1 Mechanical Models

1.1 Two-Process Model

We take our discussion and notation from Skeldon et al. (2014).

There is a baseline circadian cycle, which we will denote y_t. This is a weighted sum of sine waves of various frequencies and will fluctuate between -1 and +1. There are upper and lower bounds related to this cycle. We denote these H_t^u and H_t^l.

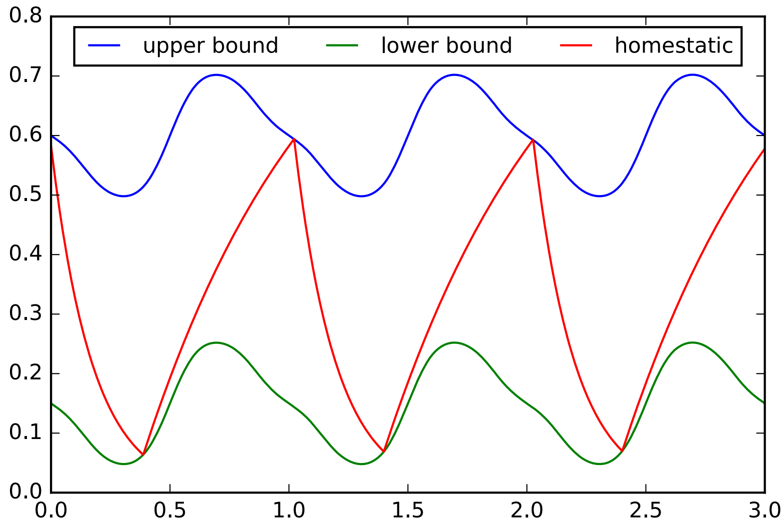
$$\begin{aligned} H_{t+1}^u &= \bar{H}_t^u + a y_t \\ H_{t+1}^l &= \bar{H}_t^l - a y_t \\ \bar{H}^u &> \bar{H}^l \end{aligned} \tag{1.1}$$

The homeostatic process which determines sleep versus waking is denoted H_t. If waking it rises toward an upper asymptote of μ and if sleeping it falls toward a lower asymptote of zero.

$$H_{t+1} = \begin{cases} H_t e^{-1/\chi_S} & \text{if sleeping} \\ \mu + (H_t - \mu) e^{-1/\chi_W} & \text{if waking} \end{cases} \tag{1.2}$$

If the individual is awake and H_t is rising over time it will eventually it the upper bound of H_t^u at this point the individual begins sleep and H_t falls until it eventually hits the lower bound, and the individual wakes up. This is illustrated in Figure 1.

Figure 1: Two-Process Model



1.2 Utility Model

In our model we use the same circadian cycle, y_t. Utility is the difference between this cycle and and index of sleep stock we denote d_t. This corresponds roughly to the negative of the homeostatic process in the previous model. We model d_t as follows.

$$Z_{t+1} = \begin{cases} Z_t(1-\delta)+1 & \text{if sleeping} \\ Z_t(1-\delta) & \text{if waking} \end{cases} \tag{1.3}$$
$$d_t = f(\phi Z_t + \xi);\ f'() > 0,\ -1 \leq d_t \leq 1$$

