ASYMPTOTIC STATIONARITY AND REGULARITY FOR NONSMOOTH OPTIMIZATION PROBLEMS

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Abstract Based on the tools of limiting variational analysis, we derive a sequential necessary optimality condition for nonsmooth mathematical programs which holds without any additional assumptions. In order to ensure that stationary points in this new sense are already Mordukhovich-stationary, the presence of a constraint qualification which we call AM-regularity is necessary. We investigate the relationship between AM-regularity and other constraint qualifications from nonsmooth optimization like metric (sub-)regularity of the underlying feasibility mapping. Our findings are applied to optimization problems with geometric and, particularly, disjunctive constraints. This way, it is shown that AM-regularity recovers recently introduced cone-continuity-type constraint qualifications, sometimes referred to as AKKT-regularity, from standard nonlinear and complementarity-constrained optimization. Finally, we discuss some consequences of AM-regularity for the limiting variational calculus.

Keywords: Asymptotic regularity, Asymptotic stationarity, Constraint qualifications, M-stationarity, Nonsmooth optimization, Variational analysis

MSC (2010): 49J52, 49J53, 90C30, 90C33

1 INTRODUCTION

Due to their inherent practical relevance in the context of solution algorithms for optimization problems, sequential necessary optimality conditions and constraint qualifications became quite popular during the last decade. A suitable theory has been developed in the context of standard nonlinear programming, see e.g. [1, 2, 5, 6, 7], complementarity-constrained programming, see [3, 35], and nonlinear semidefinite programming, see [4]. Recently, these concepts were generalized to optimization problems in Banach spaces in [14]. The main idea behind the concept is that even when a local minimizer of a given optimization problem is not stationary in classical sense (e.g., a Karush–Kuhn–Tucker point in standard nonlinear programming), it might be *asymptotically* stationary along a sequence of points converging to the point of interest without any constraint qualification. Now, the question arises which type of qualification condition is necessary in order to guarantee that an asymptotically stationary point is already stationary. This indeed leads to the concept of sequential constraint qualifications. It has been reported in [1, 3, 5, 14, 35] that such sequential constraint qualifications are comparatively weak in comparison with classical qualification conditions which makes them particularly interesting.

It is a nearby guess that *sequential* stationarity and regularity might be concepts which are quite compatible with the popular tools of limiting variational analysis, see e.g. [31, 32, 38] and the references therein. Indeed, this has been worked out for mathematical problems with complementarity constraints and the associated concept of Mordukhovich-stationarity (M-stationarity for short) in the recent

The author would like to thank Gerd Wachsmuth for pointing out the simple projection argument in the proof of Theorem 3.8.

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paper [35]. However, the ideas obviously will work for other classes of disjunctive programming like mathematical programs with vanishing, switching, or cardinality constraints as well. It is the purpose of this paper to show that the underlying concepts can be further generalized to a quite abstract class of optimization problems which covers not only all the aforementioned settings but also conic as well as cone-complementarity-constrained optimization problems and other mathematical programs with equilibrium constraints which model amongst others that the feasible points need to solve underlying (quasi-) variational inequalities. Thus, the theory is likely to possess some extensions to bilevel programming as well.

In this paper, let us consider the mathematical program

$$\text{(P)} \qquad \begin{aligned} f(x) &\to \min \\ 0 &\in \Phi(x) \end{aligned}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz continuous function and $\Phi\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a set-valued mapping whose graph is closed. Throughout the paper, let $M := \{x \in \mathbb{R}^n \,|\, 0 \in \Phi(x)\}$ denote the feasible set of (P). We assume that this set is nonempty. Let us point out that the theory of this paper stays correct whenever $\mathbb{R}^n$ and $\mathbb{R}^m$ from above are replaced by finite-dimensional Banach spaces X and Y. Particularly, the results of this manuscript extend to instances of nonlinear semidefinite programming comprising optimization problems with semidefinite cone complementarity constraints. Problems of the general form (P) have been considered in e.g. [18], [31, Section 5.2.3], or [40, Section 3]. In all these contributions, it has been pointed out that whenever $\bar{x} \in M$ is a local minimizer of (P) such that the mapping Φ enjoys the so-called *metric subregularity* property at $(\bar{x}, 0)$, see Section 2.2 for a definition and additional references to the literature, then $\bar{x}$ is indeed an M-stationary point of this problem. An easy approach to verify the presence of metric subregularity is given by checking validity of the stronger *metric regularity* property since the latter can be carried out with the aid of the so-called Mordukhovich criterion which is stated in terms of the limiting coderivative of Φ, see Section 2.3 for details. The latter, however, might be too restrictive which is why several weaker sufficient conditions for metric subregularity have been worked out in particular problem settings during the last years, see e.g. [9, 10, 21, 23, 25, 26] and the references therein.

As we will see, the sequential approach to necessary optimality conditions and constraint qualifications for (P) leads to a new regularity concept that we call *asymptotic* Mordukhovich-regularity (AM-regularity for short). The latter is weaker than metric regularity of Φ and not related to the metric subregularity of this map, see Examples 3.12 and 3.13. It, thus, puts some other light onto the previously known landscape of constraint qualifications which address (P). Furthermore, we will demonstrate that this new regularity concept ensures validity of fundamental calculus rules from limiting variational analysis like the pre-image and the intersection rule, see Theorem 3.14 and Section 5.3. Besides, we show how AM-regularity specifies in exemplary problem settings. It will turn out that it covers several sequential constraint qualifications from the literature. Throughout the manuscript, simple examples and counterexamples visualize applicability and limits of the obtained theory.

The paper is organized as follows: In Section 2, we present the notation exploited in this manuscript. Furthermore, we review the necessary essentials of set-valued and variational analysis. Section 3 is dedicated to the introduction of the asymptotic stationarity and regularity concepts of our interest. We first derive a sequential necessary optimality condition of M-stationary-type in Section 3.1 via a simple penalization argument. Based on that, we introduce the concept of AM-regularity in Section 3.2 and study its theoretical properties as well as its relationship to other constraint qualifications. In Section 4, we investigate the particular situation where the map Φ can be split in two parts where one is, again, modelled with the aid of an abstract set-valued mapping while the other one just describes that the variables need to belong to an abstract constraint set $C \subset \mathbb{R}^n$ which, in practice, can be imagined as a set of simple variational structure. We show that whenever the set-valued part of Φ possesses the so-called *Aubin property* than a weaker constraint qualification than AM-regularity is sufficient for M-stationarity of local minimizers. We discuss some applications of our results in Section 5. First, we apply the concept of AM-regularity to the broad class of mathematical problems with so-called geometric constraints in Section 5.1. The even more special class of disjunctive optimization problems, where the definition of AM-regularity can be essentially simplified, is inspected in Section 5.2. In this context, a comparison to sequential constraint qualifications from the literature will be provided. Third, we discuss some consequences of AM-regularity for the limiting variational calculus in Section 5.3. We close the paper with some concluding remarks in Section 6.

2 NOTATION AND PRELIMINARIES

2.1 BASIC NOTATION

Throughout the manuscript, we equip $\mathbb{R}^n$ with the Euclidean norm $\|\cdot\|$. For some point $x \in \mathbb{R}^n$ and a scalar $\varepsilon > 0$, $\mathbb{B}_\varepsilon(x) := \{y \in \mathbb{R}^n \,|\, \|y - x\| \le \varepsilon\}$ represents the closed ball around x of radius ε. For brevity, we exploit $\mathbb{B} := \mathbb{B}_1(0)$. Let $A \subset \mathbb{R}^n$ be a nonempty set. We use

$$\operatorname{dist}(x, A) := \inf_{y \in A} \|y - x\| \qquad \Pi(x, A) := \operatorname*{argmin}_{y \in A} \|y - x\|$$

in order to denote the distance of x to A and the associated set of projections. For brevity, we make use of $A + x = x + A := \{x + y \in \mathbb{R}^n \,|\, y \in A\}$. The set

$$A^\circ := \{z \in \mathbb{R}^n \,|\, \forall y \in A\colon\; y^\top z \le 0\}$$

is referred to as the polar cone of A. It is a nonempty, closed, convex cone. The derivative of a differentiable function $F\colon \mathbb{R}^n \to \mathbb{R}^m$ at x will be represented by $F'(x) \in \mathbb{R}^{m\times n}$ while, in case $m = 1$, we use $\nabla F(x) \in \mathbb{R}^n$ to denote its gradient at x.

2.2 PROPERTIES OF SET-VALUED MAPPINGS

Let $\Upsilon\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping. We exploit

$$\begin{aligned} \operatorname{dom}\Upsilon &:= \{x \in \mathbb{R}^n \,|\, \Upsilon(x) \neq \emptyset\} \\ \operatorname{gph}\Upsilon &:= \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \,|\, y \in \Upsilon(x)\} \\ \ker\Upsilon &:= \{x \in \mathbb{R}^n \,|\, 0 \in \Upsilon(x)\} \end{aligned}$$

in order to represent the domain, the graph, and the kernel of Υ. Frequently, we will make use of the so-called sequential outer *Painlevé–Kuratowski limit* of Υ at some point of interest $\bar{x} \in \operatorname{dom}\Upsilon$ given by

$$\limsup_{x \to \bar{x}} \Upsilon(x) := \left\{ y \in \mathbb{R}^m \,\middle|\, \begin{array}{l} \exists \{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n\, \exists \{y_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^m\colon \\ \qquad x_k \to \bar{x},\; y_k \to y,\; y_k \in \Upsilon(x_k)\,\forall k \in \mathbb{N} \end{array} \right\}.$$

For some closed set $A \subset \mathbb{R}^n$, we exploit the so-called indicator map $\Delta_A\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ given by

$$\forall x \in \mathbb{R}^n\colon \qquad \Delta_A(x) := \begin{cases} \{0\} & x \in A \\ \emptyset & x \notin A \end{cases}$$

where the dimension of the image space will be clear from the context.

In this manuscript, we will often deal with Lipschitzian properties of set-valued mappings. Recall that Υ is said to be metrically regular at some point $(\bar{x}, \bar{y}) \in \operatorname{gph}\Upsilon$ whenever there are neighbourhoods $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ of $\bar{x}$ and $\bar{y}$, respectively, and some constant $\kappa > 0$ such that

$$\forall x \in U\, \forall y \in V\colon \qquad \operatorname{dist}(x, \Upsilon^{-1}(y)) \le \kappa \operatorname{dist}(y, \Upsilon(x))$$

holds. Above, $\Upsilon^{-1}\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is the so-called inverse set-valued mapping associated with Υ given by $\Upsilon^{-1}(y) := \{x \in \mathbb{R}^n \,|\, y \in \Upsilon(x)\}$ for all $y \in \mathbb{R}^m$. Fixing $y := \bar{y}$ in the definition of metric regularity, we obtain the notion of metric subregularity, i.e., Υ is said to be metrically subregular at $(\bar{x}, \bar{y})$ if there are a neighbourhood $U \subset \mathbb{R}^n$ of $\bar{x}$ and a constant $\kappa > 0$ such that

$$\forall x \in U\colon \qquad \operatorname{dist}(x, \Upsilon^{-1}(\bar{y})) \le \kappa \operatorname{dist}(\bar{y}, \Upsilon(x))$$

is valid. We refer to κ as the modulus of metric regularity and metric subregularity, respectively. Let us recall that Υ is said to possess the Aubin property at $(\bar{x}, \bar{y})$ if there are neighbourhoods $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ of $\bar{x}$ and $\bar{y}$, respectively, as well as a constant $\kappa > 0$ such that the following estimate is valid:

$$\forall x, x' \in U\colon \quad \Upsilon(x) \cap V \subset \Upsilon(x') + \kappa \, \|x - x'\| \, \mathbb{B}. \tag{2.1}$$

It is well known that Υ possesses the Aubin property at $(\bar{x}, \bar{y})$ if and only if Υ^{-1} is metrically regular at $(\bar{y}, \bar{x})$. Fixing $x' := \bar{x}$ in the definition of the Aubin property yields the definition of calmness of Υ at $(\bar{x}, \bar{y})$. The latter is equivalent to metric subregularity of Υ^{-1} at $(\bar{y}, \bar{x})$. We refer the interested reader to [27, 29, 31, 38] for an overview of the theory and applications of metric regularity and the Aubin property. Background information about metric subregularity and calmness can be found in [10, 16, 18, 21, 25, 26, 28]. We would like to mention that polyhedral set-valued mappings, i.e., set-valued mappings whose graph can be represented as the union of finitely many convex polyhedral sets, are calm at each point of their graphs, see [36, Proposition 1]. Noting that the inverse of a polyhedral set-valued mapping is also polyhedral, such set-valued mappings are also metrically subregular at each point of their graphs.

We finalize this paragraph with the following observation: Whenever Υ possesses the Aubin property at $(\bar{x}, \bar{y}) \in \operatorname{gph} \Upsilon$, then we find $\kappa > 0$ such that for each sequence $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ with $x_k \to \bar{x}$, we have $\operatorname{dist}(\bar{y}, \Upsilon(x_k)) \le \kappa \, \|x_k - \bar{x}\|$ for sufficiently large $k \in \mathbb{N}$ from (2.1). Particularly, there exists a sequence $\{y_k\}_{k\in\mathbb{N}}$ satisfying $y_k \to \bar{y}$ and $y_k \in \Upsilon(x_k)$ for sufficiently large $k \in \mathbb{N}$. Thus, Υ is so-called inner semicontinuous at $(\bar{x}, \bar{y})$. Let us also mention that Υ is called inner semicompact at $\bar{x}$ whenever for each sequence $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ with $x_k \to \bar{x}$, there is a convergent sequence $\{y_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^m$ such that $y_k \in \Upsilon(x_k)$ holds for all sufficiently large $k \in \mathbb{N}$.

2.3 VARIATIONAL ANALYSIS

The subsequently introduced tools of variational analysis can be found in the monographs [31, 32] or [38].

For a closed set $A \subset \mathbb{R}^m$ and a point $\bar{x} \in A$, we exploit

$$\mathcal{T}_A(\bar{x}) := \limsup_{t \searrow 0} \frac{A - \bar{x}}{t} \qquad \widehat{\mathcal{N}}_A(\bar{x}) := \mathcal{T}_A(\bar{x})^\circ \qquad \mathcal{N}_A(\bar{x}) := \limsup_{x \to \bar{x}} \widehat{\mathcal{N}}_A(x)$$

in order to denote the tangent (or Bouligand) cone as well as the regular (or Fréchet) and the limiting (or Mordukhovich) normal cone to A at $\bar{x}$. For each $x \notin A$, we stipulate $\mathcal{T}_A(x) := \varnothing$, $\widehat{\mathcal{N}}_A(x) := \varnothing$, and $\mathcal{N}_A(x) := \varnothing$. By definition of the limiting normal cone, it is robust in the sense that we even have

$$\limsup_{x \to \bar{x}} \mathcal{N}_A(x) = \mathcal{N}_A(\bar{x}),$$

see [38, Proposition 6.6]. We recall that whenever A is convex, then the normal cones from above coincide with the standard normal cone of convex analysis, i.e.,

$$\widehat{\mathcal{N}}_A(\bar{x}) = \mathcal{N}_A(\bar{x}) = \{v \in \mathbb{R}^n \,|\, \forall x \in A\colon\; v^\top (x - \bar{x}) \le 0\}$$

holds true in this situation.

