Recap 1: Inhomogeneous Linear Equations

- Started to consider inhomogeneous equations (i.e. with some non-zero function on the right-hand side)

e.g. Poisson, $\phi(\boldsymbol{x})$

$$\nabla^2\phi = 4\pi G\rho(\boldsymbol{x})$$

Helmholtz, $u(\boldsymbol{x})$

$$\nabla^2 u + k^2 u = f(\boldsymbol{x})$$

1-D linear operator, $y(x)$

$$\mathcal{L}y = f(x)$$

Recap 2: Green's Functions

- For Poisson, have the principle of superposition:

$$\phi(\boldsymbol{x}) = -G \iiint d^3x' \, \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}$$

- From Fourier transform solution, saw

$$y(x) = \int_{-\infty}^{\infty} dx' \, f(x') \, g(x - x')$$

- We can generalize these results:

$$y(x) = \int_{-\infty}^{\infty} dx' \, f(x') \, G(x, x')$$

where $G(x, x')$ is the Green's function for the problem: how the source at x' influences the solution at x.

- Same basic ideas in 1, 2, or 3 dimensions.

Recap 2: Green's Functions

- Saw that G is the solution to the "point-source" problem

$$\mathcal{L}_x G(x, x') = \delta(x - x')$$

  – property of $\mathcal{L}$ and the boundary conditions, <u>not</u> of f

- How to determine?
  1. Solve the ODE directly, with boundary conditions
  2. Eigenfunction expansion gives formal expression for G
  3. Fundamental solutions
     - applications in potential theory, Helmholtz
     - method of images

Green's Functions: Example 1

$$G(x,x') = \begin{cases} A\sin kx\,, & x < x' \\ B\sin k(x-L)\,, & x > x' \end{cases}$$

- Stretched string.
- Continuity in G at $x = x'$

$$\Longrightarrow \quad A\sin kx' = B\sin k(x'-L)$$

- Jump in G': $[G']_{x'_-}^{x'_+} = 1$

$$\Longrightarrow \quad Ak\cos kx' + 1 = Bk\cos k(x'-L)$$

- Can easily solve to find

$$A = \frac{\sin k(x'-L)}{k\sin kL}$$

$$B = \frac{\sin kx'}{k\sin kL}$$

$$\Longrightarrow \quad G(x,x') = \frac{1}{k\sin kL}\begin{cases} \sin kx\sin k(x'-L), & 0 < x < x' \\ \sin kx'\sin k(x-L)\,, & x' < x < L \end{cases}$$

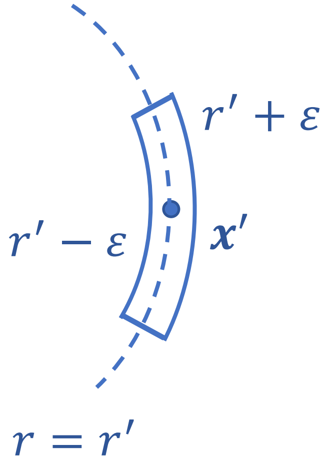
Green's Functions: Example 3

- Can do something similar in 2 (or 3)-D, e.g. membrane:

$$G = \begin{cases} \sum_m A_m J_m(kr)\cos m\theta\,, & r < r' \\ \sum_m B_m[(J_m(kr)Y_m(ka) - Y_m(kr)J_m(ka)]\cos m\theta\,, & r > r' \end{cases}$$

- Continuity at $r = r'$ applies term by term, so

$$A_m J_m(kr') = B_m[(J_m(kr')Y_m(ka) - Y_m(kr')J_M(ka)]$$



- Discontinuity in G' at $\boldsymbol{x} = \boldsymbol{x}'$:

$$\left[\frac{\partial G}{\partial r}\right]_{r'_-}^{r'_+} = \frac{1}{r'}\delta(\theta) = \frac{1}{\pi r'}\sum_m \frac{\cos m\theta}{\beta_m}$$

where $\beta_0 = 2, \beta_m = 1, m > 0$

$$\Longrightarrow \qquad A_m J'_m(kr') + \frac{1}{\pi\beta_m kr'} = B_m[(J'_m(kr')Y_m(ka) - Y'_m(kr')J_m(ka)]$$

Eigenfunction Expansion

- Formal expansion of Green's function in eigenfunctions

  eigenfunction eq.: $\mathcal{L}u_n + \omega\lambda_n u_n = 0$

  inhomogeneous eq.: $\mathcal{L}u = f$

- Expand

$$u = \sum_n a_n u_n$$

$$f = \omega \sum_n c_n u_n \,, \quad \text{where} \quad c_n = \int \omega (f/\omega) u_n^*$$

$$\Longrightarrow \quad \mathcal{L}u = \sum_n a_n \mathcal{L}u_n = \sum_n a_n(-\omega\lambda_n u_n)$$

$$= \omega \sum_n c_n u_n$$

$$\Longrightarrow \quad c_n = -\lambda_n a_n$$

$$\Longrightarrow \quad a_n = -c_n/\lambda_n$$

Eigenfunction Expansion

- Now set $f(x) = \delta(x - x')$, so $u(x) \rightarrow G(x, x')$

$$\Longrightarrow \quad c_n = \int \delta(x - x')\, u_n^*(x)\, dx = u_n^*(x')$$

$$\Longrightarrow \quad G(x, x') = \sum_n -\frac{u_n(x) u_n^*(x')}{\lambda_n}$$

- Notes:
  1. Reminiscent of the formal expansion of $\delta(x)$ in terms of the $u_n(x)$
  2. $G(x, x') = G(x', x)^*$ again

