Spherical Harmonics

- Normalized spherical harmonics are, conventionally

$$Y_l^m(\theta,\varphi) = \ (-1)^m \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\varphi}$$

$$\int_{-\pi}^{\pi}\sin\theta\, d\theta \int_0^{2\pi} d\varphi\ \ Y_l^m(\theta,\varphi)\left[\, Y_{l'}^{m'}(\theta,\varphi)\right]^* = \delta_{ll'}\delta_{mm'}$$

- <u>Laplace</u> series for <u>any</u> function on the surface of a sphere:

$$f(\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}Y_l^m(\theta,\varphi)$$

$$A_{lm} = \int_{-\pi}^{\pi}\sin\theta\, d\theta \int_0^{2\pi} d\varphi\ \ f(\theta,\varphi)[\, Y_l^m(\theta,\varphi)]^*$$

- Reminder: In spherical polars, Y_l^m couples with

  $a_{lm}r^l + b_{lm}r^{-l-1}$ in the solution to the 3-D Laplace equation

  $a_{lm}j_l(kr) + b_{lm}y_l(kr)$ in the solution to the 3-D Helmholtz equation

Spherical Harmonics

- First few (with normalization now explained):

$$Y_0^0(\theta,\varphi) = \sqrt{\frac{1}{4\pi}}$$

$$Y_1^0(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^{\pm1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi}$$

$$Y_2^0(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$$

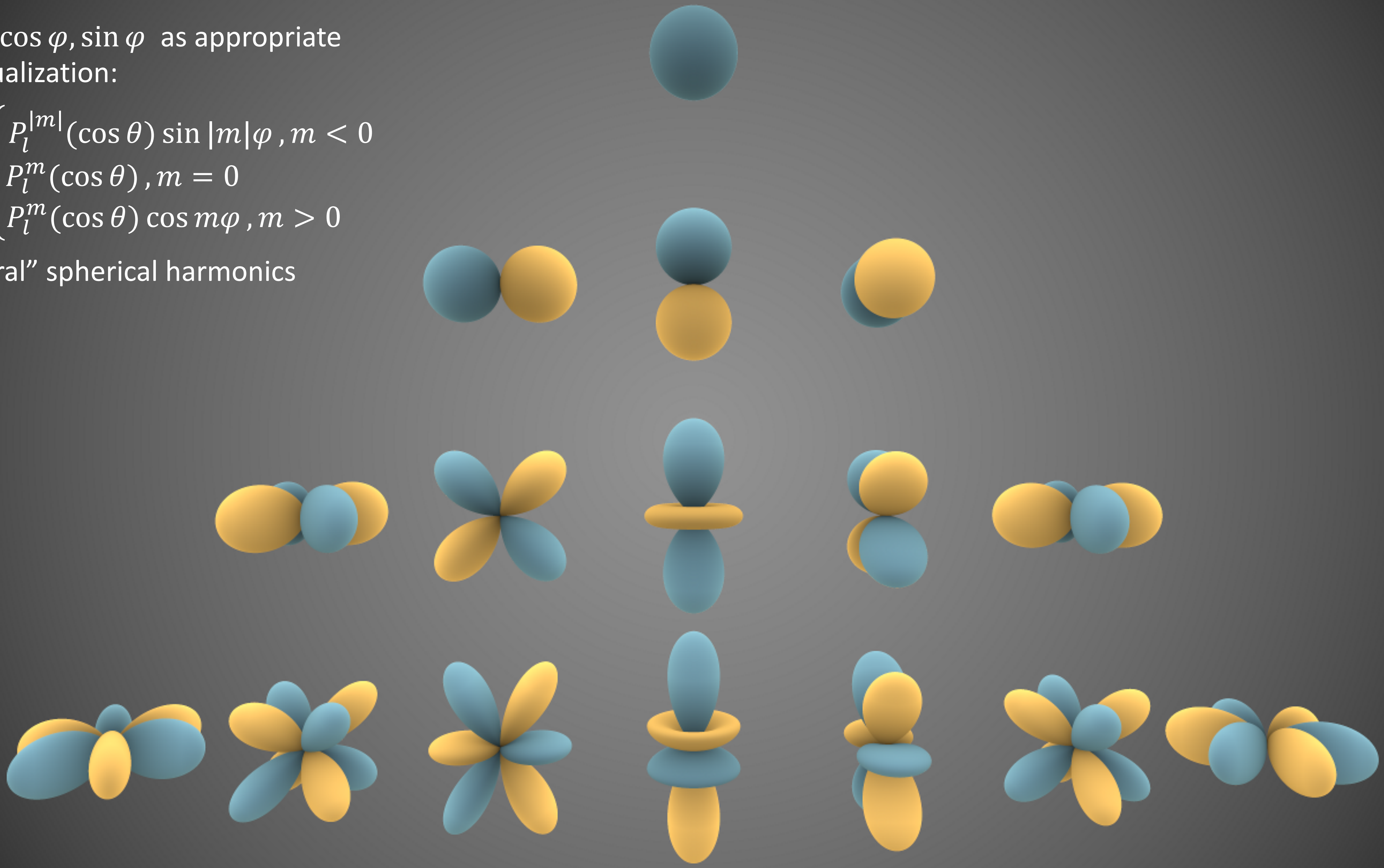
$$Y_2^{\pm1}(\theta,\varphi) = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\varphi}$$

$$Y_2^{\pm2}(\theta,\varphi) = \mp\sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\varphi}$$