For some extended real-valued, lower semicontinuous function $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, we denote its epigraph by $\operatorname{epi}\varphi := \{(x,\alpha) \in \mathbb{R}^n \times \mathbb{R} \,|\, \alpha \geq \varphi(x)\}$. Fixing $\bar{x} \in \mathbb{R}^n$ with $|\varphi(\bar{x})| < \infty$, we introduce the so-called limiting and singular subdifferential of φ at $\bar{x}$, respectively, as

$$\begin{aligned} \partial\varphi(\bar{x}) &:= \left\{v \in \mathbb{R}^n \,\middle|\, (v,-1) \in \mathcal{N}_{\operatorname{epi}\varphi}(\bar{x},\varphi(\bar{x}))\right\}, \\ \partial^\infty\varphi(\bar{x}) &:= \left\{v \in \mathbb{R}^n \,\middle|\, (v,0) \in \mathcal{N}_{\operatorname{epi}\varphi}(\bar{x},\varphi(\bar{x}))\right\}. \end{aligned}$$

It is well known that φ is locally Lipschitz continuous at $\bar{x}$ if and only if $\partial^\infty\varphi(\bar{x}) = \{0\}$ holds.

Next, for a set-valued mapping $\Upsilon\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with closed graph and some point $(\bar{x},\bar{y}) \in \operatorname{gph}\Upsilon$, we define the so-called (limiting) coderivative $D^*\Upsilon(\bar{x},\bar{y})\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ of Υ at $(\bar{x},\bar{y})$ as stated below:

$$\forall y^* \in \mathbb{R}^m\colon \quad D^*\Upsilon(\bar{x},\bar{y})(y^*) := \left\{x^* \in \mathbb{R}^n \,\middle|\, (x^*,-y^*) \in \mathcal{N}_{\operatorname{gph}\Upsilon}(\bar{x},\bar{y})\right\}.$$

In case where $\upsilon\colon \mathbb{R}^n \to \mathbb{R}^m$ is a single-valued mapping, we exploit $D^*\upsilon(\bar{x})(y^*) := D^*\upsilon(\bar{x},\upsilon(\bar{x}))(y^*)$ for all $y^* \in \mathbb{R}^m$. If υ is continuously differentiable at $\bar{x}$, $D^*\upsilon(\bar{x})(y^*) = \{\upsilon'(\bar{x})^\top y^*\}$ is valid for all $y^* \in \mathbb{R}^m$.

Using the concept of coderivatives, it is possible to characterize the presence of metric regularity or the Aubin property for Υ at $(\bar{x},\bar{y}) \in \operatorname{gph}\Upsilon$. More precisely, Υ possesses the Aubin property at $(\bar{x},\bar{y})$ if and only if

$$D^*\Upsilon(\bar{x},\bar{y})(0) = \{0\}$$

holds, see [31, Theorem 4.10]. Noting that we have

$$\mathcal{N}_{\operatorname{gph}\Upsilon^{-1}}(\bar{y},\bar{x}) = \{(y^*,x^*) \in \mathbb{R}^m \times \mathbb{R}^n \,|\, (x^*,y^*) \in \mathcal{N}_{\operatorname{gph}\Upsilon}(\bar{x},\bar{y})\}$$

from the change-or-coordinates formula of limiting normals, see [31, Theorem 1.17], while Υ is metrically regular at $(\bar{x},\bar{y})$ if and only if Υ^{-1} possesses the Aubin property at $(\bar{y},\bar{x})$, the above result also implies that Υ is metrically regular at $(\bar{x},\bar{y})$ if and only if the condition

$$\ker D^*\Upsilon(\bar{x},\bar{y}) = \{0\}$$

holds. This result can be distilled from [31, Theorem 4.18] as well. Both criteria are referred to as *Mordukhovich criterion* in the literature.

Below, we present a simple calculus rule for the coderivative of set-valued mappings of certain product structure.

Lemma 2.1. *Let $\Gamma\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping with closed graph. Furthermore, let $C \subset \mathbb{R}^n$ be a nonempty, closed set. Let $\Psi\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m \times \mathbb{R}^n$ be the set-valued mapping given by*

$$\forall x \in \mathbb{R}^n\colon \quad \Psi(x) := \begin{pmatrix} \Gamma(x) \\ x - C \end{pmatrix}.$$

For a fixed point $(\bar{x},(\bar{y},\bar{z})) \in \operatorname{gph}\Psi$, it holds

$$\forall y^* \in \mathbb{R}^m\, \forall z^* \in \mathbb{R}^n\colon \quad D^*\Psi(\bar{x},(\bar{y},\bar{z}))(y^*,z^*) = \begin{cases} D^*\Gamma(\bar{x},\bar{y})(y^*) + z^* & z^* \in \mathcal{N}_C(\bar{x}-\bar{z}) \\ \varnothing & \text{otherwise.} \end{cases}$$

Proof. Introducing a linear map $\psi\colon \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$ by $\psi(x,y,z) := (x,y,x-z)$ for all $x,z \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, we have

$$\operatorname{gph}\Psi = \{(x,y,z) \,|\, \psi(x,y,z) \in \operatorname{gph}\Gamma \times C\}.$$

Noting that the derivative of ψ is a constant invertible matrix, the desired result follows by elementary calculations from the change-of-coordinates formula from [31, Theorem 1.17] and the product rule for the computation of limiting normals, see [31, Proposition 1.2]. □

2.4 Generalized distance functions

In our analysis, we will make use of the distance function to a moving set. Therefore, let $\Gamma\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping with closed graph and consider

$$\forall x \in \mathbb{R}^n\, \forall y \in \mathbb{R}^m\colon \qquad \rho_\Gamma(x,y) := \inf_{z\in\Gamma(x)} \|y-z\|\,.$$

The function $\rho_\Gamma\colon \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ has been studied in several different publications, see e.g. [33, 37, 39] and the references therein. In contrast to the classical distance function, see [15, Section 2.4], ρ_Γ is not Lipschitz continuous in general. In fact, it does not even need to be continuous. However, as we will show below, this function is lower semicontinuous since Γ possesses a closed graph.

Lemma 2.2. *Let $\Gamma\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping with closed graph. Then the associated function ρ_Γ is lower semicontinuous.*

Proof. Suppose that there exists a point $(\bar x,\bar y) \in \mathbb{R}^n \times \mathbb{R}^m$ where Γ is not lower semicontinuous. Then we find sequences $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ and $\{y_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^m$ as well as $\alpha \geq 0$ with $x_k \to \bar x$, $y_k \to \bar y$, and $\rho_\Gamma(x_k,y_k) \to \alpha < \rho_\Gamma(\bar x,\bar y)$. Particularly, we can assume w.l.o.g. that $\Gamma(x_k) \neq \varnothing$ holds for all $k \in \mathbb{N}$. Noting that $\Gamma(x_k)$ is closed for each $k\in\mathbb{N}$, we find points $z_k \in \Pi(y_k,\Gamma(x_k))$. This yields $\rho_\Gamma(x_k,y_k) = \|y_k - z_k\|$ for all $k \in \mathbb{N}$. Due to

$$\|z_k\| \leq \|z_k - y_k\| + \|y_k\| = \rho_\Gamma(x_k,y_k) + \|y_k\|$$

and the boundedness of $\{\rho_\Gamma(x_k,y_k)\}_{k\in\mathbb{N}}$ and $\{y_k\}_{k\in\mathbb{N}}$, $\{z_k\}_{k\in\mathbb{N}}$ is bounded as well and possesses an accumulation point $\bar z$. Due to $\rho_\Gamma(x_k,y_k) \to \alpha$, we have $\alpha = \|\bar y - \bar z\|$. Observing that $z_k \in \Gamma(x_k)$ holds true for all $k \in \mathbb{N}$, the closedness of $\operatorname{gph}\Gamma$ yields $\bar z \in \Gamma(\bar x)$. Thus, we have $\rho_\Gamma(\bar x,\bar y) \leq \|\bar y - \bar z\| = \alpha$ which is a contradiction. □

Now, we want to identify situations where ρ_Γ is a locally Lipschitz continuous function. Furthermore, we aim for an upper estimate of the limiting subdifferential of this function which holds at in-set points from $\operatorname{gph}\Gamma$ but also at out-of-set points.

Lemma 2.3. *Let $\Gamma\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping with closed graph and fix a point $(\bar x,\bar y) \in \mathbb{R}^n\times\mathbb{R}^m$ such that $\bar x \in \operatorname{dom}\Gamma$. Then the following assertions hold.*

(a) Assume that Γ possesses the Aubin property at all points $(\bar x, y)$ satisfying $y \in \Pi(\bar y,\Gamma(\bar x))$. Then ρ_Γ is locally Lipschitz continuous at $(\bar x,\bar y)$.

(b) The following upper estimate for the limiting subdifferential does always hold:

$$\partial\rho_\Gamma(\bar x,\bar y) \subset \bigcup_{y\in\Pi(\bar y,\Gamma(\bar x))} \mathcal{N}_{\operatorname{gph}\Gamma}(\bar x,y).$$

Proof. (a) First, assume that $(\bar x,\bar y) \in \operatorname{gph}\Gamma$ holds. Then we clearly have $\Pi(\bar y,\Gamma(\bar x)) = \{\bar y\}$. Due to the assumptions of the lemma, Γ possesses the Aubin property at $(\bar x,\bar y)$. Thus, we can invoke [37, Theorem 2.3] in order to obtain the Lipschitz continuity of ρ_Γ at $(\bar x,\bar y)$.
Next, we assume that $(\bar x,\bar y) \notin \operatorname{gph}\Gamma$ holds. In this case, [33, Theorem 4.9, Corollary 4.10] guarantee validity of the estimate

$$\partial^\infty\rho_\Gamma(\bar x,\bar y) \subset \bigcup_{y\in\Pi(\bar y,\Gamma(\bar x))} \{(\xi,0)\,|\,\xi \in D^*\Gamma(\bar x,y)(0)\}.$$

Noting that Γ possesses the Aubin property at all points $(\bar x,y)$ with $y \in \Pi(\bar y,\Gamma(\bar x))$, the Mordukhovich criterion ensures $D^*\Gamma(\bar x,y)(0) = \{0\}$ which is why we obtain $\partial^\infty\rho_\Gamma(\bar x,\bar y) = \{(0,0)\}$ from the above formula. Due to Lemma 2.2, we already know that ρ_Γ is lower semicontinuous. Combining these two properties, we obtain that ρ_Γ is locally Lipschitz continuous at $(\bar x,\bar y)$.

(b) If we have $(\bar{x},\bar{y}) \in \operatorname{gph}\Gamma$, then [39, Proposition 2.7] guarantees

$$\mathcal{N}_{\operatorname{gph}\Gamma}(\bar{x},\bar{y}) = \bigcup_{\alpha\geq 0} \alpha\partial\rho_\Gamma(\bar{x},\bar{y}).$$

On the other hand, in case $(\bar{x},\bar{y}) \notin \operatorname{gph}\Gamma$, [33, Theorem 4.9, Corollary 4.10] can be applied in order to find the estimate

$$\partial\rho_\Gamma(\bar{x},\bar{y}) \subset \bigcup_{y\in\Pi(\bar{y},\Gamma(\bar{x}))} \left\{(\xi,v)\in\mathcal{N}_{\operatorname{gph}\Gamma}(\bar{x},y)\,\middle|\,\|v\|=1\right\}.$$

Taking both formulas together, we obtain the desired general estimate.

□

3 Asymptotic M-stationarity conditions and asymptotic regularity

3.1 Asymptotic M-stationary conditions

Let $\bar{x} \in M$ be a local minimizer of (P). Under suitable assumptions, so-called constraint qualifications, one can guarantee that this ensures the existence of a multiplier $\lambda \in \mathbb{R}^m$ such that

$$0 \in \partial f(\bar{x}) + D^*\Phi(\bar{x},0)(\lambda) \tag{3.1}$$

holds, see e.g. [31, Theorem 5.48]. We will refer to this condition as the Mordukhovich-stationarity condition (M-stationarity condition for short) of (P). Now, the question arises whether it is possible to find a milder condition which holds for each local minimizer of (P) even in the absence of a constraint qualification. A potential candidate for such a condition could be an *asymptotic* version of M-stationarity which holds along a sequence of points $\{x_k\}_{k\in\mathbb{N}}$ converging to the local minimizer of interest. However, one has to specify what *asymptotic* means in this regard. The following definition provides a potential and, as we will see later, reasonable answer to this question.

Definition 3.1. Let $\bar{x} \in M$ be a feasible point of (P). Then we call $\bar{x}$ an *asymptotically Mordukhovich-stationary point* (AM-stationary point) of (P) whenever there exist sequences $\{x_k\}_{k\in\mathbb{N}}, \{\varepsilon_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ as well as $\{y_k\}_{k\in\mathbb{N}}, \{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^m$ such that

$$\forall k\in\mathbb{N}:\quad \varepsilon_k \in \partial f(x_k) + D^*\Phi(x_k,y_k)(\lambda_k) \tag{3.2}$$

as well as $x_k \to \bar{x}$, $\varepsilon_k \to 0$, and $y_k \to 0$ hold. This implicitly requires $\{(x_k,y_k)\}_{k\in\mathbb{N}} \subset \operatorname{gph}\Phi$.

Observe that in the above definition, no convergence of the multiplier sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ is postulated. Indeed, if it would be bounded, then one could simply take the limit $k\to\infty$ along a subsequence in (3.2) in order to recover the M-stationarity conditions from (3.1), see Lemmas 3.4 and 3.9.

Using a simple penalization argument, we obtain the following result which shows that each local minimizer of (P) is an AM-stationary point without any additional assumptions.