Eigenfunction Expansion

- Write

$$G(x,x') = \sum_n -\frac{u_n(x)u_n^*(x')}{\lambda_n}$$

$$u(x) = \int_{-\infty}^{\infty} dx'\, f(x')\, G(x,x')$$

$$\Longrightarrow \qquad u(x) = \sum_n \left[-\frac{u_n(x)u_n^*(x')}{\lambda_n}\right] f(x')$$

$$= \sum_n -\frac{u_n(x)}{\lambda_n}\int_{-\infty}^{\infty} dx'\, f(x')\, u_n^*(x')$$

$$= \sum_n -\frac{u_n(x)\,(u_n, f)}{\lambda_n}$$

$\Longrightarrow$ eigenfunction expansion of the Green's function

Fundamental Solutions

- More practical way of determining and using a Green's function.
- Idea: the problem of solving $\mathcal{L}G = \delta(x - x')$ has two distinct parts
  1. the form of the singularity at $x = x'$
  2. the boundary conditions
- <u>Split</u> the problem by writing $G(x, x') = u(x, x') + v(x, x')$, where
  1. u is singular and satisfies the differential equation (with δ), but not necessarily the boundary conditions
  2. v is regular (solution to $\mathcal{L}G = 0$) and takes care of the boundary conditions (including any changes due to u)
- Why do this? Generally easy to find a suitable u, and the problem of finding v is always (in principle) solvable.

Fundamental Solutions

- Start by looking at some fundamental solutions, return later to the problem of dealing with the boundary conditions.
- Since the job of u is to handle the singular behavior at x', we may as well keep it as simple as possible.
- Four familiar cases

  1. Laplace, 2/3-D: $\nabla^2 y = 0$

     $\nabla_x^2 u = \delta(x - x')$

  2. Helmholtz, 2/3-D: $\nabla^2 y + k^2 y = 0$

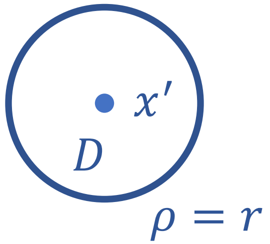
     $\nabla_x^2 u + k^2 u = \delta(x - x')$

Poisson, 2-D

- Fundamental solution satisfies

$$\nabla_x^2 u = \delta(x - x')$$

- Write $\rho = |x - x'|$, assume u is a function of ρ only, and consider a small disk D of radius r around $x = x'$:



- Integrate the equation over this disk:

$$\int_D \nabla_x^2 u \; d^2x = \int_D \delta(x - x') \; d^2x = 1$$

Note that $d^2x = \rho \, d\rho \, d\theta$, and apply Gauss's theorem in 2-D:

$$\Longrightarrow \qquad \int_D \nabla_x^2 u \; d^2x = \int_0^{2\pi} r \, d\theta \left.\frac{\partial u}{\partial \rho}\right|_{\rho = r} = 2\pi r \frac{\partial u}{\partial r}$$

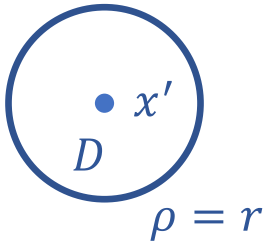
$$\Longrightarrow \qquad 2\pi r \frac{\partial u}{\partial r} = 1, \qquad \Longrightarrow \frac{\partial u}{\partial r} = \frac{1}{2\pi r}, \qquad \Longrightarrow u(x, x') = \frac{1}{2\pi} \log|x - x'|$$

Helmholtz, 2-D

- Fundamental solution now satisfies

$$\nabla_x^2 u + k^2 u = \delta(x - x')$$

- Again, write $\rho = |\mathrm{x} - \mathrm{x}'|$, assume $u(\rho)$, use the same disk D, and integrate:



$$\int_D \nabla_x^2 u \; d^2x + \int_D k^2 u \; d^2x = \int_D \delta(x - x') \;\; d^2x = 1$$

Now argue $\int_D k^2 u \; d^2x \to 0$ as $r \to 0$, so again

$$\Longrightarrow \qquad 2\pi r \frac{\partial u}{\partial r} = 1 \Longrightarrow u \sim \frac{1}{2\pi} \log r \;\; (\text{as } r \to 0)$$

- Problem: this is <u>not</u> a solution to the Helmholtz equation. $J_l(kr), Y_l(kr)$

