Legendre Polynomials

- Rodrigues formula

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l$$
$$= \frac{(-1)^l}{2^l l!}\left(\frac{d}{dx}\right)^l (1-x^2)^l$$

- Normalization

$$\int_{-1}^{1} dx\, P_l^2(x)$$
$$= \frac{(-1)^{l+m}}{2^{l+m}\, l!\, m!}\int_{-1}^{1} dx \left(\frac{d}{dx}\right)^l (1-x^2)^l \left(\frac{d}{dx}\right)^l (1-x^2)^l$$
$$\vdots$$
$$= \frac{(-1)^l}{2^{2l}\,(l!)^2}\int_{-1}^{1} dx\, (1-x^2)^l \left(\frac{d}{dx}\right)^{2l} (1-x^2)^l \qquad = \frac{2}{2l+1}$$

$= (2l)!$

$$= \frac{2^{2l+1}(l!)^2}{(2l+1)!}$$

Legendre Generating Function

- Generating function for P_l is

$$F(h,z) = (1-2hz+h^2)^{-1/2} = \sum_{l=0}^{\infty} P_l(z)h^l$$

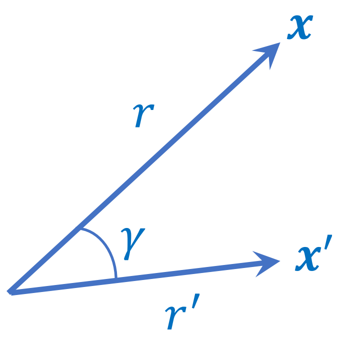
- This time, has a physical interpretation!
- Consider points $\boldsymbol{x}$ and $\boldsymbol{x}'$ in 3D space, with $r = |\boldsymbol{x}| > r' = |\boldsymbol{x}'|$

Then the Coulomb potential is

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}}$$

$$= \frac{1}{r}\left[1-2\frac{r'}{r}\cos\gamma+\left(\frac{r'}{r}\right)^2\right]^{-1/2}$$

$$= \sum_{l=0}^{\infty}\frac{r'^l}{r^{l+1}}P_l(\cos\gamma)$$



$$h = \frac{r'}{r}$$

$$z = \cos\gamma$$

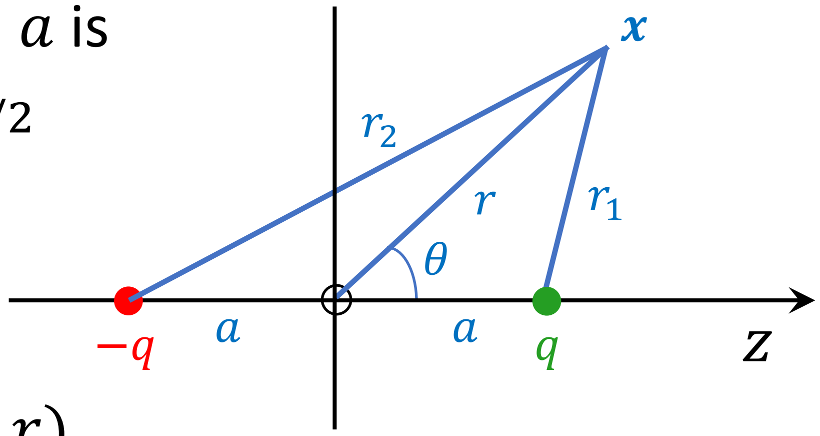
Example: Electrostatic Potential

- Potential at location $\boldsymbol{x}$ due to a charge q at $z = a$ is

$$\phi(\boldsymbol{x}) = \frac{kq}{r_1} = kq(r^2 + a^2 - 2ar\cos\theta)^{-1/2}$$

$$= kq \sum_{l=0}^{\infty} \frac{a^l}{r^{l+1}} P_l(\cos\theta)$$

$$= \frac{kq}{r} \sum_{l=0}^{\infty} \left(\frac{a}{r}\right)^l P_l(\cos\theta) \qquad (a < r)$$