Theorem 3.2. *Let $\bar{x} \in M$ be a local minimizer of (P). Then $\bar{x}$ is an AM-stationary point of (P).*

Proof. Let $\varepsilon > 0$ be chosen such that $f(x) \geq f(\bar{x})$ holds for all $x \in M\cap\mathbb{B}_\varepsilon(\bar{x})$. Consider the optimization problem

$$\begin{aligned} f(x) + \frac{k}{2}\|y\|^2 + \frac{1}{2}\|x-\bar{x}\|^2 &\to \min_{x,y} \\ (x,y) &\in \operatorname{gph}\Psi \cap (\mathbb{B}_\varepsilon(\bar{x})\times\mathbb{B}) \end{aligned} \tag{P(k)}$$

which depends on the parameter $k \in \mathbb{N}$. Observe that the objective function of this optimization problem is locally Lipschitz continuous while its feasible set is nonempty and compact. Consequently, (P(k)) possesses a global minimizer $(x_k, y_k) \in \mathbb{R}^n \times \mathbb{R}^m$ for each $k \in \mathbb{N}$. Due to $\{(x_k, y_k)\}_{k\in\mathbb{N}} \subset \mathbb{B}_\varepsilon(\bar{x}) \times \mathbb{B}$, this sequence is bounded. Choosing a subsequence (if necessary) without relabelling, we can guarantee $x_k \to \tilde{x}$ for some $\tilde{x} \in \mathbb{B}_\varepsilon(\bar{x})$ and $y_k \to \tilde{y}$ for some $\tilde{y} \in \mathbb{B}$. Noting that $(\bar{x}, 0) \in \operatorname{gph}\Phi$ is feasible to (P(k)), we find

$$\forall k \in \mathbb{N}: \quad f(x_k) + \frac{k}{2}\|y_k\|^2 + \frac{1}{2}\|x_k - \bar{x}\|^2 \leq f(\bar{x}). \tag{3.3}$$

By boundedness of $\{f(x_k)\}_{k\in\mathbb{N}}$, there is a constant $c \in \mathbb{R}$ such that $\|y_k\|^2 \leq 2(f(\bar{x}) - c)/k$ holds for all $k \in \mathbb{N}$. Consequently, $\{y_k\}_{k\in\mathbb{N}}$ converges to 0 as $k \to \infty$, i.e., we have $\tilde{y} = 0$. The closedness of $\operatorname{gph}\Phi$ now yields $(\tilde{x}, 0) \in \operatorname{gph}\Phi$. Particularly, we infer $\tilde{x} \in M \cap \mathbb{B}_\varepsilon(\bar{x})$. Now, (3.3) and the continuity of all appearing functions yield

$$\begin{aligned} f(\tilde{x}) + \frac{1}{2}\|\tilde{x} - \bar{x}\|^2 &= \lim_{k\to\infty}\left(f(x_k) + \frac{1}{2}\|x_k - \bar{x}\|^2\right) \\ &\leq \lim_{k\to\infty}\left(f(x_k) + \frac{k}{2}\|y_k\|^2 + \frac{1}{2}\|x_k - \bar{x}\|^2\right) \leq f(\bar{x}) \leq f(\tilde{x}), \end{aligned}$$

and this implies $\tilde{x} = \bar{x}$. Particularly, we have $x_k \to \bar{x}$.

Noting that (x_k, y_k) lies in the interior of $\mathbb{B}_\varepsilon(\bar{x}) \times \mathbb{B}$ for sufficiently large $k \in \mathbb{N}$, we can apply [31, Proposition 5.3] and the subdifferential sum rule from [31, Theorem 3.36] in order to obtain

$$(0, 0) \in \partial f(x_k) \times \{0\} + \{(x_k - \bar{x}, k y_k)\} + \mathcal{N}_{\operatorname{gph}\Phi}(x_k, y_k)$$

for large enough $k \in \mathbb{N}$. Setting $\lambda_k := k y_k$ and $\varepsilon_k := \bar{x} - x_k$ for any such $k \in \mathbb{N}$, we have

$$\varepsilon_k \in \partial f(x_k) + D^*\Phi(x_k, y_k)(\lambda_k),$$

and due to $\varepsilon_k \to 0$, $x_k \to \bar{x}$, and $y_k \to 0$, $\bar{x}$ is an AM-stationary point of (P). □

Note that in case where the objective function f is differentiable, one could exploit [31, Proposition 5.1] in the above proof. This way, it would be possible to replace the limiting coderivative by the regular one (i.e., one replaces the limiting normal cone to $\operatorname{gph}\Phi$ by the regular normal cone to this set in the definition of the coderivative) leading to a slightly stronger concept of asymptotic stationarity. We are, however, interested in taking the limit $k \to \infty$ in (3.2), and by definition of the limiting normal cone and its robustness, it does not matter which of these coderivative constructions is used in the definition of AM-stationarity since after taking the limit, we obtain a condition in terms of the limiting coderivative either way.

The above theorem states that in contrast to M-stationarity, AM-stationarity always provides a necessary optimality condition for optimization problems of type (P). The subsequently stated example visualizes this issue.

Example 3.3. Consider the setting

$$\forall x \in \mathbb{R}: \quad f(x) := x \qquad \Phi(x) := [x^2, \infty).$$

The uniquely determined feasible point $\bar{x} := 0$ must be the global minimizer of the associated program (P). Exploiting

$$D^*\Phi(x,y)(\lambda) = \begin{cases} \{2\lambda x\} & y = x^2,\ \lambda \geq 0 \\ \{0\} & y > x^2,\ \lambda = 0 \\ \varnothing & \text{otherwise} \end{cases}$$

for all $x, y, \lambda \in \mathbb{R}$, one can easily check that $\bar{x}$ is not an M-stationary point of this program. However, we can set

$$x_k := -\tfrac{1}{k} \qquad \varepsilon_k := 0 \qquad y_k := \tfrac{1}{k^2} \qquad \lambda_k := \tfrac{k}{2}$$

for all $k \in \mathbb{N}$ in order to see that $\bar{x}$ is an AM-stationary point of the given optimization problem. Observe that the multiplier sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ from above is not bounded.

3.2 Asymptotic regularity

We now raise the question under which additional condition a given AM-stationary point of (P) is already an M-stationary point. In order to deal with this issue, we make use of the set-valued mapping $\mathcal{M}\colon \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ given by

$$\forall x \in \mathbb{R}^n\, \forall y \in \mathbb{R}^m\colon \quad \mathcal{M}(x, y) := \bigcup_{\lambda \in \mathbb{R}^m} D^*\Phi(x, y)(\lambda).$$

By definition, we have the following result.

Lemma 3.4. *Let $\bar{x} \in M$ be a feasible point of (P). Then the following assertions hold.*

(a) If $\bar{x}$ is an AM-stationary point of (P), then we have

$$\partial f(\bar{x}) \cap \left(-\limsup_{x\to\bar{x},\, y\to 0} \mathcal{M}(x, y)\right) \neq \varnothing. \tag{3.4}$$

(b) If, on the other hand, f is continuously differentiable at $\bar{x}$ while

$$-\nabla f(\bar{x}) \in \limsup_{x\to\bar{x},\, y\to 0} \mathcal{M}(x, y)$$

holds, then $\bar{x}$ is an AM-stationary point of (P).

Proof. (a) Let $\bar{x}$ be an AM-stationary point of (P). Then we find $\{x_k\}_{k\in\mathbb{N}}, \{\varepsilon_k\}_{k\in\mathbb{N}}, \{x_k^*\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$, and $\{y_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^m$ such that $x_k \to \bar{x}, \varepsilon_k \to 0, y_k \to 0$, as well as $x_k^* \in \mathcal{M}(x_k, y_k)$ and $\varepsilon_k - x_k^* \in \partial f(x_k)$ for all $k \in \mathbb{N}$ hold. Noting that the set-valued map $x \rightrightarrows \partial f(x)$ possesses uniformly bounded image sets around $\bar{x}$ by local Lipschitz continuity of f, see [31, Corollary 1.81], the sequence $\{x_k^*\}_{k\in\mathbb{N}}$ needs to be bounded as well and, thus, possesses an accumulation point $x^* \in \mathbb{R}^n$ which, by definition, belongs to $\limsup_{x\to\bar{x},\, y\to 0} \mathcal{M}(x, y)$. On the other hand, $-x^* \in \partial f(\bar{x})$ is also true by robustness of the limiting normal cone to $\operatorname{epi} f$, i.e., by closedness of the graph associated with the normal cone mapping of this set.

(b) From the assumptions, we find $\{x_k\}_{k\in\mathbb{N}}, \{x_k^*\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$, and $\{y_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^m$ such that $x_k \to \bar{x}$, $y_k \to 0$, and $x_k^* \to -\nabla f(\bar{x})$ as well as $x_k^* \in \mathcal{M}(x_k, y_k)$ for all $k \in \mathbb{N}$ hold. Setting $\varepsilon_k := \nabla f(x_k) + x_k^*$ for each $k \in \mathbb{N}$, we have $\varepsilon_k \to 0$ by continuity of ∇f at $\bar{x}$. Thus, the definition of $\mathcal{M}$ shows that $\bar{x}$ is an AM-stationary point of (P).

□

Observe that statement (b) of the above lemma cannot be generalized to situations where f is nonsmooth at the point of interest, i.e., condition (3.4) is not necessarily sufficient for a feasible point $\bar{x} \in M$ of (P) to be AM-stationary.

Example 3.5. Let us consider the setting

$$\forall x \in \mathbb{R}\colon \quad f(x) := -|x| \qquad \Phi(x) := [-x^2, \infty)$$

and fix the feasible point $\bar{x} := 0$ of the associated problem (P). We obtain

$$\partial f(x) = \begin{cases} -1 & x > 0 \\ \{-1, 1\} & x = 0 \\ 1 & x < 0 \end{cases} \qquad D^*\Phi(x, y) = \begin{cases} \{-2\lambda x\} & y = -x^2,\ \lambda \geq 0 \\ \{0\} & y > -x^2,\ \lambda = 0 \\ \varnothing & \text{otherwise} \end{cases}$$

for all $x, y, \lambda \in \mathbb{R}$. This yields

$$\mathcal{M}(x,y) = \begin{cases} \mathbb{R}_- & x > 0,\ y = -x^2 \\ \mathbb{R}_+ & x < 0,\ y = -x^2 \\ \{0\} & y > -x^2 \text{ or } x = y = 0 \end{cases}$$

for all $x, y \in \mathbb{R}$, i.e., we find

$$\limsup_{x \to \bar{x},\ y \to 0} \mathcal{M}(x,y) = \mathbb{R}$$

in the present situation, and this shows that (3.4) holds. On the other hand, we clearly have the inclusion $\partial f(x) + \mathcal{M}(x,y) \subset (-\infty,-1] \cup [1,\infty)$ for all $x, y \in \mathbb{R}$, and this clarifies that $\bar{x}$ cannot be AM-stationary.

By definition of $\mathcal{M}$, a given feasible point $\bar{x} \in M$ of (P) is M-stationary if and only if

$$\partial f(\bar{x}) \cap (-\mathcal{M}(\bar{x},0)) \neq \emptyset$$

holds. Keeping statement (a) of Lemma 3.4 in mind, this motivates the subsequent definition.

Definition 3.6. A feasible point $\bar{x} \in M$ of (P) is said to be *asymptotically Mordukhovich-regular* (AM-regular for short) whenever

$$\limsup_{x \to \bar{x},\, y \to 0} \mathcal{M}(x,y) \subset \mathcal{M}(\bar{x},0)$$

is valid.

By definition, a feasible point $\bar{x} \in M$ of (P) is AM-regular if and only if the mapping $\mathcal{M}$ is so-called *outer* (sometimes also referred to as *upper*) semicontinuous at $(\bar{x},0)$ in the sense of set-valued mappings, see [8, 38].

Based on the above observations, the subsequent theorem follows immediately from Theorem 3.2 and Lemma 3.4. It basically says that AM-regularity is a constraint qualification for (P) ensuring M-stationarity of local minimizers.

Theorem 3.7. *Let $\bar{x} \in M$ be an AM-regular local minimizer of* (P)*. Then $\bar{x}$ is an M-stationary point of* (P).

Next, we want to embed AM-regularity into the landscape of qualification conditions which address (P). It is well known from [18, Theorem 3] or [31, Theorem 5.48] that metric subregularity of Φ at $(\bar{x},0)$ is enough to guarantee that a local minimizer $\bar{x} \in M$ of (P) is an M-stationary point of the latter. Using the concept of *directional* metric subregularity, this statement can be weakened even more, see [18, Corollary 2]. We know that polyhedral set-valued mappings are metrically subregular at all points of their graphs, i.e., this property already serves as a constraint qualification for (P). Below, we show that polyhedrality of Φ is also sufficient for the validity of AM-regularity.

Theorem 3.8. *Let Φ be a polyhedral set-valued mapping. Then each feasible point $\bar{x} \in M$ of* (P) *is AM-regular.*

Proof. Fix sequences $\{x_k\}_{k\in\mathbb{N}}, \{x_k^*\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ and $\{y_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^m$ with $x_k \to \bar{x}$, $y_k \to 0$, and $x_k^* \to x^*$ for some $x^* \in \mathbb{R}^n$ such that $x_k^* \in \mathcal{M}(x_k,y_k)$ holds for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, we find $\lambda_k \in \mathbb{R}^m$ such that $(x_k^*, -\lambda_k) \in \mathcal{N}_{\operatorname{gph}\Phi}(x_k,y_k)$ holds. Noting that $\operatorname{gph}\Phi$ is the union of finitely many convex polyhedral sets, there only exist finitely many regular and, thus, limiting normal cones to the set $\operatorname{gph}\Phi$. Particularly, we find a closed cone $\mathcal{K} \subset \mathbb{R}^n \times \mathbb{R}^m$ such that $(x_k^*, -\lambda_k) \in \mathcal{K}$ holds along a subsequence (without relabelling). By polyhedrality of Φ, $\mathcal{K}$ can be represented as the union of finitely many convex, polyhedral cones $\mathcal{K}_1, \ldots, \mathcal{K}_s \subset \mathbb{R}^n \times \mathbb{R}^m$. Again, along a subsequence (without relabelling), we have $(x_k^*, -\lambda_k) \in \mathcal{K}_i$ for some $i \in \{1, \ldots, s\}$. Let $P\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be the projection operator given by $P(x,y) := x$ for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Then $P\mathcal{K}_i$ is polyhedral and, thus, closed by polyhedrality of $\mathcal{K}_i$, i.e., from $\{x_k^*\}_{k\in\mathbb{N}} \subset P\mathcal{K}_i$ we obtain $x^* \in P\mathcal{K}_i$. This yields the existence of some $\lambda \in \mathbb{R}^m$ such that $(x^*, -\lambda) \in \mathcal{K}_i$ and, thus, $(x^*, -\lambda) \in \mathcal{K}$. The robustness of the limiting normal cone now implies $\mathcal{K} \subset \mathcal{N}_{\operatorname{gph}\Phi}(\bar{x}, 0)$ due to $x_k \to \bar{x}$ and $y_k \to 0$. Particularly, we have $x^* \in D^*\Phi(\bar{x}, 0)(\lambda)$, i.e., $x^* \in \mathcal{M}(\bar{x}, 0)$. This shows that $\bar{x}$ is an AM-regular point of (P). □

A natural consequence of the definition of AM-regularity via the Painlevé–Kuratowski limit is subsumed in the following lemma.