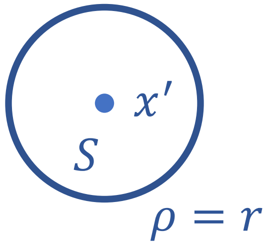
- Saw earlier that $Y_0(kr) \sim \frac{2}{\pi} \log r$ as $r \to 0$ $\qquad \Longrightarrow u(x, x') = \frac{1}{4} Y_0(k|x - x'|)$

Poisson, 3-D

- Fundamental solution satisfies

$$\nabla_x^2 u = \delta(x - x')$$

- Write $\rho = |x - x'|$, assume u is a function of ρ only, and consider a small <u>sphere</u> S of radius r around $x = x'$:



- Integrate the equation over this disk:

$$\int_S \nabla_x^2 u \; d^3x = \int_S \delta(x - x') \; d^3x = 1$$

Note that $d^3x = \rho^2 \, d\rho \, d\theta d\varphi$, and apply Gauss's theorem:

$$\Longrightarrow \qquad \int_S \nabla_x^2 u \; d^3x = \int_{-\pi}^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi \, r^2 \left.\frac{\partial u}{\partial \rho}\right|_{\rho = r} = 4\pi r^2 \frac{\partial u}{\partial r}$$

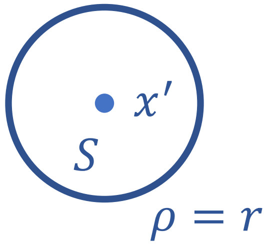
$$\Longrightarrow \qquad \frac{\partial u}{\partial r} = \frac{1}{4\pi r^2} \qquad\qquad \Longrightarrow u(x, x') = \frac{-1}{4\pi|x - x'|}$$

Helmholtz, 3-D

- Fundamental solution satisfies

$$\nabla_x^2 u + k^2 u = \delta(x - x')$$

- Same considerations as before (with $\int_S k^2 u \; d^3x \to 0$ as $r \to 0$)



$$\Longrightarrow \qquad u \sim \frac{-1}{4\pi r} \quad \text{(as } r \to 0\text{)}$$

- Again, this is <u>not</u> a solution to the Helmholtz equation.
- Solutions singular as $r \to 0$ are $y_l(kr)$.
- Again, saw earlier that $y_0(x) = -\dfrac{\cos x}{x} \sim \dfrac{-1}{x}$ as $r \to 0$

$$\Longrightarrow \qquad u(x, x') = \frac{k}{4\pi} y_0(k|x - x'|)$$

$y_0(kr) \sim \dfrac{-1}{kr}$

$\dfrac{k}{4\pi} y_0(kr) \sim \dfrac{-1}{4\pi r}$

Regular solution

- In each case, the regular solution v satisfies the homogeneous equation.
- Boundary conditions now determined by the combination of the original BCs and those forced by the behavior of u.
- Method of solution is straightforward in principle, but not necessarily easy in practice.
- Example, reconsider the circular membrane problem for $y(r, \theta)$

$$\nabla^2 y + k^2 y = 0$$

$$y(a,\ \theta) = 0$$

- Writing $G(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{4} Y_0(k|\boldsymbol{x} - \boldsymbol{x}'|) + v(\boldsymbol{x}, \boldsymbol{x}')$ means that v now has to "undo" the fact that $u \neq 0$ when $r = a$.

Regular solution

- Now,

$$\nabla^2 v + k^2 v = 0$$

$$v(a,\ \theta) = 0$$

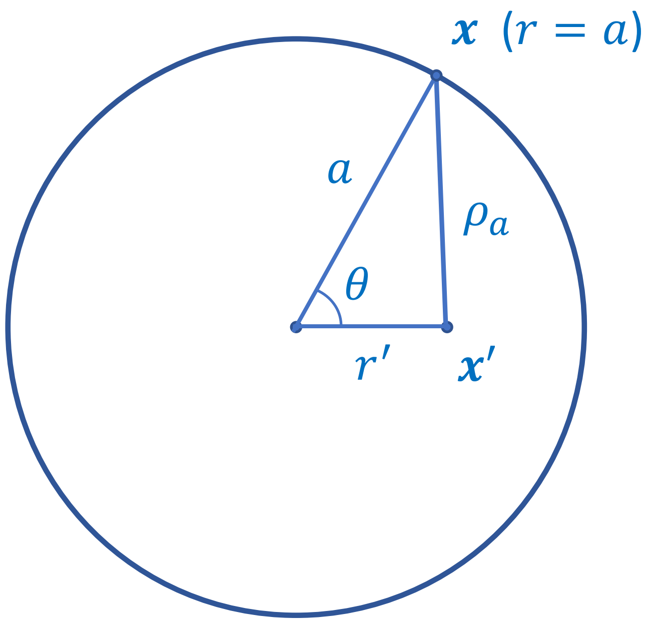
- Convenient to choose $\theta = 0$ in the direction of $\boldsymbol{x}'$
- Write general regular solution