- Now add a charge $-q$ at $z = -a$ (dipole), so

$$\phi(\boldsymbol{x}) = \frac{kq}{r_1} - \frac{kq}{r_2} = \frac{kq}{r}\left[\sum_{l=0}^{\infty} \left(\frac{a}{r}\right)^l P_l(\cos\theta) - \sum_{l=0}^{\infty} \left(\frac{-a}{r}\right)^l P_l(\cos\theta)\right]$$

even terms cancel

$$= \frac{2kq}{r} \sum_{m=0}^{\infty} \left(\frac{a}{r}\right)^{2m+1} P_{2m+1}(\cos\theta)$$

- leading term for $r \gg a$ is $\phi \sim \frac{2kqa}{r^2} P_1(\cos\theta)$

dipole potential

$D = 2qa$ is the dipole moment

Multipole Moments

- Derivation so far only works in axisymmetric distributions, but can already see a trend
  - total charge is the monopole moment M

    leading term for $r \gg a$ is $$\phi \sim \phi_M = \frac{kM}{r} P_0\cancelto{1}{(\cos\theta)}$$
  - next term is the dipole (D)

    leading term is $$\phi \sim \phi_D = \frac{kD}{r^2} P_1(\cos\theta)$$
  - next term is the quadrupole (Q)

    leading term is $$\phi \sim \phi_Q = \frac{kQ}{r^3} P_2(\cos\theta)$$
- Note: r and θ have to do with the <u>field</u> point $\boldsymbol{x}$, while the moments $M, d, Q, \ldots$ have to do with the distribution of charge within the <u>source</u> region.

Multipole Moments

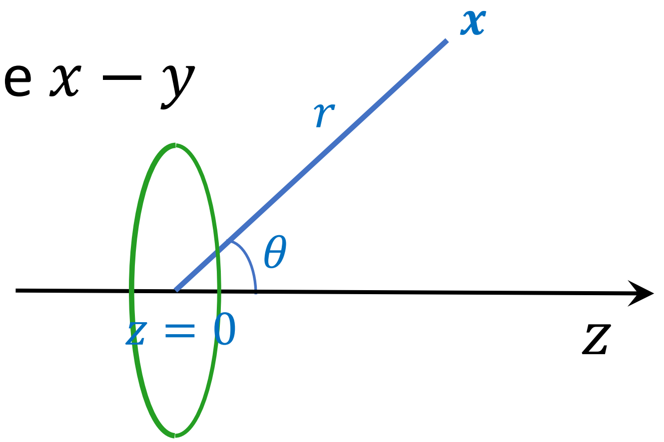
- Form of the next term:
  - expect scaling appropriate for $l = 3$

    $\Longrightarrow r^{-4} P_3(\cos\theta)$
  - moment should scale as qa^3, exact result depend on details of the geometry
  - called the <u>octupole moment</u>, $\mathcal{O}$
  - octupole potential $\phi_O = \frac{k\mathcal{O}}{r^4} P_3(\cos\theta)$
  - next is <u>hexadecapole</u>, $\phi_H = \frac{kH}{r^5} P_4(\cos\theta)\,,\ H \sim qa^4$
  - etc.

    32-pole: dotriacontapole

    64-pole: tetrahexacontapole

Example: Potential Due to a Ring of Charge



- Ring of charge, radius a, total charge q, at $z = 0$ in the $x - y$ plane, expect $\phi \to 0$ as $r \to \infty$.
- Already seen the general solution to Laplace

$$\phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} b_{lm}\, r^{-l-1}\, P_l^m(\cos\theta) e^{im\varphi}$$

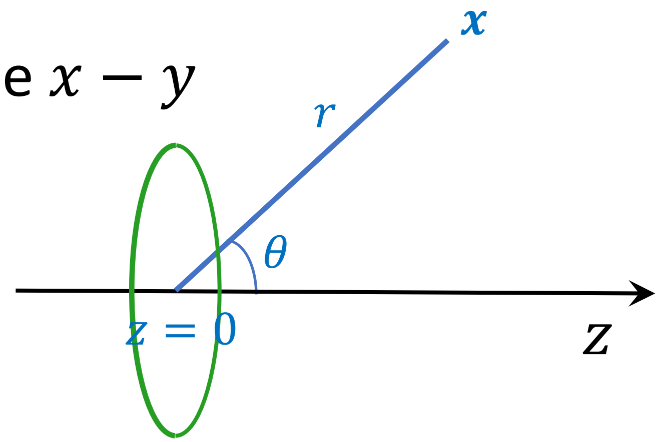
- Axisymmetric problem, expect no φ dependence, $m = 0$

$$\Longrightarrow \phi = \sum_{l=0}^{\infty} b_l\, r^{-l-1}\, P_l(\cos\theta)$$

- Along the z axis, know that $\phi = \frac{kq}{(z^2+a^2)^{1/2}}$, (same distance for all charge elements)

so $\phi = \frac{kq}{z}\left[1 + \left(\frac{a}{z}\right)^2\right]^{-1/2} = \frac{kq}{z}\left[1 - \frac{1}{2}\left(\frac{a}{z}\right)^2 + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{a}{z}\right)^4 \dots\right]$

