Energy Function

Consider a Lagrangian $L = L(q_i, \dot{q}_i, t)$

satisfies Lagrange's eqns $\dfrac{\partial L}{\partial q_i} = \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_i}\right)$

$$\frac{dL}{dt} = \frac{\partial L}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt} + \frac{\partial L}{\partial t}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt} + \frac{\partial L}{\partial t}$$

$$= \frac{d}{dt}\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial L}{\partial t}$$

$$\boxed{\frac{d}{dt}\Big(\underbrace{\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L}_{\equiv h}\Big) + \frac{\partial L}{\partial t} = 0} \qquad (1)$$

$\equiv h$ energy function

$$\boxed{\frac{dh}{dt} = -\frac{\partial L}{\partial t}}$$

$\Rightarrow$ if $L \neq L(t)$ then h is conserved

Although it is suggested by the name, 'h' coincides with the total energy only in particular circumstances.

From chapter 1 $T = T_0 + T_1 + T_2$

where $T_0 = T_0(q_i)$

$T_1 = T_1(q_i, \dot{q}_i)$ linear in 'q_i'

$T_2 = T_2(q_i, \dot{q}_i)$ quadratic in 'q_i'

In almost all cases of interest, 'L' can also be decomposed in the same way.

[certainly for forces derivable from potentials that are not dependent on 'v']

[an exception is 'L' for charged particles]

$$L(q,\dot{q},t) = L_0(q,t) + L_1(q,\dot{q},t) + L_2(q,\dot{q},t)$$

where L_1 is homogeneous function of 1st degree in $\dot{q}$

L_2 is homogeneous function of 2nd degree in $\dot{q}$

Homogeneous function: if 'f' is a homogeneous function of degree 'n' in the variable 'x_i', then

$$x_i \frac{\partial f}{\partial x_i} = n f$$

$$\text{e.g. } f(u) = u^3 : \quad u \frac{\partial f}{\partial u} = u \cdot 3u^2 = 3u^3 = 3f$$

$$\Rightarrow \quad n = 3$$

$$\dot{q} \frac{\partial L_0}{\partial \dot{q}} = 0 \cdot L_0 = 0 \qquad \dot{q} \frac{\partial L_2}{\partial \dot{q}} = 2 L_2$$

$$\dot{q} \frac{\partial L_1}{\partial \dot{q}} = 1 \cdot L_1 = L_1$$

$$h = \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

$$= L_1 + 2L_2 - L_0 - L_1 - L_2$$

$$= L_2 - L_0$$

if $\bar{r} \neq \bar{r}(t)$ then $T = T_2$

if $V \neq V(\dot{q}_i)$ then 'V' is only in the 'L_o'

$$L_2 = T_2 = T \quad \& \quad L_0 = \cancel{T_0} - V = -V$$

$$\boxed{h = T - (-V) = T + V = E}$$ total energy

Further, if $V \neq V(t) \Rightarrow L \neq L(t)$

$$\Rightarrow \frac{\partial h}{\partial t} = 0 \Rightarrow h \neq h(t)$$

NOTE:

1. Conditions for conservation of 'h' are different than conditions for 'h' = total energy

   Thus, 'h' can be conserved and *not* be the total energy.

2. 'L = T - V' is true independent of the choice of generalized coordinates, while 'h' depends on the specific set of generalized coordinates.

   For a given system, we can define different coordinates rendering different 'h'

3. For problems with $T \sim \frac{1}{2} m \dot{q}_i^2$ and $V = V(q_i)$

   $\Rightarrow h = E$ and is also conserved

Hamilton's Formulation.

Alternative formulation and statements of the previous theory (Lagrange's), but it does not carry new physics.

It is not even generally better than Lagrangian techniques on solving problems.

However, it provides for a framework for theoretical extensions in many areas of physics.

Hamilton
- Classical Perturbation theory
- Chaos
- Statiscal mechanics
- Quantum Mechanics

We will assume holonomic systems with monogenic forces (derived from a position-dependent potential and/or with a simple v-dependence).

Different point of view

Lagrangian view -

1. 'n' degrees of freedom that yield 'n' eqns of motion.
2. being 2nd order diff. eqns, we need '2n' initial conditions.
3. state of the system is a point in n-dimensional "configuration space".

Having 'n' - 'q_i's, the 'dot-q_i's are just a shorthand for their time derivatives.

$$q_i \qquad \dot{q}_i$$

Hamilton's view -

1. deals only with 1st order diff. eqns, but still need '2n' initial conditions.
2. thus, '2n' degress of freedom.
3. state of the system is a point in "phase space" in a 2n-dimensional space

Typically, one chooses half of the variables to be 'n' generalized coordinates 'q_i', and the other half using conjugate momenta 'p_i'

$$q_i \qquad\qquad p_i$$

$$p_i = \frac{\partial}{\partial \dot{q}_i} L(q_i, \dot{q}_i, t) \qquad \text{[no sum]}$$

the p's and q's are the "canonical variables".