$e^{i\varphi} \to \cos\varphi, \sin\varphi$ as appropriate for visualization:

$$Y_{lm} \sim \begin{cases} P_l^{|m|}(\cos\theta)\sin|m|\varphi\,, m < 0 \\ P_l^{m}(\cos\theta)\,, m = 0 \\ P_l^{m}(\cos\theta)\cos m\varphi\,, m > 0 \end{cases}$$

"Tesseral" spherical harmonics



Tesseral Spherical Harmonics

- Real versions are sometimes convenient:

$$Y_{lm}(\theta,\varphi) = \begin{cases} (-1)^m \left[\frac{2l+1}{2\pi}\frac{(l-|m|)!}{(l+|m|)!}\right]^{1/2} P_l^{|m|}(\cos\theta)\sin|m|\varphi\,, & m<0 \\ \sqrt{\frac{2l+1}{4\pi}}\, P_l^m(\cos\theta), & m=0 \\ (-1)^m \left[\frac{2l+1}{2\pi}\frac{(l-|m|)!}{(l+|m|)!}\right]^{1/2} P_l^{|m|}(\cos\theta)\cos|m|\varphi\,, & m>0 \end{cases}$$

$$Y_l^m(\theta,\varphi) = (-1)^m\left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\varphi}$$

$$\int_{-\pi}^{\pi}\sin\theta\, d\theta\int_0^{2\pi} d\varphi\; Y_{lm}(\theta,\varphi)[Y_{l'm'}(\theta,\varphi)]^* = \delta_{ll'}\delta_{mm'}$$

Tesseral Spherical Harmonics

- First few:

$$Y_{00}(\theta,\varphi) = \sqrt{\frac{1}{4\pi}}$$

$$Y_{1,-1}(\theta,\varphi) = -\sqrt{\frac{3}{4\pi}}\sin\theta\sin\varphi$$

$$Y_{10}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{11}(\theta,\varphi) = -\sqrt{\frac{3}{4\pi}}\sin\theta\cos\varphi$$

$$Y_{2,-2}(\theta,\varphi) = \sqrt{\frac{15}{16\pi}}\sin^2\theta\sin 2\varphi$$

$$Y_{2,-1}(\theta,\varphi) = -\sqrt{\frac{15}{4\pi}}\sin\theta\cos\theta\sin\varphi$$

$$Y_{20}(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$$

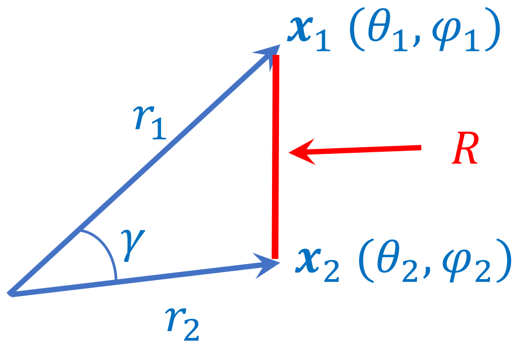
$$Y_{21}(\theta,\varphi) = -\sqrt{\frac{15}{4\pi}}\sin\theta\cos\theta\cos\varphi$$

$$Y_{22}(\theta,\varphi) = \sqrt{\frac{15}{16\pi}}\sin^2\theta\cos 2\varphi$$

The Addition Theorem

- Generalizes the Legendre generating function to the non-axisymmetric case.

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_l^m(\theta_1,\varphi_1)\left[Y_l^m(\theta_2,\varphi_2)\right]^*$$



- Corollary:

$$\frac{1}{R} = \sum_{l=0}^{\infty}\frac{r'^l}{r^{l+1}}P_l(\cos\gamma)$$

$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\underbrace{Y_l^m(\theta_1,\varphi_1)}_{\text{field point}}\underbrace{\left[Y_l^m(\theta_2,\varphi_2)\right]^*}_{\text{source point}}\begin{cases}\dfrac{r_2^l}{r_1^{l+1}}, & r_2 < r_1\\[2ex] \dfrac{r_1^l}{r_2^{l+1}}, & r_1 < r_2\end{cases}$$

Example: Multipole Expansion Again

- Potential at some point $\boldsymbol{x}$ due to a distribution of charge $\rho(\boldsymbol{x}')$ in volume V, with $r' = |\boldsymbol{x}'| \ll r = |\boldsymbol{x}|$ is

$$\phi(\boldsymbol{x}) = k \iiint_V d^3x' \, \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}$$

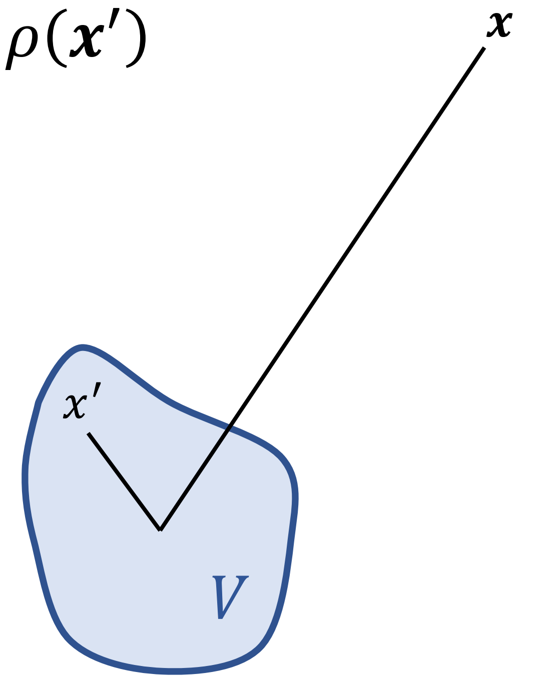
$$= k \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \left\{ \frac{Y_l^m(\theta,\varphi)}{r^{l+1}} \times \int_V \rho(\boldsymbol{x}') \, Y_l^{m*}(\theta',\varphi')(r')^l \, d^3x' \right\}$$