Example: Potential Due to a Ring of Charge



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so $\phi = \frac{kq}{z}\left[1 + \left(\frac{a}{z}\right)^2\right]^{-1/2} = \frac{kq}{z} \sum_{j=0}^{\infty} (-1)^j \frac{(2j)!}{2^{2j}(j!)^2} \left(\frac{a}{z}\right)^{2j}$

- On the axis, $\theta = 0$, $P_l(\cos\theta) = 1, z = r$, so $b_l = 0$ for l odd,

$$b_l = kq\, \frac{l!}{2^l (l/2)!^2} \quad \text{for } l \text{ even}$$

$b_0 = kq,\ b_2 = \frac{kq}{2},\ b_4 = \frac{3kq}{64}, \ldots$

Legendre Recurrence Relations

- From generating function, saw some already in derivation of the Legendre ODE.

$$P_l \;= P'_{l+1} - 2zP'_l + P'_{l-1}$$
$$zP'_l - P'_{l-1} = lP_l$$
$$(l{+}1)P_l = P'_{l+1} - zP'_l$$

- Many more, combinations of others:

$$\frac{\partial}{\partial h}: \qquad (1 - 2hz + h^2)\frac{\partial F}{\partial h} = (z-h)F$$
$$\Longrightarrow \qquad (1 - 2hz + h^2)\sum_{l=0}^{\infty} lP_l(z)h^{l-1} = (z-h)\sum_{l=0}^{\infty} P_l(z)h^l$$
$$\Longrightarrow \qquad (l+1)P_{l+1} - 2zlP_l + (l-1)P_{l-1} = zP_l - P_{l-1}$$
$$\Longrightarrow \qquad (l+1)P_{l+1} - (2l+1)zlP_l + lP_{l-1} = 0$$

- $P'_{l+1} - P'_{l-1} = (2l+1)P_l$ cf Homework 4

Associated Legendre Equation

- From the non-axisymmetric separation of Helmholtz in spherical polars $(x = \cos\theta)$

$$(1-x^2)y'' - 2xy' + \left[l(l+1) - \frac{m^2}{1-x^2}\right]y = 0$$

- Self-adjoint equation, first solution $P_l^m(x)$
- Can show

$$P_l^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x)$$

Note: first term is just $\sin^m\theta$, second is a polynomial in $\cos\theta$.

- Orthogonality

$$\int_{-1}^{1} P_j^m(x)P_l^m(x)\,dx = \frac{(l+m)!}{(l-m)!}\frac{2}{2l+1}\delta_{jl}$$

Already seen for $m = 0$; <u>same</u> m because of $e^{im\varphi}$ coupling

- Second solutions $Q_l^m(x)$ are <u>singular</u> at $x = \pm 1$.

Associated Legendre Equation

- The $P_l^m(x)$ are <u>complete:</u>

$$f(x) = \sum_{l=m}^{\infty} c_l P_l^m(x)$$

for <u>any</u> f

$$c_l = \frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}\int_{-1}^{1} f(x)\, P_l^m(x)\, dx$$

