PHYS 501 HW 6

#1 a. Let $\tilde{\phi}(k)$ and $\tilde{\rho}(k)$ be Fourier transforms of $\phi(x)$ and $\rho(x)$. Show $\tilde{\phi} = -\frac{4\pi G \tilde{\rho}}{k^2}$ and find $\phi(x)$

Poisson's equation:

$$\nabla^2 \phi = 4\pi G \rho(k)$$

Fourier transform $\nabla^2 \phi$ to get a $\tilde{\rho}(k)$:

$$4\pi G \cdot \mathcal{F}(\rho(\vec{r})) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \nabla^2 \phi(\vec{r}) \cdot e^{ikr} \, dx\,dy\,dz$$

work in cartesian 3D

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{k} = k_x\hat{i} + k_y\hat{j} + k_z\hat{k}$$

$$\nabla^2 \phi(\vec{r}) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \phi(\vec{r})$$

$$e^{ikr} = e^{ixk_x} e^{iyk_y} e^{izk_z}$$

let $\mathcal{F}(\rho(\vec{r})) = \tilde{\rho}(k)$

so $4\pi G \cdot \tilde{\rho}(k) =$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \phi(\vec{r}) e^{ixk_x} e^{iyk_y} e^{izk_z} \, dx\,dy\,dz$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(\vec{r}) \left[(ik_x)^2 + (ik_y)^2 + (ik_z)^2\right] e^{ixk_x} e^{iyk_y} e^{izk_z} \, dx\,dy\,dz$$

$$= [-k_x^2 - k_y^2 - k_z]^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(\vec{r})\, e^{i\vec{k}\vec{r}}\, dx\,dy\,dz$$

$$= [-k_x^2 - k_y^2 - k_z^2]\, \mathcal{F}(\phi(\vec{r}))$$

again saying $\mathcal{F}(\phi(\vec{r})) = \tilde{\phi}(k)$

then $4\pi G \cdot \tilde{\rho}(k) = -|\vec{k}|^2 \tilde{\phi}(k)$

so $\tilde{\phi}(k) = -\dfrac{4\pi G \tilde{\rho}(k)}{k^2}$ is Fourier transform

of $\phi(x)$ and we know that gives a $\frac{1}{\sqrt{2\pi}}$ for each dimension of the triple integral and our function will give a negative sign

$$\boxed{\phi(x) = \frac{4\pi G}{(2\pi)^{3/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\tilde{\rho}(k)}{k^2} \cdot e^{-ikr}\, dk_x\,dk_y\,dk_z}$$

b. For point mass at origin, $\rho(x) = m\delta(x)$
Find ϕ solution $\rightarrow$ need to write $\tilde{\rho}(k)$ and solve integral

using $\rho(x)$:

$$\tilde{\rho}(k) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m\delta(\vec{r})\, e^{ikr}\, dk_x\,dk_y\,dk_z = m$$

since integral of $\delta(x) = 1$

So $\phi(x) = \frac{4\pi G m}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-ikr}}{k^2} \, dk_x \, dk_y \, dk_z$

use "trick" to solve: $k_x = kr \cos\theta_k$

$$\phi(x) = \frac{4\pi G m}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk \, d\theta_k \, d\varphi_k \sin\theta_k e^{-ikr} e^{ikr\cos\theta_k}$$

$$= \frac{4\pi G m}{(2\pi)^{3/2}} \cdot \left[2\pi \left(-\frac{1}{x} \right) \right]$$

$$\boxed{\phi(x) = -\frac{Gm}{x}}$$

#2. Find Green's function for equation (solve DFQ)

$\frac{d^2y}{dx^2} - k^2 y = f(x)$ for $0 \le x \le L$
$y(0) = y(L) = 0$

$y'' - k^2 y = f(x)$

then Green's function satisfies

$G'' - k^2 G = \delta(x - x')$

$G(0, x') = 0$
$G(L, x') = 0$

for $x = x'$, $G'' - k^2 G = 0$
so will have 2 solutions: for $x < x'$, $x > x'$
want solutions that eliminate cosine terms so that $G = 0$ at $x = 0$ and $x = L$

$$G(x,x') = \begin{cases} A\sin kx & \text{for } x < x' \\ B\sin k(x-L) & \text{for } x > x' \end{cases}$$

since coefficients of cosines are 0

we know $\left[\frac{\partial G}{\partial x}\right]_{x'_-}^{x'_+} = 1$ so consider $x = x'$

G must be continuous so $A\sin kx' = B\sin k(x'-L)$

and G' will have a jump

$$\frac{d}{dx}(A\sin kx) + 1 = \frac{d}{dx}(B\sin k(x'-L))$$

$$Ak\cos kx' + 1 = Bk\cos k(x'-L)$$

solve as system of equations:

then $A = \dfrac{\sin k(x'-L)}{k\sin kL}$ $\quad B = \dfrac{\sin kx'}{k\sin kL}$

so $\boxed{G(x,x') = \dfrac{1}{k\sin kL}\begin{cases} \sin k(x'-L)\sin kx & x<x' \\ \sin kx' \sin k(x'-L) & x>x' \end{cases}}$

#3. Show $G(x,x') = -\frac{e^{ikr}}{4\pi r}$ for $\nabla^2 u + k^2 u = 0$

with boundary condition $u(x)e^{-i\omega t}$ is outgoing waves at infinity. $r = |x - x'|$

apply Green's theorem:

$$(\nabla^2 + k^2) G(x,x') = \delta(\vec{x} - \vec{x}')$$

we know at $r \neq 0$ that $(\nabla^2 + k^2) G = 0$

so this is of form $G(x,x') = C \cdot \frac{e^{\pm ikr}}{r}$

because oscillary behavior of $u(x)$ (for $x \neq x'$)

from boundary condition, $G \sim e^{+ikr}$

now consider $r \to 0$ to find C

$$(\nabla^2 + k^2) G \to C \cdot \nabla^2 \left(\frac{1}{r}\right) = C\left(-4\pi \delta(\vec{x} - \vec{x}')\right)$$
$$= \delta(\vec{x} - \vec{x}')$$

$$-4\pi C \, \delta(\vec{x} - \vec{x}') = \delta(\vec{x} - \vec{x}')$$

so $C = -\frac{1}{4\pi}$

then $\boxed{G(x,x') = -\frac{e^{ikr}}{4\pi r}}$