case
  - inhomogeneous PDE
  - more complex BCs
  - separable solution
  - appeal to continuity



Elliptic Equations

- No real characteristics, but can invoke complex solutions.
- e.g. for Laplace, $u_{xx} + u_{yy} = 0$, seek solution of the form

$$u(x,y) = f(p), \text{ where } p = x + \lambda y$$

$$\Longrightarrow \quad (1+\lambda^2) f''(p) = 0, \text{ so } \lambda = \pm i$$

$$\Longrightarrow \quad \text{general solution is } u(x,y) = f(x+iy) + g(x-iy)$$

- Still have the problem of fitting f and g to the BCs.
- Can make significant progress in 2-D by exploiting the close connection between harmonic functions ($\nabla^2 u = 0$) in $\mathbb{R}^2$and analytic functions $f(z)$ in the complex plane—can prove existence and uniqueness of solutions and demonstrate that Dirichlet/Neumann BCs on a <u>closed</u> boundary are needed.
- Will take us too far afield — return to this later.

Parabolic Equations

- One real characteristic, but just the x-axis for the diffusion equation, $u_{xx} = -\frac{1}{\kappa} u_t$.
- Separation of variables and transforms (to come) are standard methods of solution.
- Generally expect <u>open</u> boundaries (like wave equation) and Dirichlet/Neumann BCs (or mixed: specify $\alpha u + \beta u_n$).

Boundary Conditions and Domains

- Hyperbolic, elliptic, and parabolic equations significantly different mathematical properties, and generally require different combinations of boundary conditions on geometrically different (open/closed) boundaries.
- Rule of thumb:

Type	Typical Variables	Boundary Conditions	Domain
Hyperbolic	Space+time	Cauchy/mixed	Open
Elliptic	Space	Dirichlet/Neumann	Closed
Parabolic	Space+time	Dirichlet/Neumann	Open

Uniqueness

- Define linear differential operator $\mathcal{L}$

  e.g. $\mathcal{L}u \equiv \nabla^2 u - \frac{1}{c^2} u_{tt}$ for the wave equation

- Differential equation

  $$\mathcal{L}u = f(\boldsymbol{x})$$

- Boundary conditions

  $$u|_{\mathrm{bdy}} = g,\ u_n|_{\mathrm{bdy}} = h$$

$$f(x) = \frac{1}{2}u(x,0) - \frac{1}{2c}\int dx\, u_t(x,0)$$

$$\mathrm{g}(x) = \frac{1}{2}u(x,0) + \frac{1}{2c}\int dx\, u_t(x,0)$$

- Suppose there are <u>two</u> solutions, u_1 and u_2 — both satisfy PDE and BCs
- Consider $\delta u = u_1 - u_2$. Then

  $$\mathcal{L}\delta u = 0,\ \delta u|_{\mathrm{bdy}} = 0,\ \delta u_n|_{\mathrm{bdy}} = 0$$

- Then method of characteristics for hyperbolic and other means (e.g. complex) for others implies that $\delta u = 0$ everywhere $\Longrightarrow$ <u>unique</u>.

Transforms

- Getting a little ahead of ourselves, but…
- Suppose we take the diffusion equation $u_{xx} - \frac{1}{\kappa} u_t = 0$ and imagine we are looking on an infinite domain and we seek solutions of the form
$$u(x,t) = U_k(t)\, e^{ikx}$$
(really talking about part of a Fourier transform, where we can write $u(x,t) = \int dk\, U(k,t)\, e^{ikx}$, but more on this later)
- Then, substituting in, we find
$$-k^2 U_k = \frac{1}{\kappa}\frac{dU_k}{dt}$$
so $U_k(t) = U_k(0)\, e^{-\kappa k^2 t}$
- Transforms turn PDEs into ODEs, making them easier to solve, but we need to know how to transform back to the original space.

Separation of Variables

- Bread and butter method for solving PDEs.
- Uniqueness means, if it works, we're done!
- Look at the wave equation $\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$, where u is $u(x, y, z, t)$.
- Seek a <u>separable solution</u> of the form

$$u(\boldsymbol{x}, t) = \chi(\boldsymbol{x})T(t)$$

(already seen such a solution, so not unreasonable).

- Substitute in to find

$$\nabla^2 \chi T - \frac{1}{c^2}\chi\frac{d^2T}{dt^2} = 0.$$

- Divide across by $u = \chi T$:

$$\frac{\nabla^2\chi}{\chi} = \frac{1}{c^2}\frac{1}{T}\frac{d^2T}{dt^2}$$

Separation of Variables

$$\frac{\nabla^2 \chi}{\chi} = \frac{1}{c^2}\frac{1}{T}\frac{d^2 T}{dt^2} = \text{constant}, -k^2$$

<u>separation constant</u>
form is conventional

function of $\boldsymbol{x}$ only   function of t only

- Effectively splits the PDE into an ODE and a lower-dimensional PDE.
- Time dependence

  $T'' + k^2 c^2 T = 0$, and define $\omega = kc$
- Solutions $T = e^{\pm i\omega t}$
- Spatial dependence

  $\nabla^2 \chi + k^2 \chi = 0$ <u>Helmholtz equation</u>
- Wave solution: expect $k^2 > 0$

Roads to Helmholtz

- Other standard equations also get us to the same place.
- Laplace equation is already Helmholtz with $k = 0$.
- Diffusion equation

$$\nabla^2 u - \frac{1}{\kappa}\frac{\partial u}{\partial t} = 0$$

- Seek $u(\boldsymbol{x}, t) = \chi(\boldsymbol{x})T(t)$ again

$$\Longrightarrow \quad \nabla^2\chi\, T - \frac{1}{\kappa}\chi\, T' = 0$$

$$\Longrightarrow \quad \frac{\nabla^2\chi}{\chi} = \frac{1}{\kappa}\,\frac{T'}{T} \quad = \text{constant}, -l^2$$

$$\Longrightarrow \quad T' = -l^2\kappa\, T$$

$$\Longrightarrow \quad T = T_0 e^{-l^2\kappa t} \quad \text{— normal diffusion, expect } \ l^2 > 0$$

- Spatial part

$$\nabla^2\chi + l^2\chi = 0$$

- Helmholtz again!

Roads to Helmholtz

- Schrödinger equation leads us here too.
- Time-independent Schrödinger equation (assume $e^{-iEt/\hbar}$ time dependence)

$$-\frac{\hbar^2}{2m}\nabla^2\psi + \mathrm{V}\psi = E\psi$$

$$\Longrightarrow \quad \nabla^2\psi - \frac{2mV}{\hbar^2}\psi = -\frac{2mE}{\hbar^2}\psi$$

- Particle in a box, $V = 0$ inside, $V \longrightarrow \infty$ outside

$$\Longrightarrow \quad \nabla^2\psi + \frac{2mE}{\hbar^2}\psi = 0$$

- Helmholtz again, $k^2 = 2mE/\hbar$, again > 0