- Parity

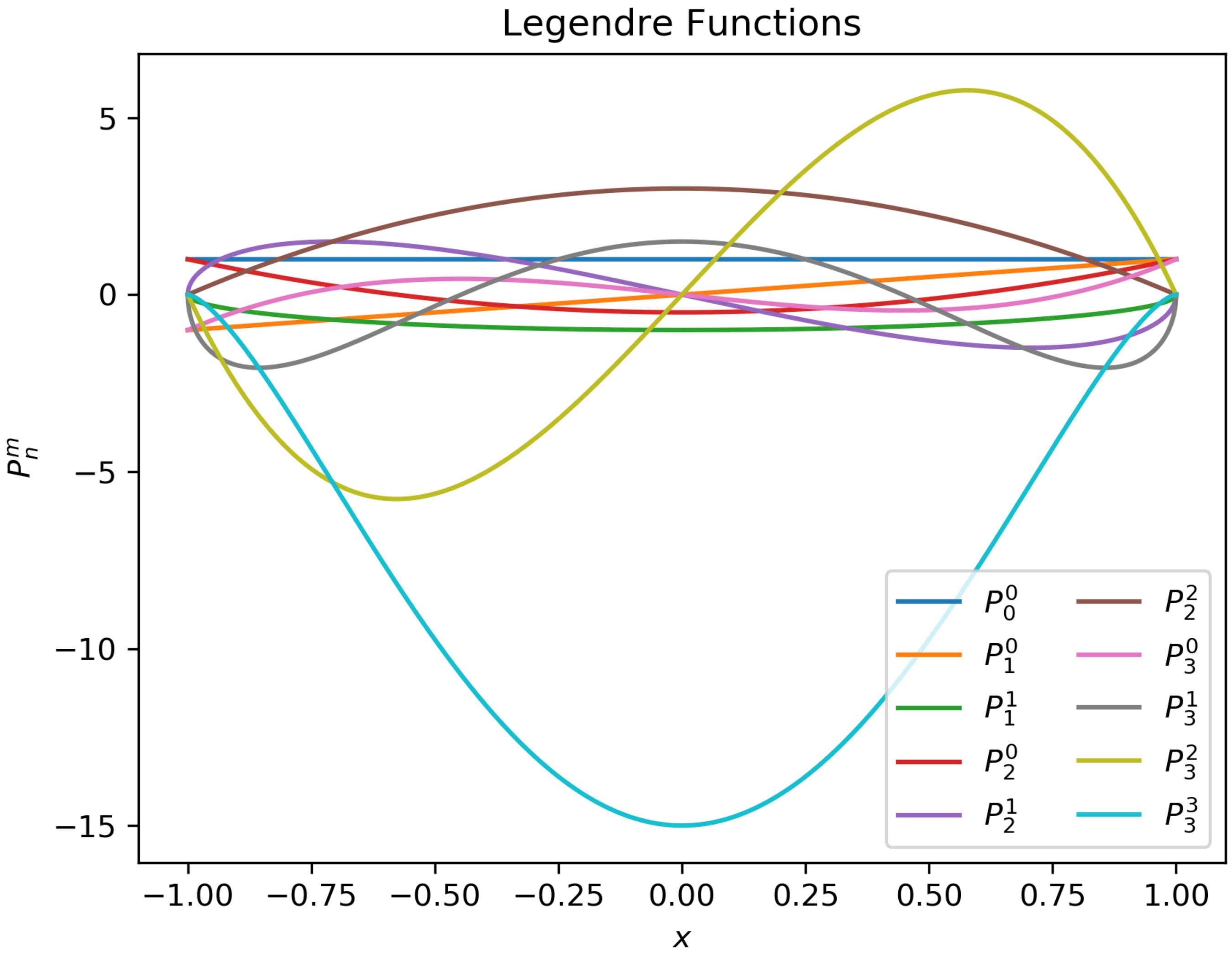
$$P_l^m(-x) = (-1)^{l+m}\, P_l^m(x)$$

- also

$$P_l^m(\pm 1) = 0, \;\; m \neq 0$$

$$P_l^0(\pm 1) = P_l(\pm 1) = (-1)^l$$

$$P_l^0(x) = P_l(x)$$


Legendre Functions
P_n^m
x
5
0
−5
−10
−15
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
P_0^0
P_1^0
P_1^1
P_2^0
P_2^1
P_2^2
P_3^0
P_3^1
P_3^2
P_3^3

Associated Legendre Equation

- Lots of recurrence relations:

$$P_l^{m+1} - \frac{2mx}{(1-x^2)^{1/2}} P_l^m + [l(l+1) - m(m-1)] P_l^{m-1} = 0$$

$$(2l+1) x P_l^m = (l+m) P_{l-1}^m + (l-m+1) P_{l+1}^m$$

$$(2l+1)(1-x^2)^{1/2} P_l^m = P_{l+1}^{m+1} - P_{l-1}^{m+1}$$

$$(1-x^2)^{1/2} (P_l^m)' = \frac{1}{2} P_l^{m+1} - \frac{1}{2}(l+1)(l-m+1) P_l^{m-1}$$

$$\vdots$$

etc.