$$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi k}{2l+1} \frac{Y_l^m(\theta,\varphi)}{r^{l+1}} M_{lm}$$

where

$$M_{lm} = \int_V d^3x' \, \rho(\boldsymbol{x}') \, (r')^l \, Y_{lm}(\theta',\varphi')$$

multipole moment

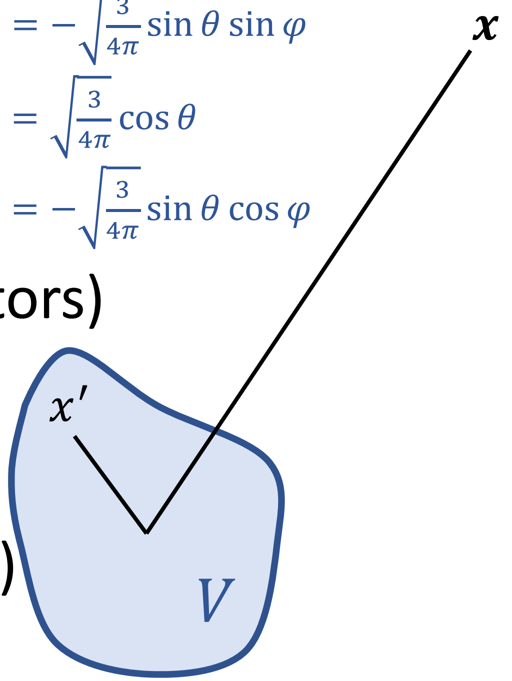


- Often define M_{lm} in terms of the tesseral harmonics Y_{lm}.

Example: Multipole Expansion Again

$$Y_{00}(\theta,\varphi) = \sqrt{\frac{1}{4\pi}}$$
$$Y_{1,-1}(\theta,\varphi) = -\sqrt{\frac{3}{4\pi}}\sin\theta\sin\varphi$$
$$Y_{10}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$
$$Y_{11}(\theta,\varphi) = -\sqrt{\frac{3}{4\pi}}\sin\theta\cos\varphi$$

$$\phi(\boldsymbol{x}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi k}{2l+1}\,\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}\,M_{lm}$$
$$M_{lm} = \int_V d^3x'\,\rho(\boldsymbol{x}')\,(r')^l\,Y_{lm}(\theta',\varphi')$$



- $l = 0,\ m = 0$ is the monopole coupling (cancel $\sqrt{4\pi}$ factors)

$$\phi_0 = \frac{k}{r}\int_V \rho(\boldsymbol{x}')\,d^3x' = \frac{kQ}{r}$$

- $l = 1, m = -1, 0, 1$ is the dipole term (cancel -1 factors)

$$\phi_1 = \frac{k}{r^2}\begin{pmatrix}\sin\theta\sin\varphi\\ \cos\theta\\ \sin\theta\cos\varphi\end{pmatrix}\cdot\int_V \rho(\boldsymbol{x}')\,r'\begin{pmatrix}\sin\theta'\sin\varphi'\\ \cos\theta'\\ \sin\theta'\cos\varphi'\end{pmatrix}d^3x'$$

- In Cartesian coordinates, the moments are

$$M = \int_V \rho(\boldsymbol{x}')\ d^3x',\ D_i = \int_V \rho(\boldsymbol{x}')x_i'\ d^3x'$$

$$x' = r'\sin\theta'\cos\varphi'$$
$$y' = r'\sin\theta'\sin\varphi'$$
$$z' = r'\cos\theta'$$

Example: Multipole Expansion Again

$$Y_{00}(\theta,\varphi) = \sqrt{\frac{1}{4\pi}}$$
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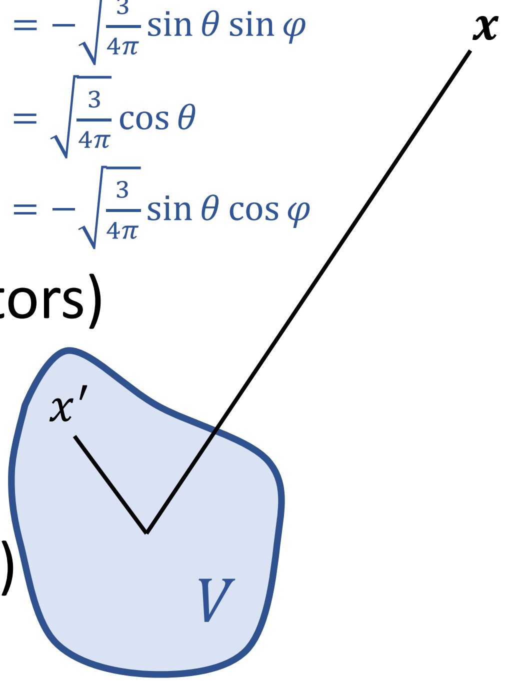
$$\phi(\boldsymbol{x}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi k}{2l+1}\,\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}}\,M_{lm}$$
$$M_{lm} = \int_V d^3x'\,\rho(\boldsymbol{x}')\,(r')^l\,Y_{lm}(\theta',\varphi')$$