$$v = \sum_{n=0}^{\infty} A_n J_n(kr) \cos n\theta$$

and expand the boundary conditions

$$0 = \frac{1}{4} Y_0(k\rho_a) + \sum_{n=0}^{\infty} A_n J_n(ka) \cos n\theta$$

where $\quad \rho_a^2(\theta) = a^2 + {r'}^2 - 2ar' \cos\theta$



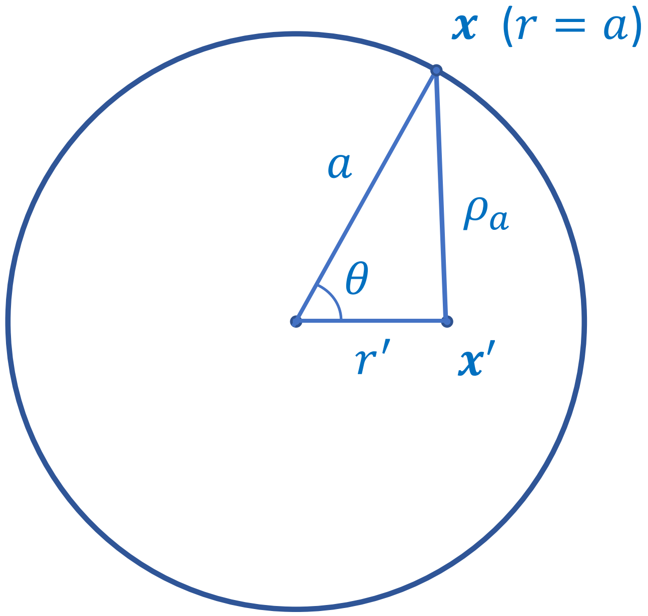
Regular solution

- Invert the Fourier series:

$$A_n J_n(ka) = -\frac{1}{\pi\beta_n}\int_0^{2\pi}\frac{1}{4}Y_0(k\rho_a)\cos n\theta'\,d\theta'$$

Same β_n as before:
$\beta_0 = 2,\ \beta_{n>0} = 1$

$\rho_a^2(\theta) = a^2 + r'^2 - 2ar'\cos\theta$



$$\Longrightarrow \qquad G(\boldsymbol{x},\boldsymbol{x}') = \frac{1}{4}Y_0(k|\boldsymbol{x}-\boldsymbol{x}'|)$$

$$-\sum_{n=0}^{\infty}\frac{J_n(kr)\cos n\theta}{4\pi J_n(ka)\beta_n}\int_0^{2\pi}Y_0(k\rho_a)\cos n\theta'\,d\theta'$$

Dealing with Boundary Conditions

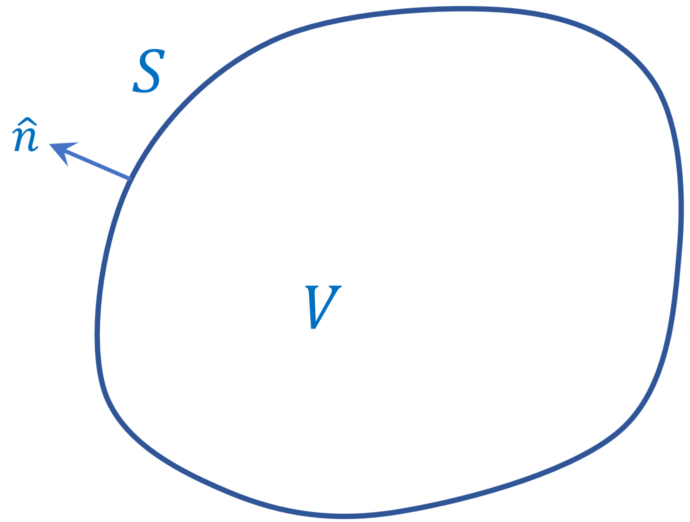
- Look again at Poisson's equation, for simplicity (could do a similar development for Helmholtz):

$$\nabla^2 \phi = \rho$$

in some volume V, with

$$\phi \ \text{ and/or } \frac{\partial \phi}{\partial n} \text{ specified on the boundary } S$$

so $$\nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{x}')$$



- Recall <u>Green's theorem</u>

$$\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, dV = \int_S (\phi \nabla \psi - \psi \nabla \phi) \,.\, d\boldsymbol{S}$$

