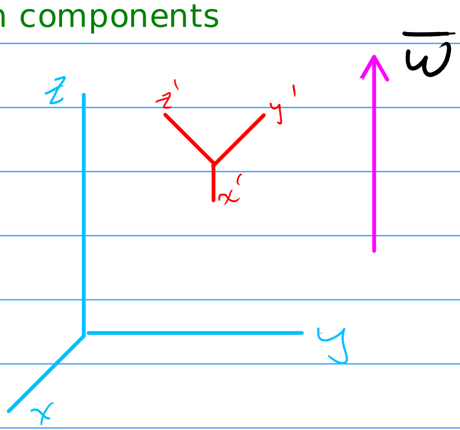
Expressing the angular velocity in terms of Cartesian components

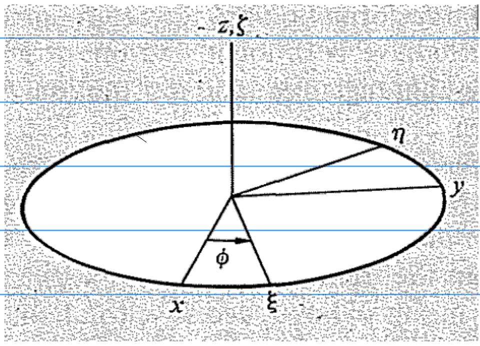
Time rate change of vectors on a rotating axis



$$\left(\frac{d}{dt}\right)_s = \left(\frac{d}{dt}\right)_r + \bar{\omega} \times$$

$$\bar{\omega} \rightarrow \omega_\varphi = \dot{\varphi} \qquad \omega_\psi = \dot{\psi}$$

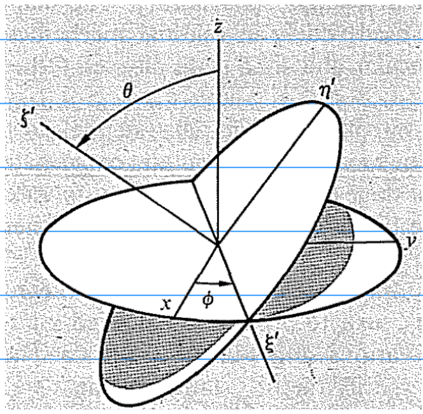
$$\omega_\theta = \dot{\theta}$$



φ: $\omega_\varphi^x = 0$

$\omega_\varphi^y = 0$

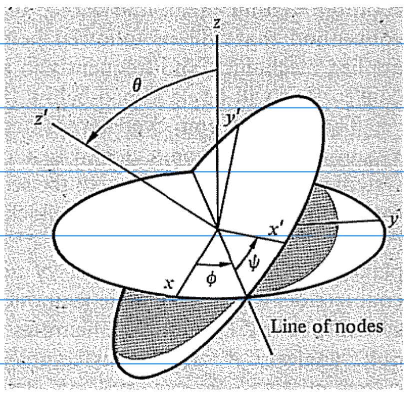
$\omega_\varphi^z = \dot{\varphi}$



θ: $\omega_\theta^{\xi'} = \dot{\theta}$

$\omega_\theta^{\eta'} = 0$

$\omega_\theta^{\zeta'} = 0$



ψ: $\omega_\psi^{x'} = 0$

$\omega_\psi^{y'} = 0$

$\omega_\psi^{z'} = \dot{\psi}$

$\psi: \quad \omega_\psi^{x'} = 0 \qquad \omega_\psi^{y'} = 0 \qquad \omega_\psi^{z'} = \dot{\psi}$

$$\theta: \begin{pmatrix} \omega_\theta^{x'} \\ \omega_\theta^{y'} \\ \omega_\theta^{z'} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{B} \begin{pmatrix} \omega_\theta^{\xi'} \\ \omega_\theta^{\eta'} \\ \omega_\theta^{\zeta'} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix}$$

$$\omega_\theta^{x'} = \dot{\theta}\cos\psi$$

$$\omega_\theta^{y'} = -\dot{\theta}\sin\psi$$

$$\omega_\theta^{z'} = 0$$

$$\varphi: \begin{pmatrix} \omega_\varphi^{x'} \\ \omega_\varphi^{y'} \\ \omega_\varphi^{z'} \end{pmatrix} = A \begin{pmatrix} 0 \\ 0 \\ \dot{\varphi} \end{pmatrix} \qquad A = BCD$$

$$\omega_\varphi^{x'} = \dot{\varphi}\sin\psi\sin\theta \qquad \omega_\varphi^{z'} = \dot{\varphi}\cos\theta$$

$$\omega_\varphi^{y'} = \dot{\varphi}\cos\psi\sin\theta$$

$$\boxed{\begin{aligned}
\omega_{x'} &= \dot{\varphi}\sin\psi\sin\theta + \dot{\theta}\cos\psi \\
\omega_{y'} &= \dot{\varphi}\cos\psi\sin\theta - \dot{\theta}\sin\psi \\
\omega_{z'} &= \dot{\varphi}\cos\theta + \dot{\psi}
\end{aligned}}$$