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- $l = 1, m = -1, 0, 1$ is the dipole term (cancel -1 factors)

$$\phi_1 = \frac{k}{r^2}\begin{pmatrix}\hat{\boldsymbol{x}}_y\\ \hat{\boldsymbol{x}}_z\\ \hat{\boldsymbol{x}}_x\end{pmatrix}\cdot\int_V \rho(\boldsymbol{x}')\begin{pmatrix}y'\\ z'\\ x'\end{pmatrix}d^3x' = \frac{k}{r^2}\,\hat{\boldsymbol{x}}\cdot\boldsymbol{D}$$

- In Cartesian coordinates, the moments are

$$M = \int_V \rho(\boldsymbol{x}')\,d^3x',\ D_i = \int_V \rho(\boldsymbol{x}')x_i'\,d^3x'$$

$$x' = r'\sin\theta'\cos\varphi'$$
$$y' = r'\sin\theta'\sin\varphi'$$
$$z' = r'\cos\theta'$$

Some details

- source point: (x', y', z')

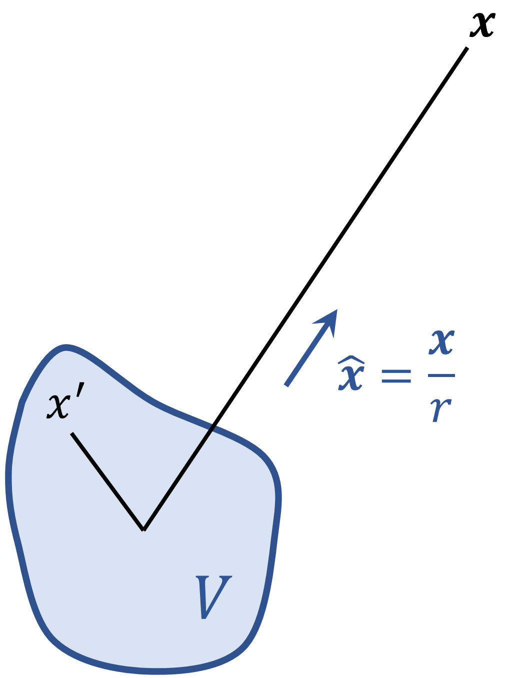
$$x' = r' \sin\theta' \cos\varphi'$$
$$y' = r' \sin\theta' \sin\varphi'$$
$$z' = r' \cos\theta'$$

so $$\begin{pmatrix} r'\sin\theta' \sin\varphi' \\ r'\cos\theta' \\ r'\sin\theta' \cos\varphi' \end{pmatrix} = \begin{pmatrix} y' \\ z' \\ x' \end{pmatrix}$$

- field point: (x, y, z)

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

$$\Longrightarrow \begin{pmatrix} \sin\theta\sin\varphi \\ \cos\theta \\ \sin\theta\cos\varphi \end{pmatrix} = \begin{pmatrix} y/r \\ z/r \\ x/r \end{pmatrix} = \begin{pmatrix} \hat{\boldsymbol{x}}_y \\ \hat{\boldsymbol{x}}_z \\ \hat{\boldsymbol{x}}_x \end{pmatrix}$$



Example: Multipole Expansion Again

$$\phi(\boldsymbol{x}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi k}{2l+1} \frac{Y_{lm}(\theta,\varphi)}{r^{l+1}} M_{lm}$$

$$M_{lm} = \int_V d^3x' \, \rho(\boldsymbol{x}') \, (r')^l \, Y_{lm}(\theta', \varphi')$$

$$Y_{2,-2}(\theta,\varphi) = \sqrt{\frac{15}{16\pi}} \sin^2\theta \sin 2\varphi$$

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$$Y_{22}(\theta,\varphi) = \sqrt{\frac{15}{16\pi}} \sin^2\theta\cos 2\varphi$$

- $l = 2, m = -2, -1, 0, 1, 2$ gives the quadrupole term

$$\phi_2 = \frac{k}{r^3} \begin{pmatrix} \frac{1}{2}\sin^2\theta \sin 2\varphi \\ \sin\theta\cos\theta \, \sin\varphi \\ \frac{1}{2}(3\cos^2\theta - 1) \\ \sin\theta\cos\theta\cos\varphi \\ \frac{1}{2}\sin^2\theta\cos 2\varphi \end{pmatrix} \cdot \int_V \rho(\boldsymbol{x}') \, (r')^2 \begin{pmatrix} \frac{3}{2}\sin^2\theta' \sin 2\varphi' \\ 3\sin\theta'\cos\theta' \, \sin\varphi' \\ \frac{1}{2}(3\cos^2\theta' - 1) \\ 3\sin\theta'\cos\theta'\cos\varphi' \\ \frac{3}{2}\sin^2\theta'\cos 2\varphi' \end{pmatrix} d^3x'$$

$$\frac{3}{2} r'^2 \sin^2\theta' \sin 2\varphi' = 3r'^2 \sin^2\theta' \sin\varphi' \cos\varphi' = 3x'y'$$

$$Q = \begin{pmatrix} 3x'^2 - r'^2 & 3x'y' & 3x'z' \\ 3x'y' & 3y'^2 - r'^2 & 3y'z' \\ 3x'z' & 3y'z' & 3z'^2 - r'^2 \end{pmatrix}$$

- In Cartesian coordinates, the (traceless) quadrupole moment is

$$Q_{ij} = \int_V \rho(\boldsymbol{x}')(3x_i'x_j' - r'^2\delta_{ij}) \, d^3x', \qquad \phi_2 = \frac{1}{2}\sum_{i,j} Q_{ij} \, \hat{\boldsymbol{x}}_i \hat{\boldsymbol{x}}_j$$

— symmetric, 5 independent components ($Q_{11} + Q_{22} + Q_{33} = 0$)