Lemma 3.9. *Let $\bar{x} \in M$ be a feasible point of* (P). *Assume that for each sequences $\{x_k\}_{k\in\mathbb{N}}, \{x_k^*\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$, and $\{y_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^m$ such that $x_k \to \bar{x}$, $y_k \to 0$, $x_k^* \to x^*$ for some $x^* \in \mathbb{R}^n$, and $x_k^* \in \mathcal{M}(x_k, y_k)$ for all $k \in \mathbb{N}$ hold, we find a bounded sequence $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^m$ such that $x_k^* \in D^*\Phi(x_k, y_k)(\lambda_k)$ holds for all $k \in \mathbb{N}$. Then $\bar{x}$ is AM-regular.*

Proof. Let $x^* \in \limsup_{x\to\bar{x},\, y\to 0} \mathcal{M}(x, y)$ be arbitrarily chosen. Then we find $\{x_k\}_{k\in\mathbb{N}}, \{x_k^*\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$, and $\{y_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^m$ such that $x_k \to \bar{x}$, $y_k \to 0$, $x_k^* \to x^*$, and $x_k^* \in \mathcal{M}(x_k, y_k)$ for all $k \in \mathbb{N}$ hold. By assumption, there is a bounded sequence $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^m$ satisfying $x_k^* \in D^*\Phi(x_k, y_k)(\lambda_k)$ for all $k \in \mathbb{N}$. Observing that $\{\lambda_k\}_{k\in\mathbb{N}}$ possesses an accumulation point $\lambda \in \mathbb{R}^m$, $x^* \in D^*\Phi(\bar{x}, 0)(\lambda)$ follows from the robustness of the limiting normal cone and the definition of the coderivative. The latter, however, yields $x^* \in \mathcal{M}(\bar{x}, 0)$, i.e., $\bar{x}$ is AM-regular. □

Employing the neighbourhood characterization of the metric regularity property, we now can state a sufficient condition for AM-regularity.

Theorem 3.10. *Let $\bar{x} \in M$ be a feasible point of* (P) *such that Φ is metrically regular at $(\bar{x}, 0)$. Then $\bar{x}$ is AM-regular.*

Proof. Exploiting [31, Theorem 4.5] and the definition of the limiting coderivative, metric regularity of Φ at $(\bar{x}, 0)$ guarantees the existence of a constant $\kappa > 0$ and a neighbourhood U of $(\bar{x}, 0)$ such that

$$\forall (x, y) \in \operatorname{gph}\Phi \cap U\ \forall x^* \in \mathbb{R}^n\ \forall \lambda \in \mathbb{R}^m\colon \quad x^* \in D^*\Phi(x, y)(\lambda) \implies \|\lambda\| \le \kappa \|x^*\| .$$

Choose sequences $\{x_k\}_{k\in\mathbb{N}}, \{x_k^*\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ and $\{y_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^m$ as well as $x^* \in \mathbb{R}^n$ such that $x_k \to \bar{x}$, $y_k \to 0$, $x_k^* \to x^*$, and $x_k^* \in \mathcal{M}(x_k, y_k)$ for all $k \in \mathbb{N}$ hold. By definition of $\mathcal{M}$, we find a sequence $\{\lambda_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^m$ such that $x_k^* \in D^*\Phi(x_k, y_k)(\lambda_k)$ holds for all $k \in \mathbb{N}$. The above considerations show that the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ needs to be bounded since $\{x_k^*\}_{k\in\mathbb{N}}$ is bounded. Now, the theorem's assertion follows from Lemma 3.9. □

Recall that the Mordukhovich criterion provides a necessary and sufficient condition for metric regularity of Φ at arbitrary points of its graph. Thus, it can be used as a sufficient condition for AM-regularity as well.

Corollary 3.11. *Let $\bar{x} \in M$ be a feasible point of* (P) *such that* $\ker D^*\Phi(\bar{x}, 0) = \{0\}$. *Then $\bar{x}$ is AM-regular.*

Clearly, there exist polyhedral set-valued mappings which are not metrically regular at all points of their graphs. In the light of Theorem 3.8, this shows that AM-regularity is generally weaker than metric regularity.

It remains to investigate the relationship between AM-regularity and metric subregularity of Φ. The subsequently stated example depicts that metric subregularity of Φ does not imply validity of AM-regularity.

Example 3.12. We set

$$\forall x \in \mathbb{R}^2\colon \quad \Phi(x) := (-x_1^2 + x_2, -x_2) - \mathbb{R}^2_-$$

and consider the point $\bar{x} := (0, 0)$.

Using the formulas from Section 5.1, we find

$$D^*\Phi(x,y)(\lambda) = \begin{cases} \{(-2x_1\lambda_1, \lambda_1 - \lambda_2)\} & \lambda \in \mathcal{N}_{\mathbb{R}^2_-}(-x_1^2 + x_2 - y_1, -x_2 - y_2) \\ \varnothing & \text{otherwise} \end{cases}$$

for all $x, y, \lambda \in \mathbb{R}^2$. This yields $\mathcal{M}(\bar{x}, 0) = \{0\} \times \mathbb{R}$. Using the sequences given by

$$\forall k \in \mathbb{N}\colon \quad x_{k,1} := -\tfrac{1}{k} \qquad x_{k,2} := 0 \qquad\qquad y_{k,1} := -\tfrac{1}{k^2} \qquad y_{k,2} := 0,$$

we have $x_k \to \bar{x}$ and $y_k \to (0,0)$ as well as $(1,0) \in \mathcal{M}(x_k, y_k)$ for all $k \in \mathbb{N}$. Due to $(1,0) \notin \mathcal{M}(\bar{x}, 0)$, $\bar{x}$ is not an AM-regular point of the associated constraint set M.

One can, however, check that Gfrerer's *second-order sufficient condition for metric subregularity* is valid at $\bar{x}$, see [21, Corollary 1], which shows that Φ is metrically subregular at $(\bar{x}, 0)$.

The next example depicts that validity of AM-regularity is not enough to ensure metric subregularity of Φ. Particularly, these conditions are independent of each other.

Example 3.13. We fix

$$\forall x \in \mathbb{R}\colon \quad \Phi(x) := \begin{cases} \mathbb{R} & x \leq 0 \\ [x^2, \infty) & x > 0. \end{cases}$$

In this case, we have $M = (-\infty, 0]$. Let us consider the point $\bar{x} := 0$.

Some calculations show

$$D^*\Phi(x,y)(\lambda) = \begin{cases} \{2x\lambda\} & x > 0,\ y = x^2,\ \lambda \geq 0 \\ \mathbb{R}_+ & x = 0,\ y \leq 0,\ \lambda = 0 \\ \{0\} & x = y = 0,\ \lambda > 0 \text{ or } x < 0,\ \lambda = 0 \text{ or } x \geq 0,\ y > x^2,\ \lambda = 0 \\ \varnothing & \text{otherwise} \end{cases}$$

for all $x, y, \lambda \in \mathbb{R}$. This yields $\mathcal{M}(\bar{x}, 0) = \mathbb{R}_+$, and since all other images of the coderivative are subsets of $\mathbb{R}_+$, we infer that $\bar{x}$ is an AM-regular point of M.

Setting $x_k := 1/k$ for each $k \in \mathbb{N}$, we find $\operatorname{dist}(x_k, M) = 1/k$ and $\operatorname{dist}(0, \Phi(x_k)) = 1/k^2$. Thus, taking the limit $k \to \infty$, it is clear that Φ is not metrically subregular at $(\bar{x}, 0)$.

The proof of the upcoming result, which provides an upper estimate of the limiting normal cone to the set M at some AM-regular point in terms of initial problem data, is inspired by [14, proof of Theorem 5.9].

Theorem 3.14. *Let $\bar{x} \in M$ be a feasible AM-regular point of* (P). *Then we have $\mathcal{N}_M(\bar{x}) \subset \mathcal{M}(\bar{x}, 0)$.*

Proof. Choose $x^* \in \mathcal{N}_M(\bar{x})$ arbitrarily. Then we find sequences $\{x_k\}_{k\in\mathbb{N}} \subset M$ and $\{x_k^*\} \subset \mathbb{R}^n$ such that $x_k \to \bar{x}$, $x_k^* \to x^*$, and $x_k^* \in \widehat{\mathcal{N}}_M(x_k)$ for all $k \in \mathbb{N}$. Using the variational description of regular normals from [31, Theorem 1.30(ii)], for each $k \in \mathbb{N}$, we find a differentiable convex function $h_k\colon \mathbb{R}^n \to \mathbb{R}$ which achieves a global minimum at x_k when restricted to M and which satisfies $\nabla h_k(x_k) = -x_k^*$. Observe that the properties of h_k already guarantee that this function is continuously differentiable for each $k \in \mathbb{N}$, see [34, Corollary of Proposition 2.8]. Applying Theorem 3.2 for fixed $k \in \mathbb{N}$, we find that x_k is an AM-stationary point of the optimization problem $\min\{h_k(x) \,|\, x \in M\}$. Thus, we find sequences $\{x_{k,\ell}\}_{\ell\in\mathbb{N}}, \{\varepsilon_{k,\ell}\}_{\ell\in\mathbb{N}} \subset \mathbb{R}^n$ and $\{y_{k,\ell}\}_{\ell\in\mathbb{N}} \subset \mathbb{R}^m$ such that $x_{k,\ell} \to x_k$, $\varepsilon_{k,\ell} \to 0$, $y_{k,\ell} \to 0$, as $\ell \to \infty$ and $\varepsilon_{k,\ell} - \nabla h_k(x_{k,\ell}) \in \mathcal{M}(x_{k,\ell}, y_{k,\ell})$ for all $\ell \in \mathbb{N}$ hold. We set $x_{k,\ell}^* := -\nabla h_k(x_{k,\ell})$ for all $\ell \in \mathbb{N}$ and obtain $x_{k,\ell}^* \to x_k^*$ as $\ell \to \infty$ by continuous differentiability of h_k. Exploiting a standard diagonal sequence argument, we, thus, find sequences $\{\bar{x}_k\}_{k\in\mathbb{N}}, \{\varepsilon_k\}_{k\in\mathbb{N}}, \{\bar{x}_k^*\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$, and $\{y_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^m$ such that $\bar{x}_k \to \bar{x}$, $\varepsilon_k \to 0$, $\bar{x}_k^* \to x^*$, $y_k \to 0$, and $\varepsilon_k + \bar{x}_k^* \in \mathcal{M}(\bar{x}_k, y_k)$ for all $k \in \mathbb{N}$ hold. Taking the limit $k \to \infty$ and exploiting the fact that $\bar{x}$ is AM-regular, we obtain $x^* \in \mathcal{M}(\bar{x}, 0)$. □

Let us recall that the assertion of Theorem 3.14 can be interpreted in the following way: Observing that $M = \Phi^{-1}(0)$ holds, under validity of AM-regularity at a given feasible point $\bar{x} \in M$ of (P), some kind of pre-image rule for the computation of the limiting normal cone to M in terms of initial problem data, i.e., the coderivative of Φ, holds. Keeping [31, Proposition 5.3] in mind, this alone is enough to show that AM-regularity is indeed a constraint qualification which guarantees validity of M-stationarity at the local minimizers of (P). A similar observation can be made in the presence of metric subregularity of Φ at $(\bar{x}, 0)$, and the upper estimate for the limiting normal cone to M can be sharpened if the precise modulus of metric subregularity is known or can be estimated from above, see [22, Proposition 4.1].

We summarize our results on the relations between all mentioned qualification conditions in Figure 1.

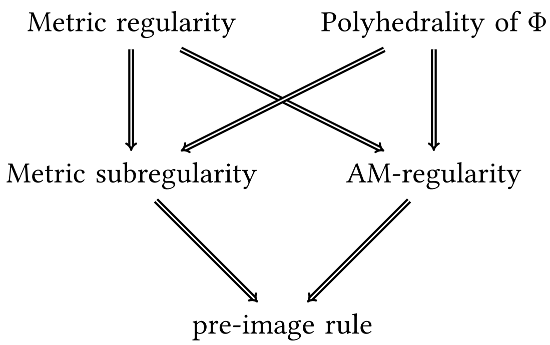

Figure 1: Relations between constraint qualifications addressing (P) which guarantee M-stationarity of associated local minimizers.

4 DECOUPLING OF ABSTRACT CONSTRAINTS

In this section, we want to investigate the particular case where the mapping Φ is given by

$$\forall x \in \mathbb{R}^n: \quad \Phi(x) := \begin{pmatrix} \Gamma(x) \\ x - C \end{pmatrix} \tag{4.1}$$

where $\Gamma\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^\ell$ is a set-valued mapping with closed graph and $C \subset \mathbb{R}^n$ is a nonempty, closed set. Roughly speaking, we assume that the abstract constraint set C is *simple* and shall be decoupled from the more enhanced constraints which are modelled with the aid of the generalized equation $0 \in \Gamma(x)$.

Exploiting the product rule for coderivative calculus from Lemma 2.1, a feasible point $\bar{x} \in M$ of problem (P) where Φ is given as in (4.1) is M-stationary if and only if there is a multiplier $\tilde{\lambda} \in \mathbb{R}^\ell$ satisfying

$$0 \in \partial f(\bar{x}) + D^*\Gamma(\bar{x}, 0)(\tilde{\lambda}) + \mathcal{N}_C(\bar{x}).$$

Furthermore, applying Definition 3.1 to the situation at hand, $\bar{x}$ is an AM-stationary point of the associated problem (P) if and only if there exist sequences $\{x_k\}_{k\in\mathbb{N}}, \{\varepsilon_k\}_{k\in\mathbb{N}}, \{z_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ as well as $\{\tilde{y}_k\}_{k\in\mathbb{N}}, \{\tilde{\lambda}_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^\ell$ satisfying $x_k \to \bar{x}$, $\varepsilon_k \to 0$, $\tilde{y}_k \to 0$, $z_k \to 0$, and

$$\forall k \in \mathbb{N}: \quad \varepsilon_k \in \partial f(x_k) + D^*\Gamma(x_k, \tilde{y}_k)(\tilde{\lambda}_k) + \mathcal{N}_C(x_k - z_k).$$

The relation $z_k := 0$ for all $k \in \mathbb{N}$ seems to be desirable since this would mean that all points x_k from above already satisfy the abstract constraint $x \in C$ hidden in the definition of Φ, i.e., some *partial* feasibility of the sequence $\{x_k\}_{k\in\mathbb{N}}$ would be guaranteed in this situation. This motivates the subsequent definition of *decoupled* AM-stationary points.