Spherical Harmonics

- Saw previously, in spherical geometry, the angular (θ, φ) pieces always occur in the combination:

$$Y_l^m(\theta, \varphi) \sim P_l^m(\cos \theta) e^{im\varphi}$$

<u>spherical harmonics</u>

- $Y_l^m(\theta, \varphi)$ satisfies the Helmholtz equation on the surface of a sphere:

$$\underbrace{\frac{1}{\sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 u}{\partial \varphi^2}}_{\nabla_\Omega^2 u} + l(l+1)u = 0$$

- 2-D <u>eigenvalue</u> equation, all the same conclusions about eigenfunctions and eigenvalues apply

Spherical Harmonics

- Normalized eigenfunctions (spherical harmonics) are, conventionally

$$Y_l^m(\theta,\varphi) = \ (-1)^m \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\varphi}$$

$$\int_{-\pi}^{\pi}\sin\theta\, d\theta \int_0^{2\pi} d\varphi\ \ Y_l^m(\theta,\varphi)\left[\,Y_{l'}^{m'}(\theta,\varphi)\right]^* = \delta_{ll'}\delta_{mm'}$$

- <u>Laplace</u> series for <u>any</u> function on the surface of a sphere:

$$f(\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}Y_l^m(\theta,\varphi)$$

$$A_{lm} = \int_{-\pi}^{\pi}\sin\theta\, d\theta \int_0^{2\pi} d\varphi\ \ f(\theta,\varphi)[\,Y_l^m(\theta,\varphi)]^*$$

- Reminder: In spherical polars, Y_l^m couples with

  $a_{lm}r^l + b_{lm}r^{-l-1}$ in the solution to the 3-D Laplace equation

  $a_{lm}j_l(kr) + b_{lm}y_l(kr)$ in the solution to the 3-D Helmholtz equation

Spherical Harmonics

- First few (with normalization now explained):

$$Y_0^0(\theta,\varphi) = \sqrt{\frac{1}{4\pi}}$$

$$Y_1^0(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^{\pm1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi}$$

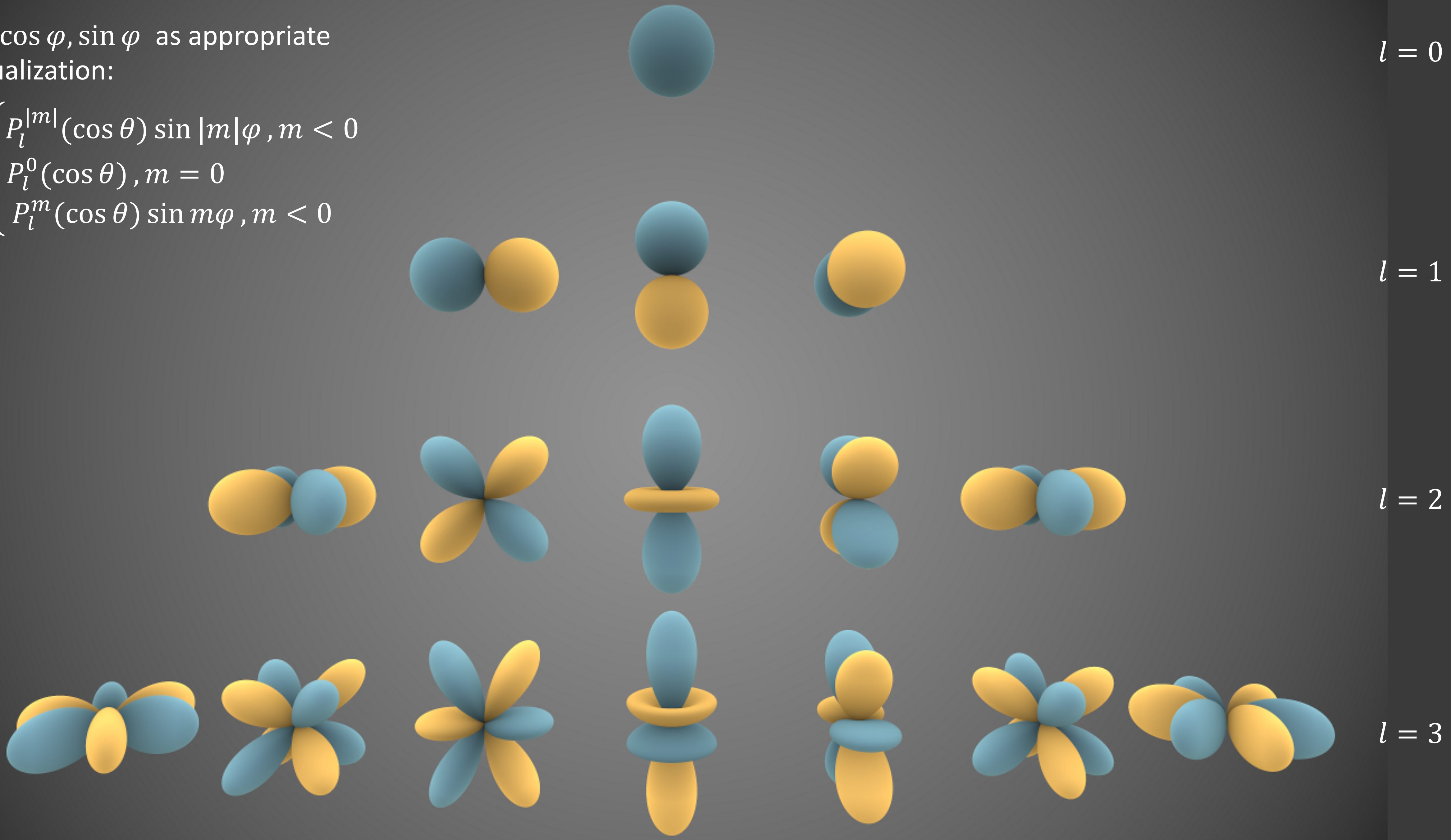
$$Y_2^0(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$$