Example: Multipole Expansion Again

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$$\phi_2 = \frac{k}{r^3} \begin{pmatrix} \frac{1}{2}\sin^2\theta\sin 2\varphi \\ \sin\theta\cos\theta\sin\varphi \\ \frac{1}{2}(3\cos^2\theta - 1) \\ \sin\theta\cos\theta\cos\varphi \\ \frac{1}{2}\sin^2\theta\cos 2\varphi \end{pmatrix} \cdot \begin{pmatrix} Q_{12} \\ Q_{23} \\ \frac{1}{2}Q_{33} \\ Q_{13} \\ \frac{1}{2}(Q_{11} - Q_{22}) \end{pmatrix}$$

$$\begin{aligned} \frac{1}{2}\sin^2\theta\sin 2\varphi &= \sin^2\theta\sin\varphi\cos\varphi \\ &= (\sin\theta\sin\varphi)(\sin\theta\cos\varphi) \\ &= \hat{\boldsymbol{x}}_y \hat{\boldsymbol{x}}_x \end{aligned}$$

$$Q = \begin{pmatrix} 3x'^2 - r'^2 & 3x'y' & 3x'z' \\ 3x'y' & 3y'^2 - r'^2 & 3y'z' \\ 3x'z' & 3y'z' & 3z'^2 - r'^2 \end{pmatrix}$$

- In Cartesian coordinates, the (traceless) quadrupole moment is

$$Q_{ij} = \int_V \rho(\boldsymbol{x}')(3x_i'x_j' - r'^2\delta_{ij}) \, d^3x' , \qquad \phi_2 = \frac{1}{2}\sum_{i,j} Q_{ij} \, \hat{\boldsymbol{x}}_i \hat{\boldsymbol{x}}_j$$

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Example: Multipole Expansion Again

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$$M_{lm} = \int_V d^3x' \, \rho(\boldsymbol{x}') \, (r')^l \, Y_{lm}(\theta', \varphi')$$

- $l = 2, m = -2, -1, 0, 1, 2$ gives the quadrupole term

$$\phi_2 = \frac{k}{r^3} \begin{pmatrix} \hat{\boldsymbol{x}}_x \hat{\boldsymbol{x}}_y \\ \hat{\boldsymbol{x}}_y \hat{\boldsymbol{x}}_z \\ \hat{\boldsymbol{x}}_z^2 - \frac{1}{2}(\hat{\boldsymbol{x}}_x^2 + \hat{\boldsymbol{x}}_y^2) \\ \hat{\boldsymbol{x}}_x \hat{\boldsymbol{x}}_y \\ \frac{1}{2}(\hat{\boldsymbol{x}}_x^2 - \hat{\boldsymbol{x}}_y^2) \end{pmatrix} \cdot \begin{pmatrix} Q_{12} \\ Q_{23} \\ \frac{1}{2} Q_{33} \\ Q_{13} \\ \frac{1}{2}(Q_{11} - Q_{22}) \end{pmatrix}$$

- In Cartesian coordinates, the (traceless) quadrupole moment is

$$Q_{ij} = \int_V \rho(\boldsymbol{x}')(3x_i' x_j' - r'^2 \delta_{ij}) \; d^3x' \,, \qquad \phi_2 = \frac{1}{2} \sum_{i,j} Q_{ij} \, \hat{\boldsymbol{x}}_i \hat{\boldsymbol{x}}_j$$

— symmetric, 5 independent components ($Q_{11} + Q_{22} + Q_{33} = 0$)

Some details

- source point: (x', y', z')

$$x' = r' \sin\theta' \cos\varphi'$$
$$y' = r' \sin\theta' \sin\varphi'$$
$$z' = r' \cos\theta'$$

so
$$\begin{pmatrix} \frac{3}{2} r'^2 \sin^2\theta' \sin 2\varphi' \\ 3 r'^2 \sin\theta' \cos\theta' \sin\varphi' \\ \frac{1}{2} r'^2 (3\cos^2\theta' - 1) \\ 3 r'^2 \sin\theta' \cos\theta' \cos\varphi' \\ \frac{3}{2} r'^2 \sin^2\theta' \cos 2\varphi' \end{pmatrix} = \begin{pmatrix} 3 r'^2 \sin^2\theta' \sin\varphi' \cos\varphi' \\ 3 r'^2 \sin\theta' \cos\theta' \sin\varphi' \\ \frac{1}{2} r'^2 (3\cos^2\theta' - 1) \\ 3 r'^2 \sin\theta' \cos\theta' \cos\varphi' \\ \frac{3}{2} r'^2 \sin^2\theta' (\cos^2\varphi' - \sin^2\varphi') \end{pmatrix} = \begin{pmatrix} 3x'y' \\ 3y'z' \\ \frac{1}{2}(3z'^2 - r'^2) \\ 3x'z' \\ \frac{1}{2}(3x'^2 - 3y'^2) \end{pmatrix} = \begin{pmatrix} Q_{12} \\ Q_{23} \\ \frac{1}{2} Q_{33} \\ Q_{13} \\ \frac{1}{2}(Q_{11} - Q_{22}) \end{pmatrix}$$