Definition 4.1. Let Φ be given as in (4.1). A feasible point $\bar{x} \in M$ of the associated problem (P) is referred to as a *decoupled asymptotically Mordukhovich-stationary point* (dAM-stationary point for short) whenever there are sequences $\{x_k\}_{k\in\mathbb{N}}, \{\varepsilon_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ as well as $\{\tilde{y}_k\}_{k\in\mathbb{N}}, \{\tilde{\lambda}_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^\ell$ satisfying $x_k \to \bar{x}$, $\varepsilon_k \to 0$, $\tilde{y}_k \to 0$, and

$$\forall k \in \mathbb{N}: \quad \varepsilon_k \in \partial f(x_k) + D^*\Gamma(x_k, \tilde{y}_k)(\tilde{\lambda}_k) + \mathcal{N}_C(x_k).$$

Let us note that, by definition, the sequence $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ from Definition 4.1 has to satisfy $\{x_k\}_{k\in\mathbb{N}} \subset \operatorname{dom}\Gamma \cap C$.

The following theorem shows that under an additional assumption on the mapping Γ, each local minimizer of (P) with Φ given as in (4.1) is already a dAM-stationary point.

Theorem 4.2. *Let $\bar{x} \in M$ be a local minimizer point of (P) where Φ is given as in (4.1). Furthermore, assume that Γ possesses the Aubin property at $(\bar{x}, 0)$. Then $\bar{x}$ is a dAM-stationary point of (P).*

Proof. To start, observe that by definition of the Aubin property, we find $\gamma, \delta > 0$ such that Γ possesses the Aubin property at all the points from $\operatorname{gph}\Gamma \cap (\mathbb{B}_\gamma(\bar{x}) \times \mathbb{B}_\delta(0))$. For small enough $\gamma, \delta > 0$, we can also guarantee $\Gamma(x) \cap \mathbb{B}_{\delta/2}(0) \neq \varnothing$ for all $x \in \mathbb{B}_\gamma(\bar{x})$ since Γ is inner semicontinuous at $(\bar{x}, 0)$.

Next, for each $k \in \mathbb{N}$, we investigate the optimization problem

$$(\mathrm{Q}(k)) \qquad \begin{aligned} f(x) + \frac{k}{2}\left(\rho_\Gamma(x,y) + \|y\|^2\right) + \frac{1}{2}\|x - \bar{x}\|^2 &\to \min_{x,y} \\ x &\in C \cap \mathbb{B}_\gamma(\bar{x}) \\ y &\in \mathbb{B}_{\delta/4}(0). \end{aligned}$$

Observing that the objective function of this problem is lower semicontinuous by Lemma 2.2 while its feasible set is nonempty and compact, (Q(k)) possesses a global minimizer $(x_k, y_k) \in \mathbb{R}^n \times \mathbb{R}^\ell$ for each $k \in \mathbb{N}$. Since $\{x_k\}_{k\in\mathbb{N}}$ and $\{y_k\}_{k\in\mathbb{N}}$ are bounded, we may pass to a subsequence (without relabelling) in order to find $\tilde{x} \in \mathbb{B}_\gamma(\bar{x})$ and $\tilde{y} \in \mathbb{B}_{\delta/4}(0)$ such that $x_k \to \tilde{x}$ and $y_k \to \tilde{y}$. Similar as in the proof of Theorem 3.2, we find $\tilde{y} = 0$. In analogous way, we obtain $0 \in \Gamma(\tilde{x})$ by lower semicontinuity of the generalized distance function. Finally, the closedness of C guarantees $\tilde{x} \in C$, i.e., $\tilde{x} \in M$. Furthermore, $\tilde{x} = \bar{x}$ can be shown as in the proof of Theorem 3.2.

For fixed $k \in \mathbb{N}$, we know $\Gamma(x_k) \cap \mathbb{B}_{\delta/2}(0) \neq \varnothing$ from the choice of γ and δ. Due to $y_k \in \mathbb{B}_{\delta/4}(0)$, we have $\varnothing \neq \Pi(y_k, \Gamma(x_k)) \subset \mathbb{B}_\delta(0)$. Particularly, Γ possesses the Aubin property at all point from $\{x_k\} \times \Pi(y_k, \Gamma(x_k))$. As a consequence, Lemma 2.3 guarantees that ρ_Γ is locally Lipschitz continuous at (x_k, y_k). For sufficiently large $k \in \mathbb{N}$, x_k is an interior point of $\mathbb{B}_\gamma(\bar{x})$ while y_k is an interior point of $\mathbb{B}_{\delta/4}(0)$. Due to these observations, we may now apply [31, Proposition 5.3], the sum rule for the limiting subdifferential, see [31, Theorem 3.36], and Lemma 2.3 in order to find $\tilde{y}_k \in \Pi(y_k, \Gamma(x_k))$ such that

$$(0,0) \in \partial f(x_k) \times \{0\} + \mathcal{N}_{\operatorname{gph}\Gamma}(x_k, \tilde{y}_k) + \{(x_k - \bar{x}, k y_k)\} + \mathcal{N}_C(x_k) \times \{0\}$$

holds for large enough $k \in \mathbb{N}$. Defining $\tilde{\lambda}_k := k y_k$ and $\varepsilon_k := \bar{x} - x_k$, this yields

$$\varepsilon_k \in \partial f(x_k) + D^*\Gamma(x_k, \tilde{y}_k)(\tilde{\lambda}_k) + \mathcal{N}_C(x_k)$$

for large enough $k \in \mathbb{N}$. The above observations guarantee $\varepsilon_k \to 0$. It remains to show $\tilde{y}_k \to 0$ in order to complete the proof. Since Γ is inner semicontinuous at $(\bar{x}, 0)$, we find a sequence $\{\bar{y}_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^\ell$ with $\bar{y}_k \to 0$ and $\bar{y}_k \in \Gamma(x_k)$ for all sufficiently large $k \in \mathbb{N}$. By definition of the projector, we have

$$\|\tilde{y}_k\| \leq \|\tilde{y}_k - y_k\| + \|y_k\| \leq \|\bar{y}_k - y_k\| + \|y_k\| \leq 2\|y_k\| + \|\bar{y}_k\| \to 0,$$

and this, finally, shows $\tilde{y}_k \to 0$. □

The subsequently stated example demonstrates that the statement of Theorem 4.2 does not remain true in general when Γ does not possess the Aubin property at the point of interest.

Example 4.3. We investigate the set-valued mapping $\Gamma\colon \mathbb{R}^2 \rightrightarrows \mathbb{R}$ given by

$$\forall x \in \mathbb{R}^2\colon \quad \Gamma(x) := \begin{cases} [0,\infty) & x_2 = 0 \\ \varnothing & \text{otherwise} \end{cases}$$

as well as the closed set $C := \{x \in \mathbb{R}^2 \,|\, x_1^2 + (x_2-1)^2 \leq 1\}$. We consider the associated optimization problem

$$\begin{aligned} x_1 &\rightarrow \min \\ 0 &\in \Gamma(x) \\ x &\in C. \end{aligned}$$

Its uniquely determined feasible point and, thus, global minimizer is $\bar{x} := (0,0)$.

Assuming that $\bar{x}$ is a dAM-stationary point of the problem of interest and keeping the relation $\operatorname{dom}\Gamma \cap C = \{\bar{x}\}$ in mind, there need to exist sequences $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^2$, $\{\tilde{y}_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$, and $\{\tilde{\lambda}_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ such that $\varepsilon_k \rightarrow (0,0)$ and $\tilde{y}_k \rightarrow 0$ as well as

$$\forall k \in \mathbb{N}\colon \quad (\varepsilon_{k,1}, \varepsilon_{k,2}) \in (1,0) + D^*\Gamma(\bar{x}, \tilde{y}_k)(\tilde{\lambda}_k) + \{0\} \times \mathbb{R}_- \tag{4.2}$$

hold true. We note, however, that

$$D^*\Gamma(\bar{x},\tilde{y})(\tilde{\lambda}) = \begin{cases} \{0\} \times \mathbb{R} & \tilde{y} = 0,\ \tilde{\lambda} \geq 0 \text{ or } \tilde{y} > 0,\ \tilde{\lambda} = 0, \\ \varnothing & \text{otherwise} \end{cases}$$

is valid for all $\tilde{y}, \tilde{\lambda} \in \mathbb{R}$. Thus, (4.2) yields $\varepsilon_{k,1} = 1$ for all $k \in \mathbb{N}$ which contradicts $\varepsilon_k \rightarrow 0$. Thus, $\bar{x}$ is not a dAM-stationary point of the problem of interest.

Note that Φ is a polyhedral set-valued mapping which does not possess the Aubin property at the reference point $(\bar{x}, 0)$.

Keeping our arguments from Section 3.2 in mind, Theorem 4.2 motivates the definition of another constraint qualification weaker than AM-regularity which ensures that a dAM-stationary point of (P) where Φ is given as in (4.1) is already an M-stationary point. For that purpose, we introduce a set-valued mapping $\widetilde{\mathcal{M}}\colon \mathbb{R}^n \times \mathbb{R}^\ell \rightrightarrows \mathbb{R}^n$ by

$$\forall x \in \mathbb{R}^n\, \forall \tilde{y} \in \mathbb{R}^\ell\colon \quad \widetilde{\mathcal{M}}(x,\tilde{y}) := \bigcup_{\tilde{\lambda} \in \mathbb{R}^\ell} D^*\Gamma(x,\tilde{y})(\tilde{\lambda}) + \mathcal{N}_C(x). \tag{4.3}$$

Definition 4.4. A feasible point $\bar{x} \in M$ of (P) where Φ is given as in (4.1) is said to be *decoupled asymptotically Mordukhovich-regular* (dAM-regular for short) whenever

$$\limsup_{x\to\bar{x},\ \tilde{y}\to 0} \widetilde{\mathcal{M}}(x,\tilde{y}) \subset \widetilde{\mathcal{M}}(\bar{x},0)$$

is valid.

For each feasible point $\bar{x} \in M$ of (P) for Φ given in (4.1), we have the relations $\mathcal{M}(\bar{x},(0,0)) = \widetilde{\mathcal{M}}(\bar{x},0)$ and

$$\limsup_{x\to\bar{x},\ \tilde{y}\to 0} \widetilde{\mathcal{M}}(x,\tilde{y}) \subset \limsup_{x\to\bar{x},\ \tilde{y}\to 0,\ z\to 0} \mathcal{M}(x,(\tilde{y},z))$$

which is why dAM-regularity is weaker than AM-regularity as promoted above. Example 4.3 shows that there are situations where dAM-regularity is strictly weaker than AM-regularity. Therein, $\bar{x}$ is a dAM-regular point. On the other hand, we have $(1,0) \in \limsup_{x\to\bar{x},\ \tilde{y}\to 0,\ z\to 0} \mathcal{M}(x,(\tilde{y},z))$ but $\mathcal{M}(\bar{x},(0,0)) = \{0\} \times \mathbb{R}$ which is why $\bar{x}$ cannot be AM-regular. The subsequent example visualizes that dAM-regularity might be strictly weaker than AM-regularity even in situations where Γ possesses the Aubin property at all points of its graph.

Example 4.5. We set $C := \mathbb{R}_+$ as well as

$$\forall x \in \mathbb{R}: \quad \Gamma(x) := [-x^2, \infty)$$

and investigate the mapping Φ from (4.1). In this situation, $M = \mathbb{R}_+$ is valid. Let us focus on the point $\bar{x} := 0$. In Example 3.5, one can find a formula for the coderivative of Γ. Using it and exploiting $\mathcal{N}_C(0) = \mathbb{R}_-$, we find

$$\widetilde{\mathcal{M}}(x, \tilde{y}) = \begin{cases} \mathbb{R}_- & x > 0,\ \tilde{y} = -x^2 \text{ or } x = 0,\ \tilde{y} \geq 0 \\ \{0\} & x > 0,\ \tilde{y} > -x^2 \\ \varnothing & \text{otherwise} \end{cases}$$

for all $x, \tilde{y} \in \mathbb{R}$. Due to $\widetilde{\mathcal{M}}(\bar{x}, 0) = \mathbb{R}_-$, $\bar{x}$ is dAM-regular. Let us set

$$x_k := -\tfrac{1}{k} \qquad \tilde{y}_k := -\tfrac{1}{k^2} \qquad z_k := -\tfrac{1}{k}$$

for all $k \in \mathbb{N}$. Then we find $\mathcal{M}(x_k, (\tilde{y}_k, z_k)) = \mathbb{R}$, i.e.,

$$\limsup_{x \to \bar{x},\ \tilde{y} \to 0,\ z \to 0} \mathcal{M}(x, (\tilde{y}, z)) = \mathbb{R},$$

and this shows that $\bar{x}$ cannot be AM-regular since $\mathcal{M}(\bar{x}, (0, 0)) = \widetilde{\mathcal{M}}(\bar{x}, 0) = \mathbb{R}_-$ holds true. Observing that Γ is the sum of the locally Lipschitzian single-valued mapping $x \mapsto -x^2$ and the constant set $\mathbb{R}_+$, Γ possesses the Aubin property at all points of its graph.

Clearly, $\bar{x} \in M$ is an M-stationary point of (P) if and only if

$$\partial f(\bar{x}) \cap \left(-\widetilde{\mathcal{M}}(\bar{x}, 0)\right) \neq \varnothing$$

is valid. Furthermore, similar as in the proof of Lemma 3.4, one can show that whenever $\bar{x}$ is a dAM-stationary point of the problem of interest, then

$$\partial f(\bar{x}) \cap \left(- \limsup_{x \to \bar{x},\ \tilde{y} \to 0} \widetilde{\mathcal{M}}(x, \tilde{y}) \right) \neq \varnothing$$

is true. Thus, Theorem 4.2 yields the following result.

Theorem 4.6. *Let $\bar{x} \in M$ be a dAM-regular local minimizer of* (P) *where Φ is given as in* (4.1)*. Furthermore, let Γ possess the Aubin property at $(\bar{x}, 0)$. Then $\bar{x}$ is an M-stationary point of* (P)*.*

Using the product rule from Lemma 2.1 as well as the result of Corollary 3.11, the condition

$$0 \in D^*\Gamma(\bar{x}, 0)(y^*) + z^*,\ z^* \in \mathcal{N}_C(\bar{x}) \quad \Longrightarrow \quad y^* = 0,\ z^* = 0 \tag{4.4}$$

is sufficient for AM-regularity and, thus, dAM-regularity of a feasible point $\bar{x} \in M$ of (P) where Φ is given as in (4.1).

Finally, we would like to mention that the assertion of Theorem 3.14 also holds true in the present setting under validity of dAM-regularity whenever Γ possesses the Aubin property at the point of interest.