- Now replace ψ by G, integrate over $\boldsymbol{x}'$, drop the boldface …

Dealing with Boundary Conditions

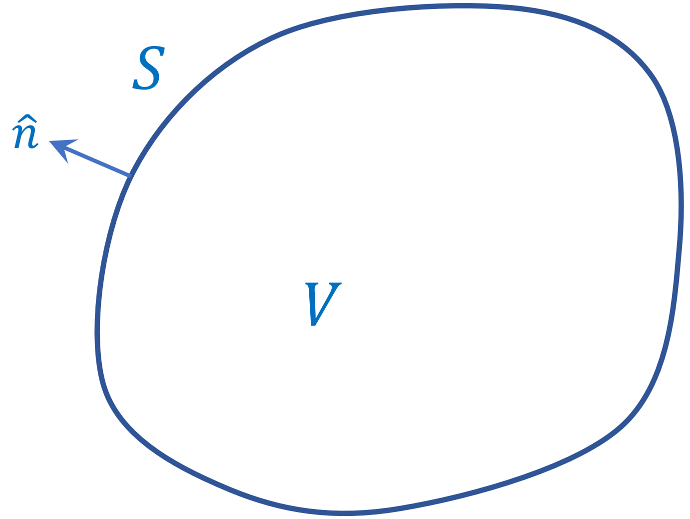
$$\Longrightarrow \qquad \int_V (\phi \nabla^2 G - G \nabla^2 \phi)\, dV' = \int_S \left(\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n} \right) dS'$$

$\delta(x - x')$ $\rho(x')$ boundary conditions

$$\Longrightarrow \qquad \phi(x) - \int_V G(x, x') \rho(x')\, dV'$$

$$= \int_S \left(\phi(x') \frac{\partial G}{\partial n} - G(x, x') \frac{\partial \phi}{\partial n} \right) dS'$$

DIrichlet Neumann



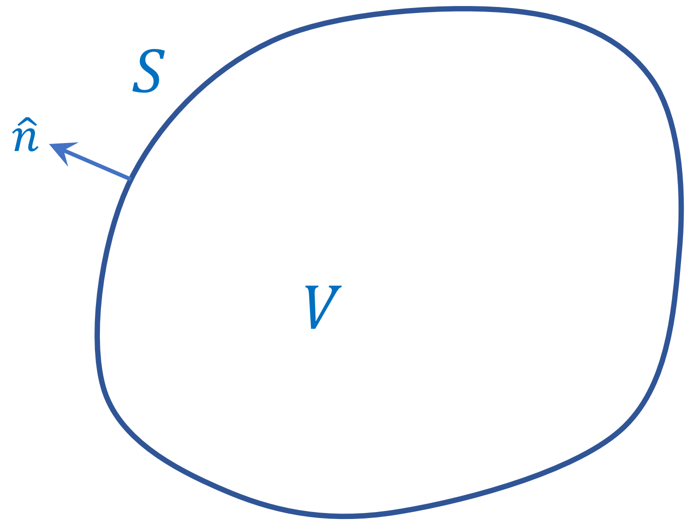
- <u>Generalizes</u> our earlier result.
- Allows us some freedom in choosing the BCs on G.
- e.g. for a Dirichlet problem, $\phi|_S$ is specified, $= f$, say

  $\left.\frac{\partial \phi}{\partial n}\right|_S$ is not specified, but can choose $G|_S = 0$

Dealing with Boundary Conditions

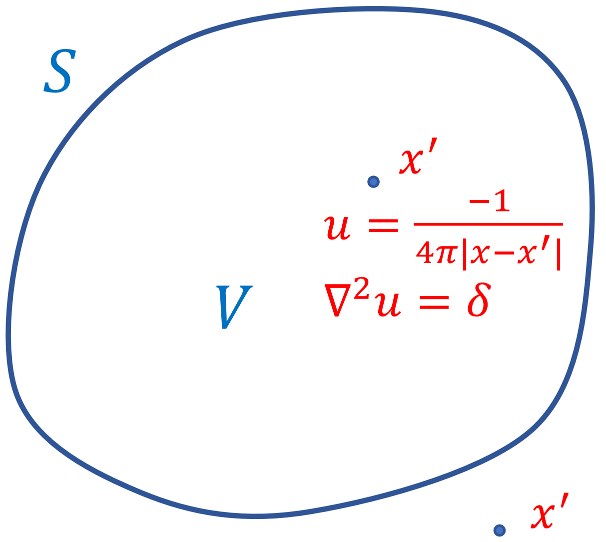
$$\Longrightarrow \qquad \phi(x) = \int_V G(x, x')\rho(x')\, dV' + \int_S f(x') \frac{\partial G}{\partial n'}\, dS'$$

- Now we have a definite boundary condition on G, tailored to the specifics of the problem.
- Still have to solve for G (may be hard).
- Can use fundamental solution and the <u>method of images</u> as a useful approach in situations with some symmetry.



Method of Images

- Sometimes we can enforce the boundary condition for G on S by adding extra charges <u>outside</u> the volume V.
- Fundamental solution u gets the singularity right.
- Any charge outside V will contribute $\nabla^2 v = 0$ inside V but its potential on S may allow us to satisfy the boundary condition $G|_S = 0$.
- Works well in simple geometries.