$$Y_2^{\pm1}(\theta,\varphi) = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\varphi}$$

$$Y_2^{\pm2}(\theta,\varphi) = \mp\sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\varphi}$$

$e^{i\varphi} \to \cos\varphi, \sin\varphi$ as appropriate for visualization:

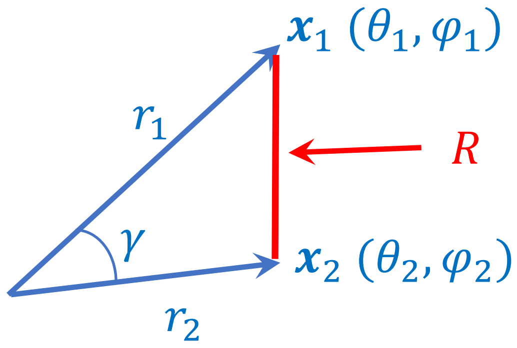
$$Y_{lm} \sim \begin{cases} P_l^{|m|}(\cos\theta)\sin|m|\varphi\,, m < 0 \\ P_l^0(\cos\theta)\,, m = 0 \\ P_l^m(\cos\theta)\sin m\varphi\,, m < 0 \end{cases}$$



The Addition Theorem

- Generalizes the Legendre generating function to the non-axisymmetric case.
- State only — proof would take us too far afield.

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_l^m(\theta_1,\varphi_1)\left[Y_l^m(\theta_2,\varphi_2)\right]^*$$



- (Basic approach: expand $P_l(\cos\gamma)$ as a Laplace series, show that the coefficients must be $[Y_l^m(\theta_2,\varphi_2)]^*$.)

$$\cos\gamma = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\varphi_1-\varphi_2)$$

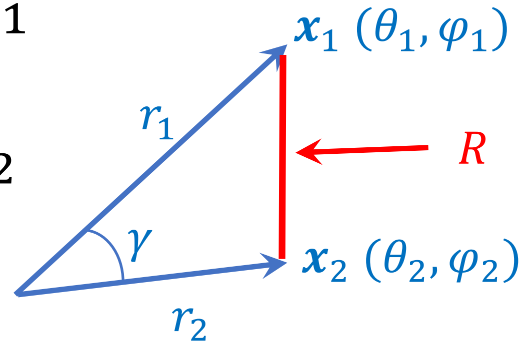
- Corollary: recall

$$\frac{1}{R} = \sum_{l=0}^{\infty}\frac{r'^{l}}{r^{l+1}}P_l(\cos\gamma)$$

$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\underbrace{Y_l^m(\theta_1,\varphi_1)}_{\text{field point}}\underbrace{\left[Y_l^m(\theta_2,\varphi_2)\right]^*}_{\text{source point}}\begin{cases}\dfrac{r_2^l}{r_1^{l+1}}, & r_2 < r_1\\[2ex] \dfrac{r_1^l}{r_2^{l+1}}, & r_1 < r_2\end{cases}$$