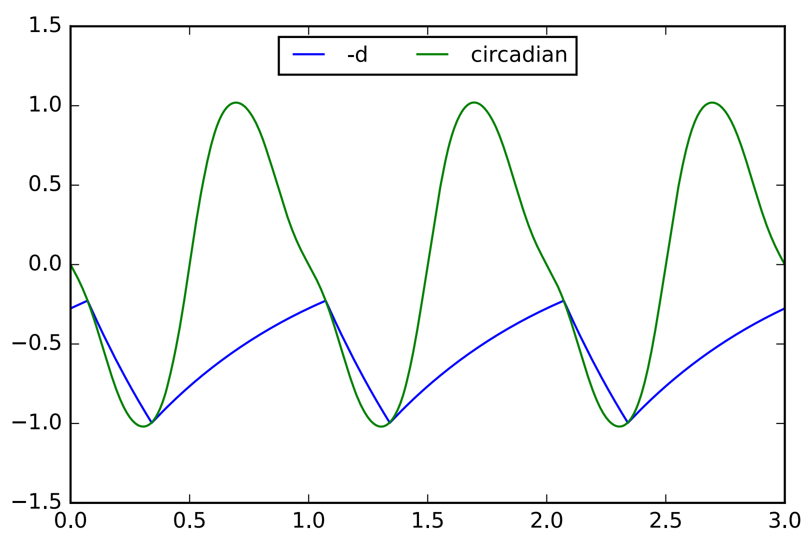
where Z_t is the stock of sleep

Utility is given by:

$$U_t = \begin{cases} -(y_t + d_t) & \text{if sleeping} \\ y_t + d_t & \text{if waking} \end{cases} \tag{1.4}$$

If $y_t = -d_t$ the individual is indifferent between sleeping and waking. If $-d_t < y_t$ the individual will be awake and d_t will be falling or $-d_t$ is rising. Eventually $-d_t$ crosses the circadian cycle and the individual chooses to sleep. From here on $-d_t$ falls until it crosses the cycle again and the individual wakes up. his is illustrated in Figure 2.

Figure 2: Utility Model



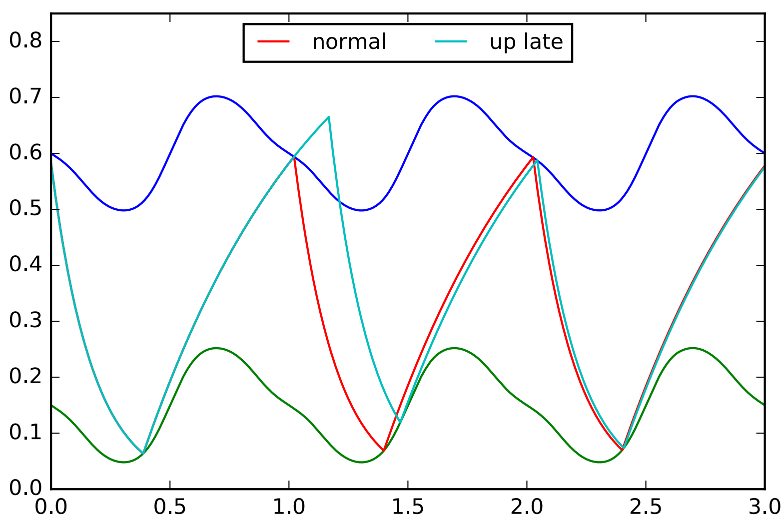
2 Predictive Differences

2.1 Sleep Deprivation

Imagine a case where an individual stays up beyond the optimal sleep time during one cycle How do the subsequent waking and sleep times respond to this shock in these two models?

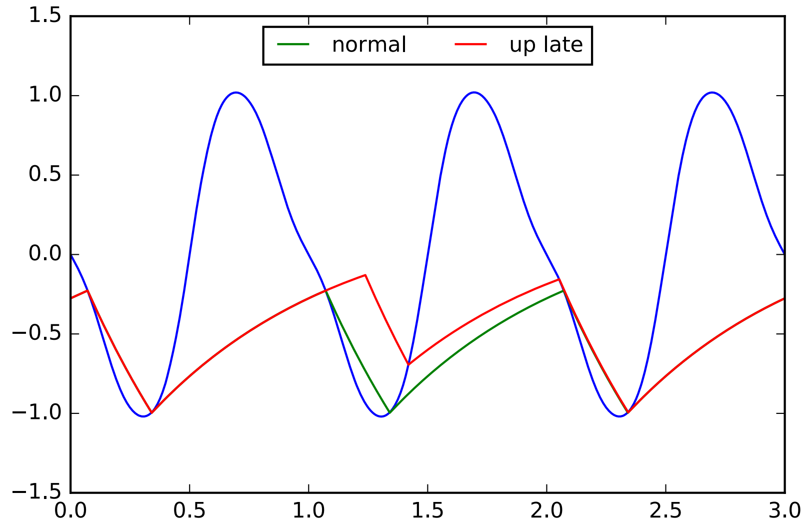
In the two-process model, H_t continues to rise as they individual stays up late. When they finally go to bed, H_t begins to fall, but will still be above the optimal path. Hence the individual wakes up later the next day. This leads to a lower level of H_t during the next day relative to the optimal path, and the individual consequently goes to sleep later than usual the next night. (But still earlier than the night before.)