Method of Images

- Simple example (3-D):

  $$\nabla^2\phi = \rho$$

  domain of interest is the space $z > 0$

  $\phi(x, y, 0)$ specified on plane $z = 0$

  Dirichlet BCs on G: $G(\boldsymbol{x}, \boldsymbol{x}') = 0$ when $z' = 0$

- Solution is

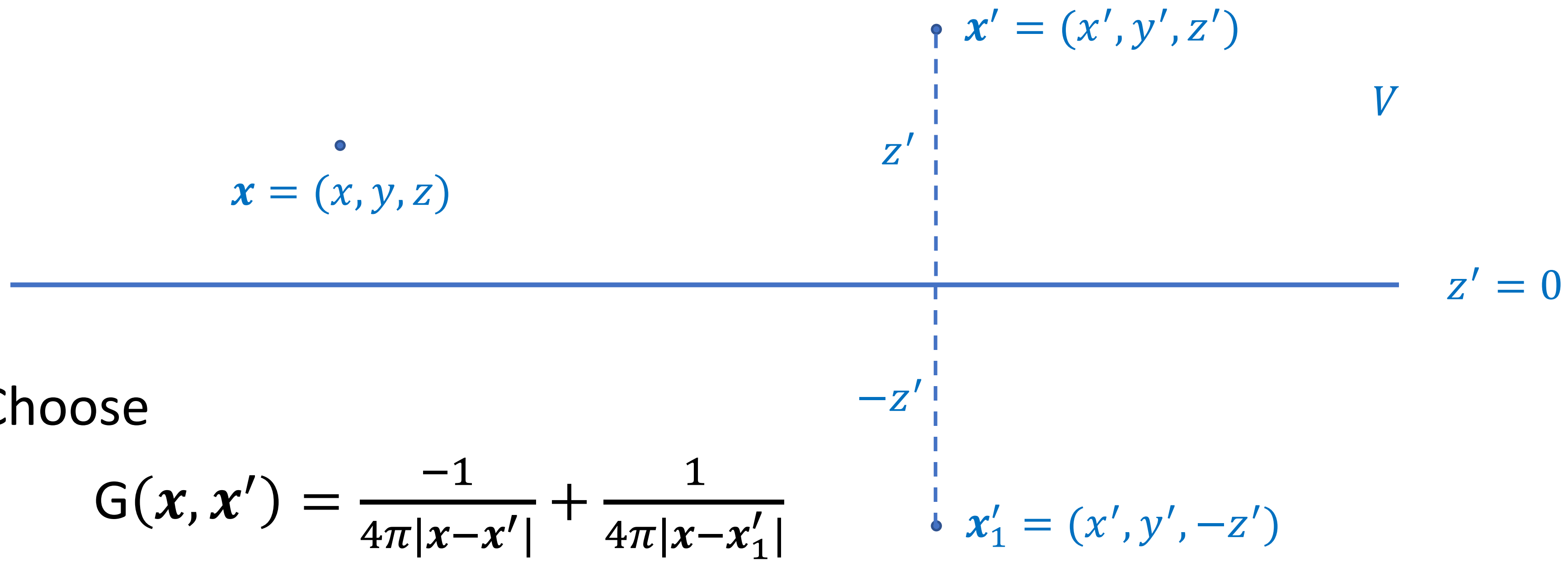
  $$\phi(\boldsymbol{x}) = \int_V G(\boldsymbol{x}, \boldsymbol{x}')\rho(\boldsymbol{x}')\, dV' + \int_S f(\boldsymbol{x}')\frac{\partial G}{\partial n'}\, dS'$$

- Fundamental solution [charge element at $\boldsymbol{x}' = (x', y', z')$] is

  $$u(\boldsymbol{x}, \boldsymbol{x}') = \frac{-1}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}$$

- Choose v to obey boundary conditions by placing an <u>image</u> charge at $\boldsymbol{x}_1' = (x', y', -z')$

Method of Images



- Choose

$$\mathrm{G}(\boldsymbol{x}, \boldsymbol{x}') = \frac{-1}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|} + \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{x}_1'|}$$

- Clearly $\nabla_x^2 G = \delta(\boldsymbol{x} - \boldsymbol{x}')$

  and $G = 0$ when $z' = 0$

$$\Longrightarrow \quad \mathrm{G}(\boldsymbol{x}, \boldsymbol{x}') = \frac{-1}{4\pi}\Big\{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{-1/2} - [(x-x')^2 + (y-y')^2 + (z+z')^2]^{-1/2}\Big\}$$

Method of Images

- Complete the calculation:

$$\left.\frac{\partial G}{\partial n'}\right|_S = \left.\frac{\partial G}{\partial z'}\right|_{z'=0}$$

$$\Longrightarrow \quad \left.\frac{\partial G}{\partial z'}\right|_{z'=0} = \frac{-1}{4\pi}\left\{-\frac{1}{2}[\ldots]^{-\frac{3}{2}}[-2(z-z')] \;+\frac{1}{2}[\ldots]^{-\frac{3}{2}}[2(z+z')]\right\}$$

$$= \frac{-1}{4\pi}\, 2z[(x-x')^2 + (y-y')^2 + z^2]^{-3/2}$$

$$\Longrightarrow \quad \phi(\boldsymbol{x}) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' \;\rho(\boldsymbol{x}')$$

$$\times\left(\frac{-1}{4\pi}\right)\Big\{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{-1/2}$$