- field point: (x, y, z)

$$x = r \sin\theta \cos\varphi$$
$$y = r \sin\theta \sin\varphi \quad \Longrightarrow$$
$$z = r \cos\theta$$

$$\begin{pmatrix} \frac{1}{2} \sin^2\theta \sin 2\varphi \\ \sin\theta \cos\theta \sin\varphi \\ \frac{1}{2}(3\cos^2\theta - 1) \\ \sin\theta \cos\theta \cos\varphi \\ \frac{1}{2} \sin^2\theta \cos 2\varphi \end{pmatrix} = \begin{pmatrix} \sin^2\theta \sin\varphi \cos\varphi \\ \sin\theta \cos\theta \sin\varphi \\ \frac{1}{2}(3\cos^2\theta - 1) \\ \sin\theta \cos\theta \cos\varphi \\ \sin^2\theta (\cos^2\varphi - \sin^2\varphi) \end{pmatrix} = \begin{pmatrix} xy/r^2 \\ yz/r^2 \\ \frac{2z^2 - (x^2 + y^2)}{2r^2} \\ xz/r^2 \\ (x^2 - y^2)/2r^2 \end{pmatrix} = \begin{pmatrix} \hat{\boldsymbol{x}}_x \hat{\boldsymbol{x}}_y \\ \hat{\boldsymbol{x}}_y \hat{\boldsymbol{x}}_z \\ \hat{\boldsymbol{x}}_z^2 - \frac{1}{2}(\hat{\boldsymbol{x}}_x^2 + \hat{\boldsymbol{x}}_y^2) \\ \hat{\boldsymbol{x}}_x \hat{\boldsymbol{x}}_y \\ \frac{1}{2}(\hat{\boldsymbol{x}}_x^2 - \hat{\boldsymbol{x}}_y^2) \end{pmatrix}$$

Fourier Transforms

- Easiest to work with the complex form of the Fourier series.
- On the range $(-L, L)$ can write, for any function f

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{i\pi n x/L}$$

where

$$c_n = \frac{1}{2L}\int_{-L}^{L} dt \, f(t) \, e^{-i\pi n t/L}$$

$$\Longrightarrow \quad f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2L}\int_{-L}^{L} dt \, f(t) \, e^{-i\pi n t/L} \, dt \, e^{i\pi n x/L}$$

$$= \frac{1}{2L}\int_{-L}^{L} dt \, f(t) \, \sum_{n=-\infty}^{\infty} e^{i\pi n(x-t)/L}$$

$$= \int_{-L}^{L} dt \, f(t) \, \frac{1}{2L}\sum_{n=-\infty}^{\infty} e^{i\pi n(x-t)/L}$$

- Aside: tempting to write

$$\frac{1}{2L}\sum_{n=-\infty}^{\infty} e^{i\pi n(x-t)/L} = \delta(x-t)$$

Delta Functions

- Operational definition of delta function $\delta(t)$, for any function f

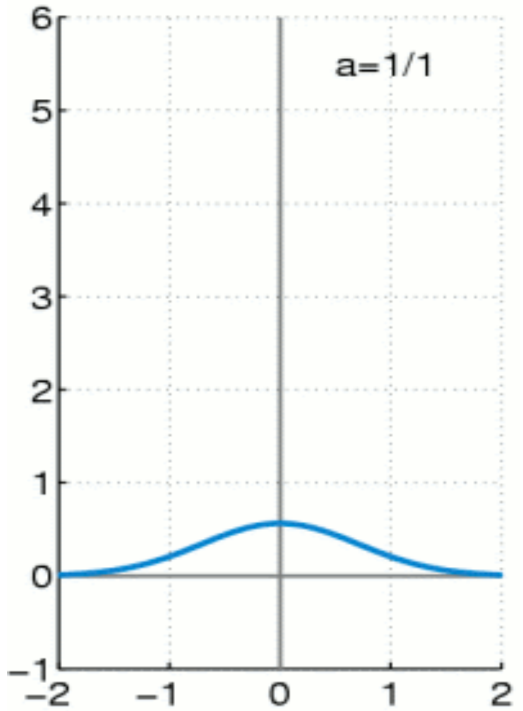
$$\int_a^b dt\, f(t)\, \delta(t-x) = \begin{cases} f(x) & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$\Longrightarrow \quad \delta(t) = 0 \ \text{ for } \ t \neq 0$$

$$\int_a^b dt\ \delta(t) = 1 \qquad\qquad \text{if} \ \ a < 0 < b$$

- Invented by Dirac in 1930, cleaned up and expanded by mathematicians in subsequent decades.
- Not really a function – e.g. series $\sum_{n=-\infty}^{\infty} e^{i\pi n(x-t)/L}$ doesn't even converge.
- Sometimes called a <u>distribution</u>.
- Properties are defined by the integral above; doesn't really make sense to talk about is unless it is inside an integral (although we do).

Delta Functions



- Can think of the delta function as a limit of a <u>delta sequence:</u>

  e.g. $\delta_a(x) = \frac{1}{\sqrt{\pi}a} e^{-x^2/a^2}$ as $a \to 0$

  or $\delta_a(x) = \begin{cases} \frac{1}{2a}, & |x| < a \\ 0, & |x| > a \end{cases}$

- With all these caveats, legitimate to treat the delta function like a function, so OK to say

$$\frac{1}{2L}\sum_{n=-\infty}^{\infty} e^{i\pi n(x-t)/L} = \delta(x-t)$$

- Nothing special about exponential/trigonometric Fourier series.
- True in general for <u>any</u> set of orthonormal eigenfunctions of a self-adjoint differential operator.