Figure 3: Sleep Deprivation in the Two-Process Model



In the utility model, $-d_t$ continues to rise as they individual stays up late. When they finally go to bed, $-d_t$ begins to fall, but will still be above the optimal path. Hence the individual wakes up later the next day, just as with the two-process model. However, this leads to a higher level of $-d_t$ during the next day relative to the optimal path, and the individual consequently goes to sleep *earlier* than usual the next night.

Figure 4: Sleep Deprivation in the Utility Model

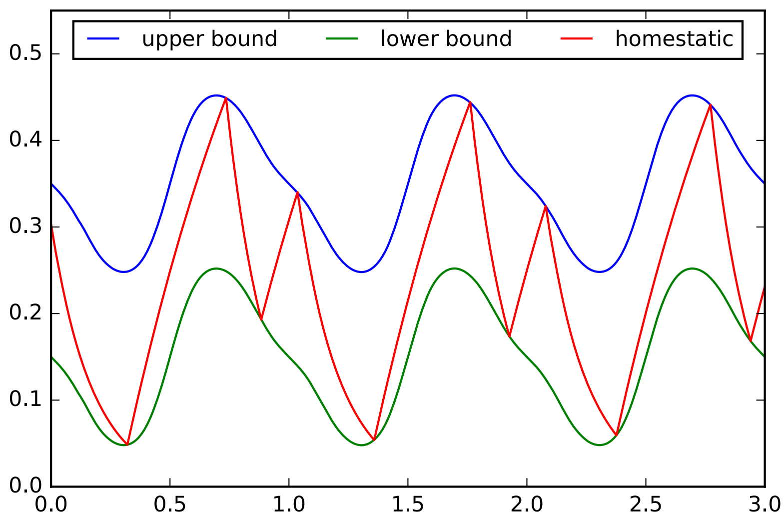


2.2 Polyphasic Sleep

Can individuals take naps during the day or be awake for extended periods in the night?

In the two-process model this is possible and the lengths of these out-of-cycle periods of sleep or wakefulness can be of short or long duration. This is illustrated in Figure 5 where the dashed line shows a cycle with a regular nap each day.

Figure 5: Polyphasic Sleep in the Two-Process Model



This is more problematic in the utility model. As shown in Figure 6 with the dashed line. It is possible with rapid deterioration of the sleep stock to have the individual fall asleep in the afternoon. However if recovery of sleep stock is slow the does not result in a nap, but in sleeping until the next day. If recovery of sleep stock is rapid, however, the individual will sleep for a short instant and then wake again, only to rapidly lose the regained stock of sleep and sleep again. This leads to a 'fuzzy' nap experience that is neither prolonged sleep or prolonged wakefulness.

Figure 7 illustrates this by showing an expanded portion of 6. When $-d$ crosses y the individual sleeps. If this will lead to a rise in the sleep stock and hence a drop in $-d$. The resulting path must lie between the vectors V_1 and V_2. If the sleep stock replenishes slowly this will lead to a path closer to V_1. As long as the slope is flatter than that of y, the individual will continue sleeping. However, if the sleep stock replenishes sufficiently quickly $-d$ will drop faster than y and the individual will immediately wake up. The only possible path is where $-d$ follows y and the individual is instantaneously switching between sleep and being awake. This is further illustrated by Figure 8 which shows the 'fuzzy' nature of the stock of sleep over time. The saw-toothed pattern is due to the discrete nature of this simulation where a time period corresponds to 15 minutes.