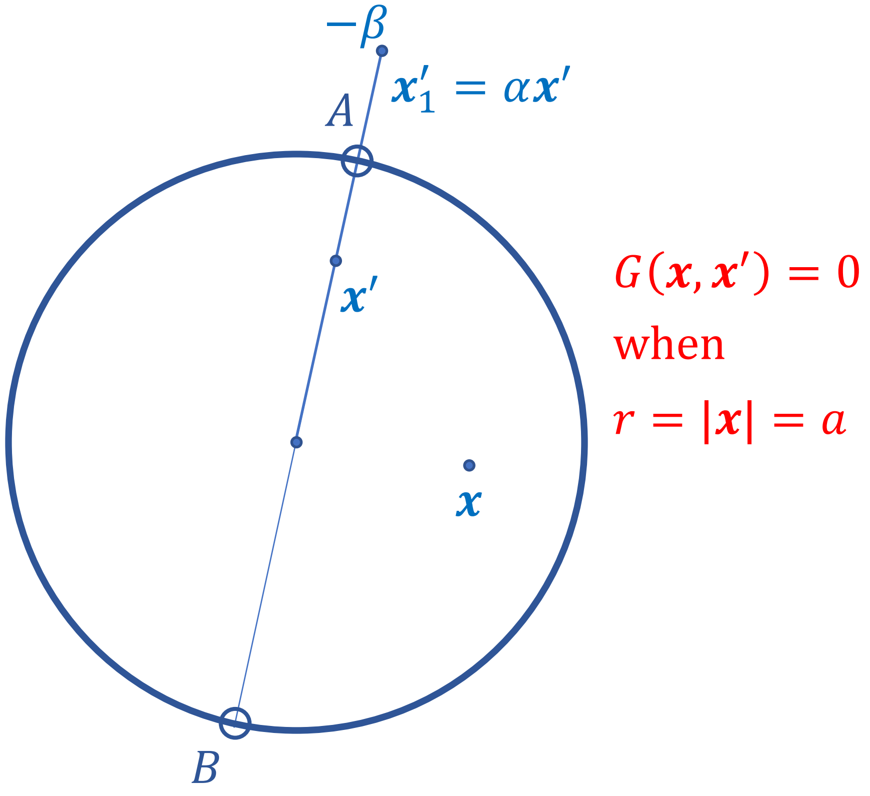
$$- [(x-x')^2 + (y-y')^2 + (z+z')^2]^{-1/2}\Big\}$$

$$+ \int_{-\infty}^{\infty} dx' \;\int_{-\infty}^{\infty} dy'\, f(x', y')$$

$$\times\left(\frac{-z}{2\pi}\right)\Big\{[(x-x')^2 + (y-y')^2 + z^2]^{-1/2}$$

Method of Images for a Sphere

- Poisson's equation in a sphere, radius a.
- For the Green's function, can show that we can satisfy the boundary condition $G = 0$ on $r = a$ with charge 1 at r' by creating an image charge $-\beta$ at $r_1' = \alpha r'$, as indicated.
- Proper choice of α and β will lead to $G = 0$ for $\boldsymbol{x}$ on the spherical boundary.
- Homework 6, problem 4 asks you you prove this at points A and B, where the line through the origin and $\boldsymbol{x}'$ intersects the sphere.



$$G(\boldsymbol{x},\boldsymbol{x}') = \frac{-1}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|} + \frac{\beta}{4\pi|\boldsymbol{x}-\boldsymbol{x}_1'|}$$

$G(\boldsymbol{x},\boldsymbol{x}') = 0$ when $r' = |\boldsymbol{x}'| = a$

$$G(\boldsymbol{x},\boldsymbol{x}') = G(\boldsymbol{x}',\boldsymbol{x})$$

Wave Equation, Again

- Wave equation on an infinite 3+1-dimensional domain:

$$\nabla^2 \phi(\boldsymbol{x}, t) - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0$$

- Green's function G satisfies

$$\nabla^2 G(\boldsymbol{x}, t) - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = \delta(\boldsymbol{x} - \boldsymbol{x}')\delta(t - t')$$

- Assert "translational invariance," so solution depends only on $\boldsymbol{x} - \boldsymbol{x}'$ and $t - t'$, so set $\boldsymbol{x}' = \boldsymbol{0}$, $t' = 0$.

- Define 4-D Fourier transforms

$$G(\boldsymbol{x}, t) = \frac{1}{(2\pi)^4}\int d^3k \int d\omega\ \tilde{G}(\boldsymbol{k}, \omega) e^{i(\boldsymbol{k}.\boldsymbol{x} - \omega t)}$$

$$\tilde{G}(\boldsymbol{k}, \omega) = \int d^3x \int dt\ G(\boldsymbol{x}, t) e^{-i(\boldsymbol{k}.\boldsymbol{x} - \omega t)}$$