From a mathematical point of view, the transition from a Lagrangian to a Hamiltonian formulation corresponds to change variables from

$$(q, \dot{q}, t) \longrightarrow (q, p, t)$$

The procedure to switch is provided by the "Legendre transformation"

Legendre transformation.

GOAL: change the set of independent variables by considering a related function

function 'f' $\quad f = f(x, y) \Rightarrow df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

$$\equiv u\,dx + v\,dy$$

$$u = \frac{\partial f}{\partial x} \qquad v = \frac{\partial f}{\partial y}$$

transform variables, $(x, y) \longrightarrow (u, y)$

define new function 'g' $g = f - ux$

$$\Rightarrow dg = df - u\,dx - x\,du = u\,\cancel{dx} + v\,dy - u\,\cancel{dx} - x\,du$$

$$= v\,dy - x\,du$$ done!

Lets apply this transformation to the Lagrangian

$$dL = \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt$$

using for the momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

and from Lagrange's eqns

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}(p_i) = \dot{p}_i$$

$$dL = \dot{p}_i\, dq_i + p_i\, d\dot{q}_i + \frac{\partial L}{\partial t} dt$$

Generate the Hamiltonian by the Legendre transformation

$$\boxed{H(q, p, t) = \dot{q}_i p_i - L(q, \dot{q}, t)} \qquad (2)$$

$$dH = \dot{q}_i\, dp_i + p_i\, d\dot{q}_i - dL$$

$$= \dot{q}_i\, dp_i + \cancel{p_i\, d\dot{q}_i} - \dot{p}_i\, dq_i - \cancel{p_i\, d\dot{q}_i} - \frac{\partial L}{\partial t} dt$$

$$= \dot{q}_i\, dp_i - \dot{p}_i\, dq_i - \frac{\partial L}{\partial t} dt$$

$$\boxed{\underbrace{\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}}_{\text{canonical eqns of Hamilton}} \qquad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}} \qquad (3)$$

NOTE: 'H' and 'h' (the energy function) are numerically equivalent, but they are functions of different variables.

All we need now is a method to construct such Hamiltonian 'H'

Formal procedure:

1. construct a Lagrangian $L(q, \dot{q}, t) = T - V$
2. find conjugate momenta $p_i = \partial L / \partial \dot{q}_i$
3. write down Hamiltonian

$$H = \dot{q} p - L$$

4. the 'q-dots' from #2 are solved in terms of the 'p's
5. 'H' is finally expressed as a function of only (q,p,t)

Algebra is greatly simplified if, as above, the Lagrangian is a sum of homogeneous terms,

$$H = \dot{q} p - L = \dot{q} p - [L_0 + L_1 + L_2]$$

$$= \dot{q} \frac{\partial L}{\partial \dot{q}} - [\quad \cdots \quad]$$

$$= L_1 + 2L_2 - L_0 - L_1 - L_2$$

$$= L_2 - L_0$$

that under the same assumptions as for the energy function above,

$$H = T - (-V)$$

$$\boxed{H = T + V = E}$$

Hamiltonian is the total energy

Special case (encompassing a large class of problems),

L_1 — linear function of generalized velocities

L_2 — quadratic function of generalized velocities if $\Rightarrow$ $L(q_i, \dot{q}_i, t) = L_0(q_j, t) + a_i(q_j, t)\,\dot{q}_i + T_i(q_j, t)\,\dot{q}_i^2$

define column matrices and matrix q, a, T

column matrices

square matrix

$$L = L_0(q, t) + \dot{q}^T a + \frac{1}{2}\dot{q}^T T \dot{q}$$

Hamiltonian

$$H = \dot{q}^T p - L$$

$$= \dot{q}^T p - L_0 - \dot{q}^T a - \frac{1}{2}\dot{q}^T T \dot{q}$$

$$= \dot{q}^T(p - a) - \frac{1}{2}\dot{q}^T T \dot{q} - L_0$$

from the generalized momenta

$$p = \frac{\partial L}{\partial \dot{q}} = T\dot{q} + a$$

using that T is symmetric

$$\Rightarrow \dot{q} = T^{-1}(p - a)$$

$$\dot{q}^T = (p^T - a^T)\,T^{-1}$$

substitute into the Hamiltonian

$$H = (p^T - a^T)T^{-1}(p - a) - \frac{1}{2}(p^T - a^T)T^{-1}\,T\,T^{-1}(p - a) - L_0$$

$$\boxed{H = \frac{1}{2}(p^T - a^T)\,T^{-1}(p - a) - L_0}$$

(4)