Theorem 4.7. *Let $\bar{x} \in M$ be a feasible dAM-regular point of* (P) *where Φ is given as in* (4.1)*. Furthermore, let Γ possess the Aubin property at $(\bar{x}, 0)$. Then we have $\mathcal{N}_M(\bar{x}) \subset \widetilde{\mathcal{M}}(\bar{x}, 0)$.*

Proof. The proof is analogous to the one of Theorem 3.14 exploiting Theorem 4.2 and the fact that Γ possesses the Aubin property at all points from $\operatorname{gph} \Gamma \cap U$ where $U \subset \mathbb{R}^n \times \mathbb{R}^\ell$ is a sufficiently small neighbourhood of $(\bar{x}, 0)$. □

5 Applications of asymptotic regularity

5.1 Asymptotic regularity for mathematical programs with geometric constraints

In this section, we assume that $\Phi\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^\ell \times \mathbb{R}^n$ is given by

$$\forall x \in \mathbb{R}^n\colon \quad \Phi(x) := \begin{pmatrix} G(x) - K \\ x - C \end{pmatrix} \tag{5.1}$$

where $G\colon \mathbb{R}^n \to \mathbb{R}^\ell$ is a locally Lipschitz continuous, single-valued mapping, while the sets $K \subset \mathbb{R}^\ell$ and $C \subset \mathbb{R}^n$ are nonempty as well as closed. Thus, the associated feasible region of (P) is given by

$$M = \{x \in C \,|\, G(x) \in K\}, \tag{5.2}$$

and this rather general description still covers numerous interesting classes of optimization problems comprising standard nonlinear problems, instances of conic programming, disjunctive programs (e.g. mathematical problems with complementarity, vanishing, switching, or cardinality constraints, see Section 5.2), and conic complementarity programming. Generally, one refers to constraint systems of this type as *geometric constraints*.

We observe that the structure of Φ is precisely the one discussed in Section 4 if we use the feasibility mapping $\Gamma\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^\ell$ given by $\Gamma(x) := G(x) - K$ for all $x \in \mathbb{R}^n$. Observing that G is a locally Lipschitz continuous map, Γ possesses the Aubin property at each point of its graph. Due to Theorems 4.2 and 4.6, the local minimizers of the underlying optimization problem are always dAM-stationary points and dAM-regularity provides a constraint qualification for the presence of M-stationarity. Using the coderivative sum rule from [31, Theorem 1.62], we have

$$\forall (x,\tilde y) \in \operatorname{gph}\Gamma\ \forall \tilde\lambda \in \mathbb{R}^\ell\colon \quad D^*\Gamma(x,\tilde y)(\tilde\lambda) = \begin{cases} D^*G(x)(\tilde\lambda) & \tilde\lambda \in \mathcal{N}_K(G(x)-\tilde y) \\ \varnothing & \text{otherwise.} \end{cases}$$

Particularly, the mapping $\widetilde{\mathcal{M}}\colon \mathbb{R}^n \times \mathbb{R}^\ell \rightrightarrows \mathbb{R}^n$ from (4.3) takes the form

$$\forall x \in \mathbb{R}^n\ \forall \tilde y \in \mathbb{R}^\ell\colon \quad \widetilde{\mathcal{M}}(x,\tilde y) = D^*G(x)\mathcal{N}_K(G(x)-\tilde y) + \mathcal{N}_C(x).$$

The latter can be used to specify the precise nature of dAM-regularity for particular classes of optimization problems with geometric constraints. We also note that validity of this constraint qualification at an arbitrary point $\bar x \in M$ yields the estimate

$$\mathcal{N}_M(\bar x) \subset D^*G(\bar x)\mathcal{N}_K(G(\bar x)) + \mathcal{N}_C(\bar x),$$

see Theorem 4.7. In the literature, metric subregularity of Φ from (4.1) at $(\bar x, 0)$ is often assumed for that purpose, see e.g. [25, Theorem 4.1] where it is shown that already metric subregularity of $\widehat\Phi\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^\ell$ given by

$$\forall x \in \mathbb{R}^n\colon \quad \widehat\Phi(x) := \begin{cases} G(x) - K & x \in C \\ \varnothing & \text{otherwise} \end{cases}$$

at the point $(\bar x, 0)$ is enough for that purpose. In the light of Section 3.2, dAM-regularity is, however, independent of the metric subregularity of Φ and, thus, provides a different approach to this pre-image rule. In case where G is smooth, $C = \mathbb{R}^n$, and K is of special structure, the fact that dAM-regularity provides a constraint qualification has been observed in [35, Theorem 3.13]. A related observation has been made in the context of semidefinite programming in [4]. Replacing the image space $\mathbb{R}^\ell$ by the Hilbert space of all real symmetric matrices, this paper's theory covers this special situation, too. Under additional assumptions on the data (e.g., convexity of K and C), related results can be obtained for optimization problems in Banach spaces as well, see [14] and Remark 5.2 below. It follows from [35, Section 4] that dAM-regularity for feasible sets of type (5.2) is not related to suitable notions of *pseudo-* and *quasinormality* which apply to feasible sets of type (5.2), see [10, Definition 3.5] and [24, Definition 4.2] as well. On the other hand, we know from our investigations in the earlier sections that this new constraint qualification is generally weaker than metric regularity of Φ from (5.1) at some point $(\bar{x},(0,0)) \in \operatorname{gph}\Phi$, and the latter is equivalent to

$$-G'(\bar{x})^\top\tilde{\lambda} \in \mathcal{N}_C(\bar{x}),\ \tilde{\lambda} \in \mathcal{N}_K(G(\bar{x})) \quad\Longrightarrow\quad \tilde{\lambda} = 0$$

in case where G is continuously differentiable at $\bar{x}$, see (4.4) as well. This condition is well known as *no nonzero abnormal multiplier constraint qualification* (NNAMCQ) or *generalized Mangasarian–Fromovitz constraint qualification* (GMFCQ) in the literature.

In the subsequently stated example, we interrelate our findings with the results from [5] where a sequential constraint qualification has been introduced for standard nonlinear programs.

Example 5.1. Fix $\ell := p+q$, $K := \mathbb{R}^p_- \times \{0\}$, as well as $C := \mathbb{R}^n$ and let the mapping $G\colon \mathbb{R}^n \to \mathbb{R}^{p+q}$ be continuously differentiable. Furthermore, let $G_1,\ldots,G_{p+q}\colon \mathbb{R}^n \to \mathbb{R}$ be the component mappings associated with G. In this situation, the mapping $\widetilde{\mathcal{M}}$ from above takes the particular form

$$\widetilde{\mathcal{M}}(x,\tilde{y}) = \left\{\sum_{i=1}^{p+q}\tilde{\lambda}_i\nabla G_i(x)\,\middle|\, \begin{aligned} \min(\tilde{\lambda}_i,\tilde{y}_i - G_i(x)) = 0 &\quad \forall i\in\{1,\ldots,p\}\\ \tilde{y}_i - G_i(x) = 0 &\quad \forall i\in\{p+1,\ldots,q\}\end{aligned}\right\}$$

for all $x \in \mathbb{R}^n$ and $\tilde{y} \in \mathbb{R}^{p+q}$. We want to compare the associated AM-regularity condition (which equals dAM-regularity due to $C = \mathbb{R}^n$) with the so-called *cone-continuity property* (CCP for short) from [5, Definition 3.1] which has been shown to be a constraint qualification for standard nonlinear problems. It is based on the mapping $\mathcal{K}\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ given by

$$\forall x\in\mathbb{R}^n\colon\quad \mathcal{K}(x) := \left\{\sum_{i=1}^{p+q}\lambda_i\nabla G_i(x)\,\middle|\,\min(\lambda_i,-G_i(\bar{x})) = 0 \quad \forall i\in\{1,\ldots,p\}\right\}$$

and demands that

$$\limsup_{x\to\bar{x}}\mathcal{K}(x) \subset \mathcal{K}(\bar{x})$$

holds at a given point $\bar{x} \in M$. Note that we have $\widetilde{\mathcal{M}}(\bar{x},0) = \mathcal{K}(\bar{x})$.

Observing that, for each $x \in \mathbb{R}^n$, we have $\mathcal{K}(x) = \widetilde{\mathcal{M}}(x, G(x) - G(\bar{x}))$ while the convergence $G(x) - G(\bar{x}) \to 0$ holds as $x \to \bar{x}$, validity of AM-regularity at $\bar{x}$ yields that CCP holds at $\bar{x}$, too. On the other hand, let CCP hold at $\bar{x}$. If $\{x_k\}_{k\in\mathbb{N}}, \{x_k^*\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ as well as $\{\tilde{y}_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^{p+q}$ are sequences with $x_k \to \bar{x}$, $x_k^* \to x^*$ for some $x^* \in \mathbb{R}^n$, and $\tilde{y}_k \to 0$ such that $x_k^* \in \widetilde{\mathcal{M}}(x_k,\tilde{y}_k)$ holds for each $k \in \mathbb{N}$, then we find a sequence $\{\tilde{\lambda}_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^{p+q}$ such that $x_k^* = \sum_{i=1}^{p+q}\tilde{\lambda}_{k,i}\nabla G_i(x_k)$ and $\min(\tilde{\lambda}_{k,i}, \tilde{y}_{k,i} - G_i(x_k)) = 0$, $i = 1,\ldots,p$, are valid for all $k \in \mathbb{N}$. Let $I(\bar{x}) := \{i \in \{1,\ldots,p\} \,|\, G_i(\bar{x}) = 0\}$ denote the set of indices associated with inequality constraints active at $\bar{x}$. For each $k \in \mathbb{N}$, we have $\tilde{\lambda}_{k,i} \geq 0$ for all $i \in \{1,\ldots,p\}$. Whenever $i \notin I(\bar{x})$ holds, $\tilde{y}_{k,i} - G_i(x_k) > 0$ is valid for sufficiently large $k \in \mathbb{N}$ due to $x_k \to \bar{x}$, $\tilde{y}_k \to 0$, and continuity of G. Thus, we have $\tilde{\lambda}_{k,i} = 0$ for sufficiently large $k \in \mathbb{N}$ and all $i \notin I(\bar{x})$. This particularly yields $\min(\tilde{\lambda}_{k,i}, -G_i(\bar{x})) = 0$ for sufficiently large $k \in \mathbb{N}$ and all $i \in \{1,\ldots,p\}$. Hence, we have shown $x_k^* \in \mathcal{K}(x_k)$ for sufficiently large $k \in \mathbb{N}$. By validity of CCP, $x^* \in \mathcal{K}(\bar{x}) = \widetilde{\mathcal{M}}(\bar{x},0)$ follows, i.e., $\bar{x}$ is AM-regular.

The above investigations show that AM-regularity is equivalent to CCP in the setting of standard nonlinear programming. Let us mention that CCP has also been referred to as AKKT-regularity in the literature which is why the latter is a particular instance of AM-regularity as well.

In the subsequent remark, we address the situation where $K \subset \mathbb{R}^\ell$ is convex and G is continuously differentiable.

Remark 5.2. Assume that $K \subset \mathbb{R}^\ell$ is convex while G is continuously differentiable. Adapting the proof of [14, Proposition 3.3], whenever $\bar{x} \in M$ is a local minimizer of the associated problem (P) where Φ is given as in (5.1), we find sequences $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ and $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ such that $x_k \to \bar{x}$, $\varepsilon_k \to 0$, and

$$\forall k \in \mathbb{N}: \quad \varepsilon_k \in \partial f(x_k) + G'(x_k)^\top (K - G(x_k))^\circ + \mathcal{N}_C(x_k)$$

hold. Note, however, that we cannot replace $(K - G(x_k))^\circ$ by $\mathcal{N}_K(G(x_k))$ in the above formula since $G(x_k)$ does not need to be an element of K in general. Consequently, this sequential concept of stationarity is slightly different from dAM-stationarity. However, it can be used in similar fashion for the derivation of a constraint qualification which guarantees that $\bar{x}$ satisfies the M-stationarity conditions of the associated optimization problem, namely

$$\limsup_{x\to\bar{x}} \widehat{\mathcal{M}}(x) \subset \widehat{\mathcal{M}}(\bar{x})$$

where $\widehat{\mathcal{M}}\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is defined by

$$\forall x \in \mathbb{R}^n: \quad \widehat{\mathcal{M}}(x) := G'(x)^\top (K - G(x))^\circ + \mathcal{N}_C(x),$$

see [14, Corollary 4.8] as well.

5.2 Asymptotic regularity in disjunctive programming

In this section, we take a closer look at so-called *mathematical programs with disjunctive constraints* which are optimization problems of the form

$$\text{(MPDC)} \qquad \begin{aligned} f(x) &\to \min \\ G(x) &\in K \end{aligned}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}$ and $G\colon \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable mappings and $K := \bigcup_{i=1}^p D_i$ is the union of finitely many convex polyhedral sets $D_1, \ldots, D_p \subset \mathbb{R}^m$. Again, we denote its feasible set by M. Such optimization problems have been dealt with e.g. in [10, 12, 17, 19, 30] in terms of first- and second-order optimality conditions as well as suitable constraint qualifications. The model (MPDC) is attractive since it covers numerous classes from structured nonlinear optimization like mathematical programs with complementarity constraints (MPCCs), mathematical programs with vanishing constraints (MPVCs), mathematical programs with switching constraints (MPSCs), or cardinality-constrained mathematical problems (CCMPs), see [30, Section 5] for an overview of these popular classes from disjunctive programming and references to the literature. Noting that (MPDC) is a particular instance of a mathematical program with geometric constraints, we are in position to apply the theory from above to the problem of interest. Noting that no abstract constraints are present in the formulation of (MPDC), we rely on AM-regularity as a constraint qualification for (P). The associated mapping $\mathcal{M}\colon \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is given by

$$\forall x \in \mathbb{R}^n\, \forall y \in \mathbb{R}^m: \quad \mathcal{M}(x, y) = G'(x)^\top \mathcal{N}_K(G(x) - y)$$

in this setting.

Our first result, which is inspired by our observations from Example 5.1, shows that we can rely on the continuity properties of a much simpler map than $\mathcal{M}$ in order to check validity of AM-regularity. The proof of this result exploits some arguments we already used to verify Theorem 3.8.