Wave Equation, Again

- In transform space

$$\left(-k^2 + \frac{\omega^2}{c^2}\right)\tilde{G} = 1$$

$$\Longrightarrow \quad \tilde{G}(\boldsymbol{k},\omega) = \frac{c^2}{\omega^2 - k^2c^2}$$

$$\Longrightarrow \quad G(\boldsymbol{x},t) = \int \frac{d^3k}{(2\pi)^3}\int \frac{d\omega}{2\pi}\, c^2 \frac{e^{i(\boldsymbol{k}.\boldsymbol{x}-\omega t)}}{\omega^2 - k^2c^2}$$

- As before, choose the z-axis in k-space to be parallel to $\boldsymbol{x}$, so we can do the angular integrals in $\boldsymbol{k}$:

$$G(\boldsymbol{x},t) = \frac{1}{(2\pi)^3}\frac{c^2}{ir}\int_{-\infty}^{\infty} k\, dk \int_{-\infty}^{\infty} d\omega\, \frac{e^{i(kr-\omega t)}}{\omega^2 - k^2c^2}$$

comes from the e^{-ikr} part
of the angular solution

Wave Equation, Again

- The ω integral in

$$G(\boldsymbol{x}, t) = \frac{1}{(2\pi)^3} \frac{c^2}{ir} \int_{-\infty}^{\infty} e^{ikr} k \, dk \int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega t}}{\omega^2 - k^2 c^2}$$

  as formulated has poles on the axis of integration, at $\omega = \pm kc$.

- For a δ-function disturbance at $\boldsymbol{x} = 0, t = 0$, expect a causal solution to have zero response for $t < 0$.
- For $t < 0$, must close the contour with a large semicircle in the <u>upper</u> half plane.
- Impose causality by shifting the poles to $\omega = \pm kc - i\gamma$ (i.e. <u>below</u> the real axis), so $G(\boldsymbol{x}, t) = 0$ for $t < 0$.

Wave Equation, Again

- For $t > 0$,

$$G(\boldsymbol{x}, t) = \frac{1}{(2\pi)^3}\frac{c^2}{ir}\int_{-\infty}^{\infty} e^{ikr} k\, dk \int_{-\infty}^{\infty} d\omega\, \frac{e^{-i\omega t}}{\omega^2 - k^2c^2}$$

$$= \frac{1}{(2\pi)^3}\frac{c}{2ir}\int_{-\infty}^{\infty} e^{ikr}\, dk \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\left(\frac{1}{\omega - kc} - \frac{1}{\omega + kc}\right)$$

$$= \frac{1}{(2\pi)^3}\frac{c}{2ir}\int_{-\infty}^{\infty} e^{ikr}\, dk\;\; 2\pi i\left(e^{-ikct} - e^{ikct}\right)$$

$$= \frac{c}{8\pi^2 r}\int_{-\infty}^{\infty} e^{ikr}\, dk\;\left(e^{-ikct} - e^{ikct}\right)$$

$$= \frac{c}{4\pi r}[\delta(r - ct) - \cancel{\delta(r + ct)}]$$

$$= \frac{c}{4\pi r}\delta(r - ct)$$

Wave Equation, Again

- Putting it back together,

$$G(\boldsymbol{x}-\boldsymbol{x}', t-t') = \begin{cases} 0, & t < t' \\ \frac{c}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}\,\delta[|\boldsymbol{x}-\boldsymbol{x}'| - c(t-t')], & t > t' \end{cases}$$

"retarded potential"
light travel time = distance/c

- e.g. $\nabla^2\phi(\boldsymbol{x},t) - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = f(\boldsymbol{x},t)$

$$\Longrightarrow \quad \phi(\boldsymbol{x},t) = \frac{-c}{4\pi}\int d^3x'dt'\, f(\boldsymbol{x}',t')\,\frac{\delta[|\boldsymbol{x}-\boldsymbol{x}'|-c(t-t')]}{|\boldsymbol{x}-\boldsymbol{x}'|}$$

$$= \frac{-1}{4\pi}\int d^3x'\,\frac{f\left(\boldsymbol{x}', t-\frac{1}{c}|\boldsymbol{x}-\boldsymbol{x}'|\right)}{|\boldsymbol{x}-\boldsymbol{x}'|}$$

Wave Equation, Again

- Suppose $f(\boldsymbol{x}', t') = \delta[\boldsymbol{x}' - \boldsymbol{\xi}(t')]$ moving source
- Then

$$\phi(\boldsymbol{x}, t) = \frac{-1}{4\pi} \int d^3x' \int dt' \, \frac{\delta[\boldsymbol{x}' - \xi(t')]\,\delta\left[t - t' - \frac{1}{c}|\boldsymbol{x} - \boldsymbol{x}'|\right]}{|\boldsymbol{x} - \boldsymbol{x}'|}$$

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$$\Longrightarrow \quad \phi(\boldsymbol{x}, t) = \frac{-1}{4\pi} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'| + \frac{1}{c}\dot{\xi}(t') \cdot (\boldsymbol{x}' - \boldsymbol{x})}$$ Lienard-Wiechert potential

where $\quad t - t' = \frac{1}{c} |\boldsymbol{x} - \boldsymbol{x}'|$