The Addition Theorem

$$\frac{1}{R} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} Y_l^m(\theta_1, \varphi_1)[Y_l^m(\theta_2, \varphi_2)]^* \begin{cases} \frac{r_2^l}{r_1^{l+1}}, & r_2 < r_1 \\ \frac{r_1^l}{r_2^{l+1}}, & r_1 < r_2 \end{cases}$$



- Allows us to generalize the earlier multipole expansion.
- Enables computation of integrals of the form

$$\mathrm{I} = \int d^3x_1 \int d^3x_2 \, \frac{f_1(r_1,\theta_1,\varphi_1)\, f_2(r_2,\theta_2,\varphi_2)}{r_{12}}$$

- e.g. potential energy of a spherical mass or charge distribution

$$f_1 = f_2 = \rho, \;\; \Phi = \frac{1}{2} k\, I$$

Example: Self-Potential Energy of a Charged Sphere

- Simplest case: uniform charge density ρ, radius R, total charge $Q = \frac{4}{3}\pi R^3 \rho$
- Then total energy is

$d\Omega = \sin\theta d\theta d\varphi$

$$\Phi = \frac{1}{2}\int d^3x_1 \int d^3x_2 \frac{k\rho^2}{r_{12}}, \text{ where } d^3x_1 = r_1^2 d\Omega_1 dr_1,\, d^3x_2 = r_2^2 d\Omega_2 dr_2$$

- Expand

$$\frac{1}{r_{12}} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1} Y_l^m(\theta_1,\varphi_1)[Y_l^m(\theta_2,\varphi_2)]^* \begin{cases} \frac{r_2^l}{r_1^{l+1}}, & r_2 < r_1 \\ \frac{r_1^l}{r_2^{l+1}}, & r_1 < r_2 \end{cases}$$

- But

$$\int d\Omega\, Y_l^m(\theta,\varphi) = \underbrace{\int d\varphi\, e^{im\varphi}}_{\text{needs } m=0} \underbrace{\int d\mu\, P_l^m(\mu)}_{P_l \Rightarrow l=0} = 4\pi\delta_{m0}\delta_{l0}Y_0^0$$

$$= \sqrt{4\pi}\,\delta_{m0}\delta_{l0}$$

Example: Self-Potential Energy of a Charged Sphere

- Only $l = 0, m = 0$ terms survive, the $d\Omega_1$ and $d\Omega_2$ integrals are independent, and we are left with $\sqrt{4\pi}$ from each.
- Then total energy is

$$\Phi = \frac{1}{2}k\rho^2 \int\int r_1^2 dr_1 r_2^2 dr_2 \ 4\pi \left(\sqrt{4\pi}\right)^2 \begin{cases} \frac{1}{r_1}, & r_2 < r_1 \\ \frac{1}{r_2}, & r_1 < r_2 \end{cases}$$

$$= 8\pi^2 k\rho^2 \int_0^R r_1^2 dr_1 \left\{\int_0^{r_1} r_2^2 dr_2 \frac{1}{r_1} + \int_{r_1}^R r_2 dr_2\right\}$$

cf Newton's theorem

$$= 8\pi^2 k\rho^2 \int_0^R r_1^2 dr_1 \left\{\frac{1}{3}r_1^2 + \frac{1}{2}(R^2 - r_1^2)\right\}$$

$$= 8\pi^2 k\rho^2 \left\{\frac{1}{15}R^5 + \frac{1}{6}R^5 - \frac{1}{10}R^5\right\}$$

$$= 8\pi^2 k \left(\frac{3Q}{4\pi R^3}\right)^2 \left\{\frac{2}{15}R^5\right\}$$

$$= \frac{3}{5}\frac{kQ^2}{R}$$