Delta Functions

- Suppose we have an orthonormal set $u_n(x)$.
- Completeness implies, for any f

$$f(x) = \sum_n a_n u_n(x)$$

where

$$a_n = \int dt\, w(t)\, f(t)\, u_n^*(t)$$

- Hence

$$f(x) = \int dt\;\, f(t)\, w(t) \sum_n u_n(x)\, u_n^*(t)$$

so, with all the caveats, we can say

$$w(t) \sum_n u_n(x)\, u_n^*(t) = \delta(x - t)$$

Fourier Transforms

- Wrote

$$f(x) = \int_{-L}^{L} dt\, f(t)\, \frac{1}{2L} \sum_{n=-\infty}^{\infty} e^{i\pi n(x-t)/L}$$

$$= \int_{-L}^{L} dt\, f(t)\, \frac{1}{2L} \sum_{n=-\infty}^{\infty} e^{i\left(\frac{\pi n}{L}\right)(x-t)}$$

- Ranges from $-\infty$ to ∞ as n varies, in steps of π/L.
- Set $\pi/L = \Delta\omega,\ \omega_n = n\Delta\omega$.

$$\frac{1}{2L} \sum_{n=-\infty}^{\infty} e^{i\left(\frac{\pi n}{L}\right)(x-t)} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \Delta\omega\, e^{i\omega_n(x-t)}$$

- Sum is a discrete approximation to $\frac{L}{\pi} \int_{-\infty}^{\infty} d\omega\, e^{i\omega(x-t)}$

- Now let $L \to \infty,\ \Delta\omega \to 0,\ \sum_n \Delta\omega \to \int d\omega$

$$\Longrightarrow \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\ f(t) \int_{-\infty}^{\infty} d\omega\, e^{i\omega(x-t)} \longrightarrow \text{Another } \delta \text{ function!}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{i\omega x} \int_{-\infty}^{\infty} dt\ f(t)\, e^{-i\omega t}$$

Fourier Transforms

- Found

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega x} \int_{-\infty}^{\infty} dt\;\, f(t)\, e^{-i\omega t}$$

- Defines Fourier transform and its inverse transform:

$$F(\omega) = \int_{-\infty}^{\infty} dt\;\, f(t)\, e^{-i\omega t}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\;\, F(\omega)\, e^{i\omega t}$$

Choose $\pm$ sign depending on context

- Points to note:

1. Often write $\tilde{f}(\omega)$ or even $\mathcal{F}f(\omega)$ in place of $F(\omega)$.
2. Signs of the exponents don't really matter, but must be opposite.
3. Who gets the 2π?

Fourier Transforms

- Found

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega x} \int_{-\infty}^{\infty} dt\; f(t)\, e^{-i\omega t}$$

- The 2π has to be there, but several schools of thought on where it goes

1. inverse: $F(\omega) = \int_{-\infty}^{\infty} dt\; f(t)\, e^{-i\omega t},\;\; f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\; F(\omega)\, e^{i\omega t}$
2. democratic: $F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\; f(t)\, e^{-i\omega t},\;\; f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\; F(\omega)\, e^{i\omega t}$
   book
3. engineering: $\omega = 2\pi f$

$$F(f) = \int_{-\infty}^{\infty} dt\; f(t)\, e^{-2\pi i f t},\;\; f(t) = \int_{-\infty}^{\infty} df\; F(f)\, e^{2\pi i f t}$$

- All equally valid – just be consistent!

Fourier Transforms

- Inverse transform is a continuous linear superposition of modes:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \;\; F(\omega)\, e^{-i\omega t}$$

- Extends to higher dimensions [function $f(\boldsymbol{x})$]:

$$F(\boldsymbol{k}) = \frac{1}{(2\pi)^{3/2}} \iiint d^3x\, f(\boldsymbol{x})\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \iiint d^3k\, F(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

Note the sign choice:
thinking of <u>waves</u> $e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega t}$

- Equivalent transforms exist for sine and cosine Fourier series, but are considerably less common.

Parseval's Theorem

- Plancherel's theorem: Let functions $f(t)$ and $F(\omega)$ be integrable on every finite interval, and suppose that

$$\int_{-\infty}^{\infty} dt\,|f(t)|^2 \quad \text{or} \quad \int_{-\infty}^{\infty} d\omega\,|F(\omega)|^2$$

is finite. Then, if either of the equations

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\ f(t)\, e^{-i\omega t}$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega$$

holds in the "mean square" sense, then so does the other and

$$\int_{-\infty}^{\infty} dt\,|f(t)|^2 = \int_{-\infty}^{\infty} d\omega\,|F(\omega)|^2$$

Parseval's theorem

total radiated energy over all time

total radiated energy over all frequencies

Parseval's Theorem

- Mean square convergence is similar to what we saw before:

$$f_A(t) = \int_{-A}^{A} d\omega\, F(\omega)\, e^{i\omega t}$$

then $$\lim_{A\to\infty} \int_{-A}^{A} [f(t) - f_A(t)]^2\, dt = 0$$

- Previously, in Fourier series, saw Bessel's inequality

$$\sum_{i=1}^{n} c_n^2 \leq \int_a^b f^2\, dx$$

and the Parseval identity for a complete set

$$\sum_{i=1}^{\infty} c_n^2 = \int_a^b f^2\, dx$$

- Parseval's theorem is Parseval's identity for a continuous transform.

