1 Simple Model with Term Structure

1.1 Households

Bonds exist with maturities from 1 to I periods into the future. The price in period t of a bond delivering 1 unit of real consumption in period $t+i$ is denoted q_{it}. The quantity of bonds held in period t, coming due in period $t+i$ is denoted b_{it}. Bonds are contracts structured so that the bond pays one unit of the final output good when the maturity goes to zero, i.e. $q_{it} = 1$.

Households solve the following dynamic program:

$$
\begin{aligned}
V(k_{t-1}, \{b_{i-1,t-1}\}_{i=1}^{I}; \Theta_t) = &\max_{x_t, \{b^{i,t+1}\}_{i=1}^{I}, l_t} [c_t^\gamma + \psi(1-l_t)^\gamma]^{\frac{\phi}{\gamma}} \\
&+ \beta E\left\{V(k_t, \{b_{it}\}_{i=1}^{I}; \Theta_{t+1})\right\}
\end{aligned}
$$

$$
c_t = w_t l_t + (1 + r_t - \delta)k_{t-1} + \sum_{i=1}^{I} q_{it} b_{i-1,t-1} - k_t - \sum_{i=1}^{I} q_{it} b_{it} \tag{1.1}
$$

where Θ_t is a set of variables known to the household.

There is one Euler equation which implicitly defines k_t, another Euler equation which implicitly defines l_t, and there are I Euler equations which implicitly define the $\{b_{it}\}_{i=1}^{I}$.

$$
u_{ct} = \beta E\left\{u_{c,t+1}(1 + r_{t+1} - \delta)\right\} \tag{1.2}
$$

$$
u_{ct} w_t = u_{lt} \tag{1.3}
$$

$$
u_{ct} q_{it} = \beta E\left\{u_{c,t+1} q_{i-1,b+1}\right\};\ \forall i \tag{1.4}
$$

$$
u_{ct} \equiv \phi[c_t^\gamma + \psi(1-l_t)^\gamma]^{\frac{\phi-\gamma}{\gamma}} c_t^{\gamma-1}
$$

$$
u_{lt} \equiv \phi[c_t^\gamma + \psi(1-l_t)^\gamma]^{\frac{\phi-\gamma}{\gamma}} \psi(1-l_t)^{\gamma-1}
$$

1.2 Firms

$$Y_t = K_{t-1}^{\alpha}(e^{z_t}L_t)^{1-\alpha} \tag{1.5}$$

$$w_t = (1-\alpha)\frac{Y_t}{L_t} \tag{1.6}$$

$$r_t = \alpha\frac{Y_t}{K_{t-1}} \tag{1.7}$$

1.3 Market Clearing

$$K_{t-1} = k_{t-1} \tag{1.8}$$

$$L_t = l_t \tag{1.9}$$

Markets for bonds of all maturities must clear each period. Demand for bonds of maturity i are denoted b_{it} above. Supply of bonds by the government will be denoted B_{it}. Narket clearing gives:

$$b_{it} = B_{it};\ \forall i \tag{1.10}$$

1.4 Exogenous Laws of Motion

$$z_{t+1} = \rho z_t + \varepsilon_t;\ \varepsilon_t \sim iid(0,\sigma^2) \tag{1.11}$$

The laws of motion for the supplies of bonds are as follows. Each period the government issues a new set of assets of maturities 1 through I, we denote these as B_{it}^N Assets of maturity 0 are retired. This gives:

$$B_{It} = B_{It}^N \tag{1.12}$$

$$B_{it} = B_{i+1,t-1} + B_{it}^N;\ \text{for } 1 \le i \le I-1 \tag{1.13}$$

If the longest maturity is 6 periods and if the government only ever issues bonds of maturites 1 and 6, then we have $I = 6$ and $B^N_{it} = 0$ if $1 < i < 6$. The remaining B^N_{1t} and B^N_{6t} will follow some exogenous law of motion that refelcts government policy.

1.5 Additional Definitions

Once we know the exogenous behavior of the supplies from above, the I Euler equations from equation (1.4) now impliicty define the market clearing prices, $\{q_{it}\}_{i=1}^{I}$, as functions of the state variables. Hence, like l_t, bond prices are "jump" variables.

The yield curve can be derived from the bond prices by calculating yield-to-maturity for simple bonds.

$$\begin{aligned} q_{it} &= \left(\frac{1}{1+r_{it}}\right)^{i} \\ r_{it} &= q_{it}^{-\frac{1}{i}} - 1 \end{aligned} \tag{1.14}$$

Investment is given by:

$$I_t = K_t - (1-\delta)K_{t-1} \tag{1.15}$$

1.6 Summary

The set of endogenous state variables is $\mathbf{X}_t = (K_t, z_t, \{B_{it}\}_{i=1}^{I})$.

The set of implicitly defined "jump" variables is $\mathbf{Y}_t = (L_t, \{q_{it}\}_{i=1}^{I})$.

The set of exogenous state variables is $\mathbf{Z}_t = (\varepsilon_t, \{B^N_{it}\}_{i=1}^{I})$.

The set of explicitly defined endogenous variables is $\mathbf{D}_t = (Y_t, I_t, w_t, r_t, c_t, \{r_{it}\}_{i=1}^{I})$.

Parameters are $(\gamma, \phi, \psi, \beta, \delta, \alpha, \rho, \sigma)$.

The characterizing equations are:

$$z_{t+1} = \rho z_t + \varepsilon_t;\ \varepsilon_t \sim iid(0, \sigma^2) \tag{1.16}$$

$$B_{It} = B_{It}^N \tag{1.17}$$

$$B_{it} = B_{i+1,t-1} + B_{it}^N;\ \text{for } 1 \leq i \leq I-1 \tag{1.18}$$

$$I_t = K_t - (1-\delta)K_{t-1} \tag{1.19}$$

$$r_{it} = q_{it}^{-\frac{1}{i}} - 1 \tag{1.20}$$

$$Y_t = K_{t-1}^{\alpha}(e^{z_t}L_t)^{1-\alpha} \tag{1.21}$$

$$w_t = (1-\alpha)\frac{Y_t}{L_t} \tag{1.22}$$

$$r_t = \alpha\frac{Y_t}{K_{t-1}} \tag{1.23}$$

$$c_t = w_t l_t + (1 + r_t - \delta)k_{t-1} + \sum_{i=1}^{I} q_{it} b_{i-1,t-1} - k_t - \sum_{i=1}^{I} q_{it} b_{it} \tag{1.24}$$

$$u_{ct} \equiv \phi[c_t^{\gamma} + \psi(1-l_t)^{\gamma}]^{\frac{\phi-\gamma}{\gamma}} c_t^{\gamma-1} \tag{1.25}$$

$$c_t^{\gamma-1} w_t = \psi(1-l_t)^{\gamma-1} \tag{1.26}$$

$$u_{ct} q_{it} = \beta E\left\{u_{c,t+1} q_{i-1,b+1}\right\};\ \forall i \tag{1.27}$$

$$u_{ct} = \beta E\left\{u_{c,t+1}(1 + r_{t+1} - \delta)\right\} \tag{1.28}$$

2 Further Enhancements to the Model

Add taxes and transfers

Add money

Add intermediate goods with monopolistic competition

Add sticky nominal prices for intermediate goods

Add capital adjustment costs