Figure 6: Polyphasic Sleep in the Utility Model

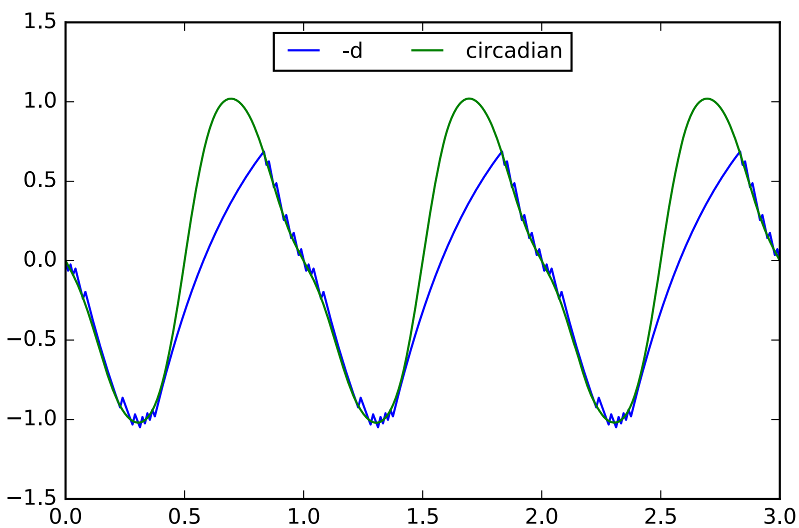


Figure 7: Enlarged Portion of Figure 6

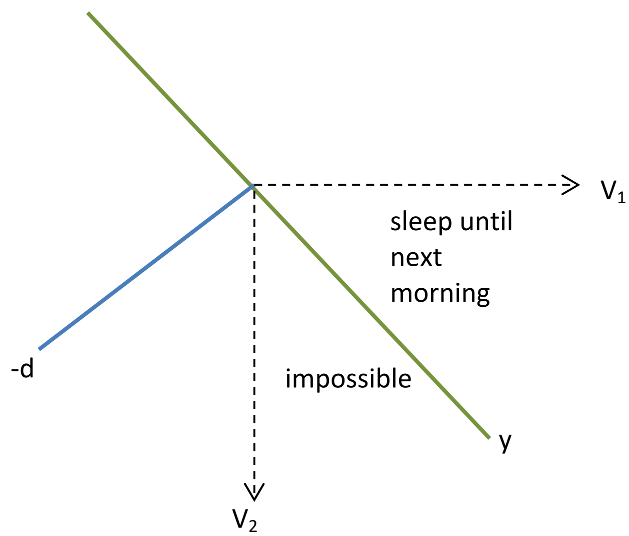
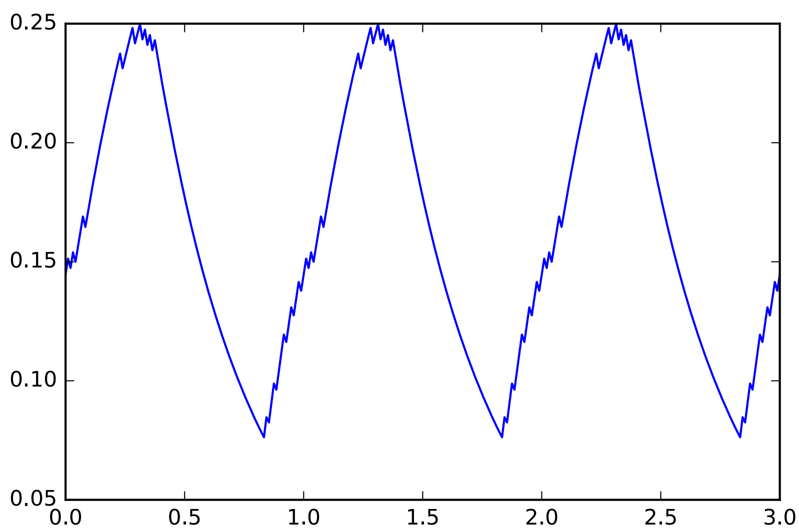