Theorem 5.3. *Fix a feasible point $\bar{x} \in M$ of (MPDC) and define a set-valued mapping $\mathcal{K}\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ by means of*

$$\forall x \in \mathbb{R}^n\colon \quad \mathcal{K}(x) := G'(x)^\top \mathcal{N}_K(G(\bar{x})).$$

Then $\bar{x}$ is AM-regular if and only if the following condition holds:

$$\limsup_{x \to \bar{x}} \mathcal{K}(x) \subset \mathcal{K}(\bar{x}). \tag{5.3}$$

Proof. We show both implications separately.
[$\Longrightarrow$] Let $\bar{x}$ be AM-regular. Then we have

$$\limsup_{x \to \bar{x}} \mathcal{K}(x) = \limsup_{x \to \bar{x}} \mathcal{M}(x, G(x) - G(\bar{x})) \subset \limsup_{x \to \bar{x},\, y \to 0} \mathcal{M}(x, y) \subset \mathcal{M}(\bar{x}, 0) = \mathcal{K}(\bar{x})$$

by continuity of G.
[$\Longleftarrow$] Assume that (5.3) holds. Furthermore, choose $\{x_k\}_{k \in \mathbb{N}}, \{x_k^*\}_{k \in \mathbb{N}} \subset \mathbb{R}^n$ and $\{y_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^m$ such that $x_k \to \bar{x}$, $x_k^* \to x^*$ for some $x^* \in \mathbb{R}^n$, $y_k \to 0$, as well as $x_k^* \in G'(x_k)^\top \mathcal{N}_K(G(x_k) - y_k)$ for all $k \in \mathbb{N}$ hold. Then we find a sequence $\{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^m$ such that $x_k^* = G'(x_k)^\top \lambda_k$ and $\lambda_k \in \mathcal{N}_K(G(x_k) - y_k)$ are valid for all $k \in \mathbb{N}$. Exploiting that K is the union of finitely many convex polyhedral sets, we can use similar arguments as in the proof of Theorem 3.8 in order to find a convex, polyhedral cone $P \subset \mathbb{R}^m$ which satisfies $P \subset \mathcal{N}_K(G(x_k) - y_k)$ and $\lambda_k \in P$ along a subsequence (without relabelling). The robustness of the limiting normal cone yields $P \subset \mathcal{N}_K(G(\bar{x}))$ due to $G(x_k) - y_k \to G(\bar{x})$ as $k \to \infty$. Thus, we have $\{\lambda_k\}_{k \in \mathbb{N}} \subset \mathcal{N}_K(G(\bar{x}))$ and, consequently, $x_k^* \in \mathcal{K}(x_k)$ for all $k \in \mathbb{N}$. By means of (5.3), we find $x^* \in \mathcal{K}(\bar{x}) = \mathcal{M}(\bar{x}, 0)$, i.e., $\bar{x}$ is AM-regular. □

The following example points out that the assertion of Theorem 5.3 does not need to be true whenever the set K is not disjunctive, i.e., in this setting, (5.3) does not provide a constraint qualification in general.

Example 5.4. We investigate the setting where $G\colon \mathbb{R} \to \mathbb{R}^2$ is given by $G(x) := (x, 0)$ for all $x \in \mathbb{R}$ and $K \subset \mathbb{R}^2$ is given by $K := \{y \in \mathbb{R}^2 \,|\, y_2 \geq y_1^2\}$. Obviously, K is not of disjunctive structure. The only feasible point of the associated constraint system $G(x) \in K$ is $\bar{x} := 0$. The mapping $\mathcal{K}$ from Theorem 5.3 is given by $\mathcal{K}(x) \equiv \{0\}$ in this situation which is why the condition (5.3) holds trivially. On the other hand, one can check

$$\forall k \in \mathbb{N}\colon \quad \mathcal{M}\left(\tfrac{1}{k}, (0, -\tfrac{1}{k^2})\right) = \mathbb{R}_+, \qquad \mathcal{M}\left(-\tfrac{1}{k}, (0, -\tfrac{1}{k^2})\right) = \mathbb{R}_-$$

and $\mathcal{M}(\bar{x}, (0,0)) = \{0\}$, i.e., $\bar{x}$ is not AM-regular.
Let us mention that one could also check the validity of the constraint qualification from Remark 5.2 which applies to the present situation since K is convex and G is continuously differentiable. The latter, however, is violated as well.

In the literature on disjunctive programs, there exist two other reasonable constraint qualifications which we will recall below, see [17, Definition 6].

Definition 5.5. Fix a feasible point $\bar{x} \in M$. We define the *linearization cone* to M at $\bar{x}$ as stated below:

$$\mathcal{L}_M(\bar{x}) := \{d \in \mathbb{R}^n \,|\, G'(\bar{x})d \in \mathcal{T}_K(G(\bar{x}))\}.$$

We say that

(a) the *generalized Abadie constraint qualification* (GACQ) holds at the point $\bar{x}$ whenever the relation $\mathcal{T}_M(\bar{x}) = \mathcal{L}_M(\bar{x})$ is valid,

(b) the *generalized Guignard constraint qualification* (GGCQ) holds at the point $\bar{x}$ whenever the relation $\widehat{\mathcal{N}}_M(\bar{x}) = \mathcal{L}_M(\bar{x})^\circ$ is valid.

Let us briefly mention that the linearization cone introduced above is, by the special structure of K, also polyhedral in the sense that it is the union of finitely many convex polyhedral cones. This is a simple consequence of

$$\mathcal{T}_K(G(\bar{x})) = \bigcup_{i\in J(\bar{x})} \mathcal{T}_{D_i}(G(\bar{x}))$$

where we used $J(\bar{x}) := \{i \in \{1,\ldots,p\} \mid G(\bar{x}) \in D_i\}$ and $\bar{x} \in M$, see [8, Table 4.1]. It has been shown in [17, Theorem 7] that whenever $\bar{x} \in M$ is a local minimizer of (MPDC) where GGCQ holds, then $\bar{x}$ is already an M-stationary point. As pointed out in [17], this result does not need to hold anymore whenever continuous differentiability of f is replaced by local Lipschitz continuity.

Clearly, one could also define GACQ and GGCQ in the situation where K is a general closed set. In this case, however, GGCQ on its own does not necessarily provide a constraint qualification ensuring M-stationarity of local minimizers. As pointed out in [11, Proposition 3], some additional metric subregularity of a linearized feasibility mapping is needed in this more general situation, see [20] as well, and the latter is inherent whenever K is of disjunctive structure due to Robinson's classical result on the inherent calmness of polyhedral set-valued mappings.

Let us now focus on (MPDC) again. Let us fix one of its feasible points $\bar{x} \in M$. It is well known that metric subregularity of the feasibility mapping $\Phi\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, given by $\Phi(x) = G(x) - K$ for all $x \in \mathbb{R}^n$ in the present situation, at $(\bar{x}, 0)$ implies validity of GACQ which, in turn, implies validity of GGCQ, see [17, formula (13)]. Noting that (MPDC) covers standard nonlinear problems while AM-regularity coincides with CCP in this setting, see Example 5.1, the considerations from [5, Section 4.2] show that validity of GACQ at $\bar{x}$ is not sufficient for AM-regularity of $\bar{x}$. In the following example, we show that validity of AM-regularity does not need to imply validity of GGCQ.

Example 5.6. Let us consider the mapping $G\colon \mathbb{R} \to \mathbb{R}^2$ given by $G(x) := (x, x^3)$ for all $x \in \mathbb{R}$ as well as the disjunctive set $K := D_1 \cup D_2$ where $D_1 := \mathbb{R}_- \times \mathbb{R}$ and $D_2 := \mathbb{R}_+ \times \mathbb{R}_-$ hold. In this situation, we have $M = (-\infty, 0]$. Let us fix $\bar{x} := 0$. One can easily check that $\mathcal{T}_K(G(\bar{x})) = K$ holds. We conclude

$$\mathcal{L}_M(\bar{x}) = \{d \in \mathbb{R} \mid (d, 0) \in K\} = \mathbb{R},$$

and this shows that GACQ and GGCQ are violated at $\bar{x}$ since we have $\mathcal{T}_M(\bar{x}) = \mathbb{R}_-$. On the other hand, we have

$$G'(x)^\top \mathcal{N}_K(G(\bar{x})) = \{\lambda_1 + 3x^2\lambda_2 \mid (\lambda_1, \lambda_2) \in (\mathbb{R}_+ \times \{0\}) \cup (\{0\} \times \mathbb{R}_+)\} = \mathbb{R}_+$$

for each $x \in \mathbb{R}$ and, thus, due to Theorem 5.3, $\bar{x}$ is AM-regular.

The above considerations show that AM-regularity for disjunctive programs is not related to the constraint qualifications GACQ and GGCQ. In the particular case of MPCCs, this already has been observed in [35, Section 4]. Due to the results of Section 5.1, AM-regularity is generally weaker than NNAMCQ, i.e.,

$$G'(\bar{x})^\top \lambda = 0,\ \lambda \in \mathcal{N}_K(G(\bar{x})) \quad\Longrightarrow\quad \lambda = 0,$$

and the later is, again, weaker than the problem-tailored version of the linear independence constraint qualification discussed in [30].

5.3 VARIATIONAL CALCULUS AND ASYMPTOTIC REGULARITY

In this section, we are going to show how the concept of asymptotic regularity can be used to establish some fundamental calculus rules for limiting normals and the limiting coderivative.

First, we show that asymptotic regularity may serve as a sufficient condition for the validity of the intersection rule for limiting normals.

Theorem 5.7. *Let $K, C \subset \mathbb{R}^n$ be closed sets and fix $\bar{x} \in K \cap C$. Suppose that the qualification condition*

$$\limsup_{x\to\bar{x},\ x'\to\bar{x}} \left(\mathcal{N}_K(x) + \mathcal{N}_C(x')\right) \subset \mathcal{N}_K(\bar{x}) + \mathcal{N}_C(\bar{x}) \tag{5.4}$$

holds. Then we have

$$\mathcal{N}_{K\cap C}(\bar{x}) \subset \mathcal{N}_K(\bar{x}) + \mathcal{N}_C(\bar{x}).$$

Proof. This result is a simple consequence of our considerations from Section 5.1 and Theorem 4.7 when fixing $G\colon \mathbb{R}^n \to \mathbb{R}^n$ to be the identity mapping. Indeed, validity of (5.4) is equivalent to the validity of dAM-regularity for the constraint system $M := K \cap C$. □

Following the structure of the books [31, 32], the intersection rule provides the fundamental basis of the overall variational calculus. Classically, validity of the intersection rule at some point $\bar{x} \in K \cap C$ is guaranteed by the so-called normal qualification condition

$$\mathcal{N}_K(\bar{x}) \cap (-\mathcal{N}_C(\bar{x})) = \{0\},$$

and the latter is equivalent to metric regularity of the mapping $x \rightrightarrows (x - K) \times (x - C)$ at $(\bar{x}, (0, 0))$. Following the arguments from Section 5.1, metric subregularity of this mapping at $(\bar{x}, (0, 0))$ is already enough to guarantee validity of the intersection rule. Apart from these classical results, Theorem 5.7 shows that the intersection rule is also valid in the presence of the *asymptotic stability condition* (5.4) which originates from the notion of asymptotic regularity. Keeping in mind our results from Section 3.2, (5.4) is independent of the aforementioned metric subregularity condition and, thus, provides a new approach to the variational calculus. Exemplary, we will show how the coderivative sum and chain rule can be derived in the presence of asymptotic stability conditions.

Let us note that validity of (5.4) is equivalent to

$$\limsup_{x\to\bar{x},\ x'\to\bar{x}} \left(\widehat{\mathcal{N}}_K(x) + \widehat{\mathcal{N}}_C(x')\right) \subset \mathcal{N}_K(\bar{x}) + \mathcal{N}_C(\bar{x})$$

by definition of the limiting normal cone. Thus, a direct proof of the intersection rule for limiting normals under validity of (5.4) can be obtained from the *fuzzy* intersection rule for regular normals, see [31, Lemma 3.1], and a simple diagonal sequence argument.

Next, we will inspect how validity of the coderivative sum rule can be guaranteed under an asymptotic stability condition. Let us mention that in [31, Theorem 3.10], [32, Theorem 3.9], or [38, Theorem 10.41], the coderivative sum rule has been derived under validity of the Mordukhovich criterion. In order to proceed, we fix set-valued mappings $S_1, S_2 \colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with closed graphs and consider their sum mapping $S\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ given by

$$\forall x \in \mathbb{R}^n\colon \quad S(x) := S_1(x) + S_2(x).$$

Furthermore, we make use of the *intermediate* mapping $\Xi\colon \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^m \times \mathbb{R}^m$ given by

$$\forall x \in \mathbb{R}^n\ \forall y \in \mathbb{R}^m\colon \quad \Xi(x, y) := \{(y_1, y_2) \in \mathbb{R}^m \times \mathbb{R}^m \,|\, y_1 + y_2 = y,\ y_1 \in S_1(x),\ y_2 \in S_2(x)\}. \tag{5.5}$$

Observe that we have $\operatorname{dom}\Xi = \operatorname{gph} S$.

Theorem 5.8. *Fix some point $(\bar{x}, \bar{y}) \in \operatorname{gph} S$. Then the following assertions hold.*

(a) Assume that there exists $(\bar{y}_1, \bar{y}_2) \in \Xi(\bar{x}, \bar{y})$ such that Ξ is inner semicontinuous at $((\bar{x}, \bar{y}), (\bar{y}_1, \bar{y}_2))$. Furthermore, let the qualification condition

$$\limsup_{\substack{(x_1,y_1)\to(\bar{x},\bar{y}_1)\\(x_2,y_2)\to(\bar{x},\bar{y}_2)\\(y_1^*,y_2^*)\to(\bar{y}_1^*,\bar{y}_2^*)}} \left(D^*S_1(x_1,y_1)(y_1^*) + D^*S_2(x_2,y_2)(y_2^*)\right) \subset D^*S_1(\bar{x},\bar{y}_1)(\bar{y}_1^*) + D^*S_2(\bar{x},\bar{y}_2)(\bar{y}_2^*) \tag{5.6}$$

hold for all $\bar{y}_1^*, \bar{y}_2^* \in \mathbb{R}^m$*. Then, for all* $y^* \in \mathbb{R}^m$*, we have*

$$D^*S(\bar{x},\bar{y})(y^*) \subset D^*S_1(\bar{x},\bar{y}_1)(y^*) + D^*S_2(\bar{x},\bar{y}_2)(y^*).$$

(b) *Assume that* Ξ *is inner semicompact at* $(\bar{x},\bar{y})$*. Furthermore, let the qualification condition* (5.6) *hold for each* $(\bar{y}_1,\bar{y}_2) \in \Xi(\bar{x},\bar{y})$ *and all* $\bar{y}_1^*, \bar{y}_2^* \in \mathbb{R}^m$*. Then, for all* $y^* \in \mathbb{R}^m$*, we have*

$$D^*S(\bar{x},\bar{y})(y^*) \subset \bigcup_{(\bar{y}_1,\bar{y}_2)\in\Xi(\bar{x},\bar{y})} \big(D^*S_1(\bar{x},\bar{y}_1)(y^*) + D^*S_2(\bar{x},\bar{y}_2)(y^*)\big).$$

Proof. The proof essentially relies on the normal cone intersection rule from Theorem 5.7 and adapts the arguments used to verify [31, Theorem 3.10].

(a) Fix $x^* \in D^*S(\bar{x},\bar{y})(y^*)$ for an arbitrarily chosen $y^* \in \mathbb{R}^m$. Mimicking the proof of [31, Theorem 3.10(i)] and exploiting the postulated inner semicontinuity of Ξ, we find the relation $(x^*,-y^*,-y^*) \in \mathcal{N}_{\Omega_1\cap\Omega_2}(\bar{x},\bar{y}_1,\bar{y}_2)$ where we used the closed sets $\Omega_1,\Omega_2 \subset \mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m$ given by $\Omega_i := \{(x,y_1,y_2)\,|\,y_i \in S_i(x)\}$, $i=1,2$.