Parseval's Theorem

- Proof is straightforward. Choose $f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \;\; F(\omega)\, e^{i\omega t}$, so

$$\int_{-\infty}^{\infty} dt\, f^*(t)\, g(t)$$

$$= \int_{-\infty}^{\infty} dt \;\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \;\; F^*(\omega)\, e^{-i\omega t} \;\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\alpha \;\; G(\alpha)\, e^{i\alpha t}$$

$$= \int_{-\infty}^{\infty} d\omega\, F^*(\omega) \int_{-\infty}^{\infty} d\alpha\, G(\alpha)\; \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} dt \;\; e^{-i(\omega-\alpha)t}}_{\delta(\omega-\alpha)}$$

$$= \int_{-\infty}^{\infty} d\omega\, F^*(\omega)\, G(\omega)$$

— inner product is preserved by a Fourier transform

- Set $f = g \;\; \Longrightarrow \;\;$ Parseval: $\qquad \int_{-\infty}^{\infty} dt\, |f(t)|^2 \; = \; \int_{-\infty}^{\infty} d\omega\, |F(\omega)|^2$

Fourier Transforms

- Before going on to examples, consider the principal use of Fourier transforms.
- Let

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\ f(t)\, e^{-i\omega t}$$

$$F_1(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\ f'(t)\, e^{-i\omega t}$$

$$= \cancel{\left[\frac{f(t)e^{-i\omega t}}{\sqrt{2\pi}}\right]_{-\infty}^{\infty}} + \frac{i\omega}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\ f(t)\, e^{-i\omega t}$$

$$= i\omega\, F(\omega)$$

- Assuming the boundary conditions cooperate, a Fourier transform

  <u>simplifies</u> the problem: $F_n(\omega) = (i\omega)^n\, F(\omega)$

  converts an ODE to an algebraic equation

  converts a PDE to an ODE

Fourier Transforms: Applications 1

- Wave equation

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}, \qquad y(x,0) = f(x)$$

- FT with respect to x (infinite domain assumed): $y(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Y(k,t)\, e^{-ikx}\, dk$

$$\Longrightarrow \frac{1}{c^2}\frac{\partial^2 Y}{\partial t^2} = (ik)^2\, Y(k,t)$$

$$\Longrightarrow \frac{1}{c^2}\frac{\partial^2 Y}{\partial t^2} = -k^2\, Y$$

$$\Longrightarrow \ddot{Y} + k^2 c^2 Y = 0, \qquad Y(k,0) = F(k)$$

PDE $\rightarrow$ ODE

$$\Longrightarrow Y(k,t) = F(k)\, e^{\pm ikct}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)\, e^{-ikX}\, dk = f(X)$$

- Transform back:

$$y(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)\, e^{-ikx \pm ikct}\, dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)\, e^{-ik(x \mp ct)}\, dk$$

$$= f(x \mp ct)$$

Fourier Transforms: Applications 2

- Simple harmonic oscillator

$$y'' + \lambda^2 y = 0$$

- Fourier transform, $y(x) \to Y(k)$

$\Longrightarrow \quad -k^2 Y + \lambda^2 Y = 0$

$\Longrightarrow \quad k^2 = \lambda^2$

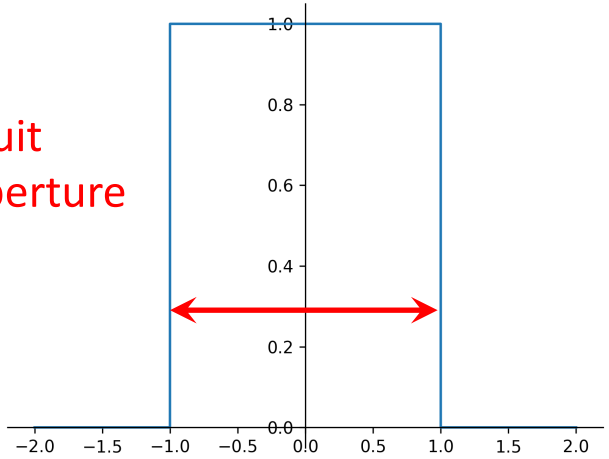
$\Longrightarrow \quad e^{\pm i\lambda x}$ solutions

Fourier Transforms Example 1: Square Pulse

- Pulse

$$f(t) = \begin{cases} 1, & |t| < a \\ 0, & |t| > a \end{cases}$$

e.g. rectangular pulse in circuit
uniformly illuminated aperture

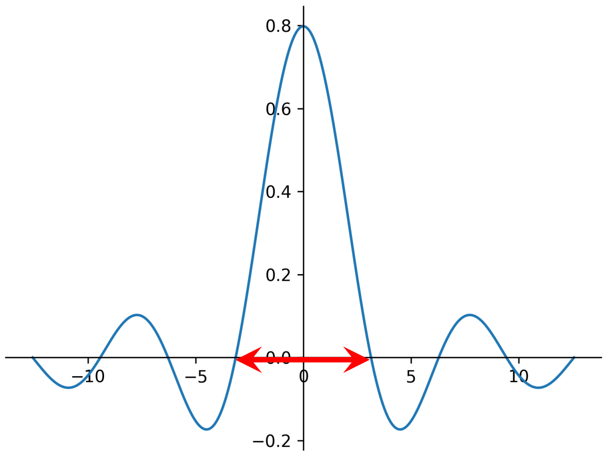


$$\Longrightarrow \quad F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} dt\ e^{-i\omega t}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\omega t}}{-i\omega} \right]_{-a}^{a}$$

$$e^{-i\omega a} - e^{i\omega a} = -2i \sin \omega a$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{\sin \omega a}{\omega}$$



- Note: width of $f(t)$ is a

  width of $F(\omega)$ is $2\pi/a$

- Inverse relation is a generic feature of transforms.

Fourier Transforms Example 1: Square Pulse

- Inverse transform

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \sqrt{\frac{2}{\pi}}\, \frac{\sin \omega a}{\omega}\, e^{i\omega t}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega\, \frac{\sin \omega a}{\omega}\, e^{i\omega t}$$

$$= ?$$