Figure 8: Stock of Sleep for Simulation in Figure 6



APPENDICES

A1 Using a Polynomial for the Circadian Cycle

Write the circadian cycle as a polynomial.

$$\begin{aligned} y(t) &= y_0 + y_1 t + y_2 t^2 + \cdots + y_N t^N \\ &= \sum_{n=1}^{N} y_n t^n \end{aligned} \tag{A.1}$$

The utility of being awake from time a_t to s_t will be given by:

$$\begin{aligned} v_w &= \int_{a_t}^{s_t} \left(\sum_{n=1}^{N} y_n t^n + d_t \right) dt \\ &= \sum_{n=1}^{N} y_n \tfrac{1}{n+1} t^{n+1} + d_t \Big|_{a_t}^{s_t} \\ &= \sum_{n=1}^{N} y_n \tfrac{1}{n+1} (s_t^{n+1} - a_t^{n+1}) + d_t (s_t - a_t) \end{aligned} \tag{A.2}$$

The utility of being asleep from time s_{t-1} to a_t is the same as that from

$s_t - 24$ to a_t:

$$
\begin{aligned}
v_s &= \int_0^{a_t} \left(\sum_{n=1}^{N} -y_n t^n - d_t \right) dt + \int_{s_t}^{24} \left(\sum_{n=1}^{N} -y_n t^n - d_t \right) dt \\
&= -\sum_{n=1}^{N} y_n \tfrac{1}{n+1} t^{n+1} + t d_t \Big|_0^{a_t} - \sum_{n=1}^{N} y_n \tfrac{1}{n+1} t^{n+1} + t d_t \Big|_{s_t}^{24} \\
&= -\sum_{n=1}^{N} y_n \tfrac{1}{n+1} (a_t^{n+1} - 0) - d_t(a_t - 0) - \sum_{n=1}^{N} y_n \tfrac{1}{n+1} (24^{n+1} - s_t^{n+1}) - d_t(24 - s_t) \\
&= \sum_{n=1}^{N} y_n \tfrac{1}{n+1} (s_t^{n+1} - a_t^{n+1} - 24^{n+1}) + d_t(s_t - a_t - 24)
\end{aligned}
\tag{A.3}
$$

Marginal utilities are:

$$
u_a = -2 \left(\sum_{n=1}^{N} y_n a_t^n + d_t \right) \tag{A.4}
$$

$$
u_s = 2 \left(\sum_{n=1}^{N} y_n s_t^n + d_t \right) \tag{A.5}
$$

A Taylor-series approximation of $-\sin([t + b]\frac{\pi}{12})$ about a central point of

$t = 12$ is:

$$
\begin{aligned}
-\sin([t+b]\tfrac{\pi}{12}) = &-\sin([12+b]\tfrac{\pi}{12}) \\
&- \tfrac{\pi}{12}\cos([12+b]\tfrac{\pi}{12})(t-12) \\
&+ \tfrac{1}{2}(\tfrac{\pi}{12})^2\sin([12+b]\tfrac{\pi}{12})(t-12)^2 \\
&+ \tfrac{1}{3!}(\tfrac{\pi}{12})^3\cos([12+b]\tfrac{\pi}{12})(t-12)^3 \\
&- \tfrac{1}{4!}(\tfrac{\pi}{12})^4\sin([12+b]\tfrac{\pi}{12})(t-12)^4 \\
&+ \ldots
\end{aligned}
\tag{A.6}
$$

References

Skeldon, Anne C., Derk-Jan Dijk, and Gianne Derks, "Mathematical Models for Sleep-Wake Dynamics: Comparison of the Two-Process Model and a Mutual Inhibition Neuronal Model," *PLOS ONE*, 2014, *9* (0), 1–16.