Let us now show that (5.6) is sufficient for the applicability of the normal cone intersection rule from Theorem 5.7 for the estimation of $\mathcal{N}_{\Omega_1\cap\Omega_2}(\bar{x},\bar{y}_1,\bar{y}_2)$ from above. Therefore, we show that (5.4) holds for the situation at hand. Choose sequences $\{x_k^*\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$, $\{y_{k,1}^*\}_{k\in\mathbb{N}}, \{y_{k,2}^*\}_{k\in\mathbb{N}} \subset \mathbb{R}^m$ as well as $\{x_{k,1}\}_{k\in\mathbb{N}}, \{x_{k,2}\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ and $\{y_{k,1}^1\}_{k\in\mathbb{N}}, \{y_{k,2}^1\}_{k\in\mathbb{N}}, \{y_{k,1}^2\}_{k\in\mathbb{N}}, \{y_{k,2}^2\}_{k\in\mathbb{N}} \subset \mathbb{R}^m$ such that $x_{k,1}\to\bar{x}$, $x_{k,2}\to\bar{x}$, $y_{k,1}^1\to\bar{y}_1$, $y_{k,2}^1\to\bar{y}_2$, $y_{k,1}^2\to\bar{y}_1$, $y_{k,2}^2\to\bar{y}_2$, $x_k^*\to x^*$ for some $x^*\in\mathbb{R}^n$, $y_{k,1}^*\to y_1^*$ as well as $y_{k,2}^*\to y_2^*$ for some $y_1^*,y_2^*\in\mathbb{R}^m$, and

$$(x_k^*, y_{k,1}^*, y_{k,2}^*) \in \mathcal{N}_{\Omega_1}(x_{k,1}, y_{k,1}^1, y_{k,2}^1) + \mathcal{N}_{\Omega_2}(x_{k,2}, y_{k,1}^2, y_{k,2}^2)$$

for all $k\in\mathbb{N}$ hold. By construction of Ω_1 and Ω_2, this guarantees the existence of sequences $\{x_{k,1}^*\}_{k\in\mathbb{N}}, \{x_{k,2}^*\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ such that $x_k^* = x_{k,1}^* + x_{k,2}^*$ as well as $(x_{k,i}^*, y_{k,i}^*) \in \mathcal{N}_{\operatorname{gph} S_i}(x_{k,i}, y_{k,i}^i)$, $i=1,2$, for all $k\in\mathbb{N}$ hold. This leads to

$$\forall k\in\mathbb{N}:\quad x_k^* \in D^*S_1(x_{k,1},y_{k,1}^1)(-y_{k,1}^*) + D^*S_2(x_{k,2},y_{k,2}^2)(-y_{k,2}^*).$$

Due to validity of (5.6), we find $x^* \in D^*S_1(\bar{x},\bar{y}_1)(-y_1^*) + D^*S_2(\bar{x},\bar{y}_2)(-y_2^*)$, i.e., there are points $x_1^*, x_2^* \in \mathbb{R}^n$ with $(x_i^*, y_i^*) \in \mathcal{N}_{\operatorname{gph} S_i}(\bar{x},\bar{y}_i)$, $i=1,2$, and $x^* = x_1^* + x_2^*$. Particularly, we have

$$(x^*, y_1^*, y_2^*) \in \mathcal{N}_{\Omega_1}(\bar{x},\bar{y}_1,\bar{y}_2) + \mathcal{N}_{\Omega_2}(\bar{x},\bar{y}_1,\bar{y}_2).$$

Due to the above considerations, we can apply Theorem 5.7 in order to obtain

$$(x^*, -y^*, -y^*) \in \mathcal{N}_{\Omega_1}(\bar{x},\bar{y}_1,\bar{y}_2) + \mathcal{N}_{\Omega_2}(\bar{x},\bar{y}_1,\bar{y}_2).$$

Now, the claim follows by definition of the sets Ω_1 and Ω_2, cf. [31, proof of Theorem 3.10].

(b) Fixing $x^* \in D^*S(\bar{x},\bar{y})(y^*)$ for an arbitrarily chosen $y^*\in\mathbb{R}^m$, the inner semicompactness of Ξ at $(\bar{x},\bar{y})$ can be used to obtain

$$(x^*, -y^*, -y^*) \in \bigcup_{(\bar{y}_1,\bar{y}_2)\in\Xi(\bar{x},\bar{y})} \mathcal{N}_{\Omega_1\cap\Omega_2}(\bar{x},\bar{y}_1,\bar{y}_2),$$

see [31, proof of Theorem 3.10(ii)] as well. Proceeding as in the proof of (a), the claim follows.

□

Finally, we would like to take a look at the coderivative chain rule. Therefore, let us consider set-valued mappings $T_1\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and $T_2\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^\ell$ as well as their composition $T\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^\ell$ given by

$$\forall x \in \mathbb{R}^n\colon \quad T(x) := \bigcup_{y \in T_1(x)} T_2(y).$$

Again, we will make use of an *intermediate* mapping $\Theta\colon \mathbb{R}^n \times \mathbb{R}^\ell \rightrightarrows \mathbb{R}^m$ which is given as stated below:

$$\forall x \in \mathbb{R}^n\, \forall z \in \mathbb{R}^\ell\colon \quad \Theta(x,z) := \{y \in T_1(x) \,|\, z \in T_2(y)\}.$$

Once more, we note that $\operatorname{gph} T = \operatorname{dom} \Theta$ is valid. Similar as in [31, Theorem 3.13] or [32, Theorem 3.11], we will derive the coderivative chain rule from the coderivative sum rule. Exploiting Theorem 5.8 for that purpose, we will see that validity of the chain rule can be guaranteed in the presence of an asymptotic stability condition. In [31, 32] or [38, Theorem 10.37], a condition related to the Mordukhovich criterion has been imposed for that purpose.

Theorem 5.9. *Fix some point $(\bar x, \bar z) \in \operatorname{gph} T$. Then the following assertions hold.*

(a) Assume that there exists $\bar y \in \Theta(\bar x, \bar z)$ such that Θ is inner semicontinuous at $((\bar x, \bar z), \bar y)$. Furthermore, let the qualification condition

$$\limsup_{\substack{(x,y^1)\to(\bar x,\bar y)\\ (y^2,z)\to(\bar y,\bar z)\\ (x^*,z^*)\to(\bar x^*,\bar z^*)}} \big(D^*T_2(y^2,z)(z^*) - (D^*T_1(x,y^1))^{-1}(x^*)\big) \subset D^*T_2(\bar y,\bar z)(\bar z^*) - (D^*T_1(\bar x,\bar y))^{-1}(\bar x^*) \tag{5.7}$$

hold for each $\bar x^* \in \mathbb{R}^n$ and $\bar z^* \in \mathbb{R}^\ell$. Then, for each $z^* \in \mathbb{R}^\ell$, we have

$$D^*T(\bar x,\bar z)(z^*) \subset \bigcup_{y^* \in D^*T_2(\bar y,\bar z)(z^*)} D^*T_1(\bar x,\bar y)(y^*).$$

(b) Assume that Θ is inner semicompact at $(\bar x, \bar z)$. Furthermore, let the qualification condition (5.7) hold for each $\bar y \in \Theta(\bar x, \bar z)$, $\bar x^* \in \mathbb{R}^n$, and $\bar z^* \in \mathbb{R}^\ell$. Then, for each $z^* \in \mathbb{R}^\ell$, we have

$$D^*T(\bar x,\bar z)(z^*) \subset \bigcup_{\bar y \in \Theta(\bar x,\bar z)} \bigcup_{y^* \in D^*T_2(\bar y,\bar z)(z^*)} D^*T_1(\bar x,\bar y)(y^*).$$

Proof. We only show validity of statement (a). Assertion (b) can be obtained in analogous way. For the proof of (a), we exploit the idea from [31, proof of Theorem 3.13] and consider the mapping $S\colon \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^\ell$ given by

$$\forall x \in \mathbb{R}^n\, \forall y \in \mathbb{R}^m\colon \quad S(x,y) := \Delta_{\operatorname{gph} T_1}(x,y) + T_2(y).$$

Then [31, Theorem 1.64] yields

$$D^*T(\bar x,\bar z)(z^*) \subset \{x^* \in \mathbb{R}^n \,|\, (x^*,0) \in D^*S((\bar x,\bar y),\bar z)(z^*)\} \tag{5.8}$$

since Θ is inner semicontinuous at $((\bar x,\bar z),\bar y)$.

In order to estimate the coderivative of S from above, we make use of Theorem 5.8. Therefore, we introduce $S_1, S_2\colon \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^\ell$ by means of

$$\forall x \in \mathbb{R}^n\, \forall y \in \mathbb{R}^m\colon \quad S_1(x,y) := \Delta_{\operatorname{gph} T_1}(x,y) \qquad S_2(x,y) := T_2(y).$$

Next, we show that (5.6) holds in the present setting. Therefore, we make use of the formulas

$$D^*S_1((x,y),z)(z^*) = \begin{cases} \mathcal{N}_{\operatorname{gph} T_1}(x,y) & z = 0 \\ \varnothing & z \neq 0 \end{cases}$$
$$D^*S_2((x,y),z)(z^*) = \{0\} \times D^*T_2(y,z)(z^*)$$

which, by elementary calculations, hold for all $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and $z, z^* \in \mathbb{R}^\ell$. In order to infer validity of (5.6), we thus need to verify validity of

$$\text{(5.9)} \qquad \begin{aligned} \limsup_{\substack{(x,y^1)\to(\bar x,\bar y)\\ (y^2,z)\to(\bar y,\bar z)\\ z^*\to\bar z^*}} & \left(\mathcal{N}_{\operatorname{gph} T_1}(x,y^1) + \{0\} \times D^*T_2(y^2,z)(z^*)\right) \\ & \subset \mathcal{N}_{\operatorname{gph} T_1}(\bar x,\bar y) + \{0\} \times D^*T_2(\bar y,\bar z)(\bar z^*) \end{aligned}$$

for all $\bar z^* \in \mathbb{R}^\ell$. Thus, for some point $\bar z^* \in \mathbb{R}^\ell$, we fix sequences $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$, $\{y_k^1\}_{k\in\mathbb{N}}, \{y_k^2\}_{k\in\mathbb{N}} \subset \mathbb{R}^m$, $\{z_k\}_{k\in\mathbb{N}}, \{z_k^*\}_{k\in\mathbb{N}} \subset \mathbb{R}^\ell$, as well as $\{x_k^*\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ and $\{y_k^*\}_{k\in\mathbb{N}} \subset \mathbb{R}^m$ such that $x_k \to \bar x$, $y_k^1 \to \bar y$, $y_k^2 \to \bar y$, $z_k \to \bar z$, $z_k^* \to \bar z^*$, $x_k^* \to x^*$ and $y_k^* \to y^*$ for some $x^* \in \mathbb{R}^n$ and $y^* \in \mathbb{R}^m$, as well as

$$(x_k^*, y_k^*) \in \mathcal{N}_{\operatorname{gph} T_1}(x_k, y_k^1) + \{0\} \times D^*T_2(y_k^2, z_k)(z_k^*)$$

for all $k \in \mathbb{N}$ hold. Keeping the definitions of the coderivative and the inverse mapping in mind, we find

$$\forall k \in \mathbb{N}\colon \quad y_k^* \in D^*T_2(y_k^2, z_k)(z_k^*) - (D^*T_1(x_k, y_k^1))^{-1}(x_k^*).$$

Inspecting (5.7), we obtain

$$y^* \in D^*T_2(\bar y,\bar z)(\bar z^*) - (D^*T_1(\bar x,\bar y))^{-1}(x^*),$$

i.e., $(x^*, y^*) \in \mathcal{N}_{\operatorname{gph} T_1}(\bar x,\bar y) + \{0\} \times D^*T_1(\bar x,\bar y)(\bar z^*)$. This shows validity of (5.9). Observing that the intermediate mapping Ξ from (5.5) is given by

$$\begin{aligned} \forall x \in \mathbb{R}^n\, \forall y \in \mathbb{R}^m\, \forall z \in \mathbb{R}^\ell\colon \quad \Xi(x,y,z) &= \{(0,z) \mid y \in T_1(x),\ z \in T_2(y)\} \\ &= \begin{cases} \{(0,z)\} & ((x,z),y) \in \operatorname{gph}\Theta \\ \varnothing & \text{otherwise} \end{cases} \end{aligned}$$

in this situation and, thus, is trivially inner semicontinuous at $((\bar x,\bar y,\bar z),(0,\bar z))$ by inner semicontinuity of Θ at $((\bar x,\bar z),\bar y)$, we can apply assertion (a) of Theorem 5.8 in order to find

$$D^*S((\bar x,\bar y),\bar z)(z^*) \subset \mathcal{N}_{\operatorname{gph} T_1}(\bar x,\bar y) + \{0\} \times D^*T_2(\bar y,\bar z)(z^*).$$

Due to (5.8), the desired estimate is obtained. □

6 CONCLUSIONS

In this paper, we introduced a new sequential constraint qualification, namely AM-regularity, for nonsmooth optimization problems. This concept has been shown to be generally weaker than metric regularity of the associated feasibility mapping while it is not related to metric subregularity of the latter. AM-regularity turned out to be a condition which is sufficient for the validity of the pre-image rule from the limiting variational calculus.

We clarified how abstract constraints can be incorporated into the framework of AM-regularity and presented some associated consequences for optimization problems with geometric constraints. Our findings were applied to mathematical programs with disjunctive constraints as well. This revealed that AM-regularity is a generalization of the so-called cone-continuity property (also referred to as AKKT-regularity) for standard nonlinear problems and mathematical programs with complementarity constraints, see [5, 35]. Keeping e.g. [1, 4, 14, 35] in mind, constraint qualifications of AM-regularity-type can be used to ensure convergence of different types of solution algorithms like augmented Lagrangian or relaxation methods to stationary points of several classes of optimization problems. It is a promising subject of future research to investigate more general algorithmic consequences of AM-regularity.

We finalized the paper by showing that asymptotic regularity provides a new approach to the limiting variational calculus. It remains to be seen whether the resulting new asymptotic stability conditions which ensure validity of the normal cone intersection rule, the coderivative sum rule, or the coderivative chain rule can be used profitably in the context of variational analysis. Following ideas from [13, 18], it might be possible to introduce a reasonable concept of *directional* AM-regularity. Such a concept may provide qualification conditions for optimization problems of type (P) and the *directional* limiting variation calculus which are even weaker that the criteria inferred from AM-regularity.

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